Version of 16.2.17

Errata and addenda for Volume 1, 2000 printing

I collect here known errors and omissions, with their discoverers, in my book *Measure Theory* (see my web page, `http://www1.essex.ac.uk/maths/people/fremlin/mt.htm`).

p 15 l 16 (Notes to §111): for '$\mathcal{A}$ and $\Sigma_{\mathcal{A}}$ belong to $\mathcal{P}X$' read '$\mathcal{A}$ and $\Sigma_{\mathcal{A}}$ belong to $\mathcal{P}(\mathcal{P}X)$'.

p 16 l 29 (112Bd) Following 'Now (X,Σ,μ) is a measure space', *add* 'I will call measures of this kind **point-supported**'.

p 17 l 35 (112Db): *add* 'I will call $\mathcal{N}$ the **null ideal** of the measure μ'.

p 17 l 28 (112Da) for '$E\subseteq\Sigma$' read '$E\in\Sigma$'. [P.Wallace Thompson]

p 18 l 17 (112D) Add new part:

(g) When f and g are real-valued functions defined on conegligible subsets of a measure space, I will write $f=_{\text{a.e.}} g$, $f\le_{\text{a.e.}} g$ or $f\ge_{\text{a.e.}} g$ to mean, respectively,

$$f=g \text{ a.e., that is, } \{x:x\in\operatorname{dom}(f)\cap\operatorname{dom}(g),\ f(x)=g(x)\} \text{ is conegligible,}$$

$$f\le g \text{ a.e., that is, } \{x:x\in\operatorname{dom}(f)\cap\operatorname{dom}(g),\ f(x)\le g(x)\} \text{ is conegligible,}$$

$$f\ge g \text{ a.e., that is, } \{x:x\in\operatorname{dom}(f)\cap\operatorname{dom}(g),\ f(x)\ge g(x)\} \text{ is conegligible.}$$

p 18 l 17 112E-112F (on image measures) have been moved to §234.

p 19 l 5 The exercises to §112 have been rearranged, as follows: 112Xd has been moved to 234E, 112Xe has been deleted (the material is now in §234), 112Xf is now 112Xd, 112Yb-112Yf are now 112Ya-112Ye.

p 19 l 18 (Part (iii) of exercise 112Xf, now 112Xd): for '$\mu(\bigcup_{n\in\mathbb{N}}\bigcap_{m\ge n}E_m)$' read '$\bigcup_{n\in\mathbb{N}}\bigcap_{m\ge n}E_m$'. (A.Andretta)

p 19 l 20 (112X) Add new exercises:

(e) Let (X,Σ,μ) be a measure space, and Φ the set of real-valued functions whose domains are conegligible subsets of X. (i) Show that $\{(f,g):f,\,g\in\Phi,\ f\le_{\text{a.e.}} g\}$ and $\{(f,g):f,\,g\in\Phi,\ f\ge_{\text{a.e.}} g\}$ are reflexive transitive relations on Φ, each the inverse of the other. (ii) Show that $\{(f,g):f,\,g\in\Phi,\ f=_{\text{a.e.}} g\}$ is their intersection, and is an equivalence relation on Φ.

(f) Let (X,Σ,μ) be a measure space, Y a set, and $\phi:X\to Y$ a function. Set $\mathrm{T}=\{F:F\subseteq Y,\ \phi^{-1}[F]\in\Sigma\}$ and $\nu F=\mu\phi^{-1}[F]$ for $F\in\mathrm{T}$. Show that ν is a measure on Y.

p 19 l 32 Exercise 112Yd (now 112Yc) is wrong as written, and should be replaced by

(d) Let X be a set and Σ a σ-algebra of subsets of X.

(i) Suppose that $\nu_0,\dots,\nu_n$ are measures on X, all with domain Σ. Set

$$\mu E=\inf\{\textstyle\sum_{i=0}^n\nu_iF_i:F_0,\dots,F_n\in\Sigma,\ E\subseteq\bigcup_{i\le n}F_i\}$$

for $E\in\Sigma$. Show that μ is a measure on X.

