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Chapter 63

Structural alterations

One of the daunting things about stochastic calculus is the amount of preliminary material required before one can approach the main theorems, let alone the interesting applications. Before proceeding to Chapter 64 we must do some more work more or less at the level of Chapters 61-62, and this is what I will try to deal with in the present chapter.

I start with a quick run through the properties of near-simple processes (§631) which got pushed out of Chapter 61 due to shortage of space. The real work of the chapter begins in §632, with 'right-continuous' filtrations $\langle\mathfrak{A}_t\rangle_{t\in T}$. For such stochastic integration structures, which include the most important examples (632D), there are useful simplifications in the theory (632C, 632F, 632I, 632J), so it is not surprising that most presentations of this material take right-continuity of the filtration as a standard hypothesis.

The integral $\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}$, as I have defined it, depends on an elaborate structure: a probability algebra $(\mathfrak{A},\bar\mu)$, a filtration $\langle\mathfrak{A}_t\rangle_{t\in T}$ and the sublattice $\mathcal{S}$. We anticipate that changing any of these will change the value of the integral. But there are many cases in which this doesn't happen. The simplest of these is 'change of law'. As long as we have a strictly positive countably additive measure on the given algebra $\mathfrak{A}$, we shall have the same integrals, as I pointed out right at the beginning in 613I. Next, there are important classes of pairs $\mathcal{S}'$, $\mathcal{S}$ of lattices for which we can expect equality of the integrals $\int_{\mathcal{S}'}$ and $\int_{\mathcal{S}}$. For these we have to work fairly hard, since it is certainly not enough just to have $\min\mathcal{S}'=\min\mathcal{S}$ and $\max\mathcal{S}'=\max\mathcal{S}$. In §633 I explore sufficient conditions to make a sublattice $\mathcal{S}'$ of $\mathcal{S}$ behave as if it had full outer (Riemann) measure, so that an integral over $\mathcal{S}$ will be the same when taken over $\mathcal{S}'$ (633K).

We are now in a position to look at the effect of replacing $(\mathfrak{A},\bar\mu,\langle\mathfrak{A}_t\rangle_{t\in T})$ with $(\mathfrak{B},\bar\mu{\restriction}\mathfrak{B},\langle\mathfrak{B}\cap\mathfrak{A}_t\rangle_{t\in T})$ where $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$. As long as we are just looking at integrals, we need only quote from §633; but if we want to understand martingales (634I), we need the theory of relative independence from Chapter 48. And there is a yet more radical change which we can consider, where the filtration $\langle\mathfrak{A}_t\rangle_{t\in T}$ is replaced by $\langle\mathfrak{A}_{\pi_r}\rangle_{r\in R}$ for some family $\langle\pi_r\rangle_{r\in R}$ of stopping times. This is what I do in §635.

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631 Near-simple processes

My presentation so far has focused on 'moderately oscillatory' integrands, with regular mentions of 'simple' processes, and an excursion into 'jump-free' processes in §§618-619. Later on, however, there will be many important results applying to an intermediate class, the 'near-simple' processes.

631A Notation We shall need a good many of the formulae introduced in Chapter 61, and as we are starting a new chapter I will give a particularly detailed list. Throughout, $(\mathfrak{A},\bar\mu,T,\langle\mathfrak{A}_t\rangle_{t\in T},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ will be a stochastic integration structure (§613 *notes*). For $u\in L^0(\mathfrak{A})$, $\theta(u)$ is $\mathbb{E}(|u|\wedge\chi 1)$ (613B). If $t\in T$, $\check t$ is the constant stopping time at t (611A). If $h:\mathbb{R}\to\mathbb{R}$ is a Borel function, $\bar h$ is the associated function from $L^0(\mathfrak{A})$ to itself (612A). If $\mathcal{S}$ is a sublattice of $\mathcal{T}$ and $\tau\in\mathcal{T}$, then $\mathcal{S}\wedge\tau=\{\sigma\wedge\tau:\sigma\in\mathcal{S}\}$, $\mathcal{S}\vee\tau=\{\sigma\vee\tau:\sigma\in\mathcal{S}\}$ and $\mathcal{I}(\mathcal{S})$ is the set of finite sublattices of $\mathcal{S}$. $[\![\ ]\!]$ will appear in formulae of the type $[\![u>\alpha]\!]$, $[\![u\le v]\!]$ where u, $v\in L^0(\mathfrak{A})$ and $\alpha\in\mathbb{R}$, as in Chapter 36; in formulae of the type $[\![\sigma>t]\!]$ and $[\![\sigma<\tau]\!]$, where σ, $\tau\in\mathcal{T}$ and $t\in T$, as in 611A and 611D; and in formulae of the type $[\![\boldsymbol{u}\ne\boldsymbol{0}]\!]=\sup_{\sigma\in\mathcal{S}}[\![u_\sigma\ne 0]\!]$ where $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}\in L^0(\mathfrak{A})^{\mathcal{S}}$, as in 612S.

