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Chapter 64

The fundamental theorem of martingales and the S-integral

To my mind, the 'Riemann-sum' integral of §613 is the natural starting point for a theory of stochastic integration, and it has a rich assortment of properties. But if you are acquainted with the Lebesgue-Stieltjes integral, you will have noticed that I have not given results corresponding to the standard convergence theorems of §123, and if you have taken the trouble to check, you will have noticed that they aren't true of the integral as presented so far. If we make the right modifications, however, we do have a kind of sequential smoothness (644C) which can, with some difficulty, be used as the basis for what I will call the 'S-integral' (645P). In fact the S-integral is much closer than the Riemann-sum integral to the standard stochastic integral developed in Protter 05.

To do this we need to know quite a lot more about stochastic processes. In §641 I describe the 'previsible version' of a near-simple process, which corresponds to the càglàd function equal except at jump points to a càdlàg function of a real variable. Looking at pointwise limits of sequences of previsible versions, we are led to the previsible processes of §642, which have the kind of measurability demanded of an integrand in the S-integral (645I). But the really important fact, if we are going to have the S-integral for martingale integrators which are not jump-free, is the fundamental theorem of martingales: under certain conditions, an integrator can be expressed not just as the sum of a virtually local martingale and a process of bounded variation, as in the Bichteler-Dellacherie theorem (627J), but as such a sum in which the virtually local martingale has small residual oscillation (643M).

With the S-integral defined, we can look at its properties, which by and large correspond to those of the Riemann-sum integral as established in chapters 61-63. Many of the details are not trivial, and I work through them in §646-648, with an S-integral version of Itô's formula (646T).

I end the chapter with a brief note (§649) left over from Chapter 63, on Riemann-sum integrals, in the classical context of progressively measurable stochastic processes defined on a probability space, which can be calculated from sample paths, one path at a time; for non-decreasing integrators, we can use a Stieltjes integral on each path to calculate the S-integral.

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641 Previsible versions

In §618 I introduced 'jump-free' processes without going into the question of what the 'jumps' were which they were free of. We now need to look at the structure of processes which are not jump-free. In the standard model of locally near-simple processes as those representable by processes with càdlàg sample paths (631D), we have direct descriptions of σ-algebras Σ_{h^-} and random variables X_{h^-} defined in terms of observations taken *before* a stopping time h, rather than at the stopping time, as in 612H. I present these descriptions in 642E, following corresponding definitions in the more abstract language I favour in this volume (641B, 641F). Once we have got hold of the previsible version $\boldsymbol{u}_<$ of a near-simple process, we have an expression for the residual oscillation of $\boldsymbol{u}$ in terms of $\boldsymbol{u}-\boldsymbol{u}_<$ (641Nb, 642Ga). For moderately oscillatory processes which are not near-simple, we do not have such a direct description of their jumps, but the construction of the previsible version still works (641L), and we have effective results on indefinite integrals (641Q) and quadratic variations (641R).

641A Notation $(\mathfrak{A},\bar\mu,T,\langle\mathfrak{A}_t\rangle_{t\in T},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ will be a stochastic integration structure, as described in the notes to §613. If $\langle\mathfrak{B}_i\rangle_{i\in I}$ is a family of closed subalgebras of $\mathfrak{A}$, $\bigvee_{i\in I}\mathfrak{B}_i$ will be the smallest closed subalgebra including every $\mathfrak{B}_i$. θ will be the standard functional defining the topology of convergence in measure on $L^0(\mathfrak{A})$ (613B). If $t \in T$ then $\check{t}$ will be the constant stopping time at t. If $\mathcal{S}$ is a sublattice of $\mathcal{T}$, $\mathcal{I}(\mathcal{S})$ will be the upwards-directed set of finite sublattices of $\mathcal{S}$; $M_{\mathrm{fa}}(\mathcal{S})$, $M_{\text{o-b}}(\mathcal{S})$, $M_{\mathrm{mo}}(\mathcal{S})$, $M_{\mathrm{lmo}}(\mathcal{S})$ and $M_{\text{n-s}}(\mathcal{S})$ will be the spaces of fully adapted, order-bounded, moderately oscillatory, locally moderately oscillatory and near-simple processes with domain $\mathcal{S}$. If $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ belongs to $M_{\text{o-b}}(\mathcal{S})$, $\sup|\boldsymbol{u}|$ will be the supremum $\sup_{\sigma \in \mathcal{S}} |u_\sigma|$ in $L^0(\mathfrak{A})$, and $\mathrm{Osclln}(\boldsymbol{u})$ will be the residual oscillation of $\boldsymbol{u}$ (618B). For sublattices I, J of $\mathcal{T}$, $I \sqcup J$ will be the sublattice of $\mathcal{T}$ generated by $I \cup J$.

641B The algebras $\mathfrak{A}_{\mathcal{S}<\tau}$ (a) Definition If $\mathcal{S}$ is a sublattice of $\mathcal{T}$ and $\tau \in \mathcal{T}$, let $\mathfrak{A}_{\mathcal{S}<\tau}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{a \cap [\![\sigma < \tau]\!] : \sigma \in \mathcal{S},\ a \in \mathfrak{A}_\sigma\}$. Note that $\mathfrak{A}_{\mathcal{S}<\tau} \subseteq \mathfrak{A}_\tau$, by 611H(c-i). I will write $\mathfrak{A}_{<\tau}$ for $\mathfrak{A}_{\mathcal{T}<\tau}$.

(b) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.

(i) If $\sigma \le \tau$ in $\mathcal{T}$ then $\mathfrak{A}_{\mathcal{S}<\sigma} \subseteq \mathfrak{A}_{\mathcal{S}<\tau}$. **P** If $\rho \in \mathcal{S}$ then $[\![\rho < \sigma]\!] \in \mathfrak{A}_\rho$ and $[\![\rho < \sigma]\!] \subseteq [\![\rho < \tau]\!]$ so

$$\begin{aligned}\{a \cap [\![\rho < \sigma]\!] : \rho \in \mathcal{S},\ a \in \mathfrak{A}_\rho\} &= \{a \cap [\![\rho < \sigma]\!] \cap [\![\rho < \tau]\!] : \rho \in \mathcal{S},\ a \in \mathfrak{A}_\rho\} \\ &\subseteq \{a \cap [\![\rho < \tau]\!] : \rho \in \mathcal{S},\ a \in \mathfrak{A}_\rho\}. \ \mathbf{Q}\end{aligned}$$

(ii) If $\sigma \in \mathcal{S}$, $\tau \in \mathcal{T}$ and $u \in L^0(\mathfrak{A}_\sigma)$, then $u \times \chi[\![\sigma < \tau]\!] \in L^0(\mathfrak{A}_{\mathcal{S}<\tau})$. **P** Write u' for $u \times \chi[\![\sigma < \tau]\!]$. For $\alpha \ge 0$, $[\![u' > \alpha]\!] = [\![u > \alpha]\!] \cap [\![\sigma < \tau]\!]$ and $[\![u' < -\alpha]\!] \cap [\![\sigma < \tau]\!]$ belong to $\mathfrak{A}_{\mathcal{S}<\tau}$. Now if $\alpha < 0$

$$[\![u' > \alpha]\!] = \sup_{\beta \in]\alpha,0]} [\![u' \ge \beta]\!] = 1 \setminus \inf_{\beta \in]\alpha,0]} [\![u' < \beta]\!]$$

again belongs to $\mathfrak{A}_{\mathcal{S}<\tau}$, so $u' \in L^0(\mathfrak{A}_{\mathcal{S}<\tau})$. **Q**

(iii) If $\mathcal{S}'$ is a sublattice of $\mathcal{T}$ covering $\mathcal{S}$, then $\mathfrak{A}_{\mathcal{S}<\tau} \subseteq \mathfrak{A}_{\mathcal{S}'<\tau}$ for every $\tau \in \mathcal{T}$. **P** If $\sigma \in \mathcal{S}$ and $a \in \mathfrak{A}_\sigma$, then $a \cap [\![\rho = \sigma]\!] \in \mathfrak{A}_\rho$ for every $\rho \in \mathcal{S}'$ (611Hc), so

$$\begin{aligned}a \cap [\![\sigma < \tau]\!] &= a \cap [\![\sigma < \tau]\!] \cap \sup_{\rho \in \mathcal{S}} [\![\rho = \sigma]\!] = \sup_{\rho \in \mathcal{S}} a \cap [\![\sigma < \tau]\!] \cap [\![\rho = \sigma]\!] \\ &= \sup_{\rho \in \mathcal{S}} a \cap [\![\rho = \sigma]\!] \cap [\![\rho < \tau]\!] \in \mathfrak{A}_{\mathcal{S}'<\tau}. \ \mathbf{Q}\end{aligned}$$

In particular, $\mathfrak{A}_{\mathcal{S}<\tau} \subseteq \mathfrak{A}_{\mathcal{S}'<\tau}$ whenever $\mathcal{S} \subseteq \mathcal{S}'$ and $\tau \in \mathcal{T}$.

(iv) Now suppose that $\mathcal{S}$ is finitely full and that $\tau \in \mathcal{S}$. Then $\mathfrak{A}_{\mathcal{S}<\tau}$ is the closed subalgebra $\mathfrak{C}$ generated by $\{[\![\sigma < \tau]\!] : \sigma \in \mathcal{S}\}$. **P** Of course $1 \cap [\![\sigma < \tau]\!] \in \mathfrak{A}_{\mathcal{S}<\tau}$ for every $\sigma \in \mathcal{S}$, so $\mathfrak{C} \subseteq \mathfrak{A}_{\mathcal{S}<\sigma}$. On the other hand, if $\sigma \in \mathcal{S}$ and $a \in \mathfrak{A}_\sigma$ then $a' = a \cap [\![\sigma < \tau]\!]$ belongs to $\mathfrak{A}_\sigma \cap \mathfrak{A}_\tau$ by 611H(c-iii). By 611I, there is a $\rho \in \mathcal{T}$ such that $a' \subseteq [\![\rho = \sigma]\!]$ and $1 \setminus a' \subseteq [\![\rho = \tau]\!]$; now $\rho \in \mathcal{S}$ and $a' = [\![\rho < \tau]\!] \in \mathfrak{C}$. So $\mathfrak{A}_{\mathcal{S}<\sigma} \subseteq \mathfrak{C}$ and we have equality. **Q**

(c) Suppose that $\tau \in \mathcal{T}$, I is a non-empty finite sublattice of $\mathcal{T}$ and $(\tau_0, \dots, \tau_n)$ linearly generates the I-cells (611L). Let $\mathfrak{B}$ be the set of those $b \in \mathfrak{A}$ such that

$b \cap [\![\tau \le \tau_0]\!]$ is either $[\![\tau \le \tau_0]\!]$ or 0, $\quad b \cap [\![\tau_n < \tau]\!] \in \mathfrak{A}_{\tau_n}$,

for every $i < n$ there is an $a \in \mathfrak{A}_{\tau_i}$ such that $b \cap [\![\tau_i < \tau]\!] \setminus [\![\tau_{i+1} < \tau]\!] = a \cap [\![\tau_i < \tau]\!] \setminus [\![\tau_{i+1} < \tau]\!]$.

Then $\mathfrak{A}_{I<\tau} = \mathfrak{B}$. **P** Writing J for $\{\tau_0, \dots, \tau_n\}$, J covers I (611Ke) so $\mathfrak{A}_{I<\tau} = \mathfrak{A}_{J<\tau}$ ((b-iii) above). Set

$$d_{-1} = [\![\tau \le \tau_0]\!] = 1 \setminus \tau_0 < \tau, \quad d_n = [\![\tau_n < \tau]\!],$$

$$d_i = [\![\tau_i < \tau]\!] \setminus [\![\tau_{i+1} < \tau]\!] \text{ for } i < n.$$

Then $\{d_{-1}, d_0, \dots, d_n\}$ is a partition of unity in $\mathfrak{A}$, and $d_i \in \mathfrak{A}_{J<\tau}$ whenever $-1 \le i \le n$.

Set $\mathfrak{B}_{-1} = \{0, d_{-1}\}$ and for $0 \le i \le n$ set $\mathfrak{B}_i = \{d_i \cap a : a \in \mathfrak{A}_{\tau_i}\}$. Then $\mathfrak{B}_i$ is always an order-closed subalgebra of the principal ideal of $\mathfrak{A}$ generated by d_i, while $\mathfrak{B} = \{b : b \cap d_i \in \mathfrak{B}_i \text{ whenever } -1 \le i \le n\}$. So $\mathfrak{B}$ is an order-closed subalgebra of $\mathfrak{A}$. Since $\mathfrak{B}_i \subseteq \mathfrak{A}_{J<\tau}$ whenever $-1 \le i \le n$, $\mathfrak{B} \subseteq \mathfrak{A}_{J<\tau}$.

If $-1 \le i \le n$, $j \le n$ and $a \in \mathfrak{A}_{\tau_j}$, then $d_i \cap a \cap [\![\tau_j < \tau]\!]$ belongs to $\mathfrak{B}_i$, because if $i < j$ then $d_i \cap [\![\tau_j < \tau]\!] = 0$, while if $j \le i$ then $a \cap [\![\tau_j < \tau]\!] \in \mathfrak{A}_{\tau_i}$. So if we write $A = \{a \cap [\![\tau_j < \tau]\!] : j \le n,\ a \in \mathfrak{A}_{\tau_j}\}$ for the generating subset of $\mathfrak{A}_{J<\tau}$, we see that $a \cap d_i \in \mathfrak{B}_i$ whenever $a \in A$ and $-1 \le i \le n$, that is, $A \subseteq \mathfrak{B}$. Accordingly $\mathfrak{A}_{J<\tau} \subseteq \mathfrak{B}$ and $\mathfrak{B} = \mathfrak{A}_{J<\tau} = \mathfrak{A}_{I<\tau}$, as claimed. **Q**

(d) If $t \in T$ then $\mathfrak{A}_{<\check{t}} = \bigvee_{s<t} \mathfrak{A}_s$. **P** If $s < t$, then $[\![\check{s} < \check{t}]\!] = 1$ so $\mathfrak{A}_s = \mathfrak{A}_{\check{s}} \subseteq \mathfrak{A}_{<\check{t}}$. On the other hand, if $\sigma \in \mathcal{T}$,

$$[\![\sigma < \check{t}]\!] = \sup_{s\in T} [\![\check{t} > s]\!] \setminus [\![\sigma > s]\!] = \sup_{s<t} 1 \setminus [\![\sigma > s]\!]$$

belongs to $\bigvee_{s<t} \mathfrak{A}_s$. **Q**

(e) If $\tau \in \mathcal{T}$ then $\mathfrak{A}_{<\tau}$ is the closed subalgebra of $\mathfrak{A}$ generated by

$$A = \{a : \text{there is a } t \in T \text{ such that } a \in \mathfrak{A}_t \text{ and } a \subseteq [\![\tau > t]\!]\}.$$

P Let $\mathfrak{B}$ be the closed subalgebra generated by A. If $t \in T$, $a \in \mathfrak{A}_t$ and $a \subseteq [\![\tau > t]\!]$, then $a \in \mathfrak{A}_{\check{t}}$ and $a \subseteq [\![\check{t} < \tau]\!]$ (611E(a-i-δ)), so $a \in \mathfrak{A}_{<\tau}$; consequently $\mathfrak{B} \subseteq \mathfrak{A}_{<\tau}$. If $\sigma \in \mathcal{T}$ and $t \in T$, then $[\![\tau > t]\!] \setminus [\![\sigma > t]\!]$ belongs to $\mathfrak{A}_t$ and is included in $[\![\tau > t]\!]$, so belongs to $A \subseteq \mathfrak{B}$; taking the supremum over t, $[\![\sigma < \tau]\!] \in \mathfrak{B}$. As σ is arbitrary and $\mathcal{T}$ is full, (b-iv) tells us that $\mathfrak{A}_{<\tau} \subseteq \mathfrak{B}$ and we have equality. **Q**

641C Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $C \subseteq \mathcal{T}$ a non-empty set with supremum τ.
(a) $\mathfrak{A}_{\mathcal{S}<\tau} = \bigvee_{\sigma\in C} \mathfrak{A}_{\mathcal{S}<\sigma}$.
(b) Now suppose that $C \subseteq \mathcal{S}$. Set $a = \inf_{\sigma\in C} [\![\sigma < \tau]\!]$. Then

$$\bigvee_{\sigma\in C} \mathfrak{A}_\sigma = \{(b \setminus a) \cup (c \cap a) : b \in \mathfrak{A}_\tau,\, c \in \mathfrak{A}_{\mathcal{S}<\tau}\}.$$

proof (a) By 641B(b-i), $\bigvee_{\sigma\in C} \mathfrak{A}_{\mathcal{S}<\sigma} \subseteq \mathfrak{A}_{\mathcal{S}<\tau}$. In the other direction, if $\rho \in \mathcal{S}$ and $a \in \mathfrak{A}_\rho$ then $a \cap [\![\rho < \tau]\!] = \sup_{\sigma\in C} a \cap [\![\rho < \sigma]\!]$ (611Eb), so belongs to $\bigvee_{\sigma\in C} \mathfrak{A}_{\mathcal{S}<\sigma}$. As ρ is arbitrary, $\mathfrak{A}_{\mathcal{S}<\tau} \subseteq \bigvee_{\sigma\in C} \mathfrak{A}_{\mathcal{S}<\sigma}$.

(b) Write $\mathfrak{B}$ for $\bigvee_{\sigma\in C} \mathfrak{A}_\sigma$ and $\mathfrak{B}'$ for $\{(b \setminus a) \cup (c \cap a) : b \in \mathfrak{A}_\tau,\, c \in \mathfrak{A}_{\mathcal{S}<\tau}\}$.

(i) Note first that

$$[\![\sigma < \tau]\!] \in \mathfrak{A}_\sigma \cap \mathfrak{A}_{\mathcal{S}<\tau} \subseteq \mathfrak{B} \cap \mathfrak{A}_{\mathcal{S}<\tau}$$

for every $\sigma \in C$, so $a \in \mathfrak{B} \cap \mathfrak{A}_{\mathcal{S}<\tau}$.

(ii) If $b \in \mathfrak{A}_\tau$, then

$$b \setminus [\![\sigma < \tau]\!] \in \mathfrak{A}_\sigma \subseteq \mathfrak{B}$$

for every $\sigma \in C$ (using 611H(c-iii) again), so $b \setminus a \in \mathfrak{B}$.

If $\rho \in \mathcal{S}$, $d \in \mathfrak{A}_\rho$ and $\rho \le \tau$, then $d \cap [\![\rho < \sigma]\!] \in \mathfrak{A}_\sigma \subseteq \mathfrak{B}$ for every $\sigma \in C$, so $\mathfrak{B}$ contains $\sup_{\sigma\in C} d \cap [\![\rho < \sigma]\!] = d \cap [\![\rho < \tau]\!]$ (611Eb again). Accordingly $\mathfrak{A}_{\mathcal{S}<\tau} \subseteq \mathfrak{B}$. By (i), it follows that $c \cap a \in \mathfrak{B}$ whenever $c \in \mathfrak{A}_{\mathcal{S}<\tau}$.

Thus $\mathfrak{B}' \subseteq \mathfrak{B}$.

(iii) $\mathfrak{B}'$ is a closed subalgebra of $\mathfrak{A}$. **P** (Cf. 314Ja.) The map $(b, c) \mapsto (b \setminus a) \cup (c \cap a) : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ is an order-continuous Boolean homomorphism, while $\mathfrak{A}_\tau \times \mathfrak{A}_{\mathcal{S}<\tau}$ is a closed subalgebra of $\mathfrak{A} \times \mathfrak{A}$, so the image $\mathfrak{B}'$ is a closed subalgebra of $\mathfrak{A}$ (314F(a-i)). **Q**

If $\sigma \in C$ and $d \in \mathfrak{A}_\sigma$ then $d \in \mathfrak{A}_\tau$ and $d \cap [\![\sigma < \tau]\!] \in \mathfrak{A}_{\mathcal{S}<\tau}$. So

$$d = (d \setminus a) \cup (d \cap a) = (d \setminus a) \cup (d \cap [\![\sigma < \tau]\!] \cap a)$$

belongs to $\mathfrak{B}'$. As σ and d are arbitrary, $\bigcup_{\sigma\in C} \mathfrak{A}_\sigma \subseteq \mathfrak{B}'$; as $\mathfrak{B}'$ is a closed subalgebra, $\mathfrak{B} \subseteq \mathfrak{B}'$ and we have equality.

641D Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\tau\rangle_{\tau\in\mathcal{S}}$ an L^1-process with a previsible variation $\boldsymbol{v}^\# = \langle v^\#_\sigma\rangle_{\sigma\in\mathcal{S}}$ (626J). Then $v^\#_\tau \in L^0(\mathfrak{A}_{\mathcal{S}<\tau})$ for every $\tau \in \mathcal{S}$.

proof Consider the formulae

$$\Delta_{c(\sigma,\sigma')}(\mathbf{1}, Pd\boldsymbol{v}) = P_\sigma v_{\sigma'} - v_\sigma,$$

$$v^\#_\tau = \text{w1lim}_{I\uparrow\mathcal{I}(\mathcal{S}\wedge\tau)} S_I(\mathbf{1}, Pd\boldsymbol{v}).$$

If $\sigma \le \sigma'$ in $\mathcal{S} \wedge \tau$, $\Delta_{c(\sigma,\sigma')}(\mathbf{1}, Pd\boldsymbol{v}) \in L^0(\mathfrak{A}_\sigma)$ and

$$[\![\Delta_{c(\sigma,\sigma')}(\mathbf{1}, Pd\boldsymbol{v}) \ne 0]\!] \subseteq [\![\sigma < \sigma']\!] \subseteq [\![\sigma < \tau]\!],$$

so $\Delta_{c(\sigma,\sigma')}(\mathbf{1}, Pd\boldsymbol{v}) \in L^0(\mathfrak{A}_{\mathcal{S}<\tau})$, by 641B(b-ii). Accordingly $S_{I\wedge\tau}(\mathbf{1}, Pd\boldsymbol{v}) \in L^0(\mathfrak{A}_{\mathcal{S}<\tau})$ for every $I \in \mathcal{I}(\mathcal{S}\wedge\tau)$. Since $L^1_{\bar\mu} \cap L^0(\mathfrak{A}_{\mathcal{S}<\tau})$ is closed in $L^1_{\bar\mu}$ for the weak topology of $L^1_{\bar\mu}$ (613B(i-ii)), $v^\#_\tau \in L^0(\mathfrak{A}_{\mathcal{S}<\tau})$.

641E Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$ a fully adapted process and I a non-empty finite sublattice of $\mathcal{S}$.

(a) For any $\tau \in \mathcal{T}$ there is an element $u_{I<\tau}$ of $L^0(\mathfrak{A}_{I<\tau})$ defined by saying that $[\![\tau \le \min I]\!] \subseteq [\![u_{I<\tau} = 0]\!]$ and

$$[\![\sigma < \tau]\!] \setminus \sup_{\sigma'\in I}([\![\sigma < \sigma']\!] \cap [\![\sigma' < \tau]\!]) \subseteq [\![u_{I<\tau} = u_\sigma]\!]$$

for every $\sigma \in I$.

(b) If $(\sigma_0, \dots, \sigma_n)$ linearly generates the I-cells, then

$$[\![\tau \le \sigma_0]\!] \subseteq [\![u_{I<\tau} = 0]\!], \quad [\![\sigma_n < \tau]\!] \subseteq [\![u_{I<\tau} = u_{\sigma_n}]\!],$$

$$[\![\sigma_i < \tau]\!] \cap [\![\tau \le \sigma_{i+1}]\!] \subseteq [\![u_{I<\tau} = u_{\sigma_i}]\!] \text{ for every } i < n.$$

(c) The process $\langle u_{I<\tau} \rangle_{\tau\in\mathcal{T}}$ is fully adapted.

(d) If J is a maximal totally ordered subset of I, then $u_{J<\tau} = u_{I<\tau}$ for every $\tau \in \mathcal{T}$.

(e) If $\tau \in \mathcal{S}$ then $u_{I<\tau} = u_{(I\wedge\tau)<\tau}$.

proof (a) For $\sigma \in I$ set

$$a_\sigma = [\![\sigma < \tau]\!] \setminus \sup_{\sigma'\in I}([\![\sigma < \sigma']\!] \cap [\![\sigma' < \tau]\!]) \in \mathfrak{A}_{I<\tau}.$$

Let $B \subseteq \mathfrak{A}_{I<\tau}$ be the set of atoms of the subalgebra generated by $\{a_\sigma : \sigma \in I\}$. If $b \in B$, either $b \cap a_\sigma = 0$ for every $\sigma \in I$ and

$$b = 1 \setminus \sup_{\sigma\in I} a_\sigma \supseteq [\![\tau \le \min I]\!]$$

or $J_b = \{\sigma : \sigma \in I,\, b \subseteq a_\sigma\}$ is non-empty. In the former case, set $v_b = 0$. In the latter case, if σ, $\sigma' \in J_b$, b is included in $[\![\sigma < \tau]\!] \cap [\![\sigma' < \tau]\!]$ so cannot meet either $[\![\sigma < \sigma']\!]$ or $[\![\sigma' < \sigma]\!]$ for any σ, $\sigma' \in J_b$, and

$$b \subseteq [\![\sigma = \sigma']\!] \subseteq [\![u_\sigma = u_{\sigma'}]\!]$$

for any σ, $\sigma' \in J_b$. Accordingly we can write v_b for the common value of $u_\sigma \times \chi b$ for any $\sigma \in J_b$. Now if $\sigma \in J_b$ and $a \in \mathfrak{A}_\sigma$, then $a \cap b = a \cap [\![\sigma < \tau]\!] \cap b$ belongs to $\mathfrak{A}_{I<\tau}$ for every $a \in \mathfrak{A}_\sigma$, so $v_b = u_\sigma \times \chi b$ belongs to $L^0(\mathfrak{A}_{I<\tau})$. We therefore have an element $v = \sum_{b\in B} v_b$ of $L^0(\mathfrak{A}_{I<\tau})$ such that $v \times \chi b = v_b \times \chi b$ for every $b \in B$ and consequently $v \times \chi a_\sigma = u_\sigma \times \chi a_\sigma$, that is, $a_\sigma \subseteq [\![v = u_\sigma]\!]$, for every $\sigma \in I$. Since also $[\![\tau \le \min I]\!] \subseteq 1 \setminus \sup_{\sigma\in I} a_\sigma \subseteq [\![v = 0]\!]$, this v has the property required of $u_{I<\tau}$, and the formula given does indeed define a member of $L^0(\mathfrak{A}_{I<\tau})$.

(b) Of course

$$[\![\tau \le \sigma_0]\!] = [\![\tau \le \min I]\!] \subseteq [\![u_{I<\tau} = 0]\!].$$

If $i < n$ and $\sigma \in I$ then

$$[\![\tau \le \sigma_{i+1}]\!] \cap [\![\sigma_i < \sigma]\!] \cap [\![\sigma < \tau]\!] \subseteq [\![\sigma_i < \sigma]\!] \cap [\![\sigma < \sigma_{i+1}]\!] = 0$$

(611Kd). So

$$[\![\sigma_i < \tau]\!] \cap [\![\tau \le \sigma_{i+1}]\!] \subseteq [\![\sigma_i < \tau]\!] \setminus \sup_{\sigma\in I}([\![\sigma_i < \sigma]\!] \cap [\![\sigma < \tau]\!]) \subseteq [\![u_{I<\tau} = u_{\sigma_i}]\!].$$

At the top end, since $\sigma_n = \max I$, $[\![\sigma_n < \sigma]\!] \cap [\![\sigma < \tau]\!] = 0$ for every $\sigma \in I$, so $[\![\sigma_n < \tau]\!] \subseteq [\![u_{I<\tau} = u_{\sigma_n}]\!]$.

(c) As $\mathfrak{A}_{I<\tau} \subseteq \mathfrak{A}_\tau$, $u_{I<\tau} \in L^0(\mathfrak{A}_\tau)$ for every $\tau \in \mathcal{T}$. If τ, $\tau' \in \mathcal{T}$, $c = [\![\tau = \tau']\!]$ and $\sigma \in I$, then $c \cap [\![\sigma < \tau]\!] = c \cap [\![\sigma < \tau']\!]$ for every $\sigma \in I$ (611E(c-iii-γ)), so

$$\begin{aligned} c \cap ([\![\sigma < \tau]\!] \setminus \sup_{\sigma'\in I}([\![\sigma < \sigma']\!] \cap [\![\sigma' < \tau]\!])) \\ &= c \cap ([\![\sigma < \tau']\!] \setminus \sup_{\sigma'\in I}([\![\sigma < \sigma']\!] \cap [\![\sigma' < \tau']\!])) \\ &\subseteq [\![u_{I<\tau} = u_\sigma]\!] \cap [\![u_{I<\tau'} = u_\sigma]\!] \subseteq [\![u_{I<\tau} = u_{I<\tau'}]\!]. \end{aligned}$$

At the same time

$$\begin{aligned} c \cap [\![\tau \le \min I]\!] &= c \cap [\![\tau' \le \min I]\!] \\ &\subseteq [\![u_{I<\tau} = 0]\!] \cap [\![u_{I<\tau'} = 0]\!] \subseteq [\![u_{I<\tau} = u_{I<\tau'}]\!]. \end{aligned}$$

So $c \subseteq [\![u_{I<\tau} = u_{I<\tau'}]\!]$; as τ and τ' are arbitrary, $\langle u_{I<\tau}\rangle_{\tau\in\mathcal{T}}$ is fully adapted.

(d) If $\langle\sigma_i\rangle_{i\le n}$ is the increasing enumeration of J, then it linearly generates both the I-cells and the J-cells, so (b) gives the result.

(e) For any $\sigma \in I$,

$$\begin{aligned}[\![\sigma < \tau]\!] \setminus \sup_{\sigma'\in I}([\![\sigma < \sigma']\!] \cap [\![\sigma' < \tau]\!])&\\ &= [\![\sigma\wedge\tau < \tau]\!] \setminus \sup_{\sigma'\in I}([\![\sigma\wedge\tau < \sigma'\wedge\tau]\!] \cap [\![\sigma'\wedge\tau < \tau]\!])\\ &= [\![\sigma\wedge\tau < \tau]\!] \setminus \sup_{\sigma'\in I\wedge\tau}([\![\sigma\wedge\tau < \sigma']\!] \cap [\![\sigma' < \tau]\!])\\ &\subseteq [\![u_{I<\tau} = u_{\sigma\wedge\tau}]\!] \cap [\![u_{(I\wedge\tau)<\tau} = u_{\sigma\wedge\tau}]\!]\end{aligned}$$

while

$$[\![\tau \le \min I]\!] = [\![\tau \le \min(I\wedge\tau)]\!] \subseteq [\![u_{I<\tau} = 0]\!] \cap [\![u_{(I\wedge\tau)<\tau} = 0]\!].$$

641F Definition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a fully adapted process. For $\tau \in \mathcal{S}$, set

$$u_{<\tau} = \lim_{I\uparrow\mathcal{I}(\mathcal{S})} u_{I<\tau}$$

when the limit exists in $L^0(\mathfrak{A})$, defining $u_{I<\tau}$ as in 641E. If $u_{<\tau}$ is defined for every $\tau \in \mathcal{S}$, I will call $\boldsymbol{u}_< = \langle u_{<\tau}\rangle_{\tau\in\mathcal{S}}$ the **previsible version** of $\boldsymbol{u}$.

641G Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.

(a) Let $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ be a fully adapted process with a previsible version $\boldsymbol{u}_< = \langle u_{<\sigma}\rangle_{\sigma\in\mathcal{S}}$.

(i) $u_{<\sigma} \in L^0(\mathfrak{A}_{\mathcal{S}<\sigma})$ for every $\sigma \in \mathcal{S}$.

(ii) $\boldsymbol{u}_<$ is fully adapted to $\langle\mathfrak{A}_t\rangle_{t\in T}$.

(iii) $[\![\boldsymbol{u}_< \ne \boldsymbol{0}]\!] \subseteq [\![\boldsymbol{u} \ne \boldsymbol{0}]\!]$.

(iv) If $z \in L^0(\mathfrak{A} \cap \bigcap_{\sigma\in\mathcal{S}}\mathfrak{A}_\sigma)$, then $z\boldsymbol{u}$ (definition: 612De) has a previsible version, which is $z\boldsymbol{u}_<$.

(v) If $\mathcal{S}$ has a least element, then $u_{<\min\mathcal{S}} = 0$.

(vi) If $\mathcal{S}'$ is a sublattice of $\mathcal{S}$ which covers $\mathcal{S}$, then $\boldsymbol{u}\restriction\mathcal{S}'$ has a previsible version, which is $\boldsymbol{u}_<\restriction\mathcal{S}'$.

(vii) Suppose that $\boldsymbol{u}$ is order-bounded.

(α) For any $\tau \in \mathcal{S}$, $|u_{<\tau}| \le \sup_{\sigma\in\mathcal{S}}(|u_\sigma| \times \chi[\![\sigma < \tau]\!])$.

(β) For any τ, $\tau' \in \mathcal{S}$,

$$|u_{<\tau'}| \times \chi[\![\tau < \tau']\!] \le \sup_{\sigma\in\mathcal{S}\vee\tau}(|u_\sigma| \times \chi[\![\sigma < \tau']\!]).$$

(γ) $\boldsymbol{u}_<$ is order-bounded and $\sup|\boldsymbol{u}_<| \le \sup|\boldsymbol{u}|$.

(viii) If $\boldsymbol{u}$ is locally order-bounded then $\boldsymbol{u}_<$ is locally order-bounded.

(b) Writing $\boldsymbol{1}^{(\mathcal{S})}$ for the constant process with value $\chi 1$ and domain $\mathcal{S}$, its previsible version $\boldsymbol{1}^{(\mathcal{S})}_<$ is defined and equal to $\langle\chi e_\sigma\rangle_{\sigma\in\mathcal{S}}$, where $e_\sigma = \sup_{\sigma'\in\mathcal{S}}[\![\sigma' < \sigma]\!]$ for $\sigma \in \mathcal{S}$.

(c) Suppose that $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a fully adapted process.

(i) $\boldsymbol{u}$ has a previsible version iff $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ has a previsible version for every $\tau \in \mathcal{S}$.

(ii) In this case, $(\boldsymbol{u}\restriction\mathcal{S}\wedge\tau)_< = \boldsymbol{u}_<\restriction\mathcal{S}\wedge\tau$ and

$$(\boldsymbol{u}\restriction\mathcal{S}\vee\tau)_< = (\boldsymbol{u}_<\restriction\mathcal{S}\vee\tau) \times \langle\chi[\![\tau < \sigma]\!]\rangle_{\sigma\in\mathcal{S}\vee\tau} = (\boldsymbol{u}_<\restriction\mathcal{S}\vee\tau) \times \boldsymbol{1}^{(\mathcal{S}\vee\tau)}_<$$

for every $\tau \in \mathcal{S}$.

(d) Suppose that $k \ge 1$ is an integer, and $h : \mathbb{R}^k \to \mathbb{R}$ is a continuous function. Take $\boldsymbol{U} = \langle\boldsymbol{u}_i\rangle_{i<k}$ where each $\boldsymbol{u}_i$ is a fully adapted process with domain $\mathcal{S}$ with a previsible version $\boldsymbol{u}_{i<}$, and set $\boldsymbol{U}_< = \langle\boldsymbol{u}_{i<}\rangle_{i<k}$ where $\boldsymbol{u}_{i<}$ is the previsible version of $\boldsymbol{u}_i$ for each i. Define $\bar h : (L^0)^k \to L^0$ and $\bar h\boldsymbol{U} = \bar h\circ\boldsymbol{U}$ as in 619E-619F. Then $\bar h\boldsymbol{U}$ has a previsible version $(\bar h\boldsymbol{U})_< = \bar h\circ(\boldsymbol{U}_<) \times \boldsymbol{1}^{(\mathcal{S})}_<$. If $h(0,\dots,0) = 0$, then $(\bar h\boldsymbol{U})_< = \bar h\circ\boldsymbol{U}_<$.

(e) Let M be the set of those order-bounded processes $\boldsymbol{u}$ with domain $\mathcal{S}$ such that $\boldsymbol{u}$ has a previsible version $\boldsymbol{u}_<$.

(i) M is an f-subalgebra of $M_{\text{o-b}}(\mathcal{S})$, and $\boldsymbol{u} \mapsto \boldsymbol{u}_< : M \to M_{\text{o-b}}(\mathcal{S})$ is an f-algebra homomorphism.

(ii) M is closed for the ucp topology on $M_{\text{o-b}}(\mathcal{S})$, and $\boldsymbol{u} \mapsto \boldsymbol{u}_< : M \to M_{\text{o-b}}(\mathcal{S})$ is continuous.

proof (a)(i) We know that $u_{I<\sigma}$, as defined in 641E, belongs to $L^0(\mathfrak{A}_{I<\sigma}) \subseteq L^0(\mathfrak{A}_{\mathcal{S}<\sigma}$ for every non-empty $I \in \mathcal{I}(\mathcal{S})$ (641Ea, 641B(b-iii)). Since $L^0(\mathfrak{A}_{\mathcal{S}<\sigma})$ is closed in $L^0(\mathfrak{A})$ (613B(i-i)), $u_{<\sigma} \in L^0(\mathfrak{A}_{\mathcal{S}<\sigma})$.

(ii) By (i), $u_{<\sigma} \in L^0(\mathfrak{A}_{\mathcal{S}<\sigma} \subseteq L^0(\mathfrak{A}_\sigma$ for every $\sigma \in \mathcal{S}$, Now suppose that σ, $\sigma' \in \mathcal{S}$ and write c for $[\![\sigma = \sigma']\!]$. For each $I \in \mathcal{I}(\mathcal{S}) \setminus \{\emptyset\}$, $u_{I<\sigma} \times \chi c = u_{I<\sigma'} \times \chi c$ (641Ec), so $u_{<\sigma} \times \chi c = u_{<\sigma'} \times \chi c$, that is, $c \subseteq [\![u_{<\sigma} = u_{<\sigma'}]\!]$. As σ and σ' are arbitrary, $\boldsymbol{u}_<$ is fully adapted.

(iii) Set

$$b = 1 \setminus [\![\boldsymbol{u} \ne \mathbf{0}]\!] = \inf_{\sigma\in\mathcal{S}} [\![u_\sigma = 0]\!].$$

Looking at the formula in 641Ea, we see that $b \subseteq [\![u_{I<\tau} = 0]\!]$ whenever $\tau \in \mathcal{S}$ and $I \in \mathcal{I}(\mathcal{S})$ is non-empty. So $b \subseteq [\![u_{<\tau} = 0]\!]$ whenever $\tau \in \mathcal{S}$ and $[\![\boldsymbol{u}_< \ne \mathbf{0}]\!] \subseteq 1 \setminus b = [\![\boldsymbol{u} \ne \mathbf{0}]\!]$.

(iv) Set $v_\sigma = z \times u_\sigma$ for $\sigma \in \mathcal{S}$, so that $\langle v_\sigma\rangle_{\sigma\in\mathcal{S}} = z\boldsymbol{u}$ is fully adapted (612D(e-i)), and define $v_{I<\sigma}$, for $\sigma \in \mathcal{S}$ and non-empty $I \in \mathcal{I}(\mathcal{S})$, as in 641E. Again referring to the formula in 641Ea, we see that $v_{I<\sigma} = z \times u_{I<\sigma}$ for all σ and I. But now we have

$$\lim_{I\uparrow\mathcal{I}(\mathcal{S})} v_{I<\sigma} = \lim_{I\uparrow\mathcal{I}(\mathcal{S})} z \times u_{I<\sigma} = z \times \lim_{I\uparrow\mathcal{I}(\mathcal{S})} u_{I<\sigma} = z \times u_{<\sigma}$$

for every $\sigma \in \mathcal{S}$, so $(z\boldsymbol{u})_<$ is defined and equal to $z\boldsymbol{u}_<$.

(v) If $I \in \mathcal{I}(\mathcal{S})$ is non-empty, then $\min\mathcal{S} \le \min I$ so $u_{I<\min\mathcal{S}} = 0$; taking the limit as I increases, $u_{<(\min\mathcal{S})} = 0$.

(vi) Take $\tau \in \mathcal{S}'$ and $\epsilon > 0$. Then there is a non-empty $I_0 \in \mathcal{I}(\mathcal{S})$ such that $\theta(u_{<\tau} - u_{I<\tau}) \le \epsilon$ whenever $I \in \mathcal{I}(\mathcal{S})$ includes I_0. Let J_0 be a finite sublattice of $\mathcal{S}'$, containing τ, such that $\bar\mu a \le \epsilon$ where

$$a = \sup_{\sigma\in I_0}(1 \setminus \sup_{\rho\in J_0} [\![\sigma = \rho]\!]).$$

Take any $J \in \mathcal{I}(\mathcal{S}')$ including J_0, and set $I = J \sqcup I_0$. Then $[\![u_{I<\tau} \ne u_{J<\tau}]\!] \subseteq a$. **P** Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}$ generated by

$$\{a\} \cup \{[\![\sigma = \rho]\!] : \sigma \in I_0,\, \rho \in J\} \cup \{[\![\sigma \le \rho]\!] : \sigma,\, \rho \in I\}$$

and let b be any atom of $\mathfrak{B}$ disjoint from a. Then for every $\sigma \in I_0$ there is a $\rho \in J$ such that b meets $[\![\sigma = \rho]\!]$, in which case $b \subseteq [\![\sigma = \rho]\!]$. The set

$$\bigcup_{\rho\in J}\{\sigma : \sigma \in \mathcal{S},\, b \subseteq [\![\sigma = \rho]\!]\}$$

is a sublattice of $\mathcal{S}$ including $I_0 \cup J$ and therefore including I. For each $\sigma \in I$, b is included in one of $[\![\tau \le \sigma]\!]$, $[\![\sigma < \tau]\!]$ because $\tau \in J_0 \subseteq I$. If $b \subseteq [\![\tau \le \sigma]\!]$ for every $\sigma \in I$, then

$$b \subseteq [\![\tau \le \min I]\!] \subseteq [\![\tau \le \min J]\!] \subseteq [\![u_{I<\tau} = 0]\!] \cap [\![u_{J<\tau} = 0]\!] \subseteq [\![u_{I<\tau} = u_{J<\tau}]\!].$$

Otherwise, set $\rho = \sup\{\sigma : \sigma \in I,\, b \subseteq [\![\sigma < \tau]\!]\}$; then $b \subseteq [\![\rho < \tau]\!]$, and if $\sigma \in I$ and $\sigma \not\le \rho$, $b \subseteq [\![\tau \le \sigma]\!]$. So $b \subseteq [\![u_{I<\tau} = u_\rho]\!]$. Next, let $\rho' \in J$ be such that $b \subseteq [\![\rho = \rho']\!]$. Then $b \subseteq [\![\rho' < \tau]\!]$, and if $\sigma' \in J$ and $\sigma' \not\le \rho'$, $b \subseteq [\![\tau \le \sigma']\!]$. So $b \subseteq [\![u_{J<\tau} = u_{\rho'}]\!]$. Putting these together, $b \subseteq [\![u_{I<\tau} = u_{J<\tau}]\!]$ in this case also. As b is arbitrary, $[\![u_{I<\tau} = u_{J<\tau}]\!]$ includes $1 \setminus a$ and $[\![u_{I<\tau} \ne u_{J<\tau}]\!] \subseteq a$. **Q**

It follows that

$$\theta(u_{<\tau} - u_{J<\tau}) \le \theta(u_{<\tau} - u_{I<\tau}) + \bar\mu\epsilon \le 2\epsilon,$$

and this is true whenever $J_0 \subseteq J \in \mathcal{I}(\mathcal{S}')$. As ϵ is arbitrary, $u_{<\tau} = \lim_{J\uparrow\mathcal{I}(\mathcal{S}')} u_{J<\tau}$. As τ is arbitrary, $(\boldsymbol{u}{\restriction}\mathcal{S}')_<$ is defined and equal to $\boldsymbol{u}_<{\restriction}\mathcal{S}'$.

(vii)($\boldsymbol{\alpha}$) In the formulae of 641Ea, we see that if $\sigma \in I \in \mathcal{I}(\mathcal{S})$ then

$$[\![\sigma < \tau]\!] \setminus \sup_{\sigma'\in I}([\![\sigma < \sigma']\!] \cap [\![\sigma' < \tau]\!]) \subseteq [\![|u_{I<\tau}| = |u_\sigma| \times \chi[\![\sigma < \tau]\!]]\!],$$

while $[\![\tau \le \min I]\!] \subseteq [\![u_{I<\tau} = 0]\!]$. So $|u_{I<\tau}| \le \sup_{\sigma\in I}(|u_\sigma| \times \chi[\![\sigma < \tau]\!])$; in the limit as $I \uparrow \mathcal{I}(\mathcal{S})$, $|u_{<\tau}| \le \sup_{\sigma\in\mathcal{S}}(|u_\sigma| \times \chi[\![\sigma < \tau]\!])$.

($\boldsymbol{\beta}$) By the same argument, we see that if $\tau \in I \in \mathcal{I}(\mathcal{S})$ then $|u_{I<\tau'}| \times \chi[\![\tau < \tau']\!] \le \sup_{\sigma\in\mathcal{S}\vee\tau} |u_\sigma| \times \chi[\![\sigma < \tau']\!]$, so $|u_{<\tau'}| \times \chi[\![\tau < \tau']\!] \le \sup_{\sigma\in\mathcal{S}\vee\tau} |u_\sigma| \times \chi[\![\sigma < \tau']\!]$.

($\boldsymbol{\gamma}$) And (α) tells us that $|\boldsymbol{u}_<|$ is bounded above by $\sup|\boldsymbol{u}|$.

(viii) Similarly, if $\boldsymbol{u}$ is locally order-bounded and $\tau \in \mathcal{S}$, then $|u_{I<\sigma}| \le \sup |\boldsymbol{u}\restriction\mathcal{S}\wedge\tau|$ whenever $\sigma \in \mathcal{S}\wedge\tau$ and $I \in \mathcal{I}(\mathcal{S})$ is non-empty, so $\sup |\boldsymbol{u}_<\restriction\mathcal{S}\wedge\tau| \le \sup |\boldsymbol{u}\restriction\mathcal{S}\wedge\tau|$. (Or see (c) below.)

(b) Set $u_\sigma = \chi 1$ for $\sigma \in \mathcal{S}$, so that $\mathbf{1}^{(\mathcal{S})} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$. This time we see that for any $\sigma \in \mathcal{S}$ and non-empty $I \in \mathcal{I}(\mathcal{S})$,

$$[\![\min I < \sigma]\!] \subseteq [\![u_{I<\sigma} = \chi 1]\!], \quad [\![\sigma \le \min I]\!] \subseteq [\![u_{I<\sigma} = 0]\!],$$

so in fact $u_{I<\sigma} = \chi[\![\min I < \sigma]\!]$. Now

$$\begin{aligned} u_{<\sigma} &= \lim_{I\uparrow\mathcal{I}(\mathcal{S})} \chi[\![\min I < \sigma]\!] = \sup_{I\in\mathcal{I}(\mathcal{S})} \chi[\![\min I < \sigma]\!] \\ &= \sup_{\sigma'\in\mathcal{S}} \chi[\![\sigma' < \sigma]\!] = \chi(\sup_{\sigma'\in\mathcal{S}} [\![\sigma' < \sigma]\!]) = \chi e_\sigma, \end{aligned}$$

as claimed.

(c)(i)($\boldsymbol{\alpha}$) Suppose that $\boldsymbol{u}$ has a previsible version $\boldsymbol{u}_<$, and $\tau \in \mathcal{S}$. If $\sigma \in \mathcal{S}\wedge\tau$ and $I \in \mathcal{I}(\mathcal{S})$ contains τ, then $u_{I<\sigma} = u_{(I\wedge\tau)<\sigma}$. So

$$\lim_{J\uparrow\mathcal{I}(\mathcal{S}\wedge\tau)} u_{J<\sigma} = \lim_{I\uparrow\mathcal{I}(\mathcal{S})} u_{(I\wedge\tau)<\sigma} = \lim_{I\uparrow\mathcal{I}(\mathcal{S})} u_{I<\sigma} = u_{<\sigma}.$$

As σ is arbitrary, $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ has a previsible version, which is in fact $\boldsymbol{u}_<\restriction\mathcal{S}\wedge\tau$.

($\boldsymbol{\beta}$) Now suppose that $\tau \in \mathcal{S}$ and $\boldsymbol{w} = \boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ has a previsible version $\boldsymbol{w}_< = \langle w_{<\sigma}\rangle_{\sigma\in\mathcal{S}\wedge\tau}$. Then $u_{I<\tau} = w_{(I\wedge\tau)<\tau}$ for every $I \in \mathcal{I}(\mathcal{S})$ containing τ, and $\lim_{I\uparrow\mathcal{I}(\mathcal{S})} u_{I<\tau} = \lim_{J\uparrow\mathcal{I}(\mathcal{S}\wedge\tau)} w_{J<\tau} = w_{<\tau}$ is defined.

So if $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ has a previsible version for every $\tau \in \mathcal{S}$, $\lim_{I\uparrow\mathcal{I}(\mathcal{S})} u_{I<\tau}$ is defined for every $\tau \in \mathcal{S}$ and $\boldsymbol{u}$ has a previsible version.

(ii)($\boldsymbol{\alpha}$) We saw in (i-α) above that $(\boldsymbol{v}\restriction\mathcal{S}\wedge\tau)_< = \boldsymbol{v}_<\restriction\mathcal{S}\wedge\tau$.

($\boldsymbol{\beta}$) Take $\sigma \in \mathcal{S}\vee\tau$, and set $c = [\![\tau < \sigma]\!]$. If I is a finite sublattice of $\mathcal{S}$ containing σ and τ, and $J = I\vee\tau$, then for $\sigma' \in I$ set

$$a_{\sigma'} = [\![\sigma' < \sigma]\!] \setminus \sup_{\sigma''\in I}([\![\sigma' < \sigma'']\!] \cap [\![\sigma'' < \sigma]\!])$$

and for $\sigma' \in J$ set

$$b_{\sigma'} = [\![\sigma' < \sigma]\!] \setminus \sup_{\sigma''\in J}([\![\sigma' < \sigma'']\!] \cap [\![\sigma'' < \sigma]\!]).$$

Observe that $b_{\sigma'} = a_{\sigma'} \subseteq c$ for every $\sigma' \in J$. So, for $\sigma' \in J$, we have

$$b_{\sigma'} \subseteq [\![u_{I<\sigma} = u_{\sigma'}]\!] \cap [\![u_{J<\sigma} = u_{\sigma'}]\!] \subseteq [\![u_{I<\sigma} = u_{J<\sigma}]\!].$$

Since

$$c = [\![\tau < \sigma]\!] = [\![\min J < \sigma]\!] = \sup_{\sigma'\in J} b_{\sigma'}$$

(see the construction in 641Ea), $c \subseteq [\![u_{I<\sigma} = u_{J<\sigma}]\!]$ and $c \subseteq [\![u_{J<\sigma} = \chi c \times u_{I<\sigma}]\!]$.

On the other hand,

$$1 \setminus c = [\![\tau = \sigma]\!] \subseteq [\![u_{J<\sigma} = 0]\!]$$

and $1 \setminus c \subseteq [\![u_{J<\sigma} = \chi c \times u_{I<\sigma}]\!]$. So $u_{J<\sigma} = \chi c \times u_{I<\sigma}$. Taking the limit as I increases through $\mathcal{I}(\mathcal{S})$,

$$\lim_{J\uparrow\mathcal{I}(\mathcal{S}\vee\tau)} u_{J<\sigma} = \lim_{I\uparrow\mathcal{I}(\mathcal{S})} u_{(I\vee\tau)<\sigma} = \chi c \times u_{<\sigma}.$$

As σ is arbitrary,

$$(\boldsymbol{u}\restriction\mathcal{S}\vee\tau)_< = (\boldsymbol{u}_<\restriction\mathcal{S}\vee\tau) \times \langle\chi[\![\tau < \sigma]\!]\rangle_{\sigma\in\mathcal{S}\vee\tau} = (\boldsymbol{u}_<\restriction\mathcal{S}\vee\tau) \times \mathbf{1}_<^{(\mathcal{S}\vee\tau)}$$

by (b) applied in $\mathcal{S}\vee\tau$.

(d) For each i, express $\boldsymbol{u}_i$ as $\langle u_{i\sigma}\rangle_{\sigma\in\mathcal{S}}$ and $\boldsymbol{u}_{i<}$ as $\langle u_{i<\sigma}\rangle_{\sigma\in\mathcal{S}}$ where $u_{i<\sigma} = \lim_{I\uparrow\mathcal{I}(\mathcal{S})} u_{i,I<\sigma}$. Write $U_{I<\sigma} = \langle u_{i,I<\sigma}\rangle_{i<k}$ for $\sigma \in \mathcal{S}$ and non-empty $I \in \mathcal{I}(\mathcal{S})$. Then

$$\lim_{I\uparrow\mathcal{I}(\mathcal{S})} U_{I<\sigma} = \langle\lim_{I\uparrow\mathcal{I}(\mathcal{S})} u_{i,I<\sigma}\rangle_{i<k} = \langle u_{i<\sigma}\rangle_{i<k}$$

is defined for any $\sigma \in \mathcal{S}$; I will call it $U_{<\sigma}$.

Set $v_\sigma = \bar{h}(\langle u_{i\sigma}\rangle_{i<k})$ and $z_\sigma = \chi 1$ for $\sigma \in \mathcal{S}$, so that $\bar{h}\boldsymbol{U} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ and $\mathbf{1}^{(\mathcal{S})} = \langle z_\sigma\rangle_{\sigma\in\mathcal{S}}$. If $\tau \in I \in \mathcal{I}(\mathcal{S})$ then

$$[\![\min I = \tau]\!] \subseteq [\![v_{I<\tau} = 0]\!] \cap [\![z_{I<\tau} = 0]\!] \subseteq [\![v_{I<\tau} = \bar{h}(U_{I<\tau}) \times z_{I<\tau}]\!],$$

$$[\![\min I < \tau]\!] \subseteq [\![v_{I<\tau} = \bar{h}(U_{I<\tau})]\!] \cap [\![z_{I<\tau} = \chi 1]\!] \subseteq [\![v_{I<\tau} = \bar{h}(U_{I<\tau}) \times z_{I<\tau}]\!]$$

while if $h(0,\ldots,0) = 0$ we have

$$\begin{aligned}[\![\min I = \tau]\!] &\subseteq [\![v_{I<\tau} = 0]\!] \cap [\![U_{I<\tau} = 0]\!]\\ &\subseteq [\![v_{I<\tau} = 0]\!] \cap [\![\bar{h}(U_{I<\tau}) = 0]\!] \subseteq [\![v_{I<\tau} = \bar{h}(U_{I<\tau})]\!].\end{aligned}$$

In the limit, we see that

$$v_{<\tau} = \lim_{I\uparrow\mathcal{I}(\mathcal{S})} \bar{h}(U_{I<\tau}) \times z_{I<\tau} = \bar{h}(U_{<\tau}) \times z_{<\tau}$$

because $\bar{h} : (L^0)^k \to L^0$ and $\times : L^0 \times L^0$ are continuous (619Ed, 613Ba), while if $h(0,\ldots,0) = 0$ then

$$v_{<\tau} = \lim_{I\uparrow\mathcal{I}(\mathcal{S})} \bar{h}(U_{I<\tau}) = \bar{h}(U_{<\tau}).$$

As τ is arbitrary,

$$(\bar{h}\boldsymbol{U})_< = \boldsymbol{v}_< = (\bar{h}\boldsymbol{U}_<) \times \mathbf{1}^{(\mathcal{S})}_<$$

and $(\bar{h}\boldsymbol{U})_< = \bar{h}\boldsymbol{U}_<$ if $h(0,\ldots,0) = 0$.

(e)(i) This is immediate from (d), applied to addition and multiplication and the lattice operations.

(ii)($\boldsymbol{\alpha}$) Suppose that $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ belongs to the closure of $M \cap M_{\text{o-b}}(\mathcal{S})$ in $M_{\text{o-b}}(\mathcal{S})$, $\sigma \in \mathcal{S}$ and $\epsilon > 0$. Then there are $\boldsymbol{w} \in M \cap M_{\text{o-b}}(\mathcal{S})$ and non-empty $I \in \mathcal{I}(\mathcal{S})$ such that

$$\theta(\sup|\boldsymbol{u} - \boldsymbol{w}|) \le \epsilon, \quad \theta|w_{J<\sigma} - w_{I<\sigma}| \le \epsilon \text{ whenever } I \subseteq J \in \mathcal{I}(\mathcal{S}).$$

Now the same calculation as in (a-vii) shows that

$$|u_{J<\sigma} - w_{J<\sigma}| \le \sup|\boldsymbol{u} - \boldsymbol{w}|$$

for every non-empty $J \in \mathcal{I}(\mathcal{S})$, so $\theta(u_{J<\sigma} - u_{I<\sigma}) \le 3\epsilon$ whenever $I \subseteq J \in \mathcal{I}(\mathcal{S})$. As ϵ is arbitrary, $\lim_{I\uparrow\mathcal{I}(\mathcal{S})} u_{I<\sigma}$ is defined; as σ is arbitrary, $\boldsymbol{u} \in M$.

($\boldsymbol{\beta}$) And (a-vii) and (d) tell us again that $\sup|\boldsymbol{v}_< - \boldsymbol{w}_<| \le \sup|\boldsymbol{v} - \boldsymbol{w}|$ for every $\boldsymbol{v}$, $\boldsymbol{w} \in M$, so $\boldsymbol{v} \mapsto \boldsymbol{v}_<$ is continuous.

641H Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\mathcal{S}'$ a sublattice of $\mathcal{S}$ which separates $\mathcal{S}$. If $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is fully adapted and $\tau \in \mathcal{S}$ is such that $u_{<\tau}$ is defined, then $u_{<\tau} = \lim_{I\uparrow\mathcal{I}(\mathcal{S}')} u_{I<\tau}$.

proof Let $\epsilon > 0$. Then there is a non-empty $I \in \mathcal{I}(\mathcal{S})$ such that $\theta(u_{<\tau} - u_{J<\tau}) \le \epsilon$ whenever $J \in \mathcal{I}(\mathcal{S})$ includes I. Next, there is a finite set $C \subseteq \mathcal{S}'$ such that

$$\textstyle\sum_{\sigma\in I} \bar{\mu}([\![\sigma < \tau]\!] \setminus \sup_{\rho\in C}([\![\sigma \le \rho]\!] \cap [\![\rho < \tau]\!])) \le \epsilon;$$

let K_0 be the sublattice of $\mathcal{S}'$ generated by C. Set

$$a = \sup_{\sigma\in I}([\![\sigma < \tau]\!] \setminus \sup_{\rho\in C}([\![\sigma \le \rho]\!] \cap [\![\rho < \tau]\!])),$$

so that $\bar{\mu}a \le \epsilon$. Take any $K \in \mathcal{I}(\mathcal{S}')$ including K_0 and set $J = I \sqcup K$. Then $1 \setminus a \subseteq [\![u_{J<\tau} = u_{K<\tau}]\!]$. **P** Setting

$$a_\sigma = [\![\sigma < \tau]\!] \setminus \sup_{\rho\in J}([\![\sigma < \rho]\!] \cap [\![\rho < \tau]\!])$$

for $\sigma \in J$, as in part(a) of the proof of 641E, we see that if σ, $\sigma' \in J$ then $a_\sigma \cap [\![\sigma = \sigma']\!] \subseteq a_{\sigma'}$. Now if $\sigma \in K$ then

$$\begin{aligned}a_\sigma &= a_\sigma \cap [\![\sigma < \tau]\!] \setminus \sup_{\rho\in K}([\![\sigma < \rho]\!] \cap [\![\rho < \tau]\!])\\ &\subseteq [\![u_{J<\tau} = u_\sigma]\!] \cap [\![u_{K<\tau} = u_\sigma]\!] \subseteq [\![u_{J<\tau} = u_{K<\tau}]\!].\end{aligned}$$

And if $\sigma \in I$ then

$$\begin{aligned} a_\sigma \setminus a &\subseteq a_\sigma \cap \sup_{\rho\in C}([\![\sigma \le \rho]\!] \cap [\![\rho < \tau]\!]) \setminus \sup_{\rho\in C}([\![\sigma < \rho]\!] \cap [\![\rho < \tau]\!]) \\ &\subseteq \sup_{\rho\in C}(a_\sigma \cap [\![\sigma = \rho]\!]) \subseteq \sup_{\rho\in C} a_\rho \subseteq [\![u_{J<\tau} = u_{K<\tau}]\!]. \end{aligned}$$

But we know that the covered envelope of $I \cup K$ is a sublattice (611M(b-i)), so includes J. So for any $\sigma \in J$ we have

$$a_\sigma \setminus a = \sup_{\rho\in I\cup K} a_\sigma \cap [\![\sigma = \rho]\!] \setminus a \subseteq \sup_{\rho\in I\cup K} a_\rho \setminus a \subseteq [\![u_{J<\tau} = u_{K<\tau}]\!].$$

And we certainly have

$$\begin{aligned} [\![\tau \le \min J]\!] &= [\![\tau \le \min J]\!] \cap [\![\tau \le \min K]\!] \\ &\subseteq [\![u_{J<\tau} = 0]\!] \cap [\![u_{K<\tau} = 0]\!] \subseteq [\![u_{J<\tau} = u_{K<\tau}]\!]. \end{aligned}$$

Since $[\![\tau \le \min J]\!] \cup \sup_{\sigma\in J} a_\sigma = 1$, we see that $1 \setminus a \subseteq [\![u_{J<\tau} = u_{K<\tau}]\!]$, as claimed. **Q**

It follows that

$$\theta(u_{K<\tau} - u_{<\tau}) \le \theta(u_{K<\tau} - u_{J<\tau}) + \theta(u_{J<\tau} - u_{<\tau}) \le \bar\mu[\![u_{K<\tau} \ne u_{J<\tau}]\!] + \epsilon$$

(as $J \supseteq I$)

$$\le \bar\mu a + \epsilon \le 2\epsilon.$$

And this is true whenever $K \in \mathcal{I}(\mathcal{S}')$ includes K_0. As ϵ is arbitrary, $u_{<\tau} = \lim_{K\uparrow\mathcal{I}(\mathcal{S}')} u_{K<\tau}$.

641I Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a simple process with breakpoint string $(\tau_0, \dots, \tau_n)$ and starting value $u_\downarrow$ (614Ba).

(a) $u_{<\tau}$ is defined and

$$\inf_{\sigma\in\mathcal{S}} [\![\tau \le \sigma]\!] \subseteq [\![u_{<\tau} = 0]\!], \quad [\![\sigma < \tau]\!] \cap [\![\tau \le \tau_0]\!] \subseteq [\![u_{<\tau} = u_\downarrow]\!] \text{ for every } \sigma \in \mathcal{S},$$

$$[\![\tau_i < \tau]\!] \cap [\![\tau \le \tau_{i+1}]\!] \subseteq [\![u_{<\tau} = u_{\tau_i}]\!] \text{ for every } i < n, \quad [\![\tau_n < \tau]\!] \subseteq [\![u_{<\tau} = u_{\tau_n}]\!]$$

for every $\tau \in \mathcal{S}$.

(b) Writing $\boldsymbol{u}_<$ for the previsible version of $\boldsymbol{u}$,

$$\sup|\boldsymbol{u}| = |u_{\tau_n}| \vee \sup|\boldsymbol{u}_<|.$$

proof (a) Take any $I \in \mathcal{I}(\mathcal{S})$ such that $\tau_i \in I$ for every $i \le n$. We have $\sigma_0 \le \dots \le \sigma_m$ and $0 \le k_0 \le \dots \le k_n \le m$ such that

$$\sigma_{k_i} = \tau_i \text{ for } i \le n,$$

$$(\sigma_0, \dots, \sigma_{k_0}) \text{ linearly generates the } I \wedge \tau_0\text{-cells},$$

$$(\sigma_{k_i}, \dots, \sigma_{k_{i+1}}) \text{ linearly generates the } I \cap [\tau_i, \tau_{i+1}]\text{-cells for each } i \le n,$$

$$(\sigma_{k_n}, \dots, \sigma_m) \text{ linearly generates the } I \vee \tau_n\text{-cells},$$

and therefore $(\sigma_0, \dots, \sigma_m)$ linearly generates the I-cells. Then

$$\begin{aligned} [\![\sigma_j < \tau]\!] \cap [\![\tau \le \sigma_{j+1}]\!] &\subseteq [\![\sigma_j < \tau_0]\!] \subseteq [\![u_{\sigma_j} = u_\downarrow]\!] \text{ if } j < k_0, \\ &\subseteq [\![\tau_i \le \sigma_j]\!] \cap [\![\sigma_j < \tau_{i+1}]\!] \subseteq [\![u_{\sigma_j} = u_{\tau_i}]\!] \\ &\qquad\qquad \text{if } i \le n \text{ and } k_i \le j < k_{i+1}, \\ &\subseteq [\![\tau_n \le \sigma_j]\!] \subseteq [\![u_{\sigma_j} = u_{\tau_n}]\!] \text{ if } k_n \le i < m. \end{aligned}$$

By 641Eb, we have

$$[\![\tau \le \sigma_0]\!] \subseteq [\![u_{I<\tau} = 0]\!],$$

$$\begin{aligned}[\![\sigma_j < \tau]\!] \cap [\![\tau \le \sigma_{j+1}]\!] &\subseteq [\![u_{I<\tau} = u_{\sigma_j}]\!] \cap [\![u_{\sigma_j} = u_\downarrow]\!] \subseteq [\![u_{I<\tau} = u_\downarrow]\!] \text{ if } j < k_0,\\ &\subseteq [\![u_{I<\tau} = u_{\sigma_j}]\!] \cap [\![u_{\sigma_j} = u_{\tau_i}]\!] \subseteq [\![u_{I<\tau} = u_{\tau_i}]\!]\\ &\qquad \text{if } i \le n \text{ and } k_i \le j < k_{i+1},\\ &\subseteq [\![u_{I<\tau} = u_{\sigma_j}]\!] \cap [\![u_{\sigma_j} = u_{\tau_n}]\!] \subseteq [\![u_{I<\tau} = u_{\tau_n}]\!] \text{ if } k_n \le i < m\end{aligned}$$

and

$$[\![\sigma_m < \tau]\!] \subseteq [\![u_{I<\tau} = u_{\sigma_m}]\!] \cap [\![\tau_n \le \sigma_m]\!] \subseteq [\![u_{I<\tau} = u_{\tau_n}]\!].$$

Since

$$[\![\sigma_0 \le \tau]\!] \cap [\![\tau < \tau_0]\!] = \sup_{j<k_0} [\![\sigma_j \le \tau]\!] \cap [\![\tau < \sigma_{j+1}]\!],$$

$$[\![\tau_i < \tau]\!] \cap [\![\tau \le \tau_{i+1}]\!] = \sup_{k_i \le j < k_{i+1}} [\![\sigma_j < \tau]\!] \cap [\![\tau \le \sigma_{j+1}]\!],$$

$$[\![\tau_n < \tau]\!] = \sup_{k_n \le j} [\![\sigma_j < \tau]\!] \cap [\![\tau \le \sigma_{j+1}]\!] \cup [\![\sigma_m < \tau]\!],$$

we get

$$[\![\tau \le \min I]\!] \subseteq [\![u_{I<\tau} = 0]\!], \quad [\![\min I < \tau]\!] \cap [\![\tau < \tau_0]\!] \subseteq [\![u_{I<\tau} = u_\downarrow]\!],$$

$$[\![\tau_i < \tau]\!] \cap [\![\tau \le \tau_{i+1}]\!] \subseteq [\![u_{I<\tau} = u_{\tau_i}]\!], \quad [\![\tau_n < \tau]\!] \subseteq [\![u_{I<\tau} = u_{\tau_n}]\!].$$

If therefore we define v by the given formula, so that

$$\inf_{\sigma\in\mathcal{S}} [\![\tau \le \sigma]\!] \subseteq [\![v = 0]\!], \quad [\![\sigma < \tau]\!] \cap [\![\tau \le \tau_0]\!] \subseteq [\![v = u_\downarrow]\!] \text{ for every } \sigma \in \mathcal{S},$$

$$[\![\sigma_i < \tau]\!] \cap [\![\tau \le \tau_{i+1}]\!] \subseteq [\![v = u_{\tau_i}]\!] \text{ for every } i < n, \quad [\![\tau_n < \tau]\!] \subseteq [\![v = u_{\tau_n}]\!],$$

and set $c = \inf_{\sigma\in\mathcal{S}} [\![\tau \le \sigma]\!]$, then

$$c \cup [\![\min I < \tau]\!] \subseteq [\![u_{I<\tau} = v]\!]$$

whenever $\{\tau_0, \dots, \tau_n\} \subseteq I \in \mathcal{I}(\mathcal{S})$. It follows that, for any $\sigma \in \mathcal{S}$,

$$c \cup [\![\sigma < \tau]\!] \subseteq [\![u_{I<\tau} = v]\!]$$

whenever $\{\sigma, \tau_0, \dots, \tau_n\} \subseteq I \in \mathcal{I}(\mathcal{S})$. Consequently we shall have

$$\lim_{I\uparrow\mathcal{I}(\mathcal{S})} u_{I<\tau} \times \chi(c \cup [\![\sigma < \tau]\!]) = v \times \chi(c \cup [\![\sigma < \tau]\!])$$

for every $\sigma \in \mathcal{S}$, and therefore

$$\lim_{I\uparrow\mathcal{I}(\mathcal{S})} u_{I<\tau} \times \chi(c \cup \sup_{\sigma\in\mathcal{S}} [\![\sigma < \tau]\!]) = v \times \chi(c \cup \sup_{\sigma\in\mathcal{S}} [\![\sigma < \tau]\!]),$$

that is,

$$\lim_{I\uparrow\mathcal{I}(\mathcal{S})} u_{I<\tau} = v.$$

(b) Of course $|u_{\tau_n}| \le \sup |\boldsymbol{u}|$, and $\sup |\boldsymbol{u}_<| \le \sup |\boldsymbol{u}|$ by 641G(a-vii), so $\sup |\boldsymbol{u}| \ge |u_{\tau_n}| \vee \sup |\boldsymbol{u}_<|$.

In the other direction, if $\sigma \in \mathcal{S}$ then

$$([\![\sigma < \tau_0]\!], [\![\tau_0 \le \sigma]\!] \cap [\![\sigma < \tau_1]\!]. \dots, [\![\sigma_{n-1} \le \sigma]\!] \cap [\![\sigma < \tau_n]\!], \cap \tau_n \le \sigma$$

is a partition of unity in $\mathfrak{A}$, so

$$\begin{aligned}|u_\sigma| &= (|u_\sigma|\times\chi[\![\sigma<\tau_0]\!])\vee\sup_{i<n}(|u_\sigma|\times\chi([\![\tau_i\le\sigma]\!]\cap[\![\sigma<\tau_{i+1}]\!]))\\ &\qquad\qquad\vee(|u_\sigma|\times\chi[\![\tau_n\le\sigma]\!])\\ &=(|u_\downarrow|\times\chi[\![\sigma<\tau_0]\!])\vee\sup_{i<n}(|u_{\tau_i}|\times\chi([\![\tau_i\le\sigma]\!]\cap[\![\sigma<\tau_{i+1}]\!]))\\ &\qquad\qquad\vee(|u_{\tau_n}|\times\chi[\![\tau_n\le\sigma]\!])\\ &=(|u_{<\tau_0}|\times\chi[\![\sigma<\tau_0]\!])\vee\sup_{i<n}(|u_{<\tau_{i+1}}|\times\chi([\![\tau_i\le\sigma]\!]\cap[\![\sigma<\tau_{i+1}]\!]))\\ &\qquad\qquad\vee(|u_{\tau_n}|\times\chi[\![\tau_n\le\sigma]\!])\\ &\le\sup|\boldsymbol{u}_<|\vee|u_{\tau_n}|.\end{aligned}$$

As σ is arbitrary, $\sup|\boldsymbol{u}|\le|u_{\tau_n}|\vee\sup|\boldsymbol{u}_<|$.

641J We can now complement 614C with a formula for an integral $\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}$ when it is the *integrator* $\boldsymbol{v}$ which is simple.

Lemma Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$, $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a simple process with starting value $v_\downarrow$ and breakpoint string $(\tau_0,\dots,\tau_n)$, and $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a fully adapted process which has a previsible version $\boldsymbol{u}_<=\langle u_{<\sigma}\rangle_{\sigma\in\mathcal{S}}$. Then

$$\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}=u_{<\tau_0}\times(v_{\tau_0}-v_\downarrow)+\sum_{i=1}^n u_{<\tau_i}\times(v_{\tau_i}-v_{\tau_{i-1}}).$$

proof (a) Suppose that $i\le n$ and that

either $i=0$, $\mathcal{S}'=\mathcal{S}\wedge\tau_0$, $\tau_0\in I\in\mathcal{I}(\mathcal{S}')$ and $v=v_\downarrow$

or $1\le i\le n$, $\mathcal{S}'=\mathcal{S}\cap[\tau_{i-1},\tau_i])$, $\tau_i\in I\in\mathcal{I}(\mathcal{S}')$ and $v=v_{\tau_{i-1}}$

then

$$S_I(\boldsymbol{u},d\boldsymbol{v})=u_{I<\tau_i}\times(v_{\tau_i}-v).$$

P Let $(\sigma_0,\dots,\sigma_k)$ linearly generate the I-cells. Then

$$\begin{aligned}S_I(\boldsymbol{u},d\boldsymbol{v})&=\sum_{j=0}^{k-1}u_{\sigma_j}\times(v_{\sigma_{j+1}}-v_{\sigma_j})\\ &=\sum_{j=0}^{k-1}u_{\sigma_j}\times(v_{\tau_i}-v)\times\chi[\![\sigma_{j+1}=\tau_i]\!]\times\chi[\![\sigma_j<\tau_i]\!]\end{aligned}$$

(because $[\![v_{\sigma_{j+1}}\ne v_{\sigma_j}]\!]\subseteq[\![\sigma_j<\tau_i]\!]\cap[\![\sigma_{j+1}=\tau_i]\!]\subseteq[\![v=v_{\sigma_j}]\!]\cap[\![v_{\tau_i}=v_{\sigma_{j+1}}]\!]$)

$$=\sum_{j=0}^{k-1}u_{I<\tau_i}\times(v_{\tau_i}-v)\times\chi[\![\sigma_{j+1}=\tau_i]\!]\times\chi[\![\sigma_j<\tau_i]\!]$$

(641Eb)

$$=u_{I<\tau_i}\times(v_{\tau_i}-v)$$

because

$$[\![u_{I<\tau_i}\ne 0]\!]\subseteq[\![\sigma_0<\tau_i]\!]=\sup\nolimits_{j<k}([\![\sigma_j<\tau_i]\!]\cap[\![\sigma_{j+1}=\tau_i]\!]).\ \mathbf{Q}$$

Taking the limit as $I\uparrow\mathcal{I}(\mathcal{S}')$, $\int_{\mathcal{S}'}\boldsymbol{u}\,d\boldsymbol{v}$ is defined and equal to $u_{<\tau_i}\times(v_{\tau_i}-v)$.

(b) At the top end, $\int_{\mathcal{S}\vee\tau_n}\boldsymbol{u}\,d\boldsymbol{v}=0$ because $\boldsymbol{v}$ is constant on $\mathcal{S}\vee\tau_n$. So

$$\begin{aligned}\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}&=\int_{\mathcal{S}\wedge\tau_0}\boldsymbol{u}\,d\boldsymbol{v}+\sum_{i=1}^n\int_{\mathcal{S}\cap[\tau_{i-1},\tau_i]}\boldsymbol{u}\,d\boldsymbol{v}+\int_{\mathcal{S}\vee\tau_n}\boldsymbol{u}\,d\boldsymbol{v}\\ &=u_{<\tau_0}\times(v_{\tau_0}-v_\downarrow)+\sum_{i=1}^n u_{<\tau_i}\times(v_{\tau_i}-v_{\tau_{i-1}}).\end{aligned}$$

641K Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$ a non-decreasing non-negative fully adapted process and τ a member of $\mathcal{S}$.

(a) If I is a non-empty finite sublattice of $\mathcal{S}$ then $u_{I<\tau} = \sup_{\rho\in I}(u_\rho \times \chi[\![\rho < \tau]\!])$.

(b) If I, $J \in \mathcal{I}(\mathcal{S})$ and $I \subseteq J$ then $u_{I<\tau} \le u_{J<\tau}$.

(c) $u_{<\tau}$ is defined and equal to $\sup_{\rho\in\mathcal{S}}(u_\rho \times \chi[\![\rho < \tau]\!])$.

proof (a) I return to the formula of 641Ea. Set $x = \sup_{\rho\in I}(u_\rho \times \chi[\![\rho < \tau]\!])$ and

$$a_\sigma = [\![\sigma < \tau]\!] \setminus \sup_{\sigma'\in I}([\![\sigma < \sigma']\!] \cap [\![\sigma' < \tau]\!])$$

for $\sigma \in I$. Let b be an atom of the subalgebra generated by $\{[\![\sigma < \rho]\!] : \sigma,\ \rho \in I \cup \{\tau\}\}$. If $b \cap a_\sigma = 0$ for every $\sigma \in I$ then

$$b \subseteq [\![u_{I<\tau} = 0]\!] \cap [\![x = 0]\!] \subseteq [\![u_{I<\tau} = x]\!].$$

If $\sigma \in I$ is such that $b \subseteq a_\sigma$ then $b \subseteq [\![\sigma < \tau]\!] \subseteq [\![u_\sigma \le x]\!]$. At the same time, if $\rho \in I$ and $b \subseteq [\![\rho < \tau]\!]$, then $b \cap [\![\sigma < \rho]\!] = 0$ so

$$b \subseteq [\![\rho \le \sigma]\!] \subseteq [\![u_\rho \le u_\sigma]\!]$$

(614Ib)

$$\subseteq [\![u_\rho \times \chi[\![\rho < \tau]\!] \le u_\sigma]\!]$$

because $u_\sigma \ge 0$; accordingly $b \subseteq [\![x \le u_\sigma]\!]$ and

$$b \subseteq a_\sigma \cap [\![x = u_\sigma]\!] \subseteq [\![u_{I<\tau} = u_\sigma]\!] \cap [\![x = u_\sigma]\!] \subseteq [\![u_{I<\tau} = x]\!]$$

by 641Ea. As b is arbitrary, $u_{I<\tau} = x$, as claimed.

(b) It follows at once that if I, $J \in \mathcal{I}(\mathcal{S})$ and $I \subseteq J$, then $u_{I<\tau} \le u_{J<\tau}$ for every $\tau \in \mathcal{S}$.

(c) If $\tau \in \mathcal{S}$,

$$u_{I<\tau} = \sup_{\sigma\in I}(u_\sigma \times \chi[\![\sigma < \tau]\!]) \le \sup_{\sigma\in I} u_\sigma \le u_\tau$$

for every $I \in \mathcal{I}(\mathcal{S})$, so $\sup_{I\in\mathcal{I}(\mathcal{S})} u_{I<\tau}$ is defined and equal to $\lim_{I\uparrow\mathcal{I}(\mathcal{S})} u_{I<\tau}$. And of course $\sup_{I\in\mathcal{I}(\mathcal{S})} u_{I<\tau} = \sup_{\rho\in\mathcal{S}}(u_\rho \times \chi[\![\rho < \tau]\!])$.

641L Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{u}$ a fully adapted process with domain $\mathcal{S}$.

(a) If $\boldsymbol{u}$ is non-decreasing and non-negative, it has a previsible version $\boldsymbol{u}_<$; $\boldsymbol{u}_<$ is non-decreasing and $\boldsymbol{u}_< \le \boldsymbol{u}$.

(b) If $\boldsymbol{u}$ is (locally) of bounded variation, it has a previsible version which is (locally) of bounded variation.

(c) If $\boldsymbol{u}$ is (locally) moderately oscillatory, it has a previsible version which is (locally) moderately oscillatory.

proof (a) Express $\boldsymbol{u}$ as $\langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$. By 641Kc, $u_{<\tau} = \sup_{\rho\in\mathcal{S}}(u_\rho \times \chi[\![\rho < \tau]\!])$ is defined for every $\tau \in \mathcal{S}$; the formula shows at once that $u_{<\tau} \le u_{<\tau'} \le u_{\tau'}$ whenever $\tau \le \tau'$ in $\mathcal{S}$.

(b) If $\boldsymbol{u}$ is of bounded variation, it is expressible as the difference $\boldsymbol{v}' - \boldsymbol{v}''$ of two order-bounded non-negative non-decreasing processes (614J-614K), and now $\boldsymbol{u}_< = \boldsymbol{v}'_< - \boldsymbol{v}''_<$ (641G(e-i)) is again the difference of two order-bounded non-negative non-decreasing processes, by (a) here and 641G(a-vii), so is of bounded variation. If $\boldsymbol{u}$ is locally of bounded variation then the same argument, applied to $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ for $\tau \in \mathcal{S}$, shows that $\boldsymbol{u}_<$ is defined and locally of bounded variation (using 641Gc).

(c) Now we know that $M_{\mathrm{mo}}(\mathcal{S})$ is the closure in $M_{\text{o-b}}(\mathcal{S})$ of the space $M_{\mathrm{bv}}(\mathcal{S})$ of processes of bounded variation (615Ea), so 641G(e-i) tells us that every moderately oscillatory process on $\mathcal{S}$ has a previsible version. Moreover, because $\boldsymbol{u}_< \in M_{\mathrm{bv}}(\mathcal{S})$ for every $\boldsymbol{u} \in M_{\mathrm{bv}}(\mathcal{S})$, by (b), and the map $\boldsymbol{u} \mapsto \boldsymbol{u}_<$ is continuous for the ucp topology (641G(e-ii)), $\boldsymbol{u}_<$ will be moderately oscillatory whenever $\boldsymbol{u}$ is.

As in (b), it follows that $\boldsymbol{u}_<$ is defined, and is locally moderately oscillatory, whenever $\boldsymbol{u}$ is locally moderately oscillatory.

641M Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$ a locally moderately oscillatory process with previsible version $\boldsymbol{u}_< = \langle u_{<\sigma}\rangle_{\sigma\in\mathcal{S}}$. Suppose that $\langle\tau_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathcal{S}$ with supremum τ (taken in $\mathcal{T}$) which belongs to $\mathcal{S}$. Set $a = \inf_{n\in\mathbb{N}} [\![\tau_n < \tau]\!]$. Then $a \subseteq [\![u_{<\tau} = \lim_{n\to\infty} u_{\tau_n}]\!]$.

proof Because $\boldsymbol{u}$ is moderately oscillatory, $\boldsymbol{u}_<$ and $w = \lim_{n\to\infty} u_{\tau_n}$ are both defined (641L). Take $\epsilon > 0$. Then there is a finite sublattice I of $\mathcal{S}$ such that $\theta(u_{<\tau} - u_{J<\tau}) \le \epsilon$ whenever $J \in \mathcal{I}(\mathcal{S})$ includes I. For each $\sigma \in I$, $[\![\sigma < \tau]\!] = \sup_{n\in\mathbb{N}} [\![\sigma < \tau_n]\!]$ (611Eb once more), while of course $\langle [\![\sigma < \tau_n]\!]\rangle_{n\in\mathbb{N}}$ is non-decreasing. There is therefore an $n \in \mathbb{N}$ such that $\bar\mu b \le \epsilon$, where

$$b = \sup_{\sigma\in I}([\![\sigma < \tau]\!] \setminus [\![\sigma < \tau_n]\!]).$$

Now take any $m \ge n$ and consider the finite sublattice J of $\mathcal{S}$ generated by $I \cup \{\tau_m, \tau\}$. It is easy to see that

$$\{\sigma : \sigma \in \mathcal{T},\ [\![\tau_m < \sigma]\!] \cap [\![\sigma < \tau]\!] \subseteq b\}$$

is full, therefore a sublattice of $\mathcal{T}$ (611M(b-i) again). As it includes $I \cup \{\tau_m, \tau\}$, it includes J. So

$$[\![u_{J<\tau} = u_{\tau_m}]\!] \supseteq [\![\tau_m < \tau]\!] \setminus \sup_{\sigma\in J}([\![\tau_m < \sigma]\!] \cap [\![\sigma < \tau]\!]) \supseteq a \setminus b.$$

Accordingly

$$\theta(\chi a \times (u_{J<\tau} - u_{\tau_m})) \le \bar\mu(a \cap [\![u_{J<\tau} \ne u_{\tau_m}]\!]) \le \bar\mu b \le \epsilon.$$

But now

$$\begin{aligned}\theta(\chi a \times (u_{<\tau} - u_{\tau_m})) &\le \theta(\chi a \times (u_{<\tau} - u_{J<\tau})) + \theta(\chi a \times (u_{J<\tau} - u_{\tau_m}))\\ &\le \theta(u_{<\tau} - u_{J<\tau}) + \epsilon \le 2\epsilon,\end{aligned}$$

and this is true for every $m \ge n$. Consequently $\theta(\chi a \times (u_{<\tau} - w)) \le 2\epsilon$. As ϵ is arbitrary. $\chi a \times (u_{<\tau} - w) = 0$ and $a \subseteq [\![u_{<\tau} = w]\!]$, as claimed.

641N The natural idea of a 'jump' in a real function f is a point t at which the left and right limits $f(t^-)$, $f(t^+)$ of the function are defined and different. (See 226B.) For a càdlàg function f, the limits exist (at least, on the interior of $\operatorname{dom} f$) with $f(t^+) = f(t)$, so we are looking at $f(t) - f(t^-)$. In a near-simple process $\boldsymbol{u}$, this corresponds to a non-zero value of $\boldsymbol{u} - \boldsymbol{u}_<$. Subject to a necessary close look at the bottom end of the domain of $\boldsymbol{u}$, we find that a jump-free process, in the sense of §618, is one for which $\boldsymbol{u}$ is essentially equal to $\boldsymbol{u}_<$; and in fact we can get an expression for the residual oscillation $\mathrm{Osclln}(\boldsymbol{u})$.

Proposition Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$, and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ an order-bounded process with a previsible version $\boldsymbol{u}_< = \langle u_{<\tau}\rangle_{\tau\in\mathcal{S}}$.

(a) For $\tau \in \mathcal{S}$ set $e_\tau = \sup_{\sigma\in\mathcal{S}} [\![\sigma < \tau]\!]$. Then $\chi e_\tau \times |u_\tau - u_{<\tau}| \le \mathrm{Osclln}(\boldsymbol{u}{\restriction}\mathcal{S} \wedge \tau)$.

(b) If $\boldsymbol{u}$ is near-simple, then $\mathrm{Osclln}(\boldsymbol{u}) = \sup_{\tau\in\mathcal{S}} \chi e_\tau \times |u_\tau - u_{<\tau}|$.

proof (a) Suppose that $\tau \in I \subseteq J$ in $\mathcal{I}(\mathcal{S} \wedge \tau)$. Take $(\sigma_0, \dots, \sigma_n)$ linearly generating the J-cells, and set $a_i = [\![\sigma_i < \sigma_{i+1}]\!] \cap [\![\sigma_{i+1} = \tau]\!]$ for $i < n$. Then

$$a_i \subseteq [\![|u_\tau - u_{J<\tau}| = |u_{\sigma_{i+1}} - u_{\sigma_i}|]\!] \subseteq [\![|u_\tau - u_{J<\tau}| \le \mathrm{Osclln}_J(\boldsymbol{u})]\!]$$

for $i < n$, while

$$[\![\min I < \tau]\!] \subseteq [\![\sigma_0 < \tau]\!] \subseteq \sup_{i<n} a_i$$

because $\sigma_n = \tau$. So

$$|u_\tau - u_{J<\tau}| \times \chi[\![\min I < \tau]\!] \le \mathrm{Osclln}_J(\boldsymbol{u}) \le \mathrm{Osclln}^*_I(\boldsymbol{u}{\restriction}\mathcal{S} \wedge \tau).$$

Letting $J \uparrow \mathcal{I}(\mathcal{S} \wedge \tau)$, we get

$$|u_\tau - u_{<\tau}| \times \chi[\![\min I < \tau]\!] \le \mathrm{Osclln}^*_I(\boldsymbol{u}{\restriction}\mathcal{S} \wedge \tau).$$

Now

$$\lim_{I\uparrow\mathcal{I}(\mathcal{S}\wedge\tau)} [\![\min I < \tau]\!] = \lim_{\sigma\downarrow\mathcal{S}} [\![\sigma < \tau]\!] = \sup_{\sigma\in\mathcal{S}} [\![\sigma < \tau]\!] = e_\tau,$$

so $\lim_{I\uparrow\mathcal{I}(\mathcal{S}\wedge\tau)} \chi[\![\min I < \tau]\!] = \chi e_\tau$, and

$$\begin{aligned}|u_\tau - u_{<\tau}| \times \chi e_\tau &= \lim_{I\uparrow\mathcal{I}(\mathcal{S}\wedge\tau)}(|u_\tau - u_{<\tau}| \times \chi[\![\min I < \tau]\!])\\ &\le \lim_{I\uparrow\mathcal{I}(\mathcal{S}\wedge\tau)} \mathrm{Osclln}^*_I(\boldsymbol{u}\restriction\mathcal{S}\wedge\tau) = \inf_{I\in\mathcal{I}(\mathcal{S}\wedge\tau)} \mathrm{Osclln}^*_I(\boldsymbol{u}\restriction\mathcal{S}\wedge\tau)\\ &= \mathrm{Osclln}(\boldsymbol{u}\restriction\mathcal{S}\wedge\tau),\end{aligned}$$

as required.

(b) Note that as $\boldsymbol{u}$ is moderately oscillatory (631Ca), it has a previsible version (641L). Write $\omega(\boldsymbol{u})$ for $\sup_{\tau\in\mathcal{S}} \chi e_\tau \times |u_\tau - u_{<\tau}|$.

(i) Suppose that $\boldsymbol{u}$ is simple, with breakpoint string $(\tau_0, \dots, \tau_n)$ in $\mathcal{S}$ and starting value $u_\downarrow$. Take $\sigma \le \sigma'$ in $\mathcal{S}$. Then, using the formulae in 641Ia,

— if $\sigma' \le \tau_0$,

$$\begin{aligned}|u_{\sigma'} - u_\sigma| &= |u_{\sigma'} - u_\sigma| \times \chi[\![\sigma < \sigma']\!] = |u_{\sigma'} - u_\downarrow| \times \chi[\![\sigma < \sigma']\!]\\ &\le |u_{\sigma'} - u_\downarrow| \times \chi e_{\sigma'} = |u_{\sigma'} - u_{<\sigma'}| \times \chi e_{\sigma'} \le \omega(\boldsymbol{u});\end{aligned}$$

— if $i < n$ and $\tau_i \le \sigma \le \sigma' \le \tau_{i+1}$,

$$\begin{aligned}|u_{\sigma'} - u_\sigma| \le |u_{\tau_{i+1}} - u_{\tau_i}| &= |u_{\tau_{i+1}} - u_{\tau_i}| \times \chi[\![\tau_i < \tau_{i+1}]\!]\\ &= |u_{\tau_{i+1}} - u_{<\tau_{i+1}}| \times \chi[\![\tau_i < \tau_{i+1}]\!]\\ &\le |u_{\tau_{i+1}} - u_{<\tau_{i+1}}| \times \chi e_{\tau_{i+1}} \le \omega(\boldsymbol{u});\end{aligned}$$

— if $\tau_n \le \sigma$, $|u_{\sigma'} - u_\sigma| = 0$.

By the first formula in 618Ca,

$$\mathrm{Osclln}(\boldsymbol{u}) \le \mathrm{Osclln}^*_I(\boldsymbol{u}) \le \omega(\boldsymbol{u}),$$

where $I = \{\tau_0, \dots, \tau_n\}$.

(ii) Generally, if $\boldsymbol{u}$ is near-simple, take any $\epsilon > 0$. Then there is a simple process $\boldsymbol{u}' = \langle u'_\sigma\rangle_{\sigma\in\mathcal{S}}$ such that $\theta(\bar u) \le \epsilon$, where $\bar u = \sup|\boldsymbol{u} - \boldsymbol{u}'|$. Now

$$\mathrm{Osclln}(\boldsymbol{u}) \le \mathrm{Osclln}(\boldsymbol{u}') + \mathrm{Osclln}(\boldsymbol{u} - \boldsymbol{u}')$$

(618B(c-ii))

$$\le \omega(\boldsymbol{u}') + 2\bar u$$

((i) above and 618B(b-ii))

$$\le \omega(\boldsymbol{u}) + \omega(\boldsymbol{u}' - \boldsymbol{u}) + 2\bar u$$

(because $\boldsymbol{v} \mapsto \boldsymbol{v} - \boldsymbol{v}_<$ is linear, by 641Ge)

$$\le \omega(\boldsymbol{u}) + 4\bar u.$$

So

$$\theta((\mathrm{Osclln}(\boldsymbol{u}) - \omega(\boldsymbol{u}))^+) \le \theta(4\bar u) \le 4\epsilon.$$

As ϵ is arbitrary, $\theta((\mathrm{Osclln}(\boldsymbol{u}) - \omega(\boldsymbol{u}))^+) = 0$ and $\mathrm{Osclln}(\boldsymbol{u}) \le \omega(\boldsymbol{u})$ in this case also.

(iii) Putting this together with (a), we see that $\omega(\boldsymbol{u}) = \mathrm{Osclln}(\boldsymbol{u})$, as claimed.

641O Corollary Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$, and $\boldsymbol{u}$ a locally jump-free process with domain $\mathcal{S}$. Then $\boldsymbol{u}_< = \boldsymbol{u} \times \mathbf{1}^{(\mathcal{S})}_<$.

proof (a) To begin with, suppose that $\boldsymbol{u}$ is jump-free, therefore near-simple (631Cb). Then $\mathrm{Osclln}(\boldsymbol{u}) = 0$ so, in the language of 641N, $\chi e_\tau \times (u_\tau \times u_{<\tau}) = 0$ for every $\tau \in \mathcal{S}$, that is, $\mathbf{1}^{(\mathcal{S})}_< \times (\boldsymbol{u} - \boldsymbol{u}_<) = 0$ (641Gb) and

$$\boldsymbol{u} \times \mathbf{1}^{(\mathcal{S})}_{<} = \boldsymbol{u}_{<} \times \mathbf{1}^{(\mathcal{S})}_{<} = (\boldsymbol{u} \times \mathbf{1}^{(\mathcal{S})})_{<}$$

(641G(e-i))

$$= \boldsymbol{u}_{<}.$$

(b) In general, take any $\tau \in \mathcal{S}$; then $\boldsymbol{u}\restriction\mathcal{S} \wedge \tau$ is jump-free, so

$$\boldsymbol{u}_{<}\restriction\mathcal{S} \wedge \tau = (\boldsymbol{u}\restriction\mathcal{S} \wedge \tau)_{<}$$

(641G(c-ii))

$$= (\boldsymbol{u}\restriction\mathcal{S} \wedge \tau) \times (\mathbf{1}^{(\mathcal{S})}\restriction\mathcal{S} \wedge \tau)_{<}$$
$$= (\boldsymbol{u}\restriction\mathcal{S} \wedge \tau) \times (\mathbf{1}^{(\mathcal{S})}_{<}\restriction\mathcal{S} \wedge \tau) = (\boldsymbol{u} \times \mathbf{1}^{(\mathcal{S})}_{<})\restriction\mathcal{S} \wedge \tau).$$

As τ is arbitrary, $\boldsymbol{u}_{<} = \boldsymbol{u} \times \mathbf{1}^{(\mathcal{S})}_{<}$.

641P Corollary Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$, and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$ a near-simple process with starting value $u_\downarrow = 0$. Then $\operatorname{Osclln}(\boldsymbol{u}) = \sup_{\tau\in\mathcal{S}} |u_\tau - u_{<\tau}|$.

proof Of course $\boldsymbol{u}$ is moderately oscillatory (631Ca again) so has a starting value (615H). In the language of 641Nb, $\operatorname{Osclln}(\boldsymbol{u}) = \sup_{\tau\in\mathcal{S}} \chi e_\tau \times |u_\tau - u_{<\tau}|$. But if $\tau \in \mathcal{S}$,

$$1 \setminus e_\tau = \inf_{\sigma\in\mathcal{S}} [\![\tau \le \sigma]\!] = \inf_{\sigma\in\mathcal{S}} [\![\tau = \sigma \wedge \tau]\!]$$
$$\subseteq [\![u_\tau = \lim_{\sigma\downarrow\mathcal{S}} u_{\sigma\wedge\tau}]\!] = [\![u_\tau = u_\downarrow]\!] = [\![u_\tau = 0]\!]$$

while of course we also have $1 \setminus e_\tau \subseteq [\![u_{I<\tau} = 0]\!]$ for every non-empty $I \in \mathcal{I}(\mathcal{S})$, so $1 \setminus e_\tau \subseteq [\![u_{<\tau} = 0]\!]$. Thus $1 \setminus e_\tau \subseteq [\![u_\tau = u_{<\tau}]\!]$ and $\chi e_\tau \times |u_\tau - u_{<\tau}| = |u_\tau - u_{<\tau}|$ for every $\tau \in \mathcal{S}$, so $\operatorname{Osclln}(\boldsymbol{u}) = \sup_{\tau\in\mathcal{S}} |u_\tau - u_{<\tau}|$.

641Q Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u}$ a locally moderately oscillatory process with previsible version $\boldsymbol{u}_{<}$, and $\boldsymbol{v}$ a local integrator with previsible version $\boldsymbol{v}_{<}$. Set $\boldsymbol{w} = ii_{\boldsymbol{v}}(\boldsymbol{u})$. Then $\boldsymbol{w}-\boldsymbol{w}_{<} = \boldsymbol{u}_{<} \times (\boldsymbol{v}-\boldsymbol{v}_{<})$.

proof Because $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are all locally moderately oscillatory (616Ib, 616J), all the previsible versions are defined (641L). Express $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{u}_{<}$, $\boldsymbol{v}_{<}$ and $\boldsymbol{w}_{<}$ as $\langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$, $\langle v_\sigma \rangle_{\sigma\in\mathcal{S}}$, $\langle u_{<\sigma} \rangle_{\sigma\in\mathcal{S}}$, $\langle v_{<\sigma} \rangle_{\sigma\in\mathcal{S}}$ and $\langle w_{<\sigma} \rangle_{\sigma\in\mathcal{S}}$.

(a) Suppose to begin with that $\boldsymbol{u}$ and $\boldsymbol{v}$ are both non-negative and non-decreasing. Then $\boldsymbol{w}$ is non-negative and non-decreasing (616Rb). Take any $\tau \in \mathcal{S}$.

(i) If $\sigma \in \mathcal{S} \wedge \tau$ then

$$u_\sigma \times (v_\tau - v_\sigma) \le \int_{\mathcal{S}\cap[\sigma,\tau]} \boldsymbol{u}\, d\boldsymbol{v} \le u_{<\tau} \times (v_\tau - v_\sigma).$$

P If J is a finite sublattice of $\mathcal{S} \cap [\sigma, \tau]$ containing both σ and τ, take a sequence $(\sigma_0, \dots, \sigma_n)$ linearly generating the J-cells. Then

$$u_\sigma \times (v_\tau - v_\sigma) = u_\sigma \times \sum_{i=0}^{n-1} v_{\sigma_{i+1}} - v_{\sigma_i} \le \sum_{i=0}^{n-1} u_{\sigma_i} \times (v_{\sigma_{i+1}} - v_{\sigma_i})$$
$$= \sum_{i=0}^{n-1} u_{\sigma_i} \times \chi[\![\sigma_i < \tau]\!] \times (v_{\sigma_{i+1}} - v_{\sigma_i})$$

(because $[\![\sigma_i = \tau]\!] \subseteq [\![\sigma_i = \sigma_{i+1}]\!] \subseteq [\![v_{\sigma_i} = v_{\sigma_{i+1}}]\!]$ for each i)

$$= \sum_{i=0}^{n-1} u_{\{\sigma_i\}<\tau} \times \chi[\![\sigma_i < \tau]\!] \times (v_{\sigma_{i+1}} - v_{\sigma_i})$$

(by the formula in 641Ea)

$$\leq \sum_{i=0}^{n-1} u_{<\tau} \times (v_{\sigma_{i+1}} - v_{\sigma_i})$$

(by the argument in part (a-ii) of the proof of 641L)

$$= u_{<\tau} \times (v_\tau - v_\sigma),$$

and

$$u_\sigma \times (v_\tau - v_\sigma) \leq S_J(\boldsymbol{u}, d\boldsymbol{v}) \leq u_{<\tau} \times (v_\tau - v_\sigma).$$

In the limit as $J \uparrow \mathcal{I}(\mathcal{S} \cap [\sigma, \tau])$,

$$u_\sigma \times (v_\tau - v_\sigma) \leq \int_{\mathcal{S}\cap[\sigma,\tau]} \boldsymbol{u}\, d\boldsymbol{v} \leq u_{<\tau} \times (v_\tau - v_\sigma). \quad \mathbf{Q}$$

(ii) Now let I be a non-empty finite sublattice of $\mathcal{S}$. Then

$$u_{I<\tau} \times (v_\tau - v_{I<\tau}) \leq (w_\tau - w_{I<\tau}) \times \chi[\![\min I < \tau]\!] \leq u_{<\tau} \times (v_\tau - v_{I<\tau}).$$

$\mathbf{P}$ Take a sequence $(\tau_0, \ldots, \tau_n)$ linearly generating the I-cells. As $[\![\tau \leq \tau_0]\!] \subseteq [\![u_{I<\tau} = 0]\!] \cap [\![\chi[\![\min I < \tau]\!] = 0]\!]$, we certainly have

$$\begin{aligned}[\![\tau \leq \tau_0]\!] \subseteq [\![u_{I<\tau} \times (v_\tau - v_{I<\tau}) \leq (w_\tau - w_{I<\tau}) \times \chi[\![\min I < \tau]\!]]\!]\\ \cap [\![(w_\tau - w_{I<\tau}) \times \chi[\![\min I < \tau]\!] \leq u_{<\tau} \times (v_\tau - v_{I<\tau})]\!].\end{aligned}$$

Set $d_i = [\![\tau_i < \tau]\!] \setminus [\![\tau_{i+1} < \tau]\!]$ for $i < n$ and $d_n = [\![\tau_n < \tau]\!]$. Then for any $i \leq n$ we have $d_i \subseteq [\![w_{I<\tau} = w_{\tau_i}]\!] = [\![w_\tau - w_{I<\tau} = w_\tau - w_{\tau_i}]\!]$ so (i) tells us that

$$\begin{aligned}d_i \subseteq [\![u_{I<\tau} \times (v_\tau - v_{I<\tau}) \leq (w_\tau - w_{I<\tau}) \times \chi[\![\min I < \tau]\!]]\!]\\ \cap [\![(w_\tau - w_{I<\tau}) \times \chi[\![\min I < \tau]\!] \leq u_{<\tau} \times (v_\tau - v_{I<\tau})]\!].\end{aligned}$$

Since $[\![\tau \leq \tau_0]\!] \cup \sup_{i\leq n} d_i = 1$, we have the result. $\mathbf{Q}$

(iii) Taking the limit as $I \uparrow \mathcal{I}(\mathcal{S})$ in (ii),

$$u_{<\tau} \times (v_\tau - v_{<\tau}) \leq (w_\tau - w_{<\tau}) \times \chi e \leq u_{<\tau} \times (v_\tau - v_{<\tau})$$

where $e = \lim_{I\uparrow\mathcal{I}(\mathcal{S})} [\![\min I < \tau]\!] = \sup_{\sigma\in\mathcal{S}} [\![\sigma < \tau]\!]$. But now $u_{<\tau} \times (v_\tau - v_{<\tau}) \times \chi e = (w_\tau - w_{<\tau}) \times \chi e$ while

$$1 \setminus e \subseteq [\![w_\tau = 0]\!] \cap [\![w_{<\tau} = 0]\!] \cap [\![u_{<\tau} = 0]\!],$$

so $u_{<\tau} \times (v_\tau - v_{<\tau}) \times \chi(1 \setminus e) = (w_\tau - w_{<\tau}) \times \chi(1 \setminus e)$; adding, we get $u_{<\tau} \times (v_\tau - v_{<\tau}) = (w_\tau - w_{<\tau})$.

(iv) As τ is arbitrary, we have $\boldsymbol{w} - \boldsymbol{w}_< = \boldsymbol{u}_< \times (\boldsymbol{v} - \boldsymbol{v}_<)$ whenever $\boldsymbol{u}$ and $\boldsymbol{v}$ are non-negative and non-decreasing.

(b) Since the operator $(\boldsymbol{u}, \boldsymbol{v}) \mapsto ii_{\boldsymbol{v}}(\boldsymbol{u})$ is bilinear and the operator $\boldsymbol{u} \mapsto \boldsymbol{u}_<$ is linear, it follows that $\boldsymbol{w} - \boldsymbol{w}_< = \boldsymbol{u}_< \times (\boldsymbol{v} - \boldsymbol{v}_<)$ whenever $\boldsymbol{u}$ and $\boldsymbol{v}$ are of bounded variation (614J-614K again). Now if $\boldsymbol{u}$ is of bounded variation, the map $\boldsymbol{v} \mapsto ii_{\boldsymbol{v}}(\boldsymbol{u}) : M_{\text{mo}}(\mathcal{S}) \to M_{\text{mo}}(\mathcal{S})$ is continuous for the ucp topology on $M_{\text{mo}}(\mathcal{S})$ (615Rb). Since the map $\boldsymbol{v} \mapsto \boldsymbol{v}_<$ is also continuous (641G(e-ii)) and the ucp topology on M_{mo} is Hausdorff, $\boldsymbol{w} - \boldsymbol{w}_< = \boldsymbol{u}_< \times (\boldsymbol{v} - \boldsymbol{v}_<)$ whenever $\boldsymbol{u}$ is of bounded variation and $\boldsymbol{v}$ is moderately oscillatory. And if $\boldsymbol{v}$ is an integrator, then $\boldsymbol{u} \mapsto ii_{\boldsymbol{v}}(\boldsymbol{u}) : M_{\text{mo}} \to M_{\text{o-b}}$ is continuous (616J) and takes values in M_{mo} (616J, 616Ib), so now we see that $\boldsymbol{w} - \boldsymbol{w}_< = \boldsymbol{u}_< \times (\boldsymbol{v} - \boldsymbol{v}_<)$ whenever $\boldsymbol{u}$ is moderately oscillatory and $\boldsymbol{v}$ is an integrator.

(c) For the general case in which $\boldsymbol{u}$ is locally moderately oscillatory and $\boldsymbol{v}$ is a local integrator, apply (b) to $\boldsymbol{u}\restriction\mathcal{S} \wedge \tau$ and $\boldsymbol{v}\restriction\mathcal{S} \wedge \tau$ to see that $u_{<\tau} \times (v_\tau - v_{<\tau}) = (w_\tau - w_{<\tau})$ for every $\tau \in \mathcal{S}$; of course this depends on the fact that $\boldsymbol{u}_<\restriction\mathcal{S} \wedge \tau = (\boldsymbol{u}\restriction\mathcal{S} \wedge \tau)$ (641Gc).

641R Corollary Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$ and $\boldsymbol{v}$ a local integrator with domain $\mathcal{S}$, starting value 0 and quadratic variation $\boldsymbol{v}^*$ (617H). Write $\boldsymbol{v}_<$, $\boldsymbol{v}^*_<$ for the previsible versions of $\boldsymbol{v}$ and $\boldsymbol{v}^*$. Then $\boldsymbol{v}^* - \boldsymbol{v}^*_< = (\boldsymbol{v} - \boldsymbol{v}_<)^2$.

proof We know that $\boldsymbol{v}^* = \boldsymbol{v}^2 - 2ii_{\boldsymbol{v}}(\boldsymbol{v})$ (617Ka). So

$$\boldsymbol{v}^* - \boldsymbol{v}^*_< = \boldsymbol{v}^2 - \boldsymbol{v}^2_< - 2\boldsymbol{v}_< \times (\boldsymbol{v} - \boldsymbol{v}_<) = (\boldsymbol{v} - \boldsymbol{v}_<)^2,$$

using 641Gd and 641Q.

641T Since the previsible version of a moderately oscillatory process is moderately oscillatory (641L), an integral $\int_{\mathcal{S}} \boldsymbol{u}_< d\boldsymbol{v}$ will often be defined. There are a couple of useful conditions which will ensure it is actually equal to $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$.

Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{S}$, $\boldsymbol{u}$ a moderately oscillatory process with previsible version $\boldsymbol{u}_<$ and $\boldsymbol{v}$ an integrator. If either $\boldsymbol{v}$ is jump-free or $\boldsymbol{u}$ is jump-free and has starting value 0, then $\int_{\mathcal{S}} \boldsymbol{u}_< d\boldsymbol{v} = \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$.

proof (a) If $\boldsymbol{u}$ is jump-free and has starting value 0, then in fact $\boldsymbol{u}_< = \boldsymbol{u}$, by 641O or 641P, and the result is trivial. So in the rest of this proof I will assume that $\boldsymbol{v}$ is jump-free. Express $\boldsymbol{u}$, $\boldsymbol{u}_<$ and $\boldsymbol{v}$ as $\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$, $\langle u_{<\sigma}\rangle_{\sigma\in\mathcal{S}}$ and $\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$.

(b) Suppose that $\boldsymbol{u}$ is non-negative and non-decreasing. Recall that if σ, τ in $\mathcal{S}$ then

$$0 \le u_\sigma \times \chi[\![\sigma < \tau]\!] = u_{\{\sigma\}<\tau} \le u_{<\tau} \le u_\tau,$$

as in the proof of 641La. So if $I \in \mathcal{I}(\mathcal{S})$ is non-empty and $(\tau_0, \dots , \tau_n)$ linearly generates the I-cells,

$$\begin{aligned}
|S_I(\boldsymbol{u} - \boldsymbol{u}_<, d\boldsymbol{v})| &= |\sum_{i=0}^{n-1}(u_{\tau_i} - u_{<\tau_i}) \times (v_{\tau_{i+1}} - v_{\tau_i})| \\
&\le \sum_{i=0}^{n-1}(u_{\tau_i} - u_{<\tau_i}) \times \chi[\![\tau_i < \tau_{i+1}]\!] \times |v_{\tau_{i+1}} - v_{\tau_i}| \\
&\le \operatorname{Osclln}_I(\boldsymbol{v}) \times \sum_{i=0}^{n-1}(u_{\tau_i} - u_{<\tau_i}) \times \chi[\![\tau_i < \tau_{i+1}]\!] \\
&\le \operatorname{Osclln}_I(\boldsymbol{v}) \times (u_{\tau_0} + \sum_{i=1}^{n-1} u_{\tau_i} - u_{\tau_{i-1}}) \\
&= \operatorname{Osclln}_I(\boldsymbol{v}) \times u_{\tau_{n-1}} \le \operatorname{Osclln}_I(\boldsymbol{v}) \times \sup|\boldsymbol{u}| \\
&\to 0 \text{ as } I \uparrow \mathcal{I}(\mathcal{S})
\end{aligned}$$

because $\boldsymbol{v}$ is jump-free. So in this case we have $\int_{\mathcal{S}}(\boldsymbol{u} - \boldsymbol{u}_<)d\boldsymbol{v} = 0$ and $\int_{\mathcal{S}} \boldsymbol{u}_< d\boldsymbol{v} = \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$.

(c) Now, as in part (b) of the proof of 641Q, the linearity of integrations and of the operator $\boldsymbol{u} \mapsto \boldsymbol{u}_<$ ensures that $\int_{\mathcal{S}} \boldsymbol{u}_< d\boldsymbol{v} = \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$ whenever $\boldsymbol{u}$ is of bounded variation and $\boldsymbol{v}$ is jump-free, and the continuity of the operators $\boldsymbol{u} \mapsto \boldsymbol{u} <$ and $\boldsymbol{u} \mapsto \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$ implies that $\int_{\mathcal{S}} \boldsymbol{u}_< d\boldsymbol{v} = \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$ whenever $\boldsymbol{u}$ is moderately oscillatory and $\boldsymbol{v}$ is jump-free.

641U For the next result of this kind, I begin with a lemma which casts a little light on the difference between discrete-time processes and continuous-time processes.

Lemma Suppose that T has no points isolated on the right. Then whenever $\tau \le \tau'$ in $\mathcal{T}$ and $\epsilon > 0$ there is a $\sigma \in [\tau, \tau']$ such that $[\![\tau < \sigma]\!] = [\![\tau < \tau']\!]$ and $\bar\mu([\![\tau < \tau']\!] \setminus [\![\sigma < \tau']\!]) \le \epsilon$.

proof Set $A = \{\sigma : \sigma \in [\tau, \tau'],\ [\![\tau < \sigma]\!] = [\![\tau < \tau']\!]\}$. A is non-empty (because $\tau' \in A$) and downwards-directed (by 611E(c-i-β)); moreover, $\sup_{\sigma\in A} [\![\sigma < \tau']\!] = [\![\tau < \tau']\!]$. **P?** Otherwise, $b = [\![\tau < \tau']\!] \setminus \sup_{\sigma\in A} [\![\sigma < \tau']\!]$ is non-zero. Now $b \subseteq \sup_{t\in T} [\![\tau' > t]\!] \setminus [\![\tau > t]\!]$ (611D); let $t \in T$ be such that $b \cap [\![\tau' > t]\!] \setminus [\![\tau > t]\!]$ is non-zero. Because t is not isolated on the right, $[\![\tau' > t]\!] = \sup_{s>t} [\![\tau' > s]\!]$ (611A(b-i)), and there is an $s > t$ such that $b \cap [\![\tau' > s]\!] \setminus [\![\tau > t]\!]$ is non-zero.

Set $a = [\![\tau' > s]\!] \setminus [\![\tau > t]\!]$. Then $a \in \mathfrak{A}_s = \mathfrak{A}_{\check s}$ (611Hb) and $a \subseteq [\![\tau' > s]\!]$, so $a \in \mathfrak{A}_{\check s} \cap \mathfrak{A}_{\tau'}$ (611H(a-i)). There is a therefore a $\sigma \in \mathcal{T}$ such that

$$a \subseteq [\![\sigma = \check s]\!], \quad 1 \setminus a \subseteq [\![\sigma = \tau']\!], \quad \check s \wedge \tau' \le \sigma \le \check s \vee \tau'$$

(611I again). We see that

$$\begin{aligned}
[\![\sigma < \tau]\!] &= [\![\sigma < \tau]\!] \cap ([\![\check s < \tau]\!] \cup [\![\tau' < \tau]\!]) = [\![\sigma < \tau]\!] \cap [\![\check s < \tau]\!] \\
&\subseteq [\![\sigma < \tau']\!] \cap [\![\check s < \tau]\!] \subseteq a \cap [\![\check s < \tau]\!] \subseteq [\![\tau > s]\!] \setminus [\![\tau > t]\!]
\end{aligned}$$

(611E(a-i-δ) once more)

$$= 0,$$

that is, $\tau \le \sigma$, while also

$$\begin{aligned}[\![\tau' < \sigma]\!] &= [\![\tau' < \sigma]\!] \cap ([\![\tau' < \check{s}]\!] \cup [\![\tau' < \tau']\!]) \\ &= [\![\tau' < \sigma]\!] \cap [\![\tau' < \check{s}]\!] \subseteq a \cap [\![\tau' < \check{s}]\!] = 0\end{aligned}$$

and $\sigma \le \tau'$. Moreover,

$$a \subseteq [\![\sigma = \check{s}]\!] \setminus [\![\tau > t]\!] = [\![\sigma = \check{s}]\!] \cap [\![\tau \le \check{t}]\!] \subseteq [\![\tau < \sigma]\!]$$

(use 611E(a-i-ζ)), while

$$[\![\tau < \tau']\!] \setminus a \subseteq [\![\tau < \tau']\!] \cap [\![\sigma = \tau']\!] \subseteq [\![\tau < \sigma]\!].$$

So $[\![\tau < \tau']\!] \subseteq [\![\tau < \sigma]\!] = [\![\tau < \tau']\!]$ and $\sigma \in A$.

On the other hand,

$$a \subseteq [\![\sigma = \check{s}]\!] \cap [\![\check{s} < \tau']\!] \subseteq [\![\sigma < \tau']\!],$$

so $a \cap b = 0$. But we chose t and s so that a should meet b. **XQ**

There is therefore a $\sigma \in A$ such that $\bar{\mu}([\![\tau < \tau']\!] \setminus [\![\sigma < \tau']\!]) \le \epsilon$, which is what we are looking for.

641V Proposition Suppose that T has no points isolated on the right. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$ a moderately oscillatory process. Then $\boldsymbol{u}_{\ll} = \boldsymbol{u}_<$.

Remark I hope it is clear that $\boldsymbol{u}_{\ll} = \langle u_{\ll\sigma} \rangle_{\sigma\in\mathcal{S}}$ is the previsible version $(\boldsymbol{u}_<)_<$ of the previsible version of $\boldsymbol{u}$, so that $\boldsymbol{u}_{\ll\tau} = \lim_{I\uparrow\mathcal{I}(\mathcal{S})} u_{<,I<\tau}$ for every $\tau \in \mathcal{S}$.

proof (a) Suppose to begin with (down to the end of (g) below) that $\mathcal{S}$ has a greatest element τ and that $\epsilon > 0$. Let $J \in \mathcal{I}(\mathcal{S})$ be such that $\tau \in J$ and

$$\theta(u_{<\tau} - u_{I<\tau}) \le \epsilon, \quad \theta(u_{\ll\tau} - u_{<,I<\tau}) \le \epsilon$$

whenever $J \subseteq I \in \mathcal{I}(\mathcal{S})$. Let J_0 be a maximal totally ordered subset of J.

(b) We need to know that if $I \in \mathcal{I}(\mathcal{S})$ includes J_0 then $\theta(u_{<\tau} - u_{I<\tau}) \le \epsilon$ and $\theta(u_{\ll\tau} - u_{<,I<\tau}) \le \epsilon$. **P** Let K be a maximal totally ordered subset of I including J_0. The set

$$\begin{aligned}L = \{\sigma : \sigma \in \mathcal{S},\, \min K \le \sigma \le \max K,\, \operatorname{med}(\rho,\sigma,\rho') \in \{\rho,\rho'\} \\ \text{whenever } \rho,\, \rho' \text{ are successive members of } K\}\end{aligned}$$

is a sublattice of $\mathcal{S}$ including J and K is a maximal totally ordered subset of L, so $u_{I<\tau} = u_{K<\tau} = u_{L<\tau}$ (641Ed) and $\theta(u_{<\tau} - u_{I<\tau}) = \theta(u_{<\tau} - u_{L<\tau}) \le \epsilon$. Similarly, $\theta(u_{\ll\tau} - u_{<,I<\tau}) \le \epsilon$. **Q**

(c) Enumerate J_0 in increasing order as $\langle \tau_i \rangle_{i\le n}$, so that $\tau_n = \tau$. For each $i < n$ set $c_i = [\![\tau_i < \tau_{i+1}]\!]$ and choose $\tau_i' \in \mathcal{T}$ such that $\tau_i \le \tau_i' \le \tau_{i+1}$, $[\![\tau_i < \tau_i']\!] = c_i$ and $\bar{\mu}(c_i \setminus [\![\tau_i' < \tau_{i+1}]\!]) \le \frac{\epsilon}{n+1}$, as in 641U. Because $\mathcal{S}$ is order-convex, $\tau_i' \in \mathcal{S}$. Set $\tau_n' = \tau$ and $a = \sup_{i<n}(c_i \setminus [\![\tau_i' < \tau_{i+1}]\!])$, so that $\bar{\mu}a \le \epsilon$.

(d) Let $i < n$. Then

$$[\![\tau_i < \tau]\!] \cap [\![\tau_i' = \tau]\!] \subseteq a,$$

$$[\![\tau_i' < \tau]\!] \cap [\![\tau_{i+1} = \tau]\!] \setminus a = [\![\tau_i' < \tau]\!] \cap [\![\tau_{i+1}' = \tau]\!] \setminus a = [\![\tau_i < \tau]\!] \cap [\![\tau_{i+1} = \tau]\!] \setminus a.$$

P

$$\begin{aligned}[\![\tau_i < \tau]\!] \cap [\![\tau_i' = \tau]\!] &= [\![\tau_i < \tau]\!] \cap [\![\tau_i' = \tau_{i+1}]\!] \cap [\![\tau_{i+1} = \tau]\!] \\ &\subseteq [\![\tau_i < \tau_{i+1}]\!] \cap [\![\tau_i' = \tau_{i+1}]\!] \subseteq a.\end{aligned}$$

Next,

$$(\llbracket\tau_i' < \tau\rrbracket \cap \llbracket\tau_{i+1} = \tau\rrbracket) \triangle (\llbracket\tau_i < \tau\rrbracket \cap \llbracket\tau_{i+1} = \tau\rrbracket)$$
$$= (\llbracket\tau_i' < \tau_{i+1}\rrbracket \cap \llbracket\tau_{i+1} = \tau\rrbracket) \triangle (\llbracket\tau_i < \tau_{i+1}\rrbracket \cap \llbracket\tau_{i+1} = \tau\rrbracket)$$
$$\subseteq \llbracket\tau_i' < \tau_{i+1}\rrbracket \triangle \llbracket\tau_i < \tau_{i+1}\rrbracket = \llbracket\tau_i < \tau_{i+1}\rrbracket \setminus \llbracket\tau_i' < \tau_{i+1}\rrbracket \subseteq a$$

so $\llbracket\tau_i' < \tau\rrbracket \cap \llbracket\tau_{i+1} = \tau\rrbracket \setminus a = \llbracket\tau_i < \tau\rrbracket \cap \llbracket\tau_{i+1} = \tau\rrbracket \setminus a$. Finally, if $i = n-1$ then $\tau_{i+1} = \tau_{i+1}' = \tau$ so surely $\llbracket\tau_i' < \tau\rrbracket \cap \llbracket\tau_{i+1} = \tau\rrbracket \setminus a = \llbracket\tau_i' < \tau\rrbracket \cap \llbracket\tau_{i+1}' = \tau\rrbracket \setminus a$. If $i < n-1$ then

$$\llbracket\tau_{i+1}' = \tau\rrbracket = \llbracket\tau_{i+1}' = \tau_{i+2}\rrbracket \cap \llbracket\tau_{i+2} = \tau\rrbracket, \quad \llbracket\tau_{i+1} = \tau\rrbracket = \llbracket\tau_{i+1} = \tau_{i+2}\rrbracket \cap \llbracket\tau_{i+2} = \tau\rrbracket,$$

so

$$\llbracket\tau_{i+1}' = \tau\rrbracket \triangle \llbracket\tau_{i+1} = \tau\rrbracket \subseteq \llbracket\tau_{i+1}' = \tau_{i+2}\rrbracket \triangle \llbracket\tau_{i+1} = \tau_{i+2}\rrbracket = \llbracket\tau_{i+1}' < \tau_{i+2}\rrbracket \triangle \llbracket\tau_{i+1} < \tau_{i+2}\rrbracket$$
$$= \llbracket\tau_{i+1} < \tau_{i+2}\rrbracket \setminus \llbracket\tau_{i+1}' < \tau_{i+2}\rrbracket \subseteq a$$

and $\llbracket\tau_i' < \tau\rrbracket \cap \llbracket\tau_{i+1} = \tau\rrbracket \setminus a = \llbracket\tau_i' < \tau\rrbracket \cap \llbracket\tau_{i+1}' = \tau\rrbracket \setminus a$. **Q**

(e) Set $L = \{\tau_i' : i \le n\}$. Then $\theta(u_{\ll\tau} - u_{<,L<\tau}) \le 2\epsilon$. **P** Set $L' = \{\tau_0, \tau_0', \tau_1, \tau_1', \dots, \tau_n\}$. Then

$$u_{<,L'<\tau} = \sum_{i=0}^{n-1} u_{<\tau_i} \times \chi(\llbracket\tau_i < \tau\rrbracket \cap \llbracket\tau_i' = \tau\rrbracket) + u_{<\tau_i'} \times \chi(\llbracket\tau_i' < \tau\rrbracket \cap \llbracket\tau_{i+1} = \tau\rrbracket)$$

(641Eb). Similarly,

$$u_{<,L<\tau} = \sum_{i=0}^{n-1} u_{<\tau_i'} \times \chi(\llbracket\tau_i' < \tau\rrbracket \cap \llbracket\tau_{i+1}' = \tau\rrbracket).$$

Since $\llbracket\tau_i < \tau\rrbracket \cap \llbracket\tau_i' = \tau\rrbracket$ and $(\llbracket\tau_i' < \tau\rrbracket \cap \llbracket\tau_{i+1} = \tau\rrbracket) \triangle (\llbracket\tau_i' < \tau\rrbracket \cap \llbracket\tau_{i+1}' = \tau\rrbracket)$ are both included in a for every $i < n$ ((d) above), $\llbracket u_{<,L'<\tau} \ne u_{<,L<\tau}\rrbracket \subseteq a$ and

$$\theta(u_{\ll\tau} - u_{<,L<\tau}) \le \theta(u_{\ll\tau} - u_{<,L'<\tau}) + \epsilon \le 2\epsilon$$

because $J_0 \subseteq L'$. **Q**

(f) Let $I \in \mathcal{I}(\mathcal{S})$ be such that $J \cup \{\tau_i' : i < n\} \subseteq I$ and $\theta(u_{<\tau_i'} - u_{I<\tau_i'}) \le \frac{\epsilon}{n+1}$ for every $i < n$. Take a maximal totally subset I_0 of I including $\{\tau_i : i \le n\} \cup \{\tau_i' : i < n\}$. For each $i < n$, set $K_i = I_0 \cap [\tau_i, \tau_i']$. If $\sigma \in K_i$, we have

$$\llbracket\sigma < \tau_i'\rrbracket \in \mathfrak{A}_\sigma \cap \mathfrak{A}_{\tau_i'} \subseteq \mathfrak{A}_{\tau_{i+1}}$$

so we can define $\tilde\sigma \in \mathcal{T}$ by saying that

$$\llbracket\sigma < \tau_i'\rrbracket \subseteq \llbracket\tilde\sigma = \sigma\rrbracket, \quad \llbracket\tau_i' \le \sigma\rrbracket \subseteq \llbracket\tilde\sigma = \tau_{i+1}\rrbracket$$

(611I once more). Now $\tau_i \le \tilde\sigma \le \tau_{i+1}$ so $\tilde\sigma \in \mathcal{S}$. If $\rho \le \sigma$ in K_i then

$$\llbracket\sigma < \tau_i'\rrbracket = \llbracket\sigma < \tau_i'\rrbracket \cap \llbracket\rho < \tau_i'\rrbracket \subseteq \llbracket\tilde\rho \le \tilde\sigma\rrbracket, \quad \llbracket\tau_i' \le \sigma\rrbracket \subseteq \llbracket\tilde\sigma = \tau_{i+1}\rrbracket \subseteq \llbracket\tilde\rho \le \tilde\sigma\rrbracket,$$

so $\tilde\rho \le \tilde\sigma$; it follows that $\tilde K_i = \{\tilde\sigma : \sigma \in K_i\}$ is totally ordered. Because $\llbracket\tau_i < \tau_i'\rrbracket = \llbracket\tau_i < \tau_{i+1}\rrbracket$, $\tilde\tau_i = \tau_i$, so $\min K_i = \tau_i = \min \tilde K_i$. Also $\max \tilde K_i = \tilde\tau_i' = \tau_{i+1}$.

