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Families of Compact Sets and Tukey's Ordering (**).

Summary. – *This note is a supplement to my paper* [4]. *Here I give a largely self-contained account of some preliminary results on the structure of families of compact sets regarded as upwards-directed sets.*

1. – DEFINITIONS. (*a*) [15] Let P and Q be partially ordered sets. A *Tukey function* from P to Q is a function $f\colon P \to Q$ such that $\{p: f(p) \leqslant q\}$ is either empty or bounded above in P for every $q \in Q$. If there is a Tukey function from P to Q write $P \leqslant Q$. If $P \leqslant Q$ and $Q \leqslant P$ write $P \equiv Q$.

(*b*) For any topological space X write $\mathcal{K}(X)$ for the family of compact subsets of X.

2. – THE PROBLEM. For which topological spaces X, Y do we have $\mathcal{K}(X) \leqslant \mathcal{K}(Y)$?

3. – REMARKS. J. W. Tukey originally introduced the relations $\leqslant$, $\equiv$ (for directed sets) as part of an investigation of net-convergence. He showed that if P and Q are upwards-directed, then $P \equiv Q$ iff there is an upwards-directed set R such that P and Q are both isomorphic to cofinal subsets of R. (We shall not need this result here.) In [4] I showed how the same idea could be used to reformulate results

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(**) Nota giunta in Redazione il 2-XII-1988.
This article was presented at the « Convegno di Analisi Reale e Teoria della Misura » held in Capri during September 12-16 1988, and was supported by M.P.I.

of T. Bartoszyński ([1]) and others on the additivity and cofinality of ideals of sets. By the additivity and cofinality of a partially ordered set P I mean

$$\operatorname{add}(P) = \min\{\#(A)\colon \emptyset \neq A \subseteq P,\ A \text{ has no upper bound in } P\}$$

(conventionally taking $\min \emptyset = \infty$),

$$\operatorname{cf}(P) = \min\{\#(B)\colon B \subseteq P \text{ is cofinal with } P\}\,.$$

In the present paper I set out to classify the spaces $\mathcal{K}(X)$ in a variety of elementary cases. The hope is that this will throw light on some of their properties; in the first place their cofinality, but also certain types of caliber property (see [4], 1I-1J).

4. – Elementary facts. (*a*) The composition of two Tukey functions is a Tukey function; $\preccurlyeq$ is transitive and reflexive; $\equiv$ is an equivalence relation.

(*b*) If P is any partially ordered set and Q is a cofinal subset of P then $P \equiv Q$.

(*c*) If P_0, P_1 are partially ordered sets and Q is an upwards-directed partially ordered set and $P_i \preccurlyeq Q$ for both i then $P_0 \times P_1 \preccurlyeq Q$.

(*d*) If $P \preccurlyeq Q$ then $\operatorname{add}(Q) \leqslant \operatorname{add}(P)$ and $\operatorname{cf}(P) \leqslant \operatorname{cf}(Q)$.

(*e*) $P \equiv \{0\}$ iff P has a greatest member.

(*f*) $P \equiv N$ iff P is upwards-directed and $\operatorname{cf}(P) = \omega$.

I trust that readers will have no difficulty in finding proofs of these facts; but see [15], [6], [4] if any obstacles arise.

5. – Proposition. (*a*) If X is any topological space and F is a closed subspace of X, then $\mathcal{K}(F) \preccurlyeq \mathcal{K}(X)$.

(*b*) If W, Z are compact Hausdorff spaces, $h\colon W \to Z$ is a continuous function, $Y \subseteq Z$ and $X = h^{-1}[Y]$, then

(i) $\mathcal{K}(X) \preccurlyeq \mathcal{K}(Y)$;

(ii) if $h[W] \supseteq Y$ then $\mathcal{K}(X) \equiv \mathcal{K}(Y)$.

Proof. (*a*) The identity map from $\mathcal{K}(F)$ to $\mathcal{K}(X)$ is a Tukey function.

(*b*) (i) $K \mapsto h[K]\colon \mathcal{K}(X) \to \mathcal{K}(Y)$ is a Tukey function.

(ii) In this case, $L \mapsto g^{-1}[L]\colon \mathcal{K}(Y) \to \mathcal{K}(X)$ is a Tukey function.

6. – Proposition. Let X be a topological space.

(*a*) $\mathcal{K}(X) \equiv \{0\}$ iff X is compact.

(*b*) If X is locally compact and paracompact but not compact then $\mathcal{K}(X) \equiv [\varkappa]^{<\omega}$ where $\varkappa$ is the Lindelöf degree of X.

Proof. (*a*) By 4 *e*) above, $\mathcal{K}(X) \equiv \{0\}$ iff $\mathcal{K}(X)$ has a greatest member; and because singleton subsets of X belong to $\mathcal{K}(X)$, the only possible greatest member of $\mathcal{K}(X)$ is X itself.

(*b*) (Recall that the Lindelöf degree $L(X)$ of X is the smallest cardinal λ such that every open cover of X has a subcover of cardinal $\leqslant \lambda$, and that $[\varkappa]^{<\omega}$ is the set of finite subsets of $\varkappa$.) Because X is locally compact, it has an open cover $\mathcal{G}$ consisting of relatively compact open sets. Because it is paracompact, $\mathcal{G}$ has a locally finite refinement $\mathcal{H}$ still covering X. Because $L(X) = \varkappa$ there is an $\mathcal{E} \subseteq \mathcal{H}$ of cardinal $\leqslant \varkappa$ still covering X. But in fact $\#(\mathcal{E})$ is precisely $\varkappa$. To see this, note first that as X is not compact, and $\mathcal{E}$ consists of relatively compact sets, $\mathcal{E}$ is surely infinite. Next, let $\mathcal{V}$ be any open cover of X; let $\mathcal{W}$ be a locally finite refinement of $\mathcal{V}$ covering X. Then any member of $\mathcal{E}$, being relatively compact, can meet only finitely many members of $\mathcal{W}$. So $\#(\mathcal{W}) \leqslant \max(\omega, \#(\mathcal{E})) = \#(\mathcal{E})$. Now there is a $\mathcal{V}' \subseteq \mathcal{V}$ such that $\#(\mathcal{V}') \leqslant \#(\mathcal{W}) \leqslant \#(\mathcal{E})$ and every member of $\mathcal{W}$ is included in some member of $\mathcal{V}'$, so that $\mathcal{V}'$ covers X. As $\mathcal{V}$ is arbitrary, $L(X) \leqslant \#(\mathcal{E})$ and $\#(\mathcal{E}) = \varkappa$.

So we may enumerate $\mathcal{E}$ as $\langle G_\xi \rangle_{\xi<\varkappa}$. Now we can define $f\colon \mathcal{K}(X) \to [\varkappa]^{<\omega}$, $g\colon [\varkappa]^{<\omega} \to \mathcal{K}(X)$ in such a way that

$$f(K) = \{\xi\colon K \cap G_\xi \neq \emptyset\}\,, \qquad \forall K \in \mathcal{K}(X)\,,$$

$$g(I) \supseteq \bigcup_{\xi \in I} G_\xi \qquad \forall I \in [\varkappa]^{<\omega}\,,$$

and see that f and g are both Tukey functions, so that $\mathcal{K}(X) \equiv [\varkappa]^{<\omega}$.

7. – The topology of $\mathcal{K}(X)$. In so far as there are any non-trivial ideas in this note, they are based on the fact that each space $\mathcal{K}(X)$ has a more or less well-known topology closely linked with its order structure. Recall that if X is any topological space, the *exponential* or *Victoris* topology on $\mathcal{K}(X)$ is that generated by sets of the form

$$\{K\colon K\in\mathcal{K}(X),\ K\cap G\neq\emptyset\}\ , \qquad \{K\colon K\in\mathcal{K}(X),\ K\subseteq G\}\ ,$$

where in both formulae G is an open subset of X. If X is Hausdorff, so is $\mathcal{K}(X)$. If X is separable, so is $\mathcal{K}(X)$. If X is metrizable, with metric ϱ, then $\mathcal{K}(X)$ is metrizable, with metric

$$\tilde{\varrho}(K, L) = \min\left(1, \max\left(\sup_{x\in K}\varrho(x, L), \sup_{y\in L}\varrho(y, K)\right)\right) .$$

(taking $\tilde{\varrho}(K, L) = 1$ if just one of K, L is empty). If (X, ϱ) is a complete metric space, so is $(\mathcal{K}(X), \tilde{\varrho})$. If X is any subset of any topological space Y, then $\mathcal{K}(X)$, with its exponential topology, is a subspace of $\mathcal{K}(Y)$, with its.

All the above is standard; see for instance [9], § 17, § 21.VII, § 21,VIII, § 33.IV. Now for something less familiar.

8. – Proposition. Let X be any topological space.

(*a*) If $\mathcal{C}\subseteq\mathcal{K}(X)$ is compact then $\bigcup\mathcal{C}\in\mathcal{K}(X)$.

(*b*) If $K_0\in\mathcal{K}(X)$ then $\{K\colon K\in\mathcal{K}(X),\ K\subseteq K_0\}$ is compact in $\mathcal{K}(X)$.