For a sublattice $\mathcal{S}$ of $\mathcal{T}$, $M_{\mathrm{fa}}(\mathcal{S})$, $M_{\mathrm{simp}}(\mathcal{S})$, $M_{\text{o-b}}(\mathcal{S})$ and $M_{\text{lo-b}}(\mathcal{S})$ and are the f-algebras of, respectively, the fully adapted processes (612I), the simple processes (612J), the order-bounded processes and the locally order-bounded processes (614E) with domain $\mathcal{S}$. I will use the symbol $\mathbf{1}$ for constant processes with value $\chi 1 \in L^0(\mathfrak{A})$ (361D, 612De). If $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}} \in M_{\mathrm{fa}}(\mathcal{S})$ and $z \in L^0(\mathfrak{A}) \cap \bigcap_{\sigma\in\mathcal{S}} L^0(\mathfrak{A}_\sigma)$, then $z\boldsymbol{u}$ is the process $\langle z \times u_\sigma \rangle_{\sigma\in\mathcal{S}}$ (612De).

For an order-bounded process $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$, $\sup|\boldsymbol{u}| = \sup_{\sigma\in\mathcal{S}} |u_\sigma| \in L^0(\mathfrak{A})$. For a finite sublattice I of $\mathcal{T}$ and fully adapted processes $\boldsymbol{u}$, $\boldsymbol{v}$ with domains including I, $S_I(\boldsymbol{u}, d\boldsymbol{v})$ is the Riemann sum defined in 613E-613F. $\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}$ and $\int_{\mathcal{S}} |d\boldsymbol{v}|$ will denote Riemann-sum integrals as in 613L, while $ii_{\boldsymbol{v}}(\boldsymbol{u})$ will be an indefinite integral as defined in 613O.

631B Definitions Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.

(a) A fully adapted process $\boldsymbol{u}$ with domain $\mathcal{S}$ is **near-simple** if it is in the closure of $M_{\mathrm{simp}}(\mathcal{S})$ for the ucp topology on $M_{\text{o-b}}(\mathcal{S})$; that is, it is order-bounded and for every $\epsilon > 0$ there is a simple process $\boldsymbol{v}$ with domain $\mathcal{S}$ such that $\theta(\sup|\boldsymbol{u} - \boldsymbol{v}|) \le \epsilon$.

(b) A fully adapted process $\boldsymbol{u}$ with domain $\mathcal{S}$ is **locally near-simple** if $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ is near-simple for every $\tau \in \mathcal{S}$.

Remarks Note that in this definition it is essential that the approximating simple process $\boldsymbol{v}$ should have the same domain as the process $\boldsymbol{u}$. We have no general assurance that if $\boldsymbol{v}$ is a simple process, and $\mathcal{S}$ is a sublattice of $\operatorname{dom}\boldsymbol{v}$, then $\boldsymbol{v}\restriction\mathcal{S}$ is near-simple (see 631Xb).

631C Proposition (a) (Locally) near-simple processes are (locally) moderately oscillatory.

(b) (Locally) jump-free processes are (locally) near-simple.

proof (a) We know that simple processes are moderately oscillatory (615E) and that the space of moderately oscillatory processes is closed in the space of order-bounded processes for the ucp topology (615F(a-iv)), so near-simple processes are moderately oscillatory. It follows at once that locally near-simple processes are locally moderately oscillatory.

(b) As declared in 612Ja, the empty process counts as simple, so we need only to look at non-trivial sublattices. Re-reading part (i) of the proof of 618Gb, we see that if $\mathcal{S}$ is a non-empty sublattice of $\mathcal{T}$, $\boldsymbol{u}$ is a jump-free process with domain $\mathcal{S}$, and $\epsilon > 0$, there is a simple process $\boldsymbol{u}'$ with domain $\mathcal{S}$ such that $\theta(\sup|\boldsymbol{u} - \boldsymbol{u}'|) \le \epsilon$. As ϵ is arbitrary, $\boldsymbol{u}$ is near-simple. As in (a) just above, it follows immediately that locally jump-free processes are locally simple.

631D Where near-simple processes come from: Theorem Let (Ω, Σ, μ) be a complete probability space and $\langle \Sigma_t \rangle_{t\ge0}$ a filtration of σ-subalgebras of Σ, all containing every negligible subset of Ω. Suppose that we are given a family $\langle X_t \rangle_{t\ge0}$ of real-valued functions on Ω such that X_t is Σ_t-measurable for every t and $t \mapsto X_t(\omega) : [0,\infty[\, \to \mathbb{R}$ is càdlàg (in the sense of 4A2A) for every $\omega \in \Omega$.