The point of this is that $\llbracket\sigma < \tau_i'\rrbracket = \llbracket\tilde\sigma < \tau_{i+1}\rrbracket$ for every $\sigma \in K_i$. It follows that $u_{K_i<\tau_i'} = u_{\tilde K_i<\tau_{i+1}}$, and this is true for every $i < n$.

Now set $K = \bigcup_{i<n} K_i$ and $\tilde K = \bigcup_{i<n} \tilde K_i$. Both of these are totally ordered sets including J_0 with $\min K = \min \tilde K = \tau_0$. Since $K \cap [\tau_i, \tau_i'] = K_i = I_0 \cap [\tau_i, \tau_i']$,

$$c_i = \llbracket\tau_i < \tau_i'\rrbracket \subseteq \llbracket u_{I_0<\tau_i'} = u_{K_i<\tau_i'}\rrbracket = \llbracket u_{I<\tau_i'} = u_{K_i<\tau_i'}\rrbracket;$$

similarly, $\tilde K \cap [\tau_i, \tau_{i+1}] = \tilde K_i$, so $c_i = \llbracket\tau_i < \tau_{i+1}\rrbracket \subseteq \llbracket u_{\tilde K<\tau_{i+1}} = u_{\tilde K_i<\tau_{i+1}}\rrbracket$; accordingly $c_i \subseteq \llbracket u_{I<\tau_i'} = u_{\tilde K<\tau_{i+1}}\rrbracket$. Next, $\llbracket\tau_{i+1} = \tau\rrbracket \subseteq \llbracket u_{\tilde K<\tau} = u_{\tilde K<\tau_{i+1}}\rrbracket$ (641Ec), so if we set $b_i = c_i \cap \llbracket\tau_{i+1} = \tau\rrbracket$, we have $b_i \subseteq \llbracket u_{I<\tau_i'} = u_{\tilde K<\tau}\rrbracket$. By the choice of I, $\theta((u_{<\tau_i'} - u_{\tilde K<\tau}) \times \chi b_i) \le \frac{\epsilon}{n+1}$.

(g) Now we find that

$$\theta(u_{\ll\tau} - u_{<\tau}) \le 2\epsilon + \theta(u_{<,L<\tau} - u_{<\tau})$$

(by (e))

$$\leq 3\epsilon + \theta(u_{<,L<\tau} - u_{\tilde{K}<\tau})$$

(by (b), because $J_0 \subseteq \tilde{K}$)

$$= 3\epsilon + \theta(u_{<,L<\tau} - u_{\tilde{K}<\tau} \times \chi[\![\tau_0 < \tau]\!])$$

(because $\tau_0 = \min \tilde{K}$, so $[\![\tau_0 = \tau]\!] \subseteq [\![u_{\tilde{K}<\tau} = 0]\!]$)

$$= 3\epsilon + \theta(u_{<,L<\tau} - \sum_{i=0}^{n-1} u_{\tilde{K}<\tau} \times \chi([\![\tau_i < \tau]\!] \cap [\![\tau_{i+1} = \tau]\!]))$$

$$\leq 4\epsilon + \theta(u_{<,L<\tau} - \sum_{i=0}^{n-1} u_{<\tau'_i} \times \chi([\![\tau_i < \tau]\!] \cap [\![\tau_{i+1} = \tau]\!]))$$

(by (f))

$$\leq 5\epsilon + \theta(u_{<,L<\tau} - \sum_{i=0}^{n-1} u_{\tau'_i} \times \chi([\![\tau'_i < \tau]\!] \cap [\![\tau'_{i+1} = \tau]\!]))$$

(because $\sup_{i<n}([\![\tau_i < \tau]\!] \cap [\![\tau_{i+1} = \tau]\!]) \bigtriangleup ([\![\tau'_i < \tau]\!] \cap [\![\tau'_{i+1} = \tau]\!]) \subseteq a$ has measure at most ϵ, by (d))

$$= 5\epsilon$$

as noted in (e).

(h) At this point, we recall that $\epsilon > 0$ was arbitrary, so we see that $u_{\ll\tau} = u_{<\tau}$. This was all on the assumption that $\tau = \max \mathcal{S}$. But generally, given that $\mathcal{S}$ is an order-convex subset of $\mathcal{T}$ and that $\tau \in \mathcal{S}$, we can apply the argument so far to $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$, $(\boldsymbol{u}\restriction\mathcal{S}\wedge\tau)_< = \boldsymbol{u}_<\restriction\mathcal{S}\wedge\tau$ and $(\boldsymbol{u}\restriction\mathcal{S}\wedge\tau)_{\ll} = \boldsymbol{u}_{\ll}\restriction\mathcal{S}\wedge\tau$ (641G(c-ii)) to see that we still have $u_{\ll\tau} = u_\tau$. Thus $\boldsymbol{u}_{\ll} = \boldsymbol{u}_<$.

641W Proposition Suppose that T has no points isolated on the right. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$, $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a moderately oscillatory process and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a near-simple integrator. Then $\int_{\mathcal{S}} \boldsymbol{u}_< d\boldsymbol{v} = \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$.

proof (a) If $\boldsymbol{v}$ is simple, we can use the formula of 641J to see that $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$ is determined by the previsible version $\boldsymbol{u}_<$. Since in this context we have $\boldsymbol{u}_{\ll} = \boldsymbol{u}_<$, by 641V, we shall have $\int_{\mathcal{S}} \boldsymbol{u}_< d\boldsymbol{v} = \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$.

(b) If $\boldsymbol{u}$ is of bounded variation, then $\boldsymbol{v} \mapsto \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$ is continuous for the ucp topology (631H(a-ii)). At the same time, $\boldsymbol{u}_<$ is of bounded variation (641Lb), so $\{\boldsymbol{v} : \boldsymbol{v} \in M_{\text{n-s}}(\mathcal{S}), \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v} = \int_{\mathcal{S}} \boldsymbol{u}_< d\boldsymbol{v}\}$ is closed for the ucp topology. By (a), it contains all simple functions, so it is the whole of $M_{\text{n-s}}(\mathcal{S})$, which is more than we need.

(c) In general, take any $\epsilon > 0$. Let $\delta > 0$ be such that $\theta(\int_{\mathcal{S}} \boldsymbol{w}\, d\boldsymbol{v}) \leq \epsilon$ whenever $I \in \mathcal{I}(\mathcal{S})$ and $\boldsymbol{w} \in M_{\text{mo}}(\mathcal{S})$ is such that $\theta(\sup|\boldsymbol{w}|) \leq \delta$ (616J, applied to $\Delta\boldsymbol{v}$). Then we have a process $\tilde{\boldsymbol{u}}$ of bounded variation such that $\theta(\sup|\boldsymbol{u} - \tilde{\boldsymbol{u}}|) \leq \delta$. As $\tilde{\boldsymbol{u}}$ is moderately oscillatory it has a previsible version $\tilde{\boldsymbol{u}}_<$. Now $\int_{\mathcal{S}} \tilde{\boldsymbol{u}}\, d\boldsymbol{v} = \int_{\mathcal{S}} \tilde{\boldsymbol{u}}_< d\boldsymbol{v}$, by (b), while $\theta(\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v} - \int_{\mathcal{S}} \tilde{\boldsymbol{u}}\, d\boldsymbol{v}) \leq \epsilon$. Next, $\sup|\boldsymbol{u}_< - \tilde{\boldsymbol{u}}_<| \leq \sup|\boldsymbol{u} - \tilde{\boldsymbol{u}}|)$ and $\theta(\int_{\mathcal{S}} \boldsymbol{u}_< d\boldsymbol{v} - \int_{\mathcal{S}} \tilde{\boldsymbol{u}}_< d\boldsymbol{v}) \leq \epsilon$. Putting these together, we see that $\theta(\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v} - \int_{\mathcal{S}} \boldsymbol{u}_< d\boldsymbol{v}) \leq 2\epsilon$. As ϵ is arbitrary, $\int_{\mathcal{S}} \boldsymbol{u}_< d\boldsymbol{v} = \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$.

641X Basic exercises (a) In 641B, show that $\mathfrak{A}_{<\min\mathcal{T}} = \{0,1\}$ and $\mathfrak{A}_{<\max\mathcal{T}} = \bigvee_{t\in T}\mathfrak{A}_t$.

(i) Give an example of a structure $(\mathfrak{A}, \bar{\mu}, T, \langle\mathfrak{A}_t\rangle_{t\in T}, \mathcal{T}, \langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$, a martingale $\langle u_\sigma\rangle_{\sigma\in\mathcal{T}}$, a $\tau \in \mathcal{T}$ and a finite sublattice I of $\mathcal{T}$ such that $u_{I<\tau}$ is not the conditional expectation of u_τ on the algebra $\mathfrak{A}_{I<\tau}$.

(b) Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$, and $\boldsymbol{v}$ a near-simple process with domain $\mathcal{S}$ and previsible version $\boldsymbol{v}_<$. Show that $\sup|\boldsymbol{v}| = \sup|\boldsymbol{v}_<| \vee |v_\uparrow|$, where $v_\uparrow = \lim_{\sigma\uparrow\mathcal{S}} v_\sigma$.

(h) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a locally moderately oscillatory process with previsible version $\boldsymbol{u}_< = \langle u_{<\sigma}\rangle_{\sigma\in\mathcal{S}}$. Suppose that A is a non-empty upwards-directed subset of $\mathcal{S}$ such that $\tau = \sup A$ belongs to $\mathcal{S}$. Set $a = \inf_{\sigma\in A}[\![\sigma < \tau]\!]$. Show that $a \subseteq [\![u_{<\tau} = \lim_{\sigma\uparrow A} u_\sigma]\!]$.

(c) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v}$, $\boldsymbol{w}$ near-simple processes with domain $\mathcal{S}$ such that $(\boldsymbol{v}-\boldsymbol{v}_<)\times(\boldsymbol{w}-\boldsymbol{w}_<)=\mathbf{0}$. Show that $\mathrm{Osclln}(\boldsymbol{v}+\boldsymbol{w})=\mathrm{Osclln}(\boldsymbol{v})\vee\mathrm{Osclln}(\boldsymbol{w})$.

(d) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v}$ a process of locally bounded variation with domain $\mathcal{S}$ and starting value 0. Let $\boldsymbol{v}^{\uparrow}=\langle\int_{\mathcal{S}\wedge\tau}|d\boldsymbol{v}|\rangle_{\tau\in\mathcal{S}}$ be the cumulative variation of $\boldsymbol{v}$ (614O). Show that $\boldsymbol{v}^{\uparrow}-\boldsymbol{v}^{\uparrow}_<=|\boldsymbol{v}-\boldsymbol{v}_<|$. (*Hint*: 614P(c-ii).)

(e) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v}$, $\boldsymbol{w}$ two local integrators with domain $\mathcal{S}$, at least one of which has starting value 0. Write $\boldsymbol{z}$ for their covariation. Show that $\boldsymbol{z}-\boldsymbol{z}_<=(\boldsymbol{v}-\boldsymbol{v}_<)\times(\boldsymbol{w}-\boldsymbol{w}_<)$.

(f) Show that if T has an element which is isolated on the right, then there are simple processes $\boldsymbol{u}$, $\boldsymbol{v}$ defined on the whole of $\mathcal{T}$ such that $\int_{\mathcal{T}}\boldsymbol{u}_<\,d\boldsymbol{v}\ne\int_{\mathcal{T}}\boldsymbol{u}\,d\boldsymbol{v}$.

(g) Suppose that T has no points isolated on the right. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$, $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a moderately oscillatory process and $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a near-simple integrator. Show that $ii_{\boldsymbol{v}}(\boldsymbol{u})=ii_{\boldsymbol{v}}(\boldsymbol{u}_<)$.

(h) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u}$ a non-decreasing fully adapted process with domain $\mathcal{S}$. Show that $\boldsymbol{u}_<$ is defined everywhere in $\mathcal{S}$.

641Y Further exercises (a) Give an example, in the context of 641B, of a structure $(\mathfrak{A},T,\langle\mathfrak{A}_t\rangle_{t\in T},\mathcal{T})$ with stopping times σ, $\tau\in\mathcal{T}$ such that $\mathfrak{A}_{<(\sigma\wedge\tau)}\ne\mathfrak{A}_{<\sigma}\cap\mathfrak{A}_{<\tau}$.

(b) Suppose that $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$ which is coordinated with $\langle\mathfrak{A}_t\rangle_{t\in T}$ in the sense of 634Fb. Set $\mathfrak{B}_t=\mathfrak{B}\cap\mathfrak{A}_t$ for $t\in T$, and let $\mathcal{T}_{\mathfrak{B}}\subseteq\mathcal{T}$ be the sublattice of stopping times for $(\mathfrak{B},\langle\mathfrak{B}_t\rangle_{t\in T})$, as in 634C. Show that if $\tau\in\mathcal{T}_{\mathfrak{B}}$ then $\mathfrak{B}_{<\tau}$, defined from τ and $\mathcal{T}_{\mathfrak{B}}$ by the formula of 641B, is equal to $\mathfrak{B}\cap\mathfrak{A}_{<\tau}$.

(d)(i) Let $\boldsymbol{u}$ be a simple process defined on a non-empty sublattice of $\mathcal{T}$. Show that $\boldsymbol{u}_<$ is previsibly simple in the sense of 612Ye. (ii) Let $\boldsymbol{u}$ be a previsibly simple process on a sublattice $\mathcal{S}$ of $\mathcal{T}$ defined from sequences $(\tau_0,\dots,\tau_n)$ and $u_*,u_0,\dots,,u_n$ by the formulae of 612Ye. Suppose that $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ is an integrator with domain $\mathcal{S}$ and that *either* $\boldsymbol{v}$ is jump-free *or* $\mathcal{S}$ is order-convex and no member of T is isolated on the right. Show that

$$\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}=u_*\times(v_{\tau_0}-v_{\downarrow})+\sum_{i=0}^{n-1}u_i\times(v_{\tau_{i+1}}-v_{\tau_i})+u_{\tau_n}\times(v_{\uparrow}-v_{\tau_n})$$

where $v_{\downarrow}=\lim_{\sigma\downarrow\mathcal{S}}v_\sigma$ and $v_{\uparrow}=\lim_{\sigma\uparrow\mathcal{S}}v_\sigma$.

641 Notes and comments While $\langle\mathfrak{A}_{<\check{t}}\rangle_{t\in T}$ is a filtration of closed subalgebras of $\mathfrak{A}$, you must not think of $<\tau$, in $\mathfrak{A}_{<\tau}$ and $u_{<\tau}$, as a stopping time. We have to re-examine all our standard formulae, as in 641Ca and 641Ya. The familiar appearance of 641G(a-ii) is not exactly an accident, but it certainly demands a proper check.

The concept of 'previsibility' is awkward at the bottom end. In the definitions in 641E-641F, I have chosen to set $u_{<\tau}=0$ if $\tau=\min\mathrm{dom}\,\boldsymbol{u}$. This will cascade through the clauses '$h(0,\dots,0)=0$' and '$g(0)=0$' in the statements of 641Gd and 642Da, '$=0$ otherwise' in the statement of 642E, '$=0$ if $h(\omega)=0$' in the statements of 642Fb and 642La, and '$=0$ if $t=0$' in the statement of 642K, all of which seem a little arbitrary. (There is also a subformula '$\chi e_\tau\times$' in the statement of 641N which is there for the same reason.) But it seems to me that there are similar difficulties, sometimes acknowledged[1] and sometimes not, in the alternative presentations I have seen. On the ideological ground that a 'previsible' process $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is one in which each value u_τ can be determined from observations of u_σ in regions $[\![\sigma<\tau]\!]$, I have taken the view that in the region $\inf_{\sigma\in\mathcal{S}}[\![\tau\le\sigma]\!]$, where we have no previous information, we must go to a default value, and that the only sensible general default value is 0. If you have any reason to choose another, you

[1] Cf. Rogers & Williams 00, intro. to Chap. IV.

should model this with a process starting at a time less than τ. In the meantime, I ask you to put up with the fact that the previsible version $\mathbf{1}^{(\mathcal{S})}_<$ of a constant process $\mathbf{1}^{(\mathcal{S})}$ might not be $\mathbf{1}^{(\mathcal{S})}$. After all, at the time of the first observation, its constancy is *not* previsible.

Up to this point it has seemed safe to write $\mathbf{1}$ for a constant process with value $\chi 1$ without declaring a domain other than the default $\mathcal{T}$ of 612D(e-i). But if (as will be necessary when we come to define the S-integral in §645) we want to talk about $\mathbf{1}_<$, it will be essential to understand exactly what its domain is. The notation $\mathbf{1}^{(\mathcal{S})}$ is inelegant but, I hope, easily interpreted.

Note that in 641W we have a fundamental difference between $T=\mathbb{N}$ and $T=[0,\infty[$ (see 641Xf). The point here is that when we come to the S-integral we shall have a basic formula

$$\oint_{\mathcal{S}} \boldsymbol{u}_< \, d\boldsymbol{v} = \int_{\mathcal{S}} \boldsymbol{u} \, d\boldsymbol{v}$$

(645R(a-i)), so it is interesting to know when this happens to be equal to the Riemann-sum integral $\int_{\mathcal{S}} \boldsymbol{u}_< \, d\boldsymbol{v}$.

In 641Q-641R I return to a theme from §617. The formulae

$$d(ii_{\boldsymbol{v}}(\boldsymbol{u})) \sim \boldsymbol{u}\, d\boldsymbol{v}, \quad d\boldsymbol{v}^* \sim (d\boldsymbol{v})^2, \quad d[\boldsymbol{v} \overset{*}{|} \boldsymbol{w}] \sim d\boldsymbol{v}\, d\boldsymbol{w}$$

reappear, transformed, in formulae for the jumps of $ii_{\boldsymbol{v}}(\boldsymbol{u})$ (641Q), $\boldsymbol{v}^*$ (641R) and $[\boldsymbol{v} \overset{*}{|} \boldsymbol{w}]$ (641Xe). Similarly, $d\boldsymbol{v}^{\uparrow} \sim |d\boldsymbol{v}|$ (616T) turns into the formula of 641Xd, a refinement of 618U.

In 641U-641W I look at some results which depend on particular assumptions concerning the totally ordered set T of the stochastic integration structure and the domain of the processes under consideration. While in a sense these are restrictive, they are in fact satisfied by all the leading examples of the theory once we have left the case $T=\mathbb{N}$. It is fair to consider them as the normal case. Elsewhere (641Nb, 641P, 641W) I call on processes to be near-simple. This is indeed a restriction (for instance, previsible versions are often not near-simple). Its necessity is clear in 641P; the previsible version can tell us only about jumps as we approach a limit from below, not as we approach from above.

Version of 1.7.24

642 Previsible processes

I continue the work of §641 with a description of the previsible version of a process defined in the standard way from a probability space and a filtration of σ-algebras (642E-642G). The other objective of the section is to make a step towards a general theory of 'previsible' processes. The point is that among such processes, starting with those of the form $\boldsymbol{u}_<$, a form of sequential convergence (the order*-convergence of 642B) has striking connections with stochastic integration. I will come to this in §644. For the moment, I present a definition of the space M_{pv} of previsible processes, with some of its elementary properties (642D) and a description in terms of suitably measurable processes in the case in which $T=[0,\infty[$ (642L).

642A Notation $(\mathfrak{A},\bar\mu,T,\langle\mathfrak{A}_t\rangle_{t\in T},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ will be a stochastic integration structure. θ will be the standard functional defining the topology of convergence in measure on $L^0(\mathfrak{A})$ (613B). If $t\in T$ then $\check t$ will be the constant stopping time at t. If $\mathcal{S}$ is a sublattice of $\mathcal{T}$, $\mathcal{I}(\mathcal{S})$ will be the upwards-directed set of finite sublattices of $\mathcal{S}$, and $\mathcal{S}\wedge\tau$ will be the lattice $\{\sigma\wedge\tau:\sigma\in\mathcal{S}\}$ for every $\tau\in\mathcal{S}$. $M_{\text{fa}}(\mathcal{S})$, $M_{\text{simp}}(\mathcal{S})$, $M_{\text{o-b}}(\mathcal{S})$, $M_{\text{n-s}}(\mathcal{S})$ and $M_{\text{mo}}(\mathcal{S})$ will be the spaces of fully adapted processes, simple processes, order-bounded processes, near-simple processes and moderately oscillatory processes with domain $\mathcal{S}$. If $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is an order-bounded process (614E), $\sup|\boldsymbol{u}|$ will be the supremum $\sup_{\sigma\in\mathcal{S}}|u_\sigma|$ in $L^0(\mathfrak{A})$, and $\operatorname{Osclln}(\boldsymbol{u})$ will be the residual oscillation of $\boldsymbol{u}$ (618B).

642B Order*-convergence in L^0 and $(L^0)^{\mathcal{S}}$ (a) In 367A, I gave a definition of order*-convergent sequence in arbitrary lattices which is more elaborate than we need here. For our present purposes, it will be enough to know that, because $\mathfrak{A}$ is a Dedekind complete Boolean algebra, a sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in $L^0(\mathfrak{A})$ is order*-convergent to $u\in L^0(\mathfrak{A})$ iff it is order-bounded and

$$u=\inf_{n\in\mathbb{N}}\sup_{i\ge n}u_i=\sup_{n\in\mathbb{N}}\inf_{i\ge n}u_i$$

(367Gb). Another way of expressing this is to say that $\langle u_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to u iff there are a non-decreasing sequence $\langle v_n\rangle_{n\in\mathbb{N}}$ and a non-increasing sequence $\langle w_n\rangle_{n\in\mathbb{N}}$ such that $\sup_{n\in\mathbb{N}} v_n = \inf_{n\in\mathbb{N}} w_n = u$ and $v_n \le u_n \le w_n$ for every n. In this case, u is the limit of $\langle u_n\rangle_{n\in\mathbb{N}}$ for the topology of convergence in measure (367Pa).

Note that $\langle u_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to u iff $\langle u_n - u\rangle_{n\in\mathbb{N}}$ is order*-convergent to 0 (367Cd), and that if $\langle u_n\rangle_{n\in\mathbb{N}}$ is order-convergent to 0 and $|v_n| \le |u_n|$ for every n, then $\langle v_n\rangle_{n\in\mathbb{N}}$ is order-convergent to 0 (367Cc, 367Bd).

If $\mathfrak{A}$ is expressed as the measure algebra of a probability space (Ω,Σ,μ), and each u_n, u is represented as $f_n^\bullet$, $f^\bullet$ where the f_n, f are measurable real-valued functions defined on Ω, then $\langle u_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to u iff $f(\omega) = \lim_{n\to\infty} f_n(\omega)$ for almost every ω (367F). Hence, or otherwise, we see that if $\langle u_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to u and $\langle v_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to v, then $\langle u_n + v_n\rangle_{n\in\mathbb{N}}$, $\langle \bar g(u_n)\rangle_{n\in\mathbb{N}}$ and $\langle u_n \times v_n\rangle_{n\in\mathbb{N}}$ are order*-convergent to $u+v$, $\bar g(u)$ and $u\times v$ respectively, for any continuous $g:\mathbb{R}\to\mathbb{R}$. More generally, if $\langle u_{in}\rangle_{n\in\mathbb{N}}$ is order*-convergent to u_i for $1\le i\le k$, and $g:\mathbb{R}^k\to\mathbb{R}$ is continuous, then $\langle \bar g(u_{1n},\dots,u_{kn})\rangle_{n\in\mathbb{N}}$, defined as in 619Eb, is order*-convergent to $\bar g(u_1,\dots,u_k)$.

(b) Now suppose that we have a sublattice $\mathcal{S}$ of $\mathcal{T}$. Then $L^0(\mathfrak{A})^{\mathcal{S}}$ is isomorphic, as f-algebra, to $L^0(\mathfrak{A}^{\mathcal{S}})$, where $\mathfrak{A}^{\mathcal{S}}$ is the simple product (315A, 364R) and is itself a Dedekind complete Boolean algebra (315D(e-i)). So we shall be able to use the same formula for sequences of processes: if $\mathcal{S}$ is a sublattice of $\mathcal{T}$, $\langle \boldsymbol{u}_n\rangle_{n\in\mathbb{N}} = \langle\langle u_{n\sigma}\rangle_{\sigma\in\mathcal{S}}\rangle_{n\in\mathbb{N}}$ is a sequence of processes with domain $\mathcal{S}$, and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is another process with domain $\mathcal{S}$, then

$$\begin{aligned}
&\langle \boldsymbol{u}_n\rangle_{n\in\mathbb{N}} \text{ is order*-convergent to } \boldsymbol{u}\\
&\qquad\iff \langle \boldsymbol{u}_n\rangle_{n\in\mathbb{N}} \text{ is order-bounded and } \boldsymbol{u} = \inf_{n\in\mathbb{N}}\sup_{i\ge n}\boldsymbol{u}_i = \sup_{n\in\mathbb{N}}\inf_{i\ge n}\boldsymbol{u}_i\\
&\qquad\iff \langle u_{n\sigma}\rangle_{n\in\mathbb{N}} \text{ is order-bounded and } u_\sigma = \inf_{n\in\mathbb{N}}\sup_{i\ge n}u_{i\sigma} = \sup_{n\in\mathbb{N}}\inf_{i\ge n}u_{i\sigma}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for every } \sigma\in\mathcal{S}\\
&\qquad\iff \langle u_{n\sigma}\rangle_{n\in\mathbb{N}} \text{ is order*-convergent to } u_\sigma \text{ for every } \sigma\in\mathcal{S}.
\end{aligned}$$

Note that if we have an order*-convergent sequence of fully adapted processes, the limit will also be fully adapted. **P** Suppose that $\langle \boldsymbol{u}_n\rangle_{n\in\mathbb{N}} = \langle\langle u_{n\sigma}\rangle_{\sigma\in\mathcal{S}}\rangle_{n\in\mathbb{N}}$ is a sequence of fully adapted processes order*-convergent to $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$. For each $\sigma\in\mathcal{S}$, $\langle u_{n\sigma}\rangle_{n\in\mathbb{N}}$ is a sequence in $L^0(\mathfrak{A}_\sigma)$ which is order*-convergent, therefore topologically convergent, to u_σ, and $u_\sigma\in L^0(\mathfrak{A}_\sigma)$ because $L^0(\mathfrak{A}_\sigma)$ is topologically closed. If σ, $\tau\in\mathcal{S}$ and $[\![\sigma=\tau]\!] = a$, then

$$u_\sigma\times\chi a = \lim_{n\to\infty} u_{n\sigma}\times\chi a = \lim_{n\to\infty} u_{n\tau}\times\chi a = u_\tau\times\chi a,$$

so $a\subseteq[\![u_\sigma = u_\tau]\!]$. Thus $\boldsymbol{u}$ is fully adapted. **Q**

(c) A topologically convergent sequence need not be order*-convergent (245Cc), but if $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in L^0 such that $\sum_{n=0}^\infty \theta(u_n)$ is finite, then $\langle u_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to 0. **P** I check first that $\{|u_n| : n\in\mathbb{N}\}$ is bounded above. If $\epsilon>0$, there is an $n\in\mathbb{N}$ such that

$$\sum_{m=n}^\infty \bar\mu[\![|u_m|\ge 1]\!] \le \sum_{m=n}^\infty \theta(u_m) \le \epsilon;$$

now there is a $k\ge 1$ such that $\sum_{m=0}^{n-1}\bar\mu[\![|u_m|>k]\!]\le\epsilon$, so that

$$\bar\mu(\sup_{m\in\mathbb{N}}[\![|u_m|>k]\!]) \le \sum_{m=0}^{n-1}\bar\mu[\![|u_m|>k]\!] + \sum_{m=n}^\infty\bar\mu[\![|u_m|\ge 1]\!] \le 2\epsilon.$$

As ϵ is arbitrary,

$$\bar\mu(\inf_{\alpha\in\mathbb{R}}\sup_{m\in\mathbb{N}}[\![|u_m|>\alpha]\!]) = \lim_{\alpha\to\infty}\bar\mu(\sup_{m\in\mathbb{N}}[\![|u_m|>\alpha]\!]) = 0$$

and $\langle |u_n|\rangle_{n\in\mathbb{N}}$ is bounded above, by the criterion in 364L(a-ii).

Accordingly we may speak of $\sup_{m\ge n}|u_m|$ for each n. Given $\epsilon>0$, let $n\in\mathbb{N}$ be such that $\sum_{m=n}^\infty\theta(u_m)\le\epsilon^2$; then

$$\bar\mu[\![\sup_{m\ge n}|u_m|>\epsilon]\!] \le \sum_{m=n}^\infty\bar\mu[\![|u_m|>\epsilon]\!] \le \frac{1}{\epsilon}\sum_{m=n}^\infty\theta(u_m)\le\epsilon,$$

so $\theta(\sup_{m\ge n}|u_m|)\le 2\epsilon$. As ϵ is arbitrary, $\inf_{n\in\mathbb{N}}\sup_{m\ge n}|u_m|=0$ and $\langle u_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to 0. **Q**

(d) The description of order*-convergence in L^0 in terms of pointwise convergence in $\mathcal{L}^0$, given in (a) above, makes it easy to see that if $\langle h_n\rangle_{n\in\mathbb{N}}$ is a sequence of Borel measurable functions from $\mathbb{R}^k$ to $\mathbb{R}$, and $h(x)=\lim_{n\to\infty}h_n(x)$ is defined in $\mathbb{R}$ for every $x\in\mathbb{R}^k$, then $\langle\bar h_n(u_1,\dots,u_k)\rangle_{n\in\mathbb{N}}$ is order*-convergent to $\bar h(u_1,\dots,u_k)$ whenever $u_1,\dots,u_k\in L^0$. (See 619Ee-619Ef.)

(e) Note that if $\langle x_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $L^0(\mathfrak{A})$ then it is order*-convergent iff it is order-bounded, and its order*-limit is then $\sup_{n\in\mathbb{N}}x_n$, which is also its topological limit. At the same time, if $\langle x_n\rangle_{n\in\mathbb{N}}$ is topologically convergent, then its topological limit is an upper bound of $\{x_n:n\in\mathbb{N}\}$, so is again the order*-limit.

We are now ready for the next definition.

642C Definition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. I will say that a process $\boldsymbol{x}$ with domain $\mathcal{S}$ is **previsible** if it belongs to the smallest subset of $(L^0)^{\mathcal{S}}$ which contains $\boldsymbol{u}_<$ for every simple process $\boldsymbol{u}$ and is closed under order*-convergence of sequences in $(L^0)^{\mathcal{S}}$.

642D Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $M_{\text{pv}}(\mathcal{S})$ the space of previsible processes with domain $\mathcal{S}$.

(a) $M_{\text{pv}}(\mathcal{S})$ is an f-subalgebra of $M_{\text{fa}}(\mathcal{S})$, and $\bar g\boldsymbol{u}\in M_{\text{pv}}(\mathcal{S})$ whenever $\boldsymbol{u}\in M_{\text{pv}}(\mathcal{S})$ and $g:\mathbb{R}\to\mathbb{R}$ is a continuous function such that $g(0)=0$.

(b) $M_{\text{pv}}(\mathcal{S})\cap M_{\text{o-b}}(\mathcal{S})$ is closed in $M_{\text{o-b}}(\mathcal{S})$ for the ucp topology. Consequently $\boldsymbol{u}_<\in M_{\text{pv}}(\mathcal{S})$ for every $\boldsymbol{u}\in M_{\text{n-s}}(\mathcal{S})$.

(c) If $\tau\in\mathcal{S}$, then $M_{\text{pv}}(\mathcal{S}\wedge\tau)=\{\boldsymbol{x}\restriction\mathcal{S}\wedge\tau:\boldsymbol{x}\in M_{\text{pv}}(\mathcal{S})\}$.

proof (a) Because $\boldsymbol{u}_<$ is defined and fully adapted for every simple process $\boldsymbol{u}$ with domain $\mathcal{S}$ (641I, 641G(a-ii)), and the limit of an order*-convergent sequence in $M_{\text{fa}}(\mathcal{S})$ belongs to $M_{\text{fa}}(\mathcal{S})$ (642Bb), $M_{\text{pv}}(\mathcal{S})\subseteq M_{\text{fa}}(\mathcal{S})$. If $g:\mathbb{R}\to\mathbb{R}$ is continuous and $g(0)=0$, then $\{\boldsymbol{x}:\boldsymbol{x}\in M_{\text{pv}}(\mathcal{S}),\ \bar g\boldsymbol{x}\in M_{\text{pv}}(\mathcal{S})\}$ is closed under order*-convergence, by the last remark in 642Ba, and contains $\boldsymbol{u}_<$ for every $\boldsymbol{u}\in M_{\text{simp}}(\mathcal{S})$, by 612La and 641Gd, so must be the whole of $M_{\text{pv}}(\mathcal{S})$. Similarly, if $\boldsymbol{u}\in M_{\text{simp}}(\mathcal{S})$, $\{\boldsymbol{x}:\boldsymbol{x}\in M_{\text{pv}}(\mathcal{S}),\ \boldsymbol{u}_<+\boldsymbol{x}\in M_{\text{pv}}(\mathcal{S})\}=M_{\text{pv}}(\mathcal{S})$; so if $\boldsymbol{x}\in M_{\text{pv}}(\mathcal{S})$, $\{\boldsymbol{u}:\boldsymbol{u}\in M_{\text{pv}}(\mathcal{S}),\ \boldsymbol{u}+\boldsymbol{x}\in M_{\text{pv}}(\mathcal{S})\}=M_{\text{pv}}(\mathcal{S})$, and $M_{\text{pv}}(\mathcal{S})$ is closed under addition. Putting these together, $M_{\text{pv}}(\mathcal{S})$ is an f-subalgebra of $M_{\text{fa}}(\mathcal{S})$.

(b)(i) Suppose that $\boldsymbol{x}=\langle x_\sigma\rangle_{\sigma\in\mathcal{S}}$ belongs to the closure of $M_{\text{pv}}(\mathcal{S})\cap M_{\text{o-b}}(\mathcal{S})$ in $M_{\text{o-b}}(\mathcal{S})$ for the ucp topology. Then for each $n\in\mathbb{N}$ there is a $\boldsymbol{x}_n\in M_{\text{pv}}(\mathcal{S})$ such that $\theta(\bar x_n)\le 2^{-n}$, where $\bar x_n=\sup|\boldsymbol{x}_n-\boldsymbol{x}|$. Now $\langle\boldsymbol{x}_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to $\boldsymbol{x}$. **P** Express each $\boldsymbol{x}_n$ as $\langle x_{n\sigma}\rangle_{\sigma\in\mathcal{S}}$. Setting $w_n=\sum_{i=0}^n\bar x_i$ for $n\in\mathbb{N}$, $\langle w_n\rangle_{n\in\mathbb{N}}$ is Cauchy for the linear space topology of convergence in measure, so is convergent to w say; $w\ge w_n$ for every n and $\langle w-w_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence with infimum 0. For any $\sigma\in\mathcal{S}$, we have $|x_{n\sigma}-x_\sigma|\le w_n$ for every n, so

$$\inf_{n\in\mathbb{N}}\sup_{i\ge n}x_{i\sigma}\le\inf_{n\in\mathbb{N}}\sup_{i\ge n}w_i+x_\sigma=x_\sigma$$

and similarly

$$\sup_{n\in\mathbb{N}}\inf_{i\ge n}x_{i\sigma}\ge x_\sigma;$$

of course

$$\sup_{n\in\mathbb{N}}\inf_{i\ge n}x_{i\sigma}\le\inf_{n\in\mathbb{N}}\sup_{i\ge n}x_{i\sigma},$$

so $\langle x_{n\sigma}\rangle_{n\in\mathbb{N}}$ is order*-convergent to x_σ. **Q**

Thus $\boldsymbol{x}\in M_{\text{pv}}(\mathcal{S})$. As $\boldsymbol{x}$ is arbitrary, $M_{\text{pv}}(\mathcal{S})\cap M_{\text{o-b}}(\mathcal{S})$ is ucp-closed.

(ii) Since $\boldsymbol{u}\mapsto\boldsymbol{u}_<:M_{\text{mo}}(\mathcal{S})\to M_{\text{o-b}}(\mathcal{S})$ is continuous (641G(e-ii)), $\{\boldsymbol{u}:\boldsymbol{u}\in M_{\text{mo}}(\mathcal{S}),\ \boldsymbol{u}_<\in M_{\text{pv}}(\mathcal{S})\}$ is relatively ucp-closed in $M_{\text{mo}}(\mathcal{S})$; as it certainly includes $M_{\text{simp}}(\mathcal{S})$, it must include the relative closure $\overline{M_{\text{simp}}(\mathcal{S})}\cap M_{\text{mo}}(\mathcal{S})=M_{\text{n-s}}(\mathcal{S})$ (631Ba, 631Ca).

(c)(i) If $\boldsymbol{v}\in M_{\text{simp}}(\mathcal{S})$ then $\boldsymbol{v}\restriction\mathcal{S}\wedge\tau\in M_{\text{simp}}(\mathcal{S}\wedge\tau)$ (612K(d-ii)) and $\boldsymbol{v}_<\restriction\mathcal{S}\wedge\tau=(\boldsymbol{v}\restriction\mathcal{S}\wedge\tau)_<$ (641G(c-ii)) belongs to $M_{\text{pv}}(\mathcal{S}\wedge\tau)$. Now

$$\{\boldsymbol{x}:\boldsymbol{x}\in M_{\mathrm{pv}}(\mathcal{S}),\,\boldsymbol{x}\restriction\mathcal{S}\in M_{\mathrm{pv}}(\mathcal{S}\wedge\tau)\}$$

is closed under order*-convergence; since it contains $\boldsymbol{v}_<$ for every $\boldsymbol{v}\in M_{\mathrm{simp}}(\mathcal{S})$, it is the whole of $M_{\mathrm{pv}}(\mathcal{S})$, that is, $\boldsymbol{x}\restriction\mathcal{S}\wedge\tau\in M_{\mathrm{pv}}(\mathcal{S}\wedge\tau)$ for every $\boldsymbol{x}\in M_{\mathrm{pv}}(\mathcal{S})$.

(ii)($\boldsymbol{\alpha}$) For $\boldsymbol{u}\in M_{\mathrm{fa}}(\mathcal{S}\wedge\tau)$, define $\tilde{\boldsymbol{u}}=\langle\tilde u_\sigma\rangle_{\sigma\in\mathcal{S}}$ by saying that $\tilde u_\sigma=u_{\sigma\wedge\tau}\times\chi[\![\sigma\le\tau]\!]$ for every $\sigma\in\mathcal{S}$. Then $\tilde{\boldsymbol{u}}$ is fully adapted. **P** As $[\![\sigma\le\tau]\!]\in\mathfrak{A}_\sigma$ (611H(c-i)) and $u_{\sigma\wedge\tau}\in L^0(\mathfrak{A}_{\sigma\wedge\tau})\subseteq L^0(\mathfrak{A}_\sigma)$, $\tilde u_\sigma\in L^0(\mathfrak{A}_\sigma)$ for every $\sigma\in\mathcal{S}$. If σ, $\sigma'\in\mathcal{S}$ and $a=[\![\sigma=\sigma']\!]$, then $a\subseteq[\![\sigma\wedge\tau=\sigma'\wedge\tau]\!]$ and $a\cap[\![\sigma\le\tau]\!]=a\cap[\![\sigma'\le\tau]\!]$ (611E(c-v-α, 611E(c-iv-α))) so

$$\tilde u_\sigma\times\chi a=u_{\sigma\wedge\tau}\times\chi([\![\sigma\le\tau]\!]\cap[\![\sigma=\sigma']\!])=u_{\sigma'\wedge\tau}\times\chi([\![\sigma'\le\tau]\!]\cap[\![\sigma=\sigma']\!])=\tilde u_{\sigma'}\times\chi a$$

and $a\subseteq[\![\tilde u_\sigma=\tilde u_{\sigma'}]\!]$. **Q**

Of course $\tilde{\boldsymbol{u}}\restriction\mathcal{S}\wedge\tau=\boldsymbol{u}$.

($\boldsymbol{\beta}$) If $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}\wedge\tau}$ belongs to $M_{\mathrm{simp}}(\mathcal{S}\wedge\tau)$, there is a $\boldsymbol{w}\in M_{\mathrm{simp}}(\mathcal{S})$ such that $\boldsymbol{w}_<=(\boldsymbol{v}_<)^\sim$. **P** There is certainly a $\boldsymbol{w}'\in M_{\mathrm{simp}}(\mathcal{S})$ extending $\boldsymbol{v}$ (631Ma, or otherwise). Now define $\boldsymbol{w}''=\langle w''_\sigma\rangle_{\sigma\in\mathcal{S}}\in M_{\mathrm{simp}}(\mathcal{S})$ by saying that

$$[\![\sigma<\tau]\!]\subseteq[\![w''_\sigma=\chi 1]\!],\quad[\![\tau\le\sigma]\!]\subseteq[\![w''_\sigma=0]\!]$$

for every $\sigma\in\mathcal{S}$ (612Ka), and set $\boldsymbol{w}=\boldsymbol{w}'\times\boldsymbol{w}''$. Expressing $\boldsymbol{w}$ as $\langle w_\sigma\rangle_{\sigma\in\mathcal{S}}$,

$$[\![\sigma<\tau]\!]\subseteq[\![w_\sigma=v_\sigma]\!],\quad[\![\tau\le\sigma]\!]\subseteq[\![w_\sigma=0]\!]$$

for every $\sigma\in\mathcal{S}$.

If $\sigma\le\tau$, then $[\![\sigma'<\sigma]\!]\subseteq[\![w_{\sigma'}=v_{\sigma'}]\!]$ for every σ', so, in the language of 641E, $w_{I<\sigma}=v_{I<\sigma}$ for every $I\in\mathcal{I}(\mathcal{S}\wedge\sigma)$ and $w_{<\sigma}=v_{<\sigma}$. Thus $\boldsymbol{w}_<\restriction\mathcal{S}\wedge\tau=\boldsymbol{v}_<$. For general $\sigma\in\mathcal{S}$, if $\sigma\wedge\tau\in I\in\mathcal{I}(\mathcal{S}\wedge\sigma)$,

$$[\![\sigma'<\sigma]\!]\setminus([\![\sigma'<\sigma\wedge\tau]\!]\cap[\![\sigma\wedge\tau<\sigma]\!])\subseteq[\![\tau\le\sigma']\!]\cup[\![\sigma\le\tau]\!]\subseteq[\![w_{\sigma'}=0]\!]\cup[\![\sigma\le\tau]\!]$$

for every $\sigma'\in I$, so $w_{I<\sigma}\times\chi[\![\tau<\sigma]\!]=0$. Taking the limit as $I\uparrow\mathcal{I}(\mathcal{S}\wedge\sigma)$, $w_{<\sigma}\times\chi[\![\tau<\sigma]\!]=0$, so

$$w_{<\sigma}=w_{<\sigma}\times\chi[\![\sigma\le\tau]\!]=w_{<(\sigma\wedge\tau)}\times\chi[\![\sigma\le\tau]\!]$$

(because $\boldsymbol{w}_<$ is fully adapted)

$$=v_{<(\sigma\wedge\tau)}\times\chi[\![\sigma\le\tau]\!].$$

Thus $\boldsymbol{w}_<=(\boldsymbol{v}_<)^\sim$, as required. **Q**

($\boldsymbol{\gamma}$) If $\langle\boldsymbol{u}_n\rangle_{n\in\mathbb{N}}$ is a sequence in $M_{\mathrm{fa}}(\mathcal{S}\wedge\tau)$ which is order*-convergent to $\boldsymbol{u}\in M_{\mathrm{fa}}(\mathcal{S}\wedge\tau)$, then $\langle\tilde{\boldsymbol{u}}_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to $\tilde{\boldsymbol{u}}$. **P** Expressing $\boldsymbol{u}_n$ as $\langle u_{n\sigma}\rangle_{\sigma\in\mathcal{S}\wedge\tau}$, etc., we have

$$\inf_{n\in\mathbb{N}}\sup_{i\ge n}\tilde u_{i\sigma}=\inf_{n\in\mathbb{N}}\sup_{i\ge n}u_{i,\sigma\wedge\tau}\times\chi[\![\sigma\le\tau]\!]=u_{\sigma\wedge\tau}\times\chi[\![\sigma\le\tau]\!]=\tilde u_\sigma$$

and similarly

$$\sup_{n\in\mathbb{N}}\inf_{i\ge n}\tilde u_{i\sigma}=\tilde u_\sigma$$

for every $\sigma\in\mathcal{S}$. **Q**

($\boldsymbol{\delta}$) So

$$\{\boldsymbol{x}:\boldsymbol{x}\in M_{\mathrm{pv}}(\mathcal{S}\wedge\tau),\,\tilde{\boldsymbol{x}}\in M_{\mathrm{pv}}(\mathcal{S})\}$$

is closed under order*-convergence and contains $\boldsymbol{v}_<$ for every $\boldsymbol{v}\in M_{\mathrm{simp}}(\mathcal{S}\wedge\tau)$, and is the whole of $M_{\mathrm{pv}}(\mathcal{S}\wedge\tau)$. So

$$M_{\mathrm{pv}}(\mathcal{S}\wedge\tau)=\{\tilde{\boldsymbol{x}}\restriction\mathcal{S}\wedge\tau:\boldsymbol{x}\in M_{\mathrm{pv}}(\mathcal{S}\wedge\tau)\}\subseteq\{\boldsymbol{x}\restriction\mathcal{S}\wedge\tau:\boldsymbol{x}\in M_{\mathrm{pv}}(\mathcal{S})\}.$$

With (i), this shows that we have equality.

642E Previsible versions in the standard model of near-simple processes I had better work through the connections between the constructions of 641B and 641E and the ideas on processes with càdlàg sample paths from which they were derived.

Proposition Let (Ω,Σ,μ) be a complete probability space and $\langle\Sigma_t\rangle_{t\in[0,\infty[}$ a right-continuous filtration of σ-subalgebras of Σ all containing every negligible subset of Ω. Suppose that we are given a family $\langle U_t\rangle_{t\ge0}$ of real-valued functions on Ω such that U_t is Σ_t-measurable for every t and $t\mapsto U_t(\omega):[0,\infty[\,\to\mathbb{R}$ is càdlàg for every $\omega\in\Omega$. Let $h:\Omega\to[0,\infty[$ be a stopping time, and Σ_{h^-} the σ-subalgebra of Σ generated by $\{E:$ there is a $t\in[0,\infty[$ such that $E\in\Sigma_t$ and $h(\omega)>t$ for every $\omega\in E\}$; define $U_{h^-}:\Omega\to\mathbb{R}$ by setting

$$\begin{aligned}U_{h^-}(\omega)&=\lim_{t\uparrow h(\omega)}U_t(\omega)\text{ if }h(\omega)>0,\\&=0\text{ otherwise.}\end{aligned}$$

Suppose that $(\mathfrak{A},\bar\mu,\langle\mathfrak{A}_t\rangle_{t\in[0,\infty[})$ and $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{T}_f}$ are defined from $(\Omega,\Sigma,\mu,\langle\Sigma_t\rangle_{t\in[0,\infty[})$ and $\langle U_t\rangle_{t\in[0,\infty[}$ as in 612H. Let τ be the stopping time represented by h (612Ha), and $\boldsymbol{u}_<=\langle u_{<\sigma}\rangle_{\sigma\in\mathcal{T}_f}$ the previsible version of $\boldsymbol{u}$. Then

(a) $\mathfrak{A}_{<\tau}=\{E^\bullet:E\in\Sigma_{h^-}\}$,

(b) $u_{<\tau}=U^\bullet_{h^-}$ in $L^0(\mathfrak{A})$.

proof Note that $\boldsymbol{u}$ is locally near-simple, by 631D, so we can speak of its previsible variation.

(a) This is a direct translation of the description of $\mathfrak{A}_{<\tau}$ in 641Be.

(b)(i) For $n\in\mathbb{N}$, define $f_n:\Omega\to\mathbb{R}$ by setting

$$\begin{aligned}f_n(\omega)&=0\text{ if }h(\omega)=0,\\&=U_{2^{-n}k}(\omega)\text{ if }k<4^n\text{ and }2^{-n}k<h(\omega)\le2^{-n}(k+1),\\&=U_{2^n}(\omega)\text{ if }2^n<h(\omega).\end{aligned}$$

Then $U_{h^-}(\omega)=\lim_{n\to\infty}f_n(\omega)$ for every ω.

(ii) In fact we know more than just that $U_{h^-}=\lim_{n\to\infty}f_n$. Given $\epsilon>0$, there is for each $\omega\in\Omega$ an $n\in\mathbb{N}$ such that $|U_{h^-}(\omega)-U_t(\omega)|\le\epsilon$ whenever $\max(0,h(\omega)-2^{-n})\le t<h(\omega)$. So there is an n such that $\mu F\ge1-\epsilon$, where

$$\begin{aligned}F&=\{\omega:h(\omega)\le2^n\text{ and }|U_{h^-}(\omega)-U_t(\omega)|\le\epsilon\\&\qquad\qquad\text{whenever }t\in\mathbb{Q}\cap\bigl[\max(0,h(\omega)-2^{-n},h(\omega))\bigr[\}\\&=\{\omega:h(\omega)\le2^n\text{ and }|U_{h^-}(\omega)-U_t(\omega)|\le\epsilon\\&\qquad\qquad\text{whenever }\max(0,h(\omega)-2^{-n})\le t<h(\omega)\}.\end{aligned}$$

Now suppose that $I\in\mathcal{I}(\mathcal{T}_f)$ includes the finite sublattice $I_n=\{(2^{-n}k)^\vee:k\le4^n\}$ of constant stopping times. Then there is a sequence $(\sigma_0,\dots,\sigma_m)$, linearly generating the I-cells, such that the totally ordered set I_n is included in $\{\sigma_j:j\le m\}$. Take stopping times $g_0,\dots,g_m:\Omega\to[0,\infty[$ representing $\sigma_0,\dots,\sigma_m$ and adjusted so that $g_j\le g_{j+1}$ for every j and whenever $\sigma_j=\check t$ then $g_j(\omega)=t$ for every ω. Observe that $\{g_j(\omega):j\le m\}\supseteq\{2^{-n}k:k\le4^n\}$ for every ω. We can now calculate $u_{I<\tau}$ from 641Eb, and find that $u_{I<\tau}=\tilde f^\bullet$, where

$$\begin{aligned}\tilde f(\omega)&=0\text{ if }h(\omega)=0,\\&=U_{g_j}(\omega)\text{ if }j<m\text{ and }g_j(\omega)<h(\omega)\le g_{j+1}(\omega),\\&=U_{g_m}(\omega)\text{ if }g_m(\omega)<h(\omega).\end{aligned}$$

If $\omega\in F$, then either $h(\omega)=0$ and $\tilde f(\omega)=0$, or there is a $k<4^n$ such that $2^{-n}k<h(\omega)$ and $|U_{h^-}(\omega)-U_t(\omega)|\le\epsilon$ whenever $2^{-n}k\le t<h(\omega)$. In the latter case, let $j\le m$ be maximal such that $g_j(\omega)<h(\omega)$. Then $2^{-n}k\le g_j(\omega)$ and

$$|\tilde f(\omega)-U_{h^-}(\omega)|=|U_{g_j}(\omega)-U_{h^-}(\omega)|=|U_{g_j(\omega)}(\omega)-U_{h^-}(\omega)|\le\epsilon.$$

So we see that $F\subseteq\{\omega:|\tilde f(\omega)-U_{h^-}(\omega)|\le\epsilon\}$ and

$$\theta(u_{I<\tau}-U^\bullet_{h^-})=\theta((\tilde f-U_{h^-})^\bullet)\le\epsilon+\mu(\Omega\setminus F)\le2\epsilon.$$

And this is true whenever $I_n \subseteq I \in \mathcal{I}(\mathcal{T}_f)$.

(iii) As ϵ is arbitrary,

$$U^\bullet_{h^-} = \lim_{I\uparrow\mathcal{I}(\mathcal{T}_f)} u_{I<\tau} = u_{<\tau},$$

as claimed.

642F I spell out what amounts to a special case (though here we do not need a right-continuous filtration) so as to be able to quote it later.

Corollary Suppose that (Ω,Σ,μ) is a complete probability space, $\langle\Sigma_t\rangle_{t\ge0}$ a filtration of σ-subalgebras of Σ such that every μ-negligible set belongs to Σ_0, and $(\mathfrak{A},\bar\mu,[0,\infty[\,,\langle\mathfrak{A}_t\rangle_{t\ge0},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ the associated real-time stochastic integration structure.

(a) Suppose that $\mathcal{S}$ is a sublattice of $\mathcal{T}_f$ containing $\check 0$ and that $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a simple process with breakpoint string $(\tau_0,\dots,\tau_n)$ in $\mathcal{S}$ starting from $\tau_0 = \check 0$. Suppose that $h_0,\dots,h_n:\Omega\to[0,\infty[$ are stopping times representing $\tau_0,\dots,\tau_n$ respectively, starting from $h_0(\omega)=0$ for every ω, and such that $h_0\le\dots\le h_n$. For $i\le n$, let $f_i:\Omega\to\mathbb{R}$ be a measurable function representing $u_{\tau_i}\in L^0(\mathfrak{A})$. If $h:\Omega\to[0,\infty[$ is any stopping time representing a member σ of $\mathcal{S}$, and we set

$$\begin{aligned} f(\omega) &= f_i(\omega) \text{ if } i<n \text{ and } h_i(\omega)\le h(\omega)<h_{i+1}(\omega),\\ &= f_n(\omega) \text{ if } h_n(\omega)\le h(\omega), \end{aligned}$$

then $f^\bullet = u_\sigma$ in $L^0(\mathfrak{A})$.

(b) Now suppose that $\boldsymbol{u}_< = \langle u_{<\sigma}\rangle_{\sigma\in\mathcal{S}}$ is the previsible version of $\boldsymbol{u}$. If $h:\Omega\to[0,\infty[$ is any stopping time representing a member σ of $\mathcal{S}$, and we set

$$\begin{aligned} f_-(\omega) &= 0 \text{ if } h(\omega)=0,\\ &= f_i(\omega) \text{ if } i<n \text{ and } h_i(\omega)<h(\omega)\le h_{i+1}(\omega),\\ &= f_n(\omega) \text{ if } h_n(\omega)<h(\omega), \end{aligned}$$

then $f_-^\bullet = u_{<\sigma}$ in $L^0(\mathfrak{A})$.

proof (a) I had better check that f is measurable; this is because all the sets $\{\omega : h_i(\omega)<h(\omega)\}$ are measurable, as is $\{\omega : h(\omega)=0\}$. Now we have

$$[\![f^\bullet = u_\sigma]\!] \supseteq \sup_{i\le n}[\![f^\bullet = f_i^\bullet]\!]\cap[\![u_{\tau_i}=u_\sigma]\!]$$

(because $f_i^\bullet = u_{\tau_i}$ for each i)

$$\supseteq \sup_{i\le n}\{\omega : f(\omega)=f_i(\omega)\}^\bullet\cap[\![\tau_i\le\sigma]\!]\cap[\![\sigma<\tau_{i+1}]\!]$$

(612H(a-iv), taking $\tau_{n+1}=\max\mathcal{T}$, so that $[\![\sigma<\tau_{n+1}]\!]=1$, because $\sigma\in\mathcal{T}_f$)

$$= \sup_{i\le n}\{\omega : f(\omega)=f_i(\omega)\}^\bullet\cap\{\omega : h_i(\omega)\le h(\omega)\}^\bullet\cap\{\omega : h(\omega)<h_{i+1}(\omega)\}^\bullet$$

(counting $h_{n+1}(\omega)$ as ∞)

$$= \sup_{i\le n}\{\omega : h_i(\omega)\le h(\omega)<h_{i+1}(\omega)\}^\bullet = \{\omega : h_0(\omega)\le h(\omega)\}^\bullet = \Omega^\bullet = 1,$$

so $f^\bullet = u_\sigma$.

(b) Similarly,

$$\begin{aligned} [\![f_-^\bullet = u_{<\sigma}]\!] &\supseteq ([\![f_-^\bullet = 0]\!]\cap[\![u_{<\sigma}=0]\!]) \cup \sup_{i\le n}([\![f_-^\bullet = f_i^\bullet]\!]\cap[\![u_{\tau_i}=u_{<\sigma}]\!])\\ &\supseteq (\{\omega : f_-(\omega)=0\}^\bullet\cap[\![\sigma=\check 0]\!])\\ &\qquad\cup\sup_{i\le n}(\{\omega : f_-(\omega)=f_i(\omega)\}^\bullet\cap[\![\tau_i<\sigma]\!]\cap[\![\sigma\le\tau_{i+1}]\!]) \end{aligned}$$

(641I, again taking $\tau_{n+1}=\max\mathcal{T}$)

$$\supseteq (\{\omega : h(\omega) = 0\}^\bullet \cap [\![\sigma = \check{0}]\!])$$
$$\cup \sup_{i \le n} (\{\omega : f_-(\omega) = f_i(\omega)\}^\bullet \cap \{\omega : h_i(\omega) < h(\omega) \le h_{i+1}(\omega)\}^\bullet)$$

(again taking $h_{n+1}(\omega) = \infty$ for every ω)

$$= \{\omega : h(\omega) = 0\}^\bullet \cup \sup_{i \le n} \{\omega : h_i(\omega) < h(\omega) \le h_{i+1}(\omega)\}^\bullet$$
$$= \{\omega : h_0(\omega) \le h(\omega)\}^\bullet = 1,$$

so $f_-^\bullet = u_{<\sigma}$.

642G Corollary Suppose that $(\Omega, \Sigma, \mu, \langle \Sigma_t \rangle_{t \in [0,\infty[})$, $(\mathfrak{A}, \bar{\mu}, \langle \mathfrak{A}_t \rangle_{t \in [0,\infty[})$, $\langle U_t \rangle_{t \ge 0}$ and $\boldsymbol{u} = \langle u_\tau \rangle_{\tau \in \mathcal{T}_f}$ are as in 631D and 642E, that is, with right-continuous filtrations and càdlàg sample paths.

(a) If $h : \Omega \to [0, \infty[$ is a stopping time representing $\tau \in \mathcal{T}_f$, and

$$f(\omega) = \sup_{0 < t \le h(\omega)} |U_t(\omega) - \lim_{s \uparrow t} U_s(\omega)| \text{ if } h(\omega) > 0$$
$$= 0 \text{ if } h(\omega) = 0,$$

then $f^\bullet = \operatorname{Osclln}(\boldsymbol{u} \restriction [\check{0}, \tau])$ in $L^0(\mathfrak{A})$.

(b) $\boldsymbol{u}$ is locally jump-free iff $\omega \mapsto U_t(\omega) : [0, \infty[\to \mathbb{R}$ is continuous for almost every ω.

proof (a)(i) I had better check that f is measurable. **P** Given $\alpha \in \mathbb{R}$, consider the set

$$H = \{(t, \omega) : \omega \in \Omega,\ 0 < t \le h(\omega),\ |U_t(\omega) - \lim_{s \uparrow t} U_s(\omega)| > \alpha\}$$
$$= \bigcup_{\substack{n \in \mathbb{N} \\ q \in \mathbb{Q} \cap [0,\infty[}} \bigcap_{q' \in \mathbb{Q}, q' \ge q} \{(t, \omega) : q < t \le h(\omega)\}$$
$$\cap (\{(t, \omega) : t \le q'\} \cup \{(t, \omega) : |U_t(\omega) - U_{q'}(\omega)| \ge \alpha + 2^{-n}\})$$
$$\in \mathcal{B} \widehat{\otimes} \Sigma$$

where $\mathcal{B}$ is the Borel σ-algebra of $[0, \infty[$, because the ordinate set $\{(t, \omega) : 0 \le t \le h(\omega)\}$ belongs to $\mathcal{B} \widehat{\otimes} \Sigma$ (252Xj[2]) and $(t, \omega) \mapsto U_t(\omega)$ is $\mathcal{B} \widehat{\otimes} \Sigma$-measurable (631D). Set $\pi_2(t, \omega) = \omega$ for $\omega \in \Omega$ and $t \ge 0$. Because μ is complete, Σ is closed under Souslin's operation (431A); because $[0, \infty[$ is Polish,

$$E = \{\omega : f(\omega) > \alpha\} = \pi_2[H]$$

belongs to Σ (423O[2]). As α is arbitrary, f is measurable. **Q**

(ii) Refining these ideas, we can learn more.

(α) Fix $\alpha > 0$ for a moment. Define H as in (i) above, so that $E = \{\omega : f(\omega) > \alpha\}$ is the projection of H on Ω. Now, for any $\omega \in E$, the càdlàg function $t \mapsto U_t(\omega) : [0, h(\omega)] \to \mathbb{R}$ can have at most finitely many jumps of size α or more, so there is a least $g(\omega) \in]0, h(\omega)]$ such that $(g(\omega), \omega) \in H$. For $\omega \in X \setminus E$, set $g(\omega) = h(\omega)$.

(β) g is a stopping time. **P** For any $s \in [0, \infty[$,

$$\{\omega : \omega \in \Omega,\ g(\omega) \le s\} = \{\omega : h(\omega) \le s\} \cup \pi_2[H_s]$$

where

$$H_s = \{(t, \omega) : \omega \in \Omega,\ 0 < t \le s,\ |U_t(\omega) - \lim_{t' \uparrow t} U_{t'}(\omega)| > \alpha\}$$
$$= \bigcup_{\substack{n \in \mathbb{N} \\ q \in \mathbb{Q} \cap [0,\infty[}} \bigcap_{q' \in \mathbb{Q}, q \le q' \le s} \{(t, \omega) : q < t \le s\}$$
$$\cap (\{(t, \omega) : t \le q'\} \cup \{(t, \omega) : |U_t(\omega) - U_{q'}(\omega)| \ge \alpha + 2^{-n}\})$$
$$\in \mathcal{B} \widehat{\otimes} \Sigma_s$$

[2] Later editions only.

because $(t,\omega)\mapsto U_t(\omega):[0,s]\times\Omega\to\mathbb{R}$ is $(\mathcal{B}\widehat{\otimes}\Sigma_s)$-measurable, as observed in part (a) of the proof of 631D. Now one of the assumptions transferred from 631D is that Σ_s contains all μ-negligible sets, so that $(\Omega,\Sigma_s,\mu\restriction\Sigma_s)$ is complete, Σ_s is closed under Souslin's operation and $\pi_2[H_s]\in\Sigma_s$. Since $\{\omega:h(\omega)\le s\}$ certainly belongs to Σ_s, $\{\omega:g(\omega)\le s\}\in\Sigma_s$. As s is arbitrary, g is a stopping time. **Q**

(γ) Let $\sigma\in\mathcal{T}$ be the stopping time represented by g. Then $\sigma\le\tau$ and

$$\begin{aligned}\chi[\![\check 0<\sigma]\!]\times|u_\sigma-u_{<\sigma}|&\le\operatorname{Osclln}(\boldsymbol{u}\restriction[\check 0,\sigma])\\&\le\operatorname{Osclln}(\boldsymbol{u}\restriction[\check 0,\tau])\end{aligned}\qquad\text{(641Na)}$$

(618D). But

$$E=\{\omega:h(\omega)>0,\ |U_{g(\omega)}(\omega)-\lim_{t'\uparrow g(\omega)}U_{t'}(\omega)|>\alpha\},$$

so

$$\chi[\![\check 0<\sigma]\!]\times[\![|u_\sigma-u_{<\sigma}|>\alpha]\!]=E^\bullet=[\![f^\bullet>\alpha]\!]$$

by 642E. It follows that $[\![f^\bullet>\alpha]\!]\subseteq[\![\operatorname{Osclln}(\boldsymbol{u}\restriction[\check 0,\tau])>\alpha]\!]$. And this is true for every $\alpha>0$.

(δ) Taking the supremum over $\alpha>0$, we get $[\![f^\bullet>0]\!]\subseteq[\![\operatorname{Osclln}(\boldsymbol{u}\restriction[\check 0,\tau])>0]\!]$. And of course $[\![f^\bullet>\alpha]\!]=1=[\![\operatorname{Osclln}(\boldsymbol{u}\restriction[\check 0,\tau])>\alpha]\!]$ if $\alpha<0$. So in fact we have $f^\bullet\le\operatorname{Osclln}(\boldsymbol{u}\restriction[\check 0,\tau])$.

(iii) In the other direction, if $\sigma\le\tau$ in $\mathcal{T}$, we have a stopping time $g:\Omega\to[0,\infty[$ representing σ, and we can suppose that $g\le h$. Now, for any $\omega\in\Omega$ such that $g(\omega)>0$,

$$|U_g(\omega)-U_{g^-}(\omega)|=\lim_{s\uparrow g(\omega)}|U_{g(\omega)}(\omega)-U_s(\omega)|\le f(\omega)$$

and

$$f^\bullet\ge|U_g^\bullet-U_{g^-}^\bullet|=|u_\sigma-u_{<\sigma}|$$

by 642Eb again. By 641Nb, $\operatorname{Osclln}(\boldsymbol{u}\restriction[\check 0,\tau])\le f^\bullet$ and we have equality.

(b) Setting

$$f_n(\omega)=\sup_{0<t\le n}|U_t(\omega)-\lim_{s\uparrow t}U_s(\omega)|$$

for $n\ge1$ and $\omega\in\Omega$, we see that

$$\begin{aligned}\omega\mapsto U_t(\omega):[0,\infty[\,\to\mathbb{R}&\text{ is continuous for almost every }\omega\\&\iff f_n=0\text{ a.e. for every }n\ge1\\&\iff\operatorname{Osclln}(\boldsymbol{u}\restriction[\check 0,\check n])=0\text{ for every }n\ge1\end{aligned}$$

by (a). Setting $v_\tau=\operatorname{Osclln}(\boldsymbol{u}\restriction[\check 0,\tau])$ for $\tau\in\mathcal{T}_f$, 618Da tells us that $\langle v_\tau\rangle_{\tau\in\mathcal{T}_f}$ is a non-decreasing fully adapted process; conseqently

$$\begin{aligned}\boldsymbol{u}\text{ is locally jump-free}&\iff v_\tau=0\text{ for every }\tau\in\mathcal{T}_f\\&\iff v_{\tau\wedge\check n}=0\text{ for every }\tau\in\mathcal{T}_f\text{ and }n\in\mathbb{N}\end{aligned}$$

(because if $\tau\in\mathcal{T}_f$ then $\sup_{n\in\mathbb{N}}[\![\tau=\tau\wedge\check n]\!]=1$)

$$\begin{aligned}&\iff v_{\check n}=0\text{ for every }n\ge1\\&\iff\operatorname{Osclln}(\boldsymbol{u}\restriction[\check 0,\check n])=0\text{ for every }n\ge1\\&\iff\omega\mapsto U_t(\omega):[0,\infty[\,\to\mathbb{R}\text{ is continuous for almost every }\omega.\end{aligned}$$

642H Previsible σ-algebras In the context of 642E-642G, we have an important σ-algebra of subsets of $[0,\infty[\,\times\Omega$.

Definitions (a) Given a probability space (Ω,Σ,μ) and a filtration $\langle\Sigma_t\rangle_{t\ge0}$ of σ-subalgebras of Σ, the **previsible σ-algebra** is the σ-algebra $\Lambda_{\rm pv}$ of subsets of $[0,\infty[\,\times\Omega$ generated by sets $]s,\infty[\,\times E$ where $s\ge0$ and $E\in\Sigma_s$.

(b) I will say that a family $\langle X_t\rangle_{t\ge 0}$ of real-valued functions on Ω is **previsibly measurable** if $(t,\omega)\mapsto X_t(\omega):[0,\infty[\,\times\Omega\to\mathbb{R}$ is Λ_{pv}-measurable.

642I Proposition Previsibly measurable processes are progressively measurable.

proof Given a probability space (Ω,Σ,μ) and a filtration $\langle\Sigma_t\rangle_{t\ge 0}$ of σ-subalgebras of Σ, consider the set

$$\Lambda=\{W:W\subseteq[0,\infty[\,\times\Omega,\ W\cap([0,t]\times\Omega)\in\mathcal{B}([0,t])\widehat{\otimes}\Sigma_t\text{ for every }t\ge 0\}$$

where $\mathcal{B}([0,t])$ is the Borel σ-algebra of $[0,t]$. Then Λ is a σ-algebra of subsets of $[0,\infty[\,\times\Omega$ containing $]s,\infty[\,\times E$ whenever $s\ge 0$ and $E\in\Sigma_s$, so $\Lambda_{\mathrm{pv}}\subseteq\Lambda$. Now a family $\langle X_t\rangle_{t\ge 0}$ of real-valued functions on Ω is a previsibly measurable process iff $(s,\omega)\mapsto X_s(\omega):[0,\infty[\,\times\Omega\to\mathbb{R}$ is Λ_{pv}-measurable; but in this case it is Λ-measurable and $(s,\omega)\mapsto X_s(\omega):[0,t]\times\Omega\to\mathbb{R}$ is $\mathcal{B}([0,t])\widehat{\otimes}\Sigma_t$-measurable for every t, that is, $\langle X_t\rangle_{t\ge 0}$ is progressively measurable.

642J Lemma Let (Ω,Σ,μ) be a complete probability space, $\langle\Sigma_t\rangle_{t\ge 0}$ a filtration of σ-subalgebras of Σ, Λ_{pv} the associated previsible σ-algebra and W a member of Λ_{pv}.

(a) If $h:\Omega\to[0,\infty]$ is a stopping time, then $\{(t,\omega):h(\omega)<t\}\in\Lambda_{\mathrm{pv}}$.

(b) $F=\pi_2[W]$ belongs to Σ, where $\pi_2(t,\omega)=\omega$ for $\omega\in\Omega$ and $t\ge 0$.

(c) Now suppose that every Σ_t contains every negligible set. If F is not negligible there is a stopping time $h:\Omega\to[0,\infty]$ such that $\{\omega:(t,h(\omega))\in W\}$ is not negligible.

proof (a)

$$\{(t,\omega):h(\omega)<t\}=\bigcup_{q\in\mathbb{Q},q\ge 0}\,]q,\infty[\,\times\{\omega:h(\omega)\le q\}.$$

(b) Take $\mathcal{A}_0\subseteq\Lambda_{\mathrm{pv}}$ to be the family

$$\{\{0\}\times\Omega\}\cup\{]s,t]\times E:s<t,\ E\in\Sigma_s\}\cup\{]s,\infty[\,\times E:E\in\Sigma_s\}.$$

The complement of any member of $\mathcal{A}_0$ is expressible as a finite union of members of $\mathcal{A}_0$, so the family $\mathcal{S}\mathrm{ous}(\mathcal{A}_0)$ of sets obtainable by Souslin's operation from sets in $\mathcal{A}_0$ includes Λ_{pv} (421F). By 423O again, $F=\pi_2[W]\in\mathcal{S}\mathrm{ous}(\Sigma)=\Sigma$ (431A again).

(c)(i) Set

$$\mathcal{A}=\{[s,t]\times E:0<s\le t,\ E\in\bigcup_{s'<s}\Sigma_{s'}\}\cup\{\{0\}\times\Omega\}$$

then

$$\mathcal{S}\mathrm{ous}(\mathcal{A})=\mathcal{S}\mathrm{ous}(\mathcal{S}\mathrm{ous}(\mathcal{A}))\supseteq\mathcal{S}\mathrm{ous}(\mathcal{A}_0)\supseteq\Lambda_{\mathrm{pv}}.$$

Note that the intersection of two members of $\mathcal{A}$ belongs to $\mathcal{A}$. (If $s_0'<s_0\le t_0$, $s_1'<s_1\le t_1$, $E_0\in\Sigma_{s_0'}$ and $E_1\in\Sigma_{s_1'}$ and $([s_0,t_0]\times E_0)\cap([s_1,t_1]\times E_1)$ is not empty, it is equal to $[\max(s_0,s_1),\min(t_0,t_1)]\times(E_0\cap E_1)$ where $\max(s_0',s_1')<\max(s_0,s_1)\le\min(t_0,t_1)$ and $E_0\cap E_1\in\Sigma_{\max(s_0',s_1')}$.)

(ii) Express W as the kernel of a Souslin scheme $\langle A_\rho\rangle_{\rho\in S^*}$ where $A_\rho\in\mathcal{A}$ for $\rho\in S^*=\bigcup_{n\ge 1}\mathbb{N}^n$. Because $\mathcal{A}$ is closed under finite intersections, we can suppose that $A_{\rho'}\subseteq A_\rho$ if $\rho\subseteq\rho'$. For each ρ, we can express A_ρ as $I_\rho\times E_\rho$ where $E_\rho\in\Sigma$ and I_ρ is a non-empty closed subinterval of $[0,\infty[$ such that $E_\rho\in\Sigma_{\min I_\rho}$. It follows that for any $K\subseteq\mathbb{N}^{\mathbb{N}}$,

$$\pi_2[\bigcup_{\phi\in K}\bigcap_{n\ge 1}A_{\phi\restriction n}]=\bigcup_{\phi\in K}\bigcap_{n\ge 1}E_{\phi\restriction n}.$$

As $F=\bigcup_{\phi\in\mathbb{N}^{\mathbb{N}}}\bigcap_{n\ge 1}E_{\phi\restriction n}$ is not negligible, there is a compact $K\subseteq\mathbb{N}^{\mathbb{N}}$ such that $\tilde F$ is not negligible, where $\tilde F=\bigcup_{\phi\in K}\bigcap_{n\ge 1}E_{\phi\restriction n}$ (431D). Set $\tilde W=\bigcup_{\phi\in K}\bigcap_{n\ge 1}A_{\phi\restriction n}$, so that $\tilde F=\pi_2[\tilde W]$. Note that if $\omega\in\tilde F$, then

$$K_\omega=\{\phi:\phi\in K,\ \omega\in E_{\phi\restriction n}\text{ for every }n\ge 1\}$$

is compact in $\mathbb{N}^{\mathbb{N}}$, so

$$\tilde W^{-1}[\{\omega\}]=\bigcup_{\phi\in K}\{t:\omega\in\bigcap_{n\ge 1}E_{\phi\restriction n},\ t\in\bigcap_{n\ge 1}I_{\phi\restriction n}\}=\bigcup_{\phi\in K_\omega}\bigcap_{n\ge 1}I_{\phi\restriction n}$$

is compact (421M).

(iii) Define $h : \Omega \to [0, \infty]$ by setting

$$h(\omega) = \inf \tilde{W}^{-1}[\{\omega\}] \text{ for every } \omega \in \Omega$$

(interpreting $\inf \emptyset$ as ∞). Then h is a stopping time. **P** Take $t \ge 0$. Set

$$L = \{\phi : \phi \in K,\ E_{\phi \restriction n} \in \Sigma_t \text{ for every } n \ge 1\},$$

$$\begin{aligned}\tilde{W}_t = \tilde{W} \cap ([0, t] \times \Omega) &= \bigcup_{\phi \in K} \bigcap_{n \ge 1} (I_{\rho \restriction n} \cap [0, t]) \times E_{\phi \restriction n} \\ &= \bigcup_{\phi \in L} \bigcap_{n \ge 1} (I_{\rho \restriction n} \cap [0, t]) \times E_{\phi \restriction n}\end{aligned}$$

because if $\phi \in K \setminus L$, there is an $n \ge 1$ such that $E_{\phi \restriction n} \notin \Sigma_t$, in which case $\min I_{\phi \restriction n} > t$ and $I_{\phi \restriction n} \cap [0, t]$ is empty. Now

$$\{\omega : h(\omega) \le t\} = \{\omega : \tilde{W}^{-1}[\{\omega\}] \cap [0, t] \ne \emptyset\}$$

(because $\tilde{W}^{-1}[\{\omega\}]$ is compact)

$$= \pi_2[\tilde{W}_t] = \bigcup_{\phi \in L} \bigcap_{n \ge 1} E_{\phi \restriction n} \in \Sigma_t$$

because $\mu \restriction \Sigma_t$ is complete so Σ_t is closed under Souslin's operation (431A once more). Thus h is a stopping time in the sense of 455La and 612H. **Q** And

$$\{\omega : (\omega, h(\omega)) \in W\} = \{\omega : h(\omega) < \infty\} = \pi_2[\tilde{W}] = \tilde{F}$$

is non-negligible.

642K Proposition Let $(\Omega, \Sigma, \mu, \langle \Sigma_t \rangle_{t \ge 0}, \langle U_t \rangle_{t \ge 0})$ be as in 642E. For $\omega \in \Omega$ set

$$\begin{aligned}U_{t^-}(\omega) &= \lim_{s \uparrow t} U_s(\omega) \text{ if } t > 0, \\ &= 0 \text{ if } t = 0.\end{aligned}$$

(a) If we take a stopping time $h : \Omega \to [0, \infty[$ and define U_{h^-} as in 642E, we have $U_{h^-}(\omega) = U_{h(\omega)^-}(\omega)$ for every ω;

(b) $\langle U_{t^-} \rangle_{t \ge 0}$ is previsibly measurable.

proof (a) We just have to read the definitions.

(b) For each $n \in \mathbb{N}$, $\omega \in \Omega$ and $t \ge 0$ set

$$\begin{aligned}f_n(t, \omega) &= 0 \text{ if } t = 0, \\ &= U_{2^{-n}k}(\omega) \text{ if } k < 4^n \text{ and } 2^{-n}k < t \le 2^{-n}(k + 1), \\ &= U_{2^n}(\omega) \text{ if } 2^n < t.\end{aligned}$$

Then

$$\begin{aligned}\{(t, \omega) : f_n(t, \omega) > \alpha\} = W \cup \bigcup_{k < 2^n} \bigl((&]2^{-k}n, \infty[\times \{\omega : U_{2^{-n}k}(\omega) > \alpha\}) \\ &\setminus (]2^{-n}(k + 1), \infty[\times \{\omega : U_{2^{-n}k}(\omega) > \alpha\}) \bigr) \\ \cup (&]2^n, \infty[\times \{\omega : U_{2^n}(\omega) > \alpha\})\end{aligned}$$

where

$$\begin{aligned}W &= ([0, \infty[\times \Omega) \setminus (]0, \infty[\times \Omega) \text{ if } \alpha < 0, \\ &= \emptyset \text{ if } \alpha \ge 0.\end{aligned}$$

These sets all belong to Λ_{pv}, so f_n is Λ_{pv}-measurable, for every $n \in \mathbb{N}$. Now $U_{t^-}(\omega) = \lim_{n \to \infty} f_n(t, \omega)$ for all ω and t, so $(t, \omega) \mapsto U_{t^-}(\omega)$ is Λ_{pv}-measurable.

642L Theorem Let (Ω,Σ,μ) be a complete probability space and $\langle\Sigma_t\rangle_{t\in[0,\infty[}$ a filtration of σ-subalgebras of Σ all containing every negligible subset of Ω. Let Λ_{pv} be the corresponding previsible σ-algebra. Suppose that $(\mathfrak{A},\bar\mu,\langle\mathfrak{A}_t\rangle_{t\in[0,\infty[})$ is defined from $(\Omega,\Sigma,\mu,\langle\Sigma_t\rangle_{t\in[0,\infty[})$ as in 612H.

(a) Write $\mathcal{L}^0=\mathcal{L}^0(\Lambda_{\text{pv}})$ for the f-algebra of Λ_{pv}-measurable functions from $[0,\infty[\,\times\Omega$ to $\mathbb{R}$. For every $\phi\in\mathcal{L}^0$, there is a fully adapted process $\boldsymbol{x}_\phi=\langle x_{\phi\sigma}\rangle_{\sigma\in\mathcal{T}_f}$ defined by saying that $x_{\phi\sigma}=\phi_h^\bullet$ whenever $h:\Omega\to[0,\infty[$ is a stopping time representing $\sigma\in\mathcal{T}_f$, where

$$\begin{aligned}\phi_h(\omega)&=0\text{ if }h(\omega)=0,\\&=\phi(h(\omega),\omega)\text{ for other }\omega\in\Omega,\end{aligned}$$

and now $\boldsymbol{x}_\phi\in M_{\text{pv}}=M_{\text{pv}}(\mathcal{T}_f)$ as defined in 642D.

(b) The map $\phi\mapsto\boldsymbol{x}_\phi:\mathcal{L}^0\to M_{\text{pv}}$ is a surjective f-algebra homomorphism with kernel

$$\{\phi:\phi\in\mathcal{L}^0\text{ and there is a }\mu\text{-conegligible set }E\text{ such that }\phi(t,\omega)=0\text{ whenever }\omega\in E\text{ and }t>0\},$$

and $\boldsymbol{x}_{g\phi}=\bar g\boldsymbol{x}_\phi$ for every $\phi\in\mathcal{L}^0$ and every Borel measurable $g:\mathbb{R}\to\mathbb{R}$ such that $g(0)=0$.

(c)(i) If $\langle\phi_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}^0$ converging pointwise to $\phi\in\mathcal{L}^0$, then $\langle\boldsymbol{x}_{\phi_n}\rangle_{n\in\mathbb{N}}$ is order*-convergent to $\boldsymbol{x}_\phi$.

(ii) If $\langle\boldsymbol{x}_n\rangle_{n\in\mathbb{N}}$ is a sequence in M_{pv} which is order*-convergent to $\boldsymbol{x}\in M_{\text{pv}}$, there is a pointwise convergent sequence $\langle\phi_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{L}^0$ such that $\boldsymbol{x}_n=\phi_n$ for every $n\in\mathbb{N}$.

Remark Of course we have an immediate identification between $\mathcal{L}^0$ here and the set of previsibly measurable processes as defined in 642Hb.

proof (a)(i) Consider the set Φ of functions $\phi:[0,\infty[\,\times\Omega\to\mathbb{R}$ such that ϕ_h is Σ_h-measurable for every stopping time $h:\Omega\to[0,\infty[$. Then Φ is a linear space closed under multiplication and pointwise limits of sequences. If $s\ge0$, $E\in\Sigma_s$ and $\phi=\chi(]s,\infty[\,\times E)$, then for any finite stopping time h and any $t\ge0$,

$$\{\omega:\phi_h(\omega)=1\}\cap\{\omega:h(\omega)\le t\}=\{\omega:\omega\in E,\,h(\omega)>s\}\cap\{\omega:h(\omega)\le t\}$$

certainly belongs to Σ_t if $t\ge s$, and otherwise is empty, so still belongs to Σ_t. As t is arbitrary, $\{\omega:\phi_h(\omega)=1\}\in\Sigma_h$ and ϕ_h is Σ_h-measurable. As h is arbitrary, $\chi(]s,\infty[\,\times E)\in\Phi$.

Of course $\chi([0,\infty[\times\Omega)\in\Phi$. Since $\{W:\chi W\in\Phi\}$ is a Dynkin class (136A) closed under finite intersection, it includes the σ-algebra generated by sets of the form $]s,\infty[\,\times E$ where $E\in\Sigma_s$, that is, $\chi W\in\Phi$ for every $W\in\Lambda_{\text{pv}}$. Consequently every Λ_{pv}-measurable real-valued function belongs to Φ, that is, $\mathcal{L}^0\subseteq\Phi$.

(ii) Now take $\phi\in\mathcal{L}^0$. If h_0, $h_1:\Omega\to[0,\infty[$ are stopping times representing the same member of $\mathcal{T}_f$, they are equal almost everywhere (612H(a-iv) again), in which case $\phi_{h_0}=_{\text{a.e.}}\phi_{h_1}$ and $\phi_{h_0}^\bullet=\phi_{h_1}^\bullet$; we therefore have, for each $\sigma\in\mathcal{T}_f$, a unique member $x_{\phi\sigma}$ of $L^0(\mathfrak{A})$ such that $x_{\phi\sigma}=\phi_h^\bullet$ whenever $h^\bullet=\sigma$. Just as in 612H(b-ii), $\boldsymbol{x}=\langle x_{\phi\sigma}\rangle_{\sigma\in\mathcal{T}_f}$ is fully adapted.

(iii) The map $\phi\mapsto\boldsymbol{x}_\phi:\mathcal{L}^0\to L^0(\mathfrak{A})^{\mathcal{T}_f}$ is a linear operator; moreover, if $g:\mathbb{R}\to\mathbb{R}$ is Borel measurable and $g(0)=0$, then $\boldsymbol{x}_{g\phi}=\bar g\boldsymbol{x}_\phi$ for every $\phi\in\mathcal{L}^0$. (In the language of (i) above,

$$x_{g\phi,\sigma}=((g\phi)_h)^\bullet=(g\phi_h)^\bullet=\bar g(x_{\phi\sigma})$$

whenever a finite stopping time h represents $\sigma\in\mathcal{T}_f$.) So $\phi\mapsto\boldsymbol{x}_\phi$ is a multiplicative Riesz homomorphism. Next, if $\langle\phi_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}^0$ converging pointwise to $\phi\in\mathcal{L}^0$, $\langle\boldsymbol{x}_{\phi_n}\rangle_{n\in\mathbb{N}}$ is order*-convergent to $\boldsymbol{x}_\phi$. (This time, if h is a finite stopping time representing $\sigma\in\mathcal{T}_f$, $\lim_{n\to\infty}(\phi_n)_h(\omega)=\phi_h(\omega)$ for every ω, so $\langle x_{\phi_n\sigma}\rangle_{n\in\mathbb{N}}=\langle(\phi_n)_h^\bullet\rangle_{n\in\mathbb{N}}$ is order*-convergent to $\phi_h^\bullet=x_{\phi\sigma}$.)

(iv) If $s\ge0$, $E\in\Sigma_s$ and $\phi=\chi(]s,\infty[\,\times E)$, then $\boldsymbol{x}_\phi\in M_{\text{pv}}$. **P** Let $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{T}_f}$ be the simple process defined by the formulae of 612Ka with $n=0$, $\tau_0=\check s$, $u_*=0$ and $u_0=\chi E^\bullet$. Then $\boldsymbol{v}_<=\langle v_{<\sigma}\rangle_{\sigma\in\mathcal{T}_f}$ is defined by saying that

$$[\![\sigma\le\check s]\!]\subseteq[\![v_{<\sigma}=0]\!],\quad[\![\check s<\sigma]\!]\subseteq[\![v_{<\sigma}=\chi E^\bullet]\!]$$

for every $\sigma\in\mathcal{T}_f$ (641I) again). On the other hand, if a finite stopping time h represents σ, then

$$\begin{aligned}\phi_h(\omega) &= 0 \text{ if } h(\omega) \le s,\\ &= \chi E(\omega) \text{ if } h(\omega) > s.\end{aligned}$$

Since

$$[\![\check{s} < \sigma]\!] = [\![\sigma > s]\!] = \{\omega : h(\omega) > s\}^\bullet$$

(611E(a-i-δ), 612H(a-i)), $\phi_h^\bullet = v_{<\sigma}$. As σ is arbitrary, $\boldsymbol{x}_\phi = \boldsymbol{v}_<$ belongs to M_{pv}. **Q**

(v) If $\phi = \chi(\{0\} \times \Omega)$, then $\boldsymbol{x}_\phi = 0$. So

$$\boldsymbol{x}_{\chi([0,\infty[\times\Omega)} = \boldsymbol{x}_{\chi(]0,\infty[\times\Omega)} \in M_{\text{pv}}.$$

(vi) By (iii) and (v), $\{W : W \in \Lambda_{\text{pv}}, \boldsymbol{x}_{\chi W} \in M_{\text{pv}}\}$ is a Dynkin class closed under finite intersections; by (iv), it includes $\{]s,\infty[\times E : s \ge 0, E \in \Sigma_s\}$, so is the whole of Λ_{pv}. Now $\{\phi : \phi \in \mathcal{L}^0, \boldsymbol{x}_\phi \in M_{\text{pv}}\}$ is a linear subspace of $\mathcal{L}^0$ closed under pointwise convergence of sequences, and includes $\{\chi W : W \in \Lambda_{\text{pv}}\}$, so is the whole of $\mathcal{L}^0$.

This concludes the proof of (a).

(b)(i) In the course of (a) above, I showed that $\phi \mapsto \boldsymbol{x}_\phi$ is a multiplicative Riesz homomorphism and that $\boldsymbol{x}_{g\phi} = \bar{g}\boldsymbol{x}_\phi$ for Borel measurable functions g such that $g(0) = 0$.

(ii)($\boldsymbol{\alpha}$) If $\phi \in \mathcal{L}^0$ and there is a negligible set F such that ϕ is zero except on $]0,\infty[\times F$, then $\phi(h(\omega),\omega) = 0$ whenever $\omega \in \Omega \setminus F$ and h is a finite stopping time, so $x_{\phi\sigma} = 0$ for every $\sigma \in \mathcal{T}_f$ and $\boldsymbol{x}_\phi = \boldsymbol{0}$.

($\boldsymbol{\beta}$) If $\phi \in \mathcal{L}^0$ and $\boldsymbol{x}_\phi = \boldsymbol{0}$, set $W = \{(t,\omega) : \omega \in \Omega, t > 0, \phi(t,\omega) \ne 0\}$ and $F = \pi_2[W]$, as in 642J. If $h : \Omega \to [0,\infty[$ is a stopping time representing $\sigma \in \mathcal{T}_f$, then

$$\{\omega : (\omega, h(\omega)) \in W\}^\bullet = [\![x_{\phi\sigma} \ne 0]\!] = 0,$$

that is, $\{\omega : (\omega, h(\omega)) \in W\}$ is negligible. By 642Jc, F is negligible, while ϕ is zero on $]0,\infty[\times (\Omega \setminus F)$. Thus the declared set is the kernel of the operator $\phi \mapsto \boldsymbol{x}_\phi$.

(iii) Let $\langle\phi_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{L}^0$ such that $\langle\boldsymbol{x}_{\phi_n}\rangle_{n\in\mathbb{N}}$ is order*-convergent in M_{pv}. Then there is a pointwise convergent sequence $\langle\phi'_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{L}^0$ such that $\boldsymbol{x}_{\phi'_n} = \boldsymbol{x}_{\phi_n}$ for every n. **P** Set

$$W = \{(t,\omega) : \omega \in \Omega, t > 0, \langle\phi_n(t,\omega)\rangle_{n\in\mathbb{N}} \text{ is not convergent}\}.$$

Then $W \in \Lambda_{\text{pv}}$. If $h : \Omega \to [0,\infty[$ is a stopping time representing $\sigma \in \mathcal{T}_f$, then

$$\{\omega : (\omega, h(\omega)) \in W\} = \{\omega : \langle\phi_n(\omega, h(\omega))\rangle_{n\in\mathbb{N}} \text{ is not convergent}\}$$

is negligible because $\langle x_{\phi_n\sigma}\rangle_{n\in\mathbb{N}}$ is order*-convergent in $L^0(\mathfrak{A})$. By 642Jc again, $F = \pi_2[W]$ is negligible. For $n \in \mathbb{N}$, $\omega \in \Omega$ and $t \ge 0$, set

$$\begin{aligned}\phi'_n(t,\omega) &= 0 \text{ if } t = 0 \text{ or } \omega \in F,\\ &= \phi_n(t,\omega) \text{ otherwise.}\end{aligned}$$

Since Σ_0 contains every negligible set, $]0,\infty] \times (\Omega \setminus F) \in \Lambda_{\text{pv}}$ and $\phi'_n \in \mathcal{L}^0$ for every n. Also, by (ii-α) above, $\boldsymbol{x}_{\phi'_n} = \boldsymbol{x}_{\phi_n}$ for every n, while $\langle\phi'_n\rangle_{n\in\mathbb{N}}$ is pointwise convergent everywhere in $[0,\infty[\times \Omega$. **Q**

(iv) Suppose that $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{T}_f}$ is a simple process.

($\boldsymbol{\alpha}$) Let $(\tau_0,\dots,\tau_k)$ be a breakpoint string for $\boldsymbol{v}$, starting from $\tau_0 = \check{0}$ (use 612Kb), and for $i \le k$ choose measurable $f_i : \Omega \to \mathbb{R}$ such that $f_i^\bullet = v_{\tau_i}$. For each $i \le k$, there is a stopping time $h_i : \Omega \to [0,\infty[$ representing τ_i. We can take $h_0 = 0$, and since $\sup_{i\le j} h_i$ is always a stopping time (455L(c-iv)), we can suppose that $h_0 \le \ldots \le h_k$. Now set

$$\begin{aligned}\phi(t,\omega) &= 0 \text{ if } t = 0,\\ &= f_i(\omega) \text{ if } i < k \text{ and } h_i(\omega) < t \le h_{i+1}(\omega),\\ &= f_k(\omega) \text{ if } h_k(\omega) < t.\end{aligned}$$

(β) $\phi \in \mathcal{L}^0$. **P** For $\alpha \in \mathbb{R}$, $i \le k$ and $q \in \mathbb{Q}$ set $E_{\alpha qi} = \{\omega : h_i(\omega) \le q,\ f_i(\omega) > \alpha\}$. Then

$$E^\bullet_{\alpha qi} = [\![v_{\tau_i} > \alpha]\!] \setminus [\![\tau_i > q]\!] \in \mathfrak{A}_q$$

because $[\![v_{\tau_i} > \alpha]\!] \in \mathfrak{A}_{\tau_i}$. (See the definition in 611G.) So $E_{\alpha qi} \in \Sigma_q$ and

$$]q, \infty[\times E_{\alpha qi} = \{(t, \omega) : h_i(\omega) \le q < t,\ f_i(\omega) > \alpha\}$$

belongs to Λ_{pv}. Taking the union over $q \in \mathbb{Q} \cap [0, \infty[$,

$$\{(t, \omega) : h_i(\omega) < t,\ f_i(\omega) > \alpha\} \in \Lambda_{\text{pv}}.$$

As already noted in 642Ja, $\{(t, \omega) : h_{i+1}(\omega) < t\} \in \Lambda_{\text{pv}}$ for every $i < k$. Accordingly

$$\begin{aligned}\{(t, \omega) &: t > 0,\ \phi(t, \omega) > \alpha\} \\ &= \bigcup_{i<k} (\{(t, \omega) : h_i(\omega) < t,\ f_i(\omega) > \alpha\} \setminus \{(t, \omega) : h_{i+1}(\omega) < t\}) \\ &\qquad\qquad \cup \{(t, \omega) : h_k(\omega) < t,\ f_k(\omega) > \alpha\} \\ &\in \Lambda_{\text{pv}}.\end{aligned}$$

Of course $\{(t, \omega) : t = 0,\ \phi(t, \omega) > \alpha\}$ is either $\{0\} \times \Omega$ or $\emptyset$, and in either case belongs to Λ_{pv}. So ϕ is Λ_{pv}-measurable. **Q**

(γ) If $h : \Omega \to [0, \infty[$ is a stopping time representing $\sigma \in \mathcal{T}_f$, then

$$\begin{aligned}\phi_h(\omega) &= 0 \text{ if } h(\omega) = 0, \\ &= f_i(\omega) \text{ if } i < k \text{ and } h_i(\omega) < h(\omega) \le h_{i+1}(\omega), \\ &= f_k(\omega) \text{ if } h_k(\omega) < h(\omega).\end{aligned}$$

So

$$[\![\sigma = \check{0}]\!] = \{\omega : h(\omega) = 0\}^\bullet \subseteq \{\omega : \phi_h(\omega) = 0\}^\bullet = [\![\phi^\bullet_h = 0]\!],$$

$$[\![\tau_k < \sigma]\!] = \{\omega : h_k(\omega) < h(\omega)\}^\bullet \subseteq \{\omega : \phi_h(\omega) = f_k(\omega)\}^\bullet = [\![\phi^\bullet_h = v_{\tau_k}]\!],$$

$$\begin{aligned}[\![\tau_i < \sigma]\!] \cap [\![\sigma \le \tau_{i+1}]\!] &= \{\omega : h_i(\omega) < h(\omega) \le h_{i+1}(\omega)\}^\bullet \\ &\subseteq \{\omega : \phi_h(\omega) = f_i(\omega)\}^\bullet = [\![\phi^\bullet_h = v_{\tau_i}]\!]\end{aligned}$$

for $i < k$. Matching this with 641I, we see that $\phi^\bullet_h = v_{<\sigma}$. As h is arbitrary, $\boldsymbol{x}_\phi = \boldsymbol{v}_<$.

(v) Consider now the set

$$\{\boldsymbol{x}_\phi : \phi \in \mathcal{L}^0\} \subseteq M_{\text{pv}}.$$

From (iv), we see that this contains $\boldsymbol{v}_<$ for every $\boldsymbol{v} \in M_{\text{simp}}(\mathcal{T}_f)$; from (iii) here and (a-iii) above, we see that it is closed under sequential order*-convergence; by the definition in 642C, it must be the whole of M_{pv}. Thus $\phi \mapsto \boldsymbol{x}_\phi : \mathcal{L}^0 \to M_{\text{pv}}$ is surjective.

This completes the proof of (b).

(c) These facts have been dealt with in (a-iii) and (b-iii) above.

642M Proposition Suppose that $\langle \mathfrak{A}_t \rangle_{t \in T}$ is right-continuous. Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ such that $\inf A \in \mathcal{S}$ whenever $A \subseteq \mathcal{S}$ is non-empty and bounded below in $\mathcal{S}$. If $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ is moderately oscillatory, there is a $\boldsymbol{u}' \in M_{\text{n-s}}(\mathcal{S})$ such that $\boldsymbol{u}_< = \boldsymbol{u}'_<$.

proof If $\mathcal{S}$ is empty, this is trivial, so let us suppose otherwise.

(a) If $\mathcal{S}$ has greatest and least members and $\delta > 0$, then there is a simple process $\boldsymbol{u}' = \langle u'_\sigma \rangle_{\sigma \in \mathcal{S}}$ such that $\theta(\sup |\boldsymbol{u}_< - \boldsymbol{u}'_<|) \le 2\delta$ and $\theta(\sup |\boldsymbol{u}'|) \le \theta(\sup |\boldsymbol{u}_<|) + 2\delta$. **P** Construct $\langle D_i \rangle_{i \in \mathbb{N}}$, $\langle y_i \rangle_{i \in \mathbb{N}}$, $\langle d_i \rangle_{i \in \mathbb{N}}$, $\langle c_{i\sigma} \rangle_{i \in \mathbb{N}, \sigma \in \mathcal{S}}$ and $\tilde{\boldsymbol{u}} = \langle \tilde{u}_\sigma \rangle_{\sigma \in \mathcal{S}}$ from $\boldsymbol{u}$ and δ as in 615M. For $i \in \mathbb{N}$, set $\tau_i = \inf D_i$, so that $\tau_i \in \mathcal{S}$ and $y_i \in L^0(\bigcap_{\sigma \in D_i} \mathfrak{A}_\sigma) = L^0(\mathfrak{A}_{\tau_i})$ (615Mb, 632C(a-iii)). As $D_0 = \mathcal{S}$ and $\tau_i \le \sigma$ whenever $i \in \mathbb{N}$ and $\sigma \in D_{i+1}$

(615Ma), $\tau_0 = \min \mathcal{S}$ and $\tau_i \le \tau_{i+1}$ for every i. Let $n \ge 1$ be such that $\bar{\mu} d_n \le \delta$ (615M(c-ii)), and take $\boldsymbol{u}' = \langle u'_\sigma \rangle_{\sigma \in \mathcal{S}}$ to be the simple process with domain $\mathcal{S}$ and breakpoint string $(\tau_0, \dots, \tau_n)$ such that $[\![\tau_i < \tau_{i+1}]\!] \subseteq [\![u'_{\tau_i} = y_i]\!]$ for $i \le n$ and $u'_{\tau_n} = 0$ (612Ka).

If $\sigma \in \mathcal{S}$ and $i < n$, then

$$[\![\sigma < \tau_{i+1}]\!] \subseteq \inf_{\tau \in D_{i+1}} [\![\sigma < \tau]\!] \subseteq 1 \setminus c_{i+1,\sigma}$$

(615M(d-i)). If $\sigma \in \mathcal{S}$, $i < n$, $\tau \in D_i$ and $I \in \mathcal{I}(\mathcal{S})$ contains τ, then for any $\sigma' \in I$

$$[\![\tau \le \sigma']\!] \cap [\![\sigma' < \sigma]\!] \cap [\![\sigma \le \tau_{i+1}]\!] \subseteq c_{i\sigma'} \setminus c_{i+1,\sigma'} \subseteq [\![\tilde{u}_{\sigma'} = y_i]\!]$$

(615Me). As σ' is arbitrary,

$$[\![\tau < \sigma]\!] \cap [\![\sigma \le \tau_{i+1}]\!] \subseteq [\![\tilde{u}_{I<\sigma} = y_i]\!],$$

defining $\tilde{u}_{I<\sigma}$ as in 641Ea. Taking the limit as $I \uparrow \mathcal{I}(\mathcal{S})$,

$$[\![\tau < \sigma]\!] \cap [\![\sigma \le \tau_{i+1}]\!] \subseteq [\![\tilde{u}_{<\sigma} = y_i]\!].$$

Now

$$[\![\tau_i < \sigma]\!] \cap [\![\sigma \le \tau_{i+1}]\!] = \sup_{\tau \in D_i} [\![\tau < \sigma]\!] \cap [\![\sigma \le \tau_{i+1}]\!]$$

(632C(a-ii))

$$\subseteq [\![\tilde{u}_{<\sigma} = y_i]\!];$$

and at the same time we have

$$[\![\tau_i < \sigma]\!] \cap [\![\sigma \le \tau_{i+1}]\!] \subseteq [\![u'_{<\sigma} = y_i]\!]$$

by 641I. So $[\![\tau_i < \sigma]\!] \cap [\![\sigma \le \tau_{i+1}]\!] \subseteq [\![u'_{<\sigma} = \tilde{u}_{<\sigma}]\!]$, and this is true whenever $i < n$. At the bottom,

$$[\![\tau_0 = \sigma]\!] \subseteq [\![u'_{<\sigma} = 0]\!] \cap [\![\tilde{u}_{<\sigma} = 0]\!] \subseteq [\![u'_{<\sigma} = \tilde{u}_{<\sigma}]\!],$$

so $[\![\sigma \le \tau_n]\!] \subseteq [\![u'_{<\sigma} = \tilde{u}_{<\sigma}]\!]$. But

$$[\![\tau_n < \sigma]\!] = \sup_{\tau \in D_n} [\![\tau < \sigma]\!] \subseteq d_n.$$

As σ is arbitrary, $[\![\boldsymbol{u}'_< \ne \tilde{\boldsymbol{u}}_<]\!] \subseteq d_n$ and $\theta(\sup |\boldsymbol{u}'_< - \tilde{\boldsymbol{u}}_<|) \le \bar{\mu} d_n \le \delta$.

We know also that

$$\sup |\tilde{\boldsymbol{u}}_< - \boldsymbol{u}_<| = \sup |(\tilde{\boldsymbol{u}} - \boldsymbol{u})_<|$$

(641G(e-i))

$$\le \sup |\tilde{\boldsymbol{u}} - \boldsymbol{u}|$$

(641G(a-vii))

$$\le \delta \chi 1$$

(615Mf), so $\theta(\sup |\boldsymbol{u}'_< - \boldsymbol{u}_<|) \le 2\delta$.

As for $\sup |\boldsymbol{u}'|$, this is $\sup |\boldsymbol{u}'_<|$ by 641Ib, because $u'_{\tau_n} = 0$. So

$$\theta(\sup |\boldsymbol{u}'|) = \theta(\sup |\boldsymbol{u}'_<|) \le \theta(\sup |\boldsymbol{u}'_< - \boldsymbol{u}_<|) + \theta(\sup |\boldsymbol{u}_<|) \le \theta(\sup |\boldsymbol{u}_<|) + 2\delta,$$

as required. **Q**

(b) With a bit more effort we have a similar result in the general case. Again, take any $\delta > 0$. By 615G, $u_\downarrow = \lim_{\sigma \downarrow \mathcal{S}} u_\sigma$ and $u_\uparrow = \lim_{\uparrow \mathcal{S}} u_\sigma$ are defined, and there are τ^*, $\tau_* \in \mathcal{S}$ such that

$$\theta(\sup_{\sigma \in \mathcal{S} \vee \tau^*} |u_\sigma - u_\uparrow|) \le \delta, \quad \theta(\sup_{\sigma \in \mathcal{S} \wedge \tau_*} |u_\sigma - u_\downarrow|) \le \delta;$$

we can suppose that $\tau_* \le \tau^*$. We now have

$$\theta(\sup_{\sigma \in \mathcal{S} \vee \tau^*} |u_\sigma - u_{\tau^*}|) \le 2\delta.$$

$\mathcal{S}_1 = \mathcal{S} \cap [\tau_*, \tau^*]$, like $\mathcal{S}$, is closed under arbitrary infima, and $\boldsymbol{u}{\restriction}\mathcal{S}_1$ is moderately oscillatory (615F(a-i)), so (a) tells us that there is a simple process $\boldsymbol{u}'_1 = \langle u'_{1\sigma}\rangle_{\sigma\in\mathcal{S}_1}$ such that $\theta(\sup|(\boldsymbol{u}{\restriction}\mathcal{S}_1)_< - \boldsymbol{u}'_{1<}|) \le 2\delta$. Let $(\tau_0, \dots, \tau_n)$ be a breakpoint string for $\boldsymbol{u}'_1$ starting from $\tau_0 = \tau_*$ and ending with $\tau_n = \tau^*$. Let $\boldsymbol{u}' = \langle u'_\sigma\rangle_{\sigma\in\mathcal{S}}$ be the simple process defined by the formula of 612Ja from $\tau_0, \dots, \tau_n$ and

$$u_* = u_\downarrow,\ u'_{1\tau_0}, \dots, u'_{1\tau_{n-1}},\ u_{\tau_n} \times \chi(\sup_{\sigma\in\mathcal{S}} [\![\tau_n < \sigma]\!]).$$

(Because $[\![\tau_n < \sigma]\!] \in \mathfrak{A}_{\tau_n}$ for every $\sigma \in \mathcal{S}$, $\sup_{\sigma\in\mathcal{S}} [\![\tau_n < \sigma]\!] \in \mathfrak{A}_{\tau_n}$ and $u_{\tau_n} \times \chi(\sup_{\sigma\in\mathcal{S}} [\![\tau_n < \sigma]\!]) \in L^0(\mathfrak{A}_{\tau_n})$.) Set

$$w = \sup_{\sigma\in\mathcal{S}\wedge\tau_0} |u_\sigma - u_\downarrow| \vee \sup_{\sigma\in\mathcal{S}_1} |u_{<\sigma} - u'_{1<\sigma}| \vee \sup_{\sigma\in\mathcal{S}\vee\tau_n} |u_\sigma - u_{\tau_n}|;$$

then $\theta(w) \le 5\delta$. Fix $\tau \in \mathcal{S}$.

(i) For any $\sigma \in \mathcal{S}$,

$$\begin{aligned}[\![\tau \le \tau_0]\!] \cap [\![\sigma < \tau]\!] &\subseteq [\![\sigma < \tau_0]\!] \\ &\subseteq [\![u_\sigma = u_{\sigma\wedge\tau_0}]\!] \cap [\![u'_\sigma = u_\downarrow]\!] \\ &\subseteq [\![|u_\sigma - u'_\sigma| = |u_{\sigma\wedge\tau_0} - u_\downarrow|]\!] \subseteq [\![|u_\sigma - u'_\sigma| \le w]\!]\end{aligned}$$

By 641G(a-vii-α), applied to $\boldsymbol{u} - \boldsymbol{u}'$, $[\![\tau \le \tau_0]\!] \subseteq [\![|u_{<\tau} - u'_{<\tau}| \le w]\!]$.

(ii) Take $i < n$ and $\sigma \in \mathcal{S}$; set $\sigma' = \operatorname{med}(\tau_i, \sigma, \tau_{i+1})$. Applying 641I to the simple processes $\boldsymbol{u}'$, $\boldsymbol{u}'_1$, we have

$$\begin{aligned}[\![\tau_i < \sigma]\!] \cap [\![\sigma \le \tau_{i+1}]\!] &\subseteq [\![u'_{<\sigma} = u'_{\tau_i}]\!] \cap [\![u'_{1<\sigma'} = u'_{1\tau_i}]\!] \cap [\![u'_{\tau_i} = u'_{1\tau_i}]\!] \cap [\![u_{<\sigma} = u_{<\sigma'}]\!] \\ &\subseteq [\![|u_{<\sigma} - u'_{<\sigma}| = |u_{<\sigma'} - u'_{1<\sigma'}|]\!] \\ &\subseteq [\![|u_{<\sigma} - u'_{<\sigma}| \le w]\!].\end{aligned}$$

(iii) At the top end, setting $\sigma' = \sigma \vee \tau_n$, we have $[\![\tau_n < \sigma']\!] \subseteq [\![u'_\rho = u_{\tau_n}]\!]$ whenever $\rho \in \mathcal{S} \vee \tau_n$. Applying 641G(a-vii-$\beta$) to $\boldsymbol{u} - \boldsymbol{u}'$,

$$[\![\tau_n < \sigma']\!] \subseteq [\![|u_{<\sigma'} - u'_{<\sigma'}| \le \sup_{\rho\in\mathcal{S}\vee\tau_n}(|u_\rho - u'_\rho| \times \chi[\![\rho < \sigma']\!])]\!],$$

so

$$\begin{aligned}[\![\tau_n < \sigma']\!] &\subseteq [\![|u_{<\sigma'} - u'_{<\sigma'}| \le \sup_{\rho\in\mathcal{S}\vee\tau_n} |u_\rho - u_{\tau_n}|]\!] \\ &\subseteq [\![|u_{<\sigma'} - u'_{<\sigma'}| \le w]\!].\end{aligned}$$

Now

$$\begin{aligned}[\![\tau_n < \sigma]\!] \subseteq [\![\tau_n < \sigma]\!] \cap [\![\sigma = \sigma']\!] &\subseteq [\![\tau_n < \sigma']\!] \cap [\![u_{<\sigma} = u_{<\sigma'}]\!] \cap [\![u'_{<\sigma} = u'_{<\sigma'}]\!] \\ &\subseteq [\![|u_{<\sigma'} - u'_{<\sigma'}| \le w]\!] \cap [\![u_{<\sigma} - u'_{<\sigma} = u_{<\sigma'} - u'_{<\sigma'}]\!] \\ &\subseteq [\![|u_{<\sigma} - u'_{<\sigma}| \le w]\!].\end{aligned}$$

(iv) Since

$$([\![\sigma \le \tau_0]\!],\ [\![\tau_0 < \sigma]\!] \cap [\![\sigma \le \tau_1]\!], \dots, [\![\tau_{n-1} < \sigma]\!] \cap [\![\sigma \le \tau_n]\!],\ [\![\tau_n < \sigma]\!])$$

is a partition of unity in $\mathfrak{A}$, we see that $|u_{<\sigma} - u'_{<\sigma}| \le w$; as σ is arbitrary, $\sup|\boldsymbol{u}_< - \boldsymbol{u}'_<| \le w$ and $\theta(\sup|\boldsymbol{u}_< - \boldsymbol{u}'_<|) \le 5\delta$.

(v) A further feature of the construction here is that $\sup|\boldsymbol{u}'| \le \sup|\boldsymbol{u}'_<|$. **P** By 641Ib, $\sup|\boldsymbol{u}'| = \sup|\boldsymbol{u}'_<| \vee |u'_{\tau_n}|$. But

$$u'_{\tau_n} = u_{\tau_n} \times \chi(\sup_{\sigma\in\mathcal{S}} [\![\tau_n < \sigma]\!]) = u'_{\tau_n} \times \chi(\sup_{\sigma\in\mathcal{S}} [\![\tau_n < \sigma]\!]).$$

And if $\sigma \in \mathcal{S}$, then $[\![\tau_n < \sigma]\!] \subseteq [\![u'_{<\sigma} = u'_{\tau_n}]\!]$, so

$$|u'_{\tau_n}| \times \chi[\![\tau_n < \sigma]\!] \le |u'_{<\sigma}| \le \sup|\boldsymbol{u}'_<|.$$

Taking the supremum over σ, $|u'_{\tau_n}| \le \sup|\boldsymbol{u}'_<|$ and $\sup|\boldsymbol{u}'| = \sup|\boldsymbol{u}'_<|$.

(c) Once more supposing that $\mathcal{S}$ is a non-empty finitely full sublattice of $\mathcal{T}$ such that $\inf A \in \mathcal{S}$ whenever $A \subseteq \mathcal{S}$ is non-empty and bounded below in $\mathcal{S}$, take any $\boldsymbol{u} \in M_{\mathrm{mo}}(\mathcal{S})$. Choose simple processes $\boldsymbol{u}_n$, $\boldsymbol{v}_n$ as follows. Start with $\boldsymbol{u}_0 = \boldsymbol{0}$, the constant process on $\mathcal{S}$ with value 0. Given that $\boldsymbol{u}_n$ is a simple process with domain $\mathcal{S}$, (b) tells us that there is a simple process $\boldsymbol{v}_n$ such that

$$\theta(\sup|\boldsymbol{u}_< - \boldsymbol{u}_{n<} - \boldsymbol{v}_{n<}|) \le 2^{-n}, \quad \sup|\boldsymbol{v}_n| = \sup|\boldsymbol{v}_{n<}|. \qquad (*)$$

Set $\boldsymbol{u}_{n+1} = \boldsymbol{u}_n + \boldsymbol{v}_n$ and continue.

The construction ensures that

$$\sup|\boldsymbol{u}_{n+1} - \boldsymbol{u}_n| = \sup|\boldsymbol{v}_n| = \sup|\boldsymbol{v}_{<n}|, \quad \theta(\sup|\boldsymbol{u}_< - \boldsymbol{u}_{(n+1)<}|) \le 2^{-n}$$

for every n. Consequently (in the notation of 615B)

$$\begin{aligned}\widehat{\theta}(\boldsymbol{u}_{n+1} - \boldsymbol{u}_n) &= \widehat{\theta}(\boldsymbol{v}_{n+1}) = \widehat{\theta}(\boldsymbol{v}_{(n+1)<}) \\ &= \widehat{\theta}(\boldsymbol{u}_< - \boldsymbol{u}_{(n+2)<} - \boldsymbol{u}_< + \boldsymbol{u}_{(n+1)<}) \le 2^{-n+1}\end{aligned}$$

for every n, and $\sum_{n=0}^{\infty} \widehat{\theta}(\boldsymbol{u}_{n+1} - \boldsymbol{u}_n) < \infty$. Because $M_{\text{o-b}}(\mathcal{S})$ is complete under the ucp uniformity (615Cc), $\langle \boldsymbol{u}_n \rangle_{n\in\mathbb{N}}$ has a limit $\boldsymbol{u}'$ say for the ucp topology, and $\boldsymbol{u}'$ is near-simple.

Since

$$\widehat{\theta}(\boldsymbol{u}'_< - \boldsymbol{u}_{n<}) \le \widehat{\theta}(\boldsymbol{u}' - \boldsymbol{u}_n) \to 0$$

as $n \to \infty$,

$$\lim_{n\to\infty} \boldsymbol{v}_{n<} = \lim_{n\to\infty}(\boldsymbol{u}_{(n+1)<} - \boldsymbol{u}_{n<}) = \boldsymbol{0}.$$

By (*) above,

$$\boldsymbol{u}_< = \lim_{n\to\infty}(\boldsymbol{u}_{n<} + \boldsymbol{v}_{n<}) = \boldsymbol{u}'_<.$$

So $\boldsymbol{u}'$ witnesses that the proposition is true.

642N Corollary Suppose that $\langle \mathfrak{A}_t \rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ such that $\inf A \in \mathcal{S}$ whenever $A \subseteq \mathcal{S}$ is non-empty and bounded below in $\mathcal{S}$. If $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$ is moderately oscillatory, $\boldsymbol{u}_<$ is a previsible process.

proof By 642M, there is a $\boldsymbol{v} \in M_{\text{n-s}}(\mathcal{S})$ such that $\boldsymbol{u}_< = \boldsymbol{v}_<$. Now there is a sequence $\langle \boldsymbol{v}_n \rangle_{n\in\mathbb{N}}$ in $M_{\mathrm{simp}}(\mathcal{S})$ such that $\lim_{n\to\infty} \theta(\sup|\boldsymbol{v}_n - \boldsymbol{v}|) = 0$; taking a subsequence if necessary, we can arrange that $\sum_{n\in\mathbb{N}} \theta(\sup|\boldsymbol{v}_n - \boldsymbol{v}|)$ is finite. Since $\sup|\boldsymbol{v}_{n<} - \boldsymbol{v}_<| \le \sup|\boldsymbol{v}_n - \boldsymbol{v}|$ for each n (641G(a-vii-γ)), $\sum_{n\in\mathbb{N}} \theta(\sup|\boldsymbol{v}_{n<} - \boldsymbol{v}_<|)$ is finite and $\langle \sup|\boldsymbol{v}_{n<} - \boldsymbol{v}_<| \rangle_{n\in\mathbb{N}}$ is order*-convergent to 0 (642Bc). It follows at once that if we express $\boldsymbol{v}_<$ as $\langle v_{<\sigma} \rangle_{\sigma\in\mathcal{S}}$ and each $\boldsymbol{v}_{n<}$ as $\langle v_{n<\sigma} \rangle_{\sigma\in\mathcal{S}}$, $\langle v_{n<\sigma} \rangle_{n\in\mathbb{N}}$ is order*-convergent to $v_{<\sigma}$ for every $\sigma \in \mathcal{S}$ and $\langle \boldsymbol{v}_{n<} \rangle_{n\in\mathbb{N}}$ is order*-convergent to $\boldsymbol{v}_<$ (642Bb). So $\boldsymbol{v}_< \in M_{\mathrm{pv}}(\mathcal{S})$ and $\boldsymbol{u}_<$ is previsible.

642O Proposition (a) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ such that there is a countable set $A \subseteq \mathcal{S}$ which separates $\mathcal{S}$. Then $\boldsymbol{u}_<$ is previsible for every $\boldsymbol{u} \in M_{\mathrm{mo}}(\mathcal{S})$.

(b) If $T \subseteq \mathbb{R}$, then for every sublattice $\mathcal{S}$ of $\mathcal{T}$ there is a countable subset of $\mathcal{S}$ which separates $\mathcal{S}$.

proof (a)(i) Consider first the case in which $\boldsymbol{u}$ is non-negative and non-increasing. If $\mathcal{S}$ is empty or a singleton, the result is trivial; so suppose otherwise. Then $A \ne \emptyset$. Write $\mathcal{S}'$ for the sublattice of $\mathcal{S}$ generated by A, so that $\mathcal{S}'$ is countable and not empty and separates $\mathcal{S}$; let $\langle I_n \rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of finite sublattices of $\mathcal{S}'$ with union $\mathcal{S}'$. For each n, enumerate a maximal totally ordered subset of I_n in increasing order as $(\tau_{n0}, \dots, \tau_{n,k(n)})$, and let $\boldsymbol{u}_n = \langle u_{n\sigma} \rangle_{\sigma\in\mathcal{S}}$ be the simple process with domain $\mathcal{S}$ defined by saying that

$$[\![\sigma < \tau_{n0}]\!] \subseteq [\![u_{n\sigma} = 0]\!], \quad [\![\tau_{n,k(n)} \le \sigma]\!] \subseteq [\![u_{n\sigma} = u_{\tau_{n,k(n)}}]\!],$$

$$[\![\tau_{ni} \le \sigma]\!] \cap [\![\sigma < \tau_{n,i+1}]\!] \subseteq [\![u_{n\sigma}]\!] = u_{\tau_{ni}} \text{ for } 0 \le i < k(n).$$

If $\tau \in \mathcal{S}$ and $n \in \mathbb{N}$, then

$$[\![\tau \le \tau_{n0}]\!] \subseteq [\![u_{I_n<\tau} = 0]\!], \quad [\![\tau_{n,k(n)} < \tau]\!] \subseteq [\![u_{I_n<\tau} = u_{\tau_{n,k(n)}}]\!],$$

$$[\![\tau_{ni} < \tau]\!] \cap [\![\tau \le \tau_{n,i+1}]\!] \subseteq [\![u_{I_n<\tau} = u_{\tau_{ni}}]\!] \text{ for every } i < k(n)$$

by 641Eb. On the other hand, 641I tells us that

$$\inf_{\sigma\in\mathcal{S}} [\![\tau \le \sigma]\!] \subseteq [\![u_{n<\tau} = 0]\!],$$

$$[\![\sigma < \tau]\!] \cap [\![\tau \le \tau_{n0}]\!] \subseteq [\![u_{n<\tau} = 0]\!] \text{ for every } \sigma \in \mathcal{S},$$

so $[\![\tau \le \tau_{n0}]\!] \subseteq [\![u_{n<\tau} = 0]\!]$, while

$$[\![\tau_k(n) < \tau]\!] \subseteq [\![u_{n<\tau} = u_{\tau_{kn}}]\!],$$

$$[\![\tau_{ni} < \tau]\!] \cap [\![\tau \le \tau_{n,i+1}]\!] \subseteq [\![u_{n<\tau} = u_{\tau_{ni}}]\!] \text{ for every } i < n.$$

So we see that $u_{I_n<\tau} = u_{n<\tau}$.

Finally, given $\tau \in \mathcal{S}$, we know from 641 that $u_{<\tau} = \lim_{I\uparrow\mathcal{I}(\mathcal{S}')} u_{I<\tau}$. But any finite subset of $\mathcal{S}'$ is included in I_n for all n large enough, so $u_{<\tau}$ is the topological limit of $\langle u_{I_n<\tau}\rangle_{n\in\mathbb{N}} = \langle u_{n<\tau}\rangle_{n\in\mathbb{N}}$. However, $\langle u_{I_n<\tau}\rangle_{n\in\mathbb{N}}$ is non-decreasing, by 641Kb. So $\langle \boldsymbol{u}_{n<\tau}\rangle_{n\in\mathbb{N}}$ order*-converges to $\boldsymbol{u}_{<\tau}$, by 642Be. As τ is arbitrary, $\langle \boldsymbol{u}_{n<}\rangle_{n\in\mathbb{N}}$ order*-converges to $\boldsymbol{u}_{<}$ (642Bb).