(*c*) A subset of $\mathcal{K}(X)$ is relatively compact in $\mathcal{K}(X)$ iff it is bounded above in $\mathcal{K}(X)$.

Proof. (*a*) Let $\mathcal{G}$ be an upwards-directed open cover of $\bigcup\mathcal{C}$ in X. For each $G\in\mathcal{G}$ set $\mathcal{V}(G) = \{K\colon K\in\mathcal{K}(X), K\subseteq G\}$. Then $\{\mathcal{V}(G)\colon G\in\mathcal{G}\}$ is an upwards-directed open cover of $\mathcal{C}$ in $\mathcal{K}(X)$. As $\mathcal{C}$ is compact, there is a $G\in\mathcal{G}$ such that $\mathcal{C}\subseteq\mathcal{V}(G)$ i.e. $\bigcup\mathcal{C}\subseteq G$. As $\mathcal{G}$ is arbitrary, $\bigcup\mathcal{C}$ is compact.

(*b*) Set $\mathcal{C} = \{K\colon K\in\mathcal{K}(X),\ K\subseteq K_0\}$ and let Φ be any ultrafilter on $\mathcal{K}(X)$ containing $\mathcal{C}$. Set

$$\mathcal{U} = \{U\colon U\subseteq X \text{ is open}, \{K\colon K\in\mathcal{K}(X),\ K\cap U=\emptyset\}\in\Phi\}\ ;$$

then $\mathcal{U}$ is an upwards-directed family of open set in X. Set

$$G=\bigcup\mathcal{U}\ , \qquad K_1 = K_0\setminus G\ ;$$

then $K_1\in\mathcal{C}$. I claim that $\Phi\to K_1$ in $\mathcal{K}(X)$. To see this, examine the sub-basic open sets for the topology of $\mathcal{K}(X)$ which contain K_1.

i) Suppose that $H\subseteq X$ is open and that $K_1\cap H\neq\emptyset$. Then $H\nsubseteq G$ so $H\notin\mathcal{U}$ i.e.

$$\{K\colon K\in\mathcal{K}(X),\ K\cap U\neq\emptyset\}\in\Phi\ .$$

(ii) Suppose that $H\subseteq X$ is open and that $K_1\subseteq H$. Then $K_0\setminus H$ is a compact subset of $\bigcup\mathcal{U}$; because $\mathcal{U}$ is upwards-directed there is a $U\in\mathcal{U}$ including $K_0\setminus H$. Now

$$\{K\colon K\subseteq H\}\supseteq\{K\colon K\subseteq K_0,\ K\cap U=\emptyset\}\in\Phi\ .$$

Thus all the standard subbasic open sets containing K_1 belong to Φ: it follows at once that $\Phi\to K_1$. Because Φ is arbitrary, $\mathcal{C}$ must be compact.

(*c*) Follows at once from (*a*), (*b*).

9. – In view of the above, it is natural to invest a little time in considering topological partially ordered sets with the property of 8 *c*).

Proposition. Let P and Q be metrizable partially ordered sets such that (i) $\{p\colon p\in P,\ p\leqslant p_0\}$ is relatively compact in P for every $p_0\in P$; (ii) every convergent sequence in Q (for the topology) has a subsequence which is bounded above in Q; (iii) $P\preccurlyeq Q$. Suppose that P, Q are embedded in metric spaces $\tilde{P}$, $\tilde{Q}$ (which need not be ordered). Then there is a closed set $R\subseteq\tilde{Q}\times\tilde{P}$ such that $R[Q] = P$.

Proof. Let $f\colon P\to Q$ be a Tukey function. Set

$$R = \overline{\{(q, p)\colon p\in P,\ f(p)\leqslant q\in Q\}}\ ,$$

the closure being taken in $\tilde{Q}\times\tilde{P}$. Then of course R is closed. If $p\in P$, then $(f(p), p)\in R$, so $R[Q]\supseteq P$. On the other hand, if $\tilde{p}\in R[Q]$, say $(q, \tilde{p})\in R$; then there is a sequence $\langle(q_n, p_n)\rangle_{n\in N}$ in $Q\times P$, converging to $(q, \tilde{p})$ in $Q\times\tilde{P}$, such that $f(p_n)\leqslant q_n$ for each $n\in N$. By hypothesis (ii), there is an infinite $I\subseteq N$ such that $\{q_n\colon n\in I\}$ is bounded above in Q. Because f is a Tukey function, $\{p_n\colon n\in I\}$ is bounded above in P, therefore (by hypothesis (i)) relatively compact in P, and its closure point $\tilde{p}$ must belong to P. As $\tilde{p}$ is arbitrary, $R[Q] = P$, as required.

Remark. What is really happening here is that $R\cap(Q\times P)$ is usco-compact; i.e. the sections $R[\{q\}]$ are compact subsets of P and $R^{-1}[F]\cap Q$ is closed in Q for every closed $F\subseteq P$.

10. – Proposition. Let P and Q be partially ordered sets with Hausdorff topologies such that a relatively compact subset of P is bounded above in P, and a subset of Q which is bounded above in Q is relatively compact in Q. Suppose that P, Q are embedded as subspaces of Hausdorff spaces $\tilde{P},\tilde{Q}$ (not necessarily ordered), and that there is a K-analytic set $S\subseteq\tilde{Q}\times\tilde{P}$ such that $S[Q]$ is a cofinal subset of P. Then $P\preccurlyeq N^N\times Q$.

Proof. Recall that $S\subseteq\tilde{Q}\times\tilde{P}$ is said to be K-analytic if there is an usco-compact relation $R\subseteq N^N\times(\tilde{Q}\times\tilde{P})$ such that $R[N^N]=S$; see [7]. For each $p\in P$ fix a $p'\in S[Q]$ such that $p\leqslant p'$; now choose $f(p)=(\alpha,q)\in N^N\times Q$ so that $(q,p')\in S$ and $(\alpha,(q,p'))\in R$. I claim that f is a Tukey function. To see this, take any $\alpha_0\in N^N$, $q_0\in Q$ and set

$$A=\{p: f(p)\leqslant(\alpha_0,q_0)\}\ ,\qquad K=\{\alpha: \alpha\in N^N,\ \alpha\leqslant\alpha_0\}\ ,$$
$$L=\overline{\{q: q\in Q,\ q\leqslant q_0\}}\ ,$$

the last closure being taken in Q; but as L is compact, it is also closed in $\tilde{Q}$. Then K is a compact subset of N^N, so $R[K]$ is a compact subset of $\tilde{Q}\times\tilde{P}$ ([7], 2.7.2), and $C=R[K][L]$ is a compact subset of $\tilde{P}$. However, $R[K]\subseteq S$ and $L\subseteq Q$, so $C\subseteq S[Q]\subseteq P$; consequently C has an upper bound $p_0\in P$. Now take any $p\in A$. Set $(\alpha,q)=f(p)$; then $\alpha\in K$ and $q\in L$, so $p'\in C$ and $p\leqslant p'\leqslant p_0$. Thus A is bounded above in P. As (α_0,q_0) is arbitrary, f is a Tukey function and $P\preccurlyeq N^N\times Q$.

Remark. Readers unfamiliar with the theory of K-analytic sets may wish to note that in the applications below S will always be actually a continuous image of N^N, so that R may be taken to be a continuous function, which simplifies the proof slightly.

11. – Lemma. Let X be a metrizable space which is not locally compact. Then $N^N\preccurlyeq\mathcal{K}(X)$.

Proof. Let $x\in X$ be a point with no compact neighbourhood. Let $\langle U_n\rangle_{n\in N}$ be a decreasing sequence enumerating a base of neighbourhoods of X. Choose for each $n\in N$ a sequence $\langle a_{ni}\rangle_{i\in N}$ in U_n with no cluster point in X. Define $f: N^N\to\mathcal{K}(X)$ by

$$f(\alpha)=\{x\}\cup\{a_{ni}: n\in N,\ i\leqslant\alpha(n)\}\ .$$

Then f is a Tukey function, so $N^N\preccurlyeq\mathcal{K}(X)$.

12. – The universality of $\mathcal{K}(\boldsymbol{Q})$. In [13] it was pointed out that the set $\mathcal{A}$ of uncountable compact subsets of the Cantor set is a universal analytic set, that is, that every zero-dimensional metrizable analytic set is homeomorphic to a closed subset of $\mathcal{A}$. Here I adapt the same ideas to show that $\mathcal{K}(\boldsymbol{Q})$ is a kind of universal coanalytic set.

Lemma. If $A\subseteq N^N$ is any coanalytic set then there is a continuous function $h: N^N\to\mathcal{K}(\boldsymbol{R})$ such that $A=h^{-1}[\mathcal{K}(\boldsymbol{Q})]$.

