Version of 3.2.14/4.5.20

Chapter 64

The fundamental theorem of martingales and the S-integral

To my mind, the 'Riemann-sum' integral of §613 is the natural starting point for a theory of stochastic integration, and it has a rich assortment of properties. But if you are acquainted with the Lebesgue-Stieltjes integral, you will have noticed that I have not given results corresponding to the standard convergence theorems of §123, and if you have taken the trouble to check, you will have noticed that they aren't true of the integral as presented so far. If we make the right modifications, however, we do have a kind of sequential smoothness (644C) which can, with some difficulty, be used as the basis for what I will call the 'S-integral' (645P). In fact the S-integral is much closer than the Riemann-sum integral to the standard stochastic integral developed in Protter 05.

To do this we need to know quite a lot more about stochastic processes. In §641 I describe the 'previsible version' of a near-simple process, which corresponds to the càglàd function equal except at jump points to a càdlàg function of a real variable. Looking at pointwise limits of sequences of previsible versions, we are led to the previsible processes of §642, which have the kind of measurability demanded of an integrand in the S-integral (645I). But the really important fact, if we are going to have the S-integral for martingale integrators which are not jump-free, is the fundamental theorem of martingales: under certain conditions, an integrator can be expressed not just as the sum of a virtually local martingale and a process of bounded variation, as in the Bichteler-Dellacherie theorem (627J), but as such a sum in which the virtually local martingale has small residual oscillation (643M).

With the S-integral defined, we can look at its properties, which by and large correspond to those of the Riemann-sum integral as established in chapters 61-63. Many of the details are not trivial, and I work through them in §646-648, with an S-integral version of Itô's formula (646T).

I end the chapter with a brief note (§649) left over from Chapter 63, on Riemann-sum integrals, in the classical context of progressively measurable stochastic processes defined on a probability space, which can be calculated from sample paths, one path at a time; for non-decreasing integrators, we can use a Stieltjes integral on each path to calculate the S-integral.

Version of 20.1.21/29.9.23

641 Previsible versions

In §618 I introduced 'jump-free' processes without going into the question of what the 'jumps' were which they were free of. We now need to look at the structure of processes which are not jump-free. In the standard model of locally near-simple processes as those representable by processes with càdlàg sample paths (631D), we have direct descriptions of σ-algebras Σ_{h^-} and random variables X_{h^-} defined in terms of observations taken *before* a stopping time h, rather than at the stopping time, as in 612H. I present these descriptions in 642E, following corresponding definitions in the more abstract language I favour in this volume (641B, 641F). Once we have got hold of the previsible version $\boldsymbol{u}_<$ of a near-simple process, we have an expression for the residual oscillation of $\boldsymbol{u}$ in terms of $\boldsymbol{u}-\boldsymbol{u}_<$ (641Nb, 642Ga). For moderately oscillatory processes which are not near-simple, we do not have such a direct description of their jumps, but the construction of the previsible version still works (641L), and we have effective results on indefinite integrals (641Q) and quadratic variations (641R).

641B The algebras $\mathfrak{A}_{\mathcal{S}<\tau}$ (a) Definition If $\mathcal{S}$ is a sublattice of $\mathcal{T}$ and $\tau\in\mathcal{T}$, let $\mathfrak{A}_{\mathcal{S}<\tau}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{a\cap[\![\sigma<\tau]\!]:\sigma\in\mathcal{S},\,a\in\mathfrak{A}_\sigma\}$. $\mathfrak{A}_{\mathcal{S}<\tau}\subseteq\mathfrak{A}_\tau$. I will write $\mathfrak{A}_{<\tau}$ for $\mathfrak{A}_{\mathcal{T}<\tau}$.

(b) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.

(i) If $\sigma \le \tau$ in $\mathcal{T}$ then $\mathfrak{A}_{\mathcal{S}<\sigma} \subseteq \mathfrak{A}_{\mathcal{S}<\tau}$.

(ii) If $\sigma \in \mathcal{S}$, $\tau \in \mathcal{T}$ and $u \in L^0(\mathfrak{A}_\sigma)$, then $u \times \chi[\![\sigma < \tau]\!] \in L^0(\mathfrak{A}_{\mathcal{S}<\tau})$.

(iii) If $\mathcal{S}'$ is a sublattice of $\mathcal{T}$ covering $\mathcal{S}$, then $\mathfrak{A}_{\mathcal{S}<\tau} \subseteq \mathfrak{A}_{\mathcal{S}'<\tau}$ for every $\tau \in \mathcal{T}$.
In particular, $\mathfrak{A}_{\mathcal{S}<\tau} \subseteq \mathfrak{A}_{\mathcal{S}'<\tau}$ whenever $\mathcal{S} \subseteq \mathcal{S}'$ and $\tau \in \mathcal{T}$.

(iv) Now suppose that $\mathcal{S}$ is finitely full and that $\tau \in \mathcal{S}$. Then $\mathfrak{A}_{\mathcal{S}<\tau}$ is the closed subalgebra $\mathfrak{C}$ generated by $\{[\![\sigma < \tau]\!] : \sigma \in \mathcal{S}\}$.

(c) Suppose that $\tau \in \mathcal{T}$, I is a non-empty finite sublattice of $\mathcal{T}$ and $(\tau_0, \dots, \tau_n)$ linearly generates the I-cells (611L). Let $\mathfrak{B}$ be the set of those $b \in \mathfrak{A}$ such that

$b \cap [\![\tau \le \tau_0]\!]$ is either $[\![\tau \le \tau_0]\!]$ or 0, $\quad b \cap [\![\tau_n < \tau]\!] \in \mathfrak{A}_{\tau_n}$,

for every $i < n$ there is an $a \in \mathfrak{A}_{\tau_i}$ such that $b \cap [\![\tau_i < \tau]\!] \setminus [\![\tau_{i+1} < \tau]\!] = a \cap [\![\tau_i < \tau]\!] \setminus [\![\tau_{i+1} < \tau]\!]$.

Then $\mathfrak{A}_{I<\tau} = \mathfrak{B}$.

(d) If $t \in T$ then $\mathfrak{A}_{<\check{t}} = \bigvee_{s<t} \mathfrak{A}_s$.

(e) If $\tau \in \mathcal{T}$ then $\mathfrak{A}_{<\tau}$ is the closed subalgebra of $\mathfrak{A}$ generated by

$$\{a : \text{there is a } t \in T \text{ such that } a \in \mathfrak{A}_t \text{ and } a \subseteq [\![\tau > t]\!]\}.$$

641C Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $C \subseteq \mathcal{T}$ a non-empty set with supremum τ.
(a) $\mathfrak{A}_{\mathcal{S}<\tau} = \bigvee_{\sigma \in C} \mathfrak{A}_{\mathcal{S}<\sigma}$.
(b) Now suppose that $C \subseteq \mathcal{S}$. Set $a = \inf_{\sigma \in C} [\![\sigma < \tau]\!]$. Then

$$\bigvee_{\sigma \in C} \mathfrak{A}_\sigma = \{(b \setminus a) \cup (c \cap a) : b \in \mathfrak{A}_\tau,\ c \in \mathfrak{A}_{\mathcal{S}<\tau}\}.$$

641D Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\tau \rangle_{\tau \in \mathcal{S}}$ an L^1-process with a previsible variation $\boldsymbol{v}^\# = \langle v^\#_\sigma \rangle_{\sigma \in \mathcal{S}}$. Then $v^\#_\tau \in L^0(\mathfrak{A}_{\mathcal{S}<\tau})$ for every $\tau \in \mathcal{S}$.

641E Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ a fully adapted process and I a non-empty finite sublattice of $\mathcal{S}$.

(a) For any $\tau \in \mathcal{T}$ there is an element $u_{I<\tau}$ of $L^0(\mathfrak{A}_{I<\tau})$ defined by saying that $[\![\tau \le \min I]\!] \subseteq [\![u_{I<\tau} = 0]\!]$ and

$$[\![\sigma < \tau]\!] \setminus \sup_{\sigma' \in I}([\![\sigma < \sigma']\!] \cap [\![\sigma' < \tau]\!]) \subseteq [\![u_{I<\tau} = u_\sigma]\!]$$

for every $\sigma \in I$.

(b) If $(\sigma_0, \dots, \sigma_n)$ linearly generates the I-cells, then

$$[\![\tau \le \sigma_0]\!] \subseteq [\![u_{I<\tau} = 0]\!], \quad [\![\sigma_n < \tau]\!] \subseteq [\![u_{I<\tau} = u_{\sigma_n}]\!],$$

$$[\![\sigma_i < \tau]\!] \cap [\![\tau \le \sigma_{i+1}]\!] \subseteq [\![u_{I<\tau} = u_{\sigma_i}]\!] \text{ for every } i < n.$$

(c) The process $\langle u_{I<\tau} \rangle_{\tau \in \mathcal{T}}$ is fully adapted.
(d) If J is a maximal totally ordered subset of I, then $u_{J<\tau} = u_{I<\tau}$ for every $\tau \in \mathcal{T}$.
(e) If $\tau \in \mathcal{S}$ then $u_{I<\tau} = u_{(I \wedge \tau)<\tau}$.

641F Definition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ a fully adapted process. For $\tau \in \mathcal{S}$, set

$$u_{<\tau} = \lim_{I \uparrow \mathcal{I}(\mathcal{S})} u_{I<\tau}$$

when the limit exists in $L^0(\mathfrak{A})$. If $u_{<\tau}$ is defined for every $\tau \in \mathcal{S}$, I will call $\boldsymbol{u}_< = \langle u_{<\tau} \rangle_{\tau \in \mathcal{S}}$ the **previsible version** of $\boldsymbol{u}$.

641G Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.

(a) Let $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ be a fully adapted process with a previsible version $\boldsymbol{u}_<=\langle u_{<\sigma}\rangle_{\sigma\in\mathcal{S}}$.

(i) $u_{<\sigma}\in L^0(\mathfrak{A}_{\mathcal{S}<\sigma})$ for every $\sigma\in\mathcal{S}$.

(ii) $\boldsymbol{u}_<$ is fully adapted to $\langle\mathfrak{A}_t\rangle_{t\in T}$.

(iii) $[\![\boldsymbol{u}_<\ne\boldsymbol{0}]\!]\subseteq[\![\boldsymbol{u}\ne\boldsymbol{0}]\!]$.

(iv) If $z\in L^0(\mathfrak{A}\cap\bigcap_{\sigma\in\mathcal{S}}\mathfrak{A}_\sigma)$, then $z\boldsymbol{u}$ has a previsible version, which is $z\boldsymbol{u}_<$.

(v) If $\mathcal{S}$ has a least element, then $u_{<\min\mathcal{S}}=0$.

(vi) If $\mathcal{S}'$ is a sublattice of $\mathcal{S}$ which covers $\mathcal{S}$, then $\boldsymbol{u}\restriction\mathcal{S}'$ has a previsible version, which is $\boldsymbol{u}_<\restriction\mathcal{S}'$.

(vii) Suppose that $\boldsymbol{u}$ is order-bounded.

(α) For any $\tau\in\mathcal{S}$, $|u_{<\tau}|\le\sup_{\sigma\in\mathcal{S}}(|u_\sigma|\times\chi[\![\sigma<\tau]\!])$.

(β) For any τ, $\tau'\in\mathcal{S}$,

$$|u_{<\tau'}|\times\chi[\![\tau<\tau']\!]\le\sup_{\sigma\in\mathcal{S}\vee\tau}(|u_\sigma|\times\chi[\![\sigma<\tau']\!]).$$

(γ) $\boldsymbol{u}_<$ is order-bounded and $\sup|\boldsymbol{u}_<|\le\sup|\boldsymbol{u}|$.

(viii) If $\boldsymbol{u}$ is locally order-bounded then $\boldsymbol{u}_<$ is locally order-bounded.

(b) Writing $\boldsymbol{1}^{(\mathcal{S})}$ for the constant process with value $\chi1$ and domain $\mathcal{S}$, its previsible version $\boldsymbol{1}^{(\mathcal{S})}_<$ is defined and equal to $\langle\chi e_\sigma\rangle_{\sigma\in\mathcal{S}}$, where $e_\sigma=\sup_{\sigma'\in\mathcal{S}}[\![\sigma'<\sigma]\!]$ for $\sigma\in\mathcal{S}$.

(c) Suppose that $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a fully adapted process.

(i) $\boldsymbol{u}$ has a previsible version iff $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ has a previsible version for every $\tau\in\mathcal{S}$.

(ii) In this case, $(\boldsymbol{u}\restriction\mathcal{S}\wedge\tau)_<=\boldsymbol{u}_<\restriction\mathcal{S}\wedge\tau$ and

$$(\boldsymbol{u}\restriction\mathcal{S}\vee\tau)_<=(\boldsymbol{u}_<\restriction\mathcal{S}\vee\tau)\times\langle\chi[\![\tau<\sigma]\!]\rangle_{\sigma\in\mathcal{S}\vee\tau}=(\boldsymbol{u}_<\restriction\mathcal{S}\vee\tau)\times\boldsymbol{1}^{(\mathcal{S}\vee\tau)}_<$$

for every $\tau\in\mathcal{S}$.

(d) Suppose that $k\ge1$ is an integer, and $h:\mathbb{R}^k\to\mathbb{R}$ is a continuous function. Take $\boldsymbol{U}=\langle\boldsymbol{u}_i\rangle_{i<k}$ where each $\boldsymbol{u}_i$ is a fully adapted process with domain $\mathcal{S}$ with a previsible version $\boldsymbol{u}_{i<}$, and set $\boldsymbol{U}_<=\langle\boldsymbol{u}_{i<}\rangle_{i<k}$ where $\boldsymbol{u}_{i<}$ is the previsible version of $\boldsymbol{u}_i$ for each i. Define $\bar h:(L^0)^k\to L^0$ and $\bar h\boldsymbol{U}=\bar h\circ\boldsymbol{U}$ as in 619E-619F. Then $\bar h\boldsymbol{U}$ has a previsible version $(\bar h\boldsymbol{U})_<=\bar h\circ(\boldsymbol{U}_<)\times\boldsymbol{1}^{(\mathcal{S})}_<$. If $h(0,\dots,0)=0$, then $(\bar h\boldsymbol{U})_<=\bar h\circ\boldsymbol{U}_<$.

(e) Let M be the set of those order-bounded processes $\boldsymbol{u}$ with domain $\mathcal{S}$ such that $\boldsymbol{u}$ has a previsible version $\boldsymbol{u}_<$.

(i) M is an f-subalgebra of $M_{\text{o-b}}(\mathcal{S})$, and $\boldsymbol{u}\mapsto\boldsymbol{u}_<:M\to M_{\text{o-b}}(\mathcal{S})$ is an f-algebra homomorphism.

(ii) M is closed for the ucp topology on $M_{\text{o-b}}(\mathcal{S})$, and $\boldsymbol{u}\mapsto\boldsymbol{u}_<:M\to M_{\text{o-b}}(\mathcal{S})$ is continuous.

641H Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\mathcal{S}'$ a sublattice of $\mathcal{S}$ which separates $\mathcal{S}$. If $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is fully adapted and $\tau\in\mathcal{S}$ is such that $u_{<\tau}$ is defined, then $u_{<\tau}=\lim_{I\uparrow\mathcal{I}(\mathcal{S}')}u_{I<\tau}$.

641I Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a simple process with breakpoint string $(\tau_0,\dots,\tau_n)$ and starting value $u_\downarrow$.

(a) $u_{<\tau}$ is defined and

$$\inf_{\sigma\in\mathcal{S}}[\![\tau\le\sigma]\!]\subseteq[\![u_{<\tau}=0]\!],\quad[\![\sigma<\tau]\!]\cap[\![\tau\le\tau_0]\!]\subseteq[\![u_{<\tau}=u_\downarrow]\!]\text{ for every }\sigma\in\mathcal{S},$$

$$[\![\tau_i<\tau]\!]\cap[\![\tau\le\tau_{i+1}]\!]\subseteq[\![u_{<\tau}=u_{\tau_i}]\!]\text{ for every }i<n,\quad[\![\tau_n<\tau]\!]\subseteq[\![u_{<\tau}=u_{\tau_n}]\!]$$

for every $\tau\in\mathcal{S}$.

(b) Writing $\boldsymbol{u}_<$ for the previsible version of $\boldsymbol{u}$,

$$\sup|\boldsymbol{u}|=|u_{\tau_n}|\vee\sup|\boldsymbol{u}_<|.$$

641J Lemma Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$, $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a simple process with starting value $v_\downarrow$ and breakpoint string $(\tau_0,\dots,\tau_n)$, and $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a fully adapted process which has a previsible version $\boldsymbol{u}_<=\langle u_{<\sigma}\rangle_{\sigma\in\mathcal{S}}$. Then

$$\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}=u_{<\tau_0}\times(v_{\tau_0}-v_\downarrow)+\sum_{i=1}^n u_{<\tau_i}\times(v_{\tau_i}-v_{\tau_{i-1}}).$$

641K Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a non-decreasing non-negative fully adapted process and τ a member of $\mathcal{S}$.

(a) If I is a non-empty finite sublattice of $\mathcal{S}$ then $u_{I<\tau} = \sup_{\rho\in I}(u_\rho \times \chi[\![\rho < \tau]\!])$.

(b) If I, $J \in \mathcal{I}(\mathcal{S})$ and $I \subseteq J$ then $u_{I<\tau} \le u_{J<\tau}$.

(c) $u_{<\tau}$ is defined and equal to $\sup_{\rho\in\mathcal{S}}(u_\rho \times \chi[\![\rho < \tau]\!])$.

641L Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{u}$ a fully adapted process with domain $\mathcal{S}$.

(a) If $\boldsymbol{u}$ is non-decreasing and non-negative, it has a previsible version $\boldsymbol{u}_<$; $\boldsymbol{u}_<$ is non-decreasing and $\boldsymbol{u}_< \le \boldsymbol{u}$.

(b) If $\boldsymbol{u}$ is (locally) of bounded variation, it has a previsible version which is (locally) of bounded variation.

(c) If $\boldsymbol{u}$ is (locally) moderately oscillatory, it has a previsible version which is (locally) moderately oscillatory.