(ii) Let N be a non-empty family of measures on X, all with domain Σ. Set

$$\mu E=\inf\{\sum_{n=0}^\infty\nu_nF_n:\langle\nu_n\rangle_{n\in\mathbb{N}} \text{ is a sequence in N},\ \langle F_n\rangle_{n\in\mathbb{N}} \text{ is a sequence in } \Sigma,\ E\subseteq\bigcup_{n\in\mathbb{N}}F_n\}$$

for $E\in\Sigma$. Show that μ is a measure on X.

(iii) Let N be a non-empty family of measures on X, all with domain Σ, and suppose that there is some $\tilde\nu\in\mathrm{N}$ such that $\tilde\nu X<\infty$. Set

$$\mu E=\inf\{\sum_{i=0}^n\nu_iF_i:n\in\mathbb{N},\,\nu_0,\dots,\nu_n\in\mathrm{N},\,F_0,\dots,F_n\in\Sigma,E\subseteq\bigcup_{i\le n}F_i\}$$

for $E\in\Sigma$. Show that μ is a measure on X.

(iv) Suppose, in (iii), that N is downwards-directed, that is, for any ν_1, $\nu_2\in\mathrm{N}$ there is a $\nu\in\mathrm{N}$ such that $\nu E\le\min(\nu_1E,\nu_2E)$ for every $E\in\Sigma$. Show that $\mu E=\inf_{\nu\in\mathrm{N}}\nu E$ for every $E\in\Sigma$.

(v) Show that in all the cases (i)-(iii) the measure μ constructed is the greatest measure with domain Σ such that $\mu E\le\inf_{\nu\in\mathrm{N}}\nu E$ for every $E\in\Sigma$.

p 20 l 11 (112Y) Add new exercise:

(f) Let X be a set and μ, ν two measures on X, with domains Σ, T respectively. Set $\Lambda=\Sigma\cap\mathrm{T}$ and define $\lambda:\Lambda\to[0,\infty]$ by setting $\lambda E=\mu E+\nu E$ for every $E\in\Lambda$. Show that (X,Λ,λ) is a measure space.

p 23 l 36 (Exercise 113Yd): we need to suppose that $\lambda\emptyset=0$. [P.W.T.]

p 24 l 16 (Exercise 113Yh): for '$\mu_*:\mathcal{P}X\to[0,\infty[$' read '$\mu_*:\mathcal{P}X\to[0,\infty]$'. [P.W.T.]

p 24 l 20 (Exercise 113Yi): for 'every $A\in\mathcal{A}$' read 'every $E\in\mathcal{A}$'.

p 24 l 29 (Exercise 113Yj): for '$\Sigma=\{E:E\in\mathrm{T},\,\theta E=\theta(E\cap A)+\theta(E\setminus A)$ for every $A\in\mathrm{T}\}$' read '$\Sigma=\{E:E\in\mathrm{T},\,\theta A=\theta(A\cap E)+\theta(A\setminus E)$ for every $A\in\mathrm{T}\}$'.

p 24 l 30 (113Y) Add new exercise:

(k) Let X, $\tau:\mathcal{P}X\to[0,\infty]$ and θ be as in 113Yd; let μ be the measure defined by Carathéodory's method from θ, and Σ the domain of μ. Suppose that $E\subseteq X$ is such that $\theta(C\cap E)+\theta(C\setminus E)\le\lambda C$ whenever $C\subseteq X$ is such that $0<\lambda C<\infty$. Show that $E\in\Sigma$.

p 26 l 30 (part (a-iv) of the proof of 114D): for '$\lim_{m\to\infty}\sum_{m=0}^M\lambda I_{k_m,l_m}$' read '$\lim_{M\to\infty}\sum_{m=0}^M\lambda I_{k_m,l_m}$'. [P.W.T.]

