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Chapter 65

Applications

At long last I turn again to some of the results for which the theory outlined in this volume was developed. I start with some relatively elementary ideas using nothing more advanced than §624, showing that locally jump-free virtual local martingales are associated with 'exponential' processes of the same kind (651C). These in turn are associated with identities for integral equations (651G, 651K) and change-of-law results (651I). Ideas at the same level take us to Lévy's characterization of Brownian motion (653F); going deeper, and using the time-changes of §635, we can represent many locally jump-free local martingales in terms of Brownian motion (653G).

The exponential processes of §651 can be thought of as solutions of a particularly simple kind of stochastic differential equation. Working very much harder, we find that we have versions of Picard's theorem, for integral equations with a Lipschitz condition on the integrand, for both the Riemann-sum integral (654G) and the S-integral (654L). A twist in the theory of exponential processes, with a refinement inspired by the theory of financial markets, leads us to the famous Black-Scholes equation (655D).

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651 Exponential processes

Associated with any jump-free integrator is an 'exponential process' (651B); if the integrator is a martingale, the exponential process may be a uniformly integrable martingale (651D-651E). This gives us an important class of non-negative martingales which we can use in change-of-law results (651J).

651A Notation As always, $(\mathfrak{A},\bar\mu,T,\langle\mathfrak{A}_t\rangle_{t\in T},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ is a stochastic integration structure. $L^0=L^0(\mathfrak{A})$; if $h:\mathbb{R}\to\mathbb{R}$ is a Borel measurable function, I write $\bar h$ for either of the corresponding operators on members of L^0 (612Ac) and on processes in L^0 (612B). Limits in L^0 will be taken with respect to the topology of convergence in measure (613B). For a fully adapted process $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$, $u_\downarrow=\lim_{\sigma\downarrow\mathcal{S}}u_\sigma$ is its starting value if this is defined (613Bk). For local integrators $\boldsymbol{v}$ and $\boldsymbol{w}$, $[\boldsymbol{v}\overset{*}{|}\boldsymbol{w}]$ is their covariation and $\boldsymbol{v}^*$ is the quadratic variation of $\boldsymbol{v}$ (617H).

$\mathbb{E}=\mathbb{E}_{\bar\mu}$ is integration with respect to $\bar\mu$ as in §§365 and 613, and $L^1_{\bar\mu}=L^1(\mathfrak{A},\bar\mu)$ is the corresponding L-space, while $\theta(w)=\mathbb{E}(|w|\wedge\chi 1)$ for $w\in L^0$ (613Ba). For $\tau\in\mathcal{T}$, $P_\tau:L^1_{\bar\mu}\to L^1_{\bar\mu}\cap L^0(\mathfrak{A}_\tau)$ is the conditional expectation operator associated with the closed subalgebra $\mathfrak{A}_\tau$. For $z\in L^1_{\bar\mu}$, $\boldsymbol{P}z$ is the martingale $\langle P_\tau z\rangle_{\tau\in\mathcal{T}}$. For $y\in L^0(\mathfrak{A})$, $y\mathbf{1}$ is the constant process on $\{\tau:y\in L^0(\mathfrak{A}_\tau)\}$ with value y (612De).

651B Theorem Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$. Suppose that $\boldsymbol{v}$ is a locally jump-free local integrator, and $\boldsymbol{u}$ a locally moderately oscillatory process, both with domain $\mathcal{S}$. Set $\boldsymbol{z}=\overline{\exp}(\boldsymbol{v}-v_\downarrow\mathbf{1}-\frac{1}{2}\boldsymbol{v}^*)$. Then $\boldsymbol{z}$ is a locally jump-free local integrator, $\boldsymbol{z}=\mathbf{1}+ii_{\boldsymbol{v}}(\boldsymbol{z})$ and $ii_{\boldsymbol{z}}(\boldsymbol{u})=ii_{\boldsymbol{v}}(\boldsymbol{u}\times\boldsymbol{z})$. If $\boldsymbol{v}$ is in fact a jump-free integrator and $\boldsymbol{u}$ a moderately oscillatory process, then $\boldsymbol{z}$ is a jump-free integrator

proof (a) Suppose, to begin with, that $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a jump-free integrator and $\boldsymbol{u}$ is moderately oscillatory.

(i) By 616Ib and 615Gb, $v_\downarrow$ is defined. By 617H-617I, $\boldsymbol{v}^*$ is defined everywhere on $\mathcal{S}$ and is an integrator; by 618T, it is jump-free; by 618Ga, $\boldsymbol{w}=\boldsymbol{v}-v_\downarrow\mathbf{1}-\frac{1}{2}\boldsymbol{v}^*$ and $\boldsymbol{z}=\overline{\exp}(\boldsymbol{w})$ are jump-free, therefore moderately oscillatory.

Express $\boldsymbol{v}^*$, $\boldsymbol{w}$ and $\boldsymbol{z}$ as $\langle v^*_\sigma\rangle_{\sigma\in\mathcal{S}}$, $\langle w_\sigma\rangle_{\sigma\in\mathcal{S}}$ and $\langle z_\sigma\rangle_{\sigma\in\mathcal{S}}$. We have $w_\downarrow=\lim_{\sigma\downarrow\mathcal{S}}v^*_\sigma=0$ (617J(b-i)).

Because $\boldsymbol{v}$ and $\boldsymbol{v}^*$ are integrators, so is $\boldsymbol{w}$; by 616N, $\boldsymbol{z}$ is an integrator. Now $v_\downarrow\mathbf{1}+\frac{1}{2}\boldsymbol{v}^*$ is a jump-free integrator of bounded variation, so

$$\boldsymbol{w}^* = \boldsymbol{v}^* - 2[\boldsymbol{v}\,{}^*_{|}\,v_\downarrow\mathbf{1} + \tfrac{1}{2}\boldsymbol{v}^*] + (v_\downarrow\mathbf{1} + \tfrac{1}{2}\boldsymbol{v}^*)^* = \boldsymbol{v}^*$$

by 624C.

(ii) We have

$$\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{z} = \int_{\mathcal{S}} \boldsymbol{u}\,d(\overline{\exp}(\boldsymbol{w}))$$

$$= \int_{\mathcal{S}} \boldsymbol{u} \times \overline{\exp}(\boldsymbol{w})\,d\boldsymbol{w} + \frac{1}{2}\int_{\mathcal{S}} \boldsymbol{u} \times \overline{\exp}(\boldsymbol{w})\,d\boldsymbol{w}^*$$

(619C, because $\exp' = \exp'' = \exp$)

$$= \int_{\mathcal{S}} \boldsymbol{u} \times \boldsymbol{z}\,d\boldsymbol{w} + \frac{1}{2}\int_{\mathcal{S}} \boldsymbol{u} \times \boldsymbol{z}\,d\boldsymbol{v}^*$$

(because $\boldsymbol{w}^* = \boldsymbol{v}^*$)

$$= \int_{\mathcal{S}} \boldsymbol{u} \times \boldsymbol{z}\,d(\boldsymbol{w} + v_\downarrow\mathbf{1} + \tfrac{1}{2}\boldsymbol{v}^*)$$

($\int_{\mathcal{S}} \boldsymbol{u} \times \boldsymbol{z}\,d(v_\downarrow\mathbf{1}) = 0$ because $v_\downarrow\Delta\mathbf{1}$ is the zero interval function)

$$= \int_{\mathcal{S}} \boldsymbol{u} \times \boldsymbol{z}\,d\boldsymbol{v}.$$

(b) In the general case, in which $\boldsymbol{v}$ is a locally jump-free local integrator and $\boldsymbol{u}$ is locally moderately oscillatory, then for any $\tau \in \mathcal{S}$ we see that $\boldsymbol{z}\restriction\mathcal{S}\wedge\tau$ is a jump-free integrator and $\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\,d\boldsymbol{z} = \int_{\mathcal{S}\wedge\tau} \boldsymbol{u} \times \boldsymbol{z}\,d\boldsymbol{v}$. So $\boldsymbol{z}$ is a locally jump-free local integrator and $ii_{\boldsymbol{z}}(\boldsymbol{u}) = ii_{\boldsymbol{v}}(\boldsymbol{u} \times \boldsymbol{z})$. Now

$$\boldsymbol{z} = z_\downarrow\mathbf{1} + ii_{\boldsymbol{z}}(\mathbf{1}) = \mathbf{1} + ii_{\boldsymbol{v}}(\boldsymbol{z})$$

because

$$z_\downarrow = \overline{\exp}(w_\downarrow) = \overline{\exp}(0) = \chi 1.$$

651C Corollary Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$, and $\boldsymbol{v}$ a locally jump-free virtually local martingale with domain $\mathcal{S}$. Then $\boldsymbol{z} = \overline{\exp}(\boldsymbol{v} - v_\downarrow\mathbf{1} - \frac{1}{2}\boldsymbol{v}^*)$ is a locally jump-free virtually local martingale.

proof By 623Kd, $\boldsymbol{v}$ is a local integrator, so 651B tells us that $\boldsymbol{z}$ is a locally jump-free local integrator (therefore locally moderately oscillatory) and $\boldsymbol{z} = \mathbf{1} + ii_{\boldsymbol{v}}(\boldsymbol{z})$. By 623O, $ii_{\boldsymbol{v}}(\boldsymbol{z})$ and therefore $\boldsymbol{z}$ are virtually local martingales.

651D It is useful to know when $\overline{\exp}(\boldsymbol{v} - v_\downarrow\mathbf{1} - \frac{1}{2}\boldsymbol{v}^*)$ is actually a martingale. The following criterion gives us a sufficient condition.

Theorem Let $\mathcal{S}$ be a non-empty finitely full sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a locally jump-free virtually local martingale with quadratic variation $\boldsymbol{v}^* = \langle v^*_\sigma\rangle_{\sigma\in\mathcal{S}}$. Let $\boldsymbol{z} = \overline{\exp}(\boldsymbol{v} - v_\downarrow\mathbf{1} - \frac{1}{2}\boldsymbol{v}^*)$ be the associated exponential process, as in 651B-651C. If $\beta = \sup_{\sigma\in\mathcal{S}} \mathbb{E}(\overline{\exp}(\frac{1}{2}(v_\sigma - v_\downarrow)))$ is finite, then $\boldsymbol{z}$ is a uniformly integrable martingale.

proof (a) Since $\boldsymbol{v} - v_\downarrow\mathbf{1}$ is a locally jump-free virtually local martingale (using 623Kg) with quadratic variation $\boldsymbol{v}^*$, it is enough to consider the case $v_\downarrow = 0$, so that $\boldsymbol{z} = \overline{\exp}(\boldsymbol{v} - \frac{1}{2}\boldsymbol{v}^*)$. Being a virtually local martingale (651C) and non-negative, $\boldsymbol{z}$ is a $\| \|_1$-bounded supermartingale (627Da) and, setting $z_\sigma = \overline{\exp}(v_\sigma - \frac{1}{2}v^*_\sigma)$,

$$\|z_\sigma\|_1 = \mathbb{E}(z_\sigma) \le \mathbb{E}(z_\downarrow) = 1$$

for every $\sigma \in \mathcal{S}$. As $\mathcal{S}$ is finitely full, $\boldsymbol{z}$ is an approximately local martingale (623K(b-iii)), so $w = \lim_{\sigma\uparrow\mathcal{S}} z_\sigma$ is defined (623La), and $\mathbb{E}(w) = \|w\|_1 \le 1$ (613Bc).

(b) The next thing to observe is that $\boldsymbol{v}$ is $\| \|_1$-bounded. $\mathbf{P}$ Again because $\mathcal{S}$ is finitely full, $\boldsymbol{v}$ also is an approximately local martingale. Take $\tau \in \mathcal{S}$ and $\epsilon > 0$. Then there is a non-empty downwards-directed subset $A \subseteq \mathcal{S}$ such that $\sup_{\rho\in A} \bar\mu[\![\rho < \tau]\!] \le \epsilon$ and $R_A(\boldsymbol{v})$, as defined in 623B, is a martingale. Express it as $\langle v_{A\sigma}\rangle_{\sigma\in\mathcal{S}}$. Then $\lim_{\sigma\downarrow\mathcal{S}} v_{A\sigma} = v_\downarrow = 0$ (623B(c-i)). By 622Ed, 0 is the $\| \|_1$-limit $1\mathrm{lim}_{\sigma\downarrow\mathcal{S}} v_{A\sigma}$ (622Ed) and $\mathbb{E}(v_{A\tau}) = \lim_{\sigma\downarrow\mathcal{S}} \mathbb{E}(v_{A\sigma}) = 0$. It follows that

$$\|v_{A\tau}\|_1 = \mathbb{E}(v_{A\tau}^+) + \mathbb{E}(v_{A\tau}^-) = 2\mathbb{E}(v_{A\tau}^+) = 2\|v_{A\tau}^+\|_1.$$

By definition, $v_{A\tau} = \lim_{\rho\downarrow A} v_{\tau\wedge\rho}$, so $v_{A\tau}^+ = \lim_{\rho\downarrow A} v_{\tau\wedge\rho}^+$ (since $x \mapsto x^+ : L^0 \to L^0$ is continuous), while $x^+ \le 2\,\overline{\exp}(\frac12 x)$ for any $x \in L^0$ because $\max(\alpha, 0) \le 2\exp(\frac12\alpha)$ for every $\alpha \in \mathbb{R}$. So

$$\|v_{\tau\wedge\rho}^+\|_1 = \mathbb{E}(v_{\tau\wedge\rho}^+) \le 2\mathbb{E}(\overline{\exp}(\tfrac12 v_{\tau\wedge\rho}) \le 2\beta$$

for every $\rho \in A$ and

$$\|v_{A\tau}\|_1 = 2\|v_{A\tau}^+\|_1 \le 2\sup_{\rho\in A}\|v_{\tau\wedge\rho}^+\|_1 \le 4\beta,$$

while

$$\theta(v_\tau - v_{A\tau}) \le \bar\mu[\![v_\tau \ne v_{A\tau}]\!] \le \bar\mu(\sup_{\rho\in A}[\![v_\tau \ne v_{\tau\wedge\rho}]\!]) \le \bar\mu(\sup_{\rho\in A}[\![\rho < \tau]\!]) \le \epsilon.$$

As ϵ is arbitrary, $\|v_\tau\|_1 \le 4\beta$; as τ is arbitrary. $\boldsymbol{v}$ is $\| \|_1$-bounded. **Q**

By 623La again, it follows that we have a limit $v = \lim_{\sigma\uparrow\mathcal{S}} v_\sigma$. In this case, $\overline{\exp}(\frac12 v) = \lim_{\sigma\uparrow\mathcal{S}} \overline{\exp}(\frac12 v_\sigma)$ (613Bb), so

$$\mathbb{E}(\overline{\exp}(\tfrac12 v)) \le \liminf_{\sigma\uparrow\mathcal{S}} \mathbb{E}(\overline{\exp}(\tfrac12 v_\sigma)) \le \beta$$

(613Bc once more).

(c) For $0 < \alpha < 1$ and $\sigma \in \mathcal{S}$ set $z_{\alpha\sigma} = \overline{\exp}(\alpha v_\sigma - \frac12\alpha^2 v_\sigma^*)$. If $q = \frac{1}{2\alpha-\alpha^2}$, then $q > 1$ and

$$\mathbb{E}(z_{\alpha\sigma}^q) \le \beta^{(2-2\alpha)/(2-\alpha)}.$$

P

$$\begin{aligned} z_{\alpha\sigma}^q &= \overline{\exp}(q\alpha v_\sigma - \tfrac12 q\alpha^2 v_\sigma^*) = \overline{\exp}(\tfrac{1}{2-\alpha} v_\sigma - \tfrac{\alpha}{2(2-\alpha)} v_\sigma^*) \\ &= \overline{\exp}(\tfrac{\alpha}{2-\alpha} v_\sigma - \tfrac{\alpha}{2(2-\alpha)} v_\sigma^*) \times \overline{\exp}(\tfrac{1-\alpha}{2-\alpha} v_\sigma) \end{aligned}$$

so, by Hölder's inequality (244Eb),

$$\begin{aligned} \mathbb{E}(z_{\alpha\sigma}^q) &\le \|\overline{\exp}(\tfrac{\alpha}{2-\alpha} v_\sigma - \tfrac{\alpha}{2(2-\alpha)} v_\sigma^*)\|_{(2-\alpha)/\alpha} \|\overline{\exp}(\tfrac{1-\alpha}{2-\alpha} v_\sigma)\|_{(2-\alpha)/(2-2\alpha)} \\ &= \big(\mathbb{E}(\overline{\exp}(v_\sigma - \tfrac12 v_\sigma^*))\big)^{\alpha/(2-\alpha)} \big(\mathbb{E}(\overline{\exp}(\tfrac12 v_\sigma))\big)^{(2-2\alpha)/(2-\alpha)} \\ &\le \big(\mathbb{E}(\overline{\exp}(\tfrac12 v_\sigma))\big)^{(2-2\alpha)/(2-\alpha)} \le \beta^{(2-2\alpha)/(2-\alpha)}. \ \mathbf{Q} \end{aligned}$$

It follows that $\{z_{\alpha\sigma} : \sigma \in \mathcal{S}\}$ is uniformly integrable (621Be). But $\alpha\boldsymbol{v}$ is a locally jump-free virtually local martingale with quadratic variation $\alpha^2\boldsymbol{v}^*$, starting from 0, so $\langle z_{\alpha\sigma}\rangle_{\sigma\in\mathcal{S}}$ is a virtually local martingale, by 651C; as $\mathcal{S}$ is finitely full, it is an approximately local martingale (623K(b-iii) again); by 623Nb, it is in fact a martingale, and $w_\alpha = \lim_{\sigma\uparrow\mathcal{S}} z_{\alpha\sigma}$ is defined, with

$$\mathbb{E}(w_\alpha) = \lim_{\sigma\uparrow\mathcal{S}} \mathbb{E}(z_{\alpha\sigma}) = \lim_{\sigma\downarrow\mathcal{S}} \mathbb{E}(z_{\alpha\sigma}) = \mathbb{E}(\lim_{\sigma\downarrow\mathcal{S}} z_{\alpha\sigma}) = 1.$$

(d) For each $\alpha \in \,]0,1[$ and $\sigma \in \mathcal{S}$,

$$\begin{aligned} z_{\alpha\sigma} &= \overline{\exp}(\alpha v_\sigma - \tfrac12\alpha^2 v_\sigma^*) = \overline{\exp}(\alpha^2 v_\sigma - \tfrac12\alpha^2 v_\sigma^*) \times \overline{\exp}(\alpha(1-\alpha)v_\sigma) \\ &= z_\sigma^{\alpha^2} \times \overline{\exp}(\alpha(1-\alpha)v_\sigma); \end{aligned}$$

taking the limit as $\sigma \uparrow \mathcal{S}$,

$$w_\alpha = w^{\alpha^2} \times \overline{\exp}(\alpha(1-\alpha)v)$$

(because multiplication and the operations $u \mapsto |u|^{\alpha^2}$, $u \mapsto \overline{\exp}(\alpha(1-\alpha)u)$ from L^0 to itself are continuous for the topology of convergence in measure). By Hölder's inequality again,

$$\begin{aligned}1 = \mathbb{E}(w_\alpha) &\le \|w^{\alpha^2}\|_{1/\alpha^2}\,\|\overline{\exp}(\alpha(1-\alpha)v)\|_{1/(1-\alpha^2)}\\ &= \big(\mathbb{E}(w)\big)^{\alpha^2}\,\big(\mathbb{E}((\overline{\exp}(\alpha(1-\alpha)v))^{1/(1-\alpha^2)})\big)^{1-\alpha^2}\\ &= \big(\mathbb{E}(w)\big)^{\alpha^2}\,\big(\mathbb{E}((\overline{\exp}(\tfrac{1}{2}v))^{2\alpha/(1+\alpha)})\big)^{1-\alpha^2}.\end{aligned}$$

Taking the limit as $\alpha \uparrow 1$, we get

$$\|w\|_1 = \mathbb{E}(w) \ge 1 \ge \sup_{\sigma\in\mathcal{S}} \|z_\sigma\|_1$$

by (a).

(e) By 623N(b-iii), it follows that $\boldsymbol{z}$ is a uniformly integrable martingale.

651E Corollary Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$ and $\boldsymbol{v}$ a locally jump-free virtually local martingale with domain $\mathcal{S}$. Let $\boldsymbol{z} = \overline{\exp}(\boldsymbol{v} - v_\downarrow\mathbf{1} - \frac{1}{2}\boldsymbol{v}^*)$ be the associated exponential process.

(a) If $\overline{\exp}(\frac{1}{2}\boldsymbol{v}^*)$ is $\|\ \|_1$-bounded, then $\boldsymbol{z}$ is a uniformly integrable martingale.

(b) If $\boldsymbol{v}^*$ is an L^∞-process, then $\boldsymbol{z}$ is a martingale.

proof (a) As in 651D, it is enough to consider the case in which $v_\downarrow = 0$.

(i) Suppose to begin with that $\mathcal{S}$ is finitely full. Express $\boldsymbol{v}$, $\boldsymbol{v}^*$ and $\boldsymbol{z}$ as $\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$, $\langle v^*_\sigma\rangle_{\sigma\in\mathcal{S}}$ and $\langle z_\sigma\rangle_{\sigma\in\mathcal{S}}$. For any non-negative u, $v \in L^0$,

$$(\mathbb{E}(\sqrt{u\times v}))^2 \le \|\sqrt{u}\|_2^2\,\|\sqrt{v}\|_2^2 = \mathbb{E}(u)\,\mathbb{E}(v).$$

Now for any $\sigma\in\mathcal{S}$ we have $\overline{\exp}(v_\sigma) = z_\sigma \times \overline{\exp}(\frac{1}{2}v^*_\sigma)$, so

$$(\mathbb{E}(\overline{\exp}(\tfrac{1}{2}v_\sigma)))^2 \le \mathbb{E}(z_\sigma)\,\mathbb{E}(\overline{\exp}(\tfrac{1}{2}v^*_\sigma)) \le \mathbb{E}(\overline{\exp}(\tfrac{1}{2}v^*_\sigma))$$

because $\boldsymbol{z}$ is a supermartingale starting at $\chi 1$, as in part (a) of the proof of 651D. Accordingly

$$\sup_{\sigma\in\mathcal{S}} \mathbb{E}(\overline{\exp}(\tfrac{1}{2}v_\sigma)) \le \sqrt{\sup_{\sigma\in\mathcal{S}} \mathbb{E}(\overline{\exp}(\tfrac{1}{2}v^*_\sigma))}$$

is finite, and $\boldsymbol{z}$ is a uniformly integrable martingale, by 651D.

(ii) For the general case, let $\hat{\mathcal{S}}_f$ be the finitely-covered envelope of $\mathcal{S}$, and $\hat{\boldsymbol{v}} = \langle\hat{v}_\tau\rangle_{\tau\in\hat{\mathcal{S}}_f}$, $\hat{\boldsymbol{v}}^*$ and $\hat{\boldsymbol{z}}$ the fully adapted extensions of $\boldsymbol{v}$, $\boldsymbol{v}^*$ and $\boldsymbol{z}$ to $\hat{\mathcal{S}}_f$. Then $\hat{\boldsymbol{v}}^*$ is the quadratic variation of $\hat{\boldsymbol{v}}$ (617N), the starting value of $\hat{\boldsymbol{v}}$ is 0 (615H) and $\hat{\boldsymbol{z}} = \overline{\exp}(\hat{\boldsymbol{v}} - \frac{1}{2}\hat{\boldsymbol{v}}^*)$ (612Qb). Also $\hat{\boldsymbol{v}}^*$ is non-decreasing, so $\overline{\exp}(\frac{1}{2}\hat{\boldsymbol{v}}^*)$ is non-decreasing (614If). Since $\mathcal{S}$ is cofinal with $\hat{\mathcal{S}}_f$ (611Pe), $\sup_{\tau\in\hat{\mathcal{S}}_f}\mathbb{E}(\overline{\exp}(\frac{1}{2}\hat{v}_\tau)) = \sup_{\sigma\in\mathcal{S}}\mathbb{E}(\overline{\exp}(\frac{1}{2}v_\sigma))$ is finite. So (i) tells us that $\hat{\boldsymbol{z}}$ is a uniformly integrable martingale and therefore $\boldsymbol{z} = \hat{\boldsymbol{z}}\restriction\mathcal{S}$ is a uniformly integrable martingale, as claimed.

(b) This is immediate from (a).

651F Corollary If $\boldsymbol{w}$ is Brownian motion (612T), then $\overline{\exp}(\boldsymbol{w} - \frac{1}{2}\boldsymbol{\iota})\restriction\mathcal{T}_b$ is a martingale.

proof As observed in 624F, $\boldsymbol{w}^*$ can be identified with the identity process $\boldsymbol{\iota}$ on $\mathcal{T}_f$, and therefore lies in L^∞ on $\mathcal{T}_b$, so we can apply 651Eb to $\boldsymbol{w}\restriction\mathcal{T}_b\wedge\tau$ for each $\tau\in\mathcal{T}_b$.

651G Theorem Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$, and $\boldsymbol{v}$, $\boldsymbol{w}$ locally jump-free local integrators with domain $\mathcal{S}$. Set $\boldsymbol{z} = \overline{\exp}(\boldsymbol{v} - v_\downarrow\mathbf{1} - \frac{1}{2}\boldsymbol{v}^*)$ and $\boldsymbol{y} = \boldsymbol{w} - [\boldsymbol{v}\overset{*}{|}\boldsymbol{w}]$. Then

$$ii_{\boldsymbol{y}\times\boldsymbol{z}}(\boldsymbol{u}) = ii_{\boldsymbol{w}}(\boldsymbol{u}\times\boldsymbol{z}) + ii_{\boldsymbol{v}}(\boldsymbol{u}\times\boldsymbol{y}\times\boldsymbol{z})$$

for any locally moderately oscillatory process $\boldsymbol{u}$ with domain $\mathcal{S}$.

proof We know that $\boldsymbol{y}$ and $\boldsymbol{z}$ are locally jump-free local integrators (618T, 617I, 616N again), so $\boldsymbol{y}\times\boldsymbol{z}$ also is (618Ga, 616Qa), and $\boldsymbol{u}\times\boldsymbol{y}$ and $\boldsymbol{u}\times\boldsymbol{z}$ are locally moderately oscillatory (616Ib again, 615F(b-iii)). Setting $\boldsymbol{z}' = ii_{\boldsymbol{v}}(\boldsymbol{z})$, we know that $\boldsymbol{z}' - \boldsymbol{z}$ is constant (651B). If $\tau\in\mathcal{S}$, then

$$\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\,d(\boldsymbol{y}\times\boldsymbol{z})=\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\times\boldsymbol{z}\,d\boldsymbol{y}+\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\times\boldsymbol{y}\,d\boldsymbol{z}+\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\,d\boldsymbol{y}d\boldsymbol{z}$$

(613M)

$$=\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\times\boldsymbol{z}\,d\boldsymbol{w}-\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\times\boldsymbol{z}\,d[\boldsymbol{v}\overset{*}{|}\boldsymbol{w}]+\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\times\boldsymbol{y}\times\boldsymbol{z}\,d\boldsymbol{v}+\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\,d\boldsymbol{z}'d\boldsymbol{y}$$

(651B, and because the interval functions $\Delta\boldsymbol{z}$ and $\Delta\boldsymbol{z}'$ are equal, so $\Delta\boldsymbol{y}\times\Delta\boldsymbol{z}=\Delta\boldsymbol{z}'\times\Delta\boldsymbol{y}$ and $\int_{\mathcal{S}\wedge\tau}\ldots d\boldsymbol{y}d\boldsymbol{z}=\int_{\mathcal{S}\wedge\tau}\ldots d\boldsymbol{z}'d\boldsymbol{y}$)

$$=\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\times\boldsymbol{z}\,d\boldsymbol{w}-\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\times\boldsymbol{z}\,d\boldsymbol{v}d\boldsymbol{w}+\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\times\boldsymbol{y}\times\boldsymbol{z}\,d\boldsymbol{v}+\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\times\boldsymbol{z}\,d\boldsymbol{v}d\boldsymbol{y}$$

(617I once more, 617Q)

$$=\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\times\boldsymbol{z}\,d\boldsymbol{w}-\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\times\boldsymbol{z}\,d\boldsymbol{v}d\boldsymbol{w}+\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\times\boldsymbol{y}\times\boldsymbol{z}\,d\boldsymbol{v}+\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\times\boldsymbol{z}\,d\boldsymbol{w}d\boldsymbol{v}-\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\times\boldsymbol{z}\,d\boldsymbol{v}d[\boldsymbol{w}\overset{*}{|}\boldsymbol{v}]$$

(because $\Delta\boldsymbol{v}\times\Delta\boldsymbol{y}=\Delta\boldsymbol{w}\times\Delta\boldsymbol{v}-\Delta[\boldsymbol{w}\overset{*}{|}\boldsymbol{v}]\times\Delta\boldsymbol{v}$)

$$=\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\times\boldsymbol{z}\,d\boldsymbol{w}+\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\times\boldsymbol{y}\times\boldsymbol{z}\,d\boldsymbol{v}-\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\times\boldsymbol{z}\,d[\boldsymbol{v}\overset{*}{|}[\boldsymbol{w}\overset{*}{|}\boldsymbol{v}]]$$

$$=\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\times\boldsymbol{z}\,d\boldsymbol{w}+\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\times\boldsymbol{y}\times\boldsymbol{z}\,d\boldsymbol{v}$$

by 624C again, because $\boldsymbol{v}$ is locally jump-free and $[\boldsymbol{w}\overset{*}{|}\boldsymbol{v}]$ is locally of bounded variation, so $[\boldsymbol{v}\overset{*}{|}[\boldsymbol{w}\overset{*}{|}\boldsymbol{v}]]=0$.

651H Corollary Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$, and $\boldsymbol{v}$, $\boldsymbol{w}$ locally jump-free virtually local martingales. Set $\boldsymbol{z}=\overline{\exp}(\boldsymbol{v}-v_{\downarrow}\boldsymbol{1}-\frac{1}{2}\boldsymbol{v}^*)$ and $\boldsymbol{y}=\boldsymbol{w}-[\boldsymbol{w}\overset{*}{|}\boldsymbol{v}]$. Then $\boldsymbol{y}\times\boldsymbol{z}$ is a virtually local martingale.

proof Since $\boldsymbol{v}^*$ and $[\boldsymbol{w}\overset{*}{|}\boldsymbol{v}]$ both start from 0, $\boldsymbol{y}\times\boldsymbol{z}$ starts from a value $w_{\downarrow}$ with finite expectation (623Kg). By 623Kd again, $\boldsymbol{v}$ and $\boldsymbol{w}$ are local integrators. Now

$$\boldsymbol{y}\times\boldsymbol{z}-w_{\downarrow}\boldsymbol{1}=ii_{\boldsymbol{y}\times\boldsymbol{z}}(\boldsymbol{1})=ii_{\boldsymbol{w}}(\boldsymbol{z})+ii_{\boldsymbol{v}}(\boldsymbol{y}\times\boldsymbol{z})$$

is a virtually local martingale by 623O again, so $\boldsymbol{y}\times\boldsymbol{z}$ also is.

651I Proposition Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$ and $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$, $\boldsymbol{w}=\langle w_\sigma\rangle_{\sigma\in\mathcal{S}}$ locally jump-free virtually local martingales such that $\boldsymbol{z}=\overline{\exp}(\boldsymbol{v}-v_{\downarrow}\boldsymbol{1}-\frac{1}{2}\boldsymbol{v}^*)$ is a uniformly integrable martingale and $v'=\lim_{\sigma\uparrow\mathcal{S}}(v_\sigma-\frac{1}{2}v_\sigma^*)$ is defined in L^0, where $\boldsymbol{v}^*=\langle v_\sigma^*\rangle_{\sigma\in\mathcal{S}}$. Set $\boldsymbol{y}=\boldsymbol{w}-[\boldsymbol{w}\overset{*}{|}\boldsymbol{v}]$. Then there is a change of law on $\mathfrak{A}$ rendering $\boldsymbol{y}$ a virtually local martingale.

proof Because $\boldsymbol{z}$ is a uniformly integrable martingale, it must be $\boldsymbol{P}z\restriction\mathcal{S}$, where

$$z=\lim_{\sigma\uparrow\mathcal{S}}\overline{\exp}(v_\sigma-v_{\downarrow}-\tfrac{1}{2}v_\sigma^*)=\overline{\exp}(v'-v_{\downarrow})$$

belongs to $L^1_{\bar\mu}$ (622J); note that $[\![z>0]\!]=1$. Now 651H tells us that $\boldsymbol{y}\times\boldsymbol{P}z=\boldsymbol{y}\times\boldsymbol{z}$ is a virtually local $\bar\mu$-martingale. Set $\gamma=\mathbb{E}_{\bar\mu}(z)$ and $\bar\nu a=\frac{1}{\gamma}\mathbb{E}_{\bar\mu}(z\times\chi a)$ for $a\in\mathfrak{A}$. Then $(\mathfrak{A},\bar\nu)$ is a probability algebra and $\boldsymbol{y}$ is a virtually local $\bar\nu$-martingale, by 625C.

651J Corollary Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$, $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a locally moderately oscillatory process such that $\gamma=\|\boldsymbol{u}\|_\infty$ is finite, and $\boldsymbol{w}=\langle w_\sigma\rangle_{\sigma\in\mathcal{S}}$ a locally jump-free virtually local martingale with quadratic variation $\boldsymbol{w}^*$ such that $\overline{\exp}(\frac{1}{2}\gamma^2\boldsymbol{w}^*)$ is a $\| \|_1$-bounded process. Set $\boldsymbol{y} = \boldsymbol{w} + ii_{\boldsymbol{w}^*}(\boldsymbol{u})$. Then there is a change of law on $\mathfrak{A}$ rendering $\boldsymbol{y}$ a virtually local martingale.

proof Set $\boldsymbol{v} = -ii_{\boldsymbol{w}}(\boldsymbol{u})$; note that its starting value is 0. By 618R and 623O once more, $\boldsymbol{v}$ is a locally jump-free virtually local martingale. Express the quadratic variations $\boldsymbol{v}^*$, $\boldsymbol{w}^*$ as $\langle v^*_\sigma\rangle_{\sigma\in\mathcal{S}}$ and $\langle w^*_\sigma\rangle_{\sigma\in\mathcal{S}}$. By 617Q(a-iii),

$$v^*_\sigma = \int_{\mathcal{S}\wedge\sigma}(-\boldsymbol{u})^2\,d\boldsymbol{w}^* \le \gamma^2\int_{\mathcal{S}\wedge\sigma}d\boldsymbol{w}^* \le \gamma^2 w^*_\sigma$$

and $\sup_{\sigma\in\mathcal{S}}\mathbb{E}(\overline{\exp}(\frac{1}{2}v^*_\sigma))$ is finite. By 651Ea, $\boldsymbol{z} = \overline{\exp}(\boldsymbol{v} - \frac{1}{2}\boldsymbol{v}^*)$ is a uniformly integrable martingale.

On the other hand,

$$ii_{\boldsymbol{w}^*}(\boldsymbol{u}) = ii_{[\boldsymbol{w}\overset{*}{|}\boldsymbol{w}]}(\boldsymbol{u}\times\boldsymbol{1}) = [ii_{\boldsymbol{w}}(\boldsymbol{u})\overset{*}{|}ii_{\boldsymbol{w}}(\boldsymbol{1})]$$

(617Q(a-ii))

$$= -[\boldsymbol{v}\overset{*}{|}\boldsymbol{w} - w_\downarrow\boldsymbol{1}] = -[\boldsymbol{v}\overset{*}{|}\boldsymbol{w}]$$

by 624C once more. So $\boldsymbol{y}$ is equal to $\boldsymbol{w} - [\boldsymbol{w}\overset{*}{|}\boldsymbol{v}]$ and can be made into a virtually local martingale by a change of law, by 651I.

Remark Of course the leading application is to the case in which $\boldsymbol{w}$ is Brownian motion and $\mathcal{S}$ is bounded above in $\mathcal{T}_b$, so that $\sup_{\sigma\in\mathcal{S}}\mathbb{E}(\overline{\exp}(\frac{1}{2}\gamma^2 w^*_\sigma))$ is finite for every γ.

651K S-integrals It is easy to find re-formulations of 651B and 651G in terms of S-integrals, as follows.

Theorem Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous, and that $\mathcal{S}$ is a non-empty order-convex sublattice of $\mathcal{T}$. Let $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ be a jump-free integrator, and $\boldsymbol{x}$ an S-integrable process with domain $\mathcal{S}$. Set $\boldsymbol{z} = \overline{\exp}(\boldsymbol{v} - v_\downarrow\boldsymbol{1} - \frac{1}{2}\boldsymbol{v}^*)$.

(a)

$$\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{z} = \oint_{\mathcal{S}}\boldsymbol{x}\times\boldsymbol{z}\,d\boldsymbol{v}.$$

(b) Suppose that $\boldsymbol{w}$ is another jump-free integrator with domain $\mathcal{S}$. Set $\boldsymbol{y} = \boldsymbol{w} - [\boldsymbol{v}\overset{*}{|}\boldsymbol{w}]$. Then

$$\oint_{\mathcal{S}}\boldsymbol{x}\,d(\boldsymbol{y}\times\boldsymbol{z}) = \oint_{\mathcal{S}}\boldsymbol{x}\times\boldsymbol{z}\,d\boldsymbol{w} + \oint_{\mathcal{S}}\boldsymbol{x}\times\boldsymbol{y}\times\boldsymbol{z}\,d\boldsymbol{v}.$$

proof (a) We know that $\boldsymbol{z}$ is jump-free (651B). So $\boldsymbol{z}\times\boldsymbol{1}^{(\mathcal{S})}_{<} = \boldsymbol{z}_<$ (641O) and $\boldsymbol{z}$ and $\boldsymbol{x}\times\boldsymbol{z}$ are S-integrable (645F, 645Ka, 645Pb). Next,

$$\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{z}) = \mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{z}\times\boldsymbol{1}^{(\mathcal{S})}_{<})$$

(646Kb)

$$= \mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{z}_<) = ii_{\boldsymbol{v}}(\boldsymbol{z})$$

(646Kc)

$$= \boldsymbol{z} - \boldsymbol{1}$$

(651B). So 646R tells us that

$$\oint_{\mathcal{S}}\boldsymbol{x}\times\boldsymbol{z}\,d\boldsymbol{v} = \oint_{\mathcal{S}}\boldsymbol{x}\,d(\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{z})) = \oint_{\mathcal{S}}\boldsymbol{x}\,d(\boldsymbol{z}-\boldsymbol{1}) = \oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{z} - \oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{1} = \oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{z}.$$

(b) Similarly,

$$\oint_{\mathcal{S}}\boldsymbol{x}\times\boldsymbol{z}\,d\boldsymbol{w} + \oint_{\mathcal{S}}\boldsymbol{x}\times\boldsymbol{y}\times\boldsymbol{z}\,d\boldsymbol{v} = \oint_{\mathcal{S}}\boldsymbol{x}\,d(\mathrm{S}ii_{\boldsymbol{w}}(\boldsymbol{z})) + \oint_{\mathcal{S}}\boldsymbol{x}\,d(\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{y}\times\boldsymbol{z}))$$

(because $\boldsymbol{y}$ and $\boldsymbol{y}\times\boldsymbol{z}$ are jump-free)

$$= \oint_{\mathcal{S}} \boldsymbol{x}\, d(ii_{\boldsymbol{w}}(\boldsymbol{z})) + \oint_{\mathcal{S}} \boldsymbol{x}\, d(ii_{\boldsymbol{v}}(\boldsymbol{y}\times\boldsymbol{z}))$$

(again because $\boldsymbol{y}\times\boldsymbol{z}$ is jump-free)

$$= \oint_{\mathcal{S}} \boldsymbol{x}\, d(ii_{\boldsymbol{w}}(\boldsymbol{z}) + ii_{\boldsymbol{v}}(\boldsymbol{y}\times\boldsymbol{z})).$$

But setting $\boldsymbol{u} = \mathbf{1}^{(\mathcal{S})}$ in 651G, we have

$$ii_{\boldsymbol{w}}(\boldsymbol{z}) + ii_{\boldsymbol{v}}(\boldsymbol{y}\times\boldsymbol{z}) = ii_{\boldsymbol{y}\times\boldsymbol{z}}(\mathbf{1}^{(\mathcal{S})}) = \boldsymbol{y}\times\boldsymbol{z} - w_{\downarrow}\mathbf{1}^{(\mathcal{S})}$$

where $w_{\downarrow}$ is the common starting value of $\boldsymbol{w}$, $\boldsymbol{y}$ and $\boldsymbol{y}\times\boldsymbol{z}$. So

$$\begin{aligned}\oint_{\mathcal{S}} \boldsymbol{x}\times\boldsymbol{z}\, d\boldsymbol{w} + \oint_{\mathcal{S}} \boldsymbol{x}\times\boldsymbol{y}\times\boldsymbol{z}\, d\boldsymbol{v} &= \oint_{\mathcal{S}} \boldsymbol{x}\, d(\boldsymbol{y}\times\boldsymbol{z}) - \oint_{\mathcal{S}} \boldsymbol{x}\, d(w_{\downarrow}\mathbf{1}^{(\mathcal{S})})\\ &= \oint_{\mathcal{S}} \boldsymbol{x}\, d(\boldsymbol{y}\times\boldsymbol{z}),\end{aligned}$$

as claimed.

651X Basic exercises (a) Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$. Suppose that $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a locally jump-free local integrator, and $y \in \bigcap_{\sigma\in\mathcal{S}} L^0(\mathfrak{A}_\sigma)$. Set $v_{\downarrow} = \lim_{\sigma\downarrow\mathcal{S}} v_\sigma$ and $\boldsymbol{z} = y\,\overline{\exp}(\boldsymbol{v} - v_{\downarrow}\mathbf{1} - \frac{1}{2}\boldsymbol{v}^*)$. Show that $\boldsymbol{z} = y\mathbf{1} + ii_{\boldsymbol{v}}(\boldsymbol{z})$.

651 Notes and comments We are trying to combine the fundamental result 623O, which tells us we have a virtually local martingale, with 625B, which will demand a true martingale if we are to identify our exponential process $\boldsymbol{z} = \overline{\exp}(\boldsymbol{v} - v_{\downarrow}\mathbf{1} - \frac{1}{2}\boldsymbol{v}^*)$ with $\boldsymbol{P}z$ for a Radon-Nikodým derivative z. So a good deal of work has been done on this, of which I offer a sample in 651D-651E. In this context Brownian motion (651F) is more than the leading example; it is the archetype of a locally jump-free martingale, in a sense which I hope to make clear in §653.

Note that $\boldsymbol{z}$ can also be regarded as a solution of the stochastic integral equation

$$z_\tau = \chi 1 + \int_{\mathcal{S}\wedge\tau} \boldsymbol{z}\, d\boldsymbol{v}$$

(651B). The correction term $-\frac{1}{2}\boldsymbol{v}^*$ in the formula for $\boldsymbol{z}$ vanishes if $\boldsymbol{v}$ is of bounded variation, leaving us with the solution $z_\tau = \overline{\exp}(v_\tau)$ corresponding to the solution of the integral equation

$$z(t) = 1 + \int_0^t z(s)v'(s)ds$$

when $v(0) = 0$. For martingales $\boldsymbol{v}$ we need the term in $\boldsymbol{v}^*$ here just as we need it in Itô's Lemma. I will return to integral equations in §654.

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652 Lévy processes

When defining the Poisson process in 612U, I referred to 455P in Volume 4. §455 is hard work; it is the longest section in the whole treatise and bristles with technical difficulties. Some of these are exacerbated by the generality which seemed natural at that point – for instance, it is meant to support the treatment of multidimensional Brownian motion in Chapter 47. The processes described in the last quarter of §455 take values in Polish groups which need not even be abelian. But these processes have always been the standard-bearers for the theory of stochastic processes I have set out to describe in the present volume, and the time has come to link the approaches.

652A Notation If $(\mathfrak{A}, \bar\mu, T, \langle\mathfrak{A}_t\rangle_{t\in T}, \mathcal{T}, \langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ is a stochastic integration structure, I write $\check t$ for the constant stopping time at t, for each $t \in T$, $\check T$ for the sublattice $\{\check t : t \in T\}$ of $\mathcal{T}$, and $\mathcal{T}_b$, $\mathcal{T}_f \subseteq \mathcal{T}$ for the ideals of bounded and finite-valued stopping times. μ_L will be Lebesgue measure on $\mathbb{R}$. For any measure μ,

$\mathcal{N}(\mu)$ will be the ideal of μ-negligible sets. For a topological space Ω, $\mathcal{B}(\Omega)$ will be the Borel σ-algebra of Ω. If $\mathcal{S}$ is a sublattice of $\mathcal{T}$, $\mathcal{I}(\mathcal{S})$ will be the set of finite sublattices of $\mathcal{S}$, and $\mathcal{S}^{2\uparrow}$ will be $\{(\sigma,\tau) : \sigma,\,\tau \in \mathcal{S},\ \sigma \le \tau\}$.

652B Independence (a) Since Chapter 27 I have given the fundamental concept of 'stochastic independence' rather cavalier treatment. So perhaps I should run through the definitions in the forms I will use here. If $(\mathfrak{A},\bar\mu)$ is a probability algebra, two subalgebras $\mathfrak{B}$, $\mathfrak{C}$ of $\mathfrak{A}$ are (stochastically) independent if $\bar\mu(b\cap c)=\bar\mu b\cdot\bar\mu c$ whenever $b\in\mathfrak{B}$ and $c\in\mathfrak{C}$; more generally, a family $\langle\mathfrak{B}_i\rangle_{i\in I}$ of subalgebras of $\mathfrak{A}$ is independent if $\bar\mu(\inf_{i\in J}b_i)=\prod_{i\in J}\bar\mu b_i$ whenever $J\subseteq I$ is finite and $b_i\in\mathfrak{B}_i$ for every $i\in J$ (325L, 458La). Turning to families in $L^0=L^0(\mathfrak{A})$, $\langle u_i\rangle_{i\in I}$ is independent if $\langle\mathfrak{B}_i\rangle_{i\in I}$ is independent where $\mathfrak{B}_i=\{[\![u_i\in E]\!] : E\subseteq\mathbb{R}$ is Borel$\}$ is the closed subalgebra generated by u_i for each $i\in I$. $u\in L^0$ is independent of an algebra $\mathfrak{C}\subseteq\mathfrak{A}$ if $\mathfrak{B}$ and $\mathfrak{C}$ are independent where $\mathfrak{B}$ is the closed subalgebra generated by u. More elaborately, a family $\langle u_i\rangle_{i\in I}$ in L^0 is independent of $\mathfrak{C}$, where $\mathfrak{C}$ is a subalgebra of $\mathfrak{A}$, if $\mathfrak{B}$ and $\mathfrak{C}$ are independent, where $\mathfrak{B}$ is the smallest closed subalgebra of $\mathfrak{A}$ including all the subalgebras generated by the u_i.

(b) Using the Monotone Class Theorem in the form 313Gc, we see that if B, $C\subseteq\mathfrak{A}$ are such that both B and C are closed under $\cap$ and $\bar\mu(b\cap c)=\bar\mu b\cdot\bar\mu c$ for all $b\in B$ and $c\in C$, then the closed subalgebras $\mathfrak{B}$, $\mathfrak{C}$ generated by B, C respectively are independent. (Show first that $\bar\mu(b\cap c)=\bar\mu b\cdot\bar\mu c$ for $b\in B$ and $c\in\mathfrak{C}$.)

(c) If $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$, the set $C=\{u : u\in L^0,\ u$ is independent of $\mathfrak{B}\}$ is closed for the topology of convergence in measure. **P** Take $v\in\overline{C}$. $\alpha\in\mathbb{R}$ and $\epsilon>0$. Write d for $[\![v>\alpha]\!]$. Then there are a $\delta>0$ such that $\bar\mu(d\setminus[\![v>\alpha+2\delta]\!])\le\epsilon$, and a $u\in C$ such that $\theta(v-u)\le\epsilon\delta$. Consider $c=[\![u>\alpha+\delta]\!]$. Then $c\setminus d\subseteq[\![u-v>\delta]\!]$ and

$$\begin{aligned}d\setminus c&\subseteq(d\setminus[\![v>\alpha+2\delta]\!])\cup([\![v>\alpha+2\delta]\!]\setminus c)\\&\subseteq(d\setminus[\![v>\alpha+2\delta]\!])\cup[\![v-u>\delta]\!],\end{aligned}$$

so

$$\bar\mu(c\mathbin{\triangle}d)\le\bar\mu(d\setminus[\![v>\alpha+2\delta]\!])+\bar\mu[\![|u-v|>\delta]\!]\le2\epsilon.$$

Now if $b\in\mathfrak{B}$,

$$\begin{aligned}|\bar\mu(b\cap d)-\bar\mu b\cdot\bar\mu d|&\le|\bar\mu(b\cap d)-\bar\mu(b\cap c)|+\bar\mu b|\bar\mu d-\bar\mu c|+|\bar\mu(b\cap c)-\bar\mu b\cdot\bar\mu c|\\&\le2\bar\mu(d\triangle c)+0\end{aligned}$$

(because u is independent of $\mathfrak{B}$)

$$\le4\epsilon.$$

As $\epsilon>0$, $\bar\mu(b\cap d)=\bar\mu b\cdot\bar\mu d$; as α and b are arbitrary, u is independent of $\mathfrak{B}$; as u is arbitrary, C is closed. **Q**

652C Definition Let $(\mathfrak{A},\bar\mu,[0,\infty[\,,\langle\mathfrak{A}_t\rangle_{t\ge0},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ be a right-continuous real-time stochastic integration structure. I will say that a fully adapted process $\langle v_\sigma\rangle_{\sigma\in\mathcal{T}_f}$ is a **Lévy process** if it is locally near-simple and

whenever s, $t\ge0$, $v_{(s+t)^\check{}}-v_{\check s}$ is independent of $\mathfrak{A}_s$ and has the same distribution (364Gb[1]) as $v_{\check t}$.

Examples (i) The identity process is a Lévy process. (By 613Ea, it is locally near-simple. And in the notation of 612F, $\iota_{\check t}=t\chi1$ for every $t\ge0$.)

(ii) Brownian motion, as described in 612T, is a Lévy process (477D(c-ii)).

(iii) The standard Poisson process, as described in 612U, is a Lévy process (455P-455R).

(iv) The Cauchy process, to be described in 652M-652O below, is a Lévy process.

[1]Formerly 364Xd.

652D Lemma If $(\mathfrak{A},\bar\mu,[0,\infty[\,,\langle\mathfrak{A}_t\rangle_{t\ge0},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ is a right-continuous real-time stochastic integration structure and $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{T}_f}$ a Lévy process, then $v_{\check 0}=\lim_{t\downarrow0}v_{\check t}=0$.

proof $v_{\check 0}$ has the same distribution as $v_{\check 0}-v_{\check 0}$, so must be 0. And $\check 0=\inf_{t>0}\check t$ in $\mathcal{T}$ (611Ce), so $\lim_{t\downarrow]0,\infty]}v_{\check t}=v_{\check 0}$ by 632E.

652E Proposition Let $(\mathfrak{A},\bar\mu,[0,\infty[\,,\langle\mathfrak{A}_t\rangle_{t\ge0},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ be a right-continuous real-time stochastic integration structure, and $\boldsymbol{v}=\langle v_\tau\rangle_{\tau\in\mathcal{T}_f}$, $\boldsymbol{w}=\langle w_\tau\rangle_{\tau\in\mathcal{T}_f}$ two Lévy processes. If $v_{\check t}=w_{\check t}$ for every $t\ge0$, then $\boldsymbol{v}=\boldsymbol{w}$.

proof Note that $\boldsymbol{v}\restriction\mathcal{T}_b$ is locally near-simple, just because $\boldsymbol{v}\restriction\mathcal{T}_b\wedge\tau=\boldsymbol{v}\restriction\mathcal{T}_f\wedge\tau$ is near-simple for every $\tau\in\mathcal{T}_b$. Similarly, $\boldsymbol{w}\restriction\mathcal{T}_b$ is near-simple. Now $\{\check t:t\ge0\}$ separates $\mathcal{T}_b$ (633Da) and is cofinal with $\mathcal{T}_b$, so $\boldsymbol{v}\restriction\mathcal{T}_b=\boldsymbol{w}\restriction\mathcal{T}_b$ by 633F. As $\mathcal{T}_f$ is the covered envelope of $\mathcal{T}_b$, $\boldsymbol{v}=\boldsymbol{w}$ (612Qa).

652F Classical Lévy processes I have already used the phrase 'Lévy process' in §455 to describe a class of processes of the form $\langle X_t\rangle_{t\ge0}$ (455Q). In one sense these are very much more general than the processes of this section, because they deal with U-valued random variables where U can be any separable metrizable topological group. But there is a crucial difference in that the whole analysis is based on a prior notion of what the distributions λ_t of the X_t are to be. In particular, both the measure $\ddot\mu$ and the filtration $\langle\hat{\ddot\Sigma}_t\rangle_{t\ge0}$ (455T) are defined from the joint distribution of $\langle X_t\rangle_{t\ge0}$ (454K), while in the present section I am dealing with processes in which the underlying probability algebra $(\mathfrak{A},\bar\mu)$ and the filtration $\langle\mathfrak{A}_t\rangle_{t\ge0}$ need not be describable from the process $\langle v_{\check t}\rangle_{t\ge0}$.

To bridge the gap, I offer the following.

Proposition Let $C_{\rm dlg}$ be the space of càdlàg real-valued functions on $[0,\infty[$, endowed with its topology of pointwise convergence. Let $\langle\lambda_t\rangle_{t>0}$ be a family of distributions (that is, Radon probability measures on $\mathbb{R}$) such that the convolution $\lambda_s*\lambda_t$ (257A) is equal to λ_{s+t} for all s, $t>0$, and $\lim_{t\downarrow0}\lambda_tG=1$ for every open subset G of $\mathbb{R}$ containing 0. Let $\ddot\mu$ be the completion regular quasi-Radon probability measure on $C_{\rm dlg}$ defined by saying that

$$\begin{aligned}\ddot\mu\{\omega:\omega\in C_{\rm dlg},\,\omega(s_0)\in E_0,\,\omega(s_i)-\omega(s_{i-1})\in E_i\text{ for }1\le i\le n\}\\=\delta_0E_0\cdot\prod_{i=1}^n\lambda_{s_i-s_{i-1}}E_i\end{aligned}\qquad(*)$$

whenever $0=s_0<\ldots<s_n$ in $[0,\infty[$ and $E_0,\ldots,E_n\subseteq\mathbb{R}$ are Borel sets (here δ_0 is the Dirac measure concentrated at 0), and $\ddot\Sigma$ its domain. For $t\ge0$, set

$$\ddot\Sigma_t=\{F:F\in\ddot\Sigma,\,\omega'\in F\text{ whenever }\omega\in F,\,\omega'\in C_{\rm dlg}\text{ and }\omega'\restriction[0,t]=\omega\restriction[0,t]\},$$

$$\hat{\ddot\Sigma}_t=\{F\triangle A:F\in\ddot\Sigma_t,\,A\in\mathcal{N}(\ddot\mu)\}$$

so that $\langle\hat{\ddot\Sigma}_t\rangle_{t\ge0}$ is a right-continuous filtration of σ-subalgebras of $\ddot\Sigma$ (455T). Let $(\mathfrak{C},\bar{\ddot\mu})$ be the measure algebra of $(C_{\rm dlg},\ddot\Sigma,\ddot\mu)$ and set $\mathfrak{C}_t=\{E^\bullet:E\in\hat{\ddot\Sigma}_t\}$ for $t\ge0$, so that $(\mathfrak{C},\bar{\ddot\mu},[0,\infty[\,,\langle\mathfrak{C}_t\rangle_{t\ge0})$ is a right-continuous stochastic integration structure (632K) with associated set $\mathcal{T}$ of stopping times and family $\langle\mathfrak{C}_\tau\rangle_{\tau\in\mathcal{T}}$ of closed subalgebras. Setting $X_t(\omega)=\omega(t)$ for $\omega\in\Omega$ and $t\ge0$, $\langle X_t\rangle_{t\ge0}$ is progressively measurable and gives rise to a locally near-simple process $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{T}_f}$ such that if $\sigma\in\mathcal{T}_f$ is represented by a stopping time $h:C_{\rm dlg}\to[0,\infty[$ (in the sense of 455L or 612Gb) adapted to $\langle\hat{\ddot\Sigma}_t\rangle_{t\ge0}$, and $X_h(\omega)=X_{h(\omega)}(\omega)$ for $\omega\in C_{\rm dlg}$, then $u_\sigma=X_h^\bullet$ in $L^0(\mu)$ (612H, 631D). Now $\boldsymbol{u}$ is a Lévy process as defined in 652C, and the distribution of $u_{\check t}$ is λ_t for every $t\ge0$.

proof (a) For the existence of such a measure $\ddot\mu$, see 455Pc and 455K. I did not discuss the question of uniqueness there, but as $\ddot\mu$ is complete and completion regular it must be the completion of its restriction $\ddot\mu\restriction\mathcal{B}\mathfrak{a}(C_{\rm dlg})$ to the Baire σ-algebra of $C_{\rm dlg}$. Now $\mathcal{B}\mathfrak{a}(C_{\rm dlg})$ is the subspace σ-algebra on $C_{\rm dlg}$ induced by $\mathcal{B}\mathfrak{a}(\mathbb{R}^{[0,\infty[}=\widehat{\bigotimes}_{[0,\infty[}\mathcal{B}(\mathbb{R})$ (4A3N), so $\mathcal{B}\mathfrak{a}(C_{\rm dlg})$ is the σ-algebra of subsets of $C_{\rm dlg}$ generated by the coordinate functionals $\omega\mapsto\omega(s)$ for $s\ge 0$, and by the Monotone Class Theorem (136C) $\ddot\mu\restriction\mathcal{B}\mathfrak{a}(C_{\mathrm{dlg}})$ is determined by the values specified in (*) above.