In this case, $\langle X_t \rangle_{t\ge0}$ is progressively measurable, and if $(\mathfrak{A}, \bar\mu, [0,\infty[, \langle \mathfrak{A}_t \rangle_{t\ge0}, \mathcal{T}, \langle \mathfrak{A}_\tau \rangle_{\tau\in\mathcal{T}})$ and $\langle x_\sigma \rangle_{\sigma\in\mathcal{T}_f}$ are defined as in 612H, then $\boldsymbol{x} = \langle x_\sigma \rangle_{\sigma\in\mathcal{T}_f}$ is locally near-simple.

proof Just as 613Ub corresponds to 618Gb, the proof here follows that of 618H. Corresponding to the weaker hypothesis and less ambitious objective, some modifications are necessary. To begin with, the formulae can be copied exactly, but rather than keep you turning back and forth, I repeat the details.

(a) As before, I start by showing that we have a progressively measurable process. **P** Take any $t \ge 0$ and $\alpha \in \mathbb{R}$. Set $Q = \{qt : q \in \mathbb{Q} \cap [0,1]\}$. Then

$$\begin{aligned}\{(s,\omega) : s \le t,\ X_s(\omega) > \alpha\}\\ = \{(s,\omega) : s \le t,\ \limsup_{q\downarrow Q\cap[s,t]} X_q(\omega) > \alpha\}\end{aligned}$$

(because $s \mapsto X_s(\omega)$ is càdlàg for every ω)

$$= \bigcup_{k\in\mathbb{N}} \bigcap_{q\in Q} \bigcup_{\substack{q'\in Q\\ q'\le q}} \{(s,\omega) : s \le t,\ s > q \text{ or } s \le q' \text{ and } X_{q'}(\omega) \ge \alpha + 2^{-k}\}$$

$$\in \mathcal{B}([0,t])\widehat{\otimes}\Sigma_t$$

where $\mathcal{B}([0,t])$ is the Borel σ-algebra of $[0,t]$. **Q**

We can therefore again apply the method of 612H to define, for each stopping time $h : \Omega \to [0,\infty[$, the σ-algebra Σ_h and the Σ_h-measurable function X_h, and we find ourselves with a stochastic integration structure $(\mathfrak{A}, \bar\mu, [0,\infty[\,, \langle\mathfrak{A}_t\rangle_{t\ge0}, \mathcal{T}, \langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ and a fully adapted process $\boldsymbol{x} = \langle x_\sigma\rangle_{\sigma\in\mathcal{T}_f}$ such that $x_{h^\bullet} = X_h^\bullet$ for every h.

(b) Let $h : \Omega \to [0,\infty[$ be a stopping time, and $\epsilon > 0$. For $\omega \in \Omega$ set

$$f(\omega) = \min(\{h(\omega)\} \cup \{t : t \ge 0,\ |X_t(\omega)| \ge \epsilon\}).$$

(Once again I can write min rather than inf because if $\{t : |X_t(\omega)| \ge \epsilon\}$ is non-empty it contains its infimum.) Then f is a stopping time. **P** For any $t \ge 0$, Σ_t contains every μ-negligible set, so $(\Omega, \Sigma_t, \mu\restriction\Sigma_t)$ is a complete probability space and Σ_t is closed under Souslin's operation (431A). Next,

$$\{(s,\omega) : s \le t,\ |X_s(\omega)| \ge \epsilon\} = \bigcap_{k\in\mathbb{N}}\{(s,\omega) : s \le t,\ |X_s(\omega)| > \epsilon - 2^{-k}\}$$

belongs to $\mathcal{B}([0,t])\widehat{\otimes}\Sigma_t$, by (a) applied to the process $(s,\omega) \mapsto |X_s(\omega)|$. So its projection $E = \{\omega : \exists\ s \in [0,t], |X_s(\omega)| \ge \epsilon\}$ belongs to Σ_t (423O[1]). Now

$$\{\omega : f(\omega) \le t\} = \{\omega : h(\omega) \le t\} \cup E \in \Sigma_t.$$

As t is arbitrary, f is a stopping time adapted to $\langle\Sigma_t\rangle_{t\ge0}$. **Q**