Proof. Write $Seq=\bigcup_{n\in N}N^n$, and for $\sigma\in Seq$ write

$$I_\sigma=\{\alpha: \sigma\subseteq\alpha\in N^N\}\ .$$

For $\sigma\in N^n$ define q_σ by setting

$$q_\emptyset=0\ ,\qquad q_{k^\frown\sigma}=1/(1+k+q_\sigma)\qquad \forall k\in N\ ,\ \sigma\in Seq\ ,$$

where $k^\frown(i_0,\ldots)=(k,i_0,\ldots)$. Observe that if $\sigma\subseteq\tau\in Seq$ then q_τ lies between q_σ and $q_{\sigma^\frown 0}$; also, that

$$\lim_{n\to\infty}\sup\{|q_\sigma-q_\tau|: \sigma\in N^n,\ \sigma\subseteq\tau\in Seq\}=0\ .$$

Now suppose that $A\subseteq N^N$ is coanalytic. Then there is a closed set $F\subseteq N^N\times N^N$ such that $A=N^N\setminus\pi_1[F]$. Define $h: N^N\to\mathcal{K}(\boldsymbol{R})$ by setting

$$T(\alpha)=\{\sigma: \sigma\in Seq,\ (I_{\alpha|\#(\sigma)}\times I_\sigma)\cap F\neq\emptyset\}$$

(where $\#(\sigma)=n$ if $\sigma\in N^n$),

$$h(\alpha)=\overline{\{q_\sigma: \sigma\in T(\alpha)\}}\subseteq[0,1]$$

for each $\alpha\in N^N$. To confirm that h is continuous, let $\tilde{\varrho}$ be the metric on $\mathcal{K}(\boldsymbol{R})$ derived from the usual metric of $\boldsymbol{R}$. Then if $\alpha,\beta\in N^N$ and

$\alpha\restriction n = \beta\restriction n$, we have

$$\sigma \in T(\alpha) \Rightarrow \sigma\restriction n \in T(\beta)\,, \qquad \sigma \in T(\beta) \Rightarrow \sigma\restriction n \in T(\alpha)\,,$$

so that

$$\varrho(h(\alpha), h(\beta)) \leqslant \sup\{|q_\sigma - q_\tau|\colon \sigma \in \mathbf{N}^n,\ \sigma \subseteq \tau \in Seq\}$$

which is small if n is large.

Observe next that we have a map $g\colon \mathbf{N}^{\mathbf{N}} \to [0,1]\setminus \mathbf{Q}$ given by

$$g(\alpha) = \lim_{n\to\infty} q_{\alpha\restriction n}$$

which is surjective (indeed, a homeomorphism). For $\alpha, \beta \in \mathbf{N}^{\mathbf{N}}$ we have

$$\begin{aligned}(\alpha,\beta)\in F &\Rightarrow (I_{\alpha\restriction n}\times I_{\beta\restriction n})\times F \neq \emptyset \quad \forall n \in \mathbf{N}\,,\\ &\Rightarrow q_{\beta\restriction n} \in h(\alpha) \quad \forall n\in\mathbf{N}\,,\\ &\Rightarrow g(\beta)\in h(\alpha)\,,\\ &\Rightarrow h(\alpha)\cap J(q_{\beta\restriction n}, q_{(\beta\restriction n)^\frown 0}) \neq \emptyset \quad \forall n\in\mathbf{N}\,,\end{aligned}$$

(where $J(s,t)$ is the set of real numbers lying strictly between s and t)

$$\begin{aligned}&\Rightarrow \forall n\in\mathbf{N}\ \exists \sigma\supseteq \beta\restriction n \text{ such that } \sigma\in T(\alpha)\,,\\ &\Rightarrow (\alpha,\beta)\in F\,.\end{aligned}$$

So

$$h^{-1}[\mathcal{K}(\mathbf{Q})] = \{\alpha\colon g(\beta)\notin h(\alpha)\ \ \forall\beta\in\mathbf{N}^{\mathbf{N}}\} = \{\alpha\colon (\alpha,\beta)\notin F\ \ \forall \beta\in\mathbf{N}^{\mathbf{N}}\} = A\,,$$

as required.

13. – Lemma if X and Y are metric spaces and $\mathcal{K}(X)\preccurlyeq\mathcal{K}(Y)$ then $d(X)\leqslant \max(\omega, d(Y))$.

Proof. Recall that for any topological space X, $d(X)$ is the least cardinal of any dense subset of X. Let ϱ be the metric of X and for each $n\in\mathbf{N}$ let $A_n\subseteq X$ be a set maximal subject to the condition that $\varrho(x,y)\geqslant 2^{-n}$ for all distinct $x, y\in A_n$. Then $\bigcup_{n\in\mathbf{N}} A_n$ is a dense subset of X so $d(X)\leqslant \sup_{n\in\mathbf{N}} \#(A_n)$. Fix a Tukey function $f\colon \mathcal{K}(X)\to\mathcal{K}(Y)$.

Suppose, if possible, that $d(X) > \max(\omega, d(Y))$. Then there is an $n\in\mathbf{N}$ such that

$$\#(A_n) > \max(\omega, d(Y)) \geqslant d(\mathcal{K}(Y)) = L(\mathcal{K}(Y))\,,$$

the Lindelöf degree of $\mathcal{K}(Y)$ given its Hausdorff metric. Accordingly there must be a $K\in\mathcal{K}(Y)$ such that

$$\{a\colon a\in A_n,\ f(\{a\})\in V\}$$

is infinite for every neighbourhood V of K in $\mathcal{K}(Y)$; so there is a sequence $\langle a_i\rangle_{i\in\mathbf{N}}$ of distinct points of A_n such that $\langle f(\{a_i\})\rangle_{i\in\mathbf{N}}\to K$ in $\mathcal{K}(Y)$. In this case, however, $\{f(\{a_i\})\colon i\in\mathbf{N}\}$ is bounded above in $\mathcal{K}(Y)$, while $\{\{a_i\}\colon i\in\mathbf{N}\}$ is not bounded above in $\mathcal{K}(X)$; which is impossible.

14. – Proposition. Let X be a metric space. Then the following are equivalent:

(i) $\mathcal{K}(X)\preccurlyeq\mathcal{K}(\mathbf{Q})$;

(ii) X is separable and $\mathcal{K}(X)$ is a PCA ($=\boldsymbol{\Sigma}^1_2$) set in its metric space completion Z;

(iii) X is separable and $\mathcal{K}(X)$ has a cofinal subset which is a PCA subset of Z.

Proof. (*a*) (i) $\Rightarrow$ (ii) If $\mathcal{K}(X)\preccurlyeq\mathcal{K}(\mathbf{Q})$ then X is separable by Lemma 13. Also, Prop. 9 tells us that there is a closed set $R\subseteq\mathcal{K}(\mathbf{R})\times Z$ such that $R[\mathcal{K}(\mathbf{Q})] = \mathcal{K}(X)$. But as $\mathcal{K}(\mathbf{R})$ and Z are Polish and $\mathcal{K}(\mathbf{Q})$ is coanalytic in $\mathcal{K}(\mathbf{R})$ (being the complement of the projection of the G_δ subset

$$\{(\alpha, K)\colon K\in\mathcal{K}(\mathbf{R}),\ \alpha\in K\setminus\mathbf{Q}\}$$

of $\mathbf{R}\times\mathcal{K}(\mathbf{R})$), $\mathcal{K}(X)$ must be PCA in Z.

(*b*) (ii) $\Rightarrow$ (iii) is trivial.

(*c*) (iii) $\Rightarrow$ (i) Let $C\subseteq\mathcal{K}(X)$ be a cofinal subset which is PCA in Z. Then there is a coanalytic set $B\subseteq Z\times\mathbf{N}^{\mathbf{N}}$ such that $C=\pi_1[B]$. Also $Z\times\mathbf{N}^{\mathbf{N}}$ is Polish so is a continuous image of $\mathbf{N}^{\mathbf{N}}$; let $g\colon\mathbf{N}^{\mathbf{N}}\to Z\times\mathbf{N}^{\mathbf{N}}$ be a continuous surjection, and A the coanalytic set $g^{-1}[B]$. By Lemma 12, there is a continuous function $h\colon\mathbf{N}^{\mathbf{N}}\to\mathcal{K}(\mathbf{R})$ such that $A = h^{-1}[\mathcal{K}(\boldsymbol{Q})]$. Now consider

$$S = \{((L, \alpha), K) : L = h(\alpha), K = \pi_1 g(\alpha)\} \subseteq (\mathcal{K}(\boldsymbol{R}) \times \boldsymbol{N}^{\boldsymbol{N}}) \times Z .$$

This S is a closed set in $(\mathcal{K}(\boldsymbol{R}) \times \boldsymbol{N}^{\boldsymbol{N}}) \times Z$ and

$$S[\mathcal{K}(\boldsymbol{Q}) \times \boldsymbol{N}^{\boldsymbol{N}}] = \{\pi_1 g(\alpha) : \alpha \in h^{-1}[\mathcal{K}(\boldsymbol{Q})]\} = C .$$

By Prop. 10 (with $P = \mathcal{K}(X)$, $\tilde{P} = Z$, $Q = \mathcal{K}(\boldsymbol{Q}) \times \boldsymbol{N}^{\boldsymbol{N}}$, $\tilde{Q} = \mathcal{K}(\boldsymbol{R}) \times \boldsymbol{N}^{\boldsymbol{N}}$), we see that $\mathcal{K}(X) \leqslant \boldsymbol{N}^{\boldsymbol{N}} \times \mathcal{K}(\boldsymbol{Q}) \times \boldsymbol{N}^{\boldsymbol{N}}$. But as also $\boldsymbol{N}^{\boldsymbol{N}} \leqslant \mathcal{K}(\boldsymbol{Q})$ (by Lemma 11), $\mathcal{K}(X) \leqslant \mathcal{K}(\boldsymbol{Q})$ (by 4 *c*)).