641M Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a locally moderately oscillatory process with previsible version $\boldsymbol{u}_< = \langle u_{<\sigma}\rangle_{\sigma\in\mathcal{S}}$. Suppose that $\langle\tau_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathcal{S}$ with supremum τ (taken in $\mathcal{T}$) which belongs to $\mathcal{S}$. Set $a = \inf_{n\in\mathbb{N}}[\![\tau_n < \tau]\!]$. Then $a \subseteq [\![u_{<\tau} = \lim_{n\to\infty} u_{\tau_n}]\!]$.

641N Proposition Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$, and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ an order-bounded process with a previsible version $\boldsymbol{u}_< = \langle u_{<\tau}\rangle_{\tau\in\mathcal{S}}$.

(a) For $\tau \in \mathcal{S}$ set $e_\tau = \sup_{\sigma\in\mathcal{S}}[\![\sigma < \tau]\!]$. Then $\chi e_\tau \times |u_\tau - u_{<\tau}| \le \mathrm{Osclln}(\boldsymbol{u}{\restriction}\mathcal{S}\wedge\tau)$.

(b) If $\boldsymbol{u}$ is near-simple, then $\mathrm{Osclln}(\boldsymbol{u}) = \sup_{\tau\in\mathcal{S}} \chi e_\tau \times |u_\tau - u_{<\tau}|$.

641O Corollary Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$, and $\boldsymbol{u}$ a locally jump-free process with domain $\mathcal{S}$. Then $\boldsymbol{u}_< = \boldsymbol{u} \times \mathbf{1}^{(\mathcal{S})}_<$.

641P Corollary Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$, and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a near-simple process with starting value $u_\downarrow = 0$. Then $\mathrm{Osclln}(\boldsymbol{u}) = \sup_{\tau\in\mathcal{S}} |u_\tau - u_{<\tau}|$.

641Q Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u}$ a locally moderately oscillatory process with previsible version $\boldsymbol{u}_<$, and $\boldsymbol{v}$ a local integrator with previsible version $\boldsymbol{v}_<$. Set $\boldsymbol{w} = ii_{\boldsymbol{v}}(\boldsymbol{u})$. Then $\boldsymbol{w}-\boldsymbol{w}_< = \boldsymbol{u}_< \times (\boldsymbol{v}-\boldsymbol{v}_<)$.

641R Corollary Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$ and $\boldsymbol{v}$ a local integrator with domain $\mathcal{S}$, starting value 0 and quadratic variation $\boldsymbol{v}^*$. Write $\boldsymbol{v}_<$, $\boldsymbol{v}^*_<$ for the previsible versions of $\boldsymbol{v}$ and $\boldsymbol{v}^*$. Then $\boldsymbol{v}^* - \boldsymbol{v}^*_< = (\boldsymbol{v} - \boldsymbol{v}_<)^2$.

641T Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{S}$, $\boldsymbol{u}$ a moderately oscillatory process with previsible version $\boldsymbol{u}_<$ and $\boldsymbol{v}$ an integrator. If either $\boldsymbol{v}$ is jump-free or $\boldsymbol{u}$ is jump-free and has starting value 0, then $\int_{\mathcal{S}} \boldsymbol{u}_< d\boldsymbol{v} = \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$.

641U Lemma Suppose that T has no points isolated on the right. Then whenever $\tau \le \tau'$ in $\mathcal{T}$ and $\epsilon > 0$ there is a $\sigma \in [\tau, \tau']$ such that $[\![\tau < \sigma]\!] = [\![\tau < \tau']\!]$ and $\bar\mu([\![\tau < \tau']\!] \setminus [\![\sigma < \tau']\!]) \le \epsilon$.

641V Proposition Suppose that T has no points isolated on the right. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a moderately oscillatory process. Then $\boldsymbol{u}_{<<} = \boldsymbol{u}_<$.

641W Proposition Suppose that T has no points isolated on the right. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$, $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a moderately oscillatory process and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a near-simple integrator. Then $\int_{\mathcal{S}} \boldsymbol{u}_< d\boldsymbol{v} = \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$.

Version of 1.7.24

642 Previsible processes

I continue the work of §641 with a description of the previsible version of a process defined in the standard way from a probability space and a filtration of σ-algebras (642E-642G). The other objective of the section is to make a step towards a general theory of 'previsible' processes. The point is that among such processes, starting with those of the form $\boldsymbol{u}_<$, a form of sequential convergence (the order*-convergence of 642B) has striking connections with stochastic integration. I will come to this in §644. For the moment, I present a definition of the space M_{pv} of previsible processes, with some of its elementary properties (642D) and a description in terms of suitably measurable processes in the case in which $T = [0, \infty[$ (642L).

642B Order*-convergence in L^0 and $(L^0)^{\mathcal{S}}$ (a) In 367A, I gave a definition of order*-convergent sequence in arbitrary lattices. For our present purposes, it will be enough to know that a sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in $L^0(\mathfrak{A})$ is order*-convergent to $u \in L^0(\mathfrak{A})$ iff it is order-bounded and

$$u = \inf_{n\in\mathbb{N}} \sup_{i\ge n} u_i = \sup_{n\in\mathbb{N}} \inf_{i\ge n} u_i.$$

Another way of expressing this is to say that $\langle u_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to u iff there are a non-decreasing sequence $\langle v_n\rangle_{n\in\mathbb{N}}$ and a non-increasing sequence $\langle w_n\rangle_{n\in\mathbb{N}}$ such that $\sup_{n\in\mathbb{N}} v_n = \inf_{n\in\mathbb{N}} w_n = u$ and $v_n \le u_n \le w_n$ for every n. In this case, u is the limit of $\langle u_n\rangle_{n\in\mathbb{N}}$ for the topology of convergence in measure.

Note that $\langle u_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to u iff $\langle u_n - u\rangle_{n\in\mathbb{N}}$ is order*-convergent to 0 (367Cd), and that if $\langle u_n\rangle_{n\in\mathbb{N}}$ is order-convergent to 0 and $|v_n| \le |u_n|$ for every n, then $\langle v_n\rangle_{n\in\mathbb{N}}$ is order-convergent to 0 (367Cc, 367Bd).

If $\mathfrak{A}$ is expressed as the measure algebra of a probability space (Ω, Σ, μ), and each u_n, u is represented as $f_n^\bullet$, $f^\bullet$ where the f_n, f are measurable real-valued functions defined on Ω, then $\langle u_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to u iff $f(\omega) = \lim_{n\to\infty} f_n(\omega)$ for almost every ω. Hence we see that if $\langle u_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to u and $\langle v_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to v, then $\langle u_n + v_n\rangle_{n\in\mathbb{N}}$, $\langle \bar g(u_n)\rangle_{n\in\mathbb{N}}$ and $\langle u_n \times v_n\rangle_{n\in\mathbb{N}}$ are order*-convergent to $u + v$, $\bar g(u)$ and $u \times v$ respectively, for any continuous $g : \mathbb{R} \to \mathbb{R}$. More generally, if $\langle u_{in}\rangle_{n\in\mathbb{N}}$ is order*-convergent to u_i for $1 \le i \le k$, and $g : \mathbb{R}^k \to \mathbb{R}$ is continuous, then $\langle \bar g(u_{1n}, \dots, u_{kn})\rangle_{n\in\mathbb{N}}$, is order*-convergent to $\bar g(u_1, \dots, u_k)$.

(b) Now suppose that we have a sublattice $\mathcal{S}$ of $\mathcal{T}$. Then $L^0(\mathfrak{A})^{\mathcal{S}}$ is isomorphic, as f-algebra, to $L^0(\mathfrak{A}^{\mathcal{S}})$, where $\mathfrak{A}^{\mathcal{S}}$ is the simple product. So if $\langle \boldsymbol{u}_n\rangle_{n\in\mathbb{N}} = \langle\langle u_{n\sigma}\rangle_{\sigma\in\mathcal{S}}\rangle_{n\in\mathbb{N}}$ is a sequence of processes with domain $\mathcal{S}$, and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is another process with domain $\mathcal{S}$, then

$$\begin{aligned}
&\langle \boldsymbol{u}_n\rangle_{n\in\mathbb{N}} \text{ is order*-convergent to } \boldsymbol{u}\\
&\qquad\iff \langle \boldsymbol{u}_n\rangle_{n\in\mathbb{N}} \text{ is order-bounded and } \boldsymbol{u} = \inf_{n\in\mathbb{N}} \sup_{i\ge n} \boldsymbol{u}_i = \sup_{n\in\mathbb{N}} \inf_{i\ge n} \boldsymbol{u}_i\\
&\qquad\iff \langle u_{n\sigma}\rangle_{n\in\mathbb{N}} \text{ is order-bounded and } u_\sigma = \inf_{n\in\mathbb{N}} \sup_{i\ge n} u_{i\sigma} = \sup_{n\in\mathbb{N}} \inf_{i\ge n} u_{i\sigma}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for every } \sigma \in \mathcal{S}\\
&\qquad\iff \langle u_{n\sigma}\rangle_{n\in\mathbb{N}} \text{ is order*-convergent to } u_\sigma \text{ for every } \sigma \in \mathcal{S}.
\end{aligned}$$

Note that if we have an order*-convergent sequence of fully adapted processes, the limit will also be fully adapted.

(c) A topologically convergent sequence need not be order*-convergent, but if $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in L^0 such that $\sum_{n=0}^\infty \theta(u_n)$ is finite, then $\langle u_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to 0.

(d) The description of order*-convergence in L^0 in terms of pointwise convergence in $\mathcal{L}^0$ makes it easy to see that if $\langle h_n\rangle_{n\in\mathbb{N}}$ is a sequence of Borel measurable functions from $\mathbb{R}^k$ to $\mathbb{R}$, and $h(x) = \lim_{n\to\infty} h_n(x)$ is defined in $\mathbb{R}$ for every $x \in \mathbb{R}^k$, then $\langle \bar h_n(u_1, \dots, u_k)\rangle_{n\in\mathbb{N}}$ is order*-convergent to $\bar h(u_1, \dots, u_k)$ whenever $u_1, \dots, u_k \in L^0$.

(e) Note that if $\langle x_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $L^0(\mathfrak{A})$ then it is order*-convergent iff it is order-bounded, and its order*-limit is then $\sup_{n\in\mathbb{N}} x_n$, which is also its topological limit. At the same time, if $\langle x_n\rangle_{n\in\mathbb{N}}$ is topologically convergent, then its topological limit is an upper bound of $\{x_n : n \in \mathbb{N}\}$, so is again the order*-limit.

642C Definition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. I will say that a process $\boldsymbol{x}$ with domain $\mathcal{S}$ is **previsible** if it belongs to the smallest subset of $(L^0)^{\mathcal{S}}$ which contains $\boldsymbol{u}_<$ for every simple process $\boldsymbol{u}$ and is closed under order*-convergence of sequences in $(L^0)^{\mathcal{S}}$.

642D Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $M_{\mathrm{pv}}(\mathcal{S})$ the space of previsible processes with domain $\mathcal{S}$.

(a) $M_{\mathrm{pv}}(\mathcal{S})$ is an f-subalgebra of $M_{\mathrm{fa}}(\mathcal{S})$, and $\bar{g}\boldsymbol{u} \in M_{\mathrm{pv}}(\mathcal{S})$ whenever $\boldsymbol{u} \in M_{\mathrm{pv}}(\mathcal{S})$ and $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $g(0) = 0$.

(b) $M_{\mathrm{pv}}(\mathcal{S}) \cap M_{\text{o-b}}(\mathcal{S})$ is closed in $M_{\text{o-b}}(\mathcal{S})$ for the ucp topology. $\boldsymbol{u}_< \in M_{\mathrm{pv}}(\mathcal{S})$ for every $\boldsymbol{u} \in M_{\text{n-s}}(\mathcal{S})$.

(c) If $\tau \in \mathcal{S}$, then $M_{\mathrm{pv}}(\mathcal{S} \wedge \tau) = \{\boldsymbol{x}{\restriction}\mathcal{S} \wedge \tau : \boldsymbol{x} \in M_{\mathrm{pv}}(\mathcal{S})\}$.

642E Previsible versions in the standard model of near-simple processes: Proposition Let (Ω, Σ, μ) be a complete probability space and $\langle \Sigma_t \rangle_{t \in [0,\infty[}$ a right-continuous filtration of σ-subalgebras of Σ all containing every negligible subset of Ω. Suppose that we are given a family $\langle U_t \rangle_{t \ge 0}$ of real-valued functions on Ω such that U_t is Σ_t-measurable for every t and $t \mapsto U_t(\omega) : [0,\infty[\to \mathbb{R}$ is càdlàg for every $\omega \in \Omega$. Let $h : \Omega \to [0,\infty[$ be a stopping time, and Σ_{h^-} the σ-subalgebra of Σ generated by $\{E :$ there is a $t \in [0,\infty[$ such that $E \in \Sigma_t$ and $h(\omega) > t$ for every $\omega \in E\}$; define $U_{h^-} : \Omega \to \mathbb{R}$ by setting

$$\begin{aligned} U_{h^-}(\omega) &= \lim_{t \uparrow h(\omega)} U_t(\omega) \text{ if } h(\omega) > 0, \\ &= 0 \text{ otherwise.} \end{aligned}$$

Suppose that $(\mathfrak{A}, \bar{\mu}, \langle \mathfrak{A}_t \rangle_{t \in [0,\infty[})$ and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{T}_f}$ are defined from $(\Omega, \Sigma, \mu, \langle \Sigma_t \rangle_{t \in [0,\infty[})$ and $\langle U_t \rangle_{t \in [0,\infty[}$ as in 612H. Let τ be the stopping time represented by h, and $\boldsymbol{u}_< = \langle u_{<\sigma} \rangle_{\sigma \in \mathcal{T}_f}$ the previsible version of $\boldsymbol{u}$. Then

(a) $\mathfrak{A}_{<\tau} = \{E^\bullet : E \in \Sigma_{h^-}\}$,

(b) $u_{<\tau} = U_{h^-}^\bullet$ in $L^0(\mathfrak{A})$.

642F Corollary Suppose that (Ω, Σ, μ) is a complete probability space, $\langle \Sigma_t \rangle_{t \ge 0}$ a filtration of σ-subalgebras of Σ such that every μ-negligible set belongs to Σ_0, and $(\mathfrak{A}, \bar{\mu}, [0,\infty[, \langle \mathfrak{A}_t \rangle_{t \ge 0}, \mathcal{T}, \langle \mathfrak{A}_\tau \rangle_{\tau \in \mathcal{T}})$ the associated real-time stochastic integration structure.

(a) Suppose that $\mathcal{S}$ is a sublattice of $\mathcal{T}_f$ containing $\check{0}$ and that $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ is a simple process with breakpoint string $(\tau_0, \dots, \tau_n)$ in $\mathcal{S}$ starting from $\tau_0 = \check{0}$. Suppose that $h_0, \dots, h_n : \Omega \to [0,\infty[$ are stopping times representing $\tau_0, \dots, \tau_n$ respectively, starting from $h_0(\omega) = 0$ for every ω, and such that $h_0 \le \ldots \le h_n$. For $i \le n$, let $f_i : \Omega \to \mathbb{R}$ be a measurable function representing $u_{\tau_i} \in L^0(\mathfrak{A})$. If $h : \Omega \to [0,\infty[$ is any stopping time representing a member σ of $\mathcal{S}$, and we set

$$\begin{aligned} f(\omega) &= f_i(\omega) \text{ if } i < n \text{ and } h_i(\omega) \le h(\omega) < h_{i+1}(\omega), \\ &= f_n(\omega) \text{ if } h_n(\omega) \le h(\omega), \end{aligned}$$

then $f^\bullet = u_\sigma$ in $L^0(\mathfrak{A})$.

(b) Now suppose that $\boldsymbol{u}_< = \langle u_{<\sigma} \rangle_{\sigma \in \mathcal{S}}$ is the previsible version of $\boldsymbol{u}$. If $h : \Omega \to [0,\infty[$ is any stopping time representing a member σ of $\mathcal{S}$, and we set

$$\begin{aligned} f_-(\omega) &= 0 \text{ if } h(\omega) = 0, \\ &= f_i(\omega) \text{ if } i < n \text{ and } h_i(\omega) < h(\omega) \le h_{i+1}(\omega), \\ &= f_n(\omega) \text{ if } h_n(\omega) < h(\omega), \end{aligned}$$

then $f_-^\bullet = u_{<\sigma}$ in $L^0(\mathfrak{A})$.

642G Corollary Suppose that $(\Omega, \Sigma, \mu, \langle \Sigma_t \rangle_{t \in [0,\infty[})$, $(\mathfrak{A}, \bar{\mu}, \langle \mathfrak{A}_t \rangle_{t \in [0,\infty[})$, $\langle U_t \rangle_{t \ge 0}$ and $\boldsymbol{u} = \langle u_\tau \rangle_{\tau \in \mathcal{T}_f}$ are as in 642E.

(a) If $h : \Omega \to [0,\infty[$ is a stopping time representing $\tau \in \mathcal{T}_f$, and

$$\begin{aligned} f(\omega) &= \sup_{0<t\le h(\omega)} |U_t(\omega) - \lim_{s\uparrow t} U_s(\omega)| \text{ if } h(\omega) > 0 \\ &= 0 \text{ if } h(\omega) = 0, \end{aligned}$$

then $f^\bullet = \mathrm{Osclln}(\boldsymbol{u}\restriction[\check{0},\tau])$ in $L^0(\mathfrak{A})$.

(b) $\boldsymbol{u}$ is locally jump-free iff $\omega \mapsto U_t(\omega) : [0,\infty[\, \to \mathbb{R}$ is continuous for almost every ω.

642H Previsible σ-algebras: Definitions (a) Given a probability space (Ω,Σ,μ) and a filtration $\langle\Sigma_t\rangle_{t\ge0}$ of σ-subalgebras of Σ, the **previsible σ-algebra** is the σ-algebra Λ_{pv} of subsets of $[0,\infty[\,\times\Omega$ generated by sets $]s,\infty[\,\times E$ where $s \ge 0$ and $E \in \Sigma_s$.

(b) I will say that a family $\langle X_t\rangle_{t\ge0}$ of real-valued functions on Ω is **previsibly measurable** if $(t,\omega) \mapsto X_t(\omega) : [0,\infty[\,\times\Omega \to \mathbb{R}$ is Λ_{pv}-measurable.