(b) Theorem 455U tells us that if $h:C_{\mathrm{dlg}}\to[0,\infty]$ is a stopping time adapted to $\langle\hat{\ddot\Sigma}_t\rangle_{t\ge0}$ and we define $\phi_h:C_{\mathrm{dlg}}\times C_{\mathrm{dlg}}\to C_{\mathrm{dlg}}$ by setting

$$\begin{aligned}\phi_h(\omega,\omega')(t)&=\omega'(t-h(\omega))+\omega(h(\omega)) \text{ if } t\ge h(\omega),\\&=\omega(t)\text{ otherwise,}\end{aligned}$$

then ϕ_h is inverse-measure-preserving for $\ddot\mu\times\ddot\mu$ and $\ddot\mu$.

(c) Turning to the conditions in 652C, set $X_t(\omega)=\omega(t)$ for $\omega\in C_{\mathrm{dlg}}$ and $t\ge0$. For $t\ge0$, X_t is continuous, so is $\ddot\Sigma$-measurable, and now X_t is $\hat{\ddot\Sigma}_t$-measurable. By 631D, $\langle X_t\rangle_{t\ge0}$ is progressively measurable so $\boldsymbol{u}$ is well-defined and locally near-simple (631D).

If $s,t\ge0$ then the distribution of $u_{(s+t)^\check{}}-u_{\check s}$ is the distribution of the random variable $\omega\mapsto\omega(s+t)-\omega(s)$. But now take h to be the constant stopping time with value s and consider $\phi_h:C_{\mathrm{dlg}}\times C_{\mathrm{dlg}}\to C_{\mathrm{dlg}}$. We have

$$\phi_h(\omega,\omega')(s+t)-\phi_h(\omega,\omega')(s)=\phi_h(\omega,\omega')(s+t)-\omega(s)=\omega'(t).$$

Because ϕ_h is inverse-measure-preserving, the distribution of $\omega\mapsto\omega(s+t)-\omega(s)$ is the same as the distribution of $(\omega,\omega')\mapsto\phi_h(\omega,\omega')(s+t)-\phi_h(\omega,\omega')(s)=\omega'(t)$, which is the same as the distribution of $\omega'\mapsto\omega'(t)$ and $u_{\check t}$.

Finally, if $s\le t$, $E\in\hat{\ddot\Sigma}_s$ and $F\subseteq\mathbb{R}$ is Borel, then take h once again to be the constant stopping time with value s, and let $E'\in\ddot\Sigma_t$ be such that $\ddot\mu(E\triangle E')=0$. Then

$$\begin{aligned}&\ddot\mu(E\cap\{\omega:\omega(t)-\omega(s)\in F\}\\&\qquad=\ddot\mu(E'\cap\{\omega:\omega(t)-\omega(s)\in F\}\\&\qquad=\ddot\mu^2\{(\omega,\omega'):\phi_h(\omega,\omega')\in E',\,\phi_h(\omega,\omega')(t)-\phi_h(\omega,\omega')(s)\in F\}\\&\qquad=\ddot\mu\{\omega:\omega\in E'\}\cdot\ddot\mu\{\omega':\omega'(t-s)\in F\}\\&\qquad=\ddot\mu\{\omega:\omega\in E\}\cdot\ddot\mu\{\omega':\omega'(t)-\omega'(s)\in F\}.\end{aligned}$$

Translating this into terms of $(\mathfrak{C},\bar{\ddot\mu},[0,\infty[\,,\langle\mathfrak{C}_t\rangle_{t\ge0})$ and $\boldsymbol{u}$,

$$\bar{\ddot\mu}(a\cap[\![u_{\check t}-u_{\check s}\in F]\!])=\bar{\ddot\mu}a\cdot\bar{\ddot\mu}[\![u_{\check t}-u_{\check s}\in F]\!]$$

for every $a\in\mathfrak{C}_s$ and Borel set $F\subseteq\mathbb{R}$; that is, $u_{\check t}-u_{\check s}$ is independent of $\mathfrak{C}_s$. So $\boldsymbol{u}$ satisfies all the clauses of 652C.

652G Sums of stopping times Let $(\mathfrak{A},\bar\mu,[0,\infty[\,,\langle\mathfrak{A}_t\rangle_{t\ge0},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ be a real-time stochastic integration structure.

(a) If $\tau\in\mathcal{T}$ and $s\ge0$ we have an element $\tau+\check s$ of $\mathcal{T}$ defined by saying that

$$\begin{aligned}[\![\tau+\check s>t]\!]&=[\![\tau>t-s]\!]\text{ if }s\le t,\\&=1\text{ if }s>t.\end{aligned}$$

(b)(i) $\tau\vee\check s\le\tau+\check s$ whenever $\tau\in\mathcal{T}$ and $s\ge0$.

(ii) If $\tau\in\mathcal{T}$ and s, $s'\ge0$ then $\tau+(s+s')^\check{}=(\tau+\check s)+\check s'$.

(iii) $\tau+\check0=\tau$ for every $\tau\in\mathcal{T}$.

(iv) If s, $s'\ge0$ then $\check s+\check s'=(s+s')^\check{}$.

(v) If τ, $\tau'\in\mathcal{T}$ and $s\ge0$ then $[\![\tau\le\tau']\!]\subseteq[\![\tau+\check s\le\tau'+\check s]\!]$.

(vi) For any $\tau\in\mathcal{T}$, $\{\tau+\check s:s\ge0\}$ separates $\mathcal{T}\vee\tau$.

(vii) Suppose that $\langle\mathfrak{A}_t\rangle_{t\ge0}$ is right-continuous. If $A\subseteq\mathcal{T}$, $s\ge0$ and we write $A+\check s$ for $\{\sigma+\check s:\sigma\in A\}$, then $\inf(A+\check s)=(\inf A)+\check s$.

proof (a) Setting

$$a_t=[\![\tau>t-s]\!]\text{ if }s\le t,\quad 1\text{ otherwise,}$$

we see that a_t belongs to $\mathfrak{A}_{t-s} \subseteq \mathfrak{A}_t$ in the first case and to $\mathfrak{A}_t$ in either case. If $0 \le t \le t'$, then

$$\begin{aligned} a_t &= [\![\tau > t-s]\!] \supseteq [\![\tau > t'-s]\!] = a_{t'} \text{ if } s \le t, \\ &= 1 \supseteq [\![\tau > t'-s]\!] = a_{t'} \text{ if } t < s \le t', \\ &= 1 = a_{t'} \text{ if } t' < s, \end{aligned}$$

so $a_t \supseteq a_{t'}$ in all cases. And

$$\begin{aligned} \sup_{t'>t} a_{t'} = \lim_{t'\downarrow t} a_{t'} &= \lim_{t'\downarrow t} [\![\tau > t'-s]\!] = [\![\tau > t-s]\!] = a_t \text{ if } s \le t, \\ &= \lim_{t'\downarrow t, t'<s} a_{t'} = 1 = a_t \text{ otherwise.} \end{aligned}$$

Thus $\langle a_t\rangle_{t\ge 0}$ satisfies the conditions of 611A(b-i) and defines a stopping time which we may call $\tau + \check{t}$.

(b)(i) For any $t \ge 0$,

$$\begin{aligned} [\![\tau \vee \check{s} > t]\!] = [\![\tau > t]\!] \cup [\![\check{s} > t]\!] &\subseteq [\![\tau > t-s]\!] = [\![\tau + \check{s} > t]\!] \text{ if } s \le t, \\ &\subseteq 1 = [\![\tau + \check{s} > t]\!] \text{ if } s \ge t. \end{aligned}$$

(ii)

$$\begin{aligned} [\![(\tau + \check{s}) + \check{s}' > t]\!] = [\![\tau + \check{s} > t - s']\!] &= [\![\tau > t - (s+s')]\!] \\ &= [\![\tau + (s+s')\check{}\,]\!] \text{ if } s + s' \le t, \\ &= 1 = [\![\tau + (s+s')\check{}\,]\!] \\ &\qquad \text{if } t < s \text{ or } s \le t \text{ and } t - s < s'. \end{aligned}$$

(iii) The formula in (a) tells us that $[\![\tau + \check{0} > t]\!] = [\![\tau > t - 0]\!] = [\![\tau > t]\!]$ for every t.

(iv) For $t \ge 0$,

$$\begin{aligned} [\![\check{s} + \check{s}' > t]\!] &= [\![\check{s} > t - s']\!] = 0 = [\![(s+s')\check{}\, > t]\!] \text{ if } s + s' \le t, \\ &\qquad \text{i.e., } s' \le t \text{ and } s \le t - s', \\ &= [\![\check{s} > t - s']\!] = 1 = [\![(s+s')\check{}\, > t]\!] \text{ if } s' \le t < s + s', \\ &\qquad \text{i.e., } s' \le t \text{ and } t - s' < s, \\ &= 1 = [\![(s+s')\check{}\, > t]\!] \text{ if } t < s'. \end{aligned}$$

(v) If $0 \le t < s$ then $[\![\tau' + \check{s} > t]\!] = 1$. If $s \le t$ then

$$[\![\tau \le \tau']\!] \cap [\![\tau + \check{s} > t]\!] = [\![\tau \le \tau']\!] \cap [\![\tau > t - s]\!] \subseteq [\![\tau' > t - s]\!] = [\![\tau' + \check{s} > t]\!].$$

So $[\![\tau + \check{s} > t]\!] \setminus [\![\tau' + \check{s} > t]\!]$ does not meet $[\![\tau \le \tau']\!]$ for any $t > 0$, $[\![\tau \le \tau']\!] \cap [\![\tau' + \check{s} < \tau + \check{s}]\!] = 0$ (see 611D) and $[\![\tau \le \tau']\!] \subseteq [\![\tau + \check{s} \le \tau' + \check{s}]\!]$.

(vi) Suppose that τ', $\tau'' \in \mathcal{T} \vee \tau$ and $0 \ne a \subseteq [\![\tau' < \tau'']\!]$. Then there are a $t \ge 0$ such that $b = a \cap [\![\tau'' > t]\!] \setminus [\![\tau' > t]\!]$ is non-zero, and a $t' > t$ such that $c = b \setminus [\![\tau' > t]\!]$ is non-zero. For $i \in \mathbb{N}$ set $s_i = i(t'-t)$. Then there is certainly an i such that $i(t'-t) > t$, in which case $[\![\tau + \check{s}_i > t]\!] = 1$, so there is a first $n \in \mathbb{N}$ such that $d = c \cap [\![\check{t} < \tau + \check{s}_n]\!]$ is non-zero. Since $[\![\tau + \check{s}_0 > t]\!] \subseteq [\![\tau' > t]\!]$ is disjoint from c, $n \ge 1$.

Now $c \subseteq [\![\tau' \le \check{t}]\!]$ and $d \subseteq [\![\check{t} < \tau + \check{s}_n]\!]$, so $d \subseteq [\![\tau' \le \tau + \check{s}_n]\!]$. At the same time,

$$c \subseteq [\![\tau + \check{s}_{n-1} \le \check{t}]\!] \subseteq [\![\tau + \check{s}_n \le \check{t}']\!]$$

by (ii) and (v) above, and as c is also included in $[\![\check{t}' < \tau'']\!]$, we have $c \subseteq [\![\tau + \check{s}_n < \tau'']\!]$ and

$$a \cap [\![\tau' \le \tau + \check{s}_n]\!] \cap [\![\tau + \check{s}_n < \tau'']\!] \supseteq d \ne 0.$$

As a is arbitrary, $[\![\tau' < \tau'']\!] \subseteq \sup_{s\ge 0}([\![\tau' \le \tau + \check{s}]\!] \cap [\![\tau + \check{s} < \tau'']\!])$. As τ' and τ'' are arbitrary, we have the result.

(vii) Write τ for $\inf A$, τ' for $\inf(A+\check{s})$. If A is empty, so is $A+\check{s}$, so $\max\mathcal{T}=\tau=\tau'=\tau+\check{s}$. Otherwise. if $0\le t<s$ then

$$[\![\tau+\check{s}>t]\!]=1=[\![\check{s}>t]\!]=[\![\tau'>t]\!]$$

because $\check{s}$ is a lower bound for $A+\check{s}$ so $\check{s}\le\tau'$. If $A\ne\emptyset$ and $s\le t$ then

$$[\![\tau'>t]\!]=\sup_{t'>t}\inf_{\sigma\in A+\check{s}}[\![\sigma>t']\!]$$

(632C(a-i))

$$=\sup_{t'>t}\inf_{\sigma\in A}[\![\sigma+\check{s}>t']\!]=\sup_{t'>t}\inf_{\sigma\in A}[\![\sigma>t'-s]\!]$$
$$=\sup_{t'>t-s}\inf_{\sigma\in A}[\![\sigma>t']\!]=[\![\tau>t-s]\!]$$

(632C(a-i) again)

$$=[\![\tau+\check{s}>t]\!].$$

Thus $[\![\tau'>t]\!]=[\![\tau+\check{s}>t]\!]$ for all $t\ge0$ and $\tau'=\tau+\check{s}$, as claimed.

652H Proposition Let $(\mathfrak{A},\bar\mu,[0,\infty[\,,\langle\mathfrak{A}_t\rangle_{t\ge0},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ be a right-continuous real-time stochastic integration structure, and $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{T}_f}$ a Lévy process.

(a) For any $s\ge0$ and $\tau\in\mathcal{T}_f$, $v_{\tau+\check{s}}-v_\tau$ is independent of $\mathfrak{A}_\tau$ and has the same distribution as $v_{\check{s}}$.

(b) For any $\tau\in\mathcal{T}_f$, $\langle v_{\tau+\check{s}}-v_\tau\rangle_{s\ge0}$ is independent of $\mathfrak{A}_\tau$ and has the same distribution as $\langle v_{\check{s}}\rangle_{s\ge0}$.

proof (a)(i) Write D for the set of those $\tau\in\mathcal{T}_f$ such that $v_{\tau+\check{s}}-v_\tau$ is independent of $\mathfrak{A}_\tau$ and has the same distribution as $v_{\check{s}}$. By the definition in 652C, $\check{T}$ is included in D.

(ii) D is full. **P** Suppose that $\tau\in\mathcal{T}$ and $\sup_{\sigma\in D}[\![\tau=\sigma]\!]=1$. Then $\tau\in\mathcal{T}_f$ by 611N(e-i), and there is a sequence $\langle\sigma_i\rangle_{i\in\mathbb{N}}$ in D such that $1=\sup_{i\in\mathbb{N}}a_i$, where $a_i=[\![\tau=\sigma_i]\!]$ for $i\in\mathbb{N}$. Set $b_i=a_i\setminus\sup_{j<i}a_j$ for $i\in\mathbb{N}$; as $a_i\in\mathfrak{A}_\tau\cap\mathfrak{A}_{\sigma_i}$ for every i, $b_i\in\mathfrak{A}_\tau$ and therefore $b_i\in\mathfrak{A}_{\sigma_i}$ (611H(c-iii)).

Now if $a\in\mathfrak{A}_\tau$ and $\alpha\in\mathbb{R}$,

$$\bar\mu(a\cap[\![v_{\tau+\check{s}}-v_\tau>\alpha]\!])=\sum_{i=0}^\infty\bar\mu(a\cap b_i\cap[\![v_{\tau+\check{s}}-v_\tau>\alpha]\!])$$
$$=\sum_{i=0}^\infty\bar\mu(a\cap b_i\cap[\![v_{\sigma_i+\check{s}}-v_{\sigma_i}>\alpha]\!])$$

(because $b_i\subseteq[\![v_\tau=v_{\check{s}}]\!]\cap[\![\tau+\check{s}=\sigma_i+\check{s}]\!]$, by 652G(b-v))

$$=\sum_{i=0}^\infty\bar\mu(a\cap b_i)\cdot\bar\mu[\![v_{\sigma_i+\check{s}}-v_{\sigma_i}>\alpha]\!]$$

(because $a\cap b_i\in\mathfrak{A}_{\sigma_i}$ and $\sigma_i\in D$, so $v_{\sigma_i+\check{s}}-v_{\sigma_i}$ is independent of $\mathfrak{A}_{\sigma_i}$)

$$=\sum_{i=0}^\infty\bar\mu(a\cap b_i)\cdot\bar\mu[\![v_{\check{s}}>\alpha]\!]$$

(because $v_{\sigma_i+\check{s}}-v_{\sigma_i}$ has the same distribution as $v_{\check{s}}$)

$$=\bar\mu a\cdot\bar\mu[\![v_{\check{s}}>\alpha]\!].$$

Taking $a=1$, this shows that $v_{\tau+\check{s}}-v_\tau$ has the same distribution as $v_{\check{s}}$. So in fact we see that

$$\bar\mu(a\cap[\![v_{\tau+\check{s}}-v_\tau>\alpha]\!])=\bar\mu a\cdot\bar\mu[\![v_{\tau+\check{s}}-v_\tau>\alpha]\!]$$

whenever $a\in\mathfrak{A}_\tau$ and $\alpha\in\mathbb{R}$, and (using the Monotone Class Theorem) that $v_{\tau+\check{s}}-v_\tau$ is independent of $\mathfrak{A}_\tau$. Thus $\tau\in D$. As τ was an arbitrary member of the covered envelope of D, D is full. **Q**

(iii) If $A\subseteq D$ is non-empty and downwards-directed, then $\inf A\in D$. **P** Write τ for $\inf A$. As $\mathcal{T}_f$ is an ideal of $\mathcal{T}$, $\tau\in\mathcal{T}_f$. $A+\check{s}=\{\sigma+\check{s}:\sigma\in A\}$ is downwards-directed (652G(b-v)) and $\inf(A+\check{s})=\tau+\check{s}$

(652G(b-vii)). Since $\langle\mathfrak{A}_t\rangle_{t\ge0}$ is right-continuous and $\boldsymbol{v}$ is locally near-simple, $v_\tau=\lim_{\sigma\downarrow A}v_\sigma$ (632F), $v_{\tau+\check{s}}=\lim_{\sigma\downarrow A}v_{\sigma+\check{s}}$ and $v_{\tau+\check{s}}-v_\tau=\lim_{\sigma\downarrow A}v_{\sigma+\check{s}}-v_\sigma$.

Since $v_{\sigma+\check{s}}-v_\sigma$ has the same distribution as $v_{\check{s}}$ for every $\sigma\in A$, $v_{\tau+\check{s}}-v_\tau$ also has the same distribition as $v_{\check{s}}$, by 454Ve[2]. Since $v_{\sigma+\check{s}}-v_\sigma$ is independent of $\mathfrak{A}_\sigma$, and therefore of the smaller algebra $\mathfrak{A}_\tau$. for every $\sigma\in A$, $v_{\tau+\check{s}}-v_\tau$ is independent of $\mathfrak{A}_\tau$, by 652Bc. So $\tau\in D$, as claimed. **Q**

(iv) $D=\mathcal{T}_f$. **P** We saw in (ii) that D is a full sublattice of $\mathcal{T}_f$. As $\check{T}$ separates $\mathcal{T}_f$ (633Ea) and is included in D, D separates $\mathcal{T}_f$. If $\tau\in\mathcal{T}_b$, then $A_\tau=\{\sigma:\tau\le\sigma\in D\}$ is non-empty and downwards-directed and has infimum τ (633Eb); by (iii) just above, $\tau\in D$. Thus $D\supseteq\mathcal{T}_b$: as D is full, $D=\mathcal{T}_f$ (611N(e-ii)). **Q**

(b) For $C\subseteq[0,\infty[$ write $\mathfrak{B}_C$ for the closed subalgebra generated by $\{v_{\tau+\check{t}}-v_\tau:t\in C\}$.

(i) If $C\subseteq[0,\infty[$ is finite then $\mathfrak{B}_C$ and $\mathfrak{A}_\tau$ are independent. **P** Induce on $\#(C)$. If C is empty then $\mathfrak{B}_C=\{0,1\}$ and the result is trivial. For the inductive step to $\#(C)=n\ge1$ enumerate C in ascending order as $(t_1,\dots,t_n)$ and set $t_0=0$. Then $\mathfrak{B}_C$ is the closed subalgebra generated by $\{v_{\tau+\check{t}_{i+1}}-v_{\tau+\check{t}_i}:i<n\}$. Suppose that $\alpha_0,\dots,\alpha_{n-1}\in\mathbb{R}$ and $a\in\mathfrak{A}_\tau$. Setting $a_0=a$ and $a_{i+1}=a_i\cap[\![v_{\tau+\check{t}_{i+1}}-v_{\tau+\check{t}_i}>\alpha_i]\!]$ for $i<n$, we see that $a_i\in\mathfrak{A}_{\tau+\check{t}_i}$, while $v_{\tau+\check{t}_{i+1}}-v_{\tau+\check{t}_i}=v_{(\tau+\check{t}_i)+(t_{i+1}-t_i)\check{}}-v_{\tau+\check{t}_i}$ is independent of $\mathfrak{A}_{\tau+\check{t}_i}$ by (a) above, so that $\bar\mu a_{i+1}=\bar\mu a_i\cdot\bar\mu[\![v_{\tau+\check{t}_{i+1}}-v_{\tau+\check{t}_i}>\alpha_i]\!]$. Consequently

$$\bar\mu(a\cap\inf_{i<n}[\![v_{\tau+\check{t}_{i+1}}-v_{\tau+\check{t}_i}>\alpha_i]\!])=\bar\mu a\cdot\prod_{i=0}^{n-1}\bar\mu[\![v_{\tau+\check{t}_{i+1}}-v_{\tau+\check{t}_i}>\alpha_i]\!].$$

In particular,

$$\bar\mu(\inf_{i<n}[\![v_{\tau+\check{t}_{i+1}}-v_{\tau+\check{t}_i}>\alpha_i]\!])=\prod_{i=0}^{n-1}\bar\mu[\![v_{\tau+\check{t}_{i+1}}-v_{\tau+\check{t}_i}>\alpha_i]\!]$$

so

$$\bar\mu(a\cap\inf_{i<n}[\![v_{\tau+\check{t}_{i+1}}-v_{\tau+\check{t}_i}>\alpha_i]\!])=\bar\mu a\cdot\bar\mu(\inf_{i<n}[\![v_{\tau+\check{t}_{i+1}}-v_{\tau+\check{t}_i}>\alpha_i]\!]).$$

As a and $\alpha_0,\dots,\alpha_{n-1}$ are arbitrary, $\mathfrak{A}_\tau$ and $\mathfrak{B}_C$ are independent (apply 313Gb to the subsets

$$\{\inf_{i<n}[\![v_{\tau+\check{t}_{i+1}}-v_{\tau+\check{t}_i}>\alpha_i]\!]:\alpha_0,\dots,\alpha_{n-1}\in\mathbb{R}\},$$

$$\{d:\bar\mu(a\cap d)=\bar\mu a\cdot\bar\mu d\text{ for every }a\in\mathfrak{A}_\tau\}$$

of $\mathfrak{A}$). **Q**

(ii) Now $\mathfrak{E}=\bigcup\{\mathfrak{B}_C:C\subseteq[0,\infty[\text{ is finite}\}$ is a subalgebra of $\mathfrak{B}_{[0,\infty[}$ and $\bar\mu(a\cap e)=\bar\mu a\cdot\bar\mu e$ whenever $a\in\mathfrak{A}_\tau$ and $e\in\mathfrak{E}$. As $\cap:\mathfrak{A}\times\mathfrak{A}\to\mathfrak{A}$ and $\bar\mu:\mathfrak{A}\to[0,1]$ are continuous for the measure-algebra topology (323Ba, 323Ca), $\bar\mu(a\cap d)=\bar\mu a\cdot\bar\mu d$ whenever $a\in\mathfrak{A}_\tau$ and d belongs to the topological closure of $\mathfrak{E}$, which is the closed subalgebra generated by $\mathfrak{E}$ (323J), that is, $\mathfrak{B}_{[0,\infty[}$. Thus $\langle v_{\tau+\check{s}}-v_\tau\rangle_{s\ge0}$ is independent of $\mathfrak{A}_\tau$.

(iii) As for the distribution of $\langle v_{\tau+\check{s}}-v_\tau\rangle_{s\ge0}$, we saw in (i) of the argument above that

$$\begin{aligned}\bar\mu(\inf_{i<n}[\![v_{\tau+\check{t}_{i+1}}-v_{\tau+\check{t}_i}>\alpha_i]\!])&=\prod_{i=0}^{n-1}\bar\mu[\![v_{\tau+\check{t}_{i+1}}-v_{\tau+\check{t}_i}>\alpha_i]\!]\\&=\prod_{i=0}^{n-1}\bar\mu[\![v_{(t_{i+1}-t_i)\check{}}>\alpha_i]\!]\end{aligned}$$

(using the other clause in (a))

$$=\prod_{i=0}^{n-1}\bar\mu[\![v_{\check{t}_{i+1}}-v_{\check{t}_i}>\alpha_i]\!]$$

whenever $0\le t_0\le\dots\le t_n$. So $\langle v_{\tau+\check{s}}-v_\tau\rangle_{s\ge0}$ has the same distribution as $\langle v_{\check{s}}\rangle_{s\ge0}$.

652I Theorem Let $(\mathfrak{A},\bar\mu,[0,\infty[\,,\langle\mathfrak{A}_t\rangle_{t\ge0},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ be a right-continuous real-time stochastic integration structure, and $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{T}_f}$ a Lévy process such that $\operatorname{Osclln}(\boldsymbol{v}\restriction[\check{0},\tau])\in L^\infty(\mathfrak{A})$ for every $\tau\in\mathcal{T}_b$. Then $\boldsymbol{v}\restriction\mathcal{T}_b$ is an L^1-process.

[2]Later editions only.

proof Take $\tau^* \in \mathcal{T}_b$; I need to show that $v_{\tau^*} \in L^1_{\bar\mu}$.

(a) Take $t^* \ge 0$ such that $\tau^* \le \check t^*$. Since $\boldsymbol{v}$ is locally near-simple, $\boldsymbol{v}\restriction\mathcal{T}\wedge\check t^*$ is near-simple. Set $\delta = \frac{1}{2}$ and apply the construction of 615M to $\boldsymbol{v}\restriction\mathcal{T}\wedge\check t^*$ and δ to obtain families $\langle D'_i\rangle_{i\in\mathbb{N}}$, $\langle y'_i\rangle_{i\in\mathbb{N}}$ and $\langle c'_{i\sigma}\rangle_{i\in\mathbb{N},\sigma\in\mathcal{T}\wedge\check t^*}$ as in 615M. Set $\tau'_1 = \inf D'_1$; then $\tau'_1 \in D'_1$ and $v_{\tau'_1} = y'_1$ (631Q), while $y'_0 = v_{\check 0} = 0$ and $c'_{0\sigma} = 1$ for every $\sigma \in \mathcal{T}\wedge\check t^*$.

Since $\tau'_1 \in D'_1$, the formula in 615Ma tells us that $[\![\tau'_1 < \check t^*]\!] \subseteq [\![|v_{\tau'_1} - 0| \ge 1]\!]$; so if $\tau'_1 = \check 0$ this must be because $\check t^* = \check 0$, $\tau^* = \check 0$, $v_{\tau^*} = 0 \in L^1_{\bar\mu}$ and we're done. So suppose from now on that $\tau'_1 \ne \check 0$. In this case there is an integer $m \ge 1$ such that $\eta \in \,]0, t^*]$ such that $a = [\![\tau'_1 > \eta]\!]$ is non-zero, where $\eta = \frac{t^*}{m}$. If $0 \le s \le \eta$ and we set $\sigma = \check s \wedge \tau'_1$, then

$$a = [\![\tau'_1 > \eta]\!] \setminus [\![\check s > \eta]\!] \subseteq [\![\check s < \tau'_1]\!] \subseteq [\![\sigma < \tau'_1]\!] = c'_{0\sigma} \setminus c'_{1\sigma} \subseteq [\![|v_\sigma| < \tfrac{1}{2}]\!]$$

(615M(d-v))

$$\subseteq [\![|v_{\check s}| < \tfrac{1}{2}]\!] \cup [\![\tau'_1 < \check s]\!]$$

so in fact $a \subseteq [\![|v_{\check s}| < \frac{1}{2}]\!]$.

(b) Now re-apply 615M to $\boldsymbol{v}\restriction\mathcal{T}\wedge\check t^*$, this time with $\delta = 1$, to get $\langle D_i\rangle_{i\in\mathbb{N}}$, $\langle y_i\rangle_{i\in\mathbb{N}}$ and $\langle c_{i\sigma}\rangle_{i\in\mathbb{N},\sigma\in\mathcal{T}\wedge\check t^*}$. Set $\tau_i = \inf D_i$ for $i \in \mathbb{N}$; this time we have $\tau_i \in D_i$ so $c_{i\sigma} = [\![\tau_i \le \sigma]\!]$ for $i \in \mathbb{N}$ and $\sigma \le \check t^*$.

For $i \in \mathbb{N}$, set $a_i = \inf_{0\le s\le\eta} [\![|v_{\tau_i+\check s} - v_{\tau_i}| < \frac{1}{2}]\!]$, so that $\bar\mu a_0 \ge \bar\mu a$ is non-zero. Since $\langle v_{\tau_i+\check s} - v_{\tau_i}\rangle_{s\ge0}$ has the same distribution as $\langle v_{\check s}\rangle_{s\ge0}$ (652Hb), $\bar\mu a_i = \bar\mu a_0$ for every $i \ge n$; write γ for this common value. Since $\langle v_{\tau_i+\check s} - v_{\tau_i}\rangle_{s\ge0}$ is independent of $\mathfrak{A}_{\tau_i}$ for each i, $\bar\mu(a_i \cap b) = \gamma\bar\mu b$ for every $i \in \mathbb{N}$ and $b \in \mathfrak{A}_{\tau_i}$; as also $a_i \in \mathfrak{A}_{\tau_{i+1}}$ for every i, $\langle a_i\rangle_{i\in\mathbb{N}}$ is independent.

(c) The point is that $a_i \cap [\![\tau_{i+1} < \check t^*]\!] \subseteq [\![\tau_i + \check\eta \le \tau_{i+1}]\!]$ for every $i \in \mathbb{N}$. **P** We know that $\{\tau_i + \check s : s \ge 0\}$ separates $\mathcal{T} \vee \tau_i$ (652G(b-vi)), so $A = \{\tau_i + \check s : 0 \le s \le \eta\}$ separates $\mathcal{T} \cap [\tau_i, \tau_i + \check\eta]$ (633C(b-iii)). Now $a_i \subseteq [\![|v_\sigma - v_{\tau_i}| \le \frac{1}{2}]\!]$ whenever $\sigma \in A$. If σ belongs to the covered envelope $\hat A$ of A, then

$$a_i = \sup_{\rho\in A} a_i \cap [\![\sigma = \rho]\!] \cap [\![|v_\rho - v_{\tau_i}| \le \tfrac{1}{2}]\!] \subseteq [\![|v_\sigma - v_{\tau_i}| \le \tfrac{1}{2}]\!].$$

Now consider $\tau = (\tau_i + \check\eta) \wedge \tau_{i+1}$ and $\hat A_\tau = \{\sigma : \tau \le \sigma \in \hat A\}$. Then $\hat A_\tau$ is downwards-directed and its infimum is τ, by 633Eb, so $v_\tau = \lim_{\sigma\downarrow\hat A_\tau} v_\sigma$ (632F),

$$|v_\tau - v_{\tau_i}| \times \chi a_i = \lim_{\sigma\downarrow\hat A_\tau} |v_\sigma - v_{\tau_i}| \times \chi a_i \le \tfrac{1}{2}\chi a_i$$

and $a_i \subseteq [\![|v_\tau - v_{\tau_i}| \le \frac{1}{2}]\!]$. On the other hand, $[\![\tau_{i+1} < \check t^*]\!] \subseteq [\![|v_{\tau_{i+1}} - v_{\tau_i}| \ge 1]\!]$ so

$$\begin{aligned} a_i \cap [\![\tau_{i+1} < \check t^*]\!] \cap [\![\tau_{i+1} < \tau_i + \check\eta]\!] &\subseteq [\![|v_\tau - v_{\tau_i}| \le \tfrac{1}{2}]\!] \cap [\![|v_{\tau_{i+1}} - v_{\tau_i}| \ge 1]\!] \cap [\![\tau_{i+1} = \tau]\!] \\ &\subseteq [\![|v_\tau - v_{\tau_i}| \le \tfrac{1}{2}]\!] \cap [\![|v_\tau - v_{\tau_i}| \ge 1]\!] = 0 \end{aligned}$$

and $a_i \cap [\![\tau_{i+1} < \check t^*]\!] \subseteq [\![\tau_i + \check\eta \le \tau_{i+1}]\!]$. **Q**

(d) If $I \subseteq n \in \mathbb{N}$ then $\inf_{i\in I} a_i \cap [\![\tau_n < \check t^*]\!] \subseteq [\![(\eta\#(I))\check{} \le \tau_n]\!]$. **P** Induce on n. For $n = 0$ this is trivial. For the inductive step to $n+1$, if $I \subseteq n+1$ and $n \notin I$ then $I \subseteq n$ and

$$\begin{aligned} \inf_{i\in I} a_i \cap [\![\tau_{n+1} < \check t^*]\!] &\subseteq \inf_{i\in I} a_i \cap [\![\tau_n < \check t^*]\!] \\ &\subseteq [\![(\eta\#(I))\check{} \le \tau_n]\!] \subseteq [\![(\eta\#(I))\check{} \le \tau_{n+1}]\!]. \end{aligned}$$

If $n \in I$ set $J = I \setminus \{n\}$; then

$$\begin{aligned} \inf_{i\in I} a_i \cap [\![\tau_{n+1} < \check t^*]\!] &= a_n \cap [\![\tau_{n+1} < \check t^*]\!] \cap \inf_{i\in J} a_i \cap [\![\tau_n < \check t^*]\!] \\ &\subseteq [\![\tau_n + \check\eta \le \tau_{n+1}]\!] \cap [\![(\eta\#(J))\check{} \le \tau_n]\!] \end{aligned}$$

(by (c) above and the inductive hypothesis)

$$\subseteq [\![(\eta\#(J))\check{} + \check{\eta} \le \tau_{n+1}]\!]$$

(652G(b-v))

$$= [\![(\eta\#(I))\check{} \le \tau_{n+1}]\!]$$

by 652G(b-iv). Thus the induction continues. **Q**

(e) Now $\bar{\mu}[\![\tau_n < \tau^*]\!] \le n^m(1-\gamma)^{n-m}$ for every $n > m$. **P** If $I \subseteq n$ and $\#(I) > m$, then

$$[\![\tau_n < \tau^*]\!] \cap \inf_{i\in I} a_i \subseteq [\![\tau_n < \check{t}^*]\!] \cap \inf_{i\in I} a_i$$

$$\subseteq [\![(\eta\#(I))\check{} \le \tau_n]\!] \cap [\![\tau_n < \check{t}^*]\!]$$

(by (c))

$$= 0$$

because $\eta\#(I) \ge t^*$. So

$$[\![\tau_n < \tau^*]\!] = \sup_{I\subseteq n}([\![\tau_n < \tau^*]\!] \cap \inf_{i\in I} a_i \cap \inf_{i\in n\setminus I} 1 \setminus a_i)$$

$$\subseteq \sup_{I\subseteq n, \#(I)\le m} \inf_{i\in n\setminus I} 1 \setminus a_i)$$

has measure at most $n^m(1-\gamma)^{n-m}$, since the a_i are independent and $\bar{\mu}(1 \setminus a_i) = 1-\gamma$ for every i, by (b). **Q**

(f) Everything I have said so far applies to any Lévy process. But now consider the hypothesis on the oscillations of $\boldsymbol{v}\restriction[\check{0},\tau]$ for $\tau \in \mathcal{T}_b$. Let $\beta \ge 0$ be such that $\mathrm{Osclln}(\boldsymbol{v}\restriction[\check{0},\check{t}^*]) \le \beta\chi 1$. Then $|y_{i+1} - y_i| \le (\beta+1)\chi 1$ for every $i \in \mathbb{N}$ (618N), so $|y_n| \le n(\beta+1)\chi 1$ for every $n \in \mathbb{N}$, and

$$[\![\tau^* < \tau_n]\!] = \sup_{i<n} [\![\tau_i \le \tau^*]\!] \setminus [\![\tau_{i+1} \le \tau^*]\!] = \sup_{i<n} c_{i\tau^*} \setminus c_{i+1,\tau^*} \subseteq \sup_{i<n} [\![|v_{\tau^*} - y_i| < 1]\!]$$

(615Md)

$$\subseteq [\![|v_{\tau^*} \le n(\beta+1)]\!];$$

since also

$$[\![\tau^* = \tau_n]\!] \subseteq [\![v_{\tau^*} = y_n]\!] \subseteq [\![|v_{\tau^*} \le n(\beta+1)]\!],$$

$[\![|v_{\tau^*} > n(\beta+1)]\!] \subseteq [\![\tau_n < \tau^*]\!]$ has measure at most $n^m(1-\gamma)^{n-m}$ for $n > m$. But $\gamma = \bar{\mu}a_0$ is greater than 0, so

$$\|v_{\tau^*}\|_1 \le \sum_{n=0}^{\infty} (\beta+1)\bar{\mu}[\![|v_{\tau^*}| > n(\beta+1)]\!]$$

(365A)

$$\le (m+1)(\beta+1) + \sum_{n=m+1}^{\infty} n^m(1-\gamma)^{n-m}$$

is finite, and $v_{\tau^*} \in L^1_{\bar{\mu}}$, as required.

652J Proposition Let $(\mathfrak{A}, \bar{\mu}, [0,\infty[\,, \langle\mathfrak{A}_t\rangle_{t\ge0}, \mathcal{T}, \langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ be a right-continuous real-time stochastic integration structure, and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{T}_f}$ a Lévy process such that $\boldsymbol{v}\restriction\mathcal{T}_b$ is an L^1-process. Then there is an $\alpha \in \mathbb{R}$ such that $\boldsymbol{v} - \alpha\boldsymbol{\iota}$ is a local martingale.

proof (a) Consider first the case in which $\mathbb{E}(v_{\check{t}}) = 0$ for every $t \ge 0$.

(i) If $0 \le s \le t$, then $v_{\check t} - v_{\check s}$ is independent of $\mathfrak{A}_s$ and has zero expectation; so $\mathbb{E}(\chi b \times (v_{\check t} - v_{\check s}) = \bar\mu b \cdot \mathbb{E}(v_{\check t} - v_{\check s}) = 0$ for every $b \in \mathfrak{A}_s$. Writing $P_{\check s}$ for the conditional expectation associated with the closed subalgebra $\mathfrak{A}_{\check s} = \mathfrak{A}_s$, $P_{\check s}(v_{\check t} - v_{\check s}) = 0$ and $P_{\check s}(v_{\check t}) = P_{\check s}(v_{\check s}) = v_{\check s}$. Thus $\boldsymbol{v}\restriction [0,\infty[\,\check{}$ is a martingale.

(ii) Now there is a local martingale $\boldsymbol{w}$ with domain $\mathcal{T}_f$ extending $\boldsymbol{v}\restriction [0,\infty[\,\check{}$ (622Ob), and $\boldsymbol{w}$ is locally near-simple (632I). Because $\boldsymbol{v}$ and $\boldsymbol{w}$ are both locally near-simple and agree on $[0,\infty[\,\check{}$, they are equal (633F again). So $\boldsymbol{v}$ is a local martingale.

(b)(i) For the general case, set $f(t) = \mathbb{E}(v_{\check t})$ for $t \ge 0$. Then $f(s+t) = f(s)+f(t)$ for all s, $t \ge 0$, because $f(s+t) - f(t) = \mathbb{E}(v_{(s+t)}\check{} = v_{\check t})$ and $v_{(s+t}\check{} = v_{\check t}$ has the same distribution as $v_{\check s}$ and therefore has the same expectation. As $f(\frac{1}{n}) = \frac{1}{n}f(1)$ for every integer $n \ge 1$, and $v_{\check 0} = 0$, $f(q) = qf(1)$ for every rational $q \ge 0$.

(ii) $f : [0,\infty[\, \to \mathbb{R}$ is Borel measurable. **P** For M, $t \ge 0$, set $f_M(t) = \mathbb{E}(\mathrm{med}(-M\chi 1, v_{\check t}, M\chi 1))$. We know that $\lim_{t\downarrow 0} v_{\check t} = 0$ for the topology of convergence in measure (652D), so

$$\lim_{t\downarrow s} v_{\check t} - v_{\check s} = \lim_{t\downarrow s} v_{\check t} - v_{\check s} = \lim_{t\downarrow s} v_{(t-s)\check{}} = 0$$

for every $s \ge 0$; similarly,

$$\lim_{t\uparrow s} v_{\check t} - v_{\check s} = -\lim_{t\uparrow s} v_{\check s} - v_{\check t} = -\lim_{t\uparrow s} v_{(s-t)\check{}} = 0$$

for every $s > 0$. Thus $t \mapsto v_{\check t} : [0,\infty[\, \to L^0$ is continuous, and it follows that $t \mapsto \mathrm{med}(-M\chi 1, v_{\check t}, M\chi 1)$ is continuous for the topology of convergence in measure. But this agrees with the norm topology of $L^1_{\bar\mu}$ on the uniformly integrable set $[-M\chi 1, M\chi 1]$, so $t \mapsto \mathrm{med}(-M\chi 1, v_{\check t}, M\chi 1)$ is $\| \|_1$-continuous and $f_M : [0,\infty[\, \to \mathbb{R}$ is continuous. Now $f(t) = \lim_{n\to\infty} f_n(t)$ for every t, so f is Borel measurable. **Q**

(iii) Consequently there is a $\delta > 0$ such that f is bounded on $[0,\delta]$. **P** There is a compact set $K \subseteq [0,\infty[$, of non-zero Lebesgue measure, such that $f\restriction K$ is continuous, therefore bounded. Now $f\restriction(K-K)\cap[0,\infty[$ is bounded and $K-K$ includes $[-\delta,\delta]$ for some $\delta > 0$ (443Dc), so $f\restriction[0,\delta]$ is bounded. **Q**

(iv) $\lim_{t\downarrow 0} f(t) = 0$. **P** Set $M = \sup_{t\in[0,\delta]} |f(t)|$. If $n \in \mathbb{N}$ and $0 \le t \le 2^{-n}\delta$, $M \ge |f(2^n t)| = 2^n|f(t)|$ and $|f(t)| \le 2^{-n}M$. **Q** Since $|f(s) - f(t)| = |f(|s-t|)|$ for all s, $t \ge 0$, f is continuous. Now $f(q) = qf(1)$ for every rational $q > 0$, so $f(t) = tf(1)$ for every $t \ge 0$.

(v) Now consider $\boldsymbol{w} = \boldsymbol{v} - f(1)\boldsymbol{\iota}$. Because both $\boldsymbol{v}$ and $\boldsymbol{\iota}$ are locally near-simple with domain $\mathcal{T}_f$ (631Ea), so is $\boldsymbol{w}$. Express $\boldsymbol{w}$ as $\langle w_\sigma\rangle_{\sigma\in\mathcal{T}_f}$. If s, $t \ge 0$, the distribution of $w_{(s+t)\check{}} - w_{\check s} = v_{(s+t)\check{}} - v_{\check s} + tf(1)\chi 1$ is the same as that of $w_{\check t}$, and $w_{(s+t)\check{}} - w_{\check s}$ is independent of $\mathfrak{A}_s$, so $\boldsymbol{w}$ is a Lévy process. For $\sigma \in \mathcal{T}_b$, $\iota_\sigma \in L^\infty(\mathfrak{A}) \subseteq L^1_{\bar\mu}$, so $w_\sigma \in L^1_{\bar\mu}$; while $\mathbb{E}(w_{\check s}) = f(s) - f(1)s = 0$ for every $s \ge 0$. By (a), $\boldsymbol{w}$ is a local martingale, and we have an expression of the right form for $\boldsymbol{v}$.

652K Theorem Let $(\mathfrak{A}, \bar\mu, [0,\infty[\,, \langle\mathfrak{A}_t\rangle_{t\ge 0}, \mathcal{T}, \langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ be a right-continuous real-time stochastic integration structure, and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{T}_f}$ a Lévy process. Then $\boldsymbol{v}$ is a semi-martingale, therefore a local integrator.

proof (a) Set $\psi(\sigma,\tau) = \mathrm{med}(-\chi 1, v_\tau - v_\sigma, \chi 1)$ for $(\sigma,\tau) \in \mathcal{T}_f^{2\uparrow}$. Applying 633M(b-i) to $\psi\restriction(\mathcal{T}\wedge\tau)^{2\uparrow}$ for $\tau \in \mathcal{T}_f$, we see that ψ is a strictly adapted interval function. By 633Mc,

$$\boldsymbol{w} = ii_\psi(\mathbf{1}) = \langle \textstyle\int_{\mathcal{T}\wedge\tau} d\psi\rangle_{\tau\in\mathcal{T}_f}$$

is defined everywhere in $\mathcal{T}_f$; by 633Mg, $\boldsymbol{w}$ is locally near-simple; by 633Me, $\mathrm{Osclln}(\boldsymbol{w}\restriction\mathcal{T}\wedge\tau) \le \chi 1$ for every $\tau \in \mathcal{T}_f$. By 633Md, $\boldsymbol{v} - \boldsymbol{w}$ is locally of bounded variation.

(b) The point is that $\boldsymbol{w}$ is a Lévy process. **P** Set $\psi_0 = \psi\restriction([0,\infty[\,\check{})^{2\uparrow}$. Since $\boldsymbol{v}\restriction[0,\infty[\,\check{}$ is moderately oscillatory, $\boldsymbol{w}_0 = ii_{\psi_0}(\mathbf{1})$ is defined everywhere on $[0,\infty[\,\check{}$, and by 633N (applied to $\psi\restriction(\mathcal{T}\wedge\check t)^{2\uparrow}$ for $t \ge 0$) $\boldsymbol{w}_0 = \boldsymbol{w}\restriction[0,\infty[\,\check{}$.

Express w as $\langle w_\sigma\rangle_{\sigma\in\mathcal{T}_f}$. If s, $t \ge 0$, then $\langle v_{(s+t)\check{}}\, v_{\check s}\rangle_{t\ge 0}$ is independent of $\mathfrak{A}_s$ (652Hb) and has the same distribution as $\langle v_{\check t}\rangle_{t\ge 0}$. Set

$$\begin{aligned}\psi'(\check t, \check t') = \psi((s+t)\check{}, (s+t')\check{})) &= \mathrm{med}(-\chi 1, v_{(s+t)\check{}} - v_{(s+t')\check{}}, \chi 1)\\ &= \mathrm{med}(-\chi 1, (v_{(s+t)\check{}} - v_{\check s}) - (v_{(s+t')\check{}} - v_{\check s}), \chi 1)\end{aligned}$$

for $0 \le t \le t'$; then ψ' and ψ_0 have the same distribution, that is,

$$\bar{\mu}(\inf_{i\le n}[\![\psi'(\check{t}_i,\check{t}'_i)>\alpha_i]\!])=\bar{\mu}(\inf_{i\le n}[\![\psi_0(\check{t}_i,\check{t}'_i)>\alpha_i]\!])$$

whenever $0\le t_i\le t'_i$ and $\alpha_i\in\mathbb{R}$ for $i\le n$ (454Vd[3]). At the same time, ψ' is independent of $\mathfrak{A}_s$, that is,

$$\bar{\mu}(a\cap\inf_{i\le n}[\![\psi'(\check{t}_i,\check{t}'_i)>\alpha_i]\!])=\bar{\mu}a\cdot\bar{\mu}(\inf_{i\le n}[\![\psi'(\check{t}_i,\check{t}'_i)>\alpha_i]\!])$$

whenever $a\in\mathfrak{A}_s$ and $0\le t_i\le t'_i$ and $\alpha_i\in\mathbb{R}$ for $i\le n$. It follows that $I\mapsto S_I(\psi',\mathbf{1}):[0,\infty[^{<\omega}\to L^0$ and $I\mapsto S_I(\psi_0,\mathbf{1})$ have the same distribution, that is,

$$\bar{\mu}(\inf_{i\le n}[\![S_{I_i}(\psi',\mathbf{1})>\alpha_i]\!])=\bar{\mu}(\inf_{i\le n}[\![S_{I_i}(\psi_0,\mathbf{1})>\alpha_i]\!])$$

whenever $I_0,\dots,I_n$ are finite subsets of $\{\check{t}:t\ge 0\}$ and $\alpha_i\in\mathbb{R}$ for $i\le n$, while $I\mapsto S_I(\psi',\mathbf{1})$ is independent of $\mathfrak{A}_s$. Now this means that if $t\ge 0$ then the limits

$$\int_{[0,\infty[^\check{}\wedge\check{t}}d\psi'=\int_{[0,\infty[^\check{}\cap[\check{s},(s+t)^\check{})}d\psi=w_{(s+t)^\check{}}-w_{\check{s}}$$

and

$$\int_{[0,\infty[^\check{}\wedge\check{t}}d\psi_0=w_{\check{t}}$$

have the same distribution (454Ve again), while $w_{(s+t)^\check{}}-w_{\check{s}}$ is independent of $\mathfrak{A}_s$ (652Bc). As s and t are arbitrary, $\boldsymbol{w}$ is a Lévy process. **Q**

(c) Since $\mathrm{Osclln}(\boldsymbol{w}\restriction\mathcal{T}\wedge\tau)\le\chi 1$ for every $\tau\in\mathcal{T}_f$, $\boldsymbol{w}\restriction\mathcal{T}_b$ is an L^1-process (652I) and there is an $\alpha\in\mathbb{R}$ such that $\boldsymbol{w}-\alpha\boldsymbol{\iota}$ is a local martingale (652J). But now our original process

$$\boldsymbol{v}=(\boldsymbol{v}-\boldsymbol{w})+\alpha\boldsymbol{\iota}+(\boldsymbol{w}-\alpha\boldsymbol{\iota})$$

is expressible as the sum of a locally order-bounded process and a local martingale, so is a semi-martingale, that is, is a local integrator (627Q).

652L Proposition Let $(\mathfrak{A},\bar{\mu},[0,\infty[\,,\langle\mathfrak{A}_t\rangle_{t\ge 0},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ be a right-continuous real-time stochastic integration structure, and $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{T}_f}$ a Lévy process. Then its quadratic variation $\boldsymbol{v}^*$ is a Lévy process.

proof Since $\boldsymbol{v}$ is a local integrator (652K), its quadratic variation is defined everywhere in $\mathcal{T}_b$. Express $\boldsymbol{v}^*$ as $\langle v^*_\tau\rangle_{\tau\in\mathcal{T}_b}$.

(a) By 631Jb, $\boldsymbol{v}^*$ is locally near-simple.

(b) We know that $\check{T}$ separates $\mathcal{T}_b$ (633Da), so if $t\ge 0$ then $\check{T}\wedge\check{t}=\{\check{s}:0\le s\le t\}$ separates $\mathcal{T}_b\wedge\check{t}$ (633D(b-i)) and contains both $\min(\mathcal{T}_b\wedge\check{t})=\check{0}$ and $\max(\mathcal{T}_b\wedge\check{t})=\check{t}$. Also $\boldsymbol{v}\restriction\check{T}\wedge\check{t}$ is an integrator (616P(b-ii)). By 633Ph, $\boldsymbol{v}^*$ extends the quadratic variation of $\boldsymbol{v}\restriction\check{T}\wedge\check{t}$; in particular,

$$v^*_{\check{t}}=\int_{\check{T}\wedge\check{t}}(d\boldsymbol{v})^2=\lim_{I\uparrow\mathcal{I}(\check{T}\wedge\check{t})}S_I(\mathbf{1},(d\boldsymbol{v})^2).$$

(c) Now if s, $t\ge 0$ and $I\in\mathcal{I}(\check{T}\wedge\check{t})=(\check{T}\wedge\check{t})^{<\omega}$, then $S_{I+\check{s}}(\mathbf{1},(d\boldsymbol{v})^2)$ is independent of $\mathfrak{A}_s$ and has the same distribution as $S_I(\mathbf{1},(d\boldsymbol{v})^2)$. **P** If $\#(I)\le 1$, then $S_{I+\check{s}}(\mathbf{1},(d\boldsymbol{v})^2)$ and $S_I(\mathbf{1},(d\boldsymbol{v})^2)$ are both zero. Otherwise, let $\langle s_i\rangle_{i\le n}$ be the increasing enumeration of I; then $\langle v_{(s+s_i)^\check{}}\rangle_{i\le n}$ is independent of $\mathfrak{A}_s$ and has the same distribution as $\langle v_{\check{s}_i}\rangle_{i\le n}$, by 652Hb, so $S_{I+\check{s}}(\mathbf{1},(d\boldsymbol{v})^2)=\sum_{i=0}^{n-1}(v_{(s+s_{i+1})^\check{}}-v_{(s+s_i)^\check{}})^2$ is independent of $\mathfrak{A}_s$ and has the same distribution as $\sum_{i=0}^{n-1}(v_{\check{s}_{i+1}}-v_{\check{s}_i})^2=S_{I+\check{s}}(\mathbf{1},(d\boldsymbol{v})^2)$. **Q**

(d) Consequently

$$\begin{aligned}v^*_{(s+t)^\check{}}-v^*_{\check{s}}&=\int_{\check{T}\wedge(s+t)^\check{}}(d\boldsymbol{v})^2-\int_{\check{T}\wedge\check{s}}(d\boldsymbol{v})^2=\int_{\check{T}\cap[\check{s},(s+t)^\check{}]}(d\boldsymbol{v})^2\\&=\lim_{I\uparrow\mathcal{I}(\check{T}\cap[\check{s},(s+t)^\check{}])}S_I(\mathbf{1},(d\boldsymbol{v})^2)=\lim_{I\uparrow\mathcal{I}(\check{T}\wedge\check{t})}S_{I+\check{s}}(\mathbf{1},(d\boldsymbol{v})^2)\end{aligned}$$

is independent of $\mathfrak{A}_s$, by 652Bc, and has the same distribution as $\int_{\check{T}\wedge\check{t}}(d\boldsymbol{v})^2=v^*_{\check{t}}$ by 454Ve. Thus $\boldsymbol{v}^*$ satisfies the conditions of 652C and is a Lévy process.

[3]Later editions only.

652M The Cauchy process I mentioned the Cauchy distribution in the exercises to §285, but I wish now to go rather more deeply into its properties, so I run over the basic facts from the beginning.