p 28 l 22 (part (a) of the proof of 114G): the **Q** following '$G'=\bigcup_{(q,q')\in K}[q,q'[$ is measurable' should be moved three lines down, to follow 'G is measurable'. [D.Werner]

p 30 l 8 (Exercise 114Yh): for '$\nu:\mathcal{B}\to[0,\infty[$' read '$\nu:\mathcal{B}\to[0,\infty]$'. [P.W.T.]

p 30 l 19 (114Y) Add new exercise:

(l) Write μ for Lebesgue measure on $\mathbb{R}$. Show that there is a countable family $\mathcal{F}$ of Lebesgue measurable subsets of $\mathbb{R}$ such that whenever μE is defined and finite, and $\epsilon>0$, there is an $F\in\mathcal{F}$ such that $\mu(E\triangle F)\le\epsilon$.

p 31 ll 33-39 (part (b) of the proof of 115B) In the formulae $\lambda(I_j\cap H_\xi)$ (twice), $\lambda(I_j\cap H_{\alpha_{r+1}})$ and $\lambda(I_j\cap H_\gamma)$, read '$\lambda_{r+1}$' for '$\lambda$'. [P.W.T.]

p 33 l 18 (part (e) of the proof of 115B, mislabeled (d)): for '$(1+\epsilon)\sum_{j=0}^n\lambda_{r+1}(I_j\cap H_{\beta_{r+1}})$' read '$(1+\epsilon)\sum_{j=0}^\infty\lambda_{r+1}(I_j\cap H_{\beta_{r+1}})$'. [Georg Meyer]

p 34 l 3 (statement of Lemma 115F): for '$i\le m$' read '$i\le r$'.

p 34 l 34 (part (a) of the proof of 115G): the **Q** following '$G'=\bigcup_{(r,s)\in K}[r,s[$ is measurable' should be moved three lines down, to follow 'G is measurable'. [D.W.]

p 35 l 8 (part (d) of the proof of 115G): for '$\inf_{\epsilon>0}\prod_{i=1}^r(\beta_i-\alpha_1+\epsilon)$' read '$\inf_{\epsilon>0}\prod_{i=1}^r(\beta_i-\alpha_i+\epsilon)$'. [G.M.]

p 35 l 37 Exercise 115Ya has been re-written, as follows:

(i) Suppose that M is a strictly positive integer and k_i, l_i are integers for $1\le i\le r$. Set $\alpha_i=k_i/M$ and $\beta_i=l_i/M$ for each i, and $I=[a,b[$. Show that $\lambda I=\#(J)/M^r$, where J is

$\{z : z \in \mathbb{Z}^r,\ \frac{1}{M}z \in I\}$. (ii) Show that if a half-open interval $I \subseteq \mathbb{R}^r$ is covered by a *finite* sequence $I_0, \dots, I_m$ of half-open intervals, and all the coordinates involved in specifying the intervals I, $I_0, \dots, I_m$ are rational, then $\lambda I \le \sum_{j=0}^{m} \lambda I_j$. (iii) Assuming the Heine-Borel theorem in the form

whenever $[a,b]$ is a closed interval in $\mathbb{R}^r$ which is covered by a sequence $\langle\,]a^{(j)}, b^{(j)}[\,\rangle_{j\in\mathbb{N}}$ of open intervals, there is an $m \in \mathbb{N}$ such that $[a,b] \subseteq \bigcup_{j\le m}]a^{(j)}, b^{(j)}[$,

prove 115B.

p 37 l 13 (introduction to Chapter 12): the '1904' (referring to Lebesgue's monograph *Leçons sur l'intégration et la recherche des fonctions primitives*) should be replaced by '1901' (referring to Lebesgue's paper *Sur une généralisation de l'intégrale définie*).

p 40 l 36 (part (g) of the proof of 121E): for 'is for the form' read 'is of the form'. [P.W.T.]

p 43 l 2 (part (b-i) of the proof of 121I): for '$E_q = D_q \cap D$' read '$D_q = E_q \cap D$'. [A.A.]

p 43 l 25 (statement of Lemma 121J): for 'Borel subsets of $\mathbb{R}$' read 'Borel subsets of $\mathbb{R}^r$'. [A.A.]