642I Proposition Previsibly measurable processes are progressively measurable.

642J Lemma Let (Ω,Σ,μ) be a complete probability space, $\langle\Sigma_t\rangle_{t\ge0}$ a filtration of σ-subalgebras of Σ, Λ_{pv} the associated previsible σ-algebra and W a member of Λ_{pv}.

(a) If $h : \Omega \to [0,\infty]$ is a stopping time, then $\{(t,\omega) : h(\omega) < t\} \in \Lambda_{\mathrm{pv}}$.

(b) $F = \pi_2[W]$ belongs to Σ, where $\pi_2(t,\omega) = \omega$ for $\omega \in \Omega$ and $t \ge 0$.

(c) Now suppose that every Σ_t contains every negligible set. If F is not negligible there is a stopping time $h : \Omega \to [0,\infty]$ such that $\{\omega : (t,h(\omega)) \in W\}$ is not negligible.

642K Proposition Let $(\Omega,\Sigma,\mu,\langle\Sigma_t\rangle_{t\ge0},\langle U_t\rangle_{t\ge0})$ be as in 642E. For $\omega \in \Omega$ set

$$\begin{aligned} U_{t^-}(\omega) &= \lim_{s\uparrow t} U_s(\omega) \text{ if } t > 0, \\ &= 0 \text{ if } t = 0. \end{aligned}$$

(a) If we take a stopping time $h : \Omega \to [0,\infty[$ and define U_{h^-} as in 642E, we have $U_{h^-}(\omega) = U_{h(\omega)^-}(\omega)$ for every ω;

(b) $\langle U_{t^-}\rangle_{t\ge0}$ is previsibly measurable.

642L Theorem Let (Ω,Σ,μ) be a complete probability space and $\langle\Sigma_t\rangle_{t\in[0,\infty[}$ a filtration of σ-subalgebras of Σ all containing every negligible subset of Ω. Let Λ_{pv} be the corresponding previsible σ-algebra. Suppose that $(\mathfrak{A},\bar\mu,\langle\mathfrak{A}_t\rangle_{t\in[0,\infty[})$ is defined from $(\Omega,\Sigma,\mu,\langle\Sigma_t\rangle_{t\in[0,\infty[})$ as in 612H.

(a) Write $\mathcal{L}^0 = \mathcal{L}^0(\Lambda_{\mathrm{pv}})$ for the f-algebra of Λ_{pv}-measurable functions from $[0,\infty[\,\times\Omega$ to $\mathbb{R}$. For every $\phi \in \mathcal{L}^0$, there is a fully adapted process $\boldsymbol{x}_\phi = \langle x_{\phi\sigma}\rangle_{\sigma\in\mathcal{T}_f}$ defined by saying that $x_{\phi\sigma} = \phi_h^\bullet$ whenever $h : \Omega \to [0,\infty[$ is a stopping time representing $\sigma \in \mathcal{T}_f$, where

$$\begin{aligned} \phi_h(\omega) &= 0 \text{ if } h(\omega) = 0, \\ &= \phi(h(\omega),\omega) \text{ for other } \omega \in \Omega, \end{aligned}$$

and now $\boldsymbol{x}_\phi \in M_{\mathrm{pv}} = M_{\mathrm{pv}}(\mathcal{T}_f)$ as defined in 642D.

(b) The map $\phi \mapsto \boldsymbol{x}_\phi : \mathcal{L}^0 \to M_{\mathrm{pv}}$ is a surjective f-algebra homomorphism with kernel

$$\begin{aligned} \{\phi : \phi \in \mathcal{L}^0 &\text{ and there is a } \mu\text{-conegligible set } E \\ &\text{such that } \phi(t,\omega) = 0 \text{ whenever } \omega \in E \text{ and } t > 0\}, \end{aligned}$$

and $\boldsymbol{x}_{g\phi} = \bar g\boldsymbol{x}_\phi$ for every $\phi \in \mathcal{L}^0$ and every Borel measurable $g : \mathbb{R} \to \mathbb{R}$ such that $g(0) = 0$.

(c)(i) If $\langle\phi_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}^0$ converging pointwise to $\phi \in \mathcal{L}^0$, then $\langle\boldsymbol{x}_{\phi_n}\rangle_{n\in\mathbb{N}}$ is order*-convergent to $\boldsymbol{x}_\phi$.

(ii) If $\langle\boldsymbol{x}_n\rangle_{n\in\mathbb{N}}$ is a sequence in M_{pv} which is order*-convergent to $\boldsymbol{x} \in M_{\mathrm{pv}}$, there is a pointwise convergent sequence $\langle\phi_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{L}^0$ such that $\boldsymbol{x}_n = \phi_n$ for every $n \in \mathbb{N}$.

642M Proposition Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ such that $\inf A\in\mathcal{S}$ whenever $A\subseteq\mathcal{S}$ is non-empty and bounded below in $\mathcal{S}$. If $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is moderately oscillatory, there is a $\boldsymbol{u}'\in M_{\text{n-s}}(\mathcal{S})$ such that $\boldsymbol{u}_<=\boldsymbol{u}'_<$.

642N Corollary Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ such that $\inf A\in\mathcal{S}$ whenever $A\subseteq\mathcal{S}$ is non-empty and bounded below in $\mathcal{S}$. If $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is moderately oscillatory, $\boldsymbol{u}_<$ is a previsible process.

642O Proposition (a) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ such that there is a countable set $A\subseteq\mathcal{S}$ which separates $\mathcal{S}$. Then $\boldsymbol{u}_<$ is previsible for every $\boldsymbol{u}\in M_{\text{mo}}(\mathcal{S})$.

(b) If $T\subseteq\mathbb{R}$, then for every sublattice $\mathcal{S}$ of $\mathcal{T}$ there is a countable subset of $\mathcal{S}$ which separates $\mathcal{S}$.

Version of 24.7.17/9.6.21

643 The fundamental theorem of martingales

I come at last to one of the most remarkable properties of martingales: under moderately restrictive conditions, a martingale can be expressed as the sum of a local martingale with small jumps and a process of locally bounded variation (643M). In fact I express the result in terms of the 'residual oscillations' introduced in §618, but these are intimately connected with 'jumps' in sample paths, if we use the standard representation of locally near-simple processes. The proof depends on the notion of 'accessibility' of a stopping time (643C).

643B Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a locally moderately oscillatory process and C a non-empty upwards-directed subset of $\mathcal{S}$ with supremum $\tau\in\mathcal{S}$. Write $\langle u_{<\sigma}\rangle_{\sigma\in\mathcal{S}}$ for the previsible version of $\boldsymbol{u}$ and a for $\inf_{\sigma\in C}[\![\sigma<\tau]\!]$.

(a) Set $w=\lim_{\sigma\uparrow C}u_\sigma$. Then

$$[\![w=u_{<\tau}]\!]\supseteq a,\quad [\![w=u_\tau]\!]\supseteq 1\setminus a.$$

(b) Now suppose that $\boldsymbol{u}$ is a martingale. Write $P_{\mathcal{S}<\tau}:L^1_{\bar\mu}\to L^1_{\bar\mu}$ for the conditional expectation associated with $\mathfrak{A}_{\mathcal{S}<\tau}$. Then $a\subseteq[\![u_{<\tau}=P_{\mathcal{S}<\tau}u_\tau]\!]$.

643C Approachability and accessibility Suppose that $\tau\in\mathcal{T}$.

(a) The **region of accessibility** of τ is

$$\operatorname{acc}(\tau)=\sup_{\emptyset\ne C\subseteq\mathcal{T}\wedge\tau}([\![\sup C=\tau]\!]\setminus\sup_{\sigma\in C}[\![\sigma=\tau]\!]).$$

(b) For $\sigma\in\mathcal{T}$, write $\sigma\ll\tau$ for $\sup_{\rho\in\mathcal{T}}([\![\sigma<\rho]\!]\cap[\![\rho<\tau]\!])$. The **region of approachability** of τ is

$$\operatorname{app}(\tau)=\inf_{\sigma\le\tau}([\![\sigma=\tau]\!]\cup[\![\sigma\ll\tau]\!])$$

so that

$$1\setminus\operatorname{app}(\tau)=\sup_{\sigma\le\tau}([\![\sigma<\tau]\!]\setminus[\![\sigma\ll\tau]\!]).$$

(c) $\operatorname{acc}(\tau)\subseteq\operatorname{app}(\tau)$.

(d)(i) If $\tau\in\mathcal{T}$ then

$$[\![\sup C=\tau]\!]\setminus\sup_{\sigma\in C}[\![\sigma=\tau]\!]=(1\setminus[\![\sup C<\tau]\!])\cap\inf_{\sigma\in C}[\![\sigma<\tau]\!]\in\mathfrak{A}_{<\tau}$$

whenever $\emptyset\ne C\subseteq\mathcal{T}\wedge\tau$, so $\operatorname{acc}(\tau)\in\mathfrak{A}_{<\tau}$.

(ii) If σ, ρ, $\tau\in\mathcal{T}$ then

$$[\![\sigma<\rho]\!]\cap[\![\rho<\tau]\!]=[\![\sigma<\rho]\!]\cap[\![\rho<\tau]\!]\cap[\![\sigma<\tau]\!]=[\![\sigma<\rho\wedge\tau]\!]\cap[\![\rho<\tau]\!]\in\mathfrak{A}_{<\tau},$$

so $[\![\sigma\ll\tau]\!]\in\mathfrak{A}_{<\tau}$ and $\operatorname{app}(\tau)\in\mathfrak{A}_{<\tau}$.

(iii) Note that $\operatorname{acc}(\min\mathcal{T})=0$, while $\operatorname{app}(\min\mathcal{T})=1$.

643D Proposition For $t \in T$, let $\check{t}$ be the constant stopping time at t. Let $T_{\text{r-i}}$ be the set of those $t \in T$ which are isolated on the right, and for $t \in T_{\text{r-i}}$ define $\check{t}^+ \in \mathcal{T}$ by saying that

$$\begin{aligned} \check{t}^+ &= \max \mathcal{T} \text{ if } t = \max T \text{ is the greatest element of } T, \\ &= \check{s} \text{ if } t \text{ is not the greatest element of } T \\ &\qquad\qquad\qquad \text{and } s \text{ is the least element of } T \text{ greater than } t. \end{aligned}$$

Then $\operatorname{app}(\tau) = 1 \setminus \sup_{t \in T_{\text{r-i}}} [\![\tau = \check{t}^+]\!]$ for every $\tau \in \mathcal{T}$.

643E Proposition Suppose that $\tau \in \mathcal{T}$. For non-empty $C \subseteq \mathcal{T} \wedge \tau$, set

$$a_C = [\![\sup C = \tau]\!] \setminus \sup_{\sigma \in C} [\![\sigma = \tau]\!].$$

(a) $a_C = \inf_{\sigma \in C} [\![\sigma < \tau]\!] \setminus [\![\sup C < \tau]\!]$ belongs to $\mathfrak{A}_{<\tau}$ whenever $\emptyset \neq C \subseteq \mathcal{T} \wedge \tau$; so $\operatorname{acc}(\tau) \in \mathfrak{A}_{<\tau}$.

(b) For every non-empty $C \subseteq \mathcal{T} \wedge \tau$ there is a non-empty upwards-directed $D \subseteq \mathcal{T} \wedge \tau$ such that $a_D = a_C$.

$$\begin{aligned} \operatorname{acc}(\tau) = \sup\{[\![\sup C = \tau]\!] \setminus \sup_{\sigma \in C} [\![\sigma = \tau]\!] : \\ C \subseteq \mathcal{T} \wedge \tau \text{ is non-empty and upwards-directed}\}. \end{aligned}$$

(c) If $\upsilon \in \mathcal{T}$, $\operatorname{acc}(\tau) \cap [\![\upsilon = \tau]\!] = \operatorname{acc}(\upsilon) \cap [\![\upsilon = \tau]\!]$.

643F Lemma Suppose that $\langle \mathfrak{A}_t \rangle_{t \in T}$ is right-continuous. Take $\tau \in \mathcal{T}$ and $\epsilon > 0$. For $I \in \mathcal{I}(\mathcal{T} \wedge \tau)$ and $\sigma \leq \tau$, set

$$d_{\sigma I} = (\operatorname{app}(\tau) \setminus \operatorname{acc}(\tau)) \cap [\![\sigma < \tau]\!] \setminus \sup_{\rho \in I}([\![\sigma < \rho]\!] \cap [\![\rho < \tau]\!]), \quad w_{\sigma I} = P_\sigma \chi d_{\sigma I}.$$

For $I \in \mathcal{I}(\mathcal{T} \wedge \tau)$ set $w_I = \sup_{\sigma \leq \tau} w_{\sigma I}$.

(a)(i) $d_{\sigma_0 I} \cap [\![\sigma_0 = \sigma_1]\!] = d_{\sigma_1 I} \cap [\![\sigma_0 = \sigma_1]\!]$ whenever $I \in \mathcal{I}(\mathcal{T} \wedge \tau)$ and σ_0, $\sigma_1 \leq \tau$.

(ii) For any $\sigma \leq \tau$, $\lim_{I \uparrow \mathcal{I}(\mathcal{T} \wedge \tau)} \bar{\mu} d_{\sigma I} = 0$.

(b) For any $I \in \mathcal{I}(\mathcal{T} \wedge \tau)$, $\langle w_{\sigma I} \rangle_{\sigma \leq \tau}$ is fully adapted.

(c) If $I \subseteq J$ in $\mathcal{I}(\mathcal{T} \wedge \tau)$, then $d_{\sigma I} \supseteq d_{\sigma J}$ and $w_{\sigma I} \geq w_{\sigma J}$ for every $\sigma \leq \tau$, and $w_I \geq w_J$.

(d) For $I \in \mathcal{I}(\mathcal{T} \wedge \tau)$, set

$$A_I = \{\sigma : \sigma \leq \tau, [\![w_{\sigma I} > \epsilon]\!] \supseteq [\![\sigma < \tau]\!]\}.$$

(i) $[\![w_I > \epsilon]\!] = \sup_{\sigma \in A_I} [\![\sigma < \tau]\!]$.

(ii) A_I is closed under $\wedge$.

(iii) Set $\bar{\sigma}_I = \inf A_I$. Then

(α) $\bar{\sigma}_I \leq \tau$;

(β) $d_{\bar{\sigma}_I I}$ is the limit $\lim_{\sigma \downarrow A_I} d_{\sigma I}$ for the measure-algebra topology of $\mathfrak{A}$;

(γ) $w_{\bar{\sigma}_I I}$ is the limit $1\lim_{\sigma \downarrow A_I} w_{\sigma I}$ for the norm topology of $L^1_{\bar{\mu}}$;

(δ) $[\![w_I > \epsilon]\!] \subseteq [\![w_{\bar{\sigma}_I I} \geq \epsilon]\!]$.

(iv) If $I \subseteq J$ in $\mathcal{I}(\mathcal{T} \wedge \tau)$ then $A_I \supseteq A_J$ and $\bar{\sigma}_I \leq \bar{\sigma}_J$.

(e) There is an $I \in \mathcal{I}(\mathcal{T} \wedge \tau)$ such that $\mathbb{E}(w_I) \leq 3\epsilon$.

643G Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ a non-negative non-decreasing $\| \|_\infty$-bounded process. Suppose that for every $\epsilon > 0$ there are an $I \in \mathcal{I}(\mathcal{S})$ and a $w \in L^0(\mathfrak{A})$ such that $\|w\|_1 \leq \epsilon$ and $P_\sigma v_\tau - v_\sigma \leq w$ whenever $\sigma \leq \tau$ in $\mathcal{S}$ and $[\![\sigma < \sigma']\!] \cap [\![\sigma' < \tau]\!] = 0$ for every $\sigma' \in I$. Let $\boldsymbol{v}^\#$ be the previsible variation of $\boldsymbol{v}$. Then $\boldsymbol{v}^\#$ is jump-free.

643H Lemma Suppose that $\langle \mathfrak{A}_t \rangle_{t \in T}$ is right-continuous. Take τ_1 in $\mathcal{T}$ and a non-negative $v \in L^0(\mathfrak{A}_{\tau_1}) \cap L^1_{\bar{\mu}}$. Set $v_\sigma = v \times \chi[\![\sigma = \tau_1]\!]$ for $\sigma \in \mathcal{T} \wedge \tau_1$. Then $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{T} \wedge \tau_1}$ is a non-negative non-decreasing submartingale. Let $\boldsymbol{v}^\# = \langle v^\#_\sigma \rangle_{\sigma \in \mathcal{T} \wedge \tau_1}$ be its previsible variation, and $\boldsymbol{v}^\#_< = \langle v^\#_{<\sigma} \rangle_{\sigma \in \mathcal{T} \wedge \tau_1}$ the previsible version of $\boldsymbol{v}^\#$. If $\tau \leq \tau_1$ then $\operatorname{app}(\tau) \setminus \operatorname{acc}(\tau) \subseteq [\![v^\#_\tau = v^\#_{<\tau}]\!]$.

643I Lemma Suppose that $\langle \mathfrak{A}_t \rangle_{t \in T}$ is right-continuous. Take $\tau_1 \in \mathcal{T}$ and a martingale $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \leq \tau_1}$. Suppose that $\epsilon > 0$ is such that $[\![\sigma < \tau_1]\!] \subseteq [\![|u_\sigma| \leq \epsilon]\!]$ for every $\sigma \leq \tau_1$. Then there is a martingale $\tilde{\boldsymbol{u}} = \langle \tilde{u}_\sigma \rangle_{\sigma \leq \tau_1}$ such that $\operatorname{Osclln}(\tilde{\boldsymbol{u}}) \leq 2\epsilon\chi 1$ and $\boldsymbol{u} - \tilde{\boldsymbol{u}}$ is of bounded variation.

643J Corollary Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Take an interval $\mathcal{S}=[\tau,\tau']$ where $\tau\le\tau'$ in $\mathcal{T}$, and a martingale $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$. Suppose that $\epsilon>0$ is such that $[\![\sigma<\tau']\!]\subseteq[\![|u_\sigma-u_\tau|\le\epsilon]\!]$ for every $\sigma\in\mathcal{S}$. Then there is a martingale $\tilde{\boldsymbol{u}}=\langle\tilde u_\sigma\rangle_{\sigma\in\mathcal{S}}$ such that $\mathrm{Osclln}(\tilde{\boldsymbol{u}})\le\epsilon\chi1$, $\tilde u_\tau=0$ and $\boldsymbol{u}-\tilde{\boldsymbol{u}}$ is of bounded variation.