15. – Theorem. Let X be a metric space.

(*a*) $\mathcal{K}(X) \equiv \{0\}$ iff X is compact.

(*b*) $\mathcal{K}(X) \equiv \boldsymbol{N}$ iff X is separable, locally compact, not compact.

(*c*) $\mathcal{K}(X) \equiv \boldsymbol{N}^{\boldsymbol{N}}$ iff X is Polish, not locally compact.

(*d*) If X is separable, not Polish, and is coanalytic in its completion then $\mathcal{K}(X) \equiv \mathcal{K}(\boldsymbol{Q})$.

Proof. (*a*) See 6 (*a*).

(*b*) (i) If X is separable and locally compact but not compact then $\mathcal{K}(X) \equiv [\boldsymbol{N}]^{<\omega} \equiv \boldsymbol{N}$ by 6 (*b*), 4 (*f*). (ii) If $\mathcal{K}(X) \equiv \boldsymbol{N}$ then X is locally compact (by Lemma 11) and σ-compact, therefore separable, but not of course compact.

(*c*) (i) If X is Polish, but not locally compact, then $\boldsymbol{N}^{\boldsymbol{N}} \leqslant \mathcal{K}(X)$ by Lemma 11. Also $\mathcal{K}(X)$ is itself Polish. So we can apply Prop. 10 with $P = \tilde{P} = \mathcal{K}(X)$, $Q = \tilde{Q} = \{0\}$, $S = \tilde{Q} \times \tilde{P}$ to see that $\mathcal{K}(X) \leqslant \boldsymbol{N}^{\boldsymbol{N}}$. (ii) If $\mathcal{K}(X) \equiv \boldsymbol{N}^{\boldsymbol{N}}$, then X is separable, by Lemma 13. Now apply Prop. 9 with $P = \mathcal{K}(X)$, $\tilde{P} = \mathcal{K}(\hat{X})$ (where $\hat{X}$ is the metric completion of X), $Q = \tilde{Q} = \boldsymbol{N}^{\boldsymbol{N}}$ to see that $\mathcal{K}(X) = \pi_2[R]$ for some closed $R \subseteq \mathcal{K}(\hat{X}) \times \boldsymbol{N}^{\boldsymbol{N}}$, and is therefore analytic. By Christensen's theorem ([2], Thm. 3.3), X must be Polish. Of course it now cannot be locally compact, because $\mathcal{K}(X) \not\leqslant \boldsymbol{N}$.

(*d*) If X is separable, not Polish, and is coanalytic in its completion $\hat{X}$, then $\{(K, x) : K \in \mathcal{K}(\hat{X}), x \in K\}$ is closed in $\mathcal{K}(\hat{X}) \times \hat{X}$, so $\{(K, x) : K \in \mathcal{K}(\hat{X}), x \in K \setminus X\}$ is analytic and $\mathcal{K}(X)$ is coanalytic in $\mathcal{K}(\hat{X})$. By Lemma 14, $\mathcal{K}(X) \leqslant \mathcal{K}(\boldsymbol{Q})$. On the other hand, by Hurewicz' theorem ([5], § 6), X has a closed subset F homeomorphic to $\boldsymbol{Q}$; so that $\mathcal{K}(\boldsymbol{Q}) \equiv \mathcal{K}(F) \leqslant \mathcal{K}(X)$ by 5 (*a*).

16. – The partially ordered set $\mathcal{K}(\boldsymbol{Q})$. In view of 15 (*d*) it seems worth taking a bit of trouble to investigate $\mathcal{K}(\boldsymbol{Q})$. Set

$$\mathfrak{d} = \mathrm{cf}\,(\boldsymbol{N}^{\boldsymbol{N}}) .$$

Theorem. (*a*) $\omega_1 \times \boldsymbol{N}^{\boldsymbol{N}} \leqslant \mathcal{K}(\boldsymbol{Q})$.

(*b*) $\mathcal{K}(\boldsymbol{Q}) \not\leqslant \omega_1 \times \boldsymbol{N}^{\boldsymbol{N}}$.

(*c*) $\mathrm{cf}\,(\mathcal{K}(\boldsymbol{Q})) = \mathfrak{d}$.

Proof. (*a*) Recall that we have functions $\partial_\xi : \mathcal{K}(\boldsymbol{Q}) \to \mathcal{K}(\boldsymbol{Q})$, for each $\xi < \omega_1$, defined by writing

$$\partial_0(K) = K ,$$
$$\partial_{\xi+1}(K) = \{x : x \in \partial_\xi(K),\ x \text{ is not isolated in } \partial_\xi(K)\} ,$$
$$\partial_\xi(K) = \bigcap_{\eta<\xi} \partial_1(K) \quad \text{for limit ordinals } \xi > 0 .$$

(See [9], § 24.IV.) It is easy to check that (i) if $K \subseteq L$ then $\partial_\xi(K) \subseteq \partial_\xi(L)$ for every ξ; (ii)

$$r(K) = \min \{\xi : \partial_\xi(K) = \emptyset\}$$

is a countable ordinal for every $K \in \mathcal{K}(\boldsymbol{Q})$; (iii) for every $\xi < \omega_1$ there is a $K \in \mathcal{K}(\boldsymbol{Q})$ such that $r(K) \geqslant \xi$ (induce on ξ). Accordingly we may choose a function $f : \omega_1 \to \mathcal{K}(\boldsymbol{Q})$ such that $r(f(\xi)) \geqslant \xi$ for every $\xi < \omega_1$, and f will be a Tukey function, so that $\omega_1 \leqslant \mathcal{K}(\boldsymbol{Q})$. By Lemma 11, $\boldsymbol{N}^{\boldsymbol{N}} \leqslant \mathcal{K}(\boldsymbol{Q})$, so $\omega_1 \times \boldsymbol{N}^{\boldsymbol{N}} \leqslant \mathcal{K}(\boldsymbol{Q})$ by 4 (*c*).

(*b*) Let $K \mapsto f(K) = (f_1(K), f_2(K)) : \mathcal{K}(\boldsymbol{Q}) \to \omega_1 \times \boldsymbol{N}^{\boldsymbol{N}}$ be any function. (i) By 14 (*c*), $\mathcal{K}(\boldsymbol{Q}) \not\leqslant \boldsymbol{N}^{\boldsymbol{N}}$, so $f_2 : \mathcal{K}(\boldsymbol{Q}) \to \boldsymbol{N}^{\boldsymbol{N}}$ cannot be a Tukey function. Let $\alpha \in \boldsymbol{N}^{\boldsymbol{N}}$ be such that $\mathcal{A} = \{K : f_2(K) \leqslant \alpha\}$ is not bounded above in $\mathcal{K}(\boldsymbol{Q})$, i.e. $\bigcup \mathcal{A}$ is not relatively compact in $\mathcal{K}(\boldsymbol{Q})$. Let $\mathcal{A}_0 \subseteq \mathcal{A}$ be a countable set such that $\bigcup \mathcal{A}_0$ is dense in $\bigcup \mathcal{A}$; then $\mathcal{A}_0$ is not bounded above in $\mathcal{K}(\boldsymbol{Q})$; but $f_1[\mathcal{A}_0]$, being countable, must be bounded above in ω_1, so that $f[\mathcal{A}_0]$ is bounded above in $\omega_1 \times \boldsymbol{N}^{\boldsymbol{N}}$. Thus f is not a Tukey function. As f is arbitrary, $\mathcal{K}(\boldsymbol{Q}) \not\leqslant \omega_1 \times \boldsymbol{N}^{\boldsymbol{N}}$.