(ii) If $\boldsymbol{u}$ is of bounded variation, then it is expressible as a difference $\boldsymbol{u}' - \boldsymbol{u}''$ where $\boldsymbol{u}'$ and $\boldsymbol{u}''$ are non-negative, non-increasing order-bounded processes (614J). So $\boldsymbol{u}_< = \boldsymbol{u}'_< - \boldsymbol{u}''_<$ (641G(e-i)) is a difference of previsible processes and is previsible (642Da). Finally, the space M of order-bounded processes with previsible versions is closed for the ucp topology of $M_{\text{o-b}}(\mathcal{S})$ (641G(e-ii)), so includes $\overline{M_{\text{bv}}(\mathcal{S})} = M_{\text{mo}}(\mathcal{S})$ (615E).

(b) The set T_0 of points of T which are isolated on the right in T is countable, so there is a countable dense subset Q of T including T_0.

(i) For $q \in Q$ and $\tau \in \mathcal{S}$, set

$$c_{\tau q} = [\![\tau \le \check{q}]\!] \setminus \sup_{\sigma\in\mathcal{S}}([\![\tau < \sigma]\!] \cap [\![\sigma \le \check{q}]\!]).$$

Note that if σ, $\tau \in \mathcal{S}$ then $c_{\sigma q} \cap c_{\tau q} \subseteq [\![\sigma = \tau]\!]$. **P**

$$\begin{aligned} c_{\sigma q} \cap c_{\tau q} &\subseteq [\![\sigma \le \check{q}]\!] \cap [\![\tau \le \check{q}]\!] \setminus ([\![\tau < \sigma]\!] \cap [\![\sigma \le \check{q}]\!]) \\ &= [\![\sigma \le \check{q}]\!] \cap [\![\tau \le \check{q}]\!] \cap ([\![\sigma \le \tau]\!] \cup [\![\check{q} < \sigma]\!]) \subseteq [\![\sigma \le \tau]\!], \end{aligned}$$

and similarly $c_{\sigma q} \cap c_{\tau q} \subseteq [\![\tau \le \sigma]\!]$. **Q** Choose a countable $C_q \subseteq \mathcal{S}$ such that $\sup_{\tau\in C_q} c_{\tau q} = \sup_{\tau\in\mathcal{S}} c_{\tau q}$.

Next, for q, $r \in Q$, set

$$b_{rq} = \sup_{\tau\in\mathcal{S}}([\![\tau > r]\!] \setminus [\![\tau > q]\!]) = \sup_{\tau\in\mathcal{S}}([\![\check{r} < \tau]\!] \cap [\![\tau \le \check{q}]\!]),$$

and choose a countable $B_{rq} \subseteq \mathcal{S}$ such that $b_{qr} = \sup_{\tau\in B_{rq}}([\![\tau > r]\!] \setminus [\![\tau > q]\!])$. Set $A = \bigcup_{q,r\in Q} B_{rq} \cup \bigcup_{q\in Q} C_q$, so that A is a countable subset of $\mathcal{S}$.

(ii) ? Suppose, if possible, that A does not separate $\mathcal{S}$. Then there are τ, $\tau' \in \mathcal{S}$ such that

$$a = [\![\tau < \tau']\!] \setminus \sup_{\rho\in A}([\![\tau \le \rho]\!] \cap [\![\rho < \tau']\!])$$

is non-zero. Note that if $\rho \in A$, then

$$[\![\tau = \rho]\!] \cap a \subseteq [\![\tau = \rho]\!] \cap [\![\tau < \tau']\!] \cap ([\![\rho < \tau]\!] \cup [\![\tau' \le \rho]\!]) = 0.$$

As $a \subseteq [\![\tau < \tau']\!]$, and $\{\check{q} : q \in Q\}$ separates $\mathcal{T}$ (633Da), there is a $q \in Q$ such that $a_1 = a \cap [\![\tau \le \check{q}]\!] \cap [\![\check{q} < \tau']\!]$ is non-zero. Now we know that

$$\begin{aligned} 0 = a_1 \cap \sup_{\rho\in C_q} [\![\rho = \tau]\!] &\supseteq a_1 \cap \sup_{\rho\in C_q} (c_{\tau q} \cap c_{\rho q}) = a_1 \cap \sup_{\sigma\in S}(c_{\tau q} \cap c_{\sigma q}) = a_1 \cap c_{\tau q} \\ &= a_1 \cap [\![\tau \le \check{q}]\!] \setminus \sup_{\sigma\in\mathcal{S}}([\![\tau < \sigma]\!] \cap [\![\sigma \le \check{q}]\!]) = a_1 \setminus \sup_{\sigma\in\mathcal{S}}([\![\tau < \sigma]\!] \cap [\![\sigma \le \check{q}]\!]) \end{aligned}$$

so there is a $\sigma \in S$ such that $a_2 = a_1 \cap [\![\tau < \sigma]\!] \cap [\![\sigma \le \check{q}]\!]$ is non-zero. Again because $\{\check{q} : q \in Q\}$ separates $\mathcal{T}$, there is an $r \in Q$ such that $a_3 = a_2 \cap [\![\tau \le \check{r}]\!] \cap [\![\check{r} < \sigma]\!]$ is non-zero. Now

$$a_3 \subseteq [\![\check{r} < \sigma]\!] \cap [\![\sigma \le \check{q}]\!] \subseteq b_{rq} = \sup_{\rho\in A}([\![\check{r} < \rho]\!] \cap [\![\rho \le \check{q}]\!])$$

and there is a $\rho \in A$ such that $a_4 = a_3 \cap [\![\check{r} < \rho]\!] \cap [\![\rho \le \check{q}]\!]$ is non-zero. But in this case

$$\begin{aligned}0 \ne a_4 &\subseteq a \cap [\![\tau \le \check{r}]\!] \cap [\![\check{r} < \rho]\!] \cap [\![\rho \le \check{q}]\!] \cap [\![\check{q} < \tau']\!] \\ &\subseteq a \cap [\![\tau \le \rho]\!] \cap [\![\rho < \tau']\!]\end{aligned}$$

which is impossible, by the definition of a. **X**

(iii) So the countable set A separates $\mathcal{S}$, as required.

642X Basic exercises (a) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\langle \boldsymbol{u}_n \rangle_{n\in\mathbb{N}} = \langle\langle u_{n\sigma}\rangle_{\sigma\in\mathcal{S}}\rangle_{n\in\mathbb{N}}$ a sequence in $(L^0)^{\mathcal{S}}$ which is order*-convergent to $\boldsymbol{u} \in (L^0)^{\mathcal{S}}$. (i) Show that if every $\boldsymbol{u}_n$ is non-decreasing, then $\boldsymbol{u}$ is non-decreasing. (ii) Show that if every $\boldsymbol{u}_n$ is fully adapted and of bounded variation and $\sup_{n\in\mathbb{N}} \int_{\mathcal{S}} |d\boldsymbol{u}_n|$ is defined in L^0, then $\boldsymbol{u}$ is of bounded variation. (iii) Give an example in which every $\boldsymbol{u}_n$ is simple, but $\boldsymbol{u}$ is neither near-simple nor an integrator. (iv) Show that if every $\boldsymbol{u}_n$ is a martingale, and $\{u_{n\sigma} : n \in \mathbb{N}, \sigma \in \mathcal{S}\}$ is uniformly integrable, then $\boldsymbol{u}$ is a martingale. (*Hint*: 367Xo[3].)

(b) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\langle \boldsymbol{u}_n \rangle_{n\in\mathbb{N}}$ a sequence of order-bounded processes with domain $\mathcal{S}$ such that $\sum_{n=0}^{\infty} \widehat{\theta}_{\mathcal{S}}(\boldsymbol{u}_n)$ is finite. Show that $\langle \boldsymbol{u}_n \rangle_{n\in\mathbb{N}}$ is order*-convergent to 0.

(c) Suppose that $T = \mathbb{N}$. Show that $M_{\text{pv}}(\mathcal{T}_f)$ can be identified with the space of sequences $\langle x_n \rangle_{n\in\mathbb{N}}$ in $L^0(\mathfrak{A})$ such that $x_0 = 0$ and $x_n \in L^0(\mathfrak{A}_{n-1})$ for $n \ge 1$.

(d) Suppose that $T = [0,\infty[$ and $\mathfrak{A} = \{0,1\}$, as in 613W, 615Xf, 616Xa, 617Xb, 618Xa, 622Xd, 626Xa and 627Xa. Let $f : [0,\infty[\to \mathbb{R}$ be a function and $\boldsymbol{u}$ the corresponding process on $\mathcal{T}_f$. (i) Show that if f is of bounded variation and is continuous on the right, $\boldsymbol{u}_<$ corresponds to the function g where $g(0) = 0$ and $g(t) = \lim_{s\uparrow t} f(s)$ for $t > 0$. (ii) Show that order*-convergence in $L^0(\mathfrak{A})$ corresponds to ordinary sequential convergence in $\mathbb{R}$. (iii) Show that $\boldsymbol{u}$ is previsible iff $f(0) = 0$ and f is Borel measurable.

(e) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and z a member of $L^0(\mathfrak{A} \cap \bigcap_{\sigma\in\mathcal{S}} \mathfrak{A}_\sigma)$. Show that if $\boldsymbol{x} \in M_{\text{pv}}(\mathcal{S})$ then $z\boldsymbol{x}$ (definition: 612De) belongs to $M_{\text{pv}}(\mathcal{S})$.

(f) Suppose that $T = [0,\infty[$ and that $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma\in\mathcal{T}_f}$ is the standard Poisson process (612U). (i) Let $\langle \tau_n \rangle_{n\in\mathbb{N}}$ be the sequence of jump times of $\boldsymbol{v}$ (612Ue-612Uf). Show that $[\![v_{<\sigma} = 0]\!] = [\![\sigma \le \tau_1]\!]$ and $[\![v_{<\sigma} = n\chi 1]\!] = [\![\tau_n < \sigma]\!] \cap [\![\sigma \le \tau_{n+1}]\!]$ for every $n \ge 1$ and $\sigma \in \mathcal{T}_f$. (ii) Show that $v_{<\check{t}} = v_{\check{t}}$ for every $t \ge 0$. (iii) Show that $t \mapsto v_{\check{t}} : T \to L^0(\mathfrak{A})$ is continuous.

(g) Show that if $\langle X_t \rangle_{t\ge0}$ is a previsibly measurable process then X_0 is constant.

642Y Further exercises (a) Suppose that $T = \omega_1$, the first uncountable ordinal. (i) Show that for any previsible process $\langle x_\sigma \rangle_{\sigma\in\mathcal{T}}$ there is a $\xi < \omega_1$ such that $x_{\check{\eta}} = x_{\max\mathcal{T}}$ whenever $\xi \le \eta < \omega_1$. (ii) Define $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma\in\mathcal{T}}$ by saying that $v_\sigma = \chi[\![\sigma = \max\mathcal{T}]\!]$ for $\sigma \in \mathcal{T}$. Show that $\boldsymbol{v}$ is a simple process and a submartingale, but is not previsible. (iii) Show that if $\mathfrak{A} = \overline{\bigcup_{\xi<\omega_1} \mathfrak{A}_\xi}$ then $\boldsymbol{v}$ is equal to its previsible variation.

(b) Give an example of a stochastic integration structure $(\mathfrak{A}, \bar{\mu}, T, \langle \mathfrak{A}_t \rangle_{t\in T}, \mathcal{T}, \langle \mathfrak{A}_\tau \rangle_{\tau\in\mathcal{T}})$ and a moderately oscillatory process $\boldsymbol{u}$ with domain $\mathcal{T}$ such that $\boldsymbol{u}_<$ is not previsible.

(c) Let $\boldsymbol{w} = \langle w_\sigma \rangle_{\sigma\in\mathcal{T}_f}$ be the local martingale of 634N, a difference of independent Poisson processes. Show that $\boldsymbol{w}$ is not a previsible process. (*Hint*: show that if τ_1 is the stopping time at which $\boldsymbol{w}$ makes its first jump, then $\mathbb{E}(z_{\tau_1} \times w_{\tau_1}) = 0$ for every $\| \|_\infty$-bounded previsible process $\boldsymbol{z} = \langle z_\sigma \rangle_{\sigma\in\mathcal{T}_f}$.)

642 Notes and comments The idea of 'jump' arises most naturally from the representation of a locally near-simple process by a classical stochastic process with càdlàg sample paths, as in 642E. There is a paradoxical element in 641N and 642G which is is of the greatest importance. When we have a classical

[3]Later editions only.

stochastic process with càdlàg sample paths, we can detect jumps directly from the basic process $\langle U_t\rangle_{t\ge 0}$. But if we turn to the corresponding process $\langle U_t^\bullet\rangle_{t\in[0,\infty[} = \langle u_{\check t}\rangle_{t\in[0,\infty[}$, we may well find that $u_{\check t} = u_{\check t-} = \lim_{s\uparrow t} u_{\check s}$ for every $t > 0$, as in 642Xf. To discuss jumps in the language I have chosen for the development of the general theory, we have to look for jumps at arbitrary stopping times, which is the idea of part (a-ii) of the proof of 642G.

You will see that I have not mentioned martingales in this section or the last, and the 'previsible variations' of §626 appear only in the example 642Ya. There is a good deal more to be said about both, of course, but most of it will have to wait until we have some further tools, starting with 643B. For the moment, however, I call your attention to 642Yc. Martingales which are not jump-free may well not be previsible.

The idea of 642L is that 'previsible processes' in the sense of 642C correspond to 'previsibly measurable processes' in the sense of 642H. I will try to use these phrases consistently to distinguish between the two concepts. Note that the correspondence is not exact because I find myself requiring previsible processes to start with 0 (see the notes to §641) but previsibly measurable processes can start with any constant function. We shall find that for all the important things we call on previsible processes to do the starting value is irrelevant.

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643 The fundamental theorem of martingales

I come at last to one of the most remarkable properties of martingales: under moderately restrictive conditions, a martingale can be expressed as the sum of a local martingale with small jumps and a process of locally bounded variation (643M). In fact I express the result in terms of the 'residual oscillations' introduced in §618, but these are intimately connected with 'jumps' in sample paths, if we use the standard representation of locally near-simple processes (631D, 642E-642G). The proof depends on the notion of 'accessibility' of a stopping time (643C).

643A Notation We have the usual foundations; $(\mathfrak{A}, \bar\mu, T, \langle\mathfrak{A}_t\rangle_{t\in T}, \mathcal{T}, \langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ is a stochastic integration structure, and $L^0(\mathfrak{A})$ is given its topology of convergence in measure, defined by the functional θ where $\theta(u) = \mathbb{E}(|u| \wedge \chi 1)$ for $u \in L^0(\mathfrak{A})$. For $t \in T$, $\check t$ is the constant stopping time at t. For $\tau \in \mathcal{T}$, $P_\tau : L^1_{\bar\mu} \to L^1_{\bar\mu}$ is the conditional expectation associated with $\mathfrak{A}_\tau$. In addition, we shall need the closed subalgebras $\mathfrak{A}_{\mathcal{S}<\tau}$ defined in 641B. For a locally moderately oscillatory process $\boldsymbol{u}$, $\boldsymbol{u}_<$ will be its previsible version (641L). For a sublattice $\mathcal{S}$ of $\mathcal{T}$, $\mathcal{I}(\mathcal{S})$ is the set of finite sublattices of $\mathcal{S}$, and if $\tau \in \mathcal{S}$, then $\mathcal{S} \wedge \tau = \{\sigma \wedge \tau : \sigma \in \mathcal{S}\}$ and $\mathcal{S} \vee \tau = \{\sigma \vee \tau : \sigma \in \mathcal{S}\}$.

643B Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a locally moderately oscillatory process and C a non-empty upwards-directed subset of $\mathcal{S}$ with supremum $\tau \in \mathcal{S}$. Write $\langle u_{<\sigma}\rangle_{\sigma\in\mathcal{S}}$ for the previsible version of $\boldsymbol{u}$ and a for $\inf_{\sigma\in C} [\![\sigma < \tau]\!]$.

(a) Set $w = \lim_{\sigma\uparrow C} u_\sigma$. Then

$$[\![w = u_{<\tau}]\!] \supseteq a, \quad [\![w = u_\tau]\!] \supseteq 1 \setminus a.$$

(b) Now suppose that $\boldsymbol{u}$ is a martingale. Write $P_{\mathcal{S}<\tau} : L^1_{\bar\mu} \to L^1_{\bar\mu}$ for the conditional expectation associated with $\mathfrak{A}_{\mathcal{S}<\tau}$. Then $a \subseteq [\![u_{<\tau} = P_{\mathcal{S}<\tau} u_\tau]\!]$.

proof Of course we know from 641L that $\boldsymbol{u}$ has a previsible version.

(a)(i) 615Ga, applied in $\mathcal{S} \wedge \tau$, tells us that w is well-defined. Let $\epsilon > 0$. Then there is a finite sublattice J of $\mathcal{S}$, containing $\min \mathcal{S}$, such that $\theta(u_{<\tau} - u_{I<\tau}) \le \epsilon$ whenever $J \subseteq I \in \mathcal{I}(\mathcal{S})$. Let $\sigma_0 \in C$ be such that $\bar\mu d \le \epsilon$, where $d = \sup_{\sigma'\in J} [\![\sigma' < \tau]\!] \setminus [\![\sigma' < \sigma_0]\!]$ (611Eb). Take $\sigma \in C$ such that $\sigma_0 \le \sigma$ and $\theta(w - u_\sigma) \le \epsilon$, and let $I \in \mathcal{I}(\mathcal{S})$ be the sublattice generated by $J \cup \{\sigma\}$. Using 611E(c-i) and 611E(c-ii), it is easy to check that

$$\{\sigma' : [\![\sigma < \sigma']\!] \cap [\![\sigma' < \tau]\!] \subseteq d\}$$

is a sublattice of $\mathcal{T}$; as it includes $J \cup \{\sigma\}$, it includes I. But this means that

$$a \setminus d \subseteq [\![\sigma < \tau]\!] \setminus d \subseteq [\![u_{I<\tau} = u_\sigma]\!],$$

and

$$\begin{aligned}\theta((w-u_{<\tau})\times\chi a) &\le \theta(w-u_\sigma)+\theta((u_\sigma-u_{I<\tau})\times\chi a)+\theta(u_{I<\tau}-u_{<\tau})\\ &\le \epsilon+\bar\mu d+\epsilon\le 3\epsilon.\end{aligned}$$

As ϵ is arbitrary, $\theta((w-u_{<\tau})\times\chi a)=0$ and $a\subseteq[\![w=u_{<\tau}]\!]$.

(ii) On the other side, if $\sigma\in C$ then $[\![\sigma=\tau]\!]\subseteq[\![u_{\sigma'}=u_\tau]\!]$ whenever $\sigma'\in C$ and $\sigma\le\sigma'$, so $[\![\sigma=\tau]\!]\subseteq[\![w=u_\tau]\!]$. As σ is arbitrary, $1\setminus a\subseteq[\![w=u_\tau]\!]$.

(b) Write $\mathfrak{B}$ for the closed subalgebra $\bigvee_{\sigma\in C}\mathfrak{A}_\sigma$ generated by $\bigcup_{\sigma\in C}\mathfrak{A}_\sigma$. By 621C(g-ii), the conditional expectation of u_τ on $\mathfrak{B}$ is the $\|\;\|_1$-limit $1\mathrm{lim}_{\sigma\uparrow C}P_\sigma u_\tau=1\mathrm{lim}_{\sigma\uparrow C}u_\sigma$, which is equal to w (613B(d-i)). If $c\in\mathfrak{A}_{\mathcal{S}<\tau}$ and $c\subseteq a$, then $c\in\mathfrak{B}$, by 641Cb, so

$$\mathbb{E}(\chi c\times P_{\mathcal{S}<\tau}u_\tau)=\mathbb{E}(\chi c\times u_\tau)=\mathbb{E}(\chi c\times w)=\mathbb{E}(\chi c\times u_{<\tau})$$

because $a\subseteq[\![w=u_{<\tau}]\!]$. As both $P_{\mathcal{S}<\tau}u_\tau$ and $u_{<\tau}$ belong to $L^0(\mathfrak{A}_{\mathcal{S}<\tau})$ (641G(a-i)), and $a\in\mathfrak{A}_{\mathcal{S}<\tau}$ (by the definition in 641Ba), this shows that $\chi a\times P_{\mathcal{S}<\tau}u_\tau=\chi a\times u_{<\tau}$, that is, $a\subseteq[\![P_{\mathcal{S}<\tau}u_\tau=u_{<\tau}]\!]$.

643C Approachability and accessibility Suppose that $\tau\in\mathcal{T}$.

(a) The **region of accessibility** of τ is

$$\mathrm{acc}(\tau)=\sup_{\emptyset\ne C\subseteq\mathcal{T}\wedge\tau}([\![\sup C=\tau]\!]\setminus\sup_{\sigma\in C}[\![\sigma=\tau]\!]).$$

(b) For $\sigma\in\mathcal{T}$, write $\sigma\ll\tau$ for $\sup_{\rho\in\mathcal{T}}([\![\sigma<\rho]\!]\cap[\![\rho<\tau]\!])$. The **region of approachability** of τ is

$$\mathrm{app}(\tau)=\inf_{\sigma\le\tau}([\![\sigma=\tau]\!]\cup[\![\sigma\ll\tau]\!])$$

so that

$$1\setminus\mathrm{app}(\tau)=\sup_{\sigma\le\tau}([\![\sigma<\tau]\!]\setminus[\![\sigma\ll\tau]\!]).$$

(c) $\mathrm{acc}(\tau)\subseteq\mathrm{app}(\tau)$. **P** Suppose that $C\subseteq\mathcal{T}\wedge\tau$ is non-empty and that $\sigma\le\tau$. Then

$$[\![\sup C=\tau]\!]\cap[\![\sigma<\tau]\!]\setminus[\![\sigma\ll\tau]\!]\subseteq[\![\sigma<\sup C]\!]\setminus[\![\sigma\ll\tau]\!]=\sup_{\rho\in C}[\![\sigma<\rho]\!]\setminus[\![\sigma\ll\tau]\!]$$

(611Eb)

$$\subseteq\sup_{\rho\in C}[\![\sigma<\rho]\!]\setminus([\![\sigma<\rho]\!]\cap[\![\rho<\tau]\!])\subseteq\sup_{\rho\in C}[\![\rho=\tau]\!]$$

and

$$([\![\sup C=\tau]\!]\setminus\sup_{\rho\in C}[\![\rho=\tau]\!])\cap([\![\sigma<\tau]\!]\setminus[\![\sigma\ll\tau]\!])=0.$$

As C and σ are arbitrary, $\mathrm{acc}(\tau)\cap(1\setminus\mathrm{app}(\tau))=0$, that is, $\mathrm{acc}(\tau)\subseteq\mathrm{app}(\tau)$. **Q**

(d)(i) If $\tau\in\mathcal{T}$ then

$$[\![\sup C=\tau]\!]\setminus\sup_{\sigma\in C}[\![\sigma=\tau]\!]=(1\setminus[\![\sup C<\tau]\!])\cap\inf_{\sigma\in C}[\![\sigma<\tau]\!]\in\mathfrak{A}_{<\tau}$$

whenever $\emptyset\ne C\subseteq\mathcal{T}\wedge\tau$, so $\mathrm{acc}(\tau)\in\mathfrak{A}_{<\tau}$.

(ii) If σ, ρ, $\tau\in\mathcal{T}$ then

$$[\![\sigma<\rho]\!]\cap[\![\rho<\tau]\!]=[\![\sigma<\rho]\!]\cap[\![\rho<\tau]\!]\cap[\![\sigma<\tau]\!]=[\![\sigma<\rho\wedge\tau]\!]\cap[\![\rho<\tau]\!]\in\mathfrak{A}_{<\tau},$$

so $[\![\sigma\ll\tau]\!]\in\mathfrak{A}_{<\tau}$ and $\mathrm{app}(\tau)\in\mathfrak{A}_{<\tau}$.

(iii) Note that $\mathrm{acc}(\min\mathcal{T})=0$, because the only non-empty subset of $\mathcal{T}\wedge\min\mathcal{T}$ is $\{\min\mathcal{T}\}$, while $\mathrm{app}(\min\mathcal{T})=1$.

643D Proposition For $t\in T$, let $\check t$ be the constant stopping time at t. Let $T_{\text{r-i}}$ be the set of those $t\in T$ which are isolated on the right, and for $t\in T_{\text{r-i}}$ define $\check t^+\in\mathcal{T}$ by saying that

$$\begin{aligned}\check t^+ &= \max\mathcal{T} \text{ if } t=\max T \text{ is the greatest element of } T,\\ &= \check s \text{ if } t \text{ is not the greatest element of } T\\ &\qquad\qquad\qquad \text{and } s \text{ is the least element of } T \text{ greater than } t.\end{aligned}$$

Then $\operatorname{app}(\tau)=1\setminus\sup_{t\in T_{\text{r-i}}}[\![\tau=\check t^+]\!]$ for every $\tau\in\mathcal{T}$.

proof (a)(i) Note that if σ, τ', $\tau''\in\mathcal{T}$ then

$$\begin{aligned}[\![\tau'\le\tau'']\!]\cap[\![\sigma\ll\tau']\!] &= \sup_{\rho\in\mathcal{T}}[\![\tau'\le\tau'']\!]\cap[\![\sigma<\rho]\!]\cap[\![\rho<\tau']\!]\\ &\subseteq \sup_{\rho\in\mathcal{T}}[\![\sigma<\rho]\!]\cap[\![\rho<\tau'']\!]=[\![\sigma\ll\tau'']\!],\end{aligned}$$

just as in 611E(c-iv-α).

(ii) If $t\in T_{\text{r-i}}$ and $\rho\in\mathcal{T}$,

$$\begin{aligned}[\![\check t<\rho]\!]\cap[\![\rho<\check t^+]\!] &= \sup_{s,s'\in T}([\![\rho>s]\!]\setminus[\![\check t>s]\!])\cap([\![\check t^+>s']\!]\setminus[\![\rho>s']\!])\\ &\subseteq \sup_{s\ge t,s'\le t}[\![\rho>s]\!]\setminus[\![\rho>s']\!]\end{aligned}$$

(because if $s<t$ then $[\![\check t>s]\!]=1$ and if $s'>t$ then $[\![\check t^+>s']\!]=0$)

$$\subseteq \sup_{s'\le s}[\![\rho>s]\!]\setminus[\![\rho>s']\!]=0.$$

So $[\![\check t\ll\check t^+]\!]=0$. And of course

$$[\![\check t<\check t^+]\!]\supseteq[\![\check t^+>t]\!]\setminus[\![\check t^+>t]\!]=1.$$

Now

$$[\![\tau=\check t^+]\!]=[\![\tau\le\check t^+]\!]\cap[\![\check t<\check t^+]\!]\setminus([\![\check t^+\le\tau]\!]\cap[\![\check t\ll\check t^+]\!])$$

(611E(c-iv-α) and (i) just above)

$$\subseteq[\![\check t<\tau]\!]\setminus[\![\check t\ll\tau]\!]\subseteq 1\setminus\operatorname{app}(\tau).$$

As t is arbitrary,

$$\operatorname{app}(\tau)\subseteq 1\setminus\sup_{t\in T_{\text{r-i}}}[\![\tau=\check t^+]\!].$$

(b) If $\sigma\le\tau$ and $b\subseteq[\![\sigma<\tau]\!]\setminus[\![\sigma\ll\tau]\!]$ is non-zero, there is a $t\in T$ such that

$$b'=b\cap[\![\tau>t]\!]\setminus[\![\sigma>t]\!]$$

is non-zero. **?** If $t\notin T_{\text{r-i}}$, $[\![\tau>t]\!]=\sup_{s>t}[\![\tau>s]\!]$ (611A(b-i)), and there is an $s>t$ such that $b'\cap[\![\tau>s]\!]\ne 0$. But now

$$\begin{aligned}b'\cap[\![\tau>s]\!] &= \subseteq b\cap[\![\tau>s]\!]\setminus[\![\sigma>t]\!]\\ &\subseteq b\cap[\![\check s<\tau]\!]\cap[\![\check t<\check s]\!]\cap[\![\sigma\le\check t]\!]\subseteq b\cap[\![\sigma\ll\tau]\!],\end{aligned}$$

contrary to the hypothesis on b. **X**

Thus $t\in T_{\text{r-i}}$. Now we know from (a-ii) that $[\![\check t<\tau]\!]\cap[\![\tau<\check t^+]\!]=0$, so $[\![\check t^+\le\tau]\!]=[\![\check t<\tau]\!]\supseteq b'$. At the same time, b' is disjoint from

$$[\![\sigma\ll\tau]\!]\supseteq[\![\sigma<\check t^+]\!]\cap[\![\check t^+<\tau]\!]\supseteq[\![\sigma\le\check t]\!]\cap[\![\check t^+<\tau]\!]\supseteq b'\cap[\![\check t^+<\tau]\!]$$

so $b'\subseteq[\![\tau\le\check t^+]\!]$ and $b'\subseteq b\cap[\![\tau=\check t^+]\!]$.

As b is arbitrary, $[\![\sigma<\tau]\!]\setminus[\![\sigma\ll\tau]\!]\subseteq\sup_{t\in T_{\text{r-i}}}[\![\tau=\check t^+]\!]$. As σ is arbitrary,

$$1\setminus\operatorname{app}(\tau)\subseteq\sup_{t\in T_{\text{r-i}}}[\![\tau=\check t^+]\!],\quad \operatorname{app}(\tau)\supseteq 1\setminus\sup_{t\in T_{\text{r-i}}}[\![\tau=\check t^+]\!].$$

Putting this together with (a), we have equality, as claimed.

643E Proposition Suppose that $\tau \in \mathcal{T}$. For non-empty $C \subseteq \mathcal{T} \wedge \tau$, set

$$a_C = [\![\sup C = \tau]\!] \setminus \sup_{\sigma\in C} [\![\sigma = \tau]\!].$$

(a) $a_C = \inf_{\sigma\in C} [\![\sigma < \tau]\!] \setminus [\![\sup C < \tau]\!]$ belongs to $\mathfrak{A}_{<\tau}$ whenever $\emptyset \ne C \subseteq \mathcal{T} \wedge \tau$; so $\mathrm{acc}(\tau) \in \mathfrak{A}_{<\tau}$.

(b) For every non-empty $C \subseteq \mathcal{T} \wedge \tau$ there is a non-empty upwards-directed $D \subseteq \mathcal{T} \wedge \tau$ such that $a_D = a_C$. Consequently

$$\mathrm{acc}(\tau) = \sup\{[\![\sup C = \tau]\!] \setminus \sup_{\sigma\in C} [\![\sigma = \tau]\!] : \\ C \subseteq \mathcal{T} \wedge \tau \text{ is non-empty and upwards-directed}\}.$$

(c) If $\upsilon \in \mathcal{T}$, $\mathrm{acc}(\tau) \cap [\![\upsilon = \tau]\!] = \mathrm{acc}(\upsilon) \cap [\![\upsilon = \tau]\!]$.

proof (a) Immediate from the definitions and the fact that $b \setminus c = (1 \setminus c) \setminus (1 \setminus b)$ for all b, $c \in \mathfrak{A}$.

(b) Set $D = \{\sigma' : \sigma' \le \sup C, [\![\sigma' < \tau]\!] \supseteq \inf_{\sigma\in C} [\![\sigma < \tau]\!]\}$. By 611E(c-ii), D is closed under $\vee$. Because $C \subseteq D$, $\sup D = \sup C$ and $\inf_{\sigma\in D} [\![\sigma < \tau]\!] = \inf_{\sigma\in C} [\![\sigma < \tau]\!]$, so $a_D = a_C$.

(c) Write c for $[\![\upsilon = \tau]\!]$. If $\emptyset \ne C \subseteq \mathcal{T} \wedge \tau$ set $D = \{\sigma \wedge \upsilon : \sigma \in C\}$, so that $\emptyset \ne D \subseteq \mathcal{T} \wedge \upsilon$, and

$$\begin{aligned} c \cap a_C &= c \cap [\![\sup C = \tau]\!] \setminus \sup_{\sigma\in C} [\![\sigma = \tau]\!] = c \cap [\![\tau \wedge \sup C = \tau]\!] \setminus \sup_{\sigma\in C} [\![\sigma \wedge \tau = \tau]\!] \\ &\subseteq [\![\upsilon \wedge \sup C = \upsilon]\!] \setminus \sup_{\sigma\in C} [\![\sigma \wedge \upsilon = \upsilon]\!] = [\![\sup D = \upsilon]\!] \setminus \sup_{\sigma\in D} [\![\sigma = \upsilon]\!] \end{aligned}$$

(611Cd)

$$\subseteq b.$$

Taking the supremum over C, $c \cap \mathrm{acc}(\tau) \subseteq \mathrm{acc}(\upsilon)$. Similarly, $c \cap \mathrm{acc}(\upsilon) \subseteq \mathrm{acc}(\tau)$ and $c \cap \mathrm{acc}(\tau) = c \cap \mathrm{acc}(\upsilon)$, as claimed.

643F Lemma Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Take $\tau \in \mathcal{T}$ and $\epsilon > 0$. For $I \in \mathcal{I}(\mathcal{T} \wedge \tau)$ and $\sigma \le \tau$, set

$$d_{\sigma I} = (\mathrm{app}(\tau) \setminus \mathrm{acc}(\tau)) \cap [\![\sigma < \tau]\!] \setminus \sup_{\rho\in I}([\![\sigma < \rho]\!] \cap [\![\rho < \tau]\!]), \quad w_{\sigma I} = P_\sigma \chi d_{\sigma I}.$$

For $I \in \mathcal{I}(\mathcal{T} \wedge \tau)$ set $w_I = \sup_{\sigma\le\tau} w_{\sigma I}$.

(a)(i) $d_{\sigma_0 I} \cap [\![\sigma_0 = \sigma_1]\!] = d_{\sigma_1 I} \cap [\![\sigma_0 = \sigma_1]\!]$ whenever $I \in \mathcal{I}(\mathcal{T} \wedge \tau)$ and σ_0, $\sigma_1 \le \tau$.

(ii) For any $\sigma \le \tau$, $\lim_{I\uparrow\mathcal{I}(\mathcal{T}\wedge\tau)} \bar\mu d_{\sigma I} = 0$.

(b) For any $I \in \mathcal{I}(\mathcal{T} \wedge \tau)$, $\langle w_{\sigma I}\rangle_{\sigma\le\tau}$ is fully adapted.

(c) If $I \subseteq J$ in $\mathcal{I}(\mathcal{T} \wedge \tau)$, then $d_{\sigma I} \supseteq d_{\sigma J}$ and $w_{\sigma I} \ge w_{\sigma J}$ for every $\sigma \le \tau$, and $w_I \ge w_J$.

(d) For $I \in \mathcal{I}(\mathcal{T} \wedge \tau)$, set

$$A_I = \{\sigma : \sigma \le \tau, [\![w_{\sigma I} > \epsilon]\!] \supseteq [\![\sigma < \tau]\!]\}.$$

(i) $[\![w_I > \epsilon]\!] = \sup_{\sigma\in A_I} [\![\sigma < \tau]\!]$.

(ii) A_I is closed under $\wedge$.

(iii) Set $\bar\sigma_I = \inf A_I$. Then

(α) $\bar\sigma_I \le \tau$;

(β) $d_{\bar\sigma_I I}$ is the limit $\lim_{\sigma\downarrow A_I} d_{\sigma I}$ for the measure-algebra topology of $\mathfrak{A}$;

(γ) $w_{\bar\sigma_I I}$ is the limit $1\mathrm{lim}_{\sigma\downarrow A_I} w_{\sigma I}$ for the norm topology of $L^1_{\bar\mu}$;

(δ) $[\![w_I > \epsilon]\!] \subseteq [\![w_{\bar\sigma_I I} \ge \epsilon]\!]$.

(iv) If $I \subseteq J$ in $\mathcal{I}(\mathcal{T} \wedge \tau)$ then $A_I \supseteq A_J$ and $\bar\sigma_I \le \bar\sigma_J$.

(e) There is an $I \in \mathcal{I}(\mathcal{T} \wedge \tau)$ such that $\mathbb{E}(w_I) \le 3\epsilon$.

proof (a)(i) Setting $b = [\![\sigma_0 = \sigma_1]\!]$, $b \cap [\![\sigma_0 < \sigma]\!] = b \cap [\![\sigma_1 < \sigma]\!]$ for every $\sigma \in \mathcal{T}$ (611E(c-v), or otherwise), so

$$
\begin{aligned}
b\cap d_{\sigma_0 I}&=(\operatorname{app}(\tau)\setminus\operatorname{acc}(\tau))\cap b\cap[\![\sigma_0<\tau]\!]\setminus\sup_{\rho\in I}(b\cap[\![\sigma_0<\rho]\!]\cap[\![\rho<\tau]\!])\\
&=(\operatorname{app}(\tau)\setminus\operatorname{acc}(\tau))\cap b\cap[\![\sigma_1<\tau]\!]\setminus\sup_{\rho\in I}(b\cap[\![\sigma_1<\rho]\!]\cap[\![\rho<\tau]\!])\\
&=b\cap d_{\sigma_1 I}.
\end{aligned}
$$

(ii) Take any $\eta>0$. Since $\operatorname{app}(\tau)\cap[\![\sigma<\tau]\!]\subseteq[\![\sigma\ll\tau]\!]$, there is a finite set $J\subseteq\mathcal{T}$ such that

$$
\begin{aligned}
\eta&\ge\bar\mu(\operatorname{app}(\tau)\cap[\![\sigma<\tau]\!]\setminus\sup_{\rho\in J}[\![\sigma<\rho]\!]\cap[\![\rho<\tau]\!])\\
&=\bar\mu(\operatorname{app}(\tau)\cap[\![\sigma<\tau]\!]\setminus\sup_{\rho\in J}[\![\sigma<\rho\wedge\tau]\!]\cap[\![\rho\wedge\tau<\tau]\!]).
\end{aligned}
$$

Now if $I\in\mathcal{I}(\mathcal{T}\wedge\tau)$ includes $\{\rho\wedge\tau:\rho\in J\}$,

$$d_{\sigma I}\subseteq\operatorname{app}(\tau)\cap[\![\sigma<\tau]\!]\setminus\sup\nolimits_{\rho\in J}[\![\sigma<\rho\wedge\tau]\!]\cap[\![\rho\wedge\tau<\tau]\!]$$

has measure at most η. As η is arbitrary, $\lim_{I\uparrow\mathcal{I}(\mathcal{T}\wedge\tau)}\bar\mu d_{\sigma I}=0$.

(b) $w_{\sigma I}=P_\sigma\chi d_{\sigma I}$ certainly belongs to $L^0(\mathfrak{A}_\sigma)$ for every $\sigma\le\tau$. If σ_0, $\sigma_1\le\tau$ and $b=[\![\sigma_0=\sigma_1]\!]$, then we saw in (a-i) that $b\cap d_{\sigma_0 I}=b\cap d_{\sigma_1 I}$, so

$$\chi b\times w_{\sigma_0 I}=\chi b\times P_{\sigma_0}\chi d_{\sigma_0 I}=\chi b\times P_{\sigma_1}\chi d_{\sigma_0 I}$$

(622Bb)

$$=P_{\sigma_1}\chi(b\cap d_{\sigma_0 I})$$

(because $b\in\mathfrak{A}_{\sigma_1}$, by 611H(c-i))

$$=P_{\sigma_1}\chi(b\cap d_{\sigma_1 I})=\chi b\times w_{\sigma_1 I},$$

and $b\subseteq[\![w_{\sigma_0 I}=w_{\sigma_1 I}]\!]$. As σ_0 and σ_1 are arbitrary, $\langle w_{\sigma I}\rangle_{\sigma\le\tau}$ is fully adapted.

(c) Immediate from the definitions. Perhaps I should note here that $w_{\sigma I}\le\chi 1$ for every σ and I, so that w_I is always defined and less than or equal to $\chi 1$.

(d)(i) If $\sigma\le\tau$, then there is a $\rho\in A_I$ such that $[\![\rho<\tau]\!]=[\![w_{\sigma I}>\epsilon]\!]\cap[\![\sigma<\tau]\!]$. **P** $b=[\![w_{\sigma I}>\epsilon]\!]\cap[\![\sigma<\tau]\!]$ belongs to $\mathfrak{A}_\sigma$, so there is a $\rho\le\tau$ such that $b\subseteq[\![\rho=\sigma]\!]$ and $1\setminus b\subseteq[\![\rho=\tau]\!]$ (611I). As $b\subseteq[\![\sigma<\tau]\!]$, $b=[\![\rho=\sigma]\!]=[\![\rho<\tau]\!]$. Now

$$[\![w_{\rho I}>\epsilon]\!]\supseteq[\![\rho=\sigma]\!]\cap[\![w_{\sigma I}>\epsilon]\!]=b=[\![\rho<\tau]\!]$$

and $\rho\in A_I$. **Q**

Accordingly

$$\sup_{\sigma\le\tau}([\![w_{\sigma I}>\epsilon]\!]\cap[\![\sigma<\tau]\!])\subseteq\sup_{\sigma\in A_I}[\![\sigma<\tau]\!]\subseteq\sup_{\sigma\in A_I}[\![w_{\sigma I}>\epsilon]\!]\subseteq[\![w_I>\epsilon]\!].$$

Now, by 364L(a-ii),

$$[\![w_I>\epsilon]\!]=\sup_{\sigma\le\tau}[\![w_{\sigma I}>\epsilon]\!]=\sup_{\sigma\le\tau}([\![w_{\sigma I}>\epsilon]\!]\cap[\![\sigma<\tau]\!])$$

(because $d_{\tau I}\subseteq[\![\tau<\tau]\!]=0$, so $w_{\tau I}=0$ and $[\![w_{\sigma I}>\epsilon]\!]\cap[\![\sigma=\tau]\!]\subseteq[\![w_{\tau I}>\epsilon]\!]=0$ for every $\sigma\le\tau$)

$$\subseteq\sup_{\sigma\in A_I}[\![\sigma<\tau]\!]\subseteq[\![w_I>\epsilon]\!]$$

and we have equality.

(ii) Suppose that σ_0, $\sigma_1\in A_I$, and set $\sigma=\sigma_0\wedge\sigma_1$, $b=[\![\sigma=\sigma_0]\!]\subseteq[\![w_{\sigma I}=w_{\sigma_0 I}]\!]$. Then

$$[\![w_{\sigma I}>\epsilon]\!]\supseteq b\cap[\![w_{\sigma_0 I}>\epsilon]\!]\supseteq b\cap[\![\sigma_0<\tau]\!]=b\cap[\![\sigma<\tau]\!].$$

Similarly,

$$[\![w_{\sigma I} > \epsilon]\!] \supseteq [\![\sigma = \sigma_1]\!] \cap [\![\sigma < \tau]\!];$$

as $[\![\sigma = \sigma_0]\!] \cup [\![\sigma = \sigma_1]\!] = 1$ (611E(a-ii-γ)), $[\![\sigma < \tau]\!] \subseteq [\![w_{\sigma I} > \epsilon]\!]$ and $\sigma \in A_I$.

(iii)(α) Of course $\tau \in A_I$. So $\bar\sigma_I \le \tau$.

(β) Because $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous, we have $[\![\bar\sigma_I < \rho]\!] = \sup_{\sigma\in A_I} [\![\sigma < \rho]\!]$ for any $\rho \in \mathcal{T}$ (632C(a-ii)); because A_I is downwards-directed, $\langle[\![\sigma < \rho]\!]\rangle_{\sigma\in A_I}$ is upwards-directed and $[\![\bar\sigma_I < \rho]\!] = \lim_{\sigma\downarrow A_I} [\![\sigma < \rho]\!]$ for the measure-algebra topology (323D(a-ii)). Because $\cap$ and $\setminus$ are continuous for the measure-algebra topology (323Ba),

$$\begin{aligned} d_{\bar\sigma_I I} &= (\mathrm{app}(\tau) \setminus \mathrm{acc}(\tau)) \cap [\![\bar\sigma_I < \tau]\!] \setminus \sup_{\rho\in I}([\![\bar\sigma_I < \rho]\!] \cap [\![\rho < \tau]\!]) \\ &= \lim_{\sigma\downarrow A_I} (\mathrm{app}(\tau) \setminus \mathrm{acc}(\tau)) \cap [\![\sigma < \tau]\!] \setminus \sup_{\rho\in I}([\![\sigma < \rho]\!] \cap [\![\rho < \tau]\!])) = \lim_{\sigma\downarrow A_I} d_{\sigma I}. \end{aligned}$$

(γ)

$$\begin{aligned} &\limsup_{\sigma\downarrow A_I} \|w_{\bar\sigma_I I} - w_{\sigma I}\|_1 \\ &\qquad\le \limsup_{\sigma\downarrow A_I} \|P_{\bar\sigma_I}\chi d_{\bar\sigma_I I} - P_\sigma \chi d_{\bar\sigma_I I}\|_1 + \limsup_{\sigma\downarrow A_I} \|P_\sigma\chi d_{\bar\sigma_I I} - P_\sigma\chi d_{\sigma I}\|_1 \\ &\qquad\le 0 + \limsup_{\sigma\downarrow A_I} \|\chi d_{\bar\sigma_I I} - \chi d_{\sigma I}\|_1 \end{aligned}$$

(621B(g-i))

$$= \limsup_{\sigma\downarrow A_I} \bar\mu(d_{\bar\sigma_I I} \triangle d_{\sigma I}) = 0.$$

Thus $w_{\bar\sigma_I I} = 1\lim_{\sigma\downarrow A_I} w_{\sigma I}$.

(δ) The point is that $[\![\sigma < \tau]\!] \subseteq [\![w_{\bar\sigma_I I} > \gamma]\!]$ whenever $\sigma \in A_I$ and $\gamma < \epsilon$. **P?** Otherwise, set $\eta = \bar\mu([\![\sigma < \tau]\!] \setminus [\![w_{\bar\sigma_I I} > \gamma]\!]) > 0$. By ($\gamma$) just above, there is a $\rho \in A_I$ such that $\rho \le \sigma$ and $\|w_{\rho I} - w_{\bar\sigma_I I}\|_1 < \eta(\epsilon - \gamma)$. In this case,

$$\begin{aligned} \eta &\le \bar\mu([\![\rho < \tau]\!] \setminus [\![w_{\bar\sigma_I I} > \gamma]\!]) \le \bar\mu([\![w_{\rho I} > \epsilon]\!] \setminus [\![w_{\bar\sigma_I I} > \gamma]\!]) \\ &\le \tfrac{1}{\epsilon-\gamma}\|w_{\rho I} - w_{\bar\sigma_I I}\|_1 < \eta \end{aligned}$$

which is absurd. **XQ**

As γ is arbitrary, $[\![\sigma < \tau]\!] \subseteq [\![w_{\bar\sigma_I I} \ge \epsilon]\!]$ for every $\sigma \in A_I$. So

$$[\![w_I > \epsilon]\!] = \sup\nolimits_{\sigma\in A_I} [\![\sigma < \tau]\!] \subseteq [\![w_{\bar\sigma_I I} \ge \epsilon]\!],$$

as claimed.

(iv) If $\sigma \in A_J$ then

$$[\![\sigma < \tau]\!] \subseteq [\![w_{\sigma J} > \epsilon]\!] \subseteq [\![w_{\sigma I} > \epsilon]\!]$$

because $w_{\sigma J} \le w_{\sigma I}$ (by (c) above). So $\sigma \in A_I$. Thus $A_J \subseteq A_I$ and $\bar\sigma_I = \inf A_I \le \inf A_J = \bar\sigma_J$.

(e) Set $\sigma^* = \sup_{I\in\mathcal{I}(\mathcal{T}\wedge\tau)} \bar\sigma_I \le \tau$. By the definition of $\mathrm{acc}(\tau)$, $[\![\sigma^* = \tau]\!] \setminus \mathrm{acc}(\tau)$ is included in $\sup_{I\in\mathcal{I}(\mathcal{T}\wedge\tau)} [\![\bar\sigma_I = \tau]\!]$; because $\langle\bar\sigma_I\rangle_{I\in\mathcal{I}(\mathcal{T}\wedge\tau)}$ and $\langle[\![\bar\sigma_I = \tau]\!]\rangle_{I\in\mathcal{I}(\mathcal{T}\wedge\tau)}$ are non-decreasing, this is the limit $\lim_{I\uparrow\mathcal{I}(\mathcal{T}\wedge\tau)} [\![\bar\sigma_I = \tau]\!]$ for the measure-algebra topology.

Putting this together with (a-ii), we see that there is an $I \in \mathcal{I}(\mathcal{T}\wedge\tau)$ such that $\sigma^* \in I$, $\bar\mu d_{\sigma^* I} \le \epsilon^2$ and $\bar\mu(([\![\sigma^* = \tau]\!] \setminus \mathrm{acc}(\tau)) \setminus [\![\bar\sigma_I = \tau]\!]) \le \epsilon^2$.

Consider

$$\epsilon\bar\mu[\![w_I > \epsilon]\!] \le \epsilon\bar\mu[\![w_{\bar\sigma_I I} \ge \epsilon]\!]$$

((d-iii-δ) above)

$$\le \epsilon\bar\mu[\![w_{\bar\sigma_I I} \times \chi[\![\bar\sigma_I < \sigma^*]\!] \ge \epsilon]\!] + \epsilon\bar\mu[\![w_{\sigma^* I} \ge \epsilon]\!]$$

(because $[\![\bar\sigma_I \ge \sigma^*]\!] = [\![\bar\sigma_I = \sigma^*]\!] \subseteq [\![w_{\bar\sigma_I I} = w_{\sigma^* I}]\!]$)

$$\le \mathbb{E}(w_{\bar\sigma_I I} \times \chi[\![\bar\sigma_I < \sigma^*]\!]) + \mathbb{E}(w_{\sigma^* I})$$

$$= \bar\mu(d_{\bar\sigma_I I} \cap [\![\bar\sigma_I < \sigma^*]\!]) + \bar\mu d_{\sigma^* I}$$

(because $w_{\bar\sigma_I I} = P_{\bar\sigma_I}\chi d_{\bar\sigma_I I}$ and $[\![\bar\sigma_I < \sigma^*]\!] \in \mathfrak{A}_{\bar\sigma_I}$)

$$\le \bar\mu(d_{\bar\sigma_I I} \cap [\![\bar\sigma_I < \sigma^*]\!] \cap [\![\sigma^* = \tau]\!]) + \epsilon^2$$

(because $\sigma^* \in I$, so $d_{\bar\sigma_I I} \cap [\![\bar\sigma_I < \sigma^*]\!] \cap [\![\sigma^* < \tau]\!] = 0$)

$$\le \bar\mu(([\![\sigma^* = \tau]\!] \setminus a) \setminus [\![\bar\sigma_I = \tau]\!]) + \epsilon^2 \le 2\epsilon^2,$$

so $\bar\mu[\![w_I > \epsilon]\!] \le 2\epsilon$ and $\mathbb{E}(w_I) \le 3\epsilon$.

643G Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a non-negative non-decreasing $\| \|_\infty$-bounded process. Suppose that for every $\epsilon > 0$ there are an $I \in \mathcal{I}(\mathcal{S})$ and a $w \in L^0(\mathfrak{A})$ such that $\|w\|_1 \le \epsilon$ and $P_\sigma v_\tau - v_\sigma \le w$ whenever $\sigma \le \tau$ in $\mathcal{S}$ and $[\![\sigma < \sigma']\!] \cap [\![\sigma' < \tau]\!] = 0$ for every $\sigma' \in I$. Let $\boldsymbol{v}^\#$ be the previsible variation of $\boldsymbol{v}$ (626M). Then $\boldsymbol{v}^\#$ is jump-free.

proof The case $\mathcal{S} = \emptyset$ is trivial; suppose that $\mathcal{S}$ is non-empty.

(a) Something is worth noting straight away. Of course we shall have $P_\sigma v_\tau - v_\sigma = P_\sigma(v_\tau - v_\sigma) \ge 0$ whenever $\sigma \le \tau$ in $\mathcal{S}$; that is, $\boldsymbol{v}$ is a submartingale and we can speak of its previsible version. Since $\boldsymbol{v}$ is $\| \|_\infty$-bounded, $P_\sigma v_\tau - v_\sigma$ is always square-integrable. Now suppose that $\sigma \le \tau$ in $\mathcal{S}$, and that

$$\gamma = \sup_{\sigma=\sigma_0\le\ldots\le\sigma_n=\tau} \|\sum_{j=0}^{n-1} P_{\sigma_j} v_{\sigma_{j+1}} - v_{\sigma_j}\|_2$$

is finite. Then $\|v^\#_\tau - v^\#_\sigma\|_2 \le \gamma$. **P** By 626K(f-ii), $v^\#_\tau - v^\#_\sigma$ is the value of the previsible version of $\boldsymbol{v}\restriction\mathcal{S} \vee \sigma$ at τ, so belongs to the weak closure of $A = \{S_I(\mathbf{1}, Pd\boldsymbol{v}) : \{\sigma,\tau\} \subseteq I \in \mathcal{I}(\mathcal{S} \cap [\sigma,\tau])\}$ in $L^1_{\bar\mu}$. If $\{\sigma,\tau\} \subseteq I \in \mathcal{I}(\mathcal{S} \cap [\sigma,\tau])$, let $(\sigma_0,\ldots,\sigma_n)$ linearly generate the I-cells, so that $S_I(\mathbf{1}, Pd\boldsymbol{v}) = \sum_{j=0}^{n-1} P_{\sigma_j} v_{\sigma_{j+1}} - v_{\sigma_j}$ and we have $\|S_I(\mathbf{1}, Pd\boldsymbol{v})\|_2 \le \gamma$. Thus $\|y\|_2 \le \gamma$ for every $y \in A$. Now if $z \in L^\infty(\mathfrak{A})$, $\|z\|_2 \le 1$ and $\eta > 0$, there is a $y \in A$ such that

$$\mathbb{E}(z \times (v^\#_\tau - v^\#_\sigma)) \le \eta + \mathbb{E}(z \times y) \le \eta + \|y\|_2 \le \eta + \gamma;$$

as η and z are arbitrary, $\|v^\#_\tau - v^\#_\sigma\|_2 \le \gamma$. **Q**

Squaring, we have

$$\mathbb{E}((v^\#_\tau - v^\#_\sigma)^2) \le \sup_{\sigma=\sigma_0\le\ldots\le\sigma_n=\tau} \mathbb{E}((\sum_{j=0}^{n-1} P_{\sigma_j} v_{\sigma_{j+1}} - v_{\sigma_j})^2).$$

Moreover, whenever $\sigma = \sigma_0 \le \ldots \le \sigma_n \le \tau$, we have

$$\begin{aligned}
\mathbb{E}((\sum_{j=0}^{n-1} P_{\sigma_j} v_{\sigma_{j+1}} - v_{\sigma_j})^2) &\le \mathbb{E}(2\sum_{j=0}^{n-1}(\sum_{k=j}^{n-1} P_{\sigma_k} v_{\sigma_{k+1}} - v_{\sigma_k}) \times (P_{\sigma_j} v_{\sigma_{j+1}} - v_{\sigma_j}))\\
&= 2\sum_{j=0}^{n-1}\sum_{k=j}^{n-1} \mathbb{E}((P_{\sigma_k} v_{\sigma_{k+1}} - v_{\sigma_k}) \times (P_{\sigma_j} v_{\sigma_{j+1}} - v_{\sigma_j}))\\
&= 2\sum_{j=0}^{n-1}\sum_{k=j}^{n-1} \mathbb{E}(P_{\sigma_j}(P_{\sigma_k} v_{\sigma_{k+1}} - v_{\sigma_k}) \times (P_{\sigma_j} v_{\sigma_{j+1}} - v_{\sigma_j}))\\
&= 2\sum_{j=0}^{n-1}\sum_{k=j}^{n-1} \mathbb{E}(P_{\sigma_j}(v_{\sigma_{k+1}} - v_{\sigma_k}) \times (P_{\sigma_j} v_{\sigma_{j+1}} - v_{\sigma_j}))\\
&= 2\sum_{j=0}^{n-1} \mathbb{E}(P_{\sigma_j}(v_\tau - v_{\sigma_j}) \times (P_{\sigma_j} v_{\sigma_{j+1}} - v_{\sigma_j})).
\end{aligned}$$

(b) Now for the main line of this proof. Let $M > 0$ be such that $\|v_\sigma\|_\infty \le M$ for every $\sigma \in \mathcal{S}$. Let $\epsilon > 0$. Let $I \in \mathcal{I}(\mathcal{S})$ and $w \in L^0(\mathfrak{A})$ be such that $I \ne \emptyset$, $\|w\|_1 \le \epsilon$ and $P_\sigma v_\tau - v_\sigma \le w$ whenever $\sigma \le \tau$ in $\mathcal{S}$ and $[\![\sigma < \sigma']\!] \cap [\![\sigma' < \tau]\!] = 0$ for every $\sigma' \in I$; we can suppose that $w \le M\chi 1$. Let $(\tau_0, \dots, \tau_n)$ linearly generate the I-cells. Then we shall have $0 \le P_\sigma v_\tau - v_\sigma \le w$ whenever

either $\sigma \le \tau \le \tau_0$
or $\tau_{i-1} \le \sigma \le \tau \le \tau_i$ where $1 \le i \le n$,
or $\tau_n \le \sigma \le \tau$.

Set $w' = \sup_{\sigma \in \mathcal{S}} P_\sigma w$. Then

$$\mathbb{E}(w') \le \|w'\|_2 \le 2 \sup_{\sigma \in \mathcal{S}} \|P_\sigma w\|_2$$

(623M, since $\boldsymbol{P}w$ is a martingale)

$$\le 2\|w\|_2 = 2\sqrt{\mathbb{E}(w^2)} \le 2\sqrt{M\mathbb{E}(w)} \le 2\sqrt{\epsilon M}.$$

(c) Suppose that $\langle k_i \rangle_{i \le n+1}$, $\langle \sigma_{ij} \rangle_{i \le n+1, j \le k_i}$ are such that $\sigma_{ij} \in \mathcal{S}$ whenever $i \le n+1$ and $j \le k_i$ and

$$\sigma_{00} \le \sigma_{01} \le \ldots \le \sigma_{0k_0} = \tau_0,$$

$$\tau_{i-1} = \sigma_{i0} \le \ldots \le \sigma_{ik_i} = \tau_i \text{ for } 1 \le i \le n,$$

$$\tau_n = \sigma_{n+1,0} \le \ldots \le \sigma_{n+1,k_{n+1}}.$$

Then, using the last formula in (a),

$$\begin{aligned}
&\mathbb{E}\Big(\sum_{i=0}^{n+1}\Big(\sum_{j=0}^{k_i-1} P_{\sigma_{ij}} v_{\sigma_{i,j+1}} - v_{\sigma_{ij}}\Big)^2\Big) \\
&\qquad \le 2\sum_{i=0}^{n+1}\sum_{j=0}^{k_i-1} \mathbb{E}((P_{\sigma_{ij}} v_{\sigma_{ik_i}} - v_{\sigma_{ij}}) \times (P_{\sigma_{ij}} v_{\sigma_{i,j+1}} - v_{\sigma_{ij}})) \\
&\qquad \le 2\sum_{i=0}^{n+1}\sum_{j=0}^{k_i-1} \mathbb{E}(w \times (P_{\sigma_{ij}} v_{\sigma_{i,j+1}} - v_{\sigma_{ij}})) \\
&\qquad = 2\sum_{i=0}^{n+1}\sum_{j=0}^{k_i-1} \mathbb{E}(P_{\sigma_{ij}} w \times (v_{\sigma_{i,j+1}} - v_{\sigma_{ij}})) \\
&\qquad \le 2\sum_{i=0}^{n+1}\sum_{j=0}^{k_i-1} \mathbb{E}(w' \times (v_{\sigma_{i,j+1}} - v_{\sigma_{ij}})) \\
&\qquad = 2\mathbb{E}(w' \times (v_{\sigma_{n+1,k_{n+1}}} - v_{\sigma_{00}})) \le 4M\mathbb{E}(w') \le 8M\sqrt{\epsilon M}.
\end{aligned}$$

(d) Putting this together with (a), we see that if $\sigma \le \tau_0$ and $\tau_n \le \tau$ then

$$\mathbb{E}\big((v^\#_{\tau_0} - v^\#_\sigma)^2 + \textstyle\sum_{i=1}^n (v^\#_{\tau_i} - v^\#_{\tau_{i-1}})^2 + (v^\#_\tau - v^\#_{\tau_n})^2\big) \le 8M\sqrt{\epsilon M}.$$

Now consider $\mathrm{Osclln}^*_I(\boldsymbol{v}^\#)$. By 618Ca, this is

$$\begin{aligned}
\sup\{|v^\#_{\sigma'} - v^\#_\sigma| : \sigma, \sigma' \in \mathcal{S} &\text{ and either } \sigma \le \sigma' \le \tau_0 \\
&\text{ or there is an } i \text{ such that } \tau_{i-1} \le \sigma \le \sigma' \le \tau_i \\
&\text{ or } \tau_n \le \sigma \le \sigma'\} \\
= \sup_{\sigma \in \mathcal{S} \wedge \tau_0} (v^\#_{\tau_0} - v^\#_\sigma) \vee \sup_{1 \le i \le n} &(v^\#_{\tau_i} - v^\#_{\tau_{i-1}}) \vee \sup_{\tau \in \mathcal{S} \vee \tau_n} (v^\#_\tau - v^\#_{\tau_n}).
\end{aligned}$$

So

$$\begin{aligned}\mathrm{Oscln}^*_I(\boldsymbol{v}^\#)^2 &= \sup_{\sigma\in\mathcal{S}\wedge\tau_0}(v^\#_{\tau_0}-v^\#_\sigma)^2 \vee \sup_{1\le i\le n}(v^\#_{\tau_i}-v^\#_{\tau_{i-1}})^2 \vee \sup_{\tau\in\mathcal{S}\vee\tau_n}(v^\#_\tau-v^\#_{\tau_n})^2\\ &\le \sup_{\substack{\sigma\in\mathcal{S}\wedge\tau_0\\ \tau\in\mathcal{S}\vee\tau_n}}\bigl((v^\#_{\tau_0}-v^\#_\sigma)^2+\sum_{1\le i\le n}(v^\#_{\tau_i}-v^\#_{\tau_{i-1}})^2+(v^\#_\tau-v^\#_{\tau_n})^2\bigr).\end{aligned}$$

As the sum here increases as σ decreases and τ increases,

$$\begin{aligned}\mathbb{E}((\mathrm{Oscln}^*_I(\boldsymbol{v}^\#))^2) &\le \sup_{\substack{\sigma\in\mathcal{S}\wedge\tau_0\\ \tau\in\mathcal{S}\vee\tau_n}}\mathbb{E}\bigl((v^\#_{\tau_0}-v^\#_\sigma)^2+\sum_{1\le i\le n}(v^\#_{\tau_i}-v^\#_{\tau_{i-1}})^2+(v^\#_\tau-v^\#_{\tau_n})^2\bigr)\\ &\le 8M\sqrt{\epsilon M}.\end{aligned}$$

Accordingly

$$\theta(\mathrm{Oscln}^*_I(\boldsymbol{v}^\#)) \le \mathbb{E}(\mathrm{Oscln}^*_I(\boldsymbol{v}^\#)) \le \|\,\mathrm{Oscln}^*_I(\boldsymbol{v}^\#)\|_2 \le \sqrt{8M\sqrt{\epsilon M}}.$$

As ϵ is arbitrary, $\inf_{I\in\mathcal{I}(\mathcal{S})}\theta(\mathrm{Oscln}^*_I(\boldsymbol{v}^\#))=0$ and $\boldsymbol{v}^\#$ is jump-free.

643H Lemma Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Take τ_1 in $\mathcal{T}$ and a non-negative $v\in L^0(\mathfrak{A}_{\tau_1})\cap L^1_{\bar\mu}$. Set $v_\sigma = v\times\chi[\![\sigma=\tau_1]\!]$ for $\sigma\in\mathcal{T}\wedge\tau_1$. Then $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{T}\wedge\tau_1}$ is a non-negative non-decreasing submartingale. Let $\boldsymbol{v}^\#=\langle v^\#_\sigma\rangle_{\sigma\in\mathcal{T}\wedge\tau_1}$ be its previsible variation, and $\boldsymbol{v}^\#_<=\langle v^\#_{<\sigma}\rangle_{\sigma\in\mathcal{T}\wedge\tau_1}$ the previsible version of $\boldsymbol{v}^\#$. If $\tau\le\tau_1$ then $\mathrm{app}(\tau)\setminus\mathrm{acc}(\tau)\subseteq[\![v^\#_\tau=v^\#_{<\tau}]\!]$.

proof Write a^* for $\mathrm{app}(\tau)\setminus\mathrm{acc}(\tau)$.

(a) By 612CF, $v_\sigma\in L^0(\mathfrak{A}_\sigma)$ for every $\sigma\le\tau_1$. If σ, $\sigma'\in\mathcal{T}\wedge\tau_1$ then

$$\begin{aligned}v_\sigma\times\chi[\![\sigma=\sigma']\!] &= v\times\chi([\![\tau_1\le\sigma]\!]\cap[\![\sigma=\sigma']\!])\\ &= v\times\chi([\![\tau_1\le\sigma']\!]\cap[\![\sigma=\sigma']\!]) = v_{\sigma'}\times\chi[\![\sigma=\sigma']\!]\end{aligned}$$

so $[\![\sigma=\sigma']\!]\subseteq[\![v_\sigma=v_{\sigma'}]\!]$; thus $\boldsymbol{v}$ is fully adapted. Because $v\in L^1_{\bar\mu}$, $\boldsymbol{v}$ is an L^1-process. If $\sigma\le\sigma'$, then $[\![\tau_1\le\sigma]\!]\subseteq[\![\tau_1\le\sigma']\!]$ so (because $v\ge 0$) $v_\sigma\le v_{\sigma'}$; thus $\boldsymbol{v}$ is non-decreasing, therefore a submartingale (626B), and has a previsible version (626M), which is itself non-negative and non-decreasing, therefore locally moderately oscillatory (616Ra, or otherwise), and has a previsible version (641L).

(b) Suppose first that $v\le\chi a^*$.

(i) Take $I\in\mathcal{I}(\mathcal{T}\wedge\tau)$, and for $\sigma\le\tau$ define $d_{\sigma I}$, $w_{\sigma I}$ and w_I from τ as in 643F. Suppose that $\sigma_0\le\sigma_1\le\tau$ and $[\![\sigma_0<\sigma]\!]\cap[\![\sigma<\sigma_1]\!]=0$ for every $\sigma\in I$. Then $P_{\sigma_0}v_{\sigma_1}-v_{\sigma_0}\le w_I$. **P**

$$[\![\sigma_0<\sigma]\!]\cap[\![\sigma<\tau]\!]\cap[\![\sigma_1=\tau]\!]\subseteq[\![\sigma_0<\sigma]\!]\cap[\![\sigma<\sigma_1]\!]=0,$$

that is, $[\![\sigma_0<\sigma]\!]\cap[\![\sigma<\tau]\!]\subseteq[\![\sigma_1<\tau]\!]$ for every $\sigma\in I$, and

$$\begin{aligned}&a^*\cap[\![\sigma_0<\tau]\!]\setminus[\![\sigma_1<\tau]\!]\\ &\qquad\subseteq a^*\cap[\![\sigma_0<\tau]\!]\setminus\sup_{\sigma\in I}([\![\sigma_0<\sigma]\!]\cap[\![\sigma<\tau]\!])\\ &\qquad= d_{\sigma_0 I}.\end{aligned}$$

Consequently

$$\begin{aligned}P_{\sigma_0}v_{\sigma_1}-v_{\sigma_0} &= P_{\sigma_0}(v\times\chi[\![\sigma_1=\tau_1]\!]-v\times\chi[\![\sigma_0=\tau_1]\!])\\ &= P_{\sigma_0}(v\times\chi([\![\tau=\tau_1]\!]\cap[\![\sigma_1=\tau]\!]\setminus[\![\sigma_0=\tau]\!]))\\ &\le P_{\sigma_0}(v\times\chi([\![\sigma_0<\tau]\!]\setminus[\![\sigma_1<\tau]\!]))\\ &\le P_{\sigma_0}\chi d_{\sigma_0 I}=w_{\sigma_0 I}\le w_I.\ \mathbf{Q}\end{aligned}$$

(ii) Now 643Fe tells us that for every $\epsilon > 0$ there is an $I \in \mathcal{I}(\mathcal{T} \wedge \tau)$ such that $\mathbb{E}(w_I) \le \epsilon$. So we can apply 643G to see that $(\boldsymbol{v} \restriction \mathcal{T} \wedge \tau)^{\#}$ is jump-free, that is, that $\boldsymbol{v}^{\#} \restriction \mathcal{T} \wedge \tau$ is jump-free (626K(f-i)). It follows that $v^{\#}_{<\tau} = v^{\#}_{\tau}$ (641O), so surely we have $a^* \subseteq [\![v^{\#}_{<\tau} = v^{\#}_{\tau}]\!]$.

(c) Next suppose that $\emptyset \ne C \subseteq \mathcal{T} \wedge \tau$ and $v \le \chi a_C$, where

$$a_C = [\![\sup C = \tau]\!] \setminus \sup_{\sigma \in C} [\![\sigma = \tau]\!] \subseteq \operatorname{acc}(\tau)$$

as in 643E.

(i) If $\rho \le \sigma \in C$,

$$v_\rho = v \times \chi[\![\rho = \tau_1]\!] \le \chi(a_C \cap [\![\rho = \tau_1]\!]) \le \chi(a_C \cap [\![\sigma = \tau]\!]) = 0$$

so $\boldsymbol{v} \restriction \mathcal{T} \wedge \sigma = 0$, $\boldsymbol{v}^{\#} \restriction \mathcal{T} \wedge \sigma = (\boldsymbol{v} \restriction \mathcal{T} \wedge \sigma)^{\#} = 0$ (626K(f-i)), $v^{\#}_{\sigma} = v^{\#}_{<\sigma} = 0$ and $[\![\sigma = \tau]\!] \subseteq [\![v^{\#}_{\tau} = v^{\#}_{<\tau}]\!]$ (because $\boldsymbol{v}^{\#}_{<}$ is fully adapted, by 641G(a-ii)).

(ii) If $\sup C \le \sigma \le \sigma' \le \tau$ then

$$[\![\sigma < \sigma']\!] \subseteq [\![\sup C < \tau]\!] \subseteq [\![v_\sigma = 0]\!] \cap [\![v_{\sigma'} = 0]\!]$$

so

$$\mathbb{E}(P_\sigma v_{\sigma'} - v_\sigma) = \mathbb{E}(v_{\sigma'} - v_\sigma) = \mathbb{E}((v_{\sigma'} - v_\sigma) \times \chi[\![\sigma < \sigma']\!]) = 0;$$

as $v_\sigma \le P_\sigma v_{\sigma'}$, $P_\sigma v_{\sigma'} - v_\sigma = 0$. It follows that $S_I(\mathbf{1}, Pd\boldsymbol{v}) = 0$ for every finite sublattice I of $[\sup C, \tau]$ and $\boldsymbol{v}^{\#}$ is constant on $[\sup C, \tau]$. Accordingly

$$[\![\sup C < \tau]\!] \subseteq [\![v^{\#}_{\tau} = v^{\#}_{\sup C}]\!] \cap [\![v^{\#}_{<\tau} = v^{\#}_{\sup C}]\!] \subseteq [\![v^{\#}_{\tau} = v^{\#}_{<\tau}]\!].$$

(iii) Putting these together,

$$a^* \subseteq 1 \setminus a_C \subseteq [\![v^{\#}_{\tau} = v^{\#}_{<\tau}]\!].$$

(d) Thirdly, suppose that $t \in T_{\text{r-i}}$ and that $v \le \chi[\![\tau = \check{t}^+]\!]$ where $\check{t}^+$ is defined as in 643D. If $\tau \wedge \check{t}^+ \le \rho \le \sigma \le \tau$ then

$$[\![\rho < \sigma]\!] \subseteq [\![\rho < \tau_1]\!] \cap [\![\check{t}^+ < \sigma]\!] \subseteq [\![v_\rho = 0]\!] \cap [\![v_\sigma = 0]\!]$$

because

$$v_\sigma \le \chi([\![\tau = \check{t}^+]\!] \cap [\![\sigma = \tau_1]\!]) \le \chi([\![\tau = \check{t}^+]\!] \cap [\![\sigma = \tau]\!]).$$

So

$$P_\rho v_\sigma - v_\rho = (P_\rho v_\sigma - v_\rho) \times \chi[\![\rho < \sigma]\!] = P_\rho((v_\sigma - v_\rho)) \times \chi[\![\rho < \sigma]\!]) = 0.$$

Thus $\boldsymbol{v}^{\#}$ is constant on $[\tau \wedge \check{t}^+, \tau]$ and $[\![\tau \wedge \check{t}^+ < \tau]\!] \subseteq [\![v^{\#}_{<\tau} = v^{\#}_{\tau}]\!]$.

If $\sigma \le \tau \wedge \check{t}$ then $[\![\tau = \check{t}^+]\!] \subseteq [\![\sigma < \tau_1]\!] \subseteq [\![v_\sigma = 0]\!]$, so $v^{\#}_{\tau \wedge \check{t}} = 0 = v^{\#}_{<\tau \wedge \check{t}}$. Now

$$\begin{aligned}[\![\tau < \check{t}^+]\!] &\subseteq [\![\tau \le \check{t}]\!] \cup ([\![\check{t} < \tau]\!] \cap [\![\tau < \check{t}^+]\!]) = [\![\tau \le \check{t}]\!] \\ &\subseteq [\![v^{\#}_{\tau} = v^{\#}_{\tau \wedge \check{t}}]\!] \cap [\![v^{\#}_{<\tau} = v^{\#}_{<\tau \wedge \check{t}}]\!] \subseteq [\![v^{\#}_{\tau} = v^{\#}_{<\tau}]\!]\end{aligned}$$

and

$$[\![v^{\#}_{\tau} = v^{\#}_{<\tau}]\!] \supseteq 1 \setminus [\![\tau = \check{t}^+]\!] \supseteq a^*.$$

(e)(i) If we think of $v \mapsto \boldsymbol{v}$, $\boldsymbol{v} \mapsto \boldsymbol{v}^{\#}$, $\boldsymbol{v}^{\#} \mapsto v^{\#}_{\tau}$ and $\boldsymbol{v}^{\#} \mapsto v^{\#}_{<\tau}$ as functions, then $v \mapsto v^{\#}_{\tau}$ and $v \mapsto v^{\#}_{<\tau}$ are additive, and the set

$$A = \{v : v \in L^0(\mathfrak{A}_{\tau_1}) \cap L^1_{\bar{\mu}},\, v \ge 0,\, a^* \subseteq [\![v^{\#}_{\tau} = v^{\#}_{<\tau}]\!]\}$$

is closed under addition and multiplication by non-negative scalars. So

$$D = \{d : d \in \mathfrak{A}_{\tau_1},\, v \in A \text{ whenever } v \in L^0(\mathfrak{A}_{\tau_1}) \cap L^1_{\bar{\mu}} \text{ and } 0 \le v \le \chi d\}$$

is an ideal of $\mathfrak{A}_{\tau_1}$. (If d_0, $d_1 \in D$, $d \in \mathfrak{A}_{\tau_1}$, $d \subseteq d_0 \cup d_1$, $v \in L^0(\mathfrak{A}_{\tau_1}) \cap L^1_{\bar\mu}$ and $0 \le v \le \chi d$, then $v = v \times \chi d_0 + v \times \chi(d_1 \setminus d_0)$ is the sum of two members of A.) We know also that $a^* \in D$ ((b) above, with 643Cd), $a_C \in D$ whenever $\emptyset \ne C \subseteq \mathcal{T} \wedge \tau$ (by (c)) and $[\![\tau = \check{t}^+]\!] \in D$ whenever $t \in T_{\text{r-i}}$ (by (d)). So

$$\sup D \supseteq a^* \cup \operatorname{acc}(\tau) \cup (1 \setminus \operatorname{app}(\tau)) = 1$$

by the definition of $\operatorname{acc}(\tau)$ and 643D, and $\sup_{d \in D} \bar\mu d = 1$.

(ii) Now suppose just that $v \in L^0(\mathfrak{A}_{\tau_1})$ is non-negative and integrable. Let $\delta > 0$ be such that $\mathbb{E}(v \times \chi a) \le \epsilon$ whenever $\bar\mu a \le \delta$. Let $d \in D$ be such that $\bar\mu(1 \setminus d) \le \delta$, and $M > 0$ such that $\mathbb{E}((v - M\chi 1)^+) \le \epsilon$. Set $w = \frac{1}{M}(v \wedge M\chi d)$. Then $w \in L^0(\mathfrak{A}_{\tau_1})$, $0 \le w \le v$ and $v - w \le (v - M\chi 1)^+ + v \times \chi(\operatorname{acc}(\tau) \setminus d)$ so $\mathbb{E}(v - w) \le 2\epsilon$. Also $w \in A$ (using (d) above), and $a^* \subseteq [\![w^\#_\tau = w^\#_{<\tau}]\!]$. We know too that $\boldsymbol{v}^\# - \boldsymbol{w}^\# = (\boldsymbol{v} - \boldsymbol{w})^\#$. Since $\boldsymbol{v} - \boldsymbol{w}$ is non-negative, 626M tells us that

$$\mathbb{E}(v^\#_\tau - w^\#_\tau) \le \mathbb{E}(v_\tau - w_\tau) \le 2\epsilon.$$

Now $w^\#_{<\tau} \le v^\#_{<\tau} \le v^\#_\tau$ and $w^\#_{<\tau} \le w^\#_\tau \le v^\#_\tau$ so

$$\mathbb{E}((v^\#_\tau - v^\#_{<\tau}) \times \chi a^*) \le \mathbb{E}(v^\#_\tau - w^\#_\tau) + \mathbb{E}((w^\#_\tau - w^\#_{<\tau}) \times \chi a^*) \le 2\epsilon.$$

As ϵ is arbitrary, $a^* \subseteq [\![v^\#_\tau = v^\#_{<\tau}]\!]$ and the proof is complete.

643I Lemma Suppose that $\langle \mathfrak{A}_t \rangle_{t \in T}$ is right-continuous. Take $\tau_1 \in \mathcal{T}$ and a martingale $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \le \tau_1}$. Suppose that $\epsilon > 0$ is such that $[\![\sigma < \tau_1]\!] \subseteq [\![|u_\sigma| \le \epsilon]\!]$ for every $\sigma \le \tau_1$. Then there is a martingale $\tilde{\boldsymbol{u}} = \langle \tilde u_\sigma \rangle_{\sigma \le \tau_1}$ such that $\operatorname{Osclln}(\tilde{\boldsymbol{u}}) \le 2\epsilon\chi 1$ and $\boldsymbol{u} - \tilde{\boldsymbol{u}}$ is of bounded variation.

proof Write v for u_{τ_1}.

(a)(i) Set $\acute v = v^+$ and $\grave v = v^-$, so that $\acute v$ and $\grave v$ are non-negative integrable members of $L^0(\mathfrak{A}_{\tau_1})$. As in 643H, set

$$\acute v_\sigma = \acute v \times [\![\sigma = \tau_1]\!], \quad \grave v_\sigma = \grave v \times [\![\sigma = \tau_1]\!], \quad v_\sigma = v \times [\![\sigma = \tau_1]\!]$$

for $\sigma \le \tau_1$,

$$\acute{\boldsymbol{v}} = \langle \acute v_\sigma \rangle_{\sigma \le \tau_1}, \quad \grave{\boldsymbol{v}} = \langle \grave v_\sigma \rangle_{\sigma \le \tau_1}, \quad \boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \le \tau_1},$$

and let $\acute{\boldsymbol{v}}^\#$, $\grave{\boldsymbol{v}}^\#$ be the previsible variations of $\acute{\boldsymbol{v}}$ and $\grave{\boldsymbol{v}}$ respectively; write $\boldsymbol{v}^\#$ for $\acute{\boldsymbol{v}}^\# - \grave{\boldsymbol{v}}^\#$, the previsible version of $\boldsymbol{v}$. Then $\boldsymbol{v} - \boldsymbol{v}^\#$ is a martingale (626Ka). Set $\tilde{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{v} + \boldsymbol{v}^\#$, so that $\tilde{\boldsymbol{u}}$ is a martingale. Note that $\acute{\boldsymbol{v}}$, $\grave{\boldsymbol{v}}$, $\acute{\boldsymbol{v}}^\#$ and $\grave{\boldsymbol{v}}^\#$ are non-negative monotonic processes so are of bounded variation; accordingly $\boldsymbol{v}$, $\boldsymbol{v}^\#$ and $\boldsymbol{u} - \tilde{\boldsymbol{u}} = \boldsymbol{v} - \boldsymbol{v}^\#$ are of bounded variation.

(ii) $u_{\tau_1} = v_{\tau_1}$, so if $\sigma \le \tau_1$ then

$$[\![\sigma = \tau_1]\!] \subseteq [\![u_\sigma - v_\sigma = 0]\!]$$

while

$$[\![\sigma < \tau_1]\!] \subseteq [\![|u_\sigma| \le \epsilon]\!] \cap [\![v_\sigma = 0]\!] \subseteq [\![|u_\sigma - v_\sigma| \le \epsilon]\!];$$

thus $|u_\sigma - v_\sigma| \le \epsilon\chi 1$. It follows at once that $|u_{<\sigma} - v_{<\sigma}| \le \epsilon\chi 1$ and that $|u_\sigma - v_\sigma - u_{<\sigma} + v_{<\sigma}| \le 2\epsilon\chi 1$ for every $\sigma \le \tau_1$.

(iii) If $\sigma \le \tau_1$ and $P_{<\sigma}$ is the conditional expectation associated with the subalgebra $\mathfrak{A}_{<\sigma}$, $|\tilde u_\sigma - P_{<\sigma}\tilde u_\sigma| \le 2\epsilon\chi 1$. **P** As $v^\#_\sigma \in L^0(\mathfrak{A}_{<\sigma})$ (641D), $P_{<\sigma} v^\#_\sigma = v^\#_\sigma$ and

$$\begin{aligned}|\tilde u_\sigma - P_{<\sigma}\tilde u_\sigma| &= |u_\sigma - v_\sigma - P_{<\sigma}u_\sigma + P_{<\sigma}v_\sigma + v^\#_\sigma - P_{<\sigma}v^\#_\sigma| \\ &\le |u_\sigma - v_\sigma| + |P_{<\sigma}(u_\sigma - v_\sigma)| \le |u_\sigma - v_\sigma| + P_{<\sigma}|u_\sigma - v_\sigma| \\ &\le \epsilon\chi 1 + P_{<\sigma}(\epsilon\chi 1) = 2\epsilon\chi 1. \ \textbf{Q}\end{aligned}$$

(b) Take $\tau \le \tau_1$.

(i) Set $a^* = \operatorname{app}(\tau) \setminus \operatorname{acc}(\tau)$. Then 643H and the last remark in (i) above tell us that

$$a^* \subseteq [\![\acute v^\#_\tau = \acute v^\#_{<\tau}]\!] \cap [\![\grave v^\#_\tau = \grave v^\#_{<\tau}]\!] \subseteq [\![v^\#_\tau = v^\#_{<\tau}]\!]$$
$$\subseteq [\![\tilde u_\tau - \tilde u_{<\tau} = u_\tau - u_{<\tau} - v_\tau + v_{<\tau}]\!] \subseteq [\![|\tilde u_\tau - \tilde u_{<\tau}| \le 2\epsilon\chi 1]\!].$$

(ii) Now suppose that C is a non-empty upwards-directed subset of $\mathcal{T} \wedge \tau$ with supremum ρ and that $a_C = [\![\rho = \tau]\!] \setminus \sup_{\sigma\in C} [\![\sigma = \tau]\!]$, as in 643F and 643H. Of course a_C is also expressible as $[\![\rho = \tau]\!] \cap \inf_{\sigma\in C} [\![\sigma < \rho]\!]$. Next, $a_C \subseteq [\![\tilde u_{<\rho} = P_{<\rho}\tilde u_\rho]\!]$ by 643Bb, because $\boldsymbol{\tilde u}$ is a martingale. Now

$$a_C \subseteq [\![\tilde u_{<\rho} = P_{<\rho}\tilde u_\rho]\!] \cap [\![\tilde u_\tau = \tilde u_\rho]\!] \cap [\![\tilde u_{<\tau} = \tilde u_{<\rho}]\!]$$

(using 641L) again)

$$\subseteq [\![|\tilde u_\tau - \tilde u_{<\tau}| = |\tilde u_\rho - P_{<\rho}\tilde u_\rho|]\!] \subseteq [\![|\tilde u_\tau - \tilde u_{<\tau}| \le 2\epsilon\chi 1]\!]$$

by (a-iii) above.

(iii) Next, suppose that $t \in T$ is isolated on the right, and define $\check t^+$ as in 643D. Set $\rho = \tau \wedge \check t^+$ and $c = [\![\check t < \rho]\!] = [\![\rho > t]\!]$ (611E(a-i-δ).

($\boldsymbol\alpha$) $c \subseteq [\![\tilde u_{<\rho} = \tilde u_{\check t}]\!]$. **P** For any $\sigma \in \mathcal{T}$,

$$[\![\check t < \sigma]\!] \cap [\![\sigma < \rho]\!] \subseteq [\![\check t < \sigma]\!] \cap [\![\sigma < \check t^+]\!] = 0.$$

So if $I \in \mathcal{I}(\mathcal{T} \wedge \tau_1)$ contains $\check t$, and $\tilde u_{I<\tau}$ is defined from $\boldsymbol{\tilde u}$ as in 641Ea, then $c = [\![\check t < \rho]\!] \subseteq [\![\tilde u_{I<\tau} = \tilde u_{\check t}]\!]$. Taking the limit as $I \uparrow \mathcal{I}(\mathcal{T} \wedge \tau_1)$, $c \subseteq [\![\tilde u_{<\rho} = \tilde u_{\check t}]\!]$. **Q**

($\boldsymbol\beta$) $c \in \mathfrak{A}_{\check t} \cap \mathfrak{A}_\rho$. Now if $a \in \mathfrak{A}$ and $a \subseteq c$, then $a \in \mathfrak{A}_{\check t} = \mathfrak{A}_t$ (611Hb) iff $a \in \mathfrak{A}_\rho$. **P** If $a \in \mathfrak{A}_{\check t}$ then $a \in \mathfrak{A}_\rho$ by 611H(a-i). In the other direction, set $\mathfrak{B} = \{b : b \in \mathfrak{A},\ c \cap b \in \mathfrak{A}_{\check t}\}$. Then $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$. If $\sigma \in \mathcal{T}$, then

$$c \cap [\![\sigma < \rho]\!] = c \cap [\![\sigma < \check t^+]\!]$$

(because $c = [\![\check t^+ \le \rho]\!] = [\![\check t^+ = \rho]\!]$)

$$= c \cap [\![\sigma \le \check t]\!] \in \mathfrak{A}_{\check t},$$

so $[\![\sigma < \rho]\!] \in \mathfrak{B}$. By the definition of $\mathfrak{A}_{<\rho}$ (641Ba), $\mathfrak{A}_{<\rho} \subseteq \mathfrak{B}$ and $c \cap a \in \mathfrak{A}_{\check t}$. **Q**

($\boldsymbol\gamma$) $c \subseteq [\![P_{<\rho}\tilde u_\rho = \tilde u_{\check t}]\!]$. **P** For $\alpha \in \mathbb{R}$,

$$[\![P_{<\rho}\tilde u_\rho \times \chi c > \alpha]\!] = [\![P_{<\rho}\tilde u_\rho > \alpha]\!] \cap c \text{ if } \alpha \ge 0,$$
$$= [\![P_{<\rho}\tilde u_\rho > \alpha]\!] \cup (1 \setminus c) \text{ if } \alpha \le 0,$$

and in either case belongs to $\mathfrak{A}_{\check t}$. So $P_{<\rho}\tilde u_\rho \times \chi c \in L^0(\mathfrak{A}_{\check t})$. Now if $a \in \mathfrak{A}_{\check t}$,

$$\mathbb{E}(((\tilde u_{\check t} \times \chi c) \times \chi a) = \mathbb{E}(\tilde u_{\check t} \times \chi(c \cap a)) = \mathbb{E}(\tilde u_{\check t^+} \times \chi(c \cap a))$$

(because $\tilde u$ is a martingale)

$$= \mathbb{E}(\tilde u_\rho \times \chi(c \cap a))$$

(because $c \subseteq [\![\rho = \check t^+]\!]$)

$$= \mathbb{E}(P_{<\rho}\tilde u_\rho \times \chi(c \cap a))$$

(because $c \cap a \in \mathfrak{A}_{<\rho}$)

$$= \mathbb{E}((P_{<\rho}\tilde u_\rho \times \chi c) \times \chi a)).$$

So $\tilde u_{\check t} \times \chi c = P_{<\rho}\tilde u_\rho \times \chi c$, that is, $c \subseteq [\![P_{<\rho}\tilde u_\rho = \tilde u_{\check t}]\!]$. **Q**

($\boldsymbol\delta$) Putting (α) and (γ) together,

$$\begin{aligned}
[\![\tau = \check{t}^{+}]\!] &= [\![\rho = \check{t}^{+}]\!] \cap [\![\rho = \tau]\!] \subseteq c \cap [\![\tilde{u}_{\tau} = \tilde{u}_{\rho}]\!] \cap [\![\tilde{u}_{<\tau} = \tilde{u}_{<\rho}]\!] \\
&\subseteq [\![\tilde{u}_{<\rho} = P_{<\rho}\tilde{u}_{\rho}]\!] \cap [\![\tilde{u}_{\tau} = \tilde{u}_{\rho}]\!] \cap [\![\tilde{u}_{<\tau} = \tilde{u}_{<\rho}]\!] \\
&\subseteq [\![|\tilde{u}_{\tau} - \tilde{u}_{<\tau}| = |\tilde{u}_{\rho} - P_{\rho}\tilde{u}_{\rho}|]\!] \subseteq [\![|\tilde{u}_{\tau} - \tilde{u}_{<\tau}| \le 2\epsilon\chi 1]\!]
\end{aligned}$$

by (a-iii) again.

(iv) Assembling (i)-(iii), we see that

$$1 = a^{*} \cup \sup_{\emptyset \ne C \subseteq \mathcal{T}\wedge\tau} a_C \cup \sup_{t \in T_{\text{r-i}}} [\![\tau = \check{t}^{+}]\!] \subseteq [\![|\tilde{u}_{\tau} - \tilde{u}_{<\tau}| \le 2\epsilon\chi 1]\!]$$

and $|\tilde{u}_{\tau} - \tilde{u}_{<\tau}| \le 2\epsilon\chi 1$. As τ is arbitrary and $\tilde{\boldsymbol{u}}$ is near-simple (632I), $\operatorname{Osclln}(\tilde{\boldsymbol{u}}) \le 2\epsilon\chi 1$ (641Nb), and we have found a suitable auxiliary martingale.

643J Corollary Suppose that $\langle \mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Take an interval $\mathcal{S} = [\tau, \tau']$ where $\tau \le \tau'$ in $\mathcal{T}$, and a martingale $\boldsymbol{u} = \langle u_{\sigma}\rangle_{\sigma\in\mathcal{S}}$. Suppose that $\epsilon > 0$ is such that $[\![\sigma < \tau']\!] \subseteq [\![|u_{\sigma} - u_{\tau}| \le \epsilon]\!]$ for every $\sigma \in \mathcal{S}$. Then there is a martingale $\tilde{\boldsymbol{u}} = \langle \tilde{u}_{\sigma}\rangle_{\sigma\in\mathcal{S}}$ such that $\operatorname{Osclln}(\tilde{\boldsymbol{u}}) \le \epsilon\chi 1$, $\tilde{u}_{\tau} = 0$ and $\boldsymbol{u} - \tilde{\boldsymbol{u}}$ is of bounded variation.

proof Set $u'_{\sigma} = P_{\sigma}(u_{\tau'} - u_{\tau})$ for $\sigma \le \tau'$, so that $\boldsymbol{u}' = \langle u'_{\sigma}\rangle_{\sigma\le\tau'}$ is a martingale. Then $[\![\sigma < \tau']\!] \subseteq [\![|u'_{\sigma}| \le \epsilon]\!]$ for every $\sigma \le \tau'$. **P** We have

$$[\![\sigma \le \tau]\!] = [\![\sigma = \sigma\wedge\tau]\!] \subseteq [\![P_{\sigma}(u_{\tau'} - u_{\tau}) = P_{\sigma\wedge\tau}(u_{\tau'} - u_{\tau})]\!]$$

(622Bb once more)

$$= [\![u'_{\sigma} = P_{\sigma}P_{\tau}(u_{\tau'} - u_{\tau})]\!]$$

(622Ba)

$$\subseteq [\![u'_{\sigma} = 0]\!] \subseteq [\![|u'_{\sigma}| \le \epsilon]\!],$$

while also

$$\begin{aligned}
[\![\tau \le \sigma]\!] \cap [\![\sigma < \tau']\!] &= [\![\sigma = \sigma\vee\tau]\!] \cap [\![\sigma\vee\tau < \tau']\!] \\
&\subseteq [\![u'_{\sigma} = P_{\sigma\vee\tau}(u_{\tau'} - u_{\tau})]\!] \cap [\![|u_{\sigma\vee\tau} - u_{\tau}| \le \epsilon]\!] \\
&\subseteq [\![u'_{\sigma} = u_{\sigma\vee\tau} - u_{\tau}]\!] \cap [\![|u_{\sigma\vee\tau} - u_{\tau}| \le \epsilon]\!] \subseteq [\![|u'_{\sigma}| \le \epsilon]\!].
\end{aligned}$$

Now

$$[\![\sigma < \tau']\!] \subseteq [\![\sigma \le \tau]\!] \cup ([\![\tau \le \sigma]\!] \cup [\![\sigma < \tau']\!]) \subseteq [\![|u'_{\sigma}| \le \epsilon]\!],$$

as claimed. **Q**

By 643I, there is a martingale $\tilde{\boldsymbol{u}}' = \langle \tilde{u}'_{\sigma}\rangle_{\sigma\le\tau'}$ such that $\operatorname{Osclln}(\tilde{\boldsymbol{u}}') \le 2\epsilon\chi 1$ and $\boldsymbol{u}' - \tilde{\boldsymbol{u}}'$ is of bounded variation. Set $\tilde{\boldsymbol{u}} = \langle \tilde{u}_{\sigma}\rangle_{\sigma\in\mathcal{S}}$ where $\tilde{u}_{\sigma} = \tilde{u}'_{\sigma} - \tilde{u}'_{\tau}$ for $\sigma \in \mathcal{S}$. Then $\tilde{\boldsymbol{u}}$ is a martingale and $\tilde{u}_{\tau} = 0$. In the language of 613Cc, $\Delta\tilde{\boldsymbol{u}} = \Delta(\tilde{\boldsymbol{u}}'\restriction\mathcal{S})$, so

$$\operatorname{Osclln}(\tilde{\boldsymbol{u}}) = \operatorname{Osclln}(\tilde{\boldsymbol{u}}'\restriction\mathcal{S}) \le \operatorname{Osclln}(\tilde{\boldsymbol{u}}') \le 2\epsilon$$

(using 618D(b-i)). We see also that $u'_{\sigma} = u_{\sigma} - u_{\tau}$ for every $\sigma \in \mathcal{S}$, so $u_{\sigma} - \tilde{u}_{\sigma} = u'_{\sigma} - \tilde{u}'_{\sigma} + u_{\tau} + \tilde{u}'_{\tau}$ for $\sigma \in \mathcal{S}$, $\Delta(\boldsymbol{u} - \tilde{\boldsymbol{u}}) = \Delta((\boldsymbol{u}' - \tilde{\boldsymbol{u}}')\restriction\mathcal{S})$ and $\int_{\mathcal{S}} |d(\boldsymbol{u} - \tilde{\boldsymbol{u}})|$ is defined in $L^0(\mathfrak{A})$ and equal to $\int_{\mathcal{S}} |d(\boldsymbol{u}' - \tilde{\boldsymbol{u}}')|$; thus $\boldsymbol{u} - \tilde{\boldsymbol{u}}$ is of bounded variation, as required.

643K Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\langle \tau_n\rangle_{n\in\mathbb{N}}$ a non-decreasing sequence in $\mathcal{S}$ such that $\mathcal{S} \subseteq \bigcup_{n\in\mathbb{N}}[\tau_0, \tau_n]$. Suppose that for each $n \in \mathbb{N}$ we are given a fully adapted process $\boldsymbol{u}_n = \langle u_{n\sigma}\rangle_{\sigma\in\mathcal{S}\cap[\tau_n,\tau_{n+1}]}$ starting from $u_{n\tau_n} = 0$.

(a) There is a unique fully adapted process $\boldsymbol{u} = \langle u_{\sigma}\rangle_{\sigma\in\mathcal{S}}$ such that

$$u_{\sigma} = u_{n\sigma} + \sum_{i=0}^{n-1} u_{i\tau_{i+1}} \text{ whenever } \sigma \in \mathcal{S},\ n \in \mathbb{N} \text{ and } \tau_n \le \sigma \le \tau_{n+1}. \qquad (*)$$

(b) If every $\boldsymbol{u}_n$ is a martingale, then $\boldsymbol{u}$ is a martingale.

(c) If every $\boldsymbol{u}_n$ is order-bounded, then $\boldsymbol{u}$ is locally order-bounded and $\operatorname{Osclln}(\boldsymbol{u}\restriction\mathcal{S}\wedge\tau_n) = \sup_{i<n}\operatorname{Osclln}(\boldsymbol{u}_i)$ for every $n \in \mathbb{N}$.

proof Set $\mathcal{S}_0 = \bigcup_{n\in\mathbb{N}} \mathcal{S}\cap[\tau_n,\tau_{n+1}]$.

(a) $\mathcal{S}_0$ is a sublattice of $\mathcal{T}$. **P** If σ, $\sigma' \in \mathcal{S}_0$, let m, $n \in \mathbb{N}$ be such that $\tau_m \le \sigma \le \tau_{m+1}$ and $\tau_n \le \sigma' \le \tau_{n+1}$. We can suppose that $m \le n$. In this case $\sigma\wedge\sigma' \in [\tau_m,\tau_{m+1}]$ and $\sigma\vee\sigma' \in [\tau_n,\tau_{n+1}]$ belong to $\mathcal{S}_0$. **Q**

The formula (*) defines a fully adapted process $\boldsymbol{u}^*$ on $\mathcal{S}_0$. **P** If $n \in \mathbb{N}$ and $\tau_n \le \sigma \le \tau_{n+1}$ then $u_{n\sigma} + \sum_{i=0}^{n-1} u_{i\tau_{i+1}} \in L^0(\mathfrak{A}_\sigma)$ because $L^0(\mathfrak{A}_{\tau_{i+1}}) \subseteq L^0(\mathfrak{A}_\sigma)$ for every $i < n$. If $m \le n$ in $\mathbb{N}$, $\sigma \in [\tau_m,\tau_{m+1}]$, $\sigma' \in [\tau_n,\tau_{n+1}]$ and $c = [\![\sigma = \sigma']\!]$, then

— if $m = n$ then $c \subseteq [\![u_{n\sigma} = u_{n\sigma'}]\!] = [\![u_{n\sigma} + \sum_{i=0}^{n-1} u_{i\tau_{i+1}} = u_{n\sigma'} + \sum_{i=0}^{n-1} u_{i\tau_{i+1}}]\!]$;
— if $m < n$ then

$$\begin{aligned} c &\subseteq [\![\sigma = \tau_{m+1}]\!] \cap \inf_{m<i<n} [\![\tau_i = \tau_{i+1}]\!] \cap [\![\sigma' = \tau_n]\!] \\ &\subseteq [\![u_{m\sigma} = u_{m\tau_{m+1}}]\!] \cap \inf_{m<i<n} [\![u_{i\tau_{i+1}} = 0]\!] \cap [\![u_{n\sigma'} = 0]\!] \\ &\subseteq [\![u_{m\sigma} + \textstyle\sum_{i=0}^{m-1} u_{i\tau_{i+1}} = u_{n\sigma'} + \sum_{i=0}^{n-1} u_{i\tau_{i+1}}]\!], \end{aligned}$$

In particular, this shows that if $\sigma = \sigma'$ belongs to $[\tau_n,\tau_{n+1}]$ for more than one n, then (*) gives a well-assigned value for u_σ; and in general we have $[\![\sigma = \sigma']\!] \subseteq [\![u_\sigma = u_{\sigma'}]\!]$, so that $\boldsymbol{u}^*$ is fully adapted. **Q**

Next, $\mathcal{S}_0$ covers $\mathcal{S}\wedge\tau_n$ for every n. **P** Induce on n. For $n = 0$ this is trivial. For the inductive step to $n+1 \ge 1$, if $\sigma \in \mathcal{S}\wedge\tau_{n+1}$ then $\{\sigma\}$ is covered by $\{\sigma\wedge\tau_n, \sigma\vee\tau_n\} \subseteq \mathcal{S}\wedge\tau_n \cup \mathcal{S}\cap[\tau_n,\tau_{n+1}]$ and therefore by $\mathcal{S}_0$. As σ is arbitrary, $\mathcal{S}\wedge\tau_{n+1}$ is covered by $\mathcal{S}_0$. **Q**

Consequently $\mathcal{S}_0$ covers $\mathcal{S}$. Writing $\tilde{\boldsymbol{u}}^*$ for the fully adapted extension of $\boldsymbol{u}^*$ to the covered envelope $\tilde{\mathcal{S}}_0$ of $\mathcal{S}_0$, we have $\mathcal{S} \subseteq \tilde{\mathcal{S}}_0$ and $\boldsymbol{u} = \tilde{\boldsymbol{u}}^*\restriction\mathcal{S}$ is a fully adapted process satisfying the condition (*). And because (*) determines the process $\boldsymbol{u}^* = \boldsymbol{u}\restriction\mathcal{S}_0$, it determines $\boldsymbol{u}$ (612R).

(b) $\boldsymbol{u}\restriction\mathcal{S}_0$ is a martingale. **P** Suppose that m, $n \in \mathbb{N}$, $\sigma \in [\tau_m,\tau_{m+1}]$, $\sigma' \in [\tau_n,\tau_{n+1}]$ and $\sigma \le \sigma'$. If $m > n$ then $\sigma = \sigma'$ and of course $P_\sigma u_{\sigma'} = u_\sigma$. If $m = n$ then

$$P_\sigma u_{\sigma'} = \textstyle\sum_{i=0}^{m-1} P_\sigma u_{i\tau_{i+1}} + P_\sigma u_{m\sigma'} = \sum_{i=0}^{m-1} u_{i\tau_{i+1}} + u_{m\sigma} = u_\sigma.$$

If $m < n$ then

$$\begin{aligned} P_\sigma u_{\sigma'} &= \sum_{i=0}^{m-1} P_\sigma u_{i\tau_{i+1}} + \sum_{i=m}^{n-1} P_\sigma u_{i\tau_{i+1}} + P_\sigma u_{n\sigma'} \\ &= \sum_{i=0}^{m-1} u_{i\tau_{i+1}} + P_\sigma u_{m\tau_{m+1}} + \sum_{i=m+1}^{n-1} P_\sigma u_{i\tau_{i+1}} + P_\sigma u_{n\sigma'} \\ &= \sum_{i=0}^{m-1} u_{i\tau_{i+1}} + u_{m\sigma} + \sum_{i=m+1}^{n-1} P_\sigma P_{\tau_i} u_{i\tau_{i+1}} + P_\sigma P_{\tau_n} u_{n\sigma'} \\ &= u_\sigma + \sum_{i=m+1}^{n-1} P_\sigma u_{i\tau_i} + P_\sigma u_{n\tau_n} = u_\sigma. \end{aligned}$$

As σ and σ' are arbitrary, $\boldsymbol{u}\restriction\mathcal{S}_0$ is a martingale. **Q**

Because $\mathcal{S}_0$ covers $\mathcal{S}$ and is cofinal with $\mathcal{S}$, $\boldsymbol{u}$ is a martingale. **P** By 622Oa, there is a martingale $\boldsymbol{u}'$ on the ideal $\mathcal{S}'$ of $\mathcal{T}$ generated by $\mathcal{S}_0$ which extends $\boldsymbol{u}\restriction\mathcal{S}_0$. Now $\mathcal{S} \subseteq \mathcal{S}'$ so $\boldsymbol{u}'\restriction\mathcal{S}$ is an extension of $\boldsymbol{u}\restriction\mathcal{S}_0$, and $\boldsymbol{u} = \boldsymbol{u}'\restriction\mathcal{S}$ is a martingale. **Q**

(c) Since $|u_{n\sigma'} - u_{n\sigma}| = |u_{\sigma'} - u_\sigma|$ whenever $\tau_n \le \sigma \le \sigma' \le \tau_{n+1}$, $\operatorname{Osclln}(\boldsymbol{u}\restriction[\tau_n,\tau_{n+1}]) = \operatorname{Osclln}(\boldsymbol{u}_n)$ for every n. Now an easy induction on n, using 618Db for the inductive step, shows that

$$\operatorname{Osclln}(\boldsymbol{u}\restriction[\tau_0,\tau_n]) = \sup\nolimits_{i<n} \operatorname{Osclln}(\boldsymbol{u}\restriction[\tau_i,\tau_{i+1}]) = \sup\nolimits_{i<n} \operatorname{Osclln}(\boldsymbol{u}_i)$$

for every n.

643L Lemma Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$ with a least element and a non-decreasing sequence $\langle\tau_n\rangle_{n\in\mathbb{N}}$ such that $\{\tau_n : n \in \mathbb{N}\}$ is cofinal with $\mathcal{S}$, and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a martingale. Then for any $\epsilon > 0$ there is a local martingale $\tilde{\boldsymbol{u}} = \langle\tilde{u}_\sigma\rangle_{\sigma\in\mathcal{S}}$ such that $\mathrm{Osclln}(\tilde{\boldsymbol{u}}\restriction\mathcal{S}\wedge\tau) \le \epsilon\chi1$ for every $\tau \in \mathcal{S}$ and $\boldsymbol{u} - \tilde{\boldsymbol{u}}$ is locally of bounded variation.

proof (a) Choose $\langle k_n\rangle_{n\in\mathbb{N}}$ and $\langle\sigma_i\rangle_{i\in\mathbb{N}}$ as follows. Start with $k_0 = 0$ and $\sigma_0 = \min\mathcal{S}$. Given $k_n \in \mathbb{N}$ and $\sigma_{k_n} \in \mathcal{S}\wedge\tau_n$, then, because $\boldsymbol{u}$ is locally near-simple (632Ia), there is a non-decreasing $\langle\sigma_{nj}\rangle_{j\in\mathbb{N}}$ in $[\sigma_{k_n}, \tau_n]$ such that $\sigma_{n0} = \sigma_{k_n}$, $\inf_{j\in\mathbb{N}}[\![\sigma_{kj} < \tau_n]\!] = 0$ and $[\![\sigma < \sigma_{n,j+1}]\!] \subseteq [\![|u_\sigma - u_{\sigma_{nj}}| \le \frac{1}{4}\epsilon]\!]$ whenever $j \in \mathbb{N}$ and $\sigma_{nj} \le \sigma \le \sigma_{n,j+1}$ (631Ra). Let $k \in \mathbb{N}$ be such that $\bar{\mu}[\![\sigma_{nk} < \tau_n]\!] \le 2^{-n}$; set $k_{n+1} = k_n + k$ and $\sigma_{k_n+i} = \sigma_{ni}$ for $i \le k$. Continue.

(b) We see that $\langle\sigma_i\rangle_{i\in\mathbb{N}}$ is a non-decreasing sequence in $\mathcal{S}$, that

$$\lim_{i\to\infty}\bar{\mu}[\![\sigma_i < \tau_n]\!] = \lim_{m\to\infty}\bar{\mu}[\![\sigma_{k_{m+1}} < \tau_n]\!] \le \lim_{m\to\infty}\bar{\mu}[\![\sigma_{k_{m+1}} < \tau_m]\!] = 0$$

for every $n \in \mathbb{N}$, and that $[\![\sigma < \sigma_{i+1}]\!] \subseteq [\![|u_\sigma - u_{\sigma_i}| \le \frac{1}{4}\epsilon]\!]$ whenever $i \in \mathbb{N}$ and $\sigma_i \le \sigma \le \sigma_{i+1}$. Set $\mathcal{S}_0 = \bigcup_{i\in\mathbb{N}}\mathcal{S}\wedge\sigma_i$. Then $\mathcal{S}_0$ is an ideal of $\mathcal{S}$ covering every τ_n and therefore covering $\mathcal{S}$.

(c) For each $i \in \mathbb{N}$, we can apply 643J to $\boldsymbol{u}\restriction[\sigma_i, \sigma_{i+1}]$ to see that there is a martingale $\tilde{\boldsymbol{u}}_i = \langle\tilde{u}_{i\sigma}\rangle_{\sigma_i\le\sigma\le\sigma_{i+1}}$ such that $\mathrm{Osclln}(\tilde{\boldsymbol{u}}_i) \le 2^{-i-1}\epsilon$, $\tilde{u}_{i\sigma_i} = 0$ and $\tilde{\boldsymbol{u}}_i - \boldsymbol{u}\restriction[\sigma_i, \sigma_{i+1}]$ is of bounded variation. By 643K, we have a martingale $\tilde{\boldsymbol{u}}' = \langle\tilde{u}'_\sigma\rangle_{\sigma\in\mathcal{S}_0}$ such that $\mathrm{Osclln}(\tilde{\boldsymbol{u}}'\restriction\mathcal{S}\wedge\sigma_i) \le \epsilon$ for every $i \in \mathbb{N}$; it follows that $\mathrm{Osclln}(\tilde{\boldsymbol{u}}'\restriction\mathcal{S}\wedge\sigma) \le \epsilon$ for every $\sigma \in \mathcal{S}_0$.

Consider the difference $(\boldsymbol{u}\restriction\mathcal{S}_0) - \tilde{\boldsymbol{u}}'$. If $i \in \mathbb{N}$ and $\sigma \in [\sigma_i, \sigma_{i+1}]$, this is given by

$$u_\sigma - \tilde{u}'_\sigma = u_\sigma - \tilde{u}_{i\sigma} - \tilde{u}'_{\sigma_i}.$$

So, for any i,

$$\int_{\mathcal{S}\wedge\sigma_i}|d(\boldsymbol{u} - \tilde{\boldsymbol{u}}')| = \sum_{j=0}^{i-1}\int_{[\sigma_j,\sigma_{j+1}]}|d(\boldsymbol{u} - \tilde{\boldsymbol{u}}')| = \sum_{j=0}^{i-1}\int_{[\sigma_j,\sigma_{j+1}]}|d(\boldsymbol{u}_i - \tilde{\boldsymbol{u}}_i)|$$

is well-defined, that is, $(\boldsymbol{u}-\tilde{\boldsymbol{u}}')\restriction\mathcal{S}\wedge\sigma_i$ is of bounded variation. Thus $\boldsymbol{u}\restriction\mathcal{S}_0-\tilde{\boldsymbol{u}}'$ is locally of bounded variation.

(e) For any $i \in \mathbb{N}$,

$$\mathrm{Osclln}(\tilde{\boldsymbol{u}}'\restriction\mathcal{S}\wedge\sigma_i) = \sup_{j<i}\mathrm{Osclln}(\tilde{\boldsymbol{u}}'\restriction[\sigma_j, \sigma_{j+1}])$$

(using 618Db repeatedly)

$$= \sup_{j<i}\mathrm{Osclln}(\tilde{\boldsymbol{u}}_j) \le \epsilon\chi1.$$

So $\mathrm{Osclln}(\boldsymbol{u}'\restriction\mathcal{S}\wedge\sigma) \le \epsilon\chi1$ for any $\sigma \in \mathcal{S}_0$ (618Da again).

(f) As $\mathcal{S}_0 \subseteq \mathcal{S}$ covers $\mathcal{S}$, they have the same covered envelope, and we have a unique fully adapted process $\tilde{\boldsymbol{u}}$ with domain $\mathcal{S}$ extending $\tilde{\boldsymbol{u}}'$. Since $\mathcal{S}_0$ is a covering ideal of $\mathcal{S}$, $\tilde{\boldsymbol{u}}$ is a local martingale. $\boldsymbol{u} - \tilde{\boldsymbol{u}}$ is locally of bounded variation because $(\boldsymbol{u} - \tilde{\boldsymbol{u}})\restriction\mathcal{S}_0$ is locally of bounded variation and we can apply 614Q(b-v). Finally,

$$\sup_{\tau\in\mathcal{S}}\mathrm{Osclln}(\tilde{\boldsymbol{u}}\restriction\mathcal{S}\wedge\tau) = \sup_{\sigma\in\mathcal{S}'}\mathrm{Osclln}(\tilde{\boldsymbol{u}}'\restriction\mathcal{S}_0\wedge\sigma) \le \epsilon\chi1$$

by 618Lc.

643M Theorem Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$ with a least element, and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a semi-martingale. Then for any $\epsilon > 0$ there is a local martingale $\tilde{\boldsymbol{v}} = \langle\tilde{v}_\sigma\rangle_{\sigma\in\mathcal{S}}$ such that $\sup_{\tau\in\mathcal{S}}\mathrm{Osclln}(\tilde{\boldsymbol{v}}\restriction\mathcal{S}\wedge\tau) \le \epsilon\chi1$ and $\boldsymbol{v} - \tilde{\boldsymbol{v}}$ is locally of bounded variation.

proof (a) By 611Pc, $\mathcal{S}$ is finitely full. Because the sum of two processes which are locally of bounded variation is again locally of bounded variation (614Q(b-iii)), it is enough to consider the case in which $\boldsymbol{v}$ is a virtually local martingale, in which case it will be a local martingale and locally near-simple (632I). Let $\mathcal{S}_0$ be a covering ideal of $\mathcal{S}$ such that $\boldsymbol{v}\restriction\mathcal{S}_0$ is a martingale. By 627N, there are a non-decreasing sequence $\langle\tau_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{S}_0$ and a non-decreasing sequence $\langle d_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that

$$d_n \in \mathfrak{A}_{\tau_n}, \quad d_n \subseteq [\![\tau_{n+1} = \tau_n]\!]$$

for every $n \in \mathbb{N}$, and

$$\sup_{n\in\mathbb{N}} d_n \cup [\![\tau \le \tau_n]\!] = 1, \quad v_\tau = \lim_{n\to\infty} v_{\tau\wedge\tau_n}$$

for every $\tau \in \mathcal{S}$.

(b) Set $\mathcal{S}' = \bigcup_{n\in\mathbb{N}} \mathcal{S} \wedge \tau_n$. Then $\boldsymbol{v}\restriction\mathcal{S}'$ is a martingale. By 643L, there is a local martingale $\tilde{\boldsymbol{u}} = \langle \tilde{u}_\sigma\rangle_{\sigma\in\mathcal{S}'}$ such that $\operatorname{Oscln}(\tilde{\boldsymbol{u}}\restriction\mathcal{S}' \wedge \sigma) \le \epsilon\chi 1$ for every $\sigma \in \mathcal{S}'$ and $\boldsymbol{v} - \tilde{\boldsymbol{u}}$ is locally of bounded variation.

(c) By 627O there are processes $\tilde{\boldsymbol{v}} = \langle \tilde{v}_\tau\rangle_{\tau\in\mathcal{S}}$ and $\boldsymbol{v}^\# = \langle v^\#_\tau\rangle_{\tau\in\mathcal{S}}$ such that

$$\tilde{v}_\tau = \lim_{n\to\infty} \tilde{u}_{\tau\wedge\tau_n}, \quad v^\#_\tau = \lim_{n\to\infty} v_{\tau\wedge\tau_n} - \tilde{u}_{\tau\wedge\tau_n}$$

for every $\tau \in \mathcal{S}$. Observe that

$$\tilde{v}_\tau + v^\#_\tau = \lim_{n\to\infty} v_{\tau\wedge\tau_n} = v_\tau$$

for every $\tau \in \mathcal{S}$. Now 627Od tells us that $\tilde{\boldsymbol{v}}$ is a local martingale; 627Oe tells us that $\boldsymbol{v} - \tilde{\boldsymbol{v}} = \boldsymbol{v}^\#$ is locally of bounded variation; and 627Of tells us that $\operatorname{Oscln}(\tilde{\boldsymbol{v}}\restriction\mathcal{S} \wedge \tau) \le \epsilon\chi 1$ for every $\tau \in \mathcal{S}$. So $\tilde{\boldsymbol{v}}$ has the required properties.

Remark Recall that a semi-martingale is the same thing as a local integrator (627Q).

643N I have more than once noted (622P, 624H, 626T) that L^2-martingales have a special place in the theory, and later in this chapter they will become of prime importance. We are now ready for a general result showing why.

Lemma Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ with a greatest element, and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ an integrator. Then for every $\epsilon > 0$ there are an L^∞-martingale $\tilde{\boldsymbol{v}}$ and a process $\boldsymbol{v}'$ of bounded variation, both with domain $\mathcal{S}$, such that $\bar{\mu}[\![\boldsymbol{v} \ne \tilde{\boldsymbol{v}} + \boldsymbol{v}']\!] \le \epsilon$.

proof (a) By 627J/627Q, $\boldsymbol{v}$ is a semi-martingale; let $\boldsymbol{v}_1 = \langle v_{1\sigma}\rangle_{\sigma\in\mathcal{S}}$ be a virtually local martingale such that $\boldsymbol{v} - \boldsymbol{v}_1$ is locally of bounded variation. As $\mathcal{S}$ has a greatest element, $\boldsymbol{v} - \boldsymbol{v}_1$ is actually of bounded variation. Let $\hat{\boldsymbol{v}}_1 = \langle \hat{v}_{1\sigma}\rangle_{\sigma\in\hat{\mathcal{S}}}$ be the fully adapted extension of $\boldsymbol{v}_1$ to the covered envelope $\hat{\mathcal{S}}$ of $\mathcal{S}$. Then there is a non-empty downwards-directed set $A \subseteq \hat{\mathcal{S}}$ such that $\sup_{\rho\in A} \bar{\mu}[\![\rho < \max\mathcal{S}]\!] \le \frac{1}{3}\epsilon$ and $\boldsymbol{v}_2 = R_A(\hat{\boldsymbol{v}}_1)$, as defined in 623B,is a martingale. Set $b = \sup_{\rho\in A} [\![\rho < \max\mathcal{S}]\!]$, so that $\bar{\mu}b \le \frac{1}{3}\epsilon$. Note that $\max\mathcal{S}$ is also the greatest element of $\hat{\mathcal{S}}$ (611M(b-ii)). If $\sigma \in \hat{\mathcal{S}}$ and $\rho \in A$, then

$$1 \setminus b \subseteq [\![\rho = \max\mathcal{S}]\!] \subseteq [\![\sigma \le \rho]\!] \subseteq [\![\hat{v}_{1,\sigma\wedge\rho} = \hat{v}_{1\sigma}]\!];$$

taking the limit as $\rho \downarrow A$,

$$1 \setminus b \subseteq [\![\hat{v}_{1,\sigma} = \lim_{\rho\downarrow A} \hat{v}_{1,\sigma\wedge\rho}]\!], \quad [\![\hat{v}_{1,\sigma} \ne \lim_{\rho\downarrow A} \hat{v}_{1,\sigma\wedge\rho}]\!] \subseteq b,$$

As σ is arbitrary, $[\![\boldsymbol{v}_2 \ne \hat{\boldsymbol{v}}_1]\!] \subseteq b$.

Express $\boldsymbol{v}_2$ as $\langle v_{2\sigma}\rangle_{\sigma\in\hat{\mathcal{S}}}$. Consider the martingale $\boldsymbol{v}_3 = \boldsymbol{P}v_{2,\max\mathcal{S}}$ (622F). This extends $\boldsymbol{v}_2$ to $\mathcal{T}$. By 643L there is a local martingale $\boldsymbol{w} = \langle w_\sigma\rangle_{\sigma\in\mathcal{T}}$ such that $\operatorname{Oscln}(\boldsymbol{w}) \le \chi 1$ and $\boldsymbol{v}_3 - \boldsymbol{w}$ is of bounded variation; replacing $\boldsymbol{w}$ by $\boldsymbol{w} - w_{\min\mathcal{T}}\mathbf{1}$ if necessary. we can arrange that $w_{\min\mathcal{T}} = 0$. Now there is a $\tau' \in \mathcal{T}$ such that $\boldsymbol{w}\restriction\mathcal{T} \wedge \tau'$ is a martingale and $b' = [\![\tau' < \max\mathcal{T}]\!]$ has measure at most $\frac{1}{3}\epsilon$. Also $\boldsymbol{w}$ is near-simple (632Ia again). By 631Ra again, $\mathrm{SL}_1(\boldsymbol{w})$ is true and there is a non-decreasing sequence $\langle\tau_i\rangle_{i\in\mathbb{N}}$ in $\mathcal{T}$ such that $\tau_0 = \min\mathcal{T}$, $\inf_{i\in\mathbb{N}} [\![\tau_i < \max\mathcal{T}]\!] = 0$ and $[\![\sigma < \tau_{i+1}]\!] \subseteq [\![|w_\sigma - w_{\tau_i}| < 1]\!]$ whenever $i \in \mathbb{N}$ and $\sigma \in [\tau_i, \tau_{i+1}]$. Accordingly $|w_{\tau_{n+1}} - w_{\tau_n}| \le 2\chi 1$ whenever $n \in \mathbb{N}$. **P** Looking at the definition in 641Ea, we see that $|w_{I<\tau_{n+1}} - w_{\tau_n}| \times \chi[\![\tau_n < \tau_{n+1}]\!] \le \chi 1$ for every finite sublattice I of $\mathcal{S}$ containing τ_n and τ_{n+1}, so that $|w_{<\tau_{n+1}} - w_{\tau_n}| \times \chi[\![\tau_n < \tau_{n+1}]\!] \le \chi 1$; while $|w_{\tau_{n+1}} - w_{<\tau_{n+1}}| \times \chi[\![\tau_n < \tau_{n+1}]\!] \le \operatorname{Oscln}(\boldsymbol{v}\restriction\mathcal{S}\wedge\tau_{n+1})$, by 641Na. So

$$\begin{aligned} |w_{\tau_{n+1}} - w_{\tau_n}| \times \chi[\![\tau_n < \tau_{n+1}]\!] &\le \chi 1 + \operatorname{Oscln}(\boldsymbol{w}\restriction\mathcal{S} \wedge \tau_{n+1}) \\ &\le \chi 1 + \operatorname{Oscln}(\boldsymbol{w}) \le 2\chi 1 \end{aligned}$$

(using 618D(b-i)). It follows at once that $\|w_{\tau_{n+1}} - w_{\tau_n}\|_\infty \le 2$. **Q**

As $w_{\tau_0} = 0$, $\|w_{\tau_n}\|_\infty \le 2n$ for every $n \in \mathbb{N}$. Next, there is an $n \in \mathbb{N}$ such that $b'' = [\![\tau_n < \max\mathcal{T}]\!]$ has measure at most $\frac{1}{3}\epsilon$. Try

$$\tilde{\boldsymbol{v}} = \boldsymbol{P}w_{\tau_n\wedge\tau'}\restriction\mathcal{S}, \quad \boldsymbol{v}' = \boldsymbol{v}_1 - \boldsymbol{v} + (\boldsymbol{w} - \boldsymbol{v}_3)\restriction\mathcal{S}, \quad c = b \cup b' \cup b''.$$

Then $\tilde{\boldsymbol{v}}$ is an L^∞-martingale, $\boldsymbol{v}'$ is of bounded variation and $\bar{\mu}c \le \epsilon$. We have

$$[\![\tilde{\boldsymbol{v}} \ne \boldsymbol{w}\restriction\mathcal{S}]\!] \subseteq [\![\tau_n \wedge \tau' < \max\mathcal{S}]\!]$$

(because $\boldsymbol{w}\restriction\mathcal{T}\wedge\tau_n\wedge\tau'$ is a martingale)

$$\subseteq [\![\tau_n < \max\mathcal{T}]\!] \cap [\![\tau' < \max\mathcal{T}]\!] \subseteq b' \cup b''.$$

Consequently

$$[\![\tilde{\boldsymbol{v}} \ne \boldsymbol{v} + \boldsymbol{v}']\!] \subseteq [\![\tilde{\boldsymbol{v}} \ne \boldsymbol{w}\restriction\mathcal{S}]\!] \cup [\![\boldsymbol{w}\restriction\mathcal{S} \ne \boldsymbol{v}_1 + (\boldsymbol{w} - \boldsymbol{v}_3)\restriction\mathcal{S}]\!]$$
$$\subseteq b' \cup b'' \cup [\![\boldsymbol{v}_3\restriction\mathcal{S} \ne \boldsymbol{v}_1]\!] \subseteq b' \cup b'' \cup [\![\boldsymbol{v}_3\restriction\hat{\mathcal{S}} \ne \hat{\boldsymbol{v}}_1]\!] \subseteq b' \cup b'' \cup [\![\boldsymbol{v}_2 \ne \hat{\boldsymbol{v}}_1]\!]$$

(because $\boldsymbol{v}_3$ extends $\boldsymbol{v}_2$)

$$\subseteq b' \cup b'' \cup b = c$$

has measure at most ϵ, as required.