643K Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\langle\tau_n\rangle_{n\in\mathbb{N}}$ a non-decreasing sequence in $\mathcal{S}$ such that $\mathcal{S}\subseteq\bigcup_{n\in\mathbb{N}}[\tau_0,\tau_n]$. Suppose that for each $n\in\mathbb{N}$ we are given a fully adapted process $\boldsymbol{u}_n=\langle u_{n\sigma}\rangle_{\sigma\in\mathcal{S}\cap[\tau_n,\tau_{n+1}]}$ starting from $u_{n\tau_n}=0$.

(a) There is a unique fully adapted process $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ such that

$$u_\sigma=u_{n\sigma}+\sum_{i=0}^{n-1}u_{i\tau_{i+1}}\text{ whenever }\sigma\in\mathcal{S},\ n\in\mathbb{N}\text{ and }\tau_n\le\sigma\le\tau_{n+1}.\qquad(*)$$

(b) If every $\boldsymbol{u}_n$ is a martingale, then $\boldsymbol{u}$ is a martingale.

(c) If every $\boldsymbol{u}_n$ is order-bounded, then $\boldsymbol{u}$ is locally order-bounded and $\mathrm{Osclln}(\boldsymbol{u}\restriction\mathcal{S}\wedge\tau_n)=\sup_{i<n}\mathrm{Osclln}(\boldsymbol{u}_i)$ for every $n\in\mathbb{N}$.

643L Lemma Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$ with a least element and a non-decreasing sequence $\langle\tau_n\rangle_{n\in\mathbb{N}}$ such that $\{\tau_n:n\in\mathbb{N}\}$ is cofinal with $\mathcal{S}$, and $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a martingale. Then for any $\epsilon>0$ there is a local martingale $\tilde{\boldsymbol{u}}=\langle\tilde u_\sigma\rangle_{\sigma\in\mathcal{S}}$ such that $\mathrm{Osclln}(\tilde{\boldsymbol{u}}\restriction\mathcal{S}\wedge\tau)\le\epsilon\chi1$ for every $\tau\in\mathcal{S}$ and $\boldsymbol{u}-\tilde{\boldsymbol{u}}$ is locally of bounded variation.

643M Theorem Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$ with a least element, and $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a semi-martingale. Then for any $\epsilon>0$ there is a local martingale $\tilde{\boldsymbol{v}}=\langle\tilde v_\sigma\rangle_{\sigma\in\mathcal{S}}$ such that $\sup_{\tau\in\mathcal{S}}\mathrm{Osclln}(\tilde{\boldsymbol{v}}\restriction\mathcal{S}\wedge\tau)\le\epsilon\chi1$ and $\boldsymbol{v}-\tilde{\boldsymbol{v}}$ is locally of bounded variation.

643N Lemma Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ with a greatest element, and $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ an integrator. Then for every $\epsilon>0$ there are an L^∞-martingale $\tilde{\boldsymbol{v}}$ and a process $\boldsymbol{v}'$ of bounded variation, both with domain $\mathcal{S}$, such that $\bar\mu[\![\boldsymbol{v}\ne\tilde{\boldsymbol{v}}+\boldsymbol{v}']\!]\le\epsilon$.

643O Lemma Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous, and that $\mathcal{S}$ is a non-empty finitely full sublattice of $\mathcal{T}$ with a greatest member such that $\inf A\in\mathcal{S}$ for every non-empty $A\subseteq\mathcal{S}$. If $\boldsymbol{v}$ is a near-simple integrator with domain $\mathcal{S}$, there are an L^∞-martingale $\tilde{\boldsymbol{v}}$ and a near-simple process $\boldsymbol{v}'$ of bounded variation, both with domain $\mathcal{S}$, such that $[\![\boldsymbol{v}\ne\tilde{\boldsymbol{v}}+\boldsymbol{v}']\!]$ has measure at most ϵ.

Version of 25.8.20

644 Pointwise convergence

It is a remarkable fact that while the Riemann-sum integral, as defined in §613, is not 'sequentially smooth' in the most natural adaptation of the definition in 436A, a variation on this concept (Theorem 644H) gives us a route to a Daniell-type integral, which I will develop in §645.

644B Definitions (a) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. A family A of processes with domain $\mathcal{S}$ is **uniformly order-bounded** if $\sup_{\boldsymbol{u}\in A}\sup|\boldsymbol{u}|$ is defined in L^0.

(b) If $\mathcal{S}$ is a sublattice of $\mathcal{T}$, $M^\uparrow_{\text{n-s}}(\mathcal{S})$ will be the family of non-negative non-decreasing near-simple processes with domain $\mathcal{S}$. Any member of $M^\uparrow_{\text{n-s}}(\mathcal{S})$ will be order-bounded; being non-decreasing, it will be an integrator.

644C Lemma Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ such that $\sup D\in\mathcal{S}$ whenever $D\subseteq\mathcal{S}$ is countable, non-empty and bounded above in $\mathcal{S}$. Let $\langle\boldsymbol{u}_n\rangle_{n\in\mathbb{N}}=\langle\langle u_{n\sigma}\rangle_{\sigma\in\mathcal{S}}\rangle_{n\in\mathbb{N}}$ be a non-increasing sequence of non-negative moderately oscillatory processes such that $\inf_{n\in\mathbb{N}}\boldsymbol{u}_{n<}$, taken in $(L^0)^{\mathcal{S}}$, is zero. Then $\inf_{n\in\mathbb{N}}\int_{\mathcal{S}}\boldsymbol{u}_n d\boldsymbol{v}=0$ for every $\boldsymbol{v}\in M^\uparrow_{\text{n-s}}(\mathcal{S})$.

644D Lemma Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ such that $\sup D\in\mathcal{S}$ whenever $D\subseteq\mathcal{S}$ is countable, non-empty and bounded above in $\mathcal{S}$. Let $\langle\boldsymbol{u}_n\rangle_{n\in\mathbb{N}}=\langle\langle u_{n\sigma}\rangle_{\sigma\in\mathcal{S}}\rangle_{n\in\mathbb{N}}$ be a uniformly order-bounded sequence of moderately oscillatory processes such that $\langle\boldsymbol{u}_{n<}\rangle_{n\in\mathbb{N}}$ is order*-convergent in $(L^0)^{\mathcal{S}}$. If $\boldsymbol{v}\in M^\uparrow_{\text{n-s}}(\mathcal{S})$, $\lim_{n\to\infty}\int_{\mathcal{S}}\boldsymbol{u}_n d\boldsymbol{v}$ is defined in L^0 for the topology of convergence in measure.

644E Corollary Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ such that $\sup D \in \mathcal{S}$ whenever $D \subseteq \mathcal{S}$ is countable, non-empty and bounded above in $\mathcal{S}$, and $\langle \boldsymbol{u}_n \rangle_{n\in\mathbb{N}}$ a uniformly order-bounded sequence of moderately oscillatory processes with domain $\mathcal{S}$ such that $\langle \boldsymbol{u}_{n<} \rangle_{n\in\mathbb{N}}$ is order*-convergent to $\mathbf{0}$ in $(L^0)^{\mathcal{S}}$. Then $\lim_{n\to\infty} \int_{\mathcal{S}} \boldsymbol{u}_n d\boldsymbol{v} = 0$ for $\boldsymbol{v} \in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$.

644F Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, A a uniformly order-bounded subset of $M_{\text{mo}}(\mathcal{S})$, and $\boldsymbol{v}$ an integrator with domain $\mathcal{S}$.

(a)(i) $\{\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v} : \boldsymbol{u} \in A\}$ is topologically bounded in L^0.

(ii) if $\mathcal{S}$ is non-empty,

$$\lim_{\tau\uparrow\mathcal{S}} \sup_{\boldsymbol{u}\in A} \theta(\int_{\mathcal{S}\vee\tau} \boldsymbol{u}\, d\boldsymbol{v}) = \lim_{\tau\downarrow\mathcal{S}} \sup_{\boldsymbol{u}\in A} \theta(\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\boldsymbol{v}) = 0.$$

(b) If $\boldsymbol{v}$ is non-decreasing, then $\{\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v} : \boldsymbol{u} \in A\}$ is order-bounded in L^0.

644G Lemma Suppose that $\langle \mathfrak{A}_t \rangle_{t\in T}$ is right-continuous, and $\mathcal{S}$ is a non-empty order-convex subset of $\mathcal{T}$. On the space $M_{\text{mo}} = M_{\text{mo}}(\mathcal{S})$ of moderately oscillatory processes, we have a linear space topology $\mathfrak{S}$ defined by functionals of the form $\boldsymbol{u} \mapsto \theta(\int_{\mathcal{S}} |\boldsymbol{u}| d\boldsymbol{v})$ where $\boldsymbol{v} \in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$. Let $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma\in\mathcal{S}}$ be a near-simple integrator. Then $\boldsymbol{u} \mapsto \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v} : M_{\text{mo}} \to L^0$ is uniformly continuous, for the uniformity induced by $\mathfrak{S}$, on any uniformly order-bounded set in M_{mo}.

644H Theorem Suppose that $\langle \mathfrak{A}_t \rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$ and $\langle \boldsymbol{u}_n \rangle_{n\in\mathbb{N}}$ a uniformly order-bounded sequence of moderately oscillatory processes with domain $\mathcal{S}$ such that $\langle \boldsymbol{u}_{n<} \rangle_{n\in\mathbb{N}}$ is order*-convergent in $(L^0)^{\mathcal{S}}$. Then $\lim_{n\to\infty} \int_{\mathcal{S}} \boldsymbol{u}_n d\boldsymbol{v}$ is defined for every near-simple integrator $\boldsymbol{v}$ with domain $\mathcal{S}$. If $\langle \boldsymbol{u}_{n<} \rangle_{n\in\mathbb{N}}$ is order*-convergent to $\boldsymbol{u}_<$, where $\boldsymbol{u}$ is moderately oscillatory, then $\lim_{n\to\infty} \int_{\mathcal{S}} \boldsymbol{u}_n d\boldsymbol{v} = \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$.

Version of 29.5.21

645 Construction of the S-integral

We are now in a position to define a sequentially smooth integral which corresponds, in a sense, to Lebesgue-Stieltjes integrals on the real line. The objective is to integrate bounded previsible processes with respect to near-simple integrators, and I set this up as a kind of Daniell integral based on ideas in §644. Since simple and near-simple and mderately oscillatory processes, as I have defined them in this volume, are often not previsible, we need to deal throughout with their previsible versions; and as our integrals take values in L^0 rather than in $\mathbb{R}$ or $\mathbb{C}$, we have to calculate with the functional θ rather than with a modulus or norm.

The key to the programme is really Lemma 644G. We saw there that (subject to certain conditions) integration with respect to an arbitrary near-simple integrator is controlled by integration with respect to appropriate non-decreasing processes. We can therefore do nearly all the work of the present section with non-decreasing integrators, which are very much easier to handle, even though our real aim is to understand integration with respect to martingales. With a non-decreasing integrator, as with an ordinary non-negative measure, integration is a positive linear operator. This makes it possible to consider upper integrals, which are what, in effect, we have in Definition 645Bb. Based on the functionals $\widehat{\theta}^{\#}_{\boldsymbol{v}}$ there, we have a linear space topology $\mathfrak{T}_{\text{S-i}}$ on a large space $M_{\text{po-b}}$ of order-bounded processes (645F). As with ordinary integration, unbounded sequences of integrands can be uncontrollable, so we have to find types of domination – preferably weaker than simply assuming uniform $\| \|_\infty$-boundedness – which will be adequate to ensure convergence of sequences of integrals. (See 645G-645L.) These bring us to a definition of what I call the ‘S-integral’ in 645P, in a form which makes it easy to check that it is bilinear in integrand and integrator (645Rb), and with the tools to show that it is sequentially smooth in the integrand (645T).

645B Definitions Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.

(a)(i) I will say that a fully adapted process $\boldsymbol{x}$ with domain $\mathcal{S}$ is **previsibly order-bounded** if there is a non-negative $\boldsymbol{u} \in M_{\text{mo}}(\mathcal{S})$ such that $|\boldsymbol{x}| \le \boldsymbol{u}_<$. $M_{\text{po-b}}(\mathcal{S})$ will be the set of previsibly order-bounded fully adapted processes $\boldsymbol{x}$ with domain $\mathcal{S}$.

(ii) I will say that a set $A \subseteq M_{\text{po-b}}(\mathcal{S})$ is **uniformly previsibly order-bounded** if there is a non-negative $\boldsymbol{u} \in M_{\text{mo}}(\mathcal{S})$ such that $|\boldsymbol{x}| \le \boldsymbol{u}_<$ for every $\boldsymbol{x} \in A$. A uniformly previsibly order-bounded set is uniformly order-bounded.

(b) For $\boldsymbol{x} \in M_{\text{po-b}}(\mathcal{S})$ and $\boldsymbol{v} \in M_{\text{n-s}}^{\uparrow}(\mathcal{S})$, write $\widehat{\theta}_{\boldsymbol{v}}^{\#}(\boldsymbol{x})$ for

$$\inf\{\sup_{n\in\mathbb{N}} \theta(\int_{\mathcal{S}} \boldsymbol{u}_n\, d\boldsymbol{v}) : \langle \boldsymbol{u}_n\rangle_{n\in\mathbb{N}} \text{ is a uniformly order-bounded non-decreasing sequence of non-negative processes in } M_{\text{mo}}(\mathcal{S}) \text{ and } |\boldsymbol{x}| \le \sup_{n\in\mathbb{N}} \boldsymbol{u}_{n<}\}.$$

(c) If $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$, $\boldsymbol{v}' = \langle v'_\sigma\rangle_{\sigma\in\mathcal{S}}$ are two fully adapted processes with domain $\mathcal{S}$, I will write $\boldsymbol{v} \preccurlyeq \boldsymbol{v}'$ if $\boldsymbol{v}' - \boldsymbol{v}$ is non-decreasing.

In this case, if $\boldsymbol{u}$ is any non-negative fully adapted process with domain $\mathcal{S}$, $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v} \le \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}'$ if both integrals are defined.

645C Lemma (a) Let X be a metrizable space. Then the set of Borel measurable real-valued functions on X is the smallest subset U of $\mathbb{R}^X$ which contains every bounded continuous real-valued function and is such that $\lim_{n\to\infty} h_n \in U$ whenever $\langle h_n\rangle_{n\in\mathbb{N}}$ is a sequence in U which has a limit in $\mathbb{R}$ at every point and which is either non-decreasing or non-increasing.

(b) If $k \ge 1$ and $h : \mathbb{R}^k \to \mathbb{R}$ is a locally bounded function, then there is a continuous non-decreasing function $g : \mathbb{R} \to [0, \infty[$ such that $|h(x)| \le g(\|x\|)$ for every $x \in \mathbb{R}^k$.

(c) If $k \ge 1$ and U is a set of real-valued functions on $\mathbb{R}^k$ such that (α) every continuous function belongs to U (β) $\lim_{n\to\infty} f_n \in U$ whenever $\langle f_n\rangle_{n\in\mathbb{N}}$ is a pointwise convergent sequence in U and $\sup_{n\in\mathbb{N}} |f_n|$ is locally bounded, then every locally bounded Borel measurable function on $\mathbb{R}^k$ belongs to U.

645D Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.

(a)(i) If $\boldsymbol{x}_0, \dots, \boldsymbol{x}_{k-1} \in M_{\text{po-b}} = M_{\text{po-b}}(\mathcal{S})$ and $h : \mathbb{R}^k \to \mathbb{R}$ is a locally bounded Borel measurable function, then $\bar{h}(\boldsymbol{x}_0, \dots, \boldsymbol{x}_{k-1}) \times \mathbf{1}^{(\mathcal{S})}_{<} \in M_{\text{po-b}}$; and if $h(0, \dots, 0) = 0$, then $\bar{h}(\boldsymbol{x}_0, \dots, \boldsymbol{x}_{k-1}) \in M_{\text{po-b}}$.

(ii) $M_{\text{po-b}}$ is an f-subalgebra of $M_{\text{o-b}}(\mathcal{S})$.

(iii) $\boldsymbol{u}_< \in M_{\text{po-b}}$ for every $\boldsymbol{u} \in M_{\text{o-b}}(\mathcal{S})$.

(iv) If $z \in L^0(\mathfrak{A} \cap \bigcap_{\sigma\in\mathcal{S}} \mathfrak{A}_\sigma)$ then $z\boldsymbol{x}$ belongs to $M_{\text{po-b}}$ for every $\boldsymbol{x} \in M_{\text{po-b}}$.

(b) Suppose that $\boldsymbol{v} \in M_{\text{n-s}}^{\uparrow}(\mathcal{S})$. Then $\widehat{\theta}_{\boldsymbol{v}}^{\#}$ is an F-seminorm and if $\boldsymbol{x}, \boldsymbol{x}' \in M_{\text{po-b}}$, $|\boldsymbol{x}| \le |\boldsymbol{x}'|$ and $\alpha \in \mathbb{R}$ then $\widehat{\theta}_{\boldsymbol{v}}^{\#}(\boldsymbol{x}) \le \widehat{\theta}_{\boldsymbol{v}}^{\#}(\boldsymbol{x}')$ and $\widehat{\theta}_{\boldsymbol{v}}^{\#}(\alpha\boldsymbol{x}) \le \max(1, |\alpha|)\widehat{\theta}_{\boldsymbol{v}}^{\#}(\boldsymbol{x})$.

(c) If now we have another $\boldsymbol{v}' \in M_{\text{n-s}}^{\uparrow}(\mathcal{S})$ and $\boldsymbol{v} \preccurlyeq \boldsymbol{v}'$ in the sense of 645Bc, $\widehat{\theta}_{\boldsymbol{v}}^{\#}(\boldsymbol{x}) \le \widehat{\theta}_{\boldsymbol{v}'}^{\#}(\boldsymbol{x})$ for every $\boldsymbol{x} \in M_{\text{po-b}}(\mathcal{S})$.

645E The topology $\mathfrak{T}_{\text{S-i}}$: Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.