(c) Set

$$\mathfrak{D}_\xi = \{K: K \in \mathcal{K}(\mathbf{Q}),\ \partial_\xi(K) = \emptyset\}$$

for $\xi < \omega_1$, where ∂_ξ is defined in (a) above. I claim that $\mathfrak{D}_\xi \preccurlyeq \mathbf{N}^\mathbf{N}$ for each $\xi < \omega_1$. To see this, construct inductively countable sets $I(\xi)$ and Tukey functions $f_\xi\colon \mathfrak{D}_\xi \to \mathbf{N}^{I(\xi)}$, as follows. (i) $I(0) = \emptyset$, $f_0\colon \mathfrak{D}_0 \to \mathbf{N}^\emptyset$ takes $\emptyset$ to $\emptyset$. (ii) Given f_ξ, enumerate $\mathbf{Q}$ as $\langle q_n\rangle_{n\in\mathbf{N}}$ and a base for the topology of $\mathbf{Q}$, consisting of (relatively) open-and-closed sets, as $\langle U_n\rangle_{n\in\mathbf{N}}$. Set

$$I(\xi+1) = \{0\} \cup (\mathbf{N}\times I(\xi)) .$$

Define $f_{\xi+1}\colon \mathfrak{D}_{\xi+1} \to \mathbf{N}^{I(\xi+1)}$ by setting

$$f_{\xi+1}(K)(0) = \text{least } n \text{ such that } \mathfrak{D}_\xi(K) \subseteq \{q_i\colon i < n\}$$

(of course $\partial_{\xi+1}(K) = \emptyset$ iff $\partial_\xi(K)$ is finite),

$$\begin{aligned} f_{\xi+1}(K)(n, i) &= f_\xi(K \cap U_n)(i) && \text{if } K \cap U_n \in \mathfrak{D}_\xi ,\\ &= 0 && \text{if } K \cap U_n \notin \mathfrak{D}_\xi , \end{aligned}$$

for each $i \in I(\xi)$, $n \in \mathbf{N}$, $K \in \mathfrak{D}_{\xi+1}$. Then it is easy to sheck that $f_{\xi+1}$ is a Tukey function. (iii) If ξ is a non-zero countable limit ordinal, let $\langle \xi(j)\rangle_{j\in\mathbf{N}}$ be a strictly increasing sequence with limit ξ, set

$$I(\xi) = \{0\} \cup \{(n. i)\colon n \in \mathbf{N},\ i \in I(\xi(n))\} ,$$

and set

$$\begin{aligned} f_\xi(K)(0) &= \min\{n\colon K \in \mathfrak{D}_{\xi(n)}\} ,\\ f_\xi(K)(n, i) &= f_{\xi(n)}(K)(i) \quad \text{if } K \in \mathfrak{D}_{\xi(n)} ,\ i \in I_{\xi(n)} ,\\ &= 0 \quad \text{otherwise} . \end{aligned}$$

Again, f_ξ is a Tukey function, so the induction proceeds.

Now $\mathrm{cf}(\mathfrak{D}_\xi) \leqslant \mathrm{cf}(\mathbf{N}^\mathbf{N}) = \mathfrak{d}$ for every $\xi < \omega_1$, so

$$\mathrm{cf}(\mathcal{K}(\mathbf{Q})) \leqslant \max(\omega_1, \mathfrak{d}) = \mathfrak{d} .$$

On the other hand, as $\mathbf{N}^\mathbf{N} \preccurlyeq \mathcal{K}(\mathbf{Q})$, $\mathrm{cf}(\mathcal{K}(\mathbf{Q})) \geqslant \mathfrak{d}$.

17. – After Theorem 15 it is natural to investigate other classes of projective set ([9], § 38; [8], chap. 7; [12]). As might be expected, we encounter undecidable questions. I begin with a miscellany of results in *ZFC*. I follow the notation of [8] and [12], writing $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$ for what [9] calls $\mathbf{L}_{2n-1}$ and $\mathbf{L}_{2n}$.

Proposition. Let $n \geqslant 1$ be an integer.

(a) A separable metric space X is $\mathbf{\Pi}^1_n$ iff $\mathcal{K}(X)$ is $\mathbf{\Pi}^1_n$.

(b) If X and Y are separable metric spaces and $\mathcal{K}(X) \preccurlyeq \mathcal{K}(Y)$ and $\mathcal{K}(Y)$ is $\mathbf{\Sigma}^1_n$ then $\mathcal{K}(X)$ is $\mathbf{\Sigma}^1_n$.

(c) If a separable metric space X is $\mathbf{\Sigma}_{n+1}$ and totally imperfect then $\mathcal{K}(X)$ is $\mathbf{\Sigma}_{n+1}$.

(d) Writing $C = \{0, 1\}^\mathbf{N}$, there is a $\mathbf{\Sigma}^1_n$ set $X \subseteq C$ such that $\mathcal{K}(X)$ is a universal $\mathbf{\Pi}^1_{n+1}$ set in the sense that for any $\mathbf{\Pi}^1_{n+1}$ set $B \subseteq C$ there is a continuous $h\colon C \to \mathcal{K}(C)$ such that $B = h^{-1}[\mathcal{K}(X)]$.

(e) If $X \subseteq C$ has the property described in (d), then $\mathcal{K}(Y) \preccurlyeq \mathcal{K}(X)$ whenever Y is a separable metric space and $\mathcal{K}(Y)$ is $\mathbf{\Sigma}^1_{n+2}$.

(f) In particular, there is an analytic set $X \subseteq C$ such that $\mathcal{K}(\mathbf{Q}) \preccurlyeq \mathcal{K}(X)$ but $\mathcal{K}(\mathbf{Q}) \not\equiv \mathcal{K}(X)$.

Proof. Note that for separable metric spaces, the properties of being $\mathbf{\Sigma}^1_n$ or $\mathbf{\Pi}^1_n$ are topological and intrinsic; see [9], § 38.VII.

(a) Let $\hat{X}$ be the metric completion of X. (i) The map $x \mapsto \{x\}\colon \hat{X} \to \mathcal{K}(\hat{X})$ is continuous; so if $\mathcal{K}(X)$ is $\mathbf{\Pi}^1_n$ in $\mathcal{K}(\hat{X})$ then X is $\mathbf{\Pi}^1_n$ in $\hat{X}$. (ii) If X is $\mathbf{\Pi}^1_n$ in $\hat{X}$ then

$$\{(x, K)\colon K \in \mathcal{K}(\hat{X}),\ x \in K \setminus X\}$$

is $\mathbf{\Sigma}^1_n$ in $\hat{X}\times\mathcal{K}(\hat{X})$ so its projection $\mathcal{K}(\hat{X})\setminus\mathcal{K}(X)$ is $\mathbf{\Sigma}^1_n$ in $\mathcal{K}(\hat{X})$ and $\mathcal{K}(X)$ is $\mathbf{\Pi}^1_n$.

(b) Let $\hat{X}$, $\hat{Y}$ be the metric completions of X, Y. By Prop. 9 there is a closed set $R \subseteq \mathcal{K}(\hat{Y})\times\mathcal{K}(\hat{X})$ such that $\mathcal{K}(X) = R[\mathcal{K}(Y)]$, which is surely $\mathbf{\Sigma}^1_n$ if $\mathcal{K}(Y)$ is.

(c) (i) We need to know the following. Let $Q \subseteq C = \{0, 1\}^\mathbf{N}$ be the countable dense set $\{c\colon c \in C,\ \{n\colon c(n) \neq 0\} \text{ is finite}\}$. Then if L is any non-empty countable compact metric space, there are a continuous function $h\colon C \to L$ and a compact set $K \subseteq Q$ such that

$h[K] = L$. (The proof is by induction on the Cantor-Bendixson rank of L, described in the proof of Theorem 16.) (ii) Now let F be the Polish space of continuous functions from C to the metric completion $\hat{X}$ of X, given the topology of uniform convergence. Consider the set

$$A = \{(f, K): f \in F, K \in \mathcal{K}(Q), f[Q] \subseteq X\} .$$

Then A is a $\mathbf{\Sigma}^1_{n+1}$ set in $F \times \mathcal{K}(C)$. Next, the map

$$(f, K) \mapsto f[K]: F \times \mathcal{K}(C) \to \mathcal{K}(\hat{X})$$

is continuous. So

$$\{f[K]: (f, K) \in A\}$$

is $\mathbf{\Sigma}^1_{n+1}$. But by (i) it is precisely the set of countable compact subsets of X (except in the trivial case $X = \emptyset$), which (because X is totally imperfect) is $\mathcal{K}(X)$ itself.

(*d*) There is a $\mathbf{\Pi}^1_{n+1}$ set $B_1 \subseteq C \times C$ which is universal in the simple sense that every $\mathbf{\Pi}^1_{n+1}$ subset of C is a section of B_1 (see [9], § 38.V). Let $h_0: C \times C \to C$ be any homeomorphism; then $B_2 = h_0 \cdot [B_1] \subseteq C$ is universal in the sense that every $\mathbf{\Pi}^1_{n+1}$ subset of C is the inverse image of B_2 under a continuous function; also B_2 is $\mathbf{\Pi}^1_{n+1}$. Let $A \subseteq C \times C$ be a $\mathbf{\Pi}^1_n$ set such that $B_2 = C \setminus \pi_1[A]$. Set $X_1 = (C \times C) \setminus A$; then X_1 is $\mathbf{\Sigma}^1_n$. Define $\varphi: C \to \mathcal{K}(C \times C)$ by setting $\varphi(c) = \{c\} \times C$; then φ is continuous and $\varphi^{-1}[\mathcal{K}(X_1)] = B_2$. If $B \subseteq C$ is any $\mathbf{\Pi}^1_{n+1}$ set, there is a continuous $h: C \to C$ such that $B = h^{-1}[B_2]$; now $\varphi h: C \to \mathcal{K}(C \times C)$ is continuous and $B = (\varphi h)^{-1}[\mathcal{K}(X_1)]$.

Thus X_1 has the required property except that it is embedded in $C \times C$ rather than in C. But of course $X = h_0[X_1]$ will give the result as stated.