643O Lemma Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous, and that $\mathcal{S}$ is a non-empty finitely full sublattice of $\mathcal{T}$ with a greatest member such that $\inf A \in \mathcal{S}$ for every non-empty $A \subseteq \mathcal{S}$. If $\boldsymbol{v}$ is a near-simple integrator with domain $\mathcal{S}$, there are an L^∞-martingale $\tilde{\boldsymbol{v}}$ and a near-simple process $\boldsymbol{v}'$ of bounded variation, both with domain $\mathcal{S}$, such that $[\![\boldsymbol{v} \ne \tilde{\boldsymbol{v}} + \boldsymbol{v}']\!]$ has measure at most ϵ.

proof By 643N, we have an L^∞-martingale $\tilde{\boldsymbol{v}}$ and a process $\boldsymbol{v}_0'$ of bounded variation, both with domain $\mathcal{S}$, such that $a = [\![\boldsymbol{v} \ne \tilde{\boldsymbol{v}} + \boldsymbol{v}_0']\!]$ has measure at most ϵ. Express $\boldsymbol{v}$, $\tilde{\boldsymbol{v}}$ and $\boldsymbol{v}_0'$ as $\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$, $\langle\tilde{v}_\sigma\rangle_{\sigma\in\mathcal{S}}$ and $\langle v_{0\sigma}'\rangle_{\sigma\in\mathcal{S}}$. By 632Ia once more, $\tilde{\boldsymbol{v}}$ is locally near-simple; as $\mathcal{S}$ has a greatest member, $\tilde{\boldsymbol{v}}$ is actually near-simple. Set $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ where $u_\sigma = \chi(\mathrm{upr}(1\setminus a, \mathfrak{A}_\sigma))$ for $\sigma \in \mathcal{S}$, and $\boldsymbol{v}' = \boldsymbol{u} \times (\boldsymbol{v} - \tilde{\boldsymbol{v}})$. Then $\boldsymbol{u}$ is near-simple, by 632G, so $\boldsymbol{v}'$ is near-simple, by 631F(a-ii). Now for any $\sigma \in \mathcal{S}$,

$$1 \setminus a \subseteq [\![v_\sigma - \tilde{v}_\sigma - v_{0\sigma}' = 0]\!] \in \mathfrak{A}_\sigma,$$

so

$$[\![u_\sigma \ne 0]\!] = \mathrm{upr}(1\setminus a, \mathfrak{A}_\sigma) \subseteq [\![v_\sigma - \tilde{v}_\sigma - v_{0\sigma}' = 0]\!]$$

and $\boldsymbol{u} \times (\boldsymbol{v} - \tilde{\boldsymbol{v}} - \boldsymbol{v}_0') = 0$, that is, $\boldsymbol{v}' = \boldsymbol{u} \times \boldsymbol{v}_0'$. By 614Q(a-ii), $\boldsymbol{v}'$ is of bounded variation. Finally,

$$[\![\boldsymbol{v} \ne \tilde{\boldsymbol{v}} + \boldsymbol{v}']\!] \subseteq [\![\boldsymbol{v} \ne \tilde{\boldsymbol{v}} + \boldsymbol{v}_0']\!] \cup [\![\boldsymbol{v}_0' \ne \boldsymbol{v}']\!] \subseteq a \cup [\![\boldsymbol{u} \ne \mathbf{1}]\!]$$
$$= a \cup \sup_{\sigma\in\mathcal{S}}(1 \setminus \mathrm{upr}(1\setminus a, \mathfrak{A}_\sigma) \subseteq a \cup \sup_{\sigma\in\mathcal{S}}(1 \setminus (1 \setminus a)) = a$$

has measure at most ϵ.

643X Basic exercises (a) Show that if $t \in T$ then the region of accessibility of $\check{t}$ is 1 if $\{s : s < t\}$ is non-empty and has supremum t, 0 otherwise.

(b) Let $\langle\tau_n\rangle_{n\in\mathbb{N}}$ be the sequence of jump times associated with the standard Poisson process, as described in 612U. Show that the region of accessibility of τ_n is 0 for every $n \ge 1$.

(c) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a fully adapted process with a previsible version $\boldsymbol{u}_< = \langle u_{<\sigma}\rangle_{\sigma\in\mathcal{S}}$. Show that whenever σ, $\tau \in \mathcal{S}$, $[\![\sigma < \tau]\!] \setminus [\![\sigma \ll \tau]\!] \subseteq [\![u_{<\tau} = u_\sigma]\!]$.

643Y Further exercises (a) Suppose that $\tau \in \mathcal{T}$ and that $C_0, \dots, C_n$ are non-empty subsets of $\mathcal{T}\wedge\tau$. Show that there are $D_0, \dots, D_n$ such that

$$\emptyset \ne D_0 \subseteq \ldots \subseteq D_n \subseteq \mathcal{T}\wedge\tau, \quad \sup_{i\le n} a_{D_i} \supseteq \sup_{i\le n} a_{C_i}.$$

(b) Show that 643L can be used in place of 621Hf or 628D to show that if the filtration is right-continuous a martingale is a local integrator; now use the construction of 633Ya to see that a martingale is still a local integrator even when the filtration is not right-continuous.

643 Notes and comments This section seems to be hard work, and I do not know of any route to Theorem 643L which doesn't use most of the ideas here, though (as usual) my own exposition is designed for line-by-line checking rather than transparency. Of course the complications involving regions of approachability are quite unnecessary if we have a real-time stochastic integration structure, in which case $\mathrm{app}(\tau) = 1$ for ecery τ, by 643D.

The proof here is derived, at some remove, from PROTTER 05, Chap. III. I note that while it depends on some deep ideas about stochastic processes, it uses little of the theory of integration developed in §§613-617, so could in principle be used in a proof of Theorem 622H, as suggested in 643Yb.

Version of 25.8.20

644 Pointwise convergence

It is a remarkable fact that while the Riemann-sum integral, as defined in §613, is not 'sequentially smooth' in the most natural adaptation of the definition in 436A (644Xb), a variation on this concept (Theorem 644H) gives us a route to a Daniell-type integral, which I will develop in §645.

644A Notation I repeat the familiar list. $(\mathfrak{A}, \bar{\mu}, T, \langle\mathfrak{A}_t\rangle_{t\in T}, \mathcal{T}, \langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ is a stochastic integration structure. $\mathcal{T}_f \subseteq \mathcal{T}$ is the ideal of finite stopping times. If $\mathcal{S}$ is a sublattice of $\mathcal{T}$, $\mathcal{I}(\mathcal{S})$ is the set of finite sublattices of $\mathcal{S}$ and $\mathcal{S}\wedge\tau = \{\sigma\wedge\tau : \sigma\in\mathcal{S}\}$, $\mathcal{S}\vee\tau = \{\sigma\vee\tau : \sigma\in\mathcal{S}\}$ for $\tau\in\mathcal{T}$. $M_{\mathrm{mo}}(\mathcal{S})$ will be the space of moderately oscillatory processes with domain $\mathcal{S}$, and $M_{\text{n-s}}(\mathcal{S})$ the space of near-simple processes. L^0 will be $L^0(\mathfrak{A})$, with the topology of convergence in measure defined by the functional θ where $\theta(w) = \mathbb{E}(|w|\wedge\chi 1)$ for $w\in L^0$. If $\mathcal{S}$ is a sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is an order-bounded fully adapted process, I write $\sup\boldsymbol{u}$ for $\sup_{\sigma\in\mathcal{S}} u_\sigma$; $\|\boldsymbol{u}\|_\infty$ will be $\sup_{\sigma\in\mathcal{S}}\|u_\sigma\|_\infty$; if $\boldsymbol{u}$ is locally moderately oscillatory, $\boldsymbol{u}_<$ will be its previsible version.

644B Definitions (a) It will be useful to have a short phrase for the following. Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. A family A of processes with domain $\mathcal{S}$ is **uniformly order-bounded** if $\sup_{\boldsymbol{u}\in A}\sup|\boldsymbol{u}|$ is defined in L^0; that is, there is a $\bar{u}\in L^0$ such that $|u_\sigma|\le\bar{u}$ for every $\sigma\in\mathcal{S}$ whenever $\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}\in A$.

(b) Similarly, a special class of integrators will be prominent in the rest of the chapter. If $\mathcal{S}$ is a sublattice of $\mathcal{T}$, $M^{\uparrow}_{\text{n-s}}(\mathcal{S})$ will be the family of non-negative non-decreasing near-simple processes with domain $\mathcal{S}$. Any member of $M^{\uparrow}_{\text{n-s}}(\mathcal{S})$, being near-simple, will be order-bounded (631Ba); being non-decreasing, it will be an integrator (616Ra).

644C Lemma (The key.) Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ such that $\sup D\in\mathcal{S}$ whenever $D\subseteq\mathcal{S}$ is countable, non-empty and bounded above in $\mathcal{S}$. Let $\langle\boldsymbol{u}_n\rangle_{n\in\mathbb{N}} = \langle\langle u_{n\sigma}\rangle_{\sigma\in\mathcal{S}}\rangle_{n\in\mathbb{N}}$ be a non-increasing sequence of non-negative moderately oscillatory processes such that $\inf_{n\in\mathbb{N}}\boldsymbol{u}_{n<}$, taken in $(L^0)^{\mathcal{S}}$, is zero. Then $\inf_{n\in\mathbb{N}}\int_{\mathcal{S}}\boldsymbol{u}_n d\boldsymbol{v} = 0$ for every $\boldsymbol{v}\in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$.

proof (a) To begin with, suppose that $\mathcal{S}$ has greatest and least elements and that $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ is jump-free.

(i) ? If $w = \inf_{n\in\mathbb{N}}\int_{\mathcal{S}}\boldsymbol{u}_n d\boldsymbol{v}$ is non-zero, there is an $\epsilon\in\,]0,1]$ such that

$$\bar{\mu}[\![2\epsilon(v_{\max\mathcal{S}} - v_{\min\mathcal{S}}) < w]\!] \ge 2\epsilon.$$

P Take $\delta > 0$ such that $b = [\![w > \delta]\!]$ is non-zero, $M\ge 1$ such that $\bar{\mu}[\![v_{\max\mathcal{S}} - v_{\min\mathcal{S}}\ge M]\!]\le\frac{1}{3}\bar{\mu}b$ and set $\epsilon = \min(\frac{\bar{\mu}b}{3}, \frac{\delta}{2M})$. **Q**

(ii) Now choose a non-decreasing sequence $\langle\tau_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{S}$ and sequences $\langle w_n\rangle_{n\in\mathbb{N}}$ in L^0, $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ inductively, as follows. The inductive hypothesis is that $w_n = \inf_{i\in\mathbb{N}}\int_{\mathcal{S}\vee\tau_n}\boldsymbol{u}_i d\boldsymbol{v}$, $a_n\subseteq[\![(1+2^{-n})\epsilon(v_{\max\mathcal{S}} - v_{\tau_n}) < w_n]\!]$ and $\bar{\mu}a_n\ge(1+3^{-n})\epsilon$.

(α) Start with $\tau_0 = \min\mathcal{S}$, $w_0 = w$ and $a_0 = [\![2\epsilon(v_{\max\mathcal{S}} - v_{\min\mathcal{S}}) < w]\!]$.

(β) Given τ_n, w_n and a_n, let $\delta \in]0,1]$ be such that

$$[\![v_{\max\mathcal{S}} - v_{\tau_n} > 0]\!] \setminus [\![2^{-n-1}\epsilon(v_{\max\mathcal{S}} - v_{\tau_n}) > \delta]\!]$$

has measure at most $3^{-n-1}\epsilon$, and set $\eta = 3^{-n-2}\epsilon\delta$. Now $ii_{\boldsymbol{v}}(\boldsymbol{u}_n)$ is jump-free (618Q). By 618E, there is a $\tau'_n \in \mathcal{S} \vee \tau_n$ such that $[\![\tau_n < \tau'_n]\!] = [\![\tau_n < \max\mathcal{S}]\!]$ and, setting $y = \int_{\mathcal{S}\wedge\tau'_n} \boldsymbol{u}_n d\boldsymbol{v} - \int_{\mathcal{S}\wedge\tau_n} \boldsymbol{u}_n d\boldsymbol{v}$, $\theta(y) \le \eta$.

(γ) Now there is an $I \in \mathcal{I}(\mathcal{S}\vee\tau'_n)$, containing τ'_n and $\max\mathcal{S}$, such that $\theta(S_K(\boldsymbol{u}_n, d\boldsymbol{v}) - S_{K'}(\boldsymbol{u}_n, d\boldsymbol{v})) \le \eta$ whenever K, $K' \in \mathcal{I}(\mathcal{S}\vee\tau'_n)$ include I, in which case $\theta(S_{I\wedge\tau}(\boldsymbol{u}_n, d\boldsymbol{v}) - \int_{\mathcal{S}\cap[\tau'_n,\tau]} \boldsymbol{u}_n d\boldsymbol{v}) \le 2\eta$ for every $\tau \in \mathcal{S}\vee\tau'_n$ (613V(ii-β)).

Take $(\rho_0, \dots, \rho_k)$ linearly generating the I-cells (611L). By 611I, as usual, there is a $\tau_{n+1} \in \mathcal{T}$ such that

$$[\![\tau_{n+1} = \rho_i]\!] \supseteq [\![u_{n\rho_i} \ge \epsilon]\!] \setminus \sup_{j<i} [\![u_{n\rho_j} \ge \epsilon]\!]$$

for $i < k$,

$$[\![\tau_{n+1} = \rho_k]\!] \supseteq 1 \setminus \sup_{j<i} [\![u_{n\rho_j} \ge \epsilon]\!],$$

$$\tau'_n = \rho_0 \le \tau_{n+1} \le \rho_k = \max\mathcal{S}$$

and $\tau_{n+1} \in \mathcal{S}$ because $\mathcal{S}$ is finitely full. Now

$$[\![\tau_{n+1} < \max\mathcal{S}]\!] \subseteq [\![u_{n\tau_{n+1}} \ge \epsilon]\!]$$

and

$$[\![\rho_i \wedge \tau_{n+1} < \rho_{i+1} \wedge \tau_{n+1}]\!] \subseteq [\![\rho_i < \tau_{n+1}]\!] \subseteq \sup_{i<j\le k} [\![\tau_{n+1} = \rho_j]\!] \subseteq [\![u_{n,\rho_i\wedge\tau_n} < \epsilon]\!]$$

for every $i < k$. Accordingly

$$S_{I\wedge\tau_{n+1}}(\boldsymbol{u}_n, d\boldsymbol{v}) = \sum_{i=0}^{k-1} u_{n,\rho_i\wedge\tau_{n+1}} \times (v_{\rho_{i+1}\wedge\tau_{n+1}} - v_{\rho_i\wedge\tau_{n+1}})$$

(because $(\rho_0 \wedge \tau_{n+1}, \dots, \rho_k \wedge \tau_{n+1})$ linearly generates the $I \wedge \tau_{n+1}$-cells, by 611Kg)

$$\le \sum_{i=0}^{k-1} \epsilon(v_{\rho_{i+1}\wedge\tau_{n+1}} - v_{\rho_i\wedge\tau_{n+1}}) = \epsilon(v_{\tau_{n+1}} - v_{\tau'_n}).$$

Setting

$$z = (\int_{\mathcal{S}\cap[\tau'_n,\tau_{n+1}]} \boldsymbol{u}_n d\boldsymbol{v} - \epsilon(v_{\tau_{n+1}} - v_{\tau'_n}))^+ \le (\int_{\mathcal{S}\cap[\tau'_n,\tau_{n+1}]} \boldsymbol{u}_n d\boldsymbol{v} - S_{I\wedge\tau_{n+1}}(\boldsymbol{u}_n, d\boldsymbol{v}))^+,$$

$\theta(z) \le 2\eta$, so $\theta(y+z) \le 3\eta$, while

$$\int_{\mathcal{S}\cap[\tau_n.\tau_{n+1}]} \boldsymbol{u}_n d\boldsymbol{v} \le y + z + \epsilon(v_{\tau_{n+1}} - v_{\tau'_n}).$$

(δ) Set $a_{n+1} = a_n \cap [\![y + z \le 2^{-n-1}\epsilon(v_{\max\mathcal{S}} - v_{\tau_n})]\!]$. Then

$$\bar\mu(a_n \setminus a_{n+1}) \le \bar\mu([\![v_{\max\mathcal{S}} - v_{\tau_n} > 0]\!] \cap [\![y + z \ge 2^{-n-1}\epsilon(v_{\max\mathcal{S}} - v_{\tau_n})]\!]$$

(because $a_n \subseteq [\![w_n > 0]\!] \subseteq [\![v_{\max\mathcal{S}} - v_{\tau_n} > 0]\!]$)

$$\le \bar\mu([\![v_{\max\mathcal{S}} - v_{\tau_n} > 0]\!] \cap [\![2^{-n-1}\epsilon(v_{\max\mathcal{S}} - v_{\tau_n}) \le \delta]\!])$$
$$+ \bar\mu[\![y + z \ge \delta]\!]$$
$$\le 3^{-n-1}\epsilon + \bar\mu[\![y + z \ge \delta]\!] \le 2 \cdot 3^{-n-1}\epsilon$$

because $\theta(y+z) \le 3^{-n-1}\epsilon\delta$. Accordingly $\bar\mu a_{n+1} \ge (1 + 3^{-n-1})\epsilon$.

(ε) For any $i \ge n$,

$$\int_{\mathcal{S}\vee\tau_{n+1}} \boldsymbol{u}_i d\boldsymbol{v} = \int_{\mathcal{S}\vee\tau_n} \boldsymbol{u}_i d\boldsymbol{v} - \int_{\mathcal{S}\cap[\tau_n,\tau_{n+1}]} \boldsymbol{u}_i d\boldsymbol{v}$$
$$\ge w_n - \int_{\mathcal{S}\cap[\tau_n,\tau_{n+1}]} \boldsymbol{u}_n d\boldsymbol{v} \ge w_n - y - z - \epsilon(v_{\tau_{n+1}} - v_{\tau'_n}).$$

So, setting $w_{n+1} = \inf_{i\in\mathbb{N}} \int_{\mathcal{S}\vee\tau_{n+1}} \boldsymbol{u}_i d\boldsymbol{v}$,

$$w_{n+1} \ge w_n - y - z - \epsilon(v_{\tau_{n+1}} - v_{\tau'_n}) \ge w_n - y - z - \epsilon(v_{\tau_{n+1}} - v_{\tau_n}).$$

Now

$$\begin{aligned} a_{n+1} &\subseteq [\![(1+2^{-n})\epsilon(v_{\max\mathcal{S}} - v_{\tau_n}) < w_n]\!] \cap [\![y + z \le 2^{-n-1}\epsilon(v_{\max\mathcal{S}} - v_{\tau_n}]\!]) \\ &\subseteq [\![(1+2^{-n-1})\epsilon(v_{\max\mathcal{S}} - v_{\tau_n}) < w_n - y - z]\!] \\ &\subseteq [\![(1+2^{-n-1})\epsilon(v_{\max\mathcal{S}} - v_{\tau_{n+1}}) < w_n - y - z - \epsilon(v_{\tau_{n+1}} - v_{\tau_n})]\!] \\ &\subseteq [\![(1+2^{-n-1})\epsilon(v_{\max\mathcal{S}} - v_{\tau_{n+1}}) < w_{n+1}]\!], \end{aligned}$$

and the induction proceeds.

(iii) At the end of the induction, set $\tau = \sup_{n\in\mathbb{N}} \tau_n$; by hypothesis, $\tau \in \mathcal{S}$. Since $\langle\tau_n\rangle_{n\in\mathbb{N}}$ is non-decreasing, $\langle w_n\rangle_{n\in\mathbb{N}}$ and $\langle v_{\max\mathcal{S}} - v_{\tau_n}\rangle_{n\in\mathbb{N}}$ are non-increasing. The construction in (ii) above arranged that $a_{n+1} \subseteq a_n$ and $\bar\mu a_n \ge \epsilon$ for every n. At the same time we have

$$a_n \subseteq [\![w_n > 0]\!] \subseteq [\![\tau_n < \max\mathcal{S}]\!] = [\![\tau_n < \tau'_n]\!] \subseteq [\![\tau_n < \tau]\!]$$

and

$$a_{n+1} \subseteq [\![u_{n\tau_{n+1}} \ge \epsilon]\!] \subseteq [\![u_{i\tau_{n+1}} \ge \epsilon]\!]$$

whenever $n \ge i$. So for any $i \in \mathbb{N}$, $\langle\tau_n\rangle_{n>i}$ is a non-decreasing sequence with supremum τ and

$$a \subseteq [\![\tau_n < \tau]\!] \cap [\![u_{i\tau_n} \ge \epsilon]\!]$$

for every $n > i$. By 641M or otherwise, $a \subseteq [\![u_{i<\tau} \ge \epsilon]\!]$; as i is arbitrary, $a \subseteq [\![\inf_{i\in\mathbb{N}} u_{i<\tau} \ge \epsilon]\!]$. But also we have

$$\bar\mu a = \inf_{n\in\mathbb{N}} \bar\mu a_n \ge \epsilon > 0$$

so $\inf_{i\in\mathbb{N}} u_{i<\tau} \ne 0$, contrary to hypothesis. **X**

(iv) Thus we see that if $\mathcal{S}$ has greatest and least elements and $\boldsymbol{v}$ is jump-free, $\inf_{n\in\mathbb{N}} \int_{\mathcal{S}} \boldsymbol{u}_n d\boldsymbol{v} = 0$.

(b) Now suppose only that $\boldsymbol{v} \in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$, while still assuming that $\mathcal{S}$ has greatest and least elements.

(i) Consider first the case in which $\boldsymbol{v}$ is actually simple. Let $(\tau_0, \dots, \tau_m)$ be a breakpoint string for $\boldsymbol{v}$ starting from $\tau_0 = \min\mathcal{S}$ and finishing with $\tau_m = \max\mathcal{S}$. Then

$$\int_{\mathcal{S}} \boldsymbol{u}_n\, d\boldsymbol{v} = \sum_{j=0}^{m-1} u_{n<\tau_{j+1}} \times (v_{\tau_{j+1}} - v_{\tau_j})$$

for every $n \in \mathbb{N}$ (641J). So

$$\lim_{n\to\infty} \int_{\mathcal{S}} \boldsymbol{u}_n\, d\boldsymbol{v} = \sum_{j=0}^{m-1} \lim_{n\to\infty} u_{n<\tau_{j+1}} \times (v_{\tau_{j+1}} - v_{\tau_j}) = 0.$$

(ii) If $\boldsymbol{v}$ is just near-simple, take $\epsilon > 0$. Set $\bar u = \sup|\boldsymbol{u}_0|$. Let $\delta > 0$ be such that $\theta(v \times \bar u) \le \epsilon$ whenever $\theta(v) \le \delta$. By 631U, there are non-negative non-decreasing processes $\boldsymbol{v}'$, $\boldsymbol{w}$, $\boldsymbol{v}''$, all with domain $\mathcal{S}$, such that $\boldsymbol{v}'$ is simple, $\boldsymbol{w}$ is jump-free, $\boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{w} + \boldsymbol{v}''$ and $\theta(\sup\boldsymbol{v}'') \le \delta$. In this case, we see that

$$\int_{\mathcal{S}} \boldsymbol{u}_n\, d\boldsymbol{v}'' \le \sup_{I\in\mathcal{I}(\mathcal{S})} S_I(\boldsymbol{u}_n, d\boldsymbol{v}'') \le \bar u \times \sup_{I\in\mathcal{I}(\mathcal{S})} S_I(\mathbf{1}, d\boldsymbol{v}'')$$

(because, expressing $\boldsymbol{v}''$ as $\langle v''_\sigma\rangle_{\sigma\in\mathcal{S}}$, $u_{n\sigma} \times (v''_\tau - v''_\sigma) \le \bar u \times (v''_\tau - v''_\sigma)$ whenever $\sigma \le \tau$ in $\mathcal{S}$)

$$\le \bar u \times \sup\boldsymbol{v}''$$

and $\theta(\sup\boldsymbol{v}'') \le \delta$, so that $\theta(\int_{\mathcal{S}} \boldsymbol{u}_n\, d\boldsymbol{v}'') \le \epsilon$, for every $n \in \mathbb{N}$. From (a) above we know that $\lim_{n\to\infty} \int_{\mathcal{S}} \boldsymbol{u}_n\, d\boldsymbol{w} = 0$, and from (i) here we see that $\lim_{n\to\infty} \int_{\mathcal{S}} \boldsymbol{u}_n\, d\boldsymbol{v}' = 0$. So

$$\limsup_{n\to\infty} \theta(\int_{\mathcal{S}} \boldsymbol{u}_n\, d\boldsymbol{v}) = \limsup_{n\to\infty} \theta(\int_{\mathcal{S}} \boldsymbol{u}_n\, d\boldsymbol{v}'') \le \epsilon.$$

As ϵ is arbitrary, $\lim_{n\to\infty} \theta(\int_{\mathcal{S}} \boldsymbol{u}_n\, d\boldsymbol{v}) = 0$.

(c) Finally, for the general case, if $\mathcal{S}$ is empty the result is of course trivial. Otherwise, given $\epsilon>0$, there are τ, $\tau'\in\mathcal{S}$ such that $\tau\le\tau'$, $\theta(\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}_0d\boldsymbol{v})\le\epsilon$ and $\theta(\int_{\mathcal{S}\vee\tau'}\boldsymbol{u}_0d\boldsymbol{v})\le\epsilon$. Consequently $\theta(\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}_nd\boldsymbol{v})\le\epsilon$ and $\theta(\int_{\mathcal{S}\vee\tau'}\boldsymbol{u}_nd\boldsymbol{v})\le\epsilon$ for every n, and

$$\inf_{n\in\mathbb{N}}\theta(\int_{\mathcal{S}}\boldsymbol{u}_nd\boldsymbol{v})\le 2\epsilon+\inf_{n\in\mathbb{N}}\theta(\int_{\mathcal{S}'}\boldsymbol{u}_nd\boldsymbol{v})=2\epsilon+\theta(\inf_{n\in\mathbb{N}}\int_{\mathcal{S}'}\boldsymbol{u}_nd\boldsymbol{v})$$

where $\mathcal{S}'=\mathcal{S}\cap[\tau,\tau']$. If $n\in\mathbb{N}$, $\boldsymbol{u}_n\restriction\mathcal{S}'$ is non-negative and moderately oscillatory (615F(a-i)), and $0\le(\boldsymbol{u}_n\restriction\mathcal{S}')_<\le\boldsymbol{u}_{n<}\restriction\mathcal{S}'$ (641Gc). So $\inf_{n\in\mathbb{N}}(\boldsymbol{u}_n\restriction\mathcal{S}')_<=0$, while of course $\langle\boldsymbol{u}_n\restriction\mathcal{S}'\rangle_{n\in\mathbb{N}}$ is non-increasing. As $\boldsymbol{v}\restriction\mathcal{S}'=\boldsymbol{v}\restriction(\mathcal{S}\vee\tau)\wedge\tau'$ is non-decreasing and near-simple (631F(a-iv)), (b) above tells us that $\inf_{n\in\mathbb{N}}\int_{\mathcal{S}'}\boldsymbol{u}_nd\boldsymbol{v}=0$. Consequently $\theta(\inf_{n\in\mathbb{N}}\int_{\mathcal{S}}\boldsymbol{u}_nd\boldsymbol{v})\le 2\epsilon$; as ϵ is arbitrary, $\inf_{n\in\mathbb{N}}\int_{\mathcal{S}}\boldsymbol{u}_nd\boldsymbol{v}=0$, and the proof is complete.

644D Lemma Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ such that $\sup D\in\mathcal{S}$ whenever $D\subseteq\mathcal{S}$ is countable, non-empty and bounded above in $\mathcal{S}$. Let $\langle\boldsymbol{u}_n\rangle_{n\in\mathbb{N}}=\langle\langle u_{n\sigma}\rangle_{\sigma\in\mathcal{S}}\rangle_{n\in\mathbb{N}}$ be a uniformly order-bounded sequence of moderately oscillatory processes such that $\langle\boldsymbol{u}_{n<}\rangle_{n\in\mathbb{N}}$ is order*-convergent in $(L^0)^{\mathcal{S}}$ (definition: 642B). If $\boldsymbol{v}\in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$, $\lim_{n\to\infty}\int_{\mathcal{S}}\boldsymbol{u}_nd\boldsymbol{v}$ is defined in L^0 for the topology of convergence in measure.

proof Let $\epsilon>0$.

(a) Writing $\bar u$ for $\sup_{n\in\mathbb{N},\sigma\in\mathcal{S}}|u_{n\sigma}|$, we have

$$|\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}|\le\sup_{I\in\mathcal{I}(\mathcal{S})}|S_I(\boldsymbol{u},d\boldsymbol{v})|\le\bar u\times\sup_{I\in\mathcal{I}(\mathcal{S})}S_I(\mathbf{1},d\boldsymbol{v})=\bar u\times\int_{\mathcal{S}}d\boldsymbol{v}$$

whenever $\boldsymbol{u}$ is a moderately oscillatory process with domain $\mathcal{S}$ and $\sup|\boldsymbol{u}|\le\bar u$. In particular, if $n\in\mathbb{N}$, the monotonic sequences $\langle\int_{\mathcal{S}}\inf_{n\le i\le m}\boldsymbol{u}_id\boldsymbol{v}\rangle_{m\ge n}$, $\langle\int_{\mathcal{S}}\sup_{n\le i\le m}\boldsymbol{u}_id\boldsymbol{v}\rangle_{m\ge n}$ are order-bounded in L^0 and have limits $\bar w_n$, $\bar z_n$ respectively in L^0 (613Ba). If we take $k_n\ge n$ such that

$$\theta(\int_{\mathcal{S}}\inf_{n\le i\le k_n}\boldsymbol{u}_id\boldsymbol{v}-\bar w_n)\le 2^{-n}\epsilon,\quad\theta(\bar z_n-\int_{\mathcal{S}}\sup_{n\le i\le k_n}\boldsymbol{u}_id\boldsymbol{v})\le 2^{-n}\epsilon,$$

and set $\boldsymbol{w}_n=\inf_{n\le i\le k_n}\boldsymbol{u}_i$, $\boldsymbol{z}_n=\sup_{n\le i\le k_n}\boldsymbol{u}_i$, then for any $m\ge n$

$$\begin{aligned}\theta(\int_{\mathcal{S}}(\boldsymbol{w}_n-\boldsymbol{u}_m)^+d\boldsymbol{v})&\le\theta(\int_{\mathcal{S}}(\boldsymbol{w}_n-\inf_{i\le\max(m,k_n)}\boldsymbol{u}_i)d\boldsymbol{v})\\&\le\theta(\int_{\mathcal{S}}\boldsymbol{w}_nd\boldsymbol{v}-\bar w_n)\le 2^{-n}\epsilon\end{aligned}$$

and similarly

$$\theta(\int_{\mathcal{S}}(\boldsymbol{u}_m-\boldsymbol{z}_n)^+d\boldsymbol{v})\le 2^{-n}\epsilon.$$

(b) Define $\langle\boldsymbol{x}_n\rangle_{n\in\mathbb{N}}$, $\langle\boldsymbol{y}_n\rangle_{n\in\mathbb{N}}$ inductively by setting $\boldsymbol{x}_0=\boldsymbol{w}_0$, $\boldsymbol{y}_0=\boldsymbol{z}_0$ and

$$\boldsymbol{x}_{n+1}=\text{med}(\boldsymbol{x}_n,\boldsymbol{w}_{n+1},\boldsymbol{y}_n),\qquad\boldsymbol{y}_{n+1}=\text{med}(\boldsymbol{x}_n,\boldsymbol{z}_{n+1},\boldsymbol{y}_n)$$

for every $n\in\mathbb{N}$. Because $\boldsymbol{w}_n\le\boldsymbol{u}_n\le\boldsymbol{z}_n$ for each n, $\boldsymbol{x}_n\le\boldsymbol{x}_{n+1}\le\boldsymbol{y}_{n+1}\le\boldsymbol{y}_n$ for each n. We also have $\boldsymbol{y}_n-\boldsymbol{x}_n\le\boldsymbol{z}_n-\boldsymbol{w}_n$ for every $n\in\mathbb{N}$. **P** If $n=0$ this is immediate. For $n>0$,

$$\begin{aligned}\boldsymbol{y}_n-\boldsymbol{x}_n&=(\boldsymbol{x}_{n-1}\vee\boldsymbol{z}_n)\wedge\boldsymbol{y}_{n-1}-(\boldsymbol{x}_{n-1}\vee\boldsymbol{w}_n)\wedge\boldsymbol{y}_{n-1}\\&\le(\boldsymbol{x}_{n-1}\vee\boldsymbol{w}_n+\boldsymbol{x}_{n-1}\vee\boldsymbol{z}_n-\boldsymbol{x}_{n-1}\vee\boldsymbol{w}_n)\wedge(\boldsymbol{y}_{n-1}+\boldsymbol{x}_{n-1}\vee\boldsymbol{z}_n-\boldsymbol{x}_{n-1}\vee\boldsymbol{w}_n)\\&\qquad-(\boldsymbol{x}_{n-1}\vee\boldsymbol{w}_n)\wedge\boldsymbol{y}_{n-1}\\&=\boldsymbol{x}_{n-1}\vee\boldsymbol{z}_n-\boldsymbol{x}_{n-1}\vee\boldsymbol{w}_n\\&\le(\boldsymbol{x}_{n-1}+\boldsymbol{z}_n-\boldsymbol{w}_n)\vee(\boldsymbol{w}_n+\boldsymbol{z}_n-\boldsymbol{w}_n)-\boldsymbol{x}_{n-1}\vee\boldsymbol{w}_n=\boldsymbol{z}_n-\boldsymbol{w}_n.\ \mathbf{Q}\end{aligned}$$

(c) Next, $\theta(\int_{\mathcal{S}}(\boldsymbol{x}_n-\boldsymbol{u}_m)^+d\boldsymbol{v})\le(2-2^{-n})\epsilon$ whenever $n\le m$. **P** Induce on n. For $n=0$ we have

$$\theta(\int_{\mathcal{S}}(\boldsymbol{x}_0-\boldsymbol{u}_m)^+d\boldsymbol{v})=\theta(\int_{\mathcal{S}}(\boldsymbol{w}_0-\boldsymbol{u}_m)^+d\boldsymbol{v})\le\epsilon$$

by (a). For the inductive step to $n+1\le m$, we have $\boldsymbol{x}_{n+1}\le\boldsymbol{x}_n\vee\boldsymbol{w}_{n+1}$ so

$$\begin{aligned}\theta(\int_{\mathcal{S}}(\boldsymbol{x}_{n+1}-\boldsymbol{u}_m)^+d\boldsymbol{v}) &\le \theta(\int_{\mathcal{S}}((\boldsymbol{x}_n-\boldsymbol{u}_m)^+ + (\boldsymbol{w}_{n+1}-\boldsymbol{u}_m)^+)d\boldsymbol{v})\\ &\le \theta(\int_{\mathcal{S}}(\boldsymbol{x}_n-\boldsymbol{u}_m)^+d\boldsymbol{v}) + \theta(\int_{\mathcal{S}}(\boldsymbol{w}_{n+1}-\boldsymbol{u}_m)^+d\boldsymbol{v})\\ &\le (2-2^{-n})\epsilon + 2^{-n-1}\epsilon = (2-2^{-n-1})\epsilon. \ \mathbf{Q}\end{aligned}$$

Similarly, $\theta(\int_{\mathcal{S}}(\boldsymbol{u}_m-\boldsymbol{y}_n)^+d\boldsymbol{v}) \le (2-2^{-n})\epsilon$ whenever $n \le m$.

(d) Set $\boldsymbol{u}'_n = \mathrm{med}(\boldsymbol{x}_n, \boldsymbol{u}_n, \boldsymbol{y}_n)$ for $n \in \mathbb{N}$. Then

$$(\boldsymbol{u}_n - \boldsymbol{u}'_n)^+ \le \boldsymbol{u}_n - \boldsymbol{u}_n \wedge \boldsymbol{y}_n = (\boldsymbol{u}_n - \boldsymbol{y}_n)^+$$

and similarly $(\boldsymbol{u}'_n - \boldsymbol{u}_n)^+ \le (\boldsymbol{x}_n - \boldsymbol{u}_n)^+$, so

$$\begin{aligned}\theta(\int_{\mathcal{S}}\boldsymbol{u}_n d\boldsymbol{v} - \int_{\mathcal{S}}\boldsymbol{u}'_n d\boldsymbol{v}) &\le \theta(\int_{\mathcal{S}}|\boldsymbol{u}_n-\boldsymbol{u}'_n|d\boldsymbol{v}) = \theta(\int_{\mathcal{S}}((\boldsymbol{u}_n-\boldsymbol{u}'_n)^+ + (\boldsymbol{u}'_n-\boldsymbol{u}_n)^+)d\boldsymbol{v})\\ &\le \theta(\int_{\mathcal{S}}(\boldsymbol{u}_n-\boldsymbol{y}_n)^+d\boldsymbol{v}) + \theta(\int_{\mathcal{S}}(\boldsymbol{x}_n-\boldsymbol{u}_n)^+d\boldsymbol{v}) \le 4\epsilon\end{aligned}$$

for every n.

(e) Observe next that all the $\boldsymbol{w}_n$, $\boldsymbol{z}_n$, $\boldsymbol{x}_n$ and $\boldsymbol{y}_n$ are moderately oscillatory processes, and that $\langle \boldsymbol{y}_n - \boldsymbol{x}_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of non-negative moderately oscillatory processes. Now for every n we have

$$\inf_{m\ge n}\boldsymbol{u}_{m<} \le \inf_{n\le i\le k_n}\boldsymbol{u}_{m<} = (\inf_{n\le i\le k_n}\boldsymbol{u}_m)_< = \boldsymbol{w}_{n<}$$

because $\boldsymbol{u}\mapsto\boldsymbol{u}_<$ is a Riesz homomorphism (641G(e-i)), and similarly $\boldsymbol{z}_{n<} \le \sup_{m\ge n} u_{m<}$. Consequently

$$\inf_{n\in\mathbb{N}}(\boldsymbol{y}_{n<} - \boldsymbol{x}_{n<}) \le \inf_{n\in\mathbb{N}}(\boldsymbol{z}_{n<} - \boldsymbol{w}_{n<})$$

(by (b))

$$\le \inf_{n\in\mathbb{N}}(\sup_{m\ge n}\boldsymbol{u}_{m<} - \inf_{m\ge n}\boldsymbol{u}_{m<}) = 0$$

because $\langle \boldsymbol{u}_{n<}\rangle_{n\in\mathbb{N}}$ is order*-convergent, so $\inf_{n\in\mathbb{N}}\sup_{m\ge n}\boldsymbol{u}_{m<} = \sup_{n\in\mathbb{N}}\inf_{m\ge n}\boldsymbol{u}_{m<}$ (642B).

(f) By 644C, $\lim_{n\to\infty}\int_{\mathcal{S}}(\boldsymbol{y}_n-\boldsymbol{x}_n)d\boldsymbol{v} = 0$. As $\boldsymbol{v}$ is non-decreasing, $\boldsymbol{u}\mapsto\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}$ is a positive linear operator, so $\lim_{n\to\infty}\int_{\mathcal{S}}\boldsymbol{y}_n d\boldsymbol{v}$ and $\lim_{n\to\infty}\int_{\mathcal{S}}\boldsymbol{x}_n d\boldsymbol{v}$ exist and are equal. Since $\boldsymbol{x}_n \le \boldsymbol{u}'_n \le \boldsymbol{y}_n$ for every n, the common limit of these integrals is also $\lim_{n\to\infty}\int_{\mathcal{S}}\boldsymbol{u}'_n d\boldsymbol{v}$. Accordingly

$$\begin{aligned}&\limsup_{m,n\to\infty}\theta(\int_{\mathcal{S}}\boldsymbol{u}_m d\boldsymbol{v} - \int_{\mathcal{S}}\boldsymbol{u}_n d\boldsymbol{v})\\ &\quad\le \sup_{m\in\mathbb{N}}\theta(\int_{\mathcal{S}}\boldsymbol{u}_m d\boldsymbol{v} - \int_{\mathcal{S}}\boldsymbol{u}'_m d\boldsymbol{v}) + \limsup_{m,n\to\infty}\theta(\int_{\mathcal{S}}\boldsymbol{u}'_m d\boldsymbol{v} - \int_{\mathcal{S}}\boldsymbol{u}'_n d\boldsymbol{v})\\ &\qquad\qquad + \sup_{n\in\mathbb{N}}\theta(\int_{\mathcal{S}}\boldsymbol{u}'_n d\boldsymbol{v} - \int_{\mathcal{S}}\boldsymbol{u}_n d\boldsymbol{v})\\ &\quad\le 4\epsilon + 0 + 4\epsilon = 8\epsilon\end{aligned}$$

by (d). As ϵ is arbitrary, $\langle\int_{\mathcal{S}}\boldsymbol{u}_n d\boldsymbol{v}\rangle_{n\in\mathbb{N}}$ is Cauchy, therefore convergent.

644E Corollary Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ such that $\sup D \in \mathcal{S}$ whenever $D \subseteq \mathcal{S}$ is countable, non-empty and bounded above in $\mathcal{S}$, and $\langle \boldsymbol{u}_n\rangle_{n\in\mathbb{N}}$ a uniformly order-bounded sequence of moderately oscillatory processes with domain $\mathcal{S}$ such that $\langle \boldsymbol{u}_{n<}\rangle_{n\in\mathbb{N}}$ is order*-convergent to $\mathbf{0}$ in $(L^0)^{\mathcal{S}}$. Then $\lim_{n\to\infty}\int_{\mathcal{S}}\boldsymbol{u}_n d\boldsymbol{v} = 0$ for $\boldsymbol{v}\in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$.

proof Apply 644D to the sequence $\langle \boldsymbol{w}_n\rangle_{n\in\mathbb{N}}$ where $\boldsymbol{w}_{2n} = \boldsymbol{u}_n$ and $\boldsymbol{w}_{2n+1} = 0$ for every n.

644F Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, A a uniformly order-bounded subset of $M_{\mathrm{mo}}(\mathcal{S})$, and $\boldsymbol{v}$ an integrator with domain $\mathcal{S}$.

(a)(i) $\{\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v} : \boldsymbol{u} \in A\}$ is topologically bounded in L^0.

(ii) if $\mathcal{S}$ is non-empty,

$$\lim_{\tau\uparrow\mathcal{S}} \sup_{\boldsymbol{u}\in A} \theta(\int_{\mathcal{S}\vee\tau} \boldsymbol{u}\, d\boldsymbol{v}) = \lim_{\tau\downarrow\mathcal{S}} \sup_{\boldsymbol{u}\in A} \theta(\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\boldsymbol{v}) = 0.$$

(b) If $\boldsymbol{v}$ is non-decreasing, then $\{\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v} : \boldsymbol{u} \in A\}$ is order-bounded in L^0.

proof (a) Let $\bar{u} \in L^0$ be such that $\sup|\boldsymbol{u}| \le \bar{u}$ for every $\boldsymbol{u} \in A$.

(i) Take any $\epsilon > 0$. Then there are an $M > 0$ such that $[\![\bar{u} \ge M]\!]$ has measure at most ϵ, and a $\delta > 0$ such that $\theta(\delta z) \le \epsilon$ for every $z \in Q_{\mathcal{S}}(d\boldsymbol{v})$. Suppose that $\boldsymbol{u} \in A$ and set $\boldsymbol{u}' = \mathrm{med}(-M\mathbf{1}^{(\mathcal{S})}, \boldsymbol{u}, M\mathbf{1}^{(\mathcal{S})})$. If $I \in \mathcal{I}(\mathcal{S})$ then $S_I(\frac{1}{M}\boldsymbol{u}', d\boldsymbol{v}) \in Q_{\mathcal{S}}(d\boldsymbol{v})$ so $\theta(\frac{\delta}{M} S_I(\boldsymbol{u}', d\boldsymbol{v})) \le \epsilon$. Also

$$[\![S_I(\boldsymbol{u}, d\boldsymbol{v}) \ne S_I(\boldsymbol{u}, d\boldsymbol{v})]\!] \subseteq [\![\boldsymbol{u} \ne \boldsymbol{u}']\!]$$

(613Gd)

$$= [\![\sup|\boldsymbol{u}| > M]\!] \subseteq [\![\bar{u} \ge M]\!]$$

has measure at most ϵ, so

$$\theta(\frac{\delta}{M} S_I(\boldsymbol{u}, d\boldsymbol{v})) \le \epsilon + \theta(\frac{\delta}{M} S_I(\boldsymbol{u}', d\boldsymbol{v})) \le 2\epsilon.$$

Taking the limit as $I \uparrow \mathcal{S}$, $\theta(\frac{\delta}{M}\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}) \le 2\epsilon$; and this is true for every $\boldsymbol{u} \in A$. As ϵ is arbitrary, $\{\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v} : \boldsymbol{u} \in A\}$ is topologically bounded.

(ii)($\boldsymbol{\alpha}$) ? Suppose, if possible, that such that $\epsilon = \frac{1}{3}\limsup_{\tau\uparrow\mathcal{S}} \sup_{\boldsymbol{u}\in A} \theta(\int_{\mathcal{S}\vee\tau} \boldsymbol{u}\, d\boldsymbol{v})$ is greater than 0. As just above, take $M \ge 1$ such that $[\![\bar{u} \ge M]\!]$ has measure at most ϵ, and set $\eta = \frac{\epsilon}{4M}$. Take $\gamma > 0$, $m \ge 1$, $r \ge m$ and $k \ge 1$ such that

$$\theta(\tfrac{1}{\gamma} z) < \eta \text{ for every } z \in Q_{\mathcal{S}}(d\boldsymbol{v}),$$

$$m\eta \ge 2\gamma, \quad 1 - \tfrac{r!}{r^m(r-m)!} \le \frac{1}{2}\eta^m, \quad 2k\eta^m \ge \eta,$$

and set $n = rk$.

Choose $\langle\tau_i\rangle_{i\le n}$ inductively, as follows. Start from any $\tau_0 \in \mathcal{S}$ such that $\sup_{\boldsymbol{u}\in A} \theta(\int_{\mathcal{S}\vee\tau_0} \boldsymbol{u}\, d\boldsymbol{v}) > 2\epsilon$. Given that $i < n$, $\tau_i \in \mathcal{S}$ and $\sup_{\boldsymbol{u}\in A} \theta(\int_{\mathcal{S}\vee\tau_i} \boldsymbol{u}\, d\boldsymbol{v}) > 2\epsilon$, take $\boldsymbol{u}_i \in A$ such that $\theta(\int_{\mathcal{S}\vee\tau_i} \boldsymbol{u}_i\, d\boldsymbol{v}) > 2\epsilon$, and set $\boldsymbol{u}_i' = \mathrm{med}(-M\mathbf{1}^{(\mathcal{S})}, \boldsymbol{u}_i, M\mathbf{1}^{(\mathcal{S})})$. As before,

$$\theta(\int_{\mathcal{S}\vee\tau_i} \boldsymbol{u}_i'\, d\boldsymbol{v}) \ge \theta(\int_{\mathcal{S}\vee\tau_i} \boldsymbol{u}_i'\, d\boldsymbol{v}) - \bar{\mu}[\![\boldsymbol{u}_i' \ne \boldsymbol{u}_i]\!] \ge \theta(\int_{\mathcal{S}\vee\tau_i} \boldsymbol{u}_i'\, d\boldsymbol{v}) - \bar{\mu}[\![\bar{u} \ge M]\!] > \epsilon.$$

Let $I_i \in \mathcal{I}(\mathcal{S}\vee\tau_i)$ be such that $\theta(S_{I_i}(\boldsymbol{u}_i', d\boldsymbol{v})) > \epsilon$, and take $\tau_{i+1} \in \mathcal{S}\vee\max I_i$ such that $\sup_{\boldsymbol{u}\in A} \theta(\int_{\mathcal{S}\vee\tau_{i+1}} \boldsymbol{u}\, d\boldsymbol{v}) > 2\epsilon$. Continue.

Then $\tau_0 \le \ldots \le \tau_n$ in $\mathcal{S}$, and if $i < n$ then $I_i \in \mathcal{I}(\mathcal{S} \cap [\tau_i, \tau_{i+1}])$ for each $i < n$ and

$$\sup\{\theta(z) : z \in Q_{\mathcal{S}\cap[\tau_i,\tau_{i+1}]}(d\boldsymbol{v})\} \ge \theta(S_{I_i}(\tfrac{1}{M}\boldsymbol{u}_i', d\boldsymbol{v})) \ge \frac{1}{M}\theta(S_{I_i}(\boldsymbol{u}_i', d\boldsymbol{v})) > \frac{\epsilon}{M} = 4\eta.$$

So by 616Hc there is a $z \in Q_{\mathcal{S}}(d\boldsymbol{v})$ such that

$$\eta \le \bar{\mu}[\![|z| \ge \gamma]\!] \le \theta(\tfrac{1}{\gamma} w),$$

and we chose γ so that this would not be possible. **X**

Thus $\lim_{\tau\uparrow\mathcal{S}} \sup_{\boldsymbol{u}\in A} \theta(\int_{\mathcal{S}\vee\tau} \boldsymbol{u}\, d\boldsymbol{v}) = 0$.

($\boldsymbol{\beta}$) The same argument works downwards. **?** If $\epsilon = \frac{1}{3}\limsup_{\tau\downarrow\mathcal{S}} \sup_{\boldsymbol{u}\in A} \theta(\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\boldsymbol{v})$ is greater than 0, take $M \ge 1$, $\eta > 0$, $\gamma > 0$, $r \ge m \ge 1$, $k \ge 1$ and $n = mk$ as before, and choose $\tau_n \ge \ldots \ge \tau_0$, $\boldsymbol{u}_{n-1}, \ldots, \boldsymbol{u}_0$ and $I_{n-1}, \ldots, I_0$ such that for $n \ge i > 0$

$$\boldsymbol{u}_{i-1} \in A, \quad \theta(\int_{\mathcal{S}\wedge\tau_i} \boldsymbol{u}_{i-1}\, d\boldsymbol{v}) > 2\epsilon, \quad I_{i-1} \in \mathcal{I}(\mathcal{S}\wedge\tau_i),$$

$$\theta(S_{I_{i-1}}(\mathrm{med}(-M\mathbf{1}^{(\mathcal{S})}, \boldsymbol{u}_{i-1}, M\mathbf{1}^{(\mathcal{S})}), d\boldsymbol{v})) > \epsilon, \quad \tau_{i-1} \le \min I_{i-1}.$$

Once again we finish with $\tau_0 \le \ldots \le \tau_n$ in $\mathcal{S}$, and $I_i \in \mathcal{I}(\mathcal{S} \cap [\tau_i, \tau_{i+1}])$ for each $i < n$, while

$$\sup\{\theta(w) : w \in Q_{\mathcal{S}\cap[\tau_i,\tau_{i+1}]}(d\boldsymbol{v})\} \ge 4\eta$$

for each $i < n$, which is impossible. **X** So $\lim_{\tau\downarrow\mathcal{S}} \sup_{\boldsymbol{u}\in A} \theta(\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\boldsymbol{v}) = 0$.

(b) A is upwards-directed; because $\boldsymbol{v}$ is non-decreasing, $B = \{\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v} : \boldsymbol{u} \in A\}$ is upwards-directed; being topologically bounded, by (a), it is bounded above in L^0 (613B(f-v)). Now $-B = B$ is bounded above, B is bounded below and B is order-bounded.

644G Lemma Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous, and $\mathcal{S}$ is a non-empty order-convex subset of $\mathcal{T}$. On the space $M_{\mathrm{mo}} = M_{\mathrm{mo}}(\mathcal{S})$ of moderately oscillatory processes, we have a linear space topology $\mathfrak{S}$ defined by functionals of the form $\boldsymbol{u} \mapsto \theta(\int_{\mathcal{S}} |\boldsymbol{u}|d\boldsymbol{v})$ where $\boldsymbol{v} \in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$. Let $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ be a near-simple integrator. Then $\boldsymbol{u} \mapsto \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v} : M_{\mathrm{mo}} \to L^0$ is uniformly continuous, for the uniformity induced by $\mathfrak{S}$, on any uniformly order-bounded set in M_{mo}.

proof For an integrator $\boldsymbol{v}$ with domain $\mathcal{S}$ and $u \in M_{\mathrm{mo}}$, write $\psi_{\boldsymbol{v}}(u) = \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$.

(a) I should begin by noting straight away that if $\boldsymbol{v} \in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$ and we set $\phi_{\boldsymbol{v}}(\boldsymbol{u}) = \theta(\psi_{\boldsymbol{v}}|\boldsymbol{u}|)$ for $\boldsymbol{u} \in M_{\mathrm{mo}}$, then $\phi_{\boldsymbol{v}}$ is an F-seminorm. **P** (i) If $\boldsymbol{u}$, $\boldsymbol{u}' \in M_{\mathrm{mo}}$, then $|\boldsymbol{u} + \boldsymbol{u}'| \le |\boldsymbol{u}| + |\boldsymbol{u}'|$. As $\boldsymbol{v}$ is non-decreasing,

$$\phi_{\boldsymbol{v}}(\boldsymbol{u} + \boldsymbol{u}') = \psi_{\boldsymbol{v}}(|\boldsymbol{u} + \boldsymbol{u}'|) \le \psi_{\boldsymbol{v}}(|\boldsymbol{u}| + |\boldsymbol{u}'|) = \psi_{\boldsymbol{v}}(|\boldsymbol{u}|) + \psi_{\boldsymbol{v}}(|\boldsymbol{u}'|) = \phi_{\boldsymbol{v}}(\boldsymbol{u}) + \phi_{\boldsymbol{v}}(\boldsymbol{u}'),$$

(ii) If $\boldsymbol{u} \in M_{\mathrm{mo}}$ and $\alpha \in \mathbb{R}$, then $\phi_{\boldsymbol{v}}(\alpha\boldsymbol{u}) = |\alpha|\phi_{\boldsymbol{v}}(\boldsymbol{u})$, so $\lim_{\alpha\to 0} \phi_{\boldsymbol{v}}(\alpha\boldsymbol{u}) = 0$ and $\phi_{\boldsymbol{v}}(\alpha\boldsymbol{u}) \le \phi_{\boldsymbol{v}}(\boldsymbol{u})$ if $|\alpha| \le 1$. **Q**

Accordingly we have a linear space topology $\mathfrak{S}$ on M_{mo} defined by $\{\phi_{\boldsymbol{v}} : \boldsymbol{v} \in M^{\uparrow}_{\text{n-s}}(\mathcal{S})\}$.

It is worth noting that if $\boldsymbol{v}$, $\boldsymbol{v}' \in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$ then $\boldsymbol{v} + \boldsymbol{v}' \in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$, while $\phi_{\boldsymbol{v}+\boldsymbol{v}'} = \phi_{\boldsymbol{v}} + \phi_{\boldsymbol{v}'}$. So if G is any $\mathfrak{S}$-neighbourhood of 0, there are $\boldsymbol{v} \in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$ and $\delta > 0$ such that $\{\boldsymbol{u} : \boldsymbol{u} \in M_{\mathrm{mo}},\ \phi_{\boldsymbol{v}}(\boldsymbol{u}) \le \delta\} \subseteq G$.

(b) From now on, I take it that $A \subseteq M_{\mathrm{mo}}$ is a uniformly order-bounded set and $\boldsymbol{v} \in M_{\text{n-s}}(\mathcal{S})$ is an integrator. Let $\bar{u} \in L^0$ be such that $\sup|\boldsymbol{u}| \le \bar{u}$ for every $\boldsymbol{u} \in A$. The argument proceeds by looking at a succession of special cases.

If $\boldsymbol{v}$ is actually non-negative and non-increasing, then of course $\psi_{\boldsymbol{v}} : A \to L^0$ is uniformly continuous, just because

$$|\psi_{\boldsymbol{v}}(\boldsymbol{u}) - \psi_{\boldsymbol{v}}(\boldsymbol{u}')| = |\psi_{\boldsymbol{v}}(\boldsymbol{u} - \boldsymbol{u}')| \le \psi_{\boldsymbol{v}}(|\boldsymbol{u} - \boldsymbol{u}'|) = \phi_{\boldsymbol{v}}(\boldsymbol{u} - \boldsymbol{u}').$$

So if $\boldsymbol{v}$ is of bounded variation, therefore expressible as the difference of members of $M^{\uparrow}_{\text{n-s}}(\mathcal{S})$ (631L), then $\psi_{\boldsymbol{v}}$ is a difference of uniformly continuous processes, therefore uniformly continuous, on A.

(c) (The key.) Suppose that $\mathcal{S}$ has greatest and least elements, $M = \|\bar{u}\|_\infty$ is finite and $\boldsymbol{v}$ is an L^∞-martingale. Then $\|v_\sigma\|_2 \le \|v_{\max\mathcal{S}}\|_2 \le \|v_{\max\mathcal{S}}\|_\infty$ for every $\sigma \in \mathcal{S}$ (621Ca) and $\boldsymbol{v}$ is actually $\|\ \|_2$-bounded. Let $\boldsymbol{v}^* = \langle v^*_\sigma\rangle_{\sigma\in\mathcal{S}}$ be the quadratic variation of $\boldsymbol{v}$. Applying 624I with $\boldsymbol{u} = \mathbf{1}$,

$$\mathbb{E}(v^*_{\max\mathcal{S}}) = \mathbb{E}((v_{\max\mathcal{S}} - v_{\min\mathcal{S}})^2) \le \mathbb{E}(v^2_{\max\mathcal{S}}) < \infty.$$

Take any $\epsilon > 0$. Let $\delta \in \,]0, 1]$ be such that $(2M + 1)(\delta + 2M\mathbb{E}(v^*_{\max\mathcal{S}} \times \chi c)) \le \epsilon^2$ whenever $\bar{\mu}c \le \delta$. Take $\boldsymbol{u}$, $\boldsymbol{w} \in A$ such that $\theta(\int_{\mathcal{S}} |\boldsymbol{u} - \boldsymbol{w}|d\boldsymbol{v}^*) \le \delta^2$. In this case, setting $c = [\![\int_{\mathcal{S}} |\boldsymbol{u} - \boldsymbol{w}|d\boldsymbol{v}^* > \delta]\!]$, $\bar{\mu}c \le \delta$. Consequently

$$\|\int_{\mathcal{S}} |\boldsymbol{u} - \boldsymbol{w}|d\boldsymbol{v}^*\|_1 \le \delta + \mathbb{E}(\chi c \times \int_{\mathcal{S}} |\boldsymbol{u} - \boldsymbol{w}|d\boldsymbol{v}^*) \le \delta + \mathbb{E}(\chi c \times 2M \int_{\mathcal{S}} d\boldsymbol{v}^*)$$

(because $|\boldsymbol{u} - \boldsymbol{w}| \le 2M\mathbf{1}$)

$$= \delta + \mathbb{E}(\chi c \times 2Mv^*_{\max\mathcal{S}}) \le \tfrac{\epsilon^2}{2M+1}.$$

We see that

$$(\boldsymbol{u}-\boldsymbol{w})^2=|\boldsymbol{u}-\boldsymbol{w}||\boldsymbol{u}+\boldsymbol{w}|\le 2M|\boldsymbol{u}-\boldsymbol{w}|,$$

so $\|\int_{\mathcal{S}}(\boldsymbol{u}-\boldsymbol{w})^2d\boldsymbol{v}^*\|_1\le\epsilon^2$. Now

$$\theta(\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}-\int_{\mathcal{S}}\boldsymbol{w}\,d\boldsymbol{v})=\theta(\int_{\mathcal{S}}(\boldsymbol{u}-\boldsymbol{w})d\boldsymbol{v})\le\|\int_{\mathcal{S}}(\boldsymbol{u}-\boldsymbol{w})d\boldsymbol{v}\|_1$$

$$\le\|\int_{\mathcal{S}}(\boldsymbol{u}-\boldsymbol{w})d\boldsymbol{v}\|_2=\sqrt{\|\int_{\mathcal{S}}(\boldsymbol{u}-\boldsymbol{w})^2d\boldsymbol{v}^*\|_1}$$

(624I again)

$$\le\epsilon.$$

As ϵ is arbitrary (and $\boldsymbol{v}^*$ is non-negative and non-decreasing and near-simple, by 631Ja), $\psi_{\boldsymbol{v}}$ is uniformly continuous on A.

(d) Next, suppose that $\mathcal{S}$ has greatest and least elements and $\|\bar{u}\|_\infty$ is finite. Take $\epsilon>0$. Since $\mathcal{S}=[\min\mathcal{S},\max\mathcal{S}]$ is finitely full, order-closed in $\mathcal{T}$ and has a greatest member, there are an L^∞-martingale $\tilde{\boldsymbol{v}}$ and a near-simple process $\boldsymbol{v}'$ of bounded variation such that $\bar{\mu}[\![\boldsymbol{v}\ne\tilde{\boldsymbol{v}}+\boldsymbol{v}']\!]\le\epsilon$ (643O). Now

$$\theta(\psi_{\boldsymbol{v}}(\boldsymbol{u})-\psi_{\tilde{\boldsymbol{v}}}(\boldsymbol{u})-\psi_{\boldsymbol{v}'}(\boldsymbol{u}))\le\bar{\mu}[\![\psi_{\boldsymbol{v}}(\boldsymbol{u})\ne\psi_{\tilde{\boldsymbol{v}}}(\boldsymbol{u})+\psi_{\boldsymbol{v}'}(\boldsymbol{u})]\!]$$
$$\le\bar{\mu}[\![\boldsymbol{v}\ne\tilde{\boldsymbol{v}}+\boldsymbol{v}']\!]\le\epsilon$$

for every $\boldsymbol{u}\in A$, while $\psi_{\tilde{\boldsymbol{v}}}+\psi_{\boldsymbol{v}'}$ is uniformly continuous on A, by (b) and (c). As ϵ is arbitrary, $\psi_{\boldsymbol{v}}$ is uniformly continuous on A.

(e) Suppose just that $\mathcal{S}$ has greatest and least elements, while $\bar{u}$ is defined in L^0 and $\boldsymbol{v}$ is a near-simple integrator. Let $\epsilon>0$. Then there is an $M\ge 0$ such that $\bar{\mu}[\![\bar{u}>M]\!]\le\epsilon$. Set $A_1=\{\boldsymbol{u}:\boldsymbol{u}\in M_{\mathrm{mo}},\|\boldsymbol{u}\|_\infty\le M\}$ and $\bar{h}(\boldsymbol{u})=\mathrm{med}(-M\mathbf{1},\boldsymbol{u},M\mathbf{1})$ for $\boldsymbol{u}\in A$. Then $|\bar{h}(\boldsymbol{u})-\bar{h}(\boldsymbol{w})|\le|\boldsymbol{u}-\boldsymbol{w}|$ for all $\boldsymbol{u}$, $\boldsymbol{w}\in A$, so $\bar{h}:A\to A_1$ is uniformly continuous; by (d), $\psi_{\boldsymbol{v}}\restriction A_1$ is uniformly continuous, so the composition $\psi_{\boldsymbol{v}}\bar{h}$ is uniformly continuous. Next,

$$\theta(\psi_{\boldsymbol{v}}(\boldsymbol{u})-\psi_{\boldsymbol{v}}(\bar{h}(\boldsymbol{u})))\le\bar{\mu}[\![\psi_{\boldsymbol{v}}(\boldsymbol{u})\ne\psi_{\boldsymbol{v}}(\bar{h}(\boldsymbol{u}))]\!]\le\bar{\mu}[\![\boldsymbol{u}\ne\bar{h}(\boldsymbol{u})]\!]\le\bar{\mu}[\![\bar{u}>M]\!]\le\epsilon$$

for every $\boldsymbol{u}\in A$. As ϵ is arbitrary, $\psi_{\boldsymbol{v}}$ is uniformly approximated on A by uniformly continuous functions and is itself uniformly continuous.

(f) Finally, for the general case, suppose only that $\mathcal{S}$ is order-convex, $A\subseteq M_{\mathrm{mo}}$ is uniformly order-bounded and that $\boldsymbol{v}$ is a near-simple integrator with domain $\mathcal{S}$. If $\mathcal{S}$ is empty then $\#(A)\le 1$ and trivially we have a uniformly continuous function. Otherwise, let $\epsilon>0$. By 644F(a-ii), there is a $\tau\in\mathcal{S}$ such that $\sup_{\boldsymbol{u}\in A}\theta(\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\,d\boldsymbol{v})\le\epsilon$; applying the other half of 644F(a-ii) to $\boldsymbol{v}\restriction\mathcal{S}\vee\tau$ and $\{\boldsymbol{u}\restriction\mathcal{S}\vee\tau:\boldsymbol{u}\in A\}$, we see that there is a $\tau'\in\mathcal{S}\vee\tau$ such that $\sup_{\boldsymbol{u}\in A}\theta(\int_{\mathcal{S}\vee\tau'}\boldsymbol{u}\,d\boldsymbol{v})\le\epsilon$. (Of course this step depends on knowing that $\boldsymbol{v}\restriction\mathcal{S}\vee\tau$ is a near-simple integrator, by 616P(b-ii) and 631F(a-iv), while $\boldsymbol{u}\restriction\mathcal{S}\vee\tau$ is moderately oscillatory for every $\boldsymbol{u}\in M_{\mathrm{mo}}$, by 615F(a-i) again; clearly $\{\boldsymbol{u}\restriction\mathcal{S}\vee\tau:\boldsymbol{u}\in A\}$ will be uniformly order-bounded if A is.)

Now consider $\mathcal{S}'=\mathcal{S}\cap[\tau,\tau']=[\tau,\tau']$, $\boldsymbol{v}\restriction\mathcal{S}'$ and $\{\boldsymbol{u}\restriction\mathcal{S}':\boldsymbol{u}\in A\}$. From (e) and the last remark in (a) above we know that there are a non-negative, non-decreasing process $\tilde{\boldsymbol{v}}'=\langle\tilde{v}'_\sigma\rangle_{\sigma\in\mathcal{S}'}$ and $\delta>0$ such that $\theta(\int_{\mathcal{S}'}\boldsymbol{u}-\boldsymbol{u}'\,d\boldsymbol{v})\le\epsilon$ whenever $\boldsymbol{u}$, $\boldsymbol{u}'\in A$ and $\int_{\mathcal{S}'}|\boldsymbol{u}-\boldsymbol{u}'|d\tilde{\boldsymbol{v}}'\le\delta$. Define $\tilde{\boldsymbol{v}}=\langle\tilde{v}_\sigma\rangle_{\sigma\in\mathcal{S}}$ by setting

$$\tilde{v}_\sigma=\tilde{v}'_{\mathrm{med}(\tau,\sigma,\tau')}\times\chi([\![\tau\le\sigma]\!]\cap[\![\sigma<\tau']\!])+\tilde{v}'_{\tau'}\times\chi[\![\tau'\le\sigma]\!]$$

for $\sigma\in\mathcal{S}$. It is straightforward to check that $\tilde{\boldsymbol{v}}$ is fully adapted (use 612C), non-negative, non-decreasing and order-bounded (with greatest value $v'_{\tau'}$), while $\tilde{\boldsymbol{v}}\restriction\mathcal{S}'=\tilde{\boldsymbol{v}}'$; and by 631F(a-iv) again $\tilde{\boldsymbol{v}}$ is near-simple.

Suppose that $\boldsymbol{u}\in A$. Then

$$\theta(\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}-\int_{\mathcal{S}'}\boldsymbol{u}\,d\boldsymbol{v})=\theta(\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\,d\boldsymbol{v}+\int_{\mathcal{S}\vee\tau'}\boldsymbol{u}\,d\boldsymbol{v})\le 2\epsilon$$

by the choice of τ and τ'. Now if $\boldsymbol{u}$, $\boldsymbol{u}'\in A$ and $\theta(\int_{\mathcal{S}}|\boldsymbol{u}-\boldsymbol{u}'|d\tilde{\boldsymbol{v}})\le\delta$, we shall have

$$\theta(\int_{\mathcal{S}'}|\boldsymbol{u}-\boldsymbol{u}'|d\tilde{\boldsymbol{v}}')=\theta(\int_{\mathcal{S}'}|\boldsymbol{u}-\boldsymbol{u}'|d\tilde{\boldsymbol{v}})\le\delta,$$

so that $\theta(\int_{\mathcal{S}'}(\boldsymbol{u}-\boldsymbol{u}')d\boldsymbol{v})\le\epsilon$ and $\theta(\int_{\mathcal{S}}(\boldsymbol{u}-\boldsymbol{u}')d\boldsymbol{v})\le 5\epsilon$. As ϵ is arbitrary, $\boldsymbol{u}\mapsto\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}:A\to L^0$ is uniformly continuous, and the proof is complete.

644H Theorem Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$ and $\langle\boldsymbol{u}_n\rangle_{n\in\mathbb{N}}$ a uniformly order-bounded sequence of moderately oscillatory processes with domain $\mathcal{S}$ such that $\langle\boldsymbol{u}_{n<}\rangle_{n\in\mathbb{N}}$ is order*-convergent in $(L^0)^{\mathcal{S}}$. Then $\lim_{n\to\infty}\int_{\mathcal{S}}\boldsymbol{u}_n d\boldsymbol{v}$ is defined for every near-simple integrator $\boldsymbol{v}$ with domain $\mathcal{S}$. If $\langle\boldsymbol{u}_{n<}\rangle_{n\in\mathbb{N}}$ is order*-convergent to $\boldsymbol{u}_<$, where $\boldsymbol{u}$ is moderately oscillatory, then $\lim_{n\to\infty}\int_{\mathcal{S}}\boldsymbol{u}_n d\boldsymbol{v}=\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}$.

proof If $\langle n_k\rangle_{k\in\mathbb{N}}$ is strictly increasing and we set $\boldsymbol{w}_k=|\boldsymbol{u}_{n_{k+1}}-\boldsymbol{u}_{n_k}|$ for each k, $\langle\boldsymbol{w}_k\rangle_{k\in\mathbb{N}}$ is a uniformly order-bounded sequence of moderately oscillatory processes such that $\langle\boldsymbol{w}_{k<}\rangle_{k\in\mathbb{N}}$ is order*-convergent to 0 (apply the last sentence of 642Ba with $g(\alpha,\beta)=|\alpha-\beta|$). So 644E tells us that $\lim_{k\to\infty}\int_{\mathcal{S}}\boldsymbol{w}_k\,d\boldsymbol{v}=0$ for every $\boldsymbol{v}\in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$. As $\langle n_k\rangle_{k\in\mathbb{N}}$ is arbitrary, $\lim_{m,n\to\infty}\theta(\int_{\mathcal{S}}|\boldsymbol{u}_n-\boldsymbol{u}_m|d\boldsymbol{v})=0$ for every $\boldsymbol{v}\in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$. By 644G, $\lim_{m,n\to\infty}\int_{\mathcal{S}}\boldsymbol{u}_n-\boldsymbol{u}_m d\boldsymbol{v}=0$ for every near-simple integrator $\boldsymbol{v}$ with domain $\mathcal{S}$, that is, $\lim_{n\to\infty}\int_{\mathcal{S}}\boldsymbol{u}_n d\boldsymbol{v}$ is defined for every near-simple integrator $\boldsymbol{v}$ with domain $\mathcal{S}$.

If we know that there is a moderately oscillatory process $\boldsymbol{u}$ such that $\langle\boldsymbol{u}_{n<}\rangle_{n\in\mathbb{N}}$ order*-converges to $\boldsymbol{u}_<$, then we can apply the trick of 644E to a sequence alternating between $\boldsymbol{u}_n-\boldsymbol{u}$ and zero to see that $\lim_{n\to\infty}\int_{\mathcal{S}}\boldsymbol{u}_n\,d\boldsymbol{v}=\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}$.

644X Basic exercises (a) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\langle\boldsymbol{u}_n\rangle_{n\in\mathbb{N}}=\langle\langle u_{n\sigma}\rangle_{\sigma\in\mathcal{S}}\rangle_{n\in\mathbb{N}}$ a uniformly order-bounded sequence in $(L^0)^{\mathcal{S}}$ which is order*-convergent to $\boldsymbol{u}\in(L^0)^{\mathcal{S}}$. Show that $\boldsymbol{u}$ is order-bounded.

>(b) Give an example of a simple process $\boldsymbol{v}$ and a non-increasing sequence $\langle\boldsymbol{u}_n\rangle_{n\in\mathbb{N}}$ of simple processes, all these processes having the same domain $\mathcal{S}$, such that $\inf_{n\in\mathbb{N}}\boldsymbol{u}_n=\mathbf{0}$ but $\langle\int_{\mathcal{S}}\boldsymbol{u}_n d\boldsymbol{v}\rangle_{n\in\mathbb{N}}$ is not convergent to 0. (*Hint*: take $T=[0,\infty[$, $\mathfrak{A}=\{0,1\}$.)

(c) In 644G, show that the topology $\mathfrak{S}$ described there is coarser than the ucp topology.

(d)(i) Suppose that U and V are linear topological spaces and $T:U\to V$ is a linear operator. Let $A\subseteq U$ be a non-empty set such that $T\restriction(A-A)$ is continuous at 0. Show that $T\restriction A$ is uniformly continuous. (ii) Use this to simplify the formulae in the proof of 644G.

(e) In 644G, suppose that $T=[0,\infty[$ and $\mathfrak{A}=\{0,1\}$, as in 613W, 615Xf, 616Xa, 617Xb, 618Xa, 622Xd, 626Xa, 627Xa and 642Xd. Show that the topology $\mathfrak{S}$ on $M_{\text{o-b}}(\mathcal{T}_f)$ corresponds to the topology on the space of bounded functions in $\tilde{C}^{\Downarrow}$ (615Xf) generated by functionals of the form $f\mapsto\int|f_<|d\nu$ where ν is a totally finite Radon measure on $[0,\infty[$, taking $f_<(0)=0$ and $f_<(t)=\lim_{s\uparrow t}f(s)$ for $t>0$.

644 Notes and comments The essence of the Lebesgue integral lies in the convergence theorems of §123, all concerning limits of integrals of sequences of functions. Here for the first time in this volume we have a corresponding result for limits of Riemann-sum integrals. The real content lies in the case in which the sequence $\langle\boldsymbol{u}_n\rangle_{n\in\mathbb{N}}$ of integrands is a non-increasing sequence of non-negative processes such that $\inf_{n\in\mathbb{N}}\boldsymbol{u}_{n<}=0$ and the integrator is non-decreasing (644C). Moving to the case of a uniformly order-bounded sequence $\langle\boldsymbol{u}_n\rangle_{n\in\mathbb{N}}$ such that $\langle\boldsymbol{u}_{n<}\rangle_{n\in\mathbb{N}}$ is order*-convergent in $(L^0)^{\mathcal{S}}$ (644D) is not quite trivial, but is really a result about positive linear operators on Riesz spaces. The point here is that $M_{\text{mo}}(\mathcal{S})$ is a Riesz subspace of $(L^0)^{\mathcal{S}}$, but is not as a rule sequentially order-closed.

To handle general integrators, the first step is to deal with L^2-martingales (part (c) of the proof of 644G). At this point it seems that we have to move up a gear, assuming right-continuity of the filtration and order-convexity of the sublattice $\mathcal{S}$ (644G) so as to apply the Fundamental Theorem of Martingales. But under these conditions we get a result more or less corresponding to the Lebesgue's Dominated Convergence Theorem (644H). When adapting this to define a second kind of integral (§645), it will be helpful to refer to a new uniformity on the space of moderately oscillatory processes (644G). The point is that this is much coarser than the uniformity corresponding to the ucp topology, but still makes integration uniformly continuous on uniformly order-bounded sets.

Version of 29.5.21

645 Construction of the S-integral

We are now in a position to define a sequentially smooth integral which corresponds, in a sense, to Lebesgue-Stieltjes integrals on the real line. The objective is to integrate bounded previsible processes with respect to near-simple integrators, and I set this up as a kind of Daniell integral (see 436Ya) based on ideas in §644. Since simple and near-simple and mderately oscillatory processes, as I have defined them in this volume, are often not previsible, we need to deal throughout with their previsible versions; and as our integrals take values in L^0 rather than in $\mathbb{R}$ or $\mathbb{C}$, we have to calculate with the functional θ rather than with a modulus or norm.

The key to the programme is really Lemma 644G. We saw there that (subject to certain conditions) integration with respect to an arbitrary near-simple integrator is controlled by integration with respect to appropriate non-decreasing processes. We can therefore do nearly all the work of the present section with non-decreasing integrators, which are very much easier to handle, even though our real aim is to understand integration with respect to martingales. With a non-decreasing integrator, as with an ordinary non-negative measure, integration is a positive linear operator. This makes it possible to consider upper integrals, which are what, in effect, we have in Definition 645Bb. Based on the functionals $\widehat{\theta}^{\#}_{\boldsymbol{v}}$ there, we have a linear space topology $\mathfrak{T}_{\text{S-i}}$ on a large space $M_{\text{po-b}}$ of order-bounded processes (645F). As with ordinary integration, unbounded sequences of integrands can be uncontrollable, so we have to find types of domination – preferably weaker than simply assuming uniform $\| \|_\infty$-boundedness – which will be adequate to ensure convergence of sequences of integrals. (See 645G-645L.) These bring us to a definition of what I call the 'S-integral' in 645P, in a form which makes it easy to check that it is bilinear in integrand and integrator (645Rb), and with the tools to show that it is sequentially smooth in the integrand (645T).

645A Notation This section brings together many ideas, and there is a correspondingly long list of notations to recall. As always, $(\mathfrak{A}, \bar{\mu}, T, \langle\mathfrak{A}_t\rangle_{t\in T}, \mathcal{T}, \langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ will be a stochastic integration structure. For a sublattice A of $\mathcal{T}$ and $\tau \in \mathcal{T}$, $A \wedge \tau = \{\sigma \wedge \tau : \sigma \in A\}$ and $A \vee \tau = \{\sigma \vee \tau : \sigma \in A\}$. $L^0 = L^0(\mathfrak{A})$ (612A) and $\theta(w) = \mathbb{E}(|w| \wedge \chi 1)$ for $w \in L^0$ (613Ba). If $k \ge 1$ and $h : \mathbb{R}^k \to \mathbb{R}$ is Borel measurable, I will write $\bar{h}$ for any of the associated functions from $(L^0)^k$ to L^0 (612A, 619E) and from $((L^0)^{\mathcal{S}})^k$ to $(L^0)^{\mathcal{S}}$ (612B, 619F).

If $\mathcal{S}$ is a sublattice of $\mathcal{T}$, $\boldsymbol{u}$, $\boldsymbol{v}$ are processes with domain $\mathcal{S}$, and I is a finite sublattice of $\mathcal{S}$, then $S_I(\boldsymbol{u}, d\boldsymbol{v})$ will be the corresponding Riemann sum (613Fb), $Q_{\mathcal{S}}(d\boldsymbol{v})$ the capped-stake variation set (616B) and $\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}$ (if it exists) the Riemann-sum integral defined in 613H/613Na. $\mathbf{1}^{(\mathcal{S})}$ will be the constant process with domain $\mathcal{S}$ and value $\chi 1$.

If $\mathcal{S}$ is a sublattice of $\mathcal{T}$, $M_{\text{o-b}}(\mathcal{S})$ is the f-algebra of order-bounded processes with domain $\mathcal{S}$ (614Fc); for $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}} \in M_{\text{o-b}}(\mathcal{S})$, $\sup|\boldsymbol{u}| = \sup_{\sigma\in\mathcal{S}} |u_\sigma|$. $M_{\text{mo}}(\mathcal{S})$ is the f-algebra of moderately oscillatory processes with domain $\mathcal{S}$ (615Fa), and $M_{\text{mo}}(\mathcal{S})^+$ its positive cone. If $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}} \in M_{\text{mo}}(\mathcal{S})$, $\boldsymbol{u}_< = \langle u_{<\sigma}\rangle_{\sigma\in\mathcal{S}}$ will always be its previsible version (641L). $M_{\text{fa}}(\mathcal{S})$, $M_{\text{simp}}(\mathcal{S})$ and $M_{\text{n-s}}(\mathcal{S})$ are the spaces of fully adapted, simple and near-simple processes with domain $\mathcal{S}$ (612I, 612L, 631Fa). $M^{\uparrow}_{\text{n-s}}(\mathcal{S})$ is the cone of non-negative non-decreasing near-simple processes with domain $\mathcal{S}$ (644Bb), and $M_{\text{bv}}(\mathcal{S})$ is the space of processes of bounded variation with domain $\mathcal{S}$ (614K).

645B Definitions Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.

(a)(i) I will say that a fully adapted process $\boldsymbol{x}$ with domain $\mathcal{S}$ is **previsibly order-bounded** if there is a non-negative $\boldsymbol{u} \in M_{\text{mo}}(\mathcal{S})$ such that $|\boldsymbol{x}| \le \boldsymbol{u}_<$. $M_{\text{po-b}}(\mathcal{S})$ will be the set of previsibly order-bounded fully adapted processes $\boldsymbol{x}$ with domain $\mathcal{S}$.

(ii) I will say that a set $A \subseteq M_{\text{po-b}}(\mathcal{S})$ is **uniformly previsibly order-bounded** if there is a non-negative $\boldsymbol{u} \in M_{\text{mo}}(\mathcal{S})$ such that $|\boldsymbol{x}| \le \boldsymbol{u}_<$ for every $\boldsymbol{x} \in A$. A uniformly previsibly order-bounded set is uniformly order-bounded in the sense of 644Bb, by 641G(a-vii-γ).

(b) For $\boldsymbol{x} \in M_{\text{po-b}}(\mathcal{S})$ and $\boldsymbol{v} \in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$, write $\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x})$ for

$$\inf\{\sup_{n\in\mathbb{N}}\theta(\int_{\mathcal{S}}\boldsymbol{u}_n\,d\boldsymbol{v}) : \langle\boldsymbol{u}_n\rangle_{n\in\mathbb{N}} \text{ is a uniformly order-bounded non-decreasing sequence}$$
$$\text{of non-negative processes in } M_{\mathrm{mo}}(\mathcal{S}) \text{ and } |\boldsymbol{x}| \le \sup_{n\in\mathbb{N}}\boldsymbol{u}_{n<}\}.$$

For the avoidance of doubt, perhaps I should say that the supremum here is to be taken in $(L^0)^{\mathcal{S}}$, so that (in the notation of 641F) we have $|x_\sigma| \le \sup_{n\in\mathbb{N}} u_{n<\sigma}$ for every $\sigma\in\mathcal{S}$.

(c) If $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$, $\boldsymbol{v}' = \langle v'_\sigma\rangle_{\sigma\in\mathcal{S}}$ are two fully adapted processes with domain $\mathcal{S}$, I will write $\boldsymbol{v} \preccurlyeq \boldsymbol{v}'$ if $\boldsymbol{v}'-\boldsymbol{v}$ is non-decreasing, that is, $v_\tau - v_\sigma \le v'_\tau - v'_\sigma$ whenever $\sigma\le\tau$ in $\mathcal{S}$, that is, $\Delta\boldsymbol{v} \le \Delta\boldsymbol{v}'$ where $\Delta\boldsymbol{v}$ and $\Delta\boldsymbol{v}'$ are the associated adapted interval functions (613Cc).

In this case, if $\boldsymbol{u}$ is any non-negative fully adapted process with domain $\mathcal{S}$, $\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v} \le \int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}'$ if both integrals are defined (616R(b-i), applied to $\boldsymbol{v}'-\boldsymbol{v}$).

645C A standard fragment of real analysis didn't quite get into §4A3.

Lemma (a) Let X be a metrizable space. Then the set of Borel measurable real-valued functions on X is the smallest subset U of $\mathbb{R}^X$ which contains every bounded continuous real-valued function and is such that $\lim_{n\to\infty} h_n \in U$ whenever $\langle h_n\rangle_{n\in\mathbb{N}}$ is a sequence in U which has a limit in $\mathbb{R}$ at every point and which is either non-decreasing or non-increasing.

(b) If $k\ge 1$ and $h:\mathbb{R}^k\to\mathbb{R}$ is a locally bounded function, then there is a continuous non-decreasing function $g:\mathbb{R}\to[0,\infty[$ such that $|h(x)|\le g(\|x\|)$ for every $x\in\mathbb{R}^k$.

(c) If $k\ge 1$ and U is a set of real-valued functions on $\mathbb{R}^k$ such that (α) every continuous function belongs to U (β) $\lim_{n\to\infty} f_n\in U$ whenever $\langle f_n\rangle_{n\in\mathbb{N}}$ is a pointwise convergent sequence in U and $\sup_{n\in\mathbb{N}}|f_n|$ is locally bounded, then every locally bounded Borel measurable function on $\mathbb{R}^k$ belongs to U.

proof (a)(i) We know that the set of Borel measurable functions is closed under pointwise limits of sequences (121E-121F) and that every continuous function is Borel measurable (4A3Cd), so every member of U is Borel measurable.

(ii) In the other direction, we can argue as follows. Write C_b for the space of bounded continuous functions on X, and let $\mathbb{V}$ be the family of subsets V of $\mathbb{R}^X$ such that $C_b(X)\subseteq V$ and V is closed under monotone sequential limits. Then $U=\bigcap\mathbb{V}$.

($\boldsymbol{\alpha}$) If $g\in C_b$, then it is easy to check that $\{h : h\in\mathbb{R}^X,\ g+h\in U\}$ belongs to $\mathbb{V}$, so includes U. Thus $g+h\in U$ for every $g\in C_b$ and $h\in U$. Next, $\{g : g\in\mathbb{R}^X,\ g+h\in U$ for every $h\in U\}$ belongs to $\mathbb{V}$, so U is closed under addition. And if $\alpha\in\mathbb{R}$, $\{h : h\in\mathbb{R}^X,\ \alpha h\in U\}\in\mathbb{V}$, so $\alpha U\subseteq U$. Thus U is a linear subspace of $\mathbb{R}^X$.

($\boldsymbol{\beta}$) If $G\subseteq X$ is open then $\alpha\chi G\in U$. **P** If $G=X$ then $\chi G\in C_b$ and we can stop. Otherwise, let ρ be a metric on X defining its topology, and set $h_n(x)=\min(1,2^n\rho(x,X\setminus G))$ for $n\in\mathbb{N}$ and $x\in X$; then $\langle h_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in C_b with limit χG, so $\chi G\in U$. **Q**

($\boldsymbol{\gamma}$) Set $\mathcal{E}=\{E : E\subseteq X,\ \chi E\in U\}$. Then $\mathcal{E}$ satisfies (i) of 136A, so is a Dynkin class; as it contains every open set, it contains every Borel set, by the Monotone Class Theorem (136B).

($\boldsymbol{\delta}$) Because U is a linear subspace of $\mathbb{R}^X$, it contains any linear combination of indicator functions of Borel sets. But any non-negative Borel measurable function is expressible as the limit of a non-decreasing sequence of such simple Borel measurable functions, so belongs to U; and now every Borel measurable function is the difference of non-negative Borel measurable functions, so belongs to U.

(b) For $n\in\mathbb{N}$, set $\gamma_n=\sup\{|h(x)| : x\in\mathbb{R}^k,\ \|x\|\le n\}$; define $g:\mathbb{R}\to[0,\infty[$ by setting

$$\begin{aligned} g(\alpha) &= \gamma_1 \text{ if } \alpha\le 0,\\ &= (\alpha-n)\gamma_{n+2}+(n+1-\alpha)\gamma_{n+1} \text{ if } n\in\mathbb{N} \text{ and } n\le\alpha\le n+1. \end{aligned}$$

Then g is continuous and non-decreasing and $|h(x)|\le g(\|x\|)$ for every x.

(c)(i) Take any continuous $g:\mathbb{R}^k\to[0,\infty[$ and set $V_g=\{f:f\in\mathbb{R}^{\mathbb{R}^k},\ \operatorname{med}(-g,f,g)\in U\}$. Then V_g contains all continuous functions and is closed under sequential pointwise convergence, so contains every Borel measurable function on $\mathbb{R}^k$, by (a).

(ii) If now $h:\mathbb{R}^k\to\mathbb{R}$ is any locally bounded Borel measurable function, there is a continuous g such that $|h|\le g$, by (b), while $h\in V_g$, so that $h=\operatorname{med}(-g,h,g)$ belongs to U.

645D Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.

(a)(i) If $\boldsymbol{x}_0,\dots,\boldsymbol{x}_{k-1}\in M_{\text{po-b}}=M_{\text{po-b}}(\mathcal{S})$ and $h:\mathbb{R}^k\to\mathbb{R}$ is a locally bounded Borel measurable function, then $\bar h(\boldsymbol{x}_0,\dots,\boldsymbol{x}_{k-1})\times\mathbf{1}^{(\mathcal{S})}_{<}\in M_{\text{po-b}}$; and if $h(0,\dots,0)=0$, then $\bar h(\boldsymbol{x}_0,\dots,\boldsymbol{x}_{k-1})\in M_{\text{po-b}}$.

(ii) $M_{\text{po-b}}$ is an f-subalgebra of $M_{\text{o-b}}(\mathcal{S})$.

(iii) $\boldsymbol{u}_<\in M_{\text{po-b}}$ for every $\boldsymbol{u}\in M_{\text{o-b}}(\mathcal{S})$.

(iv) If $z\in L^0(\mathfrak{A}\cap\bigcap_{\sigma\in\mathcal{S}}\mathfrak{A}_\sigma)$ then $z\boldsymbol{x}$ (definition: 612De) belongs to $M_{\text{po-b}}$ for every $\boldsymbol{x}\in M_{\text{po-b}}$.

(b) Suppose that $\boldsymbol{v}\in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$. Then $\widehat{\theta}^{\#}_{\boldsymbol{v}}$ is an F-seminorm and if $\boldsymbol{x},\boldsymbol{x}'\in M_{\text{po-b}}$, $|\boldsymbol{x}|\le|\boldsymbol{x}'|$ and $\alpha\in\mathbb{R}$ then $\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x})\le\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x}')$ and $\widehat{\theta}^{\#}_{\boldsymbol{v}}(\alpha\boldsymbol{x})\le\max(1,|\alpha|)\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x})$.

(c) If now we have another $\boldsymbol{v}'\in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$ and $\boldsymbol{v}\preccurlyeq\boldsymbol{v}'$ in the sense of 645Bc, $\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x})\le\widehat{\theta}^{\#}_{\boldsymbol{v}'}(\boldsymbol{x})$ for every $\boldsymbol{x}\in M_{\text{po-b}}(\mathcal{S})$.

proof (a)(i) Let $\boldsymbol{u}\in M^+_{\text{mo}}$ be such that $\sum_{i=0}^{k-1}|\boldsymbol{x}_i|\le\boldsymbol{u}_<$. By 645Cb, there is a continuous non-decreasing $g:\mathbb{R}\to[0,\infty[$ such that $|h(x)|\le g(\|x\|)$ for every $x\in\mathbb{R}^k$; it follows that

$$|\bar h(\boldsymbol{x}_0,\dots,\boldsymbol{x}_{k-1})\times\mathbf{1}^{(\mathcal{S})}_{<}|\le\bar g\circ\boldsymbol{u}_<\times\mathbf{1}^{(\mathcal{S})}_{<}=(\bar g\boldsymbol{u})_<$$

(641Gd) and $\bar h(\boldsymbol{x}_0,\dots,\boldsymbol{x}_{k-1})\times\mathbf{1}^{(\mathcal{S})}_{<}\in M_{\text{po-b}}$. If in addition $h(0,\dots,0)=0$, then

$$\bar h(\boldsymbol{x}_0,\dots,\boldsymbol{x}_{k-1})=\bar h(\boldsymbol{x}_0\times\mathbf{1}^{(\mathcal{S})}_{<},\dots,\boldsymbol{x}_{k-1}\times\mathbf{1}^{(\mathcal{S})}_{<})$$
$$=\bar h(\boldsymbol{x}_0,\dots,\boldsymbol{x}_{k-1})\times\mathbf{1}^{(\mathcal{S})}_{<}$$

(by 619G(e-ii), because $\mathbf{1}^{(\mathcal{S})}_{<}\times\mathbf{1}^{(\mathcal{S})}_{<}=(\mathbf{1}^{\mathcal{S}}\times\mathbf{1}^{\mathcal{S}})_<=\mathbf{1}^{(\mathcal{S})}_{<}$)

$$\in M_{\text{po-b}}.$$

(ii)($\boldsymbol{\alpha}$) If $\boldsymbol{x}\in M_{\text{po-b}}$, there is a non-negative $\boldsymbol{u}\in M_{\text{mo}}=M_{\text{mo}}(\mathcal{S})$ such that $|\boldsymbol{x}|\le\boldsymbol{u}_<$. We know that $\boldsymbol{u}$ is order-bounded, while $\sup|\boldsymbol{u}_<|\le\sup|\boldsymbol{u}|$, as in 641G(a-vii), so $\boldsymbol{u}_<$ and $\boldsymbol{x}$ are order-bounded.

($\boldsymbol{\beta}$) As noted in 612Bc, (i) just above is enough to ensure that $M_{\text{po-b}}$ is an f-aslgebra.

($\boldsymbol{\delta}$) As in 612Bc, (β) and (γ) are enough to ensure that $M_{\text{po-b}}$ is an f-algebra.

(ii) This is elementary; all we need to know is that $|\boldsymbol{u}|$ is moderately oscillatory and $|\boldsymbol{u}_<|=|\boldsymbol{u}|_<$ (641G(e-i)).

(iii) There is a $\boldsymbol{u}\in M^+_{\text{mo}}$ such that $|\boldsymbol{x}|\le\boldsymbol{u}_<$; now $|z|\boldsymbol{u}=|z|\mathbf{1}\times\boldsymbol{u}$ is moderately oscillatory (615F(a-iii)) and non-negative, and $|z\boldsymbol{x}|\le|z|\boldsymbol{u}_<=(|z|\boldsymbol{u})_<$ (641G(a-iv)), so $z\boldsymbol{x}$ is previsibly order-bounded.

(iv) This was covered in (i-γ) above.

(b)(i) $\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x}+\tilde{\boldsymbol{x}})\le\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x})+\widehat{\theta}^{\#}_{\boldsymbol{v}}(\tilde{\boldsymbol{x}})$ for all $\boldsymbol{x},\tilde{\boldsymbol{x}}\in M_{\text{po-b}}$. **P** For any $\epsilon>0$, there are non-decreasing uniformly order-bounded sequences $\langle\boldsymbol{u}_n\rangle_{n\in\mathbb{N}}$, $\langle\tilde{\boldsymbol{u}}_n\rangle_{n\in\mathbb{N}}$ of non-negative moderately oscillatory processes, all with domain $\mathcal{S}$, such that

$$|\boldsymbol{x}|\le\sup_{n\in\mathbb{N}}\boldsymbol{u}_{n<},\quad\sup_{n\in\mathbb{N}}\theta(\int_{\mathcal{S}}\boldsymbol{u}_n\,d\boldsymbol{v})\le\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x})+\epsilon,$$

$$|\tilde{\boldsymbol{x}}|\le\sup_{n\in\mathbb{N}}\tilde{\boldsymbol{u}}_{n<},\quad\sup_{n\in\mathbb{N}}\theta(\int_{\mathcal{S}}\tilde{\boldsymbol{u}}_n\,d\boldsymbol{v})\le\widehat{\theta}^{\#}_{\boldsymbol{v}}(\tilde{\boldsymbol{x}})+\epsilon.$$

Now $\langle\boldsymbol{u}_n+\tilde{\boldsymbol{u}}_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing uniformly order-bounded sequence of non-negative moderately oscillatory processes, $|\boldsymbol{x}+\tilde{\boldsymbol{x}}|\le\sup_{n\in\mathbb{N}}(\boldsymbol{u}_n+\tilde{\boldsymbol{u}}_n)_<$ and

$$\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x}+\tilde{\boldsymbol{x}}) \le \sup_{n\in\mathbb{N}} \theta(\int_{\mathcal{S}} (\boldsymbol{u}_n + \tilde{\boldsymbol{u}}_n)\,d\boldsymbol{v})$$
$$\le \sup_{n\in\mathbb{N}} \bigl(\theta(\int_{\mathcal{S}} \boldsymbol{u}_n\,d\boldsymbol{v}) + \theta(\int_{\mathcal{S}} \tilde{\boldsymbol{u}}_n)\,d\boldsymbol{v}\bigr) \le \widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x}) + \widehat{\theta}^{\#}_{\boldsymbol{v}}(\tilde{\boldsymbol{x}}) + 2\epsilon.$$

As ϵ is arbitrary, we have the result. **Q**

(ii) Suppose that $\boldsymbol{x} \in M_{\text{po-b}}$. Let $\boldsymbol{u} \in M^+_{\text{mo}}$ be such that $|\boldsymbol{x}| \le \boldsymbol{u}_<$. Then

$$|\alpha\boldsymbol{x}| \le |\alpha|\boldsymbol{u}_< = (|\alpha|\boldsymbol{u})_<$$

for every α, so

$$\limsup_{\alpha\to 0} \widehat{\theta}^{\#}_{\boldsymbol{v}}(\alpha\boldsymbol{x}) \le \limsup_{\alpha\to 0} \theta(\int_{\mathcal{S}} |\alpha|\boldsymbol{u}\,d\boldsymbol{v})$$
$$= \limsup_{\alpha\to 0} \theta(\alpha\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}) = 0.$$

So $\widehat{\theta}^{\#}_{\boldsymbol{v}}$ is an F-seminorm.

(iii) Immediately from the definition in 645Bb, we see that if $\boldsymbol{x}$, $\boldsymbol{x}' \in M_{\text{po-b}}$ and $|\boldsymbol{x}| \le |\boldsymbol{x}'|$ then $\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x}) \le \widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x}')$.

(iv) If $\langle\boldsymbol{u}_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in M^+_{mo} such that $|\boldsymbol{x}| \le \sup_{n\in\mathbb{N}} \boldsymbol{u}_{n<}$, then $\langle|\alpha|\boldsymbol{u}_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in M^+_{mo} and $|\alpha\boldsymbol{x}| \le \sup_{n\in\mathbb{N}} |\alpha|\boldsymbol{u}_{n<}$, so

$$\widehat{\theta}^{\#}_{\boldsymbol{v}}(\alpha\boldsymbol{x}) \le \sup_{n\in\mathbb{N}} \theta(\int_{\mathcal{S}} |\alpha|\boldsymbol{u}_n\,d\boldsymbol{v})$$
$$= \sup_{n\in\mathbb{N}} \theta(|\alpha|\int_{\mathcal{S}} \boldsymbol{u}_n\,d\boldsymbol{v}) \le \sup_{n\in\mathbb{N}} \max(1,|\alpha|)\theta(\int_{\mathcal{S}} \boldsymbol{u}_n\,d\boldsymbol{v})$$

(613Ba)

$$= \max(1,|\alpha|)\sup_{n\in\mathbb{N}} \theta(\int_{\mathcal{S}} \boldsymbol{u}_n\,d\boldsymbol{v}).$$

As $\langle\boldsymbol{u}_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\widehat{\theta}^{\#}_{\boldsymbol{v}}(\alpha\boldsymbol{x}) \le \max(1,|\alpha|)\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x})$; in particular, $\widehat{\theta}^{\#}_{\boldsymbol{v}}(\alpha\boldsymbol{x}) \le \widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x})$ if $|\alpha| \le 1$.

(v) If $\boldsymbol{x} \in M_{\text{po-b}}$, take a non-negative $\boldsymbol{u} \in M_{\text{mo}}$ such that $|\boldsymbol{x}| \le \boldsymbol{u}_<$. For any $\alpha \in \mathbb{R}$, $|\alpha\boldsymbol{x}| \le |\alpha|\boldsymbol{u}_<$ so

$$\limsup_{\alpha\to 0} \widehat{\theta}^{\#}_{\boldsymbol{v}}(\alpha\boldsymbol{x}) \le \limsup_{\alpha\to 0} \theta(\textstyle\int_{\mathcal{S}} |\alpha|\boldsymbol{u}\,d\boldsymbol{v}) = \limsup_{\alpha\to 0} \theta(|\alpha|\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}) = 0.$$

So all the conditions of 2A5B are satisfied, and $\widehat{\theta}^{\#}_{\boldsymbol{v}}$ is an F-seminorm.

(c) If $\langle\boldsymbol{u}_n\rangle_{n\in\mathbb{N}}$ is a uniformly order-bounded non-decreasing sequence of non-negative moderately oscillatory processes with domain $\mathcal{S}$ and $|\boldsymbol{x}| \le \sup_{n\in\mathbb{N}} \boldsymbol{u}_{n<}$, then $0 \le \int_{\mathcal{S}} \boldsymbol{u}_n\,d\boldsymbol{v} \le \int_{\mathcal{S}} \boldsymbol{u}_n\,d\boldsymbol{v}'$, so $\theta(\int_{\mathcal{S}} \boldsymbol{u}_n\,d\boldsymbol{v}) \le \theta(\int_{\mathcal{S}} \boldsymbol{u}_n\,d\boldsymbol{v}')$ for every n; consequently

$$\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x}) \le \sup_{n\in\mathbb{N}} \theta(\textstyle\int_{\mathcal{S}} \boldsymbol{u}_n\,d\boldsymbol{v}) \le \sup_{n\in\mathbb{N}} \theta(\int_{\mathcal{S}} \boldsymbol{u}_n\,d\boldsymbol{v}').$$

As $\langle\boldsymbol{u}_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x}) \le \widehat{\theta}^{\#}_{\boldsymbol{v}'}(\boldsymbol{x})$.

645E The topology $\mathfrak{T}_{\text{S-i}}$: Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.

(a)(i) We have a linear space topology $\mathfrak{T}_{\text{S-i}}$ on $M_{\text{po-b}} = M_{\text{po-b}}(\mathcal{S})$ defined by the functionals $\widehat{\theta}^{\#}_{\boldsymbol{v}}$ as $\boldsymbol{v}$ runs over $M^{\uparrow}_{\text{n-s}}(\mathcal{S})$.

(ii) If $\boldsymbol{x} \in G \in \mathfrak{T}_{\text{S-i}}$, there are $\boldsymbol{v} \in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$ and a $\delta > 0$ such that $\{\boldsymbol{x}' : \boldsymbol{x}' \in M_{\text{po-b}}, \widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x}'-\boldsymbol{x}) \le \delta\} \subseteq G$.

(iii) For any $\tau \in \mathcal{S}$, the coordinate projection $\langle x_\sigma\rangle_{\sigma\in\mathcal{S}} \mapsto x_\tau : M_{\text{po-b}} \to L^0$ is continuous for $\mathfrak{T}_{\text{S-i}}$ and the topology of convergence in measure on L^0.

(iv) $\mathfrak{T}_{\text{S-i}}$ is Hausdorff.

(v)(α) For any $\boldsymbol{x} \in M_{\text{po-b}}$, the map $\boldsymbol{x}' \mapsto \boldsymbol{x}' \times \boldsymbol{x} : M_{\text{po-b}} \to M_{\text{po-b}}$ is continuous.

(β) If $A \subseteq M_{\text{po-b}}$ is uniformly previsibly order-bounded, then $(\boldsymbol{x}, \boldsymbol{x}') \mapsto \boldsymbol{x} \times \boldsymbol{x}' : A \times A \to M_{\text{po-b}}$ is uniformly continuous.

(vi) If $z \in L^0(\mathfrak{A} \cap \bigcap_{\sigma \in \mathcal{S}} \mathfrak{A}_\sigma)$ then $\boldsymbol{x}' \mapsto z\boldsymbol{x}' : M_{\text{po-b}} \to M_{\text{po-b}}$ is continuous.

(b) Let $\mathfrak{S}$ be the linear space topology on $M_{\text{mo}} = M_{\text{mo}}(\mathcal{S})$ defined by the F-seminorms $\boldsymbol{u} \mapsto \theta(\int_{\mathcal{S}} |\boldsymbol{u}|\, d\boldsymbol{v})$ as $\boldsymbol{v}$ runs over $M^{\uparrow}_{\text{n-s}}(\mathcal{S})$. Then $\boldsymbol{u} \mapsto \boldsymbol{u}_< : M_{\text{mo}} \to M_{\text{po-b}}$ is continuous for $\mathfrak{S}$ and $\mathfrak{T}_{\text{S-i}}$. Consequently $\boldsymbol{u} \mapsto \boldsymbol{u}_< : M_{\text{mo}} \to M_{\text{po-b}}$ is continuous for the ucp topology on M_{mo} and $\mathfrak{T}_{\text{S-i}}$.

proof (a)(i) By 645Db and 2A5B, the functionals $\widehat{\theta}^{\#}_{\boldsymbol{v}}$, for $\boldsymbol{v} \in M^{\uparrow}_{\text{n-s}} = M^{\uparrow}_{\text{n-s}}(\mathcal{S})$, define a linear space topology on $M_{\text{po-b}}$.

(ii) By the definition of $\mathfrak{T}_{\text{S-i}}$ (2A5B, 2A3Fc) there are $\boldsymbol{v}_0, \dots, \boldsymbol{v}_n \in M^{\uparrow}_{\text{n-s}}$ and a $\delta > 0$ such that

$$\{\boldsymbol{x}' : \boldsymbol{x}' \in M_{\text{po-b}},\ \max_{i \le n} \widehat{\theta}^{\#}_{\boldsymbol{v}_i}(\boldsymbol{x}' - \boldsymbol{x}) \le \delta\} \subseteq G.$$

Set $\boldsymbol{v} = \sum_{i=0}^{n} \boldsymbol{v}_i$; then $\boldsymbol{v} \in M^{\uparrow}_{\text{n-s}}$ and $\boldsymbol{v}_j \preccurlyeq \boldsymbol{v}$ so $\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x}') \ge \widehat{\theta}^{\#}_{\boldsymbol{v}_j}(\boldsymbol{x}')$ for every $\boldsymbol{x}' \in M_{\text{po-b}}$ and $j \le n$ (645Dc). Accordingly $\{\boldsymbol{x}' : \boldsymbol{x}' \in M_{\text{po-b}}, \widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x}' - \boldsymbol{x}) \le \delta\} \subseteq G$.

(iii) Let $\boldsymbol{v}$ be the simple process defined on $\mathcal{S}$ by the formula in 612Ka, taking the breakpoint string to be (τ), the base value v_* to be 0, and $v_0 = \chi 1$. Observe that the starting value $v_\downarrow = \lim_{\sigma \downarrow \mathcal{S}} v_\sigma$ is $\chi(1 \setminus e_\tau)$ where $e_\tau = \sup_{\sigma \in \mathcal{S}} [\![\sigma < \tau]\!]$ (614Ba). Of course $\boldsymbol{v}$ is non-negative and non-decreasing.

If $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ is a moderately oscillatory process, then

$$\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v} = u_{<\tau} \times (v_\tau - v_\downarrow)$$

(641J)

$$= u_{<\tau} \times \chi e_\tau = u_{<\tau}.$$

Let $\langle \boldsymbol{u}_n \rangle_{n \in \mathbb{N}} = \langle \langle u_{n\sigma} \rangle_{\sigma \in \mathcal{S}} \rangle_{n \in \mathbb{N}}$ be any non-decreasing sequence in M^{+}_{mo} such that $|\boldsymbol{x}| \le \sup_{n \in \mathbb{N}} \boldsymbol{u}_{n<}$. Then $|x_\tau| \le \sup_{n \in \mathbb{N}} u_{n<\tau}$ so

$$\theta(x_\tau) \le \sup_{n \in \mathbb{N}} \theta(u_{n<\tau}) = \sup_{n \in \mathbb{N}} \theta(\textstyle\int_{\mathcal{S}} \boldsymbol{u}_n\, d\boldsymbol{v}).$$

As $\langle \boldsymbol{u}_n \rangle_{n \in \mathbb{N}}$ is arbitrary, $\theta(x_\tau) \le \widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x})$. This shows that $\langle x_\sigma \rangle_{\sigma \in \mathcal{S}} \mapsto x_\tau$ is continuous at $\mathbf{0}$. But as it is a linear operator, it is continuous everywhere in $M_{\text{po-b}}$.

(iv) It follows at once that $\mathfrak{T}_{\text{S-i}}$ is Hausdorff, being finer than the topology induced on $M_{\text{po-b}}$ by the product topology on $(L^0)^{\mathcal{S}}$.

(v)($\boldsymbol{\alpha}$) I noted in 645D(a-ii) that $M_{\text{po-b}}$ is closed under multiplication. Take $\boldsymbol{v} \in M^{\uparrow}_{\text{n-s}}$ and $\epsilon > 0$. There is a $\boldsymbol{u} \in M^{+}_{\text{mo}}$ such that $|\boldsymbol{x}| \le \boldsymbol{u}_<$. Let $M \ge 1$ be such that $\bar{\mu}[\![\sup |\boldsymbol{u}| \ge M]\!] \le \epsilon$; then

$$[\![\textstyle\int_{\mathcal{S}} \boldsymbol{u}' \times \boldsymbol{u}\, d\boldsymbol{v} > M \int_{\mathcal{S}} \boldsymbol{u}'\, d\boldsymbol{v}]\!] = [\![\int_{\mathcal{S}} (\boldsymbol{u}' \times \boldsymbol{u} - M\boldsymbol{u}') d\boldsymbol{v} > 0]\!] \subseteq [\![\int_{\mathcal{S}} (\boldsymbol{u}' \times \boldsymbol{u} - M\boldsymbol{u}')^+ d\boldsymbol{v} > 0]\!]$$
$$\subseteq [\![(\boldsymbol{u}' \times \boldsymbol{u} - M\boldsymbol{u}')^+ \ne \mathbf{0}]\!]$$

(613Jd)

$$\subseteq [\![\sup |\boldsymbol{u}| \ge M]\!]$$

for every $\boldsymbol{u}' \in M^{+}_{\text{mo}}$, so

$$\theta(\textstyle\int_{\mathcal{S}} \boldsymbol{u}' \times \boldsymbol{u}\, d\boldsymbol{v}) \le \epsilon + M\theta(\int_{\mathcal{S}} \boldsymbol{u}'\, d\boldsymbol{v})$$

for every $\boldsymbol{u}' \in M^{+}_{\text{mo}}$. Now suppose that $\boldsymbol{x}' \in M^{+}_{\text{mo}}$ and $\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x}') < \frac{\epsilon}{M}$. Then there is a non-decreasing uniformly order-bounded sequence $\langle \boldsymbol{u}_n \rangle_{n \in \mathbb{N}}$ in M^{+}_{mo} such that $|\boldsymbol{x}'| \le \sup_{n \in \mathbb{N}} \boldsymbol{u}_{n<}$ and $\theta(\int_{\mathcal{S}} \boldsymbol{u}_n\, d\boldsymbol{v}) \le \frac{\epsilon}{M}$ for every n. In this case, $\langle \boldsymbol{u}_n \times \boldsymbol{u} \rangle_{n \in \mathbb{N}}$ is a non-decreasing uniformly order-bounded sequence in $M^{+}_{\text{n-s}}$ such that

$$|\boldsymbol{x}' \times \boldsymbol{x}| \le |\boldsymbol{x}'| \times \boldsymbol{u}_< \le \sup_{n\in\mathbb{N}} \boldsymbol{u}_{n<} \times \boldsymbol{u}_< = \sup_{n\in\mathbb{N}}(\boldsymbol{u}_n \times \boldsymbol{u})_<,$$

so

$$\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x}' \times \boldsymbol{x}) \le \sup_{n\in\mathbb{N}} \theta(\int_{\mathcal{S}} \boldsymbol{u}_n \times \boldsymbol{u}\, d\boldsymbol{v}) \le 2\epsilon.$$

As $\boldsymbol{v}$ and ϵ are arbitrary, the linear operator $\boldsymbol{x}' \mapsto \boldsymbol{x}' \times \boldsymbol{x}$ is continuous at $\mathbf{0}$, therefore continuous.

(β) Let $\boldsymbol{u} \in M^+_{\text{mo}}$ be such that $|\boldsymbol{x}| \le \boldsymbol{u}_<$ for every $\boldsymbol{x} \in A$. Let G be a $\mathfrak{T}_{\text{S-i}}$-neighbourhood of $\mathbf{0}$. By (ii) above, there are a $\boldsymbol{v} \in M^{\uparrow}_{\text{n-s}}$ and a $\delta > 0$ such that $\boldsymbol{x} \in G$ whenever $\boldsymbol{x} \in M_{\text{po-b}}$ and $\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x}) \le \delta$. By ($\alpha$), $\boldsymbol{x} \mapsto \boldsymbol{x} \times \boldsymbol{u}_<$ is continuous, so there is a $\mathfrak{T}_{\text{S-i}}$-neighbourhood G' of 0 such that $\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x} \times \boldsymbol{u}_<) \le \frac{1}{2}\delta$ for every $\boldsymbol{x} \in G'$. Now suppose that $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}'_1$, $\boldsymbol{x}'_2 \in A$ and both $\boldsymbol{x}_1 - \boldsymbol{x}_2$ and $\boldsymbol{x}'_1 - \boldsymbol{x}'_2$ belong to G'. Then

$$\begin{aligned}|(\boldsymbol{x}_1 \times \boldsymbol{x}'_1) - (\boldsymbol{x}_2 - \boldsymbol{x}'_2)| &\le |\boldsymbol{x}_1| \times |\boldsymbol{x}'_1 - \boldsymbol{x}'_2| + |\boldsymbol{x}_1 - \boldsymbol{x}_2| \times |\boldsymbol{x}'_2| \\ &\le |\boldsymbol{x}'_1 - \boldsymbol{x}'_2| \times \boldsymbol{u}_< + |\boldsymbol{x}_1 - \boldsymbol{x}_2| \times \boldsymbol{u}_<\end{aligned}$$

and

$$\widehat{\theta}^{\#}_{\boldsymbol{v}}((\boldsymbol{x}_1 \times \boldsymbol{x}'_1) - (\boldsymbol{x}_2 \times \boldsymbol{x}'_2)) \le \widehat{\theta}^{\#}_{\boldsymbol{v}}((\boldsymbol{x}'_1 - \boldsymbol{x}'_2) \times \boldsymbol{u}_<) + \widehat{\theta}^{\#}_{\boldsymbol{v}}((\boldsymbol{x}_1 - \boldsymbol{x}_2) \times \boldsymbol{u}_<) \le \delta,$$

so $(\boldsymbol{x}_1 \times \boldsymbol{x}'_1) - (\boldsymbol{x}_2 \times \boldsymbol{x}'_2) \in G$. As G is arbitrary, $(\boldsymbol{x}, \boldsymbol{x}') \mapsto \boldsymbol{x} \times \boldsymbol{x}'$ is uniformly continuous on $A \times A$.

(vi) This is the special case of (v-α) in which $\boldsymbol{x} = z\mathbf{1}^{(\mathcal{S})}_{<}$.

(b) We have pnly to look at the definition in 645Bb; $\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{u}_<) \le \theta(\int_{\mathcal{S}} |\boldsymbol{u}| d\boldsymbol{v})$ for all relevant $\boldsymbol{u}$ and $\boldsymbol{v}$, so we shall be able to apply 2A3H to see that $\boldsymbol{u} \mapsto \boldsymbol{u}_<$ is continuous for $\mathfrak{S}$ and $\mathfrak{T}_{\text{S-i}}$. As for the ucp topology, this is finer than $\mathfrak{S}$, because if $\boldsymbol{v} \in M^{\uparrow}_{\text{n-s}}$ then $\theta(\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}) \le \theta(\sup|\boldsymbol{u}| \times \sup|\boldsymbol{v}|)$. So $\boldsymbol{u} \mapsto \boldsymbol{u}_<$ is continuous for the ucp topology and $\mathfrak{T}_{\text{S-i}}$.

645F Definitions Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.

(a) I will call the topology $\mathfrak{T}_{\text{S-i}}$ defined in 645E the **S-integration topology** on $M_{\text{po-b}}(\mathcal{S})$. As it is a linear space topology (645E(a-i)), there is an associated uniformity (3A4Ad) which I will call the **S-integration uniformity**.

(b) $M^0_{\text{S-i}}(\mathcal{S})$ will be the $\mathfrak{T}_{\text{S-i}}$-closure of $\{\boldsymbol{u}_< : \boldsymbol{u} \in M_{\text{mo}}(\mathcal{S})\}$ in $M_{\text{po-b}}(\mathcal{S})$.

(c) $M_{\text{S-i}}(\mathcal{S})$ will be the set of fully adapted processes $\boldsymbol{x}$ with domain $\mathcal{S}$ such that $\boldsymbol{x} \times \mathbf{1}^{(\mathcal{S})}_{<} \in M^0_{\text{S-i}}(\mathcal{S})$.

645G Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\mathfrak{T}_{\text{S-i}}$ the S-integration topology on $M_{\text{po-b}} = M_{\text{po-b}}(\mathcal{S})$. If $\langle \boldsymbol{x}_n \rangle_{n\in\mathbb{N}}$ is a uniformly previsibly order-bounded $\mathfrak{T}_{\text{S-i}}$-Cauchy sequence in $M_{\text{po-b}}$, then it is $\mathfrak{T}_{\text{S-i}}$-convergent.

proof (a) By the definition of 'uniformly previsibly order-bounded', there must be a non-negative $\boldsymbol{u} \in M_{\text{mo}} = M_{\text{mo}}(\mathcal{S})$ such that $|\boldsymbol{x}_n| \le \boldsymbol{u}_<$ for every $n \in \mathbb{N}$. Express each $\boldsymbol{x}_n$ as $\langle x_{n\sigma} \rangle_{\sigma\in\mathcal{S}}$. Because the coordinate projections from $M_{\text{po-b}}$ to L^0 are continuous linear operators, therefore uniformly continuous (4A5Hd), $\langle x_{n\sigma} \rangle_{n\in\mathbb{N}}$ is a Cauchy sequence in L^0 for each $\sigma \in \mathcal{S}$ (4A2Ji), with a limit x_σ because L^0 is a complete linear topological space (613Bh). Now $\boldsymbol{x} = \langle x_\sigma \rangle_{\sigma\in\mathcal{S}}$ is fully adapted (613Bl) and $|\boldsymbol{x}| \le \boldsymbol{u}_<$ (613Bm), so $\boldsymbol{x} \in M_{\text{po-b}}$.

(b) Take $\boldsymbol{v} \in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$ and $\epsilon > 0$. Then we have a non-decreasing sequence $\langle n_k \rangle_{k\in\mathbb{N}}$ such that $\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x}_m - \boldsymbol{x}_{n_k}) \le 2^{-k}\epsilon$ whenever k, $m \in \mathbb{N}$ and $n_k \le m$. For each $k \in \mathbb{N}$ choose a non-decreasing uniformly order-bounded sequence $\langle \boldsymbol{u}_{ki} \rangle_{i\in\mathbb{N}}$ in M^+_{mo} such that $|\boldsymbol{x}_{n_{k+1}} - \boldsymbol{x}_{n_k}| \le \sup_{i\in\mathbb{N}} \boldsymbol{u}_{ki<}$ and $\sup_{i\in\mathbb{N}} \theta(\int_{\mathcal{S}} \boldsymbol{u}_{ki}\, d\boldsymbol{v}) \le 2^{-k+1}\epsilon$. Set $\boldsymbol{u}'_m = 2\boldsymbol{u} \wedge \sum_{k=0}^{m} \boldsymbol{u}_{km}$ for $m \in \mathbb{N}$. Then $\langle \boldsymbol{u}'_m \rangle_{m\in\mathbb{N}}$ is a non-decreasing uniformly order-bounded sequence in M^+_{mo}. The point is that $|\boldsymbol{x} - \boldsymbol{x}_{n_0}| \le \sup_{m\in\mathbb{N}} \boldsymbol{u}'_{m<}$. **P** Note first that for each $\sigma \in \mathcal{S}$, $x_\sigma - x_{n_0\sigma} = \lim_{k\to\infty} x_{n_k\sigma} - x_{n_0\sigma}$. This is a limit for the topology of convergence in measure, but as we know that $|\boldsymbol{x}_{n_k}| \le \boldsymbol{u}_<$ for every k, we must certainly have

$$|x_\sigma - x_{n_0\sigma}| \le \sup_{k\in\mathbb{N}} 2u_{<\sigma} \wedge |x_{n_k\sigma} - x_{n_0\sigma}|$$

for every σ. (Here I am thinking of $\boldsymbol{u}$ as $\langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$ and its previsible version $\boldsymbol{u}_<$ as $\langle u_{<\sigma} \rangle_{\sigma\in\mathcal{S}}$, in the manner of 641E-641F.) Thus we have

$$|\boldsymbol{x}-\boldsymbol{x}_{n_0}| \le \sup_{k\ge 1} 2\boldsymbol{u}_< \wedge |\boldsymbol{x}_{n_k}-\boldsymbol{x}_{n_0}| \le \sup_{k\ge 1} 2\boldsymbol{u}_< \wedge \sum_{j=0}^{k-1} |\boldsymbol{x}_{n_{j+1}}-\boldsymbol{x}_{n_j}|$$

$$\le \sup_{k\ge 1} 2\boldsymbol{u}_< \wedge \sum_{j=0}^{k-1} \sup_{i\ge k} \boldsymbol{u}_{ji<} = \sup_{k\ge 1}\sup_{i\ge k} 2\boldsymbol{u}_< \wedge \sum_{j=0}^{k-1} \boldsymbol{u}_{ji<}$$

(because $\langle \boldsymbol{u}_{ji<}\rangle_{i\in\mathbb{N}}$ is non-decreasing for each j)

$$\le \sup_{k\ge 1}\sup_{i\ge k} \boldsymbol{u}'_{i<} = \sup_{m\in\mathbb{N}} \boldsymbol{u}'_{m<}. \textbf{ Q}$$

(c) Accordingly, if $n \ge n_0$,

$$\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x}-\boldsymbol{x}_n) \le \widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x}_n-\boldsymbol{x}_{n_0}) + \widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x}-\boldsymbol{x}_{n_0})$$

$$\le \epsilon + \sup_{m\in\mathbb{N}} \theta(\int_{\mathcal{S}} \boldsymbol{u}'_m\, d\boldsymbol{v}) \le \epsilon + \sup_{m\in\mathbb{N}} \sum_{k=0}^{m} \theta(\int_{\mathcal{S}} \boldsymbol{u}_{km}\, d\boldsymbol{v}) \le 5\epsilon.$$

As ϵ and $\boldsymbol{v}$ are arbitrary, $\boldsymbol{x}$ is the $\mathfrak{T}_{\text{S-i}}$-limit of $\langle \boldsymbol{x}_n\rangle_{n\in\mathbb{N}}$.

645H Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. Suppose that $\langle \boldsymbol{x}_n\rangle_{n\in\mathbb{N}}$ is a uniformly previsibly order-bounded sequence in $M^0_{\text{S-i}} = M^0_{\text{S-i}}(\mathcal{S})$ which is order*-convergent to $\boldsymbol{x} \in (L^0)^{\mathcal{S}}$. Then $\boldsymbol{x} \in M^0_{\text{S-i}}$ and $\langle \boldsymbol{x}_n\rangle_{n\in\mathbb{N}}$ converges to $\boldsymbol{x}$ for the S-integration topology $\mathfrak{T}_{\text{S-i}}$.

proof (a) The first thing to note is that if a sequence $\langle \boldsymbol{x}_n\rangle_{n\in\mathbb{N}} = \langle\langle x_{n\sigma}\rangle_{\sigma\in\mathcal{S}}\rangle_{n\in\mathbb{N}}$ in $M_{\text{fa}} = M_{\text{fa}}(\mathcal{S})$ is order*-convergent in $(L^0)^{\mathcal{S}}$ to $\boldsymbol{x} = \langle x_\sigma\rangle_{\sigma\in\mathcal{S}}$, and $\tau \in \mathcal{S}$, then $\langle x_{n\tau}\rangle_{n\in\mathbb{N}}$ is order*-convergent to x_τ in L^0, therefore convergent in the topology of convergence in measure (367Ma). So $\langle \boldsymbol{x}_n\rangle_{n\in\mathbb{N}} \to \boldsymbol{x}$ for the product topology on $(L^0)^{\mathcal{S}}$, and $\boldsymbol{x} \in M_{\text{fa}}$ (613Bl). If moreover $\langle \boldsymbol{x}_n\rangle_{n\in\mathbb{N}}$ is uniformly previsibly order-bounded, that is, there is a $\boldsymbol{u} \in M^+_{\text{mo}} = M_{\text{mo}}(\mathcal{S})^+$ such that $|\boldsymbol{x}_n| \le \boldsymbol{u}_<$ for every n, then $|\boldsymbol{x}| \le \boldsymbol{u}_<$ so $\boldsymbol{x} \in M_{\text{po-b}} = M_{\text{po-b}}(\mathcal{S})$.