p 43 l 32 (part (b) of the proof of 121J): for '121Df' read '121Ef'.

p 44 l 13 (part (a-i) of the proof of 121K): for '$\mathbb{R} \setminus E \in \mathrm{T}$' read '$\mathbb{R}^r \setminus E \in \mathrm{T}$' [G.M.]

p 44 l 22 The exercises for §121 have been re-arranged, as follows: 121Xc has become 121Xa, 121Xa has become 121Xb, 121Xb has become 121Xc, 121Yb has become 121Yc, 121Yc has become 121Ye, 121Yd has become 121Yb, 121Ye has become 121Xf and 121Yf has become 121Yd.

p 45 l 8 Exercise 121Ya is wrong, and should be re-written, as follows:

(a) Let X and Y be sets, Σ a σ-algebra of subsets of X, $\phi : X \to Y$ a function and g a real-valued function defined on a subset of Y. Set $\mathrm{T} = \{F : F \subseteq Y,\ \phi^{-1}[F] \in \Sigma\}$; then T is a σ-algebra of subsets of Y (see 111Xc). (i) Show that if g is T-measurable then $g\phi$ is Σ-measurable. (ii) Give an example in which $g\phi$ is Σ-measurable but g is not T-measurable. (iii) Show that if $g\phi$ is Σ-measurable and *either* ϕ is injective *or* $\operatorname{dom}(g\phi) \in \Sigma$ *or* $\phi[X] \subseteq \operatorname{dom} g$, then g is T-measurable. [P.W.T.]

p 44 l 34 (Exercise 121Xb, now 121Xc): for '$\limsup_{n\in\mathbb{N}}$' read '$\limsup_{n\to\infty}$'. [P.W.T.]

p 45 l 14 (Exercise 121Yb, now 121Yc): add new part

Suppose that $\mathcal{A} \subseteq \mathrm{T}$ is such that T is the σ-algebra of subsets of Y generated by $\mathcal{A}$ (111Gb). Show that $\phi : X \to Y$ is (Σ, T)-measurable iff $\phi^{-1}[A] \in \Sigma$ for every $A \in \mathcal{A}$.

p 45 l 39 (part (iv) of 121Yf, now 121Yd): for 'any continuous function from a subset of $\mathbb{R}^s$ to $\mathbb{R}^r$ is measurable' read 'any continuous function from a subset of $\mathbb{R}^s$ to $\mathbb{R}^r$ is Borel measurable'.

p 46 l 32 (Definition 122Aa): for '121A' read '121C'.

p 47 l 13 (part (a) of the proof of 122C): for '$i \le n$' read '$k \le n$'. [P.W.T.]

p 47 l 29 (statement of Corollary 122D): the clause 'and $f : X \to \mathbb{R}$ a simple function' should be deleted. [D.W.]

p 50 l 24 (Remark 122Na): for 'it can be expressed as $f_1 - f_2$ where f_1 and f_2 are simple functions' read 'it can be expressed as $f_1 - f_2$ where f_1 and f_2 are non-negative simple functions'.

p 51 l 3 (part (a) of the proof of 122O): for '$f+g = (f_1+g_1)-(f_2-g_2)$' read '$f+g = (f_1+g_1)-(f_2+g_2)$'. [P.W.T.]

p 53 l 48 Exercise 122Yg is wrong as stated; it works if we assume that f is defined everywhere on X. [T.Helineva]

p 54 l 34 (Notes on §122): for '122Xb' read '122Yb'.

p 56 l 24 (part (a-iv) of the proof of 123A): in the formula '$g \le f_n + M\chi G_n + M\chi(X \setminus E) + \epsilon\chi H$', the '$\le$' should be '$\le_{\text{a.e.}}$'. An improvement in the whole sentence would be

