Fixed-point sets for contraction mappings

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Theorem Let (X,ρ) be a non-empty complete metric space, and $\mathcal{S}$ a non-empty family of functions from X to itself such that

(i) $\mathcal{S}$ is uniformly contractive, that is, there is an $\alpha<1$ such that $\rho(Sx,Sy)\le\alpha\rho(x,y)$ whenever $S\in\mathcal{S}$ and x, $y\in X$,

(ii) $\{d_S:S\in\mathcal{S}\}$ has finite diameter, where d_S is the fixed point of S for each $S\in\mathcal{S}$.

Then there is a unique non-empty closed set K of finite diameter such that $S[K]\subseteq K$ for each $S\in\mathcal{S}$ and $\bigcup_{S\in\mathcal{S}}S[K]$ is dense in K. If $\mathcal{S}$ is finite, then K is compact and $K=\bigcup_{S\in\mathcal{S}}S[K]$.

proof (a) Start by fixing $c\in X$. Set $\gamma=\sup_{S\in\mathcal{S}}\rho(c,d_S)$, $R=\frac{1}{1-\alpha}(1+3\gamma)$. Then whenever $x\in X$, $S\in\mathcal{S}$ and $\rho(x,c)\ge R-\gamma$, $\rho(Sx,c)\le\rho(x,c)-1$. **P**

$$\begin{aligned}\rho(Sx,c)\le\rho(Sx,Sd_S)+\rho(d_S,c)&\le\alpha\rho(x,d_S)+\gamma\le\alpha\rho(x,c)+2\gamma\\&=\rho(x,c)-(1-\alpha)\rho(x,c)+2\gamma\\&\le\rho(x,c)-(1-\alpha)R+3\gamma\le\rho(x,c)-1.\ \mathbf{Q}\end{aligned}$$

In particular, if we set $B=\{x:\rho(x,c)\le R\}$, $S[B]\subseteq B$ for every $S\in\mathcal{S}$. **P** If $x\in B$ and $\rho(x,c)\ge R-\gamma$, $\rho(Sx,c)\le\rho(x,c)\le R$. If $x\in B$ and $\rho(x,c)\le R-\gamma$, then

$$\rho(Sx,c)\le\alpha\rho(x,d_S)+\gamma\le\alpha R+\gamma\le R.\ \mathbf{Q}$$

(b) For $k\in\mathbb{N}$, let G_k be the set of products $S_kS_{k-1}\dots S_1S_0$ where $S_0,\dots,S_k\in\mathcal{S}$. Note that if $T\in G_k$ then $\rho(Tx,Ty)\le\alpha^{k+1}\rho(x,y)$ for all x, $y\in X$. Set $G=\bigcup_{k\in\mathbb{N}}G_k$; for $T\in G$ let d_T be the fixed point of T. Then $d_T\in B$. **P?** Otherwise, express T as $S_k\dots S_1S_0$ where $S_0,\dots,S_k\in\mathcal{S}$, and set $x_0=d_T$, $x_{j+1}=S_jx_j$ for $j\le k$. Then $\rho(x_{j+1},c)<\rho(x_j,c)$ whenever $x_j\ne B$, and $x_{j+1}\in B$ whenever $x_j\in B$. So $\rho(x_j,c)<\rho(x_0,c)$ for every $j\ge1$ and we cannot have $Td_T=x_{j+1}=x_0$. **XQ**

Set $K=\overline{\{d_T:T\in G\}}\subseteq B$.

(c) If $S\in\mathcal{S}$ and $T\in G$ then $Sd_T\in K$. **P** Let $\epsilon>0$. Let m be so large that $2R\alpha^{m+1}\le\epsilon$ and consider $T_1=ST^m\in G_m$. We have $d_{T_1}\in K$, while

$$\rho(Sd_T,d_{T_1})=\rho(ST^md_T,ST^md_{T_1})\le\alpha^{m+1}\rho(d_T,d_{T_1})\le\epsilon.$$

As ϵ is arbitrary, $Sd_T\in K$. **Q** It follows that $S[K]\subseteq K$. At the same time, if $T=S_k\dots S_1S_0\in G$, then $x=S_{k-1}\dots S_1S_0d_T\in K$, so $d_T=S_kx\in K$; accordingly $\bigcup_{S\in\mathcal{S}}S[K]\supseteq\{d_T:T\in G\}$ is dense in K.

(d) Now suppose that L is another closed non-empty set of finite diameter such that $\bigcup_{S\in\mathcal{S}}S[L]$ is dense in L; then $\bigcup_{T\in G_m}T[L]$ is dense in L for every $m\in\mathbb{N}$. Note first that

$L \subseteq B$. **P** Set $\beta = \sup_{x \in L} \rho(x, c)$. Then $\rho(S_i x, c) \le \max(R, \beta - 1)$ for every $x \in L$ and $i \le n$, so $\beta \le \max(R, \beta - 1)$ and $\beta \le R$. **Q** Let $\epsilon > 0$. Let m be so large that $2R\alpha^{m+1} \le \epsilon$. If $x \in L$, then there are $T \in G_m$, $y \in L$ such that $\rho(x, Ty) \le \epsilon$. Now

$$\rho(x, d_T) \le \epsilon + \rho(Ty, Td_T) \le \epsilon + \alpha^{m+1}\rho(y, d_T) \le 2\epsilon.$$

Thus $\rho(x, K) \le 2\epsilon$; as ϵ is arbitrary, $x \in K$; as x is arbitrary, $L \subseteq K$.

On the other hand, if $T \in G$ and $x \in L$, then all the iterates $T^k x$ belong to L, so their limit d_T belongs to L. So $K \subseteq L$. Thus K is unique in the sense declared.

(e) If $\mathcal{S}$ is finite, K is compact. **P** Let $\epsilon > 0$. Let m be so large that $2R\alpha^{m+1} \le \epsilon$, and let A be the finite set $\{d_T : T \in G_m\}$. If $x \in K$, then there are $T \in G_m$, $y \in K$ such that $\rho(x, Ty) \le \epsilon$, and now

$$\rho(x, d_T) \le \epsilon + \rho(Ty, Td_T) \le \epsilon + \alpha^{m+1}\rho(y, d_T) \le 2\epsilon.$$

As ϵ is arbitrary, K is totally bounded, therefore compact, because X is complete. **Q**

Accordingly, in this case, $\bigcup_{S \in \mathcal{S}} S[K]$ is compact, therefore closed, and must be the whole of K.