(c) Again suppose that $h : \Omega \to [0,\infty[$ is a stopping time and $\epsilon > 0$. Define g_n and $X_t^{(n)}$, for $n \in \mathbb{N}$ and $t \ge 0$, by setting

$$g_0(\omega) = 0 \text{ for every } \omega \in \Omega$$

and

$$\begin{aligned} X_t^{(n)}(\omega) &= 0 \text{ if } g_n(\omega) > t, \\ &= X_t(\omega) - X_{g_n}(\omega) \text{ if } g_n(\omega) \le t, \end{aligned}$$

$$g_{n+1}(\omega) = \inf(\{h(\omega)\} \cup \{t : t \ge 0,\ |X_t^{(n)}(\omega)| \ge \epsilon\}$$

for $n \in \mathbb{N}$, $t \ge 0$ and $\omega \in \Omega$, We see immediately that $t \mapsto X_t^{(n)}(\omega)$ is always càdlàg. Also we can see by induction on n that every g_n is a stopping time and every $X_t^{(n)}$ is Σ_t-measurable. **P** For $n = 0$ this is trivial, since of course $X_t - X_0$ is Σ_t-measurable and $t \mapsto X_t(\omega) - X_0(\omega)$ is always càdlàg. For the inductive step to $n \ge 1$, g_n is a stopping time, by (b) applied to $\langle X_t^{(n-1)}\rangle_{t\ge0}$. Next, setting $F = \{\omega : g_n(\omega) \le t\}$, we have

$$\begin{aligned} F \cap \{\omega : g_n(\omega) \le s\} &= \{\omega : g_n(\omega) \le s\} \in \Sigma_s \subseteq \Sigma_t \text{ if } s \le t, \\ &= F \in \Sigma_t \text{ if } t \le s \end{aligned}$$

so $F \in \Sigma_{g_n}$. If $E \in \Sigma_{g_n}$ then $E \cap F \in \Sigma_t$, so $X_{g_n} \times \chi F$ is Σ_t-measurable; while $F \in \Sigma_t$ so $X_t \times \chi F$ is Σ_t-measurable. Consequently $X_t^{(n)} = (X_t - X_{g_n}) \times \chi F$ is Σ_t-measurable, and the induction continues. **Q**

We therefore have a non-decreasing sequence $\langle g_n\rangle_{n\in\mathbb{N}}$ of stopping times such that, for any $n \in \mathbb{N}$ and $\omega \in \Omega$,

— if $n = 0$ then $g_n(\omega) = 0$,
— $g_n(\omega) \le h(\omega)$,
— $|X_t(\omega) - X_{g_n}(\omega)| < \epsilon$ whenever $g_n(\omega) \le t < g_{n+1}(\omega)$,
— if $g_{n+1}(\omega) < h(\omega)$ then $|X_{g_{n+1}}(\omega) - X_{g_n}(\omega)| \ge \epsilon$.

It is at this point that the argument begins to diverge from that of 618H. But we still see that for any ω, $\langle g_n(\omega)\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence bounded above by $h(\omega)$, so

[1]Later editions only.

$$\lim_{n\to\infty} X_{g_n}(\omega) = \lim_{n\to\infty} X_{g_n(\omega)}(\omega)$$

is defined and finite, in which case there must be some n such that $g_{n+1}(\omega) = h(\omega)$.

(d) Translating (c) into the language of the stochastic integration structure $(\mathfrak{A}, \bar{\mu}, T, \langle\mathfrak{A}_t\rangle_{t\in T}, \mathcal{T}, \langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$, we see that given any $\tau \in \mathcal{T}_f$ and $\epsilon > 0$ we have a non-decreasing sequence $\langle\tau_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{T}_f$ such that, for every $n \in \mathbb{N}$,

— if $n = 0$ then $\tau_n = \check{0} = \min \mathcal{T}_f$,

— $\tau_n \le \tau$,

— $[\![\tau_n \le \sigma]\!] \cap [\![\sigma < \tau_{n+1}]\!] \subseteq [\![|x_\sigma - x_{\tau_n}| < \epsilon]\!]$ for every $\sigma \in \mathcal{T}_f$,

and

$$\sup_{n\in\mathbb{N}} [\![\tau_n = \tau]\!] = 1.$$

But this means that if we take $n \in \mathbb{N}$ such that $c = 1 \setminus [\![\tau_n = \tau]\!]$ has measure at most ϵ, and take $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{T}\wedge\tau}$ to be the simple function defined from $\tau_0, \dots, \tau_n$ and $x_{\tau_0}, \dots, x_{\tau_n}$, then $[\![\tau_i \le \sigma]\!] \cap [\![\sigma < \tau_{i+1}]\!] \subseteq [\![|x_\sigma - v_\sigma| < \epsilon]\!]$ for every $\sigma \in \mathcal{T} \wedge \tau$ and $i < n$. So $[\![|x_\sigma - v_\sigma| \ge \epsilon]\!] \subseteq c$ for every $\sigma \in \mathcal{T}\wedge\tau$, and $\theta(\sup|(\boldsymbol{x}{\restriction}\mathcal{T}\wedge\tau) - \boldsymbol{v}|) \le 2\epsilon$. As ϵ is arbitrary, $\boldsymbol{x}{\restriction}\mathcal{T}\wedge\tau$ is near-simple; as τ is arbitrary, $\boldsymbol{x}$ is locally near-simple.