(a)(i) We have a linear space topology $\mathfrak{T}_{\text{S-i}}$ on $M_{\text{po-b}} = M_{\text{po-b}}(\mathcal{S})$ defined by the functionals $\widehat{\theta}_{\boldsymbol{v}}^{\#}$ as $\boldsymbol{v}$ runs over $M_{\text{n-s}}^{\uparrow}(\mathcal{S})$.

(ii) If $\boldsymbol{x} \in G \in \mathfrak{T}_{\text{S-i}}$, there are $\boldsymbol{v} \in M_{\text{n-s}}^{\uparrow}(\mathcal{S})$ and a $\delta > 0$ such that $\{\boldsymbol{x}' : \boldsymbol{x}' \in M_{\text{po-b}}, \widehat{\theta}_{\boldsymbol{v}}^{\#}(\boldsymbol{x}' - \boldsymbol{x}) \le \delta\} \subseteq G$.

(iii) For any $\tau \in \mathcal{S}$, the coordinate projection $\langle x_\sigma\rangle_{\sigma\in\mathcal{S}} \mapsto x_\tau : M_{\text{po-b}} \to L^0$ is continuous for $\mathfrak{T}_{\text{S-i}}$ and the topology of convergence in measure on L^0.

(iv) $\mathfrak{T}_{\text{S-i}}$ is Hausdorff.

(v)(α) For any $\boldsymbol{x} \in M_{\text{po-b}}$, the map $\boldsymbol{x}' \mapsto \boldsymbol{x}' \times \boldsymbol{x} : M_{\text{po-b}} \to M_{\text{po-b}}$ is continuous.

(β) If $A \subseteq M_{\text{po-b}}$ is uniformly previsibly order-bounded, then $(\boldsymbol{x}, \boldsymbol{x}') \mapsto \boldsymbol{x} \times \boldsymbol{x}' : A \times A \to M_{\text{po-b}}$ is uniformly continuous.

(vi) If $z \in L^0(\mathfrak{A} \cap \bigcap_{\sigma\in\mathcal{S}} \mathfrak{A}_\sigma)$ then $\boldsymbol{x}' \mapsto z\boldsymbol{x}' : M_{\text{po-b}} \to M_{\text{po-b}}$ is continuous.

(b) Let $\mathfrak{S}$ be the linear space topology on $M_{\text{mo}} = M_{\text{mo}}(\mathcal{S})$ defined by the F-seminorms $\boldsymbol{u} \mapsto \theta(\int_{\mathcal{S}} |\boldsymbol{u}|\, d\boldsymbol{v})$ as $\boldsymbol{v}$ runs over $M_{\text{n-s}}^{\uparrow}(\mathcal{S})$. Then $\boldsymbol{u} \mapsto \boldsymbol{u}_< : M_{\text{mo}} \to M_{\text{po-b}}$ is continuous for $\mathfrak{S}$ and $\mathfrak{T}_{\text{S-i}}$. Consequently $\boldsymbol{u} \mapsto \boldsymbol{u}_< : M_{\text{mo}} \to M_{\text{po-b}}$ is continuous for the ucp topology on M_{mo} and $\mathfrak{T}_{\text{S-i}}$.

645F Definitions Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.

(a) I will call the topology $\mathfrak{T}_{\text{S-i}}$ defined in 645E the **S-integration topology** on $M_{\text{po-b}}(\mathcal{S})$. As it is a linear space topology, there is an associated uniformity which I will call the **S-integration uniformity**.

(b) $M^0_{\text{S-i}}(\mathcal{S})$ will be the $\mathfrak{T}_{\text{S-i}}$-closure of $\{\boldsymbol{u}_< : \boldsymbol{u} \in M_{\text{mo}}(\mathcal{S})\}$ in $M_{\text{po-b}}(\mathcal{S})$.

(c) $M_{\text{S-i}}(\mathcal{S})$ will be the set of fully adapted processes $\boldsymbol{x}$ with domain $\mathcal{S}$ such that $\boldsymbol{x} \times \mathbf{1}^{(\mathcal{S})}_< \in M^0_{\text{S-i}}(\mathcal{S})$.

645G Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\mathfrak{T}_{\text{S-i}}$ the S-integration topology on $M_{\text{po-b}}(\mathcal{S})$. If $\langle \boldsymbol{x}_n \rangle_{n\in\mathbb{N}}$ is a uniformly previsibly order-bounded $\mathfrak{T}_{\text{S-i}}$-Cauchy sequence in $M_{\text{po-b}}$, then it is $\mathfrak{T}_{\text{S-i}}$-convergent.

645H Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. Suppose that $\langle \boldsymbol{x}_n \rangle_{n\in\mathbb{N}}$ is a uniformly previsibly order-bounded sequence in $M^0_{\text{S-i}}(\mathcal{S})$ which is order*-convergent to $\boldsymbol{x} \in (L^0)^{\mathcal{S}}$. Then $\boldsymbol{x} \in M^0_{\text{S-i}}$ and $\langle \boldsymbol{x}_n \rangle_{n\in\mathbb{N}}$ converges to $\boldsymbol{x}$ for the S-integration topology.

645I Corollary Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. If $\boldsymbol{x} \in M_{\text{po-b}}(\mathcal{S})$ is a previsible process, then $\boldsymbol{x} \in M^0_{\text{S-i}}$.

645J Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $k \ge 1$ an integer and $h : \mathbb{R}^k \to \mathbb{R}$ a locally bounded Borel measurable function. Write $M^0_{\text{S-i}}$, $M_{\text{S-i}}$ for $M^0_{\text{S-i}}(\mathcal{S})$, $M_{\text{S-i}}(\mathcal{S})$.

(a) If $\boldsymbol{X} \in (M^0_{\text{S-i}})^k$, then $\bar{h}\boldsymbol{X} \times \mathbf{1}^{(\mathcal{S})}_< \in M^0_{\text{S-i}}$; if $h(0,\dots,0) = 0$, then $\bar{h}\boldsymbol{X} \in M^0_{\text{S-i}}$.

(b) $\bar{h}\boldsymbol{X} \in M_{\text{S-i}}$ for every $\boldsymbol{X} \in M^k_{\text{S-i}}$.

645K Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $k \ge 1$ an integer, $h : \mathbb{R}^k \to \mathbb{R}$ a locally bounded Borel measurable function and $z \in L^0(\mathfrak{A} \cap \bigcap_{\sigma\in\mathcal{S}} \mathfrak{A}_\sigma)$.

(a) $M^0_{\text{S-i}} = M^0_{\text{S-i}}(\mathcal{S})$ is an f-subalgebra of $M_{\text{po-b}}(\mathcal{S})$ and $z\boldsymbol{x} \in M^0_{\text{S-i}}$ for every $\boldsymbol{x} \in M^0_{\text{S-i}}$.

(b) $M_{\text{S-i}} = M_{\text{S-i}}(\mathcal{S})$ is an f-subalgebra of $M_{\text{o-b}}(\mathcal{S})$ and $z\boldsymbol{x} \in M_{\text{S-i}}$ for every $\boldsymbol{x} \in M_{\text{S-i}}$.

645L Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. Give $M_{\text{po-b}}(\mathcal{S})$ its S-integration topology $\mathfrak{T}_{\text{S-i}}$. Suppose that $\boldsymbol{x} \in M^0_{\text{S-i}}(\mathcal{S})$.

(a) If $\boldsymbol{u}^* \in M_{\text{mo}} = M_{\text{mo}}(\mathcal{S})$ is such that $|\boldsymbol{x}| \le \boldsymbol{u}^*_<$, then $A = \{\boldsymbol{u} : \boldsymbol{u} \in M_{\text{mo}},\ |\boldsymbol{u}| \le \boldsymbol{u}^*\}$ is uniformly order-bounded and

$$\boldsymbol{x} \in \overline{\{\boldsymbol{u}_< : \boldsymbol{u} \in A\}} \subseteq \overline{\{\boldsymbol{u}_< : \boldsymbol{u} \in M_{\text{mo}},\ \sup|\boldsymbol{u}| \le \sup|\boldsymbol{u}^*|\}}.$$

(b) There is a $\boldsymbol{w}^* \in M_{\text{mo}}$ such that $\boldsymbol{x} \in \overline{\{\boldsymbol{u}_< : \boldsymbol{u} \in M_{\text{simp}}(\mathcal{S}),\ |\boldsymbol{u}| \le \boldsymbol{w}^*\}}$.

645N Lemma Suppose that $\langle \mathfrak{A}_t \rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$. If $\boldsymbol{u} \in M_{\text{mo}}(\mathcal{S})$ and $\boldsymbol{v} \in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$, then $\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{u}_<) = \theta(\int_{\mathcal{S}} |\boldsymbol{u}| d\boldsymbol{v})$.

645O Theorem Suppose that $\langle \mathfrak{A}_t \rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$, and give $M_{\text{po-b}}(\mathcal{S})$ its S-integration topology $\mathfrak{T}_{\text{S-i}}$. If $\boldsymbol{x} \in M_{\text{S-i}}(\mathcal{S})$ and $\boldsymbol{v} \in M_{\text{n-s}}(\mathcal{S})$ is an integrator, then there is a unique $z \in L^0$ such that whenever $A \subseteq M_{\text{mo}}(\mathcal{S})$ is uniformly order-bounded and $\epsilon > 0$ there is a $\mathfrak{T}_{\text{S-i}}$-neighbourhood G of $\boldsymbol{x} \times \mathbf{1}^{(\mathcal{S})}_<$ such that $\theta(z - \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}) \le \epsilon$ whenever $\boldsymbol{u} \in A$ and $\boldsymbol{u}_< \in G$.

645P Definition Suppose that $\langle \mathfrak{A}_t \rangle_{t\in T}$ is right-continuous and that $\mathcal{S}$ is an order-convex sublattice of $\mathcal{T}$.

(a) If $\boldsymbol{x} \in M_{\text{S-i}}(\mathcal{S})$ and $\boldsymbol{v} \in M_{\text{n-s}}(\mathcal{S})$ is an integrator, I will say that the element z of L^0 defined as in Theorem 645O is $\oint_{\mathcal{S}} \boldsymbol{x}\, d\boldsymbol{v}$, the **S-integral** of $\boldsymbol{x}$ with respect to $\boldsymbol{v}$.

(b) In these circumstances I will say that members of $M_{\text{S-i}}(\mathcal{S})$ are **S-integrable**.

(d) If $\boldsymbol{x}$ is a fully adapted process with domain $\mathcal{S}$, I will say that it is **locally S-integrable** if $\boldsymbol{x}\restriction\mathcal{S}\wedge\tau \in M_{\text{S-i}}(\mathcal{S}\wedge\tau)$ for every $\tau \in \mathcal{S}$.

(e) I shall allow myself to write $\oint_{\mathcal{S}} \boldsymbol{x}\,d\boldsymbol{v}$ for $\oint_{\mathcal{S}}(\boldsymbol{x}\restriction\mathcal{S})\,d(\boldsymbol{v}\restriction\mathcal{S})$ whenever $\boldsymbol{x}$, $\boldsymbol{v}$ are fully adapted processes such that $\mathcal{S}\subseteq\operatorname{dom}\boldsymbol{x}\cap\operatorname{dom}\boldsymbol{v}$, $\boldsymbol{x}\restriction\mathcal{S}$ is S-integrable and $\boldsymbol{v}\restriction\mathcal{S}$ is a near-simple integrator.

645R Theorem Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$.

(a) Suppose that $\boldsymbol{u}\in M_{\text{mo}}(\mathcal{S})$ and $\boldsymbol{v}\in M_{\text{n-s}}=M_{\text{n-s}}(\mathcal{S})$ is an integrator.

(i) $\boldsymbol{u}_<\in M^0_{\text{S-i}}(\mathcal{S})$ and $\oint_{\mathcal{S}}\boldsymbol{u}_<d\boldsymbol{v}=\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}$.

(ii) If *either* $\boldsymbol{v}$ is jump-free *or* T has no points isolated on the right, $\oint_{\mathcal{S}}\boldsymbol{u}_<d\boldsymbol{v}=\int_{\mathcal{S}}\boldsymbol{u}_<\,d\boldsymbol{v}$.

(b) If $\boldsymbol{x}$, $\boldsymbol{x}'\in M_{\text{S-i}}=M_{\text{S-i}}(\mathcal{S})$, $\boldsymbol{v}$, $\boldsymbol{v}'\in M_{\text{n-s}}$ are integrators, and $\alpha\in\mathbb{R}$, then

$$\oint_{\mathcal{S}}\boldsymbol{x}+\boldsymbol{x}'\,d\boldsymbol{v}=\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{v}+\oint_{\mathcal{S}}\boldsymbol{x}'\,d\boldsymbol{v},\qquad\oint_{\mathcal{S}}\boldsymbol{x}\,d(\boldsymbol{v}+\boldsymbol{v}')=\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{v}+\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{v}',$$

$$\oint_{\mathcal{S}}\alpha\boldsymbol{x}\,d\boldsymbol{v}=\oint_{\mathcal{S}}\boldsymbol{x}\,d(\alpha\boldsymbol{v})=\alpha\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{v}.$$

(c)(i) If $\boldsymbol{x}\in M_{\text{S-i}}$, $\boldsymbol{v}\in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$ and $\boldsymbol{x}\ge0$, then $\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{v}\ge0$;

(ii) if $\boldsymbol{x}\in M_{\text{S-i}}$ and $\boldsymbol{v}$ is a constant process with domain $\mathcal{S}$, then $\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{v}=0$.

645S Theorem Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous and that $\mathcal{S}$ is an order-convex sublattice of $\mathcal{T}$. Give $M^0_{\text{S-i}}=M^0_{\text{S-i}}(\mathcal{S})$ and L^0 the S-integration topology $\mathfrak{T}_{\text{S-i}}$ and the topology of convergence in measure respectively, with their associated uniformities. If $\boldsymbol{v}\in M_{\text{n-s}}(\mathcal{S})$ is an integrator, then $\boldsymbol{x}\mapsto\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{v}:M^0_{\text{S-i}}\to L^0$ is uniformly continuous on any uniformly previsibly order-bounded subset of $M^0_{\text{S-i}}$.

645T Theorem Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$. Suppose that $\langle\boldsymbol{x}_n\rangle_{n\in\mathbb{N}}$ is a sequence in $M_{\text{S-i}}(\mathcal{S})$ such that $\langle\boldsymbol{x}_n\times\mathbf{1}^{(\mathcal{S})}_<\rangle_{n\in\mathbb{N}}$ is uniformly previsibly order-bounded and $\langle\boldsymbol{x}_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to $\boldsymbol{x}$ in $(L^0)^{\mathcal{S}}$. Then $\boldsymbol{x}$ is S-integrable and $\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{v}=\lim_{n\to\infty}\oint_{\mathcal{S}}\boldsymbol{x}_n\,d\boldsymbol{v}$ for every integrator $\boldsymbol{v}\in M_{\text{n-s}}(\mathcal{S})$.

Version of 21.2.22

646 Basic properties of the S-integral

Having defined the S-integral as an adaptation of the Riemann-sum integral for previsible processes, it is natural to look for parallels to the properties of the Riemann-sum integral set out in Chapters 61-63. After a few easy remarks (646B-646D), I embark on the question of splitting a domain $\mathcal{S}$ into $\mathcal{S}\wedge\tau$ and $\mathcal{S}\vee\tau$ (646J). This leads naturally to an examination of indefinite S-integrals (646K), which I approach through a result on capped-stake variation sets for martingale integrators (646P). We have a change-of-variable theorem (646R), a formula for jumps in an indefinite S-integral (646S) and a version of Itô's formula (646T).

646B Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. If $\boldsymbol{x}=\langle x_\sigma\rangle_{\sigma\in\mathcal{S}}\in M^0_{\text{S-i}}(\mathcal{S})$ then $x_\tau\in L^0(\mathfrak{A}_{\mathcal{S}<\tau})$ for every $\tau\in\mathcal{S}$.

646C Proposition Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$, $\boldsymbol{x}$ an S-integrable process and $\boldsymbol{v}$ a near-simple integrator, both with domain $\mathcal{S}$. Then $[\![\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{v}\ne0]\!]\subseteq[\![\boldsymbol{v}\ne\mathbf{0}]\!]$.

646D Proposition Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$, $\boldsymbol{x}$ a member of $M_{\text{S-i}}(\mathcal{S})$, and $\boldsymbol{v}$ a near-simple integrator with domain $\mathcal{S}$. If $z\in L^0(\mathfrak{A}\cap\bigcap_{\sigma\in\mathcal{S}}\mathfrak{A}_\sigma)$, then

$$\oint_{\mathcal{S}}z\boldsymbol{x}\,d\boldsymbol{v}=\oint_{\mathcal{S}}\boldsymbol{x}\,d(z\boldsymbol{v})=z\times\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{v}.$$

646E Lemma Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous, $\mathcal{S}$ is an order-convex sublattice of $\mathcal{T}$, and $\boldsymbol{v}\in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$. If $\boldsymbol{x}\in M^0_{\text{S-i}}(\mathcal{S})$, then $\theta(\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{v})\le\hat\theta^{\#}_{\boldsymbol{v}}(\boldsymbol{x})$.

646F Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and τ a member of $\mathcal{S}$. Suppose that $\boldsymbol{u}'=\langle u'_\sigma\rangle_{\sigma\in\mathcal{S}\wedge\tau}$ and $\boldsymbol{u}''=\langle u''_\sigma\rangle_{\sigma\in\mathcal{S}\vee\tau}$ are families in L^0. Define $R(\boldsymbol{u}',\boldsymbol{u}'')\in(L^0)^{\mathcal{S}}$ by saying that $R(\boldsymbol{u}',\boldsymbol{u}'')=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ where

$$u_\sigma=u'_{\sigma\wedge\tau}\times\chi[\![\sigma<\tau]\!]+u''_{\sigma\vee\tau}\times\chi[\![\tau\le\sigma]\!]$$

for $\sigma \in \mathcal{S}$.

(a)(i) $R(\boldsymbol{u}', \boldsymbol{u}'')\restriction\mathcal{S} \wedge \tau = \boldsymbol{u}' + \boldsymbol{z}$, where $\boldsymbol{z} = \langle (u''_\tau - u'_\tau) \times \chi[\![\sigma = \tau]\!]\rangle_{\sigma \in \mathcal{S}\wedge\tau}$.