(a) If $t > 0$, then

$$\int_{-\infty}^{\infty} \frac{t}{\pi(t^2+\xi^2)} d\xi = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d}{d\xi} \arctan(1+\tfrac{\xi}{t}) d\xi = 1,$$

so we have a distribution λ_t on $\mathbb{R}$ with density function $\xi \mapsto \frac{t}{\pi(t^2+\xi^2)}$ (271H), the **Cauchy distribution** with centre 0 and scale parameter t.

(b)(i) For $t > 0$ and $\eta \in \mathbb{R}$,

$$\int_0^{\infty} e^{-t\xi} \cos \eta\xi \, d\xi = \frac{1}{t^2+\eta^2} \int_0^{\infty} \frac{d}{d\xi}(\eta e^{-t\xi} \sin \eta\xi - te^{-t\xi} \cos \eta\xi) d\xi = \frac{t}{t^2+\eta^2}.$$

So

$$\int_{-\infty}^{\infty} e^{-i\eta\xi} e^{-t|\xi|} d\xi = \int_{-\infty}^{\infty} e^{-t|\xi|} \cos \eta\xi \, d\xi = \frac{2t}{t^2+\eta^2},$$

the Fourier transform of $\xi \mapsto e^{-t|\xi|}$ is $\eta \mapsto \frac{2t}{\sqrt{2\pi}(t^2+\eta^2)}$ and the inverse Fourier transform of $\eta \mapsto \frac{2t}{\sqrt{2\pi}(t^2+\eta^2)}$ is $\xi \mapsto e^{-t|\xi|}$ (283J), that is,

$$e^{-t|\xi|} = \int_{-\infty}^{\infty} e^{i\xi\eta} \frac{t}{\pi(t^2+\eta^2)} d\eta = \int_{-\infty}^{\infty} e^{i\xi\eta} \lambda_t(d\eta)$$

for $\xi \in \mathbb{R}$ (235K), and the characteristic function of λ_t is $\xi \mapsto e^{-t|\xi|}$ (285Aa).

(ii) The argument gives us another integral: $\int_0^{\infty} \frac{1-\cos\eta}{\eta^2} d\eta = \frac{\pi}{2}$. **P** For any $t > 0$,

$$\int_0^{\infty} \frac{\cos \eta}{t^2+\eta^2} d\eta = \frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{i\eta}}{t^2+\eta^2} d\eta = \frac{\pi}{2t} e^{-t}$$

and

$$\int_0^{\infty} \frac{1}{t^2+\eta^2} d\eta = \frac{1}{t} \int_0^{\infty} \frac{1}{1+\eta^2} d\eta = \frac{\pi}{2t}.$$

Because $\cos \eta \le 1$, $\sin \eta \le \eta$, $1 - \cos \eta \le \frac{1}{2}\eta^2$ and $0 \le \frac{1-\cos\eta}{\eta^2} \le \min(\frac{1}{2}, \frac{1}{\eta^2})$ for every $\eta > 0$, $\int_0^{\infty} \frac{1-\cos\eta}{\eta^2} d\eta$ is defined in $\mathbb{R}$ and equal to

$$\lim_{t\downarrow 0} \int_0^{\infty} \frac{1-\cos\eta}{t^2+\eta^2} d\eta = \lim_{t\downarrow 0} \frac{\pi}{2t}(1 - e^{-t}) = \frac{\pi}{2}. \textbf{ Q}$$

(c) If s, $t > 0$ the characteristic function of $\lambda_s * \lambda_t$ is $\xi \mapsto e^{-s|\xi|} e^{-t|\xi|} = e^{-(s+t)|\xi|}$ (285R), that is, it is equal to the characteristic function of λ_{s+t}, and $\lambda_s * \lambda_t = \lambda_{s+t}$ (285Ma).

We can therefore apply the construction of 652F with the family $\langle \lambda_t \rangle_{t>0}$ to obtain a probability space $(C_{\mathrm{dlg}}, \ddot{\Sigma}, \ddot{\mu})$, a stochastic integration structure $(\mathfrak{C}, \bar{\ddot{\mu}}, [0,\infty[\,, \langle \mathfrak{C}_t \rangle_{t\ge 0})$ and a classical Lévy process $\boldsymbol{u}$, the **Cauchy process**.

(d) Writing $C([0,\infty[) \subseteq C_{\mathrm{dlg}}$ for the set of continuous functions from $[0,\infty[$ to $\mathbb{R}$, $\ddot{\mu} C([0,\infty[) = 0$. **P** Writing $X_t(\omega) = \omega(t)$ for $\omega \in C_{\mathrm{dlg}}$ and $t \ge 0$ as usual,

$$\ddot{\mu}\{\omega : |X_t(\omega)| \le \epsilon\} = 1 - 2\int_{\epsilon}^{\infty} \frac{t}{\pi(t^2+\xi^2)} d\xi \le 1 - \frac{2}{\pi} \int_{\epsilon}^{\infty} \frac{t}{2\xi^2} d\xi = 1 - \frac{t}{\pi\epsilon}$$

whenever $0 < t \le \epsilon$. So if $n > \frac{1}{\epsilon}$ is an integer and we set $F_n(\epsilon) = \{\omega : |X_{t_{i+1}}(\omega) - X_{t_i}(\omega)| \le \epsilon$ for every $i < n\}$ where $t_i = \frac{i}{n}$ for $i \le n$,

$$\ddot{\mu} F_n(\epsilon) = \ddot{\mu}\{\omega : |X_{t_1}| \le \epsilon\}^n \le (1 - \frac{1}{n\pi\epsilon})^n \le e^{-1/\pi\epsilon}$$

because the $X_{t_{i+1}} - X_{t_i}$ are independent with the same distribution as X_{t_1} and $\ln(1 - \frac{1}{n\pi\epsilon}) \le -\frac{1}{n\pi\epsilon}$. But now

$$C([0,\infty[) \subseteq \bigcup_{m\in\mathbb{N}}\bigcap_{n\ge m} F_n(\epsilon)$$

has measure at most $\liminf_{n\to\infty} \ddot{\mu}F_n(\epsilon) \le e^{-1/\pi\epsilon}$. As ϵ is arbitrary, $C([0,\infty[)$ is negligible. **Q**

652N Alternative description of the Cauchy process (a) Let μ_0 be the indefinite-integral measure over μ_L corresponding to the function $\xi \mapsto \frac{1}{\pi\xi^2} : \mathbb{R}\setminus\{0\} \to [0,\infty[$ (234J), μ_1 the subspace measure on $[0,\infty[$ induced by μ_L, and $\mu = \mu_0 \times \mu_1$ the c.l.d. product measure on $S = \mathbb{R}\times[0,\infty[$ (251F). We see that

μ is atomless (234Nf, 251Xt),

$\mu(\{\xi\}\times[0,\infty[) = \mu(\mathbb{R}\times\{t\}) = 0$ whenever $\xi\in\mathbb{R}$ and $t \ge 0$,

$\mu(\{\xi : |\xi| \ge \delta\}\times[0,t]) = \frac{2t}{\pi\delta}$ is finite whenever $\delta > 0$ and $t \in [0,\infty[$,

$\mu([0,\delta]\times[\alpha,\beta]) = \mu([-\delta,0]\times[\alpha,\beta]) = \infty$ whenever $\delta > 0$ and $0 \le \alpha < \beta$.

(b) Let ν be the Poisson point process on (S,μ) with intensity 1 (495E[4]), so that ν is a probability measure on $\Omega = \mathcal{P}S$, and Σ its domain. Then for ν-almost every $\varpi \subseteq S$,

$\xi \ne 0$, $|\xi| \ne |\xi'|$ and $s \ne s'$ whenever (ξ,s), $(\xi',s') \in \varpi$ are distinct

(applying 495H[5] to $(\xi,s)\mapsto|\xi|$ and $(\xi,s)\mapsto s$),

$\varpi\cap\{(\xi,s) : |\xi|\ge\delta,\ s\le t\}$ is finite whenever $\delta > 0$ and $t\in[0,\infty[$ are rational,

$\varpi\cap([0,\delta]\times[\alpha,\beta])$ and $\varpi\cap([-\delta,0]\times[\alpha,\beta])$ are infinite
whenever $\delta > 0$ and $0\le\alpha<\beta$ and α, β and δ are rational.

Write Ω_0 for the set of $\varpi\subseteq S$ with these properties. Note that if $\varpi\in\Omega_0$ then in fact

$\varpi\cap\{(\xi,s) : |\xi|\ge\delta,\ s\le t\}$ is finite whenever $\delta > 0$ and $t\in[0,\infty[$,

$\varpi\cap([0,\delta]\times[\alpha,\beta])$ and $\varpi\cap([-\delta,0]\times[\alpha,\beta])$ are infinite
whenever $\delta > 0$ and $0\le\alpha<\beta$.

(c) For $t \ge 0$ set

$$\Sigma_t = \{H\triangle A : H\in\Sigma,\ H = \{\varpi : \varpi\subseteq S,\ \varpi\cap(\mathbb{R}\times[0,t])\in H\},\ A\in\mathcal{N}(\mu)\}.$$

Then $\langle\Sigma_t\rangle_{t\ge0}$ is a filtration of σ-subalgebras of Σ. Moreover, it is right-continuous. **P** Take any $t\ge0$, and set $s_n = t + 2^{-n}$ for $n\in\mathbb{N}$. Setting $S_0 = \mathbb{R}\times[0,t]$, $S_1 = \mathbb{R}\times]t+1,\infty[$ and $S_n = \mathbb{R}\times]s_{n-2},s_{n-1}]$ for $n\ge2$, we have an isomorphism $\psi : \mathcal{P}S \to \prod_{n\in\mathbb{N}}\mathcal{P}S_n$ where $\psi(\varpi) = \langle\varpi\cap S_n\rangle_{n\in\mathbb{N}}$ for $\varpi\subseteq S$, and ψ is a measure space isomorphism between ν and the product measure $\tilde\lambda = \prod_{n\in\mathbb{N}}\nu_n$, where each ν_n is the Poisson point process with intensity 1 on S_n endowed with the subspace measure induced by μ (495F). Suppose that $H\in\bigcap_{s>t}\Sigma_s$ and consider $\psi[H]$. For each $n\in\mathbb{N}$, there is an $H_n\in\Sigma$ such that $H_n = \{\varpi : \varpi\subseteq S,\ \varpi\cap(\mathbb{R}\times[0,s_n])\in H_n\}$ and $H\triangle H_n$ is ν-negligible. Setting $H'_n = \bigcap_{m\ge n}H_m$, we see that $H'_n = \{\varpi : \varpi\subseteq S,\ \varpi\cap(\mathbb{R}\times[0,s_n])\in H'_n\}$ and $H\triangle H'_n$ is negligible for each n. Now each $\psi[H'_n]\subseteq\prod_{n\in\mathbb{N}}\mathcal{P}S_n$ is determined by coordinates in $\{0,n+2,n+3,\dots\}$, so $\bigcap_{m\in\mathbb{N}}\psi[H'_m] = \bigcap_{m\ge n}\psi[H'_m]$ is determined by coordinates in $\{0,n+2,n+3,\dots\}$ for each n, and there is therefore a $W\subseteq\prod_{n\in\mathbb{N}}\mathcal{P}S_n$ determined by the single coordinate $\{0\}$ such that $W\triangle\bigcap_{m\in\mathbb{N}}\psi[H'_m]$ is $\tilde\lambda$-negligible, by 254Rd. But now $H\triangle\psi^{-1}[W]$ is ν-negligible, while $\psi^{-1}[W] = \{\varpi : \varpi\subseteq S,\ \varpi\cap(\mathbb{R}\times[0,t])\in\psi^{-1}[W]\}$. So $H\in\Sigma_t$. As t and H are arbitrary, $\langle\Sigma_t\rangle_{t\ge0}$ is right-continuous. **Q**

(d) Suppose that we have an $m\ge0$ and family $\langle E_\alpha\rangle_{0\le\alpha<1}$ of Borel subsets of $[-1,1]\times[0,m]$ such that

$$E_\alpha\subseteq E_\beta \text{ whenever } 0\le\alpha\le\beta<1,$$

[4] In earlier editions I used the word 'density'.
[5] Later editions only.

$$\text{if } 0 \le \alpha < 1 \text{ and } (\xi, t) \in E_\alpha \text{ then } (-\xi, t) \in E_\alpha,$$

$$\text{if } 0 \le \alpha < 1 \text{ there is a } \delta > 0 \text{ such that } |\xi| \ge \delta \text{ whenever } (\xi, t) \in E_\alpha.$$

For $0 \le \alpha < 1$ set

$$X_\alpha(\varpi) = \sum_{(\xi,s)\in\varpi\cap E_\alpha} \xi \quad \text{if } \varpi \in \Omega_0,$$

(this is legitimate because $\varpi \cap E_\alpha$ is finite)

$$= 0 \quad \text{if } \varpi \in \Omega \setminus \Omega_0.$$

Then

(i) $\lim_{\alpha\uparrow 1} X_\alpha(\varpi)$ is defined in $\mathbb{R}$ for almost every ϖ,

(ii) setting $h(\varpi) = \sup_{\alpha<1} |X_\alpha(\varpi)|$ for $\varpi \in \Omega$, $(\int h\,d\nu)^2$ is defined and is not greater than $\sup_{\alpha<1} \int_{E_\alpha} \xi^2 \mu(d(\xi, s))$.

P(i)(α) If

$$\mathrm{T}_\alpha = \{H : H \in \Sigma,\, H = \{\varpi : \varpi \in \Omega,\, \varpi \cap E_\alpha \in H\}\},$$

$$\hat{\mathrm{T}}_\alpha = \{H \triangle A : H \in \mathrm{T}_\alpha,\, A \in \mathcal{N}(\nu)\},$$

$$\mathrm{T}'_\alpha = \{H : H \in \Sigma,\, H = \{\varpi : \varpi \setminus E_\alpha \in H\}\},$$

$$\hat{\mathrm{T}}'_\alpha = \{H \triangle A : H \in \mathrm{T}'_\alpha,\, A \in \mathcal{N}(\nu)\},$$

then X_α is $\hat{\mathrm{T}}_\alpha$-measurable. We have

$$\textstyle\int_{E_\alpha} |\xi| \mu(d(\xi, s)) \le \mu E_\alpha < \infty,$$

$$\int X_\alpha d\nu = \int_{E_\alpha} \xi\, \mu(d(\xi, s))$$

(495Oa[6])

$$= \int_{E_\alpha} -\xi\, \mu(d(\xi, s))$$

(because $(\xi, t) \mapsto (-\xi, t) : E_\alpha \to R_\alpha$ is a μ-preserving bijection)

$$= 0.$$

If $0 < \beta < \alpha$, then

$$X_\beta(\varpi) - X_\alpha(\varpi) = \textstyle\sum_{(\xi,s)\in\varpi\cap E_\beta\setminus E_\alpha} \xi$$

for $\varpi \in \Omega_0$. So $X_\beta - X_\alpha$ is $\hat{\mathrm{T}}'_\alpha$-measurable By 495F[7], T'_α and T_α are independent, so $\hat{\mathrm{T}}'_\alpha$ and $\hat{\mathrm{T}}_\alpha$ are independent. Since $\int X_\beta - X_\alpha d\nu = 0$, $\int_H X_\beta - X_\alpha d\nu = 0$ for every $H \in \hat{\mathrm{T}}_\alpha$, so X_α is a conditional expectation of X_β on $\hat{\mathrm{T}}_\alpha$. But this means that $\langle X_\alpha \rangle_{\alpha<1}$ is a martingale with respect to the ordering $\le$ on $[0, 1[$.

(β) Next, 495Ob[7] tells us that

$$\int X_\alpha^2 d\nu = \int_{E_\alpha} \xi^2 \mu(d(\xi, s)) + (\int_{E_\alpha} \xi\, \mu(d(\xi, s)))^2$$
$$= \int_{E_\alpha} \xi^2 \mu(d(\xi, s)) \le m \int_{[-1,1]} \xi^2 \mu_0(d\xi) = \frac{2m}{\pi}$$

[6] Formerly 495L.
[7] Later editions only.

for every $\alpha \in [0,1[$. So $\{X_\alpha : 0 \le \alpha < 1\}$ is $\| \|_2$-bounded, therefore uniformly integrable (621Be). By 622J, $v = \lim_{\alpha\uparrow 1} X^\bullet_\alpha$ is defined in $L^0(\nu)$ for the topology of convergence in measure and $\inf_{\alpha<1}\sup_{\alpha\le\beta<1}|v - X^\bullet_\beta| = 0$.

(γ) If we take any Σ-measurable function $g : \Omega \to \mathbb{R}$ such that $g^\bullet = v$ and write Q for $\mathbb{Q} \cap [0,1[$, we know that

$$\inf_{q\in Q}\sup_{q'\in Q,q'\ge q}|g^\bullet - X^\bullet_{q'}| = 0 \text{ in } L^0(\mathfrak{A})$$

and because Q is countable, this translates directly to

$$\inf_{q\in Q}\sup_{q'\in Q,q'\ge q}|g(\varpi) - X_{q'}(\varpi)| = 0 \text{ for } \nu\text{-almost every } \varpi \subseteq \mathcal{S}$$

if we calculate the suprema and infimum in $[0,\infty]$. But now observe that, at least for $\varpi \in \Omega_0$,

$$\sup_{q'\in Q,q'\ge q}|h(\varpi) - X_{tq'}(\varpi)| = \sup_{q\le\beta<1}|g(\varpi) - X_\beta(\varpi)|$$

for every $q \in Q$, because if $q \le \beta < 1$ and $\varpi \in \Omega_0$, there is a $q' \in Q$ such that $\beta < q' < 1$ and $\varpi \cap (E_{q'} \setminus E_\beta) = \emptyset$ so that $X_\beta(\varpi) = X_{q'}(\varpi)$. And now we see that for ν-almost every $\varpi \in \Omega_0$,

$$\inf_{\alpha<1}\sup_{\alpha\le\beta<1}|g(\varpi) - X_\beta(\varpi)| = 0$$

and $g(\varpi) = \lim_{\alpha\uparrow 1} X_\alpha(\varpi)$.

(ii) As noted in (i-γ) just above, if $\varpi \in \Omega_0$ and $0 \le \alpha < 1$ there is a rational $q \in [\alpha, 1[$ such that $X_\alpha(\varpi) = X_q(\varpi)$, so $h(\varpi) = \sup_{q\in\mathbb{Q},0\le q<1}|X_q(\varpi)|$ for every $\varpi \in \Omega_0$. Now 623M tells us that $\sup_{0\le\alpha<1}|X^\bullet_\alpha|$ is defined in $L^0(\nu)$ and

$$\begin{aligned}(\int h\,d\nu)^2 &= \|\sup_{q\in\mathbb{Q},0\le q<1}|X^\bullet_q|\|_1^2 \le \|\sup_{0\le\alpha<1}|X^\bullet_\alpha|\|_1^2 \\ &\le \sup_{0\le\alpha<1}\|X^\bullet_\alpha\|_2^2 = \sup_{\alpha<1}\int_{E_\alpha}\xi^2\mu(d(\xi,s))\end{aligned}$$

by (i-β). **Q**

(e)(i) For m, $n \in \mathbb{N}$, $\alpha \in [0,1[$ and $\varpi \in \Omega$ set

$$E_{mn\alpha} = \{(\xi,s) : 2^{-n-1} < |\xi| \le 2^{-n},\ 0 \le s \le \alpha m\} \subseteq S,$$

$$\begin{aligned}Y_{mn\alpha}(\varpi) &= \sum_{(\xi,s)\in\varpi\cap E_{mn\alpha}}\xi \text{ if } \varpi \in \Omega_0, \\ &= 0 \text{ otherwise.}\end{aligned}$$

Then

$$\int_{E_{mn\alpha}}\xi^2\mu(d(\xi,s)) = \alpha m\int_{F_n}\xi^2\mu_0(d\xi)$$

(where $F_n = \{\eta : 2^{-n-1} < |\eta| \le 2^{-n}\}$)

$$= \frac{\alpha m}{\pi}\mu_L F_n = \frac{2^{-n}\alpha m}{\pi}.$$

If we set

$$h_{mn}(\varpi) = \sup_{\alpha\in\mathbb{Q}\cap[0,1[}|Y_{mn\alpha}(\varpi)| \in [0,\infty]$$

for $\varpi \in \Omega$, $\mathbb{E}(h_{mn}) \le 2^{-n/2}\sqrt{\frac{m}{\pi}}$, by (d-ii). But this means that

$$\Omega_2 = \{\varpi : \varpi \in \Omega_0,\ \sum_{n=0}^\infty h_{mn}(\varpi) < \infty \text{ for every } m \in \mathbb{N}\}$$

is ν-conegligible. At the same time, for $\varpi \in \Omega_2$,

$$|Y_{mn\alpha}(\varpi)| \le \sup_{q\in\mathbb{Q}\cap[0,1[}|Y_{mnq}(\varpi)| \le h_{mn}(\varpi)$$

whenever m, $n \in \mathbb{N}$ and $\alpha \in [0,1[$, just as in (d-i-γ) and (d-ii) above.

(ii) For $t \ge 0$ and $n \in \mathbb{N}$, set

$$E_{tn} = \{(\xi,s) : 2^{-n-1} < |\xi| \le 2^{-n},\ 0 \le s \le t\} \subseteq S,$$

$$\begin{aligned} Y_{tn}(\varpi) &= \sum_{(\xi,s)\in\varpi\cap E_{tn}} \xi \text{ if } \varpi \in \Omega_0,\\ &= 0 \text{ otherwise.}\end{aligned}$$

If $\varpi \in \Omega_2$ then, applying (i) with $m = \lfloor t+1 \rfloor$ and $\alpha = \frac{t}{m}$, we see that $\sum_{n=0}^{\infty} Y_{tn}(\varpi)$ is defined (as an unconditional sum) and finite. Since $\varpi \cap (]1,\infty[\times [0,t])$ and $\varpi \cap (]-\infty,-1[\times [0,t])$ are finite,

$$Z_t(\varpi) = \sum_{n=0}^{\infty} Y_{tn}(\varpi) + \sum_{(\xi,s)\in\varpi,|\xi|>1,s\le t}\xi$$

is defined. We see also that, for any $m \in \mathbb{N}$,

$$Z_t(\varpi) = \lim_{n\to\infty}\sum_{k=0}^{n} Y_{tk}(\varpi) + \sum_{(\xi,s)\in\varpi,|\xi|>1,s\le t}\xi$$

uniformly for $t \in [0,m]$. Since the functions

$$t \mapsto \sum_{k=0}^{n} Y_{tk}(\varpi) + \sum_{(\xi,s)\in\varpi,|\xi|>1,s\le t}\xi$$

are càdlàg for every $n \in \mathbb{N}$, $t \mapsto Z_t(\varpi) : [0,m] \to \mathbb{R}$ is càdlàg. Since m is arbitrary, $t \mapsto Z_t(\varpi) : [0,\infty[\to \mathbb{R}$ is càdlàg.

For completeness, I will set $Z_t(\varpi) = 0$ for $t \ge 0$ and $\varpi \in \Omega \setminus \Omega_2$, so that $t \mapsto Z_t(\varpi)$ is càdlàg for every $\varpi \in \Omega$.

(f) If $t \ge 0$, the characteristic function of Z_t is $\eta \mapsto e^{-t|\eta|}$. **P** For $n \in \mathbb{N}$, set

$$F_{tn} = \{(\xi,s) : 2^{-n} \le |\xi| \le 2^{n},\ s \le t\} \subseteq S,$$

$$\begin{aligned} g_n(\xi,s) &= \xi \text{ if } (\xi,s) \in F_{tn},\\ &= 0 \text{ for other } (\xi,s) \in S,\end{aligned}$$

$$\begin{aligned} Z_{tn}(\varpi) &= \sum_{(\xi,s)\in\varpi\cap F_{tn}} \xi \quad \text{if } \varpi \cap F_{tn} \text{ is finite,}\\ &= 0 \text{ for other } \varpi \subseteq S.\end{aligned}$$

Then $Z_t(\varpi) = \lim_{n\to\infty} Z_{tn}(\varpi)$ for every $\varpi \in \Omega_2$. Now the characteristic function of Z_{tn} is

$$\eta \mapsto \int_{\mathcal{P}S} e^{i\eta Z_{tn}}d\nu = \exp\Bigl(\int_S (e^{i\eta g_n} - 1)d\mu\Bigr)$$

by 495P[8]. Next,

$$\begin{aligned} \int_S (e^{i\eta g_n} - 1)d\mu &= \int_0^t \Bigl(\int_{-2^n}^{-2^{-n}} \frac{1}{\pi\xi^2}(e^{i\eta\xi} - 1)d\xi + \int_{2^{-n}}^{2^n} \frac{1}{\pi\xi^2}(e^{i\eta\xi} - 1)d\xi\Bigr)dt\\ &= \frac{1}{\pi}\int_0^t \int_{2^{-n}}^{2^n} \frac{1}{\xi^2}(e^{i\eta\xi} + e^{-i\eta\xi} - 2)d\xi dt\\ &= \frac{2t}{\pi}\int_{2^{-n}}^{2^n} \frac{1}{\xi^2}(\cos\eta\xi - 1)d\xi\\ &= -\frac{2t}{\pi}\int_{2^{-n}}^{2^n} \frac{1}{\xi^2}(1 - \cos|\eta|\xi)d\xi.\end{aligned}$$

[8]Formerly 495M.

As in (b-ii) of the proof of 652M, $\int_0^\infty \frac{1}{\xi^2}(1-\cos|\eta|\xi)d\xi = |\eta|\int_0^\infty \frac{1}{\xi^2}(1-\cos\xi)d\xi$ is finite, so we can apply Lebesgue's Dominated Convergence Theorem again to see that

$$\begin{aligned}\lim_{n\to\infty}\int_S(e^{i\eta g_n}-1)d\mu &= -\lim_{n\to\infty}\frac{2t}{\pi}\int_{2^{-n}}^{2^n}\frac{1}{\xi^2}(1-\cos|\eta|\xi)d\xi = -\frac{2t}{\pi}\int_0^\infty\frac{1}{\xi^2}(1-\cos|\eta|\xi)d\xi\\ &= -\frac{2t|\eta|}{\pi}\int_0^\infty\frac{1}{\xi^2}(1-\cos\xi)d\xi = -\frac{2t|\eta|}{\pi}\cdot\frac{\pi}{2}\end{aligned}$$

(652M(b-ii))

$$= -t|\eta|,$$

and

$$\begin{aligned}\int_{\mathcal{P}S}e^{i\eta Z_t}d\nu &= \lim_{n\to\infty}\int_{\mathcal{P}S}e^{i\eta Z_{tn}}d\nu = \lim_{n\to\infty}\exp(\int_S(e^{i\eta g_n}-1)d\mu)\\ &= \exp(\lim_{n\to\infty}\int_S(e^{i\eta g_n}-1)d\mu) = \exp(-t|\eta|),\end{aligned}$$

as claimed. **Q**

By 285Mb, it follows that (for $t>0$) the distribution of Z_t is the Cauchy distribution λ_t as defined in 652M.

(g)(i) If s, $t>0$, $Z_{s+t}-Z_s$ has the same distribution as Z_t. **P** Setting $\phi_s(\xi,t')=(\xi,s+t')$ for $\xi\in\mathbb{R}$ and $t'\ge 0$, ϕ_s is a measure-preserving bijection between (S,μ) and $\mathbb{R}\times[s,\infty[$ with its subspace measure, so $\varpi\mapsto\phi_s^{-1}[\varpi]:\mathcal{P}S\to\mathcal{P}S$ is ν-inverse-measure-preserving (495F), while (in the notation of (e) above)

$$\textstyle\sum_{(\xi,t')\in\varpi\cap F_{s+t,n}\setminus F_{sn}}\xi = \sum_{(\xi,t')\in\phi_s^{-1}[\varpi]\cap F_{tn}\setminus F_{0n}}\xi$$

for every $\varpi\in\Omega_0$. So $Z_{s+t,n}-Z_{sn}$ and $Z_{tn}-Z_{t0}=Z_{tn}$ have the same distribution for each n. Since $\langle Z_{t'n}\rangle_{n\in\mathbb{N}}$ converges in measure to $Z_{t'}$ for every $t'\ge 0$, 454U[9] tells us that $Z^\bullet_{s+t}-Z^\bullet_s$ and $Z^\bullet_t$ have the same distribution, that is, $Z_{s+t}-Z_s$ and Z_t have the same distribution. **Q**

(ii) If s, $t\ge 0$ and we take Σ'_s to be

$$\{H\triangle A: H\in\Sigma,\, H=\{\varpi:\varpi\subseteq S,\,\varpi\cap(\mathbb{R}\times]s,\infty[)\in H\},\, A\in\mathcal{N}(\mu)\},$$

then Σ'_s and Σ_s, as defined in (c), are independent, by 495F again, while $Z_{s+t}-Z_s$ is Σ'_s-measurable. So $Z_{s+t}-Z_s$ is independent of Σ_s.

(h) Accordingly we have a second representation of the Cauchy process based on (Ω,ν) rather than on $(C_{\mathrm{dlg}},\ddot\mu)$. We saw in (e-ii) that $t\mapsto Z_t(\varpi)$ is càdlàg for every $\varpi\in\Omega$. Taking $\mathfrak{A}_t$ to be $\{E^\bullet:E\in\Sigma_t\}$ for $t\ge 0$, 631D tells us that in the stochastic integration structure $(\mathfrak{A},\bar\nu,[0,\infty[\,,\langle\mathfrak{A}_t\rangle_{t\ge 0})$ we have a locally near-simple process $\boldsymbol{z}=\langle z_\sigma\rangle_{\sigma\in\mathcal{T}_f}$ such that $z_{\check t}=Z^\bullet_t$ for every $t\ge 0$. Because $\langle\Sigma_t\rangle_{t\ge 0}$ is right continuous ((c) above), so is $\langle\mathfrak{A}_t\rangle_{t\ge 0}$. Thus $\boldsymbol{z}$ is a Lévy process as defined in 652B.

(i) We can use this new representation to get some information about typical sample paths in the process described in 652Mc.

If $\varpi\in\Omega_0$ and $t>0$ we have a canonical enumeration $\langle(\tilde\xi_{nt}(\varpi),\tilde s_{nt}(\varpi))\rangle_{n\in\mathbb{N}}$ of the countably infinite set $\varpi\cap(\mathbb{R}\times[0,t])$ such that $|\tilde\xi_{n+1,t}(\varpi)|<|\tilde\xi_{nt}(\varpi)|$ for every $n\in\mathbb{N}$. Now we find that for any $t\ge 0$, the conditional sum

$$\textstyle\sum_{n=0}^\infty\tilde\xi_{nt}(\varpi)=\lim_{n\to\infty}\sum_{i=0}^n\tilde\xi_{it}(\varpi)$$

[9] Later editions only.

is defined in $\mathbb{R}$ and equal to $Z_t(\varpi)$ for almost every ϖ. **P** For $0 \le \alpha < 1$, set $E_\alpha = ([-1,-1+\alpha] \cup [1-\alpha,1]) \times [0,t]$. By (d-i),

$$\Omega_{1t} = \{\varpi : \varpi \in \Omega_0,\ \lim_{\alpha\uparrow 1} \sum_{(\xi,s)\in\varpi\cap E_\alpha} \xi \text{ is defined in } \mathbb{R}\}$$

is ν-conegligible. Now if $\varpi \in \Omega_{1t} \cap \Omega_2$ and $\epsilon > 0$, $\varpi \cap ((]-\infty,-1[\cup]1,\infty[) \times [0,t]$ is finite, so there is a first $m \in \mathbb{N}$ such that $|\tilde{\xi}_{mt}|(\varpi) \le 1$, Next, there is a $\beta \in [0,1[$ such that

$$\epsilon \ge |\sum_{(\xi,s)\in\varpi\cap E_{\alpha'}} \xi - \sum_{(\xi,s)\in\varpi\cap E_\alpha} \xi| = |\sum_{(\xi,s)\in\varpi\cap(E_{\alpha'}\setminus E_\alpha)} \xi|$$

whenever $\beta \le \alpha \le \alpha' < 1$. Taking $k \ge m$ such that $|\tilde{\xi}_{kt}(\varpi)| \le \beta$, we see that if $k \le n \le n'$ then

$$|\sum_{i=0}^{n'} \tilde{\xi}_{it}(\varpi) - \sum_{i=0}^{n} \tilde{\xi}_{it}(\varpi)| = |\sum_{(\xi,s)\in\varpi\cap(E_{\alpha'}\setminus E_\alpha)} \xi| \le \epsilon$$

where $\alpha = 1 - |\tilde{\xi}_{nt}|$ and $\alpha' = 1 - |\tilde{\xi}_{n't}|$. As ϵ is arbitrary, $\lim_{n\to\infty} \sum_{i=0}^{n} \tilde{\xi}_{it}(\varpi)$ is defined.

At the same time,

$$\begin{aligned} Z_t(\varpi) &= \lim_{n\to\infty} \sum_{\substack{(\xi,s)\in\varpi, s\le t\\ |\xi|>2^{-n}}} \xi = \sum_{(\xi,s)\in\varpi,\xi>1} \xi + \lim_{n\to\infty} \sum_{\substack{(\xi,s)\in\varpi, s\le t\\ 2^{-n}<|\xi|\le 1}} \xi \\ &= \sum_{(\xi,s)\in\varpi,\xi>1} \xi + \lim_{\alpha\uparrow 1} \sum_{\substack{(\xi,s)\in\varpi, s\le t\\ 1-\alpha<|\xi|\le 1}} \xi = \lim_{n\to\infty} \sum_{i=0}^{n} \tilde{\xi}_{it}(\varpi). \ \mathbf{Q} \end{aligned}$$

(j) By (e-ii), we have a function $\phi : \Omega \to C_{\text{dlg}}$ defined by saying that

$$\begin{aligned} \phi(\varpi)(t) &= Z_t(\varpi) \text{ if } \varpi \in \Omega_2, \\ &= 0 \text{ otherwise.} \end{aligned}$$

Now ϕ is inverse-measure-preserving for ν and the measure $\ddot{\mu}$ defined in 652Mc. **P** If $0 = s_0 < \ldots < s_n$ in $[0,\infty[$ and $\alpha_i \in \mathbb{R}$ for $i \le n$, then

$$\begin{aligned} \nu\phi^{-1}\{\omega : \omega \in C_{\text{dlg}},\ \omega(s_0) \le \alpha_0,\ \omega(s_i) - \omega(s_{i-1}) \le \alpha_i \text{ for } 1 \le i \le n\} \\ = \nu\{\varpi : \varpi \in \Omega_2,\ Z_{s_0}(\varpi) \le \alpha_0,\ Z_{s_i}(\varpi) - Z_{s_{i-1}}(\varpi) \le \alpha_i \text{ for } 1 \le i \le n\} \\ = 0 \text{ if } \alpha_0 < 0, \\ = \prod_{1\le i\le n} \nu\{\varpi : Z_{s_i}(\varpi) - Z_{s_{i-1}}(\varpi) \le \alpha_i\} \end{aligned}$$

(by (g-ii))

$$= \prod_{1\le i\le n} \nu\{\varpi : Z_{s_i - s_{i-1}}(\varpi) \le \alpha_i\}$$

(by (g-i))

$$= \prod_{1\le i\le n} \lambda_{s_i - s_{i-1}}\,]-\infty,\alpha_i] \text{ otherwise}$$

(by (f)), and in either case is equal to

$$\ddot{\mu}\{\omega : \omega(s_0) \le \alpha_0,\ \omega(s_i) - \omega(s_{i-1}) \le \alpha_i \text{ for } 1 \le i \le n\}.$$

Since $\mathcal{H} = \{H : H \in \ddot{\Sigma},\ \nu\phi^{-1}[H] \text{ is defined and equal to } \ddot{\mu}H\}$ is a Dynkin class, it contains all sets of the form $\{\omega : \omega \in C_{\text{dlg}},\ \omega(s_0) \in E_0,\ \omega(s_i) - \omega(s_{i-1}) \in E_i \text{ for } 1 \le i \le n\}$ where $n \in \mathbb{N}$ and $E_i \subseteq \mathbb{R}$ is Borel for every $i \le n$, and therefore the σ-algebra generated by these, which contains all sets of the form $\{\omega : \omega \in C_{\text{dlg}},\ \omega(s_i) \in E_i \text{ for } i \le n\}$ where $n \in \mathbb{N}$ and $E_i \subseteq \mathbb{R}$ is Borel for every $i \le n$, and therefore is the subspace σ-algebra defined by the Baire σ-algebra of $\mathbb{R}^{[0,\infty[}$. Since $\ddot{\mu}$ is a completion regular quasi-Radon probability measure, every $H \in \ddot{\Sigma}$ is expressible as $H_0 \triangle A$ where $H_0 \in \mathcal{H}$ and A is included in a $\ddot{\mu}$-negligible member of $\mathcal{H}$; because ν is complete, $\ddot{\Sigma} \subseteq \mathcal{H}$, that is, ϕ is inverse-measure-preserving. **Q**

(k) If $\varpi \in \Omega_2$ and $t \ge 0$, then

$$\begin{aligned}\phi(\varpi)(t) - \lim_{s\uparrow t}\phi(\varpi)(s) &= \xi \text{ if } (\xi,t) \in \varpi,\\ &= 0 \text{ if there is no } \xi \text{ such that } (\xi,t) \in \varpi.\end{aligned}$$

P Take $m \in \mathbb{N}$ such that $t \le m$, and $\epsilon > 0$. Let $k \in \mathbb{N}$ be such that $\sum_{n=k}^{\infty} h_{mn}(\varpi) \le \epsilon$, where h_{mn} is defined as in (e-i) above. For $s \ge 0$ set $f(s) = \sum_{(\xi,s')\in\varpi,|\xi|>2^{-k},s'\le s}\xi$; as $\varpi \in \Omega_0$, this is a finite sum. Then

$$|\phi(\varpi)(s) - f(s)| = |Z_s(\varpi) - f(s)| \le \epsilon$$

for every $s \le m$, while

$$\begin{aligned}f(t) - \lim_{s\uparrow t} f(s) &= \xi \text{ if } (\xi,t) \in \varpi,\\ &= 0 \text{ if there is no } \xi \text{ such that } (\xi,t) \in \varpi.\end{aligned}$$

Now

$$\begin{aligned}&\limsup_{s\uparrow t} |\phi(\varpi)(t) - \xi - \phi(\varpi)(s)|\\ &\qquad \le |\phi(\varpi)(t) - f(t)| + \limsup_{s\uparrow t}|f(t) - f(s) - \xi| + \limsup_{s\uparrow t}|f(s) - \phi(\varpi)(s)|\\ &\qquad \le 2\sup_{s\le t}|\phi(\varpi)(s) - f(s)| \le 2\epsilon\end{aligned}$$

if $(\xi,t) \in \varpi$, and

$$\limsup_{s\uparrow t}|\phi(\varpi)(t) - \phi(\varpi)(s)| \le 2\epsilon$$

if there is no ξ such that $(\xi,t) \in \varpi$. As ϵ is arbitrary, we have the result. **Q**

(l) If $\omega \in C_{\mathrm{dlg}}$, I will write $\mathrm{Jump}(\omega)$ for

$$\{(\xi,t) : \xi \in \mathbb{R}\setminus\{0\},\ t > 0,\ \omega(t) = \lim_{s\uparrow t}\omega(s) + \xi\}.$$

Observe that $\varpi = \mathrm{Jump}(\phi(\varpi))$ for every $\varpi \in \Omega_2$, by (k).

For $E \subseteq S$, set

$$A_1(E) = \{\omega : E \cap \mathrm{Jump}(\omega) \ne \emptyset\}.$$

If $E \in \mathcal{B}(S)$, $A_1(E) \in \ddot\Sigma$. **P** Since $(\omega,t) \mapsto \omega(t) : C_{\mathrm{dlg}} \times [0,\infty[\to \mathbb{R}$ is $\ddot\Sigma\widehat{\otimes}\mathcal{B}([0,\infty[)$-measurable (631D),

$$\{(\omega,\xi,t,s) : |\omega(t) - \omega(s) - \xi| \le 2^{-n}\}$$

belongs to $\ddot\Sigma\widehat{\otimes}\mathcal{B}(S \times [0,\infty[)$ for every $n \in \mathbb{N}$,

$$\begin{aligned}\{(\omega,\xi,t,\langle s_n\rangle_{n\in\mathbb{N}}) : (\xi,t) \in E,\ t - 2^{-n} \le s_n < t&\\ \text{and } |\omega(t) - \omega(s_n) - \xi| \le 2^{-n} \text{ for every } n \in \mathbb{N}\}&\end{aligned}$$

belongs to

$$\ddot\Sigma\widehat{\otimes}\mathcal{B}(S \times [0,\infty[^{\mathbb{N}})$$

and its projection $A_1(E)$ can be obtained by Souslin's operation from members of $\ddot\Sigma$ (423O), so belongs to $\ddot\Sigma$ (431A again). **Q** Now (k) tells us that

$$\begin{aligned}\ddot\mu A_1(E) = \nu\phi^{-1}[A_1(E)] &= \nu(\Omega_2 \cap \phi^{-1}[A_1(E)])\\ &= \nu\{\varpi : \varpi \in \Omega_2,\ E \cap \varpi \ne \emptyset\}\\ &= 1 - e^{-\mu E} \text{ if } \mu E \text{ is finite,}\\ &= 1 \text{ if } \mu E = \infty.\end{aligned}$$

(m) Observe now that μ is atomless and (S,μ) is a countably separated measure space (343D), while $\ddot\mu$ is complete. So Jump : $C_{\text{dlg}} \to \mathcal{P}S$ leads us to a complete image measure $\ddot\mu\,\text{Jump}^{-1}$ which agrees with ν on sets of the form $\{S : S\cap E \ne \emptyset\}$ where $\mu E < \infty$, and therefore extends ν (495I[10]); that is, Jump is inverse-measure-preserving for $\ddot\mu$ and ν. We can therefore refer to (b) above to see that, for $\ddot\mu$-almost every $\omega \in C_{\text{dlg}}$,

$$\xi \ne 0,\ |\xi| \ne |\xi'| \text{ and } s \ne s' \text{ whenever } (\xi,s),\ (\xi',s') \in \text{Jump}(\omega) \text{ are distinct,}$$

$$\text{Jump}(\omega) \cap \{(\xi,s) : |\xi| \ge \delta,\ s \le t\} \text{ is finite whenever } \delta > 0 \text{ and } t \in [0,\infty[,$$

$$\begin{gathered}\text{Jump}(\omega) \cap ([0,\delta]\times[\alpha,\beta]) \text{ and } \varpi \cap ([-\delta,0]\times[\alpha,\beta]) \text{ are infinite}\\ \text{whenever } \delta > 0 \text{ and } 0 \le \alpha < \beta,\end{gathered}$$

so that for $\ddot\mu$-almost every ω we have, for every $t > 0$, a canonical enumeration $\langle(\xi_{nt}(\omega), s_{nt}(\omega))\rangle_{n\in\mathbb{N}}$ of the countably infinite set $\text{Jump}(\omega)\cap(\mathbb{R}\times[0,t])$ such that $|\xi_{n+1,t}(\omega)| < |\xi_{nt}(\omega)|$ for every $n \in \mathbb{N}$; we take $\xi_{nt}(\omega) = \tilde\xi_{nt}(\text{Jump}(\omega))$, $s_{nt}(\omega) = \tilde s_{nt}(\text{Jump}(\omega))$ whenever $\text{Jump}(\omega) \in \Omega_0$ and $n \in \mathbb{N}$.

If $\varpi \in \Omega_2$, then $\varpi = \text{Jump}(\phi(\varpi))$, so $\xi_{nt}(\phi(\varpi)) = \tilde\xi_{nt}(\varpi)$ and $s_{nt}(\phi(\varpi)) = \tilde s_{nt}(\varpi)$ for all $t > 0$ and $n \in \mathbb{N}$. For any particular $t > 0$, $\Omega_2 \cap \Omega_{1t}$ is conegligible and

$$\sum_{n=0}^\infty \xi_{nt}(\phi(\varpi)) = \sum_{n=0}^\infty \tilde\xi_{nt}(\varpi) = Z_t(\varpi) = \phi(\varpi)(t).$$

So $\omega(t) = \sum_{n=0}^\infty \xi_{nt}(\omega)$ whenever $\omega \in \phi[\Omega_2\cap\Omega_{1t}]$. On the other hand, $\omega \mapsto \xi_{nt}(\omega) = \tilde\xi_{nt}(\text{Jump}(\omega))$ is $\ddot\Sigma$-measurable for each n because Jump is $(\ddot\Sigma,\Sigma)$-measurable and $\tilde\xi_{nt}$ is Σ-measurable. But this implies that $\{\omega : \omega(t) = \sum_{n=0}^\infty \xi_{nt}(\omega)\}$ belongs to $\ddot\Sigma$; as it includes the set $\phi[\Omega_2\cap\Omega_{1t}]$ of full outer measure, it is conegligible. So we see that $\omega(t) = \sum_{n=0}^\infty \xi_{nt}(\omega)$ for $\ddot\mu$-almost every $\omega \in C_{\text{dlg}}$.

652O Third construction for the Cauchy process (a)(i) Write $\mu_W = \mu_{W2}$ for two-dimensional Wiener measure on the space $\Omega = C([0,\infty[;\mathbb{R}^2)_0$ of continuous functions from $[0,\infty[$ to $\mathbb{R}^2$ starting at zero. Recall that Ω can be identified with $C([0,\infty[)_0^2$ and μ_W with μ_{W1}^2, where μ_{W1} is one-dimensional Wiener measure on $C([0,\infty[)_0$. For $\omega \in \Omega$, I will write ω_0, ω_1 for its coordinates in $C([0,\infty[)_0$, so that $\omega(t) = (\omega_0(t),\omega_1(t))$ for $t \ge 0$. Write Σ for the domain of μ_W. For $t \ge 0$, let Σ_t be the σ-algebra of sets $F \in \Sigma$ such that $\omega' \in F$ whenever $\omega \in F$, $\omega' \in \Omega$ and $\omega'\restriction[0,t] = \omega\restriction[0,t]$.

(ii) For $t \ge 0$, let h_t be the Brownian hitting time to the closed set $[t,\infty[\times\mathbb{R}$ (477I), so that $h_t(\omega) = \inf\{s : \omega_0(s) \ge t\}$, counting $\inf\emptyset$ as ∞, and $\omega_0(h_t(\omega)) = t$ if $h_t(\omega)$ is finite. Evidently $t \mapsto h_t(\omega)$ is non-decreasing. Set

$$\Omega'_0 = \{\omega : \omega \in C([0,\infty[)_0,\ \omega_0[[0,\infty[] = \mathbb{R}\}, \quad \Omega' = \Omega'_0 \times C([[0,\infty[)_0;$$

then Ω'_0 is μ_{W1}-conegligible in $C([[0,\infty[)_0$ (478Ma), so Ω' is μ_W-conegligible in Ω. For $\omega \in \Omega'$, $h_t(\omega) < \infty$ and $\omega(h_t(\omega)) = t$ for every $t \ge 0$. Of course $h_0(\omega) = 0$ for every $\omega \in \Omega$.

If $t \ge 0$ and $\alpha > 0$, then

$$\begin{aligned}\mu_W\{\omega : \omega\in\Omega,\ h_t(\omega) \le \alpha\} &= \mu_W\{\omega : \omega \in \Omega,\ \max_{s\le\alpha}\omega_0(s) \ge t\}\\ &= \mu_{W1}\{\omega_0 : \omega_0 \in C([0,\infty[)_0,\ \max_{s\le\alpha}\omega_0(s) \ge t\}\end{aligned}$$

(identifying μ_W with μ_{W1}^2)

$$= \frac{2}{\sqrt{2\pi}}\int_{t/\sqrt\alpha}^\infty e^{-\xi^2/2}d\xi$$

by 477J. Accordingly the distribution of h_t is absolutely continuous and has a Radon-Nikodým derivative, with respect to Lebesgue measure,

$$\alpha \mapsto \frac{d}{d\alpha}\frac{2}{\sqrt{2\pi}}\int_{t/\sqrt\alpha}^\infty e^{-\eta^2/2}d\eta = \frac{2}{\sqrt{2\pi}}\frac{t}{2\alpha\sqrt\alpha}e^{-t^2/2\alpha} = \frac{t}{\alpha\sqrt{2\pi\alpha}}e^{-t^2/2\alpha}$$

[10] Formerly 495Xa.

for $\alpha > 0$.

(iii) Each h_t is a stopping time adapted to $\langle\Sigma_s\rangle_{s\ge0}$ (477I(c-iii)), so we can speak of the σ-algebra $\Sigma_{h_t} = \{E : E \in \Sigma,\ E \cap \{\omega : h_t(\omega) \le s\} \in \Sigma_s$ for every $s \ge 0\}$, as in 612G.

(b)(i) For $t \ge 0$ and $\omega \in \Omega$, set

$$\begin{aligned} Z_t(\omega) &= \omega_1(h_t(\omega)) \text{ if } h_t(\omega) < \infty, \\ &= 0 \text{ otherwise.} \end{aligned}$$

Since $(f, s) \mapsto f(s) : C([0,\infty[) \times [0,\infty[\to \mathbb{R}$ is continuous for the topology of uniform convergence on compact sets (4A2G(g-iii)), $(\omega, s) \mapsto \omega_0(s) : \Omega \times [0,\infty[\to \mathbb{R}$ is continuous. Now h_t is lower semi-continuous (477I(c-ii)), therefore Borel measurable (4A3Ce), so $Z_t : \Omega \to \mathbb{R}$ is Borel measurable, therefore Σ-measurable.

(ii) In fact Z_t is always Σ_{h_t}-measurable. **P** Take $\alpha \in \mathbb{R}$ and set $E = \{\omega : Z_t(\omega) > \alpha\}$. If $s \ge 0$, $\omega \in E$, $h_t(\omega) \le s$ and $\omega' \in \Omega$ are such that $\omega'{\restriction}[0, s] = \omega{\restriction}[0, s]$, then $\omega'_0(h_t(\omega)) = \omega_0(h_t(\omega)) = t$ and $\omega'_0(r) = \omega_0(r) < t$ for $r < h_t(\omega)$, so $h_t(\omega') = h_t(\omega) \le s$,

$$Z_t(\omega') = \omega'_1(h_t(\omega)) = \omega_1(h_t(\omega)) = Z_t(\omega) > \alpha,$$

and $\omega' \in E$. Thus $E \cap \{\omega : h_t(\omega) \le s\} \in \Sigma_s$; as s is arbitrary, $E \in \Sigma_{h_t}$; as α is arbitrary, Z_t is Σ_{h_t}-measurable. **Q**

(iii) For any $t > 0$, the distribution of Z_t is the Cauchy distribution λ_t of 652M. **P** Take any $a \in \mathbb{R}$, and set $f = \chi\,]-\infty, a]$. Then

$$\begin{aligned} \mu_W\{\omega : Z_t(\omega) \le a\} &= \int_\Omega f Z_t d\mu_W = \int_{\Omega'} f(\omega_1(h_t(\omega)))\mu_W(d\omega) \\ &= \int_{\Omega'} f(\omega_1(h_t(\omega_0, \mathbf{0})))\mu_W(d\omega) \end{aligned}$$

(where $\mathbf{0} \in C([0,\infty[)_0$ is the constant function with value 0, because $h_t(\omega)$ is calculated from ω_0)

$$= \int_{\Omega'_0} \int_{C([0,\infty[)_0} f(\omega_1(h_t(\omega_0, \mathbf{0})))\mu_{W1}(d\omega_1)\mu_{W1}(d\omega_0)$$

(identifying μ_W with μ_{W1}^2)

$$= \int_{\Omega'_0} \int_{\mathbb{R}} f(\alpha)\frac{1}{\sqrt{2\pi h_t(\omega_0, \mathbf{0})}} e^{-\alpha^2/2h_t(\omega_0,\mathbf{0})} d\alpha\, \mu_{W1}(d\omega_0)$$

(by 271Ic, because $\omega_1 \mapsto \omega_1(h_t(\omega_0, \mathbf{0}))$ is normally distributed with expectation 0 and variance $h_t(\omega_0, \mathbf{0})$)

$$\begin{aligned} &= \int_{\Omega'_0} \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi h_t(\omega_0,\mathbf{0})}} e^{-\alpha^2/2h_t(\omega_0,\mathbf{0})} d\alpha\, \mu_{W1}(d\omega_0) \\ &= \int_{-\infty}^{a} \int_{\Omega'_0} \frac{1}{\sqrt{2\pi h_t(\omega_0,\mathbf{0})}} e^{-\alpha^2/2h_t(\omega_0,\mathbf{0})} \mu_{W1}(d\omega_0) d\alpha \\ &= \int_{-\infty}^{a} \int_{\Omega'} \frac{1}{\sqrt{2\pi h_t(\omega_0,\mathbf{0})}} e^{-\alpha^2/2h_t(\omega_0,\mathbf{0})} \mu_W(d\omega) d\alpha \end{aligned}$$

(because $\omega \mapsto \omega_0 : \Omega \to C([0,\infty[)_0$ is inverse-measure-preserving)

$$\begin{aligned} &= \int_{-\infty}^{a} \int_{\Omega'} \frac{1}{\sqrt{2\pi h_t(\omega)}} e^{-\alpha^2/2h_t(\omega)} \mu_W(d\omega) d\alpha \\ &= \int_{-\infty}^{a} \int_{0}^{\infty} \frac{1}{\sqrt{2\pi\beta}} e^{-\alpha^2/2\beta} \frac{t}{\beta\sqrt{2\pi\beta}} e^{-t^2/2\beta} d\beta\, d\alpha \end{aligned}$$

(by (a-ii))

$$= \frac{t}{2\pi} \int_{-\infty}^{a} \frac{2}{\alpha^2+t^2} \int_{0}^{\infty} e^{-\gamma} d\gamma\, d\alpha$$

(substituting $\gamma = \frac{\alpha^2+t^2}{2\beta}$)

$$= \int_{-\infty}^{a} \frac{t}{\pi(\alpha^2+t^2)} d\alpha = \lambda_t\,]-\infty, a]\,.$$

As a is arbitrary, we have the result. **Q**

(c) Suppose that $0 \le t \le t'$.

(i) By 455U/477G we have a function $\phi : \Omega \times \Omega \to \Omega$ defined by setting

$$\begin{aligned}\phi(\omega, \omega')(s) &= \omega(h_t(\omega)) + \omega'(s - h_t(\omega)) \text{ if } s \ge h_t(\omega),\\ &= \omega(s) \text{ if } s \le h_t(\omega)\end{aligned}$$

which is inverse-measure-preserving for $\mu_W \times \mu_W$ and μ_W.

(ii) For almost every ω, $\omega' \in \Omega$, $h_t(\omega)$ and $h_{t'-t}(\omega')$ are finite, and for such pairs

$$h_{t'}(\phi(\omega, \omega')) = \min\{s : \phi(\omega, \omega')_0(s) \ge t'\}$$

(writing $\phi(\omega, \omega')_0(s)$ for the first component of $\phi(\omega, \omega')(s)$)

$$= \min\{s : s > h_t(\omega),\ \phi(\omega, \omega')_0(s) \ge t'\}$$

(because if $s \le h_t(\omega)$ then $\phi(\omega, \omega')_0(s) = \omega_0(s) \le t\}$)

$$\begin{aligned} &= \min\{s : s > h_t(\omega),\ t + \omega'_0(s - h_t(\omega)) \ge t'\}\\ &= h_t(\omega) + \min\{s : \omega'_0(s) \ge t' - t\} = h_t(\omega) + h_{t'-t}(\omega'),\\ Z_{t'}(\phi(\omega, \omega')) &= \phi(\omega, \omega')_1(h_t(\omega) + h_{t'-t}(\omega'))\\ &= \omega_1(h_t(\omega)) + \omega'_1(h_{t'-t}(\omega')) = Z_t(\phi(\omega, \omega')) + Z_{t'-t}(\omega').\end{aligned}$$

Because ϕ is inverse-measure-preserving, $Z_{t'} - Z_t$ has the same distribution as

$$(\omega, \omega') \mapsto Z_{t'}(\phi(\omega, \omega')) - Z_t(\phi(\omega, \omega')) = Z_{t'-t}(\omega')$$

which has the same distribution as $Z_{t'-t}$.