(*e*) Let $\hat{Y}$ be the metric completion of Y, and let $D_1 \subseteq \mathcal{K}(\hat{Y}) \times N^N$ be a $\mathbf{\Pi}^1_{n+1}$ set such that $\pi_1[D_1] = \mathcal{K}(Y)$. Let $h_1: C \to \mathcal{K}(\hat{Y}) \times N^N$ be a Borel measurable surjection; then $D_2 = h_1^{-1}[D_1]$ is $\mathbf{\Pi}^1_{n+1}$. So there is a continuous function $h_2: C \to \mathcal{K}(C)$ such that $D_2 = h_2^{-1}[\mathcal{K}(X)]$. Let S be

$$\{(K, L): \exists c \in C \text{ such that } L = \pi_1 h_1(c),\ K = h_2(c)\}$$

which is an analytic subset of $\mathcal{K}(C) \times \mathcal{K}(\hat{Y})$. We see that $S[\mathcal{K}(X)] = \mathcal{K}(Y)$, so that $\mathcal{K}(Y) \preccurlyeq \mathcal{K}(X) \times N^N$, by Prop. 9. Also, of course, $N^N \preccurlyeq \mathcal{K}(X)$, since otherwise $\mathcal{K}(X)$ has countable cofinality and must be σ-compact. So $\mathcal{K}(Y) \preccurlyeq \mathcal{K}(X)$, as claimed.

(*f*) Taking $n = 1$ in (*d*)-(*e*), we get an analytic set $X \subseteq C$ such that $\mathcal{K}(X)$ is a universal *CPCA* set; in particular, $\mathcal{K}(X)$ is not *PCA*, so $\mathcal{K}(X) \not\preccurlyeq \mathcal{K}(\mathbf{Q})$. On the other hand, $\mathcal{K}(\mathbf{Q}) \preccurlyeq \mathcal{K}(X)$ by (*e*).

18. – If we assume the axiom of projective determinacy, we can now sort out projective separable metric spaces.

THEOREM [*PD*]. Let X and Y be projective separable metric spaces. Then $\mathcal{K}(X) \preccurlyeq \mathcal{K}(Y)$ iff

(i) X is compact or

(ii) X is locally compact and Y is not compact or

(iii) X is Polish, Y is not locally compact or

(iv) X is coanalytic, Y is not Polish or

(v) for some $n \geq 1$, X is $\mathbf{\Pi}^1_{n+1}$ and Y is not $\mathbf{\Pi}^1_n$.

PROOF. (*a*) I shall actually use the following consequence of *PD*: if X and Y are projective subsets of C, then either there is a continuous $h: C \to C$ such that $X = h^{-1}[Y]$, or there is a continuous $h: C \to C$ such that $Y = C \setminus h^{-1}[X]$. (Apply *PD* to

$$A = (X \times Y) \cup ((C \setminus X) \times (C \setminus Y)) \subseteq C \times C ,$$

as in [11], 5.2.)

(*b*) Now suppose that $X \subseteq C$ and that $\mathcal{K}(X)$ is $\mathbf{\Sigma}_{n+1}$, where $n \geq 1$; then X is $\mathbf{\Pi}^1_n$. To see this, take a $\mathbf{\Sigma}^1_n$ set $Y \subseteq C$ such that $\mathcal{K}(Y)$ is a universal $\mathbf{\Pi}^1_{n+1}$ set in the sense of 17 (*d*). Then $\mathcal{K}(Y)$ is surely not $\mathbf{\Sigma}^1_{n+1}$, so $\mathcal{K}(Y) \not\preccurlyeq \mathcal{K}(X)$, by 17 (*b*). Consequently there can be no continuous $h: C \to C$ such that $Y = h^{-1}[X]$, by 5 (*b*). There must therefore be a continuous $h: C \to C$ such that $X = C \setminus h^{-1}[Y]$, which is $\mathbf{\Pi}^1_n$.

(*c*) Next, if $X, Y \subseteq C$ are projective, $\mathcal{K}(X)$ is $\mathbf{\Pi}^1_{n+1}$ and $\mathcal{K}(Y)$ is not $\mathbf{\Pi}^1_n$, then $\mathcal{K}(X) \preccurlyeq \mathcal{K}(Y)$. To see this, note that $\mathcal{K}(C) \setminus \{\emptyset\}$ is homeomorphic to C, being a zero-dimensional compact metric space without isolated points, and that $\mathcal{K}(X)$, $\mathcal{K}(Y)$ are projective subsets of $\mathcal{K}(C)$ containing $\emptyset$ (using 17 (*a*)). Consequently either there is a continuous $h\colon \mathcal{K}(C) \to \mathcal{K}(C)$ such that $\mathcal{K}(X) = h^{-1}[\mathcal{K}(Y)]$, or there is a continuous $h\colon \mathcal{K}(C) \to \mathcal{K}(C)$ such that

$$\mathcal{K}(Y) = (\mathcal{K}(C) \setminus h^{-1}[\mathcal{K}(X)]) \cup \{\emptyset\} .$$

But the latter is impossible because by (*c*) just above $\mathcal{K}(Y)$ is not $\boldsymbol{\Sigma}^1_{n+1}$.

So we must have a continuous $h\colon \mathcal{K}(C) \to \mathcal{K}(C)$ such that $\mathcal{K}(X) = h^{-1}[\mathcal{K}(Y)]$. By Prop. 10, with $S = \{(h(K), K)\colon K \in \mathcal{K}(C)\}$, we see that $\mathcal{K}(X) \leqslant \mathcal{K}(Y) \times \boldsymbol{N}^{\boldsymbol{N}}$. As in 18 (*e*), $\boldsymbol{N}^{\boldsymbol{N}} \leqslant \mathcal{K}(Y)$, so $\mathcal{K}(X) \leqslant \mathcal{K}(Y)$, as claimed.

(*d*) It follows that if X, $Y \subseteq C$, X is coanalytic and Y is projective but not G_δ, then $\mathcal{K}(X) \leqslant \mathcal{K}(Y)$. For if Y is coanalytic, then $\mathcal{K}(X) \leqslant \mathcal{K}(\boldsymbol{Q}) \leqslant \mathcal{K}(Y)$, by Theorem 15; while if Y is not coanalytic, then $\mathcal{K}(Y)$ cannot be coanalytic, and we can use (*d*) with $n = 1$.

(*e*) We now have enough to see that if X, Y are subsets of C and satisfy any of (i)-(v) in the statement of this theorem, then $\mathcal{K}(X) \leqslant \mathcal{K}(Y)$. For (i)-(iii) are covered by Lemma 11 and Theorem 15; (iv) by (*c*) here; and (v) by (*d*).

(*f*) Now suppose that X, Y are projective subsets of C and that $\mathcal{K}(X) \leqslant \mathcal{K}(Y)$. If X is coanalytic, then by Theorem 15 we must have one of the cases (i)-(iv). If X is not coanalytic, then for some $n \geqslant 1$ we have $X \square \boldsymbol{\Pi}^1_{n+1}$ but not $\boldsymbol{\Pi}^1_n$; in this case $\mathcal{K}(X)$ is not $\boldsymbol{\Sigma}^1_{n+1}$ (by (*c*)) so $\mathcal{K}(Y)$ is not $\boldsymbol{\Sigma}^1_{n+1}$ (by 17 (*b*)) and Y cannot be $\boldsymbol{\Pi}^1_n$ (by 17 (*a*)).

(*g*) Thus the theorem is proved when X and Y are both subsets of C. But in the general case they have metrizable compactifications $\tilde{X}$, $\tilde{Y}$ with continuous surjections $g\colon C \to \tilde{X}$, $h\colon C \to \tilde{Y}$ (passing over the trivial case in which one of them is empty). Setting $X' = g^{-1}[X]$, $Y' = h^{-1}[Y]$ we have $\mathcal{K}(X) \leqslant \mathcal{K}(Y)$ iff $\mathcal{K}(X') \leqslant \mathcal{K}(Y')$, by 5 (*b*). But in terms of the classification required in this theorem (compact/locally compact/Polish/$\boldsymbol{\Pi}^1_n$) X is identical to X' and Y is identical to Y'. So we have the general result as stated.

19. – Remarks. This theorem shows that under projective determinacy the Tukey types of $\mathcal{K}(X)$ for projective X are well-ordered of order type ω; moreover, except for minor complications at the beginning, the Tukey type of $\mathcal{K}(X)$ depends precisely on the first n such that X is $\boldsymbol{\Pi}^1_n$. In particular, analytic non-Borel sets are classifieñ with PCA sets at the next level of complexity above $\boldsymbol{Q}$ and the coanalytic sets.

Of course one expects PD to have a drastic, and simplifying, effect on the theory of projective sets. To show that very little of Theorem 19 can be looked for in ZFC I offer the following.

20. – Proposition. (*a*) $[V = L]$ There is a non-coanalytic set $X \subseteq \boldsymbol{R}$ such that $\mathcal{K}(X) \leqslant \mathcal{K}(\boldsymbol{Q})$.

(*b*) $[\mathfrak{p} > \omega_1 + \boldsymbol{L}]$ There is an analytic set $X \subseteq \boldsymbol{R}$ such that $\omega_1 \not\leqslant \mathcal{K}(X)$ (so that $\mathcal{K}(\boldsymbol{Q}) \not\leqslant \mathcal{K}(X)$).