(ii) $R(\boldsymbol{u}', \boldsymbol{u}'')\restriction\mathcal{S} \vee \tau = \boldsymbol{u}''$.

(b) Regarded as an operator from $(L^0)^{\mathcal{S}\wedge\tau} \times (L^0)^{\mathcal{S}\vee\tau}$ to $(L^0)^{\mathcal{S}}$, R is linear, positive and order-continuous.

(c) If $\boldsymbol{u}'$ and $\boldsymbol{u}''$ are fully adapted, then $R(\boldsymbol{u}', \boldsymbol{u}'')$ is fully adapted.

(d) If $\boldsymbol{u}'$ and $\boldsymbol{u}''$ are order-bounded, then $R(\boldsymbol{u}', \boldsymbol{u}'')$ is order-bounded and $\sup|R(\boldsymbol{u}', \boldsymbol{u}'')| \le \sup|\boldsymbol{u}'| \vee \sup|\boldsymbol{u}''|$.

(e) Suppose that $\boldsymbol{u}'$ and $\boldsymbol{u}''$ are moderately oscillatory.

(i) $R(\boldsymbol{u}', \boldsymbol{u}'')$ is moderately oscillatory.

(ii) $\boldsymbol{u}'_< = R(\boldsymbol{u}', \boldsymbol{u}'')_<\restriction\mathcal{S} \wedge \tau$.

(f) If $\boldsymbol{u}'$ and $\boldsymbol{u}''$ are near-simple, $R(\boldsymbol{u}', \boldsymbol{u}'')$ is near-simple.

(g) Suppose that $\boldsymbol{u}'$ and $\boldsymbol{u}''$ are moderately oscillatory, and that $\boldsymbol{v}$ is an integrator with domain $\mathcal{S}$. Then

$$\int_{\mathcal{S}} R(\boldsymbol{u}', \boldsymbol{u}'')d\boldsymbol{v} = \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}'\,d\boldsymbol{v} + \int_{\mathcal{S}\vee\tau} \boldsymbol{u}''\,d\boldsymbol{v}.$$

646G Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and τ a member of $\mathcal{S}$. Suppose that $\boldsymbol{v}' = \langle v'_\sigma\rangle_{\sigma\in\mathcal{S}\wedge\tau} \in M^\uparrow_{\text{n-s}}(\mathcal{S}\wedge\tau)$ and $\boldsymbol{v}'' = \langle v''_\sigma\rangle_{\sigma\in\mathcal{S}\vee\tau} \in M^\uparrow_{\text{n-s}}(\mathcal{S}\vee\tau)$.

(a) There is a $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}} \in M^\uparrow_{\text{n-s}}(\mathcal{S})$ such that $\boldsymbol{v}'' = \boldsymbol{v}\restriction\mathcal{S}\vee\tau$.

(b) There is a $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}} \in M^\uparrow_{\text{n-s}}(\mathcal{S})$ such that

$$\boldsymbol{v}' = \boldsymbol{v}\restriction\mathcal{S}\wedge\tau, \quad \boldsymbol{v}'' \equiv \boldsymbol{v}\restriction\mathcal{S}\vee\tau.$$

(c) If $\boldsymbol{w} \in M^\uparrow_{\text{n-s}}(\mathcal{S})$, there is a $\boldsymbol{v}^* \in M^\uparrow_{\text{n-s}}(\mathcal{S})$ such that

$$\boldsymbol{w} \preccurlyeq \boldsymbol{v}^*, \quad \boldsymbol{v}' \preccurlyeq \boldsymbol{v}^*\restriction\mathcal{S}\wedge\tau, \quad \boldsymbol{v}'' \preccurlyeq \boldsymbol{v}^*\restriction\mathcal{S}\vee\tau.$$

646H Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and τ a member of $\mathcal{S}$. For $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}} \in (L^0)^{\mathcal{S}}$ define $R^*(\boldsymbol{u}) \in (L^0)^{\mathcal{S}\vee\tau}$ by saying that

$$R^*(\boldsymbol{u}) = \langle u_\sigma \times \chi[\![\tau < \sigma]\!]\rangle_{\sigma\in\mathcal{S}\vee\tau} = (\boldsymbol{u}\restriction\mathcal{S}\vee\tau) \times \mathbf{1}^{(\mathcal{S}\vee\tau)}_<.$$

(a)(i) $R^* : (L^0)^{\mathcal{S}} \to (L^0)^{\mathcal{S}\vee\tau}$ is an order-continuous f-algebra homomorphism.

(ii) If $\boldsymbol{u} \in (L^0)^{\mathcal{S}}$ is fully adapted, then $R^*(\boldsymbol{u})$ is fully adapted.

(iii) If $\boldsymbol{u} \in M_{\text{o-b}}(\mathcal{S})$, then $R^*(\boldsymbol{u}) \in M_{\text{o-b}}(\mathcal{S}\vee\tau)$ and $\sup|R^*(\boldsymbol{u})| \le \sup|\boldsymbol{u}|$.

(iv) If $\boldsymbol{u} \in M_{\text{mo}}(\mathcal{S})$ then $R^*(\boldsymbol{u}) \in M_{\text{mo}}(\mathcal{S}\vee\tau)$.

(v) If $\boldsymbol{x}, \boldsymbol{u} \in (L^0)^{\mathcal{S}}$, $\boldsymbol{x}\restriction\mathcal{S}\wedge\tau \le \boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ and $R^*(\boldsymbol{x}) \le \boldsymbol{u}\restriction\mathcal{S}\vee\tau$, then $\boldsymbol{x} \le \boldsymbol{u}$.

(b) If $\boldsymbol{u} \in M_{\text{mo}}(\mathcal{S})$, then $(\boldsymbol{u}\restriction\mathcal{S}\vee\tau)_< = R^*(\boldsymbol{u}_<)$.

(c) Suppose that $\boldsymbol{x} \in M_{\text{po-b}}(\mathcal{S})$. Write $\boldsymbol{x}'$ for $\boldsymbol{x}\restriction\mathcal{S}\wedge\tau$.

(i) $\boldsymbol{x}' \in M_{\text{po-b}}(\mathcal{S}\wedge\tau)$ and $R^*(\boldsymbol{x}) \in M_{\text{po-b}}(\mathcal{S}\vee\tau)$.

(ii) If $\boldsymbol{v} \in M^\uparrow_{\text{n-s}}(\mathcal{S})$, and we set $\boldsymbol{v}' = \boldsymbol{v}\restriction\mathcal{S}\wedge\tau$ and $\boldsymbol{v}'' = \boldsymbol{v}\restriction\mathcal{S}\vee\tau$, then

$$\max(\widehat{\theta}^\#_{\boldsymbol{v}'}(\boldsymbol{x}'), \widehat{\theta}^\#_{\boldsymbol{v}''}R^*(\boldsymbol{x})) \le \widehat{\theta}^\#_{\boldsymbol{v}}(\boldsymbol{x}) \le \widehat{\theta}^\#_{\boldsymbol{v}'}(\boldsymbol{x}') + \widehat{\theta}^\#_{\boldsymbol{v}''}R^*(\boldsymbol{x}).$$

(d) If $\boldsymbol{x} \in M_{\text{S-i}}(\mathcal{S})$ then $\boldsymbol{x}\restriction\mathcal{S}\wedge\tau \in M_{\text{S-i}}(\mathcal{S}\wedge\tau)$ and $\boldsymbol{x}\restriction\mathcal{S}\vee\tau \in M_{\text{S-i}}(\mathcal{S}\vee\tau)$.

646I Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and τ a member of $\mathcal{S}$. Define $R^* : (L^0)^{\mathcal{S}} \to (L^0)^{\mathcal{S}\vee\tau}$ as in 646H. Take $\boldsymbol{x} \in (L^0)^{\mathcal{S}}$.

(a) If $\boldsymbol{x}\restriction\mathcal{S}\wedge\tau \in M_{\text{po-b}}(\mathcal{S}\wedge\tau)$ and $R^*(\boldsymbol{x}) \in M_{\text{po-b}}(\mathcal{S}\vee\tau)$, then $\boldsymbol{x} \in M_{\text{po-b}}(\mathcal{S})$.

(b) If $\boldsymbol{x}\restriction\mathcal{S}\wedge\tau \in M^0_{\text{S-i}}(\mathcal{S}\wedge\tau)$ and $R^*(\boldsymbol{x}) \in M^0_{\text{S-i}}(\mathcal{S}\vee\tau)$, then $\boldsymbol{x} \in M^0_{\text{S-i}}(\mathcal{S})$.

(c) If $\boldsymbol{x}\restriction\mathcal{S}\wedge\tau \in M_{\text{S-i}}(\mathcal{S}\wedge\tau)$ and $\boldsymbol{x}\restriction\mathcal{S}\vee\tau \in M_{\text{S-i}}(\mathcal{S}\vee\tau)$, then $\boldsymbol{x} \in M_{\text{S-i}}(\mathcal{S})$.

646J Theorem Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$, and τ a member of $\mathcal{S}$. If $\boldsymbol{x} \in (L^0)^{\mathcal{S}}$, then $\boldsymbol{x}$ is an S-integrable process iff $\boldsymbol{x}\restriction\mathcal{S}\wedge\tau$ and $\boldsymbol{x}\restriction\mathcal{S}\vee\tau$ are both S-integrable, and in this case

$$\oint_{\mathcal{S}} \boldsymbol{x}\,d\boldsymbol{v} = \oint_{\mathcal{S}\wedge\tau} \boldsymbol{x}\,d\boldsymbol{v} + \oint_{\mathcal{S}\vee\tau} \boldsymbol{x}\,d\boldsymbol{v}$$

for every near-simple integrator $\boldsymbol{v}$ with domain $\mathcal{S}$.

646K Indefinite S-integrals Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous and that $\mathcal{S}$ is an order-convex sublattice of $\mathcal{T}$.

(a) Suppose that $\boldsymbol{x}$ is a locally S-integrable process and that $\boldsymbol{v}$ is a locally near-simple local integrator, both with domain $\mathcal{S}$. Then we can define $z_\tau = \oint_{\mathcal{S}\wedge\tau}\boldsymbol{x}\,d\boldsymbol{v}$ for $\tau\in\mathcal{S}$. Now the **indefinite S-integral of $\boldsymbol{x}$ with respect to $\boldsymbol{v}$** is $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x}) = \langle z_\tau\rangle_{\tau\in\mathcal{S}}$.

(b) It will more than once be useful to note that, in the context of (a) just above, $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x}) = \mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x}\times\mathbf{1}^{(\mathcal{S})}_{<})$.

(c) If $\boldsymbol{u}$ is a locally moderately oscillatory process with domain $\mathcal{S}$ then $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{u}_<) = ii_{\boldsymbol{v}}(\boldsymbol{u})$.

646L Lemma Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous and that $\mathcal{S}$ is an order-convex sublattice of $\mathcal{T}$. Let $\boldsymbol{v}$ be a near-simple integrator and $\boldsymbol{x}$ an S-integrable process, both with domain $\mathcal{S}$. For $\tau\in\mathcal{S}$, set $z_\tau = \oint_{\mathcal{S}\wedge\tau}\boldsymbol{x}\,d\boldsymbol{v}$. Suppose that $\boldsymbol{u}^*\in M_{\mathrm{mo}}(\mathcal{S})^+$ is such that $|\boldsymbol{x}|\le\boldsymbol{u}^*$. Then for any $\epsilon>0$ there is a $\mathfrak{T}_{\mathrm{S\text{-}i}}$-neighbourhood G of $\boldsymbol{x}$ such that $\theta(z_\tau - \int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\,d\boldsymbol{v})\le\epsilon$ whenever $\boldsymbol{u}\in M_{\mathrm{mo}}(\mathcal{S})$, $|\boldsymbol{u}|\le\boldsymbol{u}^*$, $\boldsymbol{u}_<\in G$ and $\tau\in\mathcal{S}$.

646M Proposition Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous and that $\mathcal{S}$ is a non-empty order-convex sublattice of $\mathcal{T}$. Let $\boldsymbol{v}$ be a near-simple integrator and $\boldsymbol{x}$ an S-integrable process, both with domain $\mathcal{S}$; set $z = \oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{v}$ and $z_\tau = \oint_{\mathcal{S}\wedge\tau}\boldsymbol{x}\,d\boldsymbol{v}$ for $\tau\in\mathcal{S}$. Then $\lim_{\tau\uparrow\mathcal{S}} z_\tau = z$ and $\lim_{\tau\downarrow\mathcal{S}} z_\tau = 0$.

646N Theorem Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous and that $\mathcal{S}$ is an order-convex sublattice of $\mathcal{T}$. Let $\boldsymbol{v}$ be a near-simple integrator and $\boldsymbol{x}$ an S-integrable process, both with domain $\mathcal{S}$. Then $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})$ is a near-simple integrator.

646O Lemma Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous and that $\mathcal{S}$ is an order-convex sublattice of $\mathcal{T}$. Let $\boldsymbol{v}$ be a near-simple integrator with domain $\mathcal{S}$, and $A\subseteq M^0_{\mathrm{S\text{-}i}}(\mathcal{S})$ a uniformly previsibly order-bounded set. Then $\mathrm{S}ii_{\boldsymbol{v}}\restriction A$ is uniformly continuous with respect to the S-integration uniformity on A and and the ucp uniformity on $M_{\mathrm{o\text{-}b}}(\mathcal{S})$.

646P Lemma Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous and that $\mathcal{S}$ is an order-convex sublattice of $\mathcal{T}$ with a least element. Let $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ be a martingale. Then $\mathcal{S}_{\boldsymbol{v}} = \{\tau : \tau\in\mathcal{S},\ Q_{\mathcal{S}\wedge\tau}(d\boldsymbol{v})$ is uniformly integrable$\}$ is a covering ideal of $\mathcal{S}$.

646Q Theorem Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous and that $\mathcal{S}$ is an order-convex sublattice of $\mathcal{T}$ with a least element. Let $\boldsymbol{v}$ be a near-simple integrator and $\boldsymbol{x}$ an S-integrable process, both with domain $\mathcal{S}$.
(a) If $\boldsymbol{v}$ is a martingale, then $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})$ is a local martingale.
(b) If $\boldsymbol{v}$ is jump-free, then $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})$ is jump-free.
(c) If $\boldsymbol{v}$ is of bounded variation, then $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})$ is of bounded variation.

646R Theorem Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous and that $\mathcal{S}$ is an order-convex sublattice of $\mathcal{T}$. Let $\boldsymbol{v}$ be a near-simple integrator and $\boldsymbol{x}$, $\boldsymbol{x}'$ two S-integrable processes, all with domain $\mathcal{S}$; write $\boldsymbol{z}$ for the indefinite S-integral $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})$. Then $\oint_{\mathcal{S}}\boldsymbol{x}'\,d\boldsymbol{z} = \oint_{\mathcal{S}}\boldsymbol{x}'\times\boldsymbol{x}\,d\boldsymbol{v}$.

Mnemonic $d(\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})) = \boldsymbol{x}\,d\boldsymbol{v}$.

646S Proposition Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous and that $\mathcal{S}$ is an order-convex sublattice of $\mathcal{T}$. Let $\boldsymbol{v}$ be a near-simple integrator and $\boldsymbol{x}$ an S-integrable process, both with domain $\mathcal{S}$. Then $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x}) - \mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})_< = \boldsymbol{x}\times(\boldsymbol{v}-\boldsymbol{v}_<)\times\mathbf{1}^{(\mathcal{S})}_<$.

646T Itô's Formula, fourth form Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous, and that $\mathcal{S}$ is an order-convex sublattice of $\mathcal{T}$ with a least element. Let $\boldsymbol{v}$ be a jump-free integrator with domain $\mathcal{S}$, and $\boldsymbol{v}^*$ its quadratic variation; let $h:\mathbb{R}\to\mathbb{R}$ be a differentiable function such that its derivative h' is locally Lipschitz.

If $h'' : \mathbb{R} \to \mathbb{R}$ is a locally bounded Borel measurable function Lebesgue-almost-everywhere equal to the derivative of h', then

$$\oint_{\mathcal{S}} \boldsymbol{x}\, d(\bar{h}\boldsymbol{v}) = \oint_{\mathcal{S}} \boldsymbol{x} \times \bar{h}'\boldsymbol{v}\, d\boldsymbol{v} + \frac{1}{2} \oint_{\mathcal{S}} \boldsymbol{x} \times \bar{h}''\boldsymbol{v}\, d\boldsymbol{v}^*$$

for every $\boldsymbol{x} \in M_{\text{S-i}}(\mathcal{S})$.

Version of 29.3.23

647 Changing the filtration II

The answer (647J) to a natural question left over from §646 leads us to a new construction to add to those in the second half of Chapter 63.

647B Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\mathfrak{T}$ the linear space topology on $M_{\text{po-b}} = M_{\text{po-b}}(\mathcal{S})$ defined by the F-seminorms $\widehat{\theta}^{\#}_{\boldsymbol{v}}$ where $\boldsymbol{v} \in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$ is $\| \, \|_\infty$-bounded. Then $\mathfrak{T}$ is the S-integration topology on $M_{\text{po-b}}$.

647C Lemma Suppose that $\mathfrak{D}$ is a closed subalgebra of $\mathfrak{A}$, and $b \in \mathfrak{A}$; let $\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{b\} \cup \mathfrak{D}$.

(a) If $c \in \mathfrak{B}$, then $b \cap c = b \cap \operatorname{upr}(b \cap c, \mathfrak{D})$.

(b) If $u \in L^0(\mathfrak{B})$, there are u', $u'' \in L^0(\mathfrak{D})$ such that $u = u' \times \chi b + u'' \times \chi(1 \setminus b)$.

647D Construction For most of the rest of this section, b will be a fixed member of $\mathfrak{A}$. For $t \in T$, let $\mathfrak{B}_t$ be the subalgebra of $\mathfrak{A}$ generated by $\{b\} \cup \mathfrak{A}_t$; then $\mathfrak{B}_t$ is a closed subalgebra. $\langle \mathfrak{B}_t \rangle_{t \in T}$ is a filtration.