(iii) If $\omega \in E \in \Sigma_{h_t}$ and $\omega' \in \Omega$ is such that $\omega'\restriction[0, h_t(\omega)] = \omega\restriction[0, h_t(\omega)]$, then $\omega' \in E$. **P** If $h_t(\omega) = \infty$ this is trivial. Otherwise, setting $s = h_t(\omega)$, we see that $\omega \in E \cap \{\tilde\omega : h(\tilde\omega) \le s\} \in \Sigma_s$ while $\omega'\restriction[0, s] = \omega\restriction[0, s]$ so $\omega' \in E \cap \{\tilde\omega : h(\tilde\omega) \le s\} \subseteq E$. **Q**

Now if $\omega \in E \in \Sigma_{h_t}$ and $\omega' \in \Omega$ then $\phi(\omega, \omega')\restriction[0, h_t(\omega)] = \omega\restriction[0, h_t(\omega)]$ so $\phi(\omega, \omega') \in E$. The same applies to the complement of E, so we see that $\phi^{-1}[E] = E \times \Omega$ for every $E \in \Sigma_{h_t}$. And it follows that $Z_{t'} - Z_t$ is independent of Σ_{h_t}. **P** If $E \in \Sigma_{h_t}$ and $\alpha \in \mathbb{R}$, set $F = \{\omega : (Z_{t'} - Z_t)(\omega) > \alpha\}$; then

$$\begin{aligned}\mu_W(E \cap F) &= \mu_W^2(\phi^{-1}[E] \cap \phi^{-1}[F]) = \mu_W^2((E \times \Omega) \cap \phi^{-1}[F])\\ &= \mu_W^2((E \times \Omega) \cap (\Omega \times \{\omega' : Z_{t'-t}(\omega') > \alpha\}))\end{aligned}$$

(by (i) above)

$$= \mu_W^2((E \times \Omega) \cap (\Omega \times F)) = \mu_W E \cdot \mu_W F.$$

As E and α are arbitrary we have the result. **Q**

(d) From (c), we can easily confirm that $\langle Z_t\rangle_{t\ge0}$ here has the same distribution as the process $\langle Z_t\rangle_{t\ge0}$ of 652N. But the filtration $\langle\Sigma_{h_t}\rangle_{t\ge0}$ is not right-continuous and the paths $\langle Z_t(\omega)\rangle_{t\ge0}$ are not càdlàg, so to get a Lévy process as described in 652C there is still some work to do.

(i) If $0 \le s \le t$, $E \in \Sigma_{h_s}$ and $r \ge 0$, then

$$E \cap \{\omega : h_t(\omega) \le r\} = (E \cap \{\omega : h_s(\omega) \le r\}) \cap \{\omega : h_t(\omega) \le r\}$$

is the intersection of two members of Σ_r and belongs to Σ_r; so $E \in \Sigma_{h_t}$. Thus $\Sigma_{h_s} \subseteq \Sigma_{h_t}$.

(ii) For $t\ge 0$, set

$$\mathrm{T}_t=\{E\triangle A:E\in\bigcap_{s>t}\Sigma_{h_s},\,A\in\mathcal{N}(\mu_W)\}.$$

Then $\langle\mathrm{T}_t\rangle_{t\ge 0}$ is a filtration of σ-subalgebras of Σ, and is right-continuous. **P** If $t\ge 0$ and $E\in\bigcap_{s>t}\mathrm{T}_s$, then for each $n\in\mathbb{N}$ there is an $A_n\in\mathcal{N}(\mu_W)$ such that $E\triangle A_n\in\bigcap_{s>t+2^{-n}}\Sigma_{h_s}$; using (i) just above, we see that $F=\bigcap_{n\in\mathbb{N}}E\triangle A_n\in\bigcap_{s>t}\Sigma_{h_s}$ and $E\triangle F\subseteq\bigcup_{n\in\mathbb{N}}A_n$ is negligible. **Q** Note also that $\Sigma_{h_t}\subseteq\mathrm{T}_t$ for every t, by (i).

(iii) For $t\ge 0$ and $\omega\in\Omega$, set $\tilde h_t(\omega)=\inf_{s>t}h_s(\omega)=\lim_{s\downarrow t}h_s(\omega)$. Because $h_t\le h_s$ whenever $t\le s$, $h_t\le\tilde h_t$ for every t and $t\mapsto\tilde h_t(\omega):[0,\infty[\,\to[0,\infty]$ is non-decreasing and càdlàg for every ω. We find also that $\tilde h_t=_{\text{a.e.}}h_t$ for every $t\ge 0$. **P** $\tilde h_t(\omega)$ is always $\lim_{n\to\infty}h_{t+2^{-n}}(\omega)$, and

$$\{\omega:\tilde h_t(\omega)\le\alpha\}=\bigcap_{m\in\mathbb{N}}\bigcup_{n\in\mathbb{N}}\{\omega:h_{t+2^{-n}}(\omega)\le\alpha+2^{-m}\}$$

has measure

$$\lim_{m\to\infty}\lim_{n\to\infty}\frac{2}{\sqrt{2\pi}}\int_{(t+2^{-n})/\sqrt{\alpha+2^{-m}}}^{\infty}e^{-\xi^2/2}d\xi=\frac{2}{\sqrt{2\pi}}\int_{t/\sqrt{\alpha}}^{\infty}e^{-\xi^2/2}d\xi$$

whenever $t\ge 0$ and $\alpha>0$, by (a-ii) above. So $\{\omega:\tilde h_t(\omega)\le\alpha\}$ and $\{\omega:h_t(\omega)\le\alpha\}$ have the same measure for every α; as the former is always included in the latter, $\tilde h_t=_{\text{a.e.}}h_t$. **Q**

(iv) For $t\ge 0$ and $\omega\in\Omega$, set

$$\tilde Z_t(\omega)=\omega_1(\tilde h_t(\omega))\text{ if }\omega\in\Omega',$$
$$=0\text{ otherwise.}$$

Then $\tilde Z_t=_{\text{a.e.}}Z_t$ for every t, and $\langle\tilde Z_t(\omega)\rangle_{t\ge 0}$ is càdlàg for every ω. **P** If $\omega\notin\Omega'$ this is trivial. Otherwise, $t\mapsto\tilde h_t(\omega):[0,\infty[\,\to[0,\infty[$ is càdlàg and $\omega_1:[0,\infty[\,\to\mathbb{R}$ is continuous so $t\mapsto\omega_1(\tilde h_t(\omega)):[0,\infty[\,\to\mathbb{R}$ is càdlàg. **Q**

Next, $\tilde Z_t$ is T_t-measurable for every $t\ge 0$. **P** $\tilde Z_t=_{\text{a.e.}}\lim_{n\to\infty}Z_{t+2^{-n}}$ and if $s>t$ then $\lim_{n\to\infty}Z_{t+2^{-n}}$ is Σ_{h_s}-measurable by (b-ii) above. So in fact $\lim_{n\to\infty}Z_{t+2^{-n}}$ is $(\bigcap_{s>t}\Sigma_{h_s})$-measurable and $\tilde Z_t$ is T_t-measurable. **Q**

(v) If $0\le t\le t'$ then $\tilde Z_{t'}-\tilde Z_t$ is independent of T_t. **P** For $n\in\mathbb{N}$, $Z_{t'+2^{-n}}-Z_{t+2^{-n}}$ is independent of $\Sigma_{h_{t+2^{-n}}}$ and therefore of $\bigcap_{s>t}\Sigma_{h_s}$ and of T_t. Now $\tilde Z_{t'}-\tilde Z_t=_{\text{a.e.}}\lim_{n\to\infty}Z_{t'+2^{-n}}-Z_{t+2^{-n}}$ so is in the closure of $\{Z_{t'+2^{-n}}-Z_{t+2^{-n}}:n\in\mathbb{N}\}$ for the topology of convergence in measure; by 652Bc, applied with $\mathfrak{B}=\{E^\bullet:E\in\mathrm{T}_t\}$, $\tilde Z_{t'}-\tilde Z_t$ is independent of T_t. **Q**

(vi) Putting these together, we see that if $(\mathfrak{A},\bar\mu_W)$ is the measure algebra of (Ω,Σ,μ_W) and $\mathfrak{B}_t=\{E^\bullet:E\in\mathrm{T}_t\}\subseteq\mathfrak{A}$ for $t\ge 0$, we have a right-continuous real-time stochastic integration structure $(\mathfrak{A},\bar\mu_W,[0,\infty[\,,\langle\mathfrak{B}_t\rangle_{t\ge 0})$; and 613D, applied to $\langle\mathrm{T}_t\rangle_{t\ge 0}$ and $\langle\tilde Z_t\rangle_{t\ge 0}$, tells us that $\langle\tilde Z_t\rangle_{t\ge 0}$ is progressively measurable and corresponds to a locally near-simple process $\langle\tilde z_\sigma\rangle_{\sigma\in\mathcal{T}_f}$ which is a Lévy process in the sense of 652C.

652X Basic exercises (a) Suppose that $(\mathfrak{A},\bar\mu,[0,\infty[\,,\langle\mathfrak{A}_t\rangle_{t\ge 0},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ is a right-continuous real-time stochastic integration structure and $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{T}_f}$ a Lévy process. Show that if $\tau\in\mathcal{T}_f$ and $\langle t_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $[0,\infty[$, then $\langle v_{\tau+\check t_{n+1}}-v_{\tau+\check t_n}\rangle_{n\in\mathbb{N}}$ is independent.

(b) Show that the formula in 652Ga can be adapted to describe the sum of any two stopping times, and explore the properties of this operation. (*Hint*: 364C.)

(c) Let $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{T}_f}$ be a Lévy process such that $\alpha=\mathbb{E}(v_{\check 1})$ is finite. Show that $\boldsymbol{v}-\alpha\boldsymbol{\iota}$ is a local martingale. (*Hint*: 632Ma.)

(d) In 652Ni, show that $\tilde\xi_{ns}$ is a stopping time adapted to $\langle\Sigma_t\rangle_{t\ge 0}$ for all $n\in\mathbb{N}$ and $s>0$.

(e) In 652Od, show that $\tilde h_t$ is always the Brownian hitting time to $]t,\infty[\,\times\mathbb{R}$, so is a stopping time adapted to the right-continuous filtration $\langle\bigcap_{r>s}\Sigma_r\rangle_{s\ge 0}$.

(f) Suppose that $(\mathfrak{A},\bar{\mu},[0,\infty[\,,\langle\mathfrak{A}_t\rangle_{t\ge0},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ is a real-time stochastic integration structure. (i) Show that if $s\ge0$ then $\tau\mapsto\tau+\check{s}:\mathcal{T}\to\mathcal{T}$ is a lattice homomorphism. (ii) Show that if $\tau\in\mathcal{T}$ then $t\mapsto\tau+\check{t}:[0,\infty[\,\to\mathcal{T}_f$ is order-continuous.

652Y Further exercises (a) Let $\ddot{\mu}$ be the measure on C_{dlg} corresponding to the Poisson process, as in 612U. Show that it is not a Radon measure for the topology of pointwise convergence on C_{dlg}.

(b) Let $(\mathfrak{C},\bar{\nu},[0,\infty[\,,\langle\mathfrak{C}_t\rangle_{t\ge0},\mathcal{T},\langle\mathfrak{C}_\tau\rangle_{\tau\in\mathcal{T}})$ be the real-time stochastic integration structure of 612T, and $\boldsymbol{w}=\langle w_\tau\rangle_{\tau\in\mathcal{T}_f}$ Brownian motion. Set $\mathfrak{A}_t=\mathfrak{C}_{2t}$ for $t\ge0$, so that $(\mathfrak{C},\bar{\nu},[0,\infty[\,,\langle\mathfrak{A}_t\rangle_{t\ge0})$ is a stochastic integration structure, and $\mathcal{S}=\{\check{t}:t\ge0\}$. Show that with respect to the structure $(\mathfrak{C},\bar{\nu},[0,\infty[\,,\langle\mathfrak{A}_t\rangle_{t\ge0})$, $\langle w_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a fully adapted process but is not a local integrator.

(c) Let $f:[0,\infty[\,\to\mathbb{R}$ be a càdlàg function. For $t>0$, write $f(t^-)$ for $\lim_{s\uparrow t}f(s)$. For finite sets $I\subseteq[0,\infty[$, define α_I by saying that $\alpha_\emptyset=0$ and

$$\alpha_I=\sum_{i=0}^{n-1}f(s_{i+1})-f(s_i)-\mathrm{med}(-1,f(s_{i+1})-f(s_i),1)$$

if $\#(I)=n>0$ and $\langle s_i\rangle_{i<n}$ is the increasing enumeration of I. Take $t\ge0$, and write Q_t for $(\mathbb{Q}\cap[0,t])\cup\{t\}$. Show that

(i) For every $\epsilon>0$, $\{s:0<s\le t,\ |f(s)-f(s^-)|\ge\epsilon\}$ is finite;

(ii) $\alpha=\sum_{0<s\le t}f(s)-f(s^-)-\mathrm{med}(-1,f(s)-f(s^-),1)$ is defined;

(iii) For every $\epsilon>0$, there is a finite set $J\subseteq Q_t$ such that $|\alpha_I-\alpha|\le\epsilon$ whenever $I\subseteq[0,t]$ is finite and $J\subseteq I$.

(d) Let (Ω,Σ,μ) be a probability space and $\langle X_t\rangle_{t\ge0}$ a family of random variables defined everywhere on Ω such that $t\mapsto X_t(\omega):[0,\infty[\,\to\mathbb{R}$ is càdlàg for every $\omega\in\Omega$. For $\omega\in\Omega$ and $t>0$ set $X_{t^-}(\omega)=\lim_{s\uparrow t}X_s(\omega)$. For $\omega\in\Omega$ and $t\ge0$ and non-empty sets $I\subseteq[0,\infty[$, set

$$Y_t(\omega)=\sum_{0<s\le t}X_s(\omega)-X_{s^-}(\omega)-\mathrm{med}(-1,X_s(\omega)-X_{s^-}(\omega),1),$$

$$W_I=\sum_{i=0}^{n-1}X_{s_{i+1}}-X_{s_i}-\mathrm{med}(-\chi\Omega,X_{s_{i+1}}-X_{s_i},\chi\Omega)$$

where $\langle s_i\rangle_{i\le n}$ is the increasing enumeration of I. for $\omega\in\Omega$, write $V(\omega)$ for the total variation of the function $s\mapsto Y_s(\omega):[0,t]\to\mathbb{R}$. Take any $t\ge0$. Show that

(i) $Y_t:\Omega\to\mathbb{R}$ is measurable;

(ii) writing $\mathcal{I}_t$ for the set of finite subsets of $[0,t]$, $Y_t^\bullet=\lim_{I\uparrow\mathcal{I}_t}W_I^\bullet$ for the topology of convergence in measure on $L^0(\mu)$;

(iii) V is measurable and finite-valued.

(e) Let $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{T}_f}$ be a a classical Lévy process. For finite sets $I\subseteq[0,\infty[$ define w_I by saying that $w_I=0$ if I is empty and otherwise $w_I=\sum_{i=0}^{n-1}u_{\check{s}_{i+1}}-u_{\check{s}_i}-\mathrm{med}(-\chi1,u_{\check{s}_{i+1}}-u_{\check{s}_i},\chi1)$ where $\langle s_i\rangle_{i\le n}$ is the increasing enumeration of I. For $t\ge0$ let $\mathcal{I}_t$ be the set of finite subsets of $[0,t]$. Show that

(i) $y_{\check{t}}=\lim_{I\uparrow\mathcal{I}_t}w_I$ is defined for every $t\ge0$;

(ii) $\boldsymbol{y}=\langle y_\sigma\rangle_{\sigma\in[0,\infty[^\vee}$ is locally of bounded variation;

(iii) $u_{\check{t}}-y_{\check{t}}$ has finite expectation for every $t\ge0$;

(iv) $\mathbb{E}(u_{\check{t}}-y_{\check{t}})=t\mathbb{E}(u_{\check{1}}-y_{\check{1}})$ for every $t\ge0$.

652Z Problem Is the Cauchy process, as described in 652Mc, a local martingale?

652 Notes and comments The 'classical' Lévy processes of 652F are the natural translation of the (real-valued) Lévy processes of §455 into the language of this volume. What I call a Lévy process in 652C offers a generalization. We then find ourselves doing some work in 652G-652H to show that the processes of 652C have a weak form of the Markov property of 455U. For classical Lévy processes, it is enough to express the ideas of 633M-633N in terms of càdlàg functions (652Yc-652Ye), giving us an easier route to 652I-652K. The calculations are not trivial, but they are natural once you have come to terms with the fact that we really do not have continuous sample paths except in very few cases, and consequently have to understand the jumps $X_s(\omega)-X_{s^-}(\omega)$, as in 652Yd; compare 641N.

However, the step from 'classical Lévy process' to 'Lévy process', as defined here, is more than a formality. The point is that the independent-increment property changes from

whenever $0 \le t_0 \le \ldots \le t_n$ then $X_{t_1} - X_{t_0}, \ldots , X_{t_n} - X_{t_{n-1}}$ are independent (455Q)

to

$v_{\check{t}} - v_{\check{s}}$ is independent of $\mathfrak{A}_s$ whenever $0 \le s \le t$.

The former version translates into

$v_{\check{t}} - v_{\check{s}}$ is independent of $\{v_{\check{s}'} : 0 \le s' \le s\}$ whenever $0 \le s \le t$;

it comes to the same thing for classical Lévy processes because in that context, in effect, $\mathfrak{A}_s$ is defined as the closed subalgebra generated by $\{v_{\check{s}'} : 0 \le s' \le s\}$. If we want to move to more complex structures (for instance, in order to look at an interaction between two different Lévy processes, as in multidimensional Brownian motion) we need to make the distinction. We cannot take it for granted that a general Lévy process $\langle v_\tau\rangle_{\tau\in\mathcal{T}_b}$ will copy every feature of the associated classical Lévy process defined from the distributions λ_t of $v_{\check{t}}$. The culminating result of §455 was the strong Markov property of classical Lévy processes (Theorem 455U). Applied to Brownian motion it was the basis of large parts of §§477-479. I have not found a corresponding property of Lévy processes as defined in 652C.

Lévy processes have been studied intensively, and in this section I have been content to stop at the semi-martingale property (652K), since this belongs to the basic classification scheme for stochastic processes which I am following in this volume. But you should be aware that the analysis in 652G-652K amounts to a few baby steps towards a general description of Lévy processes, the Lévy-Khintchine and Lévy-Itô decompositions (Sato 13, §8 and §§19-20). As a very special case, I offer a detailed analysis of the Cauchy distribution (652M-652O). Regarded as a classical Lévy process, as in 652M, this is straightforward enough to have been an exercise in §455. But the Lévy-Itô decomposition seeks to describe it in terms of its 'Lévy measure' μ in 652N, and to reach the final formula

$$\omega(t) = \sum_{n=0}^{\infty} \xi_{nt}(\omega) \text{ for } \ddot{\mu}\text{-almost every } \omega$$

in 652Nm I think we must do a good deal more work. Note that the sum here is a conditional sum. We really do have to take the terms in an appropriate order, biggest jumps first. Nearly always in this volume we take things as they arrive, in temporal order. But in the present case, if $0 < s \le t$, $\omega(s) = \sum_{n=0}^{\infty} \xi_{ns}(\omega)$ is the sum of a subsequence of $\langle\xi_{nt}(\omega)\rangle_{n\in\mathbb{N}}$, summing those jumps which occurred at or before the time s, not (for instance) an initial segment. When we look at the whole jump set $\mathrm{Jump}(\omega)$ for a measure-generic $\omega \in C_{\mathrm{dlg}}$, all its points are isolated, but its closure in $[-\infty, \infty] \times [0, \infty]$ is $\mathrm{Jump}(\omega) \cup \{(0, t) : t \in [0, \infty]\} \cup \{(\alpha, \infty) : \alpha \in [-\infty, \infty]\}$, and it has no useful natural enumeration.

I dare say you have learnt how to calculate the integrals of 652Mb by contour integration. I offer the alternative route through Fourier transforms just because I gave what I hope was a correct and reasonably complete argument for the Fourier transform inversion theorem in §283, and I have omitted contour integration entirely from this treatise because of the difficulty of presenting the Jordan Curve Theorem in the style I have chosen. But if you look at the contours required for the results here you will have no difficulty in describing approximating polygons and triangulations in elementary geometric terms, so that a more or less elementary version of the Residue Theorem is adequate.

I go through the details of 652O partly because they are striking (and include a solution to 478Ym[11]), but mostly to offer a different view of the jumps in the Cauchy process. I do not think it is obvious from the formula in 652Ma that the Cauchy process is discontinuous (652Md). In fact the analysis in 652N shows that it is a 'pure jump' process, that is, almost every sample path is expressible as the sum of its jumps – provided, that is, that we can express the sum in the right way, because it is typically not of bounded variation, and we do not have a saltus function of the type in 226B. The method of 652N clearly has a potential for generalization to other probability measures on $\mathbb{R} \times [0, \infty[$. But in the absence of the general theory of Lévy processes and Lévy measures, as described in Sato 13, it is bound to look a bit arbitrary, even though an elementary scaling argument shows that *if* there is an expression in terms of a Poisson point process as in 652Nc, then the underlying measure must look like the measure μ of 652Na. Also, of course, this exposition leans rather heavily on §495. On the other hand, once we have worked through the formulae of 652O, we can see that the jumps in the paths $\langle\tilde{Z}_t(\omega)\rangle_{t\ge0}$ there correspond to jumps in $g_t(\omega_0) = \inf\{s : \omega_0(s) > t\}$ for one-dimensional Brownian paths ω_0. Now we know that almost every ω_0

[11] Later editions only.

is continuous and nowhere differentiable (477K). It follows that it will have many local maxima, and we can expect it to happen (countably and densely) often that the hitting time to a value t is a strict local maximum of ω_0, so that $\lim_{s\uparrow t} g_s(\omega_0)$ will be strictly less than $g_t(\omega_0)$.

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653 Brownian processes

In 624F, we saw that the quadratic variation of Brownian motion is the identity process. In fact, under suitable conditions, this characterizes Brownian motion among local martingales (653F). Elaborating on the argument, we can show that (again under suitable conditions) general locally jump-free local martingales can be described in terms of Brownian motion and a time-change of the type considered in §635.

653A Notation As always, $(\mathfrak{A}, \bar{\mu}, T, \langle \mathfrak{A}_t \rangle_{t \in T}, \mathcal{T}, \langle \mathfrak{A}_\tau \rangle_{\tau \in \mathcal{T}})$ will be a stochastic integration structure; $L^0 = L^0(\mathfrak{A})$ will be given its topology of convergence in measure, defined by the F-norm θ where $\theta(w) = \mathbb{E}(|w| \wedge \chi 1_{\mathfrak{A}})$ for $w \in L^0$. $L^1_{\bar{\mu}} \subseteq L^0$ will be the L-space defined by $\bar{\mu}$.

For $t \in T$, $\check{t} \in \mathcal{T}$ will be the constant stopping time at t; for $\tau \in \mathcal{T}$, P_τ will be the conditional expectation associated with $\mathfrak{A}_\tau$. If $\mathcal{S}$ is a sublattice of $\mathcal{T}$, $\mathcal{I}(\mathcal{S})$ will be the set of finite sublattices of $\mathcal{S}$.

I will use the symbols $\int$ for Riemann-sum integrals and $\oint$ for S-integrals. If $\mathcal{S}$ is a sublattice of $\mathcal{T}$ then $\mathbf{1}^{(\mathcal{S})}$ will be the constant process with domain $\mathcal{S}$ and value $\chi 1_{\mathfrak{A}}$. When $T = [0, \infty[$, $\boldsymbol{\iota} = \langle \iota_\tau \rangle_{\tau \in \mathcal{T}_f}$ will be the identity process.

653B Distributions (a) If $k \ge 1$ and $u_1, \dots, u_k \in L^0$, we have a sequentially order-continuous function $E \mapsto [\![(u_1, \dots, u_k) \in E]\!]$ from the Borel σ-algebra $\mathcal{B}_k$ of $\mathbb{R}^k$ to $\mathfrak{A}$ (619E). This leads us to a Borel probability measure $E \mapsto \bar{\mu}[\![u \in E]\!] : \mathcal{B}_k \to [0, 1]$; the completion of this measure is a Radon probability measure ν_U on $\mathbb{R}^k$ (256C), which I will call the **distribution** of $U = (u_1, \dots, u_k)$ (cf. 364Yo).

Note that if $(\mathfrak{A}, \bar{\mu})$ is the measure algebra of a probability space (Ω, Σ, μ), and the identification of $L^0(\mathfrak{A})$ with $L^0(\mu)$ represents u_i as $f_i^\bullet$, where $f_i : \Omega \to \mathbb{R}$ is measurable for $1 \le i \le k$, then the distribution ν_U is just the joint distribution of the sequence $(f_1, \dots, f_k)$ of random variables as described in 271B. If $h : \mathbb{R}^k \to \mathbb{R}$ is a bounded Borel measurable function, then $\mathbb{E}(\bar{h}(U)) = \int h \, d\nu_U$. **P** Set

$$W = \{h : h \text{ is a bounded Borel measurable function,}$$
$$\mathbb{E}(\bar{h}(U)) \text{ and } \int h \, d\nu_U \text{ are defined, finite and equal}\}.$$

If $h = \chi E$ where $E \in \mathcal{B}_k$, then $h \in W$ by the definitions of $\bar{h}$ (619Eb) and ν_U. Since W is a linear space closed under limits of uniformly bounded monotonic sequences (use 271E and 619Ef), it contains all bounded Borel measurable functions. **Q**

(b) We can now speak of the corresponding characteristic function φ_{ν_U} where

$$\varphi_{\nu_U}(y) = \int e^{iy \centerdot x} \nu_U(dx) = \int \cos(y \centerdot x) \nu_U(dx) + i \int \sin(y \centerdot x) \nu_U(dx)$$
$$= \mathbb{E}(\overline{\cos}(\eta_1 u_1 + \ldots + \eta_k u_k)) + i \mathbb{E}(\overline{\sin}(\eta_1 u_1 + \ldots + \eta_k u_k))$$

for $y = (\eta_1, \dots, \eta_k) \in \mathbb{R}^k$, and I will call this the characteristic function of $U = (u_1, \dots, u_k)$, following 285Ab. If now $V \in (L^0)^k$ has the same characteristic function as U, it must have the same distribution as U, by 285M.

(c) If $u_1, \dots, u_k$ are stochastically independent (367W), then the distribution ν_U of $U = (u_1, \dots, u_k)$ is the product of the distributions ν_{u_i} of u_i (272G).

653C Lemma Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ a locally jump-free virtually local martingale such that its quadratic variation $\boldsymbol{v}^*$ is an L^∞-process. Writing $v_\downarrow$ for the starting value of $\boldsymbol{v}$,

$$\boldsymbol{z}_1=\overline{\sin}(\boldsymbol{v}-v_{\downarrow}\mathbf{1}^{(\mathcal{S})})\times\overline{\exp}(\tfrac{1}{2}\boldsymbol{v}^*),\quad \boldsymbol{z}_2=\overline{\cos}(\boldsymbol{v}-v_{\downarrow}\mathbf{1}^{(\mathcal{S})})\times\overline{\exp}(\tfrac{1}{2}\boldsymbol{v}^*)$$

are martingales.

proof I follow the line of argument in 651B-651E.

(a) To begin with (down to the end of (c)) I will suppose that $\mathcal{S}$ is finitely full. As in 651C, $\boldsymbol{v}$ is a local integrator; so $v_{\downarrow}$ is defined. As in 651B, $\boldsymbol{v}^*$ has domain $\mathcal{S}$ and is a locally jump-free local integrator (617L). As in 651D, $\boldsymbol{v}-v_{\downarrow}\mathbf{1}^{(\mathcal{S})}$ is a virtually local martingale with quadratic variation $\boldsymbol{v}^*$, so it will be enough to deal with the case $v_{\downarrow}=0$.

(b)(i) We can apply 619K with $h(x,y)=\sin x\exp(\frac{1}{2}y)$ and $\boldsymbol{V}=(\boldsymbol{v},\boldsymbol{v}^*)$ to see that

$$\begin{aligned}\boldsymbol{z}_1=ii_{\bar h\boldsymbol{V}}(\mathbf{1}^{(\mathcal{S})})&=ii_{\boldsymbol{v}}(\overline{\cos}(\boldsymbol{v})\times\overline{\exp}(\tfrac{1}{2}\boldsymbol{v}^*))\\&\quad+\frac{1}{2}ii_{\boldsymbol{v}^*}(\overline{\sin}(\boldsymbol{v})\times\overline{\exp}(\tfrac{1}{2}\boldsymbol{v}^*))\\&\quad-\frac{1}{2}ii_{\boldsymbol{v}^*}(\overline{\sin}(\boldsymbol{v})\times\overline{\exp}(\tfrac{1}{2}\boldsymbol{v}^*))\\&\quad+\frac{1}{2}ii_{[\boldsymbol{v}\overset{*}{|}\boldsymbol{v}^*]}(\overline{\cos}(\boldsymbol{v})\times\overline{\exp}(\tfrac{1}{2}\boldsymbol{v}^*))\\&\quad+\frac{1}{8}ii_{(\boldsymbol{v}^*)^*}(\overline{\sin}(\boldsymbol{v})\times\overline{\exp}(\tfrac{1}{2}\boldsymbol{v}^*))\\&=ii_{\boldsymbol{v}}(\overline{\cos}(\boldsymbol{v})\times\overline{\exp}(\tfrac{1}{2}\boldsymbol{v}^*))=ii_{\boldsymbol{v}}(\boldsymbol{z}_2)\end{aligned}$$

because $[\boldsymbol{v}\overset{*}{|}\boldsymbol{v}^*]$ and $(\boldsymbol{v}^*)^*$ are both the zero process, by 624C.

(ii) Similarly, taking $g(x,y)=\cos x\exp(\frac{1}{2}y)$, the starting value of $\boldsymbol{z}_2=\bar g\boldsymbol{V}$ is $\chi 1$, so

$$\boldsymbol{z}_2=\mathbf{1}^{(\mathcal{S})}+ii_{\bar g\boldsymbol{V}}(\mathbf{1}^{(\mathcal{S})})=\mathbf{1}^{(\mathcal{S})}-ii_{\boldsymbol{v}}(\overline{\sin}(\boldsymbol{v})\times\overline{\exp}(\tfrac{1}{2}\boldsymbol{v}^*))=\mathbf{1}^{(\mathcal{S})}-ii_{\boldsymbol{v}}(\boldsymbol{z}_1).$$

(c) By 623O, $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ are virtually local martingales; because $\mathcal{S}$ is finitely full, they are approximately local martingales (623K(b-iii)). Express $\boldsymbol{v}^*$ as $\langle v^*_\sigma\rangle_{\sigma\in\mathcal{S}}$. If $\tau\in\mathcal{S}$, then there is an $M\ge 0$ such that $v^*_\tau\le M\mathbf{1}^{(\mathcal{S})}$, so that $\sup(|\boldsymbol{z}_1|\restriction\mathcal{S}\wedge\tau)$ and $\sup(|\boldsymbol{z}_2|\restriction\mathcal{S}\wedge\tau)$ are both at most $e^M\mathbf{1}^{(\mathcal{S})}$. But this means that $\boldsymbol{z}_1\restriction\mathcal{S}\wedge\tau$ and $\boldsymbol{z}_2\restriction\mathcal{S}\wedge\tau$ are uniformly integrable; as τ is arbitrary, $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ are martingales (623Na).

(d) Thus the result is true if $\mathcal{S}$ is finitely full. In general, write $\hat{\mathcal{S}}_f$ for the finitely-covered envelope of $\mathcal{S}$, and $\hat{\boldsymbol{v}}$, $\hat{\boldsymbol{v}}^*$, $\hat{\boldsymbol{z}}_1$ and $\hat{\boldsymbol{z}}_2$ for the fully additive extensions of $\boldsymbol{v}$, $\boldsymbol{v}^*$, $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ to $\hat{\mathcal{S}}_f$. As in (a-ii) of the proof of 651E,

$$\hat{\boldsymbol{z}}_1=\overline{\sin}(\hat{\boldsymbol{v}}-v_{\downarrow}\mathbf{1}^{(\hat{\mathcal{S}}_f)})\times\overline{\exp}(\tfrac{1}{2}\hat{\boldsymbol{v}}^*),\quad \hat{\boldsymbol{z}}_2=\overline{\cos}(\hat{\boldsymbol{v}}-v_{\downarrow}\mathbf{1}^{(\hat{\mathcal{S}}_f)})\times\overline{\exp}(\tfrac{1}{2}\hat{\boldsymbol{v}}^*),$$

while $\langle\hat v^*_\tau\rangle_{\tau\in\hat{\mathcal{S}}_f}=\hat{\boldsymbol{v}}^*$ is an L^∞-process because if $\tau\in\hat{\mathcal{S}}_f$ there are a $\sigma\in\mathcal{S}$ and an $M\ge 0$ such that $\tau\le\sigma$ and $\hat v^*_\sigma\le M\mathbf{1}^{\hat{\mathcal{S}}_f}$, in which case $\hat v^*_\tau\le M\mathbf{1}^{\hat{\mathcal{S}}_f}$. So (a)-(c) tell us that $\hat{\boldsymbol{z}}_1$ and $\hat{\boldsymbol{z}}_2$ are martingales, and it follows at once that $\boldsymbol{z}_1=\hat{\boldsymbol{z}}_1\restriction\mathcal{S}$ and $\boldsymbol{z}_2=\hat{\boldsymbol{z}}_2\restriction\mathcal{S}$ are martingales.

653D Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ with least element τ and greatest element τ', and $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a locally jump-free virtually local martingale with quadratic variation $\boldsymbol{v}^*=\langle v^*_\sigma\rangle_{\sigma\in\mathcal{S}}$. If $v_\tau=0$ and $v^*_{\tau'}=\gamma\chi 1$ for some $\gamma>0$, then $v_{\tau'}$ has a normal distribution with mean 0 and variance γ and is independent of $\mathfrak{A}_\tau$.

proof (a) Take any non-zero $a\in\mathfrak{A}_\tau$. Then

$$\mathbb{E}(\chi a\times\overline{\sin}(v_{\tau'}))=\mathbb{E}(\chi a\times P_\tau(\overline{\sin}(v_{\tau'})))$$

(because $a\in\mathfrak{A}_\tau$)

$$\begin{aligned}&=e^{-\gamma/2}\mathbb{E}(\chi a\times P_\tau(\overline{\sin}(v_{\tau'})\times\overline{\exp}(\tfrac{1}{2}v^*_{\tau'})))\\&=e^{-\gamma/2}\mathbb{E}(\chi a\times\overline{\sin}(v_\tau)\times\overline{\exp}(\tfrac{1}{2}v^*_\tau))\end{aligned}$$

(653C)

$$= 0,$$

and similarly

$$\mathbb{E}(\chi a \times \overline{\cos}(v_{\tau'})) = e^{-\gamma/2}\mathbb{E}(\chi a \times \overline{\cos}(v_\tau)) = e^{-\gamma/2}\bar{\mu}a.$$

Applying the same argument to the virtually local martingale $\alpha\boldsymbol{v}$ with quadratic variation $\alpha^2\boldsymbol{v}^*$, we see that

$$\mathbb{E}(\chi a \times \overline{\sin}(\alpha v_{\tau'})) = 0, \quad \mathbb{E}(\chi a \times \overline{\cos}(\alpha v_{\tau'})) = e^{-\gamma\alpha^2/2}\bar{\mu}a$$

for any $\alpha \in \mathbb{R}$.

(b) Setting $a = 1$, this means that the characteristic function

$$\alpha \mapsto \mathbb{E}(\overline{\cos}(\alpha v_{\tau'})) + i\mathbb{E}(\overline{\sin}(\alpha v_{\tau'}))$$

of the distribution of $v_{\tau'}$ is just $\alpha \mapsto e^{-\gamma\alpha^2/2}$, which is the characteristic function of the normal distribution with mean 0 and variance γ (285E); by 285Ma, $v_{\tau'}$ must be normally distributed with mean 0 and variance γ.

(c) Next, $\bar{\mu}(a \cap [\![v_{\tau'} > \beta]\!]) = \bar{\mu}a \cdot \bar{\mu}[\![v_{\tau'} > \beta]\!]$ whenever $a \in \mathfrak{A}_\tau$ and $\beta \in \mathbb{R}$. **P** If $a = 0$ this is trivial. Otherwise, take $\mathfrak{B}$ to be the principal ideal of $\mathfrak{A}$ generated by a. Set $\bar{\nu} = \frac{1}{\bar{\mu}a}\bar{\mu}\restriction\mathfrak{B}$, and consider $w \in L^0(\mathfrak{B})$ defined by saying that $[\![w > \alpha]\!] = a \cap [\![v_{\tau'} > \alpha]\!]$ for every $\alpha \in \mathbb{R}$. Now we see that

$$\mathbb{E}_{\bar{\nu}}(\overline{\sin}(\alpha w)) = \frac{1}{\bar{\mu}a}\mathbb{E}_{\bar{\mu}}(\chi a \times \overline{\sin}(\alpha v_{\tau'})) = 0,$$

$$\mathbb{E}_{\bar{\nu}}(\overline{\cos}(\alpha w)) = \frac{1}{\bar{\mu}a}\mathbb{E}_{\bar{\mu}}(\chi a \times \overline{\cos}(\alpha v_{\tau'})) = e^{-\gamma\alpha^2/2}$$

for $\alpha \in \mathbb{R}$. But this means that w and $v_{\tau'}$ have the same characteristic function and the same distribution, so that

$$\bar{\mu}(a \cap [\![v_{\tau'} > \beta]\!]) = \bar{\mu}a \cdot \bar{\nu}[\![w > \beta]\!] = \bar{\mu}a \cdot \bar{\mu}[\![v_{\tau'} > \beta]\!]. \textbf{ Q}$$

As β is arbitrary, $v_{\tau'}$ is independent of $\mathfrak{A}_\tau$, as required.

653E Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ with least element τ and greatest element τ', and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a locally jump-free virtually local martingale with quadratic variation $\boldsymbol{v}^* = \langle v^*_\sigma\rangle_{\sigma\in\mathcal{S}}$, starting from $v_\tau = 0$. If $\tau = \tau_0 \le \ldots \le \tau_k$ in $\mathcal{S}$ and $v^*_{\tau_j} = \gamma_j\chi 1$ for $j \le k$, where $0 = \gamma_0 \le \gamma_1 \le \ldots \le \gamma_k$, then $(v_{\tau_0}, \ldots, v_{\tau_k})$ has a centered Gaussian distribution with covariance matrix $\langle\gamma_{\min(j,l)}\rangle_{j,l\le k}$ (definition: 456A).

proof For $j \le k$, write w_j for v_{τ_j}.

(a) If $0 \le j < k$ and $\gamma_j < \gamma_{j+1}$, $w_{j+1} - w_j$ has a normal distribution with mean 0 and variance $\gamma_{j+1} - \gamma_j$ and is independent of $\mathfrak{A}_{\tau_j}$. **P** Write $\mathcal{S}_j$ for $\mathcal{S}\cap[\tau_j, \tau_{j+1}]$ and set $u_\sigma = v_\sigma - w_j$ for $\sigma \in \mathcal{S}_j$. Then $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}_j}$ is a jump-free local martingale, being the difference of the jump-free local martingale $\langle v_\sigma\rangle_{\sigma\in\mathcal{S}_j}$ and the constant process $\langle w_j\rangle_{\sigma\in\mathcal{S}_j}$. Also, defining stopping-time intervals $c(\sigma, \sigma')$ as in 611J and the interval function $\Delta\boldsymbol{u}$ as in 613Cc, we have

$$(\Delta\boldsymbol{u})(c(\sigma, \sigma')) = u_{\sigma'} - u_\sigma = (\Delta\boldsymbol{v})(c(\sigma, \sigma'))$$

when $\sigma \le \sigma'$ in $\mathcal{S}_j$, so if we write $\boldsymbol{u}^* = \langle u^*_\sigma\rangle_{\sigma\in\mathcal{S}_j}$ for the quadratic variation of $\boldsymbol{u}$,

$$\begin{aligned} u^*_{\tau_{j+1}} &= \int_{\mathcal{S}_j} (d\boldsymbol{u})^2 = \int_{\mathcal{S}_j} (d\boldsymbol{v})^2 \\ &= \int_{\mathcal{S}\wedge\tau_{j+1}} (d\boldsymbol{v})^2 - \int_{\mathcal{S}\wedge\tau_j} (d\boldsymbol{v})^2 = v^*_{\tau_{j+1}} - v^*_{\tau_j} = (\gamma_{j+1} - \gamma_j)\chi 1. \end{aligned}$$

By 653D, $w_{j+1} - w_j = u_{\tau_{j+1}}$ is normally distributed with mean 0 and variance $\gamma_{j+1} - \gamma_j$ and independent of $\mathfrak{A}_{\tau_j}$. **Q**

(b) On the other hand, if $0 \le j < k$ and $\gamma_j = \gamma_{j+1}$, $w_{j+1} = w_j$, by 624D, so trivially $w_{j+1} - w_j$ is independent of $\mathfrak{A}_{\tau_j}$, and in the language of §456 has a centered Gaussian distribution with covariance matrix (0).

(c) Now $\langle w_{j+1} - w_j\rangle_{j<k}$ is stochastically independent. **P** Suppose that $\alpha_j \in \mathbb{R}$ for $j < k$. Then, for each $j < k$, $\inf_{j'<j} [\![w_{j'+1} - w_{j'} > \alpha_{j'}]\!] \in \mathfrak{A}_{\tau_j}$, so

$$\begin{aligned}\bar\mu(\inf_{j'\le j} [\![w_{j'+1} - w_{j'} > \alpha_{j'}]\!]) &= \bar\mu([\![w_{j+1} - w_j > \alpha_j]\!] \cap \inf_{j'\le j} [\![w_{j'+1} - w_{j'} > \alpha_{j'}]\!])\\ &= \bar\mu([\![w_{j+1} - w_j > \alpha_j]\!]) \cdot \bar\mu(\inf_{j'\le j} [\![w_{j'+1} - w_{j'} > \alpha_{j'}]\!]).\end{aligned}$$

Inducing on j, we see that

$$\bar\mu(\inf_{j'\le j} [\![w_{j'+1} - w_{j'} > \alpha_{j'}]\!]) = \prod_{j'\le j} \bar\mu([\![w_{j'+1} - w_{j'} > \alpha_{j'}]\!])$$

for $j < k$, and in particular

$$\bar\mu(\inf_{j<k} [\![w_{j+1} - w_j > \alpha_j]\!]) = \prod_{j<k} \bar\mu([\![w_{j+1} - w_j > \alpha_j]\!]).$$

As $\alpha_0, \dots , \alpha_{k-1}$ are arbitrary, $\langle w_{j+1} - w_j\rangle_{j<k}$ is stochastically independent. **Q**

(d) It follows that the distribution of $\langle w_{j+1} - w_j\rangle_{j<k}$ is the product of the distributions of the $w_{j+1} - w_j$ (653Bc), and is a centered Gaussian distribution with covariance matrix $\langle\beta_{jj'}\rangle_{j,j'<k}$, where $\beta_{jj} = \gamma_{j+1} - \gamma_j$ and $\beta_{jj'} = 0$ if $j \ne j'$ (see 456Be).

(e) Consequently

$$\langle w_j\rangle_{j\le k} = \langle \sum_{j'=0}^{j-1} w_{j'+1} - w_{j'}\rangle_{j\le k}$$

has a centered Gaussian distribution (456Ba), and its covariance matrix is

$$\begin{aligned}\big\langle\mathbb{E}(w_j \times w_l)\big\rangle_{j,l\le k} &= \big\langle \sum_{j'=0}^{j-1}\sum_{l'=0}^{l-1} \mathbb{E}((w_{j'+1} - w_{j'}) \times (w_{l'+1} - w_{l'}))\big\rangle_{j,l\le k}\\ &= \big\langle \sum_{j'=0}^{j-1}\sum_{l'=0}^{l-1} \beta_{j'l'}\big\rangle_{j,l\le k}\\ &= \big\langle \sum_{j'=0}^{\min(j,l)-1} \gamma_{j'+1} - \gamma_{j'}\big\rangle_{j,l\le k} = \langle\gamma_{\min(j,l)}\rangle_{j,l\le k}\end{aligned}$$

as claimed.

653F Theorem ('Lévy's characterization of Brownian motion') Let $(\mathfrak{A}, \bar\mu, [0, \infty[\,, \langle\mathfrak{A}_t\rangle_{t\ge0}, \mathcal{T}, \langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ be a right-continuous real-time stochastic integration structure and $\boldsymbol{v} = \langle v_\tau\rangle_{\tau\in\mathcal{T}_f}$ a locally jump-free local martingale such that

(α) the quadratic variation of $\boldsymbol{v}$ is the identity process,

(β) $\mathfrak{A}$ is the closed subalgebra of itself defined by $\{v_{\check t} : t \ge 0\}$,

(γ) for each $t \ge 0$, $\mathfrak{A}_t$ is the closed subalgebra of $\mathfrak{A}$ defined by $\{v_{\check s} : s \le t\}$.

Then $(\mathfrak{A}, \bar\mu, [0, \infty[\,, \langle\mathfrak{A}_t\rangle_{t\ge0}, \mathcal{T}, \langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}}, \boldsymbol{v})$ is isomorphic to Brownian motion as defined in 612T.

proof (a) Let $(\mathfrak{C}, \bar\nu, \langle\mathfrak{C}_r\rangle_{r\ge0}, \mathcal{Q}, \boldsymbol{w})$ be Brownian motion as described in 612T, writing $\mathcal{Q}$ for the set of stopping times associated with the filtration $\langle\mathfrak{C}_r\rangle_{r\ge0}$. For $t \ge 0$ and $\alpha \in \mathbb{R}$, set $a_{t\alpha} = [\![v_{\check t} > \alpha]\!]$ and $c_{t\alpha} = [\![w_{\check t} > \alpha]\!]$. Then

$$\bar\mu(\inf_{i\le n} a_{t_i\alpha_i}) = \bar\nu(\inf_{i\le n} c_{t_i\alpha_i})$$

whenever $t_0, \dots , t_n \ge 0$ and $\alpha_0, \dots , \alpha_n \in \mathbb{R}$. **P** Of course we can suppose that $0 = t_0 \le \dots \le t_n$. In this case, Lemma 653E tells us that $(v_{\check t_0}, \dots , v_{\check t_n})$ has a centered Gaussian distribution with covariance matrix $\langle\min(t_i, t_j)\rangle_{i,j\le n}$. But of course this is also the distribution of $(w_{\check t_0}, \dots , w_{\check t_n})$ (477Db, 456Bb). So

$$\bar\mu(\inf_{i\le n} a_{t_i\alpha_i}) = \bar\mu(\inf_{i\le n} [\![v_{\check t_i} > \alpha_i]\!]) = \bar\mu(\inf_{i\le n} [\![w_{\check t_i} > \alpha_i]\!]) = \bar\nu(\inf_{i\le n} c_{t_i\alpha_i}).$$ **Q**

(b) It follows that there is a measure algebra isomorphism $\phi : \mathfrak{C} \to \mathfrak{A}$ such that $\phi(c_{t\alpha}) = a_{t\alpha}$ whenever $t \ge 0$ and $\alpha \in \mathbb{R}$. **P** Of course (a) tells us that if (s, α) and (t, β) are such that $c_{s\alpha} = c_{t\beta}$, then $a_{s\alpha} = a_{t\beta}$. So (a) can be interpreted as saying that if $C = \{c_{t\alpha} : r \ge 0,\ \alpha \in \mathbb{R}\}$, we have a function $\phi_0 : C \to \mathfrak{A}$ defined by saying that $\phi_0(c_{t\alpha}) = a_{t\alpha}$ for all t and α, and $\bar\nu(\inf_{i\le n} d_i) = \bar\mu(\inf_{i\le n} \phi_0 d_i)$ for all $d_0, \dots, d_n \in C$. By 324P, ϕ_0 has an extension to a measure-preserving Boolean homomorphism $\phi : \mathfrak{D} \to \mathfrak{A}$, where $\mathfrak{D}$ is the closed subalgebra of $\mathfrak{C}$ generated by $\{c_{t\alpha} : t \ge 0,\ \alpha \in \mathbb{R}\}$; but this is $\mathfrak{C}$ itself, by 612Td. Moreover, $\phi[\mathfrak{C}]$ will be a closed subalgebra of $\mathfrak{A}$ including $\phi_0[C] = \{a_{t\alpha} : t \ge 0,\ \alpha \in \mathbb{R}\}$ (324Kb), so is the whole of $\mathfrak{A}$. Thus ϕ is a surjective measure-preserving Boolean homomorphism; being measure-preserving, it is surely injective, so is an isomorphism. **Q**

(c) Let $T_\phi : L^0(\mathfrak{C}) \to L^0(\mathfrak{A})$ be the f-algebra isomorphism corresponding to ϕ (612Af). Because $\phi(c_{t\alpha}) = a_{t\alpha}$ for all t and α, $T_\phi w_{\check t} = v_{\check t}$ for every $t \ge 0$.

(d) $\phi[\mathfrak{C}_t] = \mathfrak{A}_t$ for every $t \ge 0$. **P** By 612T(d-ii), $\mathfrak{C}_t$ is the closed subalgebra defined by $\{w_{\check s} : s \le t\}$, so $\phi[\mathfrak{C}_t]$ is the closed subalgebra defined by $\{T_\phi w_{\check s} : s \le t\} = \{v_{\check s} : s \le t\}$ and is equal to $\mathfrak{A}_t$. **Q**

Thus ϕ is an isomorphism between $(\mathfrak{C}, \bar\nu, \langle\mathfrak{C}_t\rangle_{t\ge0})$ and $(\mathfrak{A}, \bar\mu, \langle\mathfrak{A}_t\rangle_{t\ge0})$ and therefore corresponds to an isomorphism between $(\mathfrak{C}, \bar\nu, \langle\mathfrak{C}_t\rangle_{t\ge0}, \mathcal{Q}, \langle\mathfrak{C}_\rho\rangle_{\rho\in\mathcal{Q}})$ and $(\mathfrak{A}, \bar\mu, \langle\mathfrak{A}_t\rangle_{t\ge0}, \mathcal{T}, \langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$.

(e) It follows that if we define $\hat\phi : \mathcal{Q} \to \mathcal{T}$ as in 634B, so that $[\![\hat\phi(\rho) > t]\!] = \phi([\![\rho > t]\!])$ for $\rho \in \mathcal{Q}$ and $t \ge 0$, and set

$$v'_\tau = T_\phi w_{\hat\phi^{-1}(\tau)}$$

for $\tau \in \mathcal{T}_f$, then $\boldsymbol{v}' = \langle v'_\tau\rangle_{\tau\in\mathcal{T}_f}$, like $\boldsymbol{w}$, will be locally near-simple. At the same time, (c) tells us that if $\mathcal{T}_c$ is the lattice of constant stopping times, $\boldsymbol{v}'{\restriction}\mathcal{T}_c = \boldsymbol{v}{\restriction}\mathcal{T}_c$. Now $\mathcal{T}_c$ is a separating cofinal sublattice of the ideal $\mathcal{T}_b$ of bounded stopping times (633Da), so $\boldsymbol{v}'{\restriction}\mathcal{T}_b = \boldsymbol{v}{\restriction}\mathcal{T}_b$ (633F); as $\mathcal{T}_f$ is the covered envelope of $\mathcal{T}_b$, $\boldsymbol{v}' = \boldsymbol{v}$ (612Qa). Thus

$$v_\tau = T_\phi w_{\hat\phi^{-1}(\tau)} \text{ for } \tau \in \mathcal{T}_f, \quad v_{\hat\phi(\rho)} = T_\phi w_\rho \text{ for } \rho \in \mathcal{Q}_f,$$

and $\hat\phi$ induces an isomorphism between $(\mathfrak{C}, \bar\nu, \langle\mathfrak{C}_t\rangle_{t\ge0}, \mathcal{Q}, \boldsymbol{w})$ and $(\mathfrak{A}, \bar\mu, \langle\mathfrak{A}_t\rangle_{t\ge0}, \mathcal{T}, \boldsymbol{v})$.

653G Theorem Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$ with a least member, $\boldsymbol{v} = \langle v_\tau\rangle_{\tau\in\mathcal{S}}$ a locally jump-free local martingale such that $v_{\min\mathcal{S}} = 0$, and $\boldsymbol{v}^* = \langle v^*_\tau\rangle_{\tau\in\mathcal{S}}$ the quadratic variation of $\boldsymbol{v}$. Suppose that for every $n \in \mathbb{N}$ there is a $\tau \in \mathcal{S}$ such that $v^*_\tau \ge n\chi 1_{\mathfrak{A}}$. Let $(\mathfrak{C}, \bar\nu, \langle\mathfrak{C}_r\rangle_{r\ge0}, \mathcal{Q}, \boldsymbol{w})$ be Brownian motion as described in 612T, again writing $\mathcal{Q}$ for the set of stopping times associated with $\langle\mathfrak{C}_r\rangle_{r\ge0}$. Express $\boldsymbol{w}$ as $\langle w_\sigma\rangle_{\sigma\in\mathcal{Q}_f}$. Then there are ϕ, $\hat\pi$ and $\mathcal{Q}'$ such that

$\phi : \mathfrak{C} \to \mathfrak{A}$ is a measure-preserving Boolean homomorphism,

$\hat\pi : \mathcal{Q} \to \mathcal{T}$ is a right-continuous lattice homomorphism,

$\mathcal{Q}' = \{\rho : \rho \in \mathcal{Q}_f,\ \hat\pi(\rho) \in \mathcal{S}\}$ is an ideal in $\mathcal{Q}$ including the ideal $\mathcal{Q}_b$ of bounded stopping times,

taking $T_\phi : L^0(\mathfrak{C}) \to L^0(\mathfrak{A})$ to be the f-algebra homomorphism associated with ϕ and $\langle\iota_\rho\rangle_{\rho\in\mathcal{Q}_f}$ to be the identity process on $\mathcal{Q}_f$, $v_{\hat\pi(\rho)} = T_\phi(w_\rho)$ and $v^*_{\hat\pi(\rho)} = T_\phi(\iota_\rho)$ for every $\rho \in \mathcal{Q}'$,

if $\boldsymbol{u} = \langle u_\tau\rangle_{\tau\in\mathcal{S}}$ and $\boldsymbol{z} = \langle z_\rho\rangle_{\rho\in\mathcal{Q}'}$ are locally moderately oscillatory processes such that $T_\phi(z_\rho) = u_{\hat\pi(\rho)}$ for every $\rho \in \mathcal{Q}'$, then $\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\,d\boldsymbol{v} = T_\phi(\int_{\mathcal{Q}\wedge\rho} \boldsymbol{z}\,d\boldsymbol{w})$ whenever $\tau \in \mathcal{S}$ and $\rho \in \mathcal{Q}'$ are such that $v^*_\tau = T_\phi(\iota_\rho)$.

proof (a)(i) Because $\boldsymbol{v}$ is jump-free, $\boldsymbol{v}^*$ is jump-free (618T). It follows that for every $r \ge 0$ there is a $\tau \in \mathcal{S}$ such that $v^*_\tau = r\chi 1_{\mathfrak{A}}$. **P** If $r = 0$ we can take $\tau = \min\mathcal{S}$. Otherwise, by hypothesis, there is a $\tau' \in \mathcal{S}$ such that $v^*_{\tau'} \ge r\chi 1_{\mathfrak{A}}$. By 631Rb, applied to the jump-free process $\boldsymbol{v}^*{\restriction}\mathcal{S}\wedge\tau'$, there is a $\tau \in [\min\mathcal{S}, \tau']$ such that $\|v^*_\tau\|_\infty \le r$ and $[\![v^*_\tau < r]\!] \subseteq [\![\tau = \tau']\!]$; but this means that $[\![v^*_\tau < r]\!] = 0$ and $v^*_\tau = r\chi 1_{\mathfrak{A}}$. **Q**

(ii) For $r \ge 0$, set

$$\pi_r = \inf\{\tau : \tau \in \mathcal{S},\ v^*_\tau \ge s\chi 1_{\mathfrak{A}} \text{ for some } s > r\};$$

because $\mathcal{S}$ is order-convex and has a least element, $\pi_r \in \mathcal{S}$. By 632H, $v^*_{\pi_r} \ge r\chi 1_{\mathfrak{A}}$; by (i) just above, $v^*_{\pi_r} \le s\chi 1_{\mathfrak{A}}$ whenever $r < s$, so $v^*_{\pi_r} = r\chi 1_{\mathfrak{A}}$ for every $r \ge 0$. Of course $\pi_r \le \pi_s$ whenever $0 \le r \le s$, while $\pi_r = \inf_{s>r} \pi_s$ for every $r \ge 0$, that is, $\langle\pi_r\rangle_{r\ge0}$ is right-continuous in the sense of 635Cd.