Secondly, if a sequence in $M_{\text{po-b}}$ is simultaneously order*-convergent and $\mathfrak{T}_{\text{S-i}}$-convergent, the limits must be the same, because the coordinate projections are $\mathfrak{T}_{\text{S-i}}$-continuous (645E(a-iii)), that is, $\mathfrak{T}_{\text{S-i}}$ is finer than the topology induced by the product topology on $(L^0)^{\mathcal{S}}$. And if a sequence in $M^0_{\text{S-i}}$ is $\mathfrak{T}_{\text{S-i}}$-convergent in $M_{\text{po-b}}$, its limit belongs to $M^0_{\text{S-i}}$, because $M^0_{\text{S-i}}$ is defined to be a $\mathfrak{T}_{\text{S-i}}$-closure.

(b) So all we need to prove is that, under the conditions of the theorem, $\langle \boldsymbol{x}_n\rangle_{n\in\mathbb{N}}$ is $\mathfrak{T}_{\text{S-i}}$-convergent in $M_{\text{po-b}}$; and by 645G it will in fact be enough to show that it is $\mathfrak{T}_{\text{S-i}}$-Cauchy. I seek to do this successively more complex sequences $\langle \boldsymbol{x}_n\rangle_{n\in\mathbb{N}}$. First, suppose it is of the form $\langle \boldsymbol{u}_{n<}\rangle_{n\in\mathbb{N}}$ where $\langle \boldsymbol{u}_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in M^+_{mo} with an upper bound $\boldsymbol{u} \in M_{\text{mo}}$. Then $\langle \boldsymbol{x}_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with an upper bound $\boldsymbol{u}_<$, so is order*-convergent to its supremum $\boldsymbol{x}$ in $(L^0)^{\mathcal{S}}$, which is a fully adapted process (612Ia).

As $0 \le \boldsymbol{x} \le \boldsymbol{u}_<$, $\boldsymbol{x} \in M_{\text{po-b}}$. If $\boldsymbol{v} \in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$, then $\langle \int_{\mathcal{S}} \boldsymbol{u}_n\, d\boldsymbol{v}\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in L^0 with an upper bound $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$, so has a limit in L^0 (613Ba) and is Cauchy. For any $m \in \mathbb{N}$,

$$0 \le \boldsymbol{x}-\boldsymbol{x}_m \le \sup_{n\ge m} \boldsymbol{u}_{n<} - \boldsymbol{u}_{m<},$$

so

$$\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x}-\boldsymbol{x}_m) \le \sup_{n\ge m} \theta(\int_{\mathcal{S}} \boldsymbol{u}_n - \boldsymbol{u}_m\, d\boldsymbol{v}).$$

Thus

$$\lim_{m\to\infty} \widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x}-\boldsymbol{x}_m) \le \lim_{m\to\infty} \sup_{n\ge m} \theta(\int_{\mathcal{S}} \boldsymbol{u}_n - \boldsymbol{u}_m\, d\boldsymbol{v}) = 0;$$

as $\boldsymbol{v}$ is arbitary, $\langle \boldsymbol{x}_n\rangle_{n\in\mathbb{N}}$ is $\mathfrak{T}_{\text{S-i}}$-convergent to $\boldsymbol{x}$.

(c) Suppose next that $\langle \boldsymbol{x}_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $(M^0_{\text{S-i}})^+$, and that it is bounded above by $\boldsymbol{u}_<$ where $\boldsymbol{u} \in M^+_{\text{mo}}$. Then again it is order*-convergent to its supremum $\boldsymbol{x}$, and $\boldsymbol{x} \in M_{\text{po-b}}$. Take $\boldsymbol{v} \in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$. and $\epsilon > 0$. For each $n \in \mathbb{N}$, let $\boldsymbol{u}_n \in M_{\text{mo}}$ be such that $\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x}_n - \boldsymbol{u}_{n<}) \le 2^{-n}\epsilon$. For each n, set $\boldsymbol{u}'_n = \text{med}(\boldsymbol{0}, \sup_{i\le n} \boldsymbol{u}_i, \boldsymbol{u})$; then

$$|\boldsymbol{x}_n - \boldsymbol{u}'_{n<}| = |\operatorname{med}(\mathbf{0}, \sup_{i\le n} \boldsymbol{x}_i, \boldsymbol{u}_<) - \operatorname{med}(\mathbf{0}, \sup_{i\le n} \boldsymbol{u}_{i<}, \boldsymbol{u}_<)|$$

(641Ge)

$$\le |\sup_{i\le n} \boldsymbol{x}_i - \sup_{i\le n} \boldsymbol{u}_{i<}| \le \sum_{i=0}^{n} |\boldsymbol{x}_i - \boldsymbol{u}_{i<}|,$$

so

$$\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x}_n - \boldsymbol{u}'_{n<}) \le \sum_{i=0}^{n} \widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x}_i - \boldsymbol{u}_{i<}) \le 2\epsilon.$$

Now we know from (b) that $\langle \boldsymbol{u}'_{n<}\rangle_{n\in\mathbb{N}}$ is $\mathfrak{T}_{\text{S-i}}$-convergent, so there is an $m \in \mathbb{N}$ such that $\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{u}'_{n<} - \boldsymbol{u}'_{m<}) \le \epsilon$ for every $n \ge m$; in which case $\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x}_{n<} - \boldsymbol{x}_{m<}) \le 5\epsilon$ for every $n \ge m$. As $\boldsymbol{v}$ and ϵ are arbitrary, $\langle \boldsymbol{x}_n\rangle_{n\in\mathbb{N}}$ is Cauchy, therefore convergent, and the limit must be $\boldsymbol{x}$.

(d) Applying (c) to $\langle \boldsymbol{x}_n - \boldsymbol{x}_0\rangle_{n\in\mathbb{N}}$, we see that if $\langle \boldsymbol{x}_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing previsibly order-bounded sequence in $M^0_{\text{S-i}}$, it will be $\mathfrak{T}_{\text{S-i}}$-convergent; it follows at once that if $\langle \boldsymbol{x}_n\rangle_{n\in\mathbb{N}}$ is a non-increasing previsibly order-bounded sequence in $M^0_{\text{S-i}}$, it is $\mathfrak{T}_{\text{S-i}}$-convergent.

(e) Now let $\langle \boldsymbol{x}_n\rangle_{n\in\mathbb{N}}$ be any previsibly order-bounded sequence in $M^0_{\text{S-i}}$ which is order*-convergent to $\boldsymbol{x} \in (L^0)^{\mathcal{S}}$. For each $n \in \mathbb{N}$, set $\overline{\boldsymbol{x}}_n = \sup_{i\ge n} \boldsymbol{x}_i$, the supremum being taken in $(L^0)^{\mathcal{S}}$. Then $\overline{\boldsymbol{x}}_n$ is the order*-convergent limit of the non-decreasing sequence $\langle \sup_{i\le m} \boldsymbol{x}_{n+i}\rangle_{m\in\mathbb{N}}$ in $M^0_{\text{S-i}}$, which of course is bounded above in $M_{\text{po-b}}$ because $\langle \boldsymbol{x}_i\rangle_{i\in\mathbb{N}}$ is, so $\overline{\boldsymbol{x}}_n \in M^0_{\text{S-i}}$, by (d). We now see that $\langle \overline{\boldsymbol{x}}_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $M^0_{\text{S-i}}$, bounded below in $M_{\text{po-b}}$, which is order*-convergent to $\boldsymbol{x}$. So $\boldsymbol{x} \in M^0_{\text{S-i}}$ and is the $\mathfrak{T}_{\text{S-i}}$-limit of $\langle \overline{\boldsymbol{x}}_n\rangle_{n\in\mathbb{N}}$.

Similarly, $\underline{\boldsymbol{x}}_n = \inf_{i\ge n} \boldsymbol{x}_i$ belongs to $M^0_{\text{S-i}}$ for every n, and $\boldsymbol{x}$ is the $\mathfrak{T}_{\text{S-i}}$-limit of $\langle \underline{\boldsymbol{x}}_n\rangle_{n\in\mathbb{N}}$.

To see that $\boldsymbol{x}$ is in fact the $\mathfrak{T}_{\text{S-i}}$-limit of the original sequence $\langle \boldsymbol{x}_n\rangle_{n\in\mathbb{N}}$, take any $\boldsymbol{v} \in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$. If $n \in \mathbb{N}$, then $|\boldsymbol{x}_n - \boldsymbol{x}| \le \overline{\boldsymbol{x}}_n - \underline{\boldsymbol{x}}_n$, so

$$\begin{aligned}\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x}_n - \boldsymbol{x}) &\le \widehat{\theta}^{\#}_{\boldsymbol{v}}(\overline{\boldsymbol{x}}_n - \underline{\boldsymbol{x}}_n)\\ &\le \widehat{\theta}^{\#}_{\boldsymbol{v}}(\overline{\boldsymbol{x}}_n - \boldsymbol{x}) + \widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x} - \underline{\boldsymbol{x}}_n) \to 0\end{aligned}$$

as $n \to \infty$. As $\boldsymbol{v}$ is arbitrary, $\langle \boldsymbol{x}_n\rangle_{n\in\mathbb{N}}$ is topologically convergent to $\boldsymbol{x}$. This completes the proof.

645I Corollary Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. If $\boldsymbol{x} \in M_{\text{po-b}}(\mathcal{S})$ is a previsible process (642C), then $\boldsymbol{x} \in M^0_{\text{S-i}} = M^0_{\text{S-i}}(\mathcal{S})$.

proof Let $\boldsymbol{u} \in M_{\text{mo}}(\mathcal{S})^+$ be such that $|\boldsymbol{x}| \le \boldsymbol{u}_<$. The set

$$\{\boldsymbol{x}' : \boldsymbol{x}' \in (L^0)^{\mathcal{S}},\ \operatorname{med}(-\boldsymbol{u}_<, \boldsymbol{x}', \boldsymbol{u}_<) \in M^0_{\text{S-i}}\}$$

contains $\boldsymbol{u}'_<$ for every $\boldsymbol{u}' \in M_{\text{n-s}}(\mathcal{S})$ and is closed under order*-convergence, by 645H, so contains $\boldsymbol{x} = \operatorname{med}(-\boldsymbol{u}_<, \boldsymbol{x}, \boldsymbol{u}_<)$.

645J Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $k \ge 1$ an integer and $h : \mathbb{R}^k \to \mathbb{R}$ a locally bounded Borel measurable function. Write $M^0_{\text{S-i}}$, $M_{\text{S-i}}$ for $M^0_{\text{S-i}}(\mathcal{S})$, $M_{\text{S-i}}(\mathcal{S})$.

(a) If $\boldsymbol{X} \in (M^0_{\text{S-i}})^k$, then $\bar{h}\boldsymbol{X} \times \mathbf{1}^{(\mathcal{S})}_< \in M^0_{\text{S-i}}$; if $h(0, \dots, 0) = 0$, then $\bar{h}\boldsymbol{X} \in M^0_{\text{S-i}}$.

(b) $\bar{h}\boldsymbol{X} \in M_{\text{S-i}}$ for every $\boldsymbol{X} \in M^k_{\text{S-i}}$.

proof Express $\boldsymbol{X}$ as $\langle \boldsymbol{x}_i\rangle_{i<k}$. For any $\boldsymbol{x} \in M_{\text{fa}}(\mathcal{S})$, write $R(\boldsymbol{x})$ for $\boldsymbol{x} \times \mathbf{1}^{(\mathcal{S})}_<$. By 645D(a-i), $R(\bar{h}\boldsymbol{X}) \in M_{\text{po-b}} = M_{\text{po-b}}(\mathcal{S})$.

(a)(i) Let $g : \mathbb{R}^k \to \mathbb{R}$ be a continuous function.

(α) If $\boldsymbol{v} \in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$ and $\epsilon > 0$, there is an $M \ge 0$ such that $\widehat{\theta}^{\#}_{\boldsymbol{v}}(R(\bar{g}\boldsymbol{X}) - R(\bar{g}\boldsymbol{X}')) \le \epsilon$, where $\boldsymbol{X}' = \langle \operatorname{med}(-M\mathbf{1}^{(\mathcal{S})}_<, \boldsymbol{x}_i, M\mathbf{1}^{(\mathcal{S})}_<)\rangle_{i<k}$. **P** There is a $\boldsymbol{u} \in M^+_{\text{mo}} = M_{\text{mo}}(\mathcal{S})^+$ such that $|\boldsymbol{x}_i| \le \boldsymbol{u}_<$ for every $i < k$. Set $\bar{u} = \sup|\boldsymbol{u}|$, and let $M \ge 0$ be such that $\bar{\mu}[\![\bar{u} \ge M]\!] \le \epsilon$. In this case, setting $\boldsymbol{x}'_i = \operatorname{med}(-M\mathbf{1}^{(\mathcal{S})}_<, \boldsymbol{x}_i, M\mathbf{1}^{(\mathcal{S})}_<)$, we have

$$|\boldsymbol{x}_i - \boldsymbol{x}'_i| = (|\boldsymbol{x}_i| - M\mathbf{1}^{(\mathcal{S})}_<)^+ \le (\boldsymbol{u}_< - M\mathbf{1}^{(\mathcal{S})}_<)^+ = ((\boldsymbol{u} - M\mathbf{1}^{(\mathcal{S})})^+)_<.$$

So, writing $\boldsymbol{X}'$ for $\langle \boldsymbol{x}'_i\rangle_{i<k}$,

$$[\![R(\bar{g}\boldsymbol{X}) \ne R(\bar{g}\boldsymbol{X}')]\!] \subseteq [\![\bar{g}\boldsymbol{X} \ne \bar{g}\boldsymbol{X}']\!] \subseteq \sup_{i<k} [\![\boldsymbol{x}_i \ne \boldsymbol{x}'_i]\!]$$

(612Sc, 619Ec)

$$\subseteq [\![((\boldsymbol{u} - M\mathbf{1}^{(\mathcal{S})})^+)_< \ne 0]\!].$$

Next, $|\boldsymbol{x}'_i| \le |\boldsymbol{x}_i|$ so $\boldsymbol{x}'_i \in M_{\text{po-b}}$ for every i, $R(\bar{g}\boldsymbol{X}') \in M_{\text{po-b}}$, $R(\bar{g}\boldsymbol{X}) - R(\bar{g}\boldsymbol{X}') \in M_{\text{po-b}}$ and there is a $\tilde{\boldsymbol{u}}$ in M^+_{mo} such that $|R(\bar{g}\boldsymbol{X}) - R(\bar{g}\boldsymbol{X}')| \le \tilde{\boldsymbol{u}}_<$. Setting $\tilde{\boldsymbol{u}}_n = \tilde{\boldsymbol{u}} \wedge n(\boldsymbol{u} - M\mathbf{1}^{(\mathcal{S})})^+$ for $n \in \mathbb{N}$, $\langle \tilde{\boldsymbol{u}}_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing uniformly order-bounded sequence in M^+_{mo} and

$$|R(\bar{g}\boldsymbol{X}) - R(\bar{g}\boldsymbol{X}')| \le \sup\nolimits_{n\in\mathbb{N}} \tilde{\boldsymbol{u}}_< \wedge n(\boldsymbol{u}_< - M\mathbf{1}^{(\mathcal{S})}_<)^+ = \sup\nolimits_{n\in\mathbb{N}} \tilde{\boldsymbol{u}}_{n<}.$$

Accordingly

$$\begin{aligned}\widehat{\theta}^{\#}_{\boldsymbol{v}}(R(\bar{g}\boldsymbol{X}) - R(\bar{g}\boldsymbol{X}')) &\le \sup_{n\in\mathbb{N}} \theta(\int_{\mathcal{S}} \tilde{\boldsymbol{u}}_n\, d\boldsymbol{v}) \le \sup_{n\in\mathbb{N}} \bar{\mu}[\![\textstyle\int_{\mathcal{S}} \tilde{\boldsymbol{u}}_n\, d\boldsymbol{v} \ne 0]\!]\\ &\le \sup_{n\in\mathbb{N}} \bar{\mu}[\![\tilde{\boldsymbol{u}}_n \ne \mathbf{0}]\!] \le \bar{\mu}[\![(\boldsymbol{u} - M\mathbf{1}^{(\mathcal{S})})^+ \ne 0]\!] = \bar{\mu}[\![\bar{u} > M]\!] \le \epsilon,\end{aligned}$$

as required. **Q**

(β) $R(\bar{g}\boldsymbol{X}) \in M^0_{\text{S-i}}$. **P** Take $\boldsymbol{v} \in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$ and $\epsilon > 0$. By (α), there is an $M \ge 0$ such that, setting $\boldsymbol{X}' = \langle \boldsymbol{x}'_i\rangle_{i<k}$ where $\boldsymbol{x}'_i = \text{med}(-M\mathbf{1}^{(\mathcal{S})}_<, \boldsymbol{x}_i, M\mathbf{1}^{(\mathcal{S})}_<)$ for $i<k$, we have $\widehat{\theta}^{\#}_{\boldsymbol{v}}(R(\bar{g}\boldsymbol{X}) - R(\bar{g}\boldsymbol{X}')) \le \epsilon$. Next, let $\delta > 0$ be such that $\widehat{\theta}^{\#}_{\boldsymbol{v}}(\delta\mathbf{1}^{(\mathcal{S})}_<) \le \epsilon$ (645Db), and $\eta > 0$ such that $|g(\beta) - g(\beta')| \le \delta$ whenever $\beta, \beta' \in [-M, M]^k$ and $\|\beta - \beta'\|_\infty \le \eta$; setting $K = 1 + \frac{2}{\eta}\sup_{\|\beta\|_\infty \le M} |g(\beta)|$, we shall have $|g(\beta) - g(\beta')| \le \delta + K\|\beta - \beta'\|_\infty$ whenever $\beta, \beta' \in [-M, M]^k$, while $K \ge 1$. For each $i < k$, because $\boldsymbol{x}_i \in M^0_{\text{S-i}}$, there is a $\boldsymbol{u}_i \in M_{\text{mo}} = M_{\text{mo}}(\mathcal{S})$ such that $\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x}_i - \boldsymbol{u}_{i<}) \le \frac{\epsilon}{K}$; set $\boldsymbol{U}' = \langle \boldsymbol{u}'_i\rangle_{i<k}$ and $\boldsymbol{U}'_< = \langle \boldsymbol{u}'_{i<}\rangle_{i<k}$ where $\boldsymbol{u}'_i = \text{med}(-M\mathbf{1}^{(\mathcal{S})}, \boldsymbol{u}_i, M\mathbf{1}^{(\mathcal{S})})$ for $i < k$. Note that $\boldsymbol{u}'_{i<} = \text{med}(-M\mathbf{1}^{(\mathcal{S})}_<, \boldsymbol{u}_{i<}, M\mathbf{1}^{(\mathcal{S})}_<)$ for $i < k$.

Since $|\boldsymbol{x}'_i - \boldsymbol{u}'_{i<}| \le |\boldsymbol{x}_i - \boldsymbol{u}_{i<}|$, $\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x}'_i - \boldsymbol{u}'_{i<}) \le \frac{\epsilon}{K}$ for each i; since $\|\boldsymbol{x}'_i\|_\infty$ and $\|\boldsymbol{u}'_{i<}\|_\infty$ are both less than or equal to M for each i,

$$|R(\bar{g}\boldsymbol{X}') - \bar{g}\boldsymbol{U}'_<| = |R(\bar{g}\boldsymbol{X}') - R(\bar{g}\boldsymbol{U}'_<)|$$

(641Gd)

$$\begin{aligned}&= R(|\bar{g}\boldsymbol{X}' - \bar{g}\boldsymbol{U}'_<|) \le R(\delta\mathbf{1}^{(\mathcal{S})} + K\sum_{i=0}^{k-1} |\boldsymbol{x}'_i - \boldsymbol{u}_{i<}|)\\ &\le \delta\mathbf{1}^{(\mathcal{S})}_< + K\sum_{i=0}^{k-1} |\boldsymbol{x}'_i - \boldsymbol{u}_{i<}|\end{aligned}$$

and

$$\begin{aligned}\widehat{\theta}^{\#}_{\boldsymbol{v}}(R(\bar{g}\boldsymbol{X}) - \bar{g}\boldsymbol{U}')_<) &\le \widehat{\theta}^{\#}_{\boldsymbol{v}}(R(\bar{g}\boldsymbol{X}) - R(\bar{g}\boldsymbol{X}')) + \widehat{\theta}^{\#}_{\boldsymbol{v}}(R(\bar{g}\boldsymbol{X}) - \bar{g}\boldsymbol{U}'_<)\\ &\le \epsilon + \widehat{\theta}^{\#}_{\boldsymbol{v}}(\delta\mathbf{1}^{(\mathcal{S})}_<) + K\sum_{i=0}^{k-1} \widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x}'_i - \boldsymbol{u}_{i<})\\ &\le \epsilon + \epsilon + k\epsilon = (k+2)\epsilon.\end{aligned}$$

As $\boldsymbol{v}$ and ϵ are arbitrary, $R(\bar{g}\boldsymbol{X}) \in M^0_{\text{S-i}}$. **Q**

(ii) Now consider the space U of Borel measurable functions $f:\mathbb{R}^k\to\mathbb{R}$ such that $R(\bar f\boldsymbol{X})\in M^0_{\text{S-i}}$. Suppose that $\langle f_n\rangle_{n\in\mathbb{N}}$ is a pointwise convergent sequence in U with limit f, and that $g=\sup_{n\in\mathbb{N}}|f_n|$ is locally bounded. In this case, $|\bar f_n\boldsymbol{X}|\le\bar g\boldsymbol{X}$ for each n, while $R(\bar g\boldsymbol{X})\in M_{\text{po-b}}$ (645D(a-i)), so $\langle R(\bar f_n\boldsymbol{X})\rangle_{n\in\mathbb{N}}$ is a uniformly previsibly order-bounded sequence in $M^0_{\text{S-i}}$. Also $\langle\bar f_n\boldsymbol{X}\rangle_{n\in\mathbb{N}}$ is order*-convergent in $(L^0)^{\mathcal{S}}$ to $\bar f\boldsymbol{X}$, by 642Bd applied at each coordinate in $\mathcal{S}$. So $\langle R(\bar f_n\boldsymbol{X})\rangle_{n\in\mathbb{N}}$ is order*-convergent to $R(\bar f\boldsymbol{X})$, and 645H tells us that $R(\bar f\boldsymbol{X})\in M^0_{\text{S-i}}$, that is, $f\in U$.

Since we know from (i) that every continuous function from $\mathbb{R}^k$ to $\mathbb{R}$ belongs to U, we see that our locally bounded Borel measurable function h belongs to U, by 645Cc, and $\bar h\boldsymbol{X}\times\mathbf{1}^{(\mathcal{S})}_{<}\in M^0_{\text{S-i}}$.

(iii) If $h(0,\ldots,0)=0$ then, as in part (a-i) of the proof of 645D, we can use 619Ge to see that

$$\bar h\boldsymbol{X}=\bar hR(\boldsymbol{X})=R(\bar h\boldsymbol{X})=R(\bar hR(\boldsymbol{X}))=R(\bar h\boldsymbol{X})\in M^0_{\text{S-i}}.$$

(b) If $\boldsymbol{X}\in M^k_{\text{S-i}}$ then $R(\boldsymbol{X})=(\boldsymbol{x}_1\times\mathbf{1}^{(\mathcal{S})}_{<},\ldots,\boldsymbol{x}_k\times\mathbf{1}^{(\mathcal{S})}_{<})$ belongs to $(M^0_{\text{S-i}})^k$ and $R(\bar hR(\boldsymbol{X}))\in M^0_{\text{S-i}}$, by (a); but now 619G(e-i) tells us that $R(\bar h\boldsymbol{X})=R(\bar hR(\boldsymbol{X}))$, so $\bar h\boldsymbol{X}\in M_{\text{S-i}}$.

645K Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $k\ge1$ an integer, $h:\mathbb{R}^k\to\mathbb{R}$ a locally bounded Borel measurable function and $z\in L^0(\mathfrak{A}\cap\bigcap_{\sigma\in\mathcal{S}}\mathfrak{A}_\sigma)$.

(a) $M^0_{\text{S-i}}=M^0_{\text{S-i}}(\mathcal{S})$ is an f-subalgebra of $M_{\text{po-b}}(\mathcal{S})$ and $z\boldsymbol{x}\in M^0_{\text{S-i}}$ for every $\boldsymbol{x}\in M^0_{\text{S-i}}$.

(b) $M_{\text{S-i}}=M_{\text{S-i}}(\mathcal{S})$ is an f-subalgebra of $M_{\text{o-b}}(\mathcal{S})$ and $z\boldsymbol{x}\in M_{\text{S-i}}$ for every $\boldsymbol{x}\in M_{\text{S-i}}$.

proof (a) Because the S-integration topology on $M_{\text{po-b}}$ is a linear space topology (645E(a-i)), and $\{\boldsymbol{u}_<:\boldsymbol{u}\in M_{\text{mo}}(\mathcal{S})\}$ is a linear subspace of $M_{\text{po-b}}=M_{\text{po-b}}(\mathcal{S})$, its closure $M^0_{\text{S-i}}$ is a linear subspace (2A5Ec). Now 645Ja and 612Bc, as usual, show that $M^0_{\text{S-i}}$ is an f-subalgebra of $M_{\text{po-b}}$. Since $z\mathbf{1}^{\mathcal{S}}\in M_{\text{mo}}(\mathcal{S})$, $z\mathbf{1}^{(\mathcal{S})}_{<}=(z\mathbf{1}^{\mathcal{S}})_<$ (641G(a-iv)) belongs to $M^0_{\text{S-i}}$, and if $\boldsymbol{x}\in M^0_{\text{S-i}}$ then

$$z\boldsymbol{x}=(z\mathbf{1}^{\mathcal{S}})\times\boldsymbol{x}=z\mathbf{1}^{\mathcal{S}}\times(\mathbf{1}^{(\mathcal{S})}_{<}\times\boldsymbol{x})=z\mathbf{1}^{(\mathcal{S})}_{<}\times\boldsymbol{x}\in M^0_{\text{S-i}}$$

because $M^0_{\text{S-i}}$ is closed under $\times$.

(b)(i) If $\boldsymbol{x}=\langle x_\sigma\rangle_{\sigma\in\mathcal{S}}$ belongs to $M_{\text{S-i}}$, then $\boldsymbol{x}\times\mathbf{1}^{(\mathcal{S})}_{<}\in M^0_{\text{S-i}}\subseteq M_{\text{po-b}}$ is order-bounded. Setting $e_\sigma=\sup_{\sigma'\in\mathcal{S}}[\![\sigma'<\sigma]\!]$ for $\sigma\in\mathcal{S}$, as in 641Gb, we have

$$\sup_{\sigma\in\mathcal{S}}|x_\sigma\times\chi(1\setminus e_\sigma)|=\sup|\boldsymbol{x}\times\mathbf{1}^{(\mathcal{S})}_{<}|=\bar w$$

say defined in L^0. Set $e=\inf_{\sigma\in\mathcal{S}}e_\sigma$.

Take any $\epsilon>0$. If σ, $\tau\in\mathcal{S}$ and $\sigma\le\tau$, then $e_\sigma\subseteq e_\tau$, so there is a $\tau\in\mathcal{S}$ such that $\bar\mu(e_\tau\setminus e)\le\epsilon$. Now if $\sigma\in\mathcal{S}$,

$$[\![\tau\le\sigma]\!]\cup e\subseteq[\![\sigma=\tau]\!]\cup e_\sigma\subseteq[\![|x_\sigma|=|x_\tau|]\!]\cup[\![|x_\sigma|\le w]\!]\subseteq[\![|x_\sigma|\le|x_\tau|\vee w]\!].$$

$$[\![\sigma<\tau]\!]\setminus e\subseteq e_\tau\setminus e.$$

But this means that $\{x_\sigma\times\chi(1\setminus(e_\tau\setminus e)):\sigma\in\mathcal{S}\}$ is order-bounded in L^0, while $\bar\mu(1\setminus(e_\tau\setminus e))\ge1-\epsilon$. As ϵ is arbitrary, $\{x_\sigma:\sigma\in\mathcal{S}\}$ is order-bounded, by 613Bp, and $\boldsymbol{x}\in M_{\text{o-b}}(\mathcal{S})$.

(ii) $M_{\text{S-i}}=\{\boldsymbol{x}:\boldsymbol{x}\in M_{\text{fa}}(\mathcal{S}),\ \boldsymbol{x}\times\mathbf{1}^{(\mathcal{S})}_{<}\in M^0_{\text{S-i}}\}$ is an f-subalgebra just because $\boldsymbol{y}\mapsto\boldsymbol{y}\times\mathbf{1}^{(\mathcal{S})}_{<}:(L^0)^{\mathcal{S}}\to(L^0)^{\mathcal{S}}$ is a multiplicative Riesz homomorphism and $M^0_{\text{S-i}}$ is an f-subalgebra of $M_{\text{po-b}}$. Similarly, if $\boldsymbol{x}=\langle x_\sigma\rangle_{\sigma\in\mathcal{S}}\in M_{\text{S-i}}$, $(z\boldsymbol{x})\times\mathbf{1}^{(\mathcal{S})}_{<}=z(\boldsymbol{x}\times\mathbf{1}^{(\mathcal{S})}_{<})$ belongs to $M^0_{\text{S-i}}$ so $z\boldsymbol{x}\in M_{\text{S-i}}$.

645L Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. Give $M_{\text{po-b}}(\mathcal{S})$ its S-integration topology $\mathfrak{T}_{\text{S-i}}$. Suppose that $\boldsymbol{x}\in M^0_{\text{S-i}}(\mathcal{S})$.

(a) If $\boldsymbol{u}^*\in M_{\text{mo}}=M_{\text{mo}}(\mathcal{S})$ is such that $|\boldsymbol{x}|\le\boldsymbol{u}^*_<$, then $A=\{\boldsymbol{u}:\boldsymbol{u}\in M_{\text{mo}},\ |\boldsymbol{u}|\le\boldsymbol{u}^*\}$ is uniformly order-bounded and

$$\boldsymbol{x}\in\overline{\{\boldsymbol{u}_<:\boldsymbol{u}\in A\}}\subseteq\overline{\{\boldsymbol{u}_<:\boldsymbol{u}\in M_{\text{mo}},\ \sup|\boldsymbol{u}|\le\sup|\boldsymbol{u}^*|\}}.$$

(b) There is a $\boldsymbol{w}^*\in M_{\text{mo}}$ such that $\boldsymbol{x}\in\overline{\{\boldsymbol{u}_<:\boldsymbol{u}\in M_{\text{simp}}(\mathcal{S}),\ |\boldsymbol{u}|\le\boldsymbol{w}^*\}}$.

proof (a) Setting $\bar u=\sup|\boldsymbol{u}^*|$, of course $\bar u=\sup_{\boldsymbol{u}\in A}\sup|\boldsymbol{u}|$, so A is uniformly order-bounded.

If G is a neighbourhood of $\boldsymbol{x}$ in $M_{\text{po-b}}=M_{\text{po-b}}(\mathcal{S})$, there are a $\boldsymbol{v}\in M^{\uparrow}_{\text{n-s}}=M^{\uparrow}_{\text{n-s}}(\mathcal{S})$ and a $\delta>0$ such that

$$\{\boldsymbol{x}' : \boldsymbol{x}' \in M_{\text{po-b}},\, \widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x}' - \boldsymbol{x}) \le \delta\}$$

is included in G (645E(a-ii)). Since $\boldsymbol{x} \in M^0_{\text{S-i}} = M^0_{\text{S-i}}(\mathcal{S})$, there is a $\boldsymbol{u} \in M_{\text{mo}}$ such that $\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{u}_< - \boldsymbol{x}) \le \delta$. Consider $\boldsymbol{u}' = \text{med}(-\boldsymbol{u}^*, \boldsymbol{u}, \boldsymbol{u}^*)$. Then $\boldsymbol{u}' \in M_{\text{mo}}$ and $\boldsymbol{u}' \in A$. Next, because the operation of taking previsible version is a lattice homomorphism (641Ge again), $\boldsymbol{u}'_< = \text{med}(-\boldsymbol{u}^*_<, \boldsymbol{u}_<, \boldsymbol{u}^*_<)$. Since $|\boldsymbol{x}| \le \boldsymbol{u}^*_<$, $|\boldsymbol{x} - \boldsymbol{u}'_<| \le |\boldsymbol{x} - \boldsymbol{u}_<|$ and

$$\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x} - \boldsymbol{u}'_<) \le \widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x} - \boldsymbol{u}_<) \le \delta,$$

so $\boldsymbol{u}'_< \in G$.

Thus G meets $\{\boldsymbol{u}_< : \boldsymbol{u} \in A\}$. As G is arbitrary, $\boldsymbol{x} \in \overline{\{\boldsymbol{u}_< : \boldsymbol{u} \in A\}}$.

(b) As $\boldsymbol{x} \in M_{\text{po-b}}$, there is certainly a $\boldsymbol{u}^* \in M_{\text{mo}}$ such that $|\boldsymbol{x}| \le \boldsymbol{u}^*$; again write $\bar{u}$ for $\sup|\boldsymbol{u}^*|$, and now let $\boldsymbol{w}^*$ be the non-negative non-decreasing process defined from $\bar{u}$ as in 614Ie, so that $|\boldsymbol{u}| \le \boldsymbol{w}^*$ whenever $\boldsymbol{u} \in M_{\text{fa}}(\mathcal{S})$ and $\sup|\boldsymbol{u}| \le \bar{u}$. Take any $\boldsymbol{v} \in M^{\uparrow}_{\text{n-s}}$ and $\epsilon > 0$. By (a), there is a $\boldsymbol{u} \in M_{\text{mo}}$ such that $|\boldsymbol{u}| \le \boldsymbol{u}^*$ and $\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x} - \boldsymbol{u}_<) \le \epsilon$.

Let $\delta > 0$ be such that $\theta(\int_{\mathcal{S}} |\boldsymbol{z}| d\boldsymbol{v}) \le \epsilon$ whenever $\boldsymbol{z} \in M_{\text{mo}}$ and $\theta(\sup|\boldsymbol{z}|) \le \delta$ (616J, applied to $\Delta\boldsymbol{v}$). By 615O, we have a process $\boldsymbol{w} \in M_{\text{bv}}(\mathcal{S})$ such that $\theta(\sup|\boldsymbol{w} - \boldsymbol{u}|) \le \delta$ and $\sup|\boldsymbol{w}| \le \sup|\boldsymbol{u}|$. We can express $\boldsymbol{w}$ as $\boldsymbol{w}' - \boldsymbol{w}''$ where $\boldsymbol{w}$, $\boldsymbol{w}''$ are order-bounded non-negative non-decreasing processes (614J).

Turn now to the construction in 617B. For any $I \in \mathcal{I}(\mathcal{S})$ we have a simple process $\boldsymbol{w}_I = \langle w_{I\sigma} \rangle_{\sigma \in \mathcal{S}}$ defined by saying that $\boldsymbol{w}_I$ has a breakpoint string in I, $\boldsymbol{w}_I$ and $\boldsymbol{w}$ agree on I, and $1 \setminus \sup_{\tau \in I} [\![\tau \le \sigma]\!] \subseteq [\![w_{I\sigma} = 0]\!]$ for every $\sigma \in \mathcal{S}$; and we have corresponding processes $\boldsymbol{w}'_I$, $\boldsymbol{w}''_I$. Evidently $\boldsymbol{w}_I = \boldsymbol{w}'_I - \boldsymbol{w}''_I$. Now because $\boldsymbol{w}'$ is non-negative and non-decreasing, we must have $0 \le \boldsymbol{w}'_I \le \boldsymbol{w}'$ for every $I \in \mathcal{I}(\mathcal{S})$. Next, 617B(b-ii) tells us that

$$\int_{\mathcal{S}} \boldsymbol{w}' d\boldsymbol{v} = \lim_{I \uparrow \mathcal{I}} \int_{\mathcal{S}} \boldsymbol{w}'_I d\boldsymbol{v},$$

that is,

$$0 = \lim_{I \uparrow \mathcal{I}(\mathcal{S})} \int_{\mathcal{S}} \boldsymbol{w}' - \boldsymbol{w}'_I \, d\boldsymbol{v} = \lim_{I \uparrow \mathcal{I}(\mathcal{S})} \int_{\mathcal{S}} |\boldsymbol{w}' - \boldsymbol{w}'_I| d\boldsymbol{v}.$$

Consequently

$$\lim_{I \uparrow \mathcal{I}(\mathcal{S})} \int_{\mathcal{S}} |\boldsymbol{w} - \boldsymbol{w}_I| d\boldsymbol{v} \le \lim_{I \uparrow \mathcal{I}(\mathcal{S})} \int_{\mathcal{S}} (|\boldsymbol{w}' - \boldsymbol{w}'_I| + |\boldsymbol{w}'' - \boldsymbol{w}''_I|) d\boldsymbol{v} = 0,$$

and there is an $I \in \mathcal{I}(\mathcal{S})$ such that $\int_{\mathcal{S}} |\boldsymbol{w} - \boldsymbol{w}_I| d\boldsymbol{v} \le \epsilon$, while $\boldsymbol{w}_I$ is a simple process.

At this point note that $\sup|\boldsymbol{w}_I| \le \sup|\boldsymbol{w} \restriction I| \le \sup|\boldsymbol{u}|$, so $|\boldsymbol{w}_I| \le \boldsymbol{w}^*$, while

$$\begin{aligned}\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x} - \boldsymbol{w}_{I<}) &\le \widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x} - \boldsymbol{u}_<) + \widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{u}_< - \boldsymbol{w}_{I<}) \le \epsilon + \int_{\mathcal{S}} |\boldsymbol{u} - \boldsymbol{w}_I| d\boldsymbol{v} \\ &\le \epsilon + \int_{\mathcal{S}} |\boldsymbol{u} - \boldsymbol{w}| d\boldsymbol{v} + \int_{\mathcal{S}} |\boldsymbol{w} - \boldsymbol{w}_I| d\boldsymbol{v} \le 2\epsilon + \int_{\mathcal{S}} |\boldsymbol{u} - \boldsymbol{w}| d\boldsymbol{v} \le 3\epsilon\end{aligned}$$

by the choice of δ and $\boldsymbol{w}$. As ϵ and $\boldsymbol{v}$ are arbitrary, $\boldsymbol{x} \in \overline{\{\boldsymbol{u}_< : \boldsymbol{u} \in M_{\text{simp}}(\mathcal{S}),\, |\boldsymbol{u}| \le \boldsymbol{w}^*\}}$.

645N I have spent all this time on $M^0_{\text{S-i}}$ and $\mathfrak{T}_{\text{S-i}}$ because these can be described and investigated in a very general context. But for the new integral, we need to restrict ourselves in a way which is already familiar.

Lemma Suppose that $\langle \mathfrak{A}_t \rangle_{t \in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$. If $\boldsymbol{u} \in M_{\text{mo}}(\mathcal{S})$ and $\boldsymbol{v} \in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$, then $\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{u}_<) = \theta(\int_{\mathcal{S}} |\boldsymbol{u}| d\boldsymbol{v})$.

proof (a) Setting $\boldsymbol{u}_n = |\boldsymbol{u}|$ for every $n \in \mathbb{N}$, $\langle \boldsymbol{u}_n \rangle_{n \in \mathbb{N}}$ is a uniformly order-bounded non-decreasing sequence in M^+_{mo} and $|\boldsymbol{u}_<| \le \sup_{n \in \mathbb{N}} \boldsymbol{u}_{n<}$, so

$$\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{u}_<) \le \sup_{n \in \mathbb{N}} \theta(\int_{\mathcal{S}} \boldsymbol{u}_n d\boldsymbol{v}) = \theta(\int_{\mathcal{S}} |\boldsymbol{u}| d\boldsymbol{v}).$$

(b) Now suppose that $\langle \boldsymbol{u}_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing uniformly order-bounded sequence in M^+_{mo} such that $|\boldsymbol{u}_<| \le \sup_{n \in \mathbb{N}} \boldsymbol{u}_{n<}$. Then $\langle \boldsymbol{u}_{n<} \rangle_{n \in \mathbb{N}}$ is non-decreasing, so $\langle |\boldsymbol{u}_<| \wedge \boldsymbol{u}_{n<} \rangle_{n \in \mathbb{N}}$ order*-converges to $|\boldsymbol{u}_<|$, that is, $\langle (|\boldsymbol{u}| \wedge \boldsymbol{u}_n)_< \rangle_{n \in \mathbb{N}}$ order*-converges to $|\boldsymbol{u}|_<$. Now $\langle |\boldsymbol{u}| \wedge \boldsymbol{u}_n \rangle_{n \in \mathbb{N}}$ is uniformly order-bounded, so Theorem 644H tells us that $\int_{\mathcal{S}} |\boldsymbol{u}| \, d\boldsymbol{v} = \lim_{n \to \infty} \int_{\mathcal{S}} |\boldsymbol{u}| \wedge \boldsymbol{u}_n \, d\boldsymbol{v}$ and

$$\theta(\int_{\mathcal{S}}|\boldsymbol{u}|\,d\boldsymbol{v}) = \lim_{n\to\infty}\theta(\int_{\mathcal{S}}|\boldsymbol{u}|\wedge\boldsymbol{u}_n\,d\boldsymbol{v}) \le \lim_{n\to\infty}\theta(\int_{\mathcal{S}}\boldsymbol{u}_n\,d\boldsymbol{v}).$$

As $\langle\boldsymbol{u}_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\theta(\int_{\mathcal{S}}|\boldsymbol{u}|d\boldsymbol{v}) \le \widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{u}_<)$.

645O Theorem Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$, and give $M_{\text{po-b}}(\mathcal{S})$ its S-integration topology $\mathfrak{T}_{\text{S-i}}$. If $\boldsymbol{x}\in M_{\text{S-i}}(\mathcal{S})$ and $\boldsymbol{v}\in M_{\text{n-s}}(\mathcal{S})$ is an integrator, then there is a unique $z\in L^0$ such that whenever $A\subseteq M_{\text{mo}}(\mathcal{S})$ is uniformly order-bounded and $\epsilon>0$ there is a $\mathfrak{T}_{\text{S-i}}$-neighbourhood G of $\boldsymbol{x}\times\boldsymbol{1}^{(\mathcal{S})}_{<}$ such that $\theta(z-\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v})\le\epsilon$ whenever $\boldsymbol{u}\in A$ and $\boldsymbol{u}_<\in G$.

proof Write $R(\boldsymbol{x})\in M^0_{\text{S-i}}(\mathcal{S})$ for $\boldsymbol{x}\times\boldsymbol{1}^{(\mathcal{S})}_{<}$.

(a) Let $\mathcal{A}$ be the family of uniformly order-bounded subsets of $M_{\text{mo}}=M_{\text{mo}}(\mathcal{S})$ such that $R(\boldsymbol{x})\in\overline{\{\boldsymbol{u}_<:\boldsymbol{u}\in A\}}$. By 645L, $\mathcal{A}$ is not empty.

Define $T:M_{\text{mo}}\to L^0$ by setting $T\boldsymbol{u}=\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}$ for $\boldsymbol{u}\in M_{\text{mo}}$. For $A\in\mathcal{A}$, let $\mathcal{F}_A$ be the filter on M_{mo} generated by sets of the form $\{\boldsymbol{u}:\boldsymbol{u}\in A,\ \boldsymbol{u}_<\in G\}$ where G is a $\mathfrak{T}_{\text{S-i}}$-neighbourhood of $R(\boldsymbol{x})$. Then the image filter $T[[\mathcal{F}_A]]$ on L^0 is Cauchy. **P** Let $\epsilon>0$. By 644G, $T\restriction A$ is uniformly continuous for the uniformity induced by the topology $\mathfrak{S}$ on M_{mo} described there, so there are a $\boldsymbol{v}'\in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$ and a $\delta>0$ such that $\theta(T\boldsymbol{u}-T\boldsymbol{u}')\le\epsilon$ whenever $\boldsymbol{u}$, $\boldsymbol{u}'\in A$ and $\int_{\mathcal{S}}|\boldsymbol{u}-\boldsymbol{u}'|d\boldsymbol{v}'\le\delta$. Next, $G=\{\boldsymbol{y}:\boldsymbol{y}\in M_{\text{po-b}}=M_{\text{po-b}}(\mathcal{S}),\ \widehat{\theta}^{\#}_{\boldsymbol{v}'}(\boldsymbol{y}-R(\boldsymbol{x}))\le\frac{1}{2}\delta\}$ is a neighbourhood of $R(\boldsymbol{x})$, so $F=\{\boldsymbol{u}:\boldsymbol{u}\in A,\ \boldsymbol{u}_<\in G\}$ belongs to $\mathcal{F}_A$. If $\boldsymbol{u}$, $\boldsymbol{u}'\in F$, then

$$\begin{aligned}\int_{\mathcal{S}}|\boldsymbol{u}-\boldsymbol{u}'|d\boldsymbol{v}' &= \widehat{\theta}^{\#}_{\boldsymbol{v}'}((\boldsymbol{u}-\boldsymbol{u}')_<) \\ &= \widehat{\theta}^{\#}_{\boldsymbol{v}'}(\boldsymbol{u}_<-\boldsymbol{u}'_<)\le\delta,\end{aligned}$$

(645N)

and $\theta(T\boldsymbol{u}-T\boldsymbol{u}')\le\epsilon$. This shows that $T[[\mathcal{F}_A]]$ contains a set of diameter at most ϵ for the metric defined by θ. As ϵ is arbitrary, $T[[\mathcal{F}_A]]$ is Cauchy. **Q**

(b) It follows that

$$z_A=\lim T[[\mathcal{F}_A]]=\lim_{\boldsymbol{u}\to\mathcal{F}_A}\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}$$

is defined. If A, $A'\in\mathcal{A}$ and $A\subseteq A'$, then $\mathcal{F}_{A'}\subseteq\mathcal{F}_A$ and $\lim_{\boldsymbol{u}\to\mathcal{F}_{A'}}\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}$ is defined, so this is also $\lim_{\boldsymbol{u}\to\mathcal{F}_A}\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}$, that is, $z_{A'}=z_A$. If A, A' are any two members of $\mathcal{A}$, then $A\cup A'\in\mathcal{A}$ so $z_A=z_{A\cup A'}=z_{A'}$. We can therefore define $z\in L^0$ by saying that $z=z_A$ for every $A\in\mathcal{A}$.

(c) Suppose that $\epsilon>0$ and $A\subseteq M_{\text{mo}}$ is uniformly order-bounded. If $A\notin\mathcal{A}$, then $G=M_{\text{po-b}}\setminus\overline{\{\boldsymbol{u}_<:\boldsymbol{u}\in A\}}$ is a neighbourhood of $R(\boldsymbol{x})$, and certainly $\theta(z-\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v})\le\epsilon$ whenever $\boldsymbol{u}\in A$ and $\boldsymbol{u}_<\in G$. If $A\in\mathcal{A}$, then there is an $F\in\mathcal{F}_A$ such that $\theta(z_A-\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v})\le\epsilon$ whenever $\boldsymbol{u}\in F$. Now F must include some set of the form $\{\boldsymbol{u}:\boldsymbol{u}\in A,\ \boldsymbol{u}_<\in G\}$ where G is a neighbourhood of $R(\boldsymbol{x})$, so we see that

$$\theta(z-\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v})=\theta(z_A-\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v})\le\epsilon$$

whenever $\boldsymbol{u}\in A$ and $\boldsymbol{u}_<\in G$.

(d) Thus we have found a z with the given properties. To see that it is unique, recall from (a) that there is an $A\in\mathcal{A}$, and observe that $z=\lim_{\boldsymbol{u}\to\mathcal{F}_A}\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}$.

645P Definition Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous and that $\mathcal{S}$ is an order-convex sublattice of $\mathcal{T}$.

(a) If $\boldsymbol{x}\in M_{\text{S-i}}(\mathcal{S})$ and $\boldsymbol{v}\in M_{\text{n-s}}(\mathcal{S})$ is an integrator, I will say that the element z of L^0 defined as in Theorem 645O is $\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{v}$, the **S-integral** of $\boldsymbol{x}$ with respect to $\boldsymbol{v}$.

(b) In these circumstances I will say that members of $M_{\text{S-i}}(\mathcal{S})$ are **S-integrable**.

(c) Note that if $\boldsymbol{x}$ is a fully adapted process with domain $\mathcal{S}$, $\boldsymbol{x}$ is S-integrable iff $\boldsymbol{x}\times\mathbf{1}^{(\mathcal{S})}_{<}$ is S-integrable, and in this case $\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{v}=\oint_{\mathcal{S}}\boldsymbol{x}\times\mathbf{1}^{(\mathcal{S})}_{<}\,d\boldsymbol{v}$ for every near-simple integrator $\boldsymbol{v}$ with domain $\mathcal{S}$.

(d) If $\boldsymbol{x}$ is a fully adapted process with domain $\mathcal{S}$, I will say that it is **locally S-integrable** if $\boldsymbol{x}\restriction\mathcal{S}\wedge\tau\in M_{\text{S-i}}(\mathcal{S}\wedge\tau)$ for every $\tau\in\mathcal{S}$.

(e) Following my practice with the Riemann-sum integral (613H), I shall allow myself to write $\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{v}$ for $\oint_{\mathcal{S}}(\boldsymbol{x}\restriction\mathcal{S})\,d(\boldsymbol{v}\restriction\mathcal{S})$ whenever $\boldsymbol{x}$, $\boldsymbol{v}$ are fully adapted processes such that $\mathcal{S}\subseteq\operatorname{dom}\boldsymbol{x}\cap\operatorname{dom}\boldsymbol{v}$, $\boldsymbol{x}\restriction\mathcal{S}$ is S-integrable and $\boldsymbol{v}\restriction\mathcal{S}$ is a near-simple integrator.

645Q Law-independence It is a while since I mentioned law-independence, but the question of which features of a structure $(\mathfrak{A},\bar\mu,T,\langle\mathfrak{A}_t\rangle_{t\in T})$ really depend on the measure $\bar\mu$, rather than just the measurable algebra $\mathfrak{A}$ and the filtration $\langle\mathfrak{A}_t\rangle_{t\in T}$, never goes away. I have done my best to express concepts and theorems in terms which make it easy to see that they are law-independent. I have found however that some results seem most naturally expressed in terms of the functionals θ of 613Ba, and these need to be checked. In many cases there is an obvious re-statement of a theorem in terms of the topology of convergence in measure, which is safely law-independent. In the second half of §634, I introduced 'coordinated' subalgebras, which are surely not law-independent, but they have been practically invisible since. I remind you that anything involving martingales is not expected to be law-independent (though the property of being a semi-martingale is), but the regions of accessibility and approachability in 643C are law-independent.

In 645E, however, we have a new formula involving θ. Just as ucp topologies are defined in terms of functionals $\boldsymbol{u}\mapsto\widehat\theta(\boldsymbol{u})=\theta(\sup|\boldsymbol{u}|)$ (615B), 645E uses functionals

$$\boldsymbol{u}\mapsto\widehat\theta^{\#}_{\boldsymbol{v}}(\boldsymbol{u})=\inf_{\langle\boldsymbol{u}_n\rangle_{n\in\mathbb{N}}}\sup_n\theta(\textstyle\int_{\mathcal{S}}\boldsymbol{u}_n\,d\boldsymbol{v}).$$

But changing the measure on $\mathfrak{A}$ just produces a new F-norm ϑ on L^0 which is equivalent to θ, so the F-norms $\widehat\theta^{\#}_{\boldsymbol{v}}$ and $\widehat\vartheta^{\#}_{\boldsymbol{v}}$ will be equivalent and induce the same topology $\mathfrak{T}_{\text{S-i}}$ on $M_{\text{po-b}}$. Thus $\mathfrak{T}_{\text{S-i}}$, the spaces $M^0_{\text{S-i}}$ and $M_{\text{S-i}}$ and the S-integral are law-independent.

645R Theorem Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$.

(a) Suppose that $\boldsymbol{u}\in M_{\text{mo}}(\mathcal{S})$ and $\boldsymbol{v}\in M_{\text{n-s}}=M_{\text{n-s}}(\mathcal{S})$ is an integrator.

(i) $\boldsymbol{u}_<\in M^0_{\text{S-i}}(\mathcal{S})$ and $\oint_{\mathcal{S}}\boldsymbol{u}_<d\boldsymbol{v}=\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}$.

(ii) If *either* $\boldsymbol{v}$ is jump-free *or* T has no points isolated on the right, $\oint_{\mathcal{S}}\boldsymbol{u}_<d\boldsymbol{v}=\int_{\mathcal{S}}\boldsymbol{u}_<\,d\boldsymbol{v}$.

(b) If $\boldsymbol{x}$, $\boldsymbol{x}'\in M_{\text{S-i}}=M_{\text{S-i}}(\mathcal{S})$, $\boldsymbol{v}$, $\boldsymbol{v}'\in M_{\text{n-s}}$ are integrators, and $\alpha\in\mathbb{R}$, then

$$\oint_{\mathcal{S}}\boldsymbol{x}+\boldsymbol{x}'\,d\boldsymbol{v}=\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{v}+\oint_{\mathcal{S}}\boldsymbol{x}'\,d\boldsymbol{v},\quad\oint_{\mathcal{S}}\boldsymbol{x}\,d(\boldsymbol{v}+\boldsymbol{v}')=\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{v}+\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{v}',$$

$$\oint_{\mathcal{S}}\alpha\boldsymbol{x}\,d\boldsymbol{v}=\oint_{\mathcal{S}}\boldsymbol{x}\,d(\alpha\boldsymbol{v})=\alpha\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{v}.$$

(c)(i) If $\boldsymbol{x}\in M_{\text{S-i}}$, $\boldsymbol{v}\in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$ and $\boldsymbol{x}\ge0$, then $\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{v}\ge0$;

(ii) if $\boldsymbol{x}\in M_{\text{S-i}}$ and $\boldsymbol{v}$ is a constant process with domain $\mathcal{S}$, then $\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{v}=0$.

proof (a)(i) Directly from the definition in 645Fb we see that $\boldsymbol{u}_<\in M^0_{\text{S-i}}=M^0_{\text{S-i}}(\mathcal{S})$. Write $\mathfrak{T}_{\text{S-i}}$ for the S-integration topology on $M_{\text{po-b}}=M_{\text{po-b}}(\mathcal{S})$. Set $z=\oint_{\mathcal{S}}\boldsymbol{u}_<\,d\boldsymbol{v}$. If $\epsilon>0$, then $A=\{\boldsymbol{u}\}$ is surely a uniformly order-bounded subset of $M_{\text{mo}}=M_{\text{mo}}(\mathcal{S})$, so there is a $\mathfrak{T}_{\text{S-i}}$-neighbourhood G of $\boldsymbol{u}_<$ in $M_{\text{po-b}}$ such that $\theta(z-\int_{\mathcal{S}}\boldsymbol{u}'\,d\boldsymbol{v})\le\epsilon$ whenever $\boldsymbol{u}'\in A$ and $\boldsymbol{u}'_<\in G$; that is, $\theta(z-\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v})\le\epsilon$. As ϵ is arbitrary, $z=\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}$.

(ii) Now 641T and 641W tell us that this will be equal to $\int_{\mathcal{S}}\boldsymbol{u}_<d\boldsymbol{v}$ if either $\boldsymbol{v}$ is jump-free or T has no points isolated on the right.

(b) For $\boldsymbol{x}\in M_{\text{S-i}}$ write $R(\boldsymbol{x})$ for $\boldsymbol{x}\times\mathbf{1}^{(\mathcal{S})}_{<}$.

(i) We know from 645Kb that $\alpha\boldsymbol{x}+\boldsymbol{x}'\in M_{\text{S-i}}$. Set

$$z=\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{v},\quad z'=\oint_{\mathcal{S}}\boldsymbol{x}'\,d\boldsymbol{v},\quad\tilde z=\oint_{\mathcal{S}}\alpha\boldsymbol{x}+\boldsymbol{x}'\,d\boldsymbol{v}.$$

By 645L, there are uniformly order-bounded sets A, $A'\subseteq M_{\text{n-s}}$ such that $R(\boldsymbol{x})\in\overline{\{\boldsymbol{u}_<:\boldsymbol{u}\in A\}}$ and $R(\boldsymbol{x}')\in\overline{\{\boldsymbol{u}_<:\boldsymbol{u}\in A'\}}$. In this case, $\alpha A+A'$ is uniformly order-bounded and $R(\alpha\boldsymbol{x}+\boldsymbol{x}')\in\overline{\{\boldsymbol{u}_<:\boldsymbol{u}\in\alpha A+A'\}}$, because R is a linear operator and $\mathfrak{T}_{\text{S-i}}$ is a linear space topology.

Take any $\epsilon > 0$. Then there are neighbourhoods G, G' and $\tilde{G}$ of $R(\boldsymbol{x})$, $R(\boldsymbol{x}')$ and $R(\alpha\boldsymbol{x}+\boldsymbol{x}')$ respectively such that

$$\theta(z - \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}) \le \epsilon \text{ for every } \boldsymbol{u} \in A \text{ such that } \boldsymbol{u}_< \in G,$$

$$\theta(z' - \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}) \le \epsilon \text{ for every } \boldsymbol{u} \in A' \text{ such that } \boldsymbol{u}_< \in G',$$

$$\theta(\tilde{z} - \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}) \le \epsilon \text{ for every } \boldsymbol{u} \in \alpha A + A' \text{ such that } \boldsymbol{u}_< \in \tilde{G}.$$

Again because $\mathfrak{T}_{\text{S-i}}$ is a linear space topology, we can suppose, shrinking G and G' if necessary, that $\alpha G+G' \subseteq \tilde{G}$. Now we know that there are $\boldsymbol{u} \in A$, $\boldsymbol{u}' \in A'$ such that $\boldsymbol{u}_< \in G$ and $\boldsymbol{u}'_< \in G'$. Set $\tilde{\boldsymbol{u}} = \alpha\boldsymbol{u} + \boldsymbol{u}'$; then $\tilde{\boldsymbol{u}} \in \alpha A + A'$ and $\tilde{\boldsymbol{u}}_< \in \alpha G + G' \subseteq \tilde{G}$. So

$$\theta(z - \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}) \le \epsilon, \quad \theta(z' - \int_{\mathcal{S}} \boldsymbol{u}'\, d\boldsymbol{v}) \le \epsilon, \quad \theta(\tilde{z} - \int_{\mathcal{S}} \tilde{\boldsymbol{u}}\, d\boldsymbol{v}) \le \epsilon.$$

But $\int_{\mathcal{S}} \tilde{\boldsymbol{u}}\, d\boldsymbol{v} = \alpha \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}+\int_{\mathcal{S}} \boldsymbol{u}'\, d\boldsymbol{v}$, so $\theta(\tilde{z}-\alpha z-z') \le 3\epsilon$. As ϵ is arbitrary, $\tilde{z} = \alpha z+z'$, that is, $\oint_{\mathcal{S}} \alpha\boldsymbol{x}+\boldsymbol{x}'\, d\boldsymbol{v} = \alpha \oint_{\mathcal{S}} \boldsymbol{x}\, d\boldsymbol{v} + \oint_{\mathcal{S}} \boldsymbol{x}'\, d\boldsymbol{v}$.

(ii) To see that $\oint_{\mathcal{S}}$ is linear in the integrator as well as in the integrand, repeat the method, with slight variations, as follows. Start with

$$z = \oint_{\mathcal{S}} \boldsymbol{x}\, d\boldsymbol{v}, \quad z' = \oint_{\mathcal{S}} \boldsymbol{x}\, d\boldsymbol{v}', \quad \tilde{z} = \oint_{\mathcal{S}} \boldsymbol{x}\, d(\alpha\boldsymbol{v} + \boldsymbol{v}').$$

Let $A \subseteq M_{\text{n-s}}$ be a uniformly order-bounded set such that $R(\boldsymbol{x}) \in \overline{\{\boldsymbol{u}_< : \boldsymbol{u} \in A\}}$.

Take any $\epsilon > 0$. Then there is a neighbourhood G of $R(\boldsymbol{x})$ such that

$$\theta(z - \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}) \le \epsilon\ , \quad \theta(z' - \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}') \le \epsilon, \quad \theta(\tilde{z} - \int_{\mathcal{S}} \boldsymbol{u}\, d(\alpha\boldsymbol{v} + \boldsymbol{v}')) \le \epsilon$$

for every $\boldsymbol{u} \in A$ such that $\boldsymbol{u}_< \in G$. Since there is a $\boldsymbol{u} \in A$ such that $\boldsymbol{u}_< \in G$, and

$$\alpha \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v} + \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}' = \int_{\mathcal{S}} \boldsymbol{u}\, d(\alpha\boldsymbol{v} + \boldsymbol{v}'),$$

$\theta(\tilde{z} - \alpha z - z') \le 3\epsilon$. As ϵ is arbitrary, $\tilde{z} = \alpha z + z'$, that is, $\oint_{\mathcal{S}} \boldsymbol{x}\, d(\alpha\boldsymbol{v} + \boldsymbol{v}') = \alpha \oint_{\mathcal{S}} \boldsymbol{x}\, d\boldsymbol{v} + \oint_{\mathcal{S}} \boldsymbol{x}\, d\boldsymbol{v}'$. So we have both halves of the result claimed.

(c)(i) Since $\boldsymbol{x} \times \mathbf{1}^{(\mathcal{S})}_< \ge 0$, and $\oint_{\mathcal{S}} \boldsymbol{x}\, d\boldsymbol{v} = \oint_{\mathcal{S}} \boldsymbol{x} \times \mathbf{1}^{(\mathcal{S})}_< d\boldsymbol{v}$, it will be enough to deal with the case in which $\boldsymbol{x} \in M^0_{\text{S-i}}$. Let $\boldsymbol{u}^* \in M^+_{\text{mo}}$ be such that $|\boldsymbol{x}| \le \boldsymbol{u}^*_<$, and set $A = \{\boldsymbol{u} : \boldsymbol{u} \in M_{\text{mo}},\ |\boldsymbol{u}| \le \boldsymbol{u}^*\}$. By 645L, $\boldsymbol{x} \in \overline{\{\boldsymbol{u}_< : \boldsymbol{u} \in A\}}$. Let $\epsilon > 0$. Then there are a $\boldsymbol{w} \in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$ and a $\delta > 0$ such that $\theta(\oint_{\mathcal{S}} \boldsymbol{x}\, d\boldsymbol{v} - \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}) \le \epsilon$ whenever $\boldsymbol{u} \in A$ and $\widehat{\theta}^{\#}_{\boldsymbol{w}}(\boldsymbol{x} - \boldsymbol{u}_<) \le \delta$. Now $|\boldsymbol{u}| \in A$ and

$$|\boldsymbol{x} - |\boldsymbol{u}|_<| = ||\boldsymbol{x}| - |\boldsymbol{u}_<|| \le |\boldsymbol{x} - \boldsymbol{u}_<|, \quad \widehat{\theta}^{\#}_{\boldsymbol{w}}(\boldsymbol{x} - |\boldsymbol{u}|_<) \le \widehat{\theta}^{\#}_{\boldsymbol{w}}(\boldsymbol{x} - \boldsymbol{u}_<) \le \delta$$

(645Db), so $\theta(\oint_{\mathcal{S}} \boldsymbol{x}\, d\boldsymbol{v} - \int_{\mathcal{S}} |\boldsymbol{u}| d\boldsymbol{v}) \le \epsilon$. But $\int_{\mathcal{S}} |\boldsymbol{u}| d\boldsymbol{v} \in (L^0)^+$ (616R(b-i) again). As ϵ is arbitrary and $(L^0)^+$ is closed (613Ba), $\oint_{\mathcal{S}} \boldsymbol{x}\, d\boldsymbol{v} \in (L^0)^+$, as claimed.

(ii) If $\boldsymbol{v}$ is constant, it is an integrator (616P(b-i)), and $\oint_{\mathcal{S}} \boldsymbol{x}\, d\boldsymbol{v}$ is defined. Also $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v} = 0$ for every $\boldsymbol{u} \in M_{\text{mo}}$ (613Lc), so $\oint_{\mathcal{S}} \boldsymbol{x}\, d\boldsymbol{v} = 0$.

645S Theorem Suppose that $\langle \mathfrak{A}_t \rangle_{t \in T}$ is right-continuous and that $\mathcal{S}$ is an order-convex sublattice of $\mathcal{T}$. Give $M^0_{\text{S-i}} = M^0_{\text{S-i}}(\mathcal{S})$ and L^0 the S-integration topology $\mathfrak{T}_{\text{S-i}}$ and the topology of convergence in measure respectively, with their associated uniformities. If $\boldsymbol{v} \in M_{\text{n-s}}(\mathcal{S})$ is an integrator, then $\boldsymbol{x} \mapsto \oint_{\mathcal{S}} \boldsymbol{x}\, d\boldsymbol{v} : M^0_{\text{S-i}} \to L^0$ is uniformly continuous on any uniformly previsibly order-bounded subset of $M^0_{\text{S-i}}$.

proof Let $A \subseteq M^0_{\text{S-i}}$ be uniformly previsibly order-bounded. Then there is a $\boldsymbol{u}^* \in M_{\text{n-s}}(\mathcal{S})^+$ such that $A \subseteq \{\boldsymbol{x} : |\boldsymbol{x}| \le \boldsymbol{u}^*_<\}$. Set $B = \{\boldsymbol{u} : \boldsymbol{u} \in M_{\text{n-s}}(\mathcal{S}),\ |\boldsymbol{u}| \le \boldsymbol{u}^*\}$; note that B and $B - B$ are uniformly order-bounded. Take $\epsilon > 0$. Then there is a $\mathfrak{T}_{\text{S-i}}$-neighbourhood G of $\mathbf{0}$ in $M^0_{\text{S-i}}$ such that $\theta(\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}) \le \epsilon$ whenever $\boldsymbol{u} \in B - B$ and $\boldsymbol{u}_< \in G$ (645O). Let H be a neighbourhood of 0 in $M^0_{\text{S-i}}$ such that $H + H - H \subseteq G$. If $\boldsymbol{x}$, $\boldsymbol{x}' \in A$ and $\boldsymbol{x}' - \boldsymbol{x} \in H$, then there are $\boldsymbol{u}$, $\boldsymbol{u}' \in B$ such that

$$\boldsymbol{x} - \boldsymbol{u}_< \in H, \quad \theta(\oint_{\mathcal{S}} \boldsymbol{x}\, d\boldsymbol{v} - \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}) \le \epsilon,$$

$$\boldsymbol{x}' - \boldsymbol{u}' \in H, \quad \theta(\oint_{\mathcal{S}} \boldsymbol{x}'\, d\boldsymbol{v} - \int_{\mathcal{S}} \boldsymbol{u}'\, d\boldsymbol{v}) \le \epsilon$$

(645L, 645O). In this case, $\boldsymbol{u}-\boldsymbol{u}'\in B-B$ and $\boldsymbol{u}_<-\boldsymbol{u}'_<\in G$, so

$$\theta(\oint_{\mathcal{S}}\boldsymbol{x}'\,d\boldsymbol{v}-\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{v})\le 2\epsilon+\theta(\int_{\mathcal{S}}\boldsymbol{u}'\,d\boldsymbol{v}-\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v})=2\epsilon+\theta(\int_{\mathcal{S}}\boldsymbol{u}'-\boldsymbol{u}\,d\boldsymbol{v})\le 3\epsilon.$$

As ϵ is arbitrary, S-integration with respect to $\boldsymbol{v}$ is uniformly continuous on A.

645T Theorem Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$. Suppose that $\langle\boldsymbol{x}_n\rangle_{n\in\mathbb{N}}$ is a sequence in $M_{\text{S-i}}(\mathcal{S})$ such that $\langle\boldsymbol{x}_n\times\mathbf{1}^{(\mathcal{S})}_{<}\rangle_{n\in\mathbb{N}}$ is uniformly previsibly order-bounded and $\langle\boldsymbol{x}_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to $\boldsymbol{x}$ in $(L^0)^{\mathcal{S}}$. Then $\boldsymbol{x}$ is S-integrable and $\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{v}=\lim_{n\to\infty}\oint_{\mathcal{S}}\boldsymbol{x}_n\,d\boldsymbol{v}$ for every integrator $\boldsymbol{v}\in M_{\text{n-s}}(\mathcal{S})$.

proof Write $R(\boldsymbol{x})$ for $\boldsymbol{x}\times\mathbf{1}^{(\mathcal{S})}_{<}$, etc., as usual. By 645H, $\langle R(\boldsymbol{x}_n)\rangle_{n\in\mathbb{N}}$ is $\mathfrak{T}_{\text{S-i}}$-convergent to $R(\boldsymbol{x})$. Since $\{R(\boldsymbol{x})\}\cup\{R(\boldsymbol{x}_n):n\in\mathbb{N}\}$ is uniformly previsibly order-bounded, 645S tells us that

$$\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{v}=\oint_{\mathcal{S}}R(\boldsymbol{x})\,d\boldsymbol{v}=\lim_{n\to\infty}\oint_{\mathcal{S}}R(\boldsymbol{x}_n)\,d\boldsymbol{v}=\lim_{n\to\infty}\oint_{\mathcal{S}}\boldsymbol{x}_n\,d\boldsymbol{v}.$$

645X Basic exercises >(a) Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous, and that $\mathcal{S}$ is an order-convex sublattice of $\mathcal{T}$. Show that if $A\subseteq M_{\text{n-s}}(\mathcal{S})$ is uniformly order-bounded, then it is bounded above and below in $M_{\text{n-s}}(\mathcal{S})$. (*Hint*: show that if $\bar u\ge 0$ in L^0, and we set $u_\sigma=\sup\{u:u\in L^0(\mathfrak{A}_\sigma),\ u\le\bar u\}$ for $\sigma\in\mathcal{S}$, then $\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ satisfies the conditions of 632F.)

(b) Suppose that $T=[0,\infty[$ and $\mathfrak{A}=\{0,1\}$, as in 613W, 615Xf, 616Xa, 617Xb, 618Xa, 622Xd, 626Xa, 627Xa, 642Xd and 644Xe. (i) Show that $M_{\text{po-b}}(\mathcal{T}_f)$ corresponds to the space V of bounded functions $f:[0,\infty[\,\to\mathbb{R}$ such that $f(0)=0$. (ii) Show that the S-integration topology on $M_{\text{po-b}}(\mathcal{T}_f)$ corresponds to the topology on V generated by functionals $f\mapsto\overline{\int}f\,d\nu$ where ν is a totally finite Radon measure on $[0,\infty[$. (iii) Show that $M^0_{\text{S-i}}(\mathcal{T}_f)$ corresponds to the space $W=\{f:f\in V,\ f\text{ is universally measurable}\}$. (iv) Show that if $f\in W$ corresponds to $\boldsymbol{x}\in M^0_{\text{S-i}}$, $g:[0,\infty[\,\to\mathbb{R}$ is a non-decreasing function of bounded variation which is continuous on the right and $\boldsymbol{v}$ is the process corresponding to g, then $\oint_{\mathcal{T}_f}\boldsymbol{x}\,d\boldsymbol{v}=\int f\,d\nu_g$ where ν_g is the Lebesgue-Stieltjes measure on $[0,\infty[$ defined from g.

(c) Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice, and $\langle\boldsymbol{u}_n\rangle_{n\in\mathbb{N}}$ a uniformly previsibly order-bounded sequence of jump-free processes with domain $\mathcal{S}$ which is order*-convergent to a jump-free process $\boldsymbol{u}$. Show that $\lim_{n\to\infty}\int_{\mathcal{S}}\boldsymbol{u}_n d\boldsymbol{v}=\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}$ for every near-simple integrator $\boldsymbol{v}$ with domain $\mathcal{S}$.

645Y Further exercises (a) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. (i) Show that $\preccurlyeq$ (645Bc) is transitive and reflexive, that is, is a pre-order on $M_{\text{fa}}=M_{\text{fa}}(\mathcal{S})$ in the sense of 511A. (ii) Show that if $\equiv$ is the associated equivalence relation, then the set $M_{\text{fa}}/\!\equiv$ of equivalence classes can be thought of as a partially ordered linear space in the sense of §351. (iii) Taking $M_{\text{bv}}\subseteq M_{\text{fa}}$ to be the space of processes of bounded variation, show that if $\boldsymbol{v}\in M_{\text{bv}}$ and $\boldsymbol{v}'\equiv\boldsymbol{v}$ then $\boldsymbol{v}'\in M_{\text{bv}}$. (iv) Show that the image of M_{bv} in $M/\!\equiv$ is a Dedekind complete Riesz space in the sense of §353.

(b) Find an example of a sublattice $\mathcal{S}$ with a least element and an order-bounded fully adapted process $\boldsymbol{x}=\langle x_\sigma\rangle_{\sigma\in\mathcal{S}}$ such that $x_{\min\mathcal{S}}=0$ but $\boldsymbol{x}\notin M_{\text{po-b}}(\mathcal{S})$.

645 Notes and comments In 645P we have at last arrived at something like the standard stochastic integral as described in Rogers & Williams 00 and Protter 05. In particular, we have a dominated convergence theorem (645T). The S-integral is *not* an extension of the Riemann-sum stochastic integral of §613. They are linked by the formula $\oint\boldsymbol{u}_<\,d\boldsymbol{v}=\int\boldsymbol{u}\,d\boldsymbol{v}$ of 645Ra. But integrands for the Riemann-sum integral need not be previsible, and integrators need not be near-simple. It is true that we shall often have $\oint\boldsymbol{u}_<\,d\boldsymbol{v}=\int\boldsymbol{u}_<\,d\boldsymbol{v}$ (645R(a-ii)), but this is a touch accidental, and the theorems I have offered concerning the Riemann-sum integral are mostly unconcerned with this phenomenon.

I said in the introduction to this section that I was trying to define an integral which would look like a Lebesgue-Stieltjes integral. For a non-decreasing integrator, the S-integral is indeed of this type, at least if $T=[0,\infty[$. But the purpose of this volume is to look at integration with respect to martingales, and here the analogy is too weak to be useful. Only if we suppress martingales altogether, as in 645Xb or 649L, do we get a direct correspondence. The wonder is that we can devise any kind of sequentially smooth integral with a martingale integrator.

The definition of $M^0_{\text{S-i}}$ in 645F corresponds to a kind of universal measurability, as in 645Xb, but with an extra boundedness condition. Typically, theorems involving the S-integral, when they speak of uniformly order-bounded families, require families included in sets $\{\boldsymbol{x} : |\boldsymbol{x}| \times \mathbf{1}_< \le \boldsymbol{u}_<\}$ where $\boldsymbol{u}$ is moderately oscillatory, rather than just $\{\boldsymbol{x} : \sup |\boldsymbol{x}| \le \bar{w}\}$.

This section is based on a discussion of the properties of previsible versions $\boldsymbol{u}_<$ of moderately oscillatory processes $\boldsymbol{u}$. It is therefore worth noting that in the context of the principal results here, it would have been enough to start from near-simple processes $\boldsymbol{u}$ (642M).

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646 Basic properties of the S-integral

Having defined the S-integral as an adaptation of the Riemann-sum integral for previsible processes (645O, 645Pa, 645I), it is natural to look for parallels to the properties of the Riemann-sum integral set out in Chapters 61-63. After a few easy remarks (646B-646D), I embark on the question of splitting a domain $\mathcal{S}$ into $\mathcal{S} \wedge \tau$ and $\mathcal{S} \vee \tau$ (646J). This leads naturally to an examination of indefinite S-integrals (646K), which I approach through a result on capped-stake variation sets for martingale integrators (646P). We have a change-of-variable theorem (646R), a formula for jumps in an indefinite S-integral (646S) and a version of Itô's formula (646T).

646A Notation As in §645, I shall be calling on a substantial part of the special notation developed in this volume. $(\mathfrak{A}, \bar{\mu}, T, \langle \mathfrak{A}_t \rangle_{t \in T}, \mathcal{T}, \langle \mathfrak{A}_\tau \rangle_{\tau \in \mathcal{T}})$ will be a stochastic integration structure, with its associated Riesz space $L^0 = L^0(\mathfrak{A})$, endowed with the linear space topology of convergence in measure and the defining F-seminorm θ where $\theta(w) = \mathbb{E}(|w| \wedge \chi 1)$ for $w \in L^0$ (613Ba). I write $L^1_{\bar{\mu}} = L^1(\mathfrak{A}, \bar{\mu})$ for the L-space of members of L^0 with finite expectation. For $\sigma \in \mathcal{T}$, $P_\sigma : L^1_{\bar{\mu}} \to L^1_{\bar{\mu}}$ will be the conditional expectation associated with $\mathfrak{A}_\sigma$. We have the algebras $\mathfrak{A}_{\mathcal{S}<\tau}$ for sublattices $\mathcal{S}$ of $\mathcal{T}$ and $\tau \in \mathcal{T}$ (641B). If $\mathcal{S}$ is a sublattice of $\mathcal{T}$ and $\tau \in \mathcal{S}$, I write $\mathcal{S} \wedge \tau$ and $\mathcal{S} \vee \tau$ for $\{\sigma \wedge \tau : \sigma \in \mathcal{S}\}$ and $\{\sigma \vee \tau : \sigma \in \mathcal{S}\}$ respectively, and $\mathcal{I}(\mathcal{S})$ for the upwards-directed set of finite sublattices of $\mathcal{S}$. $\mathbf{1}^{(\mathcal{S})}$ will be the constant process with domain $\mathcal{S}$ and value $\chi 1$.

For a process $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}} \in (L^0)^{\mathcal{S}}$, I write $[\![\boldsymbol{u} \ne \mathbf{0}]\!]$ for $\sup_{\sigma \in \mathcal{S}} [\![u_\sigma \ne 0]\!]$ (612Sb); if $\boldsymbol{u}$ is order-bounded, I write $\sup |\boldsymbol{u}|$ for $\sup_{\sigma \in \mathcal{S}} |u_\sigma|$; if $\sup |\boldsymbol{u}| \in L^\infty(\mathfrak{A})$, I write $\|\boldsymbol{u}\|_\infty$ for $\| \sup |\boldsymbol{u}| \|_\infty = \sup_{\sigma \in \mathcal{S}} \|u_\sigma\|_\infty$. If $\boldsymbol{u}$ and $\boldsymbol{v}$ are fully adapted processes and I is a finite sublattice of $\operatorname{dom} \boldsymbol{u} \cap \operatorname{dom} \boldsymbol{v}$, then $S_I(\boldsymbol{u}, d\boldsymbol{v})$ is the Riemann sum described in 613Fb. If $\boldsymbol{v}$ is a fully adapted process and $\mathcal{S}$ is a sublattice of $\operatorname{dom} \boldsymbol{v}$, $Q_{\mathcal{S}}(d\boldsymbol{v})$ is the capped-stake variation set described in 616B. For processes $\boldsymbol{v}$, $\boldsymbol{v}'$ with the same domain I say that $\boldsymbol{v} \preccurlyeq \boldsymbol{v}'$ if $\boldsymbol{v}' - \boldsymbol{v}$ is non-decreasing (645Bc), and in this context $\boldsymbol{v} \equiv \boldsymbol{v}'$ if $\boldsymbol{v}' - \boldsymbol{v}$ is constant. An L^2-process is a process $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ such that $u_\sigma^2 \in L^1_{\bar{\mu}}$ for every $\sigma \in \mathcal{S}$ (622Ca). If $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ is a process and $z \in L^0(\mathfrak{A} \cap \bigcap_{\sigma \in \mathcal{S}} \mathfrak{A}_\sigma)$, then $z\boldsymbol{u} = \langle z \times u_\sigma \rangle_{\sigma \in \mathcal{S}}$ (612De). For a moderately oscillatory process $\boldsymbol{u}$, $\boldsymbol{u}_<$ is its previsible version (641L).

For a sublattice $\mathcal{S}$ of $\mathcal{T}$, $M_{\text{fa}}(\mathcal{S})$ is the space of fully adapted processes with domain $\mathcal{S}$, $M_{\text{simp}}(\mathcal{S}) \subseteq M_{\text{fa}}(\mathcal{S})$ the space of simple processes (612L), $M_{\text{o-b}}(\mathcal{S})$ the space of order-bounded processes (614Fc), $M_{\text{mo}}(\mathcal{S})$ the space of moderately oscillatory processes (615Fa), $M_{\text{n-s}}(\mathcal{S})$ the space of near-simple processes (631Fa), $M^{\uparrow}_{\text{n-s}}(\mathcal{S})$ the cone of non-negative non-decreasing near-simple processes (644Bb), $M_{\text{j-f}}(\mathcal{S})$ the space of jump-free processes (618G) and $M_{\text{po-b}}(\mathcal{S})$ the space of previsibly order-bounded processes (645Ba). On $M_{\text{o-b}}(\mathcal{S})$ we have the ucp topology (615B) and on $M_{\text{po-b}}(\mathcal{S})$ we have the S-integration topology $\mathfrak{T}_{\text{S-i}}$; $M^0_{\text{S-i}}(\mathcal{S})$ is the $\mathfrak{T}_{\text{S-i}}$-closure of $\{\boldsymbol{u}_< : \boldsymbol{u} \in M_{\text{mo}}(\mathcal{S})\}$ (645F), and $M_{\text{S-i}}(\mathcal{S})$ is $\{\boldsymbol{x} : \boldsymbol{x} \in M_{\text{o-b}}(\mathcal{S}),\ \boldsymbol{x} \times \mathbf{1}^{(\mathcal{S})}_< \in M^0_{\text{S-i}}(\mathcal{S})\}$ (645Fc). If $\boldsymbol{v} \in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$, $\widehat{\theta}^{\#}_{\boldsymbol{v}}$ is the corresponding F-seminorm on $M_{\text{po-b}}(\mathcal{S})$ (645Bb).

Finally, we shall be looking at both Riemann-sum integrals $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$ (613L) and S-integrals $\oint_{\mathcal{S}} \boldsymbol{x}\, d\boldsymbol{v}$ (645P). For the former case, our default assumption is that $\boldsymbol{u}$ is moderately oscillatory and $\boldsymbol{v}$ is an integrator (616K); in the latter, that $\langle \mathfrak{A}_t \rangle_{t \in T}$ is right-continuous (632B), $\mathcal{S}$ is order-convex, $\boldsymbol{x} \in M_{\text{S-i}}(\mathcal{S})$ and $\boldsymbol{v}$ is a near-simple integrator.

646B Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. If $\boldsymbol{x} = \langle x_\sigma \rangle_{\sigma \in \mathcal{S}} \in M^0_{\text{S-i}}(\mathcal{S})$ then $x_\tau \in L^0(\mathfrak{A}_{\mathcal{S}<\tau})$ for every $\tau \in \mathcal{S}$.

proof Define $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ by saying that $v_\sigma = \chi[\![\tau \le \sigma]\!]$ for $\sigma \in \mathcal{S}$. Then $\boldsymbol{v}$ is a non-negative order-bounded non-decreasing simple process with breakpoint string (τ), so belongs to $M^{\uparrow}_{\text{n-s}}(\mathcal{S})$. If $\epsilon > 0$, there is a $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}} \in M_{\text{mo}}(\mathcal{S})$ such that $\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x} - \boldsymbol{u}_<) < \epsilon$. Now there is a uniformly order-bounded non-decreasing sequence $\langle \boldsymbol{u}_n\rangle_{n\in\mathbb{N}} = \langle\langle u_{n\sigma}\rangle_{\sigma\in\mathcal{S}}\rangle_{n\in\mathbb{N}}$ of non-negative moderately oscillatory processes such that $|\boldsymbol{x} - \boldsymbol{u}_<| \le \sup_{n\in\mathbb{N}} \boldsymbol{u}_{n<}$ and $\sup_{n\in\mathbb{N}} \theta(\int_{\mathcal{S}} \boldsymbol{u}_n\, d\boldsymbol{v}) \le \epsilon$. In this case, $|x_\tau - u_{<\tau}| \le \sup_{n\in\mathbb{N}} u_{n<\tau}$, while $\langle u_{n<\tau}\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence and $\sup_{n\in\mathbb{N}} \theta(u_{n<\tau}) \le \epsilon$ (using 641J). So $\theta(x_\tau - u_{<\tau}) \le \epsilon$. We know also that $u_{<\tau} \in L^0(\mathfrak{A}_{\mathcal{S}<\tau})$ (641G(a-i)). As ϵ is arbitrary and $L^0(\mathfrak{A}_{\mathcal{S}<\tau})$ is closed, $x_\tau \in L^0(\mathfrak{A}_{\mathcal{S}<\tau})$.

646C Proposition Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$, $\boldsymbol{x}$ an S-integrable process and $\boldsymbol{v}$ a near-simple integrator, both with domain $\mathcal{S}$. Then $[\![\oint_{\mathcal{S}} \boldsymbol{x}\, d\boldsymbol{v} \ne 0]\!] \subseteq [\![\boldsymbol{v} \ne \boldsymbol{0}]\!]$.

proof Set $a = \inf_{\sigma\in\mathcal{S}} [\![v_\sigma = 0]\!]$ and $z = \oint_{\mathcal{S}} \boldsymbol{x}\, d\boldsymbol{v} = \oint_{\mathcal{S}} \boldsymbol{x} \times \boldsymbol{1}^{(\mathcal{S})}_{<}\, d\boldsymbol{v}$. By the definition of the S-integral in 645O-645P, there is for any $\epsilon > 0$ a $\boldsymbol{u} \in M_{\text{mo}}(\mathcal{S})$ such that $\theta(z - \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}) \le \epsilon$, so that $\theta(z \times \chi a - \chi a \times \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}) \le \epsilon$. But $\chi a \times \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v} = 0$ (613Ld), so $\theta(z \times \chi a) \le \epsilon$. As ϵ is arbitrary, $\theta(z \times \chi a) = 0$ and $[\![z \ne 0]\!] \subseteq 1 \setminus a = [\![\boldsymbol{v} \ne \boldsymbol{0}]\!]$.

Remark It seems to be harder to match the other half of 613Ld; see 647J. But we have the following easy remark.

646D Proposition Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$, $\boldsymbol{x}$ a member of $M_{\text{S-i}}(\mathcal{S})$, and $\boldsymbol{v}$ a near-simple integrator with domain $\mathcal{S}$. If $z \in L^0(\mathfrak{A} \cap \bigcap_{\sigma\in\mathcal{S}} \mathfrak{A}_\sigma)$, then

$$\oint_{\mathcal{S}} z\boldsymbol{x}\, d\boldsymbol{v} = \oint_{\mathcal{S}} \boldsymbol{x}\, d(z\boldsymbol{v}) = z \times \oint_{\mathcal{S}} \boldsymbol{x}\, d\boldsymbol{v}.$$

proof (a) To begin with, suppose that $\boldsymbol{x} \in M^0_{\text{S-i}} = M^0_{\text{S-i}}(\mathcal{S})$. In this case, $z\boldsymbol{x} \in M^0_{\text{S-i}}$ (645Ka), and if $\boldsymbol{u}^* \in M^+_{\text{mo}} = M_{\text{mo}}(\mathcal{S})^+$ is such that $|\boldsymbol{x}| \le \boldsymbol{u}^*_<$, then $|z\boldsymbol{x}| \le (|z|\boldsymbol{u}^*)_<$ (see the proof of 645D(a-iv)). Also $z\boldsymbol{v}$ is a near-simple integrator (631F(a-v), 616P(b-iv)). Let $\epsilon > 0$. Then there is a $\delta > 0$ such that $\theta(z \times u) \le \epsilon$ whenever $\theta(u) \le \delta$. Setting $A = \{\boldsymbol{u} : \boldsymbol{u} \in M_{\text{mo}},\ |\boldsymbol{u}| \le \boldsymbol{u}^*\}$ and $A' = \{\boldsymbol{u} : \boldsymbol{u} \in M_{\text{mo}},\ |\boldsymbol{u}| \le |z|\boldsymbol{u}^*\}$, we have $\mathfrak{T}_{\text{S-i}}$-neighbourhoods G, G' of $\boldsymbol{x}$, $z\boldsymbol{x}$ respectively such that

$$\theta(\oint_{\mathcal{S}} \boldsymbol{x}\, d(z\boldsymbol{v}) - \int_{\mathcal{S}} \boldsymbol{u}\, d(z\boldsymbol{v})) \le \epsilon \text{ whenever } \boldsymbol{u} \in A \text{ and } \boldsymbol{u}_< \in G,$$

$$\theta(\oint_{\mathcal{S}} \boldsymbol{x}\, d\boldsymbol{v} - \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}) \le \delta \text{ whenever } \boldsymbol{u} \in A \text{ and } \boldsymbol{u}_< \in G,$$

$$\theta(\oint_{\mathcal{S}} z\boldsymbol{x}\, d\boldsymbol{v} - \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}) \le \epsilon \text{ whenever } \boldsymbol{u} \in A' \text{ and } \boldsymbol{u}_< \in G';$$

and because $\boldsymbol{x}' \mapsto z\boldsymbol{x}'$ is continuous (645E(a-vi)), we can suppose that $z\boldsymbol{x}' \in G'$ whenever $\boldsymbol{x}' \in G$. Now there is a $\boldsymbol{u} \in A$ such that $\boldsymbol{u}_< \in G$, in which case we shall have

$$\theta(\oint_{\mathcal{S}} \boldsymbol{x}\, d(z\boldsymbol{v}) - \int_{\mathcal{S}} \boldsymbol{u}\, d(z\boldsymbol{v})) \le \epsilon,$$

$$\theta(z \times \oint_{\mathcal{S}} \boldsymbol{x}\, d\boldsymbol{v} - z \times \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}) \le \epsilon,$$

$$\theta(\oint_{\mathcal{S}} z\boldsymbol{x}\, d\boldsymbol{v} - \int_{\mathcal{S}} z\boldsymbol{u}\, d\boldsymbol{v}) \le \epsilon,$$

and moreover

$$\int_{\mathcal{S}} \boldsymbol{u}\, d(z\boldsymbol{v}) = z \times \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v} = \int_{\mathcal{S}} z\boldsymbol{u}\, d\boldsymbol{v}$$

(613L(b-ii)). So

$$\theta(\oint_{\mathcal{S}} \boldsymbol{x}\, d(z\boldsymbol{v}) - z \times \oint_{\mathcal{S}} \boldsymbol{x}\, d\boldsymbol{v}) \le 2\epsilon, \quad \theta(\oint_{\mathcal{S}} z\boldsymbol{x}\, d\boldsymbol{v} - z \times \oint_{\mathcal{S}} \boldsymbol{x}\, d\boldsymbol{v}) \le 2\epsilon;$$

as ϵ is arbitrary, the three expressions are equal.

(b) For the general case of $\boldsymbol{x} \in M_{\text{S-i}}(\mathcal{S})$, we have

$$\oint_{\mathcal{S}} z\boldsymbol{x}\, d\boldsymbol{v} = \oint_{\mathcal{S}} (z\boldsymbol{x}) \times \boldsymbol{1}^{(\mathcal{S})}_{<}\, d\boldsymbol{v} = \oint_{\mathcal{S}} z(\boldsymbol{x} \times \boldsymbol{1}^{(\mathcal{S})}_{<}) d\boldsymbol{v},$$

$$\oint_{\mathcal{S}} \boldsymbol{x}\, d(z\boldsymbol{v}) = \oint_{\mathcal{S}} \boldsymbol{x} \times \boldsymbol{1}^{(\mathcal{S})}_{<}\, d(z\boldsymbol{v}), \quad z \times \oint_{\mathcal{S}} \boldsymbol{x}\, d\boldsymbol{v} = z \times \oint_{\mathcal{S}} \boldsymbol{x} \times \boldsymbol{1}^{(\mathcal{S})}_{<}\, d\boldsymbol{v},$$

while $\boldsymbol{x} \times \boldsymbol{1}^{(\mathcal{S})}_{<} \in M^0_{\text{S-i}}$, so (a) tells us that these three terms are equal.

646E It will be convenient to have a partial expression of the continuity of S-integration in a more precise form than those offered in §645.

Lemma Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous, $\mathcal{S}$ is an order-convex sublattice of $\mathcal{T}$, and $\boldsymbol{v}\in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$. If $\boldsymbol{x}\in M^0_{\text{S-i}}(\mathcal{S})$, then $\theta(\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{v})\le\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x})$.

proof Let $\epsilon>0$. Then there is an order-bounded non-decreasing sequence $\langle\boldsymbol{u}_n\rangle_{n\in\mathbb{N}}$ in $M_{\text{mo}}(\mathcal{S})^+$ such that $|\boldsymbol{x}|\le\sup_{n\in\mathbb{N}}\boldsymbol{u}_{n<}$ and $\sup_{n\in\mathbb{N}}\theta(\int_{\mathcal{S}}\boldsymbol{u}_n d\boldsymbol{v})\le\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x})+\epsilon$. We are supposing that $\boldsymbol{x}$ is previsibly order-bounded, so there is a $\boldsymbol{u}\in M_{\text{mo}}(\mathcal{S})^+$ such that $|\boldsymbol{x}|\le\boldsymbol{u}_<$. Set $\boldsymbol{u}'_n=\boldsymbol{u}\wedge\boldsymbol{u}_n$ for each $n\in\mathbb{N}$. Then

$$|\boldsymbol{x}|\le\boldsymbol{u}_<\wedge\sup_{n\in\mathbb{N}}\boldsymbol{u}_{n<}=\sup_{n\in\mathbb{N}}(\boldsymbol{u}_<\wedge\boldsymbol{u}_{n<})=\sup_{n\in\mathbb{N}}\boldsymbol{u}'_{n<}$$

(641G(e-i)), so

$$0\le|\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{v}|\le\oint_{\mathcal{S}}|\boldsymbol{x}|d\boldsymbol{v}\le\oint_{\mathcal{S}}(\sup_{n\in\mathbb{N}}\boldsymbol{u}'_{n<})d\boldsymbol{v}$$

(because $\oint_{\mathcal{S}}..d\boldsymbol{v}$ is a positive linear operator, by 645Rc)

$$=\sup_{n\in\mathbb{N}}\oint_{\mathcal{S}}\boldsymbol{u}'_{n<}d\boldsymbol{v}$$

(645T, because $\langle\boldsymbol{u}'_{n<}\rangle_{n\in\mathbb{N}}$ is uniformly previsibly order-bounded and order*-convergent to its supremum)

$$=\sup_{n\in\mathbb{N}}\int_{\mathcal{S}}\boldsymbol{u}'_n d\boldsymbol{v}\le\sup_{n\in\mathbb{N}}\int_{\mathcal{S}}\boldsymbol{u}_n d\boldsymbol{v}$$

and

$$\theta(\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{v})\le\theta(\sup_{n\in\mathbb{N}}\int_{\mathcal{S}}\boldsymbol{u}_n d\boldsymbol{v})=\sup_{n\in\mathbb{N}}\theta(\int_{\mathcal{S}}\boldsymbol{u}_n d\boldsymbol{v})\le\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x})+\epsilon.$$

As ϵ is arbitrary, we have the result.

646F Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and τ a member of $\mathcal{S}$. Suppose that $\boldsymbol{u}'=\langle u'_\sigma\rangle_{\sigma\in\mathcal{S}\wedge\tau}$ and $\boldsymbol{u}''=\langle u''_\sigma\rangle_{\sigma\in\mathcal{S}\vee\tau}$ are families in L^0. Define $R(\boldsymbol{u}',\boldsymbol{u}'')\in(L^0)^{\mathcal{S}}$ by saying that $R(\boldsymbol{u}',\boldsymbol{u}'')=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ where

$$u_\sigma=u'_{\sigma\wedge\tau}\times\chi[\![\sigma<\tau]\!]+u''_{\sigma\vee\tau}\times\chi[\![\tau\le\sigma]\!]$$

for $\sigma\in\mathcal{S}$.

(a)(i) $R(\boldsymbol{u}',\boldsymbol{u}'')\restriction\mathcal{S}\wedge\tau=\boldsymbol{u}'+\boldsymbol{z}$, where $\boldsymbol{z}=\langle(u''_\tau-u'_\tau)\times\chi[\![\sigma=\tau]\!]\rangle_{\sigma\in\mathcal{S}\wedge\tau}$.

(ii) $R(\boldsymbol{u}',\boldsymbol{u}'')\restriction\mathcal{S}\vee\tau=\boldsymbol{u}''$.

(b) Regarded as an operator from $(L^0)^{\mathcal{S}\wedge\tau}\times(L^0)^{\mathcal{S}\vee\tau}$ to $(L^0)^{\mathcal{S}}$, R is linear, positive and order-continuous.

(c) If $\boldsymbol{u}'$ and $\boldsymbol{u}''$ are fully adapted, then $R(\boldsymbol{u}',\boldsymbol{u}'')$ is fully adapted.

(d) If $\boldsymbol{u}'$ and $\boldsymbol{u}''$ are order-bounded, then $R(\boldsymbol{u}',\boldsymbol{u}'')$ is order-bounded and $\sup|R(\boldsymbol{u}',\boldsymbol{u}'')|\le\sup|\boldsymbol{u}'|\vee\sup|\boldsymbol{u}''|$.

(e) Suppose that $\boldsymbol{u}'$ and $\boldsymbol{u}''$ are moderately oscillatory.

(i) $R(\boldsymbol{u}',\boldsymbol{u}'')$ is moderately oscillatory.

(ii) $\boldsymbol{u}'_<=R(\boldsymbol{u}',\boldsymbol{u}'')_<\restriction\mathcal{S}\wedge\tau$.

(f) If $\boldsymbol{u}'$ and $\boldsymbol{u}''$ are near-simple, $R(\boldsymbol{u}',\boldsymbol{u}'')$ is near-simple.

(g) Suppose that $\boldsymbol{u}'$ and $\boldsymbol{u}''$ are moderately oscillatory, and that $\boldsymbol{v}$ is an integrator with domain $\mathcal{S}$. Then

$$\int_{\mathcal{S}}R(\boldsymbol{u}',\boldsymbol{u}'')d\boldsymbol{v}=\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}'\,d\boldsymbol{v}+\int_{\mathcal{S}\vee\tau}\boldsymbol{u}''\,d\boldsymbol{v}.$$

proof Throughout the proof I will write $\boldsymbol{u}$ for $R(\boldsymbol{u}',\boldsymbol{u}'')$.

(a)(i) For $\sigma\in\mathcal{S}\wedge\tau$,

$$u'_{\sigma\wedge\tau}\times\chi[\![\sigma<\tau]\!]+u''_{\sigma\vee\tau}\times\chi[\![\tau\le\sigma]\!]=u'_\sigma+(u''_\tau-u'_\tau)\times\chi[\![\sigma=\tau]\!].$$

(ii) For $\sigma\in\mathcal{S}\vee\tau$,

$$u'_{\sigma\wedge\tau}\times\chi[\![\sigma<\tau]\!]+u''_{\sigma\vee\tau}\times\chi[\![\tau\le\sigma]\!]=u''_\sigma.$$

(b) We just have to observe that, for any $\sigma \in \mathcal{S}$,

$$(\boldsymbol{u}', \boldsymbol{u}'') \mapsto (u'_{\sigma\wedge\tau}, u''_{\sigma\vee\tau}) : (L^0)^{\mathcal{S}\wedge\tau} \times (L^0)^{\mathcal{S}\vee\tau} \to L^0 \times L^0$$

is linear, positive and order-continuous, and so is

$$(u', u'') \mapsto u' \times \chi[\![\sigma < \tau]\!] + u'' \times \chi[\![\tau \le \sigma]\!] : L^0 \times L^0 \to L^0.$$

(c)(i) If $\sigma \in \mathcal{S}$, then $u'_{\sigma\wedge\tau} \in L^0(\mathfrak{A}_{\sigma\wedge\tau}) \subseteq L^0(\mathfrak{A}_\sigma)$; $[\![\sigma < \tau]\!] \in \mathfrak{A}_\sigma$ so $\chi[\![\sigma < \tau]\!] \in L^0(\mathfrak{A}_\sigma)$; and

$$u''_{\sigma\vee\tau} \times \chi[\![\tau \le \sigma]\!] = u''_{\sigma\vee\tau} \times \chi[\![\sigma \vee \tau = \sigma]\!]$$

(611E(a-ii-β))

$$\in L^0(\mathfrak{A}_\sigma)$$

by 612C. So $u_\sigma \in L^0(\mathfrak{A}_\sigma)$.

(ii) If σ, $\sigma' \in \mathcal{S}$ then

$$[\![\sigma = \sigma']\!] \subseteq [\![\sigma \wedge \tau = \sigma' \wedge \tau]\!] \cap [\![\sigma \vee \tau = \sigma' \vee \tau]\!] \setminus (([\![\sigma < \tau]\!] \triangle [\![\sigma' < \tau]\!]) \cup ([\![\tau \le \sigma]\!] \triangle [\![\tau \le \sigma']\!]))$$

(611Ec, *passim*)

$$\subseteq [\![u'_{\sigma\wedge\tau} = u'_{\sigma'\wedge\tau}]\!] \cap [\![u''_{\sigma\vee\tau} = u''_{\sigma'\vee\tau}]\!] \cap [\![\chi[\![\sigma < \tau]\!] = \chi[\![\sigma' < \tau]\!]]\!] \cap [\![\chi[\![\tau \le \sigma]\!] = \chi[\![\tau \le \sigma']\!]]\!]$$
$$\subseteq [\![u_\sigma = u_{\sigma'}]\!].$$

So $R(\boldsymbol{u}', \boldsymbol{u}'') = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$ is fully adapted.

(d) Setting $\bar{u} = \sup |\boldsymbol{u}'| \vee \sup |\boldsymbol{u}''|$, we have

$$\begin{aligned}|u_\sigma| &= |u'_{\sigma\wedge\tau} \times \chi[\![\sigma < \tau]\!] + u''_{\sigma\vee\tau} \times \chi[\![\tau \le \sigma]\!]| \\ &= (|u'_{\sigma\wedge\tau}| \times \chi[\![\sigma < \tau]\!]) \vee (|u''_{\sigma\vee\tau}| \times \chi[\![\tau \le \sigma]\!]) \\ &\le (\bar{u} \times \chi[\![\sigma < \tau]\!]) \vee (\bar{u} \times \chi[\![\tau \le \sigma]\!]) = \bar{u}\end{aligned}$$

for every $\sigma \in \mathcal{S}$.

(e)(i) Use (a). The process $\boldsymbol{z} = \langle (u''_\tau - u'_\tau) \times \chi[\![\sigma = \tau]\!] \rangle_{\sigma\in\mathcal{S}\wedge\tau}$ is simple (612Ka), so $\boldsymbol{u}\restriction\mathcal{S} \wedge \tau = \boldsymbol{u}' + \boldsymbol{z}$ is moderately oscillatory, while $\boldsymbol{u}\restriction\mathcal{S} \vee \tau = \boldsymbol{u}''$ is also moderately oscillatory. By 615F(a-v), $\boldsymbol{u}$ is moderately oscillatory.

(ii) By the formula in 641Ia with $\tau_0 = \tau$, $\boldsymbol{z}_< = \boldsymbol{0}$, so

$$\boldsymbol{u}_<\restriction\mathcal{S} \wedge \tau = (\boldsymbol{u}\restriction\mathcal{S} \wedge \tau)_<$$

(641G(c-ii))

$$= \boldsymbol{u}'_< + \boldsymbol{z}_<$$

(641G(e-i))

$$= \boldsymbol{u}'.$$

(f) Again, if $\boldsymbol{u}'$ and $\boldsymbol{u}''$ are near-simple, so are $\boldsymbol{u}\restriction\mathcal{S} \vee \tau = \boldsymbol{u}''$ and $\boldsymbol{u}\restriction\mathcal{S} \wedge \tau = \boldsymbol{u}' + \boldsymbol{z}$, because $\boldsymbol{z}$ is simple; so $R(\boldsymbol{u}', \boldsymbol{u}'')$ is near-simple, by 631F(a-iv).

(g) Since $\boldsymbol{u}$ is moderately oscillatory, $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$ is defined and equal to $\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\boldsymbol{v} + \int_{\mathcal{S}\vee\tau} \boldsymbol{u}\, d\boldsymbol{v}$ (613J(c-i)). Now $\int_{\mathcal{S}\vee\tau} \boldsymbol{u}\, d\boldsymbol{v} = \int_{\mathcal{S}\vee\tau} \boldsymbol{u}''\, d\boldsymbol{v}$ because $\boldsymbol{u}'' = \boldsymbol{u}\restriction\mathcal{S} \vee \tau$. As for $\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\boldsymbol{v}$, observe that if $v_\downarrow = \lim_{\sigma\downarrow\mathcal{S}} v_\sigma$ then

$$\int_{\mathcal{S}\wedge\tau} \boldsymbol{z}\,d\boldsymbol{v} = \lim_{\sigma\downarrow\mathcal{S}} u''_\tau \times \chi[\![\sigma=\tau]\!] \times (v_\tau - v_\downarrow) + (u''_\tau - u'_\tau)\times(v_\tau - v_\tau)$$

(614C)

$$= u''_\tau \times \chi(\inf_{\sigma\in\mathcal{S}} [\![\sigma=\tau]\!]) \times (v_\tau - v_\downarrow)$$

$$= 0$$

because $\inf_{\sigma\in\mathcal{S}} [\![\sigma=\tau]\!] \subseteq [\![v_\downarrow = v_\tau]\!]$. So

$$\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\,d\boldsymbol{v} = \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}'\,d\boldsymbol{v} + \int_{\mathcal{S}\wedge\tau} \boldsymbol{z}\,d\boldsymbol{v} = \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}'\,d\boldsymbol{v}$$

and

$$\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v} = \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\,d\boldsymbol{v} + \int_{\mathcal{S}\vee\tau} \boldsymbol{u}\,d\boldsymbol{v} = \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}'\,d\boldsymbol{v} + \int_{\mathcal{S}\vee\tau} \boldsymbol{u}''\,d\boldsymbol{v},$$

as claimed.

646G Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and τ a member of $\mathcal{S}$. Suppose that $\boldsymbol{v}' = \langle v'_\sigma\rangle_{\sigma\in\mathcal{S}\wedge\tau} \in M^{\uparrow}_{\text{n-s}}(\mathcal{S}\wedge\tau)$ and $\boldsymbol{v}'' = \langle v''_\sigma\rangle_{\sigma\in\mathcal{S}\vee\tau} \in M^{\uparrow}_{\text{n-s}}(\mathcal{S}\vee\tau)$.

(a) There is a $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}} \in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$ such that $\boldsymbol{v}'' = \boldsymbol{v}\restriction\mathcal{S}\vee\tau$.

(b) There is a $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}} \in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$ such that

$$\boldsymbol{v}' = \boldsymbol{v}\restriction\mathcal{S}\wedge\tau, \quad \boldsymbol{v}'' \equiv \boldsymbol{v}\restriction\mathcal{S}\vee\tau.$$

(c) If $\boldsymbol{w} \in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$, there is a $\boldsymbol{v}^* \in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$ such that

$$\boldsymbol{w} \preccurlyeq \boldsymbol{v}^*, \quad \boldsymbol{v}' \preccurlyeq \boldsymbol{v}^*\restriction\mathcal{S}\wedge\tau, \quad \boldsymbol{v}'' \preccurlyeq \boldsymbol{v}^*\restriction\mathcal{S}\vee\tau.$$

proof (a) In the language of 646F, set $\boldsymbol{v} = R(\boldsymbol{0},\boldsymbol{v}'')$, where $\boldsymbol{0}$ is the zero process with domain $\mathcal{S}\wedge\tau$. By 646Ff, $\boldsymbol{v}$ is a near-simple process; the defining formula

$$v_\sigma = v''_{\sigma\vee\tau} \times [\![\tau \le \sigma]\!]$$

shows that $\boldsymbol{v}\restriction\mathcal{S}\vee\tau = \boldsymbol{v}''$ and that $\boldsymbol{v}$ is non-negative and non-decreasing. Now we know that $\boldsymbol{v}''$ is order-bounded (616Ib), so $\sup|\boldsymbol{v}| = \sup|\boldsymbol{v}''|$ is defined. Thus $\boldsymbol{v}$ is order-bounded; being monotonic, it is an integrator (616Ra).

(b) Set

$$\tilde{\boldsymbol{v}}' = \boldsymbol{v}'' + (v'_\tau - v''_\tau)\boldsymbol{1}\restriction\mathcal{S}\vee\tau = \langle v'_\tau + v''_\sigma - v''_\tau\rangle_{\sigma\in\mathcal{S}\vee\tau},$$

so that $\tilde{\boldsymbol{v}}' \in M^{\uparrow}_{\text{n-s}}(\mathcal{S}\vee\tau)$.

Now set $\boldsymbol{v} = R(\boldsymbol{v}',\tilde{\boldsymbol{v}}')$, where $R : (L^0)^{\mathcal{S}\wedge\tau} \times (L^0)^{\mathcal{S}\vee\tau} \to (L^0)^{\mathcal{S}}$ is defined as in 646F. By 646Ff, $\boldsymbol{v}$ is near-simple. $\boldsymbol{v}$ is non-negative because $\boldsymbol{v}'$ and $\tilde{\boldsymbol{v}}'$ are non-negative. $\boldsymbol{v}$ is non-decreasing because if $\sigma \le \sigma'$ in $\mathcal{S}$, $a = [\![\sigma' < \tau]\!]$, $b = [\![\sigma < \tau]\!] \setminus a$ and $c = [\![\tau \le \sigma]\!]$ then

$$\begin{aligned}
v_\sigma &= v'_{\sigma\wedge\tau} \times \chi a + v'_{\sigma\wedge\tau} \times \chi b + (v'_\tau - v''_\tau + v''_{\sigma\vee\tau}) \times \chi c \\
&\le v'_{\sigma'\wedge\tau} \times \chi a + v'_\tau \times \chi b + (v'_\tau - v''_\tau + v''_{\sigma'\vee\tau}) \times \chi c \\
&\le v'_{\sigma'\wedge\tau} \times \chi a + (v'_\tau - v''_\tau + v''_{\sigma'\vee\tau}) \times \chi b + (v'_\tau - v''_\tau + v''_{\sigma'\vee\tau}) \times \chi c = v_{\sigma'}.
\end{aligned}$$

If $\sigma \in \mathcal{S}\wedge\tau$, then

$$\begin{aligned}
v_\sigma &= v'_{\sigma\wedge\tau} \times \chi[\![\sigma < \tau]\!] + (v'_\tau - v''_\tau + v''_{\sigma\vee\tau}) \times \chi[\![\tau \le \sigma]\!] \\
&= v'_{\sigma\wedge\tau} \times \chi[\![\sigma < \tau]\!] + (v'_\tau - v''_\tau + v''_{\sigma\vee\tau}) \times \chi[\![\tau = \sigma]\!] \\
&= v'_{\sigma\wedge\tau} \times \chi[\![\sigma < \tau]\!] + v'_\tau \times \chi[\![\tau = \sigma]\!] = v'_\sigma,
\end{aligned}$$

so $\boldsymbol{v}' = \boldsymbol{v}\restriction\mathcal{S}\wedge\tau$. If $\sigma \le \sigma'$ in $\mathcal{S}\vee\tau$ then of course

$$v_{\sigma'} - v_\sigma = (v'_\tau - v''_\tau + v''_{\sigma'}) - (v'_\tau - v''_\tau + v''_\sigma) = v''_{\sigma'} - v''_\sigma,$$

so $\boldsymbol{v}\restriction\mathcal{S}\vee\tau \equiv \boldsymbol{v}''$. Finally, as in (a), $\boldsymbol{v}$ is order-bounded, therefore an integrator.

(c) Take $\boldsymbol{v}$ as in (b) and set $\boldsymbol{v}^* = \boldsymbol{v} + \boldsymbol{w}$.

646H Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and τ a member of $\mathcal{S}$. For $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}} \in (L^0)^{\mathcal{S}}$ define $R^*(\boldsymbol{u}) \in (L^0)^{\mathcal{S}\vee\tau}$ by saying that

$$R^*(\boldsymbol{u}) = \langle u_\sigma \times \chi[\![\tau < \sigma]\!]\rangle_{\sigma\in\mathcal{S}\vee\tau} = (\boldsymbol{u}\restriction\mathcal{S}\vee\tau) \times \mathbf{1}^{(\mathcal{S}\vee\tau)}_{<}.$$

(a)(i) $R^* : (L^0)^{\mathcal{S}} \to (L^0)^{\mathcal{S}\vee\tau}$ is an order-continuous f-algebra homomorphism.

(ii) If $\boldsymbol{u} \in (L^0)^{\mathcal{S}}$ is fully adapted, then $R^*(\boldsymbol{u})$ is fully adapted.

(iii) If $\boldsymbol{u} \in M_{\text{o-b}}(\mathcal{S})$, then $R^*(\boldsymbol{u}) \in M_{\text{o-b}}(\mathcal{S}\vee\tau)$ and $\sup|R^*(\boldsymbol{u})| \le \sup|\boldsymbol{u}|$.

(iv) If $\boldsymbol{u} \in M_{\text{mo}}(\mathcal{S})$ then $R^*(\boldsymbol{u}) \in M_{\text{mo}}(\mathcal{S}\vee\tau)$.

(v) If $\boldsymbol{x}$, $\boldsymbol{u} \in (L^0)^{\mathcal{S}}$, $\boldsymbol{x}\restriction\mathcal{S}\wedge\tau \le \boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ and $R^*(\boldsymbol{x}) \le \boldsymbol{u}\restriction\mathcal{S}\vee\tau$, then $\boldsymbol{x} \le \boldsymbol{u}$.

(b) If $\boldsymbol{u} \in M_{\text{mo}}(\mathcal{S})$, then $(\boldsymbol{u}\restriction\mathcal{S}\vee\tau)_< = R^*(\boldsymbol{u}_<)$.

(c) Suppose that $\boldsymbol{x} \in M_{\text{po-b}}(\mathcal{S})$. Write $\boldsymbol{x}'$ for $\boldsymbol{x}\restriction\mathcal{S}\wedge\tau$.

(i) $\boldsymbol{x}' \in M_{\text{po-b}}(\mathcal{S}\wedge\tau)$ and $R^*(\boldsymbol{x}) \in M_{\text{po-b}}(\mathcal{S}\vee\tau)$.

(ii) If $\boldsymbol{v} \in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$, and we set $\boldsymbol{v}' = \boldsymbol{v}\restriction\mathcal{S}\wedge\tau$ and $\boldsymbol{v}'' = \boldsymbol{v}\restriction\mathcal{S}\vee\tau$, then

$$\max(\widehat{\theta}^{\#}_{\boldsymbol{v}'}(\boldsymbol{x}'), \widehat{\theta}^{\#}_{\boldsymbol{v}''}R^*(\boldsymbol{x})) \le \widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x}) \le \widehat{\theta}^{\#}_{\boldsymbol{v}'}(\boldsymbol{x}') + \widehat{\theta}^{\#}_{\boldsymbol{v}''}R^*(\boldsymbol{x}).$$

(d) If $\boldsymbol{x} \in M_{\text{S-i}}(\mathcal{S})$ then $\boldsymbol{x}\restriction\mathcal{S}\wedge\tau \in M_{\text{S-i}}(\mathcal{S}\wedge\tau)$ and $\boldsymbol{x}\restriction\mathcal{S}\vee\tau \in M_{\text{S-i}}(\mathcal{S}\vee\tau)$.

proof (a)(i) Immediate from the facts that $[\![\tau < \sigma]\!] \in \mathfrak{A}_\sigma$ for all τ and σ, and if $a \in \mathfrak{A}$ then $u \mapsto u \times \chi a$ is an order-continuous f-algebra homomorphism from L^0 to itself.

(ii) If $\boldsymbol{u}$ is fully adapted, then $\boldsymbol{u}\restriction\mathcal{S}\vee\tau$ is fully adapted (612Dc). Next, $\boldsymbol{x} = \langle\chi[\![\tau < \sigma]\!]\rangle_{\sigma\in\mathcal{S}\vee\tau}$ is fully adapted, because if σ, $\sigma' \in \mathcal{S}\vee\tau$ then

$$[\![\chi[\![\tau < \sigma]\!] \ne \chi[\![\tau < \sigma']\!]]\!] = [\![\tau < \sigma]\!] \mathbin{\triangle} [\![\tau < \sigma']\!]$$

does not meet $[\![\sigma = \sigma']\!]$, by 611E(c-iv-β). So $R^*(\boldsymbol{u}) = (\boldsymbol{u}\restriction\mathcal{S}\vee\tau) \times \boldsymbol{x}$ is fully adapted.

(iii) This is just because $|u_\sigma \times \chi[\![\tau < \sigma]\!]| \le |u_\sigma| \le \sup|\boldsymbol{u}|$ for every $\sigma \in \mathcal{S}\wedge\tau$.

(iv) $\mathbf{1}^{(\mathcal{S}\vee\tau)}$ is constant therefore moderately oscillatory, so $\mathbf{1}^{(\mathcal{S}\vee\tau)}_<$ is moderately oscillatory (641L), while $\boldsymbol{u}\restriction\mathcal{S}\vee\tau$ is moderately oscillatory (615F(a-i)); so the product $R^*(\boldsymbol{u}) = (\boldsymbol{u}\restriction\mathcal{S}\vee\tau) \times \mathbf{1}^{(\mathcal{S}\vee\tau)}_<$ is moderately oscillatory (615F(a-iii)).

(v) Express $\boldsymbol{x}$, $\boldsymbol{u}$ as $\langle x_\sigma\rangle_{\sigma\in\mathcal{S}}$ and $\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$. For $\sigma \in \mathcal{S}$,

$$x_\sigma \times \chi[\![\sigma \le \tau]\!] = x_{\sigma\wedge\tau} \times \chi[\![\sigma \le \tau]\!] \le u_{\sigma\wedge\tau} \times \chi[\![\sigma \le \tau]\!] = u_\sigma \times \chi[\![\sigma \le \tau]\!]$$

and

$$x_\sigma \times \chi[\![\tau < \sigma]\!] = x_{\sigma\vee\tau} \times \chi[\![\tau < \sigma]\!] \le u_{\sigma\vee\tau} \times \chi[\![\tau < \sigma]\!]$$

(because $R^*(\boldsymbol{x}) \le \boldsymbol{u}\restriction\mathcal{S}\vee\tau$)

$$= u_\sigma \times \chi[\![\tau < \sigma]\!],$$

so $x_\sigma \le u_\sigma$.

(b) This is covered by 641G(c-ii).

(c)(i) Let $\boldsymbol{u} \in M_{\text{mo}}(\mathcal{S})^+$ be such that $|\boldsymbol{x}| \le \boldsymbol{u}_<$. Then $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau \in M_{\text{mo}}(\mathcal{S}\wedge\tau)$ and $\boldsymbol{u}\restriction\mathcal{S}\vee\tau \in M_{\text{mo}}(\mathcal{S}\vee\tau)$ (615F(a-i) again), while $(\boldsymbol{u}\restriction\mathcal{S}\wedge\tau)_< = \boldsymbol{u}_<\restriction\mathcal{S}\wedge\tau$ and $(\boldsymbol{u}\restriction\mathcal{S}\vee\tau)_< = R^*(\boldsymbol{u}_<)$, by 641Gc. So

$$|\boldsymbol{x}\restriction\mathcal{S}\wedge\tau| \le (\boldsymbol{u}\restriction\mathcal{S}\wedge\tau)_<, \quad \boldsymbol{x}\restriction\mathcal{S}\wedge\tau \in M_{\text{po-b}}(\mathcal{S}\wedge\tau),$$

$$|R^*(\boldsymbol{x})| = R^*(|\boldsymbol{x}|) \le R^*(\boldsymbol{u}_<) = (\boldsymbol{u}\restriction\mathcal{S}\vee\tau)_<, \quad R^*(\boldsymbol{x}) \in M_{\text{po-b}}(\mathcal{S}\vee\tau).$$

(ii) Observe first that $\boldsymbol{v}'$ and $\boldsymbol{v}''$ are non-negative non-decreasing near-simple integrators (see 631F(a-iv) and 616P(b-ii)).

($\boldsymbol{\alpha}$) Let $\langle\boldsymbol{u}_n\rangle_{n\in\mathbb{N}}$ be a uniformly order-bounded non-decreasing sequence of non-negative moderately oscillatory processes with domain $\mathcal{S}$ such that $|\boldsymbol{x}| \le \sup_{n\in\mathbb{N}}\boldsymbol{u}_{n<}$. Setting $\boldsymbol{x}' = \boldsymbol{x}\restriction\mathcal{S}\wedge\tau$, $\boldsymbol{v}' = \boldsymbol{v}\restriction\mathcal{S}\wedge\tau$ and $\boldsymbol{u}'_n = \boldsymbol{u}_n\restriction\mathcal{S}\wedge\tau$ for each n, we have $\boldsymbol{u}'_{n<} = \boldsymbol{u}_{n<}\restriction\mathcal{S}\wedge\tau$ for each n (641Gc again), so

$$|\boldsymbol{x}'| \le \sup_{n\in\mathbb{N}} \boldsymbol{u}_{n<}\restriction\mathcal{S}\wedge\tau = \sup_{n\in\mathbb{N}} \boldsymbol{u}'_{n<},$$

while $\langle\boldsymbol{u}'_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of non-negative moderately oscillatory processes (615F(a-i) once more); also $\sup|\boldsymbol{u}'_n| \le \sup|\boldsymbol{u}_n|$ for each n, so $\langle\boldsymbol{u}'_n\rangle_{n\in\mathbb{N}}$ is uniformly order-bounded. Accordingly

$$\begin{aligned}\widehat{\theta}^{\#}_{\boldsymbol{v}'}(\boldsymbol{x}') &\le \sup_{n\in\mathbb{N}} \theta(\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}'_n\,d\boldsymbol{v}') \\ &= \sup_{n\in\mathbb{N}} \theta(\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}_n\,d\boldsymbol{v}) \le \sup_{n\in\mathbb{N}} \theta(\int_{\mathcal{S}} \boldsymbol{u}_n\,d\boldsymbol{v})\end{aligned}$$

because $\boldsymbol{v}$ is non-decreasing and $\boldsymbol{u}_n$ is non-negative, so

$$0 \le \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}_n\,d\boldsymbol{v} = \int_{\mathcal{S}} \boldsymbol{u}_n\,d\boldsymbol{v} - \int_{\mathcal{S}\vee\tau} \boldsymbol{u}_n\,d\boldsymbol{v} \le \int_{\mathcal{S}} \boldsymbol{u}_n\,d\boldsymbol{v}$$

for every n. As $\langle\boldsymbol{u}_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\widehat{\theta}^{\#}_{\boldsymbol{v}'}(\boldsymbol{x}') \le \widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x})$.

(β) The same line of argument works in $\mathcal{S}\vee\tau$. Again, let $\langle\boldsymbol{u}_n\rangle_{n\in\mathbb{N}}$ be a uniformly order-bounded non-decreasing sequence of non-negative moderately oscillatory processes with domain $\mathcal{S}$ such that $|\boldsymbol{x}| \le \sup_{n\in\mathbb{N}} \boldsymbol{u}_{n<}$. This time, setting $\boldsymbol{v}'' = \boldsymbol{v}\restriction\mathcal{S}\vee\tau$ and $\boldsymbol{u}''_n = \boldsymbol{u}_n\restriction\mathcal{S}\vee\tau$ for each n, we have $\boldsymbol{u}''_{n<} = R^*(\boldsymbol{u}_{n<})$ for each n ((b) above), so

$$|R^*(\boldsymbol{x})| \le \sup_{n\in\mathbb{N}} R^*(\boldsymbol{u}_{n<})$$

(recall that R^* is an order-continuous lattice homomorphism with domain $(L^0)^{\mathcal{S}}$)

$$= \sup_{n\in\mathbb{N}} \boldsymbol{u}''_{n<}.$$

As in (α), $\langle\boldsymbol{u}''_n\rangle_{n\in\mathbb{N}}$ is a uniformly order-bounded non-decreasing sequence of non-negative near-simple processes. Accordingly

$$\begin{aligned}\widehat{\theta}^{\#}_{\boldsymbol{v}''}R^*(\boldsymbol{x}) &\le \sup_{n\in\mathbb{N}} \theta(\int_{\mathcal{S}\vee\tau} \boldsymbol{u}''_n\,d\boldsymbol{v}'') \\ &= \sup_{n\in\mathbb{N}} \theta(\int_{\mathcal{S}\vee\tau} \boldsymbol{u}_n\,d\boldsymbol{v}) \le \sup_{n\in\mathbb{N}} \theta(\int_{\mathcal{S}} \boldsymbol{u}_n\,d\boldsymbol{v}).\end{aligned}$$

As $\langle\boldsymbol{u}_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\widehat{\theta}^{\#}_{\boldsymbol{v}''}R^*(\boldsymbol{x}) \le \widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x})$.