647E Notation From now on, we shall have the two filtrations $\langle \mathfrak{A}_t \rangle_{t \in T}$ and $\langle \mathfrak{B}_t \rangle_{t \in T}$, giving stochastic integration structures $\mathbb{A} = (\mathfrak{A}, \bar{\mu}, T, \langle \mathfrak{A}_t \rangle_{t \in T}, \mathcal{T}_{\mathbb{A}}, \langle \mathfrak{A}_\sigma \rangle_{\sigma \in \mathcal{T}_{\mathbb{A}}})$ and $\mathbb{B} = (\mathfrak{A}, \bar{\mu}, T, \langle \mathfrak{B}_t \rangle_{t \in T}, \mathcal{T}_{\mathbb{B}}, \langle \mathfrak{B}_\tau \rangle_{\tau \in \mathcal{T}_{\mathbb{B}}})$. For the various spaces of processes, I write $\mathbb{A}M^0_{\text{S-i}}(\mathcal{S})$, $\mathbb{B}M_{\text{mo}}(\mathcal{S})$, $\mathbb{A}M_{\text{po-b}}(\mathcal{S})$ etc. When we come to S-integration, I talk of F-seminorms $\mathbb{A}\widehat{\theta}^{\#}_{\boldsymbol{v}}$, the S-integration topology $\mathbb{B}\mathfrak{T}_{\text{S-i}}$ and S-integrals $\mathbb{A}\oint$.

647F Proposition (a)(i) $\mathcal{T}_{\mathbb{A}}$ is a sublattice of $\mathcal{T}_{\mathbb{B}}$.

(ii) $\min \mathcal{T}_{\mathbb{A}} = \min \mathcal{T}_{\mathbb{B}}$ and $\max \mathcal{T}_{\mathbb{A}} = \max \mathcal{T}_{\mathbb{B}}$.

(b) For any $\sigma \in \mathcal{T}_{\mathbb{A}}$, $\mathfrak{B}_\sigma$ is the subalgebra of $\mathfrak{A}$ generated by $\{b\} \cup \mathfrak{A}_\sigma$.

(c) If σ, $\tau \in \mathcal{T}_{\mathbb{A}}$, then $[\![\sigma < \tau]\!]$ and $[\![\sigma = \tau]\!]$ are the same in either structure.

(d) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}_{\mathbb{A}}$.

(i) If $\boldsymbol{u}$ is an $\mathbb{A}$-fully adapted process with domain $\mathcal{S}$, it is $\mathbb{B}$-fully adapted.

(ii) $\mathbb{A}M_{\text{simp}}(\mathcal{S}) \subseteq \mathbb{B}M_{\text{simp}}(\mathcal{S})$.

(iii) $\mathbb{A}M_{\text{o-b}}(\mathcal{S}) \subseteq \mathbb{B}M_{\text{o-b}}(\mathcal{S})$, and the ucp topology on $\mathbb{A}M_{\text{o-b}}(\mathcal{S})$ is the subspace topology induced by the ucp topology on $\mathbb{B}M_{\text{o-b}}(\mathcal{S})$.

(iv) $\mathbb{A}M_{\text{n-s}}(\mathcal{S}) \subseteq \mathbb{B}M_{\text{n-s}}(\mathcal{S})$.

(v) $\mathbb{A}M_{\text{bv}}(\mathcal{S}) \subseteq \mathbb{B}M_{\text{bv}}(\mathcal{S})$.

(vi) $\mathbb{A}M_{\text{mo}}(\mathcal{S}) \subseteq \mathbb{B}M_{\text{mo}}(\mathcal{S})$.

(e) If $\boldsymbol{u}$, $\boldsymbol{v}$ are $\mathbb{A}$-fully adapted processes with domain $\mathcal{S}$, and $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$ is defined in either of the structures $\mathbb{A}$, $\mathbb{B}$, then it is defined in the other, with the same value.

(f) If $\mathcal{S}$ is a sublattice of $\mathcal{T}_{\mathbb{A}}$ and $\boldsymbol{v}$ an $\mathbb{A}$-integrator with domain $\mathcal{S}$, then $\boldsymbol{v}$ is a $\mathbb{B}$-integrator.

(g) If $\mathcal{S}$ is a sublattice of $\mathcal{T}_{\mathbb{A}}$ and $\boldsymbol{u}$ belongs to $\mathbb{A}M_{\text{mo}}(\mathcal{S}) \subseteq \mathbb{B}M_{\text{mo}}(\mathcal{S})$, then its previsible version $\boldsymbol{u}_<$ is the same when calculated in either of the structures $\mathbb{A}$, $\mathbb{B}$.

647G Proposition Suppose that $\langle \mathfrak{A}_t \rangle_{t \in T}$ is right-continuous.

(a) $\langle \mathfrak{B}_t \rangle_{t \in T}$ is right-continuous.

(b) $\mathcal{T}_{\mathbb{A}}$ is an order-closed sublattice of $\mathcal{T}_{\mathbb{B}}$.

(c) If $\tau \in \mathcal{T}_{\mathbb{B}}$, there are σ, $\sigma' \in \mathcal{T}_{\mathbb{A}}$ such that $b \subseteq [\![\tau = \sigma]\!]$ and $1 \setminus b \subseteq [\![\tau = \sigma']\!]$. $\mathcal{T}_{\mathbb{A}}$ covers $\mathcal{T}_{\mathbb{B}}$.

647H Lemma Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous, and that $\mathcal{S}$ is an order-convex sublattice of $\mathcal{T}_{\mathbb{A}}$. Let $\boldsymbol{v}\in\mathbb{B}M^{\uparrow}_{\text{n-s}}(\mathcal{S})$ be $\|\;\|_\infty$-bounded. Then there is a $\|\;\|_\infty$-bounded $\boldsymbol{w}\in\mathbb{A}M^{\uparrow}_{\text{n-s}}(\mathcal{S})$ such that $\boldsymbol{v}\preccurlyeq\boldsymbol{w}$.

647I Proposition Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}_{\mathbb{A}}$, and $\hat{\mathcal{S}}^{(\mathbb{B})}$ the covered envelope of $\mathcal{S}$ in $\mathcal{T}_{\mathbb{B}}$.

(a)(i) $\hat{\mathcal{S}}^{(\mathbb{B})}$ is order-convex in $\mathcal{T}_{\mathbb{B}}$.

(ii) For any $\mathbb{A}$-fully adapted process $\boldsymbol{x}$ with domain $\mathcal{S}$, there is a unique $\mathbb{B}$-fully adapted process $\hat{\boldsymbol{x}}$ with domain $\hat{\mathcal{S}}^{(\mathbb{B})}$ extending $\boldsymbol{x}$.

(b)(i) If $\boldsymbol{x}\in\mathbb{A}M_{\text{o-b}}(\mathcal{S})$ then $\hat{\boldsymbol{x}}\in\mathbb{B}M_{\text{o-b}}(\hat{\mathcal{S}}^{(\mathbb{B})})$ and $\sup|\hat{\boldsymbol{x}}|=\sup|\boldsymbol{x}|$.

(ii) If $\boldsymbol{v}\in\mathbb{A}M_{\text{n-s}}(\mathcal{S})$ then $\hat{\boldsymbol{v}}\in\mathbb{B}M_{\text{n-s}}(\hat{\mathcal{S}}^{(\mathbb{B})})$.

(iii) If $\boldsymbol{v}\in\mathbb{A}M_{\text{bv}}(\mathcal{S})$ then $\hat{\boldsymbol{v}}\in\mathbb{B}M_{\text{bv}}(\hat{\mathcal{S}}^{(\mathbb{B})})$.

(iv) If $\boldsymbol{v}$ is an $\mathbb{A}$-integrator with domain $\mathcal{S}$, then $\hat{\boldsymbol{v}}$ is a $\mathbb{B}$-integrator.

(v) If $\boldsymbol{u}\in\mathbb{A}M_{\text{mo}}(\mathcal{S})$ then $\hat{\boldsymbol{u}}\in\mathbb{B}M_{\text{mo}}(\hat{\mathcal{S}}^{(\mathbb{B})})$ and the previsible version of $\hat{\boldsymbol{u}}$ is the image of the previsible version of $\boldsymbol{u}$.

(c)(i) If $\boldsymbol{w}\in\mathbb{A}M_{\text{po-b}}(\mathcal{S})$, then $\hat{\boldsymbol{w}}\in\mathbb{B}M_{\text{po-b}}(\hat{\mathcal{S}}^{(\mathbb{B})})$.

(ii) The map $\boldsymbol{w}\mapsto\hat{\boldsymbol{w}}:\mathbb{A}M_{\text{po-b}}(\mathcal{S})\to\mathbb{B}M_{\text{po-b}}(\hat{\mathcal{S}}^{(\mathbb{B})})$ is continuous for the S-integration topologies $\mathbb{A}\mathfrak{T}_{\text{S-i}}$ and $\mathbb{B}\mathfrak{T}_{\text{S-i}}$.

(iii) If $\boldsymbol{x}\in\mathbb{A}M_{\text{S-i}}(\mathcal{S})$, then $\hat{\boldsymbol{x}}\in\mathbb{B}M_{\text{S-i}}(\hat{\mathcal{S}}^{(\mathbb{B})})$.

(d) If $\boldsymbol{x}\in\mathbb{A}M_{\text{S-i}}(\mathcal{S})$, then $\mathbb{B}\oint_{\hat{\mathcal{S}}^{(\mathbb{B})}}\hat{\boldsymbol{x}}\,d\hat{\boldsymbol{v}}=\mathbb{A}\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{v}$ for every $\mathbb{A}$-integrator $\boldsymbol{v}\in\mathbb{A}M_{\text{n-s}}(\mathcal{S})$.

647J Theorem Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous and $\mathcal{S}$ is an order-convex sublattice of $\mathcal{T}$. If $\boldsymbol{w}$ is an S-integrable process with domain $\mathcal{S}$, and $\boldsymbol{v}$ is a near-simple integrator with domain $\mathcal{S}$, then $[\![\oint_{\mathcal{S}}\boldsymbol{w}\,d\boldsymbol{v}\neq 0]\!]\subseteq[\![\boldsymbol{w}\neq\boldsymbol{0}]\!]$.

Version of 4.5.20/29.9.23

648 Changing the algebra II

In §634, I looked at questions involving pairs $(\mathfrak{A},\mathfrak{B})$ where $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$, and the corresponding stochastic integration structures $\mathbb{A}$ and $\mathbb{B}$. In particular, we can relate Riemann-sum integrals calculated in the two structures (634Eg). Unsurprisingly, there is a corresponding result for S-integration (648G), though it seems to need a good deal more work.

648A Notation As usual, $\mathbb{A}=(\mathfrak{A},\bar\mu,T,\langle\mathfrak{A}_t\rangle_{t\in T},\mathcal{T}_{\mathbb{A}},\langle\mathfrak{A}_\sigma\rangle_{\sigma\in\mathcal{T}_{\mathbb{A}}})$ will be a stochastic integration structure. For nearly the whole section, we shall have a closed subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ with the associated structure $\mathbb{B}=(\mathfrak{B},\bar\mu\restriction\mathfrak{B},T,\langle\mathfrak{B}_t\rangle_{t\in T},\mathcal{T}_{\mathbb{B}},\langle\mathfrak{B}_\sigma\rangle_{\sigma\in\mathcal{T}_{\mathbb{B}}})$ as in 634C. As in §634, I will use formulations like '$\boldsymbol{x}$ is $\mathbb{A}$-previsibly-order-bounded', '$\boldsymbol{v}'\in\mathbb{B}M_{\text{n-s}}(\mathcal{S}')$', '$\mathbb{A}\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{v}$' to indicate which structure is being considered at any particular moment.

If $E\subseteq\mathbb{R}$ is a Borel set and $h:E\to\mathbb{R}$ is a Borel measurable function, $\bar h$ is the corresponding function from $\{u:u\in L^0(\mathfrak{A}),\ [\![u\in E]\!]=1\}$ to $L^0(\mathfrak{A})$ (612Ac). If $\boldsymbol{u}$ is a moderately oscillatory process, $\boldsymbol{u}_<$ will denote its previsible version (641F). I use the symbol $\int$ for Riemann-sum integrals (613H, 613L) and $\oint$ for S-integrals (645P); $\mathbb{E}$ will be the ordinary integral on $L^1_{\bar\mu}=L^1(\mathfrak{A},\bar\mu)$, and θ the associated F-norm on $L^0(\mathfrak{A})$ (613Ba). The F-seminorms $\widehat{\theta}^{\#}_{\boldsymbol{v}}$ will be those of 645B. $S_I(\boldsymbol{u},d\boldsymbol{v})$ will be the Riemann sum (613Fb). $\mathbf{1}^{(\mathcal{S})}$ will be the constant process on $\mathcal{S}$ with value $\chi 1$.

We shall have the usual spaces of processes: M_{fa} (fully adapted, 612I), M_{mo} (moderately oscillatory, 615Fa), $M_{\text{n-s}}$ (near-simple, 631Ba), $M_{\text{po-b}}$ (previsibly order-bounded, 645Ba), $M^{\uparrow}_{\text{n-s}}$ (non-decreasing non-negative near-simple, 644Bb), $M^0_{\text{S-i}}$ and $M_{\text{S-i}}$ (S-integrable, 645F). If $\mathcal{S}$ is a sublattice of $\mathcal{T}_{\mathbb{A}}$, $\mathcal{I}(\mathcal{S})$ will be the directed set of finite sublattices of $\mathcal{S}$.

648B Lemma Let E be a Borel subset of $\mathbb{R}$; write Q_E for $\{u:u\in L^0(\mathfrak{A}),\ [\![u\in E]\!]=1\}$. Let $h:E\to\mathbb{R}$ be a continuous function. Suppose that $\mathcal{S}$ is a finitely full sublattice of $\mathcal{T}_{\mathbb{A}}$ and $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a moderately oscillatory process such that $\overline{\{u_\sigma:\sigma\in\mathcal{S}\}}\subseteq Q_E$, the closure being for the topology of convergence in measure on $L^0(\mathfrak{A})$. Then $\bar h\boldsymbol{u}=\langle\bar h(u_\sigma)\rangle_{\sigma\in\mathcal{S}}$ is a moderately oscillatory process.

648C Lemma Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}_{\mathbb{A}}$, $\boldsymbol{w}$ a $\| \|_2$-bounded martingale with domain $\mathcal{S}$, and $\boldsymbol{x}$ a $\| \|_\infty$-bounded S-integrable process with domain $\mathcal{S}$. If $\boldsymbol{w}^*$ is the quadratic variation of $\boldsymbol{w}$, $\|(\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{w})^2\|_1 \le \|\oint_{\mathcal{S}}\boldsymbol{x}^2 d\boldsymbol{w}^*\|_1 < \infty$.

648D Lemma Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathfrak{B}$ be a closed subalgebra of $\mathfrak{A}$ which is coordinated with $\langle\mathfrak{A}_t\rangle_{t\in T}$, and $\mathbb{B} = (\mathfrak{B}, \bar\mu\restriction\mathfrak{B}, T, \langle\mathfrak{B}_t\rangle_{t\in T}, \mathcal{T}_{\mathbb{B}}, \langle\mathfrak{B}_\sigma\rangle_{\sigma\in\mathcal{T}_{\mathbb{B}}})$ the corresponding stochastic integration structure, where $\mathfrak{B}_t = \mathfrak{B}\cap\mathfrak{A}_t$ for $t \in T$. Let $\mathcal{S}'$ be a relatively order-convex sublattice of $\mathcal{T}_{\mathbb{B}}$ and $\mathcal{S}$ its order-convex hull in $\mathcal{T}_{\mathbb{A}}$. If $\boldsymbol{x} \in \mathbb{A}M^0_{\text{S-i}}(\mathcal{S})$ and $\boldsymbol{x}\restriction\mathcal{S}' \in L^0(\mathfrak{B})^{\mathcal{S}'}$, then $\boldsymbol{x}\restriction\mathcal{S}' \in \mathbb{B}M^0_{\text{S-i}}(\mathcal{S})$.

648E Lemma Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathfrak{B}$ be a closed subalgebra of $\mathfrak{A}$ which is coordinated with $\langle\mathfrak{A}_t\rangle_{t\in T}$, and $\mathbb{B} = (\mathfrak{B}, \bar\mu\restriction\mathfrak{B}, T, \langle\mathfrak{B}_t\rangle_{t\in T}, \mathcal{T}_{\mathbb{B}}, \langle\mathfrak{B}_\sigma\rangle_{\sigma\in\mathcal{T}_{\mathbb{B}}})$ the corresponding stochastic integration structure, where $\mathfrak{B}_t = \mathfrak{B}\cap\mathfrak{A}_t$ for $t \in T$. Let $\mathcal{S}'$ be a relatively order-convex sublattice of $\mathcal{T}_{\mathbb{B}}$ and $\mathcal{S}$ its order-convex hull in $\mathcal{T}_{\mathbb{A}}$. If $\boldsymbol{u} \in \mathbb{A}M_{\text{n-s}}(\mathcal{S})$ and $\boldsymbol{u}\restriction\mathcal{S}' \in L^0(\mathfrak{B})^{\mathcal{S}'}$ then $\boldsymbol{u}' = \boldsymbol{u}\restriction\mathcal{S}'$ is near-simple in either $\mathbb{A}$ or $\mathbb{B}$.

648F Lemma Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathfrak{B}$ be a closed subalgebra of $\mathfrak{A}$ which is coordinated with $\langle\mathfrak{A}_t\rangle_{t\in T}$, and $\mathbb{B} = (\mathfrak{B}, \bar\mu\restriction\mathfrak{B}, T, \langle\mathfrak{B}_t\rangle_{t\in T}, \mathcal{T}_{\mathbb{B}}, \langle\mathfrak{B}_\sigma\rangle_{\sigma\in\mathcal{T}_{\mathbb{B}}})$ the corresponding stochastic integration structure, where $\mathfrak{B}_t = \mathfrak{B}\cap\mathfrak{A}_t$ for $t \in T$. Let $\mathcal{S}'$ be a relatively order-convex sublattice of $\mathcal{T}_{\mathbb{B}}$ and $\mathcal{S}$ its order-convex hull in $\mathcal{T}_{\mathbb{A}}$. Let $\boldsymbol{x} \in \mathbb{A}M^0_{\text{S-i}}(\mathcal{S})$, $\boldsymbol{v} \in \mathbb{A}M^{\uparrow}_{\text{n-s}}(\mathcal{S})$ be such that $\boldsymbol{x}' = \boldsymbol{x}\restriction\mathcal{S}'$ and $\boldsymbol{v}' = \boldsymbol{v}\restriction\mathcal{S}'$ belong to $L^0(\mathfrak{B})^{\mathcal{S}'}$. If $\mathbb{B}\widehat{\theta}^{\#}_{\boldsymbol{v}'}(\boldsymbol{x}') = 0$ then $\mathbb{A}\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{v} = 0$.