631E Proposition (a) If $T = [0, \infty[$, then the identity process on $\mathcal{T}_f$ is locally near-simple.

(b) Brownian motion is locally near-simple.

(c) The Poisson process is locally near-simple.

proof By 618Ja and 618Jc, the identity process and Brownian motion are locally jump-free, so by 631Cb they are locally near-simple. As for the Poisson process $\boldsymbol{v}$, I defined it in 612Ub in terms of a probability measure on $\Omega = C_{\text{dlg}}$ and a process $\langle X_t\rangle_{t\ge0}$ with $X_t(\omega) = \omega(t)$ for every ω, so that $t \mapsto X_t(\omega)$ is surely càdlàg for every ω, and 631D tells us that $\boldsymbol{v}$ is locally near-simple.

631F Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.

(a) Write $M_{\text{n-s}} = M_{\text{n-s}}(\mathcal{S})$ for the set of near-simple processes with domain $\mathcal{S}$.

(i) If $h : \mathbb{R} \to \mathbb{R}$ is continuous, then $\bar{h}\boldsymbol{u} \in M_{\text{n-s}}$ for every $\boldsymbol{u} \in M_{\text{n-s}}$.

(ii) $M_{\text{n-s}}$ is an f-subalgebra of $M_{\text{o-b}} = M_{\text{o-b}}(\mathcal{S})$.

(iii) $M_{\text{n-s}}$ is complete for the ucp uniformity.

(iv) If $\tau \in \mathcal{S}$ and $\boldsymbol{u} \in M_{\text{fa}}(\mathcal{S})$, then $\boldsymbol{u}$ is near-simple iff $\boldsymbol{u}{\restriction}\mathcal{S}\wedge\tau$ and $\boldsymbol{u}{\restriction}\mathcal{S}\vee\tau$ are both near-simple.

(v) If $\boldsymbol{u} \in M_{\text{n-s}}$ and $z \in L^0(\mathfrak{A} \cap \bigcap_{\sigma\in\mathcal{S}} \mathfrak{A}_\sigma)$, then $z\boldsymbol{u}$ belongs to $M_{\text{n-s}}$.

(vi) If $\boldsymbol{u}$ is near-simple it is locally near-simple.

(b) Write $M_{\text{ln-s}} = M_{\text{ln-s}}(\mathcal{S})$ for the set of locally near-simple processes with domain $\mathcal{S}$.

(i) If $h : \mathbb{R} \to \mathbb{R}$ is continuous, then $\bar{h}\boldsymbol{u} \in M_{\text{ln-s}}$ for every $\boldsymbol{u} \in M_{\text{ln-s}}$.

(ii) $M_{\text{ln-s}}$ is an f-subalgebra of the space $M_{\text{lo-b}} = M_{\text{lo-b}}(\mathcal{S})$ of locally order-bounded processes with domain $\mathcal{S}$.

(iii) If $\boldsymbol{u} \in M_{\text{fa}}(\mathcal{S})$ and $\tau \in \mathcal{S}$, then $\boldsymbol{u}$ is locally near-simple iff $\boldsymbol{u}{\restriction}\mathcal{S}\wedge\tau$ and $\boldsymbol{u}{\restriction}\mathcal{S}\vee\tau$ are both locally near-simple.

(iv) If $\boldsymbol{u} \in M_{\text{ln-s}}$ and $z \in L^0(\mathfrak{A} \cap \bigcap_{\sigma\in\mathcal{S}} \mathfrak{A}_\sigma)$, then $z\boldsymbol{u} \in M_{\text{ln-s}}$.

(v) If $\boldsymbol{u} \in M_{\text{fa}}(\mathcal{S})$ and $\{\sigma : \sigma \in \mathcal{S}, \boldsymbol{u}{\restriction}\mathcal{S}\wedge\sigma \text{ is near-simple}\}$ covers $\mathcal{S}$, then $\boldsymbol{u} \in M_{\text{ln-s}}$.

(c) Suppose that $\boldsymbol{u}$ is a moderately oscillatory process with domain $\mathcal{S}$.

(i) If $\boldsymbol{u}{\restriction}\mathcal{S}\cap[\tau, \tau']$ is near-simple whenever $\tau \le \tau'$ in $\mathcal{S}$, then $\boldsymbol{u}$ is near-simple.