(b)(i) For $r\ge 0$ and $\alpha\in\mathbb{R}$, set $a_{r\alpha}=[\![v_{\pi_r}>\alpha]\!]$, $c_{r\alpha}=[\![w_{\check r}>\alpha]\!]$. Then

$$\bar\mu(\inf_{i\le n}a_{r_i\alpha_i})=\bar\nu(\inf_{i\le n}c_{r_i\alpha_i})$$

whenever $r_0,\dots,r_n\ge 0$ and $\alpha_0,\dots,\alpha_n\in\mathbb{R}$. **P** Of course we can suppose that $r_0\le\ldots\le r_n$. In this case, Lemma 653E tells us that $(0,v_{\pi_{r_0}},\dots,v_{\pi_{r_n}})$ has a centered Gaussian distribution with covariance matrix $\langle\min(r_i,r_j)\rangle_{-1\le i,j\le n}$, counting 0 as r_{-1}. But of course this is also the distribution of $(0,w_{\check r_0},\dots,w_{\check r_n})$. So

$$\bar\mu(\inf_{i\le n}a_{r_i\alpha_i})=\bar\mu(\inf_{i\le n}[\![v_{\pi_{r_i}}>\alpha_i]\!])=\bar\mu(\inf_{i\le n}[\![w_{\check r_i}>\alpha_i]\!])=\bar\nu(\inf_{i\le n}c_{r_i\alpha_i}).\ \mathbf{Q}$$

(ii) It follows that there is a measure-preserving Boolean homomorphism $\phi:\mathfrak{C}\to\mathfrak{A}$ such that $\phi(c_{r\alpha})=a_{r\alpha}$ whenever $r\ge 0$ and $\alpha\in\mathbb{R}$. **P** Of course (i) tells us that if (r,α) and (s,β) are such that $c_{r\alpha}=c_{s\beta}$, then $a_{r\alpha}=a_{s\beta}$. So (i) can be interpreted as saying that if $C=\{c_{r\alpha}:r\ge 0,\,\alpha\in\mathbb{R}\}$, we have a function $\phi_0:C\to\mathfrak{A}$ defined by saying that $\phi_0(c_{r\alpha})=a_{r\alpha}$ for all r and α, and $\bar\mu(\phi_0(\inf_{i\le n}d_i))=\bar\nu(\inf_{i\le n}\phi_0(d_i))$ for all $d_0,\dots,d_n\in C$. By 324P again, ϕ_0 has an extension to a measure-preserving Boolean homomorphism $\phi:\mathfrak{D}\to\mathfrak{A}$, where $\mathfrak{D}$ is the closed subalgebra of $\mathfrak{C}$ generated by $\{c_{r\alpha}:r\ge 0,\,\alpha\in\mathbb{R}\}$; but this is $\mathfrak{C}$ itself, by 612Td. **Q**

Looking at the associated homomorphism $T_\phi:L^0(\mathfrak{C})\to L^0(\mathfrak{A})$, we see that

$$[\![T_\phi(w_{\check r})>\alpha]\!]=\phi([\![w_{\check r}>\alpha]\!])=[\![v_{\pi_r}>\alpha]\!]$$

whenever $r\ge 0$ and $\alpha\in\mathbb{R}$, so $T_\phi(w_{\check r})=v_{\pi_r}$ for every r.

(c)(i) Returning to (a-ii), 635B-635C tell us that if we set $\mathfrak{B}_r=\mathfrak{A}_{\pi_r}$ for $r\ge 0$, we have a right-continuous filtration $\langle\mathfrak{B}_r\rangle_{r\in[0,\infty[}$ of closed subalgebras of $\mathfrak{A}$, and that if $\mathcal{R}$ is the lattice of stopping times adapted to $\langle\mathfrak{B}_r\rangle_{r\in[0,\infty[}$, we have a lattice homomorphism $\pi:\mathcal{R}\to\mathcal{T}$ such that $\pi(\check r)=\pi_r$ for every $r\ge 0$ and π is right-continuous. Note that in the formula '$\pi(\check r)$' here, $\check r$ must be interpreted as a constant stopping time in $\mathcal{R}$, which in a formal sense is not the same thing as a constant stopping time in $\mathcal{Q}$, as examined in (b). Recall that $[\![\sigma=\sigma']\!]\subseteq[\![\pi(\sigma)=\pi(\sigma')]\!]$ for all σ, $\sigma'\in\mathcal{R}$ (635Cc).

Note next that $\pi(\sigma)\in\mathcal{S}$ for every $\sigma\in\mathcal{R}_b$. **P** The set $\mathcal{R}'=\{\sigma:\sigma\in\mathcal{R},\,\pi(\sigma)\in\mathcal{S}\}$ is an order-convex sublattice of $\mathcal{R}$ (635D(a-i), 635E(h-i)). We know that $\pi_r\in\mathcal{S}$, so $\check r\in\mathcal{R}'$ for every $r\ge 0$; since $\min\mathcal{R}=\check 0$ belongs to $\mathcal{R}'$, $\mathcal{R}_b\subseteq\mathcal{R}'$. **Q** Moreover,

$$v_{\pi(\check r)}=v_{\pi_r}=T_\phi(w_{\check r})$$

for every $r\ge 0$, by (b-ii). (I see that I am employing the notation $\check r$ in both senses here.)

(ii) If $r\ge 0$ then $\mathfrak{C}_r$ is the closed subalgebra of $\mathfrak{C}$ generated by $\{c_{s\alpha}:s\in[0,r],\,\alpha\in\mathbb{R}\}$ (612T(d-ii)). Consequently $\phi[\mathfrak{C}_r]$ is the closed subalgebra of $\mathfrak{A}$ generated by $\{a_{s\alpha}:s\in[0,r],\,\alpha\in\mathbb{R}\}$ (314H) and is included in $\mathfrak{A}_{\pi_r}$. Being measure-preserving, ϕ is order-continuous and injective. We therefore have an injective lattice homomorphism $\hat\phi:\mathcal{Q}\to\mathcal{R}$ associated with ϕ (634B again), which is order-continuous because $\langle\mathfrak{C}_r\rangle_{r\in[0,\infty[}$ is right-continuous (632C(a-i), 634Be). As noted in 634B(c-i) and 634B(b-iii), $[\![\hat\phi(\rho)=\hat\phi(\rho')]\!]=\phi([\![\rho=\rho']\!])$ for all ρ, $\rho'\in\mathcal{Q}$, and $\hat\phi(\rho)\in\mathcal{R}_f$ whenever $\rho\in\mathcal{Q}_f$. As in 634Bf, $\iota_{\hat\phi(\rho)}=T_\phi(\iota_\rho)$ for $\rho\in\mathcal{Q}_f$.

(d)(i) Set $D=\{\sigma:\sigma\in\mathcal{R}_b,\,v^*_{\pi(\sigma)}=\iota_\sigma\}$. Then $D=\mathcal{R}_b$. **P** If $\sigma=\check r$, $\pi(\sigma)=\pi_r$ and $\iota_\sigma=r\chi 1_{\mathfrak{A}}$, so $\sigma\in D$ by (a-ii). If $\sigma\in\mathcal{R}_b$ and $\sigma'\in D$, then

$$[\![v^*_{\pi(\sigma)}=\iota_\sigma]\!]\supseteq[\![v^*_{\pi(\sigma)}=v^*_{\pi(\sigma')}]\!]\cap[\![\iota_\sigma=\iota_{\sigma'}]\!]\supseteq[\![\pi(\sigma)=\pi(\sigma')]\!]\cap[\![\sigma=\sigma']\!]=[\![\sigma=\sigma']\!]$$

by 635Cc. So if $\sigma\in\mathcal{R}_b$ belongs to the full envelope of D, $\sigma\in D$; in particular, as $\mathcal{R}_b$ is a sublattice of $\mathcal{R}$, D is finitely full. If $A\subseteq D$ is non-empty and downwards-directed and $\sigma=\inf A$, then $\pi(\sigma)=\inf_{\sigma'\in A}\pi(\sigma')$ and

$$v^*_{\pi(\sigma)}=\inf_{\sigma'\in A}v^*_{\pi(\sigma')}=\inf_{\sigma'\in A}\iota_{\sigma'}=\iota_\sigma$$

because $\boldsymbol{v}^*$ is an order-continuous lattice homomorphism (632H). By 633G, $D=\mathcal{R}_b$. **Q**

It follows that π is injective. **P** If σ, σ' are distinct members of $\mathcal{R}$, there is an $r\ge 0$ such that $\sigma\wedge\check r\ne\sigma'\wedge\check r$; now

$$v^*_{\pi(\sigma)\wedge\pi_r}=v^*_{\pi(\sigma\wedge\check r)}=\iota_{\sigma\wedge\check r}\ne\iota_{\sigma'\wedge\check r}\ne v^*_{\pi(\sigma')\wedge\pi_r}$$

so $\pi(\sigma)\wedge\pi_r\ne\pi(\sigma')\wedge\pi_r$ and $\pi(\sigma)\ne\pi(\sigma')$. **Q**

(ii) Set $\hat{\pi} = \pi\hat{\phi} : \mathcal{Q} \to \mathcal{T}$. As both $\hat{\phi}$ and π are injective right-continuous lattice homomorphisms, so is $\hat{\pi}$. For $r \ge 0$, $\hat{\pi}(\check{r}) = \pi_r$; in particular, $\hat{\pi}(\min \mathcal{Q}) = \min \mathcal{S}$; consequently $\mathcal{Q}' = \hat{\pi}^{-1}[\mathcal{S}]$ is an order-convex sublattice of $\mathcal{Q}_f$ containing $\min \mathcal{Q}$, that is, is an ideal in $\mathcal{Q}_f$. Also $\mathcal{Q}_b \subseteq \mathcal{Q}'$.

Now $v_{\hat{\pi}(\rho)} = T_\phi(w_\rho)$ for every $\rho \in \mathcal{Q}_b$. **P** This time, set $D' = \{\rho : \rho \in \mathcal{Q}_b,\ v_{\hat{\pi}(\rho)} = T_\phi(w_\rho)\}$. If $\rho \in \mathcal{Q}_b$ and $\rho' \in D'$, then

$$\begin{aligned}[\![v_{\hat{\pi}(\rho)} = T_\phi(w_\rho)]\!] &\supseteq [\![v_{\hat{\pi}(\rho)} = v_{\hat{\pi}(\rho')}]\!] \cap [\![T_\phi(w_\rho) = T_\phi(w_{\rho'})]\!] \\ &\supseteq [\![\pi(\hat{\phi}(\rho)) = \pi(\hat{\phi}(\rho'))]\!] \cap \phi([\![w_\rho = w_{\rho'}]\!]) \\ &\supseteq [\![\hat{\phi}(\rho) = \hat{\phi}(\rho')]\!] \cap \phi([\![w_\rho = w_{\rho'}]\!]) = \phi([\![\rho = \rho']\!]).\end{aligned}$$

So if $\rho \in \mathcal{Q}_b$ belongs to the full envelope of D',

$$[\![v_{\hat{\pi}(\rho)} = T_\phi(w_\rho)]\!] \supseteq \sup_{\rho' \in D'} \phi([\![\rho = \rho']\!]) = \phi(\sup_{\rho' \in D'} [\![\rho = \rho']\!]) = \phi(1_{\mathfrak{C}}) = 1_{\mathfrak{A}}$$

and $v_{\hat{\pi}(\rho))} = T_\phi(w_\rho)$, that is, $\rho \in D'$. In particular, D' is finitely full. By (d), it contains all constant stopping times. If $A \subseteq D'$ is non-empty and downwards-directed and has infimum ρ, then $\{\hat{\pi}(\rho') : \rho' \in A\}$ is downwards-directed and has infimum $\hat{\pi}(\rho)$, so

$$v_{\hat{\pi}(\rho)} = \lim_{\rho' \downarrow A} v_{\hat{\pi}(\rho')}$$

(because $\boldsymbol{v}$ is locally near-simple)

$$= \lim_{\rho' \downarrow A} T_\phi(w_{\rho'}) = T_\phi(\lim_{\rho' \downarrow A} w_{\rho'})$$

(because T_ϕ is continuous for the topologies of convergence in measure on $L^0(\mathfrak{C})$ and $L^0(\mathfrak{A})$ (613Bn))

$$= T_\phi(w_\rho)$$

because $\boldsymbol{w}$ is locally near-simple. So $\inf A \in D'$. Consequently $D' = \mathcal{Q}_b \subseteq D'$, by 633G again. **Q**

(e) We are at last ready to look at some integrals.

(i) Taking $\boldsymbol{u}$ and $\boldsymbol{z}$ as in the last clause of the statement of this theorem, $S_{\hat{\pi}[I]}(\boldsymbol{u}, d\boldsymbol{v}) = T_\phi(S_I(\boldsymbol{z}, d\boldsymbol{w}))$ for every $I \in \mathcal{I}(\mathcal{Q}_b)$. **P** Because $\hat{\pi}{\restriction}I : I \to \hat{\phi}[I]$ is an injective lattice homomorphism, $\hat{\pi}[J]$ will be a maximal totally ordered subset of $\hat{\pi}[I]$ whenever J is a maximal totally ordered subset of I. So if $\langle \rho_i \rangle_{i \le n}$ linearly generates the I-cells, $\langle \hat{\pi}(\rho_i) \rangle_{i \le n}$ will linearly generate the $\hat{\pi}[I]$-cells (see 611L), and

$$\begin{aligned}S_{\hat{\pi}[I]}(\boldsymbol{u}, d\boldsymbol{v}) &= \sum_{i=0}^{n-1} u_{\hat{\pi}(\rho_i)} \times (v_{\hat{\pi}(\rho_{i+1})} - v_{\hat{\pi}(\rho_i)}) = \sum_{i=0}^{n-1} T_\phi(z_{\rho_i}) \times (T_\phi(w_{\rho_{i+1}}) - T_\phi(w_{\rho_i})) \\ &= T_\phi(\sum_{i=0}^{n-1} z_{\rho_i} \times (w_{\rho_{i+1}} - w_{\rho_i})) = T_\phi(S_I(\boldsymbol{z}, d\boldsymbol{w})). \ \mathbf{Q}\end{aligned}$$

(ii) Set $\mathcal{S}' = \hat{\pi}[\mathcal{Q}']$. Then $\int_{\mathcal{S}' \wedge \hat{\pi}(\rho)} \boldsymbol{u}\, d\boldsymbol{v} = T_\phi(\int_{\mathcal{Q} \wedge \rho} \boldsymbol{z}\, d\boldsymbol{w})$ for every $\rho \in \mathcal{Q}_b$. **P** Because $\hat{\pi} : \mathcal{Q}' \to \mathcal{S}$ is an injective lattice homomorphism, $\hat{\pi}{\restriction}\mathcal{Q} \wedge \rho$ is a lattice isomorphism between $\mathcal{Q} \wedge \rho = \mathcal{Q}' \wedge \rho = \mathcal{Q}_b \wedge \rho$ and $\mathcal{S}' \wedge \hat{\pi}(\rho)$, so

$$\begin{aligned}\int_{\mathcal{S}' \wedge \hat{\pi}(\rho)} \boldsymbol{u}\, d\boldsymbol{v} &= \lim_{J \uparrow \mathcal{I}(\mathcal{S}' \wedge \hat{\pi}(\rho))} S_J(\boldsymbol{u}, d\boldsymbol{v}) = \lim_{I \uparrow \mathcal{I}(\mathcal{Q} \wedge \rho)} S_{\hat{\pi}[I]}(\boldsymbol{u}, d\boldsymbol{v}) \\ &= \lim_{I \uparrow \mathcal{I}(\mathcal{Q} \wedge \rho)} T_\phi(S_I(\boldsymbol{z}, d\boldsymbol{w})) = T_\phi(\lim_{I \uparrow \mathcal{I}(\mathcal{Q} \wedge \rho)} S_I(\boldsymbol{z}, d\boldsymbol{w})) = T_\phi(\int_{\mathcal{Q} \wedge \rho} \boldsymbol{z}\, d\boldsymbol{w}). \ \mathbf{Q}\end{aligned}$$

(iii) Again suppose that $\rho \in \mathcal{Q}_b$. Then $\mathcal{S}' \wedge \hat{\pi}(\rho)$ $\boldsymbol{v}$-separates $\mathcal{S} \wedge \hat{\pi}(\rho))$. **P** Suppose that $\tau \le \tau' \le \hat{\pi}(\rho)$ and $v_\tau \ne v_{\tau'}$. Then $v^*_\tau \ne v^*_{\tau'}$, by 624E applied to $\boldsymbol{v}{\restriction}[\tau, \tau']$. So there is an $r \ge 0$ such that $[\![v^*_\tau < r]\!] \cap [\![v^*_{\tau'} > r]\!] \ne 0$. But $v^*_{\pi_r} = r\chi 1_{\mathfrak{A}}$, so $[\![v^*_\tau < v^*_{\pi_r}]\!] \cap [\![v^*_{\tau'} > v^*_{\pi_r}]\!] \ne 0$. Since $[\![\pi_r \le \tau]\!] \subseteq [\![v^*_{\pi_r} \le v^*_\tau]\!]$

(614Ib), $[\![\tau < \pi_r]\!] \supseteq [\![v^*_\tau < v^*_{\pi_r}]\!]$; similarly, $[\![\pi_r < \tau']\!] \supseteq [\![v^*_{\pi_r} < v^*_{\tau'}]\!]$. It follows that, setting $\bar\tau = \pi_r \wedge \hat\pi(\rho)$, $[\![\tau \le \bar\tau]\!] \cap [\![\bar\tau < \tau']\!] \ne 0$. And

$$\bar\tau = \pi_r \wedge \hat\pi(\rho) = \hat\pi(\check r \wedge \rho) \in \mathcal{S}' \wedge \hat\pi(\rho).$$

Thus the conditions of 633Sa are satisfied, and $\mathcal{S}' \wedge \hat\pi(\rho)$ $\boldsymbol{v}$-separates $\mathcal{S} \wedge \hat\pi(\rho)$). **Q**

Of course $\mathcal{S}'$ contains $\hat\pi(\rho)$, so is cofinal with $\mathcal{S} \wedge \hat\pi(\rho)$, and we can use 633Ka to see that

$$T_\phi(\int_{\mathcal{Q}\wedge\rho} \boldsymbol{z}\, d\boldsymbol{w}) = \int_{\mathcal{S}'\wedge\hat\pi(\rho)} \boldsymbol{u}\, d\boldsymbol{v} = \int_{\mathcal{S}\wedge\hat\pi(\rho)} \boldsymbol{u}\, d\boldsymbol{v}.$$

(iv) Now suppose that we are given $\rho \in \mathcal{Q}_b$ and $\tau \in \mathcal{S}$ such that $v^*_\tau = T_\phi(\iota_\rho)$, as in the statement of the theorem. Then $v^*_\tau = v^*_{\hat\pi(\rho)}$, so $\boldsymbol{v}$ is constant on $[\tau \wedge \hat\pi(\rho), \tau \vee \hat\pi(\rho)]$ (624E) and

$$\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\boldsymbol{v} = \int_{\mathcal{S}\wedge\tau\wedge\hat\pi(\rho)} \boldsymbol{u}\, d\boldsymbol{v} = \int_{\mathcal{S}\wedge\hat\pi(\rho)} \boldsymbol{u}\, d\boldsymbol{v} = T_\phi(\int_{\mathcal{Q}\wedge\rho} \boldsymbol{z}\, d\boldsymbol{w}).$$

(v) Finally, suppose we are told only that $\rho \in \mathcal{Q}'$, $\tau \in \mathcal{S}$ and $v^*_\tau = T_\phi(\iota_\rho)$. Set $\tau' = \sup_{r\ge 0} \tau \wedge \pi_r$. Then

$$v^*_{\tau'} = \sup_{r\ge0} v^*_\tau \wedge v^*_{\pi_r} = \sup_{r\ge0} v^*_\tau \wedge r\chi 1\mathfrak{A} = v^*_\tau,$$

so $\boldsymbol{v}$ is constant on $[\tau', \tau]$. Now

$$v^*_{\tau\wedge\pi_r} = v^*_\tau \wedge r\chi 1_{\mathfrak{A}} = T_\phi(\iota_\rho) \wedge r\chi 1_{\mathfrak{A}} = T_\phi(\iota_\rho \wedge r\chi 1_{\mathfrak{C}}) = T_\phi(\iota_{\rho\wedge\check r})$$

for each r, so

$$\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\boldsymbol{v} = \int_{\mathcal{S}\wedge\tau'} \boldsymbol{u}\, d\boldsymbol{v} = \lim_{r\to\infty} \int_{\mathcal{S}\wedge\tau\wedge\pi_r} \boldsymbol{u}\, d\boldsymbol{v}$$

(because $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is locally jump-free (618R))

$$= \lim_{r\to\infty} T_\phi(\int_{\mathcal{Q}\wedge\rho\wedge\check r} \boldsymbol{z}\, d\boldsymbol{w}) = T_\phi(\lim_{r\to\infty} \int_{\mathcal{Q}\wedge\rho\wedge\check r} \boldsymbol{z}\, d\boldsymbol{w}) = T_\phi(\int_{\mathcal{Q}\wedge\rho} \boldsymbol{z}\, d\boldsymbol{w}).$$

This completes the proof.

653H Remarks This theorem has been a strenuous journey through a tangle of technicalities. It is attempting to say that any locally jump-free local martingale is a kind of time-changed version of Brownian motion, and that integration with respect to such a martingale can sometimes be reduced to integration with respect to Brownian motion. The key is in part (b) of the proof. If we pick the stopping times π_r correctly, we can get a process $\langle v_{\pi_r}\rangle_{r\ge0}$ which has the same distribution as $\langle w_r\rangle_{r\ge0}$ and can therefore be thought of as a version of Brownian motion. The measure-preserving homomorphism ϕ is supposed to give a structural foundation for this thought. Being an injective Boolean homomorphism, it represents $\mathfrak{C}$ as a closed subalgebra of $\mathfrak{A}$. We have to check that $\mathfrak{C}_r$ can now be identified with a subalgebra of $\mathfrak{A}_{\pi_r}$, and we find that this identifies the bounded stopping times adapted to $\langle\mathfrak{C}_r\rangle_{r\in[0,\infty[}$ with a $\boldsymbol{v}$-separating sublattice of $\operatorname{dom}\boldsymbol{v}$, so that the image of $\boldsymbol{w}$ determines $\boldsymbol{v}$. But here we come to something important. $\mathfrak{A}$ can be a much larger algebra than $\mathfrak{C}$. I do not mean just that it can have arbitrarily large Maharam type, which is what I usually meant by 'large' measure algebra in Volumes 3 and 5. From the point of view of the stochastic processes it supports, what matters more is that there can be large subalgebras of $\mathfrak{A}$ which are stochastically independent of the image of $\mathfrak{C}$. For instance, $\mathfrak{A}$ may have been set up to model a family of more or less independent martingales – e.g., Brownian motion in $\mathbb{R}^n$ where $n > 1$ – and there does not have to be any natural way of simultaneously reducing them to a single copy of one-dimensional Brownian motion.

It is this potential complication which makes the final clause on integration so involved; while ϕ, $\hat\pi$ and $\mathcal{Q}'$ all have constructive definitions, there is no straightforward way of getting from $\boldsymbol{u}$ to $\boldsymbol{z}$ or vice versa, and τ does not have to be $\hat\pi(\rho)$. Given the process $\boldsymbol{u}$, for instance, we should like to set $z_\rho = T_\phi^{-1} u_{\hat\pi(\rho)}$, but this won't work unless (at least) $u_{\hat\pi(\rho)} \in L^0(\phi[\mathfrak{C}_\rho])$ for enough ρ. (See 653Xb.) Or given the process $\boldsymbol{z}$, there will be an indeterminacy in $\boldsymbol{u}$ on any interval $[\tau, \tau']$ on which $\boldsymbol{v}^*$ (and therefore $\boldsymbol{v}$) is constant. Moreover, even if all these difficulties have been resolved, there is no promise that the required calculations, beginning with the π_r of part (a) of the proof, will be manageable; and in many important cases, we want to compute an integral $\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\boldsymbol{v}$ where τ is unconnected with $\phi[\mathfrak{C}]$, and this theorem gives no help at all.

Nevertheless, it tells us that it will always be worth looking at the possibility of describing a given jump-free martingale in terms of Brownian motion (see 653Yb), and there are useful special cases in which simplifications are possible, as in the following.

653I Corollary Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$ with a least member, $\boldsymbol{v} = \langle v_\tau\rangle_{\tau\in\mathcal{S}}$ a locally jump-free local martingale such that $v_{\min\mathcal{S}} = 0$, and $\boldsymbol{v}^* = \langle v^*_\tau\rangle_{\tau\in\mathcal{S}}$ the quadratic variation of $\boldsymbol{v}$. Suppose that for every $n \in \mathbb{N}$ there is a $\tau \in \mathcal{S}$ such that $v^*_\tau \ge n\chi 1_{\mathfrak{A}}$. Let $(\mathfrak{C}, \bar\nu, \langle\mathfrak{C}_r\rangle_{r\in[0,\infty[}, \mathcal{Q}, \boldsymbol{w})$ be Brownian motion. Then there are ϕ and $\hat\pi$ such that

$\phi : \mathfrak{C} \to \mathfrak{A}$ is a measure-preserving Boolean homomorphism,

$\hat\pi : \mathcal{Q} \to \mathcal{T}$ is a lattice homomorphism,

if $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous, then, taking $T_\phi : L^0(\mathfrak{C}) \to L^0(\mathfrak{A})$ to be the f-algebra homomorphism associated with ϕ, $\int_{\mathcal{S}\wedge\hat\pi(\rho)} \bar f(\boldsymbol{v}, \boldsymbol{v}^*)\,d\boldsymbol{v} = T_\phi(\int_{\mathcal{Q}\wedge\rho} \bar f(\boldsymbol{w}, \boldsymbol{\iota})\,d\boldsymbol{w})$ whenever $\rho \in \mathcal{Q}_f \cap \hat\pi^{-1}[\mathcal{S}]$.

proof In 653G, set $\boldsymbol{u} = \bar f(\boldsymbol{v}, \boldsymbol{v}^*)$, $\tau = \hat\pi(\rho)$ and $\boldsymbol{z} = \bar f(\boldsymbol{w}\restriction\mathcal{Q}', \boldsymbol{\iota}\restriction\mathcal{Q}')$, where $\mathcal{Q}' = \{\rho' : \rho' \in \mathcal{Q}_f,\ \hat\pi(\rho') \in \mathcal{S}\}$. Because f is continuous, these are near-simple, and

$$T_\phi z_{\rho'} = T_\phi(\bar f(w_{\rho'}, \iota_{\rho'})) = \bar f(T_\phi w_{\rho'}, T_\phi \iota_{\rho'})$$

(619Eg)

$$= \bar f(v_{\hat\pi(\rho')}, v^*_{\hat\pi(\rho')}) = u_{\hat\pi(\rho')}$$

for every $\rho' \in \mathcal{Q}'$, so the conditions of 653G are satisfied and we can read the result off.

653J Corollary Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$ with a least member, $\boldsymbol{v} = \langle v_\tau\rangle_{\tau\in\mathcal{S}}$ a locally jump-free local martingale such that $v_{\min\mathcal{S}} = 0$, and $\boldsymbol{v}^* = \langle v^*_\tau\rangle_{\tau\in\mathcal{S}}$ the quadratic variation of $\boldsymbol{v}$. Suppose that for every $n \in \mathbb{N}$ there is a $\tau \in \mathcal{S}$ such that $v^*_\tau \ge n\chi 1_{\mathfrak{A}}$. Let $(\mathfrak{C}, \bar\nu, \langle\mathfrak{C}_r\rangle_{r\in[0,\infty[}, \mathcal{Q}, \boldsymbol{w})$ be Brownian motion. Then there are ϕ and $\hat\pi$ such that

$\phi : \mathfrak{C} \to \mathfrak{A}$ is a measure-preserving Boolean homomorphism,

$\hat\pi : \mathcal{Q} \to \mathcal{T}$ is a lattice homomorphism,

if $h : \mathbb{R}^2 \to \mathbb{R}$ is locally bounded and Borel measurable then, taking $T_\phi : L^0(\mathfrak{C}) \to L^0(\mathfrak{A})$ to be the f-algebra homomorphism associated with ϕ, $\oint_{\mathcal{S}\wedge\hat\pi(\rho)} \bar h(\boldsymbol{v}, \boldsymbol{v}^*)\,d\boldsymbol{v} = T_\phi(\oint_{\mathcal{Q}\wedge\rho} \bar h(\boldsymbol{w}, \boldsymbol{\iota})\,d\boldsymbol{w})$ whenever $\rho \in \mathcal{Q}_f \cap \hat\pi^{-1}[\mathcal{S}]$.

proof (a) Take ϕ, $\hat\pi$, T_ϕ, τ and ρ as in 653I. Let H be the set of locally bounded Borel measurable real functions on $\mathbb{R}^2$. Because $\boldsymbol{v}$ and $\boldsymbol{w}$ are locally jump-free and start at zero, they are equal to their previsible versions $\boldsymbol{v}_<$ and $\boldsymbol{w}_<$ (641O) and are surely locally S-integrable (directly from the definitions in 645F and 645P), so $\bar h(\boldsymbol{v}, \boldsymbol{v}^*)$ and $\bar h(\boldsymbol{w}, \boldsymbol{\iota})$ are locally S-integrable (645J) and the S-integrals $\oint_{\mathcal{S}\wedge\hat\pi(\rho)} \bar h(\boldsymbol{v}, \boldsymbol{v}^*)\,d\boldsymbol{v}$ and $\oint_{\mathcal{Q}\wedge\rho} \bar h(\boldsymbol{w}, \boldsymbol{\iota})\,d\boldsymbol{w}$ are defined. Write H_0 for the set of those $h \in H$ such that $\oint_{\mathcal{S}\wedge\hat\pi(\rho)} \bar h(\boldsymbol{v}, \boldsymbol{v}^*)\,d\boldsymbol{v} = T_\phi(\oint_{\mathcal{Q}\wedge\rho} \bar h(\boldsymbol{w}, \boldsymbol{\iota})\,d\boldsymbol{w})$.

(b) If $h : \mathbb{R}^2 \to \mathbb{R}$ is continuous, then $h \in H_0$. **P** $\bar h(\boldsymbol{v}, \boldsymbol{v}^*)$ and $\bar h(\boldsymbol{w}, \boldsymbol{\iota})$ are locally jump-free (619Gd), so are equal to $\bar h(\boldsymbol{v}, \boldsymbol{v}^*)_<$ and $\bar h(\boldsymbol{w}, \boldsymbol{\iota})_<$ except perhaps at $\min\mathcal{S}$, $\min\mathcal{Q}$ respectively; that is, in the language of §645,

$$(\bar h(\boldsymbol{v}, \boldsymbol{v}^*)\restriction\mathcal{S}\wedge\hat\pi(\rho))_< = (\bar h(\boldsymbol{v}, \boldsymbol{v}^*)\restriction\mathcal{S}\wedge\hat\pi(\rho)) \times \mathbf{1}_<^{(\mathcal{S}\wedge\hat\pi(\rho))},$$

$$(\bar h(\boldsymbol{w}, \boldsymbol{\iota})\restriction\mathcal{Q}\wedge\rho)_< = (\bar h(\boldsymbol{w}, \boldsymbol{\iota})\restriction\mathcal{Q}\wedge\rho) \times \mathbf{1}_<^{(\mathcal{Q}\wedge\rho)}$$

(using 641G(c-ii) to see that the shift to $\mathcal{S}\wedge\hat\pi(\rho)$ and $\mathcal{Q}\wedge\rho$ makes no difference). We therefore have

$$\oint_{\mathcal{S}\wedge\hat\pi(\rho)} \bar h(\boldsymbol{v}, \boldsymbol{v}^*)\,d\boldsymbol{v} = \oint_{\mathcal{S}\wedge\hat\pi(\rho)} \bar h(\boldsymbol{v}, \boldsymbol{v}^*)_<\,d\boldsymbol{v} = \int_{\mathcal{S}\wedge\hat\pi(\rho)} \bar h(\boldsymbol{v}, \boldsymbol{v}^*)\,d\boldsymbol{v} = T_\phi(\int_{\mathcal{Q}\wedge\rho} \bar h(\boldsymbol{w}, \boldsymbol{\iota})\,d\boldsymbol{w})$$

(by 653I)

$$= T_\phi(\oint_{\mathcal{Q}\wedge\rho} \bar h(\boldsymbol{w},\boldsymbol{\iota})_< \, d\boldsymbol{w}) = T_\phi(\oint_{\mathcal{Q}\wedge\rho} \bar h(\boldsymbol{w},\boldsymbol{\iota})\, d\boldsymbol{w})$$

and $h \in H_0$. **Q**

(c) If $\langle h_n\rangle_{n\in\mathbb{N}}$ is a sequence in H_0 such that $h(\alpha) = \lim_{n\to\infty} h_n(\alpha)$ for every $\alpha \in \mathbb{R}$ and $\sup_{n\in\mathbb{N}} |h_n|$ is locally bounded, then $h \in H_0$. **P**

$$\langle(\bar h_n(\boldsymbol{v},\boldsymbol{v}^*)\restriction\mathcal{S}\wedge\hat\pi(\rho)) \times \mathbf{1}^{(\mathcal{S}\wedge\hat\pi(\rho))}_<\rangle_{n\in\mathbb{N}}$$

is uniformly previsibly order-bounded, because there is a continuous function $g : \mathbb{R} \to [0,\infty[$ such that $|h_n(\alpha,\beta)| \le g(|\alpha| + |\beta|)$ for all n, α and β (645Cb), and

$$\begin{aligned}|(\bar h_n(\boldsymbol{v},\boldsymbol{v}^*)\restriction\mathcal{S}\wedge\hat\pi(\rho)) \times \mathbf{1}^{(\mathcal{S}\wedge\hat\pi(\rho))}_<| &\le (\bar g(|\boldsymbol{v}| + |\boldsymbol{v}^*|)\restriction\mathcal{S}\wedge\hat\pi(\rho)) \times \mathbf{1}^{(\mathcal{S}\wedge\hat\pi(\rho))}_< \\ &= (\bar g(|\boldsymbol{v}| + |\boldsymbol{v}^*|)\restriction\mathcal{S}\wedge\hat\pi(\rho))_<\end{aligned}$$

for every n. Also $\langle\bar h_n(\boldsymbol{v},\boldsymbol{v}^*)\restriction\mathcal{S}\wedge\hat\pi(\rho)\rangle_{n\in\mathbb{N}}$ is order*-convergent to $\bar h(\boldsymbol{v},\boldsymbol{v}^*)\restriction\mathcal{S}\wedge\hat\pi(\rho)$ (642Bd). So

$$\oint_{\mathcal{S}\wedge\hat\pi(\rho)} \bar h(\boldsymbol{v},\boldsymbol{v}^*)\, d\boldsymbol{v} = \lim_{n\to\infty} \oint_{\mathcal{S}\wedge\hat\pi(\rho)} \bar h_n(\boldsymbol{v},\boldsymbol{v}^*)\, d\boldsymbol{v}$$

by 645T. Similarly,

$$\oint_{\mathcal{Q}\wedge\rho} \bar h(\boldsymbol{w},\boldsymbol{\iota})\, d\boldsymbol{w} = \lim_{n\to\infty} \oint_{\mathcal{Q}\wedge\rho} \bar h_n(\boldsymbol{w},\boldsymbol{\iota})\, d\boldsymbol{w}.$$

Accordingly

$$\oint_{\mathcal{S}\wedge\hat\pi(\rho)} \bar h(\boldsymbol{v},\boldsymbol{v}^*)\, d\boldsymbol{v} = \lim_{n\to\infty} T_\phi(\oint_{\mathcal{Q}\wedge\rho} \bar h_n(\boldsymbol{w},\boldsymbol{\iota})\, d\boldsymbol{w}) = T_\phi(\oint_{\mathcal{Q}\wedge\rho} \bar h(\boldsymbol{w},\boldsymbol{\iota})\, d\boldsymbol{w})$$

and $h \in H_0$. **Q**

(d) By 645Cc we see that $H_0 = H$, as required.

653K Brownian processes The lemmas 653D-653E, and Theorem 653F, have a direct simplicity which has vanished completely in Theorem 653G and its corollaries. In the generality claimed for 653G, with an arbitrary time-set T and a largely arbitrary locally jump-free local martingale $\boldsymbol{v}$, we have to expect complications. But suppose (as I am sure many readers are already doing) that $T = [0,\infty[$, that $\mathcal{S}$ contains all bounded stopping times, and that $\boldsymbol{v}^* = \boldsymbol{\iota}$ is the identity process. In this case 653E already tells us what the distribution of $\boldsymbol{v}$ is on the lattice of constant stopping times. Working through the proof of 653G in this case, we see that

$$\iota_{\pi_r} = v^*_{\pi_r} = r\chi 1_{\mathfrak{A}} = \iota_{\check r}$$

so that $\pi_r = \check r$ (here to be interpreted as a member of $\mathcal{T}$) for every $r \ge 0$. Consequently $\mathfrak{B}_r = \mathfrak{A}_r$ for every r and $\mathcal{R} = \mathcal{T}$; moreover, the formula in 635B ensures that $\pi(\sigma) = \sigma$ for every $\sigma \in \mathcal{R}$. Accordingly $\hat\pi$, as defined in (d-ii) of the proof of 653G, is just the natural embedding $\hat\phi$ of $\mathcal{Q}$ into $\mathcal{T}$ corresponding to the embedding $\phi : \mathfrak{C} \to \mathfrak{A}$ as in 634B. Because $\phi[\mathfrak{C}]$, in the interesting cases, is smaller than $\mathfrak{A}$, the process $\boldsymbol{v}$, defined on an ideal of $\mathcal{T}$, can be far from isomorphic to the process $\boldsymbol{w}$. Nevertheless, it has enough in common to make it seem useful to have a phrase to describe it. I will therefore use the following definition. Note that (unlike the processes considered in 653G-653J) I do not require these processes to be strongly unbounded. You will see that the abstraction here corresponds to the difference between 'Lévy process' as defined in 652C and the classical version in 652F.

Definition A **Brownian-type process** is a locally jump-free virtually local martingale $\boldsymbol{v}$, defined on a lattice $\mathcal{S}$ of stopping times based on a real-time stochastic integration structure, such that the quadratic variation of $\boldsymbol{v}$ is $\boldsymbol{\iota}\restriction\mathcal{S}$.

653X Basic exercises (a) In 653F, show that (γ) is not a consequence of the other hypotheses.

(b) In 653G, let $\boldsymbol{u} = \langle u_\tau\rangle_{\tau\in\mathcal{S}}$ be a near-simple fully adapted process. Show that there is a fully adapted process $\boldsymbol{z} = \langle z_\rho\rangle_{\rho\in\mathcal{Q}}$ such that $T_\phi(z_\rho) = u_{\hat\pi(\rho)}$ for every $\rho \in \mathcal{Q}$ iff $u_{\pi_r} \in L^0(\phi[\mathfrak{C}_r])$ for every $r \ge 0$, and in this case $\boldsymbol{z}$ is near-simple.

(c) In 653B, show that if $\langle u_i \rangle_{i \in I}$ is any family in L^0, there is an associated sequentially order-continuous Boolean homomorphism ϕ from the Baire σ-algebra $\mathcal{B}\mathfrak{a}(\mathbb{R}^I)$ to $\mathfrak{A}$ such that $\phi\{x : x(i) \in E\} = [\![u_i \in E]\!]$ whenever $i \in I$ and E is a Borel subset of $\mathbb{R}$, so that we have a corresponding probability measure $W \mapsto \bar{\mu}\phi(W) : \mathcal{B}\mathfrak{a}(\mathbb{R}^I) \to [0,1]$ (compare 454J).

653Y Further exercises (a) Suppose that $\langle \mathfrak{A}_t \rangle_{t \in T}$ is right-continuous, that $\mathcal{S}$ is a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ is a locally jump-free local martingale. Show that there are

a stochastic integration structure $(\mathfrak{D}, \bar{\lambda}, T^*, \langle \mathfrak{D}_t \rangle_{t \in T^*})$ such that T is an initial segment of T^* and $T^* \setminus T$ is order-isomorphic to $[0, \infty[$,

an ideal $\mathcal{S}^*$ of the lattice $\mathcal{T}^*$ of stopping times adapted to $\langle \mathfrak{D}_t \rangle_{t \in T^*}$,

a locally jump-free local martingale $\boldsymbol{z} = \langle z_\tau \rangle_{\tau \in \mathcal{S}^*}$ such that for every $r \ge 0$ there is a $\tau \in \mathcal{S}^*$ such that $z^*_\tau \ge r\chi 1_{\mathfrak{D}}$, where $\boldsymbol{z}^*$ is the quadratic variation of $\boldsymbol{z}$,

and a measure-preserving Boolean homomorphism $\phi : \mathfrak{A} \to \mathfrak{D}$ such that $z_{\pi(\sigma)}$ is defined and equal to $T_\phi(v_\sigma)$ for every $\sigma \in \mathcal{S}$, where for $\sigma \in \mathcal{T}$

$$\begin{aligned} [\![\pi(\sigma) > t]\!] &= \phi([\![\sigma > t]\!]) \text{ if } t \in T, \\ &= 0 \text{ if } t \in T^* \setminus T, \end{aligned}$$

and $T_\phi : L^0(\mathfrak{A}) \to L^0(\mathfrak{D})$ is the f-algebra homomorphism defined from ϕ. Show that this can be done in such a way that $\pi[\mathcal{S}]$ $\boldsymbol{z}$-separates $\mathcal{S}^*$.

(b) Use 653Ya and 653G to show that 651F implies 651C.

653 Notes and comments Clearly 653C is just the complex-valued version of 651C expressed in terms of its real and imaginary parts separated. We could omit this step altogether, and go straight from 651C to 653D, if we re-worked the theory so far with the complex linear space $L^0_{\mathbb{C}}$ (241J, 366M) in place of the real linear space L^0.

The argument for 653F depends heavily on the fact that we can define an isomorphism of a structure $(\mathfrak{A}, \bar{\mu}, \langle \mathfrak{A}_t \rangle_{t \ge 0}, \mathcal{T}, \boldsymbol{v})$ in terms of a measure-preserving Boolean isomorphism. I have tried to present this neither too pedantically nor too glibly. The point of the careful elaboration of Chapter 61 is to show how the processes we are looking at are definable in terms of stochastic integration structures $(\mathfrak{A}, \bar{\mu}, T, \langle \mathfrak{A}_t \rangle_{t \in T})$ as described in the notes to §613, so that isomorphisms of these structures will carry faithful copies of the processes with them.

You will see that the main results of §635 are an attempt to generalize part of the proof of 653G. A conspicuous difficulty in applying 653G is the requirement that the quadratic variation of the martingale $\boldsymbol{v}$ should be unbounded in a strong sense. However this is not the real obstacle, since with a different kind of time-change any martingale has a strongly unbounded end-extension (653Ya).

In 653I-653J I look at functions of two variables and processes $\boldsymbol{u} = \bar{f}(\boldsymbol{v}, \boldsymbol{v}^*)$, $\boldsymbol{z} = \bar{f}(\boldsymbol{w}, \boldsymbol{\iota})$. In fact what really matters is that $\boldsymbol{u}$ is calculated from $\boldsymbol{v}$ by the same 'legitimate' method as $\boldsymbol{z}$ is calculated from $\boldsymbol{w}$. Introducing the quadratic variations $\boldsymbol{v}^*$ and $\boldsymbol{w}^* = \boldsymbol{\iota}$ is permitted because

$$\boldsymbol{v}^* = \boldsymbol{v}^2 - 2ii_{\boldsymbol{v}}(\boldsymbol{v}), \quad \boldsymbol{\iota} = \boldsymbol{w}^2 - 2ii_{\boldsymbol{w}}(\boldsymbol{w})$$

and indefinite integration is a legitimate method. But to go farther with this idea we should need a definition of 'legitimate method' which included a way of matching operations in different stochastic integration structures.

Version of 26.9.24

654 Picard's theorem

The general theory of solutions of ordinary differential equations begins with a classical existence and uniqueness theorem: if h is a continuous function of two variables which is Lipschitz in the first variable, then the differential equation

$$x'(t) = h(x(t), t), \quad x(0) = x_\star$$

or, equivalently, the integral equation

$$x(t) = x_\star + \int_0^t h(x(s), s)ds$$

has a unique solution. In this section I present corresponding results for stochastic integral equations of this type, first for the Riemann-sum integral (654G) and then for the S-integral (654L).

654A Notation $(\mathfrak{A}, \bar\mu, T, \langle\mathfrak{A}_t\rangle_{t\in T}, \mathcal{T}, \langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ will be a stochastic integration structure, and $\mathbb{E}$ the integral on $L^1(\mathfrak{A}, \bar\mu)$; L^0, L^2, L^∞ and θ will be $L^0(\mathfrak{A})$, $L^2(\mathfrak{A}, \bar\mu)$, $L^\infty(\mathfrak{A})$ and the standard F-norm defining the topology of convergence in measure on L^0 (613B).

For a sublattice $\mathcal{S}$ of $\mathcal{T}$ and $\tau \in \mathcal{S}$, I write $\mathcal{S} \wedge \tau$ for $\{\sigma : \sigma \in \mathcal{S}, \sigma \le \tau\}$, $\mathcal{S} \vee \tau$ for $\{\sigma : \sigma \in \mathcal{S}, \tau \le \sigma\}$ and $\mathcal{I}(\mathcal{S})$ for the upwards-directed set of finite sublattices of $\mathcal{S}$. $\mathbf{1}^{(\mathcal{S})}$ will be the constant process with domain $\mathcal{S}$ and value $\chi 1$.

If $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a process, $\sup|\boldsymbol{u}| = \sup_{\sigma\in\mathcal{S}}|u_\sigma|$ if this is defined in L^0, and $\|\boldsymbol{u}\|_\infty = \sup_{\sigma\in\mathcal{S}}\|u_\sigma\|_\infty$, counting this as ∞ if any u_σ does not belong to L^∞. If $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a process and $z \in L^0(\mathfrak{A} \cap \bigcap_{\sigma\in\mathcal{S}}\mathfrak{A}_\sigma)$, then $z\boldsymbol{u} = \langle z \times u_\sigma\rangle_{\sigma\in\mathcal{S}}$. If $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ are processes, then $[\![\boldsymbol{u} \ne \boldsymbol{v}]\!] = \sup_{\sigma\in\mathcal{S}}[\![u_\sigma \ne v_\sigma]\!]$.

If $h : \mathbb{R}^k \to \mathbb{R}$ is Borel measurable, then I write $\bar h$ for any of the corresponding functions from $(L^0)^k$ to L^0 and from $((L^0)^{\mathcal{S}})^k$ to $(L^0)^{\mathcal{S}}$ for a set $\mathcal{S}$, as in 619E-619G.

If $\mathcal{S}$ is a sublattice of $\mathcal{T}$, $M_{\text{o-b}}(\mathcal{S})$ is the space of order-bounded fully adapted processes with domain $\mathcal{S}$, $M_{\text{n-s}}(\mathcal{S})$ the space of near-simple processes with domain $\mathcal{S}$, $M_{\text{mo}}(\mathcal{S})$ the space of moderately oscillatory processes with domain $\mathcal{S}$, and for $\boldsymbol{u} \in M_{\text{mo}}(\mathcal{S})$ $\boldsymbol{u}_<$ is its previsible version (641F). $M_{\text{po-b}}(\mathcal{S})$ is the solid linear subspace of $M_{\text{o-b}}(\mathcal{S})$ generated by $\{\boldsymbol{u}_< : \boldsymbol{u} \in M_{\text{n-s}}(\mathcal{S})\}$. The ucp topology on $M_{\text{o-b}}(\mathcal{S})$ is the linear space topology defined by the F-norm $\boldsymbol{u} \mapsto \theta(\sup|\boldsymbol{u}|)$ as in 615B. $M^0_{\text{S-i}}(\mathcal{S})$ is the closure of $\{\boldsymbol{u}_< : \boldsymbol{u} \in M_{\text{n-s}}(\mathcal{S})\}$ in $M_{\text{po-b}}(\mathcal{S})$ for the S-integration topology (645E-645F), and $M_{\text{S-i}}(\mathcal{S}) = \{\boldsymbol{x} : \boldsymbol{x} \in M_{\text{o-b}}(\mathcal{S}), \boldsymbol{x} \times \mathbf{1}^{(\mathcal{S})}_< \in M^0_{\text{S-i}}(\mathcal{S})\}$.

$S_I(\boldsymbol{u}, d\boldsymbol{v})$ will denote a Riemann sum (613Fb); $\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}$ and $\int_{\mathcal{S}}\boldsymbol{u}\,|d\boldsymbol{v}|$ will be Riemann-sum integrals (613H), with associated indefinite integrals $ii_{\boldsymbol{v}}(\boldsymbol{u})$ (613O); and $\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{v}$ and $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})$ will refer to the S-integral and indefinite S-integral defined in 645P and 646K.

654B Lemma (a) Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$ such that $\inf_{\tau\in\mathcal{S}}\sup_{\sigma\in\mathcal{S}}[\![\tau < \sigma]\!] = 0$, and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a fully adapted process.

(a) If $\boldsymbol{u}$ is locally order-bounded, it is order-bounded.

(b) If $\boldsymbol{u}$ is locally moderately oscillatory, it is moderately oscillatory.

(c) If $\boldsymbol{u}$ is locally near-simple, it is near-simple.

proof For $\tau \in \mathcal{S}$, set $a_\tau = \sup_{\sigma\in\mathcal{S}}[\![\tau < \sigma]\!]$. Then $a_{\tau'} \subseteq a_\tau$ whenever $\tau \le \tau'$ in $\mathcal{S}$, while we are supposing that $\inf_{\tau\in\mathcal{S}} a_\tau = 0$, so $\inf_{\tau\in\mathcal{S}}\bar\mu a_\tau = 0$.

(a) Suppose that $\boldsymbol{u}$ is locally order-bounded. Set $A = \{u_\sigma : \sigma \in \mathcal{S}\}$. Let $\epsilon > 0$. Then there is a $\tau \in \mathcal{S}$ such that $\bar\mu a_\tau \le \epsilon$. For any $\sigma \in \mathcal{S}$,

$$[\![u_\sigma \ne u_{\sigma\wedge\tau}]\!] \subseteq [\![\tau < \sigma]\!] \subseteq a_\tau,$$

so $u_\sigma \times \chi(1 \setminus a_\tau) = u_{\sigma\wedge\tau} \times \chi(1 \setminus a_\tau)$. Accordingly

$$\{u \times \chi(1 \setminus a_\tau) : u \in A\} \subseteq \{u_\sigma \times \chi(1 \setminus a_\tau) : \sigma \in \mathcal{S} \wedge \tau\}$$

is order-bounded, because $\boldsymbol{u}$ is locally order-bounded, while $\bar\mu(1 \setminus a_\tau) \ge 1-\epsilon$. By 613Bp, A is order-bounded in L^0, that is, $\boldsymbol{u}$ is an order-bounded process.

(b) Take $\epsilon > 0$. Let $\tau \in \mathcal{S}$ be such that $\bar\mu a_\tau \le \epsilon$. Set $\boldsymbol{u}' = \langle u'_\sigma\rangle_{\sigma\in\mathcal{S}}$ where $u'_\sigma = u_{\sigma\wedge\tau}$ for $\sigma \in \mathcal{S}$. Then $\boldsymbol{u}'$ is fully adapted (612Ib). $\boldsymbol{u}'{\restriction}\mathcal{S} \wedge \tau$ is moderately oscillatory because it is equal to $\boldsymbol{u}{\restriction}\mathcal{S} \wedge \tau$, and $\boldsymbol{u}'{\restriction}\mathcal{S} \vee \tau$ is moderately oscillatory because it is constant. So $\boldsymbol{u}'$ is moderately oscillatory (615F(a-v)). If $\sigma \in \mathcal{S}$,

$$[\![u'_\sigma \ne u_\sigma]\!] = [\![u_{\sigma\wedge\tau} \ne u_\sigma]\!] \subseteq [\![\sigma \wedge \tau < \sigma]\!] = [\![\tau < \sigma]\!] \subseteq a_\tau,$$

while $\boldsymbol{u}$ is order-bounded by (a) above, so

$$\sup|\boldsymbol{u} - \boldsymbol{u}'| \le 2\sup|\boldsymbol{u}| \times \chi a_\tau$$

and $\theta(\sup|\boldsymbol{u} - \boldsymbol{u}'|) \le \bar\mu a_\tau \le \epsilon$. As ϵ is arbitrary, $\boldsymbol{u}$ belongs to the topological closure of $M_{\text{mo}}(\mathcal{S})$ in $M_{\text{o-b}}(\mathcal{S})$ and is moderately oscillatory (615F(a-iv)).

(c) If $\boldsymbol{u}$ is locally near-simple, then it is locally moderately oascillatory (by 631Ca) and moderately oscillatory (by (b) above), so is near-simple by 631F(c-ii).

654C Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $h : \mathbb{R}^k \to \mathbb{R}$ a continuous function. Then $\bar{h}(\boldsymbol{u}_1, \ldots , \boldsymbol{u}_k) \in M_{\text{o-b}} = M_{\text{o-b}}(\mathcal{S})$ whenever $\boldsymbol{u}_1, \ldots , \boldsymbol{u}_k \in M_{\text{o-b}}$, and $\bar{h} : M_{\text{o-b}}^k \to M_{\text{o-b}}$ is continuous for the ucp topology on $M_{\text{o-b}}$.

proof We can repeat the argument of 615Ca. The result is trivial if $\mathcal{S}$ is empty; suppose otherwise. Express $\boldsymbol{u}_i$ as $\langle u_{i\sigma}\rangle_{\sigma\in\mathcal{S}}$ for each i.

(a) For $\xi \in \mathbb{R}$, set $g(\xi) = \sup\{|h(x)| : x \in \mathbb{R}^k,\ \|x\|_\infty \le |\xi|\}$; then g is continuous, and $|\bar{h}(v_1, \ldots , v_k)| \le \bar{g}(v)$ whenever $v_1, \ldots , v_k, v \in L^0$ and $|v_i| \le v$ for $1 \le i \le k$.

Suppose that $\boldsymbol{u}_i = \langle u_{i\sigma}\rangle_{\sigma\in\mathcal{S}}$ belongs to $M_{\text{o-b}}$ for $1 \le i \le k$. Set $\bar{u} = \sup_{1\le i\le k} \sup |\boldsymbol{u}_i|$. Then $|\bar{h}(u_{1\sigma}, \ldots , u_{k\sigma})| \le g(\bar{u})$ for every $\sigma \in \mathcal{S}$, so $\{\bar{h}(u_{1\sigma}, \ldots , u_{k\sigma}) : \sigma \in \mathcal{S}\}$ is order-bounded in L^0, and $\bar{h}(\boldsymbol{u}_1, \ldots , \boldsymbol{u}_k) \in M_{\text{o-b}}$.