648G Theorem Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathfrak{B}$ be a closed subalgebra of $\mathfrak{A}$ which is coordinated with $\langle\mathfrak{A}_t\rangle_{t\in T}$, and $\mathbb{B} = (\mathfrak{B}, \bar\mu\restriction\mathfrak{B}, T, \langle\mathfrak{B}_t\rangle_{t\in T}, \mathcal{T}_{\mathbb{B}}, \langle\mathfrak{B}_\sigma\rangle_{\sigma\in\mathcal{T}_{\mathbb{B}}})$ the corresponding stochastic integration structure, where $\mathfrak{B}_t = \mathfrak{B}\cap\mathfrak{A}_t$ for $t \in T$. Let $\mathcal{S}'$ be a relatively order-convex sublattice of $\mathcal{T}_{\mathbb{B}}$ and $\mathcal{S}$ its order-convex hull in $\mathcal{T}_{\mathbb{A}}$. Let $\boldsymbol{x} \in \mathbb{A}M_{\text{S-i}}(\mathcal{S})$ and an $\mathbb{A}$-near-simple $\mathbb{A}$-integrator $\boldsymbol{w}$ with domain $\mathcal{S}$ be such that $\boldsymbol{x}' = \boldsymbol{x}\restriction\mathcal{S}'$, $\boldsymbol{w}' = \boldsymbol{w}\restriction\mathcal{S}'$ belong to $L^0(\mathfrak{B})^{\mathcal{S}'}$. Then $\mathbb{B}\oint_{\mathcal{S}'}\boldsymbol{x}'d\boldsymbol{w}' = \mathbb{A}\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{w}$.

648Z Problem In 648G, can we drop the hypothesis that '$\mathfrak{B}$ is coordinated with $\langle\mathfrak{A}_t\rangle_{t\in T}$'?

Version of 30.9.14/28.1.20

649 Pathwise integration

The integrals of §613 and §645 are defined in terms of convergence in L^0. The most important applications are associated with processes of the form $\langle X_t(\omega)\rangle_{t\ge0,\omega\in\Omega}$ with paths $\langle X_t(\omega)\rangle_{t\ge0}$. It turns out that in the case of the Riemann-sum integral, we can often, with some effort, define integrals 'pathwise'. I do not think that this approach gives a good picture of the theory as a whole, but it is surely worth knowing what can be done.

The S-integral is rather different; I do not see any way of giving a pathwise description of the S-integral with respect to Brownian motion, for instance. But for non-decreasing integrators we have an effective approach through Stieltjes integrals, which I have hinted at in earlier sections. I now give a detailed account of the method (649H, 649L).

649B Theorem Suppose that $\mathcal{S}$ is a sublattice of $\mathcal{T}$ with a least element. Let $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ be an integrator and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a near-simple process. Suppose that we have, for each $n \in \mathbb{N}$, a non-decreasing sequence $\langle\tau_{ni}\rangle_{i\in\mathbb{N}}$ in $\mathcal{S}$ such that $\tau_{n0} = \min\mathcal{S}$, $\inf_{i\in\mathbb{N}}[\![\tau_{ni} < \sup\mathcal{S}]\!] = 0$ and, for each $i \in \mathbb{N}$,

$$[\![\sigma < \tau_{n,i+1}]\!] \subseteq [\![|u_\sigma - u_{\tau_{n,i}}| \le 2^{-n}]\!] \text{ for every } \sigma \in [\tau_{n,i}, \tau_{n,i+1}],$$

$$[\![\tau_{n,i+1} < \sup\mathcal{S}]\!] \subseteq [\![|u_{\tau_{n,i+1}} - u_{\tau_{n,i}}| \ge 2^{-n}]\!].$$

Then

$$z_n = \lim_{k\to\infty}\sum_{i=0}^{k-1} u_{\tau_{n,i}} \times (v_{\tau_{n,i+1}} - v_{\tau_{n,i}})$$

is defined for each n, and $\langle z_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to $\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}$.

649C Corollary Let (Ω,Σ,μ) be a complete probability space and $\langle\Sigma_t\rangle_{t\in[0,\infty[}$ a right-continuous filtration of σ-subalgebras of Σ, all containing every negligible subset of Ω; suppose that $(\mathfrak{A},\bar\mu)$ and $\langle\mathfrak{A}_t\rangle_{t\ge 0}$ are the corresponding measure algebra and filtration of closed subalgebras. Let $\langle U_t\rangle_{t\ge0}$, $\langle V_t\rangle_{t\ge0}$ be stochastic processes such that $t\mapsto U_t(\omega):[0,\infty[\,\to\mathbb{R}$ is càdlàg for every $\omega\in\Omega$, and $(t,\omega)\mapsto V_t(\omega):[0,\infty[\,\times\Omega\to\mathbb{R}$ is progressively measurable; let $\boldsymbol{u}$, $\boldsymbol{v}$ be the corresponding fully adapted processes with domain $\mathcal{T}_f$. Suppose that $\boldsymbol{v}$ is a local integrator.

Let $h:\Omega\to[0,\infty[$ be a stopping time, and $\tau^*=h^\bullet$ the corresponding stopping time in $\mathcal{T}_f$. For $n\in\mathbb{N}$ and $\omega\in\Omega$, define $h_{ni}(\omega)$, for $i\in\mathbb{N}$, by setting $h_{n0}(\omega)=0$ and then

$$h_{n,i+1}(\omega)=\inf(\{h(\omega)\}\cup\{t:t\ge h_{ni}(\omega),\,|U_t(\omega)-U_{h_{ni}(\omega)}(\omega)|>2^{-n}\})$$

for $i\in\mathbb{N}$.

In this case,

(a) every h_{ni} is a stopping time adapted to $\langle\Sigma_t\rangle_{t\ge0}$,

(b)

$$f_n(\omega)=\sum_{i=0}^\infty U_{h_{ni}}(\omega)\big(V_{h_{n,i+1}}(\omega)-V_{h_{ni}}(\omega)\big)$$

is defined for all $n\in\mathbb{N}$ and $\omega\in\Omega$,

(c) $f=\lim_{n\to\infty}f_n$ is defined in $\mathbb{R}$ almost everywhere in Ω, and $f^\bullet=\int_{[\check 0,\tau^*]}\boldsymbol{u}\,d\boldsymbol{v}$.

649D (a) Definition A filter $\mathcal{F}$ on $\mathbb{N}$ is **measure-converging** if whenever (Ω,Σ,μ) is a probability space, $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in Σ, and $\lim_{n\to\infty}\mu E_n=1$, then $\bigcup_{A\in\mathcal{F}}\bigcap_{n\in A}E_n$ is conegligible.

(b) Suppose that $\mathcal{F}$ is a measure-converging filter on $\mathbb{N}$, (Ω,Σ,μ) is a probability space, and $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}^0=\mathcal{L}^0(\mu)$ which converges in measure to $f\in\mathcal{L}^0$. Then $\lim_{n\to\mathcal{F}}f_n=_{\text{a.e.}}f$.

649E Proposition Suppose that $\mathcal{F}$ is a measure-converging filter on $\mathbb{N}$. Let (Ω,Σ,μ) be a complete probability space, $\langle\Sigma_t\rangle_{t\ge0}$ a right-continuous filtration of σ-subalgebras of Σ all containing every negligible set; suppose that $(\mathfrak{A},\bar\mu)$, $\langle\mathfrak{A}_t\rangle_{t\ge0}$ are the corresponding probability algebra and filtration of closed subalgebras. Let $\langle U_t\rangle_{t\ge0}$, $\langle V_t\rangle_{t\ge0}$ be stochastic processes on Ω, adapted to $\langle\Sigma_t\rangle_{t\ge0}$, such that the paths $t\mapsto U_t(\omega)$, $t\mapsto V_t(\omega)$ are càdlàg for every ω; let $\boldsymbol{u}$, $\boldsymbol{v}$ be the corresponding locally near-simple processes defined on $\mathcal{T}_f$, and suppose that $\boldsymbol{v}$ is a local integrator. Let h, $h':\Omega\to[0,\infty[$ be stopping times corresponding to τ, $\tau'\in\mathcal{T}_f$, with $h(\omega)\le h'(\omega)$ for every ω. Enumerate $\mathbb{Q}\cap[0,\infty[$ as $\langle q_n\rangle_{n\in\mathbb{N}}$, starting with $q_0=0$, and for $n\in\mathbb{N}$ let $\langle q_{ni}\rangle_{i\le n}$ be the increasing enumeration of $\{q_i:i\le n\}$. Set

$$f_n(\omega)=\sum_{i=0}^{n-1}U_{\operatorname{med}(h(\omega),q_{ni},h'(\omega))}(\omega)\big(V_{\operatorname{med}(h(\omega),q_{n,i+1},h'(\omega))}(\omega)-V_{\operatorname{med}(h(\omega),q_{ni},h'(\omega))}(\omega)\big)$$

for $\omega\in\Omega$. Then $f(\omega)=\lim_{n\to\mathcal{F}}f_n(\omega)$ is defined for almost every ω, f is Σ-measurable and $f^\bullet=\int_{[\tau,\tau']}\boldsymbol{u}\,d\boldsymbol{v}$.

649F Lemma Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a locally near-simple process. Let $\tilde{\mathcal{S}}$ be the full ideal of $\mathcal{T}$ generated by $\mathcal{S}$. Then there is a locally near-simple process $\tilde{\boldsymbol{u}}$ with domain $\tilde{\mathcal{S}}$ extending $\boldsymbol{u}$. If $\boldsymbol{u}$ is non-negative and non-decreasing, we can arrange that $\tilde{\boldsymbol{u}}$ should be non-negative and non-decreasing.

649G Lemma Suppose that $(\mathfrak{A},\bar\mu,[0,\infty[\,,\langle\mathfrak{A}_t\rangle_{t\ge0},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ is a real-time integration structure and $\mathcal{S}$ is a non-empty sublattice of $\mathcal{T}$. There is a non-decreasing sequence $\langle\tau_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{S}$ such that $\sup_{n\in\mathbb{N}}[\![\tau\le\tau_n]\!]=1$ for every $\tau\in\mathcal{S}$.

649H Theorem Suppose that $(\mathfrak{A},\bar\mu,[0,\infty[\,,\langle\mathfrak{A}_t\rangle_{t\ge0},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ is a right-continuous real-time stochastic integration structure.

(a)(i) There is a complete probability space (Ω,Σ,μ) such that $(\mathfrak{A},\bar\mu)$ can be identified with the measure algebra of (Ω,Σ,μ).

(ii) For $E\in\Sigma$, write $E^\bullet$ for the corresponding member of $\mathfrak{A}$; for $t\ge0$ set $\Sigma_t=\{E:E\in\Sigma,\,E^\bullet\in\mathfrak{A}_t\}$. Then $\langle\Sigma_t\rangle_{t\ge0}$ is a right-continuous filtration of σ-algebras all containing every negligible subset of Ω.

(iii) Members of $\mathcal{T}$ can be represented by stopping times $h:\Omega\to[0,\infty]$ as in 612H, with the corresponding identification of the algebras $\mathfrak{A}_\tau$ as in 612H(a-iii).

(b) Now suppose that $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a locally near-simple process with non-empty domain $\mathcal{S}\subseteq\mathcal{T}_f$. Then there are a progressively measurable stochastic process $\langle U_t\rangle_{t\ge0}$ and a non-decreasing sequence $\langle h_n\rangle_{n\in\mathbb{N}}$ of finite-valued stopping times, all adapted to $\langle\Sigma_t\rangle_{t\ge0}$, such that

(α) h_n represents a stopping time $\tau_n\in\mathcal{S}$ for every $n\in\mathbb{N}$, and $\sup_{n\in\mathbb{N}}[\![\sigma\le\tau_n]\!]=1$ for every $\sigma\in\mathcal{S}$,

(β) $U_g^\bullet=u_\sigma$ whenever $g:\Omega\to[0,\infty[$ is a stopping time representing $\sigma\in\mathcal{S}$,

(γ) $t\mapsto U_t(\omega):[0,h_n(\omega)]\to\mathbb{R}$ is càdlàg for every $\omega\in\Omega$ and $n\in\mathbb{N}$.

649I Scholium If, in 649Hb, $\boldsymbol{u}$ is a non-negative non-decreasing process, then we can arrange that $\langle U_t\rangle_{t\ge0}$ is non-decreasing.

649J Lemma Let (Ω,Σ,μ) be a complete probability space, $\langle\Sigma_t\rangle_{t\ge0}$ a right-continuous filtration of σ-subalgebras of Σ all containing every negligible set, and $\langle U_t\rangle_{t\ge0}$ a progressively measurable stochastic process. Let $h:\Omega\to[0,\infty[$ be a stopping time such that $t\mapsto U_t(\omega):[0,h(\omega)]\to\mathbb{R}$ is càdlàg for every $\omega\in\Omega$. For $\omega\in\Omega$ and $t\ge0$ set

$$\begin{aligned}U_{<t}(\omega)&=\lim_{s\uparrow t}U_s(\omega)\text{ if }0<t\le h(\omega),\\&=0\text{ otherwise.}\end{aligned}$$

(a) $\langle U_{<t}\rangle_{t\ge0}$ is a previsibly measurable stochastic process, therefore progressively measurable.

(b) Let $(\mathfrak{A},\bar\mu,[0,\infty[\,,\langle\mathfrak{A}_t\rangle_{t\ge0},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ be the real-time stochastic integration structure defined from $(\Omega,\Sigma,\mu,\langle\Sigma_t\rangle_{t\ge0})$, and $\tau\in\mathcal{T}_f$ the stopping time represented by h. If $\boldsymbol{u}$, $\boldsymbol{z}$ are the fully adapted processes defined from U and $U_<$, then $\boldsymbol{u}{\restriction}\mathcal{T}\wedge\tau$ is near-simple and its previsible version is $\boldsymbol{z}{\restriction}\mathcal{T}\wedge\tau$.

(c) Now suppose that $\langle V_t\rangle_{t\ge0}$ is another progressively measurable stochastic process, this time non-negative and non-decreasing, such that $t\mapsto V_t(\omega):[0,h(\omega)]\to\mathbb{R}$ is càdlàg for every $\omega\in\Omega$. Let $\boldsymbol{v}$ be the process defined by $\langle V_t\rangle_{t\ge0}$. For $\omega\in\Omega$ let ν_ω be the Radon measure on $[0,h(\omega)]$ such that $\nu_\omega[0,t]=V_t(\omega)$ for every $t\ge0$, and set

$$e(\omega)=\int_{[0,h(\omega)]}U_{<t}(\omega)\nu_\omega(dt).$$

Then $e:\Omega\to\mathbb{R}$ is Σ_h-measurable and $e^\bullet=\int_{\mathcal{T}\wedge\tau}\boldsymbol{u}\,d\boldsymbol{v}$.

649K Lemma Let (Ω,Σ,μ) be a probability space and $\langle\Sigma_t\rangle_{t\ge0}$ a filtration of σ-subalgebras of Σ. Let Λ_{pv} be the corresponding previsible σ-algebra and write $\mathcal{L}$ for the smallest subset of $\mathbb{R}^{[0,\infty[\times\Omega}$ such that

constant functions belong to $\mathcal{L}$,

scalar multiples of functions in $\mathcal{L}$ belong to $\mathcal{L}$,

if $\phi\in\mathcal{L}$ and $\psi\in\mathbb{R}^{[0,\infty[\times\Omega}$ and $|\phi|\wedge|\psi|=0$, then $\phi+\psi\in\mathcal{L}$ iff $\psi\in\mathcal{L}$,

$\chi(]s,\infty[\times E)\in\mathcal{L}$ whenever $s\ge0$ and $E\in\Sigma_s$,

$\lim_{n\to\infty}\phi_n\in\mathcal{L}$ whenever $\langle\phi_n\rangle_{n\in\mathbb{N}}$ is a pointwise convergent sequence in $\mathcal{L}$.

Then $\mathcal{L}$ is the set of all Λ_{pv}-measurable real-valued functions on $[0,\infty[\times\Omega$.

649L Theorem Let (Ω,Σ,μ) be a complete probability space, $\langle\Sigma_t\rangle_{t\ge0}$ a right-continuous filtration of σ-subalgebras of Σ all containing every negligible set, $(\mathfrak{A},\bar\mu,[0,\infty[\,,\langle\mathfrak{A}_t\rangle_{t\ge0},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ the corresponding real-time stochastic integration structure, $\langle X_t\rangle_{t\ge0}$ a previsibly measurable stochastic process and $\langle V_t\rangle_{t\ge0}$ a non-negative non-decreasing stochastic process. Let $h:\Omega\to[0,\infty[$ be a stopping time such that $t\mapsto X_t(\omega)$ is bounded on $[0,h(\omega)]$ and $t\mapsto V_t(\omega):[0,h(\omega)]\to\mathbb{R}$ is càdlàg for every $\omega\in\Omega$, and write τ for the corresponding stopping time in $\mathcal{T}$. Let $\boldsymbol{x}$, $\boldsymbol{v}$ be the processes defined by $\langle X_t\rangle_{t\ge0}$ and $\langle V_t\rangle_{t\ge0}$. For $\omega\in\Omega$ let ν_ω be the Radon measure on $[0,h(\omega)]$ such that $\nu_\omega[0,t]=V_t(\omega)$ for every $t\ge0$, and set

$$e(\omega)=\int_{]0,h(\omega)]}X_t(\omega)\nu_\omega(dt).$$

Then $e:\Omega\to\mathbb{R}$ is Σ-measurable and $e^\bullet=\oint_{\mathcal{T}\wedge\tau}\boldsymbol{x}\,d\boldsymbol{v}$.