(ii) If $\boldsymbol{u}$ is locally near-simple it is near-simple.

proof (a)(i) By 615Ca, $\boldsymbol{u} \mapsto \bar{h}\boldsymbol{u} : M_{\text{o-b}}(\mathcal{S}) \to M_{\text{o-b}}(\mathcal{S})$ is continuous, while $\bar{h}\boldsymbol{u} \in M_{\text{simp}}(\mathcal{S})$ for every $\boldsymbol{u} \in M_{\text{simp}}(\mathcal{S})$, by 612La. Since $M_{\text{n-s}}(\mathcal{S})$ is the closure of $M_{\text{simp}}(\mathcal{S})$, $\bar{h}\boldsymbol{v} \in M_{\text{n-s}}(\mathcal{S})$ for every $\boldsymbol{v} \in M_{\text{n-s}}(\mathcal{S})$.

(ii) Similarly, as $M_{\text{simp}}(\mathcal{S})$ is closed under the operations $+$, $\times$ and $\boldsymbol{u} \mapsto |\boldsymbol{u}|$, and these are continuous on $M_{\text{o-b}}(\mathcal{S})$, $M_{\text{n-s}}(\mathcal{S})$ also is closed under these operations, and is an f-subalgebra of $M_{\text{o-b}}(\mathcal{S})$.

(iii) By definition, $M_{\text{n-s}}$ is a closed subset of $M_{\text{o-b}}$. As $M_{\text{o-b}}$ is complete under its ucp uniformity (615Cc), so is $M_{\text{n-s}}$ (3A4Fd[2]).

[2]Later editions only.

(iv)(α) The operation $\boldsymbol{u}\mapsto\boldsymbol{u}\restriction\mathcal{S}\wedge\tau:M_{\text{o-b}}(\mathcal{S})\to M_{\text{o-b}}(\mathcal{S}\wedge\tau)$ is linear, and also continuous, as

$$\theta(\sup|\boldsymbol{u}\restriction\mathcal{S}\wedge\tau|)\le\theta(\sup|\boldsymbol{u}|)$$

for every $\boldsymbol{u}\in M_{\text{o-b}}(\mathcal{S})$. Since $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau\in M_{\text{simp}}(\mathcal{S}\wedge\tau)$ for every $\boldsymbol{u}\in M_{\text{simp}}(\mathcal{S})$ (612K(d-ii)), $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau\in\overline{M_{\text{simp}}(\mathcal{S}\wedge\tau)}=M_{\text{n-s}}(\mathcal{S})$ for every $\boldsymbol{u}\in\overline{M_{\text{simp}}(\mathcal{S})}=M_{\text{n-s}}(\mathcal{S})$. Similarly, using 612K(d-iii), $\boldsymbol{u}\restriction\mathcal{S}\vee\tau\in M_{\text{n-s}}(\mathcal{S}\vee\tau)$ for every $\boldsymbol{u}\in M_{\text{n-s}}(\mathcal{S})$.

(β) Now suppose that $\boldsymbol{u}\in M_{\text{fa}}(\mathcal{S})$ is such that $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ and $\boldsymbol{u}\restriction\mathcal{S}\vee\tau$ are near-simple. Let $\epsilon>0$. Then there are simple processes $\boldsymbol{u}'=\langle u'_\sigma\rangle_{\sigma\in\mathcal{S}\wedge\tau}$ and $\boldsymbol{u}''=\langle u''_\sigma\rangle_{\sigma\in\mathcal{S}\vee\tau}$ such that $\theta(\bar u')$, $\theta(\bar u'')$ are both less than or equal to ϵ, where $\bar u'=\sup|\boldsymbol{u}'-\boldsymbol{u}\restriction\mathcal{S}\wedge\tau|$ and $\bar u''=\sup|\boldsymbol{u}''-\boldsymbol{u}\restriction\mathcal{S}\vee\tau|$.

Let $(\tau'_0,\dots,\tau'_m)$ and $(\tau''_0,\dots,\tau''_n)$ be breakpoint strings for $\boldsymbol{u}'$, $\boldsymbol{u}''$ respectively; we can suppose that $\tau'_m=\tau''_0=\tau$. Let $\boldsymbol{v}$ be the simple process defined by the formula in 612Ka from the breakpoint string $(\tau'_0,\dots,\tau,\dots,\tau''_n)$ and the values $u'_\downarrow,u'_{\tau'_0},\dots,u'_{\tau'_{m-1}},u''_\tau,\dots,u''_{\tau''_n}$, where $u'_\downarrow$ is the starting value of $\boldsymbol{u}'$ (614Ba). If $\sigma\in\mathcal{S}$ then