(γ) Thus

$$\max(\widehat{\theta}^{\#}_{\boldsymbol{v}'}(\boldsymbol{x}'), \widehat{\theta}^{\#}_{\boldsymbol{v}''}R^*(\boldsymbol{x})) \le \widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x}).$$

As for the other inequality, let $\langle\boldsymbol{u}'_n\rangle_{n\in\mathbb{N}}$, $\langle\boldsymbol{u}''_n\rangle_{n\in\mathbb{N}}$ be uniformly order-bounded non-decreasing sequences in $M_{\mathrm{mo}}(\mathcal{S}\wedge\tau)^+$, $M_{\mathrm{mo}}(\mathcal{S}\vee\tau)^+$ respectively such that $|\boldsymbol{x}'| \le \sup_{n\in\mathbb{N}} \boldsymbol{u}'_{n<}$ and $|R^*(\boldsymbol{x})| \le \sup_{n\in\mathbb{N}} \boldsymbol{u}''_{n<}$. Set $\boldsymbol{u}_n = R(\boldsymbol{u}'_n, \boldsymbol{u}''_n)$ for each n, where R is the operator defined in 646F; then $\langle\boldsymbol{u}_n\rangle_{n\in\mathbb{N}}$ is a uniformly order-bounded non-decreasing sequence in $M_{\mathrm{mo}}(\mathcal{S})$ (646Fb, 646Fd, 646F(e-i)). Now $|\boldsymbol{x}| \le \sup_{n\in\mathbb{N}} \boldsymbol{u}_{n<}$. **P** Express $\boldsymbol{x}$ as $\langle x_\sigma\rangle_{\sigma\in\mathcal{S}}$, $\boldsymbol{u}_n$ as $\langle u_{n\sigma}\rangle_{\sigma\in\mathcal{S}}$, $\boldsymbol{u}_{n<}$ as $\langle u_{n<\sigma}\rangle_{\sigma\in\mathcal{S}}$ and $\boldsymbol{u}''_{n<}$ as $\langle u''_{n<\sigma}\rangle_{\sigma\in\mathcal{S}\wedge\tau}$ for each n. If $\sigma\in\mathcal{S}$, then

$$\begin{aligned}|x_\sigma| \times \chi[\![\sigma \le \tau]\!] &= |x_{\sigma\wedge\tau}| \times \chi[\![\sigma \le \tau]\!] \le \sup_{n\in\mathbb{N}} u_{n<(\sigma\wedge\tau)} \times \chi[\![\sigma \le \tau]\!] \\ &= \sup_{n\in\mathbb{N}} u_{n<\sigma} \times \chi[\![\sigma \le \tau]\!]\end{aligned}$$

because

$$|\boldsymbol{x}'| \le \sup_{n\in\mathbb{N}} \boldsymbol{u}'_{n<} = \sup_{n\in\mathbb{N}} \boldsymbol{u}_{n<}\restriction\mathcal{S}\wedge\tau$$

(641Gc once more) and $\boldsymbol{u}_{n<}$ is fully adapted. We also have

$$|x_\sigma| \times \chi[\![\tau < \sigma]\!] = |x_{\sigma\vee\tau}| \times \chi[\![\tau < \sigma]\!] \le \sup_{n\in\mathbb{N}} u''_{n<(\sigma\vee\tau)} \times \chi[\![\tau < \sigma]\!]$$

(because $|R^*(\boldsymbol{x})| \le \sup_{n\in\mathbb{N}} \boldsymbol{u}''_{n<}$)

$$= \sup_{n\in\mathbb{N}} u''_{n<(\sigma\vee\tau)} \times \chi[\![\tau < \sigma]\!] = \sup_{n\in\mathbb{N}} u''_{n<\sigma} \times \chi[\![\tau < \sigma]\!]$$

because $\boldsymbol{u}''_{n<} = R^*(\boldsymbol{u}_{n<})$ for every n, by (b) above. So $|x_\sigma| \le \sup_{n\in\mathbb{N}} u_{n<\sigma}$; as σ is arbitrary, $|\boldsymbol{x}| \le \sup_{n\in\mathbb{N}} \boldsymbol{u}_{n<}$. **Q**

Accordingly

$$\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x}) \le \sup_{n\in\mathbb{N}} \theta\Bigl(\int_{\mathcal{S}} \boldsymbol{u}_n\, d\boldsymbol{v}\Bigr) \le \sup_{n\in\mathbb{N}}\Bigl(\theta\Bigl(\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}'_n\, d\boldsymbol{v}\Bigr) + \theta\Bigl(\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}''_n\, d\boldsymbol{v}\Bigr)\Bigr)$$

(646Fg)

$$= \sup_{n\in\mathbb{N}}\Bigl(\theta\Bigl(\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}'_n\, d\boldsymbol{v}'\Bigr) + \theta\Bigl(\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}''_n\, d\boldsymbol{v}''\Bigr)\Bigr)$$

$$= \sup_{n\in\mathbb{N}} \theta\Bigl(\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}'_n\, d\boldsymbol{v}'\Bigr) + \sup_{n\in\mathbb{N}} \theta\Bigl(\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}''_n\, d\boldsymbol{v}''\Bigr)$$

because these are all non-decreasing sequences. As $\langle \boldsymbol{u}'_n\rangle_{n\in\mathbb{N}}$ and $\langle \boldsymbol{u}''_n\rangle_{n\in\mathbb{N}}$ are arbitrary,

$$\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x}) \le \widehat{\theta}^{\#}_{\boldsymbol{v}'}(\boldsymbol{x}') + \widehat{\theta}^{\#}_{\boldsymbol{v}''} R^*(\boldsymbol{x}).$$

(d)(i) Here write $\boldsymbol{y}$ for $\boldsymbol{x} \times \mathbf{1}^{(\mathcal{S})}_{<}$ and $\boldsymbol{y}'$ for

$$\boldsymbol{y}\restriction\mathcal{S}\wedge\tau = (\boldsymbol{x}\restriction\mathcal{S}\wedge\tau) \times (\mathbf{1}^{(\mathcal{S})}_{<}\restriction\mathcal{S}\wedge\tau) = (\boldsymbol{x}\restriction\mathcal{S}\wedge\tau) \times \mathbf{1}^{(\mathcal{S}\wedge\tau)}_{<}$$

(using 641G(c-ii) again). As $\boldsymbol{y} \in M^0_{\text{S-i}}(\mathcal{S})$, we know from (c-i) that $\boldsymbol{y}' \in M_{\text{po-b}}(\mathcal{S}\wedge\tau)$. Take $\boldsymbol{v}' \in M^{\uparrow}_{\text{n-s}}(\mathcal{S}\wedge\tau)$ and $\epsilon > 0$. Then we have a $\boldsymbol{v} \in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$ such that $\boldsymbol{v}' = \boldsymbol{v}\restriction\mathcal{S}\wedge\tau$ (646Gb). Let $\boldsymbol{u} \in M_{\text{mo}}(\mathcal{S})$ be such that $\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{y} - \boldsymbol{u}_<) \le \epsilon$, and set $\boldsymbol{u}' = \boldsymbol{u}\restriction\mathcal{S}\wedge\tau$. Then (c-ii) tells us that

$$\widehat{\theta}^{\#}_{\boldsymbol{v}'}(\boldsymbol{y}' - \boldsymbol{u}'_<) \le \widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{y} - \boldsymbol{u}_<) \le \epsilon.$$

As $\boldsymbol{v}'$ and ϵ are arbitrary, $\boldsymbol{y}' \in M^0_{\text{S-i}}(\mathcal{S}\wedge\tau)$ and $\boldsymbol{x}\restriction\mathcal{S}\wedge\tau \in M_{\text{S-i}}(\mathcal{S}\wedge\tau)$.

(ii) As for $(\boldsymbol{x}\restriction\mathcal{S}\vee\tau) \times \mathbf{1}^{(\mathcal{S}\vee\tau)}_{<} = R^*(\boldsymbol{x})$, the point is that this is equal to $R^*(\boldsymbol{x} \times \mathbf{1}^{(\mathcal{S})}_{<})$. **P**

$$R^*(\boldsymbol{x} \times \mathbf{1}^{(\mathcal{S})}_{<}) = R^*(\boldsymbol{x}) \times R^*(\mathbf{1}^{(\mathcal{S})}_{<}) = R^*(\boldsymbol{x}) \times \mathbf{1}^{(\mathcal{S}\vee\tau)}_{<}$$

((b) above)

$$= R^*(\boldsymbol{x}).\ \mathbf{Q}$$

Since $\boldsymbol{x} \times \mathbf{1}^{(\mathcal{S})}_{<} \in M^0_{\text{S-i}}(\mathcal{S})$, (c-i) tells us that $R^*(\boldsymbol{x}) \in M_{\text{po-b}}(\mathcal{S}\vee\tau)$. Take $\boldsymbol{v}'' \in M^{\uparrow}_{\text{n-s}}(\mathcal{S}\vee\tau)$ and $\epsilon > 0$. Then we have a $\boldsymbol{v} \in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$ such that $\boldsymbol{v}'' = \boldsymbol{v}\restriction\mathcal{S}\vee\tau$ (646Ga). Let $\boldsymbol{u} \in M_{\text{mo}}(\mathcal{S})$ be such that $\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x} - \boldsymbol{u}_<) \le \epsilon$. Then $\boldsymbol{u}'' = \boldsymbol{u}\restriction\mathcal{S}\vee\tau$ is moderately oscillatory and

$$\widehat{\theta}^{\#}_{\boldsymbol{v}''}(R^*(\boldsymbol{x}) - \boldsymbol{u}''_<) = \widehat{\theta}^{\#}_{\boldsymbol{v}''}(R^*(\boldsymbol{x}) - R^*(\boldsymbol{u}_<)) \le \widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{x} - \boldsymbol{u}_<) \le \epsilon$$

by (b) and (c-ii). As $\boldsymbol{v}''$ and ϵ are arbitrary, $R^*(\boldsymbol{x}) \in M^0_{\text{S-i}}(\mathcal{S}\vee\tau)$, so $\boldsymbol{x}\restriction\mathcal{S}\vee\tau \in M_{\text{S-i}}(\mathcal{S}\vee\tau)$.

646I Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and τ a member of $\mathcal{S}$. Define $R^* : (L^0)^{\mathcal{S}} \to (L^0)^{\mathcal{S}\vee\tau}$ as in 646H. Take $\boldsymbol{x} \in (L^0)^{\mathcal{S}}$.

(a) If $\boldsymbol{x}\restriction\mathcal{S}\wedge\tau \in M_{\text{po-b}}(\mathcal{S}\wedge\tau)$ and $R^*(\boldsymbol{x}) \in M_{\text{po-b}}(\mathcal{S}\vee\tau)$, then $\boldsymbol{x} \in M_{\text{po-b}}(\mathcal{S})$.

(b) If $\boldsymbol{x}\restriction\mathcal{S}\wedge\tau \in M^0_{\text{S-i}}(\mathcal{S}\wedge\tau)$ and $R^*(\boldsymbol{x}) \in M^0_{\text{S-i}}(\mathcal{S}\vee\tau)$, then $\boldsymbol{x} \in M^0_{\text{S-i}}(\mathcal{S})$.

(c) If $\boldsymbol{x}\restriction\mathcal{S}\wedge\tau \in M_{\text{S-i}}(\mathcal{S}\wedge\tau)$ and $\boldsymbol{x}\restriction\mathcal{S}\vee\tau \in M_{\text{S-i}}(\mathcal{S}\vee\tau)$, then $\boldsymbol{x} \in M_{\text{S-i}}(\mathcal{S})$.

proof Write $\boldsymbol{x}'$ for $\boldsymbol{x}\restriction\mathcal{S}\wedge\tau$.

(a) There are $\boldsymbol{u}' \in M_{\text{mo}}(\mathcal{S}\wedge\tau)^+$ and $\boldsymbol{u}'' \in M_{\text{mo}}(\mathcal{S}\vee\tau)^+$ such that $|\boldsymbol{x}'| \le \boldsymbol{u}'_<$ and $|R^*(\boldsymbol{x})| \le \boldsymbol{u}''_<$. Set $\boldsymbol{u} = R(\boldsymbol{u}', \boldsymbol{u}'')$, as defined in 646F. Then $\boldsymbol{u} \in M_{\text{mo}}(\mathcal{S})^+$ (646F(e-i)),

$$|\boldsymbol{x}'| \le \boldsymbol{u}'_< = \boldsymbol{u}_<\restriction\mathcal{S}\wedge\tau$$

(646F(e-ii)) and

$$|R^*(\boldsymbol{x})| \le \boldsymbol{u}''_< = (\boldsymbol{u}\restriction\mathcal{S}\vee\tau)_< = R^*(\boldsymbol{u}_<) \le \boldsymbol{u}_<\restriction\mathcal{S}\vee\tau$$

(646Fa, 646Hb). Putting these together, $|\boldsymbol{x}| \le \boldsymbol{u}_<$ (646H(a-v)) and $\boldsymbol{x} \in M_{\text{po-b}}(\mathcal{S})$.

(b) Take $\boldsymbol{v} \in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$ and $\epsilon > 0$. Then $\boldsymbol{v}' = \boldsymbol{v}\restriction\mathcal{S}\wedge\tau$ and $\boldsymbol{v}'' = \boldsymbol{v}\restriction\mathcal{S}\vee\tau$ are non-negative non-decreasing near-simple integrators, so there are $\boldsymbol{u}' \in M_{\text{mo}}(\mathcal{S}\wedge\tau)$ and $\boldsymbol{u}'' \in M_{\text{mo}}(\mathcal{S}\vee\tau)$ such that

$$\widehat{\theta}^{\#}_{\boldsymbol{v}'}(\boldsymbol{u}'_< - \boldsymbol{x}') \le \epsilon, \quad \widehat{\theta}^{\#}_{\boldsymbol{v}''}(\boldsymbol{u}''_< - R^*(\boldsymbol{x})) \le \epsilon.$$

Set $\boldsymbol{u} = R(\boldsymbol{u}', \boldsymbol{u}'')$. Then $\boldsymbol{u}'_< = \boldsymbol{u}_<\restriction\mathcal{S}\wedge\tau$ (646F(e-ii)) and $R^*(\boldsymbol{u}_<) = \boldsymbol{u}''_<$ (646Hb), so

$$\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{u}_< - \boldsymbol{x}) \le \widehat{\theta}^{\#}_{\boldsymbol{v}'}((\boldsymbol{u}_< - \boldsymbol{x})\restriction\mathcal{S}\wedge\tau) + \widehat{\theta}^{\#}_{\boldsymbol{v}''}R^*(\boldsymbol{u}_< - \boldsymbol{x})$$

(646H(c-ii))

$$= \widehat{\theta}^{\#}_{\boldsymbol{v}'}(\boldsymbol{u}'_< - \boldsymbol{x}') + \widehat{\theta}^{\#}_{\boldsymbol{v}''}(\boldsymbol{u}''_< - R^*(\boldsymbol{x})) \le 2\epsilon.$$

As ϵ is arbitrary, $\boldsymbol{x} \in M^0_{\text{S-i}}(\mathcal{S})$.

(c) $(\boldsymbol{x} \times \mathbf{1}^{(\mathcal{S})}_<)\restriction\mathcal{S}\wedge\tau = \boldsymbol{x}' \times \mathbf{1}^{(\mathcal{S}\wedge\tau)}_<$ belongs to $M^0_{\text{S-i}}(\mathcal{S}\wedge\tau)$ and $R^*(\boldsymbol{x} \times \mathbf{1}^{(\mathcal{S})}_<) \in M^0_{\text{S-i}}(\mathcal{S}\vee\tau)$. **P** Setting $e_\sigma = \sup_{\sigma'\in\mathcal{S}} [\![\sigma' < \sigma]\!]$ for $\sigma \in \mathcal{S}$, $\mathbf{1}^{(\mathcal{S})}_< = \langle\chi e_\sigma\rangle_{\sigma\in\mathcal{S}}$ (641Gb), and

$$R^*(\mathbf{1}^{(\mathcal{S})}_<) = \langle\chi e_\sigma \times \chi[\![\tau < \sigma]\!]\rangle_{\sigma\in\mathcal{S}\vee\tau} = \langle\chi[\![\tau < \sigma]\!]\rangle_{\sigma\in\mathcal{S}\vee\tau} = \mathbf{1}^{(\mathcal{S}\vee\tau)}_<$$

because $[\![\tau < \sigma]\!] \subseteq e_\sigma$ for every σ. Now

$$R^*(\boldsymbol{x} \times \mathbf{1}^{(\mathcal{S})}_<) = R^*(\boldsymbol{x}) \times R^*(\mathbf{1}^{(\mathcal{S})}_<)$$

(646H(a-i))

$$= (\boldsymbol{x}\restriction\mathcal{S}\vee\tau) \times \mathbf{1}^{(\mathcal{S}\vee\tau)}_< \times \mathbf{1}^{(\mathcal{S}\vee\tau)}_< = (\boldsymbol{x}\restriction\mathcal{S}\vee\tau) \times \mathbf{1}^{(\mathcal{S}\vee\tau)}_<$$

belongs to $M^0_{\text{S-i}}(\mathcal{S}\vee\tau)$. **Q**

Now (b) tells us that $\boldsymbol{x} \times \mathbf{1}^{(\mathcal{S})}_< \in M^0_{\text{S-i}}(\mathcal{S})$, that is, $\boldsymbol{x} \in M_{\text{S-i}}(\mathcal{S})$.

646J Theorem Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$, and τ a member of $\mathcal{S}$. If $\boldsymbol{x} \in (L^0)^{\mathcal{S}}$, then $\boldsymbol{x}$ is an S-integrable process iff $\boldsymbol{x}\restriction\mathcal{S}\wedge\tau$ and $\boldsymbol{x}\restriction\mathcal{S}\vee\tau$ are both S-integrable, and in this case

$$\oint_{\mathcal{S}} \boldsymbol{x}\,d\boldsymbol{v} = \oint_{\mathcal{S}\wedge\tau} \boldsymbol{x}\,d\boldsymbol{v} + \oint_{\mathcal{S}\vee\tau} \boldsymbol{x}\,d\boldsymbol{v}$$

for every near-simple integrator $\boldsymbol{v}$ with domain $\mathcal{S}$.

proof (a) By 646Hd and 646Ic, $\boldsymbol{x}$ is S-integrable iff $\boldsymbol{x}\restriction\mathcal{S}\wedge\tau$ and $\boldsymbol{x}\restriction\mathcal{S}\vee\tau$ are S-integrable.

(b)(i) Now suppose that $\boldsymbol{x}$, $\boldsymbol{x}' = \boldsymbol{x}\restriction\mathcal{S}\wedge\tau$ and $\boldsymbol{x}'' = \boldsymbol{x}\restriction\mathcal{S}\vee\tau$ are all S-integrable, and that $\boldsymbol{v}$ is a near-simple integrator with domain $\mathcal{S}$. Then $\boldsymbol{v}' = \boldsymbol{v}\restriction\mathcal{S}\wedge\tau$ and $\boldsymbol{v}'' = \boldsymbol{v}\restriction\mathcal{S}\vee\tau$ are near-simple integrators (631F(a-iv) and 616P(b-ii) again), so we can form the integrals

$$z = \oint_{\mathcal{S}} \boldsymbol{x}\,d\boldsymbol{v}, \quad z' = \oint_{\mathcal{S}\wedge\tau} \boldsymbol{x}\,d\boldsymbol{v}, \quad z'' = \oint_{\mathcal{S}\vee\tau} \boldsymbol{x}\,d\boldsymbol{v},$$

and setting

$$\boldsymbol{y} = x \times \mathbf{1}^{(\mathcal{S})}_<. \quad \boldsymbol{y}' = \boldsymbol{y}\restriction\mathcal{S}\wedge\tau = (\boldsymbol{x}\restriction\mathcal{S}\wedge\tau) \times \mathbf{1}^{(\mathcal{S}\wedge\tau)}_<,$$

$$R^*(\boldsymbol{y}) = (\boldsymbol{y}\restriction\mathcal{S}\vee\tau) \times \mathbf{1}^{(\mathcal{S}\vee\tau)}_< = (\boldsymbol{x}\restriction\mathcal{S}\vee\tau) \times \mathbf{1}^{(\mathcal{S}\vee\tau)}_<$$

as in 646H-646I, we have

$$z = \oint_{\mathcal{S}} \boldsymbol{y}\,d\boldsymbol{v}, \quad z' = \oint_{\mathcal{S}\wedge\tau} \boldsymbol{y}'\,d\boldsymbol{v}', \quad z'' = \oint_{\mathcal{S}\vee\tau} R^*(\boldsymbol{y})d\boldsymbol{v}''.$$

By 645La, there is a non-negative $\bar{u} \in L^0$ such that $\boldsymbol{y}$ belongs to the closure of $A = \{\boldsymbol{u}_< : \boldsymbol{u} \in M_{\text{mo}}(\mathcal{S}),\ \sup|\boldsymbol{u}| \le \bar{u}\}$ for the S-integration topology on $M^0_{\text{S-i}}(\mathcal{S})$. Set $A' = \{\boldsymbol{u}_< : \boldsymbol{u} \in M_{\text{mo}}(\mathcal{S}\wedge\tau),\ \sup|\boldsymbol{u}| \le \bar{u}\}$ and $A'' = \{\boldsymbol{u}_< : \boldsymbol{u} \in M_{\text{mo}}(\mathcal{S}\vee\tau),\ \sup|\boldsymbol{u}| \le \bar{u}\}$.

(ii) Take $\epsilon > 0$. Then we have $\boldsymbol{v}$, $\boldsymbol{v}'$, $\boldsymbol{v}''$ and $\delta > 0$ such that $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$, $\boldsymbol{v}' = \langle v'_\sigma \rangle_{\sigma \in \mathcal{S} \wedge \tau}$ and $\boldsymbol{v}'' = \langle v''_\sigma \rangle_{\sigma \in \mathcal{S} \vee \tau}$ are all non-negative non-decreasing near-simple integrators,

$$\theta(z - \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}) \le \epsilon \text{ whenever } \boldsymbol{u} \in A \text{ and } \widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{y} - \boldsymbol{u}_<) \le \delta,$$

$$\theta(z' - \int_{\mathcal{S} \wedge \tau} \boldsymbol{u}\, d\boldsymbol{v}') \le \epsilon \text{ whenever } \boldsymbol{u} \in A' \text{ and } \widehat{\theta}^{\#}_{\boldsymbol{v}'}(\boldsymbol{y}' - \boldsymbol{u}_<) \le \delta,$$

$$\theta(z'' - \int_{\mathcal{S} \vee \tau} \boldsymbol{u}\, d\boldsymbol{v}'') \le \epsilon \text{ whenever } \boldsymbol{u} \in A'' \text{ and } \widehat{\theta}^{\#}_{\boldsymbol{v}''}(R^*(\boldsymbol{y}) - \boldsymbol{u}''_<) \le \delta.$$

Now there is a $\boldsymbol{w} = \langle w^*_\sigma \rangle_{\sigma \in \mathcal{S}} \in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$ such that

$$\boldsymbol{v} \preccurlyeq \boldsymbol{w}, \quad \boldsymbol{v}' \preccurlyeq \boldsymbol{w} \restriction \mathcal{S} \wedge \tau, \quad \boldsymbol{v}'' \preccurlyeq \boldsymbol{w} \restriction \mathcal{S} \vee \tau$$

(646Gc). We chose $\bar{u}$ so that there would be a $\boldsymbol{u} \in A$ such that $\widehat{\theta}^{\#}_{\boldsymbol{w}}(\boldsymbol{y} - \boldsymbol{u}_<) \le \delta$. In this case, setting

$$\boldsymbol{w}' = \boldsymbol{w} \restriction \mathcal{S} \wedge \tau, \quad \boldsymbol{w}'' = \boldsymbol{w} \restriction \mathcal{S} \vee \tau,$$

$$\boldsymbol{u}' = \boldsymbol{u} \restriction \mathcal{S} \wedge \tau \in A', \quad \boldsymbol{u}'' = \boldsymbol{u} \restriction \mathcal{S} \vee \tau \in A'',$$

we have

$$\widehat{\theta}_{\boldsymbol{v}}(\boldsymbol{y} - \boldsymbol{u}_<) \le \widehat{\theta}^{\#}_{\boldsymbol{w}}(\boldsymbol{y} - \boldsymbol{u}_<) \le \delta$$

(645Dc),

$$\widehat{\theta}_{\boldsymbol{v}'}(\boldsymbol{y}' - \boldsymbol{u}'_<) \le \widehat{\theta}^{\#}_{\boldsymbol{w}'}(\boldsymbol{y}' - \boldsymbol{u}'_<) \le \widehat{\theta}^{\#}_{\boldsymbol{w}}(\boldsymbol{y} - \boldsymbol{u}_<) \le \delta$$

(646H(c-ii)) and

$$\widehat{\theta}_{\boldsymbol{v}''}(R^*(\boldsymbol{y}) - \boldsymbol{u}''_<) \le \widehat{\theta}^{\#}_{\boldsymbol{w}''}(R^*(\boldsymbol{y}) - \boldsymbol{u}''_<) \le \widehat{\theta}^{\#}_{\boldsymbol{w}}(\boldsymbol{y} - \boldsymbol{u}_<) \le \delta$$

by the other half of 646H(c-ii). So

$$\theta(z - \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}) \le \epsilon, \quad \theta(z' - \int_{\mathcal{S} \wedge \tau} \boldsymbol{u}'\, d\boldsymbol{v}') \le \epsilon, \quad \theta(z'' - \int_{\mathcal{S} \vee \tau} \boldsymbol{u}''\, d\boldsymbol{v}'') \le \epsilon.$$

But

$$\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v} = \int_{\mathcal{S} \wedge \tau} \boldsymbol{u}\, d\boldsymbol{v} + \int_{\mathcal{S} \vee \tau} \boldsymbol{u}\, d\boldsymbol{v}$$

(613J(c-i) again)

$$= \int_{\mathcal{S} \wedge \tau} \boldsymbol{u}'\, d\boldsymbol{v}' + \int_{\mathcal{S} \vee \tau} \boldsymbol{u}''\, d\boldsymbol{v}'',$$

so $\theta(z - z' - z'') \le 3\epsilon$. As ϵ is arbitrary, $z = z' + z''$, that is,

$$\oint_{\mathcal{S}} \boldsymbol{x}\, d\boldsymbol{v} = \oint_{\mathcal{S} \wedge \tau} \boldsymbol{x}\, d\boldsymbol{v} + \oint_{\mathcal{S} \vee \tau} \boldsymbol{x}\, d\boldsymbol{v},$$

as claimed.

646K Indefinite S-integrals Suppose that $\langle \mathfrak{A}_t \rangle_{t \in T}$ is right-continuous and that $\mathcal{S}$ is an order-convex sublattice of $\mathcal{T}$.

(a) Suppose that $\boldsymbol{x}$ is a locally S-integrable process and that $\boldsymbol{v}$ is a locally near-simple local integrator, both with domain $\mathcal{S}$. Then we can define $z_\tau = \oint_{\mathcal{S} \wedge \tau} \boldsymbol{x}\, d\boldsymbol{v}$ for $\tau \in \mathcal{S}$ (646Hd, 646J). Now the **indefinite S-integral of $\boldsymbol{x}$ with respect to $\boldsymbol{v}$** is $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x}) = \langle z_\tau \rangle_{\tau \in \mathcal{S}}$.

(b) It will more than once be useful to note that, in the context of (a) just above, $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x}) = \mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x} \times \boldsymbol{1}^{(\mathcal{S})}_<)$. **P** If $\tau \in \mathcal{S}$ then

$$\oint_{\mathcal{S} \wedge \tau} \boldsymbol{x}\, d\boldsymbol{v} = \oint_{\mathcal{S} \wedge \tau} \boldsymbol{x} \times \boldsymbol{1}^{(\mathcal{S} \wedge \tau)}_<\, d\boldsymbol{v} = \oint_{\mathcal{S} \wedge \tau} \boldsymbol{x} \times \boldsymbol{1}^{(\mathcal{S})}_< \times \boldsymbol{1}^{(\mathcal{S} \wedge \tau)}_<\, d\boldsymbol{v}$$

(because $\boldsymbol{1}^{(\mathcal{S})}_< \restriction \mathcal{S} \wedge \tau = \boldsymbol{1}^{(\mathcal{S} \wedge \tau)}_<$)

$$= \oint_{\mathcal{S}\wedge\tau} \boldsymbol{x} \times \mathbf{1}^{(\mathcal{S})}_{<}\, d\boldsymbol{v}. \ \mathbf{Q}$$

(c) If $\boldsymbol{u}$ is a locally moderately oscillatory process with domain $\mathcal{S}$ then $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{u}_<) = ii_{\boldsymbol{v}}(\boldsymbol{u})$. **P** If $\tau \in \mathcal{S}$ then

$$\oint_{\mathcal{S}\wedge\tau} \boldsymbol{u}_< d\boldsymbol{v} = \oint_{\mathcal{S}\wedge\tau} (\boldsymbol{u}_< \restriction \mathcal{S}\wedge\tau) d\boldsymbol{v} = \oint_{\mathcal{S}\wedge\tau} (\boldsymbol{u}\restriction \mathcal{S}\wedge\tau)_<\, d\boldsymbol{v}$$

(641G(c-ii) once more)

$$= \int_{\mathcal{S}\wedge\tau} (\boldsymbol{u}\restriction \mathcal{S}\wedge\tau) d\boldsymbol{v}$$

(645R(a-i))

$$= \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\boldsymbol{v}. \ \mathbf{Q}$$

646L Lemma Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous and that $\mathcal{S}$ is an order-convex sublattice of $\mathcal{T}$. Let $\boldsymbol{v}$ be a near-simple integrator and $\boldsymbol{x}$ an S-integrable process, both with domain $\mathcal{S}$. For $\tau \in \mathcal{S}$, set $z_\tau = \oint_{\mathcal{S}\wedge\tau} \boldsymbol{x}\, d\boldsymbol{v}$. Suppose that $\boldsymbol{u}^* \in M_{\mathrm{mo}}(\mathcal{S})^+$ is such that $|\boldsymbol{x}| \le \boldsymbol{u}^*$. Then for any $\epsilon > 0$ there is a $\mathfrak{T}_{\text{S-i}}$-neighbourhood G of $\boldsymbol{x}$ such that $\theta(z_\tau - \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\boldsymbol{v}) \le \epsilon$ whenever $\boldsymbol{u} \in M_{\mathrm{mo}}(\mathcal{S})$, $|\boldsymbol{u}| \le \boldsymbol{u}^*$, $\boldsymbol{u}_< \in G$ and $\tau \in \mathcal{S}$.

proof (a) I should begin by noting that z_τ is defined for every $\tau \in \mathcal{S}$, by 646J. Define $R_\tau : (L^0)^{\mathcal{S}} \to (L^0)^{\mathcal{S}}$, for $\tau \in \mathcal{S}$, by setting

$$R_\tau(\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}) = \langle u_\sigma \times \chi[\![\sigma \le \tau]\!]\rangle_{\sigma\in\mathcal{S}}$$

for $\langle u_\sigma\rangle_{\sigma\in\mathcal{S}} \in (L^0)^{\mathcal{S}}$. As in 646F and 646H, R_τ is an order-continuous f-algebra homomorphism and $R_\tau(\boldsymbol{u})$ is fully adapted whenever $\boldsymbol{u} \in (L^0)^{\mathcal{S}}$ is fully adapted. If $\boldsymbol{y}$ is any S-integrable process with domain $\mathcal{S}$ and $\tau \in \mathcal{S}$ then

$$R_\tau(\boldsymbol{y})\restriction\mathcal{S}\wedge\tau = \boldsymbol{y}\restriction\mathcal{S}\wedge\tau, \quad (R_\tau(\boldsymbol{y})\restriction\mathcal{S}\vee\tau) \times \mathbf{1}^{(\mathcal{S}\vee\tau)}_{<} = \mathbf{0},$$

so

$$\oint_{\mathcal{S}} R_\tau(\boldsymbol{y}) d\boldsymbol{v} = \oint_{\mathcal{S}\wedge\tau} R_\tau(\boldsymbol{y}) d\boldsymbol{v} + \oint_{\mathcal{S}\vee\tau} R_\tau(\boldsymbol{y}) d\boldsymbol{v}$$

(646J)

$$= \oint_{\mathcal{S}\wedge\tau} R_\tau(\boldsymbol{y}) d\boldsymbol{v} + \oint_{\mathcal{S}\vee\tau} R_\tau(\boldsymbol{y}) \times \mathbf{1}^{(\mathcal{S}\vee\tau)}_{<}\, d\boldsymbol{v}$$
$$= \oint_{\mathcal{S}\wedge\tau} \boldsymbol{y}\, d\boldsymbol{v}.$$

In particular, $z_\tau = \oint_{\mathcal{S}} R_\tau(\boldsymbol{x}) d\boldsymbol{v}$.

If $\boldsymbol{u} \in M_{\mathrm{mo}}(\mathcal{S})$ then

$$\oint_{\mathcal{S}} R_\tau(\boldsymbol{u}_<) d\boldsymbol{v} = \oint_{\mathcal{S}\wedge\tau} (\boldsymbol{u}_< \restriction \mathcal{S}\wedge\tau) d\boldsymbol{v} = \oint_{\mathcal{S}\wedge\tau} (\boldsymbol{u}\restriction \mathcal{S}\wedge\tau)_<\, d\boldsymbol{v}$$
$$= \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\boldsymbol{v}$$

by 645R(a-i) again.

(b) Let $\epsilon > 0$. By 645S, there are a $\boldsymbol{v}^* \in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$ and a $\delta > 0$ such that $\theta(\oint_{\mathcal{S}} \boldsymbol{y}\,d\boldsymbol{v} - \oint_{\mathcal{S}} \boldsymbol{y}'\,d\boldsymbol{v}) \le \epsilon$ whenever $\boldsymbol{y}$, $\boldsymbol{y}' \in M^0_{\text{S-i}}(\mathcal{S})$, $|\boldsymbol{y}| \le \boldsymbol{u}^*_<$, $|\boldsymbol{y}'| \le \boldsymbol{u}^*_<$ and $\widehat{\theta}^{\#}_{\boldsymbol{v}^*}(\boldsymbol{y} - \boldsymbol{y}') \le \delta$. Set $G = \{\boldsymbol{y} : \boldsymbol{y} \in M_{\text{po-b}}(\mathcal{S}), \widehat{\theta}^{\#}_{\boldsymbol{v}^*}(\boldsymbol{y} - \boldsymbol{x}) \le \delta\}$, so that G is a $\mathfrak{T}_{\text{S-i}}$-neighbourhood of $\boldsymbol{x}$. If $\boldsymbol{u} \in M_{\text{n-s}}(\mathcal{S})$, $|\boldsymbol{u}| \le \boldsymbol{u}^*$ and $\boldsymbol{u}_< \in G$, then, for any $\tau \in \mathcal{S}$,

$$|R_\tau(\boldsymbol{x}) - R_\tau(\boldsymbol{u}_<)| = |R_\tau(\boldsymbol{x} - \boldsymbol{u}_<)| \le |\boldsymbol{x} - \boldsymbol{u}_<|,$$

so

$$\widehat{\theta}^{\#}_{\boldsymbol{v}^*}(R_\tau(\boldsymbol{x}) - R_\tau(\boldsymbol{u}_<)) \le \widehat{\theta}^{\#}_{\boldsymbol{v}^*}(\boldsymbol{x} - \boldsymbol{u}_<) \le \delta$$

and

$$\theta(z_\tau - \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\,d\boldsymbol{v}) = \theta(\oint_{\mathcal{S}} R_\tau(\boldsymbol{x})d\boldsymbol{v} - \oint_{\mathcal{S}} R_\tau(\boldsymbol{u}_<)d\boldsymbol{v}) \le \epsilon,$$

as required.

646M Proposition Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous and that $\mathcal{S}$ is a non-empty order-convex sublattice of $\mathcal{T}$. Let $\boldsymbol{v}$ be a near-simple integrator and $\boldsymbol{x}$ an S-integrable process, both with domain $\mathcal{S}$; set $z = \oint_{\mathcal{S}} \boldsymbol{x}\,d\boldsymbol{v}$ and $z_\tau = \oint_{\mathcal{S}\wedge\tau} \boldsymbol{x}\,d\boldsymbol{v}$ for $\tau \in \mathcal{S}$. Then $\lim_{\tau\uparrow\mathcal{S}} z_\tau = z$ and $\lim_{\tau\downarrow\mathcal{S}} z_\tau = 0$.

proof By 646Kb, $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x}) = \mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x} \times \mathbf{1}^{(\mathcal{S})}_<)$, while $\oint_{\mathcal{S}} \boldsymbol{x}\,d\boldsymbol{v} = \oint_{\mathcal{S}} \boldsymbol{x} \times \mathbf{1}^{(\mathcal{S})}_<\,d\boldsymbol{v}$; so we may assume that $\boldsymbol{x}$ itself belongs to $M^0_{\text{S-i}}(\mathcal{S})$. Let $\boldsymbol{u}^* \in M^+_{\text{mo}} = M_{\text{mo}}(\mathcal{S})^+$ be such that $|\boldsymbol{x}| \le \boldsymbol{u}^*_<$, and take $\epsilon > 0$. By 646L, there is a $\mathfrak{T}_{\text{S-i}}$-neighbourhood G of $\boldsymbol{x}$ such that $\theta(z_\tau - \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\,d\boldsymbol{v}) \le \epsilon$ whenever $\tau \in \mathcal{S}$, $\boldsymbol{u} \in M_{\text{mo}}$, $|\boldsymbol{u}| \le \boldsymbol{u}^*$ and $\boldsymbol{u}_< \in G$. At the same time, there is a $\mathfrak{T}_{\text{S-i}}$-neigbourhood G' of $\boldsymbol{x}$ such that $\theta(z - \int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}) \le \epsilon$ whenever $\boldsymbol{u} \in M_{\text{mo}}$, $|\boldsymbol{u}| \le \boldsymbol{u}^*$ and $\boldsymbol{u}_< \in G'$. By 645La, there is a $\boldsymbol{u} \in M_{\text{mo}}$ such that $|\boldsymbol{u}| \le \boldsymbol{u}^*$ and $\boldsymbol{u}_< \in G \cap G'$. So now we have

$$\theta(z_\tau - \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\,d\boldsymbol{v}) \le \epsilon \text{ whenever } \tau \in \mathcal{S}, \quad \theta(z - \int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}) \le \epsilon.$$

Accordingly

$$\begin{aligned}
\limsup_{\tau\uparrow\mathcal{S}} \theta(z - z_\tau) &\le \theta(z - \int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}) + \limsup_{\tau\uparrow\mathcal{S}} \theta(\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v} - \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\,d\boldsymbol{v}) \\
&\qquad + \limsup_{\tau\uparrow\mathcal{S}} \theta(\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\,d\boldsymbol{v} - z_\tau) \\
&\le \epsilon + 0 + \epsilon
\end{aligned}$$

(613J(f-ii))

$$\begin{aligned}
&= 2\epsilon, \\
\limsup_{\tau\downarrow\mathcal{S}} \theta(z_\tau) &\le \limsup_{\tau\downarrow\mathcal{S}} \theta(\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\,d\boldsymbol{v}) + \limsup_{\tau\downarrow\mathcal{S}} \theta(\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\,d\boldsymbol{v} - z_\tau) \\
&\le 0 + \epsilon
\end{aligned}$$

(613J(f-i))

$$= \epsilon.$$

As ϵ is arbitrary, we have the result.

646N Theorem Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous and that $\mathcal{S}$ is an order-convex sublattice of $\mathcal{T}$. Let $\boldsymbol{v}$ be a near-simple integrator and $\boldsymbol{x}$ an S-integrable process, both with domain $\mathcal{S}$. Then $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})$ is a near-simple integrator.

proof (a) As in 646M, we may assume that $\boldsymbol{x}$ belongs to $M^0_{\text{S-i}}(\mathcal{S})$. Again set $z_\tau = \oint_{\mathcal{S}\wedge\tau} \boldsymbol{x}\,d\boldsymbol{v}$ for $\tau \in \mathcal{S}$, and take $\epsilon > 0$ and $\boldsymbol{u}^* \in M^+_{\text{mo}} = M_{\text{mo}}(\mathcal{S})^+$ such that $|\boldsymbol{x}| \le \boldsymbol{u}^*_<$. Also as in 646M, using 646L and 645L, we have a $\boldsymbol{u} \in M_{\text{mo}}$ such that $|\boldsymbol{u}| \le \boldsymbol{u}^*$ and $\theta(z_\tau - \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\,d\boldsymbol{v}) \le \epsilon$ for every $\tau \in \mathcal{S}$. Since $\mathcal{S}$ is finitely full (611O), $\theta(\sup|\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x}) - ii_{\boldsymbol{v}}(\boldsymbol{u})|) \le 2\sqrt{\epsilon}$ (615Db); while $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is near-simple, by 631I. As ϵ is arbitrary, $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})$ is fully adapted (by 613Bl) and near-simple.

(b) To see that $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})$ is an integrator, we can use 617E, as follows. Consider $C = \{\int \boldsymbol{u}\,d\boldsymbol{v} : \boldsymbol{u} \in M_{\text{mo}}, |\boldsymbol{u}| \le \boldsymbol{u}^*\}$. Then C is topologically bounded in L^0. $\mathbf{P}$ Set $\bar{u} = \sup|\boldsymbol{u}^*|$. Let $\epsilon > 0$. Then there is an

$M > 0$ such that $\bar\mu[\![\bar u \ge M]\!] \le \epsilon$. Let $\delta > 0$ be such that $\theta(M\delta z) \le \epsilon$ for every $z \in Q_{\mathcal{S}}(d\boldsymbol{v})$. If $\boldsymbol{u} \in M_{\mathrm{mo}}$, $I \in \mathcal{I}(\mathcal{S})$ and $\|\boldsymbol{u}\|_\infty \le M$, then $\frac{1}{M}S_I(\boldsymbol{u},\, d\boldsymbol{v}) \in Q_{\mathcal{S}}(d\boldsymbol{v})$ so $\theta(\delta S_I(\boldsymbol{u}, d\boldsymbol{v})) \le \epsilon$; taking the limit as $I \uparrow \mathcal{I}(\mathcal{S})$, $\theta(\delta\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}) \le \epsilon$. Generally, if $\boldsymbol{u} \in M_{\mathrm{mo}}$ and $|\boldsymbol{u}| \le \boldsymbol{u}^*$, set $\boldsymbol{u}' = \operatorname{med}(-M\mathbf{1}^{(\mathcal{S})}, \boldsymbol{u}, M\mathbf{1}^{(\mathcal{S})})$; then $\theta(\delta\int_{\mathcal{S}}\boldsymbol{u}'\,d\boldsymbol{v}) \le \epsilon$ and $[\![\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v} \ne \int_{\mathcal{S}}\boldsymbol{u}'\,d\boldsymbol{v}]\!] \subseteq [\![\boldsymbol{u} \ne \boldsymbol{u}']\!]$ has measure at most ϵ, so $\theta(\delta\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}) \le 2\epsilon$. Thus $\theta(\delta z) \le 2\epsilon$ for every $z \in C$. As ϵ is arbitrary, C is topologically bounded. **Q**

(c) Now $Q_{\mathcal{S}}(d\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x}))$ is included in the topological closure $\overline{C}$ of C. **P** If $z \in Q_{\mathcal{S}}(d\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x}))$, there are an $I \in \mathcal{I}(\mathcal{S})$ and a $\boldsymbol{u} \in M_{\mathrm{fa}}(I)$ such that $\|\boldsymbol{u}\|_\infty \le 1$ and $z = S_I(\boldsymbol{u}, d\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x}))$. Take $\epsilon > 0$. Then there is a $\boldsymbol{u}' \in M_{\mathrm{mo}}$ such that $|\boldsymbol{u}'| \le \boldsymbol{u}^*$ and $\theta(z_\tau - \int_{\mathcal{S}\wedge\tau}\boldsymbol{u}'\,d\boldsymbol{v}) \le \frac{\epsilon}{1+\#(I)}$ for every $\tau \in \mathcal{S}$, as in (a) above, that is, such that $\theta(z_\tau - z'_\tau) \le \frac{1}{1+\#(I)}\epsilon$ for every τ, where $\langle z'_\tau\rangle_{\tau\in\mathcal{S}} = \boldsymbol{z}'$ is the indefinite integral $ii_{\boldsymbol{v}}(\boldsymbol{u}')$. In this case, $\theta(S_I(\boldsymbol{u}, d\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})) - S_I(\boldsymbol{u}, d\boldsymbol{z}')) \le \epsilon$. By 627Ha, there is a $\boldsymbol{w} \in M_{\mathrm{simp}}(\mathcal{S})$ such that $\|\boldsymbol{w}\|_\infty \le 1$ and $S_I(\boldsymbol{u}, d\boldsymbol{z}') = \int_{\mathcal{S}}\boldsymbol{w}\,d\boldsymbol{z}'$. Now we know from 617E that $\int_{\mathcal{S}}\boldsymbol{w}\,d\boldsymbol{z}' = \int_{\mathcal{S}}\boldsymbol{w}\times\boldsymbol{u}'\,d\boldsymbol{v}$, which belongs to C because $|\boldsymbol{w}\times\boldsymbol{u}'| \le |\boldsymbol{u}'| \le \boldsymbol{u}^*$. And $\theta(z - \int_{\mathcal{S}}\boldsymbol{w}\,d\boldsymbol{z}') \le \epsilon$. As z and ϵ are arbitrary, $Q_{\mathcal{S}}(d\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})) \subseteq \overline{C}$. **Q**

As the closure of a topologically bounded set in L^0 is topologically bounded (613B(f-iii)), $Q_{\mathcal{S}}(d\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x}))$ is topologically bounded and $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})$ is an integrator.

646O Lemma Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous and that $\mathcal{S}$ is an order-convex sublattice of $\mathcal{T}$. Let $\boldsymbol{v}$ be a near-simple integrator with domain $\mathcal{S}$, and $A \subseteq M^0_{\text{S-i}} = M^0_{\text{S-i}}(\mathcal{S})$ a uniformly previsibly order-bounded set. Then $\mathrm{S}ii_{\boldsymbol{v}}{\restriction}A$ is uniformly continuous with respect to the S-integration uniformity on A and and the ucp uniformity on $M_{\text{o-b}}(\mathcal{S})$.

proof (a) It is enough to consider the case in which $A = \{\boldsymbol{x} : \boldsymbol{x} \in M^0_{\text{S-i}},\ |\boldsymbol{x}| \le \boldsymbol{u}^*_<\}$ where $\boldsymbol{u}^* \in M_{\mathrm{mo}}(\mathcal{S})^+$. By 646N, we know that $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x}) \in M_{\text{n-s}}(\mathcal{S})$ for every $\boldsymbol{x} \in A$.

(b) Take $\epsilon > 0$. Then there is a $\mathfrak{T}_{\text{S-i}}$-neighbourhood G of 0 in $M^0_{\text{S-i}}$ such that $\theta(\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{v} - \oint_{\mathcal{S}}\boldsymbol{y}\,d\boldsymbol{v}) \le \epsilon^2$ whenever $\boldsymbol{x}, \boldsymbol{y} \in A$ and $\boldsymbol{x} - \boldsymbol{y} \in G$ (645S); and we may suppose that G is of the form $\{\boldsymbol{x} : \boldsymbol{x} \in M^0_{\text{S-i}},\ \hat\theta^\#_{\boldsymbol{w}}(\boldsymbol{x}) \le \delta\}$ where $\delta > 0$ and $\boldsymbol{w} \in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$. Fix $\tau \in \mathcal{S}$ for the moment. Defining $R_\tau : (L^0)^{\mathcal{S}} \to (L^0)^{\mathcal{S}}$ as in the proof of 646L, $R_\tau(\boldsymbol{x}) \in M^0_{\text{S-i}}$ and $|R_\tau(\boldsymbol{x})| \le |\boldsymbol{x}|$ for $\boldsymbol{x} \in M^0_{\text{S-i}}$, so $R_\tau(\boldsymbol{x}) \in A$ for every $\boldsymbol{x} \in A$; also $R_\tau(\boldsymbol{x}) \in G$ for every $\boldsymbol{x} \in G$. Consequently $\theta(\oint_{\mathcal{S}}R_\tau(\boldsymbol{x})d\boldsymbol{v} - \oint_{\mathcal{S}}R_\tau(\boldsymbol{y})d\boldsymbol{v}) \le \epsilon^2$ whenever $\boldsymbol{x}, \boldsymbol{y} \in A$, $\boldsymbol{x} - \boldsymbol{y} \in G$ and $\tau \in \mathcal{S}$. But we also have $\oint_{\mathcal{S}}R_\tau(\boldsymbol{x})d\boldsymbol{v} = \oint_{\mathcal{S}\wedge\tau}\boldsymbol{x}\,d\boldsymbol{v}$ for $\boldsymbol{x} \in M^0_{\text{S-i}}$ and $\tau \in \mathcal{S}$, as observed in part (a) of the proof of 646L. So we get

$$\theta(\oint_{\mathcal{S}\wedge\tau}\boldsymbol{x}\,d\boldsymbol{v} - \oint_{\mathcal{S}\wedge\tau}\boldsymbol{y}\,d\boldsymbol{v}) \le \epsilon^2$$

whenever $\boldsymbol{x}, \boldsymbol{y} \in A$, $\boldsymbol{x} - \boldsymbol{y} \in G$ and $\tau \in \mathcal{S}$.

(c) Now we know that $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})$ is near-simple, therefore order-bounded, for every $\boldsymbol{x} \in M^0_{\text{S-i}}$; while $\mathcal{S}$ is order-convex, therefore finitely full. So we can apply 615Db again to see that $\theta(\sup|\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x}) - \mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{y})|) \le 2\epsilon$ whenever $\boldsymbol{x}, \boldsymbol{y} \in A$ and $\boldsymbol{x} - \boldsymbol{y} \in G$. As ϵ is arbitrary, $\mathrm{S}ii_{\boldsymbol{v}}{\restriction}A$ is uniformly continuous.

646P Lemma Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous and that $\mathcal{S}$ is an order-convex sublattice of $\mathcal{T}$ with a least element. Let $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ be a martingale. Then $\mathcal{S}_{\boldsymbol{v}} = \{\tau : \tau \in \mathcal{S},\ Q_{\mathcal{S}\wedge\tau}(d\boldsymbol{v})$ is uniformly integrable$\}$ is a covering ideal of $\mathcal{S}$.

proof (a) Note first that if $\tau \in \mathcal{S}_{\boldsymbol{v}}$ and $\sigma \in \mathcal{S}\wedge\tau$, then $Q_{\mathcal{S}\wedge\sigma}(d\boldsymbol{v}) \subseteq Q_{\mathcal{S}\wedge\tau}(d\boldsymbol{v})$ is uniformly integrable, so $\sigma \in \mathcal{S}_{\boldsymbol{v}}$. If $\tau, \tau' \in \mathcal{S}_{\boldsymbol{v}}$, then $Q_{\mathcal{S}\wedge(\tau\vee\tau')}(d\boldsymbol{v}) \subseteq Q_{\mathcal{S}\wedge\tau}(d\boldsymbol{v}) + Q_{\mathcal{S}\wedge\tau'}(d\boldsymbol{v})$. **P** If $z \in Q_{\mathcal{S}\wedge(\tau\vee\tau')}(d\boldsymbol{v})$ there is a simple process $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}\wedge(\tau\vee\tau')}$ such that $\|\boldsymbol{u}\|_\infty \le 1$ and $z = \int_{\mathcal{S}\wedge(\tau\vee\tau')}\boldsymbol{u}\,d\boldsymbol{v}$ (627Ha again). By 612K(d-i), there is a breakpoint string for $\boldsymbol{u}$ of the form $(\tau_0, \dots, \tau_m, \dots, \tau_n)$ where $\tau_0 = \min\mathcal{S}$, $\tau_m = \tau$ and $\tau_n = \tau\vee\tau'$. Now

$$z = \int_{\mathcal{S}\wedge(\tau\vee\tau')}\boldsymbol{u}\,d\boldsymbol{v} = \sum_{i=0}^{n-1} u_{\tau_i} \times (v_{\tau_{i+1}} - v_{\tau_i})$$

(614C, because $\tau_0 = \min\mathcal{S}$ and $\tau_n = \tau\vee\tau'$)

$$= \sum_{i=0}^{m-1} u_{\tau_i} \times (v_{\tau_{i+1}} - v_{\tau_i}) + \sum_{i=m}^{n-1} u_{\tau_i} \times (v_{\tau_{i+1}} - v_{\tau_i}) \in Q_{\mathcal{S}\wedge\tau}(d\boldsymbol{v}) + Q_{\mathcal{S}\wedge\tau'}(d\boldsymbol{v}). \textbf{Q}$$

So $Q_{\mathcal{S}\wedge(\tau\vee\tau')}(d\boldsymbol{v})$ is included in the sum of two uniformly integrable sets and is uniformly integrable (621B(c-i)). As τ and τ' are arbitrary, $\mathcal{S}_{\boldsymbol{v}}$ is closed under $\vee$ and is an ideal in $\mathcal{S}$.

(b) Suppose that $\mathcal{S}$ has a greatest element and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ is an L^2-martingale. Then $\|v_\sigma\|_2 \le \|v_{\max\mathcal{S}}\|_2$ for every $\sigma \in \mathcal{S}$ and $\boldsymbol{v}$ is L^2-bounded. Let $\boldsymbol{v}^* = \langle v^*_\sigma\rangle_{\sigma\in\mathcal{S}}$ be its quadratic variation. If $\boldsymbol{u}$ is a simple process with domain $\mathcal{S}$ and $\|\boldsymbol{u}\|_\infty \le 1$, then

$$\|\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}\|_2^2 \le \|\int_{\mathcal{S}} \boldsymbol{u}^2\, d\boldsymbol{v}^*\|_1 \le \|\int_{\mathcal{S}} d\boldsymbol{v}^*\|_1 = \|v^*_{\max\mathcal{S}}\|_1 < \infty$$

by 624I and 624G. So

$$Q_{\mathcal{S}}(d\boldsymbol{v}) \subseteq \{\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v} : \boldsymbol{u} \in M_{\mathrm{simp}}(\mathcal{S}),\ \|\boldsymbol{u}\|_\infty \le 1\}$$

(627Ha once more) is a $\|\ \|_2$-bounded subset of L^2 and is uniformly integrable (621Be).

(c) Next, suppose that $\mathcal{S}$ has a greatest element and that $\boldsymbol{v}$ is such that $v_{\min\mathcal{S}} = 0$ and $\mathrm{Osclln}(\boldsymbol{v}) \le \chi 1$. As $\boldsymbol{v}$ is locally near-simple (632Ia), therefore near-simple, there is a non-decreasing sequence $\langle\tau_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{S}$ such that $\tau_0 = \min\mathcal{S}$, $\inf_{n\in\mathbb{N}} [\![\tau_n < \max\mathcal{S}]\!] = 0$ and $[\![|v_\sigma - v_{\tau_n}| \ge 1]\!] \subseteq [\![\sigma = \tau_{n+1}]\!]$ whenever $n \in \mathbb{N}$ and $\sigma \in [\tau_n, \tau_{n+1}]$ (631Ra). Now $|v_\tau - v_{\tau_n}| \le 2\chi 1$ whenever $n \in \mathbb{N}$ and $\tau_n \le \tau \le \tau_{n+1}$. **P** Applying the method of 641E to $\boldsymbol{v}' = \boldsymbol{v}\restriction[\tau_n, \tau_{n+1}]$, we see that if $\tau \in [\tau_n, \tau_{n+1}]$ then $|v_{I<\tau} - v_{\tau_n}| \times \chi[\![\tau_n < \tau]\!] \le \chi 1$ for every finite sublattice I of $[\tau_n, \tau]$ containing τ_n, so that $|v_{<\tau} - v_{\tau_n}| \times \chi[\![\tau_n < \tau]\!] \le \chi 1$; while $|v_\tau - v_{<\tau}| \times \chi[\![\tau_n < \tau]\!] \le \mathrm{Osclln}(\boldsymbol{v}')$, by 641Na. So

$$|v_\tau - v_{\tau_n}| \times \chi[\![\tau_n < \tau]\!] \le \chi 1 + \mathrm{Osclln}(\boldsymbol{v}') \le \chi 1 + \mathrm{Osclln}(\boldsymbol{v}) \le 2\chi 1$$

(using 618D(b-ii)), whenever $\tau_n \le \tau \le \tau_{n+1}$. It follows at once that $|v_\tau - v_{\tau_n}| \le 2\chi 1$ for every $\tau \in [\tau_n, \tau_{n+1}]$. **Q**

Consequently $|v_\tau| \le 2n\chi 1$ whenever $n \in \mathbb{N}$ and $\min\mathcal{S} \le \tau \le \tau_n$, and $\boldsymbol{v}\restriction[\min\mathcal{S}, \tau_n]$ is an L^2-martingale for every n. By (b), $\tau_n \in \mathcal{S}_{\boldsymbol{v}}$ for every n; since

$$\sup_{n\in\mathbb{N}} [\![\tau \le \tau_n]\!] \supseteq \sup_{n\in\mathbb{N}} [\![\max\mathcal{S} = \tau_n]\!] = 1$$

for every $\tau \in \mathcal{S}$, $\mathcal{S}_{\boldsymbol{v}}$ is a covering ideal of $\mathcal{S}$.

(d) If $\mathcal{S}$ has a greatest element and $\boldsymbol{v}$ is such that $\mathrm{Osclln}(\boldsymbol{v}) \le \chi 1$, but we are told nothing about $v_{\min\mathcal{S}}$, then we can apply (c) to $\boldsymbol{v}' = \langle v_\sigma - v_{\min\mathcal{S}}\rangle_{\sigma\in\mathcal{S}}$ to see that $\mathcal{S}_{\boldsymbol{v}} = \mathcal{S}_{\boldsymbol{v}'}$ is a covering ideal of $\mathcal{S}$ because $Q_{\mathcal{S}\wedge\tau}(d\boldsymbol{v}) = Q_{\mathcal{S}\wedge\tau}(d\boldsymbol{v}')$ for every τ.

(e) Now suppose that $\boldsymbol{v}$ is of bounded variation and $\bar{v} = \int_{\mathcal{S}} |d\boldsymbol{v}|$ has finite expectation. If $\boldsymbol{u} \in M_{\mathrm{simp}}(\mathcal{S})$ and $\|\boldsymbol{u}\|_\infty \le 1$, then $|S_I(\boldsymbol{u}, d\boldsymbol{v})| \le S_I(\boldsymbol{1}^{(\mathcal{S})}, |d\boldsymbol{v}|) \le \bar{v}$. So $|z| \le \bar{v}$ for every $z \in Q_{\mathcal{S}}(d\boldsymbol{v})$ and $Q_{\mathcal{S}}(d\boldsymbol{v})$ is uniformly integrable.

(f) Suppose that $\mathcal{S}$ has a greatest element and $\boldsymbol{v}$ is of bounded variation, but we do not know whether $\bar{v} = \int_{\mathcal{S}} |d\boldsymbol{v}|$ has finite expectation. Set $z_\tau = \int_{\mathcal{S}\wedge\tau} |d\boldsymbol{v}|$ for $\tau \in \mathcal{S}$. Because $\boldsymbol{v}$ is a martingale, it is near-simple (632Ia again), and $\boldsymbol{z} = \langle z_\tau\rangle_{\tau\in\mathcal{S}}$ is near-simple (631K). By 631Ra again, there is a non-decreasing sequence $\langle\tau_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{S}$ such that $\tau_0 = \min\mathcal{S}$, $\inf_{n\in\mathbb{N}} [\![\tau_n < \max\mathcal{S}]\!] = 0$ and $[\![|z_\sigma - z_{\tau_n}| \ge 1]\!] \subseteq [\![\sigma = \tau_{n+1}]\!]$ whenever $n \in \mathbb{N}$ and $\sigma \in [\tau_n, \tau_{n+1}]$. In this case,

$$z_{\tau_{n+1}} - z_{\tau_n} \le 2\chi 1 + |v_{\tau_{n+1}} - v_{\tau_n}|$$

for every n. **P** Suppose that $\tau_n = \sigma_0 \le \ldots \le \sigma_k = \tau_{n+1}$. For $j \le k$, set

$$a_j = [\![\textstyle\sum_{i=0}^{j-1} |v_{\sigma_{i+1}} - v_{\sigma_i}| > 2\chi 1 + |v_{\tau_{n+1}} - v_{\tau_n}|]\!], \quad b_j = a_{j+1} \setminus a_j \text{ if } j < k.$$

Then, for any $j < k$,

$$\begin{aligned} b_j &\subseteq [\![v_{\sigma_{j+1}} \ne v_{\sigma_j}]\!] \subseteq [\![\sigma_j < \tau_{n+1}]\!] \\ &\subseteq [\![z_{\sigma_j} - z_{\tau_n} < \chi 1]\!] \subseteq [\![|v_{\sigma_j} - v_{\tau_n}| < \chi 1]\!] \cap [\![\textstyle\sum_{i=0}^{j-1} |v_{\sigma_{i+1}} - v_{\sigma_i}| < \chi 1]\!]; \end{aligned}$$

since also $b_j \subseteq a_{j+1}$,

$$b_j \subseteq [\![|v_{\sigma_{j+1}} - v_{\sigma_j}| > \chi 1 + |v_{\tau_{n+1}} - v_{\tau_n}|]\!].$$

Now observe that

$$[\![\sigma_{j+1} < \tau_{n+1}]\!] \subseteq [\![z_{\sigma_{j+1}} - z_{\tau_n} < \chi 1]\!] \subseteq [\![\sum_{i=0}^{j} |v_{\sigma_{i+1}} - v_{\sigma_i}| < \chi 1]\!]$$

is disjoint from a_{j+1} and therefore from b_j, so

$$b_j \subseteq [\![\sigma_{j+1} = \tau_{n+1}]\!] \subseteq [\![|v_{\sigma_{j+1}} - v_{\sigma_j}| = |v_{\tau_{n+1}} - v_{\sigma_j}|]\!].$$

Putting these together,

$$\begin{aligned} b_j &\subseteq [\![|v_{\sigma_j} - v_{\tau_n}| < \chi 1]\!] \cap [\![|v_{\sigma_{j+1}} - v_{\sigma_j}| > \chi 1 + |v_{\tau_{n+1}} - v_{\tau}|]\!] \\ &\qquad \cap [\![|v_{\sigma_{j+1}} - v_{\sigma_j}| = |v_{\tau_{n+1}} - v_{\sigma_j}|]\!] \\ &\subseteq [\![|v_{\sigma_j} - v_{\tau_n}| < \chi 1]\!] \cap [\![|v_{\tau_{n+1}} - v_{\sigma_j}| > \chi 1 + |v_{\tau_{n+1}} - v_{\tau_n}|]\!] = 0. \end{aligned}$$

This is true for every $j < k$, so $a_k = 0$ and $\sum_{i=0}^{j-1} |v_{\sigma_{i+1}} - v_{\sigma_i}| \le 2\chi 1 + |v_{\tau_{n+1}} - v_{\tau_n}|$. As $\sigma_0, \dots, \sigma_k$ are arbitrary, $z_{\tau_{n+1}} - z_{\tau_n} \le 2\chi 1 + |v_{\tau_{n+1}} - v_{\tau_n}|$. **Q**

Since $\boldsymbol{v}$, being a martingale, is an L^1-process, we see that $z_{\tau_n} \in L^1_{\bar\mu}$ for every n. Consequently (e) tells us that $Q_{\mathcal{S}\wedge\tau_n}(d\boldsymbol{v})$ is uniformly integrable and $\tau_n \in \mathcal{S}_{\boldsymbol{v}}$ for every n. As $\sup_{n\in\mathbb{N}} [\![\tau_n = \max\mathcal{S}]\!] = 1$, $\mathcal{S}_{\boldsymbol{v}}$ is a covering ideal of $\mathcal{S}$.

(g) Next, suppose only that $\mathcal{S}$ has a greatest member. By the fundamental theorem of martingales (643L), there is a local martingale $\hat{\boldsymbol{v}}$ with domain $\mathcal{S}$ such that $\mathrm{Osclln}(\hat{\boldsymbol{v}}) \le \chi 1$ and $\tilde{\boldsymbol{v}} = \boldsymbol{v} - \hat{\boldsymbol{v}}$ has bounded variation. Now, for any $\tau \in \mathcal{S}$,

$$Q_{\mathcal{S}\wedge\tau}(d\boldsymbol{v}) \subseteq Q_{\mathcal{S}\wedge\tau}(d\hat{\boldsymbol{v}}) + Q_{\mathcal{S}\wedge\tau}(d\tilde{\boldsymbol{v}})$$

(616Dc), so $\mathcal{S}_{\boldsymbol{v}} \supseteq \mathcal{S}_{\hat{\boldsymbol{v}}} \cap \mathcal{S}_{\tilde{\boldsymbol{v}}}$ includes the intersection of two covering ideals and (being itself an ideal) is a covering ideal of $\mathcal{S}$ (611Nc).

(h) Finally, if we suppose only that $\mathcal{S}$ is order-convex and has a least member, we know from (g) that $\mathcal{S}_{\boldsymbol{v}} \cap (\mathcal{S} \wedge \tau)$ covers $\mathcal{S} \wedge \tau$ for every $\tau \in \mathcal{S}$, so $\mathcal{S}_{\boldsymbol{v}}$ covers $\mathcal{S}$, and we're done.

646Q Theorem Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous and that $\mathcal{S}$ is an order-convex sublattice of $\mathcal{T}$ with a least element. Let $\boldsymbol{v}$ be a near-simple integrator and $\boldsymbol{x}$ an S-integrable process, both with domain $\mathcal{S}$.

(a) If $\boldsymbol{v}$ is a martingale, then $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})$ is a local martingale.

(b) If $\boldsymbol{v}$ is jump-free, then $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})$ is jump-free.

(c) If $\boldsymbol{v}$ is of bounded variation, then $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})$ is of bounded variation.

proof Since $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x}) = \mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x} \times \mathbf{1}^{(\mathcal{S})}_{<})$ (646Kb), and $\boldsymbol{x} \times \mathbf{1}^{(\mathcal{S})}_{<}$ belongs to $M^0_{\text{S-i}}(\mathcal{S})$, we can suppose throughout that $\boldsymbol{x} \in M^0_{\text{S-i}}(\mathcal{S})$. Set $z_\tau = \oint_{\mathcal{S}\wedge\tau} \boldsymbol{x}\, d\boldsymbol{v}$ for $\tau \in \mathcal{S}$.

(a)(i) Write

$$\mathcal{S}_{\boldsymbol{v}} = \{\tau : \tau \in \mathcal{S},\ Q_{\mathcal{S}\wedge\tau}(d\boldsymbol{v}) \text{ is uniformly integrable}\},$$

$$\mathcal{S}' = \{\tau : \tau \in \mathcal{S},\ \|\boldsymbol{x}\restriction\mathcal{S}\wedge\tau\|_\infty < \infty\}.$$

Then $\mathcal{S}'$ is an ideal of $\mathcal{S}$, because $(\mathcal{S}\wedge\tau) \cup (\mathcal{S}\wedge\tau')$ covers $\mathcal{S}\wedge(\tau\vee\tau')$ so $\sup|\boldsymbol{x}\restriction\mathcal{S}\wedge(\tau\vee\tau')| = \sup|\boldsymbol{x}\restriction\mathcal{S}\wedge\tau| \vee \sup|\boldsymbol{x}\restriction\mathcal{S}\wedge\tau'|$ for all $\tau, \tau' \in \mathcal{S}$. As in part (a) of the proof of 646P, $\mathcal{S}_{\boldsymbol{v}}$ is an ideal in $\mathcal{S}$, so $\mathcal{S}_{\boldsymbol{v}} \cap \mathcal{S}'$ also is.

(ii) $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})\restriction\mathcal{S}_{\boldsymbol{v}}\cap\mathcal{S}'$ is a martingale. **P** Take $\sigma \le \tau$ in $\mathcal{S}_{\boldsymbol{v}}\cap\mathcal{S}'$ and work in $\mathcal{S}\wedge\tau$. There is an $M \ge 0$ such that $|\boldsymbol{x}\restriction\mathcal{S}\wedge\tau| \le M\mathbf{1}$; now $|\boldsymbol{x}\restriction\mathcal{S}\wedge\tau| \le M\mathbf{1}^{(\mathcal{S}\wedge\tau)}_{<}$. By 645La once more, $\boldsymbol{x}\restriction\mathcal{S}\wedge\tau$ belongs to the $\mathfrak{T}_{\text{S-i}}$-closure of $\{\boldsymbol{u}_< : \boldsymbol{u} \in A\}$ where $A = \{\boldsymbol{u} : \boldsymbol{u} \in M_{\mathrm{mo}}(\mathcal{S}\wedge\tau),\ |\boldsymbol{u}| \le M\mathbf{1}\}$; by 646L there is for every $\delta > 0$ a $\boldsymbol{u} \in A$ such that $\theta(z_{\sigma'} - \int_{\mathcal{S}\wedge\sigma'} \boldsymbol{u}\, d\boldsymbol{v}) \le \delta$ for every $\sigma' \in \mathcal{S}\wedge\tau$. Set $C = MQ_{\mathcal{S}\wedge\tau}(d\boldsymbol{v})$, so that C is uniformly integrable. If $\boldsymbol{u} \in M_{\mathrm{mo}}(\mathcal{S}\wedge\tau)$ and $\|\boldsymbol{u}\|_\infty \le M$, then $\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\boldsymbol{v} = \lim_{I\uparrow\mathcal{I}(\mathcal{S}\wedge\tau)} S_I(\boldsymbol{u}, d\boldsymbol{v})$ belongs to the closure $\overline{C}$ of C for the topology of convergence in measure. But now we see that z_τ also belongs to $\overline{C}$; and the same applies to z_σ, since of course $C \supseteq MQ_{\mathcal{S}\wedge\sigma}(d\boldsymbol{v})$.

Because C is uniformly integrable, $\overline{C} \subseteq L^1_{\bar\mu}$ and the norm topology of $L^1_{\bar\mu}$ agrees with the topology of convergence in measure on C (621B(c-ii)). Given $\epsilon > 0$, there is a $\delta > 0$ such that

$$\|z_\tau - u\|_1 \le \epsilon \text{ whenever } u \in \overline{C} \text{ and } \theta(z_\tau - u) \le \delta,$$

$$\|z_\sigma - u\|_1 \le \epsilon \text{ whenever } u \in \overline{C} \text{ and } \theta(z_\sigma - u) \le \delta.$$

We therefore have a $\boldsymbol{u} \in M_{\mathrm{mo}}(\mathcal{S} \wedge \tau)$ such that

$$\|z_\tau - \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\boldsymbol{v}\|_1 \le \epsilon, \quad \|z_\sigma - \int_{\mathcal{S}\wedge\sigma} \boldsymbol{u}\, d\boldsymbol{v}\|_1 \le \epsilon.$$

At this point, we repeat the trick in a different way. As $\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\boldsymbol{v} = \lim_{I\uparrow\mathcal{I}(\mathcal{S}\wedge\tau)} S_I(\boldsymbol{u}, d\boldsymbol{v})$ for the topology of convergence in measure, and $S_I(\boldsymbol{u}, d\boldsymbol{v}) \in C$ for every $I \in \mathcal{I}(\mathcal{S} \wedge \tau)$, $\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\boldsymbol{v} = 1\mathrm{lim}_{I\uparrow\mathcal{I}(\mathcal{S}\wedge\tau)} S_I(\boldsymbol{u}, d\boldsymbol{v})$, the limit for the norm topology on $L^1_{\bar\mu}$. There is therefore an $I_0 \in \mathcal{I}(\mathcal{S} \wedge \tau)$ such that

$$\|\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\boldsymbol{v} - S_I(\boldsymbol{u}, d\boldsymbol{v})\|_1 \le \epsilon$$

whenever $I_0 \subseteq I \in \mathcal{I}(\mathcal{S} \wedge \tau)$. Similarly, there is a $J_0 \in \mathcal{I}(\mathcal{S} \wedge \sigma)$ such that

$$\|\int_{\mathcal{S}\wedge\sigma} \boldsymbol{u}\, d\boldsymbol{v} - S_J(\boldsymbol{u}, d\boldsymbol{v})\|_1 \le \epsilon$$

whenever $J_0 \subseteq J \in \mathcal{I}(\mathcal{S} \wedge \sigma)$. Let K be the sublattice of $\mathcal{S} \wedge \tau$ generated by $I_0 \cup J_0 \cup \{\tau\}$. Then $K \wedge \sigma \supseteq J_0$ so

$$\|\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\boldsymbol{v} - S_K(\boldsymbol{u}, d\boldsymbol{v})\|_1 \le \epsilon, \quad \|\int_{\mathcal{S}\wedge\sigma} \boldsymbol{u}\, d\boldsymbol{v} - S_{K\wedge\sigma}(\boldsymbol{u}, d\boldsymbol{v})\|_1 \le \epsilon$$

and

$$\|z_\tau - S_K(\boldsymbol{u}, d\boldsymbol{v})\|_1 \le 2\epsilon, \quad \|z_\sigma - S_{K\wedge\sigma}(\boldsymbol{u}, d\boldsymbol{v})\|_1 \le 2\epsilon.$$

Now

$$\begin{aligned} P_\sigma(u_{\sigma'} \times (v_{\tau'} - v_{\sigma'})) &= P_\sigma P_{\sigma'}(u_{\sigma'} \times (v_{\tau'} - v_{\sigma'})) = P_\sigma(u_{\sigma'} \times (P_{\sigma'} v_{\tau'} - P_{\sigma'} v_{\sigma'})) \\ &= P_\sigma(u_{\sigma'} \times (v_{\sigma'} - v_{\sigma'})) = 0 \end{aligned}$$

whenever $\sigma \le \sigma' \le \tau' \le \tau$, so $P_\sigma S_{K\vee\sigma}(\boldsymbol{u}, d\boldsymbol{v}) = 0$ and

$$P_\sigma S_K(\boldsymbol{u}, d\boldsymbol{v}) = P_\sigma S_{K\wedge\sigma}(\boldsymbol{u}, d\boldsymbol{v}) = S_{K\wedge\sigma}(\boldsymbol{u}, d\boldsymbol{v})$$

(using 613G(a-ii)). Since P_σ is $\|\,\|_1$-reducing,

$$\|P_\sigma z_\tau - z_\sigma\|_1 \le 4\epsilon.$$

As ϵ is arbitrary, $P_\sigma z_\tau = z_\sigma$; as σ and τ are arbitrary, $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x}) \restriction \mathcal{S}'$ is a martingale. **Q**

(iii) $\mathcal{S}'$ is a covering ideal of $\mathcal{S}$. **P** There is a non-negative moderately oscillatory process $\boldsymbol{u}^* = \langle u^*_\sigma \rangle_{\sigma\in\mathcal{S}}$ such that $|\boldsymbol{x}| \le \boldsymbol{u}^*_<$, and by 642M we can suppose that $\boldsymbol{u}^*$ is near-simple. Take $\tau \in \mathcal{S}$ and $\epsilon > 0$. There is an $M > 0$ such that $\bar\mu[\![\sup |\boldsymbol{u}^*| \ge M]\!] \le \epsilon$. Because $\boldsymbol{u}^*$ is near-simple, there is a $\tau' \in \mathcal{S} \wedge \tau$ such that $[\![u^*_\sigma \ge M]\!] \subseteq [\![\sigma = \tau']\!]$ for every $\sigma \in \mathcal{S} \wedge \tau'$ and $[\![\tau' < \tau]\!] \subseteq [\![u^*_\tau \ge M]\!]$ (631Ra once more, applied in $\mathcal{S} \wedge \tau$ with $\delta = M$). So $\bar\mu[\![\tau' < \tau]\!] \le \epsilon$. Next, for $\sigma \in \mathcal{S} \wedge \tau'$, $u^*_{I<\sigma} \le M\chi 1$ for every $I \in \mathcal{I}(\mathcal{S} \wedge \sigma)$, because $[\![\sigma' < \sigma]\!] \subseteq [\![\sigma' < \tau']\!] \subseteq [\![u^*_{\sigma'} \le M]\!]$ for every $\sigma' \in \mathcal{S} \wedge \sigma$. So $u^*_{<\sigma} \le M\chi 1$ for $\sigma \le \tau'$, and

$$\sup |\boldsymbol{x} \restriction \mathcal{S} \wedge \tau'| \le \sup |\boldsymbol{u}^*_< \restriction \mathcal{S} \wedge \tau'| \le M\chi 1.$$

Thus $\tau' \in \mathcal{S}'$; as ϵ is arbitrary, $\mathcal{S}'$ covers $\{\tau\}$; as τ is arbitrary, $\mathcal{S}'$ covers $\mathcal{S}$. **Q**

(iv) By 646P, $\mathcal{S}_{\boldsymbol{v}}$ is a covering ideal of $\mathcal{S}$. So $\mathcal{S}_{\boldsymbol{v}} \cap \mathcal{S}'$ covers $\mathcal{S}$ (611Nc again). Since $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x}) \restriction \mathcal{S}_{\boldsymbol{v}} \cap \mathcal{S}'$ is a martingale, $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})$ is a local martingale.

(b) Let $\boldsymbol{u}^* \in M_{\mathrm{mo}} = M_{\mathrm{mo}}(\mathcal{S})^+$ be such that $|\boldsymbol{x}| \le \boldsymbol{u}^*_<$ and set $A = \{\boldsymbol{y} : \boldsymbol{y} \in M^0_{\text{S-i}}(\mathcal{S}),\ |\boldsymbol{y}| \le \boldsymbol{u}^*_<\}$, $B = \{\boldsymbol{u} : \boldsymbol{u} \in M_{\mathrm{mo}},\ |\boldsymbol{u}| \le \boldsymbol{u}^*\}$. Then $\boldsymbol{x} \in \overline{\{\boldsymbol{u}_< : \boldsymbol{u} \in B\}}$, taking the closure for the S-integration topology (645La yet again). By 646O, $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})$ belongs to the closure of $\{\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{u}_<) : \boldsymbol{u} \in B\}$ for the ucp topology on $M_{\text{o-b}}(\mathcal{S})$. But observe that if $\boldsymbol{u} \in M_{\mathrm{mo}}$ then

$$\oint_{\mathcal{S}\wedge\tau} (\boldsymbol{u}_< \restriction \mathcal{S} \wedge \tau)\, d\boldsymbol{v} = \oint_{\mathcal{S}\wedge\tau} (\boldsymbol{u} \restriction \mathcal{S} \wedge \tau)_<\, d\boldsymbol{v} = \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\boldsymbol{v}$$

so $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{u}_<) = ii_{\boldsymbol{v}}(\boldsymbol{u})$. And we know that $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is jump-free for every $\boldsymbol{u} \in M_{\mathrm{mo}}$, by 618Q. So $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})$ belongs to the closure of $M_{\text{j-f}}(\mathcal{S})$ in $M_{\text{o-b}}(\mathcal{S})$ and is jump-free by 618Ga.

(c) Take $\boldsymbol{u}^*$, A and B as in (b) just above, and set $\bar u = \sup|\boldsymbol{u}^*| \times \int_{\mathcal{S}}|d\boldsymbol{v}|$. Then $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is of bounded variation, with

$$\int_{\mathcal{S}}|d(ii_{\boldsymbol{v}}(\boldsymbol{u}))| \le \sup|\boldsymbol{u}| \times \int_{\mathcal{S}}|d\boldsymbol{v}| \le \bar u$$

for every $\boldsymbol{u} \in B$ (614T). As before,

$$\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x}) \in \overline{\{\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{u}_<) : \boldsymbol{u} \in B\}} = \overline{\{ii_{\boldsymbol{v}}(\boldsymbol{u}) : \boldsymbol{u} \in B\}}$$

where the closures are taken in the ucp topology. But this means that $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})$ belongs to the closure of $\{\boldsymbol{u} : \boldsymbol{u} \in M_{\mathrm{bv}}(\mathcal{S}),\ \int_{\mathcal{S}}|d\boldsymbol{u}| \le \bar u\}$ for the product topology on $(L^0)^{\mathcal{S}}$, and $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})$ is of bounded variation, by 614N.

646R Theorem Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous and that $\mathcal{S}$ is an order-convex sublattice of $\mathcal{T}$. Let $\boldsymbol{v}$ be a near-simple integrator and $\boldsymbol{x}$, $\boldsymbol{x}'$ two S-integrable processes, all with domain $\mathcal{S}$; write $\boldsymbol{z}$ for the indefinite S-integral $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})$. Then $\oint_{\mathcal{S}}\boldsymbol{x}'\,d\boldsymbol{z} = \oint_{\mathcal{S}}\boldsymbol{x}' \times \boldsymbol{x}\,d\boldsymbol{v}$.

proof (a) Of course it will be enough to deal with the case of non-empty $\mathcal{S}$. We know from 646N that $\boldsymbol{z}$ is a near-simple integrator, and from 645Kb that $\boldsymbol{x} \times \boldsymbol{x}'$ is S-integrable, so the integrals here are well-defined. Since

$$\boldsymbol{z} = \mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x}) = \mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x} \times \mathbf{1}^{(\mathcal{S})}_<), \qquad \oint_{\mathcal{S}}\boldsymbol{x}'\,d\boldsymbol{z} = \oint_{\mathcal{S}}\boldsymbol{x}' \times \mathbf{1}^{(\mathcal{S})}_<\,d\boldsymbol{z}$$

and

$$\oint_{\mathcal{S}}\boldsymbol{x}' \times \boldsymbol{x}\,d\boldsymbol{v} = \oint_{\mathcal{S}}\boldsymbol{x}' \times \boldsymbol{x} \times \mathbf{1}^{(\mathcal{S})}_<\,d\boldsymbol{v} = \oint_{\mathcal{S}}(\boldsymbol{x}' \times \mathbf{1}^{(\mathcal{S})}_<) \times (\boldsymbol{x} \times \mathbf{1}^{(\mathcal{S})}_<)d\boldsymbol{v}$$

(646Kb again), we can suppose throughout that $\boldsymbol{x}$ and $\boldsymbol{x}'$ belong to $M^0_{\text{S-i}} = M^0_{\text{S-i}}(\mathcal{S})$. There is a $\boldsymbol{u}^* \in M^+_{\mathrm{mo}} = M_{\mathrm{mo}}(\mathcal{S})^+$ such that $|\boldsymbol{x}| \vee |\boldsymbol{x}'| \le \boldsymbol{u}^*_<$; by 642M and 645Lb, we can suppose that $\boldsymbol{u}^* \in M_{\text{n-s}}(\mathcal{S})^+$ and that $\boldsymbol{x}'$ belongs to the $\mathfrak{T}_{\text{S-i}}$-closure of $\{\boldsymbol{u}_< : \boldsymbol{u} \in M_{\mathrm{simp}}(\mathcal{S}),\ |\boldsymbol{u}| \le \boldsymbol{u}^*\}$.

(b) Take any simple process $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ and $\epsilon > 0$. Write $\bar u$ for $\sup|\boldsymbol{u}|$.

(i) There is a $\mathfrak{T}_{\text{S-i}}$-neighbourhood G_0 of $\boldsymbol{x}$ such that whenever $\check{\boldsymbol{u}} \in M_{\mathrm{mo}}$ is such that $|\check{\boldsymbol{u}}| \le \boldsymbol{u}^*$ and $\check{\boldsymbol{u}}_< \in G_0$ then $\theta(\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{z} - \int_{\mathcal{S}}\boldsymbol{u}\,d\check{\boldsymbol{z}}) \le \epsilon$, where $\check{\boldsymbol{z}} = ii_{\boldsymbol{v}}(\check{\boldsymbol{u}})$. **P** Let $(\tau_0,\dots,\tau_n)$ be a breakpoint string for $\boldsymbol{u}$. Take $\delta > 0$ such that $\theta(\bar u \times z) \le \frac{\epsilon}{2n+2}$ whenever $\theta(z) \le \delta$. By 646L, there is a $\mathfrak{T}_{\text{S-i}}$-neighbourhood G_0 of $\boldsymbol{x}$ such that whenever $\check{\boldsymbol{u}} \in M_{\mathrm{mo}}$ is such that $|\check{\boldsymbol{u}}| \le \boldsymbol{u}^*$ and $\check{\boldsymbol{u}}_< \in G_0$ then $\theta(\check z_\tau - z_\tau) \le \delta$ for every $\tau \in \mathcal{S}$, where $\langle\check z_\tau\rangle_{\tau\in\mathcal{S}} = \check{\boldsymbol{z}} = ii_{\boldsymbol{v}}(\check{\boldsymbol{u}})$ and $\langle z_\tau\rangle_{\tau\in\mathcal{S}} = \boldsymbol{z}$. In this case,

$$\begin{aligned}\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{z} - \int_{\mathcal{S}}\boldsymbol{u}\,d\check{\boldsymbol{z}} = u_\downarrow \times (z_{\tau_0} - \check z_{\tau_0}) &+ \sum_{i=0}^{n-1} u_{\tau_i} \times ((z_{\tau_{i+1}} - \check z_{\tau_{i+1}}) - (z_{\tau_i} - \check z_{\tau_i}))\\ &+ u_{\tau_n} \times ((z_\uparrow - \check z_\uparrow) - (z_{\tau_n} - \check z_{\tau_n}))\end{aligned}$$

where $u_\downarrow = \lim_{\sigma\downarrow\mathcal{S}} u_\sigma$, $z_\uparrow = \lim_{\sigma\uparrow\mathcal{S}} z_\sigma$ and $\check z_\uparrow = \lim_{\sigma\uparrow\mathcal{S}}\check z_\sigma$ (614C again, since $\lim_{\sigma\downarrow\mathcal{S}} z_\sigma = \lim_{\sigma\downarrow\mathcal{S}}\check z_\sigma = 0$, by 646M and 613J(f-i) again). Consequently

$$\begin{aligned}\theta(\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{z} - \int_{\mathcal{S}}\boldsymbol{u}\,d\check{\boldsymbol{z}}) &\le (2n+2)\sup_{\sigma,\tau\in\mathcal{S}}\theta(u_\sigma \times (z_\tau - \check z_\tau))\\ &\le (2n+2)\sup_{\tau\in\mathcal{S}}\theta(\bar u \times (z_\tau - \check z_\tau)) \le \epsilon.\end{aligned}$$

So this G_0 will serve. **Q**

(ii) There is a $\mathfrak{T}_{\text{S-i}}$-neighbourhood G_1 of $\boldsymbol{x}$ such that whenever $\check{\boldsymbol{u}} \in M_{\mathrm{mo}}$ is such that $|\check{\boldsymbol{u}}| \le \boldsymbol{u}^*$ and $\check{\boldsymbol{u}}_< \in G_1$ then $\theta(\oint_{\mathcal{S}}\boldsymbol{u}_< \times \check{\boldsymbol{u}}_<\,d\boldsymbol{v} - \oint_{\mathcal{S}}\boldsymbol{u}_< \times \boldsymbol{x}\,d\boldsymbol{v}) \le \epsilon$. **P** Set $C = \{\boldsymbol{y} : \boldsymbol{y} \in M^0_{\text{S-i}},\ |\boldsymbol{y}| \le \boldsymbol{u}^*_<\}$ and $D = \{\boldsymbol{y} : \boldsymbol{y} \in M^0_{\text{S-i}},\ |\boldsymbol{y}| \le (|\boldsymbol{u}| \times \boldsymbol{u}^*)_<\}$. The operators $\boldsymbol{y} \mapsto \boldsymbol{u}_< \times \boldsymbol{y} : C \to D$ and $\boldsymbol{y} \mapsto \oint_{\mathcal{S}}\boldsymbol{y}\,d\boldsymbol{v} : D \to L^0$ are continuous when C and D are given the S-integration topology and L^0 is given the topology of convergence in measure (645E(a-v-α), 645S), so $\boldsymbol{y} \mapsto \oint_{\mathcal{S}}\boldsymbol{u}_< \times \boldsymbol{y}\,d\boldsymbol{v} : C \to L^0$ is continuous and there is a neighbourhood G_1 of $\boldsymbol{x}$ such that $\theta(\oint_{\mathcal{S}}\boldsymbol{u}_< \times \boldsymbol{y}\,d\boldsymbol{v} - \oint_{\mathcal{S}}\boldsymbol{u}_< \times \boldsymbol{x}\,d\boldsymbol{v}) \le \epsilon$ whenever $\boldsymbol{y} \in G_1 \cap C$; this G_1 serves. **Q**

(iii) Now there is a $\check{\boldsymbol{u}} \in M_{\mathrm{mo}}$ such that $|\check{\boldsymbol{u}}| \le \boldsymbol{u}^*$ and $\check{\boldsymbol{u}}_< \in G_0 \cap G_1$ (645La, as always). In this case, writing $\check{\boldsymbol{z}} = ii_{\boldsymbol{v}}(\check{\boldsymbol{u}})$ as before, $\theta(\int_{\mathcal{S}} \boldsymbol{u}\,d\check{\boldsymbol{z}} - \int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{z}) \le \epsilon$ and $\theta(\oint_{\mathcal{S}} \boldsymbol{u}_< \times \check{\boldsymbol{u}}_<\,d\boldsymbol{v} - \oint_{\mathcal{S}} \boldsymbol{u}_< \times \boldsymbol{x}\,d\boldsymbol{v}) \le \epsilon$. But again we know from 617E that $\int_{\mathcal{S}} \boldsymbol{u}\,d\check{\boldsymbol{z}} = \int_{\mathcal{S}} \boldsymbol{u} \times \check{\boldsymbol{u}}\,d\boldsymbol{v}$, and from 645R(a-i) that $\oint_{\mathcal{S}} \boldsymbol{u}_< \times \check{\boldsymbol{u}}_<\,d\boldsymbol{v} = \int_{\mathcal{S}} \boldsymbol{u} \times \check{\boldsymbol{u}}\,d\boldsymbol{v}$. So we have

$$\begin{aligned}\theta(\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{z} - \oint_{\mathcal{S}} \boldsymbol{u}_< \times \boldsymbol{x}\,d\boldsymbol{v})&\\ \le \theta(\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{z} - \int_{\mathcal{S}} \boldsymbol{u}\,d\check{\boldsymbol{z}}) &+ \theta(\oint_{\mathcal{S}} \boldsymbol{u}_< \times \check{\boldsymbol{u}}_<\,d\boldsymbol{v} - \oint_{\mathcal{S}} \boldsymbol{u}_< \times \boldsymbol{x}\,d\boldsymbol{v}) \le 2\epsilon.\end{aligned}$$

(iv) As ϵ is arbitrary, $\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{z} = \oint_{\mathcal{S}} \boldsymbol{u}_< \times \boldsymbol{x}\,d\boldsymbol{v}$.

(c) We know that, setting $C = \{\boldsymbol{y} : \boldsymbol{y} \in M^0_{\text{S-i}},\ |\boldsymbol{y}| \le \boldsymbol{u}^*_<\}$ and $D = \{\boldsymbol{y} : \boldsymbol{y} \in M^0_{\text{S-i}},\ |\boldsymbol{y}| \le (\boldsymbol{u}^* \times \boldsymbol{u}^*)_<\}$,

$$\boldsymbol{y} \mapsto \boldsymbol{y} \times \boldsymbol{x} : C \to D$$

is $(\mathfrak{T}_{\text{S-i}}, \mathfrak{T}_{\text{S-i}})$-continuous and

$$\boldsymbol{y} \mapsto \oint_{\mathcal{S}} \boldsymbol{y}\,d\boldsymbol{z} : C \to L^0, \quad \boldsymbol{y} \mapsto \oint_{\mathcal{S}} \boldsymbol{y}\,d\boldsymbol{v} : D \to L^0$$

are $\mathfrak{T}_{\text{S-i}}$-continuous (645E(a-v-$\alpha$), 645S again), so

$$B = \{\boldsymbol{y} : \boldsymbol{y} \in C,\ \oint_{\mathcal{S}} \boldsymbol{y} \times \boldsymbol{x}\,d\boldsymbol{v} = \oint_{\mathcal{S}} \boldsymbol{y}\,d\boldsymbol{z}\}$$

is (relatively) closed in C for $\mathfrak{T}_{\text{S-i}}$; and it includes $C' = \{\boldsymbol{u}_< : \boldsymbol{u} \in M_{\text{simp}}(\mathcal{S}),\ |\boldsymbol{u}| \le \boldsymbol{u}^*\}$, by (b). But we chose $\boldsymbol{u}^*$ so that $\boldsymbol{x}'$ belongs to C and is in the $\mathfrak{T}_{\text{S-i}}$-closure of C', so $\boldsymbol{x}' \in B$, as required.

Mnemonic $d(\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})) = \boldsymbol{x}\,d\boldsymbol{v}$.

646S Proposition Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous and that $\mathcal{S}$ is an order-convex sublattice of $\mathcal{T}$. Let $\boldsymbol{v}$ be a near-simple integrator and $\boldsymbol{x}$ an S-integrable process, both with domain $\mathcal{S}$. Then $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x}) - \mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})_< = \boldsymbol{x} \times (\boldsymbol{v} - \boldsymbol{v}_<) \times \mathbf{1}^{(\mathcal{S})}_<$.

proof (a) To begin with, suppose that $\boldsymbol{x} \in M^0_{\text{S-i}} = M^0_{\text{S-i}}(\mathcal{S})$. By 646N, $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})$ is near-simple, so we can speak of its previsible version $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})_<$. Let $\boldsymbol{u}^* \in M^+_{\text{mo}} = M_{\text{mo}}(\mathcal{S})^+$ be such that $|\boldsymbol{x}| \le \boldsymbol{u}^*_<$. Set $C = \{\boldsymbol{y} : \boldsymbol{y} \in M^0_{\text{S-i}},\ |\boldsymbol{y}| \le \boldsymbol{u}^*_<\}$, $C' = \{\boldsymbol{u}_< : \boldsymbol{u} \in M_{\text{mo}},\ |\boldsymbol{u}| \le \boldsymbol{u}^*\}$ and $B = \{\boldsymbol{y} : \boldsymbol{y} \in C,\ \mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{y}) - \mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{y})_< = \boldsymbol{y} \times (\boldsymbol{v} - \boldsymbol{v}_<)\}$. If $\boldsymbol{u} \in M_{\text{mo}}$ and $|\boldsymbol{u}| \le \boldsymbol{u}^*$, then

$$\begin{aligned}\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{u}_<) - \mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{u}_<)_< &= ii_{\boldsymbol{v}}(\boldsymbol{u}) - ii_{\boldsymbol{v}}(\boldsymbol{u})_< \\ &= \boldsymbol{u}_< \times (\boldsymbol{v} - \boldsymbol{v}_<) \qquad \text{(646Kc)}\end{aligned}$$

by 641Q, so $\boldsymbol{u}_< \in B$; thus $C' \subseteq B$. We know that $\mathrm{S}ii_{\boldsymbol{v}} : C \to M_{\text{n-s}}(\mathcal{S})$ is continuous for $\mathfrak{T}_{\text{S-i}}$ and the ucp topology (646O), while $\boldsymbol{u} \mapsto \boldsymbol{u}_< : M_{\text{mo}} \to M_{\text{mo}}$ is continuous for the ucp topology (641G(e-ii)), so $\boldsymbol{y} \mapsto \mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{y}) - \mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{y})_<$ is continous for $\mathfrak{T}_{\text{S-i}}$ and the ucp topology and therefore for $\mathfrak{T}_{\text{S-i}}$ and the product topology on $(L^0)^{\mathcal{S}}$. At the same time, the embedding $M^0_{\text{S-i}} \hookrightarrow (L^0)^{\mathcal{S}}$ is continuous (645E(a-iii)), so $\boldsymbol{y} \mapsto \boldsymbol{y} \times (\boldsymbol{v} - \boldsymbol{v}_<) : M^0_{\text{S-i}} \to (L^0)^{\mathcal{S}}$ is continuous and B is closed in C. Since B includes the dense subset C' of C (see 645La), $B = C$, so

$$\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x}) - \mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})_< = \boldsymbol{x} \times (\boldsymbol{v} - \boldsymbol{v}_<) = \boldsymbol{x} \times \mathbf{1}^{(\mathcal{S})}_< \times (\boldsymbol{v} - \boldsymbol{v}_<).$$

(b) For the general case of S-integrable $\boldsymbol{x}$, set $\boldsymbol{y} = \boldsymbol{x} \times \mathbf{1}^{(\mathcal{S})}_< \in M^0_{\text{S-i}}(\mathcal{S})$; then $\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x}) = \mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{y})$, so

$$\mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x}) - \mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x})_< = \mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{y}) - \mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{y})_< = \boldsymbol{y} \times (\boldsymbol{v} - \boldsymbol{v}_<) = \boldsymbol{x} \times (\boldsymbol{v} - \boldsymbol{v}_<) \times \mathbf{1}^{(\mathcal{S})}_<.$$

646T Itô's Formula, fourth form Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous, and that $\mathcal{S}$ is an order-convex sublattice of $\mathcal{T}$ with a least element. Let $\boldsymbol{v}$ be a jump-free integrator with domain $\mathcal{S}$, and $\boldsymbol{v}^*$ its quadratic variation; let $h : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that its derivative h' is locally Lipschitz, that is, Lipschitz on every bounded set. If $h'' : \mathbb{R} \to \mathbb{R}$ is a locally bounded Borel measurable function Lebesgue-almost-everywhere equal to the derivative of h', then

$$\oint_{\mathcal{S}} \boldsymbol{x}\, d(\bar{h}\boldsymbol{v}) = \oint_{\mathcal{S}} \boldsymbol{x} \times \bar{h}'\boldsymbol{v}\, d\boldsymbol{v} + \frac{1}{2}\oint_{\mathcal{S}} \boldsymbol{x} \times \bar{h}''\boldsymbol{v}\, d\boldsymbol{v}^*$$

for every $\boldsymbol{x} \in M_{\text{S-i}} = M_{\text{S-i}}(\mathcal{S})$.

proof (a) To begin with (down to the end of (b)), suppose that h' is continuously differentiable and that its derivative is everywhere equal to h''. As before, we need only to prove the result for $\boldsymbol{x} \in M^0_{\text{S-i}}(\mathcal{S})$.

We need to know that if $\boldsymbol{y} \in M_{\text{po-b}}(\mathcal{S})$ then $\boldsymbol{y} \times \bar{h}'\boldsymbol{v} = \boldsymbol{y} \times (\bar{h}'\boldsymbol{v})_<$ and $\boldsymbol{y} \times \bar{h}''\boldsymbol{v} = \boldsymbol{y} \times (\bar{h}''\boldsymbol{v})_<$. **P** By 618Ga, $\bar{h}'\boldsymbol{v}$ is jump-free. Set $e_\sigma = \sup_{\tau\in\mathcal{S}} [\![\tau < \sigma]\!]$ for $\sigma \in \mathcal{S}$. Expressing $\bar{h}'\boldsymbol{v}$ as $\langle v'_\sigma\rangle_{\sigma\in\mathcal{S}}$ and $\boldsymbol{y}$ as $\langle y_\sigma\rangle_{\sigma\in\mathcal{S}}$, we see from 641Na that $(v'_\sigma - v'_{<\sigma}) \times \chi e_\sigma = 0$, while $y_\sigma \times \chi(1 \setminus e_\sigma) = 0$, for every $\sigma \in \mathcal{S}$; so $y_\sigma \times v'_\sigma = y_\sigma \times v'_{\sigma<}$ for every σ, and $\boldsymbol{y} \times \bar{h}'\boldsymbol{v} = \boldsymbol{y} \times (\bar{h}'\boldsymbol{v})_<$. Similarly, $\boldsymbol{y} \times \bar{h}''\boldsymbol{v} = \boldsymbol{y} \times (\bar{h}''\boldsymbol{v})_<$. **Q**

It follows that if $\boldsymbol{u} \in M_{\text{mo}} = M_{\text{mo}}(\mathcal{S})$ then

$$(\boldsymbol{u} \times \bar{h}'\boldsymbol{v})_< = \boldsymbol{u}_< \times (\bar{h}'\boldsymbol{v})_< = \boldsymbol{u}_< \times \bar{h}'\boldsymbol{v}, \quad (\boldsymbol{u} \times \bar{h}''\boldsymbol{v})_< = \boldsymbol{u}_< \times \bar{h}''\boldsymbol{v}.$$

(b) Now for the main argument. Let $\boldsymbol{u} \in M^+_{\text{mo}}$ be such that $|\boldsymbol{x}| \le \boldsymbol{u}_<$, and set

$$\boldsymbol{u}^* = \boldsymbol{u} \times (|\bar{h}'\boldsymbol{v}| \vee |\bar{h}''\boldsymbol{v}|),$$

$$A = \{\boldsymbol{y} : \boldsymbol{y} \in M^0_{\text{S-i}},\ |\boldsymbol{y}| \le \boldsymbol{u}_<\}, \quad A^* = \{\boldsymbol{y} : \boldsymbol{y} \in M^0_{\text{S-i}},\ |\boldsymbol{y}| \le \boldsymbol{u}^*_<\}.$$

By 645Ja, $\boldsymbol{y} \times \bar{h}'\boldsymbol{v} = \boldsymbol{y} \times (\bar{h}'\boldsymbol{v})_<$ belongs to $M^0_{\text{S-i}}$ for every $\boldsymbol{y} \in A$, and $\boldsymbol{y} \mapsto \boldsymbol{y} \times \bar{h}'\boldsymbol{v} : M^0_{\text{S-i}} \to M^0_{\text{S-i}}$ is $\mathfrak{T}_{\text{S-i}}$-continuous (645E(a-v)), while $\boldsymbol{y} \mapsto \oint_{\mathcal{S}} \boldsymbol{y}\, d\boldsymbol{v}$ is continuous on A^* (645S). So $\boldsymbol{y} \mapsto \oint_{\mathcal{S}} \boldsymbol{y} \times \bar{h}'\boldsymbol{v} d\boldsymbol{v} : A \to L^0$ is continuous. Similarly, $\boldsymbol{y} \mapsto \oint_{\mathcal{S}} \boldsymbol{y} \times \bar{h}''\boldsymbol{v} d\boldsymbol{v} : A \to L^0$ and $\boldsymbol{y} \mapsto \oint_{\mathcal{S}} \boldsymbol{y}\, d(\bar{h}\boldsymbol{v}) : A \to L^0$ are continuous (recalling from 616O that $\bar{h}\boldsymbol{v}$ is an integrator, and from 631F(a-i) that it is near-simple).

Now we know from 619D that if $\boldsymbol{u}' \in M_{\text{mo}}$ then

$$\int_{\mathcal{S}} \boldsymbol{u}'\, d(\bar{h}\boldsymbol{v}) = \int_{\mathcal{S}} \boldsymbol{u}' \times \bar{h}'\boldsymbol{v}\, d\boldsymbol{v} + \frac{1}{2}\int_{\mathcal{S}} \boldsymbol{u}' \times \bar{h}''\boldsymbol{v}\, d\boldsymbol{v}^*,$$

so that

$$\oint_{\mathcal{S}} \boldsymbol{u}'_<\, d(\bar{h}\boldsymbol{v}) = \oint_{\mathcal{S}} \boldsymbol{u}'_< \times \bar{h}'\boldsymbol{v} d\boldsymbol{v} + \frac{1}{2}\oint_{\mathcal{S}} \boldsymbol{u}'_< \times \bar{h}''\boldsymbol{v} d\boldsymbol{v}^*.$$

And we know also that $\boldsymbol{x}$ belongs to the closure of $A' = \{\boldsymbol{u}'_< : \boldsymbol{u}' \in M_{\text{mo}},\ |\boldsymbol{u}'| \le \boldsymbol{u}\}$ in $M^0_{\text{S-i}}$. Since we are looking at continuous operators into a Hausdorff space, we can conclude that

$$\oint_{\mathcal{S}} \boldsymbol{x}\, d(\bar{h}\boldsymbol{v}) = \oint_{\mathcal{S}} \boldsymbol{x} \times \bar{h}'\boldsymbol{v}\, d\boldsymbol{v} + \frac{1}{2}\oint_{\mathcal{S}} \boldsymbol{x} \times \bar{h}''\boldsymbol{v}\, d\boldsymbol{v}^*.$$

(c) Now turn everything round, and start from locally bounded Borel measurable functions $g : \mathbb{R} \to \mathbb{R}$. For such a function, define $I(g) : \mathbb{R} \to \mathbb{R}$ by setting

$$\begin{aligned} I(g)(\alpha) &= \int_0^\alpha g(\beta)d\beta \text{ if } \alpha \ge 0,\\ &= -\int_\alpha^0 g(\beta)d\beta \text{ if } \alpha \le 0, \end{aligned}$$

the integrals here being with respect to Lebesgue measure, of course. Then $I(g)$ will be continuous, indeed locally Lipschitz, and its derivative will be almost everywhere equal to g; $I(g)$ will be the derivative of $I^2(g)$ everywhere. Fix an $\boldsymbol{x} \in M_{\text{S-i}}$, and let Φ be the set of locally bounded Borel measurable $g : \mathbb{R} \to \mathbb{R}$ such that

$$\oint_{\mathcal{S}} \boldsymbol{x}\, d(\overline{I^2(g)}\boldsymbol{v}) = \oint_{\mathcal{S}} \boldsymbol{x} \times \overline{I(g)}\boldsymbol{v}\, d\boldsymbol{v} + \frac{1}{2}\oint_{\mathcal{S}} \boldsymbol{x} \times \bar{g}\boldsymbol{v}\, d\boldsymbol{v}^*.$$

Then (a)-(b) tell us that Φ contains all continuous functions from $\mathbb{R}$ to itself. Now the point is that if $\langle g_n\rangle_{n\in\mathbb{N}}$ is a sequence in Φ converging pointwise to a function $g : \mathbb{R} \to \mathbb{R}$, and if $\langle g_n\rangle_{n\in\mathbb{N}}$ is locally uniformly bounded in the sense that $\sup_{n\in\mathbb{N},|\alpha|\le M} |g_n(\alpha)|$ is finite for every $M \ge 0$, then $g \in \Phi$. **P** Because $g(\alpha) = \lim_{n\to\infty} g_n(\alpha)$ for every α, $\langle \boldsymbol{x} \times \bar{g}_n\boldsymbol{v}\rangle_{n\in\mathbb{N}}$ is order*-convergent to $\boldsymbol{x} \times \bar{g}\boldsymbol{v}$. Next, $\sup_{n\in\mathbb{N}} |g_n|$ is locally bounded, so is bounded above by a continuous function (645Cb), and $\{|\bar{g}_n\boldsymbol{v}| : n \in \mathbb{N}\}$ is bounded above by a jump-free process (618Ga again). Accordingly $\{\bar{g}_n\boldsymbol{v} \times \mathbf{1}^{(\mathcal{S})}_{<} : n \in \mathbb{N}\}$ is uniformly previsibly order-bounded (use 641O), so $\{\boldsymbol{x} \times \bar{g}_n\boldsymbol{v} \times \mathbf{1}^{(\mathcal{S})}_{<} : n \in \mathbb{N}\}$ is uniformly previsibly order-bounded (because $M_{\text{po-b}}(\mathcal{S})$ is closed under multiplication, by 645D(a-ii)). By 645T,

$$\oint_{\mathcal{S}} \boldsymbol{x} \times \bar{g}\boldsymbol{v}\, d\boldsymbol{v}^* = \oint_{\mathcal{S}} \boldsymbol{x} \times \bar{g}\boldsymbol{v} \times \mathbf{1}^{(\mathcal{S})}_{<}\, d\boldsymbol{v}^*$$
$$= \lim_{n\to\infty} \oint_{\mathcal{S}} \boldsymbol{x} \times \bar{g}_n\boldsymbol{v} \times \mathbf{1}^{(\mathcal{S})}_{<}\, d\boldsymbol{v}^* = \lim_{n\to\infty} \oint_{\mathcal{S}} \boldsymbol{x} \times \bar{g}_n\boldsymbol{v}\, d\boldsymbol{v}^*.$$

Next, $\langle I(g_n)\rangle_{n\in\mathbb{N}}$ and $\langle I^2(g_n)\rangle_{n\in\mathbb{N}}$ are also locally uniformly bounded sequences converging pointwise to $I(g)$ and $I^2(g)$ respectively. So

$$\oint_{\mathcal{S}} \boldsymbol{x} \times \overline{I(g)}\boldsymbol{v}\, d\boldsymbol{v} = \lim_{n\to\infty} \oint_{\mathcal{S}} \boldsymbol{x} \times \overline{I(g_n)}\boldsymbol{v}\, d\boldsymbol{v},$$

$$\oint_{\mathcal{S}} \boldsymbol{x} \times \overline{I^2(g)}\boldsymbol{v}\, d\boldsymbol{v} = \lim_{n\to\infty} \oint_{\mathcal{S}} \boldsymbol{x} \times \overline{I^2(g_n)}\boldsymbol{v}\, d\boldsymbol{v}$$
$$= \lim_{n\to\infty} \oint_{\mathcal{S}} \boldsymbol{x} \times \overline{I(g_n)}\boldsymbol{v}\, d\boldsymbol{v} + \frac{1}{2}\oint_{\mathcal{S}} \boldsymbol{x} \times \bar{g}_n\boldsymbol{v}\, d\boldsymbol{v}^*$$
$$= \oint_{\mathcal{S}} \boldsymbol{x} \times \overline{I(g)}\boldsymbol{v}\, d\boldsymbol{v} + \frac{1}{2}\oint_{\mathcal{S}} \boldsymbol{x} \times \bar{g}\boldsymbol{v}\, d\boldsymbol{v}^*$$

and $g \in \Phi$. **Q**

Consequently Φ contains all locally bounded Borel measurable functions (645Cc).

(d) Returning to the stated hypothesis, in which $\boldsymbol{x} \in M_{\text{S-i}}$, $h : \mathbb{R} \to \mathbb{R}$ is differentiable, its derivative h' is locally Lipschitz, and h'' is a locally bounded Borel measurable function equal almost everywhere to the derivative of h', set $\beta = h(0)$ and $\gamma = h'(0)$; then

$$h'(\alpha) = \gamma + I(h'')(\alpha), \quad h(\alpha) = \beta + \gamma\alpha + I^2(h'')(\alpha)$$

for every α. Setting $g = h''$, we have

$$\bar{h}'\boldsymbol{v} = \gamma\mathbf{1} + \overline{I(g)}\boldsymbol{v}, \quad \bar{h}\boldsymbol{v} = \beta\mathbf{1} + \gamma\boldsymbol{v} + \overline{I^2(g)}\boldsymbol{v}.$$

Now

$$\oint_{\mathcal{S}} \boldsymbol{x}\, d(\bar{h}\boldsymbol{v}) = \beta\oint_{\mathcal{S}} \boldsymbol{x}\, d\mathbf{1} + \gamma\oint_{\mathcal{S}} \boldsymbol{x}\, d\boldsymbol{v} + \oint_{\mathcal{S}} \boldsymbol{x}\, d(\overline{I^2(g)}\boldsymbol{v})$$
$$= 0 + \gamma\oint_{\mathcal{S}} \boldsymbol{x}\, d\boldsymbol{v} + \oint_{\mathcal{S}} \boldsymbol{x} \times \overline{I(g)}\boldsymbol{v}\, d\boldsymbol{v} + \frac{1}{2}\oint_{\mathcal{S}} \boldsymbol{x} \times \bar{g}\boldsymbol{v}\, d\boldsymbol{v}^*$$

(using 645R(c-ii) and (c) above)

$$= \oint_{\mathcal{S}} \boldsymbol{x} \times \bar{h}'\boldsymbol{v}\, d\boldsymbol{v} + \frac{1}{2}\oint_{\mathcal{S}} \boldsymbol{x} \times \bar{h}''\boldsymbol{v}\, d\boldsymbol{v}^*$$

as required.

646X Basic exercises (a) In 646F, show that $R(\bar{h}\boldsymbol{u}', \bar{h}\boldsymbol{u}'') = \bar{h}R(\boldsymbol{u}', \boldsymbol{u}'')$ whenever $\boldsymbol{u}' \in (L^0)^{\mathcal{S}\wedge\tau}$, $\boldsymbol{u}'' \in (L^0)^{\mathcal{S}\vee\tau}$ and $h : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function.

(b) In 646H, show that $R^*(\bar{h}\boldsymbol{u}) = \bar{h}R^*(\boldsymbol{u})$ whenever $\boldsymbol{u} \in (L^0)^{\mathcal{S}}$ and $h : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function with $h(0) = 0$.

(c) Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous, $\mathcal{S}$ is an order-convex sublattice of $\mathcal{T}$ and $\bar{u} \in (L^0)^+$. Show that $\{\boldsymbol{u} : \boldsymbol{u} \in M_{\text{n-s}}(\mathcal{S}),\ \sup|\boldsymbol{u}| \le \bar{u}\}$ has a greatest member defined by setting $u_\sigma = \sup\{u : u \in L^0(\mathfrak{A}_\sigma), u \le \bar{u}\}$ for $\sigma \in \mathcal{S}$. (*Hint*: use 611H and 632F.)

(d) In 646R, show that $\mathrm{S}ii_{\boldsymbol{z}}(\boldsymbol{x}) = \mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x}' \times \boldsymbol{x})$.

(e) In 646T, show that $\mathrm{S}ii_{\bar{h}\boldsymbol{v}}(\boldsymbol{x}) = \mathrm{S}ii_{\boldsymbol{v}}(\boldsymbol{x} \times \bar{h}'\boldsymbol{v}) + \frac{1}{2}\mathrm{S}ii_{\boldsymbol{v}^*}(\boldsymbol{x} \times \bar{h}''\boldsymbol{v})$.

646 Notes and comments Working through the basic properties of the Riemann-sum integral as set out in Chapter 61, we find that (as we naturally hope) most of them seem to be shared by the S-integral. I checked bilinearity in 645Rb. Some of the others are straightforward (646B, 646C, 646M, 646Qc. 646S, 646T), some take more thought (645K, 646N, 646Qb, 646R) and some are apparently much harder (646J, 646Qa, 647J).

Even in terms of the concepts used in this presentation, I have not taken the shortest road to 646J, and surely it is going to need a proper look at the definition of the S-integral, but all the same it seems to take more pages than it should. The difficulty is that in the S-integral we are committed to a special class of integrands, so that when we split the sublattice $\mathcal{S}$ into $\mathcal{S} \wedge \tau$ and $\mathcal{S} \vee \tau$ the breakpoint τ has to be analysed from both sides; this is what the operators R of 646F and its approximate inverse $\boldsymbol{x} \mapsto (\boldsymbol{x}{\restriction}\mathcal{S} \wedge \tau, R^*(\boldsymbol{x}))$ are doing, and the problem is that these are not true inverses of each other.

Coming to the properties of indefinite S-integrals (646N, 646Q), we find ourselves facing interesting questions. When showing that an indefinite integral with respect to a martingale is a virtually local martingale, there were already significant obstacles for the Riemann-sum integral, but the methods of §621 turned out to be adequate. For the S-integral, I think we need to go much deeper, with another appeal to the fundamental theorem of martingales. In fact this allows us to bypass Theorem 623O altogether, as well as giving a striking new fact about martingale integrators in 646P. Concerning the value of $[\![\oint_{\mathcal{S}} \boldsymbol{x}\,d\boldsymbol{v} \ne 0]\!]$, we have an easy argument to show that it is included in $[\![\boldsymbol{v} \ne 0]\!]$ (646C), but to show that it is included in $[\![\boldsymbol{x} \ne 0]\!]$ (corresponding to the result for the Riemann-sum integral in 613Ld), we apparently need a special construction, which I will come to in the next section.

Naturally, the proof I give of the change-of-variable result 646R is based on the corresponding result 617E for the Riemann-sum integral; the application of 617E is hidden in (b-iii) of the proof of 646R. But here again we have a not-quite-trivial check to make on the limiting process we need to use. It goes a bit more smoothly if we start from 645Lb rather than 645La. As for Itô's formula, the obvious extension (part (b) of the proof of 646T) is straightforward in view of what has been done before. But with a little bit of analysis we can use the sequential convergence properties of the S-integral to extend the result to a larger class of functions h. I am rather supposing that you recognise the relevance of Rademacher's theorem (262Q); if $h' : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz, it will be differentiable almost everywhere, and its derivative will have an extension to a locally bounded Borel measurable function (225J).

The definition of $M^0_{\text{S-i}}$ in 645F leaves the nature of S-integrable processes unclear. Of course you could say the same for universal measurability as defined in 434D; it is markedly less easy to form a mental picture of a 'typical' universally measurable function than of a typical Lebesgue integrable function, because universal measurability is not easily described in terms of universally negligible sets (439Xe). But we can say a little bit, as in 646B.

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647 Changing the filtration II

The answer (647J) to a natural question left over from §646 leads us to a new construction to add to those in the second half of Chapter 63.

647A Notation As usual, $(\mathfrak{A}, \bar{\mu})$ will be a probability algebra, and $\langle \mathfrak{A}_t \rangle_{t \in T}$ a filtration of closed subalgebras of $\mathfrak{A}$, with the associated lattice $\mathcal{T}$ of stopping times and family $\langle \mathfrak{A}_\tau \rangle_{\tau \in \mathcal{T}}$ of subalgebras. If $\mathfrak{C}$ is a closed subalgebra of $\mathfrak{A}$ and $a \in \mathfrak{A}$, $\mathrm{upr}(a, \mathfrak{C})$ is the upper envelope of a in $\mathfrak{C}$. $L^0 = L^0(\mathfrak{A})$ will be given its topology of convergence in measure.

If $\mathcal{S}$ is a sublattice of $\mathcal{T}$ and $\tau \in \mathcal{S}$, $\mathcal{S} \wedge \tau = \{\sigma \wedge \tau : \sigma \in \mathcal{S}\}$ and $\mathcal{I}(\mathcal{S})$ is the set of finite sublattices of $\mathcal{S}$. We shall be looking at some of the usual spaces of fully adapted processes; if $\mathcal{S}$ is a sublattice of $\mathcal{T}$, then $M_{\text{fa}}(\mathcal{S})$ is the space of fully adapted processes with domain $\mathcal{S}$, M_{simp} the space of simple processes, $M_{\text{o-b}}$

the space of order-bounded processes, M_{mo} the space of moderately oscillatory processes, $M_{\text{n-s}}$ the space of near-simple processes, $M_{\text{n-s}}^{\uparrow}$ the space of non-negative non-decreasing near-simple processes, M_{bv} the space of processes of finite variation, $M_{\text{po-b}}$ the space of previsibly order-bounded processes (645B), $M_{\text{S-i}}^{0}$ the closure of $\{\boldsymbol{u}_{<} : \boldsymbol{u} \in M_{\text{mo}}\}$ for the S-integration topology $\mathfrak{T}_{\text{S-i}}$ on $M_{\text{po-b}}$, and $M_{\text{S-i}}$ the space $\{\boldsymbol{x} : \boldsymbol{x} \times \mathbf{1}^{(\mathcal{S})}_{<} \in M_{\text{S-i}}^{0}\}$ (645F). If $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is order-bounded, $\sup|\boldsymbol{u}| = \sup_{\sigma\in\mathcal{S}}|u_\sigma|$. If $\boldsymbol{u}$ and $\boldsymbol{v}$ are fully adapted processes, I write $\boldsymbol{u} \preccurlyeq \boldsymbol{v}$ if $\boldsymbol{v} - \boldsymbol{u}$ is non-decreasing. For $\boldsymbol{u} \in M_{\text{mo}}$, $\boldsymbol{u}_{<} \in M_{\text{po-b}}$ is its previsible version. $\int$ will denote the Riemann-sum integral of §613 and $\oint$ the S-integral of §645.

647B Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\mathfrak{T}$ the linear space topology on $M_{\text{po-b}} = M_{\text{po-b}}(\mathcal{S})$ defined by the F-seminorms $\widehat{\theta}^{\#}_{\boldsymbol{v}}$ (645Db) where $\boldsymbol{v} \in M_{\text{n-s}}^{\uparrow}(\mathcal{S})$ is $\| \|_\infty$-bounded. Then $\mathfrak{T}$ is the S-integration topology on $M_{\text{po-b}}$.

proof Because the S-integration topology $\mathfrak{T}_{\text{S-i}}$ is defined by a larger family of F-seminorms, $\mathfrak{T}$ is coarser than $\mathfrak{T}_{\text{S-i}}$. In the other direction, let G be a $\mathfrak{T}_{\text{S-i}}$-neighbourhood of 0 in $M_{\text{po-b}}$. Then there are a $\boldsymbol{v} \in M_{\text{n-s}}^{\uparrow}(\mathcal{S})$ and an $\epsilon > 0$ such that G includes $\{\boldsymbol{w} : \boldsymbol{w} \in M_{\text{po-b}}, \widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{w}) \le 3\epsilon\}$ (645E(a-ii)). Set $\bar v = \sup|\boldsymbol{v}|$, let $M \ge 0$ be such that $\bar\mu[\![\bar v \ge M]\!] \le \epsilon$ and set $\boldsymbol{v}' = \boldsymbol{v} \wedge M\mathbf{1}$. Then $G' = \{\boldsymbol{w} : \boldsymbol{w} \in M_{\text{po-b}}, \widehat{\theta}^{\#}_{\boldsymbol{v}'}(\boldsymbol{w}) \le \epsilon\}$ is a $\mathfrak{T}$-neighbourhood of 0. Now $G' \subseteq G$. **P** The point is that, for any $\boldsymbol{u} \in M_{\text{mo}} = M_{\text{mo}}(\mathcal{S})$,

$$[\![\textstyle\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}' \ne \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}]\!] \subseteq [\![\boldsymbol{v}' \ne \boldsymbol{v}]\!]$$

(613Jd)

$$= [\![\bar v > M]\!]$$

has measure at most ϵ, and

$$\theta(\textstyle\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}) \le \epsilon + \theta(\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}').$$

If $\boldsymbol{w} \in G'$, there is a uniformly order-bounded non-decreasing sequence $\langle \boldsymbol{u}^{(n)}\rangle_{n\in\mathbb{N}}$ in M_{mo}^{+} such that $|\boldsymbol{w}| \le \sup_{n\in\mathbb{N}} \boldsymbol{u}^{(n)}_{<}$ and $\sup_{n\in\mathbb{N}} \theta(\int_{\mathcal{S}} \boldsymbol{u}^{(n)} d\boldsymbol{v}') \le 2\epsilon$. In this case,

$$\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{w}) \le \sup_{n\in\mathbb{N}} \theta(\textstyle\int_{\mathcal{S}} \boldsymbol{u}^{(n)} d\boldsymbol{v}) \le 3\epsilon$$

and $\boldsymbol{w} \in G$. As $\boldsymbol{w}$ is arbitrary, $G' \subseteq G$. **Q**

Thus G is a $\mathfrak{T}$-neighbourhood of 0, and 0 has the same neighbourhoods for $\mathfrak{T}$ and $\mathfrak{T}_{\text{S-i}}$. As these are both linear space topologies, they coincide.

647C I give a couple of perfectly elementary facts which were not spelt out in Volume 3.

Lemma Suppose that $\mathfrak{D}$ is a closed subalgebra of $\mathfrak{A}$, and $b \in \mathfrak{A}$; let $\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{b\} \cup \mathfrak{D}$.

(a) If $c \in \mathfrak{B}$, then $b \cap c = b \cap \operatorname{upr}(b \cap c, \mathfrak{D})$.