Then, for any $x \in E$,

$$g(x) \leq f_n(x) + \epsilon\chi H(x) + M\chi G_n(x),$$

so

$$g \leq_{\text{a.e.}} f_n + M\chi G_n + \epsilon\chi H$$

and

$$\int g \leq \int f_n + M\mu G_n + \epsilon\mu H \leq c + \epsilon(M + \mu H).$$

[P.W.T.]

p 57 l 1 (proof of 123B) The assertion that g_n is integrable needs more support. The sentence now reads

Set $g_n = \inf_{m\geq n} f'_m$; then each g_n is measurable (121Fc), non-negative and defined on the conegligible set $\bigcap_{m\geq n} E_m$, and $g_n \leq_{\text{a.e.}} f_n$; by 122Re and 122Ra, $|f_n|$ belongs to U, as defined in 122H, while $|g_n| \leq_{\text{a.e.}} |f_n|$, so g_n is integrable (122P) with $\int g_n \leq \inf_{m\geq n} \int f_m \leq c$.

[A.-P.Fortin]

p ?? l ?? (Exercise 123Xd): for '$\int \limsup_{n\to\infty} f_n \geq \limsup_{n\to\infty} f_n$' read '$\int \limsup_{n\to\infty} f_n \geq \limsup_{n\to\infty} \int f_n$'.
T.H.

p ?? l ?? Exercise 123Ya is wrong, and should read

Let (X, Σ, μ) be a measure space, Y any set and $\phi : X \to Y$ any function; let $\mu\phi^{-1}$ be the image measure on Y (112Xf). Show that if $h : Y \to \mathbb{R}$ is $\mu\phi^{-1}$-integrable then $h\phi$ is μ-integrable, and the integrals are then equal.

[T.H.]

p 60 l 29 Definition 131B has been extended, as follows:

When $X = \mathbb{R}^r$, where $r \geq 1$, and μ is Lebesgue measure on $\mathbb{R}^r$, I will call a subspace measure μ_H **Lebesgue measure on** H.

p 61 l 4 (part (a) of the proof of 131E): for '$\sum_{i=0}^n a_i\mu_H E_i = \sum_{i=0}^n \mu E_i$' read '$\sum_{i=0}^n a_i\mu_H E_i = \sum_{i=0}^n a_i\mu E_i$'.
[P.W.T.]

p 61 l 18 (part (d) of the proof of 131E): for '$g \leq f$ μ_H-a.e.' read '$g \leq |f|$ μ_H-a.e.'.

p 62 l 21 Exercise 131Ya is wrong as written; it can be salvaged by adding the hypothesis that $\operatorname{dom} f_n$ is measurable for every n. [P.W.T.]

p 64 l 13 (132D) For '$E \subseteq \Sigma$' read '$E \in \Sigma$'. [P.W.T.]

p 64 l 15 (132D, definition of 'measurable envelope'): for '215Yc' read '216Yc'.

p 64 l 20 (Lemma 132E): add new part

(d) Let $\langle A_n\rangle_{n\in\mathbb{N}}$ be a sequence of subsets of X. Suppose that each A_n has a measurable envelope E_n. Then $\bigcup_{n\in\mathbb{N}} E_n$ is a measurable envelope of $\bigcup_{n\in\mathbb{N}} A_n$.

Parts (d) and (e) become (e) and (f).

p 65 l 1 Proposition 132G has been moved to 234F.

p 65 l 34 Delete exercise 132Xf; rename exercises 132Xg-132Xk as 132Xf-132Xj. Add a new exercise:

(k) Let (X, Σ, μ) be a measure space and μ^* the outer measure defined from μ. Show that $\mu^*(A \cup B) + \mu^*(A \cap B) \leq \mu^*A + \mu^*B$ for all $A, B \subseteq X$.

p l (132Y) Add new exercise:

(g) Let (X, Σ, μ) be a measure space. Suppose that $A \subseteq B \subseteq C \subseteq X$ and that $\mu^*(B \setminus A) = \mu^*B$. Show that $\mu^*(C \setminus A) = \mu^*C$.

p 68 l 36 (Upper and lower integrals) The definition in 133I should be re-written, as follows:

Let (X, Σ, μ) be a measure space and f a real-valued function defined almost everywhere in X. Its **upper integral** is

$$\overline{\int} f = \inf\{\int g : \int g \text{ is defined in the sense of 133A, } f \leq_{\text{a.e.}} g\},$$

allowing ∞ for $\inf\{\infty\}$ and $-\infty$ for $\inf\mathbb{R}$. Similarly, the **lower integral** of f is

$$\underline{\int} f = \sup\{\int g : \int g \text{ is defined, } f \geq_{\text{a.e.}} g\},$$

allowing $-\infty$ for $\sup\{-\infty\}$ and ∞ for $\sup\mathbb{R}$.

p 69 l 1 (Proposition 133J) Add new facts:

(a)(i) If If $\overline{\int} f$ is finite, and g is an integrable function such that $f \leq_{\text{a.e.}} g$ and $\int g = \overline{\int} f$, then

$$A = \{x : x \in \operatorname{dom} f \cap \operatorname{dom} g,\ g(x) \leq f(x) + \epsilon\}$$

has full outer measure for every $\epsilon > 0$.

(ii) If If $\underline{\int} f$ is finite, and h is an integrable function such that $f \geq_{\text{a.e.}} h$ and $\int h = \underline{\int} f$, then

$$\{x : x \in \operatorname{dom} f \cap \operatorname{dom} h,\ f(x) \leq h(x) + \epsilon\}$$

has full outer measure for every $\epsilon > 0$.

(e) $\mu^* A = \overline{\int} \chi A$ for every $A \subseteq X$.

p 69 l 31 (part (iii)(α) of the proof of 133J): for '$cf_1 \leq cf$ a.e.' read '$cf \leq cf_1$ a.e.'

p 70 l 18 (part (a) of the proof of 133K): for '$f_n(x) \leq f_{n+1}(x)$ for every n' read '$f_n(x) \leq g_n(x)$ for every n'.

p 70 l 26 Add new result:

***133L**Proposition Let (X, Σ, μ) be a measure space and f a real-valued function defined almost everywhere in X. Suppose that h_1, h_2 are non-negative virtually measurable functions defined almost everywhere in X. Then

$$\overline{\int} f \times (h_1 + h_2) = \overline{\int} f \times h_1 + \overline{\int} f \times h_2.$$

p 70 l 27 (133X) Add new exercise:

(a) Let (X, Σ, μ) be a measure space, and $f : X \to [0, \infty[$ a measurable function. Show that

$$\int f d\mu = \sup_{n\in\mathbb{N}} 2^{-n} \textstyle\sum_{k=1}^{4^n} \mu\{x : f(x) \geq 2^{-n}k\} = \lim_{n\to\infty} 2^{-n} \sum_{k=1}^{4^n} \mu\{x : f(x) \geq 2^{-n}k\}$$

Exercises 133Xa-133Xe have been re-numbered 133Xb-133Xf.

p 71 l 19 (Exercise 133Xe, now 133Xf): for '$\underline{\int} \limsup_{n\to\infty} \geq \limsup_{n\to\infty} \underline{\int} f_n$' read '$\underline{\int} \limsup_{n\to\infty} f_n \geq \limsup_{n\to\infty} \underline{\int} f_n$'. [P.W.T.]

p 71 l 27 (Exercise 133Yc): for '$\hat{f}(s) = \frac{1}{(\sqrt{2\pi})^r} \int_{-\infty}^{\infty} e^{-is\cdot x} f(x) dx$' read '$\hat{f}(s) = \frac{1}{(\sqrt{2\pi})^r} \int e^{-is\cdot x} f(x) dx$'. [P.W.T.]

p 72 l 4 (133Y) The old Exercise 133Yf is wrong; a corrected version is now 133J(a-i).

p 72 l 12 (notes on §133): the reference '252Yf' should be changed to '252Ym'.