(b) Now take $\boldsymbol{v}_i = \langle v_{i\sigma}\rangle_{\sigma\in\mathcal{S}} \in M_{\text{o-b}}$ for $1 \le i \le k$ and $\epsilon > 0$. Set $\bar{v} = \sup_{\sigma\in\mathcal{S},1\le i\le k} |v_{i\sigma}|$, and let $M \ge 0$ be such that $\bar{\mu}[\![\bar{v} > M]\!] \le \epsilon$. Let $\delta \in\ ]0, 1]$ be such that $|h(x) - h(y)| \le \epsilon$ whenever $y \in [-M - 1, M + 1]^k$ and $\|x - y\|_\infty \le \delta$. Then for any $w_1, \ldots , w_k,\ w'_1, \ldots , w'_k \in L^0$,

$$\begin{aligned}[\![|\bar{h}(w'_1, \ldots , w'_k) - \bar{h}(w_1, \ldots , w_k)| > \epsilon]\!]\\ \subseteq \sup_{1\le i\le k} [\![|w_i| > M]\!] \cup [\![|w'_i - w_i| > \delta]\!].\end{aligned}$$

Take any $\boldsymbol{u}_1, \ldots , \boldsymbol{u}_k \in M_{\text{o-b}}$ such that $\theta(\sup |\boldsymbol{u}_i - \boldsymbol{v}_i|) \le \delta\epsilon$ for every i. Set $\bar{u} = \sup_{\sigma\in\mathcal{S},1\le i\le k} |u_{i\sigma} - v_{i\sigma}|$ and $\bar{w} = \sup_{\sigma\in\mathcal{S}} |\bar{h}(u_{1\sigma}, \ldots , u_{k\sigma}) - \bar{h}(v_{1\sigma}, \ldots , v_{k\sigma})|$. Then $\bar{\mu}[\![\bar{u} > \delta]\!] \le k\epsilon$, so

$$\begin{aligned}[\![\bar{w} > \epsilon]\!] &= \sup_{\sigma\in\mathcal{S}} [\![|\bar{h}(u_{1\sigma}, \ldots , u_{k\sigma}) - \bar{h}(v_{1\sigma}, \ldots , v_{k\sigma})| > \epsilon]\!]\\ &\subseteq \sup_{\sigma\in\mathcal{S},1\le i\le k} [\![|v_{i\sigma}| > M]\!] \cup \sup_{\sigma\in\mathcal{S},i\le k} [\![|u_{i\sigma} - v_{i\sigma}| > \delta]\!] = [\![\bar{v} > M]\!] \cup [\![\bar{u} > \delta]\!]\end{aligned}$$

has measure at most $(k + 1)\epsilon$, and

$$\theta(\sup |\bar{h}(\boldsymbol{u}_1, \ldots , \boldsymbol{u}_k) - \bar{h}(\boldsymbol{v}_1, \ldots , \boldsymbol{v}_k)|) = \theta(\bar{w}) \le (k + 2)\epsilon.$$

As $\boldsymbol{v}_1, \ldots , \boldsymbol{v}_k$ and ϵ are arbitrary, $\bar{h} : M_{\text{o-b}}^k \to M_{\text{o-b}}^k$ is continuous.

654D Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ with a greatest element, and define $\boldsymbol{z} = \langle z_\sigma\rangle_{\sigma\in\mathcal{S}}$ by setting $z_\sigma = \chi[\![\sigma < \max \mathcal{S}]\!]$ for $\sigma \in \mathcal{S}$.

(a) Suppose that $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a moderately oscillatory process and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ an integrator. Then

$$\boldsymbol{z} \times ii_{\boldsymbol{v}}(\boldsymbol{u}) = \boldsymbol{z} \times ii_{\boldsymbol{v}}(\boldsymbol{z} \times \boldsymbol{u}) = \boldsymbol{z} \times ii_{\boldsymbol{z}\times\boldsymbol{v}}(\boldsymbol{u})$$

and $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v} = \int_{\mathcal{S}} \boldsymbol{z} \times \boldsymbol{u}\, d\boldsymbol{v}$.

(b) Let $\boldsymbol{w} = \langle w_\sigma\rangle_{\sigma\in\mathcal{S}}$ be a process of bounded variation, and $\boldsymbol{w}^\uparrow = \langle w^\uparrow_\sigma\rangle_{\sigma\in\mathcal{S}}$ its cumulative variation (614O). Then $\int_{\mathcal{S}} |d(\boldsymbol{z} \times \boldsymbol{w})| \le \sup |\boldsymbol{z} \times (\boldsymbol{w}^\uparrow + |\boldsymbol{w}|)|$.

(c) Suppose that $\mathcal{S}$ has a least member and that $\boldsymbol{w} = \langle w_\sigma\rangle_{\sigma\in\mathcal{S}}$ is an order-bounded fully adapted process starting from $w_{\min \mathcal{S}} = 0$. Then $\sup |\boldsymbol{w}| \le \sup |\boldsymbol{z} \times \boldsymbol{w}| + \operatorname{Osclln}(\boldsymbol{w})$, where $\operatorname{Osclln}(\boldsymbol{w})$ is the residual oscillation of $\boldsymbol{w}$ (618B).

(d) Suppose that $\boldsymbol{w} = \langle w_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a fully adapted process and that $\alpha \ge 0$ is such that $[\![\sigma < \max \mathcal{S}]\!] \subseteq [\![|w_\sigma| \le \alpha]\!]$ for $\sigma \in \mathcal{S}$. Then $\boldsymbol{w}$ is order-bounded, $\|\sup |\boldsymbol{z} \times \boldsymbol{w}|\|_\infty \le \alpha$ and $\|\boldsymbol{w}\|_\infty \le \alpha + \|\operatorname{Osclln}(\boldsymbol{w})\|_\infty$.

(e) If $\boldsymbol{u} \in M_{\text{mo}}(\mathcal{S})$ then $\boldsymbol{u}_< = (\boldsymbol{z} \times \boldsymbol{u})_<$.

proof (a)(i) I should begin by pointing out that $\boldsymbol{z}$ is the simple process with domain $\mathcal{S}$, breakpoint string $(\max \mathcal{S})$, starting value $\chi 1$ and value 0 at $\max \mathcal{S}$. So in particular it is of bounded variation (614Q(a-iii), or otherwise), hence an integrator (616Ra); consequently $\boldsymbol{z} \times \boldsymbol{u}$ is an integrator (616Pa, or otherwise) and moderately oscillatory (616Ib). Thus all the indefinite integrals here have domain $\mathcal{S}$.

Take $\sigma \in \mathcal{S}$. If $I \in \mathcal{I}(\mathcal{S} \wedge \sigma)$ then

$$S_I(\boldsymbol{u},d\boldsymbol{v})\times z_\sigma=S_I(\boldsymbol{z}\times\boldsymbol{u},d\boldsymbol{v})\times z_\sigma=S_I(\boldsymbol{u},d(\boldsymbol{z}\times\boldsymbol{v}))\times z_\sigma.$$

P If I is empty, this is trivial. Otherwise, take $(\tau_0,\ldots,\tau_n)$ linearly generating the I-cells. Then

$$\begin{aligned}S_I(\boldsymbol{u},d\boldsymbol{v})\times z_\sigma&=\sum_{i=0}^{n-1}u_{\tau_i}\times(v_{\tau_{i+1}}-v_{\tau_i})\times z_\sigma\\&=\sum_{i=0}^{n-1}u_{\tau_i}\times z_{\tau_i}\times(v_{\tau_{i+1}}-v_{\tau_i})\times z_\sigma\end{aligned}$$

(because $\tau_i\le\sigma$, so $[\![\sigma<\max\mathcal{S}]\!]\subseteq[\![\tau_i<\max\mathcal{S}]\!]$ and $z_{\tau_i}\times z_\sigma=z_\sigma$)

$$=S_I(\boldsymbol{z}\times\boldsymbol{u},d\boldsymbol{v})\times z_\sigma;$$

similarly,

$$\begin{aligned}S_I(\boldsymbol{u},d\boldsymbol{v})\times z_\sigma&=\sum_{i=0}^{n-1}u_{\tau_i}\times(v_{\tau_{i+1}}\times z_{\tau_{i+1}}-v_{\tau_i}\times z_{\tau_i})\times z_\sigma\\&=S_I(\boldsymbol{u},d(\boldsymbol{z}\times\boldsymbol{v}))\times z_\sigma.\ \mathbf{Q}\end{aligned}$$

Taking the limit as $I\uparrow\mathcal{I}(\mathcal{S}\wedge\sigma)$,

$$z_\sigma\times\int_{\mathcal{S}\wedge\sigma}\boldsymbol{u}\,d\boldsymbol{v}=z_\sigma\times\int_{\mathcal{S}\wedge\sigma}\boldsymbol{z}\times\boldsymbol{u}\,d\boldsymbol{v}=z_\sigma\times\int_{\mathcal{S}\wedge\sigma}\boldsymbol{u}\,d(\boldsymbol{z}\times\boldsymbol{v}).$$

As σ is arbitrary,

$$\boldsymbol{z}\times ii_{\boldsymbol{v}}(\boldsymbol{u})=\boldsymbol{z}\times ii_{\boldsymbol{v}}(\boldsymbol{z}\times\boldsymbol{u})=\boldsymbol{z}\times ii_{\boldsymbol{z}\times\boldsymbol{v}}(\boldsymbol{u}).$$

(ii) Repeating part of the calculation when $\sigma=\max\mathcal{S}$, we see that if $\tau_0\le\tau_1$ in $\mathcal{S}$, then

$$[\![u_{\tau_0}\times(v_{\tau_1}-v_{\tau_0})\ne0]\!]\subseteq[\![\tau_0<\tau_1]\!]\subseteq[\![\tau_0<\max\mathcal{S}]\!]=[\![z_{\tau_0}=\chi1]\!],$$

so, in the language of 613F,

$$\begin{aligned}\Delta_{c(\tau_0,\tau_1)}(\boldsymbol{u},d\boldsymbol{v})&=u_{\tau_0}\times(v_{\tau_1}-v_{\tau_0})\\&=u_{\tau_0}\times z_{\tau_0}\times(v_{\tau_1}-v_{\tau_0})=\Delta_{c(\tau_0,\tau_1)}(\boldsymbol{z}\times\boldsymbol{u},d\boldsymbol{v}).\end{aligned}$$

It follows immediately that $S_I(\boldsymbol{u},d\boldsymbol{v})=S_I(\boldsymbol{z}\times\boldsymbol{u},d\boldsymbol{v})$ for every $I\in\mathcal{I}(\mathcal{S})$ and $\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}=\int_{\mathcal{S}}\boldsymbol{z}\times\boldsymbol{u}\,d\boldsymbol{v}$.

(b) $\boldsymbol{z}\times\boldsymbol{w}$ is of bounded variation (614Q(a-ii)), so the integral $\int_{\mathcal{S}}|d(\boldsymbol{z}\times\boldsymbol{w})|$ is defined. Suppose that $\sigma_0\le\ldots\le\sigma_n=\max\mathcal{S}$ and $\sigma_i\in\mathcal{S}$ for each $i\le n$. Set $a_0=[\![\sigma_0=\max\mathcal{S}]\!]$ and $a_j=[\![\sigma_{j-1}<\sigma_j]\!]\cap[\![\sigma_j=\max\mathcal{S}]\!]$ for $1\le j\le n$. Then $\langle a_j\rangle_{j\le n}$ is a partition of unity in $\mathfrak{A}$. Set $y=\sum_{i=0}^{n-1}|w_{\sigma_{i+1}}\times z_{\sigma_{i+1}}-w_{\sigma_i}\times z_{\sigma_i}|$. Then

$$a_0\subseteq\inf_{i\le n}[\![z_{\sigma_i}=0]\!]\subseteq[\![y=0]\!]$$

and if $0\le j<n$

$$\begin{aligned}a_{j+1}&=[\![\sigma_j<\sigma_{j+1}]\!]\cap[\![\sigma_{j+1}=\max\mathcal{S}]\!]\\&\subseteq\inf_{i\le j}[\![z_{\sigma_i}=\chi1]\!]\cap\inf_{j<i\le n}[\![z_{\sigma_i}=0]\!]\\&\subseteq\inf_{i<j}[\![w_{\sigma_{i+1}}\times z_{\sigma_{i+1}}-w_{\sigma_i}\times z_{\sigma_i}=w_{\sigma_{i+1}}-w_{\sigma_i}]\!]\\&\qquad\cap[\![w_{\sigma_{j+1}}\times z_{\sigma_{j+1}}-w_{\sigma_j}\times z_{\sigma_j}=-w_{\sigma_j}\times z_{\sigma_j}]\!]\cap[\![z_{\sigma_j}=\chi1]\!]\\&\qquad\cap\inf_{j<i<n}[\![w_{\sigma_{i+1}}\times z_{\sigma_{i+1}}-w_{\sigma_i}\times z_{\sigma_i}=0]\!]\\&\subseteq[\![y=\textstyle\sum_{i=0}^{j-1}(|w_{\sigma_{i+1}}-w_{\sigma_i}|+|w_{\sigma_j}|]\!]\subseteq[\![y\le w^{\uparrow}_{\sigma_j}+|w_{\sigma_j}|]\!];\end{aligned}$$

again because $a_{j+1}\subseteq[\![z_{\sigma_j}=\chi1]\!]$,

$$a_{j+1}\subseteq[\![y\le(w^{\uparrow}_{\sigma_j}+|w_{\sigma_j}|)\times z_{\sigma_j}]\!]\subseteq[\![y\le\sup|\boldsymbol{z}\times(\boldsymbol{w}^{\uparrow}+|\boldsymbol{w}|)|]\!].$$

As $\sup_{j\le n} a_j = 1$, $y \le \sup|\boldsymbol{z}\times(\boldsymbol{w}^{\uparrow}+|\boldsymbol{w}|)|$; as $\sigma_0,\dots,\sigma_n$ are arbitrary, $\int_{\mathcal{S}}|d(\boldsymbol{z}\times\boldsymbol{w})| \le \sup|\boldsymbol{z}\times(\boldsymbol{w}^{\uparrow}+|\boldsymbol{w}|)|$.

(c) Write $\bar{w}$ for $\sup|\boldsymbol{z}\times\boldsymbol{w}|$. Suppose that $\sigma\in\mathcal{S}$ and that $\mathcal{I}\in\mathcal{I}(\mathcal{S}\wedge\sigma)$ contains both $\min\mathcal{S}$ and σ. Then, defining $\mathrm{Osclln}_I(\boldsymbol{w})$ as in 618B, $|w_\sigma| \le \bar{w}+\mathrm{Osclln}_I(\boldsymbol{w})$. **P** Let $(\tau_0,\dots,\tau_n)$ linearly generate the I-cells, so that $\tau_0=\min\mathcal{S}$, $\tau_n=\sigma$ and $\mathrm{Osclln}_I(\boldsymbol{w})=\sup_{i<n}|w_{\tau_{i+1}}-w_{\tau_i}|$. Set $a_0=[\![\tau_0=\sigma]\!]$ and $a_i=[\![\tau_{i-1}<\sigma]\!]\cap[\![\tau_i=\sigma]\!]$ for $1\le i\le n$, so that $\sup_{i\le n}a_i=1$. For $1\le i\le n$,

$$\begin{aligned}a_i &\subseteq [\![w_\sigma=w_{\tau_i}]\!]\cap[\![z_{\tau_{i-1}}=\chi 1]\!]\\&\subseteq [\![|w_\sigma|\le|w_{\tau_i}-w_{\tau_{i-1}}|+|w_{\tau_{i-1}}\times z_{\tau_{i-1}}|]\!]\subseteq[\![|w_\sigma|\le\mathrm{Osclln}_I(\boldsymbol{w})+\bar{w}]\!];\end{aligned}$$

also, of course,

$$a_0\subseteq[\![w_\sigma=w_{\min\mathcal{S}}]\!]\subseteq[\![w_\sigma=0]\!]\subseteq[\![|w_\sigma|\le\mathrm{Osclln}_I(\boldsymbol{w})+\bar{w}]\!].$$

As $\sup_{i\le n}a_i=1$, $|w_\sigma|\le\mathrm{Osclln}_I(\boldsymbol{w})+\bar{w}$. **Q**

So if $I\in\mathcal{I}(\mathcal{S})$ contains $\min\mathcal{S}$ and σ,

$$|w_\sigma|\le\mathrm{Osclln}_{I\wedge\sigma}(\boldsymbol{w})+\bar{w}\le\mathrm{Osclln}_I(\boldsymbol{w})+\bar{w};$$

it follows that $|w_\sigma|\le\mathrm{Osclln}^*_I(\boldsymbol{w})+\bar{w}$ for every $I\in\mathcal{S}$, and taking the infimum over I we have $|w_\sigma|\le\mathrm{Osclln}(\boldsymbol{w})+\bar{w}$. As σ is arbitrary, $\sup|\boldsymbol{w}|\le\mathrm{Osclln}(\boldsymbol{w})+\bar{w}$, as claimed.

(d) If $\sigma\in\mathcal{S}$ then

$$[\![\sigma<\max\mathcal{S}]\!]\subseteq[\![|w_\sigma|\le\alpha]\!]\subseteq[\![|w_\sigma\times z_\sigma|\le\alpha]\!],$$

$$[\![\max\mathcal{S}\le\sigma]\!]\subseteq[\![w_\sigma=w_{\max\mathcal{S}}]\!]\cap[\![z_\sigma=0]\!]\subseteq[\![|w_\sigma\times z_\sigma|\le\alpha]\!].$$

So $|w_\sigma|\le\alpha\chi1\vee|w_{\max\mathcal{S}}|$ and $|w_\sigma\times z_\sigma|\le\alpha\chi1$; as σ is arbitrary, $\boldsymbol{w}$ is order-bounded, $\sup|\boldsymbol{z}\times\boldsymbol{w}|\le\alpha\chi1$ and $\|\sup|\boldsymbol{z}\times\boldsymbol{w}|\|_\infty\le\alpha$.

Now (c) tells us that

$$\|\sup|\boldsymbol{w}|\|_\infty\le\|\sup|\boldsymbol{z}\times\boldsymbol{w}|\|_\infty+\|\mathrm{Osclln}(\boldsymbol{w})\|_\infty\le\alpha+\|\mathrm{Osclln}(\boldsymbol{w})\|_\infty.$$

(e) Looking at the definition of $\boldsymbol{u}_<$ in 641E-641F, we see that if $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ and $\boldsymbol{z}\times\boldsymbol{u}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ then whenever $\sigma\in I\in\mathcal{I}(\mathcal{S})$ we have a disjoint family $\langle a_{\sigma'}\rangle_{\sigma'\in I}$ such that

$$\begin{aligned}a_{\sigma'}&\subseteq[\![\sigma'<\sigma]\!]\cap[\![u_{I<\sigma}=u_{\sigma'}]\!]\cap[\![v_{I<\sigma}=v_{\sigma'}]\!]\\&\subseteq[\![u_{\sigma'}=v_{\sigma'}]\!]\cap[\![u_{I<\sigma}=u_{\sigma'}]\!]\cap[\![v_{I<\sigma}=v_{\sigma'}]\!]\subseteq[\![u_{I<\sigma}=v_{I<\sigma}]\!]\end{aligned}$$

for every $\sigma'\in I$, while

$$1\setminus\sup_{\sigma'\in I}a_{\sigma'}\subseteq[\![u_{I<\sigma}=0]\!]\cap[\![v_{I<\sigma}=0]\!]\subseteq[\![u_{I<\sigma}=v_{I<\sigma}]\!].$$

So $u_{I<\sigma}=v_{I<\sigma}$; as I is arbitrary, $u_{<\sigma}=v_{<\sigma}$; as σ is arbitrary, $\boldsymbol{u}_<=(\boldsymbol{z}\times\boldsymbol{u})_<$.

654E Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. Write M^0_{mo} for the space of moderately oscillatory processes $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ with starting value 0. For an integrator $\boldsymbol{v}$, write $\boldsymbol{v}^*$ for its quadratic variation. Suppose that $\boldsymbol{u}\in M_{\mathrm{mo}}=M_{\mathrm{mo}}(\mathcal{S})$ and that $\boldsymbol{w}$, $\boldsymbol{w}'\in M^0_{\mathrm{mo}}$ are such that $\boldsymbol{w}$ is a virtually local martingale and $\boldsymbol{w}'$ is of bounded variation; set $\boldsymbol{v}=\boldsymbol{w}+\boldsymbol{w}'$. Then

$$\|\sup|ii_{\boldsymbol{v}}(\boldsymbol{u})|\|_2\le2\Bigl(\sqrt{\|\boldsymbol{w}^*\|_\infty}+\|\textstyle\int_{\mathcal{S}}|d\boldsymbol{w}'|\|_\infty\Bigr)\|\sup|\boldsymbol{u}|\|_2.$$

proof (a) To begin with (down to the end of (c) below), suppose that $\mathcal{S}$ is full. For a fully adapted process $\boldsymbol{u}$ with domain $\mathcal{S}$, let $j(\boldsymbol{u})$ be the infimum of sums $\sqrt{\|\sup|\boldsymbol{z}^*|\|_1}+\|\int_{\mathcal{S}}|d\boldsymbol{z}'|\|_2$ where $\boldsymbol{z}$, $\boldsymbol{z}'\in M^0_{\mathrm{mo}}$, $\boldsymbol{z}$ is a virtually local martingale, $\boldsymbol{z}'$ is of bounded variation, and $|\boldsymbol{u}|\le|\boldsymbol{z}|+|\boldsymbol{z}'|$, counting $j(\boldsymbol{u})$ as ∞ if there are no such $\boldsymbol{z}$ and $\boldsymbol{z}'$. (By 623Kd, we shall be able to speak of the quadratic variation $\boldsymbol{z}^*$.)

We find that $\|\sup|\boldsymbol{u}|\|_2\le2j(\boldsymbol{u})$ for every $\boldsymbol{u}\in M_{\mathrm{mo}}$. **P** If $j(\boldsymbol{u})=\infty$, this is trivial. Otherwise, let $\epsilon>0$. Take $\boldsymbol{z}$, $\boldsymbol{z}'\in M^0_{\mathrm{mo}}$ such that $\boldsymbol{z}$ is a virtually local martingale, $\boldsymbol{z}'$ is of bounded variation, $|\boldsymbol{u}|\le|\boldsymbol{z}|+|\boldsymbol{z}'|$ and $\sqrt{\|\sup|\boldsymbol{z}^*|\|_1}+\|\int_{\mathcal{S}}|d\boldsymbol{z}'|\|_2\le j(\boldsymbol{u})+\epsilon$. Now, expressing $\boldsymbol{z}$ as $\langle z_\sigma\rangle_{\sigma\in\mathcal{S}}$,

$$\|\sup|\boldsymbol{z}|\|_2 \le 2\sup_{\sigma\in\mathcal{S}}\|z_\sigma\|_2$$

(623M, because $\boldsymbol{z}$ is an approximately local martingale, by 623J)

$$\le 2\sup_{\sigma\in\mathcal{S}}\sqrt{\|z^*_\sigma\|_1}$$

(624H)

$$\le 2\sqrt{\|\sup|\boldsymbol{z}^*|\|_1}.$$

On the other side, $\sup|\boldsymbol{z}'| \le \int_{\mathcal{S}}|d\boldsymbol{z}'|$ because $\boldsymbol{z}'$ starts from 0, so $\|\sup|\boldsymbol{z}'|\|_2 \le \|\int_{\mathcal{S}}|d\boldsymbol{z}'|\|_2$. Putting these together,

$$\|\sup|\boldsymbol{u}|\|_2 \le \|\sup|\boldsymbol{z}| + \sup|\boldsymbol{z}'|\|_2 \le \|\sup|\boldsymbol{z}|\|_2 + \|\sup|\boldsymbol{z}'|\|_2$$
$$\le 2\sqrt{\|\sup|\boldsymbol{z}^*|\|_1} + \|\int_{\mathcal{S}}|d\boldsymbol{z}'|\|_2 \le 2(j(\boldsymbol{u})+\epsilon).$$

As ϵ is arbitrary, we have the result. **Q**

(b) Take $\boldsymbol{u}$, $\boldsymbol{w}$, $\boldsymbol{w}'$ and $\boldsymbol{v}$ as in the statement of the lemma. Then $j(ii_{\boldsymbol{v}}(\boldsymbol{u})) \le \|\sup|\boldsymbol{u}|\|_2(\sqrt{\|\boldsymbol{w}^*\|_\infty} + \|\int_{\mathcal{S}}|d\boldsymbol{w}'|\|_\infty)$. **P** If $\boldsymbol{u}=\boldsymbol{0}$ then $ii_{\boldsymbol{v}}(\boldsymbol{u})=0$ and the result is trivial. If $\boldsymbol{u}\ne\boldsymbol{0}$ and either $\sup|\boldsymbol{w}^*|$ or $\int_{\mathcal{S}}|d\boldsymbol{w}'|$ is not in L^∞, the result is again trivial. So suppose that both $\sup|\boldsymbol{w}^*|$ and $\int_{\mathcal{S}}|d\boldsymbol{w}'|$ belong to L^∞. Setting $\boldsymbol{y}=ii_{\boldsymbol{w}}(\boldsymbol{u})$ and $\boldsymbol{y}'=ii_{\boldsymbol{w}'}(\boldsymbol{u})$, $ii_{\boldsymbol{v}}(\boldsymbol{u})=\boldsymbol{y}+\boldsymbol{y}'$. Now $\boldsymbol{y}$ is a virtually local martingale (623O), $\boldsymbol{y}'$ is of bounded variation (614T), and both start from 0.

Next, $\boldsymbol{y}^* = ii_{\boldsymbol{w}^*}(\boldsymbol{u}^2)$ (617Q(a-iii)), so

$$\|\sup|\boldsymbol{y}^*|\|_1 = \|\sup|ii_{\boldsymbol{w}^*}(\boldsymbol{u}^2)|\|_1 = \|\int_{\mathcal{S}}|d(ii_{\boldsymbol{w}^*}(\boldsymbol{u}^2))|\|_1$$

(because $ii_{\boldsymbol{w}^*}(\boldsymbol{u}^2)$ is non-decreasing and starts from 0)

$$\le \|\sup|\boldsymbol{u}^2| \times \int_{\mathcal{S}} d\boldsymbol{w}^*\|_1$$

(614T)

$$\le \|\sup|\boldsymbol{u}^2|\|_1\,\|\int_{\mathcal{S}} d\boldsymbol{w}^*\|_\infty = \|(\sup|\boldsymbol{u}|)^2\|_1\|\boldsymbol{w}^*\|_\infty$$

(because $\boldsymbol{w}^*$ is non-decreasing and starts from 0), and

$$\sqrt{\|\sup|\boldsymbol{y}^*|\|_1} \le \|\sup|\boldsymbol{u}|\|_2\|\sqrt{\boldsymbol{w}^*}\|_\infty.$$

On the other side,

$$\textstyle\int_{\mathcal{S}}|d\boldsymbol{y}'| \le \sup|\boldsymbol{u}| \times \int_{\mathcal{S}}|d\boldsymbol{w}'|$$

(614T again), so $\|\int_{\mathcal{S}}|d\boldsymbol{y}'|\|_2 \le \|\sup|\boldsymbol{u}|\|_2\|\int_{\mathcal{S}}|d\boldsymbol{w}'|\|_\infty$. Putting these together,

$$j(ii_{\boldsymbol{v}}(\boldsymbol{u})) \le \sqrt{\|\sup|\boldsymbol{y}^*|\|_1} + \|\int_{\mathcal{S}}|d\boldsymbol{y}'|\|_2$$
$$\le \|\sup|\boldsymbol{u}|\|_2\big(\sqrt{\|\boldsymbol{w}^*\|_\infty} + \|\int_{\mathcal{S}}|d\boldsymbol{w}'|\|_\infty\big),$$

as claimed. **Q**

(c) Combining this with (a), we see that

$$\|\sup|ii_{\boldsymbol{v}}(\boldsymbol{u})|\|_2 \le 2j(ii_{\boldsymbol{v}}(\boldsymbol{u})) \le 2(\sqrt{\|\boldsymbol{w}^*\|_\infty} + \|\textstyle\int_{\mathcal{S}}|d\boldsymbol{w}'|\|_\infty)\|\sup|\boldsymbol{u}|\|_2.$$

(d) This proves the result when $\mathcal{S}$ is full. For the general case, let $\hat{\mathcal{S}}$ be the covered envelope of $\mathcal{S}$ and $\hat{\boldsymbol{u}}$, $\hat{\boldsymbol{w}}$, $\hat{\boldsymbol{w}}'$, $\hat{\boldsymbol{v}}$, $\hat{\boldsymbol{w}}^*$ the fully adapted extensions of $\boldsymbol{u}$, $\boldsymbol{w}$, $\boldsymbol{w}'$, $\boldsymbol{v}$, $\boldsymbol{w}^*$ to $\hat{\mathcal{S}}$. Then $\hat{\boldsymbol{u}}$, $\hat{\boldsymbol{w}}$ and $\hat{\boldsymbol{w}}'$ belong to $M_{\mathrm{mo}}(\hat{\mathcal{S}})$

(615F(a-vi)), while $\hat{\boldsymbol{w}}$ is a virtually local martingale (623J), $\hat{\boldsymbol{w}}$ and $\hat{\boldsymbol{w}}'$ have starting value 0 (615H), $\hat{\boldsymbol{w}}'$ is of bounded variation (614Q(a-iv-β)), $\hat{\boldsymbol{v}} = \hat{\boldsymbol{w}} + \hat{\boldsymbol{w}}'$ and $\hat{\boldsymbol{w}}^*$ is the quadratic variation of $\hat{\boldsymbol{w}}$ (617N). By (a)-(c) above,

$$\|\sup|ii_{\hat{\boldsymbol{v}}}(\hat{\boldsymbol{u}})|\|_2 \le 2(\sqrt{\|\hat{\boldsymbol{w}}^*\|_\infty} + \|\textstyle\int_{\hat{\mathcal{S}}}|d\hat{\boldsymbol{w}}'|\|_\infty)\|\sup|\hat{\boldsymbol{u}}|\|_2.$$

Now we know that $ii_{\hat{\boldsymbol{v}}}(\hat{\boldsymbol{u}})$ is the fully adapted extension of $ii_{\boldsymbol{v}}(\boldsymbol{u})$ (616Q(c-ii)), so that $\sup|ii_{\hat{\boldsymbol{v}}}(\hat{\boldsymbol{u}})| = \sup|ii_{\boldsymbol{v}}(\boldsymbol{u})|$ (614Ga); at the same time,

$$\|\hat{\boldsymbol{w}}^*\|_\infty = \|\sup|\hat{\boldsymbol{w}}^*|\|_\infty = \|\sup|\boldsymbol{w}^*|\|_\infty = \|\boldsymbol{w}^*\|_\infty$$

and $\sup|\hat{\boldsymbol{u}}| = \sup|\boldsymbol{u}|$. Finally, $\int_{\hat{\mathcal{S}}}|d\hat{\boldsymbol{w}}'| = \int_{\mathcal{S}}|d\boldsymbol{w}'|$ by 614Q(a-iv-β) again. So

$$\|\sup|ii_{\boldsymbol{v}}(\boldsymbol{u})|\|_2 \le 2(\sqrt{\|\boldsymbol{w}^*\|_\infty} + \|\textstyle\int_{\mathcal{S}}|d\boldsymbol{w}'|\|_\infty)\|\sup|\boldsymbol{u}|\|_2,$$

as required.

654F Lemma (The key.) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ with a greatest element. Suppose that $h : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and that $K \ge 0$ is such that $|h(\alpha,\beta) - h(\alpha',\beta)| \le K|\alpha - \alpha'|$ for all α, α', $\beta \in \mathbb{R}$; let $\boldsymbol{w} = \langle w_\sigma\rangle_{\sigma\in\mathcal{S}}$, $\boldsymbol{w}' = \langle w'_\sigma\rangle_{\sigma\in\mathcal{S}}$ be processes with domain $\mathcal{S}$ such that $\boldsymbol{w}$ is a virtually local martingale, $\boldsymbol{w}'$ is of bounded variation and both start fom 0. Write $\boldsymbol{w}^*$ for the quadratic variation of $\boldsymbol{w}$, $\boldsymbol{w}'^{\uparrow}$ for the cumulative variation of $\boldsymbol{w}'$, and $\boldsymbol{z}$ for $\langle\chi[\![\sigma < \max\mathcal{S}]\!]\rangle_{\sigma\in\mathcal{S}}$. Suppose that

$$2K(\sqrt{\|\boldsymbol{w}^*\|_\infty} + 2\|\boldsymbol{z}\times\boldsymbol{w}'^{\uparrow}\|_\infty) < 1. \qquad (*)$$

Set $\boldsymbol{v} = \boldsymbol{w} + \boldsymbol{w}'$. Then for any $\boldsymbol{u}_\star$, $\boldsymbol{y} \in M_{\mathrm{mo}} = M_{\mathrm{mo}}(\mathcal{S})$ there is a unique $\boldsymbol{u} \in M_{\mathrm{mo}}$ such that

$$\boldsymbol{u} = \boldsymbol{u}_\star + ii_{\boldsymbol{v}}(\bar{h}(\boldsymbol{u},\boldsymbol{y})).$$

proof The proof will proceed in three steps, the first two being with a variation on the hypothesis (*).

(a) Suppose that $\|\boldsymbol{u}_\star\|_\infty$ and $\|\boldsymbol{y}\|_\infty$ are both finite, and that instead of the hypothesis (*) we suppose that

$$2K\gamma < 1 \text{ where } \gamma = \sqrt{\|\boldsymbol{w}^*\|_\infty} + \|\textstyle\int_{\mathcal{S}}|d\boldsymbol{w}'|\|_\infty. \qquad (**)$$

(i) Define a sequence $\langle\boldsymbol{u}_n\rangle_{n\ge1}$ of processes with domain $\mathcal{S}$ by saying that $\boldsymbol{u}_0 = \boldsymbol{u}_\star$ and

$$\boldsymbol{u}_{n+1} = \boldsymbol{u}_\star + ii_{\boldsymbol{v}}(\bar{h}(\boldsymbol{u}_n,\boldsymbol{y}))$$

for every $n \in \mathbb{N}$. Then every $\boldsymbol{u}_n$ belongs to M_{mo} (induce on n, using 619Gc and 616J in the inductive step). Set $\boldsymbol{z}_n = \boldsymbol{u}_{n+1} - \boldsymbol{u}_n$ for $n \ge 0$. Then

$$\begin{aligned}\boldsymbol{z}_{n+1} = \boldsymbol{u}_{n+2} - \boldsymbol{u}_{n+1} &= ii_{\boldsymbol{v}}(\bar{h}(\boldsymbol{u}_{n+1},\boldsymbol{y})) - ii_{\boldsymbol{v}}(\bar{h}(\boldsymbol{u}_n,\boldsymbol{y}))\\ &= ii_{\boldsymbol{v}}(\bar{h}(\boldsymbol{u}_{n+1},\boldsymbol{y}) - \bar{h}(\boldsymbol{u}_n,\boldsymbol{y}))\end{aligned}$$

for every n. Now 654E tells us that

$$\|\sup|\boldsymbol{z}_{n+1}|\|_2 \le 2\gamma\|\sup|\bar{h}(\boldsymbol{u}_{n+1},\boldsymbol{y}) - \bar{h}(\boldsymbol{u}_n,\boldsymbol{y})|\|_2 \le 2K\gamma\|\sup|\boldsymbol{u}_{n+1} - \boldsymbol{u}_n|\|_2$$

(by the Lipschitz condition on h)

$$= 2K\gamma\|\sup|\boldsymbol{z}_n|\|_2$$

for $n \in \mathbb{N}$. At the beginning of the iteration,

$$\begin{aligned}\|\sup|\boldsymbol{z}_0|\|_2 = \|\sup|\boldsymbol{u}_1 - \boldsymbol{u}_\star|\|_2 &= \|\sup|ii_{\boldsymbol{v}}(\bar{h}(\boldsymbol{u}_\star,\boldsymbol{y}))|\|_2\\ &\le 2\gamma\|\sup|\bar{h}(\boldsymbol{u}_\star,\boldsymbol{y})|\|_2 \le 2\gamma\|\bar{h}(\boldsymbol{u}_\star,\boldsymbol{y})\|_\infty\end{aligned}$$

is finite because both $\|\boldsymbol{u}_\star\|_\infty$ and $\|\boldsymbol{y}\|_\infty$ are finite and h is locally bounded. As $2K\gamma < 1$, $\sum_{n=0}^\infty\|\sup|\boldsymbol{z}_n|\|_2$ is finite.

(ii) Of course

$$\theta(u)=\mathbb{E}(|u|\wedge\chi 1)\le\|u\|_1\le\|u\|_2$$

for every $u\in L^0$. So $\sum_{n=0}^\infty\theta(\sup|\boldsymbol{z}_n|)<\infty$ and $\langle\boldsymbol{u}_n\rangle_{n\in\mathbb{N}}$ is Cauchy for the ucp uniformity. As M_{mo} is complete under the ucp uniformity (615F(a-iv)), $\boldsymbol{u}=\lim_{n\to\infty}\boldsymbol{u}_n$ is defined in M_{mo}. Because h is continuous, $\bar h(\boldsymbol{u},\boldsymbol{y})=\lim_{n\to\infty}\bar h(\boldsymbol{u}_n,\boldsymbol{y})$ (619H). By 616J, applied to the integrating interval function $\Delta\boldsymbol{v}$ (616Ic),

$$\boldsymbol{u}_\star+ii_{\boldsymbol{v}}(\bar h(\boldsymbol{u},\boldsymbol{y}))=\lim_{n\to\infty}\boldsymbol{u}_\star+ii_{\boldsymbol{v}}(\bar h(\boldsymbol{u}_n,\boldsymbol{y}))=\lim_{n\to\infty}\boldsymbol{u}_{n+1}=\boldsymbol{u}.$$

(iii) Thus we have an appropriate $\boldsymbol{u}$. It will be helpful to know that $\sup|\boldsymbol{u}|\in L^2$. **P** Since $\sum_{n=0}^\infty\|\sup|\boldsymbol{z}_n|\|_2<\infty$, $z=\sum_{n=0}^\infty\sup|\boldsymbol{z}_n|$ is defined in L^2; now $|\boldsymbol{u}_\star|+z\ge\sup|\boldsymbol{u}_n|$ for every $n\in\mathbb{N}$, so $\sup|\boldsymbol{u}|\le|\boldsymbol{u}_\star|+z$ is square-integrable. **Q**

(b) When we come to prove that our solutions are unique, the following formulation will be useful. As in (a), assume that (**) is true. Suppose that

$$\boldsymbol{u}_{1\star},\,\boldsymbol{u}'_{1\star},\,\boldsymbol{y}_1,\,\boldsymbol{y}'_1,\,\boldsymbol{u}_1,\,\boldsymbol{u}'_1\in M_{\text{mo}},$$

$$\boldsymbol{u}_1=\boldsymbol{u}_{1\star}+ii_{\boldsymbol{v}}(\bar h(\boldsymbol{u}_1,\boldsymbol{y}_1)),\quad \boldsymbol{u}'_1=\boldsymbol{u}'_{1\star}+ii_{\boldsymbol{v}}(\bar h(\boldsymbol{u}'_1,\boldsymbol{y}'_1));$$

suppose moreover that $\sup|\boldsymbol{u}_1|\in L^2$, and set $a=[\![\boldsymbol{u}_{1\star}\ne\boldsymbol{u}'_{1\star}]\!]\cup[\![\boldsymbol{y}_1\ne\boldsymbol{y}'_1]\!]$. Then $[\![\boldsymbol{u}_1\ne\boldsymbol{u}'_1]\!]\subseteq a$. **P** Let $\epsilon>0$. Let $M\ge 0$ be such that $\bar\mu[\![\sup|\boldsymbol{u}'_1|>M]\!]\le\epsilon$ and set $a'=a\cup[\![\sup|\boldsymbol{u}'_1|>M]\!]$. Set $\tilde{\boldsymbol{u}}_0=\text{med}(-M\,\mathbf{1}^{(\mathcal{S})},\boldsymbol{u}'_1,M\,\mathbf{1}^{(\mathcal{S})})$ and $\tilde{\boldsymbol{u}}_{n+1}=\boldsymbol{u}_{1\star}+ii_{\boldsymbol{v}}(\bar h(\tilde{\boldsymbol{u}}_n,\boldsymbol{y}_1))$ for $n\in\mathbb{N}$. Then

$$\begin{aligned}\|\sup|\tilde{\boldsymbol{u}}_{n+1}-\boldsymbol{u}_1|\|_2&=\|\sup|ii_{\boldsymbol{v}}(\bar h(\tilde{\boldsymbol{u}}_n,\boldsymbol{y}_1)-\bar h(\boldsymbol{u}_1,\boldsymbol{y}_1))|\|_2\\&\le 2\gamma\|\sup|\bar h(\tilde{\boldsymbol{u}}_n,\boldsymbol{y}_1)-\bar h(\boldsymbol{u}_1,\boldsymbol{y}_1)|\|_2\le 2K\gamma\|\sup|\tilde{\boldsymbol{u}}_n-\boldsymbol{u}_1|\|_2\end{aligned}$$

for each n, while $\|\sup|\tilde{\boldsymbol{u}}_0-\boldsymbol{u}_1|\|_2\le M+\|\sup|\boldsymbol{u}_1|\|_2$ is finite. So $\lim_{n\to\infty}\|\sup|\tilde{\boldsymbol{u}}_n-\boldsymbol{u}_1|\|_2=0$ and $\boldsymbol{u}'_1=\lim_{n\to\infty}\tilde{\boldsymbol{u}}_n$ for the ucp topology.

At the same time, we find that $[\![\tilde{\boldsymbol{u}}_n\ne\boldsymbol{u}'_1]\!]\subseteq a'$ for every n. To see this, induce on n. At the start, $[\![\tilde{\boldsymbol{u}}_0\ne\boldsymbol{u}'_1]\!]=[\![\sup|\boldsymbol{u}'_1|>M]\!]\subseteq a'$. For the inductive step,

$$\begin{aligned}[\![\tilde{\boldsymbol{u}}_{n+1}\ne\boldsymbol{u}'_1]\!]&=[\![\boldsymbol{u}_{1\star}+ii_{\boldsymbol{v}}(\bar h(\tilde{\boldsymbol{u}}_n,\boldsymbol{y}_1))\ne\boldsymbol{u}'_{1\star}+ii_{\boldsymbol{v}}(\bar h(\boldsymbol{u}'_1,\boldsymbol{y}'_1))]\!]\\&\subseteq[\![\boldsymbol{u}_{1\star}\ne\boldsymbol{u}'_{1\star}]\!]\cup[\![\bar h(\tilde{\boldsymbol{u}}_n,\boldsymbol{y}_1)\ne\bar h(\boldsymbol{u}'_1,\boldsymbol{y}'_1)]\!]\subseteq a\cup[\![\tilde{\boldsymbol{u}}_n\ne\boldsymbol{u}'_1]\!]\subseteq a',\end{aligned}$$

so the induction proceeds.

It follows that $[\![\boldsymbol{u}_1\ne\boldsymbol{u}'_1]\!]\setminus a$ has measure at most ϵ. As ϵ is arbitrary, $[\![\boldsymbol{u}_1\ne\boldsymbol{u}'_1]\!]\subseteq a$. **Q**

(c) For the second stage of the existence proof, continue to assume (**), but drop the $\|\;\|_\infty$-boundedness conditions on $\boldsymbol{u}_\star$ and $\boldsymbol{y}$, and suppose only that $\boldsymbol{u}_\star$, $\boldsymbol{y}\in M_{\text{mo}}$.

(i) In this case, set

$$\boldsymbol{u}_{\star k}=\text{med}(-k\mathbf{1}^{(\mathcal{S})},\boldsymbol{u}_\star,k\mathbf{1}^{(\mathcal{S})}),\quad \boldsymbol{y}_k=\text{med}(-k\,\mathbf{1}^{(\mathcal{S})},\boldsymbol{y},k\,\mathbf{1}^{(\mathcal{S})}),$$

$$a_k=[\![\sup|\boldsymbol{u}_\star|\ge k]\!]\cup[\![\sup|\boldsymbol{y}|\ge k]\!]$$

for $k\in\mathbb{N}$. By (a), there is for each k a $\boldsymbol{u}_k\in M_{\text{mo}}$ such that $\boldsymbol{u}_k=\boldsymbol{u}_{\star k}+ii_{\boldsymbol{v}}(\bar h(\boldsymbol{u}_k,\boldsymbol{y}_k))$ and $\|\sup|\boldsymbol{u}_k|\|_2<\infty$. By (b), $[\![\boldsymbol{u}_k\ne\boldsymbol{u}_l]\!]\subseteq a_k$ for $l\ge k$. Since $a_0=1$ and $\langle a_k\rangle_{k\in\mathbb{N}}$ is a non-increasing sequence with infimum 0,

$$\sup_{k\in\mathbb{N}}\chi(a_k\setminus\chi a_{k+1})\times\sup_{j\le k+1}\sup|\boldsymbol{u}_j|=\sup_{k\in\mathbb{N}}\sup|\boldsymbol{u}_k|=z$$

is defined in L^0 and $\sup|\boldsymbol{u}-\boldsymbol{u}_k|\le z\times\chi a_k$ for each k. This shows that $\boldsymbol{u}=\lim_{k\to\infty}\boldsymbol{u}_k$ for the ucp topology, so $\boldsymbol{u}\in M_{\text{mo}}$ and

$$\boldsymbol{u}=\lim_{k\to\infty}\boldsymbol{u}_{\star k}+ii_{\boldsymbol{v}}(\bar h(\boldsymbol{u}_k,\boldsymbol{y}_k))=\boldsymbol{u}_\star+ii_{\boldsymbol{v}}(\bar h(\boldsymbol{u},\boldsymbol{y}))$$

as in (a-ii), but this time noting that $\bar h(\boldsymbol{u},\boldsymbol{y})=\lim_{k\to\infty}\bar h(\boldsymbol{u}_k,\boldsymbol{y}_k)$ for the ucp topology.

(ii) To see that $\boldsymbol{u}$ is unique, let $\boldsymbol{u}_1\in M_{\text{mo}}$ be such that $\boldsymbol{u}_1=\boldsymbol{u}_\star+ii_{\boldsymbol{v}}(\bar h(\boldsymbol{u}_1,\boldsymbol{y}))$. This time, (b) tells us that $[\![\boldsymbol{u}_1\ne\boldsymbol{u}_k]\!]\subseteq a_k$ for each k. So $\boldsymbol{u}_1=\boldsymbol{u}$.

(d) We are now ready to tackle the result under the given hypothesis (*).

(i) Setting $\boldsymbol{w}'_1 = \boldsymbol{z} \times \boldsymbol{w}'$, we see from 614Q(a-ii), or otherwise, that $\boldsymbol{w}'_1$ is of bounded variation, and from 654Db that

$$\begin{aligned} \| \int_{\mathcal{S}} |d\boldsymbol{w}'_1| \|_\infty &\le \| \sup |\boldsymbol{z} \times (\boldsymbol{w}'^{\uparrow} + |\boldsymbol{w}'|)| \|_\infty \\ &= \|\boldsymbol{z} \times (\boldsymbol{w}'^{\uparrow} + |\boldsymbol{w}'|)\|_\infty \le 2\|\boldsymbol{z} \times \boldsymbol{w}'^{\uparrow}\|_\infty \end{aligned}$$

because $\boldsymbol{w}'$ starts from 0. We therefore have

$$2K(\sqrt{\|\boldsymbol{w}^*\|_\infty} + \|\int_{\mathcal{S}} |d\boldsymbol{w}'_1|\|_\infty) < 1,$$

and if we set $\boldsymbol{v}_1 = \boldsymbol{w} + \boldsymbol{w}'_1$, (c) tells us that there is a near-simple process $\boldsymbol{u}_1$ such that $\boldsymbol{u}_1 = \boldsymbol{u}_\star + ii_{\boldsymbol{v}_1}(\bar{h}(\boldsymbol{u}_1, \boldsymbol{y}))$. Set $\boldsymbol{w}'_2 = \boldsymbol{w}' - \boldsymbol{w}'_1$ and $\boldsymbol{u} = \boldsymbol{u}_1 + ii_{\boldsymbol{w}'_2}(\bar{h}(\boldsymbol{u}_1, \boldsymbol{y}))$.

We see that

$$\boldsymbol{z} \times \boldsymbol{v}_1 = \boldsymbol{z} \times \boldsymbol{w} + \boldsymbol{z} \times \boldsymbol{w}'_1 = \boldsymbol{z} \times \boldsymbol{w} + \boldsymbol{z} \times \boldsymbol{z} \times \boldsymbol{w}' = \boldsymbol{z} \times \boldsymbol{w} + \boldsymbol{z} \times \boldsymbol{w}' = \boldsymbol{z} \times \boldsymbol{v},$$

$$\boldsymbol{z} \times \boldsymbol{u} = \boldsymbol{z} \times \boldsymbol{u}_1 + \boldsymbol{z} \times ii_{\boldsymbol{w}'_2}(\bar{h}(\boldsymbol{u}_1, \boldsymbol{y})) = \boldsymbol{z} \times \boldsymbol{u}_1 + \boldsymbol{z} \times ii_{\boldsymbol{z} \times \boldsymbol{w}'_2}(\bar{h}(\boldsymbol{u}_1, \boldsymbol{y}))$$

(654Da)

$$= \boldsymbol{z} \times \boldsymbol{u}_1$$

because $\boldsymbol{z} \times \boldsymbol{w}'_2 = 0$. Note also that

$$\boldsymbol{z} \times \bar{h}(\boldsymbol{z} \times \boldsymbol{u}, \boldsymbol{y}) = \boldsymbol{z} \times \bar{h}(\boldsymbol{z} \times \boldsymbol{z} \times \boldsymbol{u}, \boldsymbol{z} \times \boldsymbol{y})$$

(619Ge)

$$= \boldsymbol{z} \times \bar{h}(\boldsymbol{z} \times \boldsymbol{u}, \boldsymbol{z} \times \boldsymbol{y}) = \boldsymbol{z} \times \bar{h}(\boldsymbol{u}, \boldsymbol{y}).$$

Consequently

$$\boldsymbol{z} \times ii_{\boldsymbol{v}}(\bar{h}(\boldsymbol{u}, \boldsymbol{y})) = \boldsymbol{z} \times ii_{\boldsymbol{z} \times \boldsymbol{v}}(\boldsymbol{z} \times \bar{h}(\boldsymbol{u}, \boldsymbol{y}))$$

(654Da again)

$$\begin{aligned} &= \boldsymbol{z} \times ii_{\boldsymbol{z} \times \boldsymbol{v}}(\boldsymbol{z} \times \bar{h}(\boldsymbol{z} \times \boldsymbol{u}, \boldsymbol{y})) = \boldsymbol{z} \times ii_{\boldsymbol{z} \times \boldsymbol{v}_1}(\boldsymbol{z} \times \bar{h}(\boldsymbol{z} \times \boldsymbol{u}_1, \boldsymbol{y})) \\ &= \boldsymbol{z} \times ii_{\boldsymbol{v}_1}(\bar{h}(\boldsymbol{u}_1, \boldsymbol{y})) = \boldsymbol{z} \times (\boldsymbol{u}_1 - \boldsymbol{u}_\star) = \boldsymbol{z} \times (\boldsymbol{u} - \boldsymbol{u}_\star), \end{aligned}$$

that is,

$$\boldsymbol{z} \times (\boldsymbol{u} - \boldsymbol{u}_\star - ii_{\boldsymbol{v}}(\bar{h}(\boldsymbol{u}, \boldsymbol{y}))) = 0.$$

Expressing $\boldsymbol{u}$ and $\boldsymbol{u}_\star$ as $\langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ and $\langle u_{\star\sigma} \rangle_{\sigma \in \mathcal{S}}$, this means that for any $\sigma \in \mathcal{S}$ we have

$$[\![\sigma < \max \mathcal{S}]\!] \subseteq [\![u_\sigma = u_{\star\sigma} + \int_{\mathcal{S} \wedge \sigma} \bar{h}(\boldsymbol{u}, \boldsymbol{y}) d\boldsymbol{v}]\!].$$

At the top end, we can calculate

$$\begin{aligned} u_{\max \mathcal{S}} &= u_{\star, \max \mathcal{S}} + \int_{\mathcal{S}} \bar{h}(\boldsymbol{u}_1, \boldsymbol{y}) d\boldsymbol{v}_1 + \int_{\mathcal{S}} \bar{h}(\boldsymbol{u}_1, \boldsymbol{y}) d\boldsymbol{w}'_2 \\ &= u_{\star, \max \mathcal{S}} + \int_{\mathcal{S}} \bar{h}(\boldsymbol{u}_1, \boldsymbol{y}) d\boldsymbol{v} = u_{\star, \max \mathcal{S}} + \int_{\mathcal{S}} \boldsymbol{z} \times \bar{h}(\boldsymbol{u}_1, \boldsymbol{y}) d\boldsymbol{v} \end{aligned}$$

(by the other part of 654Da)

$$= u_{\star, \max \mathcal{S}} + \int_{\mathcal{S}} \boldsymbol{z} \times \bar{h}(\boldsymbol{z} \times \boldsymbol{u}_1, \boldsymbol{y}) d\boldsymbol{v}$$

and, unwinding,

$$= u_{\star,\max\mathcal{S}} + \int_{\mathcal{S}} \boldsymbol{z} \times \bar{h}(\boldsymbol{z} \times \boldsymbol{u}, \boldsymbol{y}) d\boldsymbol{v}$$
$$= u_{\star,\max\mathcal{S}} + \int_{\mathcal{S}} \boldsymbol{z} \times \bar{h}(\boldsymbol{u}, \boldsymbol{y}) d\boldsymbol{v} = u_{\star,\max\mathcal{S}} + \int_{\mathcal{S}} \bar{h}(\boldsymbol{u}, \boldsymbol{y}) d\boldsymbol{v}.$$

But this means that, for any $\sigma \in \mathcal{S}$,

$$[\![\sigma = \max\mathcal{S}]\!] \subseteq [\![u_\sigma = u_{\star\sigma} + \int_{\mathcal{S}\wedge\sigma} \bar{h}(\boldsymbol{u}, \boldsymbol{y}) d\boldsymbol{v}]\!]$$

and $u_\sigma = u_{\star\sigma} + \int_{\mathcal{S}\wedge\sigma} \bar{h}(\boldsymbol{u}, d\boldsymbol{y}) d\boldsymbol{v}$. As σ is arbitrary, $\boldsymbol{u} = \boldsymbol{u}_\star + ii_{\boldsymbol{v}}(\bar{h}(\boldsymbol{u}, d\boldsymbol{y}))$.