(b) If $u \in L^0(\mathfrak{B})$, there are u', $u'' \in L^0(\mathfrak{D})$ such that $u = u' \times \chi b + u'' \times \chi(1 \setminus b)$.

proof (a) Of course $b \cap c \subseteq b \cap \operatorname{upr}(b \cap c, \mathfrak{D})$. In the other direction, we know that $\mathfrak{B} = \{(a \cap b) \cup (a' \setminus b) : a, a' \in \mathfrak{D}\}$ (314Ja), so that there is an $a \in \mathfrak{D}$ such that $b \cap c = b \cap a$. In this case,

$$b \cap \operatorname{upr}(b \cap c, \mathfrak{D}) = b \cap \operatorname{upr}(b \cap a, \mathfrak{D}) = b \cap a \cap \operatorname{upr}(b, \mathfrak{D})$$

(313Sc)

$$= b \cap a = b \cap c,$$

as claimed.

(b) Let $\mathfrak{C}$ be the principal ideal of $\mathfrak{B}$ generated by b. Then the maps $d \mapsto d \cap b : \mathfrak{D} \to \mathfrak{C}$ and $d \mapsto d \cap b : \mathfrak{B} \to \mathfrak{C}$ are both order-continuous surjective Boolean homomorphisms. So the corresponding Riesz homomorphisms $T' : L^0(\mathfrak{D}) \to L^0(\mathfrak{C})$ and $T : L^0(\mathfrak{B}) \to L^0(\mathfrak{C})$ (364P) are surjective, and there is a $u' \in L^0(\mathfrak{D})$ such that $T'u' = Tu$. Now if $\alpha \in \mathbb{R}$,

$$b \cap [\![u' > \alpha]\!] = [\![T'u' > \alpha]\!]$$

(by the defining formula for T' in 364Pa)

$$= [\![Tu > \alpha]\!] = b \cap [\![u > \alpha]\!].$$

So

$$\begin{aligned}[\![u' \times \chi b > \alpha]\!] &= b \cap [\![u' > \alpha]\!] = [\![u \times \chi b > \alpha]\!] \text{ if } \alpha \ge 0,\\ &= b \cap [\![u' > \alpha]\!] \cup (1 \setminus b) = [\![u \times \chi b > \alpha]\!] \text{ if } \alpha < 0,\end{aligned}$$

and $u' \times \chi b = u \times \chi b$.

Repeating the argument with $1 \setminus b$ in the place of b, we obtain an appropriate u''.

647D Construction For most of the rest of this section, b will be a fixed member of $\mathfrak{A}$. For $t \in T$, let $\mathfrak{B}_t$ be the subalgebra of $\mathfrak{A}$ generated by $\{b\} \cup \mathfrak{A}_t$; then $\mathfrak{B}_t = \{(a \cap b) \cup (a' \setminus b) : a, a' \in \mathfrak{A}\}$ is a closed subalgebra (312N, 314J). If $s \le t$ then $\{b\} \cup \mathfrak{A}_s \subseteq \mathfrak{B}_t$ so $\mathfrak{B}_s \subseteq \mathfrak{B}_t$; accordingly $\langle \mathfrak{B}_t \rangle_{t \in T}$ is a filtration.

647E Notation From now on, therefore, we shall have the two filtrations $\langle \mathfrak{A}_t \rangle_{t \in T}$ and $\langle \mathfrak{B}_t \rangle_{t \in T}$, giving stochastic integration structures $\mathbb{A} = (\mathfrak{A}, \bar{\mu}, T, \langle \mathfrak{A}_t \rangle_{t \in T}, \mathcal{T}_{\mathbb{A}}, \langle \mathfrak{A}_\sigma \rangle_{\sigma \in \mathcal{T}_{\mathbb{A}}})$ and $\mathbb{B} = (\mathfrak{A}, \bar{\mu}, T, \langle \mathfrak{B}_t \rangle_{t \in T}, \mathcal{T}_{\mathbb{B}}, \langle \mathfrak{B}_\tau \rangle_{\tau \in \mathcal{T}_{\mathbb{B}}})$. For the various spaces of processes, I will write $\mathbb{A}M^0_{\text{S-i}}(\mathcal{S})$, $\mathbb{B}M_{\text{mo}}(\mathcal{S})$, $\mathbb{A}M_{\text{po-b}}(\mathcal{S})$ etc. When we come to S-integration, I will talk of F-seminorms $\mathbb{A}\widehat{\theta}^{\#}_{\boldsymbol{v}}$, the S-integration topology $\mathbb{B}\mathfrak{T}_{\text{S-i}}$ and S-integrals $\mathbb{A}\oint$. (As we shall see in 647Fe, there is no need for such distinctions in regard to Riemann-sum integrals.)

647F Proposition (a)(i) $\mathcal{T}_{\mathbb{A}}$ is a sublattice of $\mathcal{T}_{\mathbb{B}}$.

(ii) $\min \mathcal{T}_{\mathbb{A}} = \min \mathcal{T}_{\mathbb{B}}$ and $\max \mathcal{T}_{\mathbb{A}} = \max \mathcal{T}_{\mathbb{B}}$.

(b) For any $\sigma \in \mathcal{T}_{\mathbb{A}}$, $\mathfrak{B}_\sigma$ is the subalgebra of $\mathfrak{A}$ generated by $\{b\} \cup \mathfrak{A}_\sigma$.

(c) If σ, $\tau \in \mathcal{T}_{\mathbb{A}}$, then $[\![\sigma < \tau]\!]$ and $[\![\sigma = \tau]\!]$ are the same in either structure.

(d) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}_{\mathbb{A}}$.

(i) If $\boldsymbol{u}$ is an $\mathbb{A}$-fully adapted process with domain $\mathcal{S}$, it is $\mathbb{B}$-fully adapted.

(ii) $\mathbb{A}M_{\text{simp}}(\mathcal{S}) \subseteq \mathbb{B}M_{\text{simp}}(\mathcal{S})$.

(iii) $\mathbb{A}M_{\text{o-b}}(\mathcal{S}) \subseteq \mathbb{B}M_{\text{o-b}}(\mathcal{S})$, and the ucp topology on $\mathbb{A}M_{\text{o-b}}(\mathcal{S})$ is the subspace topology induced by the ucp topology on $\mathbb{B}M_{\text{o-b}}(\mathcal{S})$.

(iv) $\mathbb{A}M_{\text{n-s}}(\mathcal{S}) \subseteq \mathbb{B}M_{\text{n-s}}(\mathcal{S})$.

(v) $\mathbb{A}M_{\text{bv}}(\mathcal{S}) \subseteq \mathbb{B}M_{\text{bv}}(\mathcal{S})$.

(vi) $\mathbb{A}M_{\text{mo}}(\mathcal{S}) \subseteq \mathbb{B}M_{\text{mo}}(\mathcal{S})$.

(e) If $\boldsymbol{u}$, $\boldsymbol{v}$ are $\mathbb{A}$-fully adapted processes with domain $\mathcal{S}$, and $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$ is defined in either of the structures $\mathbb{A}$, $\mathbb{B}$, then it is defined in the other, with the same value.

(f) If $\mathcal{S}$ is a sublattice of $\mathcal{T}_{\mathbb{A}}$ and $\boldsymbol{v}$ an $\mathbb{A}$-integrator with domain $\mathcal{S}$, then $\boldsymbol{v}$ is a $\mathbb{B}$-integrator.

(g) If $\mathcal{S}$ is a sublattice of $\mathcal{T}_{\mathbb{A}}$ and $\boldsymbol{u}$ belongs to $\mathbb{A}M_{\text{mo}}(\mathcal{S}) \subseteq \mathbb{B}M_{\text{mo}}(\mathcal{S})$, then its previsible version $\boldsymbol{u}_<$ is the same when calculated in either of the structures $\mathbb{A}$, $\mathbb{B}$.

proof (a)(i) The definition in 611A(b-i) makes it plain that $\mathcal{T}_{\mathbb{A}} \subseteq \mathcal{T}_{\mathbb{B}}$ just because $\mathfrak{A}_t \subseteq \mathfrak{B}_t$ for every t. The formulae of 611Cb and 611Cc show that $\mathcal{T}_{\mathbb{A}}$ and $\mathcal{T}_{\mathbb{B}}$ can both be regarded as sublattices of $\mathfrak{A}^T$, so that $\mathcal{T}_{\mathbb{A}}$ is a sublattice of $\mathcal{T}_{\mathbb{B}}$.

(ii) Immediate from the formulae in 611Cf.

(b) Write $\mathfrak{B}'_\sigma$ for the subalgebra of $\mathfrak{A}$ generated by $\{b\} \cup \mathfrak{A}_\sigma$. If $a \in \mathfrak{A}_\sigma$, then $a \setminus [\![\sigma > t]\!] \in \mathfrak{A}_t \subseteq \mathfrak{B}_t$ for every $t \in T$, so $a \in \mathfrak{B}_\sigma$. Next, b and $[\![\sigma > t]\!]$, and therefore $b \setminus [\![\sigma > t]\!]$, belong to $\mathfrak{B}_t$ for every t, so $b \in \mathfrak{B}_\sigma$; accordingly $\mathfrak{B}'_\sigma \subseteq \mathfrak{B}_\sigma$.

In the other direction, take $c \in \mathfrak{B}_\sigma$. For each $t \in T$, set $a_t = \text{upr}(b \cap c \setminus [\![\sigma > t]\!], \mathfrak{A}_t)$. Because $c \setminus [\![\sigma > t]\!] \in \mathfrak{B}_t$, $b \cap a_t = b \cap c \setminus [\![\sigma > t]\!]$ (647Ca). So $b \cap a_s \subseteq b \cap a_t$ if $s \le t$. Also, because $[\![\sigma > t]\!] \in \mathfrak{A}_t$, $a_t = \text{upr}(b \cap c, \mathfrak{A}_t) \setminus [\![\sigma > t]\!]$ is disjoint from $[\![\sigma > t]\!]$ for each t. Set $a = \sup_{t \in T} a_t$. If $s \in T$, then

$$\begin{aligned} a \setminus [\![\sigma > s]\!] &= \sup_{t\in T}(\mathrm{upr}(b\cap c, \mathfrak{A}_t) \setminus [\![\sigma > t]\!]) \setminus [\![\sigma > s]\!] \\ &= \sup_{t<s}(\mathrm{upr}(b\cap c, \mathfrak{A}_t) \setminus [\![\sigma > t]\!]) \cup \sup_{t\ge s}(\mathrm{upr}(b\cap c, \mathfrak{A}_t) \setminus [\![\sigma > s]\!]) \\ &= \sup_{t\le s}(\mathrm{upr}(b\cap c, \mathfrak{A}_t) \setminus [\![\sigma > t]\!]) \end{aligned}$$

(because $\mathfrak{A}_s \subseteq \mathfrak{A}_t$ so $\mathrm{upr}(b\cap c, \mathfrak{A}_t) \subseteq \mathrm{upr}(b\cap c, \mathfrak{A}_s)$ if $t \ge s$)

$$\in \mathfrak{A}_s.$$

Thus $a \in \mathfrak{A}_\sigma$ and $b \cap a \in \mathfrak{B}'_\sigma$, while

$$b \cap a = \sup_{t\in T} b \cap c \setminus [\![\sigma > t]\!] = b \cap c \setminus \inf_{t\in T} [\![\sigma > t]\!].$$

On the other hand,

$$(b \cap c \cap \inf_{t\in T} [\![\sigma > t]\!]) \setminus [\![\sigma > s]\!] = 0 \in \mathfrak{A}_s$$

for every $s \in T$, so $b \cap c \cap \inf_{t\in T} [\![\sigma > t]\!]$ belongs to $\mathfrak{A}_\sigma \subseteq \mathfrak{B}'_\sigma$. Accordingly $b \cap c \in \mathfrak{B}'_\sigma$.

Similarly, $c \setminus b \in \mathfrak{B}'_\sigma$. So $c \in \mathfrak{B}'_\sigma$. As c is arbitrary, $\mathfrak{B}'_\sigma = \mathfrak{B}_\sigma$, as claimed.

(c) The defining formulae

$$[\![\sigma < \tau]\!] = \sup_{t\in T} [\![\tau > t]\!] \setminus [\![\sigma > t]\!], \quad [\![\sigma = \tau]\!] = 1 \setminus ([\![\sigma < \tau]\!] \cup [\![\tau < \sigma]\!])$$

do not refer to the filtrations.

(d)(i) $\mathcal{S}$ is a sublattice of $\mathcal{T}_{\mathbb{B}}$ because $\mathcal{T}_{\mathbb{A}}$ is. Expressing $\boldsymbol{u}$ as $\langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$, $u_\sigma \in L^0(\mathfrak{A}_\sigma) \subseteq L^0(\mathfrak{B}_\sigma)$ for every $\sigma \in \mathcal{S}$. If σ, $\tau \in \mathcal{S}$, then $[\![\sigma = \tau]\!] \subseteq [\![u_\sigma = u_\tau]\!]$ in either structure, so $\boldsymbol{u}$ is $\mathbb{B}$-fully adapted.

(ii) Of course $\bigcap_{\sigma\in\mathcal{S}} \mathfrak{A}_\sigma \subseteq \bigcap_{\sigma\in\mathcal{S}} \mathfrak{B}_\sigma$. So if $\boldsymbol{u}$ satisfies the conditions of 612J with respect to $\langle \mathfrak{A}_\sigma \rangle_{\sigma\in\mathcal{S}}$, it satisfies them with respect to $\langle \mathfrak{B}_\sigma \rangle_{\sigma\in\mathcal{S}}$.

(iii) 'Order-boundedness' of a process depends only on the structure of L^0, not on the filtration (614Ea), so $\mathbb{A}M_{\text{o-b}}(\mathcal{S}) \subseteq \mathbb{B}M_{\text{o-b}}(\mathcal{S})$. Similarly, if we think of $\widehat{\theta}$ (615B) as an F-seminorm defined on the space of order-bounded families in $(L^0)^{\mathcal{S}}$, the ucp topologies on $\mathbb{A}M_{\text{o-b}}(\mathcal{S})$, $\mathbb{B}M_{\text{o-b}}(\mathcal{S})$ are defined by the restrictions of $\widehat{\theta}$ to these spaces, so must agree on the smaller space $\mathbb{A}M_{\text{o-b}}(\mathcal{S})$.

(iv) Now the closure $\mathbb{A}M_{\text{n-s}}(\mathcal{S})$ of $\mathbb{A}M_{\text{simp}}(\mathcal{S})$ in $\mathbb{A}M_{\text{o-b}}(\mathcal{S})$ (631Ba) must be included in the closure $\mathbb{B}M_{\text{n-s}}(\mathcal{S})$ of $\mathbb{B}M_{\text{simp}}(\mathcal{S})$ in $\mathbb{B}M_{\text{o-b}}(\mathcal{S})$ just because $\mathbb{A}M_{\text{simp}}(\mathcal{S}) \subseteq \mathbb{B}M_{\text{simp}}(\mathcal{S})$.

(v) As in (iii), the definition of 'bounded variation' in 614J-614K refers to the lattice structure of $\mathcal{S}$, but not to the filtration.

(vi) And now, as in (iv), the closure $\mathbb{A}M_{\text{mo}}(\mathcal{S})$ of $\mathbb{A}M_{\text{bv}}(\mathcal{S})$ (615E) must be included in the closure $\mathbb{B}M_{\text{mo}}(\mathcal{S})$ of $\mathbb{B}M_{\text{bv}}(\mathcal{S})$.

(e) Again, working through the definitions (§613), we see that the filtration is nowhere referred to, and that $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$ is determined by the lattice structure of $\mathcal{S}$, the processes $\boldsymbol{u}$, $\boldsymbol{v} \in (L^0)^{\mathcal{S}}$ and the topology of convergence in measure on L^0.

(f) We know that the capped-stake variation set $B = Q^{(\mathbb{A})}_{\mathcal{S}}(d\boldsymbol{v})$, calculated with reference to the family $\langle \mathfrak{A}_\sigma \rangle_{\sigma\in\mathcal{S}}$, is topologically bounded. Now

$$Q^{(\mathbb{B})}_{\mathcal{S}}(d\boldsymbol{v}) \subseteq \{z' \times \chi b + z'' \times \chi(1 \setminus b) : z', z'' \in B\}.$$

P If $\mathcal{S}$ is empty, this is trivial. Otherwise, if $z \in Q^{(\mathbb{B})}_{\mathcal{S}}(d\boldsymbol{v})$, there are $\tau_0 \le \ldots \le \tau_n$ in $\mathcal{S}$ and $u_0, \ldots, u_{n-1}$ such that $u_i \in L^0(\mathfrak{B}_{\tau_i})$ and $\|u_i\|_\infty \le 1$ for each i, and $z = \sum_{i=0}^{n-1} u_i \times (v_{\tau_{i+1}} - v_{\tau_i})$. For each i, $\mathfrak{B}_{\tau_i}$ is the algebra generated by $\mathfrak{A}_{\tau_i} \cup \{b\}$, so there are u'_i, $u''_i \in L^0(\mathfrak{A}_{\tau_i})$ such that $u_i = u'_i \times \chi b + u''_i \times \chi(1 \setminus b)$ (647Cb), and replacing these by $\mathrm{med}(-\chi 1, u'_i, \chi 1)$ and $\mathrm{med}(-\chi 1, u''_i, \chi 1)$ if necessary, we can suppose that $|u'_i|$, $|u''_i| \le \chi 1$. In this case,

$$z' = \sum_{i=0}^{n-1} u'_i \times (v_{\tau_{i+1}} - v_{\tau_i}), \quad z'' = \sum_{i=0}^{n-1} u''_i \times (v_{\tau_{i+1}} - v_{\tau_i})$$

belong to B (616C(ii)). And $z = z' \times \chi b + z'' \times \chi(1 \setminus b)$. **Q**

Since $z' \mapsto z' \times \chi b : L^0 \to L^0$ is linear and continuous, $\{z' \times \chi b : z' \in B\}$ is topologically bounded (3A5N(b-v)). Similarly, $\{z' \times \chi(1 \setminus b) : z' \in B\}$ is topologically bounded; so $Q_{\mathcal{S}}^{(\mathbb{B})}(d\boldsymbol{v})$ is included in the sum of two topologically bounded sets and is topologically bounded, that is, $\boldsymbol{v}$ is a $\mathbb{B}$-integrator.

(g) Once again, the definition in 641E-641F depends only on the lattice structure of $\mathcal{S}$, the regions $[\![\sigma < \tau]\!]$ for σ, $\tau \in \mathcal{S}$ and the topology of convergence in measure on L^0, which are unchanged by the change in filtration.

647G Proposition Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous.

(a) $\langle\mathfrak{B}_t\rangle_{t\in T}$ is right-continuous.

(b) $\mathcal{T}_{\mathbb{A}}$ is an order-closed sublattice of $\mathcal{T}_{\mathbb{B}}$.

(c) If $\tau \in \mathcal{T}_{\mathbb{B}}$, there are σ, $\sigma' \in \mathcal{T}_{\mathbb{A}}$ such that $b \subseteq [\![\tau = \sigma]\!]$ and $1 \setminus b \subseteq [\![\tau = \sigma']\!]$. In particular, $\mathcal{T}_{\mathbb{A}}$ covers $\mathcal{T}_{\mathbb{B}}$.

proof (a) If $t \in T$ is not isolated on the right, take $c \in \bigcap_{s>t} \mathfrak{B}_s$. Then for every $s > t$ there are a_s, $a'_s \in \mathfrak{A}_s$ such that $c \cap b = a_s \cap b$ and $c \setminus b = a'_s \setminus b$. Set

$$a = \sup_{s'>t} \inf_{t<s\le s'} a_s, \quad a' = \sup_{s'>t} \inf_{t<s\le s'} a'_s;$$

then $a \in \bigcap_{s'>t} \mathfrak{A}_s = \mathfrak{A}_t$ and

$$c \cap b = \sup_{s'>t} \inf_{t<s\le s'} (a_s \cap b) = (\sup_{s'>t} \inf_{t<s\le s'} a_s) \cap b$$

(by the distributive laws 313B, as usual)

$$= a \cap b,$$

and similarly $a' \in \mathfrak{A}_t$ and $c \setminus b = a' \setminus b$, so $c = (a \cap b) \cup (a' \setminus b)$ belongs to $\mathfrak{B}_t$. As c and t are arbitrary, $\langle\mathfrak{B}_t\rangle_{t\in T}$ is right-continuous.

(b) Use 611Cb in its full strength and 632C(a-i) to see that if $A \subseteq \mathcal{T}_{\mathbb{A}}$ is non-empty, we get formulae defining $\sup A$ and $\inf A$ in both $\mathcal{T}_{\mathbb{A}}$ and $\mathcal{T}_{\mathbb{B}}$.

(c) For $t \in T$, set $a_t = \operatorname{upr}([\![\tau > t]\!] \cap b, \mathfrak{A}_t)$; then $a_t \cap b = [\![\tau > t]\!] \cap b$, by 647Ca, because $[\![\tau > t]\!] \in \mathfrak{B}_t$. If $s \le t$ in T, then

$$[\![\tau > t]\!] \cap b \subseteq [\![\tau > s]\!] \cap b \subseteq a_s \in \mathfrak{A}_s \subseteq \mathfrak{A}_t,$$

so $a_t \subseteq a_s$. If $t \in T$ is not isolated on the right in T, set $a = \sup_{s>t} a_s$; then $a = \sup_{t<s\le s'} a_s \in \mathfrak{A}_{s'}$ for every $s' > t$, so (because the filtration is right-continuous) $a \in \mathfrak{A}_t$, while

$$[\![\tau > t]\!] \cap b = \sup_{s>t} [\![\tau > s]\!] \cap b \subseteq a$$

so $a_t \subseteq a$ and $a_t = \sup_{s>t} a_s$. But this means that the conditions of 611A(b-i) are satisfied by $\langle a_t\rangle_{t\in T}$ and we have a $\sigma \in \mathcal{T}_{\mathbb{A}}$ such that $[\![\sigma > t]\!] = a_t$ for every t. Next, for each t,

$$[\![\sigma > t]\!] \cap b = a_t \cap b = [\![\tau > t]\!] \cap b.$$

So if we calculate $[\![\sigma < \tau]\!] \cap b$ from the formulae in 611D, we see that it is

$$\sup_{t\in T}(b \cap [\![\tau > t]\!] \setminus [\![\sigma > t]\!]) = \sup_{t\in T}(([\![\tau > t]\!] \cap b) \setminus ([\![\sigma > t]\!] \cap b)) = 0,$$

and similarly $[\![\tau < \sigma]\!] \cap b = 0$, so $b \subseteq [\![\tau = \sigma]\!]$.

Similarly, we have a $\sigma' \in \mathcal{T}_{\mathbb{A}}$ defined by setting $[\![\sigma' > t]\!] = \operatorname{upr}([\![\tau > t]\!] \setminus b, \mathfrak{A}_t)$ for every $t \in T$, and $1 \setminus b \subseteq [\![\tau = \sigma']\!]$.

647H Lemma Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous, and that $\mathcal{S}$ is an order-convex sublattice of $\mathcal{T}_{\mathbb{A}}$. Let $\boldsymbol{v} \in \mathbb{B}M^{\uparrow}_{\text{n-s}}(\mathcal{S})$ be $\| \|_\infty$-bounded. Then there is a $\| \|_\infty$-bounded $\boldsymbol{w} \in \mathbb{A}M^{\uparrow}_{\text{n-s}}(\mathcal{S})$ such that $\boldsymbol{v} \preccurlyeq \boldsymbol{w}$.

Remark Perhaps I should make it clear that when I write '$\mathcal{S}$ is an order-convex sublattice of $\mathcal{T}_{\mathbb{A}}$' I mean that whenever τ, $\tau' \in \mathcal{S}$, $\sigma \in \mathcal{T}_{\mathbb{A}}$ and $\tau \le \sigma \le \tau'$, then $\sigma \in \mathcal{S}$. We do not expect $\mathcal{S}$ to be order-convex when regarded as a sublattice of $\mathcal{T}_{\mathbb{B}}$.

proof (a) Express $\boldsymbol{v}$ as $\langle v_\tau\rangle_{\tau\in\mathcal{S}}$. For the time being (down to the end of (f) below) suppose that $[\![\boldsymbol{v}\ne\mathbf{0}]\!]\subseteq b$. Set $M=\|\boldsymbol{v}\|_\infty$. For $\tau\in\mathcal{S}$ set $\tilde v_\tau=\min\{v:v\in L^0(\mathfrak{A}_\tau),\,v\ge v_\tau\}$; this is defined because $\{v:v\in L^0(\mathfrak{A}_\tau),\,v\ge v_\tau\}$ is non-empty (it contains $M\chi 1$, for instance) and downwards-directed, and $L^0(\mathfrak{A}_\tau)$ is order-closed in L^0.

(b)(i) Of course $0\le\tilde v_\tau\le M\chi(\operatorname{upr}(b,\mathfrak{A}_\tau))$, so $[\![\tilde v_\tau>0]\!]\subseteq\operatorname{upr}(b,\mathfrak{A}_\tau)$, for every $\tau\in\mathcal{S}$.

(ii) Next, $[\![v_\tau=\tilde v_\tau]\!]\supseteq b$ for each τ. **P** (Cf. 121I, 364Xp.) If $\alpha\ge 0$, then $[\![v_\tau>\alpha]\!]$ belongs to $\mathfrak{B}_\tau$ and is included in b, so by 647Fb is of the form $b\cap a$ for some $a\in\mathfrak{A}_\tau$. In this case, $M\chi a+\alpha\chi(1\setminus a)$ belongs to $L^0(\mathfrak{A}_\tau)$ and is greater than or equal to v_τ, so is greater than or equal to $\tilde v_\tau$, and $[\![\tilde v_\tau>\alpha]\!]\subseteq a$. Consequently

$$[\![\tilde v_\tau\times\chi b>\alpha]\!]\subseteq b\cap a=[\![v_\tau>\alpha]\!]=[\![v_\tau\times\chi b>\alpha]\!].$$

Of course

$$[\![\tilde v_\tau\times\chi b>\alpha]\!]=1=[\![v_\tau\times\chi b>\alpha]\!]$$

for every $\alpha<0$, because $v_\tau\ge 0$. As α is arbitrary, $\tilde v_\tau\times\chi b\le v_\tau\times\chi b$. Since $v_\tau\le\tilde v_\tau$, $\tilde v_\tau\times\chi b\le v_\tau\times\chi b$ and $\tilde v_\tau\times\chi b=v_\tau\times\chi b$, that is, $b\subseteq[\![\tilde v_\tau=v_\tau]\!]$. **Q**

(c) $\tilde{\boldsymbol{v}}=\langle\tilde v_\tau\rangle_{\tau\in\mathcal{S}}$ is $\mathbb{A}$-fully adapted. **P** We arranged in (a) that $\tilde v_\tau\in L^0(\mathfrak{A}_\tau)$ for every $\tau\in\mathcal{S}$. Suppose that σ, τ in $\mathcal{S}$ and $a=[\![\sigma=\tau]\!]$. Then $a\in\mathfrak{A}_\sigma\cap\mathfrak{A}_\tau$ (611H(c-i)) and $a\subseteq[\![v_\sigma=v_\tau]\!]$. Now

$$\tilde v_\sigma\times\chi a\ge v_\sigma\times\chi a=v_\tau\times\chi a$$

so $\tilde v_\sigma\times\chi a+M\chi(1\setminus a)\ge v_\tau$; at the same time, $\chi(1\setminus a)$ and $\tilde v_\sigma\times\chi a=\tilde v_\sigma\times\chi[\![\sigma\le\tau]\!]\times\chi a$ belong to $L^0(\mathfrak{A}_\tau)$ (612C), so $\tilde v_\sigma\times\chi a+M\chi(1\setminus a)\in L^0(\mathfrak{A}_\tau)$ and $\tilde v_\sigma\times\chi a+M\chi(1\setminus a)\ge\tilde v_\tau$. Thus $\tilde v_\sigma\times\chi a\ge\tilde v_\tau\times\chi a$. Similarly, $\tilde v_\tau\times\chi a\ge\tilde v_\sigma\times\chi a$ and the two are equal, that is, $a\subseteq[\![\tilde v_\sigma=\tilde v_\tau]\!]$. As σ and τ are arbitrary, $\tilde{\boldsymbol{v}}$ is fully adapted. **Q**

(d) $\tilde{\boldsymbol{v}}$ is of bounded variation. **P** Suppose that $\tau_0\le\ldots\le\tau_n$ in $\mathcal{S}$. For $1\le k\le n$, set $a_k=[\![\tilde v_{\tau_k}<\tilde v_{\tau_{k-1}}]\!]$. If $1\le k\le i\le n$, then $a_k\in\mathfrak{A}_{\tau_i}$ and

$$a_k\cap b\subseteq[\![\tilde v_{\tau_k}=v_{\tau_k}]\!]\cap[\![\tilde v_{\tau_k}<\tilde v_{\tau_{k-1}}]\!]\cap[\![\tilde v_{\tau_{k-1}}=v_{\tau_{k-1}}]\!]\subseteq[\![v_{\tau_k}<v_{\tau_{k-1}}]\!]=0,$$

so $a_k\cap\operatorname{upr}(b,\mathfrak{A}_{\tau_i})=0$ and $a_k\subseteq[\![\tilde v_{\tau_i}=0]\!]$; consequently $a_k\cap a_i=0$ if $k<i$. Turning this round, we see that $a_k\cap a_i=0$ if $1\le i<k$, so that

$$\begin{aligned}a_k&\subseteq\inf_{1\le i<k}[\![\tilde v_{\tau_{i-1}}\le\tilde v_{\tau_i}]\!]\subseteq[\![\textstyle\sum_{i=1}^{k-1}|\tilde v_{\tau_i}-\tilde v_{\tau_{i-1}}|=\tilde v_{\tau_{k-1}}-\tilde v_{\tau_0}]\!]\\&\subseteq[\![\textstyle\sum_{i=1}^{k-1}|\tilde v_{\tau_i}-\tilde v_{\tau_{i-1}}|\le M\chi 1]\!],\end{aligned}$$

while on the other side

$$a_k\subseteq\inf_{k\le i\le n}[\![\tilde v_{\tau_i}=0]\!]\subseteq[\![\textstyle\sum_{i=k+1}^{n}|\tilde v_{\tau_i}-\tilde v_{\tau_{i-1}}|=0]\!].$$

Putting these together,

$$a_k\subseteq[\![\textstyle\sum_{i=1}^{n}|\tilde v_{\tau_i}-\tilde v_{\tau_{i-1}}|\le M\chi 1+|\tilde v_{\tau_k}-\tilde v_{\tau_{k-1}}|]\!]\subseteq[\![\textstyle\sum_{i=1}^{n}|\tilde v_{\tau_i}-\tilde v_{\tau_{i-1}}|\le 2M\chi 1]\!].$$

On the other hand,

$$\begin{aligned}1\setminus\sup_{1\le i\le n}a_i&\subseteq\inf_{1\le i\le n}[\![\tilde v_{\tau_{i-1}}\le\tilde v_{\tau_i}]\!]\subseteq[\![\textstyle\sum_{i=1}^{n}|\tilde v_{\tau_i}-\tilde v_{\tau_{i-1}}|=\tilde v_{\tau_n}-\tilde v_{\tau_0}]\!]\\&\subseteq[\![\textstyle\sum_{i=1}^{n}|\tilde v_{\tau_i}-\tilde v_{\tau_{i-1}}|\le M\chi 1]\!],\end{aligned}$$

so in fact we have

$$[\![\textstyle\sum_{i=1}^{n}|\tilde v_{\tau_i}-\tilde v_{\tau_{i-1}}|\le 2M\chi 1]\!]=1$$

and $\sum_{i=1}^{n}|\tilde v_{\tau_i}-\tilde v_{\tau_{i-1}}|\le 2M\chi 1$. As $\tau_0,\ldots,\tau_n$ are arbitrary, $\tilde{\boldsymbol{v}}$ is of bounded variation, with $\int_{\mathcal{S}}|d\tilde{\boldsymbol{v}}|\le 2M\chi 1$. **Q**

(e) $\tilde{\boldsymbol{v}}$ is $\mathbb{A}$-near-simple. **P** Suppose that $\tau_0\le\tau_1$ in $\mathcal{S}$. If $A\subseteq\mathcal{S}\cap[\tau_0,\tau_1]$ is non-empty and downwards-directed and has infimum τ in $\mathcal{T}_{\mathbb{A}}$, then $\tau\in A$ because A is order-convex in $\mathcal{T}_{\mathbb{A}}$. We know that $v=\lim_{\sigma\downarrow A}\tilde v_\sigma$ is defined, because $\tilde{\boldsymbol{v}}$ is of $\mathbb{A}$-bounded variation, therefore $\mathbb{A}$-moderately-oscillatory (616Ra).

Of course $v \ge 0$. Now $v \in \bigcap_{\sigma\in A} L^0(\mathfrak{A}_\sigma) = L^0(\mathfrak{A}_\tau)$ (632C(a-iii)), while

$$v \ge v \times \chi b = \lim_{\sigma\downarrow A} \tilde v_\sigma \times \chi b = \lim_{\sigma\downarrow A} v_\sigma = v_\tau$$

by 632F, because $\boldsymbol{v}$ is $\mathbb{B}$-near-simple and $\mathcal{T}_{\mathbb{A}}$ is order-closed in $\mathcal{T}_{\mathbb{B}}$, so τ is still the infimum of A in $\mathcal{T}_{\mathbb{B}}$. It follows that $\tilde v_\tau \le v$, while

$$b \subseteq [\![v = v_\tau]\!] \cap [\![v_\tau = \tilde v_\tau]\!] \subseteq [\![v = \tilde v_\tau]\!].$$

Set $a = [\![\tilde v_\tau < v]\!] \subseteq [\![v > 0]\!]$. Then $a \in \mathfrak{A}_\tau$ and $a \cap b = 0$. But this means that, for every $\sigma \in A$, a is disjoint from $\operatorname{upr}(b, \mathfrak{A}_\sigma)$ and $\tilde v_\sigma \times \chi a = 0$. Consequently $v \times \chi a = \lim_{\sigma\downarrow A} \tilde v_\sigma \times \chi a = 0$, and a must be 0. Thus $v \le \tilde v_\tau$ and $v = \tilde v_\tau$. As $\tilde{\boldsymbol{v}} \restriction \mathcal{S} \cap [\tau_0, \tau_1]$ is $\mathbb{A}$-moderately-oscillatory, it is $\mathbb{A}$-locally-near-simple, by 632F again, and in fact $\mathbb{A}$-near-simple, since $\max(\mathcal{S} \cap [\tau_0, \tau_1]) = \tau_1$ belongs to $\mathcal{S} \cap [\tau_0, \tau_1]$.

Since the whole process $\tilde{\boldsymbol{v}}$ is $\mathbb{A}$-moderately-oscillatory, it is $\mathbb{A}$-near-simple, by 631Fc. **Q**

(f) Setting $w_\tau = \int_{\mathcal{S}\wedge\tau} |d\tilde{\boldsymbol{v}}|$ for $\tau \in \mathcal{S}$, $\boldsymbol{w} = \langle w_\tau \rangle_{\tau\in\mathcal{S}}$ is $\mathbb{A}$-near-simple (631K), and of course it is non-negative and non-decreasing; moreover, $\|\boldsymbol{w}\|_\infty = \| \int_{\mathcal{S}} |d\tilde{\boldsymbol{v}}| \|_\infty$ is finite. If $\sigma \le \tau$ in $\mathcal{S}$, then

$$b \subseteq [\![v_\tau - v_\sigma = \tilde v_\tau - \tilde v_\sigma]\!], \quad 1 \setminus b \subseteq [\![v_\tau - v_\sigma = 0]\!]$$

so

$$v_\tau - v_\sigma \le |\tilde v_\tau - \tilde v_\sigma| \le w_\tau - w_\sigma;$$

as σ and τ are arbitrary, $\boldsymbol{v} \preccurlyeq \boldsymbol{w}$.

(g) This deals with the case when $[\![\boldsymbol{v} \ne \boldsymbol{0}]\!] \subseteq b$. Of course just the same arguments apply when $[\![\boldsymbol{v} \ne \boldsymbol{0}]\!] \subseteq 1 \setminus b$. For the general case, consider $\boldsymbol{v}' = (\chi b)\boldsymbol{v} = \langle \chi b \times v_\tau \rangle_{\tau\in\mathcal{S}}$ and $\boldsymbol{v}'' = \boldsymbol{v} - \boldsymbol{v}'$. These are both $\mathbb{B}$-near-simple non-decreasing non-negative processes bounded above by a multiple of $\chi 1$, while $[\![\boldsymbol{v}' \ne \boldsymbol{0}]\!] \subseteq b$ and $[\![\boldsymbol{v}'' \ne \boldsymbol{0}]\!] \subseteq 1 \setminus b$. So we have $\| \|_\infty$-bounded processes $\boldsymbol{w}', \boldsymbol{w}'' \in \mathbb{A}M^{\uparrow}_{\text{n-s}}(\mathcal{S})$ such that $\boldsymbol{v}' \preccurlyeq \boldsymbol{w}'$ and $\boldsymbol{v}'' \preccurlyeq \boldsymbol{w}''$. Setting $\boldsymbol{w} = \boldsymbol{w}' + \boldsymbol{w}''$, $\boldsymbol{w} \in \mathbb{A}M^{\uparrow}_{\text{n-s}}(\mathcal{S})$ is $\| \|_\infty$-bounded and $\boldsymbol{v} \preccurlyeq \boldsymbol{w}$, as required.

647I Proposition Suppose that $\langle \mathfrak{A}_t \rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}_{\mathbb{A}}$, and $\hat{\mathcal{S}}^{(\mathbb{B})}$ the covered envelope of $\mathcal{S}$ in $\mathcal{T}_{\mathbb{B}}$.

(a)(i) $\hat{\mathcal{S}}^{(\mathbb{B})}$ is order-convex in $\mathcal{T}_{\mathbb{B}}$.

(ii) For any $\mathbb{A}$-fully adapted process $\boldsymbol{x}$ with domain $\mathcal{S}$, there is a unique $\mathbb{B}$-fully adapted process $\hat{\boldsymbol{x}}$ with domain $\hat{\mathcal{S}}^{(\mathbb{B})}$ extending $\boldsymbol{x}$.

(b)(i) If $\boldsymbol{x} \in \mathbb{A}M_{\text{o-b}}(\mathcal{S})$ then $\hat{\boldsymbol{x}} \in \mathbb{B}M_{\text{o-b}}(\hat{\mathcal{S}}^{(\mathbb{B})})$ and $\sup|\hat{\boldsymbol{x}}| = \sup|\boldsymbol{x}|$.

(ii) If $\boldsymbol{v} \in \mathbb{A}M_{\text{n-s}}(\mathcal{S})$ then $\hat{\boldsymbol{v}} \in \mathbb{B}M_{\text{n-s}}(\hat{\mathcal{S}}^{(\mathbb{B})})$.

(iii) If $\boldsymbol{v} \in \mathbb{A}M_{\text{bv}}(\mathcal{S})$ then $\hat{\boldsymbol{v}} \in \mathbb{B}M_{\text{bv}}(\hat{\mathcal{S}}^{(\mathbb{B})})$.

(iv) If $\boldsymbol{v}$ is an $\mathbb{A}$-integrator with domain $\mathcal{S}$, then $\hat{\boldsymbol{v}}$ is a $\mathbb{B}$-integrator.

(v) If $\boldsymbol{u} \in \mathbb{A}M_{\text{mo}}(\mathcal{S})$ then $\hat{\boldsymbol{u}} \in \mathbb{B}M_{\text{mo}}(\hat{\mathcal{S}}^{(\mathbb{B})})$ and the previsible version of $\hat{\boldsymbol{u}}$ is the image of the previsible version of $\boldsymbol{u}$.

(c)(i) If $\boldsymbol{w} \in \mathbb{A}M_{\text{po-b}}(\mathcal{S})$, then $\hat{\boldsymbol{w}} \in \mathbb{B}M_{\text{po-b}}(\hat{\mathcal{S}}^{(\mathbb{B})})$.

(ii) The map $\boldsymbol{w} \mapsto \hat{\boldsymbol{w}} : \mathbb{A}M_{\text{po-b}}(\mathcal{S}) \to \mathbb{B}M_{\text{po-b}}(\hat{\mathcal{S}}^{(\mathbb{B})})$ is continuous for the S-integration topologies $\mathbb{A}\mathfrak{T}_{\text{S-i}}$ and $\mathbb{B}\mathfrak{T}_{\text{S-i}}$.

(iii) If $\boldsymbol{x} \in \mathbb{A}M_{\text{S-i}}(\mathcal{S})$, then $\hat{\boldsymbol{x}} \in \mathbb{B}M_{\text{S-i}}(\hat{\mathcal{S}}^{(\mathbb{B})})$.

(d) If $\boldsymbol{x} \in \mathbb{A}M_{\text{S-i}}(\mathcal{S})$, then $\mathbb{B}\oint_{\hat{\mathcal{S}}^{(\mathbb{B})}} \hat{\boldsymbol{x}}\, d\hat{\boldsymbol{v}} = \mathbb{A}\oint_{\mathcal{S}} \boldsymbol{x}\, d\boldsymbol{v}$ for every $\mathbb{A}$-integrator $\boldsymbol{v} \in \mathbb{A}M_{\text{n-s}}(\mathcal{S})$.

proof (a)(i) Suppose that $\tau, \tau' \in \hat{\mathcal{S}}^{(\mathbb{B})}$ and $\tilde\tau \in \mathcal{T}_{\mathbb{B}}$ are such that $\tau \le \tilde\tau \le \tau'$. Take $a \in \mathfrak{A} \setminus \{0\}$. Because $\mathcal{T}_{\mathbb{A}}$ covers $\mathcal{T}_{\mathbb{B}}$ (647Gc), there are $\sigma, \sigma' \in \mathcal{S}$ and $\tilde\sigma \in \mathcal{T}_{\mathbb{A}}$ such that $a' = a \cap [\![\sigma = \tau]\!] \cap [\![\sigma' = \tau']\!] \cap [\![\tilde\sigma = \tilde\tau]\!]$ is non-zero. Now $\operatorname{med}(\sigma, \tilde\sigma, \sigma') \in \mathcal{S}$, because $\mathcal{S}$ is order-convex in $\mathcal{T}_{\mathbb{A}}$ and $\sigma \wedge \sigma' \le \operatorname{med}(\sigma, \tilde\sigma, \sigma') \le \sigma \vee \sigma'$, and

$$0 \ne a' \subseteq a \cap [\![\operatorname{med}(\tau, \tilde\tau, \tau') = \operatorname{med}(\sigma, \tilde\sigma, \sigma')]\!] \subseteq a \cap [\![\tilde\tau = \operatorname{med}(\sigma, \tilde\sigma, \sigma')]\!].$$

As a is arbitrary, $\tilde\tau \in \hat{\mathcal{S}}^{(\mathbb{B})}$; thus $\hat{\mathcal{S}}^{(\mathbb{B})}$ is order-convex in $\mathcal{T}_{\mathbb{B}}$.

(ii) We just have to remember that $\boldsymbol{u}$ is $\mathbb{B}$-fully adapted, as noted in 647F(d-i), and apply 612Qa.

(b)(i) By 647F(d-iii), $\boldsymbol{x}$ is $\mathbb{B}$-order-bounded, so $\hat{\boldsymbol{x}}$ is $\mathbb{B}$-order-bounded, by 614Ga.

(ii) Argue as in (i), but using 647F(d-iv) for the first step and 631Ga for the second.

(iii) Use 647F(d-v) and 614Q(a-iv-β).

(iv) 647Ff and 616I.

(v) By 647F(d-vi) and 615F(a-vi), $\hat{\boldsymbol{u}}$ is $\mathbb{B}$-moderately-oscillatory. Concerning its previsible version $\hat{\boldsymbol{u}}_<$, we know from 641G(a-v), applied in $\mathbb{B}$, that $\hat{\boldsymbol{u}}_<\restriction\mathcal{S}$ is the previsible version of $\boldsymbol{u}\restriction\mathcal{S}$, which is the same whether calculated in $\mathbb{A}$ or $\mathbb{B}$, as noted in 647Fg; so $\hat{\boldsymbol{u}}_<$ must be the $\mathbb{B}$-fully-adapted extension of the $\mathbb{A}$-previsible version $\boldsymbol{u}_<$ of $\boldsymbol{u}$, that is. $\hat{\boldsymbol{u}}_<$ is the image of $\boldsymbol{u}_<$ in $\mathbb{B}M_{\text{fa}}(\hat{\mathcal{S}}^{(\mathbb{B})})$.

(c)(i) $\boldsymbol{w}$ is $\mathbb{A}$-fully adapted, so $\hat{\boldsymbol{w}}$ is $\mathbb{B}$-fully adapted, by (a-ii) just above. Let $\boldsymbol{u}\in\mathbb{A}M_{\text{mo}}(\mathcal{S})^+$ be such that $|\boldsymbol{w}|\le\boldsymbol{u}_<$, and consider $\hat{\boldsymbol{u}}$. We saw in (b-v) that, writing $\hat{\boldsymbol{u}}_<$ for the previsible version of $\hat{\boldsymbol{u}}$, we have $\hat{\boldsymbol{u}}_<\restriction\mathcal{S}=\boldsymbol{u}_<$, so $0=(|\boldsymbol{w}|-\boldsymbol{u}_<)^+=(|\hat{\boldsymbol{w}}|-\hat{\boldsymbol{u}}_<)^+\restriction\mathcal{S}$ (612Qb), and $|\hat{\boldsymbol{w}}|\le\hat{\boldsymbol{u}}_<$. Thus $\hat{\boldsymbol{w}}\in\mathbb{B}M_{\text{po-b}}(\mathcal{S})$, as claimed.

(ii) Let G be a $\mathbb{B}\mathfrak{T}_{\text{S-i}}$-neighbourhood of $\mathbf{0}$ in $\mathbb{B}M_{\text{po-b}}(\hat{\mathcal{S}}^{(\mathbb{B})})$. Then there are a $\|\,\|_\infty$-bounded $\boldsymbol{v}\in\mathbb{B}M^{\uparrow}_{\text{n-s}}(\hat{\mathcal{S}}^{(\mathbb{B})})$ and an $\epsilon>0$ such that $\boldsymbol{w}\in G$ whenever $\boldsymbol{w}\in\mathbb{B}M_{\text{po-b}}(\hat{\mathcal{S}}^{(\mathbb{B})})$ and $\mathbb{B}\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{w})\le\epsilon$ (put 645E(a-ii) and 647B together). Now $\hat{\boldsymbol{v}}$ is $\mathbb{B}$-near-simple ((b-ii) above) and is non-negative and non-decreasing (614Q(a-iv-α)) and bounded above by a constant process, so by 647H there is a $\tilde{\boldsymbol{v}}\in\mathbb{A}M^{\uparrow}_{\text{n-s}}(\mathcal{S})$ such that $\boldsymbol{v}\restriction\mathcal{S}\preccurlyeq\tilde{\boldsymbol{v}}$. In this case, $G'=\{\boldsymbol{w}:\boldsymbol{w}\in\mathbb{A}M_{\text{po-b}}(\mathcal{S}),\ \mathbb{A}\widehat{\theta}^{\#}_{\tilde{\boldsymbol{v}}}(\boldsymbol{w})<\epsilon\}$ is an $\mathbb{A}\mathfrak{T}_{\text{S-i}}$-neighbourhood of $\mathbf{0}$ in $\mathbb{A}M_{\text{po-b}}(\mathcal{S})$. If $\boldsymbol{w}\in G'$, $\hat{\boldsymbol{w}}\in G$. **P** There is a non-decreasing uniformly order-bounded sequence $\langle\boldsymbol{u}^{(n)}\rangle_{n\in\mathbb{N}}$ in $\mathbb{A}M_{\text{mo}}(\mathcal{S})^+$ such that $|\boldsymbol{w}|\le\sup_{n\in\mathbb{N}}\boldsymbol{u}^{(n)}_<$ and $\theta(\int_{\mathcal{S}}\boldsymbol{u}^{(n)}d\tilde{\boldsymbol{v}})\le\epsilon$ for every $n\in\mathbb{N}$. Because $\boldsymbol{v}\restriction\mathcal{S}\preccurlyeq\tilde{\boldsymbol{v}}$,

$$\theta(\int_{\hat{\mathcal{S}}^{(\mathbb{B})}}\hat{\boldsymbol{u}}^{(n)}d\boldsymbol{v})=\theta(\int_{\mathcal{S}}\boldsymbol{u}^{(n)}d\boldsymbol{v})$$

(613T, work8ing in $\mathbb{B}$)

$$\le\theta(\int_{\mathcal{S}}\boldsymbol{u}^{(n)}d\tilde{\boldsymbol{v}})$$

(645Bc, working in $\mathbb{A}$)

$$\le\epsilon$$

for every n. Next, I said in 645Bb that the supremum $\sup_{n\in\mathbb{N}}\boldsymbol{u}^{(n)}_<$ is to be taken in $(L^0)^{\mathcal{S}}$; but of course the space $\mathbb{B}M_{\text{fa}}(\mathcal{S})$ of $\mathbb{B}$-fully adapted processes with domain $\mathcal{S}$ is order-closed in $(L^0)^{\mathcal{S}}$ (612Ia), so $\sup_{n\in\mathbb{N}}\boldsymbol{u}^{(n)}_<\in\mathbb{B}M_{\text{fa}}(\mathcal{S})$ and is the supremum of $\{\boldsymbol{u}^{(n)}_<:n\in\mathbb{N}\}$ in $\mathbb{B}M_{\text{fa}}(\mathcal{S})$. Now the map $\boldsymbol{u}\mapsto\hat{\boldsymbol{u}}$ is an order-isomorphism between $\mathbb{B}M_{\text{fa}}(\mathcal{S})$ and the space $\mathbb{B}M_{\text{fa}}(\hat{\mathcal{S}}^{(\mathbb{B})})$ of $\mathbb{B}$-fully adapted processes with domain $\hat{\mathcal{S}}^{(\mathbb{B})}$ (612Qb again), and $\sup_{n\in\mathbb{N}}(\boldsymbol{u}^{(n)}_<)\hat{}=(\sup_{n\in\mathbb{N}}\boldsymbol{u}^{(n)}_<)\hat{}$. Consequently

$$|\hat{\boldsymbol{w}}|\le(\sup_{n\in\mathbb{N}}\boldsymbol{u}^{(n)}_<)\hat{}=\sup_{n\in\mathbb{N}}(\boldsymbol{u}^{(n)}_<)\hat{}=\sup_{n\in\mathbb{N}}\hat{\boldsymbol{u}}^{(n)}_<,$$

while $\langle\hat{\boldsymbol{u}}^{(n)}\rangle_{n\in\mathbb{N}}$ is a non-decreasing uniformly order-bounded sequence in $\mathbb{B}M_{\text{mo}}(\hat{\mathcal{S}}^{(\mathbb{B})})^+$. Accordingly $\mathbb{B}\widehat{\theta}^{\#}_{\boldsymbol{v}}(\hat{\boldsymbol{w}})\le\epsilon$ and $\hat{\boldsymbol{w}}\in G$. **Q**

As G is arbitrary, $\boldsymbol{w}\mapsto\hat{\boldsymbol{w}}$ is continuous at 0; being a linear operator, it is continuous.

(iii)($\boldsymbol{\alpha}$) Suppose to begin with that $\boldsymbol{x}\in\mathbb{A}M^0_{\text{S-i}}(\mathcal{S})$. Because $\hat{\boldsymbol{u}}\in\mathbb{B}M_{\text{mo}}(\hat{\mathcal{S}}^{(\mathbb{B})})$ and $(\boldsymbol{u}_<)\hat{}=(\hat{\boldsymbol{u}})_<$ for every $\boldsymbol{u}\in\mathbb{A}M_{\text{mo}}(\mathcal{S})$, and $\boldsymbol{y}\mapsto\hat{\boldsymbol{y}}:\mathbb{A}M_{\text{po-b}}(\mathcal{S})\to\mathbb{B}M_{\text{po-b}}(\hat{\mathcal{S}}^{(\mathbb{B})})$ is continuous, $\hat{\boldsymbol{x}}$ will belong to

$$\overline{\{\hat{\boldsymbol{u}}_<:\boldsymbol{u}\in\mathbb{A}M_{\text{mo}}(\mathcal{S})\}}\subseteq\overline{\{\boldsymbol{u}'_<:\boldsymbol{u}'\in\mathbb{B}M_{\text{mo}}(\hat{\mathcal{S}}^{(\mathbb{B})})\}}=\mathbb{B}M^0_{\text{S-i}}(\hat{\mathcal{S}}^{(\mathbb{B})}).$$

($\boldsymbol{\beta}$) Now take any $\boldsymbol{x}\in\mathbb{A}M_{\text{S-i}}(\mathcal{S})$. Then $\boldsymbol{x}\times\mathbf{1}^{(\mathcal{S})}_<\in\mathbb{A}M^0_{\text{S-i}}(\mathcal{S})$, and $(\boldsymbol{x}\times\mathbf{1}^{(\mathcal{S})}_<)\hat{}\in\mathbb{B}M^0_{\text{S-i}}(\hat{\mathcal{S}}^{(\mathbb{B})})$. The point is that $\mathbf{1}^{(\mathcal{S})}_<=\mathbf{1}^{(\hat{\mathcal{S}}^{(\mathbb{B})})}_<\restriction\mathcal{S}$, just because $\mathbf{1}^{(\hat{\mathcal{S}}^{(\mathbb{B})})}=(\mathbf{1}^{(\mathcal{S})})\hat{}$. It follows that $\hat{\boldsymbol{x}}\times\mathbf{1}^{(\hat{\mathcal{S}}^{(\mathbb{B})})}_<$ is a $\mathbb{B}$-fully adapted process with domain $\hat{\mathcal{S}}^{(\mathbb{B})}$ extending $\boldsymbol{x}\times\mathbf{1}^{(\mathcal{S})}_<$ and must be equal to $(\boldsymbol{x}\times\mathbf{1}^{(\mathcal{S})}_<)\hat{}\in\mathbb{B}M^0_{\text{S-i}}(\hat{\mathcal{S}}^{(\mathbb{B})})$. Consequently $\hat{\boldsymbol{x}}\in\mathbb{B}M_{\text{S-i}}(\hat{\mathcal{S}}^{(\mathbb{B})})$ in this case too.

(d)(i) Because $\hat{\mathcal{S}}^{(\mathbb{B})}$ is order-convex in $\mathcal{T}_{\mathbb{B}}$, by (a-i), and $\langle\mathfrak{B}_t\rangle_{t\in T}$ is right-continuous (647Ga), we can speak of both S-integrals (645P). Write z for $\mathbb{B}\oint_{\hat{\mathcal{S}}^{(\mathbb{B})}}\hat{\boldsymbol{x}}\,d\hat{\boldsymbol{v}}$ and z' for $\mathbb{A}\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{v}$.

(ii) Suppose to begin with that $\boldsymbol{x}\in\mathbb{A}M^0_{\text{S-i}}(\mathcal{S})$. Let $\epsilon>0$. We know that there is a $\boldsymbol{u}^*\in\mathbb{A}M_{\text{mo}}(\mathcal{S})^+$ such that $|\boldsymbol{x}|\le\boldsymbol{u}^*_<$, in which case $\hat{\boldsymbol{u}}^*\in\mathbb{B}M_{\text{mo}}(\hat{\mathcal{S}}^{(\mathbb{B})})^+$ and $|\hat{\boldsymbol{x}}|\le\hat{\boldsymbol{u}}^*_<$. There is a neighbourhood G of $\hat{\boldsymbol{x}}$ in

$\mathbb{B}M^0_{\text{S-i}}(\hat{\mathcal{S}}^{(\mathbb{B})})$ such that $\theta(z-\int_{\hat{\mathcal{S}}^{(\mathbb{B})}}\tilde{\boldsymbol{u}}\,d\hat{\boldsymbol{v}})\le\epsilon$ whenever $\tilde{\boldsymbol{u}}\in\mathbb{B}M_{\text{mo}}(\hat{\mathcal{S}}^{(\mathbb{B})})$, $|\tilde{\boldsymbol{u}}|\le\hat{\boldsymbol{u}}^*$ and $\tilde{\boldsymbol{u}}_<\in G$; similarly, there is a neighbourhood G' of $\boldsymbol{x}$ in in $\mathbb{A}M^0_{\text{S-i}}(\mathcal{S})$ such that $\theta(z'-\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v})\le\epsilon$ whenever $\boldsymbol{u}\in\mathbb{A}M_{\text{mo}}(\hat{\mathcal{S}}^{(\mathbb{B})})$, $|\boldsymbol{u}|\le\boldsymbol{u}^*$ and $\boldsymbol{u}_<\in G'$. And since the map $\boldsymbol{x}'\mapsto\hat{\boldsymbol{x}}'$ is continuous, we can suppose that $\hat{\boldsymbol{x}}'\in G$ whenever $\boldsymbol{x}'\in G'$.

By 645La, there is a $\boldsymbol{u}\in\mathbb{A}M_{\text{mo}}(\mathcal{S})$ such that $|\boldsymbol{u}|\le\boldsymbol{u}^*$ and $\boldsymbol{u}_<\in G'$. In this case, $\theta(z-\int_{\hat{\mathcal{S}}^{(\mathbb{B})}}\hat{\boldsymbol{u}}\,d\hat{\boldsymbol{v}})\le\epsilon$ and $\theta(z'-\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v})\le\epsilon$, while $\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}=\int_{\hat{\mathcal{S}}^{(\mathbb{B})}}\hat{\boldsymbol{u}}\,d\hat{\boldsymbol{v}}$ by 613T. So $\theta(z-z')\le 2\epsilon$. As ϵ is arbitrary, $z=z'$, as required.

(iii) And of course we now have

$$\mathbb{B}\oint_{\hat{\mathcal{S}}^{(\mathbb{B})}}\hat{\boldsymbol{x}}\,d\hat{\boldsymbol{v}}=\mathbb{B}\oint_{\hat{\mathcal{S}}^{(\mathbb{B})}}\hat{\boldsymbol{x}}\times\mathbf{1}^{(\hat{\mathcal{S}}^{(\mathbb{B})})}_<\,d\hat{\boldsymbol{v}}=\mathbb{A}\oint_{\mathcal{S}}\boldsymbol{x}\times\mathbf{1}^{(\mathcal{S})}_<\,d\hat{\boldsymbol{v}}=\mathbb{A}\oint_{\mathcal{S}}\boldsymbol{x}\,d\hat{\boldsymbol{v}}$$

for every $\boldsymbol{x}\in\mathbb{A}M_{\text{S-i}}(\mathcal{S})$.

647J Theorem Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous and $\mathcal{S}$ is an order-convex sublattice of $\mathcal{T}$. If $\boldsymbol{w}$ is an S-integrable process with domain $\mathcal{S}$, and $\boldsymbol{v}$ is a near-simple integrator with domain $\mathcal{S}$, then $[\![\oint_{\mathcal{S}}\boldsymbol{w}\,d\boldsymbol{v}\ne 0]\!]\subseteq[\![\boldsymbol{w}\ne\mathbf{0}]\!]$.

proof I stopped speaking of $\mathcal{T}_{\mathbb{A}}$ and $\mathbb{A}M^0_{\text{S-i}}$ and $\mathbb{A}\oint$ because I wish to regard this result as a fact about S-integration expressible in the language of §§645-646. But of course the idea is to apply the ideas of 647D-647I, so I immediately set $b=[\![\boldsymbol{w}\ne\mathbf{0}]\!]$ and move to the structures considered in 647D-647E. By 647Id,

$$\oint_{\mathcal{S}}\boldsymbol{w}\,d\boldsymbol{v}=\mathbb{A}\oint_{\mathcal{S}}\boldsymbol{w}\,d\boldsymbol{v}=\mathbb{B}\oint_{\hat{\mathcal{S}}^{(\mathbb{B})}}\hat{\boldsymbol{w}}\,d\hat{\boldsymbol{v}},$$

where $\hat{\mathcal{S}}^{(\mathbb{B})}$ is the covered envelope of $\mathcal{S}$ in $\mathcal{T}_{\mathbb{B}}$ and $\boldsymbol{u}\mapsto\hat{\boldsymbol{u}}$ is the canonical isomorphism between $\mathbb{B}M_{\text{fa}}(\mathcal{S})$ and $\mathbb{B}M_{\text{fa}}(\hat{\mathcal{S}}^{(\mathbb{B})})$ described in 612Qb. Now $[\![\hat{\boldsymbol{w}}\ne\mathbf{0}]\!]=[\![\boldsymbol{w}\ne\mathbf{0}]\!]$ (612S(c-ii)). At the same time, $\chi b\in L^0(\mathfrak{B}_{\min\hat{\mathcal{S}}^{(\mathbb{B})}})$. So

$$\mathbb{B}\oint_{\hat{\mathcal{S}}^{(\mathbb{B})}}\hat{\boldsymbol{w}}\,d\hat{\boldsymbol{v}}=\mathbb{B}\oint_{\hat{\mathcal{S}}^{(\mathbb{B})}}(\chi b)\hat{\boldsymbol{w}}\,d\hat{\boldsymbol{v}}=\chi b\times\mathbb{B}\oint_{\hat{\mathcal{S}}^{(\mathbb{B})}}\hat{\boldsymbol{w}}\,d\hat{\boldsymbol{v}}$$

(646D), and

$$[\![\mathbb{A}\oint_{\mathcal{S}}\boldsymbol{w}\,d\boldsymbol{v}\ne 0]\!]=[\![\chi b\times\mathbb{B}\oint_{\mathcal{S}^{(\mathbb{B})}}\hat{\boldsymbol{w}}\,d\hat{\boldsymbol{v}}\ne 0]\!]\subseteq b,$$

as claimed.

647X Basic exercises (a) Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous and that $\mathcal{S}$ is an order-convex sublattice of $\mathcal{T}$. Let $\boldsymbol{v}$ be a near-simple process of bounded variation and $\boldsymbol{w}$ an S-integrable process, both with domain $\mathcal{S}$. Show that $\int_{\mathcal{S}}|d(Sii_{\boldsymbol{v}}(\boldsymbol{w}))|\le\sup|\boldsymbol{w}|\times\int_{\mathcal{S}}|d\boldsymbol{v}|$. (*Hint*: apply 647J to $\boldsymbol{w}-\text{med}(-M\mathbf{1}_<,\boldsymbol{w},M\mathbf{1}_<)$ for $M\ge 0$.)

647 Notes and comments In §646 we found that most of the standard properties of the Riemann-sum integral transferred without much trouble to the S-integral; only the result $\int_{\mathcal{S}}=\int_{\mathcal{S}\wedge\tau}+\int_{\mathcal{S}\vee\tau}$ gave difficulty, and this was attributable to a technical problem at the least member τ of $\mathcal{S}\vee\tau$. We did of course need to know a little more about martingales (646P), and this fact seems to depend on the fundamental theorem of martingales. However, there was a curious gap when we came to look at $[\![\oint\boldsymbol{w}\,d\boldsymbol{v}\ne 0]\!]$. A straightforward calculation (613Jd) showed that, for the Riemann-sum integral, $[\![\int\boldsymbol{u}\,d\boldsymbol{v}\ne 0]\!]\subseteq[\![\boldsymbol{u}\ne\mathbf{0}]\!]\cap[\![\boldsymbol{v}\ne\mathbf{0}]\!]$. There was no difficulty in showing that $[\![\oint\boldsymbol{w}\,d\boldsymbol{v}\ne 0]\!]\subseteq[\![\boldsymbol{v}\ne\mathbf{0}]\!]$ (646C). But it seems to be much harder to confirm that $[\![\oint\boldsymbol{w}\,d\boldsymbol{v}\ne 0]\!]\subseteq[\![\boldsymbol{w}\ne\mathbf{0}]\!]$ (647J).

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648 Changing the algebra II

In §634, I looked at questions involving pairs $(\mathfrak{A},\mathfrak{B})$ where $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$, and the corresponding stochastic integration structures $\mathbb{A}$ and $\mathbb{B}$. In particular, we can relate Riemann-sum integrals calculated in the two structures (634Eg). Unsurprisingly, there is a corresponding result for S-integration (648G), though it seems to need a good deal more work.

648A Notation As usual, $\mathbb{A} = (\mathfrak{A}, \bar{\mu}, T, \langle \mathfrak{A}_t \rangle_{t \in T}, \mathcal{T}_{\mathbb{A}}, \langle \mathfrak{A}_\sigma \rangle_{\sigma \in \mathcal{T}_{\mathbb{A}}})$ will be a stochastic integration structure. For nearly the whole section, we shall have a closed subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ with the associated structure $\mathbb{B} = (\mathfrak{B}, \bar{\mu}\restriction\mathfrak{B}, T, \langle \mathfrak{B}_t \rangle_{t \in T}, \mathcal{T}_{\mathbb{B}}, \langle \mathfrak{B}_\sigma \rangle_{\sigma \in \mathcal{T}_{\mathbb{B}}})$ as in 634C. As in §634, I will use formulations like '$\boldsymbol{x}$ is $\mathbb{A}$-previsibly-order-bounded', '$\boldsymbol{v}' \in \mathbb{B}M_{\text{n-s}}(\mathcal{S}')$', '$\mathbb{A}\oint_{\mathcal{S}} \boldsymbol{x}\, d\boldsymbol{v}$' to indicate which structure is being considered at any particular moment.

If $E \subseteq \mathbb{R}$ is a Borel set and $h : E \to \mathbb{R}$ is a Borel measurable function, $\bar{h}$ is the corresponding function from $\{u : u \in L^0(\mathfrak{A}), [\![u \in E]\!] = 1\}$ to $L^0(\mathfrak{A})$ (612Ac). If $\boldsymbol{u}$ is a moderately oscillatory process, $\boldsymbol{u}_<$ will denote its previsible version (641F). I use the symbol $\int$ for Riemann-sum integrals (613H, 613L) and $\oint$ for S-integrals (645P); $\mathbb{E}$ will be the ordinary integral on $L^1_{\bar{\mu}} = L^1(\mathfrak{A}, \bar{\mu})$, and θ the associated F-norm on $L^0(\mathfrak{A})$ (613Ba). The F-seminorms $\widehat{\theta}^{\#}_{\boldsymbol{v}}$ will be those of 645B. $S_I(\boldsymbol{u}, d\boldsymbol{v})$ will be the Riemann sum (613Fb). $\mathbf{1}^{(\mathcal{S})}$ will be the constant process on $\mathcal{S}$ with value $\chi 1$.

We shall have the usual spaces of processes: M_{fa} (fully adapted, 612I), M_{mo} (moderately oscillatory, 615Fa), $M_{\text{n-s}}$ (near-simple, 631Ba), $M_{\text{po-b}}$ (previsibly order-bounded, 645Ba), $M^{\uparrow}_{\text{n-s}}$ (non-decreasing non-negative near-simple, 644Bb), $M^0_{\text{S-i}}$ and $M_{\text{S-i}}$ (S-integrable, 645F). If $\mathcal{S}$ is a sublattice of $\mathcal{T}_{\mathbb{A}}$, $\mathcal{I}(\mathcal{S})$ will be the directed set of finite sublattices of $\mathcal{S}$.

648B Lemma Let E be a Borel subset of $\mathbb{R}$; write Q_E for $\{u : u \in L^0(\mathfrak{A}), [\![u \in E]\!] = 1\}$. Let $h : E \to \mathbb{R}$ be a continuous function. Suppose that $\mathcal{S}$ is a finitely full sublattice of $\mathcal{T}_{\mathbb{A}}$ and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ a moderately oscillatory process such that $\overline{\{u_\sigma : \sigma \in \mathcal{S}\}} \subseteq Q_E$, the closure being for the topology of convergence in measure on $L^0(\mathfrak{A})$. Then $\bar{h}\boldsymbol{u} = \langle \bar{h}(u_\sigma) \rangle_{\sigma \in \mathcal{S}}$ is a moderately oscillatory process.

proof (Compare 615F(a-ii).)

(a)(i) If $\sigma \in \mathcal{S}$ and $\alpha \in \mathbb{R}$, $[\![\bar{h}(u_\sigma) > \alpha]\!] = [\![u_\sigma \in h^{-1}[\,]\alpha, \infty[\,]]\!] \in \mathfrak{A}_\sigma$, so $\bar{h}(u_\sigma) \in L^0(\mathfrak{A}_\sigma)$.

(ii) If $\sigma, \tau \in \mathcal{S}$,

$$[\![\sigma = \tau]\!] \subseteq [\![u_\sigma = u_\tau]\!] \subseteq [\![\bar{h}(u_\sigma) = \bar{h}(u_\tau)]\!]$$

by 612A(d-iii). Thus $\bar{h}\boldsymbol{u}$ is fully adapted.

(b) If $\langle \sigma_n \rangle_{n \in \mathbb{N}}$ is a monotonic sequence in $\mathcal{S}$, $u = \lim_{n\to\infty} u_{\sigma_n}$ is defined in $L^0(\mathfrak{A})$ and belongs to $\overline{\{u_\sigma : \sigma \in \mathcal{S}\}} \subseteq Q_E$. Since $\bar{h} : Q_E \to L^0(\mathfrak{A})$ is continuous (367S/613Bb), $\lim_{n\to\infty} \bar{h}(u_{\sigma_n}) = \bar{h}(u)$ is defined. As $\mathcal{S}$ is finitely full, this is enough to ensure that $\bar{h}\boldsymbol{u}$ is moderately oscillatory (615N(iii)).

648C Lemma Suppose that $\langle \mathfrak{A}_t \rangle_{t \in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}_{\mathbb{A}}$, $\boldsymbol{w}$ a $\| \|_2$-bounded martingale with domain $\mathcal{S}$, and $\boldsymbol{x}$ a $\| \|_\infty$-bounded S-integrable process with domain $\mathcal{S}$. If $\boldsymbol{w}^*$ is the quadratic variation of $\boldsymbol{w}$, $\|(\oint_{\mathcal{S}} \boldsymbol{x}\, d\boldsymbol{w})^2\|_1 \le \|\oint_{\mathcal{S}} \boldsymbol{x}^2 d\boldsymbol{w}^*\|_1 < \infty$.

proof This is trivial if $\mathcal{S} = \emptyset$, so suppose otherwise. Express $\boldsymbol{w}$ as $\langle w_\sigma \rangle_{\sigma \in \mathcal{S}}$.

(a) I had better run through the check that the S-integrals are defined. Since $\boldsymbol{w}$ is $\| \|_2$-bounded, it is $\| \|_1$-bounded and an integrator and moderately oscillatory (622G). Because the filtration is right-continuous, $\boldsymbol{w}$ is locally near-simple (632Ia), therefore near-simple (631F(c-ii)), and $\boldsymbol{w}^*$ is also near-simple (631J) as well as being an integrator (617I). And $\boldsymbol{x}^2 \in M^0_{\text{S-i}}(\mathcal{S})$ by 645Ka. So we can form $\oint_{\mathcal{S}} \boldsymbol{x}\, d\boldsymbol{w}$ and $\oint_{\mathcal{S}} \boldsymbol{x}^2 d\boldsymbol{w}^*$.

(b) For the time being (down to the end of (h) below), suppose that $\boldsymbol{x} \in M^0_{\text{S-i}}(\mathcal{S})$. Set $M = \|\boldsymbol{x}\|_\infty$, $K = \sup_{\sigma \in \mathcal{S}} \|w_\sigma\|_2$, $A = \{\boldsymbol{y} : \boldsymbol{y} \in M^0_{\text{S-i}}(\mathcal{S}), \|\boldsymbol{y}\|_\infty \le M\}$, $A_2 = \{\boldsymbol{y} : \boldsymbol{y} \in M^0_{\text{S-i}}(\mathcal{S}), \|\boldsymbol{y}\|_\infty \le M^2\}$. Note that if $\boldsymbol{y} \in A$ then $|\boldsymbol{y}| \le M\mathbf{1}^{(\mathcal{S})}_<$. **P** We know that $|\boldsymbol{y}| \le M\mathbf{1}^{(\mathcal{S})}$ and also that there is a non-negative $\boldsymbol{u} \in M_{\text{mo}}(\mathcal{S})$ such that $|\boldsymbol{y}| \le \boldsymbol{u}_<$. Now

$$\boldsymbol{u}_< = (\boldsymbol{u} \times \mathbf{1}^{\mathcal{S})})_< = \boldsymbol{u}_< \times \mathbf{1}^{(\mathcal{S})}_<,$$

so

$$|\boldsymbol{y}| \times (\mathbf{1}^{(\mathcal{S})} - \mathbf{1}^{\mathcal{S}}_<) \le \boldsymbol{u}_< \times (\mathbf{1}^{(\mathcal{S})} - \mathbf{1}^{\mathcal{S}}_<) = 0$$

and

$$|\boldsymbol{y}| = |\boldsymbol{y}| \times \mathbf{1}^{(\mathcal{S})} = |\boldsymbol{y}| \times \mathbf{1}^{(\mathcal{S})}_< \le M\mathbf{1}^{\mathcal{S})} \times \mathbf{1}^{(\mathcal{S})}_< = M\mathbf{1}^{(\mathcal{S})}_<. \ \mathbf{Q}$$

So A is uniformly previsibly order-bounded. Similarly, A_2 is uniformly previsibly order-bounded.

(c) By 645S, $\boldsymbol{y} \mapsto \oint_{\mathcal{S}} \boldsymbol{y}\,d\boldsymbol{w} : A \to L^0(\mathfrak{A})$ and $\boldsymbol{y} \mapsto \oint_{\mathcal{S}} \boldsymbol{y}\,d\boldsymbol{w}^* : A_2 \to L^0(\mathfrak{A})$ are continuous when $M^0_{\text{S-i}}(\mathcal{S})$ is given its S-integration topology and $L^0(\mathfrak{A})$ is given the topology of convergence in measure. Since multiplication on $L^0(\mathfrak{A})$ is continuous, $\boldsymbol{y} \mapsto (\oint_{\mathcal{S}} \boldsymbol{y}\,d\boldsymbol{w})^2 : A \to L^0(\mathfrak{A})$ is continuous. Since $\boldsymbol{y} \mapsto \boldsymbol{y}^2 : A \mapsto A_2$ is continuous (645E(a-v-β)), $\boldsymbol{y} \mapsto \oint_{\mathcal{S}} \boldsymbol{y}^2 d\boldsymbol{w}^* : A \to L^0(\mathfrak{A})$ is continuous.

(d) If $\boldsymbol{u} \in M_{\text{mo}}(\mathcal{S})$ is $\| \|_\infty$-bounded,

$$\|(\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{w})^2\|_1 = \mathbb{E}((\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{w})^2) = \mathbb{E}(\int_{\mathcal{S}} \boldsymbol{u}^2 d\boldsymbol{w}^*)$$

(624I)

$$= \|\int_{\mathcal{S}} \boldsymbol{u}^2 d\boldsymbol{w}^*\|_1$$

by 614Ig, because $\boldsymbol{u}^2$ is non-negative and $\boldsymbol{w}^*$ is non-decreasing. And by 622P we have $\|\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{w}\|_2 \le K\|\boldsymbol{u}\|_\infty$, so $\|(\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{w})^2\|_1 \le K^2\|\boldsymbol{u}\|_\infty^2$.

(e) Set $B = \{\boldsymbol{u} : \boldsymbol{u} \in M_{\text{mo}}(\mathcal{S}),\ \|\boldsymbol{u}\|_\infty \le M\}$. Then A is the closure of $\{\boldsymbol{u}_< : \boldsymbol{u} \in B\}$ for the S-integration topology of $M^0_{\text{S-i}}(\mathcal{S})$. **P** If $\boldsymbol{u} \in B$, $-M\mathbf{1}^{(\mathcal{S})} \le \boldsymbol{u} \le M\mathbf{1}^{\mathcal{S}}$ so $-M\mathbf{1}^{(\mathcal{S})}_< \le \boldsymbol{u}_< \le M\mathbf{1}^{\mathcal{S}}_<$ and $\boldsymbol{u}_< \in A$. On the other hand, if $\boldsymbol{y} \in A$, $\boldsymbol{v} \in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$ and $\epsilon > 0$, there is certainly a $\boldsymbol{u} \in M_{\text{mo}}(\mathcal{S})$ such that $\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{y} - \boldsymbol{u}_<) \le \epsilon$, by the definition of $M^0_{\text{S-i}}(\mathcal{S})$. Now, setting $\tilde{\boldsymbol{u}} = \text{med}(-M\mathbf{1}^{(\mathcal{S})}, \boldsymbol{u}, M\mathbf{1}^{(\mathcal{S})})$,

$$|\boldsymbol{y} - \tilde{\boldsymbol{u}}_<| = |\,\text{med}(-M\mathbf{1}^{(\mathcal{S})}_<, \boldsymbol{y}, M\mathbf{1}^{(\mathcal{S})}_<) - \text{med}(-M\mathbf{1}^{(\mathcal{S})}_<, \boldsymbol{u}_<, M\mathbf{1}^{(\mathcal{S})}_<)| \le |\boldsymbol{y} - \boldsymbol{u}_<|$$

so

$$\widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{y} - \tilde{\boldsymbol{u}}_<) \le \widehat{\theta}^{\#}_{\boldsymbol{v}}(\boldsymbol{y} - \boldsymbol{u}_<) \le \epsilon$$

(645Db), while $\tilde{\boldsymbol{u}} \in B$. As $\boldsymbol{y}$, $\boldsymbol{v}$ and ϵ are arbitrary, $A \subseteq \overline{\{\boldsymbol{u}_< : \boldsymbol{u} \in B\}}$ (645E(a-ii)). **Q**

(f) It follows that if $\boldsymbol{y} \in A$, then $(\oint_{\mathcal{S}} \boldsymbol{y}\,d\boldsymbol{w})^2$ belongs to the closure, for the topology of convergence in measure, of

$$\{(\oint_{\mathcal{S}} \boldsymbol{u}_< d\boldsymbol{w})^2 : \boldsymbol{u} \in B\} = \{(\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{w})^2 : \boldsymbol{u} \in B\} \subseteq C$$

where $C = \{z : z \in L^0(\mathfrak{A}),\ \|z\|_1 \le K^2M^2\}$. So $(\oint_{\mathcal{S}} \boldsymbol{y}\,d\boldsymbol{w})^2 \in \overline{C} = C$ (613Bc). Because $\| \|_1 : L^0(\mathfrak{A}) \to [0, \infty]$ is lower semi-continuous (613Bc), $\boldsymbol{y} \mapsto \|(\oint_{\mathcal{S}} \boldsymbol{y}\,d\boldsymbol{w})^2\|_1 : A \to \mathbb{R}$ is lower semi-continuous (4A2B(d-i)).

(g) Concerning $\oint_{\mathcal{S}} \boldsymbol{y}^2 d\boldsymbol{w}^*$ we have a little more. Note first that, setting $\boldsymbol{u} = \mathbf{1}^{(\mathcal{S})}$ in (d), we see that

$$\|\int_{\mathcal{S}} d\boldsymbol{w}^*\|_1 = \|(\int_{\mathcal{S}} d\boldsymbol{w})^2\|_1 \le K^2$$

is finite. Now if $\boldsymbol{u} \in B$, $0 \le \boldsymbol{u}^2 \le M^2\mathbf{1}^{(\mathcal{S})}$, so

$$0 \le \int_{\mathcal{S}} \boldsymbol{u}^2 d\boldsymbol{w}^* \le M^2\bar{w}$$

where $\bar{w} = \int_{\mathcal{S}} d\boldsymbol{w}^*$. Setting $D = \{z : z \in L^0(\mathfrak{A}),\ 0 \le z \le M^2\bar{w}\}$, D is closed for the topology of convergence in measure and includes

$$\{\int_{\mathcal{S}} \boldsymbol{u}^2 d\boldsymbol{w}^* : u \in B\} = \{\oint_{\mathcal{S}} (\boldsymbol{u}^2)_< d\boldsymbol{w}^* : u \in B\} = \{\oint_{\mathcal{S}} \boldsymbol{u}^2_< d\boldsymbol{w}^* : u \in B\}$$

so contains $\oint_{\mathcal{S}} \boldsymbol{y}^2 d\boldsymbol{w}^*$ for every $y \in A$. In particular, $\oint_{\mathcal{S}} \boldsymbol{x}^2 d\boldsymbol{w}^* \in D$, so

$$\|\oint_{\mathcal{S}} \boldsymbol{x}^2 d\boldsymbol{w}^*\|_1 \le M^2\|\bar{w}\|_1 \le M^2K^2 < \infty.$$

Observe next that D is uniformly integrable, so the $\| \|_1$-topology on D agrees with the topology of convergence in measure (621B(c-ii)), and $z \mapsto \|z\|_1 : D \to \mathbb{R}$ is continuous. But this means that $\boldsymbol{y} \mapsto \|\oint_{\mathcal{S}} \boldsymbol{y}^2 d\boldsymbol{w}^*\|_1 : A \to \mathbb{R}$ is continuous.

(h) Putting (f) and (g) together, we see that

$$\boldsymbol{y} \mapsto \|(\oint_{\mathcal{S}} \boldsymbol{y}\,d\boldsymbol{w})^2\|_1 - \|\oint_{\mathcal{S}} \boldsymbol{y}^2 d\boldsymbol{w}^*\|_1 : A \to \mathbb{R}$$

is lower semi-continuous (4A2B(d-iii)). And as

$$\|(\oint_{\mathcal{S}}\boldsymbol{u}_<\,d\boldsymbol{w})^2\|_1-\|\oint_{\mathcal{S}}\boldsymbol{u}_<^2d\boldsymbol{w}^*\|_1=\|(\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{w})^2\|_1-\|\int_{\mathcal{S}}\boldsymbol{u}^2d\boldsymbol{w}^*\|_1=0$$

for every $\boldsymbol{u}\in B$, $\|(\oint_{\mathcal{S}}\boldsymbol{y}\,d\boldsymbol{w})^2\|_1-\|\oint_{\mathcal{S}}\boldsymbol{y}^2d\boldsymbol{w}^*\|_1\le 0$ for every $\boldsymbol{y}\in A$. In particular, $\|(\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{w})^2\|_1\le\|\oint_{\mathcal{S}}\boldsymbol{x}^2d\boldsymbol{w}^*\|_1$.

(i) This deals with the result when $\boldsymbol{x}\in M^0_{\text{S-i}}(\mathcal{S})$. For the general case in which $\boldsymbol{x}\in M_{\text{S-i}}(\mathcal{S})$, apply (b)-(h) to $\boldsymbol{x}\times\mathbf{1}^{(\mathcal{S})}_<$.

648D We come now to the main work of the section in which we are working in both the original stochastic integration structure $\mathbb{A}$ and an embedded structure $\mathbb{B}$.

Lemma Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathfrak{B}$ be a closed subalgebra of $\mathfrak{A}$ which is coordinated with $\langle\mathfrak{A}_t\rangle_{t\in T}$, and $\mathbb{B}=(\mathfrak{B},\bar\mu\restriction\mathfrak{B},T,\langle\mathfrak{B}_t\rangle_{t\in T},\mathcal{T}_{\mathbb{B}},\langle\mathfrak{B}_\sigma\rangle_{\sigma\in\mathcal{T}_{\mathbb{B}}})$ the corresponding stochastic integration structure, where $\mathfrak{B}_t=\mathfrak{B}\cap\mathfrak{A}_t$ for $t\in T$. Let $\mathcal{S}'$ be a relatively order-convex sublattice of $\mathcal{T}_{\mathbb{B}}$ and $\mathcal{S}$ its order-convex hull in $\mathcal{T}_{\mathbb{A}}$. If $\boldsymbol{x}\in\mathbb{A}M^0_{\text{S-i}}(\mathcal{S})$ and $\boldsymbol{x}\restriction\mathcal{S}'\in L^0(\mathfrak{B})^{\mathcal{S}'}$, then $\boldsymbol{x}\restriction\mathcal{S}'\in\mathbb{B}M^0_{\text{S-i}}(\mathcal{S})$.

proof (a) We need the following basic facts: $\mathcal{S}'$ is $\mathbb{B}$-finitely full (611Pc), $\mathcal{S}'$ separates $\mathcal{S}$ (633Da and 633D(b-i), because constant processes belong to $\mathcal{T}_{\mathbb{B}}$), $\mathcal{S}'$ is cofinal with $\mathcal{S}$ and $\inf A\in\mathcal{S}$ whenever $A\subseteq\mathcal{S}$ is non-empty and has a lower bound in $\mathcal{S}$. Recall also that we have a multiplicative Riesz homomorphism $\Psi:\mathbb{A}M_{\text{n-s}}(\mathcal{S}')\to\mathbb{A}M_{\text{n-s}}(\mathcal{S})$ such that $\Psi(\boldsymbol{u}')$ extends $\boldsymbol{u}'$ for every $\boldsymbol{u}'\in\mathbb{A}M_{\text{n-s}}(\mathcal{S})$ (631Mb), while $\mathbb{B}M_{\text{n-s}}(\mathcal{S}')=\mathbb{A}M_{\text{n-s}}(\mathcal{S}')\cap L^0(\mathfrak{B})^{\mathcal{S}'}$ (634Eb). It follows that $\mathbb{B}M^{\uparrow}_{\text{n-s}}(\mathcal{S}')=\mathbb{A}M^{\uparrow}_{\text{n-s}}(\mathcal{S}')\cap L^0(\mathfrak{B})^{\mathcal{S}'}$.

(b)(i) Let $P:L^1_{\bar\mu}\to L^1_{\bar\mu}$ be the conditional expectation operator associated with $\mathfrak{B}$. If $\boldsymbol{y}=\langle y_\sigma\rangle_{\sigma\in\mathcal{S}}$ belongs to $(L^1_{\bar\mu})^{\mathcal{S}}$ set $P'(\boldsymbol{y})=\langle P(y_\sigma)\rangle_{\sigma\in\mathcal{S}'}$.

(ii) If $\boldsymbol{u}\in\mathbb{A}M_{\text{fa}}(\mathcal{S})$ is $\|\,\|_\infty$-bounded then $P'(\boldsymbol{u})\in\mathbb{B}M_{\text{fa}}(\mathcal{S}')$. **P** Express $\boldsymbol{u}$ as $\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$. If $\sigma\in\mathcal{S}'$ then $u_\sigma\in L^0(\mathfrak{A}_\sigma)$. Writing P_σ for the conditional expectation associated with the closed subalgebra $\mathfrak{A}_\sigma$, we know that $\mathfrak{B}$ and $\mathfrak{A}_\sigma$ are relatively independent over their intersection $\mathfrak{B}_\sigma$ (634H) so that PP_σ is the conditional expectation operator corresponding to $\mathfrak{B}_\sigma$. Now $P(u_\sigma)=PP_\sigma(u_\sigma)\in L^0(\mathfrak{B}_\sigma)$. Thus $P'(\boldsymbol{u})\in\prod_{\sigma\in\mathcal{S}'}L^0(\mathfrak{B}_\sigma)$.

If σ, $\tau\in\mathcal{S}'$ and $b=[\![\sigma=\tau]\!]$, then $b\in\mathfrak{B}$, so

$$P(u_\sigma)\times\chi b=P(u_\sigma\times\chi b)=P(u_\tau\times\chi b)=P(u_\tau)\times\chi b$$

and $b\subseteq[\![P(u_\sigma)=P(u_\tau)]\!]$. So $P'(\boldsymbol{u})$ is $\mathbb{B}$-fully adapted. **Q**

(iii) If $\boldsymbol{u}\in\mathbb{A}M_{\text{mo}}(\mathcal{S})$ is $\|\,\|_\infty$-bounded, then $P'(\boldsymbol{u})\in\mathbb{B}M_{\text{mo}}(\mathcal{S}')$ and $P'(\boldsymbol{u})_<=P'(\boldsymbol{u}_<)$. **P** We have just seen that $P'(\boldsymbol{u})$ is $\mathbb{B}$-fully adapted. Write M for $\|\boldsymbol{u}\|_\infty$. On the uniformly integrable set $A=\{z:z\in L^0(\mathfrak{A}),\ \|z\|_\infty\le M\}$ the topology of convergence in measure coincides with the norm topology of $L^1_{\bar\mu}$; as $P:A\to A$ is $\|\,\|_1$-continuous, it is continuous for the topology of convergence in measure. If $\langle\sigma_n\rangle_{n\in\mathbb{N}}$ is a monotonic sequence in $\mathcal{S}'$, $\langle u_{\sigma_n}\rangle_{n\in\mathbb{N}}$ is a sequence in A which converges for the topology of convergence in measure to a member of A, so $\langle(P(u_{\sigma_n})\rangle_{n\in\mathbb{N}}$ converges for the topology of convergence in measure. As $\langle\sigma_n\rangle_{n\in\mathbb{N}}$ is arbitrary and $\mathcal{S}'$ is $\mathbb{B}$-finitely-full, $P'(\boldsymbol{u})$ is $\mathbb{B}$-moderately-oscillatory (615N(iii) again).

Express $\boldsymbol{u}$ as $\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ and $P'(\boldsymbol{u})$ as $\langle w_\sigma\rangle_{\sigma\in\mathcal{S}'}$. For $\tau\in\mathcal{S}'$ and non-empty $I\in\mathcal{I}(\mathcal{S}')$, define $u_{I<\tau}$ from $\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ and $w_{I<\tau}$ from $\langle w_\sigma\rangle_{\sigma\in\mathcal{S}'}$ as in 641E. Now for $\sigma\in I$ set

$$b_\sigma=[\![\sigma<\tau]\!]\setminus\sup_{\sigma'\in I}[\![\sigma<\sigma']\!]\cap[\![\sigma'<\tau]\!],$$

and set $b=[\![\tau\le\min I]\!]$. Because $\mathcal{S}'\subseteq\mathcal{T}_{\mathbb{B}}$, b_σ and b belong to $\mathfrak{B}$, while

$$b_\sigma\subseteq[\![u_{I<\tau}=u_\sigma]\!]\cap[\![w_{I<\tau}=w_\sigma]\!],$$

$$b\subseteq[\![u_{I<\tau}=0]\!]\cap[\![w_{I<\tau}=0]\!].$$

Accordingly

$$b_\sigma\subseteq[\![P(u_{I<\tau})=w_\sigma]\!]\cap[\![w_{I<\tau}=w_\sigma]\!]\subseteq[\![P(u_{I<\tau})=w_{I<\tau}]\!],$$

$$b\subseteq[\![P(u_{I<\tau})=w_{I<\tau}]\!].$$

As $b\cup\sup_{\sigma\in I}b_\sigma=1$, $w_{I<\tau}=P(u_{I<\tau})$. As I is arbitrary and P is continuous on $\{z:z\in L^0(\mathfrak{A}),\ |z|\le M\chi 1\}$,

$$w_{<\tau} = \lim_{I\uparrow\mathcal{I}(\mathcal{S}')} w_{I<\tau} = P(\lim_{I\uparrow\mathcal{I}(\mathcal{S}')} u_{I<\tau});$$

as τ is arbitrary, $(P'(\boldsymbol{u}))_< = P(\boldsymbol{u}\restriction\mathcal{S}')_<$. But $\mathcal{S}'$ separates $\mathcal{S}$, so $(\boldsymbol{u}\restriction\mathcal{S}')_< = \boldsymbol{u}_<\restriction\mathcal{S}'$ (641Hb) and

$$(P'(\boldsymbol{u}))_< = P(\boldsymbol{u}_<\restriction\mathcal{S}') = P'(\boldsymbol{u})_<. \quad \mathbf{Q}$$

(c) $\boldsymbol{x}\restriction\mathcal{S}' \in \mathbb{B}M_{\text{po-b}}(\mathcal{S}')$.

$\mathbf{P}$ Let $\boldsymbol{u} \in \mathbb{A}M_{\text{mo}}(\mathcal{S})^+$ be such that $|\boldsymbol{x}| \le \boldsymbol{u}_<$. Consider the functions $g : \mathbb{R} \to]-1,1[$, $h :]-1,1[$ given by the formulae

$$g(\alpha) = \frac{\alpha}{1+|\alpha|} \text{ for } \alpha \in \mathbb{R}, \quad h(\beta) = \frac{\beta}{1-|\beta|} \text{ for } \beta \in]-1,1[.$$

Then $g = h^{-1}$, $h = g^{-1}$ are the two halves of an order-preserving homeomorphism between $\mathbb{R}$ and $]-1,1[$. Set

$$Q_{\mathfrak{A}} = \{u : u \in L^0(\mathfrak{A}),\ [\![|u| < 1]\!] = 1\}, \quad Q_{\mathfrak{B}} = \{u : u \in L^0(\mathfrak{B}),\ [\![|u| < 1]\!] = 1\}.$$

I will write

$$\bar{g} : L^0(\mathfrak{A})^{\mathcal{S}} \to Q_{\mathfrak{A}}^{\mathcal{S}}, \quad \bar{g}' : L^0(\mathfrak{B})^{\mathcal{S}'} \to Q_{\mathfrak{B}}^{\mathcal{S}'} \quad \bar{h} : Q_{\mathfrak{B}}^{\mathcal{S}'} \to L^0(\mathfrak{B})^{\mathcal{S}'}$$

for the induced functions as in 612A-612B, so that $\bar{h}$ is the inverse of $\bar{g}'$.

By 615F(a-ii), $\bar{g}\boldsymbol{u}$ is $\mathfrak{A}$-moderately-oscillatory, and we have

$$|\bar{g}\boldsymbol{x}| = \bar{g}|\boldsymbol{x}|$$

(because $|g(\alpha)| = g(|\alpha|)$ for $\alpha \in \mathbb{R}$)

$$\le \bar{g}\boldsymbol{u}_<$$

(by 612A(d-iii), because g is order-preserving)

$$= (\bar{g}\boldsymbol{u})_<$$

by 641Gd, while setting $\bar{u} = \sup|\boldsymbol{u}|$, $\sup \bar{g}\boldsymbol{u} \le \bar{g}(\bar{u})$. Since $\bar{g}\boldsymbol{u} \in Q_{\mathfrak{A}}^{\mathcal{S}}$ is $\|\ \|_\infty$-bounded, we can form $P'(\bar{g}\boldsymbol{u})$; by (b), $P'(\bar{g}\boldsymbol{u})$ is $\mathbb{B}$-moderately-oscillatory and $(P'(\bar{g}\boldsymbol{u}))_< = P'((\bar{g}\boldsymbol{u})_<)$. Note that $P(u) \in Q_{\mathfrak{B}}$ for every $u \in Q_{\mathfrak{A}}$, so $P'(\bar{g}\boldsymbol{u}) \in Q_{\mathfrak{B}}^{\mathcal{S}'}$ and $\bar{g}\boldsymbol{u} \in [-P\bar{g}(\bar{u}), P\bar{g}(\bar{u})]^{\mathcal{S}'}$, while $[-P\bar{g}(\bar{u}), P\bar{g}(\bar{u})]$ is a topologically closed subset of $Q_{\mathfrak{B}}$. By 648B, applied in $\mathbb{B}$, $\bar{h}P'(\bar{g}\boldsymbol{u}) \in L^0(\mathfrak{B})^{\mathcal{S}'}$ is a $\mathbb{B}$-moderately-oscillatory process.

Still working in $\mathbb{B}$, we know that the previsible version $(\bar{h}P'(\bar{g}\boldsymbol{u}))_<$ is defined. To compute it, note that we have $\bar{g}'(\bar{h}P'(\bar{g}\boldsymbol{u}))_< = (\bar{g}'\bar{h}P'(\bar{g}\boldsymbol{u}))_< = (P'(\bar{g}\boldsymbol{u}))_<$. So

$$(\bar{h}P'(\bar{g}\boldsymbol{u}))_< = \bar{h}\bar{g}'((\bar{h}P'(\bar{g}\boldsymbol{u}))_<) = \bar{h}P'(\bar{g}\boldsymbol{u})_<.$$

Now we see that

$$|\boldsymbol{x}\restriction\mathcal{S}'| = |\bar{h}\bar{g}'\boldsymbol{x}\restriction\mathcal{S}'| = |\bar{h}P'(\bar{g}\boldsymbol{x})|$$

(because $\boldsymbol{x}\restriction\mathcal{S}' \in L^0(\mathfrak{B})^{\mathcal{S}'}$ and $(\bar{g}\boldsymbol{x})\restriction\mathcal{S}' \in L^\infty(\mathfrak{B})^{\mathcal{S}'}$)

$$\le \bar{h}P'(|\bar{g}\boldsymbol{x}|) = \bar{h}P'(\bar{g}|\boldsymbol{x}|) \le \bar{h}P'(\bar{g}\boldsymbol{u}_<)$$

(because P' is a positive linear operator and $\bar{h}$ is order-preserving)

$$= \bar{h}P'(\bar{g}\boldsymbol{u}_<) = (\bar{h}P'(\bar{g}\boldsymbol{u}))_<$$

while $\bar{h}P'(\bar{g}\boldsymbol{u})$ is non-negative and $\mathbb{B}$-moderately-oscillatory. So $\boldsymbol{x}\restriction\mathcal{S}'$ is $\mathbb{B}$-previsibly-order-bounded. $\mathbf{Q}$

(d) Suppose that $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a non-negative non-decreasing $\|\ \|_\infty$-bounded $\mathbb{A}$-fully-adapted process such that $v_\sigma \in L^0(\mathfrak{B})$ for every $\sigma \in \mathcal{S}'$, and that $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a $\|\ \|_\infty$-bounded $\mathbb{A}$-moderately-oscillatory process. Then $\|\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}\|_\infty \le \|\boldsymbol{u}\|_\infty\|\boldsymbol{v}\|_\infty$ and $P(\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}) = \int_{\mathcal{S}'} P'(\boldsymbol{u})d\boldsymbol{v}$.

$\mathbf{P}$ Because $\boldsymbol{u}$ is $\mathbb{A}$-moderately-oscillatory, $P'(\boldsymbol{u})$ is $\mathbb{B}$-moderately oscillatory, while $\boldsymbol{v}$ is $\mathbb{A}$-fully adapted, $\boldsymbol{v}' = \boldsymbol{v}\restriction\mathcal{S}'$ is $\mathbb{B}$-fully adapted and both are of bounded variation, the integrals are defined. As $\mathcal{S}'$ is cofinal with $\mathcal{S}$ and separates $\mathcal{S}$, while $\inf A \in \mathcal{S}$ whenever $A \subseteq \mathcal{S}$ is non-empty and bounded below in $\mathcal{S}$, 633K tells us that $\int_{\mathcal{S}'} \boldsymbol{u}\,d\boldsymbol{v} = \int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}$.

Write M, M' for $\|\boldsymbol{u}\|_\infty$ and $\|\boldsymbol{v}\|_\infty$. If $\sigma_0 \le \ldots \le \sigma_n$ in $\mathcal{S}'$,

$$\begin{aligned}\|\sum_{i=0}^{n-1} u_{\sigma_i} \times (v_{\sigma_{i+1}} - v_{\sigma_i})\|_\infty &\le \|\sum_{i=0}^{n-1} |u_{\sigma_i}| \times |v_{\sigma_{i+1}} - v_{\sigma_i}|\|_\infty \\ &\le \|\sum_{i=0}^{n-1} M|v_{\sigma_{i+1}} - v_{\sigma_i}|\|_\infty \\ &= M\|v_{\sigma_n} - v_{\sigma_0}\|_\infty \le M\|v_{\sigma_n}\|_\infty \le MM'.\end{aligned}$$

At the same time

$$P(\textstyle\sum_{i=0}^{n-1} u_{\sigma_i} \times (v_{\sigma_{i+1}} - v_{\sigma_i})) = \sum_{i=0}^{n-1} P(u_{\sigma_i}) \times (v_{\sigma_{i+1}} - v_{\sigma_i})$$

because $v_{\sigma_{i+1}} - v_{\sigma_i} \in L^\infty(\mathfrak{B})$ for every i.

Accordingly $\|S_I(\boldsymbol{u}, d\boldsymbol{v})\|_\infty \le MM'$ and $P(S_I(\boldsymbol{u}, d\boldsymbol{v})) = S_I(P\boldsymbol{u}, d\boldsymbol{v})$ for every $I \in \mathcal{I}(\mathcal{S}')$. Taking the limit as $I \uparrow \mathcal{I}(\mathcal{S}')$, $\|\int_{\mathcal{S}'} \boldsymbol{u}\,d\boldsymbol{v}\|_\infty \le MM'$. At the same time, since P is continuous on $\{y : \|y\|_\infty \le MM'\}$,

$$\begin{aligned}P(\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}) = P(\int_{\mathcal{S}'} \boldsymbol{u}\,d\boldsymbol{v}) &= \lim_{I\uparrow\mathcal{I}(\mathcal{S}')} P(S_I(\boldsymbol{u}, d\boldsymbol{v})) \\ &= \lim_{I\uparrow\mathcal{I}(\mathcal{S}')} S_I(P\boldsymbol{u}, d\boldsymbol{v}) = \int_{\mathcal{S}'} P\boldsymbol{u}\,d\boldsymbol{v} = \int_{\mathcal{S}'} P'(\boldsymbol{u})\,d\boldsymbol{v}. \ \mathbf{Q}\end{aligned}$$

(e) Suppose that $\boldsymbol{y} \in \mathbb{A}M_{\text{po-b}}(\mathcal{S})$ is $\|\ \|_\infty$-bounded, that $\boldsymbol{v} \in \mathbb{A}M^\uparrow_{\text{n-s}}(\mathcal{S})$ is $\|\ \|_\infty$-bounded, and that $\boldsymbol{v}' = \boldsymbol{v}\restriction\mathcal{S}'$ belongs to $L^0(\mathfrak{B})^{\mathcal{S}'}$. Then $P'(\boldsymbol{y}) \in \mathbb{B}M_{\text{po-b}}(\mathcal{S}')$, $\boldsymbol{v}' \in \mathbb{B}M_{\text{n-s}}(\mathcal{S}')$ and $\mathbb{B}\widehat{\theta}^\#_{\boldsymbol{v}'}(P'(\boldsymbol{y})) \le \max(1, \|\boldsymbol{y}\|_\infty\|\boldsymbol{v}\|_\infty)\mathbb{A}\widehat{\theta}^\#_{\boldsymbol{v}}(\boldsymbol{y})$.

P Set $M = \|\boldsymbol{y}\|_\infty$, $M' = \|\boldsymbol{v}\|_\infty$ and $\gamma = \mathbb{A}\widehat{\theta}^\#_{\boldsymbol{v}}(\boldsymbol{y})$. Then $|\boldsymbol{y}| \le M\mathbf{1}^{(\mathcal{S})}_<$ so

$$|P'(\boldsymbol{y})| \le P'(|\boldsymbol{y}|) \le MP'(\mathbf{1}^{(\mathcal{S})}_<) = MP'(\mathbf{1}^{(\mathcal{S})})_< = M\mathbf{1}^{(\mathcal{S}')}_<$$

by (b-iii). So $\boldsymbol{y}'$ is $\mathbb{B}$-previsibly-order-bounded.

Let $\epsilon > 0$. Then there is a uniformly order-bounded non-decreasing sequence $\langle\boldsymbol{u}_n\rangle_{n\in\mathbb{N}}$ in $\mathbb{A}M_{\text{mo}}(\mathcal{S})^+$ such that $|\boldsymbol{y}| \le \sup_{n\in\mathbb{N}} \boldsymbol{u}_{n<}$ and $\sup_{n\in\mathbb{N}} \theta(\int_{\mathcal{S}} \boldsymbol{u}_n d\boldsymbol{v}) \le \gamma + \epsilon$. Replacing $\boldsymbol{u}_n$ by $\boldsymbol{u}_n \wedge M\mathbf{1}^{(\mathcal{S})}$ if necessary, we can arrange that $\|\boldsymbol{u}_n\|_\infty \le M$ for every n. Expressing $\boldsymbol{y}$ as $\langle y_\sigma\rangle_{\sigma\in\mathcal{S}}$ and $\boldsymbol{u}_{n<}$ as $\langle u_{n<\sigma}\rangle_{\sigma\in\mathcal{S}}$ for each n, we see that for $\sigma \in \mathcal{S}'$

$$|P(y_\sigma)| \le P(|y_\sigma|) \le P(\sup_{n\in\mathbb{N}} u_{n<\sigma}) = P(\operatorname{1lim}_{n\to\infty} u_{n<\sigma})$$

(taking the $\|\ \|_1$-limit)

$$= \operatorname{1lim}_{n\to\infty} P(u_{n<\sigma}) = \sup_{n\in\mathbb{N}} P(u_{n<\sigma}).$$

So

$$|P'(\boldsymbol{y})| \le \textstyle\sup_{n\in\mathbb{N}} P'(\boldsymbol{u}_{n<}) = \sup_{n\in\mathbb{N}} P'(\boldsymbol{u}_n)_<$$

by (b-iii) above, while $\langle P'(\boldsymbol{u}_n)\rangle_{n\in\mathbb{N}}$ is a uniformly order-bounded non-decreasing sequence of non-negative $\mathbb{B}$-moderately-oscillatory processes. So

$$\mathbb{B}\widehat{\theta}^\#_{\boldsymbol{v}'}(P'(\boldsymbol{y})) \le \sup_{n\in\mathbb{N}} \theta(\int_{\mathcal{S}'} P'(\boldsymbol{u}_n)d\boldsymbol{v}') = \sup_{n\in\mathbb{N}} \theta(P(\int_{\mathcal{S}} \boldsymbol{u}_n d\boldsymbol{v}))$$

(by (d) above)

$$\begin{aligned}&\le \sup_{n\in\mathbb{N}} \mathbb{E}(P(\int_{\mathcal{S}} \boldsymbol{u}_n d\boldsymbol{v})) = \sup_{n\in\mathbb{N}} \mathbb{E}(\int_{\mathcal{S}} \boldsymbol{u}_n d\boldsymbol{v}) \\ &\le \max(1, MM') \sup_{n\in\mathbb{N}} \mathbb{E}(\tfrac{1}{\max(1,MM')} \int_{\mathcal{S}} \boldsymbol{u}_n d\boldsymbol{v}) \\ &= \max(1, MM') \sup_{n\in\mathbb{N}} \theta(\tfrac{1}{\max(1,MM')} \int_{\mathcal{S}} \boldsymbol{u}_n d\boldsymbol{v})\end{aligned}$$

(because $\|\int_{\mathcal{S}} \boldsymbol{u}_n d\boldsymbol{v}\|_\infty \le MM'$, also noted in (d))

$$\leq \max(1, MM') \sup_{n\in\mathbb{N}} \theta(\int_{\mathcal{S}} \boldsymbol{u}_n d\boldsymbol{v}) \leq \max(1, MM')(\gamma + \epsilon).$$

As ϵ is arbitrary, $\mathbb{B}\widehat{\theta}^{\#}_{\boldsymbol{v}'}(P'(\boldsymbol{y})) \leq \max(1, MM')\gamma$, as claimed. **Q**

(f) If $\boldsymbol{x}$ is $\| \|_\infty$-bounded, then $\boldsymbol{x}' \in \mathbb{B}M^0_{\text{S-i}}(\mathcal{S})$. **P** By (c), or otherwise, $\boldsymbol{x}'$ is $\mathbb{B}$-previsibly-order-bounded. Take $\boldsymbol{v}' \in \mathbb{B}M^{\uparrow}_{\text{n-s}}(\mathcal{S}')$ and $\epsilon > 0$. Then there is an $M' \geq 0$ such that $\bar{\mu}[\![\sup \boldsymbol{v}' > M']\!] \leq \frac{1}{2}\epsilon$. Set $\boldsymbol{w}' = \boldsymbol{v}' \wedge M'\mathbf{1}^{(\mathcal{S}')}$, so that $\boldsymbol{w}' \in \mathbb{B}M^{\uparrow}_{\text{n-s}}(\mathcal{S}')$ and $\|\boldsymbol{w}'\|_\infty \leq M'$, while $\bar{\mu}[\![\boldsymbol{w}' \neq \boldsymbol{v}']\!] \leq \frac{1}{2}\epsilon$. As noted in (a), $\boldsymbol{w}'$ belongs to $\mathbb{A}M_{\text{n-s}}(\mathcal{S}')$ and has an extension $\boldsymbol{w} = \Psi(\boldsymbol{w}')$ belonging to $\mathbb{A}M_{\text{n-s}}(\mathcal{S})$. Because Ψ is a Riesz homomorphism, $0 \leq \boldsymbol{w} \leq M'\mathbf{1}^{(\mathcal{S})}$, and by 631M(b-iv) $\boldsymbol{w}$ is non-decreasing. So $\boldsymbol{w} \in \mathbb{A}M^{\uparrow}_{\text{n-s}}(\mathcal{S})$.

We can therefore speak of the F-seminorm $\mathbb{A}\widehat{\theta}^{\#}_{\boldsymbol{w}}$. Set $M = \|\boldsymbol{x}\|_\infty$, and let $\delta > 0$ be such that $\max(1, 2MM')\delta < \frac{1}{2}\epsilon$. As $\boldsymbol{x} \in \mathbb{A}M^0_{\text{S-i}}(\mathcal{S})$, there is a $\boldsymbol{u} \in \mathbb{A}M_{\text{mo}}(\mathcal{S})$ such that $\mathbb{A}\widehat{\theta}^{\#}_{\boldsymbol{w}}(\boldsymbol{x}-\boldsymbol{u}_<) \leq \delta$. Replacing $\boldsymbol{u}$ by $\operatorname{med}(-M\mathbf{1}^{(\mathcal{S})}, \boldsymbol{u}, M\mathbf{1}^{(\mathcal{S})})$ if necessary, we can arrange that $\|\boldsymbol{u}\|_\infty \leq M$, so that $\|\boldsymbol{x} - \boldsymbol{u}_<\|_\infty \leq 2M$. Now $P'(\boldsymbol{u}) \in \mathbb{B}M_{\text{mo}}(\mathcal{S}')$ and

$$\mathbb{B}\widehat{\theta}^{\#}_{\boldsymbol{w}'}(\boldsymbol{x}' - (P'(\boldsymbol{u}))_<) = \mathbb{B}\widehat{\theta}^{\#}_{\boldsymbol{w}'}(\boldsymbol{x}' - P'(\boldsymbol{u}_<)) = \mathbb{B}\widehat{\theta}^{\#}_{\boldsymbol{w}'}(P'(\boldsymbol{x} - \boldsymbol{u}_<))$$
$$\leq (1 + 2MM')\mathbb{A}\widehat{\theta}^{\#}_{\boldsymbol{w}}(\boldsymbol{x} - \boldsymbol{u}_<)$$

(by (e) above)

$$< (1 + 2MM')\delta \leq \frac{1}{2}\epsilon.$$

Working in $\mathbb{B}$, we have a uniformly previsibly order-bounded non-decreasing sequence $\langle \boldsymbol{u}'_n \rangle_{n\in\mathbb{N}}$ in $\mathbb{B}M_{\text{mo}}(\mathcal{S}')^+$ such that $|\boldsymbol{x}' - (P'(\boldsymbol{u}))_<| \leq \sup_{n\in\mathbb{N}} \boldsymbol{u}'_{n<}$ and $\sup_{n\in\mathbb{N}} \theta(\int_{\mathcal{S}'} \boldsymbol{u}'_n d\boldsymbol{w}') \leq \frac{1}{2}\epsilon$. For each n,

$$\theta(\int_{\mathcal{S}'} \boldsymbol{u}'_n d\boldsymbol{v}' - \int_{\mathcal{S}'} \boldsymbol{u}'_n d\boldsymbol{w}') \leq \bar{\mu}[\![\textstyle\int_{\mathcal{S}'} \boldsymbol{u}'_n d\boldsymbol{v}' \neq \int_{\mathcal{S}'} \boldsymbol{u}'_n d\boldsymbol{w}']\!] \leq \bar{\mu}[\![\boldsymbol{v}' \neq \boldsymbol{w}']\!]$$

(613Ld)

$$\leq \frac{1}{2}\epsilon,$$

so

$$\theta(\textstyle\int_{\mathcal{S}'} \boldsymbol{u}'_n d\boldsymbol{v}') \leq \frac{1}{2}\epsilon + \theta(\int_{\mathcal{S}'} \boldsymbol{u}'_n d\boldsymbol{w}') \leq \epsilon.$$

Accordingly

$$\mathbb{B}\widehat{\theta}^{\#}_{\boldsymbol{v}'}(\boldsymbol{x}' - P'(\boldsymbol{u})_<) \leq \sup_{n\in\mathbb{N}} \theta(\textstyle\int_{\mathcal{S}'} \boldsymbol{u}'_n d\boldsymbol{v}') \leq \epsilon.$$

As $\boldsymbol{v}'$ and ϵ are arbitrary, $\boldsymbol{x}' \in \mathbb{B}M^0_{\text{S-i}}(\mathcal{S}')$. **Q**

(g) Finally, in the general case in which $\boldsymbol{x} \in \mathbb{A}M^0_{\text{S-i}}(\mathcal{S})$ and $\boldsymbol{x}' \in L^0(\mathfrak{B})^{\mathcal{S}'}$, set $\boldsymbol{x}_n = \operatorname{med}(-M\mathbf{1}^{(\mathcal{S})}_<, \boldsymbol{x}, M\mathbf{1}^{(\mathcal{S})}_<)$ and $\boldsymbol{x}'_n = \boldsymbol{x}_n{\restriction}\mathcal{S}'$ for each $n \in \mathbb{N}$. Then $\boldsymbol{x}_n \in \mathbb{A}M^0_{\text{S-i}}(\mathcal{S})$ is $\| \|_\infty$-bounded and $\boldsymbol{x}'_n = \operatorname{med}(-M\mathbf{1}^{(\mathcal{S}')}_<, \boldsymbol{x}', M\mathbf{1}^{(\mathcal{S}')}_<)$ belongs to $L^0(\mathfrak{B})^{\mathcal{S}'}$; by (f), $\boldsymbol{x}'_n \in \mathbb{B}M^0_{\text{S-i}}(\mathcal{S}')$, for every $n \in \mathbb{N}$. Writing $\boldsymbol{x}'$ for $\boldsymbol{x}{\restriction}\mathcal{S}'$, $\boldsymbol{x}'$ is $\mathbb{B}$-previsibly-order-bounded (by (c)) and $|\boldsymbol{x}'_n| \leq |\boldsymbol{x}'|$ for every n, so $\langle \boldsymbol{x}'_n \rangle_{n\in\mathbb{N}}$ is uniformly $\mathbb{B}$-previsibly-order-bounded, while it is order*-convergent to $\boldsymbol{x}'$ in $L^0(\mathfrak{B})^{\mathcal{S}'}$; by 645H, $\boldsymbol{x}' \in \mathbb{B}M^0_{\text{S-i}}(\mathcal{S}')$.

648E Lemma Suppose that $\langle \mathfrak{A}_t \rangle_{t\in T}$ is right-continuous. Let $\mathfrak{B}$ be a closed subalgebra of $\mathfrak{A}$ which is coordinated with $\langle \mathfrak{A}_t \rangle_{t\in T}$, and $\mathbb{B} = (\mathfrak{B}, \bar{\mu}{\restriction}\mathfrak{B}, T, \langle \mathfrak{B}_t \rangle_{t\in T}, \mathcal{T}_{\mathbb{B}}, \langle \mathfrak{B}_\sigma \rangle_{\sigma\in\mathcal{T}_{\mathbb{B}}})$ the corresponding stochastic integration structure, where $\mathfrak{B}_t = \mathfrak{B} \cap \mathfrak{A}_t$ for $t \in T$. Let $\mathcal{S}'$ be a relatively order-convex sublattice of $\mathcal{T}_{\mathbb{B}}$ and $\mathcal{S}$ its order-convex hull in $\mathcal{T}_{\mathbb{A}}$. If $\boldsymbol{u} \in \mathbb{A}M_{\text{n-s}}(\mathcal{S})$ and $\boldsymbol{u}{\restriction}\mathcal{S}' \in L^0(\mathfrak{B})^{\mathcal{S}'}$ then $\boldsymbol{u}' = \boldsymbol{u}{\restriction}\mathcal{S}'$ is near-simple in either $\mathbb{A}$ or $\mathbb{B}$.

proof $\boldsymbol{u}$ is $\mathbb{A}$-moderately-oscillatory (631Ca) and $\mathcal{S}$ is $\mathbb{A}$-finitely-full (611Pc, as before), so $\boldsymbol{u}$ is $\Downarrow$-convergent (615N) and $\boldsymbol{u}{\restriction}\mathcal{S}'$ is $\Downarrow$-convergent, working in either $\mathbb{A}$ or $\mathbb{B}$. If $A \subseteq \mathcal{S}'$ is non-empty and has a lower bound in $\mathcal{S}'$, then A has a greatest lower bound in $\mathcal{T}_{\mathbb{B}}$ which is also its infimum in $\mathcal{T}_{\mathbb{A}}$, because $\mathcal{T}_{\mathbb{B}}$ is an order-closed sublattice of $\mathcal{T}_{\mathbb{A}}$ (634C(f-ii)); and because $\mathcal{S}'$ is order-convex in $\mathcal{T}_{\mathbb{B}}$, this common infimum belongs to $\mathcal{S}$.

We know that $\boldsymbol{u}'$ is $\mathbb{A}$-moderately-oscillatory (615F(a-i)), therefore $\mathbb{B}$-moderately-oscillatory (634Ee). If $A \subseteq \mathcal{S}'$ is downwards-directed and has a lower bound in $\mathcal{S}'$, then (as we have just seen) there is a common value $\inf A \in \mathcal{S}'$ of its infimum taken in either $\mathcal{T}_{\mathbb{B}}$ or $\mathcal{T}_{\mathbb{A}}$. By 632H, $u_{\inf A} = \lim_{\sigma\downarrow A} u_\sigma$, because we are supposing that $\boldsymbol{u}$ is $\mathbb{A}$-near-simple. Finally, the filtration $\langle\mathfrak{B}_t\rangle_{t\in T}$ is right-continuous (634C(f-i)). As A is arbitary, we can apply 632H in the other direction, working in the structure $\mathbb{B}$, to see that $\boldsymbol{u}'$ is $\mathbb{B}$-near-simple, therefore $\mathbb{A}$-near-simple (634Eb).

648F Lemma Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathfrak{B}$ be a closed subalgebra of $\mathfrak{A}$ which is coordinated with $\langle\mathfrak{A}_t\rangle_{t\in T}$, and $\mathbb{B} = (\mathfrak{B}, \bar\mu\restriction\mathfrak{B}, T, \langle\mathfrak{B}_t\rangle_{t\in T}, \mathcal{T}_{\mathbb{B}}, \langle\mathfrak{B}_\sigma\rangle_{\sigma\in\mathcal{T}_{\mathbb{B}}})$ the corresponding stochastic integration structure, where $\mathfrak{B}_t = \mathfrak{B}\cap\mathfrak{A}_t$ for $t \in T$. Let $\mathcal{S}'$ be a relatively order-convex sublattice of $\mathcal{T}_{\mathbb{B}}$ and $\mathcal{S}$ its order-convex hull in $\mathcal{T}_{\mathbb{A}}$. Let $\boldsymbol{x} \in \mathbb{A}M^0_{\text{S-i}}(\mathcal{S})$, $\boldsymbol{v} \in \mathbb{A}M^{\uparrow}_{\text{n-s}}(\mathcal{S})$ be such that $\boldsymbol{x}' = \boldsymbol{x}\restriction\mathcal{S}'$ and $\boldsymbol{v}' = \boldsymbol{v}\restriction\mathcal{S}'$ belong to $L^0(\mathfrak{B})^{\mathcal{S}'}$. If $\mathbb{B}\widehat{\theta}^{\#}_{\boldsymbol{v}'}(\boldsymbol{x}') = 0$ then $\mathbb{A}\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{v} = 0$.

proof We know that $\mathcal{S}' \supseteq \{\text{med}(\sigma,\check t,\tau) : \sigma, \tau \in \mathcal{S}', t \in T\}$ $\mathbb{A}$-separates $\mathcal{S}$ (633D). By 648D, $\boldsymbol{x}' \in \mathbb{B}M^0_{\text{S-i}}(\mathcal{S}')$, while $\boldsymbol{v}'$ is non-negative, non-decreasing and order-bounded, and by 648E is near-simple in either structure. Thus $\boldsymbol{v}' \in \mathbb{B}M^{\uparrow}_{\text{n-s}}(\mathcal{S}')$, and $\mathbb{B}\widehat{\theta}^{\#}_{\boldsymbol{v}'}(\boldsymbol{x}')$ is surely defined.

? Suppose, if possible, that $\epsilon = \frac{1}{4}\theta(\mathbb{A}\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{v})$ is greater than 0. Working in $\mathbb{B}$, we have an order-bounded non-decreasing sequence $\langle\boldsymbol{u}'_n\rangle_{n\in\mathbb{N}}$ in $\mathbb{B}M_{\text{mo}}(\mathcal{S}')^+$ such that $\boldsymbol{x}' \le \sup_{n\in\mathbb{N}}\boldsymbol{u}'_{n<}$ and $\sup_{n\in\mathbb{N}}\theta(\int_{\mathcal{S}'}\boldsymbol{u}'_n d\boldsymbol{v}') \le \epsilon$. Next, working in $\mathbb{A}$, there are a $\boldsymbol{w} \in \mathbb{A}M_{\text{mo}}(\mathcal{S})^+$ such that $\int_{\mathcal{S}}\boldsymbol{w}\,d\boldsymbol{v} \ge 3\epsilon$ and $\mathbb{A}\widehat{\theta}^{\#}_{\boldsymbol{v}}(|\boldsymbol{x}-\boldsymbol{w}_<|) < \epsilon$, so there is an order-bounded non-decreasing sequence $\langle\boldsymbol{w}_n\rangle_{n\in\mathbb{N}}$ in $\mathbb{A}M_{\text{mo}}(\mathcal{S})^+$ such that $|\boldsymbol{x}-\boldsymbol{w}_<| \le \sup_{n\in\mathbb{N}}\boldsymbol{w}_{n<}$ and $\sup_{n\in\mathbb{N}}\theta(\int_{\mathcal{S}}\boldsymbol{w}_n d\boldsymbol{v}) \le \epsilon$. Observe that

$$\boldsymbol{x} \ge \boldsymbol{w}_< - |\boldsymbol{x}-\boldsymbol{w}_<| \ge \inf_{n\in\mathbb{N}}(\boldsymbol{w}_< - \boldsymbol{w}_{n<}) = \inf_{n\in\mathbb{N}}(\boldsymbol{w}-\boldsymbol{w}_n)_<.$$

Setting $\boldsymbol{w}' = \boldsymbol{w}\restriction\mathcal{S}'$, $\boldsymbol{w}'_n = \boldsymbol{w}_n\restriction\mathcal{S}'$ we have $(\boldsymbol{w}'-\boldsymbol{w}'_n)_< = (\boldsymbol{w}-\boldsymbol{w}_n)_<\restriction\mathcal{S}'$ for each n (641Hb again). So

$$\inf_{n\in\mathbb{N}}(\boldsymbol{w}'-\boldsymbol{w}'_n)_< \le \boldsymbol{x}' \le \sup_{n\in\mathbb{N}}\boldsymbol{u}'_n, \quad \boldsymbol{w}'_< \le \sup_{n\in\mathbb{N}}\boldsymbol{w}'_{n<} + \boldsymbol{u}'_{n<},$$

$\langle(\boldsymbol{w}'\wedge(\boldsymbol{w}'_n+\boldsymbol{u}'_n))_<\rangle_{n\in\mathbb{N}}$ is order*-convergent to $\boldsymbol{w}'_<$ in $L^0(\mathfrak{B})^{\mathcal{S}'}$ and $\int_{\mathcal{S}'}\boldsymbol{w}'d\boldsymbol{v}' = \lim_{n\to\infty}\int_{\mathcal{S}'}(\boldsymbol{w}'\wedge(\boldsymbol{w}'_n+\boldsymbol{u}'_n))d\boldsymbol{v}'$ by 644H. On the other hand, 633Ka tells us that

$$\int_{\mathcal{S}'}\boldsymbol{w}'d\boldsymbol{v}' = \int_{\mathcal{S}}\boldsymbol{w}\,d\boldsymbol{v}, \quad \int_{\mathcal{S}'}\boldsymbol{w}'_n d\boldsymbol{v}' = \int_{\mathcal{S}}\boldsymbol{w}_n d\boldsymbol{v}$$

for every $n \in \mathbb{N}$, so $\theta(\int_{\mathcal{S}'}\boldsymbol{w}'d\boldsymbol{v}') \ge 3\epsilon$ while

$$\begin{aligned}\theta(\int_{\mathcal{S}'}(\boldsymbol{w}'\wedge(\boldsymbol{w}'_n+\boldsymbol{u}_n))d\boldsymbol{v}') &\le \theta(\int_{\mathcal{S}'}\boldsymbol{w}'_n d\boldsymbol{v}') + \theta(\int_{\mathcal{S}'}\boldsymbol{u}_n d\boldsymbol{v}')\\ &\le \theta(\int_{\mathcal{S}}\boldsymbol{w}_n d\boldsymbol{v}) + \epsilon \le 2\epsilon\end{aligned}$$

for every n. **X** Thus $\mathbb{A}\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{v} = 0$.