$$\begin{aligned}[\![\sigma<\tau]\!]&\subseteq[\![u_\sigma=u_{\sigma\wedge\tau}]\!]\cap[\![u'_{\sigma\wedge\tau}=v_{\sigma\wedge\tau}]\!]\cap[\![v_{\sigma\wedge\tau}=v_\sigma]\!]\\&\subseteq[\![|u_\sigma-v_\sigma|\le\bar u']\!],\\ [\![\tau\le\sigma]\!]&\subseteq[\![u_\sigma=u_{\sigma\vee\tau}]\!]\cap[\![u'_{\sigma\vee\tau}=v_{\sigma\vee\tau}]\!]\cap[\![v_{\sigma\vee\tau}=v_\sigma]\!]\\&\subseteq[\![|u_\sigma-v_\sigma|\le\bar u'']\!],\end{aligned}$$

so

$$\theta(\sup|\boldsymbol{u}-\boldsymbol{v}|)\le\theta(\bar u'\vee\bar u'')\le2\epsilon;$$

as ϵ is arbitrary, $\boldsymbol{u}$ is near-simple.

(v) This follows from (ii) because $z\mathbf{1}\restriction\mathcal{S}$ is simple and $z\boldsymbol{u}=(z\mathbf{1}\restriction\mathcal{S})\times\boldsymbol{u}$.

(vi) This is immediate from (iv).

(b)(i) follows from (a-i) because $(\bar h\boldsymbol{u})\restriction\mathcal{S}\wedge\tau=\bar h(\boldsymbol{u}\restriction\mathcal{S}\wedge\tau)$ for every $\tau\in\mathcal{S}$.

(ii) Similarly, restriction respects the algebraic and lattice operations on $M_{\text{o-b}}(\mathcal{S})$.

(iii)(α) If $\boldsymbol{u}$ is locally near-simple, then $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ is simple, therefore locally near-simple. Also, if $\tau'\in\mathcal{S}\vee\tau$,

$$(\boldsymbol{u}\restriction\mathcal{S}\vee\tau)\restriction(\mathcal{S}\vee\tau)\wedge\tau'=(\boldsymbol{u}\restriction\mathcal{S}\wedge\tau')\restriction(\mathcal{S}\wedge\tau')\vee\tau$$

is near-simple, so $\boldsymbol{u}\restriction(\mathcal{S}\vee\tau)$ is locally near-simple.

(β) Suppose that $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ and $\boldsymbol{u}\restriction\mathcal{S}\vee\tau$ are locally near-simple. Take any $\tau'\in\mathcal{S}\vee\tau$. Then $(\boldsymbol{u}\restriction\mathcal{S}\wedge\tau')\restriction(\mathcal{S}\wedge\tau')\wedge\tau=\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ and $(\boldsymbol{u}\restriction\mathcal{S}\vee\tau')\restriction(\mathcal{S}\wedge\tau')\vee\tau=(\boldsymbol{u}\restriction\mathcal{S}\vee\tau)\restriction(\mathcal{S}\vee\tau)\wedge\tau'$ are near-simple, so $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau'$ is near-simple.

In general, if τ' is an arbitrary member of $\mathcal{S}$,

$$\boldsymbol{u}\restriction\mathcal{S}\wedge\tau'=(\boldsymbol{u}\restriction\mathcal{S}\wedge(\tau'\vee\tau))\restriction(\mathcal{S}\wedge(\tau'\vee\tau))\wedge\tau'$$

is near-simple, so $\boldsymbol{u}$ is locally near-simple.

(iv) follows from (ii) here or from (a-v).

(v) Write A for $\{\sigma:\sigma\in\mathcal{S},\ \boldsymbol{u}\restriction\mathcal{S}\wedge\sigma\}$ is near-simple$\}$.

(α) If $\sigma\in A$ then $\boldsymbol{v}_\sigma=\langle u_{\rho\wedge\sigma}\rangle_{\rho\in\mathcal{S}}$ is near-simple. **P** $\boldsymbol{v}_\sigma\restriction\mathcal{S}\wedge\sigma=\boldsymbol{u}\restriction\mathcal{S}\wedge\sigma$ is near-simple, by (a-iv), while $\boldsymbol{v}_\sigma\restriction\mathcal{S}\vee\sigma$ is constant with value u_σ, therefore simple. By (a-iv) in the other direction, $\boldsymbol{v}_\sigma$ is near-simple. **Q**