Proof. (*a*) The point is that if $V = L$ then there is a function $f\colon \boldsymbol{N}^{\boldsymbol{N}} \to \boldsymbol{N}^{\boldsymbol{N}}$ such that the graph Γ of f is a totally imperfect coanalytic set; see [12], 5A.6. Let $A \subseteq \boldsymbol{N}^{\boldsymbol{N}}$ be any PCA set which is not $CPCA$ and consider $B = \Gamma \cap (A \times \boldsymbol{N}^{\boldsymbol{N}})$. Because $\pi_1[\Gamma \setminus B] = \boldsymbol{N}^{\boldsymbol{N}} \setminus A$ is not PCA, $\Gamma \setminus B$ cannot be PCA, and B cannot be coanalytic. On the other hand, B is certainly a totally imperfect PCA set, so $\mathcal{K}(B) \leqslant \mathcal{K}(\boldsymbol{Q})$, by 17 (*e*).

Now $\boldsymbol{N}^{\boldsymbol{N}} \times \boldsymbol{N}^{\boldsymbol{N}}$ is homeomorphic to $\boldsymbol{N}^{\boldsymbol{N}}$, so can be embedded as a G_δ set in $\boldsymbol{R}$, anf this copies B onto a non-coanalytic subset X of $\boldsymbol{R}$ such that $\mathcal{K}(X) \leqslant \mathcal{K}(\boldsymbol{Q})$.

(*b*) Recall that $\boldsymbol{L}$ is the assertion « every subset of $\boldsymbol{R}$ of cardinal ω_1 is coanalytic », while « $\mathfrak{p} > \omega_1$ » is a consequence of MA + not-CH; see [3], § 23, or [10], § 3.2.

We need to know the following: if Z is a separable metric space, $\mathcal{E}$ is a family of compact subsets of Z, $\mathcal{F}$ a family of closed subsets of Z, $(\bigcup \mathcal{E}) \cap (\bigcup \mathcal{F}) = \emptyset$, and $\#(\mathcal{E} \cup \mathcal{F}) < \mathfrak{p}$, *then* there is a sequence $\langle H_n \rangle_{n \in N}$ of closed subsets of Z such that $\left(\bigcup_{n \in N} H_n\right) \cap (\bigcup \mathcal{E}) = \emptyset$ and every member of $\mathcal{F}$ is a subset of some H_n. This is proved, though not formally stated, in [3], 23A.

Now let $X \subseteq \boldsymbol{R}$ be any analytic set such that $\#(\boldsymbol{R} \setminus X) = \omega_1$, and $f\colon \omega_1 \to \mathcal{K}(X)$ any function. Applying the fact above with $Z = \boldsymbol{R}$, $\mathcal{F} = \{f(\xi)\colon \xi < \omega_1\}$, $\mathcal{E} = \{\{x\}\colon x \in \boldsymbol{R} \setminus X\}$ we see that we have a sequence $\langle H_n \rangle_{n \in N}$ of closed subsets of $\boldsymbol{R}$ such that every H_n is included in X and every $f(\xi)$ is included in some H_n. Now there must be $m, n \in \boldsymbol{N}$ such that

$$\{\xi\colon f(\xi) \subseteq H_n \cap [-m, m]\}$$

is uncountable. So f cannot be a Tukey function.

21. – Other partially ordered sets. In [4] I examined a variety of partially ordered sets. Most do not seem to have much connexion with those discussed here. Some which are fairly close are $\mathcal{E}$, the set of subsets of $\boldsymbol{R}$ with Lebesgue negligible closures; $\mathcal{F}$, the set of nowhere dense subsets of $\boldsymbol{N}^{\boldsymbol{N}}$; $\mathcal{Z}$, the set of subsets of $\boldsymbol{N}$ with zero asymptotic density; and ℓ^1, the set of summable real sequences. In [4] I showed that $\boldsymbol{N}^{\boldsymbol{N}} \preccurlyeq \mathcal{E} \preccurlyeq \mathcal{F} \preccurlyeq \ell^1$, that $\mathcal{E} \preccurlyeq \mathcal{Z} \preccurlyeq \ell^1$, that $\mathcal{E} \not\preccurlyeq \boldsymbol{N}^{\boldsymbol{N}}$ and that $\mathcal{Z} \not\preccurlyeq \mathcal{F}$. In seeking relations between these and $\mathcal{K}(X)$ spaces, the following proposition is useful.

22. – Proposition. For any partially ordered set P, the following are equivalent:

(i) there is a separable metric space X such that $P \preccurlyeq \mathcal{K}(X)$;

(ii) there is a cofinal subset Q of P carrying a separable metrizable topology such that every compact subset of Q is bounded above (in Q or in P);

(iii) there is a cofinal subset Q of P carrying a separable metrizable topology such that every countable compact subset of Q is bounded above.

Proof. (*a*) (i) $\Rightarrow$ (ii) Take a separable metric space X and a Tukey function $f\colon P \to \mathcal{K}(X)$. Then $\operatorname{cf}(P) \leqslant \operatorname{cf}(\mathcal{K}(X)) \leqslant \#(\mathcal{K}(X)) \leqslant \mathfrak{c}$; take a cofinal subset Q of P with $\#(Q) \leqslant \mathfrak{c}$, and let $g\colon Q \to \boldsymbol{R}$ be any injective function. Let $\mathfrak{S}$ be the coarsest topology on Q rendering both $f\restriction Q$ and g continuous; then $(Q, \mathfrak{S})$ is homeomorphic to a subspaces of $\mathcal{K}(X) \times \boldsymbol{R}$, so is separable and metrizable. If $A \subseteq Q$ is $\mathfrak{S}$-compact, then $f[A]$ is compact in $\mathcal{K}(X)$, therefore bounded above; because f is a Tukey function, A is bounded above in P (or Q).

(*b*) (ii) $\Rightarrow$ (iii) is trivial.

(*c*) (iii) $\Rightarrow$ (i) Take a suitable cofinal $Q \subseteq P$ carrying such a separable metrizable topology $\mathfrak{S}$. Then there is a set $Y \subseteq Q$ such that Y and $Q \setminus Y$ are both totally imperfect. Let $\mathfrak{T}$ be the topology on Q generated by $\mathfrak{S} \cup \{Y, Q \setminus Y\}$. Then $(Q, \mathfrak{T})$ has no uncountable compact subset. Let $h\colon P \to Q$ be any function such that $p \leqslant h(p)$ for every $p \in P$, and define $f\colon P \to \mathcal{K}(Q, \mathfrak{T})$ by setting $f(p) = \{h(p)\}$ for $p \in P$. If $K \in \mathcal{K}(Q)$ then K is countable and $\mathfrak{S}$-compact, so bounded above in Q; let q_0 be an upper bound for K; then if $f(p) \subseteq K$ we must have $p \leqslant h(p) \leqslant q_0$. This shows that f is a Tukey function, so $P \preccurlyeq \mathcal{K}(Q)$.

23. – Proposition. (*a*) There is a separable metric space X such that $\mathcal{F} \preccurlyeq \mathcal{K}(X)$.

(*b*) There is no separable metric space X such that $\mathcal{Z} \preccurlyeq \mathcal{K}(X)$.

(*c*) If X is a metric space, $\mathcal{K}(X) \preccurlyeq \ell^1$ iff X is Polish; in particular, $\mathcal{K}(\boldsymbol{Q}) \not\preccurlyeq \ell^1$.

(*d*) $[V = L]$ $\mathcal{F} \preccurlyeq \mathcal{K}(\boldsymbol{Q})$.

(*e*) $[\operatorname{cov}(\mathcal{N}) > \omega_1]$ $\mathcal{E} \not\preccurlyeq \mathcal{K}(\boldsymbol{Q})$.

Proof. For the definitions of $\mathcal{E}$, $\mathcal{F}$, $\mathcal{Z}$, ℓ^1 see § 21 above.

(*a*) I use Prop. 22 (iii) $\Rightarrow$ (i). Let Q be the set of closed nowhere dense subsets of $\boldsymbol{N}^{\boldsymbol{N}}$. Give Q the topology $\mathfrak{S}$ generated by sets of the form

$$\{F\colon F \in Q, F \cap I_\sigma \neq \emptyset\}, \quad \{F\colon F \in Q, F \cap I_\sigma = \emptyset\}$$

where for $\sigma \in Seq = \bigcup_{n \in \boldsymbol{N}} \boldsymbol{N}^n$ I write $I_\sigma = \{\alpha\colon \sigma \subseteq \alpha \in \boldsymbol{N}^{\boldsymbol{N}}\}$. Then $\mathfrak{S}$ is a separable metrizable topology on Q.