p 74 l 29 (part (c-i) of the proof of 134D): for '$\mu(F \cap E_n \cap C) + \mu(F \cap E_n \setminus C)$' read '$\mu^*(F \cap E_n \cap C) + \mu^*(F \cap E_n \setminus C)$'. [P.W.T.]

p 75 l 32 (part (b-i) of the proof of 134F): for '$\sum_{m=0}^{\infty} \mu(G_m \setminus E_m)$' read '$\sum_{m=n}^{\infty} \mu(G_m \setminus E_m)$'.

p 78 ll 24-25 (part (a) of 134I): for 'I_{jn}' read 'I_{nj}' (three times).

p 81 l 8 (part (a) of the proof of 134L): for '$\inf_{y\in[a,b],0<|y-x|\leq\delta} f(y)$' read '$\inf_{y\in[a,b],|y-x|\leq\delta} f(y)$'. Similarly, on the next line, read '$h(x) = \sup_{y\in[a,b],|y-x|\leq\delta} f(y)$'.

p 81 l 32 (134X) Add new exercises:

(c) Show that if $C \subseteq \mathbb{R}$ is any non-negligible set, it has a non-measurable subset.

(e) Let $E \subseteq \mathbb{R}^r$ be a measurable set, and $\epsilon > 0$. (i) Show that there is an open set $G \supseteq E$ such that $\mu(G \setminus E) \leq \epsilon$. (ii) Show that there is a closed set $F \subseteq E$ such that $\mu(E \setminus F) \leq \epsilon$.

(i) (i) Show that there is a disjoint sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of subsets of $[0,1]^r$ all of outer measure 1. (ii) Show that there is a function $f:[0,1]^r\to\,]0,1[$ such that $\underline{\int} f=0$ and $\overline{\int} f=1$.

(j) Let f be a measurable real function and g a real function such that $\operatorname{dom} g\setminus\operatorname{dom} f$ and $\{x:x\in\operatorname{dom} g\cap\operatorname{dom} f,\ g(x)\ne f(x)\}$ are both negligible. Show that g is measurable.

Consequently some previous exercises have been renamed, as follows: 134Xc is now 134Xd, 134Xd is now 134Xg, 134Xf is now 134Xh. [P.W.T.]

p 89 l 14 Concerning upper and lower integrals of functions taking infinite values, a restatement of the definitions is formally required:

Let (X,Σ,μ) be a measure space and f a $[-\infty,\infty]$-valued function defined almost everywhere in X. Its **upper integral** is

$$\overline{\int} f=\inf\{\int g:\int g \text{ is defined in the sense of 135F and } f\le_{\text{a.e.}} g\},$$

allowing ∞ for $\inf\{\infty\}$ and $-\infty$ for $\inf\,]-\infty,\infty]$ or $\inf[-\infty,\infty]$. Similarly, the **lower integral** of f is

$$\underline{\int} f=\sup\{\int g:\int g \text{ is defined, } f\ge_{\text{a.e.}} g\}.$$

p 89 l 19 Add new paragraph on subspace measures:

135I Proposition Let (X,Σ,μ) be a measure space, and $H\in\Sigma$; write Σ_H for the subspace σ-algebra on H and μ_H for the subspace measure. For any $[-\infty,\infty]$-valued function f defined on a subset of H, write $\tilde f$ for the extension of f defined by saying that $\tilde f(x)=f(x)$ if $x\in\operatorname{dom} f$, 0 if $x\in X\setminus H$.

(a) Suppose that f is a $[-\infty,\infty]$-valued function defined on a subset of H.

(i) $\operatorname{dom} f$ is μ_H-conegligible iff $\operatorname{dom}\tilde f$ is μ-conegligible.

(ii) f is μ_H-virtually measurable iff $\tilde f$ is μ-virtually measurable.

(iii) $\int_H f d\mu_H=\int_X \tilde f d\mu$ if either is defined in $[-\infty,\infty]$.

(b) Suppose that h is a $[-\infty,\infty]$-valued function defined almost everywhere in X. Then $\int_H (h\restriction H)d\mu_H=\int h\times\chi H\,d\mu$ if either is defined in $[-\infty,\infty]$.