648G Theorem Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathfrak{B}$ be a closed subalgebra of $\mathfrak{A}$ which is coordinated with $\langle\mathfrak{A}_t\rangle_{t\in T}$, and $\mathbb{B} = (\mathfrak{B}, \bar\mu\restriction\mathfrak{B}, T, \langle\mathfrak{B}_t\rangle_{t\in T}, \mathcal{T}_{\mathbb{B}}, \langle\mathfrak{B}_\sigma\rangle_{\sigma\in\mathcal{T}_{\mathbb{B}}})$ the corresponding stochastic integration structure, where $\mathfrak{B}_t = \mathfrak{B}\cap\mathfrak{A}_t$ for $t \in T$. Let $\mathcal{S}'$ be a relatively order-convex sublattice of $\mathcal{T}_{\mathbb{B}}$ and $\mathcal{S}$ its order-convex hull in $\mathcal{T}_{\mathbb{A}}$. Let $\boldsymbol{x} \in \mathbb{A}M_{\text{S-i}}(\mathcal{S})$ and an $\mathbb{A}$-near-simple $\mathbb{A}$-integrator $\boldsymbol{w}$ with domain $\mathcal{S}$ be such that $\boldsymbol{x}' = \boldsymbol{x}\restriction\mathcal{S}'$, $\boldsymbol{w}' = \boldsymbol{w}\restriction\mathcal{S}'$ belong to $L^0(\mathfrak{B})^{\mathcal{S}'}$. Then $\mathbb{B}\oint_{\mathcal{S}'}\boldsymbol{x}'d\boldsymbol{w}' = \mathbb{A}\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{w}$.

proof (a) For the time being (down to the end of (h) below), suppose that $\mathcal{S}'$ has a greatest member and $\boldsymbol{x} \in \mathbb{A}M^0_{\text{S-i}}(\mathcal{S})$ is $\| \|_\infty$-bounded.

(i) We know from 648D above that $\boldsymbol{x}' \in \mathbb{B}M^0_{\text{S-i}}(\mathcal{S}')$, while $\boldsymbol{w}'$ is a $\mathbb{B}$-near-simple $\mathbb{B}$-integrator (648E, 634Ib). So the integral $\mathbb{B}\oint_{\mathcal{S}'}\boldsymbol{x}'d\boldsymbol{w}'$ is defined.

(ii) Conversely, by 634Eb again, $\mathbb{B}M_{\text{n-s}}(\mathcal{S}') = \mathbb{A}M_{\text{n-s}}(\mathcal{S}')\cap L^0(\mathfrak{B})^{\mathcal{S}'}$ is an f-subalgebra of $\mathbb{A}M_{\text{n-s}}(\mathcal{S}')$. By 631M we have a multiplicative Riesz homomorphism $\Psi : \mathbb{A}M_{\text{n-s}}(\mathcal{S}') \to \mathbb{A}M_{\text{n-s}}(\mathcal{S})$ such that $\Psi(\boldsymbol{u})$ extends $\boldsymbol{u}$ for every $\boldsymbol{u} \in \mathbb{A}M_{\text{n-s}}(\mathcal{S}')$. By 633F, $\Psi(\boldsymbol{u})$ is the only $\mathbb{A}$-near-simple process with domain $\mathcal{S}$ extending $\boldsymbol{u}$.

(b) Let $M \ge 0$ be such that $-M\mathbf{1}^{(\mathcal{S})}_{<} \le \boldsymbol{x} \le M\mathbf{1}^{(\mathcal{S})}_{<}$. Write Y for the set of those $\boldsymbol{y} \in \mathbb{A}M^0_{\text{S-i}}(\mathcal{S})$ such that $|\boldsymbol{y}| \le 3M\mathbf{1}^{(\mathcal{S})}_{<}$, $\boldsymbol{y}\restriction\mathcal{S}' \in L^0(\mathfrak{B})^{\mathcal{S}'}$ and $\mathbb{B}\oint_{\mathcal{S}'}(\boldsymbol{y}\restriction\mathcal{S}')d(\boldsymbol{z}\restriction\mathcal{S}') = \mathbb{A}\oint_{\mathcal{S}}\boldsymbol{y}\,d\boldsymbol{z}$ whenever $\boldsymbol{z} \in \mathbb{A}M_{\text{n-s}}(\mathcal{S})$ is an $\mathbb{A}$-integrator and $\boldsymbol{z}\restriction\mathcal{S}' \in L^0(\mathfrak{B})^{\mathcal{S}'}$.

(c) $\boldsymbol{u}_< \in Y$ whenever $\boldsymbol{u} \in \mathbb{A}M_{\text{mo}}(\mathcal{S})$, $|\boldsymbol{u}| \le 3M\mathbf{1}^{(\mathcal{S})}$ and $\boldsymbol{u}' = \boldsymbol{u}\restriction\mathcal{S}'$ belongs to $L^0(\mathfrak{B})^{\mathcal{S}'}$. **P** Certainly $\boldsymbol{u}_< \in \mathbb{A}M^0_{\text{S-i}}(\mathcal{S})$ and $|\boldsymbol{u}_<| \le 3M\mathbf{1}^{(\mathcal{S})}_{<}$. By 641Hb once more, $\boldsymbol{u}_<\restriction\mathcal{S}' = \boldsymbol{u}'_<$ belongs to $L^0(\mathfrak{B})^{\mathcal{S}'}$. If $\boldsymbol{z}$ is an $\mathbb{A}$-near-simple $\mathbb{A}$-integrator with domain $\mathcal{S}$ and $\boldsymbol{z}' = \boldsymbol{z}\restriction\mathcal{S}'$ belongs to $L^0(\mathfrak{B})^{\mathcal{S}'}$, then $\boldsymbol{z}'$ is a $\mathbb{B}$-near-simple $\mathbb{B}$-integrator (648E, 634Ib as before). Now

$$\mathbb{B}\oint_{\mathcal{S}'}(\boldsymbol{u}_<\restriction\mathcal{S}')d\boldsymbol{z}' = \mathbb{B}\oint_{\mathcal{S}'}\boldsymbol{u}'_<d\boldsymbol{z}' = \mathbb{B}\int_{\mathcal{S}'}\boldsymbol{u}'d\boldsymbol{z}'$$

(645R(a-i))

$$= \mathbb{A}\int_{\mathcal{S}'}\boldsymbol{u}\,d\boldsymbol{z}$$

(634Eg)

$$= \mathbb{A}\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{z}$$

(633Ka, because $\mathcal{S}'$ separates $\mathcal{S}$, by 633D(b-i))

$$= \mathbb{A}\oint_{\mathcal{S}}\boldsymbol{u}_<d\boldsymbol{z}.$$

As $\boldsymbol{z}$ is arbitrary, $\boldsymbol{u}_< \in Y$. **Q**

(d) If $\langle\boldsymbol{y}_n\rangle_{n\in\mathbb{N}}$ is a sequence in Y which is order*-convergent to $\boldsymbol{y}$ in $L^0(\mathfrak{A})^{\mathcal{S}}$, then $\boldsymbol{y} \in Y$. **P** Because $|\boldsymbol{y}_n| \le 3M\mathbf{1}^{(\mathcal{S})}_{<}$ for every n, $\langle\boldsymbol{y}_n\rangle_{n\in\mathbb{N}}$ is uniformly $\mathbb{A}$-previsibly-order-bounded and $|\boldsymbol{y}| \le 3M\mathbf{1}^{(\mathcal{S})}_{<}$. Writing $\boldsymbol{y}' = \boldsymbol{y}\restriction\mathcal{S}'$, $\boldsymbol{y}'_n = \boldsymbol{y}_n\restriction\mathcal{S}'$ for each $n \in \mathbb{N}$, we see that $\langle\boldsymbol{y}'_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to $\boldsymbol{y}'$ in $L^0(\mathfrak{A})^{\mathcal{S}'}$; as $\boldsymbol{y}'_n \in L^0(\mathfrak{B})^{\mathcal{S}'}$ for every n and $L^0(\mathfrak{B})$ is topologically closed in $L^0(\mathfrak{A})$, $\boldsymbol{y}' \in L^0(\mathfrak{B})^{\mathcal{S}'}$. Also $|\boldsymbol{y}'_n| \le 3M\mathbf{1}^{(\mathcal{S}')}_{<}$ for every n, so $\langle\boldsymbol{y}'_n\rangle_{n\in\mathbb{N}}$ is uniformly $\mathbb{B}$-previsibly-order-bounded. By 645H, working first in $\mathbb{A}$ and then in $\mathbb{B}$, $\boldsymbol{y} \in \mathbb{A}M^0_{\text{S-i}}(\mathcal{S})$ and $\boldsymbol{y}' \in \mathbb{B}M^0_{\text{S-i}}(\mathcal{S}')$. Moreover, if $\boldsymbol{z}$ is an $\mathbb{A}$-near-simple $\mathbb{A}$-integrator with domain $\mathcal{S}$ and $\boldsymbol{z}\restriction\mathcal{S}' \in L^0(\mathfrak{B})^{\mathcal{S}'}$, then

$$\mathbb{B}\oint_{\mathcal{S}'}\boldsymbol{y}'d(\boldsymbol{z}\restriction\mathcal{S}') = \lim_{n\to\infty}\mathbb{B}\oint_{\mathcal{S}'}\boldsymbol{y}'_nd(\boldsymbol{z}\restriction\mathcal{S}')$$

(645S)

$$= \lim_{n\to\infty}\mathbb{A}\oint_{\mathcal{S}}\boldsymbol{y}_nd\boldsymbol{z} = \mathbb{A}\oint_{\mathcal{S}}\boldsymbol{y}\,d\boldsymbol{z}.$$

So $\boldsymbol{y} \in Y$. **Q**

(e) If $\boldsymbol{v}' \in \mathbb{B}M_{\text{n-s}}(\mathcal{S}')^+$ is non-decreasing and order-bounded, there is a $\boldsymbol{y} \in Y$ such that $\mathbb{B}\widehat{\theta}^{\#}_{\boldsymbol{v}'}(\boldsymbol{x}'-\boldsymbol{y}\restriction\mathcal{S}') = 0$.

P (i) For each $n \in \mathbb{N}$ take $\acute{\boldsymbol{u}}'_n \in \mathbb{B}M_{\text{mo}}(\mathcal{S}')$ such that $\mathbb{B}\widehat{\theta}^{\#}_{\boldsymbol{v}'}(|\boldsymbol{x}' - \acute{\boldsymbol{u}}'_{n<}|) \le 2^{-n}$, and a non-decreasing sequence $\langle\acute{\boldsymbol{u}}'_{nm}\rangle_{m\in\mathbb{N}}$ in $\mathbb{B}M_{\text{mo}}(\mathcal{S}')^+$ such that $\theta(\int_{\mathcal{S}'}\acute{\boldsymbol{u}}'_{nm}d\boldsymbol{v}') \le 2^{-n+1}$ for every m and $|\boldsymbol{x}'-\acute{\boldsymbol{u}}'_{n<}| \le \sup_{m\in\mathbb{N}}\acute{\boldsymbol{u}}'_{nm<}$. By 642M, working in $\mathbb{B}$, we have $\grave{\boldsymbol{u}}'_n$, $\grave{\boldsymbol{u}}'_{nm}$ in $\mathbb{B}M_{\text{n-s}}(\mathcal{S}')$ such that $\grave{\boldsymbol{u}}'_{n<} = \acute{\boldsymbol{u}}'_{n<}$, $\grave{\boldsymbol{u}}'_{nm<} = \acute{\boldsymbol{u}}'_{nm}$ for all n, $m \in \mathbb{N}$. Setting

$$\boldsymbol{u}'_n = \text{med}(-M\mathbf{1}^{(\mathcal{S}')}, \grave{\boldsymbol{u}}'_n, M\mathbf{1}^{(\mathcal{S}')}), \quad \boldsymbol{u}'_{nm} = (\sup_{i\le m}\grave{\boldsymbol{u}}'_{ni})^+ \wedge 2M\mathbf{1}^{(\mathcal{S}')}$$

for m, $n \in \mathbb{N}$, we see that $\boldsymbol{u}'_n$, $\boldsymbol{u}'_{nm} \in \mathbb{B}M_{\text{n-s}}(\mathcal{S}')$, $0 \le \boldsymbol{u}'_n \le 2M\mathbf{1}^{(\mathcal{S}')}$ and

$$\begin{aligned}|\boldsymbol{x}' - \boldsymbol{u}'_{n<}| &\le |\boldsymbol{x}' - \grave{\boldsymbol{u}}'_{n<}| \wedge 2M\mathbf{1}^{(\mathcal{S}')}_{<} = |\boldsymbol{x}' - \acute{\boldsymbol{u}}'_{n<}| \wedge 2M\mathbf{1}^{(\mathcal{S}')}_{<}\\ &\le \sup_{m\in\mathbb{N}}\acute{\boldsymbol{u}}'_{nm<} \wedge 2M\mathbf{1}^{(\mathcal{S}')}_{<} = \sup_{m\in\mathbb{N}}\grave{\boldsymbol{u}}'_{nm<} \wedge 2M\mathbf{1}^{(\mathcal{S}')}_{<} = \sup_{m\in\mathbb{N}}\boldsymbol{u}'_{nm<}\end{aligned}$$

for all m, $n \in \mathbb{N}$. Moreover, because $\boldsymbol{v}'$ is non-decreasing,

$$\int_{\mathcal{S}'} \boldsymbol{u}'_{nm} d\boldsymbol{v}' \le \int_{\mathcal{S}'} (\sup_{i\le n} \grave{\boldsymbol{u}}'_{ni})^+ d\boldsymbol{v}' = \int_{\mathcal{S}'} (\sup_{i\le n} \acute{\boldsymbol{u}}'_{ni})^+ d\boldsymbol{v}'$$

(using 645R(a-i), because $(\sup_{i\le n} \grave{\boldsymbol{u}}'_{ni})^+_< = (\sup_{i\le n} \acute{\boldsymbol{u}}'_{ni})^+_<$)

$$= \int_{\mathcal{S}'} \acute{\boldsymbol{u}}'_{nm} d\boldsymbol{v}' \le 2^{-n+1}$$

for all m, $n \in \mathbb{N}$.

(ii) Looking at the function Ψ of 631M and (a-ii) above, $\Psi(\mathbf{1}^{(\mathcal{S}')})$ must be $\mathbf{1}^{(\mathcal{S})}$. We can speak of $\boldsymbol{u}_n = \Psi(\boldsymbol{u}'_n)$ and $\boldsymbol{u}_{nm} = \Psi(\boldsymbol{u}'_{nm})$ for n, $m \in \mathbb{N}$, and we shall have $|\boldsymbol{u}_n| \le M\mathbf{1}^{(\mathcal{S})}$, $0 \le \boldsymbol{u}_{nm} \le 2M\mathbf{1}^{(\mathcal{S})}$ for all n and m, while $\langle \boldsymbol{u}_{nm}\rangle_{m\in\mathbb{N}}$ is non-decreasing for each n. Now consider

$$\boldsymbol{z}_{nm} = \inf_{i\le n}(\boldsymbol{u}_i + \boldsymbol{u}_{im}) \in \mathbb{A}M_{\text{n-s}}(\mathcal{S})$$

for n, $m \in \mathbb{N}$. Then $\boldsymbol{z}_{nm}$ always lies between $-M\mathbf{1}^{(\mathcal{S})}$ and $3M\mathbf{1}^{(\mathcal{S})}$, while $\langle \boldsymbol{z}_{nm}\rangle_{m\in\mathbb{N}}$ is non-decreasing for each n and $\langle \boldsymbol{z}_{nm}\rangle_{n\in\mathbb{N}}$ is non-increasing for each m. So if we set

$$\boldsymbol{y}_n = \sup_{m\in\mathbb{N}} \boldsymbol{z}_{nm<} \in \mathbb{A}M^0_{\text{S-i}}(\mathcal{S})$$

for each n, $\boldsymbol{y}_n$ will be the order*-limit of $\langle \boldsymbol{z}_{nm<}\rangle_{m\in\mathbb{N}}$. By (c)-(d) above, $\boldsymbol{y}_n \in Y$ for each n, while $\langle \boldsymbol{y}_n\rangle_{n\in\mathbb{N}}$ is non-increasing, so $\boldsymbol{y} = \inf_{n\in\mathbb{N}} \boldsymbol{y}_n$ also belongs to Y.

(iii) Write $\boldsymbol{z}'_{nm} = \boldsymbol{z}_{nm}\restriction\mathcal{S}'$, $\boldsymbol{y}'_n = \boldsymbol{y}_n\restriction\mathcal{S}'$ and $\boldsymbol{y}' = \boldsymbol{y}\restriction\mathcal{S}'$. We always have $\boldsymbol{x}' \le \boldsymbol{u}'_{n<} + \sup_{m\in\mathbb{N}} \boldsymbol{u}'_{nm<}$. So if $n \in \mathbb{N}$,

$$\boldsymbol{x}' \le \inf_{i\le n}(\boldsymbol{u}'_{i<} + \sup_{m\in\mathbb{N}} \boldsymbol{u}'_{im<}) = \sup_{m\in\mathbb{N}} \inf_{i\le n}(\boldsymbol{u}'_{i<} + \boldsymbol{u}'_{im<})$$

(using the distributive law 352Ea repeatedly)

$$= \sup_{m\in\mathbb{N}} \boldsymbol{z}'_{nm<} = \boldsymbol{y}'_n.$$

(The point here is that, by 641Hb yet again, $(\boldsymbol{z}_{nm}\restriction\mathcal{S}')_< = (\boldsymbol{z}_{nm})_<\restriction\mathcal{S}'$, so we can use the formula $\boldsymbol{z}'_{nm<}$ without inhibitions.) Accordingly $\boldsymbol{x}' \le \boldsymbol{y}'$. Now, for any $n \in \mathbb{N}$,

$$\begin{aligned} 0 \le \boldsymbol{y}' - \boldsymbol{x}' &\le \boldsymbol{y}'_n - \boldsymbol{u}'_{n<} + |\boldsymbol{u}'_{n<} - \boldsymbol{x}'| \\ &= \sup_{m\in\mathbb{N}}(\boldsymbol{w}'_{nm<} - \boldsymbol{u}'_{n<}) + |\boldsymbol{u}'_{n<} - \boldsymbol{x}'| \le \sup_{m\in\mathbb{N}} \boldsymbol{u}'_{nm<} + |\boldsymbol{u}'_{n<} - \boldsymbol{x}'| \end{aligned}$$

so

$$\begin{aligned} \mathbb{B}\widehat{\theta}^{\#}_{\boldsymbol{v}'}(\boldsymbol{y}' - \boldsymbol{x}') &\le \mathbb{B}\widehat{\theta}^{\#}_{\boldsymbol{v}'}(\sup_{m\in\mathbb{N}} \boldsymbol{u}'_{nm<}) + \mathbb{B}\widehat{\theta}^{\#}_{\boldsymbol{v}'}(\boldsymbol{u}'_{n<} - \boldsymbol{x}') \\ &\le \sup_{m\in\mathbb{N}} \int_{\mathcal{S}'} \boldsymbol{u}'_{nm} d\boldsymbol{v}' + 2^{-n} \le 2^{-n+1} + 2^{-n}. \end{aligned}$$

As n is arbitrary, $\mathbb{B}\widehat{\theta}^{\#}_{\boldsymbol{v}'}(\boldsymbol{y}' - \boldsymbol{x}') = 0$, as required. **Q**

(f) If $\boldsymbol{v} \in M^{\uparrow}_{\text{n-s}}(\mathcal{S})$ and $\boldsymbol{v}' = \boldsymbol{v}\restriction\mathcal{S}'$ belongs to $L^0(\mathfrak{B})^{\mathcal{S}'}$, then $\mathbb{B}\oint_{\mathcal{S}'} \boldsymbol{x}' d\boldsymbol{v}' = \mathbb{A}\oint_{\mathcal{S}'} \boldsymbol{x}\, d\boldsymbol{v}$. **P** By 648E again, $\boldsymbol{v}'$ is $\mathbb{B}$-near-simple, and of course it is non-negative and non-decreasing. By (e), there is a $\boldsymbol{y} \in Y$ such that $\mathbb{B}\widehat{\theta}^{\#}_{\boldsymbol{v}'}(\boldsymbol{x}' - \boldsymbol{y}') = 0$, writing $\boldsymbol{y}'$ for $\boldsymbol{y}\restriction\mathcal{S}'$. Now

$\mathbb{B}\oint_{\mathcal{S}'} \boldsymbol{x}' - \boldsymbol{y}'\, d\boldsymbol{v}' = 0$ by 646E,

$\mathbb{B}\oint_{\mathcal{S}'} \boldsymbol{y}' d\boldsymbol{v}' = \mathbb{A}\oint_{\mathcal{S}} \boldsymbol{y}\, d\boldsymbol{v}$ because $\boldsymbol{y} \in Y$,

$\mathbb{A}\oint_{\mathcal{S}} \boldsymbol{x} - \boldsymbol{y}\, d\boldsymbol{v} = 0$ by 648F,

so $\mathbb{B}\oint_{\mathcal{S}'} \boldsymbol{x}' d\boldsymbol{v}' = \mathbb{A}\oint_{\mathcal{S}} \boldsymbol{x}\, d\boldsymbol{v}$. **Q**

(g) If $\boldsymbol{z}$ is an $\mathbb{A}$-L^2-martingale with domain $\mathcal{S}$ and $\boldsymbol{z}' = \boldsymbol{z}\restriction\mathcal{S}'$ belongs to $L^0(\mathfrak{B})^{\mathcal{S}'}$, then $\mathbb{B}\oint_{\mathcal{S}'}\boldsymbol{x}'d\boldsymbol{z}' = \mathbb{A}\oint_{\mathcal{S}'}\boldsymbol{x}\,d\boldsymbol{z}$. **P** We are supposing that $\mathcal{S}'$ has a greatest member, which will also be the greatest member of $\mathcal{S}$. By 632Ia again, $\boldsymbol{z}$ is $\mathbb{A}$-locally-near-simple, therefore $\mathbb{A}$-near-simple; by 366J, it is $\| \|_2$-bounded, therefore $\| \|_1$-bounded and an $\mathbb{A}$-integrator (622G). Write $\boldsymbol{z}^*$ for the $\mathbb{A}$-quadratic variation of $\boldsymbol{z}$, which is non-negative, non-decreasing, locally near-simple (631Jb) and an integrator (617I again), therefore belongs to $\mathbb{A}M^{\uparrow}_{\text{n-s}}(\mathcal{S})$. The $\mathbb{A}$-quadratic variation of $\boldsymbol{z}'$ is $\boldsymbol{z}^*\restriction\mathcal{S}'$ (633Ph), and this is equal to its $\mathbb{B}$-quadratic variation (634Ib).

By (e) again, there is a $\boldsymbol{y}\in Y$ such that, setting $\boldsymbol{y}' = \boldsymbol{y}\restriction\mathcal{S}'$, $\mathbb{B}\widehat{\theta}^{\#}_{\boldsymbol{z}^*\restriction\mathcal{S}'}(\boldsymbol{x}'-\boldsymbol{y}') = 0$. Since $|\boldsymbol{x}-\boldsymbol{y}| \le 4M\mathbf{1}^{(\mathcal{S})}$, $(\boldsymbol{x}'-\boldsymbol{y}')^2 \le 4M|\boldsymbol{x}'-\boldsymbol{y}'|$ and $\mathbb{B}\widehat{\theta}^{\#}_{\boldsymbol{z}^*\restriction\mathcal{S}'}(\boldsymbol{x}'-\boldsymbol{y}')^2 = 0$. By 646E and 648F,

$$\mathbb{B}\oint_{\mathcal{S}'}(\boldsymbol{x}'-\boldsymbol{y}')^2 d(\boldsymbol{z}^*\restriction\mathcal{S}') = \mathbb{A}\oint_{\mathcal{S}}(\boldsymbol{x}-\boldsymbol{y})^2 d\boldsymbol{z}^* = 0;$$

by 648C,

$$\|(\mathbb{B}\oint_{\mathcal{S}'}(\boldsymbol{x}'-\boldsymbol{y}')d\boldsymbol{z}')^2\|_1 = \|(\mathbb{A}\oint_{\mathcal{S}}(\boldsymbol{x}-\boldsymbol{y})d\boldsymbol{z})^2\|_1 = 0,$$

$$\mathbb{B}\oint_{\mathcal{S}'}(\boldsymbol{x}'-\boldsymbol{y}')d\boldsymbol{z}' = \mathbb{A}\oint_{\mathcal{S}}(\boldsymbol{x}-\boldsymbol{y})d\boldsymbol{z} = 0$$

and

$$\mathbb{B}\oint_{\mathcal{S}'}\boldsymbol{x}'d\boldsymbol{z}' = \mathbb{B}\oint_{\mathcal{S}'}\boldsymbol{y}'d\boldsymbol{z}' = \mathbb{A}\oint_{\mathcal{S}}\boldsymbol{y}\,d\boldsymbol{z} = \mathbb{A}\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{z}. \text{ **Q**}$$

(h) Now turn to the given $\mathbb{A}$-near-simple $\mathbb{A}$-integrator $\boldsymbol{w}$. Let $\epsilon > 0$. Working in $\mathbb{B}$, 643O tells us that we have an L^∞-bounded $\mathbb{B}$-martingale $\tilde{\boldsymbol{w}}'$ and a $\mathbb{B}$-near-simple process $\boldsymbol{v}'$ of bounded variation, both with domain $\mathcal{S}'$, such that $a = [\![\boldsymbol{w}' \ne \boldsymbol{v}' + \tilde{\boldsymbol{w}}']\!]$ has measure at most ϵ. Now $\boldsymbol{v}'$ can be expressed as $\boldsymbol{v}'_1 - \boldsymbol{v}'_2$ where $\boldsymbol{v}'_1$, $\boldsymbol{v}'_2 \in \mathbb{B}M^{\uparrow}_{\text{n-s}}(\mathcal{S}')$ (631L). Let $\boldsymbol{v}_1 = \Psi(\boldsymbol{v}'_1)$, $\boldsymbol{v}_2 = \Psi(\boldsymbol{v}'_2)$ be the $\mathbb{A}$-near-simple processes extending $\boldsymbol{v}'_1$ and $\boldsymbol{v}'_2$ to $\mathcal{S}$. Then $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ belong to $\mathbb{A}M^{\uparrow}_{\text{n-s}}(\mathcal{S})$. Since $\tilde{\boldsymbol{w}}'$ is also an $\mathbb{A}$-martingale (634Ia), and $\mathcal{S}'$ is cofinal with $\mathcal{S}$, there is a $\mathbb{A}$-martingale $\tilde{\boldsymbol{w}}$ with domain $\mathcal{S}$ which extends $\tilde{\boldsymbol{w}}'$. As in (g), $\tilde{\boldsymbol{w}}$ is $\mathbb{A}$-near-simple, so must be $\Psi(\tilde{\boldsymbol{w}}')$, and in particular is $\| \|_\infty$-bounded.

Writing $\boldsymbol{z}$ for $\boldsymbol{v}_1 - \boldsymbol{v}_2 + \tilde{\boldsymbol{w}}$ and $\boldsymbol{z}'$ for $\boldsymbol{z}\restriction\mathcal{S}' = \boldsymbol{v}' + \tilde{\boldsymbol{w}}'$, we have

$$\begin{aligned}\mathbb{B}\oint_{\mathcal{S}'}\boldsymbol{x}'d\boldsymbol{z}' &= \mathbb{B}\oint_{\mathcal{S}'}\boldsymbol{x}'d\boldsymbol{v}'_1 - \mathbb{B}\oint_{\mathcal{S}'}\boldsymbol{x}'d\boldsymbol{v}'_2 + \mathbb{B}\oint_{\mathcal{S}'}\boldsymbol{x}'d\tilde{\boldsymbol{w}}'\\ &= \mathbb{A}\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{v}_1 - \mathbb{A}\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{v}_2 + \mathbb{A}\oint_{\mathcal{S}}\boldsymbol{x}\,d\tilde{\boldsymbol{w}}\end{aligned}$$

(by (f) and (g))

$$= \mathbb{A}\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{z}.$$

So

$$\begin{aligned}[\![\mathbb{B}\oint_{\mathcal{S}'}\boldsymbol{x}'d\boldsymbol{w}' \ne \mathbb{A}\oint_{\mathcal{S}'}\boldsymbol{x}\,d\boldsymbol{w}]\!] &\subseteq [\![\mathbb{B}\oint_{\mathcal{S}'}\boldsymbol{x}'d\boldsymbol{w}' \ne \mathbb{B}\oint_{\mathcal{S}'}\boldsymbol{x}'d\boldsymbol{z}']\!] \cup [\![\mathbb{A}\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{w} \ne \mathbb{A}\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{z}]\!]\\ &\subseteq [\![\boldsymbol{w}' \ne \boldsymbol{z}']\!] \cup [\![\boldsymbol{w} \ne \boldsymbol{z}]\!]\end{aligned}$$

(646C)

$$= [\![\boldsymbol{w}'-\boldsymbol{z}' \ne 0]\!] \cup [\![\Psi(\boldsymbol{w}'-\boldsymbol{z}') \ne 0]\!] = [\![\boldsymbol{w}'-\boldsymbol{z}' \ne 0]\!]$$

(631M(b-v))

$$= a$$

has measure at most ϵ. As ϵ is arbitrary, $\mathbb{B}\oint_{\mathcal{S}'}\boldsymbol{x}'d\boldsymbol{w}' = \mathbb{A}\oint_{\mathcal{S}'}\boldsymbol{x}\,d\boldsymbol{w}$.

(i) This deals with the case in which $\mathcal{S}'$ has a greatest member and $\boldsymbol{x} \in \mathbb{A}M^0_{\text{S-i}}(\mathcal{S})$ is $\| \|_\infty$-bounded. If we know just that $\mathcal{S}'$ has a greatest member, set

$$\boldsymbol{x}_n = \operatorname{med}(-n\mathbf{1}^{(\mathcal{S})}_{<}, \boldsymbol{x}, n\mathbf{1}^{(\mathcal{S})}_{<})$$

for each n. Then each $\boldsymbol{x}_n \in \mathbb{A}M^0_{\text{S-i}}(\mathcal{S})$ is $\| \|_\infty$-bounded. Because $\boldsymbol{x}\times\mathbf{1}^{(\mathcal{S})}_{<} \in \mathbb{A}M^0_{\text{S-i}}(\mathcal{S})$ is $\mathbb{A}$-previsibly-order-bounded, $\langle\boldsymbol{x}_n\rangle_{n\in\mathbb{N}}$ is $\mathbb{A}$-uniformly-previsibly-order-bounded, while $\langle\boldsymbol{x}_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to $\boldsymbol{x}\times\mathbf{1}^{(\mathcal{S})}_{<}$.

By 648D, $(\boldsymbol{x}\times\mathbf{1}^{(\mathcal{S})}_{<})\restriction\mathcal{S}'\in\mathbb{B}M^0_{\text{S-i}}(\mathcal{S}')$ is $\mathbb{B}$-previsibly-order-bounded so $\langle\boldsymbol{x}_n\restriction\mathcal{S}'\rangle_{n\in\mathbb{N}}$ is $\mathbb{B}$-uniformly-previsibly-order-bounded, while it is order*-convergent to $(\boldsymbol{x}\times\mathbf{1}^{(\mathcal{S})}_{<})\restriction\mathcal{S}'$. Accordingly

$$\mathbb{A}\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{w}=\mathbb{A}\oint_{\mathcal{S}}\boldsymbol{x}\times\mathbf{1}^{(\mathcal{S})}_{<}\,d\boldsymbol{w}=\lim_{n\to\infty}\mathbb{A}\oint_{\mathcal{S}}\boldsymbol{x}_n d\boldsymbol{w}$$

(645T)

$$=\lim_{n\to\infty}\mathbb{B}\oint_{\mathcal{S}}(\boldsymbol{x}_n\restriction\mathcal{S}')d\boldsymbol{w}'$$

(by (a)-(h))

$$=\mathbb{B}\oint_{\mathcal{S}'}(\boldsymbol{x}\restriction\mathcal{S}')\times\mathbf{1}^{(\mathcal{S}')}_{<}d\boldsymbol{w}'=\mathbb{B}\oint_{\mathcal{S}'}\boldsymbol{x}'d\boldsymbol{w}'.$$

(j) Finally, for the general case of a relatively order-convex sublattice $\mathcal{S}'$ of $\mathcal{T}_{\mathbb{B}}$, we know from 646M that

$$\mathbb{B}\oint_{\mathcal{S}'}\boldsymbol{x}'d\boldsymbol{w}'=\lim_{\tau\uparrow\mathcal{S}'}\mathbb{B}\oint_{\mathcal{S}'\wedge\tau}\boldsymbol{x}'d\boldsymbol{w}',$$

$$\mathbb{A}\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{w}'=\lim_{\tau\uparrow\mathcal{S}}\mathbb{A}\oint_{\mathcal{S}\wedge\tau}\boldsymbol{x}\,d\boldsymbol{w}=\lim_{\tau\uparrow\mathcal{S}'}\mathbb{A}\oint_{\mathcal{S}\wedge\tau}\boldsymbol{x}\,d\boldsymbol{w}$$

because $\mathcal{S}'$ is cofinal with $\mathcal{S}$. But for every $\tau\in\mathcal{S}'$, $\mathcal{S}\wedge\tau$ is the order-convex hull of $\mathcal{S}'\wedge\tau$, so we can apply (b)-(i) to see that $\mathbb{B}\oint_{\mathcal{S}'\wedge\tau}\boldsymbol{x}'d\boldsymbol{w}'=\mathbb{A}\oint_{\mathcal{S}\wedge\tau}\boldsymbol{x}\,d\boldsymbol{w}$. In the limit, we have $\mathbb{B}\oint_{\mathcal{S}'}\boldsymbol{x}'d\boldsymbol{w}'=\mathbb{A}\oint_{\mathcal{S}}\boldsymbol{x}\,d\boldsymbol{w}$.

648X Basic exercises (a) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra and $\mathfrak{B}$ a closed subalgebra of $\mathfrak{A}$. Show that there is a continuous order-preserving projection from $L^0(\mathfrak{A})$ onto $L^0(\mathfrak{B})$.

648Z Problem In 648G, can we drop the hypothesis that '$\mathfrak{B}$ is coordinated with $\langle\mathfrak{A}_t\rangle_{t\in T}$'?

648 Notes and comments We have just gone through an uncommonly dense argument aiming at an expected result. Some of the complications (e.g., the shift from the moderately oscillatory processes $\acute{\boldsymbol{u}}'_n$ to the near-simple processes $\grave{\boldsymbol{u}}'_n$ in part (e) of the proof of 648G) arise from the idiosyncratic formulations I have chosen. Others come from my extension of the S-integral from previsible processes, as described in §642, to the S-integrable processes of §645; in effect, this is part (f) of the proof of 648G, relying on 648F. I note also that part (d) of the proof of 648D duplicates the idea of 647B.

Periodically, in this volume, I have looked at the question of law-independence. I noted in 645Q that the S-integral is law-independent. In 648G the hypotheses and conclusion are all law-independent *except* for the requirement that $\mathfrak{B}$ should be coordinated with $\langle\mathfrak{A}_t\rangle_{t\in T}$. It would be sufficient to suppose that

there is a $\bar{\nu}$ such that $(\mathfrak{A},\bar{\nu})$ is a probability algebra and $\mathfrak{B}$ is coordinated with $(\mathfrak{A},\bar{\nu},\langle\mathfrak{A}_t\rangle_{t\in T})$,

obtaining a version which demands only that $\mathfrak{A}$ should be a measurable algebra. But an affirmative answer to 648Z would show that no such manoeuvre is called for.

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649 Pathwise integration

The integrals of §613 and §645 are defined in terms of convergence in L^0. The most important applications are associated with processes of the form $\langle X_t(\omega)\rangle_{t\ge0,\omega\in\Omega}$ with paths $\langle X_t(\omega)\rangle_{t\ge0}$. It turns out that in the case of the Riemann-sum integral, we can often, with some effort, define integrals 'pathwise'. I do not think that this approach gives a good picture of the theory as a whole, but it is surely worth knowing what can be done.

The S-integral is rather different; I do not see any way of giving a pathwise description of the S-integral with respect to Brownian motion, for instance. But for non-decreasing integrators we have an effective approach through Stieltjes integrals, which I have hinted at in earlier sections. I now give a detailed account of the method (649H, 649L).

649A Notation We need only fragments of the standard framework. $(\mathfrak{A},\bar\mu,T,\langle\mathfrak{A}_t\rangle_{t\in T},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$, if not explicitly introduced, will be a stochastic integration structure. L^0 will be the Riesz space $L^0(\mathfrak{A})$ with the topology of convergence in measure. If $\mathcal{S}$ is a sublattice of $\mathcal{T}$, $\mathcal{I}(\mathcal{S})$ will be the set of finite sublattices of $\mathcal{S}$, $M_{\text{simp}}(\mathcal{S})$ will be the space of simple processes with domain $\mathcal{S}$ and $M_{\text{n-s}}(\mathcal{S})$ will be the space of near-simple processes with domain $\mathcal{S}$; if $\tau\in\mathcal{S}$ then $\mathcal{S}\wedge\tau$ will be $\{\sigma\wedge\tau:\sigma\in\mathcal{S}$; if $\boldsymbol{v}$ is a fully adapted process with domain including $\mathcal{S}$, $Q_{\mathcal{S}}(d\boldsymbol{v})$ will be the capped-stake variation set of $\boldsymbol{v}\restriction\mathcal{S}$. When $\boldsymbol{v}$ is near-simple, $\boldsymbol{v}_<$ will be its previsible version. I use $\int$ to denote Riemann-sum integrals and $\oint$ to denote S-integrals.

649B Theorem Suppose that $\mathcal{S}$ is a sublattice of $\mathcal{T}$ with a least element. Let $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ be an integrator and $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a near-simple process. Suppose that we have, for each $n\in\mathbb{N}$, a non-decreasing sequence $\langle\tau_{ni}\rangle_{i\in\mathbb{N}}$ in $\mathcal{S}$ such that $\tau_{n0}=\min\mathcal{S}$, $\inf_{i\in\mathbb{N}}[\![\tau_{ni}<\sup\mathcal{S}]\!]=0$ and, for each $i\in\mathbb{N}$,

$$[\![\sigma<\tau_{n,i+1}]\!]\subseteq[\![|u_\sigma-u_{\tau_{n,i}}|\le 2^{-n}]\!]\text{ for every }\sigma\in[\tau_{n,i},\tau_{n,i+1}],$$

$$[\![\tau_{n,i+1}<\sup\mathcal{S}]\!]\subseteq[\![|u_{\tau_{n,i+1}}-u_{\tau_{n,i}}|\ge 2^{-n}]\!].$$

Then

$$z_n=\lim_{k\to\infty}\sum_{i=0}^{k-1}u_{\tau_{n,i}}\times(v_{\tau_{n,i+1}}-v_{\tau_{n,i}})$$

is defined for each n, and $\langle z_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to $\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}$.

proof (a) Let $n\in\mathbb{N}$.

(i) Set $z_{nk}=\sum_{i=0}^{k-1}u_{\tau_{ni}}\times(v_{\tau_{n,i+1}}-v_{\tau_{ni}})$ for $k\in\mathbb{N}$. If $k\le l$, then

$$\begin{aligned}[\![z_{nk}\ne z_{nl}]\!]&\subseteq\sup_{k\le i<l}[\![v_{\tau_{n,i+1}}\ne v_{\tau_{ni}}]\!]\subseteq\sup_{k\le i<l}[\![\tau_{n,i+1}\ne\tau_{ni}]\!]\\&\subseteq\sup_{k\le i\le l}[\![\tau_{ni}\ne\sup\mathcal{S}]\!]\subseteq[\![\tau_{nk}<\sup\mathcal{S}]\!].\end{aligned}$$

But as $\langle[\![\tau_{nk}<\sup\mathcal{S}]\!]\rangle_{k\in\mathbb{N}}$ is a non-increasing sequence with infimum 0, this means that $\langle z_{nk}\rangle_{k\in\mathbb{N}}$ is actually order*-convergent to some $z_n\in L^0$, with $[\![z_{nk}\ne z_n]\!]\subseteq[\![\tau_{nk}<\sup\mathcal{S}]\!]$ for every k. Thus the topological limit z_n is always defined.

(ii) Define $\boldsymbol{u}_n=\langle u_{n\sigma}\rangle_{\sigma\in\mathcal{S}}$ by saying that, for $\sigma\in\mathcal{S}$,

$$[\![\tau_{ni}=\sigma]\!]\cup([\![\tau_{ni}\le\sigma]\!]\cap[\![\sigma<\tau_{n,i+1}]\!])\subseteq[\![u_\sigma=u_{\tau_{ni}}]\!]$$

for every $i\in\mathbb{N}$; because

$$\langle[\![\tau_{ni}=\sigma]\!]\cup([\![\tau_{ni}\le\sigma]\!]\cap[\![\sigma<\tau_{n,i+1}]\!])\rangle_{i\in\mathbb{N}}$$

is always disjoint, with supremum

$$\sup_{i\in\mathbb{N}}[\![\sigma\le\tau_{ni}]\!]\supseteq\sup_{i\in\mathbb{N}}[\![\tau_{ni}=\sup\mathcal{S}]\!]=1,$$

$\boldsymbol{u}_n$ is a fully adapted process. Next, $\boldsymbol{u}_n\restriction\mathcal{S}\wedge\tau_{nk}$ is simple, with breakpoint string $(\tau_{n0},\dots,\tau_{nk})$, and $z_{nk}=\int_{\mathcal{S}\wedge\tau_{nk}}\boldsymbol{u}_n d\boldsymbol{v}$ for each k (614C). We know also that if $k\in\mathbb{N}$ and $\sigma\in\mathcal{S}\wedge\tau_{nk}$,

$$1=\sup_{i<k}([\![\tau_{ni}\le\sigma]\!]\cap[\![\sigma<\tau_{n,i+1}]\!])\cup[\![\sigma=\tau_{nk}]\!]\subseteq[\![|u_\sigma-u_{n\sigma}|\le 2^{-n}]\!].$$

But this means that if $I\in\mathcal{I}(\mathcal{S}\wedge\tau_{nk})$, $S_I(\boldsymbol{u}-\boldsymbol{u}_n,d\boldsymbol{v})\in 2^{-n}Q_{\mathcal{S}}(d\boldsymbol{v})$.

(iii) The integral $\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}$ is $\lim_{k\to\infty}\int_{\mathcal{S}\wedge\tau_{nk}}\boldsymbol{u}\,d\boldsymbol{v}$. **P** We know that $\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}=\lim_{\sigma\uparrow\mathcal{S}}\int_{\mathcal{S}\wedge\sigma}\boldsymbol{u}\,d\boldsymbol{v}$ (613J(f-ii)). Similarly, if $k\in\mathbb{N}$, $\int_{\mathcal{S}\wedge\tau_{nk}}\boldsymbol{u}\,d\boldsymbol{v}=\lim_{\sigma\uparrow\mathcal{S}}\int_{\mathcal{S}\wedge\tau_{nk}\wedge\sigma}\boldsymbol{u}\,d\boldsymbol{v}$. But this means that

$$\begin{aligned}[\![\textstyle\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}\ne\int_{\mathcal{S}\wedge\tau_{nk}}\boldsymbol{u}\,d\boldsymbol{v}]\!]&\subseteq\sup_{\sigma\in\mathcal{S}}[\![\textstyle\int_{\mathcal{S}\wedge\sigma}\boldsymbol{u}\,d\boldsymbol{v}\ne\int_{\mathcal{S}\wedge\tau_{nk}\wedge\sigma}\boldsymbol{u}\,d\boldsymbol{v}]\!]\\&\subseteq\sup_{\sigma\in\mathcal{S}}[\![\sigma\ne\tau_{nk}\wedge\sigma]\!]\subseteq\sup_{\sigma\in\mathcal{S}}[\![\tau_{nk}<\sigma]\!]=[\![\tau_{nk}<\sup\mathcal{S}]\!]\end{aligned}$$

(611Eb). As $\inf_{k\in\mathbb{N}}\sup_{l\ge k}[\![\tau_{nk}<\sup\mathcal{S}]\!]=0$, $\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}=\lim_{k\to\infty}\int_{\mathcal{S}\wedge\tau_{nk}}\boldsymbol{u}\,d\boldsymbol{v}$. **Q**

(b) Let C be the solid convex hull of $Q_{\mathcal{S}}(d\boldsymbol{v})$, and $\overline{C}$ its topological closure. Then C is topologically bounded (627M) so $\overline{C}$ is topologically bounded (3A5N(b-ii)); also $\overline{C}$ is convex (2A5Eb) and solid (613B(f-vi)). Now we see from (a-ii) that if n, $k \in \mathbb{N}$ then $S_I(\boldsymbol{u}-\boldsymbol{u}_n, d\boldsymbol{v}) \in 2^{-n}C$ for every $I \in \mathcal{I}([\min\mathcal{S}, \tau_{nk}])$, so

$$\int_{\mathcal{S}\wedge\tau_{nk}} \boldsymbol{u}\,d\boldsymbol{v} - z_{nk} = \int_{\mathcal{S}\wedge\tau_{nk}} \boldsymbol{u}-\boldsymbol{u}_n d\boldsymbol{v} \in 2^{-n}\overline{C};$$

letting $k \to \infty$, and using (a-iii),

$$\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v} - z_n \in 2^{-n}\overline{C}.$$

Because $\overline{C}$ is solid and convex,

$$\sup_{l\le n\le m} |z_n - \int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}| \in \sum_{n=l}^{m} 2^{-n}\overline{C} \subseteq 2^{-l+1}\overline{C}$$

whenever $l \le m$. By 613B(f-v), $\{\sup_{l\le n\le m} |z_n - \int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}| : m \ge l\}$ is bounded above in L^0, and its supremum w_l belongs to $2^{-l+1}\overline{C}$ (613Ba).

Repeating the argument, $w'_p = \sup_{l\ge p} w_l$ is defined, and belongs to $2^{-p+2}\overline{C}$, for every $p \in \mathbb{N}$. As $\overline{C}$ is topologically bounded, $\lim_{p\to\infty} \theta(w'_p) = 0$ and $\inf_{p\in\mathbb{N}} w'_p = 0$. But $|z_n - \int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}| \le w_p \le w'_p$ whenever $n \ge p$, while $\langle w'_p\rangle_{p\in\mathbb{N}}$ is non-increasing and has infimum 0, so $\langle z_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to $\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}$, as claimed.

649C From 649B, we see that, under the hypotheses there, we have a chance of expressing a Riemann-sum integral as an order*-limit of order*-limits of explicitly defined Riemann sums. The hypotheses are elaborate, but they correspond to a version of SL_1 in 631Oa, so they will be satisfied in a useful number of cases. The really important feature of the result, however, is that (under appropriate conditions) the τ_{ni}, and hence the z_n, can be determined pathwise, as in the following form of the theorem.

Corollary (Bichteler 81, 7.14, or Karandikar 95) Let (Ω, Σ, μ) be a complete probability space and $\langle\Sigma_t\rangle_{t\in[0,\infty[}$ a right-continuous filtration of σ-subalgebras of Σ, all containing every negligible subset of Ω; suppose that $(\mathfrak{A}, \bar\mu)$ and $\langle\mathfrak{A}_t\rangle_{t\ge0}$ are the corresponding measure algebra and filtration of closed subalgebras. Let $\langle U_t\rangle_{t\ge0}$, $\langle V_t\rangle_{t\ge0}$ be stochastic processes such that $t \mapsto U_t(\omega) : [0,\infty[\to \mathbb{R}$ is càdlàg for every $\omega \in \Omega$, and $(t,\omega) \mapsto V_t(\omega) : [0,\infty[\times \Omega \to \mathbb{R}$ is progressively measurable; let $\boldsymbol{u}$, $\boldsymbol{v}$ be the corresponding fully adapted processes with domain $\mathcal{T}_f$, as in 612H and 631D. Suppose that $\boldsymbol{v}$ is a local integrator.

Let $h : \Omega \to [0,\infty[$ be a stopping time, and $\tau^* = h^\bullet$ the corresponding stopping time in $\mathcal{T}_f$. For $n \in \mathbb{N}$ and $\omega \in \Omega$, define $h_{ni}(\omega)$, for $i \in \mathbb{N}$, by setting $h_{n0}(\omega) = 0$ and then

$$h_{n,i+1}(\omega) = \inf(\{h(\omega)\} \cup \{t : t \ge h_{ni}(\omega),\, |U_t(\omega) - U_{h_{ni}(\omega)}(\omega)| > 2^{-n}\})$$

for $i \in \mathbb{N}$.

In this case,

(a) every h_{ni} is a stopping time adapted to $\langle\Sigma_t\rangle_{t\ge0}$,

(b)

$$f_n(\omega) = \sum_{i=0}^{\infty} U_{h_{ni}}(\omega)\big(V_{h_{n,i+1}}(\omega) - V_{h_{ni}}(\omega)\big)$$

is defined for all $n \in \mathbb{N}$ and $\omega \in \Omega$,

(c) $f = \lim_{n\to\infty} f_n$ is defined in $\mathbb{R}$ almost everywhere in Ω, and $f^\bullet = \int_{[\check{0},\tau^*]} \boldsymbol{u}\,d\boldsymbol{v}$.

proof (a) Induce on i. If $i = 0$ this is trivial. For the inductive step to $i+1$, given $t \ge 0$,

$$\begin{aligned}\{\omega : h_{n,i+1}(\omega) < t\} &= \{\omega : h(\omega) < t\} \cup \bigcup_{s<t} \{\omega : h_{ni}(\omega) \le s,\, |U_s(\omega) - U_{h_{ni}(\omega)}(\omega)| > 2^{-n}\} \\ &= \{\omega : h(\omega) < t\} \cup \bigcup_{q\in\mathbb{Q}, q<t} \{\omega : h_{ni}(\omega) \le q,\, |U_q(\omega) - U_{h_{ni}(\omega)}(\omega)| > 2^{-n}\}\end{aligned}$$

because $s \mapsto U_s(\omega)$ is càdlàg for every ω. Now the inductive hypothesis tells us that h_{ni} is a stopping time, and we saw in 631D that $(t,\omega) \mapsto U_t(\omega)$ is progressively measurable, so $U_{h_{ni}}$ is $\Sigma_{h_{ni}}$-measurable (455Le), where I write $U_{h_{ni}}$ for $\omega \mapsto U_{h_{ni}(\omega)}(\omega)$, as in 455L and 612H. For $\alpha \in \mathbb{R}$, set

$$E_\alpha = \{\omega : U_{h_{ni}(\omega)}(\omega) > \alpha\} = \{\omega : U_{h_{ni}}(\omega) > \alpha\} \in \Sigma_{h_{ni}};$$

if $q < t$, then $E_\alpha \cap \{\omega : h_{ni}(\omega) \le q\}$ belongs to $\Sigma_q \subseteq \Sigma_t$. But this means that $U_{h_{ni}} \times \chi\{\omega : h_{ni}(\omega) \le q\}$, and therefore $(U_q - U_{h_{ni}}) \times \chi\{\omega : h_{ni}(\omega) \le q\}$, are Σ_t-measurable for every $q < t$, and $\bigcup_{q\in\mathbb{Q}, q<t}\{\omega : h_{ni}(\omega) \le q,\ |U_q(\omega) - U_{h_{ni}(\omega)}(\omega)| > 2^{-n}\} \in \Sigma_t$.

It follows at once that $\{\omega : h_{n,i+1}(\omega) < t\}$ belongs to Σ_t for every t. But we are supposing that $\langle\Sigma_t\rangle_{t\ge0}$ is right-continuous, so h_{ni} is a stopping time adapted to $\langle\Sigma_t\rangle_{t\ge0}$, by 455Lb.

(b) Accordingly I will allow myself to use the notations $U_{h_{ni}}$, $V_{h_{ni}}$ for all n, $i \in \mathbb{N}$.

(i) $|U_{h_{n,i+1}}(\omega) - U_{h_{ni}}(\omega)| \ge 2^{-n}$ whenever n, $i \in \mathbb{N}$, $\omega \in \Omega$ and $h_{n,i+1}(\omega) < h(\omega)$. **P?** Otherwise, because $t \mapsto U_t(\omega)$ is everywhere continuous on the right, there is a $t > h_{n,i+1}(\omega)$ such that $|U_s(\omega) - U_{h_{ni}}(\omega)| \le 2^{-n}$ for every $s \in [h_{n,i+1}(\omega), t]$; but now $h_{n,i+1}(\omega)$ must be greater than or equal to $\min(t, h(\omega))$. **XQ**

(ii) Note also that, for each $\omega \in \Omega$ and $n \in \mathbb{N}$, $h_{ni}(\omega) = h(\omega)$ for all but finitely many i. **P?** Otherwise, $\langle h_{ni}(\omega)\rangle_{i\in\mathbb{N}}$ is a strictly increasing sequence in $[0, h(\omega)]$, while $\langle U_{h_{ni}(\omega)}(\omega)\rangle_{i\in\mathbb{N}}$ is not convergent. **XQ** Accordingly

$$f_n(\omega) = \sum_{i=0}^\infty U_{h_{ni}(\omega)}\big(V_{h_{n,i+1}}(\omega) - V_{h_{ni}}(\omega)\big)$$

is defined, as required.

(c)(i) Set $\mathcal{S} = [\check{0}, \tau^*] \subseteq \mathcal{T}$. We know from 631D that $\boldsymbol{u}$ is locally near-simple, so $\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}$ is defined. For n, $i \in \mathbb{N}$, set $\tau_{ni} = h_{ni}^\bullet \in \mathcal{T}_f$, as in 612Ha. Then $\langle\tau_{ni}\rangle_{n,i\in\mathbb{N}}$ satisfies the hypotheses of 649B. **P** Take $n \in \mathbb{N}$. Because $\langle h_{ni}\rangle_{i\in\mathbb{N}}$ is non-decreasing, so is $\langle\tau_{ni}\rangle_{i\in\mathbb{N}}$. Of course $\tau_{n0} = \check{0} = \min\mathcal{S}$. We have $\sup S = \tau^* = h^\bullet$, so

$$\inf_{i\in\mathbb{N}} [\![\tau_{ni} < \sup\mathcal{S}]\!] = \big(\bigcap_{i\in\mathbb{N}}\{\omega : h_{ni}(\omega) < h(\omega)\}\big)^\bullet = 0$$

by (b-ii). If $i \in \mathbb{N}$ and $\sigma \in [\tau_{ni}, \tau_{n,i+1}]$ then we can express σ as $g^\bullet$ where g is a stopping time and $h_{ni} \le g \le h_{n,i+1}$; in this case $u_\sigma = U_g^\bullet$ (612H(b-i)) and $|U_g(\omega) - U_{h_{ni}}(\omega)| \le 2^{-n}$ whenever $g(\omega) < h_{n,i+1}(\omega)$, so $[\![\sigma < \tau_{n,i+1}]\!] \subseteq [\![|u_\sigma - u_{\tau_{ni}}| \le 2^{-n}]\!]$. And (b) above tells us that $[\![\tau_{n,i+1} < \tau^*]\!] \subseteq [\![|u_{\tau_{n,i+1}} - u_{\tau_{ni}}| \ge 2^{-n}]\!]$. **Q**

(ii) If $n \in \mathbb{N}$,

$$f_n(\omega) = \lim_{k\to\infty} \sum_{i=0}^{k-1} U_{h_{ni}}(\omega)\big(V_{h_{n,i+1}}(\omega) - V_{h_{ni}}(\omega)\big)$$

for every ω. So

$$f_n^\bullet = \lim_{k\to\infty} \sum_{i=0}^{k-1} u_{\tau_{ni}} \times (v_{\tau_{n,i+1}} - v_{\tau_{ni}})$$

can be identified with z_n as described in 649B. As $\langle z_n\rangle_{n\in\mathbb{N}} \to^* \int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}$, 642Ba tells us that $\langle f_n\rangle_{n\in\mathbb{N}}$ is convergent a.e., and if $f = \lim_{n\to\infty} f_n$ then $f^\bullet = \int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}$, as required.

649D In the presence of a special kind of filter on $\mathbb{N}$, we have a quite different way of calculating stochastic integrals by looking at one path at a time.

(a) Definition A filter $\mathcal{F}$ on $\mathbb{N}$ is **measure-converging** (538Ag) if whenever (Ω, Σ, μ) is a probability space, $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in Σ, and $\lim_{n\to\infty} \mu E_n = 1$, then $\bigcup_{A\in\mathcal{F}} \bigcap_{n\in A} E_n$ is conegligible.

(b) Suppose that $\mathcal{F}$ is a measure-converging filter on $\mathbb{N}$, (Ω, Σ, μ) is a probability space, and $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}^0 = \mathcal{L}^0(\mu)$ which converges in measure to $f \in \mathcal{L}^0$. Then $\lim_{n\to\mathcal{F}} f_n =_{\text{a.e.}} f$ (538N(a-iii)).

Remark It seems still to be unknown whether ZFC is enough to prove the existence of a measure-converging filter (see 538Z). However, the continuum hypothesis, for instance, is more than sufficient to ensure that measure-converging filters exist (538Ng).

649E Proposition (cf. Nutz p11) Suppose that $\mathcal{F}$ is a measure-converging filter on $\mathbb{N}$. Let (Ω, Σ, μ) be a complete probability space, $\langle\Sigma_t\rangle_{t\ge0}$ a right-continuous filtration of σ-subalgebras of Σ all containing every negligible set; suppose that $(\mathfrak{A}, \bar{\mu})$, $\langle\mathfrak{A}_t\rangle_{t\ge0}$ are the corresponding probability algebra and filtration of closed subalgebras. Let $\langle U_t\rangle_{t\ge0}$, $\langle V_t\rangle_{t\ge0}$ be stochastic processes on Ω, adapted to $\langle\Sigma_t\rangle_{t\ge0}$, such that the paths

$t \mapsto U_t(\omega)$, $t \mapsto V_t(\omega)$ are càdlàg for every ω; let $\boldsymbol{u}$, $\boldsymbol{v}$ be the corresponding locally near-simple processes defined on $\mathcal{T}_f$ (631D), and suppose that $\boldsymbol{v}$ is a local integrator. Let h, $h' : \Omega \to [0,\infty[$ be stopping times corresponding to τ, $\tau' \in \mathcal{T}_f$, with $h(\omega) \le h'(\omega)$ for every ω. Enumerate $\mathbb{Q} \cap [0,\infty[$ as $\langle q_n \rangle_{n\in\mathbb{N}}$, starting with $q_0 = 0$, and for $n \in \mathbb{N}$ let $\langle q_{ni} \rangle_{i \le n}$ be the increasing enumeration of $\{q_i : i \le n\}$. Set

$$f_n(\omega) = \sum_{i=0}^{n-1} U_{\mathrm{med}(h(\omega),q_{ni},h'(\omega))}(\omega)\big(V_{\mathrm{med}(h(\omega),q_{n,i+1},h'(\omega))}(\omega) - V_{\mathrm{med}(h(\omega),q_{ni},h'(\omega))}(\omega)\big)$$

for $\omega \in \Omega$. Then $f(\omega) = \lim_{n\to\mathcal{F}} f_n(\omega)$ is defined for almost every ω, f is Σ-measurable and $f^\bullet = \int_{[\tau,\tau']} \boldsymbol{u}\, d\boldsymbol{v}$.

proof Write $\mathcal{S}'$ for $\{\mathrm{med}(\tau,\check{q},\tau') : q \in \mathbb{Q} \cap [0,\infty[\} \cup \{\tau'\}$, as in 633L. For $n \in \mathbb{N}$, set $I_n = \{\mathrm{med}(\tau,\check{q}_i,\tau') : i \le n\}$, $I'_n = I_n \cup \{\tau'\}$. Note that $\tau = \mathrm{med}(\tau,\check{q}_0,\tau')$ belongs to I_n, and that if we set

$$a_n = \sup_{i\le n} [\![\tau' = \mathrm{med}(\tau,\check{q}_i,\tau')]\!] \supseteq \sup_{i\le n} [\![\tau' \le \check{q}_i]\!];$$

then $a_n \subseteq \sup_{\sigma\in I_n} [\![\rho = \sigma]\!]$ for every $\rho \in I'_n$. Because $\tau' \in \mathcal{T}_f$, $\lim_{n\to\infty} \bar{\mu} a_n = 1$.

Now

$$\int_{[\tau,\tau]} \boldsymbol{u}\, d\boldsymbol{v} = \int_{\mathcal{S}'} \boldsymbol{u}\, d\boldsymbol{v}$$

(633L)

$$= \lim_{n\to\infty} S_{I'_n}(\boldsymbol{u}, d\boldsymbol{v})$$

(because if $J \subseteq \mathcal{S}'$ is finite there is an $m \in \mathbb{N}$ such that $J \subseteq I'_n$ for every $n \ge m$)

$$= \lim_{n\to\infty} S_{I_n}(\boldsymbol{u}, d\boldsymbol{v})$$

(because $[\![S_{I'_n}(\boldsymbol{u}, d\boldsymbol{v}) = S_{I_n}(\boldsymbol{u}, d\boldsymbol{v})]\!] \supseteq a_n$, by 613S)

$$= \lim_{n\to\infty} f_n^\bullet = (\lim_{n\to\mathcal{F}} f_n)^\bullet$$

because $\mathcal{F}$ is a measure-converging filter.

649F I have given a number of results (612H, 614U, 631D, 649C, 649E) on the ways in which classical stochastic processes, based on probability spaces, give rise to the processes considered in this volume, based on probability algebras. To get full value from these, we need to know which processes can be represented in this way. I have held off so far because there are technical complications which I feel are irrelevant to the ideas I really want to express, but I think it is time I gave a result which handles a reasonable proportion of cases. I begin with a minor extension of ideas from §§631 and 633.

Lemma Suppose that $\langle \mathfrak{A}_t \rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$ a locally near-simple process. Let $\tilde{\mathcal{S}}$ be the full ideal of $\mathcal{T}$ generated by $\mathcal{S}$. Then there is a locally near-simple process $\tilde{\boldsymbol{u}}$ with domain $\tilde{\mathcal{S}}$ extending $\boldsymbol{u}$. If $\boldsymbol{u}$ is non-negative and non-decreasing, we can arrange that $\tilde{\boldsymbol{u}}$ should be non-negative and non-decreasing.

proof If $\mathcal{S} = \emptyset$ then $\tilde{\mathcal{S}} = \emptyset$ and there is nothing to prove. So I suppose from now on that $\mathcal{S}$ is non-empty.

(a) Set $\sigma_1 = \inf \mathcal{S}$ and $\mathcal{S}_1 = \mathcal{S} \cup \{\sigma_1\}$. Then $\mathcal{S}_1$ is a sublattice of $\mathcal{S}$ and the starting value $u_\downarrow = \lim_{\sigma\downarrow\mathcal{S}} u_\sigma$ is defined and belongs to $L^0(\bigcap_{\sigma\in\mathcal{S}} \mathfrak{A}_\sigma)$ (631Ca, applied to $\mathcal{S} \wedge \tau$ for any $\tau \in \mathcal{S}$). Because $\langle \mathfrak{A}_t \rangle_{t\in T}$ is right-continuous, $u_\downarrow \in L^0(\mathfrak{A}_{\sigma_1})$ (632C(a-iii)); moreover, if $\tau \in \mathcal{S}$, then $[\![\sigma_1 < \tau]\!] = \sup_{\sigma\in\mathcal{S}\wedge\tau} [\![\sigma < \tau]\!]$ (632C(a-ii)), so

$$[\![\sigma_1 = \tau]\!] = \inf_{\sigma\in\mathcal{S}\wedge\tau} [\![\sigma = \tau]\!] \subseteq \inf_{\sigma\in\mathcal{S}\wedge\tau} [\![u_\sigma = u_\tau]\!] \subseteq [\![u_\downarrow = u_\tau]\!].$$

So we have a fully adapted process $\boldsymbol{u}_1 = \langle u_{1\sigma} \rangle_{\sigma\in\mathcal{S}_1}$ defined by saying that $u_{1,\sigma_1} = u_\downarrow$ and $u_{1\sigma} = u_\sigma$ for every $\sigma \in \mathcal{S}$.

Next, $\boldsymbol{u}_1$ is locally near-simple. **P** Take any $\tau \in \mathcal{S}$ and $\epsilon > 0$. Then there is a simple process $\boldsymbol{u}' = \langle u'_\sigma \rangle_{\sigma\in\mathcal{S}\wedge\tau}$ such that $\theta(\boldsymbol{u}) \le \epsilon$ where $\boldsymbol{u} = \sup_{\sigma\in\mathcal{S}\wedge\tau} |u_\sigma - u'_\sigma|$). Let $(\tau_0, \dots, \tau_n)$ be a breakpoint string for $\boldsymbol{u}'$ ending with $\tau_n = \tau$. Consider the simple process $\boldsymbol{u}'_1 = \langle u'_{1\sigma} \rangle_{\sigma\in\mathcal{S}_1\wedge\tau}$ with breakpoint string $(\sigma_1, \tau_0, \dots, \tau_n)$ and

$$[\![u'_{1\sigma} = u'_{\tau_i}]\!] \supseteq [\![\tau_i \le \sigma]\!] \cap [\![\sigma < \tau_{i+1}]\!] \text{ for } i < n,$$

$$[\![u'_{1\sigma} = u_\downarrow]\!] \supseteq [\![\sigma < \tau_0]\!]$$

for every $\sigma \in \mathcal{S}_1$, while $u'_{1\tau_n} = u'_{\tau_n}$. Let $u'_\downarrow$ be the starting value of $\boldsymbol{u}'$ (614Ba). Since $u'_\downarrow = \lim_{\sigma\downarrow\mathcal{S}\wedge\tau} u'_\sigma$ and $u_\downarrow = \lim_{\sigma\downarrow\mathcal{S}\wedge\tau} u_\sigma$, $|u'_\downarrow - u_\downarrow| \le \bar{u}$. Now we see that, for $\sigma \in \mathcal{S}$,

$$\begin{aligned}[\![\sigma < \tau_0]\!] &\subseteq [\![u'_\sigma = u'_\downarrow]\!] \cap [\![u'_{1\sigma} = u_\downarrow]\!] \cap [\![|u_\sigma - u'_\sigma| \le \bar{u}]\!]\\ &\subseteq [\![u'_\sigma = u'_\downarrow]\!] \cap [\![u'_{1\sigma} = u_\downarrow]\!] \cap [\![|u_\sigma - u'_\sigma| \le \bar{u}]\!] \cap [\![u_{1\sigma} = u_\sigma]\!] \subseteq [\![|u_{1\sigma} - u'_{1\sigma}| \le 2\bar{u}]\!],\end{aligned}$$

while

$$\begin{aligned}[\![\tau_i \le \sigma]\!] \cap [\![\sigma < \tau_{i+1}]\!] &\subseteq [\![u'_{1\sigma} = u'_{\tau_i}]\!] \cap [\![u'_\sigma = u'_{\tau_i}]\!] \cap [\![u_{1\sigma} = u_\sigma]\!] \cap [\![|u_\sigma - u'_\sigma| \le \bar{u}]\!]\\ &\subseteq [\![|u_{1\sigma} - u'_{1\sigma}| \le \bar{u}]\!]\end{aligned}$$

for $i < n$ and, of course,

$$|u_{1\tau} - u'_{1\tau}| = |u_\tau - u'_\tau| \le \bar{u}.$$

Thus we have $|u_{1\sigma} - u'_{1\sigma}| \le 2\bar{u}$ for every $\sigma \in \mathcal{S}$. At the bottom end,

$$[\![\sigma_1 < \tau_0]\!] \subseteq [\![u'_{1\sigma_1} = u_\downarrow]\!] = [\![u'_{1\sigma_1} = u_{1\sigma_1}]\!],$$

$$[\![\sigma_1 = \tau_0]\!] \subseteq [\![u'_{1\sigma_1} = u'_{\tau_0}]\!] \cap [\![u_{1\sigma_1} = u_{\tau_0}]\!] \subseteq [\![|u_{1\sigma_1} - u'_{1\sigma_1}| \le \bar{u}]\!].$$

Assembling these,

$$\theta(\sup_{\sigma\in\mathcal{S}_1} |u_{1\sigma} - u'_{1\sigma}|) \le \theta(2\bar{u}) \le 2\epsilon.$$

As τ and ϵ is arbitrary, $\boldsymbol{u}_1$ is locally near-simple. **Q**

(b) Write σ_2 for $\min \mathcal{T}$ and $\mathcal{S}_2$ for the lattice $\mathcal{S}_1 \cup \{\sigma_2\} = \mathcal{S} \cup \{\sigma_2, \sigma_1\}$. We have a fully adapted process $\boldsymbol{u}_2 = \langle u_{2\sigma}\rangle_{\sigma\in\mathcal{S}_2}$ defined by saying that $u_{2\sigma} = u_{1\sigma}$ for $\sigma \in \mathcal{S}_1$ and

$$[\![u_{2\sigma_2} = 0]\!] \supseteq [\![\sigma_2 < \sigma_1]\!], \quad [\![u_{2\sigma_2} = u_\downarrow]\!] \supseteq [\![\sigma_2 = \sigma_1]\!].$$

Since $\boldsymbol{u}_2{\restriction}\mathcal{S}_2 \wedge \sigma_1$ is simple while $\boldsymbol{u}_2{\restriction}\mathcal{S}_2 \vee \sigma_1 = \boldsymbol{u}_1$ is locally near-simple, $\boldsymbol{u}_2$ is locally near-simple (631F(a-iv)). And of course $\boldsymbol{u}_2$ extends $\boldsymbol{u}$. Moreover, if $\boldsymbol{u}$ is non-negative and non-decreasing,

$$0 \le u_{2\sigma_2} \le u_\downarrow = \inf_{\sigma\in\mathcal{S}} u_\sigma$$

so $\boldsymbol{u}_2$ is non-negative and non-decreasing.

(c) We are now in a position to turn to 631M. $\mathcal{S}_2$ is coinitial with $\tilde{\mathcal{S}}$, just because $\min \tilde{\mathcal{S}} = \min \mathcal{T}$ belongs to $\mathcal{S}_2$. So we have a function $\Psi^* : M_{\text{ln-s}}(\mathcal{S}_2) \to M_{\text{ln-s}}(\tilde{\mathcal{S}})$ as described in 631Mc. Set $\tilde{\boldsymbol{u}} = \Psi^*(\boldsymbol{u}_2)$. Then $\tilde{\boldsymbol{u}}$ is locally near-simple and extends $\boldsymbol{u}_2$ (631M(c-iii)), so extends $\boldsymbol{u}$. If $\boldsymbol{u}$, and therefore $\boldsymbol{u}_2$, are non-negative and non-decreasing, so is $\tilde{\boldsymbol{u}}$, by 631M(c-iii) again. So we have an extension of the kind we need.

649G Lemma Suppose that $(\mathfrak{A}, \bar{\mu}, [0,\infty[\,, \langle\mathfrak{A}_t\rangle_{t\ge0}, \mathcal{T}, \langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ is a real-time integration structure and $\mathcal{S}$ is a non-empty sublattice of $\mathcal{T}$. There is a non-decreasing sequence $\langle\tau_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{S}$ such that $\sup_{n\in\mathbb{N}} [\![\tau \le \tau_n]\!] = 1$ for every $\tau \in \mathcal{S}$.

proof For $q \in \mathbb{Q} \cap [0,\infty[$, $[\![\sup \mathcal{S} > q]\!] = \sup_{\sigma\in\mathcal{S}} [\![\sigma > q]\!]$ (611Cb), so there is a countable set $A_q \subseteq \mathcal{S}$ such that $[\![\sup \mathcal{S} > q]\!] = \sup_{\sigma\in A_q} [\![\sigma > q]\!]$, because $\mathfrak{A}$ is ccc (322G, 316E). Accordingly there is a (non-empty) countable set $A \subseteq \mathcal{S}$ such that $[\![\sup \mathcal{S} > q]\!] = \sup_{\sigma\in A} [\![\sigma > q]\!]$ for every rational $q \ge 0$. At the same time, there is a countable $B \subseteq \mathcal{S}$ such that $\sup_{\sigma\in\mathcal{S}} [\![\sigma = \sup \mathcal{S}]\!] = \sup_{\sigma\in B} [\![\sigma = \sup \mathcal{S}]\!]$. Taking a sequence $\langle\sigma_n\rangle_{n\in\mathbb{N}}$ running over $A \cup B$, and setting $\tau_n = \sup_{i\le n} \sigma_i$ for $n \in \mathbb{N}$, $\langle\tau_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathcal{S}$ and

$$[\![\sup \mathcal{S} > q]\!] = \sup_{n\in\mathbb{N}} [\![\tau_n > q]\!] \text{ for every } q \in \mathbb{Q} \cap [0,\infty[,$$

$$\sup_{\sigma\in\mathcal{S}} [\![\sigma = \sup \mathcal{S}]\!] = \sup_{n\in\mathbb{N}} [\![\tau_n = \sup \mathcal{S}]\!].$$

Now take any $\tau \in \mathcal{S}$ and non-zero $a \in \mathfrak{A}$. If $a' = a \cap [\![\tau = \sup \mathcal{S}]\!]$ is non-zero, there is an $n \in \mathbb{N}$ such that $a' \cap [\![\tau_n = \sup \mathcal{S}]\!] \ne 0$, and now $a \cap [\![\tau_n = \tau]\!] \ne 0$. Otherwise, there is a $t \ge 0$ such that $a'' = a \cap [\![\sup \mathcal{S} > t]\!] \setminus [\![\tau > t]\!]$ is non-zero. Let $q > t$ be rational and such that $a'' \cap [\![\sup \mathcal{S} > q]\!] \ne 0$; then there is an $n \in \mathbb{N}$ such that

$$0 \neq a'' \cap [\![\tau_n > q]\!] \subseteq [\![\tau < \tau_n]\!]$$

and $a \cap [\![\tau \le \tau_n]\!] \neq 0$.

As a is arbitrary, $\sup_{n\in\mathbb{N}} [\![\tau \le \tau_n]\!] = 1$ and we have a suitable sequence $\langle\tau_n\rangle_{n\in\mathbb{N}}$.

649H Theorem Suppose that $(\mathfrak{A}, \bar\mu, [0,\infty[\,, \langle\mathfrak{A}_t\rangle_{t\ge0}, \mathcal{T}, \langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ is a right-continuous real-time stochastic integration structure.

(a)(i) There is a complete probability space (Ω,Σ,μ) such that $(\mathfrak{A},\bar\mu)$ can be identified with the measure algebra of (Ω,Σ,μ).

(ii) For $E\in\Sigma$, write $E^\bullet$ for the corresponding member of $\mathfrak{A}$; for $t\ge0$ set $\Sigma_t = \{E : E\in\Sigma,\ E^\bullet\in\mathfrak{A}_t\}$. Then $\langle\Sigma_t\rangle_{t\ge0}$ is a right-continuous filtration of σ-algebras all containing every negligible subset of Ω.

(iii) Members of $\mathcal{T}$ can be represented by stopping times $h:\Omega\to[0,\infty]$ as in 612H, with the corresponding identification of the algebras $\mathfrak{A}_\tau$ as in 612H(a-iii).

(b) Now suppose that $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a locally near-simple process with non-empty domain $\mathcal{S}\subseteq\mathcal{T}_f$. Then there are a progressively measurable stochastic process $\langle U_t\rangle_{t\ge0}$ and a non-decreasing sequence $\langle h_n\rangle_{n\in\mathbb{N}}$ of finite-valued stopping times, all adapted to $\langle\Sigma_t\rangle_{t\ge0}$, such that

(α) h_n represents a stopping time $\tau_n\in\mathcal{S}$ for every $n\in\mathbb{N}$, and $\sup_{n\in\mathbb{N}}[\![\sigma\le\tau_n]\!]=1$ for every $\sigma\in\mathcal{S}$,

(β) $U_g^\bullet = u_\sigma$ whenever $g:\Omega\to[0,\infty[$ is a stopping time representing $\sigma\in\mathcal{S}$,

(γ) $t\mapsto U_t(\omega):[0,h_n(\omega)]\to\mathbb{R}$ is càdlàg for every $\omega\in\Omega$ and $n\in\mathbb{N}$.

proof (a) Really this is just the Loomis-Sikorski theorem (321J); I remarked in 321K that the construction there always gives a complete measure space, and in this context, of course, it gives a probability space. To see that $\langle\Sigma_t\rangle_{t\ge0}$ is a right-continuous filtration we need only look at the definitions in 611Aa and 632B, and surely every negligible set belongs to every Σ_t. Now 612H tells us all we need to know.

I remark that because every Σ_t contains every negligible set, the same is true of Σ_h, as defined in 612H(a-iii), for every stopping time h.

(b)(i) For the time being (down to the end of (v)) I will suppose that $\mathcal{S}$ is an ideal of $\mathcal{T}_f$. By 649G, we have a non-decreasing sequence $\langle\tau_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{S}$, starting with $\tau_0=\min\mathcal{T}=\check0$, such that $\sup_{n\in\mathbb{N}}[\![\sigma\le\tau_n]\!]=1$ for every $\sigma\in\mathcal{S}$. Now each τ_n can be represented by a stopping time h_n; since $\langle h_n(\omega)\rangle_{n\in\mathbb{N}}$ must be a non-decreasing sequence in $[0,\infty[$ for almost every ω, and negligible sets all belong to Σ_0, we can adjust the h_n, if necessary, to arrange that $\langle h_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of finite stopping times.

(ii) Set $S^*=\bigcup_{k\ge1}\mathbb{N}^k$. Choose $\langle\tau_r\rangle_{r\in S^*}$, $\langle h_r\rangle_{r\in S^*}$ and $\langle Y_r\rangle_{r\in S^*}$ inductively, as follows. Start with $\tau_{<n>}=\tau_n$ and $h_{<n>}=h_n$, for $n\in\mathbb{N}$, where I write $<n>$ for the member of $\mathbb{N}^1$ with value n. Now choose a Σ_{h_n}-measurable real-valued $Y_{<n>}$ such that $Y_{<n>}^\bullet=u_{\tau_n}$; do this in such a way that $Y_{<n+1>}(\omega)=Y_{<n>}(\omega)$ whenever $h_{n+1}(\omega)=h_n(\omega)$.

Given $r\in\mathbb{N}^k$ let r' be its successor in the lexicographic ordering of $\mathbb{N}^k$, so that $r'\restriction k-2=r\restriction k-2$ and $r'(k-1)=r(k-1)+1$; let $\langle\tau_{r^\frown<n>}\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $[\tau_r,\tau_{r'}]$ such that

$$\tau_{r^\frown<0>}=\tau_r,\quad \sup_{n\in\mathbb{N}}[\![\tau_{r^\frown<n>}=\tau_{r'}]\!]=1,$$

$$[\![\sigma<\tau_{r^\frown<n+1>}]\!]\subseteq[\![|u_\sigma-u_{\tau_{r^\frown<n+1>}}|\le2^{-n}]\!] \text{ for every } \sigma\in[\tau_{r^\frown<n>},\tau_{r^\frown<n+1>}]$$

for each $n\in\mathbb{N}$ (631Ra). Now choose a sequence $\langle h_{r^\frown<n>}\rangle_{n\in\mathbb{N}}$ of stopping times such that $h_{r^\frown<n>}$ represents $\tau_{r^\frown<n>}$ for each n; as we shall necessarily have

$$h_{r^\frown<0>}=_{\text{a.e.}}h_r,\quad \sup_{n\in\mathbb{N}}h_{r^\frown<n>}=_{\text{a.e.}}h_{r'},$$

$$h_{r^\frown<n>}\le_{\text{a.e.}}h_{r^\frown<n+1>} \text{ for each } n,$$

we can adjust the functions on a negligible set so that

$$h_{r^\frown<>}=h_r,\quad \sup_{n\in\mathbb{N}}h_{r^\frown<n>}=h_{r'},$$

$$h_{r^\frown<n>}\le h_{r^\frown<n+1>} \text{ for each } n.$$

Finally, choose $\langle Y_{r^\frown<n>}\rangle_{n\in\mathbb{N}}$ such that $Y_{r^\frown<0>}=Y_r$ and, for each n,

$Y_{r^\frown<n>}:\Omega\to\mathbb{R}$ is a $\Sigma_{h_{r^\frown<n>}}$-measurable function and $Y^\bullet_{r^\frown<n>}=u_{\tau_{r^\frown<n>}}$,

$Y_{r^\frown<n+1>}(\omega) = Y_{r^\frown<n>}(\omega)$ whenever $h_{r^\frown<n+1>}(\omega) = h_{r^\frown<n>}(\omega)$,
$Y_{r^\frown<n>}(\omega) = Y_{r'}(\omega)$ whenever $h_{r^\frown<n>}(\omega) = h_{r'}(\omega)$,
$|Y_{r^\frown<n>}(\omega) - Y_r(\omega)| \le 2^{-k}$ whenever $h_{r^\frown<n>}(\omega) < h_{r'}(\omega)$.

(The point here is that $\mathfrak{A}_{\tau_{r^\frown<n>}}$ is always $\{E^\bullet : E \in \Sigma_{r^\frown<n>}$, by 612H(a-iii), so that we can manage the first clause in this list, and the rest can be achieved by adjustments on negligible sets.)

Note that the construction ensures that if r_0, $r_1 \in S^*$ and $h_{r_0}(\omega) = h_{r_1}(\omega)$, we shall have $Y_{r_0}(\omega) = Y_{r_1}(\omega)$.

(iii) At the end of the induction, define $U_t(\omega)$, for $t \ge 0$ and $\omega \in \Omega$, as follows. If $h_r(\omega) < t$ for every $r \in S^*$, set $U_t(\omega) = 0$. Otherwise, we have a sequence $\gamma \in \mathbb{N}^\mathbb{N}$ such that $h_{\gamma\restriction k}(\omega) \le t < h_{(\gamma\restriction k)'}(\omega)$ for every $k \ge 1$. In this case, $|Y_{\gamma\restriction k+1}(\omega) - Y_{\gamma\restriction k}(\omega)| \le 2^{-k}$ for every k, so $\lim_{k\to\infty} Y_{\gamma\restriction k}(\omega)$ is defined; take this for $U_t(\omega)$. Note that this will ensure that $|U_t(\omega) - Y_r(\omega)| \le 2^{-k-1}$ whenever $k \ge 1$, $r \in \mathbb{N}^k$ and $h_r(\omega) \le t < h_{r'}(\omega)$.

Now $(t,\omega) \mapsto U_t(\omega)$ is progressively measurable. **P** Take $t_0 \in [0,\infty[$ and $\alpha \in \mathbb{R}$, and set $H = \{(t,\omega) : t \le t_0,\, \omega \in \Omega,\, U_t(\omega) > \alpha\}$. Then

$$H = H_0 \cup \bigcup_{k\ge1} \bigcup_{r\in\mathbb{N}^k} \{(t,\omega) : t \le t_0,\, h_r(\omega) \le t < h_{r'}(\omega),\, Y_r(\omega) > t + 2^{-k-1}\}$$

where $H_0 = \{(t,\omega) : h_r(\omega) < t \le t_0\}$ if $\alpha < 0$, else $\emptyset$. Now if $r \in S^*$ and we set $E_r = \{\omega : h_r(\omega) \le t_0\}$, $E_r \in \Sigma_{t_0}$ and $h_r\restriction E_r$ is Σ_{t_0}-measurable, so $\{(t,\omega) : h_r(\omega) \le t \le t_0\}$ and $\{(t,\omega) : h_r(\omega) < t \le t_0\}$ belong to $\mathcal{B}\widehat{\otimes}\Sigma_{t_0}$, where $\mathcal{B}$ is the Borel σ-algebra of $\mathbb{R}$. It follows that $H_0 \in \mathcal{B}\widehat{\otimes}\Sigma_{t_0}$. Next, given $r \in S^*$, Y_r is Σ_{h_r}-measurable, so $Y_r\restriction E_r$ is Σ_{t_0}-measurable, since $E \cap E_r \in \Sigma_{t_0}$ for every $E \in \Sigma_{h_r}$. Next, if $k \in \mathbb{N}$ and $r \in \mathbb{N}^k$

$$\{(t,\omega) : h_r(\omega) \le t \le t_0,\, Y_r(\omega) > t + 2^{-k-1}\}$$

belongs to $\mathcal{B}\widehat{\otimes}\Sigma_{h_r}$ and is included in $[0,t_0] \times E_r$, it belongs to $\mathcal{B}\widehat{\otimes}\Sigma_{t_0}$; while

$$\{(t,\omega) : h_{r'}(\omega) < t \le t_0$$

also belongs to $\mathcal{B}\widehat{\otimes}\Sigma_{t_0}$, so the difference

$$\{(t,\omega) : t \le t_0,\, h_r(\omega) \le t < h_{r'}(\omega),\, Y_r(\omega) > t + 2^{-k-1}\}$$

belongs to Σ_{t_0}. Taking the union of these, $H \in \Sigma_{t_0}$. As t_0 and α are arbitrary, $(t,\omega) \mapsto U_t(\omega)$ is progressively measurable. **Q**

(iv) Accordingly we have a fully adapted process $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{T}_f}$ defined by saying that $v_\sigma = U_g^\bullet$ whenever $g : \Omega \to [0,\infty[$ is a stopping time representing $\sigma \in \mathcal{T}_f$. Since $U_{h_r(\omega)} = Y_r(\omega)$ whenever $r \in S^*$ and $\omega \in \Omega$, $U_{h_r} = Y_r$ and $v_{\tau_r} = u_{\tau_r}$ for every $r \in S^*$.

We know also that if $k \ge 1$, $r \in \mathbb{N}^k$ and g is a finite-valued stopping time representing $\sigma \in \mathcal{T}$, then

$$|U_{g(\omega)}(\omega) - Y_r(\omega)| \le 2^{-k-1} \text{ whenever } h_r(\omega) \le g(\omega) < h_{r'}(\omega).$$

Translating this into terms of $\mathcal{T}$ and $\boldsymbol{v}$,

$$[\![\tau_r \le \sigma]\!] \cap [\![\sigma < \tau_{r'}]\!] \subseteq [\![|v_\sigma - u_{\tau_r}| \le 2^{-k-1}]\!]$$

Since, by the choice of the τ_r, we also have

$$[\![\tau_r \le \sigma]\!] \cap [\![\sigma < \tau_{r'}]\!] \subseteq [\![|u_\sigma - u_{\tau_r}| \le 2^{-k}]\!],$$

we get

$$[\![\tau_r \le \sigma]\!] \cap [\![\sigma < \tau_{r'}]\!] \subseteq [\![|v_\sigma - u_\sigma| \le 2^{-k-2}]\!].$$

If $\sigma \in \mathcal{S}$, so that $\sup_{n\in\mathbb{N}} [\![\sigma \le \tau_n]\!] = 1$, then

$$1 = \sup_{r\in\mathbb{N}^k} [\![\sigma = \tau_r]\!] \cup \sup_{r\in\mathbb{N}^k}([\![\tau_r < \sigma]\!] \cap [\![\sigma < \tau_{r'}]\!]) \subseteq [\![|v_\sigma = u_\sigma| \le 2^{-k-2}]\!]$$

for every $k \in \mathbb{N}$, so that $v_\sigma = u_\sigma$.

Thus $\boldsymbol{v}$ extends $\boldsymbol{u}$.

(v) If $n \in \mathbb{N}$ and $\omega \in \Omega$ then $t \mapsto U_t(\omega) : [0,h_n(\omega)] \to \mathbb{R}$ is càdlàg.

P$(\boldsymbol{\alpha})$ Suppose that $\langle t_i\rangle_{i\in\mathbb{N}}$ is a strictly decreasing sequence in $[0,h_n(\omega)]$. Set $t = \lim_{i\to\infty} t_i$. For each $k \ge 1$, there is an $r \in \mathbb{N}^k$ such that $h_r(\omega) \le t < h_{r'}(\omega)$. But in this case there is a $j \in \mathbb{N}$ such that $h_r(\omega) \le$

$t_i < h_{r'}(\omega)$ for every $i \ge j$, so that $|U_{t_i}(\omega) - Y_r(\omega)| \le 2^{-k-1}$ for every $i \ge j$ and $|U_t(\omega) - Y_r(\omega)| \le 2^{-k-1}$. As k is arbitrary, $U_t(\omega) = \lim_{i\to\infty} U_{t_i}(\omega)$.

(β) If now $\langle t_i\rangle_{i\in\mathbb{N}}$ is a strictly increasing sequence in $[0, h_n(\omega)]$, then by the choice of $\langle h_r\rangle_{r\in S^*}$ we find that

for every $k \ge 1$ there are an $r \in \mathbb{N}^k$ and a $j \in \mathbb{N}$ such that $h_r(\omega) \le t_i < h_{r'}(\omega)$ for every $i \ge j$.

But this means that for every $k \ge 1$ there are an $r \in \mathbb{N}^k$ and a $j \in \mathbb{N}$ such that such that $|U_{t_i}(\omega) - Y_r(\omega)| \le 2^{-k-1}$ for every $i \ge j$; it follows at once that $\lim_{i\to\infty} U_{t_i}(\omega)$ is defined. Putting these together, $t \mapsto U_t(\omega) : [0, h_n(\omega)] \to \mathbb{R}$ is càdlàg. **Q**

(vi) This proves the result when $\mathcal{S}$ is an ideal. In general, we still have a non-decreasing sequence $\langle\tau_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{S}$ such that $\sup_{n\in\mathbb{N}} [\![\sigma \le \tau_n]\!] = 1$ for every $\sigma \in \mathcal{S}$. Now $\tilde{\mathcal{S}} = \{\sigma : \sup_{n\in\mathbb{N}} [\![\sigma \le \tau_n]\!] = 1\}$ is a full ideal of $\mathcal{T}$ included in $\mathcal{T}_f$, so is the full ideal generated by $\mathcal{S}$, and we have a locally near-simple process $\tilde{\boldsymbol{u}}$ with domain $\tilde{\mathcal{S}}$ extending $\boldsymbol{u}$ (649F). Applying (i)-(v) to $\tilde{\boldsymbol{u}}$, we get a suitable process $\langle U_t\rangle_{t\ge 0}$.

649I Scholium If, in 649Hb, $\boldsymbol{u}$ is a non-negative non-decreasing process, then we can arrange that $\langle U_t\rangle_{t\ge0}$ is non-decreasing. **P** Working through the proof, we see that in part (b-ii), when we come to choose $\langle Y_r\rangle_{r\in S^*}$, we have $0 \le u_{\tau_n} \le u_{\tau_{n+1}}$, so can require that $0 \le Y_{<n>} \le Y_{<n+1>}$, for each n. Similarly, when we come to choose $\langle Y_{r^\frown<n>}\rangle_{n\in\mathbb{N}}$, we shall have $Y_r \le Y_{r'}$ and

$$u_{\tau_r} \le u_{\tau_{r^\frown<n>}} \le u_{\tau_{r^\frown<n+1>}} \le u_{\tau_{r'}}$$

for each n, so we can arrange that

$$Y_r \le Y_{r^\frown<n>} \le Y_{r^\frown<n+1>} \le Y_{r'}$$

for each n. Looking at the definition of U_t in (b-iii), we see that we shall now necessarily get a non-negative non-decreasing process.

Of course I ought to check also that we can still use the idea in (b-vi). But I noted in 649F that if we start from a non-negative non-decreasing process $\boldsymbol{u}$, then the extension $\tilde{\boldsymbol{u}}$ can be made to be non-negative and non-decreasing. So this part of the argument also works. **Q**

649J Lemma Let (Ω, Σ, μ) be a complete probability space, $\langle\Sigma_t\rangle_{t\ge0}$ a right-continuous filtration of σ-subalgebras of Σ all containing every negligible set, and $\langle U_t\rangle_{t\ge0}$ a progressively measurable stochastic process. Let $h : \Omega \to [0, \infty[$ be a stopping time such that $t \mapsto U_t(\omega) : [0, h(\omega)] \to \mathbb{R}$ is càdlàg for every $\omega \in \Omega$. For $\omega \in \Omega$ and $t \ge 0$ set

$$\begin{aligned} U_{<t}(\omega) &= \lim_{s\uparrow t} U_s(\omega) \text{ if } 0 < t \le h(\omega), \\ &= 0 \text{ otherwise.} \end{aligned}$$

(a) $\langle U_{<t}\rangle_{t\ge0}$ is a previsibly measurable stochastic process, therefore progressively measurable.

(b) Let $(\mathfrak{A}, \bar\mu, [0,\infty[, \langle\mathfrak{A}_t\rangle_{t\ge0}, \mathcal{T}, \langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ be the real-time stochastic integration structure defined from $(\Omega, \Sigma, \mu, \langle\Sigma_t\rangle_{t\ge0})$, and $\tau \in \mathcal{T}_f$ the stopping time represented by h (612H(a-i)). If $\boldsymbol{u}$, $\boldsymbol{z}$ are the fully adapted processes defined from U and $U_<$ as in 612Hb, then $\boldsymbol{u}{\restriction}\mathcal{T}\wedge\tau$ is near-simple and its previsible version is $\boldsymbol{z}{\restriction}\mathcal{T}\wedge\tau$.

(c) Now suppose that $\langle V_t\rangle_{t\ge0}$ is another progressively measurable stochastic process, this time non-negative and non-decreasing, such that $t \mapsto V_t(\omega) : [0, h(\omega)] \to \mathbb{R}$ is càdlàg for every $\omega \in \Omega$. Let $\boldsymbol{v}$ be the process defined by $\langle V_t\rangle_{t\ge0}$. For $\omega \in \Omega$ let ν_ω be the Radon measure on $[0, h(\omega)]$ such that $\nu_\omega[0,t] = V_t(\omega)$ for every $t \ge 0$, and set

$$e(\omega) = \int_{[0,h(\omega)]} U_{<t}(\omega)\nu_\omega(dt).$$

Then $e : \Omega \to \mathbb{R}$ is Σ_h-measurable and $e^\bullet = \int_{\mathcal{T}\wedge\tau} \boldsymbol{u}\, d\boldsymbol{v}$.

proof (a) If $\alpha \in \mathbb{R}$, then

$$\begin{aligned}\{(t,\omega):U_{<t}(\omega)>\alpha\}\\ &= H_0\cup\bigcup_{\substack{q\in\mathbb{Q}\\ q>\alpha}}\bigcup_{q'\in\mathbb{Q}}\bigcap_{\substack{q''\in\mathbb{Q}\\ q''\ge q'}}\\ &\qquad\{(t,\omega):q'<t\le h(\omega),\ \text{ either } t\le q'' \text{ or } U_{q''}(\omega)\ge q\}\end{aligned} \tag{$*$}$$

where

$$\begin{aligned}H_0&=(\{0\}\times\Omega)\cup\{(t,\omega):h(\omega)<t\} \text{ if } \alpha<0, &&(**)\\ &=\emptyset \text{ if } \alpha\ge 0.\end{aligned}$$

Now as $U_{q''}$ is always $\Sigma_{q''}$-measurable, the sets $\{(t,\omega):q'<t,\,U_{q''}(\omega)\ge q\}$ always belong to the previsible σ-algebra Λ_{pv} (642Ha). Also

$$\{(t,\omega):h(\omega)<t\}=\bigcup_{q\in\mathbb{Q}}[q,\infty[\,\times\{\omega:h(\omega)\le q\}$$

belongs to Λ_{pv}, so $\{(t,\omega):t\le h(\omega)\}$ also does, and of course $[0,\infty[\,\times\Omega$ and $\{0\}\times\Omega$ also do. So all the elements of the formulae (*) and (**) correspond to sets in Λ_{pv} and $\{(t,\omega):U_{<t}(\omega)>\alpha\}\in\Lambda_{\mathrm{pv}}$. As α is arbitrary, $\langle U_{<t}\rangle_{t\ge0}$ is previsibly measurable, therefore progressively measurable (642I).

(b)(i) $\boldsymbol{u}\restriction\mathcal{T}\wedge\tau$ is near-simple. **P** This can be proved by a simple adaptation of the argument in 631D; or alternatively, we can set

$$\begin{aligned}U'_t(\omega)&=U_t(\omega) \text{ if } t<h(\omega),\\ &=U_{h(\omega)}(\omega) \text{ if } t\ge h(\omega).\end{aligned}$$

Then, for any $t\ge0$ and $\alpha\in\mathbb{R}$,

$$\{\omega:U'_t(\omega)>\alpha\}=\{\omega:h(\omega)>t,\,U_t(\omega)>\alpha\}\cup\{\omega:h(\omega)\le t,\,U_{h(\omega)}(t)>\alpha\}$$

belongs to Σ_t. Evidently $t\mapsto U'_t(\omega)$ is càdlàg for every ω. We can therefore apply 631D as written to show that the process $\boldsymbol{u}'$ defined from $\langle U'_t\rangle_{t\ge0}$ is locally near-simple, so that $\boldsymbol{u}\restriction\mathcal{T}\wedge\tau=\boldsymbol{u}'\restriction\mathcal{T}\wedge\tau$ is near-simple. **Q**

(ii) Let $\epsilon>0$. Then there is a simple process $\boldsymbol{w}=\langle w_\sigma\rangle_{\sigma\in\mathcal{T}\wedge\tau}$ such that $\theta(\bar u)\le\epsilon$ where $\bar u=\sup|\boldsymbol{w}-\boldsymbol{u}\restriction\mathcal{T}\wedge\tau|)$. Take a breakpoint sequence $(\sigma_0,\dots,\sigma_n)$ for $\boldsymbol{w}$ such that $\check 0=\sigma_0\le\ldots\le\sigma_n=\tau$, and choose stopping times $g_0,\dots,g_n$ such that $0=g_0\le\ldots\le g_n=h$ and g_i represents σ_i for each $i\le n$; for each i, let f_i be a Σ_{g_i}-measurable function such that $f_i^\bullet=w_{\sigma_i}$. Set

$$\begin{aligned}W_t(\omega)&=f_i(\omega) \text{ if } i<n \text{ and } g_i(\omega)\le t<g_{i+1}(\omega),\\ &=f_n(\omega) \text{ if } g_n(\omega)\le t.\end{aligned}$$

Then $\langle W_t\rangle_{t\ge0}$ is a progressively measurable stochastic process representing $\boldsymbol{w}$, and we have

$$\begin{aligned}W_{<t}(\omega)&=0 \text{ if } t=0,\\ &=f_i(\omega) \text{ if } i<n \text{ and } g_i(\omega)<t\le g_{i+1}(\omega),\\ &=g_n(\omega) \text{ if } g_n(\omega)<t.\end{aligned}$$

Like $\langle U_{<t}\rangle_{t\ge0}$, this is a previsibly measurable process; let $\boldsymbol{w}'$ be the process it represents. Then $\boldsymbol{w}'\restriction\mathcal{T}\wedge\tau=\boldsymbol{w}_<$, by 641Ia.

Now consider the previsibly measurable process $\langle U_{<t}-W_{<t}\rangle_{t\ge0}$. If $g:\Omega\to[0,\infty[$ is a stopping time and $g\le h$, and we write $U_{<g}(\omega)$, $W_{<g}(\omega)$ for $U_{<g(\omega)}(\omega)$ and $W_{<g(\omega)}(\omega)$, then

$$\begin{aligned}U_{<g}(\omega)-W_{<g}(\omega)&=\lim_{s\uparrow g(\omega)}U_s(\omega)-W_s(\omega) \text{ if } g(\omega)>0,\\ &=0 \text{ otherwise.}\end{aligned}$$

If $\bar{f} : \Omega \to [0, \infty[$ is such that $\bar{f}^\bullet = \bar{u}$, then

$$\{\omega : |U_{<g}(\omega) - W_{<g}(\omega)| > \bar{f}(\omega)\} \subseteq \bigcup_{q \in \mathbb{Q}, q \geq 0} \{\omega : |U_{q \wedge h}(\omega) - W_{q \wedge h}(\omega)| > \bar{f}(\omega)\}$$

where I write $q \wedge h$ for the stopping time $\omega \mapsto \min(q, h(\omega))$, so is negligible, and $|U^\bullet_{<g} - W^\bullet_{<g}| \leq \bar{u}$. Thus

$$\sup |\boldsymbol{z} - \boldsymbol{w}_<| = \sup |\boldsymbol{z} - \boldsymbol{w}' {\restriction} \mathcal{T} \wedge \tau| \leq \bar{u}.$$

On the other hand,

$$\sup |\boldsymbol{w}_< - (\boldsymbol{u} {\restriction} \mathcal{T} \wedge \tau)_<| \leq \sup |\boldsymbol{w} - \boldsymbol{u} {\restriction} \mathcal{T} \wedge \tau| \leq \bar{u}$$

by 641G(a-vii). So

$$\theta(\sup |\boldsymbol{z} - (\boldsymbol{u} {\restriction} \mathcal{T} \wedge \tau)_<|) \leq \theta(2\bar{u}) \leq 2\epsilon;$$

as ϵ is arbitrary, $\boldsymbol{z} = (\boldsymbol{u} {\restriction} \mathcal{T} \wedge \tau)_<$.

(c)(i) Of course ν_ω is the Stieltjes measure associated with the non-decreasing function $t \mapsto V_t(\omega)$. The description in 114Xa is fully adequate for our needs here, and you should have no difficulty in filling in the arguments sketched there. If you are willing to use the full resources of the numbered theorems in previous volumes, however, and the fact that the function is càdlàg, the quickest route to the present case may be to apply 416K to the finitely additive functional ν'_ω defined on the ring T of subsets of $[0, h(\omega)]$ generated by the half-open intervals of the form $[0, \alpha[$ by saying that

$$\nu'_\omega\, [0, \alpha[= \lim_{t \uparrow \alpha} V_t(\omega)$$

if $0 < \alpha \leq h(\omega)$

Because $t \mapsto U_{<t}(\omega)$ is bounded and Borel measurable on $[0, h(\omega)]$, the integral $e(\omega)$ is always defined. Let $f : \Omega \to \mathbb{R}$ be a Σ-measurable function such that $f^\bullet = \int_{\mathcal{S} \wedge \tau} \boldsymbol{u}\, d\boldsymbol{v}$.

(ii) Much as in (b-ii), take $\epsilon > 0$ and a simple process $\boldsymbol{w} = \langle w_\sigma \rangle_{\sigma \in \mathcal{T} \wedge \tau}$ such that $\theta(\bar{u}) \leq \epsilon^2$ where $\bar{u} = \sup |\boldsymbol{w} - \boldsymbol{u} {\restriction} \mathcal{T} \wedge \tau|$. Let $M > 0$ be such that $F = \{\omega : V_h(\omega) > M\}$ has measure at most ϵ, and take a finite sublattice I of $\mathcal{T} \wedge \tau$ such that $\theta(S_J(\boldsymbol{u}, d\boldsymbol{v}) - \int_{\mathcal{S} \wedge \tau} \boldsymbol{u}\, d\boldsymbol{v}) \leq \frac{\epsilon^2}{M}$ whenever $I \subseteq J \in \mathcal{I}(\mathcal{T} \wedge \tau)$. Let $J \supseteq I \cup \{\check{0}, \tau\}$ be a finite sublattice of $\mathcal{T} \wedge \tau$ which includes a breakpoint string for $\boldsymbol{w}$; let $(\sigma_0, \dots, \sigma_n)$ linearly generate the J-cells, so that $\check{0} = \sigma_0 \leq \dots \leq \sigma_n = \tau$, $(\sigma_0, \dots, \sigma_n)$ is a breakpoint sequence for $\boldsymbol{w}$ (612Kb), and $S_J(\boldsymbol{u}, d\boldsymbol{v}) = \sum_{i=1}^n u_{\sigma_{i-1}} \times (v_{\sigma_i} - v_{\sigma_{i-1}})$. Now choose $g_0, \dots, g_n, f_0, \dots, f_n$ and define $\langle W_{<t} \rangle_{t \geq 0}$ as in (b-ii).

This time, we calculate

$$\hat{f}(\omega) = \int_{[0, h(\omega)]} W_{<t}(\omega) \nu_\omega(dt) = \sum_{i=1}^n f_i(\omega) \nu_\omega(]g_{i-1}(\omega), g_i(\omega)])$$

(because $f_0(\omega) = 0$)

$$= \sum_{i=1}^n f_i(\omega)(V_{g_i}(\omega) - V_{g_{i-1}}(\omega))$$

for each ω. So $\hat{f}^\bullet = S_J(\boldsymbol{u}\, d\boldsymbol{v})$ and $\int_\Omega \min(1, |f(\omega) - \hat{f}(\omega)|) \mu(d\omega) \leq \epsilon^2$; accordingly $F' = \{\omega : |f(\omega) - \hat{f}(\omega)| \geq \epsilon\}$ has measure at most ϵ.

At the same time, again taking $\bar{f} : \Omega \to [0, \infty[$ such that $\bar{f}^\bullet = \bar{u}$, the set

$$E = \{\omega : |U_{<t}(\omega) - W_{<t}(\omega)| \leq \bar{f}(\omega) \text{ for every } t \in [0, h(\omega)]$$

is conegligible. **P** Set

$$E' = \{\omega : |U_q(\omega) - f_i(\omega)| \leq \bar{f}(\omega) \text{ whenever } i < n \text{ and } q \in \mathbb{Q} \cap [g_i(\omega), g_{i+1}(\omega)[$$
$$\supseteq \{\omega : |U_{q \wedge h}(\omega) - U'_{q \wedge h}(\omega)| \leq \bar{f}(\omega) \text{ for every } q \in \mathbb{Q} \cap [0, \infty[\}.$$

Because $|u_{\check{q} \wedge \tau} - w_{\check{q} \wedge \tau}| \leq \bar{u}$ for every $q \geq 0$, E' is conegligible. If $\omega \in E'$ and $t \in [0, h(\omega)]$ then

$$|U_{<t}(\omega) - W_{<t}(\omega)| = 0 \text{ if } t = 0,$$
$$= \lim_{q\in\mathbb{Q},q\uparrow t} |U_q(\omega) - f_i(\omega)| \text{ if } i < n \text{ and } g_i(\omega) < t \le g_{i+1}(\omega),$$

and in either case is at most $\bar f(\omega)$. So $E \supseteq E'$ is conegligible. **Q**

We also know that $\int_\Omega \min(1, \bar f(\omega))\mu(d\omega) \le \frac{\epsilon^2}{M}$, so that $F'' = \{\omega : \bar f(\omega) > \frac{\epsilon}{M}\}$ has measure at most ϵ. Now take any $\omega \in E \setminus (F \cup F' \cup F'')$. Then $|U_{<t}(\omega) - W_{<t}(\omega)| \le \bar f(\omega)$ for every $t \in [0, h(\omega)]$, so

$$\begin{aligned}|e(\omega) - f(\omega)| &\le |e(\omega) - \hat f(\omega)| + |\hat f(\omega) - f(\omega)| \le \bar f(\omega)\nu_\omega([0, h(\omega)]) + \epsilon\\ &\le M\bar f(\omega) + \epsilon \le 2\epsilon,\end{aligned}$$

while $F \cup F' \cup F''$ has measure at most 3ϵ. As ϵ is arbitrary, $e =_{\text{a.e.}} f$ and $e^\bullet = f^\bullet = \int_{\mathcal{T}\wedge\tau} \boldsymbol{u}\, d\boldsymbol{v}$.

649K Lemma Let (Ω, Σ, μ) be a probability space and $\langle\Sigma_t\rangle_{t\ge0}$ a filtration of σ-subalgebras of Σ. Let Λ_{pv} be the corresponding previsible σ-algebra and write $\mathcal{L}$ for the smallest subset of $\mathbb{R}^{[0,\infty[\times\Omega}$ such that

constant functions belong to $\mathcal{L}$,

scalar multiples of functions in $\mathcal{L}$ belong to $\mathcal{L}$,

if $\phi \in \mathcal{L}$ and $\psi \in \mathbb{R}^{[0,\infty[\times\Omega}$ and $|\phi| \wedge |\psi| = 0$, then $\phi + \psi \in \mathcal{L}$ iff $\psi \in \mathcal{L}$,

$\chi(]s, \infty[\times E) \in \mathcal{L}$ whenever $s \ge 0$ and $E \in \Sigma_s$,

$\lim_{n\to\infty} \phi_n \in \mathcal{L}$ whenever $\langle\phi_n\rangle_{n\in\mathbb{N}}$ is a pointwise convergent sequence in $\mathcal{L}$.

Then $\mathcal{L}$ is the set of all Λ_{pv}-measurable real-valued functions on $[0, \infty[\times \Omega$.

proof (a) Writing $\mathcal{L}^0(\Lambda_{\text{pv}})$ for the set of Λ_{pv}-measurable functions, this is a linear space containing $]s, \infty[\times E$ whenever $s \ge 0$ and $E \in \Sigma_s$ and is closed under sequential pointwise convergence, so includes $\mathcal{L}$.

(b) In the other direction, write Λ for $\{W : W \subseteq [0, \infty[\times \Omega, \chi W \in \mathcal{L}\}$. Then Λ is a Dynkin class (136A). Also $\mathcal{I} = \{]s, \infty[\times E : s \ge 0, E \in \Sigma_s\}$ is a subset of Λ closed under finite intersections (if $E \in \Sigma_s$ and $F \in \Sigma_t$ then $E \cap F \in \Sigma_{\max(s,t)}$ and $(]s, \infty[\times E) \cap (]t, \infty[\times F) =]\max(s, t), \infty[\times (E \cap F))$, so Λ includes the σ-algebra generated by $\mathcal{I}$ (136B), which is Λ_{pv}. Thus $\chi W \in \mathcal{L}$ for every $W \in \Lambda_{\text{pv}}$.

It follows at once that $\sum_{i=0}^n \alpha_i\chi W_i \in \mathcal{L}$ whenever $\alpha_0, \dots, \alpha_n \in \mathbb{R}$ and $W_0, \dots, W_n \in \Lambda_{\text{pv}}$, therefore that $f \in \mathcal{L}$ whenever $f : \Omega \to [0, \infty[$ is Λ_{pv}-measurable, and finally that $\mathcal{L}^0(\Lambda_{\text{pv}}) \subseteq \mathcal{L}$. So we have equality.

649L Theorem Let (Ω, Σ, μ) be a complete probability space, $\langle\Sigma_t\rangle_{t\ge0}$ a right-continuous filtration of σ-subalgebras of Σ all containing every negligible set, $(\mathfrak{A}, \bar\mu, [0, \infty[\,, \langle\mathfrak{A}_t\rangle_{t\ge0}, \mathcal{T}, \langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ the corresponding real-time stochastic integration structure, $\langle X_t\rangle_{t\ge0}$ a previsibly measurable stochastic process and $\langle V_t\rangle_{t\ge0}$ a non-negative non-decreasing stochastic process. Let $h : \Omega \to [0, \infty[$ be a stopping time such that $t \mapsto X_t(\omega)$ is bounded on $[0, h(\omega)]$ and $t \mapsto V_t(\omega) : [0, h(\omega)] \to \mathbb{R}$ is càdlàg for every $\omega \in \Omega$, and write τ for the corresponding stopping time in $\mathcal{T}$. Let $\boldsymbol{x}$, $\boldsymbol{v}$ be the processes defined by $\langle X_t\rangle_{t\ge0}$ and $\langle V_t\rangle_{t\ge0}$. For $\omega \in \Omega$ let ν_ω be the Radon measure on $[0, h(\omega)]$ such that $\nu_\omega[0, t] = V_t(\omega)$ for every $t \ge 0$, and set

$$e(\omega) = \int_{]0,h(\omega)]} X_t(\omega)\nu_\omega(dt).$$

Then $e : \Omega \to \mathbb{R}$ is Σ-measurable and $e^\bullet = \oint_{\mathcal{T}\wedge\tau} \boldsymbol{x}\, d\boldsymbol{v}$.

proof (a) Let Λ_{pv} be the previsible σ-algebra derived from $\langle\Sigma_t\rangle_{t\ge0}$, and $\mathcal{L}^0 = \mathcal{L}^0(\Lambda_{\text{pv}})$ the space of Λ_{pv}-measurable real-valued functions on $[0, \infty[\times \Omega$. By the definition in 642Hb, we have a one-to-one correspondence between $\mathcal{L}^0$ and the space of previsibly measurable processes, matching $\phi \in \mathcal{L}^0$ with $\langle X_t\rangle_{t\ge0}$ where $X_t(\omega) = \phi(t, \omega)$ for all t and ω; and in this case $\langle X_t\rangle_{t\ge0}$ corresponds to the process $\boldsymbol{x}_\phi$ as defined in 642L. Recall from 642L that $\phi \mapsto \boldsymbol{x}_\phi$ is an f-algebra homomorphism taking pointwise convergent sequences to order*-convergent sequences.

(b) For $M \ge 0$, $\phi \in \mathcal{L}^0$ and $\omega \in \Omega$, set

$$e_{\phi M}(\omega) = \int_{]0,h(\omega)]} \operatorname{med}(-M, \phi(t, \omega), M)\nu_\omega(dt).$$

(This is always defined because $t \mapsto \phi(t, \omega)$ is always Borel measurable.) Let $\mathcal{L}$ be the set of those $\phi \in \mathcal{L}^0$ such that $e_{\phi M}$ is measurable and

$$e^\bullet_{\phi M} = \oint_{\mathcal{T}\wedge\tau} \operatorname{med}(-M\mathbf{1}_<, \boldsymbol{x}_\phi, M\mathbf{1}_<)d\boldsymbol{v}$$

for every $M \geq 0$. Then $\mathcal{L}$ satisfies the conditions of 649K.

P(i) If ϕ is constant with value α, then

$$e_{\phi M}(\omega) = \operatorname{med}(-M, \alpha, M)(V_{h(\omega)}(\omega) - V_0(\omega))$$

for every ω, so

$$e^\bullet_{\phi M} = \operatorname{med}(-M, \alpha, M)(V^\bullet_h - V^\bullet_0) = \operatorname{med}(-M, \alpha, M)v_\tau - v_{\check{0}},$$

while

$$\begin{aligned}\oint_{\mathcal{T}\wedge\tau} \operatorname{med}(-M\mathbf{1}_<, \boldsymbol{x}_\phi, M\mathbf{1}_<)d\boldsymbol{v} &= \oint_{\mathcal{T}\wedge\tau} \operatorname{med}(-M, \alpha, M)\mathbf{1}_<\,d\boldsymbol{v}\\ &= \int_{\mathcal{T}\wedge\tau} \operatorname{med}(-M, \alpha, M)\mathbf{1}\,d\boldsymbol{v}\\ &= \operatorname{med}(-M, \alpha, M)(v_\tau - v_{\check{0}}).\end{aligned}$$

So these are equal, for every M, and $\phi \in \mathcal{L}$.

(ii) Of course a scalar multiple of a member of $\mathcal{L}$ belongs to $\mathcal{L}$ because $\phi \mapsto e_\phi$ and $\mathrm{S}ii_{\boldsymbol{v}}$ are linear. Similarly, if $|\phi| \wedge |\psi| = 0$ then $|\boldsymbol{x}_\phi| \wedge |\boldsymbol{x}_\psi| = 0$, so

$$e_{\phi+\psi,M} = e_{\phi M} + e_{\psi M},$$

$$\operatorname{med}(-M\mathbf{1}_<, \boldsymbol{x}_{\phi+\psi}, M\mathbf{1}_<) = \operatorname{med}(-M\mathbf{1}_<, \boldsymbol{x}_\phi, M\mathbf{1}_<) + \operatorname{med}(-M\mathbf{1}_<, \boldsymbol{x}_\psi, M\mathbf{1}_<)$$

and if $\phi \in \mathcal{L}$ then $\phi + \psi \in \mathcal{L}$ iff $\psi \in \mathcal{L}$.

(iii) As in (i), if $\phi = \chi(]s, \infty[\times E)$, we can compute

$$\begin{aligned}e_{\phi M}(\omega) &= \min(1, M)(V_{h(\omega)}(\omega) - V_s(\omega)) \text{ if } \omega \in E \text{ and } h(\omega) \leq s,\\ &= 0 \text{ otherwise,}\end{aligned}$$

so

$$e^\bullet_{\phi M} = \min(1, M)\chi a \times (v_\tau - v_{\tau\wedge\check{s}})$$

where $a = E^\bullet$. At the same time, $\boldsymbol{x}_\phi = \boldsymbol{u}_<$ where $\boldsymbol{u}$ is the simple process with breakpoint $\check{s}$ and value 0 below $\check{s}$, value χa from $\check{s}$ onwards. So

$$\oint_{\mathcal{T}\wedge\tau} \operatorname{med}(-M\mathbf{1}_<, \boldsymbol{x}_\phi, M\mathbf{1}_<)d\boldsymbol{v} = \oint_{\mathcal{T}\wedge\tau} \min(1, M)\boldsymbol{u}_< d\boldsymbol{v} = \int_{\mathcal{T}\wedge\tau} \min(1, M)\boldsymbol{u}\,d\boldsymbol{v}$$

(645R(a-i))

$$\begin{aligned}&= \int_{(\mathcal{T}\wedge\tau)\vee\check{s}} \min(1, M)\mathbf{1}\,d\boldsymbol{v}\\ &= \min(1, M)\chi a \times (v_\tau - v_{\tau\wedge\check{s}})\end{aligned}$$

and again we have equality, so $\phi \in \mathcal{L}$.

(iv) And finally, for the key point, if $\langle\phi_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}$ converging pointwise to ϕ, then $\langle\boldsymbol{x}_{\phi_n}\rangle_{n\in\mathbb{N}}$ is order*-convergent to $\boldsymbol{x}_\phi$ (642L(c-i)), so $\langle\operatorname{med}(-M\mathbf{1}_<, \boldsymbol{x}_{\phi_n}, M\mathbf{1}_<)\rangle_{n\in\mathbb{N}}$ is always a previsibly order-bounded sequence which is order*-convergent to $\operatorname{med}(-M\mathbf{1}_<, \boldsymbol{x}_\phi, M\mathbf{1}_<)$, and

$$\oint_{\mathcal{T}\wedge\tau} \operatorname{med}(-M\mathbf{1}_<, \boldsymbol{x}_\phi, M\mathbf{1}_<)d\boldsymbol{v} = \lim_{n\to\infty} \oint_{\mathcal{T}\wedge\tau} \operatorname{med}(-M\mathbf{1}_<, \boldsymbol{x}_{\phi_n}, M\mathbf{1}_<)d\boldsymbol{v}$$

by 645T. At the same time, of course, $\langle\operatorname{med}(-M, \phi_n(t, \omega), M)\rangle_{n\in\mathbb{N}} \to \operatorname{med}(-M, \phi(t, \omega), M)$ for every t and ω, so by the ordinary dominated convergence theorem $e_{\phi M}(\omega) = \lim_{n\to\infty} e_{\phi_n M}(\omega)$ for every ω and

$$e^\bullet_{\phi M} = \lim_{n\to\infty} e^\bullet_{\phi_n M} = \lim_{n\to\infty} \oint_{\mathcal{T}\wedge\tau} \operatorname{med}(-M\mathbf{1}_<, \boldsymbol{x}_{\phi_n}, M\mathbf{1}_<)d\boldsymbol{v}$$
$$= \oint_{\mathcal{T}\wedge\tau} \operatorname{med}(-M\mathbf{1}_<, \boldsymbol{x}_\phi, M\mathbf{1}_<)d\boldsymbol{v},$$

so that $\phi \in \mathcal{L}$. **Q**

(c) By 649K, $\mathcal{L} = \mathcal{L}^0$. Now suppose that $\langle X_t\rangle_{t\ge 0}$ is a previsibly measurable stochastic such that $t \mapsto X_t(\omega)$ is bounded on $[0, h(\omega)]$ for each ω, and that $e(\omega) = \int_{]0,h(\omega)]} X_t(\omega)\nu_\omega(dt)$ for every ω. Set $\phi(t,\omega) = X_t(\omega)$ for $t \ge 0$ and $\omega \in \Omega$; then $\phi \in \mathcal{L}^0$ so $\phi \in \mathcal{L}$. If $M \ge 0$, then $\{(t,\omega) : t \le h(\omega), |\phi(t,\omega)| > M\}$ belongs to Λ_{pv}, so its projection $F_M = \{\omega : |\phi(t,\omega)| > M \text{ for some } t \le h(\omega)\}$ belongs to Σ (642Jb). Because $t \mapsto X_t(\omega)$ is bounded on $[0, h(\omega)]$ for each ω, $\bigcap_{M\in\mathbb{N}} F_M = \emptyset$.

If $M \in \mathbb{N}$ and $\omega \in \Omega \setminus F_M$,

$$e_{\phi M}(\omega) = \int_{]0,h(\omega)]} \phi(t,\omega)\nu_\omega(dt) = e(\omega).$$

So $e = \lim_{M\to\infty} e_{\phi M}$ is measurable. Similarly,

$$[\![\operatorname{med}(-M\mathbf{1}_<, \boldsymbol{x}_\phi, M\mathbf{1}_<) \ne \boldsymbol{x}_\phi \times \mathbf{1}_<]\!] \subseteq F^\bullet_M$$

for every M, so

$$[\![\, \oint_{\mathcal{T}\wedge\tau} \operatorname{med}(-M\mathbf{1}_<, \boldsymbol{x}_\phi, M\mathbf{1}_<)d\boldsymbol{v} \ne \oint_{\mathcal{T}\wedge\tau} \boldsymbol{x}_\phi \times \mathbf{1}_< \, d\boldsymbol{v}]\!]$$

is included in $F^\bullet_M$, for every M (647J). But $\inf_{M\in\mathbb{N}} F^\bullet_M = 0$, so

$$\oint_{\mathcal{T}\wedge\tau} \boldsymbol{x}_\phi d\boldsymbol{v} = \oint_{\mathcal{T}\wedge\tau} \boldsymbol{x}_\phi \times \mathbf{1}_< \, d\boldsymbol{v}$$

(645Pc)

$$= \lim_{M\to\infty} \oint_{\mathcal{T}\wedge\tau} \operatorname{med}(-M\mathbf{1}_<, \boldsymbol{x}_\phi, M\mathbf{1}_<)d\boldsymbol{v} = \lim_{M\to\infty} e^\bullet_{\phi M} = e^\bullet,$$

as required.

649X Basic exercises (a) In 649H, show that if $\boldsymbol{u}$ is locally of bounded variation then we can arrange that $t \mapsto U_t(\omega) : [0, h_n(t)] \to \mathbb{R}$ is of bounded variation for every $\omega \in \Omega$ and every n.

(b) State and prove a form of 649L which will cover S-integrals with respect to processes which are locally of bounded variation. (*Hint*: §437.)

649Y Further exercises (a) In 649C, show that the formula

$$h_{n,i+1}(\omega) = \inf(\{h(\omega)\} \cup \{t : t \ge h_{ni}(\omega), |U_t(\omega) - U_{h_{ni}(\omega)}(\omega)| \ge 2^{-n}\})$$

also works.

649 Notes and comments Both 649C and 649E refer to the Riemann-sum integral; there seem to be insuperable difficulties in devising any kind of path-by-path definition for the general S-integral. Bichteler's construction (649C) depends on some deep arguments, not here but in the proof of 627M. The method of 649E demands rather less understanding of stochastic processes (in a formal sense, I don't think we have to know even that martingales are local integrators), but in the context of this volume measure-converging filters are black magic; even the full axiom of choice is a bit strong for what we want to do here, and while I expect most readers would prefer to do measure theory with dependent choice at least, I think the techniques of Chapter 56 ought to be enough for a sufficiently determined purist to set out nearly all the ideas here in a form based on ZF alone. The advantage of 649E, if there is one, is that (at least for integrals between constant stopping times) we have a formula in terms of U_t, V_t alone, without even seeking to define U_h and V_h for non-constant stopping times h. I suppose that in principle this offers an opportunity to use the formula as a definition of $\int_t^{t'} U\,dV$ without troubling about right-continuity of the filtration $\langle\Sigma_t\rangle_{t\ge0}$ or any hypothesis on the paths of the processes U and V. But I do not see much use for a stochastic integral which does not allow for integration over intervals determined by arbitrary stopping times, and for these I think we need something like the hypotheses of 649E.

Most of this section, by page-count, has been devoted to an elaborate analysis of the relationship between S-integration with respect to a non-decreasing process and pathwise Lebesgue-Stieltjes integration. Of course this should be regarded as one of the starting points for the theory of the S-integral, not its culmination. The whole point of the S-integral is that it provides a common extension of the natural integral with respect to integrators of bounded variation and Itô's integral with respect to Browniam motion.

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