If $K \subseteq Q$ is countable and compact, it is bounded above in Q. To see this, let $\langle F_n \rangle_{n \in \boldsymbol{N}}$ be a sequence running over K (passing over the trivial case $K = \emptyset$) and write $F = \bigcup_{n \in \boldsymbol{N}} F_n$. For any $\sigma \in Seq$, choose inductively a sequence $\langle \sigma_n \rangle_{n \in \boldsymbol{N}}$ in Seq such that

$$\sigma_0 = \sigma, \quad \sigma_{n+1} \supseteq \sigma_n, \quad F_n \cap I_{\sigma_{n+1}} = \emptyset \quad \forall n \in \boldsymbol{N}.$$

Now

$$\{\{E\colon E \in Q, E \cap I_{\sigma_n} = \emptyset\}\colon n \in \boldsymbol{N}\}$$

is an open cover of K, so has a finite subcover, and there is an $n \in \boldsymbol{N}$ such that $E \cap I_{\sigma_n} = \emptyset$ for every $E \in K$. Accordingly $F \cap I_{\sigma_n} = \emptyset$, while $\sigma_n \supseteq \sigma$. As σ is arbitrary, F is nowhere dense, and is an upper bound for K in $\mathcal{F}$.

So by Prop. 22, there is a separable metric X such that $\mathcal{F} \preccurlyeq \mathcal{K}(X)$.

(*b*) (Compare [4], 3Mb.) Let X be any separable metric space and $f\colon \mathcal{Z} \to \mathcal{K}(X)$ any function. Let $\tilde{X}$ be a metrizable compactification of X and let $\langle U_n \rangle_{n \in \boldsymbol{N}}$ be a sequence running over a base for the topology of $\tilde{X}$ which is closed under finite unions and contains $\emptyset$.

For each $n \in \boldsymbol{N}$ let A_n be

$$\{i: i \in \boldsymbol{N}, i \in a \in \mathfrak{Z} \Rightarrow f(a) \cap U_n \neq \emptyset\} .$$

Then there is an $a \in \mathfrak{Z}$ such that $a \cap A_n \neq \emptyset$ whenever A_n is infinite. Let $\langle n(j) \rangle_{j \in \boldsymbol{N}}$ be a sequence in $\boldsymbol{N}$ such that $\langle U_{n(j)} \rangle_{j \in \boldsymbol{N}}$ is an increasing sequence with union $\tilde{X} \setminus f(a)$. Then $\langle A_{n(j)} \rangle_{j \in \boldsymbol{N}}$ is an increasing sequence of finite sets (because they are all disjoint from a). For each $i \in \boldsymbol{N}$ choose $a_i \in \mathfrak{Z}$ such that $i \in a_i$ and $f(a_i) \cap U_{n(j)} = \emptyset$ whenever $j \leqslant i$ and $i \notin A_{n(j)}$. Then $\{i: f(a_i) \cap U_{n(j)} \neq \emptyset\}$ is finite for each j so $\bigcup_{i \in \boldsymbol{N}} f(a_i)$ can have no cluster point in $U_{n(j)} \setminus \bigcup_{i \in \boldsymbol{N}} f(a_i)$. Consequently

$$K = f(a) \cup \bigcup_{i \in \boldsymbol{N}} f(a_i)$$

belongs to $\mathcal{K}(X)$. But of course $\{a_i: i \in \boldsymbol{N}\}$ is not bounded above in $\mathfrak{Z}$. So f cannot have been a Tukey function.

(*c*) Apply Prop. 9 with $P = \mathcal{K}(X)$, $\tilde{P} = \mathcal{K}(\hat{X})$ (where $\hat{X}$ is the metric completion of X), $Q = \tilde{Q} = \ell^1$ to see that if $\mathcal{K}(X) \leqslant \ell^1$ there is a closed set $R \subseteq \ell^1 \times \mathcal{K}(\hat{X})$ such that $R[\ell^1] = \mathcal{K}(X)$. So $\mathcal{K}(X)$ is analytic and X must be Polish, by Christensen's theorem ([2], Thm. 3.3).

(*d*) We need to re-examine the proofs of Prop. 22 and (*a*) above. The topology $\mathfrak{S}$ on Q described in (*a*) corresponds to a homeomorphism h between Q and a subspace Z of $\{0, 1\}^{Seq}$, writing

$$h(F)(\sigma) = 1 \quad \text{if } F \cap I_\sigma \neq \emptyset, \quad 0 \text{ otherwise} .$$

Now Z is precisely the set of those $z \in \{0, 1\}^{Seq}$ such that

(i) for every $\sigma \in Seq$, $z(\sigma) = \max_{i \in \boldsymbol{N}} z(\sigma^\frown i)$,

(ii) for every $\sigma \in Seq$, there is a $\tau \supseteq \sigma$ such that $z(\tau) = 0$.

Thus Z is a Borel set in $\{0, 1\}^{Seq}$. Next, as already remarked in the proof of 20 (*a*) above, $V = L$ implies that there is a function $f: \boldsymbol{N}^{\boldsymbol{N}} \to \boldsymbol{N}^{\boldsymbol{N}}$ with totally imperfect coanalytic graph; as Z and Q are homeomorphic to a Borel subset of $\boldsymbol{N}^{\boldsymbol{N}}$, there is a function $g: Q \to \boldsymbol{N}^{\boldsymbol{N}}$ with totally imperfect coanalytic graph X. The map $q \mapsto \{(q, g(q))\}: Q \to \mathcal{K}(X)$ is now a Tukey function by the arguments of Prop. 22 and (*a*) above. So $Q \leqslant \mathcal{K}(X)$. On the other hand, $\mathcal{K}(X) \leqslant \mathcal{K}(\boldsymbol{Q})$ by Theorem 15, while $\mathcal{F} \equiv Q$ by 4 (*b*). So $\mathcal{F} \leqslant \mathcal{K}(\boldsymbol{Q})$, as claimed.

(*e*) (Compare [4], 3Ma.) By « cov ($\mathcal{N}$) » I mean the smallest cardinal of any cover of $\boldsymbol{R}$ by Lebesgue negligible sets; so the assumption here is that $\boldsymbol{R}$ cannot be covered by ω_1 negligible sets. Let $f: \mathcal{E} \to \mathcal{K}(\boldsymbol{Q})$ be any function. Consider the sets $\mathfrak{D}_\xi$ of part (*c*) of the proof of Theorem 15. Set

$$A_\xi = \bigcup \{E: E \in \mathcal{E}, f(E) \in \mathfrak{D}_\xi\} .$$

Then $\boldsymbol{R} = \bigcup_{\xi < \omega_1} A_\xi$ so there must be some $\xi < \omega_1$ such that $\mu^*(A_\xi) > 0$, where μ is Lebesgue measure.

Take $s \in \boldsymbol{R}$, $m \in \boldsymbol{N}$ such that $0 < s < \mu^*(A_\xi \cap [-m, m])$. Let $g: \mathfrak{D}_\xi \to \boldsymbol{N}^{\boldsymbol{N}}$ be a Tukey function. Define $\alpha(n)$ inductively, for $n \in \boldsymbol{N}$, so that $\mu^*(C_n) > s$ for every $n \in \boldsymbol{N}$, where

$$C_n = [-m, m] \cap \bigcup \{E: f(E) \in \mathfrak{D}_\xi,\ g(f(E))(i) \leqslant \alpha(i)\ \forall i \leqslant n\} .$$

In this case, $\mu(\bar{C}_n) > s$ for each $n \in \boldsymbol{N}$, and $\mu(C) \geqslant s$, where $C = \bigcap_{n \in \boldsymbol{N}} \bar{C}_n$. Let $\langle t_n \rangle_{n \in \boldsymbol{N}}$ enumerate a dense subset of C, and for each $n \in \boldsymbol{N}$ choose $t'_n \in C_n$ such that $|t_n - t'_n| \leqslant 2^{-n}$, $E_n \in \mathcal{E}$ such that $t'_n \in E_n$ and $f(E_n) \in \mathfrak{D}_\xi$ and $g(f(E_n))(i) \leqslant \alpha(i)$ for every $i \leqslant n$. In this case $\{g(f(E_n)): n \in \boldsymbol{N}\}$ is bounded above in $\boldsymbol{N}^{\boldsymbol{N}}$, so $\{f(E_n): n \in \boldsymbol{N}\}$ is bounded above in $\mathfrak{D}_\xi$, therefore in $\mathcal{K}(\boldsymbol{Q})$. But $\{E_n: n \in \boldsymbol{N}\}$ cannot be bounded above in $\mathcal{E}$, because $\overline{\bigcup_{n \in \boldsymbol{N}} E_n}$ contains every non-isolated point of C, so has measure at least s. Thus f is not a Tukey function. As f is arbitrary, $\mathcal{E} \not\leqslant \mathcal{K}(\boldsymbol{Q})$.

24. – Problems. (*a*) Characterize the (metric) spaces X such that $\mathcal{K}(\boldsymbol{Q}) \leqslant \mathcal{K}(X)$.

(*b*) Is it relatively consistent with ZFC to suppose that there is an analytic non-Borel subset X of $\boldsymbol{R}$ such that $\mathcal{K}(X) \leqslant \mathcal{K}(\boldsymbol{Q})$?

Acknowledgements. I should like to thank J. P. R. Christensen and M. Laczkovich for valuable conversations.

Added in proof. The result of § 12 is given in [17], § IV,2, where it is attributed to Hurewicz. Theorem 16 (*c*) may be found in [16].

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