(c) If h is a $[-\infty,\infty]$-valued function and $\int_X h\,d\mu$ is defined in $[-\infty,\infty]$, then $\int_H(h\restriction H)d\mu_H$ is defined in $[-\infty,\infty]$.

(d) Suppose that h is a $[-\infty,\infty]$-valued function defined almost everywhere in X. Then

$$\overline{\int}_H (h\restriction H)d\mu_H=\overline{\int}_X h\times\chi H d\mu.$$

p 87 l 13 (Exercise 135Xc): add new part

(iv) a real-valued function h defined on a subset of $[-\infty,\infty]$ is Borel measurable iff $h\phi^{-1}$ is Borel measurable.

p 87 l 16 (Exercise 135Xd): for '134Ea', '134Ef' and '134Eg' read '135Ea', '135Ef' and '135Eg'.

p 87 l 23 (135Y) Add new exercises:

135Ya Let X be a set and Σ a σ-algebra of subsets of X. Show that if $f:X\to[0,\infty]$ is Σ-measurable, there is a sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in Σ such that $f=\sum_{n=0}^{\infty}\frac{1}{n+1}\chi E_n$.

(b) Let (X,Σ,μ) be a measure space, and f, g two $[-\infty,\infty]$-valued functions, defined on subsets of X, such that $\int f$ and $\int g$ are both defined in $[-\infty,\infty]$. (i) Show that $\int f\vee g$ and $\int f\wedge g$ are defined in $[-\infty,\infty]$, where $(f\vee g)(x)=\max(f(x),g(x))$, $(f\wedge g)(x)=\min(f(x),g(x))$ for $x\in\operatorname{dom} f\cap\operatorname{dom} g$. (ii) Show that $\int f\vee g+\int f\wedge g=\int f+\int g$ in the sense that if one of the sums is defined in $[-\infty,\infty]$ so is the other, and they are then equal.

p 90 l 33 Add new result:

***136H Proposition** Let (X,Σ,μ) be a measure space such that $\mu X<\infty$, and $\mathcal{E}$ a subalgebra of Σ; let Σ' be the σ-algebra of subsets of X generated by $\mathcal{E}$. If $F\in\Sigma'$ and $\epsilon>0$, there is an $E\in\mathcal{E}$ such that $\mu(E\triangle F)\le\epsilon$.

p 91 l 26 Exercise 136Xl seems a little harder than the others, and I have moved it to 136Yc.

p 90 l 44 (Hint for exercise 136Xc): for '$\mu_n I = \mu(E \cap I_n)$' read '$\mu_n E = \mu(E \cap I_n)$'. [P.W.T.]

p 94 l 21 (part (c) of 1A1B); for '(i)-(ii) above' read '(a)-(b) above'. [P.W.T.]

p 97 l 1 Add new paragraph:

> **1A1J Notation** For definiteness, I remark here that I will say that a family $\mathcal{A}$ of sets is a **partition** of a set X whenever $\mathcal{A}$ is a disjoint cover of X, that is, $X = \bigcup \mathcal{A}$ and $A \cap A' = \emptyset$ for all distinct A, $A' \in \mathcal{A}$; in particular, the empty set may or may not belong to $\mathcal{A}$. Similarly, an indexed family $\langle A_i \rangle_{i \in I}$ is a partition of X if $\bigcup_{i \in I} A_i = X$ and $A_i \cap A_j = \emptyset$ for all distinct i, $j \in I$; again, one or more of the A_i may be empty.

p 99 l 15 (statement of 1A3Bb): for '$\liminf_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = u$' read '$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = u$'.

p 99 l 24 (proof of part (a) of 1A3B): for '$\lim_{n\to\infty} \inf_{m\ge n} a_m = \limsup_{n\to\infty} a_n$' read '$\lim_{n\to\infty} \inf_{m\ge n} a_m = \liminf_{n\to\infty} a_n$'. [K.Yates]

Back cover: for 'Simón' read 'Simon'.