(ii) To show uniqueness we can run the argument backwards, as follows. Suppose that $\tilde{\boldsymbol{u}}$ is a near-simple process such that $\tilde{\boldsymbol{u}} = \boldsymbol{u}_\star + ii_{\boldsymbol{v}}(\bar{h}(\tilde{\boldsymbol{u}}, \boldsymbol{y}))$. Set $\tilde{\boldsymbol{u}}' = \boldsymbol{u}_\star + ii_{\boldsymbol{v}_1}(\bar{h}(\tilde{\boldsymbol{u}}, \boldsymbol{y}))$. Then

$$\boldsymbol{z} \times \tilde{\boldsymbol{u}}' = \boldsymbol{z} \times \boldsymbol{u}_\star + \boldsymbol{z} \times ii_{\boldsymbol{z}\times\boldsymbol{v}_1}(\bar{h}(\tilde{\boldsymbol{u}}, \boldsymbol{y}))$$
$$= \boldsymbol{z} \times \boldsymbol{u}_\star + \boldsymbol{z} \times ii_{\boldsymbol{z}\times\boldsymbol{v}}(\bar{h}(\tilde{\boldsymbol{u}}, \boldsymbol{y})) = \boldsymbol{z} \times \tilde{\boldsymbol{u}}$$

so that

$$\boldsymbol{z} \times \tilde{\boldsymbol{u}}' = \boldsymbol{z} \times \boldsymbol{u}_\star + \boldsymbol{z} \times ii_{\boldsymbol{v}_1}(\bar{h}(\tilde{\boldsymbol{u}}, \boldsymbol{y})) = \boldsymbol{z} \times \boldsymbol{u}_\star + \boldsymbol{z} \times ii_{\boldsymbol{v}_1}(\bar{h}(\boldsymbol{z} \times \tilde{\boldsymbol{u}}, \boldsymbol{y}))$$
$$= \boldsymbol{z} \times \boldsymbol{u}_\star + \boldsymbol{z} \times ii_{\boldsymbol{v}_1}(\bar{h}(\boldsymbol{z} \times \tilde{\boldsymbol{u}}', \boldsymbol{y})) = \boldsymbol{z} \times \boldsymbol{u}_\star + \boldsymbol{z} \times ii_{\boldsymbol{v}_1}(\bar{h}(\tilde{\boldsymbol{u}}', \boldsymbol{y})),$$

while

$$u_{\star,\max\mathcal{S}} + \int_{\mathcal{S}} \bar{h}(\tilde{\boldsymbol{u}}, \boldsymbol{y}) d\boldsymbol{v}_1 = u_{\star,\max\mathcal{S}} + \int_{\mathcal{S}} \boldsymbol{z} \times \bar{h}(\tilde{\boldsymbol{u}}, \boldsymbol{y}) d\boldsymbol{v}_1$$
$$= u_{\star,\max\mathcal{S}} + \int_{\mathcal{S}} \boldsymbol{z} \times \bar{h}(\tilde{\boldsymbol{u}}', \boldsymbol{y}) d\boldsymbol{v}_1 = u_{\star,\max\mathcal{S}} + \int_{\mathcal{S}} \bar{h}(\tilde{\boldsymbol{u}}', \boldsymbol{y}) d\boldsymbol{v}_1.$$

As in (i) just above, this is enough to show that

$$\tilde{\boldsymbol{u}}' = \boldsymbol{u}_\star + ii_{\boldsymbol{v}_1}(\bar{h}(\tilde{\boldsymbol{u}}', \boldsymbol{y})),$$

so that $\tilde{\boldsymbol{u}}' = \boldsymbol{u}_1$, because we know from (c-ii) that we have a unique solution to the equation defining $\boldsymbol{u}_1$. Consequently

$$\boldsymbol{z} \times \tilde{\boldsymbol{u}} = \boldsymbol{z} \times \tilde{\boldsymbol{u}}' + \boldsymbol{z} \times ii_{\boldsymbol{w}_2'}(\bar{h}(\tilde{\boldsymbol{u}}, d\boldsymbol{y})) = \boldsymbol{z} \times \boldsymbol{u}_1 = \boldsymbol{z} \times \boldsymbol{u};$$

and once again

$$u_{\star,\max\mathcal{S}} + \int_{\mathcal{S}} \bar{h}(\tilde{\boldsymbol{u}}, \boldsymbol{y}) d\boldsymbol{v} = u_{\star,\max\mathcal{S}} + \int_{\mathcal{S}} \bar{h}(\boldsymbol{z} \times \tilde{\boldsymbol{u}}, \boldsymbol{y}) d\boldsymbol{v}$$
$$= u_{\star,\max\mathcal{S}} + \int_{\mathcal{S}} \bar{h}(\boldsymbol{z} \times \boldsymbol{u}, \boldsymbol{y}) d\boldsymbol{v} = u_{\star,\max\mathcal{S}} + \int_{\mathcal{S}} \bar{h}(\boldsymbol{u}, \boldsymbol{y}) d\boldsymbol{v},$$

so $\tilde{\boldsymbol{u}}$ and $\boldsymbol{u}$ agree at $\max\mathcal{S}$ and are equal. Thus the solution assembled in (i) is unique.

This completes the proof under the hypothesis (*).

654G Theorem Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$ with a least member. Suppose that $h : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and that $K \ge 0$ is such that $|h(\alpha, \beta) - h(\alpha', \beta)| \le K|\alpha - \alpha'|$ for all α, α', $\beta \in \mathbb{R}$. Let $\boldsymbol{v}$ be a locally near-simple local integrator with domain $\mathcal{S}$. Then for any locally moderately oscillatory processes $\boldsymbol{u}_\star$, $\boldsymbol{y}$ with domain $\mathcal{S}$ there is a unique locally moderately oscillatory process $\boldsymbol{u}$ with domain $\mathcal{S}$ such that

$$\boldsymbol{u} = \boldsymbol{u}_\star + ii_{\boldsymbol{v}}(\bar{h}(\boldsymbol{u}, \boldsymbol{y})).$$

proof (a) Let $\epsilon > 0$ be such that $2K(\sqrt{\epsilon} + 3\epsilon) < 1$. By the Bichteler-Dellacherie theorem (627J), $\boldsymbol{v}$ is a semi-martingale; by the Fundamental Theorem of Martingales (643M), we can express it as $\boldsymbol{w} + \boldsymbol{w}'$ where $\boldsymbol{w} = \langle w_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a local martingale, $\boldsymbol{w}' = \langle w'_\sigma\rangle_{\sigma\in\mathcal{S}}$ is locally of bounded variation and $\mathrm{Osclln}(\boldsymbol{w}\restriction\mathcal{S}\wedge\tau) \le \epsilon\chi 1$ for every $\tau \in \mathcal{S}$.

(b) For the time being (down to the end of (f) below), suppose that $\mathcal{S}$ has a greatest member and that $\boldsymbol{w}$ is a martingale. Then $\boldsymbol{u}_\star$ and $\boldsymbol{y}$ are moderately oscillatory, $\boldsymbol{v}$ is a near-simple integrator and $\operatorname{Osclln}(\boldsymbol{w}) \le \epsilon\chi 1$. Express $\boldsymbol{u}_\star$ as $\langle u_{\star\sigma}\rangle_{\sigma\in\mathcal{S}}$.

We know that $\boldsymbol{w}$ is locally near-simple (632Ia), therefore near-simple, so $\boldsymbol{w}'$ also is near-simple, and its cumulative variation $\boldsymbol{w}'^{\uparrow} = \langle w'^{\uparrow}_\sigma\rangle_{\sigma\in\mathcal{S}}$ is near-simple (631K). Let $\boldsymbol{w}^* = \langle w^*_\sigma\rangle_{\sigma\in\mathcal{S}}$ be the quadratic variation of $\boldsymbol{w}$; this too is near-simple (631Ja).

(c) By 631Ra, there is a non-decreasing sequence $\langle\tau_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{S}$ such that $\tau_0 = \min\mathcal{S}$, $\inf_{n\in\mathbb{N}}[\![\tau_n < \max\mathcal{S}]\!] = 0$ and whenever $n \in \mathbb{N}$ and $\sigma \in [\tau_n, \tau_{n+1}]$,

$$[\![|w'^{\uparrow}_\sigma + w^*_\sigma - w'^{\uparrow}_{\tau_n} - w^*_{\tau_n}| \ge \epsilon]\!] \subseteq [\![\sigma = \tau_{n+1}]\!].$$

(d) For $n \in \mathbb{N}$, write $\mathcal{S}_n = \mathcal{S} \wedge \tau_n$. Then we have a sequence $\langle\boldsymbol{u}_n\rangle_{n\in\mathbb{N}}$ of moderately oscillatory processes such that

$$\boldsymbol{u}_n = \boldsymbol{u}_\star{\restriction}\mathcal{S}_n + ii_{\boldsymbol{v}\restriction\mathcal{S}_n}(\bar h(\boldsymbol{u}_n, \boldsymbol{y}{\restriction}\mathcal{S}_n))$$

for each n. **P** Induce on n. The induction starts with $\boldsymbol{u}_0 = \boldsymbol{u}_\star{\restriction}\mathcal{S}_0$ where $\mathcal{S}_0 = \{\min\mathcal{S}\}$. For the inductive step to $n+1$, given $\boldsymbol{u}_n = \langle u_{n\sigma}\rangle_{\sigma\in\mathcal{S}_n}$, set

$$\tilde{\mathcal{S}}_n = [\tau_n, \tau_{n+1}], \quad \tilde{\boldsymbol{u}}_{\star n} = \boldsymbol{u}_\star{\restriction}\tilde{\mathcal{S}}_n + (u_{n\tau_n} - u_{\star\tau_n})\mathbf{1}^{(\tilde{\mathcal{S}}_n)},$$

$$\tilde{\boldsymbol{w}} = \boldsymbol{w}{\restriction}\tilde{\mathcal{S}}_n - w_{\tau_n}\mathbf{1}^{(\tilde{\mathcal{S}}_n)}, \quad \tilde{\boldsymbol{w}}' = \boldsymbol{w}'{\restriction}\tilde{\mathcal{S}}_n - w'_{\tau_n}\mathbf{1}^{(\tilde{\mathcal{S}}_n)}, \quad \tilde{\boldsymbol{v}} = \tilde{\boldsymbol{w}} + \tilde{\boldsymbol{w}}'.$$

Then $\tilde{\boldsymbol{w}}$ and $\tilde{\boldsymbol{w}}'$ are near-simple processes with domain $\tilde{\mathcal{S}}_n$ (631Fa), $\tilde{\boldsymbol{w}}$ is a martingale (622Db, 622Ea), $\tilde{\boldsymbol{w}}'$ is of bounded variation (614Lc) and both start with value 0 at $\min\tilde{\mathcal{S}}_n = \tau_n$. The quadratic variation $\tilde{\boldsymbol{w}}^* = \langle\tilde w^*_\sigma\rangle_{\sigma\in\tilde{\mathcal{S}}_n}$ of $\tilde{\boldsymbol{w}}$ is $\boldsymbol{w}^*{\restriction}\tilde{\mathcal{S}}_n - w^*_{\tau_n}\mathbf{1}^{(\tilde{\mathcal{S}}_n)}$ (use both halves of 617Kb), and the cumulative variation $\tilde{\boldsymbol{w}}'^{\uparrow} = \langle\tilde w'^{\uparrow}_\sigma\rangle_{\sigma\in\tilde{\mathcal{S}}_n}$ of $\tilde{\boldsymbol{w}}'$ is $\boldsymbol{w}'^{\uparrow}{\restriction}\tilde{\mathcal{S}}_n - w'^{\uparrow}_{\tau_n}\mathbf{1}^{(\tilde{\mathcal{S}}_n)}$ (614Pb).

We know that, for $\sigma \in \tilde{\mathcal{S}}_n$,

$$[\![\sigma < \tau_{n+1}]\!] \subseteq [\![|w'^{\uparrow}_\sigma - w'^{\uparrow}_{\tau_n}| < \epsilon]\!] \subseteq [\![|\tilde w'^{\uparrow}_\sigma| \le \epsilon]\!],$$

so 654Dd tells us that $\|\sup|\boldsymbol{z}\times\tilde{\boldsymbol{w}}'^{\uparrow}|\|_\infty \le \epsilon$, where $\boldsymbol{z} = \langle\chi[\![\sigma < \tau_{n+1}]\!]\rangle_{\sigma\in\tilde{\mathcal{S}}_n}$. Similarly,

$$[\![\sigma < \tau_{n+1}]\!] \subseteq [\![|w^*_\sigma - w^*_{\tau_n}| < \epsilon]\!] \subseteq [\![|\tilde w^*_\sigma| \le \epsilon]\!],$$

and by the other half of 654Dd,

$$\|\tilde{\boldsymbol{w}}^*\|_\infty \le \epsilon + \|\operatorname{Osclln}(\tilde{\boldsymbol{w}}^*)\|_\infty = \epsilon + \|(\operatorname{Osclln}(\tilde{\boldsymbol{w}}))^2\|_\infty$$

(618Sb)

$$= \epsilon + \|\operatorname{Osclln}(\tilde{\boldsymbol{w}})\|_\infty^2 = \epsilon + \|\operatorname{Osclln}(\boldsymbol{w}{\restriction}\tilde{\mathcal{S}}_n)\|_\infty^2 \le \epsilon + \|\operatorname{Osclln}(\boldsymbol{w})\|_\infty^2 \le \epsilon + \epsilon^2.$$

Accordingly

$$2K\bigl(\sqrt{\|\tilde{\boldsymbol{w}}^*\|_\infty} + 2\|\boldsymbol{z}\times\tilde{\boldsymbol{w}}'^{\uparrow}\|_\infty\bigr) \le 2K(\sqrt{\epsilon+\epsilon^2} + 2\epsilon) \le 2K(\sqrt{\epsilon} + 3\epsilon) < 1$$

by the choice of ϵ. We can therefore apply 654F to see that there is a moderately oscillatory process $\tilde{\boldsymbol{u}}_n$ with domain $\tilde{\mathcal{S}}_n$ such that $\tilde{\boldsymbol{u}}_n = \tilde{\boldsymbol{u}}_{\star n} + ii_{\tilde{\boldsymbol{v}}}(\bar h(\tilde{\boldsymbol{u}}_n, \tilde{\boldsymbol{y}}))$.

Since the processes $\boldsymbol{u}_n$ and $\tilde{\boldsymbol{u}}_n$ take the same value at τ_n, they have a common extension $\boldsymbol{u}_n \cup \tilde{\boldsymbol{u}}_n$ to $\mathcal{S}_n \cup \tilde{\mathcal{S}}_n$, which is a covering sublattice of $\mathcal{S}_{n+1}$ (if $\sigma \in \mathcal{S}_{n+1} = [\tau_0, \tau_{n+1}]$ then $\{\sigma\}$ is covered, in the sense of 611M, by $\{\sigma\wedge\tau_n, \sigma\vee\tau_n\} \subseteq \mathcal{S}_n \cup \tilde{\mathcal{S}}_n$). We therefore have a fully adapted process $\boldsymbol{u}_{n+1}$ with domain $\mathcal{S}_{n+1}$ extending $\boldsymbol{u}_n \cup \tilde{\boldsymbol{u}}_n$. By 615F(a-v), $\boldsymbol{u}_n \cup \tilde{\boldsymbol{u}}_n$ is moderately oscillatory; by 615F(a-vi), $\boldsymbol{u}_{n+1}$ is moderately oscillatory. Now if $\sigma \in \mathcal{S}_{n+1}$, we have

$$\int_{(\mathcal{S}_{n+1}\wedge\sigma)\wedge\tau_n} \bar h(\boldsymbol{u}_{n+1}, \boldsymbol{y})d\boldsymbol{v} = \int_{\mathcal{S}_n\wedge(\sigma\wedge\tau_n)} \bar h(\boldsymbol{u}_n, \boldsymbol{y}{\restriction}\mathcal{S}_n)d(\boldsymbol{v}{\restriction}\mathcal{S}_n) = u_{n,\sigma\wedge\tau_n} - u_{\star,\sigma\wedge\tau_n},$$

$$\int_{(\mathcal{S}_{n+1}\wedge\sigma)\vee\tau_n} \bar h(\boldsymbol{u}_{n+1}, \boldsymbol{y})d\boldsymbol{v} = \int_{\tilde{\mathcal{S}}_n\wedge(\sigma\vee\tau_n)} \bar h(\tilde{\boldsymbol{u}}_n, \tilde{\boldsymbol{y}})d\tilde{\boldsymbol{v}} = \tilde u_{\sigma\vee\tau_n} - u_{\star,\sigma\vee\tau_n} - u_{n\tau_n} + u_{\star\tau_n}.$$

By 613J(c-i),

$$\begin{aligned}\int_{\mathcal{S}_{n+1}\wedge\sigma}\bar{h}(\boldsymbol{u}_{n+1},\boldsymbol{y})d\boldsymbol{v} &= u_{n,\sigma\wedge\tau_n}-u_{\star,\sigma\wedge\tau_n}+\tilde{u}_{\sigma\vee\tau_n}-u_{\star,\sigma\vee\tau_n}-u_{n\tau_n}+u_{\star\tau_n}\\ &= u_{n+1,\sigma\wedge\tau_n}+u_{n+1,\sigma\vee\tau_n}-u_{n+1,\tau_n}-u_{\star,\sigma\wedge\tau_n}-u_{\star,\sigma\vee\tau_n}+u_{\star\tau_n}\\ &= u_{n+1,\sigma}-u_{\star\sigma}\end{aligned}$$

by 612D(f-i). As σ is arbitrary,

$$\boldsymbol{u}_{n+1}=\boldsymbol{u}_\star\restriction\mathcal{S}_{n+1}+ii_{\boldsymbol{v}}(\bar{h}(\boldsymbol{u}_{n+1},\boldsymbol{y}))$$

and the induction continues. **Q**

(e) At the end of the induction set $\mathcal{S}_\infty=\bigcup_{n\in\mathbb{N}}\mathcal{S}_n$; as $\boldsymbol{u}_{n+1}$ extends $\boldsymbol{u}_n$ for every n, we have a process $\boldsymbol{u}_\infty=\bigcup_{n\in\mathbb{N}}\boldsymbol{u}_n$ with domain $\mathcal{S}_\infty$. If σ, $\sigma'\in\mathcal{S}_\infty$, there is an $n\in\mathbb{N}$ such that $\mathcal{S}_n$ contains both, so $\boldsymbol{u}_\infty$ is fully adapted. It is locally moderately oscillatory because if $\sigma\in\mathcal{S}_\infty$ there is an $n\in\mathbb{N}$ such that $\sigma\in\mathcal{S}_n$ and $\boldsymbol{u}_\infty\restriction\mathcal{S}_\infty\wedge\sigma=\boldsymbol{u}_n\restriction\mathcal{S}_n\wedge\sigma$ is near-simple. Because $\tau_n\in\mathcal{S}_\infty$ and

$$\inf_{n\in\mathbb{N}}\sup_{\sigma\in\mathcal{S}_\infty}[\![\tau_n<\sigma]\!]\subseteq\inf_{n\in\mathbb{N}}[\![\tau_n<\max\mathcal{S}]\!]=0,$$

$\boldsymbol{u}_\infty$ is moderately oscillatory (654Bb).

Again because $\inf_{n\in\mathbb{N}}[\![\tau_n<\max\mathcal{S}]\!]=0$, $\mathcal{S}_\infty$ covers $\mathcal{S}$, so $\mathcal{S}_\infty$ and $\mathcal{S}$ have the same covered envelope, and $\boldsymbol{u}_\infty$ has an extension to a process $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ with domain $\mathcal{S}$ (612Qa). Because $\boldsymbol{u}\restriction\mathcal{S}_\infty$ is locally moderately oscillatory, $\boldsymbol{u}$ is locally moderately oscillatory (615F(b-v)), therefore moderately oscillatory.

Now take any $n\in\mathbb{N}$ and $\sigma\in\mathcal{S}_n$. Then

$$u_\sigma=u_{n\sigma}=u_{\star\sigma}+\int_{\mathcal{S}_n\wedge\sigma}\bar{h}(\boldsymbol{u}_n,\boldsymbol{y})d\boldsymbol{v}=u_{\star\sigma}+\int_{\mathcal{S}\wedge\sigma}\bar{h}(\boldsymbol{u},\boldsymbol{y})d\boldsymbol{v}.$$

So $\boldsymbol{u}$ agrees with $\boldsymbol{u}_\star+ii_{\boldsymbol{v}}(\bar{h}(\boldsymbol{u},\boldsymbol{v}))$ on $\mathcal{S}_\infty$; as $\mathcal{S}_\infty$ covers $\mathcal{S}$, $\boldsymbol{u}=\boldsymbol{u}_\star+ii_{\boldsymbol{v}}(\bar{h}(\boldsymbol{u},\boldsymbol{v}))$.

(f) Now suppose that $\boldsymbol{u}'=\langle u'_\sigma\rangle_{\sigma\in\mathcal{S}}$ is another moderately oscillatory process such that $\boldsymbol{u}'=\boldsymbol{u}_\star+ii_{\boldsymbol{v}}(\bar{h}(\boldsymbol{u}',\boldsymbol{y}))$. Then $\boldsymbol{u}'\restriction\mathcal{S}_n=\boldsymbol{u}_n$ for every $n\in\mathbb{N}$. **P** Induce on n. If $n=0$ then $\mathcal{S}_n=\{\min\mathcal{S}\}$ and both $\boldsymbol{u}'$ and $\boldsymbol{u}_\star$ take the value $u_{\star,\min\mathcal{S}}$ at $\min\mathcal{S}$. For the inductive step to $n+1$, the inductive hypothesis tells us that $u'_{\tau_n}=u_{n\tau_n}$. Now, for $\sigma\in\tilde{\mathcal{S}}_n=[\tau_n,\tau_{n+1}]$,

$$\begin{aligned}u'_\sigma &= u_{\star\sigma}+\int_{\mathcal{S}\wedge\sigma}\bar{h}(\boldsymbol{u}',\boldsymbol{y})d\boldsymbol{v}\\ &= u_{\star\sigma}+\int_{(\mathcal{S}\wedge\sigma)\wedge\tau_n}\bar{h}(\boldsymbol{u}',\boldsymbol{y})d\boldsymbol{v}+\int_{(\mathcal{S}\wedge\sigma)\vee\tau_n}\bar{h}(\boldsymbol{u}',\boldsymbol{y})d\boldsymbol{v}\\ &= u_{\star\sigma}+u_{n\tau_n}-u_{\star\tau_n}+\int_{\tilde{\mathcal{S}}_n\wedge\sigma}\bar{h}(\boldsymbol{u}',\boldsymbol{y})d\boldsymbol{v}.\end{aligned}$$

As σ is arbitrary,

$$\boldsymbol{u}'\restriction\tilde{\mathcal{S}}_n=\boldsymbol{u}_\star\restriction\tilde{\mathcal{S}}_n+(u_{n\tau_n}-u_{\star\tau_n})\mathbf{1}^{\tilde{\mathcal{S}}_n}+ii_{\boldsymbol{v}\restriction\tilde{\mathcal{S}}_n}(\bar{h}(\boldsymbol{u}'\restriction\tilde{\mathcal{S}}_n,\boldsymbol{y}\restriction\tilde{\mathcal{S}}_n)).$$

But 654F assured us that this equation had a unique solution, so $\boldsymbol{u}'\restriction\tilde{\mathcal{S}}_n$ must be identical with $\tilde{\boldsymbol{u}}_n$ as found in (c), and therefore agrees with $\boldsymbol{u}_{n+1}$ and with $\boldsymbol{u}$ on $\tilde{\mathcal{S}}_n$. As $\mathcal{S}_n\cup\tilde{\mathcal{S}}_n$ covers $\mathcal{S}_{n+1}$, $\boldsymbol{u}'$ agrees with $\boldsymbol{u}$ on $\mathcal{S}_{n+1}$. So the induction continues. **Q**

Accordingly $\boldsymbol{u}'$ agrees with $\boldsymbol{u}$ on $\mathcal{S}_\infty$ and therefore on its full envelope $\mathcal{S}$. Thus $\boldsymbol{u}$ is uniquely defined.

(g) So we have the result in the special case in which $\mathcal{S}$ has a greatest member and $\boldsymbol{w}$ is a martingale. For the general case, let $\mathcal{S}'$ be a covering ideal of $\mathcal{S}$ such that $\boldsymbol{w}\restriction\mathcal{S}'$ is a martingale. Then (b)-(f) tell us that for every $\tau\in\mathcal{S}'$ there is a unique near-simple process $\boldsymbol{u}_\tau$ with domain $\mathcal{S}\wedge\tau$ such that $\boldsymbol{u}_\tau=\boldsymbol{u}_\star\restriction\mathcal{S}\wedge\tau+ii_{\boldsymbol{v}\restriction\mathcal{S}\wedge\tau}(\bar{h}(\boldsymbol{u}_\tau,\boldsymbol{y}\restriction\mathcal{S}\wedge\tau))$. If $\tau\le\tau'$ in $\mathcal{S}'$ then $\boldsymbol{u}_{\tau'}\restriction\mathcal{S}\wedge\tau$ satisfies the same equation as $\boldsymbol{u}_\tau$, so must be equal to $\boldsymbol{u}_\tau$. As in (e), with $\mathcal{S}'$ in place of $\mathcal{S}_\infty$, there is therefore a fully adapted process $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ with domain $\mathcal{S}$ extending every $\boldsymbol{u}_\tau$, and such that $\boldsymbol{u}=\boldsymbol{u}_\star+ii_{\boldsymbol{v}}(\bar{h}(\boldsymbol{u},d\boldsymbol{y}))$. Because $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau=\boldsymbol{u}_\tau$ is near-simple for every $\tau\in\mathcal{S}'$, $\boldsymbol{u}$ is locally near-simple (631F(b-v)). And as in (e), we see that if $\boldsymbol{u}'$ has the same property, then $\boldsymbol{u}'\restriction\mathcal{S}_\tau=\boldsymbol{u}_\tau$ for every $\tau\in\mathcal{S}'$, so that $\boldsymbol{u}'=\boldsymbol{u}$.

This completes the proof.

654H Lemma Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous, and that $\mathcal{S}=[\min\mathcal{S},\max\mathcal{S}]$ is an interval in $\mathcal{T}$. Define $\boldsymbol{z}=\langle z_\sigma\rangle_{\sigma\in\mathcal{S}}$ by setting $z_\sigma=\chi[\![\sigma<\max\mathcal{S}]\!]$ for $\sigma\in\mathcal{S}$. Suppose that $\boldsymbol{x}\in M_{\text{S-i}}=M_{\text{S-i}}(\mathcal{S})$, and that $\boldsymbol{v}\in M_{\text{n-s}}=M_{\text{n-s}}(\mathcal{S})$ is an integrator.

(a) $\boldsymbol{z}\times\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})=\boldsymbol{z}\times\mathrm{S}ii_{\boldsymbol{z}\times\boldsymbol{v}}(\boldsymbol{x})$.

(b) $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})_<=\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{z}\times\boldsymbol{x})_<$.

proof (a) For any $\sigma\in\mathcal{S}$,

$$[\![\textstyle\oint_{\mathcal{S}\wedge\sigma}\boldsymbol{x}\,d\boldsymbol{v}\ne\oint_{\mathcal{S}\wedge\sigma}\boldsymbol{x}\,d(\boldsymbol{z}\times\boldsymbol{v})]\!]\subseteq[\![\boldsymbol{v}\restriction\mathcal{S}\wedge\sigma\ne(\boldsymbol{z}\times\boldsymbol{v})\restriction\mathcal{S}\wedge\sigma]\!]$$

(646C)

$$\subseteq[\![\max\mathcal{S}\le\sigma]\!]\subseteq[\![\chi[\![\sigma<\max\mathcal{S}]\!]=0]\!].$$

So $\boldsymbol{z}\times(\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})-\mathrm{S}ii_{\boldsymbol{z}\times\boldsymbol{v}}(\boldsymbol{x}))=0$ and we have the result.

(b) For $\sigma\in\mathcal{S}$, set $u_\sigma=\oint_{\mathcal{S}\wedge\sigma}\boldsymbol{x}\,d\boldsymbol{v}$ and $u'_\sigma=\oint_{\mathcal{S}\wedge\sigma}\boldsymbol{z}\times\boldsymbol{x}\,d\boldsymbol{v}$. Then

$$\begin{aligned}[\![z_\sigma\times u_\sigma\ne z_\sigma\times u'_\sigma]\!]&=[\![z_\sigma=\chi1]\!]\cap[\![u_\sigma\ne u'_\sigma]\!]\\&\subseteq[\![\sigma<\max\mathcal{S}]\!]\cap[\![\boldsymbol{x}\restriction\mathcal{S}\wedge\sigma\ne(\boldsymbol{z}\times\boldsymbol{x})\restriction\mathcal{S}\wedge\sigma]\!]\end{aligned}$$

(647J)

$$\begin{aligned}&\subseteq[\![\sigma<\max\mathcal{S}]\!]\cap\sup_{\sigma'\in\mathcal{S}\wedge\sigma}[\![z_{\sigma'}=0]\!]\\&=[\![\sigma<\max\mathcal{S}]\!]\cap\sup_{\sigma'\in\mathcal{S}\wedge\sigma}[\![\sigma'=\max\mathcal{S}]\!]=0.\end{aligned}$$

So $z_\sigma\times u_\sigma=z_\sigma\times u'_\sigma$; as σ is arbitrary, $\boldsymbol{z}\times\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})=\boldsymbol{z}\times\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{z}\times\boldsymbol{x})$. Putting this together with 654De,

$$\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})_<=(\boldsymbol{z}\times\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x}))_<=(\boldsymbol{z}\times\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{z}\times\boldsymbol{x}))_<=\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{z}\times\boldsymbol{x})_<.$$

654I Now for an S-integral version of 654E.

Lemma Suppose $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous and that $\mathcal{S}=[\min\mathcal{S},\max\mathcal{S}]$ is an interval in $\mathcal{T}$. Let $M^0_{\text{n-s}}$ be the space of near-simple processes $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ with domain $\mathcal{S}$ such that $u_{\min\mathcal{S}}=0$. For a near-simple integrator $\boldsymbol{v}$, write $\boldsymbol{v}^*$ for its quadratic variation. Suppose that $\boldsymbol{w}$, $\boldsymbol{w}'\in M^0_{\text{n-s}}$ are such that $\boldsymbol{w}$ is a martingale and $\boldsymbol{w}'$ is of bounded variation; set $\boldsymbol{v}=\boldsymbol{w}+\boldsymbol{w}'$. If $\boldsymbol{x}\in M^0_{\text{S-i}}=M^0_{\text{S-i}}(\mathcal{S})$, $\boldsymbol{u}\in M_{\text{mo}}=M_{\text{mo}}(\mathcal{S})$ and $|\boldsymbol{x}|\le\boldsymbol{u}_<$, then

$$\|\sup|\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})|\|_2\le2(\sqrt{\|\boldsymbol{w}^*\|_\infty}+\|\textstyle\int_{\mathcal{S}}|d\boldsymbol{w}'|\|_\infty)\|\sup|\boldsymbol{u}|\|_2.$$

proof (a) Write γ for $2(\sqrt{\|\boldsymbol{w}^*\|_\infty}+\|\int_{\mathcal{S}}|d\boldsymbol{w}'|\|_\infty)\|\sup|\boldsymbol{u}|\|_2$. If this is infinite, we can stop, so suppose that γ is finite.

The set $\{u:u\in L^2,\ \|u\|_2\le\gamma\}$ is closed in L^0 for the topology of convergence in measure (613Bc). Next, $\boldsymbol{y}\mapsto\sup|\boldsymbol{y}|:M_{\text{o-b}}\to L^0$ is continuous for the ucp topology on $M_{\text{o-b}}=M_{\text{o-b}}(\mathcal{S})$ and the topology of convergence in measure on L^0 (615C(b-ii)), so $\{\boldsymbol{y}:\|\sup|\boldsymbol{y}|\|_2\le\gamma\}$ is closed in $M_{\text{o-b}}$ for the ucp topology.

(b) By 646O, $\mathrm{S}ii_{\boldsymbol{v}}$ is continuous on $A=\{\boldsymbol{x}:\boldsymbol{x}\in M^0_{\text{S-i}},\ |\boldsymbol{x}|\le\boldsymbol{u}_<\}$ for the S-integration topology on A and the ucp topology on $M_{\text{o-b}}$, so

$$C=\{\boldsymbol{x}':\boldsymbol{x}'\in A,\ \|\sup|\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x}')|\|_2\le\gamma\}$$

is relatively closed in A for the S-integration topology. We know also that, writing A_0 for $\{\boldsymbol{u}':\boldsymbol{u}'\in M_{\text{mo}},\ |\boldsymbol{u}'|\le\boldsymbol{u}\}$, $\{\boldsymbol{u}'_<:\boldsymbol{u}'\in A_0\}$ is dense in A (645La), $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{u}'_<)=ii_{\boldsymbol{v}}(\boldsymbol{u}')$ for every $\boldsymbol{u}'\in A_0$ (646Kc), and that

$$\|\sup|ii_{\boldsymbol{v}}(\boldsymbol{u}')|\|_2\le2(\sqrt{\|\boldsymbol{w}^*\|_\infty}+\|\textstyle\int_{\mathcal{S}}|d\boldsymbol{w}'|\|_\infty)\|\sup|\boldsymbol{u}'|\|_2\le\gamma$$

for every $\boldsymbol{u}'\in A_0$ (654E). So $\boldsymbol{u}'_<\in C$ for every $\boldsymbol{u}'\in A_0$, and C is dense in A. But this means that C is actually equal to A; consequently $\boldsymbol{x}\in C$ and $\|\sup|\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})|\|_2\le\gamma$, as claimed.

654J Lemma Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}=[\min\mathcal{S},\max\mathcal{S}]$ be an interval in $\mathcal{T}$. Suppose that $h:\mathbb{R}^2\to\mathbb{R}$ is a locally bounded Borel measurable function and that $K\ge 0$ is such that $|h(\alpha,\beta)-h(\alpha',\beta)|\le K|\alpha-\alpha'|$ for all α, α', $\beta\in\mathbb{R}$; let $\boldsymbol{w}=\langle w_\sigma\rangle_{\sigma\in\mathcal{S}}$, $\boldsymbol{w}'=\langle w'_\sigma\rangle_{\sigma\in\mathcal{S}}$ be near-simple processes with domain $\mathcal{S}$ such that $\boldsymbol{w}$ is a martingale, $\boldsymbol{w}'$ is of bounded variation and $w_{\min\mathcal{S}}=w'_{\min\mathcal{S}}=0$. Write $\boldsymbol{w}^*$ for the quadratic variation of $\boldsymbol{w}$, $\boldsymbol{w}'^{\uparrow}$ for the cumulative variation of $\boldsymbol{w}'$, and $\boldsymbol{z}$ for $\langle\chi[\![\sigma<\max\mathcal{S}]\!]\rangle_{\sigma\in\mathcal{S}}$. Suppose that $2K(\sqrt{\|\boldsymbol{w}^*\|_\infty}+2\|\boldsymbol{z}\times\boldsymbol{w}'^{\uparrow}\|_\infty)<1$. Set $\boldsymbol{v}=\boldsymbol{w}+\boldsymbol{w}'$. Then for any $\boldsymbol{x}_\star$, $\boldsymbol{y}\in M_{\text{S-i}}=M_{\text{S-i}}(\mathcal{S})$ there is a unique process $\boldsymbol{x}\in M_{\text{S-i}}$ such that

$$\boldsymbol{x}=\boldsymbol{x}_\star+\mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x},\boldsymbol{y}))_<.$$

Remark I ought to say at once that by $\mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x},\boldsymbol{y}))_<$ I mean the previsible version $(\mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x},\boldsymbol{y})))_<$ of the indefinite S-integral $\mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x},\boldsymbol{y}))$. I think it looks better without the extra brackets.

proof We can follow the same line of argument as in 654F.

(a) Suppose that $\|\boldsymbol{x}_\star\|_\infty$ and $\|\boldsymbol{y}\|_\infty$ are both finite, and that $2K\gamma<1$ where $\gamma=\sqrt{\|\boldsymbol{w}^*\|_\infty}+\|\int_{\mathcal{S}}|d\boldsymbol{w}'|\|_\infty$.

(i) Define $\langle\boldsymbol{x}_n\rangle_{n\in\mathbb{N}}$, $\langle\boldsymbol{u}_n\rangle_{n\in\mathbb{N}}$ by saying that $\boldsymbol{x}_0=\boldsymbol{x}_\star$ and

$$\boldsymbol{u}_n=\mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x}_n,\boldsymbol{y})),\quad \boldsymbol{x}_{n+1}=\boldsymbol{x}_\star+\boldsymbol{u}_{n<}$$

for $n\in\mathbb{N}$. Then $\bar h(\boldsymbol{x}_n,\boldsymbol{y})\in M_{\text{S-i}}$ (645Jb) and $\boldsymbol{u}_n$ belongs to $M_{\text{n-s}}=M_{\text{n-s}}(\mathcal{S})$ (646N), so $\boldsymbol{x}_{n+1}\in M_{\text{S-i}}$, for any $n\ge 0$.

For $n\ge 0$,

$$|\bar h(\boldsymbol{x}_{n+1},\boldsymbol{y})-\bar h(\boldsymbol{x}_n,\boldsymbol{y})|\le K|\boldsymbol{x}_{n+1}-\boldsymbol{x}_n|=K|(\boldsymbol{u}_n-\boldsymbol{u}_{n-1})_<|$$

(counting $\boldsymbol{u}_{-1}$ as $\mathbf{0}$)

$$=|K(\boldsymbol{u}_n-\boldsymbol{u}_{n-1})|_<$$

(641Gd), so

$$\begin{aligned}\|\sup|\boldsymbol{u}_{n+1}-\boldsymbol{u}_n|\|_2&=\|\sup|\mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x}_{n+1},\boldsymbol{y})-\bar h(\boldsymbol{x}_n,\boldsymbol{y}))|\|_2\\&\le 2\gamma\|\sup|K(\boldsymbol{u}_n-\boldsymbol{u}_{n-1})|\|_2\end{aligned}$$

(654I)

$$=2\gamma K\|\sup|\boldsymbol{u}_n-\boldsymbol{u}_{n-1}|\|_2$$

for $n\in\mathbb{N}$. At the beginning of the iteration, because h is locally bounded and both $\boldsymbol{x}_\star$ and $\boldsymbol{y}$ are bounded, $K'=\|\bar h(\boldsymbol{x}_\star,\boldsymbol{y})\|_\infty$ is finite, $|\bar h(\boldsymbol{x}_\star,\boldsymbol{y})\times\mathbf{1}^{(\mathcal{S})}_<|\le K'\mathbf{1}^{(\mathcal{S})}_<$ and

$$\begin{aligned}\|\sup|\boldsymbol{u}_0-\boldsymbol{u}_{-1}|\|_2&=\|\sup|\mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x}_\star,\boldsymbol{y}))|\|_2\\&=\|\sup|\mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x}_\star,\boldsymbol{y})\times\mathbf{1}^{(\mathcal{S})}_<)|\|_2\le 2\gamma\|K'\chi 1\|_2=2K'\gamma\end{aligned}$$

is finite. As $2K\gamma<1$, $\sum_{n=0}^\infty\|\sup|\boldsymbol{u}_{n+1}-\boldsymbol{u}_n|\|_2$ is finite and $\|\sup|u_n|\|_2<\infty$ for every n.

(ii) As in 654F, it follows that $\langle\boldsymbol{u}_n\rangle_{n\in\mathbb{N}}$ is Cauchy for the ucp uniformity, and has a limit $\boldsymbol{u}$ in $M_{\text{n-s}}$. This time, we need to check that $\lim_{n\to\infty}\|\sup|\boldsymbol{u}_n-\boldsymbol{u}|\|_2=0$. **P** For any n,

$$\sup|\boldsymbol{u}-\boldsymbol{u}_n|\le\sum_{i=n+1}^\infty\sup|u_i-u_{i-1}|,$$

$$\|\sup|\boldsymbol{u}-\boldsymbol{u}_n|\|_2\le\sum_{i=n+1}^\infty\|\sup|u_i-u_{i-1}|\|_2\to\infty$$

as $n\to\infty$. **Q** In particular, $\sup|\boldsymbol{u}|$ is square-integrable.

Set $\boldsymbol{x}=\boldsymbol{x}_\star+\boldsymbol{u}_<$. For $n\ge 0$, $\boldsymbol{x}-\boldsymbol{x}_{n+1}=(\boldsymbol{u}-\boldsymbol{u}_n)_<$, so

$$|\bar h(\boldsymbol{x},\boldsymbol{y})-\bar h(\boldsymbol{x}_{n+1},\boldsymbol{y})|\le K|\boldsymbol{u}-\boldsymbol{u}_n|_<,$$

$$\begin{aligned}\|\sup|\mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x},\boldsymbol{y})))-\boldsymbol{u}_{n+1}|\|_2&=\|\sup|\mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x},\boldsymbol{y})-\bar h(\boldsymbol{x}_{n+1},\boldsymbol{y}))|\|_2\\&\le 2K\gamma\|\sup|\boldsymbol{u}-\boldsymbol{u}_n|\|_2\to 0\end{aligned}$$

as $n \to \infty$. Accordingly, taking limits for the ucp topology,

$$\boldsymbol{x} - \boldsymbol{x}_\star = \boldsymbol{u}_< = (\lim_{n\to\infty} \boldsymbol{u}_n)_< = \mathrm{S}ii_{\boldsymbol{v}}(\bar{h}(\boldsymbol{x}, \boldsymbol{y}))_<$$

and $\boldsymbol{x} = \boldsymbol{x}_\star + \mathrm{S}ii_{\boldsymbol{v}}(\bar{h}(\boldsymbol{x}, \boldsymbol{y}))_<$.

(b) Once again assume that $2K\gamma < 1$ where $\gamma = \sqrt{\|\boldsymbol{w}^*\|_\infty} + \|\int_{\mathcal{S}} |d\boldsymbol{w}'|\|_\infty$. Suppose that

$$\boldsymbol{x}_{0\star},\, \boldsymbol{x}_{1\star},\, \boldsymbol{y}_0,\, \boldsymbol{y}_1,\, \boldsymbol{x}_0,\, \boldsymbol{x}_1 \in M_{\text{S-i}}$$

are such that

$$\boldsymbol{x}_0 = \boldsymbol{x}_{0\star} + \mathrm{S}ii_{\boldsymbol{v}}(\bar{h}(\boldsymbol{x}_0, \boldsymbol{y}_0))_<, \quad \boldsymbol{x}_1 = \boldsymbol{x}_{1\star} + \mathrm{S}ii_{\boldsymbol{v}}(\bar{h}(\boldsymbol{x}_1, \boldsymbol{y}_1))_<.$$

Set

$$\boldsymbol{u}_0 = \mathrm{S}ii_{\boldsymbol{v}}(\bar{h}(\boldsymbol{x}_0, \boldsymbol{y}_0)), \quad \boldsymbol{u}_1 = \mathrm{S}ii_{\boldsymbol{v}}(\bar{h}(\boldsymbol{x}_1, \boldsymbol{y}_1)),$$

and suppose that $\sup|\boldsymbol{u}_0| \in L^2$. Set $a = [\![\boldsymbol{x}_{0\star} \ne \boldsymbol{x}_{1\star}]\!] \cup [\![\boldsymbol{y}_0 \ne \boldsymbol{y}_1]\!]$. Then $[\![\boldsymbol{u}_0 \ne \boldsymbol{u}_1]\!] \subseteq a$. **P** Let $\epsilon > 0$. Let $M \ge 0$ be such that $\bar{\mu}[\![\sup|\boldsymbol{u}_1| > M]\!] \le \epsilon$ and set $a' = a \cup [\![\sup|\boldsymbol{u}_1| > M]\!]$.

Now repeat the inductive construction of (a-i), adjusted as follows. Set $\tilde{\boldsymbol{u}}_0 = \mathrm{med}(-M\mathbf{1}^{(\mathcal{S})}, \boldsymbol{u}_1, M\mathbf{1}^{(\mathcal{S})})$ and

$$\tilde{\boldsymbol{x}}_{n+1} = \boldsymbol{x}_{0\star} + \tilde{\boldsymbol{u}}_{n<}, \quad \tilde{\boldsymbol{u}}_{n+1} = \mathrm{S}ii_{\boldsymbol{v}}(\bar{h}(\tilde{\boldsymbol{x}}_{n+1}, \boldsymbol{y}_0))$$

for $n \in \mathbb{N}$. Then

$$|\bar{h}(\tilde{\boldsymbol{x}}_{n+1}, \boldsymbol{y}_0) - \bar{h}(\boldsymbol{x}_0, \boldsymbol{y}_0)| \le K|\tilde{\boldsymbol{x}}_{n+1} - \boldsymbol{x}_0| = K|\tilde{\boldsymbol{u}}_n - \boldsymbol{u}_0|_<,$$

$$\begin{aligned}\|\sup|\tilde{\boldsymbol{u}}_{n+1} - \boldsymbol{u}_0|\|_2 &= \|\sup|\mathrm{S}ii_{\boldsymbol{v}}(\bar{h}(\tilde{\boldsymbol{x}}_{n+1}, \boldsymbol{y}_0) - \bar{h}(\boldsymbol{x}_0, \boldsymbol{y}_0))|\|_2 \\ &\le 2K\gamma\|\sup|\tilde{\boldsymbol{u}}_n - \boldsymbol{u}_0|\|_2\end{aligned}$$

for each n, while $\|\sup|\tilde{\boldsymbol{u}}_0 - \boldsymbol{u}_0|\|_2 \le M + \|\sup|\boldsymbol{u}_0|\|_2$ is finite. So $\lim_{n\to\infty}\|\sup|\tilde{\boldsymbol{u}}_n - \boldsymbol{u}_0|\|_2 = 0$ and $\boldsymbol{u}_0 = \lim_{n\to\infty}\tilde{\boldsymbol{u}}_n$ for the ucp topology.

At the same time, we find that $[\![\tilde{\boldsymbol{u}}_n \ne \boldsymbol{u}_1]\!] \subseteq a'$ for every n. To see this, induce on n. At the start, $[\![\tilde{\boldsymbol{u}}_0 \ne \boldsymbol{u}_1]\!] = [\![\sup|\boldsymbol{u}_1| > M]\!] \subseteq a'$. For the inductive step,

$$[\![\tilde{\boldsymbol{u}}_{n+1} \ne \boldsymbol{u}_1]\!] = [\![\mathrm{S}ii_{\boldsymbol{v}}(\bar{h}(\tilde{\boldsymbol{x}}_{n+1}, \boldsymbol{y}_0) - \bar{h}(\boldsymbol{x}_1, \boldsymbol{y}_1)) \ne \mathbf{0}]\!] \subseteq [\![\bar{h}(\tilde{\boldsymbol{x}}_{n+1}, \boldsymbol{y}_0) - \bar{h}(\boldsymbol{x}_1, \boldsymbol{y}_1) \ne \mathbf{0}]\!]$$

(647J again)

$$\subseteq [\![\tilde{\boldsymbol{x}}_{n+1} \ne \boldsymbol{x}_1]\!] \cup [\![\boldsymbol{y}_0 \ne \boldsymbol{y}_1]\!]$$
$$\subseteq [\![\boldsymbol{x}_{0\star} \ne \boldsymbol{x}_{1\star}]\!] \cup [\![\tilde{\boldsymbol{u}}_{n<} \ne \boldsymbol{u}_{1<}]\!] \cup a \subseteq [\![\tilde{\boldsymbol{u}}_n \ne \boldsymbol{u}_1]\!] \cup a$$

(641G(a-iii))

$$\subseteq a'$$

by the inductive hypothesis, so the induction proceeds.

It follows that $[\![\boldsymbol{u}_0 \ne \boldsymbol{u}_1]\!] \setminus a \subseteq a' \setminus a$ has measure at most ϵ. As ϵ is arbitrary, $[\![\boldsymbol{u}_0 \ne \boldsymbol{u}_1]\!] \subseteq a$. **Q**

(c) For the second stage of the existence proof, continue to assume that $2K\gamma < 1$ where $\gamma = \sqrt{\|\boldsymbol{w}^*\|_\infty} + \|\int_{\mathcal{S}} |d\boldsymbol{w}'|\|_\infty$, but drop the $\|\;\|_\infty$-boundedness conditions on $\boldsymbol{x}_\star$ and $\boldsymbol{y}$, and suppose only that $\boldsymbol{x}_\star, \boldsymbol{y} \in M_{\text{S-i}}$.

(i) In this case, set

$$\boldsymbol{x}_{k*} = \mathrm{med}(-k\mathbf{1}^{(\mathcal{S})}_<, \boldsymbol{x}_\star, k\mathbf{1}^{(\mathcal{S})}_<), \quad \boldsymbol{y}_k = \mathrm{med}(-k\mathbf{1}^{(\mathcal{S})}_<, \boldsymbol{y}, k\mathbf{1}^{(\mathcal{S})}_<),$$

$$a_k = [\![\sup|\boldsymbol{x}_\star| > k]\!] \cup [\![\sup|\boldsymbol{y}| > k]\!]$$

for $k \in \mathbb{N}$. (Recall that we know from 645Kb that $\boldsymbol{x}_\star$ and $\boldsymbol{y}$ are order-bounded.) By (a), there is for each k a $\boldsymbol{u}_k \in M_{\text{n-s}}$ such that $\sup|\boldsymbol{u}_k|$ is square-integrable and, setting $\boldsymbol{x}_k = \boldsymbol{x}_{k*} + \boldsymbol{u}_{k<}$, $\boldsymbol{u}_k = \mathrm{S}ii_{\boldsymbol{v}}(\bar{h}(\boldsymbol{x}_k, \boldsymbol{y}_k))$. By (b), $[\![\boldsymbol{u}_k \ne \boldsymbol{u}_l]\!] \subseteq a_k$ for $l \ge k$. Since $\langle a_k\rangle_{k\in\mathbb{N}}$ is a non-increasing sequence with infimum 0,

$$z = \sup_{k\in\mathbb{N}}\sup|\boldsymbol{u}_k| = \sup|\boldsymbol{u}_0| \vee \sup_{k\in\mathbb{N}}\chi(a_k \setminus \chi a_{k+1}) \times \sup_{j\le k+1}\sup|\boldsymbol{u}_j|$$

is defined in L^0 and $\sup|\boldsymbol{u}_k-\boldsymbol{u}_l|\le z\times\chi a_k$ whenever $k\le l$. Accordingly $\boldsymbol{u}=\lim_{k\to\infty}\boldsymbol{u}_k$ is defined for the ucp topology, and $\boldsymbol{u}$ is near-simple (631Ba). Next, setting $\boldsymbol{x}=\boldsymbol{x}_\star+\boldsymbol{u}_<$, we have, for any k,

$$[\![\boldsymbol{x}\ne\boldsymbol{x}_k]\!]\subseteq[\![\boldsymbol{x}_\star\ne\boldsymbol{x}_k]\!]\cup[\![\boldsymbol{u}_<\ne\boldsymbol{u}_{k<}]\!]\subseteq a_k\cup[\![\boldsymbol{u}\ne\boldsymbol{u}_k]\!]=a_k,$$

$$\begin{aligned}[\![\boldsymbol{u}_{k<}\ne \mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x},\boldsymbol{y}))_<]\!]&\subseteq[\![\boldsymbol{u}_k\ne \mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x},\boldsymbol{y}))]\!]=[\![\mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x}_k,\boldsymbol{y}_k))\ne \mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x},\boldsymbol{y}))]\!]\\&\subseteq[\![\bar h(\boldsymbol{x}_k,\boldsymbol{y}_k)\ne\bar h(\boldsymbol{x},\boldsymbol{y})]\!]\subseteq[\![\boldsymbol{x}_k\ne\boldsymbol{x}]\!]\cup[\![\boldsymbol{y}_k\ne\boldsymbol{y}]\!]\subseteq a_k\end{aligned}$$

and

$$[\![\boldsymbol{x}\ne\boldsymbol{x}_\star+\mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x},\boldsymbol{y}))_<]\!]\subseteq[\![\boldsymbol{x}\ne\boldsymbol{x}_k]\!]\cup[\![\boldsymbol{x}_\star\ne\boldsymbol{x}_{k*}]\!]\cup[\![\mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x},\boldsymbol{y}))_<\ne\boldsymbol{u}_{k<}]\!]\subseteq a_k.$$

As k is arbitrary, $\boldsymbol{x}=\boldsymbol{x}_\star+\mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x},\boldsymbol{y}))_<$.

(ii) To see that $\boldsymbol{x}$ is unique, let $\boldsymbol{x}'\in M_{\text{S-i}}$ be such that $\boldsymbol{x}'=\boldsymbol{x}_\star+\mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x}',\boldsymbol{y}))_<$. This time, (b) tells us that, setting $\boldsymbol{u}'=\mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x}',\boldsymbol{y}))$, $[\![\boldsymbol{u}'\ne\boldsymbol{u}_k]\!]\subseteq a_k$ for each k. So $\boldsymbol{u}'=\boldsymbol{u}$ and $\boldsymbol{x}'=\boldsymbol{x}$.

(d) Now turn to the given hypothesis that $2K(\sqrt{\|\boldsymbol{w}^*\|_\infty}+2\|\boldsymbol{z}\times\boldsymbol{w}'^{\uparrow}\|_\infty)<1$.

(i) As in 654F, set $\boldsymbol{w}'_1=\boldsymbol{z}\times\boldsymbol{w}'$ and $\boldsymbol{v}_1=\boldsymbol{w}+\boldsymbol{w}'_1$. Then, as before, we have $2K(\sqrt{\|\boldsymbol{w}^*\|_\infty}+\|\int_{\mathcal{S}}|d\boldsymbol{w}'_1|\|_\infty)<1$. So (c-i) tells us that there is an S-integrable process $\boldsymbol{x}_1$ such that $\boldsymbol{x}_1=\boldsymbol{x}_\star+\mathrm{S}ii_{\boldsymbol{v}_1}(\bar h(\boldsymbol{x}_1,\boldsymbol{y}))_<$. Set $\boldsymbol{w}'_2=\boldsymbol{w}'-\boldsymbol{w}'_1$ and $\boldsymbol{x}=\boldsymbol{x}_1+\mathrm{S}ii_{\boldsymbol{w}'_2}(\bar h(\boldsymbol{x}_1,\boldsymbol{y}))_<$. We see that

$$\boldsymbol{v}_1+\boldsymbol{w}'_2=\boldsymbol{w}+\boldsymbol{w}'=\boldsymbol{v},$$

$$\boldsymbol{x}=\boldsymbol{x}_\star+\mathrm{S}ii_{\boldsymbol{v}_1}(\bar h(\boldsymbol{x}_1,\boldsymbol{y}))_<+\mathrm{S}ii_{\boldsymbol{w}'_2}(\bar h(\boldsymbol{x}_1,\boldsymbol{y}))_<=\boldsymbol{x}_\star+\mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x}_1,\boldsymbol{y}))_<,$$

$$\boldsymbol{z}\times\boldsymbol{x}=\boldsymbol{z}\times\boldsymbol{x}_1+\boldsymbol{z}\times\mathrm{S}ii_{\boldsymbol{w}'_2}(\bar h(\boldsymbol{x}_1,\boldsymbol{y}))_<=\boldsymbol{z}\times\boldsymbol{x}_1+\boldsymbol{z}\times\mathrm{S}ii_{\boldsymbol{z}\times\boldsymbol{w}'_2}(\bar h(\boldsymbol{x}_1,\boldsymbol{y}))_<$$

(654Ha)

$$=\boldsymbol{z}\times\boldsymbol{x}_1$$

because $\boldsymbol{z}\times\boldsymbol{w}'_2=0$. Consequently

$$\mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x},\boldsymbol{y}))_<=\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{z}\times\bar h(\boldsymbol{x},\boldsymbol{y}))_<$$

(654Hb))

$$=\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{z}\times\bar h(\boldsymbol{z}\times\boldsymbol{x},\boldsymbol{y}))_<$$

(619Ge again, as in the proof of 654F)

$$=\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{z}\times\bar h(\boldsymbol{z}\times\boldsymbol{x}_1,\boldsymbol{y}))_<=\mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x}_1,\boldsymbol{y}))_<$$

and $\boldsymbol{x}=\boldsymbol{x}_\star+\mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x},\boldsymbol{y}))_<$, as required.

(ii) To show uniqueness we can run the argument backwards, as follows. Suppose that $\boldsymbol{x}'\in M_{\text{S-i}}$ is such that $\boldsymbol{x}'=\boldsymbol{x}_\star+\mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x}',\boldsymbol{y}))_<$. Set $\boldsymbol{x}'_1=\boldsymbol{x}_\star+\mathrm{S}ii_{\boldsymbol{v}_1}(\bar h(\boldsymbol{x}',\boldsymbol{y}))_<$. Then

$$\begin{aligned}\boldsymbol{z}\times\boldsymbol{x}'_1&=\boldsymbol{z}\times\boldsymbol{x}_\star+\boldsymbol{z}\times\mathrm{S}ii_{\boldsymbol{z}\times\boldsymbol{v}_1}(\bar h(\boldsymbol{x}',\boldsymbol{y}))_<\\&=\boldsymbol{z}\times\boldsymbol{x}_\star+\boldsymbol{z}\times\mathrm{S}ii_{\boldsymbol{z}\times\boldsymbol{v}}(\bar h(\boldsymbol{x}',\boldsymbol{y}))_<=\boldsymbol{z}\times\boldsymbol{x}'\end{aligned}$$

so that $\mathrm{S}ii_{\boldsymbol{v}_1}(\bar h(\boldsymbol{x}'_1,\boldsymbol{y}))_<=\mathrm{S}ii_{\boldsymbol{v}_1}(\bar h(\boldsymbol{x}',\boldsymbol{y}))_<$ and $\boldsymbol{x}'_1=\boldsymbol{x}_\star+\mathrm{S}ii_{\boldsymbol{v}_1}(\bar h(\boldsymbol{x}'_1,\boldsymbol{y}))_<$. But this implies that $\boldsymbol{x}'_1=\boldsymbol{x}_1$, by the uniqueness established in (c-ii), and $\boldsymbol{z}\times\boldsymbol{x}'_1=\boldsymbol{z}\times\boldsymbol{x}_1$. Now

$$\begin{aligned}\boldsymbol{x}'&=\boldsymbol{x}'_1+\mathrm{S}ii_{\boldsymbol{w}'_2}(\bar h(\boldsymbol{x}',\boldsymbol{y}))_<=\boldsymbol{x}'_1+\mathrm{S}ii_{\boldsymbol{w}'_2}(\bar h(\boldsymbol{x}'_1,\boldsymbol{y}))_<\\&=\boldsymbol{x}_1+\mathrm{S}ii_{\boldsymbol{w}'_2}(\bar h(\boldsymbol{x}_1,\boldsymbol{y}))_<=\boldsymbol{x}.\end{aligned}$$

Thus the solution offered in (i) is unique.

654K The key step in the next stage involves re-starting an S-integral, so I separate it out.

Lemma Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous, and that $\mathcal{S}\subseteq\mathcal{T}$ is an order-convex sublattice with a least member. Let $h:\mathbb{R}^2\to\mathbb{R}$ be a locally bounded Borel measurable function. Take processes $\boldsymbol{x}$, $\boldsymbol{x}_\star$, $\boldsymbol{y}\in M_{\text{S-i}}(\mathcal{S})$ and an integrator $\boldsymbol{v}\in M_{\text{n-s}}(\mathcal{S})$. Set $\boldsymbol{u}=\mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x},\boldsymbol{y}))$ and express $\boldsymbol{u}$ as $\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$, $\boldsymbol{u}_<$ as $\langle u_{<\sigma}\rangle_{\sigma\in\mathcal{S}}$.

Fix $\tau\in\mathcal{S}$. Set

$$\mathcal{S}'=\mathcal{S}\wedge\tau,\quad \boldsymbol{x}'=\boldsymbol{x}\restriction\mathcal{S}',\quad \boldsymbol{v}'=\boldsymbol{v}\restriction\mathcal{S}',\quad \boldsymbol{y}'=\boldsymbol{y}\restriction\mathcal{S}',\quad \boldsymbol{x}'_\star=\boldsymbol{x}_\star\restriction\mathcal{S}',$$

$$\mathcal{S}''=\mathcal{S}\vee\tau,\quad \boldsymbol{x}''=\boldsymbol{x}\restriction\mathcal{S}'',\quad \boldsymbol{v}''=\boldsymbol{v}\restriction\mathcal{S}'',\quad \boldsymbol{y}''=\boldsymbol{y}\restriction\mathcal{S}''\quad \boldsymbol{x}''_\star=\boldsymbol{x}_\star\restriction\mathcal{S}''+\tilde{\boldsymbol{x}}_*$$

where $\tilde{\boldsymbol{x}}_*=u_{<\tau}\mathbf{1}^{(\mathcal{S}'')}+(u_\tau-u_{<\tau})\mathbf{1}^{(\mathcal{S}'')}_<$.

(a) $\mathrm{S}ii_{\boldsymbol{v}'}(\bar h(\boldsymbol{x}',\boldsymbol{y}'))=\mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x},\boldsymbol{y}))\restriction\mathcal{S}'$, so $\mathrm{S}ii_{\boldsymbol{v}'}(\bar h(\boldsymbol{x}',\boldsymbol{y}'))_<=\mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x},\boldsymbol{y}))_<\restriction\mathcal{S}'$.

(b) $\boldsymbol{x}''_\star\in M_{\text{S-i}}(\mathcal{S}'')$.

(c) $\boldsymbol{x}=\boldsymbol{x}_\star+\mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x},\boldsymbol{y}))_<$ if and only if

$$\boldsymbol{x}'=\boldsymbol{x}'_\star+\mathrm{S}ii_{\boldsymbol{v}'}(\bar h(\boldsymbol{x}',\boldsymbol{y}'))_<\quad\text{and}\quad \boldsymbol{x}''=\boldsymbol{x}''_\star+\mathrm{S}ii_{\boldsymbol{v}''}(\bar h(\boldsymbol{x}'',\boldsymbol{y}''))_<.$$

proof (a)(i) We should start by noting that $\bar h(\boldsymbol{x},\boldsymbol{y})$ is S-integrable, by 645J as usual, so we can talk about the indefinite S-integral $\mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x},\boldsymbol{y}))$. Similarly, because $\boldsymbol{x}'$, $\boldsymbol{y}'$, $\boldsymbol{x}''$ and $\boldsymbol{y}''$ are S-integrable (646Hd) and $\boldsymbol{v}'$, $\boldsymbol{v}''$ are near-simple integrators (631F(a-iv), 616P(b-ii)), we have the indefinite S-integrals $\mathrm{S}ii_{\boldsymbol{v}'}(\bar h(\boldsymbol{x}',\boldsymbol{y}'))$ and $\mathrm{S}ii_{\boldsymbol{v}''}(\bar h(\boldsymbol{x}'',\boldsymbol{y}''))$.

For $\sigma\in\mathcal{S}$ write $\mathcal{S}_\sigma$ for $\mathcal{S}\wedge\sigma$.

(ii) If $\sigma\in\mathcal{S}'$ then

$$\begin{aligned}S\!\!\!\!\!\int_{\mathcal{S}_\sigma}\bar h(\boldsymbol{x},\boldsymbol{y})\,d\boldsymbol{v}&=S\!\!\!\!\!\int_{\mathcal{S}_\sigma}\bar h(\boldsymbol{x}\restriction\mathcal{S}_\sigma,\boldsymbol{y}\restriction\mathcal{S}_\sigma)\,d(\boldsymbol{v}\restriction\mathcal{S}_\sigma)\\&=S\!\!\!\!\!\int_{\mathcal{S}_\sigma}\bar h(\boldsymbol{x}'\restriction\mathcal{S}_\sigma,\boldsymbol{y}'\restriction\mathcal{S}_\sigma)\,d(\boldsymbol{v}'\restriction\mathcal{S}_\sigma)\\&=S\!\!\!\!\!\int_{\mathcal{S}_\sigma}\bar h(\boldsymbol{x}',\boldsymbol{y}')\,d\boldsymbol{v}'.\end{aligned}$$

So $\mathrm{S}ii_{\boldsymbol{v}'}(\bar h(\boldsymbol{x}',\boldsymbol{y}'))=\mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x},\boldsymbol{y}))\restriction\mathcal{S}'$. By 641G(c-ii), it follows that $\mathrm{S}ii_{\boldsymbol{v}'}(\bar h(\boldsymbol{x}',\boldsymbol{y}'))_<=\mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x},\boldsymbol{y}))_<\restriction\mathcal{S}'$.

(b) Here I should point out that $\tilde{\boldsymbol{x}}_*\times\mathbf{1}^{(\mathcal{S}'')}_<=(u_\tau\mathbf{1}^{(\mathcal{S}'')})_<$ belongs to $M^0_{\text{S-i}}(\mathcal{S}'')$ so $\boldsymbol{x}''_\star\in M_{\text{S-i}}(\mathcal{S}'')$.

(c)(i) For $\sigma\in\mathcal{S}''$, set $u''_\sigma=S\!\!\!\!\!\int_{[\tau,\sigma]}\bar h(\boldsymbol{x},\boldsymbol{y})\,d\boldsymbol{v}$, so that $\mathrm{S}ii_{\boldsymbol{v}''}(\bar h(\boldsymbol{x}'',\boldsymbol{y}''))=\langle u''_\sigma\rangle_{\sigma\in\mathcal{S}''}$ and $\mathrm{S}ii_{\boldsymbol{v}''}(\bar h(\boldsymbol{x}'',\boldsymbol{y}''))_<=\langle u''_{<\sigma}\rangle_{\sigma\in\mathcal{S}''}$.

If $\sigma\in\mathcal{S}''$, then

$$u_\sigma=S\!\!\!\!\!\int_{\mathcal{S}_\sigma}\bar h(\boldsymbol{x},\boldsymbol{y})\,d\boldsymbol{v}=S\!\!\!\!\!\int_{\mathcal{S}'}\bar h(\boldsymbol{x},\boldsymbol{y})\,d\boldsymbol{v}+S\!\!\!\!\!\int_{[\tau,\sigma]}\bar h(\boldsymbol{x},\boldsymbol{y})\,d\boldsymbol{v}$$

(646J, because $\mathcal{S}_\sigma\wedge\tau=\mathcal{S}'$ and $\mathcal{S}_\sigma\vee\tau=[\tau,\sigma]$)

$$=u_\tau+S\!\!\!\!\!\int_{[\tau,\sigma]}\bar h(\boldsymbol{x},\boldsymbol{y})\,d\boldsymbol{v}=u_\tau+u''_\sigma.$$

We know that $(\boldsymbol{u}\restriction\mathcal{S}'')_<=(\boldsymbol{u}_<\restriction\mathcal{S}'')\times\mathbf{1}^{(\mathcal{S}'')}_<$ (641G(c-ii)), so that

$$[\![\tau<\sigma]\!]\subseteq[\![u_{<\sigma}=u_\tau+u''_{<\sigma}]\!]\cap[\![\tilde x_{*\sigma}=u_\tau]\!]\subseteq[\![u_{<\sigma}=\tilde x_{*\sigma}+u''_{<\sigma}]\!],$$

where $\tilde{\boldsymbol{x}}_*=\langle\tilde x_{*\sigma}\rangle_{\sigma\in\mathcal{S}''}$, while

$$[\![\tau=\sigma]\!]\subseteq[\![u_{<\sigma}=u_{<\tau}]\!]\cap[\![u''_{<\sigma}=0]\!]\cap[\![\tilde x_{*\sigma}=u_{<\tau}]\!]\subseteq[\![u_{<\sigma}=\tilde x_{*\sigma}+u''_{<\sigma}]\!].$$

Thus $u_{<\sigma}=\tilde x_{*\sigma}+u''_{<\sigma}$; as σ is arbitrary, $\mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x},\boldsymbol{y}))_<\restriction\mathcal{S}''=\tilde{\boldsymbol{x}}_*+\mathrm{S}ii_{\boldsymbol{v}''}(\bar h(\boldsymbol{x}'',\boldsymbol{y}''))_<$.

(ii) Writing $\boldsymbol{u}'$ for $\mathrm{S}ii_{\boldsymbol{v}'}(\bar h(\boldsymbol{x}',\boldsymbol{y}'))$ and $\boldsymbol{u}''$ for $\mathrm{S}ii_{\boldsymbol{v}''}(\bar h(\boldsymbol{x}'',\boldsymbol{y}''))$, we know from (a) that $\boldsymbol{u}'_<=\boldsymbol{u}_<\restriction\mathcal{S}'$ and from (i) just above that $\tilde{\boldsymbol{x}}_*+\boldsymbol{u}''_<=\boldsymbol{u}_<\restriction\mathcal{S}''$. Since $\mathcal{S}'\cup\mathcal{S}''$ covers $\mathcal{S}$, and $\boldsymbol{x}$ and $\boldsymbol{x}_\star+\boldsymbol{u}_<$ are fully adapted,

$$\begin{aligned}\boldsymbol{x} = \boldsymbol{x}_\star + \boldsymbol{u}_< &\iff \boldsymbol{x}{\restriction}\mathcal{S}' = \boldsymbol{x}_\star{\restriction}\mathcal{S}' + \boldsymbol{u}_<{\restriction}\mathcal{S}' \text{ and } \boldsymbol{x}{\restriction}\mathcal{S}'' = \boldsymbol{x}_\star{\restriction}\mathcal{S}'' + \boldsymbol{u}_<{\restriction}\mathcal{S}''\\ &\iff \boldsymbol{x}' = \boldsymbol{x}'_\star + \boldsymbol{u}'_< \text{ and } \boldsymbol{x}'' = \boldsymbol{x}_\star{\restriction}\mathcal{S}'' + \tilde{\boldsymbol{x}}_* + \boldsymbol{u}''_<\\ &\iff \boldsymbol{x}' = \boldsymbol{x}'_\star + \boldsymbol{u}'_< \text{ and } \boldsymbol{x}'' = \boldsymbol{x}''_\star + \boldsymbol{u}''_<,\end{aligned}$$

that is,

$$\begin{aligned}&\boldsymbol{x} = \boldsymbol{x}_\star + \mathrm{S}ii_{\boldsymbol{v}}(\bar{h}(\boldsymbol{x},\boldsymbol{y}))_<\\ &\qquad\iff \boldsymbol{x}' = \boldsymbol{x}'_\star + \mathrm{S}ii_{\boldsymbol{v}'}(\bar{h}(\boldsymbol{x}',\boldsymbol{y}'))_< \text{ and } \boldsymbol{x}'' = \boldsymbol{x}''_\star + \tilde{\boldsymbol{x}}_* + \mathrm{S}ii_{\boldsymbol{v}''}(\bar{h}(\boldsymbol{x}'',\boldsymbol{y}''))_<.\end{aligned}$$

654L With this in hand, we can embark on a full-strength version of Theorem 654G.

Theorem Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$ with a least member. Suppose that $h:\mathbb{R}^2\to\mathbb{R}$ is a locally bounded Borel measurable function and that $K\ge 0$ is such that $|h(\alpha,\beta)-h(\alpha',\beta)|\le K|\alpha-\alpha'|$ for all α, α', $\beta\in\mathbb{R}$. Let $\boldsymbol{v}$ be a locally near-simple local integrator with domain $\mathcal{S}$. Then for any locally S-integrable processes $\boldsymbol{x}_\star = \langle x_{\star\sigma}\rangle_{\sigma\in\mathcal{S}}$, $\boldsymbol{y}$ with domain $\mathcal{S}$ there is a unique locally S-integrable process $\boldsymbol{x}$ with domain $\mathcal{S}$ such that

$$\boldsymbol{x} = \boldsymbol{x}_\star + \mathrm{S}ii_{\boldsymbol{v}}(\bar{h}(\boldsymbol{x},\boldsymbol{y}))_<.$$

proof (a) Let $\epsilon>0$ be such that $2K(\sqrt{\epsilon}+3\epsilon)<1$. As in part (a) of the proof of 654G, we can express $\boldsymbol{v}$ as $\boldsymbol{w}+\boldsymbol{w}'$ where $\boldsymbol{w}=\langle w_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a local martingale, $\boldsymbol{w}'=\langle w'_\sigma\rangle_{\sigma\in\mathcal{S}}$ is of bounded variation and $\mathrm{Osclln}(\boldsymbol{w})\le\epsilon\chi 1$. For the time being (down to the end of (d) below), suppose that that $\boldsymbol{w}$ is a martingale and that $\mathcal{S}$ has a greatest member, so that $\boldsymbol{v}$ is near-simple and $\boldsymbol{x}_\star$ and $\boldsymbol{y}$ are S-integrable.

We know that $\boldsymbol{w}$ and $\boldsymbol{w}'$ are near-simple, so the cumulative variation $\boldsymbol{w}'^{\uparrow}=\langle w'^{\uparrow}_\sigma\rangle_{\sigma\in\mathcal{S}}$ of $\boldsymbol{w}'$ and the quadratic variation $\boldsymbol{w}^*=\langle w^*_\sigma\rangle_{\sigma\in\mathcal{S}}$ of $\boldsymbol{w}$ are near-simple.

As in part (c) of the proof of 654G, there is a non-decreasing sequence $\langle\tau_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{S}$ such that $\tau_0=\min\mathcal{S}$, $\inf_{n\in\mathbb{N}}[\![\tau_n<\max\mathcal{S}]\!]=0$ and whenever $n\in\mathbb{N}$ and $\sigma\in[\tau_n,\tau_{n+1}]$,

$$[\![w'^{\uparrow}_\sigma+w^*_\sigma-w'^{\uparrow}_{\tau_n}-w^*_{\tau_n}\ge\epsilon]\!]\subseteq[\![\sigma=\tau_{n+1}]\!].$$

(b) For $n\in\mathbb{N}$, write $\mathcal{S}_n=\mathcal{S}\wedge\tau_n=[\min\mathcal{S},\tau_n]$. Then we have a sequence $\langle\boldsymbol{x}_n\rangle_{n\in\mathbb{N}}$ of S-integrable processes such that

$$\boldsymbol{x}_n\in M_{\text{S-i}}(\mathcal{S}_n),\quad \boldsymbol{x}_n=\boldsymbol{x}_\star{\restriction}\mathcal{S}_n+\mathrm{S}ii_{\boldsymbol{v}\restriction\mathcal{S}_n}(\bar{h}(\boldsymbol{x}_n,\boldsymbol{y}{\restriction}\mathcal{S}_n))_<$$

for each n. **P** Induce on n. As in part (d) of the proof of 654G, the induction starts with $\boldsymbol{x}_0=\boldsymbol{x}_\star{\restriction}\{\min\mathcal{S}\}$. For the inductive step to $n+1$, given $\boldsymbol{x}_n$, set $\boldsymbol{u}=\mathrm{S}ii_{\boldsymbol{v}\restriction\mathcal{S}_n}(\bar{h}(\boldsymbol{x}_n,\boldsymbol{y}{\restriction}\mathcal{S}_n))$ and express $\boldsymbol{u}$ as $\langle u_\sigma\rangle_{\sigma\in\mathcal{S}_n}$; set

$$\tilde{\mathcal{S}}_n=[\tau_n,\tau_{n+1}],\quad \tilde{\boldsymbol{x}}_{n*}=\boldsymbol{x}_\star{\restriction}\tilde{\mathcal{S}}_n+u_{<\tau_n}\mathbf{1}^{(\tilde{\mathcal{S}}_n)}+(u_{\tau_n}-u_{<\tau_n})\mathbf{1}^{(\tilde{\mathcal{S}}_n)}_<,$$

$$\tilde{\boldsymbol{w}}=\boldsymbol{w}{\restriction}\tilde{\mathcal{S}}_n-w_{\tau_n}\mathbf{1}^{(\tilde{\mathcal{S}}_n)},\quad \tilde{\boldsymbol{w}}'=\boldsymbol{w}'{\restriction}\tilde{\mathcal{S}}_n-w'_{\tau_n}\mathbf{1}^{(\tilde{\mathcal{S}}_n)},\quad \tilde{\boldsymbol{v}}=\tilde{\boldsymbol{w}}+\tilde{\boldsymbol{w}}'.$$

Then $\tilde{\boldsymbol{w}}$ and $\tilde{\boldsymbol{w}}'$ are near-simple processes with domain $\tilde{\mathcal{S}}_n$, $\tilde{\boldsymbol{w}}$ is a martingale, $\tilde{\boldsymbol{w}}'$ is of bounded variation and both start with value 0 at $\min\tilde{\mathcal{S}}_n=\tau_n$. The quadratic variation $\tilde{\boldsymbol{w}}^*$ of $\tilde{\boldsymbol{w}}$ is $\boldsymbol{w}^*{\restriction}\tilde{\mathcal{S}}_n-w^*_{\tau_n}\mathbf{1}^{(\tilde{\mathcal{S}}_n)}$, and the cumulative variation $\tilde{\boldsymbol{w}}'^{\uparrow}$ of $\tilde{\boldsymbol{w}}'$ is $\boldsymbol{w}'^{\uparrow}{\restriction}\tilde{\mathcal{S}}_n-w^{\uparrow}_{\tau_n}\mathbf{1}^{(\tilde{\mathcal{S}}_n)}$.

As before, setting $\boldsymbol{z}=\langle\chi[\![\sigma<\tau_{n+1}]\!]\rangle_{\sigma\in\tilde{\mathcal{S}}_n}$,

$$2K\bigl(\sqrt{\|\tilde{\boldsymbol{w}}^*\|_\infty}+2\|\boldsymbol{z}\times\tilde{\boldsymbol{w}}'^{\uparrow}\|_\infty\bigr)\le 2K(\sqrt{\epsilon}+3\epsilon)<1.$$

We can therefore apply 654J to see that there is a unique S-integrable process $\tilde{\boldsymbol{x}}_n$ with domain $\tilde{\mathcal{S}}_n$ such that $\tilde{\boldsymbol{x}}_n=\tilde{\boldsymbol{x}}_{n*}+\mathrm{S}ii_{\tilde{\boldsymbol{v}}}(\bar{h}(\tilde{\boldsymbol{x}}_n,\tilde{\boldsymbol{y}}))_<$.

Since the processes $\boldsymbol{x}_n$ and $\tilde{\boldsymbol{x}}_n$ take the same value $x_{\star\tau_n}+u_{<\tau_n}$ at τ_n, we have a fully adapted process $\boldsymbol{x}_{n+1}$ with domain $\mathcal{S}_{n+1}$ extending them both. By 646J again, $\boldsymbol{x}_{n+1}$ is S-integrable. Now $\mathrm{S}ii_{\boldsymbol{v}\restriction\mathcal{S}_n}(\bar{h}(\boldsymbol{x}_{n+1}{\restriction}\mathcal{S}_n,\boldsymbol{y}{\restriction}\mathcal{S}_n))=\boldsymbol{u}$, and 654Kc tells us that

$$\boldsymbol{x}_{n+1}=\boldsymbol{x}_\star{\restriction}\mathcal{S}_{n+1}+\mathrm{S}ii_{\boldsymbol{v}\restriction\mathcal{S}_{n+1}}(\bar{h}(\boldsymbol{x}_{n+1},\boldsymbol{y}{\restriction}\mathcal{S}_{n+1}))_<,$$

so the induction continues. **Q**

(c) At the end of the induction, as in part (e) of the proof of 654G, we have a process $\boldsymbol{x} = \langle x_\sigma\rangle_{\sigma\in\mathcal{S}}$ extending every $\boldsymbol{x}_n$. Next, setting

$$\boldsymbol{u}_n = \mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x}\restriction\mathcal{S}_n, \boldsymbol{y}\restriction\mathcal{S}_n)) = \mathrm{S}ii_{\boldsymbol{v}\restriction\mathcal{S}_n}(\bar h(\boldsymbol{x}_n, \boldsymbol{y}\restriction\mathcal{S}_n)),$$

for each n, we see that $\boldsymbol{u}_n$ is near-simple (646N again) and $\boldsymbol{u}_n = \boldsymbol{u}_{n+1}\restriction\mathcal{S}_n$ for each n. Accordingly, just as in the proof of 654G, there is a near-simple $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ such that $\boldsymbol{u}_n = \boldsymbol{u}\restriction\mathcal{S}_n$ for every n.

Now $\boldsymbol{u}_<$ is S-integrable (645R(a-i)) so $\boldsymbol{x}_\star + \boldsymbol{u}_<$ is S-integrable. But

$$\begin{aligned}\boldsymbol{x}\restriction\mathcal{S}_n &= \boldsymbol{x}_\star\restriction\mathcal{S}_n + \boldsymbol{u}_{n<} = \boldsymbol{x}_\star\restriction\mathcal{S}_n + (\boldsymbol{u}\restriction\mathcal{S}_n)_< \\ &= \boldsymbol{x}_\star\restriction\mathcal{S}_n + \boldsymbol{u}_<\restriction\mathcal{S}_n = (\boldsymbol{x}_\star + \boldsymbol{u}_<)\restriction\mathcal{S}_n\end{aligned}$$

for every n, by 641G(c-ii) again. Since $\bigcup_{n\in\mathbb{N}}\mathcal{S}_n$ covers $\mathcal{S}$, $\boldsymbol{x} = \boldsymbol{x}_\star + \boldsymbol{u}_<$ is S-integrable. Consequently $\bar h(\boldsymbol{x}, \boldsymbol{y})$ is S-integrable (645J again) and we can speak of $\mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x}, \boldsymbol{y}))_<$. Since

$$\begin{aligned}\boldsymbol{x}\restriction\mathcal{S}_n &= \boldsymbol{x}_\star\restriction\mathcal{S}_n + \mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x}\restriction\mathcal{S}_n, \boldsymbol{y}\restriction\mathcal{S}_n))_< \\ &= \boldsymbol{x}_\star\restriction\mathcal{S}_n + (\mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x}, \boldsymbol{y}))\restriction\mathcal{S}_n)_< = \boldsymbol{x}_\star\restriction\mathcal{S}_n + \mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x}, \boldsymbol{y}))_<\restriction\mathcal{S}_n\end{aligned}$$

for each n,

$$\boldsymbol{x} = \boldsymbol{x}_\star + \mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x}, \boldsymbol{y}))_<.$$

(d) If $\boldsymbol{x}'$ is another S-integrable process such that $\boldsymbol{x}' = \boldsymbol{x}_\star + \mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x}', \boldsymbol{y}))_<$, then we see that

$$\boldsymbol{x}'\restriction\mathcal{S}_n = \boldsymbol{x}_\star\restriction\mathcal{S}_n + \mathrm{S}ii_{\boldsymbol{v}\restriction\mathcal{S}_n}(\bar h(\boldsymbol{x}'\restriction\mathcal{S}_n, \boldsymbol{y}\restriction\mathcal{S}_n))_<$$

for each n. Applying 654Kc in the reverse direction to $\tilde{\mathcal{S}}_n = \mathcal{S}_{n+1}\vee\tau_n$, we see by induction that $\boldsymbol{x}'\restriction\tilde{\mathcal{S}}_n$ satisfies the same equation as $\tilde{\boldsymbol{x}}_n$ for each n. By the uniqueness guaranteed in 654J, $\boldsymbol{x}'\restriction\tilde{\mathcal{S}}_n = \tilde{\boldsymbol{x}}_n$ for each n, and $\boldsymbol{x}' = \boldsymbol{x}$. Thus we have both existence and uniqueness in the case in which $\mathcal{S}$ has a greatest member.

(e) For the general case, as in part (g) of the proof of 654G, we have a covering ideal $\mathcal{S}'$ of $\mathcal{S}$ such that $\boldsymbol{w}\restriction\mathcal{S}'$ is a martingale. For each $\tau\in\mathcal{S}'$ we have a partial solution $\boldsymbol{x}_\tau$ with domain $\mathcal{S}\wedge\tau$. These have a common extension to a process $\boldsymbol{x}$ with domain $\mathcal{S}$, which will be the solution we seek on the whole lattice $\mathcal{S}$.

654X Basic exercises (a) In 654G, suppose that $\boldsymbol{u}'_\star$, $\boldsymbol{y}'$ and $\boldsymbol{u}'$ are other locally moderately oscillatory processes with domain $\mathcal{S}$ such that $\boldsymbol{u}' = \boldsymbol{u}'_\star + ii_{\boldsymbol{v}}(\bar h(\boldsymbol{u}', \boldsymbol{y}'))$. Show that $[\![\boldsymbol{u}' \ne \boldsymbol{u}]\!] \subseteq [\![\boldsymbol{u}'_\star \ne \boldsymbol{u}_\star]\!] \cup [\![\boldsymbol{y}' \ne \boldsymbol{y}]\!]$.

(b) Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$ with a least member. Suppose that $h : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function such that $|h(\alpha,\beta) - h(\alpha',\beta)| \le |\beta||\alpha-\alpha'|$ for α, α', $\beta\in\mathbb{R}$. Show that for any locally moderately oscillatory processes $\boldsymbol{u}_\star$, $\boldsymbol{y}$ with domain $\mathcal{S}$ and any integrator $\boldsymbol{v}\in M_{\text{n-s}}(\mathcal{S})$ there is a unique $\boldsymbol{u}\in M_{\text{n-s}}(\mathcal{S})$ such that $\boldsymbol{u} = \boldsymbol{u}_\star + ii_{\boldsymbol{v}}(\bar h(\boldsymbol{u}, \boldsymbol{y}))$. (*Hint*: for $K\ge 0$ set $\boldsymbol{y}_K = \mathrm{med}(-K\mathbf{1}^{(\mathcal{S})}, \boldsymbol{y}, K\mathbf{1}^{(\mathcal{S})})$.)

(c) In 654G, show that if $\boldsymbol{v}$ and $\boldsymbol{u}_\star$ are locally jump-free then $\boldsymbol{u}$ is locally jump-free.

(d) Show how 654G can be deduced from 654L.

(e) Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$ with a least member. Suppose that $k\ge 1$, that $h : \mathbb{R}^{k+1}\to\mathbb{R}$ is a locally bounded Borel measurable function and that $K\ge 0$ is such that $|h(\alpha,\beta_1,\dots,\beta_k) - h(\alpha',\beta_1,\dots,\beta_k))| \le K|\alpha-\alpha'|$ for all α, α', $\beta_1,\dots,\beta_k\in\mathbb{R}$. Let $\boldsymbol{v}$ be a locally near-simple local integrator with domain $\mathcal{S}$. (i) Show that if h is continuous and $\boldsymbol{u}_\star$, $\boldsymbol{y}_1,\dots,\boldsymbol{y}_k$ are locally near-simple processes with domain $\mathcal{S}$ there is a unique locally near-simple process $\boldsymbol{u}$ with domain $\mathcal{S}$ such that $\boldsymbol{u} = \boldsymbol{u}_\star + ii_{\boldsymbol{v}}(\bar h(\boldsymbol{u}, \boldsymbol{y}_1,\dots,\boldsymbol{y}_k))$. (ii) Show that if $\boldsymbol{x}_\star$, $\boldsymbol{y}_1,\dots,\boldsymbol{y}_k$ are locally S-integrable processes with domain $\mathcal{S}$ there is a unique locally S-integrable process $\boldsymbol{x}$ with domain $\mathcal{S}$ such that $\boldsymbol{x} = \boldsymbol{x}_\star + \mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{x}, \boldsymbol{y}_1,\dots,\boldsymbol{y}_k))_<$. (iii) Show that if $\boldsymbol{u}_\star$ is a locally near-simple process with domain $\mathcal{S}$ and $\boldsymbol{y}_1,\dots,\boldsymbol{y}_k$ are locally S-integrable processes with domain $\mathcal{S}$ there is a unique locally near-simple process $\boldsymbol{u}$ with domain $\mathcal{S}$ such that $\boldsymbol{u} = \boldsymbol{u}_\star + \mathrm{S}ii_{\boldsymbol{v}}(\bar h(\boldsymbol{u}_<, \boldsymbol{y}_1,\dots,\boldsymbol{y}_k))$.

(f) In 654L, show that if $\boldsymbol{v}$ and $\boldsymbol{x}_\star$ are locally jump-free then $\boldsymbol{x}$ is locally jump-free.

654Y Further exercises (a) Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous, and that $\mathcal{S}$ is an order-convex sublattice of $\mathcal{T}$ with a least element. Write $M_{\text{n-s}}$ for $M_{\text{n-s}}(\mathcal{S})$. Let H_1, $H_2 : M_{\text{n-s}} \to M_{\text{n-s}}$ and $K \ge 0$ be such that, for both j, (α) if $\boldsymbol{u}$, $\boldsymbol{u}' \in M_{\text{n-s}}$, $\sigma \in \mathcal{S}$ and $\boldsymbol{u}\restriction\mathcal{S}\wedge\sigma = \boldsymbol{u}'\restriction\mathcal{S}\wedge\sigma$ then $H_j(\boldsymbol{u})\restriction\mathcal{S}\wedge\sigma = H_j(\boldsymbol{u}')\restriction\mathcal{S}\wedge\sigma$ (β) $\sup|H_j(\boldsymbol{u}) - H_j(\boldsymbol{u}')| \le K\sup|\boldsymbol{u}-\boldsymbol{u}'|$ for all $\boldsymbol{u}$, $\boldsymbol{u}' \in M_{\text{n-s}}$. Show that for any near-simple integrator $\boldsymbol{v}$ with domain $\mathcal{S}$ and any $\boldsymbol{u}_\star \in M_{\text{n-s}}(\mathcal{S})$ there is a unique $\boldsymbol{u} \in M_{\text{n-s}}$ such that $\boldsymbol{u} = \boldsymbol{u}_\star + H_1(ii_{\boldsymbol{v}}(H_2(\boldsymbol{u})))$.

654 Notes and comments Rather against the general tendency of this treatise, I have given a substantial amount of space to an independent proof of a theorem (654G) which can be regarded as a corollary of a later result in the same section (654L, 654Xd). We could save a little paper by going straight to 654L. But I have tried, for once, to make the path gentler, though longer, by introducing those ideas which can be used in the context of the Riemann-sum integral, before tackling the extra technical difficulties of the S-integral.

There is a switch between the equations $\boldsymbol{u} = \boldsymbol{u}_\star + ii_{\boldsymbol{v}}(\bar{h}(\boldsymbol{u},\boldsymbol{y}))$ and $\boldsymbol{x} = \boldsymbol{x}_\star + \mathrm{S}ii_{\boldsymbol{v}}(\bar{h}(\boldsymbol{x},\boldsymbol{y}))_<$. In the first, we are dealing throughout with (locally) moderately oscillatory processes. In the second, while $\boldsymbol{v}$ remains a locally near-simple local integrator, the other variables are locally S-integrable. So we need a locally S-integrable indefinite integral, and the definition in §646 led to a locally near-simple indefinite S-integral, just as the definition in §613 led to a locally near-simple indefinite Riemann-sum integral. I have therefore turned to what one might call a previsible indefinite S-integral. (There is an immediate suggestion that we could look at previsible definite S-integrals, but I shall not pursue this here.) A re-shuffling of the notion is in 654Xe(iii).

The idea of the proof here is to follow the standard method in elementary courses on ordinary differential equations, using a contraction mapping on a suitable metric space. The trick is to find an appropriate metric. In 654E I show that the norm $\boldsymbol{u} \mapsto \|\sup|\boldsymbol{u}|\|_2$ will serve in a special case which turns out to be adequate. In fact I think that the F-norm j of the proof of that lemma is closer to the heart of the matter, as its formula carries a hint of the way in which we have to treat martingales and processes of bounded variation differently.

Of course the method can be adapted to go a great deal farther. I think that we can reach 654Ya without adding any substantial new ideas to those in the proof of 654G. I note also that the Lipschitz condition can be significantly relaxed (654Xb, and see Protter 05, §V.3).

Version of 13.2.21

655 The Black-Scholes model

This volume is supposed to be an introduction to stochastic integration for a mathematically sophisticated but otherwise untutored readership. You will find it difficult to persuade anyone else to take your efforts seriously if you do not have something to say about its most famous applications, starting with Black & Scholes 73. I will therefore take the space to give a very short account of the simplest model derived by their method, even though the mathematical content is no more than direct quotes from the work so far, and all the interesting ideas relate to the theory of financial markets.

655A Stochastic differential equations In §§651 and 653, I expressed every result in terms of integral equations; so that Theorem 651B, for instance, reads

$$\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{z} = \int_{\mathcal{S}} \boldsymbol{u}\times\boldsymbol{z}\,d\boldsymbol{v}.$$

But the mnemonic for it, in the style of §617, would be

$$d\boldsymbol{z} \sim \boldsymbol{z}\,d\boldsymbol{v},$$

and some authors are happy to express this in the form $\frac{d\boldsymbol{z}}{d\boldsymbol{v}} = \boldsymbol{z}$. Similarly, where in Theorem 654G I write

$$\boldsymbol{u} = \boldsymbol{u}_\star + ii_{\boldsymbol{v}}(\bar{h}(\boldsymbol{u},\boldsymbol{y})),$$

others might write

$$u_{\min\mathcal{S}} = u_{\star,\min\mathcal{S}}, \quad d\boldsymbol{u} \sim d\boldsymbol{u}_\star + \bar{h}(\boldsymbol{u},\boldsymbol{y})d\boldsymbol{v}$$

or perhaps

$$u_{\min\mathcal{S}} = u_{\star,\min\mathcal{S}}, \quad \frac{d\boldsymbol{u}}{d\boldsymbol{v}} = \frac{d\boldsymbol{u}_\star}{d\boldsymbol{v}} + \bar{h}(\boldsymbol{u},\boldsymbol{y}).$$

I myself prefer to regard these alternative expressions as potentially suggestive abbreviations. For instance, they erase the distinction between the Riemann-sum integral and the S-integral; as we have seen in 654L, this can be a helpful stimulus, but does not release us from the obligation to find sufficient conditions on the inputs $\boldsymbol{v}$, $\boldsymbol{x}_\star$, $\boldsymbol{y}$ and h there.

655B A model of stock prices For the rest of this section, I will suppose that $(\mathfrak{A},\bar{\mu},[0,\infty[\,,\langle\mathfrak{A}_t\rangle_{t\ge0},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ is a right-continuous real-time stochastic integration structure and $\mathcal{S}$ is a non-empty ideal of $\mathcal{T}$, that is, an order-convex sublattice with least member $\check{0} = \min\mathcal{T}$. Let $\boldsymbol{\iota}$ be the identity process (612F) and $\boldsymbol{w}$ a Brownian-type process on $\mathcal{S}$, that is, a locally jump-free local martingale with domain $\mathcal{S}$ and quadratic variation $\boldsymbol{w}^* = \boldsymbol{\iota}\restriction\mathcal{S}$ (653K). Consider the differential equation

$$d\boldsymbol{u} \sim \alpha\boldsymbol{u}d\boldsymbol{\iota} + \beta\boldsymbol{u}d\boldsymbol{w}, \quad u_{\check{0}} = u_\star$$

or

$$\boldsymbol{u} = u_\star\mathbf{1} + \alpha ii_{\boldsymbol{\iota}}(\boldsymbol{u}) + \beta ii_{\boldsymbol{w}}(\boldsymbol{u}) = u_\star\mathbf{1} + ii_{\tilde{\boldsymbol{w}}}(\boldsymbol{u})$$

where $\tilde{\boldsymbol{w}} = \alpha\boldsymbol{\iota} + \beta\boldsymbol{w}$ and $u_\star \in L^0(\mathfrak{A}_0)$. Then $\tilde{\boldsymbol{w}}$ is a locally jump-free local integrator and its quadratic variation $\tilde{\boldsymbol{w}}^*$ is

$$[\tilde{\boldsymbol{w}}\overset{*}{\restriction}\tilde{\boldsymbol{w}}] = \alpha^2[\boldsymbol{\iota}\overset{*}{\restriction}\boldsymbol{\iota}] + 2\alpha\beta[\boldsymbol{\iota}\overset{*}{\restriction}\boldsymbol{w}] + \beta^2[\boldsymbol{w}\overset{*}{\restriction}\boldsymbol{w}] = \beta^2\boldsymbol{\iota}$$

by 624C, because $\boldsymbol{\iota}$ is locally of bounded variation and both $\boldsymbol{\iota}$ and $\boldsymbol{w}$ are locally jump-free. So the equation has solution

$$\boldsymbol{u} = u_\star\,\overline{\exp}(\tilde{\boldsymbol{w}} - \tfrac12\beta^2\boldsymbol{\iota})$$

(put 651B and 613L(b-ii) together, as in 651Xa), which is a locally jump-free local integrator and is unique (654G, with $h(x,y) = x$ and $\boldsymbol{u}_\star = u_\star\mathbf{1}$).

655C A model for options Now suppose that we have an 'option' in a 'stock' whose value is accurately modelled by the process $\boldsymbol{u}$. Our objective is to find a rational approach leading to a way of determining the value $\boldsymbol{v}$ of this option. We assume that $\boldsymbol{v}$ depends on the time and the value of $\boldsymbol{u}$ at that time; that is, that there is a function h such that $\boldsymbol{v} = \bar{h}(\boldsymbol{u},\boldsymbol{\iota})$. (If $\boldsymbol{u}$ and $\boldsymbol{v}$ are represented, in the manner of 631D, by real-valued processes $\langle U_t\rangle_{t\ge0}$ and $\langle V_t\rangle_{t\ge0}$ with càdlàg sample paths, we are supposing that $V_t(\omega) = h(U_t(\omega),t)$ for most pairs (ω,t).) The terms of the option will give us some information about the function h; for instance, a call option, allowing us to buy a quantity c of the stock for a strike price x_1 at expiry time t_1, will then have value $h(x,t_1) = c\max(x-x_1,0)$, because we shall be able to buy the stock at price x_1 and sell it at price x; if $x \le x_1$, we just do nothing.

We assume also that h is twice continuously differentiable, with partial derivatives h_1, h_2 and second partial derivatives $h_{11},\dots,h_{22}$. Then we have

$$\begin{aligned}\boldsymbol{v} = \bar{h}(\boldsymbol{u},\boldsymbol{\iota}) &= \bar{h}(u_\star,0) + ii_{\bar{h}(\boldsymbol{u},\boldsymbol{\iota})}(\mathbf{1})\\ &= \bar{h}(u_\star,0) + ii_{\boldsymbol{u}}(\bar{h}_1(\boldsymbol{u},\boldsymbol{\iota})) + ii_{\boldsymbol{\iota}}(\bar{h}_2(\boldsymbol{u},\boldsymbol{\iota})) + \frac12 ii_{\boldsymbol{u}^*}(\bar{h}_{11}(\boldsymbol{u},\boldsymbol{\iota}))\end{aligned}$$

by 619K, because $\boldsymbol{u}$ and $\boldsymbol{\iota}$ are jump-free local integrators, and $\boldsymbol{\iota}^* = [\boldsymbol{\iota}\overset{*}{\restriction}\boldsymbol{u}] = \mathbf{0}$. In the differential form, this becomes

$$d\boldsymbol{v} \sim \bar{h}_1(\boldsymbol{u},\boldsymbol{\iota})d\boldsymbol{u} + \bar{h}_2(\boldsymbol{u},\boldsymbol{\iota})d\boldsymbol{\iota} + \frac12\bar{h}_{11}(\boldsymbol{u},\boldsymbol{\iota})d\boldsymbol{u}^*.$$

To get an expression for the term $ii_{\boldsymbol{u}^*}(\dots)$, observe that $\boldsymbol{u}^* = (ii_{\tilde{\boldsymbol{w}}}(\boldsymbol{u}))^*$ (617Kc) and we can use 617Q(a-iii) to see that

$$ii_{\boldsymbol{u}^*}(\bar{h}_{11}(\boldsymbol{u},\boldsymbol{\iota})) = ii_{\tilde{\boldsymbol{w}}^*}(\bar{h}_{11}(\boldsymbol{u},\boldsymbol{\iota})\times\boldsymbol{u}^2) = \beta^2 ii_{\boldsymbol{\iota}}(\bar{h}_{11}(\boldsymbol{u},\boldsymbol{\iota})\times\boldsymbol{u}^2).$$

So we have

$$\boldsymbol{v} = \bar{h}(u_\star, 0) + ii_{\boldsymbol{u}}(\bar{h}_1(\boldsymbol{u}, \boldsymbol{\iota})) + ii_{\boldsymbol{\iota}}(\bar{h}_2(\boldsymbol{u}, \boldsymbol{\iota}) + \frac{1}{2}\beta^2\boldsymbol{u}^2 \times \bar{h}_{11}(\boldsymbol{u}, \boldsymbol{\iota}))$$

or, using the mnemonic for 617Q in the form $d\boldsymbol{u}^* \sim \boldsymbol{u}^2 d\boldsymbol{w}^* \sim \beta^2\boldsymbol{u}^2 d\boldsymbol{\iota}$,

$$d\boldsymbol{v} \sim \bar{h}_1(\boldsymbol{u}, \boldsymbol{\iota})d\boldsymbol{u} + (\bar{h}_2(\boldsymbol{u}, \boldsymbol{\iota}) + \frac{1}{2}\beta^2\boldsymbol{u}^2 \times \bar{h}_{11}(\boldsymbol{u}, \boldsymbol{\iota}))d\boldsymbol{\iota}.$$

655D Hedging and a risk-free portfolio Still supposing that there is such a function h, consider a hedged version of the option. In addition to the option, we 'hedge' by a suitably varying quantity $\bar{g}(\boldsymbol{u}, \boldsymbol{\iota})$ in the stock $\boldsymbol{u}$ to give us a portfolio $\tilde{\boldsymbol{v}} = \boldsymbol{v} - ii_{\boldsymbol{u}}(\bar{g}(\boldsymbol{u}, \boldsymbol{\iota}))$ in which the (so far unknown) value of $\boldsymbol{v}$ is modified by the accumulated losses and gains of our hedging strategy. Let me try to explain the intuition behind this. Actually I should perhaps begin with an explanation of the minus sign. The idea of a 'hedge' is that we can 'sell the stock short', that is, sell stock we don't necessarily possess; in a strictly regulated market, this will be to some extent controlled by a requirement that we should borrow some stock from a legitimate holder, and then sell that; but in any case, we take cash now, and promise to buy the stock back soon *at the price then ruling*, either because we have to return it to its original owner, or because we never delivered it. This is not an option, it is a contract. From the point of view of our counterparty, it is just like buying real stock. Of course they have to trust us, but it is part of the theory of 'perfect markets' that the agents do trust each other. There is a question about who decides the buy-back time, but our counterparty doesn't much mind, because they will be able to buy the stock in the market whenever we ask (remember, we shall be paying the price at that time, and this is a perfect market, so there will always be buyers and sellers at that price). The idea behind this is that if we possess a call option, we stand to make money if the stock rises in value, and not if it falls; by going short, we hedge our bet to make our prospects more even.

We allow g to take negative values; this is because we can 'go long', that is, buy some stock with the intention of selling it again if our strategy calls on us to do so. Note that we believe that $\boldsymbol{u}$ is jump-free, so can imagine adjusting the hedge rapidly compared with changes in $\boldsymbol{u}$.

We shall have

$$\tilde{\boldsymbol{v}} = \bar{h}(u_\star, 0) + ii_{\boldsymbol{u}}(\bar{h}_1(\boldsymbol{u}, \boldsymbol{\iota}) - \bar{g}(\boldsymbol{u}, \boldsymbol{\iota})) + ii_{\boldsymbol{\iota}}(\bar{h}_2(\boldsymbol{u}, \boldsymbol{\iota}) + \frac{1}{2}\beta^2\boldsymbol{u}^2 \times \bar{h}_{11}(\boldsymbol{u}, \boldsymbol{\iota}))$$

or

$$\tilde{v}_{\breve{0}} = \bar{h}(u_\star, 0), \quad d\tilde{\boldsymbol{v}} \sim (\bar{h}_1(\boldsymbol{u}, \boldsymbol{\iota}) - \bar{g}(\boldsymbol{u}, \boldsymbol{\iota}))d\boldsymbol{u} + (\bar{h}_2(\boldsymbol{u}, \boldsymbol{\iota}) + \frac{1}{2}\beta^2\boldsymbol{u}^2 \times \bar{h}_{11}(\boldsymbol{u}, \boldsymbol{\iota}))d\boldsymbol{\iota}.$$

So if we set $g = h_1$, we get

$$\tilde{\boldsymbol{v}} = \bar{h}(u_\star, 0) + ii_{\boldsymbol{\iota}}(\bar{h}_2(\boldsymbol{u}, \boldsymbol{\iota}) + \frac{1}{2}\beta^2\boldsymbol{u}^2 \times \bar{h}_{11}(\boldsymbol{u}, \boldsymbol{\iota})),$$

$$d\tilde{\boldsymbol{v}} \sim (\bar{h}_2(\boldsymbol{u}, \boldsymbol{\iota}) + \frac{1}{2}\beta^2\boldsymbol{u}^2 \times \bar{h}_{11}(\boldsymbol{u}, \boldsymbol{\iota}))d\boldsymbol{\iota}.$$

Now this is 'risk-free'; for a short time interval $[\sigma, \sigma']$,

$$\tilde{v}_{\sigma'} - \tilde{v}_\sigma \doteq (\bar{h}_2(u_\sigma, \sigma) + \frac{1}{2}\beta^2\bar{h}_{11}(u_\sigma, \sigma)) \times (\sigma' - \sigma)$$

is well approximated by something calculable from knowledge of the stopping times σ, σ' and the situation at the starting time σ, but not requiring any foreknowledge of the evolution of $\boldsymbol{u}$ or $\boldsymbol{w}$. Consequently we can compare it with other risk-free investments. Suppose that we can be sure of being able to borrow, or lend, cash, at an interest rate ρ, with complete safety. We are supposed to be operating in a perfect market, in which every agent knows just what we know about $\boldsymbol{w}$ and $\boldsymbol{u}$, and can do the same calculations, so that the process $\bar{h}(\boldsymbol{u}, \boldsymbol{\iota})$ describes the evolution of the market price, either buying or selling, of the option. We therefore expect $\tilde{v}_{\sigma'} - \tilde{v}_\sigma$, the agreed expected profit from holding the portfolio $\tilde{v}$ from time σ to time σ', to be very close to the expected income over that time period if we sell our option and our holding in the stock $\boldsymbol{u}$, and invest the net proceeds in a bond at interest rate ρ.

At this point I need to remark that these net proceeds will not be the current value $\tilde{v}_\sigma$. The process $\tilde{\boldsymbol{v}}$ takes past gains and losses into account in the term $ii_{\boldsymbol{u}}(\bar{g}(\boldsymbol{u}, \boldsymbol{\iota}))$. But these are water under the bridge. Our holding at, and immediately after, the time σ is $v_\sigma - \bar{g}(u_\sigma, \sigma) \times u_\sigma$, because if g is positive, that is, we are shorting the stock, we shall have to buy it back at once to liquidate our position, while if g is negative, that is, we are holding some stock, we will sell it. So we expect that

$$\tilde{v}_{\sigma'} - \tilde{v}_\sigma \doteqdot \rho(v_\sigma - \bar{h}_1(u_\sigma, \sigma) \times u_\sigma) \times (\sigma' - \sigma),$$

that is,

$$d\tilde{\boldsymbol{v}} \sim \rho(\boldsymbol{v} - \bar{h}_1(\boldsymbol{u}, \boldsymbol{\iota}) \times \boldsymbol{u}) d\boldsymbol{\iota},$$

and matching the two formulae for $d\tilde{\boldsymbol{v}}$ we get

$$\bar{h}_2(\boldsymbol{u}, \boldsymbol{\iota}) + \frac{1}{2}\beta^2 \boldsymbol{u}^2 \times \bar{h}_{11}(\boldsymbol{u}, \boldsymbol{\iota}) = \rho(\boldsymbol{v} - \boldsymbol{u} \times \bar{h}_1(\boldsymbol{u}, \boldsymbol{\iota})) = \rho(\bar{h}(\boldsymbol{u}, \boldsymbol{\iota}) - \boldsymbol{u} \times \bar{h}_1(\boldsymbol{u}, \boldsymbol{\iota})).$$

To ensure this, we shall have to have

$$h_2(x, t) + \frac{1}{2}\beta^2 x^2 h_{11}(x, t) = \rho(h(x, t) - x h_1(x, t))$$

for all relevant x and t, that is,

$$\frac{\partial h}{\partial t} + \frac{1}{2}\beta^2 x^2 \frac{\partial^2 h}{\partial x^2} + \rho x \frac{\partial h}{\partial x} - \rho h = 0$$

which is the **Black-Scholes equation** for the evolution of the value $h(x, t)$ of an option in a stock with price x at time t.

655 Notes and comments For obvious reasons, an enormous amount of work has been done on adaptations and correction terms for the Black-Scholes equation. But I will indulge myself by saying why I think that most of this effort has been wasted, at least from the point of view of those who have paid for it. Up to the end of 655C we are enjoying ourselves with some pretty mathematics, and the finance theory of 655D, in the abstract, offers us some interesting new challenges. But look at the 'supposes' and 'assumes' there. Why should we believe that the value of a stock can be described by the formulae of 655B? Empirical evidence from stock exchange records shows that many stocks have followed this pattern for years at a time; but companies do quite often go bankrupt or get taken over, and the model so far has no place for either. Can we be sure that the value of an option depends only on the time and on the present value of the stock? The argument I presented above pays no attention to risk-aversion, because if the rest of the theory were sound, there would be no risk; but even hedge funds are, in practice, risk-averse, and only intermittently do they have blind faith in the theory. Why should the function h, if it exists, be twice continuously differentiable? The model I have described assumes that the parameters α, β and ρ are constant. All of these we can hope to do something about – but only if we have a convincing account of how they will change in the future. There are ways of inferring what the majority of currently active traders expect them to do, but are we supposed to believe that 'the market' is always right about such things, when we read daily of gross blunders, and when interest rates, in particular, are influenced so heavily by government action? I wrote 'we can imagine adjusting the hedge rapidly compared with changes in $\boldsymbol{u}$'. A very large amount of money has been, and is being, spent on speeding up trading processes to make this true. But what if there is enough trading in the options on a stock to make the corresponding hedges a significant influence on the price of the stock?

And then there is all the stuff about perfect markets. One of the greatest advantages claimed for open markets is that prices give a rapid and trustworthy way for information to percolate through the system; but in more than one financial market in 2012-2014, the volume of trading, the bulk of it between rival pricing models, offered large rewards for fraudulent manipulation of price indices, which duly occurred. On most trading days, to be sure, and for most stocks and many options, there are active agents and (among themselves) good information and low transaction costs; but local market failures, when trading in individual stocks is suspended, are common, and more general failures, in which whole sectors are briefly paralyzed, occur in most decades.

One of these failures is particularly associated with the Nobel prizewinners Myron Scholes and Robert Merton, who belonged to the limited partnership Long-Term Capital Management which came spectacularly to grief in 1998. For an account of the rise and fall of LTCM, see Lowenstein 01. Their business model amounted to betting when the odds seemed right. Typically they were accepting other peoples' hedges, acting as a kind of unregulated insurance company. Of course they could do this only because *not* everyone took exactly the same view of the options involved, and the margins on the bets were necessarily narrow, so the success of the fund depended on betting very heavily. They reached the level at which their own transactions were having a substantial effect, and at which quite possibly temporary variations in market prices were leading to changes in the value of their assets sufficient to alarm not only their investors but their counterparties and creditors and the Federal Reserve Bank of New York. With hindsight we can see that they made a string of bad decisions; but a common feature of these was that they assumed too readily that they could wait for anomalies to evaporate. In particular, they tried to take advantage of a reluctance on the part of other investors to hold Russian bonds, both private and governmental. This anomaly did not evaporate, and culminated in defaults. LTCM seem to have been using pricing models which took inadequate account of such risks.

One would have hoped that the catastrophe would have led to more realistic assessments by financiers. But that didn't happen. Lehman Brothers under Richard Fuld were acting for LTCM all through the crisis of 1998. In 2008 it was Fuld's turn to destroy his company by obstinately hanging on through warning signals. Nobody that year was imprudently exposed to Russia. They had found ways of being imprudently exposed to mortgages.

To put it bluntly: I do not think that this can be fixed by tinkering. Option pricing began because people had good reasons for wanting to buy options, and Black, Scholes, Merton and others offered formulae with reasonably plausible justifications, at least in comparison with what had gone before. But the many mathematical refinements which have been developed since have been no substitute for good judgement. The enormous structures which have been built on these ideas in some countries are, I believe, cancerous growths, and should be starved into remission before radical surgery becomes forced.

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References for Volume 6

Bain A. [09] *Stochastic Calculus.* `http://www.chiark.greenend.org.uk/~alanb/`, 25.5.09.

Bichteler K. [79] 'Stochastic integrators', Bull. Amer. Math. Soc. (2) 1 (1979) 761-765. [627I.]

Bichteler K. [81] 'Stochastic integration and L^p theory of martingales', Ann. Prob. 9 (1981) 49-89. [649C.]

Black F. & Scholes M. [73] 'The pricing of options and corporate liabilities', J. Political Economy 81 (1973) 637-659. [§655 *intro.*]

Burkholder D.L. [66] 'Martingale Transforms', Ann. of Math. Stat. 37 (1966) 1494-1504. [621H.]

Dellacherie C. & Meyer P.-A. [82] *Probabilities and Potential B. Theory of Martingales.* Elsevier, 1982 (North-Holland Mathematics Studies 72). [627I.]

Itô K. [44] 'Stochastic Integral', Proc. Imperial Acad. Tokyo 20 (1944) 519-524. [Vol. 6 *intro.*, Chap. 62 *intro.*]

Karandikar R.L. [95] 'On pathwise stochastic integration', Stochastic Processes and their Applications 57 (1995) 11-18. [649C.]

Karatzas I. & Shreve S. [91] *Brownian Motion and Stochastic Calculus.* Springer, 1991. [621J.]

Lowenstein R. [01] *When Genius Failed: the rise and fall of Long-Term Capital Management.* Fourth Estate, 2001. [§655 *notes*.]

Nutz M. [p11] 'Pathwise construction of stochastic integrals', 12.6.12; `www.math.columbia.edu/~mnutz/docs/pathwInt.pdf`. [649E.]

Protter P.E. [05] *Stochastic Integration and Differential Equations.* Springer, 2005. [§627 *notes*, 634Yd, §64 *intro.*, §643 *notes*, §645 *notes*, §654 *notes*.]

Rogers L.C.J. & Williams D. [00] *Diffusions, Markov Processes and Martingales,* vol. II, *Itô Calculus.* Cambridge U.P, 2000. [§641 *notes*, §645 *notes*.]

Sato K.-I. [13] *Lévy processes and infinitely divisible distributions.* Cambridge University Press, 2013. [§652 *notes*.]