(β) A is closed under $\vee$. **P** If $\sigma,\tau\in A$, then $u_{\rho\wedge(\sigma\vee\tau)}=u_{\rho\wedge\sigma}+u_{\rho\wedge\tau}-u_{\rho\wedge\sigma\wedge\tau}$ for every $\rho\in\mathcal{S}\wedge(\sigma\vee\tau)$, by 612D(f-i), that is, $\boldsymbol{v}_{\sigma\vee\tau}=\boldsymbol{v}_\sigma+\boldsymbol{v}_\tau-\boldsymbol{v}_{\sigma\wedge\tau}$. By (α) here, $\boldsymbol{v}_\sigma$, $\boldsymbol{v}_\tau$ and $\boldsymbol{v}_{\sigma\wedge\tau}$ are all near-simple, so the linear combination $\boldsymbol{v}_{\sigma\vee\tau}$ and the restriction $\boldsymbol{u}\restriction\mathcal{S}\wedge(\sigma\vee\tau)=\boldsymbol{v}_{\sigma\vee\tau}\restriction\mathcal{S}\wedge(\sigma\vee\tau)$ is near-simple. So $\sigma\vee\tau\in A$. **Q**

(γ) Now suppose that $\tau \in \mathcal{S}$. Then $\{\tau\}$ is covered by A so for any $\epsilon > 0$ there are $\sigma_0, \dots, \sigma_n \in A$ such that $\sup_{i \le n} [\![\sigma_i = \tau]\!]$ has measure at least $1 - \epsilon$. Now $\sigma = \sup_{i \le n} \sigma_i$ belongs to A, by (β), and $a = [\![\sigma < \tau]\!]$ has measure at most ϵ. By (α), $\boldsymbol{v}_\sigma$ is near-simple, while for $\rho \in \mathcal{S} \wedge \tau$

$$[\![u_\rho \ne v_{\sigma\rho}]\!] \subseteq [\![\rho \ne \rho \wedge \sigma]\!] = [\![\sigma < \rho]\!] \subseteq [\![\sigma < \tau]\!] = a.$$

Thus $[\![\boldsymbol{u} \restriction \mathcal{S} \wedge \tau \ne \boldsymbol{v}_\sigma \restriction \mathcal{S} \wedge \tau]\!] \subseteq a$ has measure at most ϵ, while $\boldsymbol{v}_\sigma \restriction \mathcal{S} \wedge \tau$ is near-simple. In particular, $\boldsymbol{v}_\sigma \restriction \mathcal{S} \wedge \tau$ is order-bounded; as ϵ is arbitrary, $\boldsymbol{u} \restriction \mathcal{S} \wedge \tau$ is order-bounded, by 613Bp applied to $\{u_\sigma : \sigma \in \mathcal{S} \wedge \tau\}$. But now we see from the same formula that for every $\epsilon > 0$ there is a $\boldsymbol{v} \in M_{\text{n-s}}(\mathcal{S} \wedge \tau)$ such that $\theta(\sup |\boldsymbol{u} \restriction \mathcal{S} \wedge \tau - \boldsymbol{v}|) \le \epsilon$; as $M_{\text{n-s}}(\mathcal{S} \wedge \tau)$ is closed in $M_{\text{o-b}}(\mathcal{S} \wedge \tau)$ in the ucp topology, $\boldsymbol{u} \restriction \mathcal{S} \wedge \tau$ is near-simple. As τ is arbitrary, $\boldsymbol{u}$ is locally near-simple, as claimed.

(c)(i) If $\mathcal{S}$ is empty, the result is trivial. Otherwise, since $\boldsymbol{u}$ is moderately oscillatory, it is surely order-bounded. Let $\epsilon > 0$. By 615Ga, $u_\uparrow = \lim_{\sigma \uparrow \mathcal{S}} u_\sigma$ is defined and there is a $\tau' \in \mathcal{S}$ such that $\theta(w') \le \epsilon$ where $w' = \sup_{\sigma \in \mathcal{S} \vee \tau'} |u_\sigma - u_\uparrow|$. Similarly, applying 615Gb to $A = \mathcal{S} \wedge \tau'$, $u_\downarrow = \lim_{\sigma \downarrow \mathcal{S}} u_\sigma$ is defined and there is a $\tau \in \mathcal{S} \wedge \tau'$ such that $\theta(w) \le \epsilon$ where $w = \sup_{\sigma \in \mathcal{S} \wedge \tau} |u_\sigma - u_\downarrow|$.

Now $\boldsymbol{u} \restriction \mathcal{S} \cap [\tau, \tau']$ is near-simple, so there is a simple process $\boldsymbol{v}' = \langle v'_\sigma \rangle_{\sigma \in \mathcal{S} \wedge \tau}$ such that $\theta(w'') \le \epsilon$ where $w'' = \sup_{\sigma \in \mathcal{S} \wedge \tau} |u_\sigma - v'_\sigma|$. Take a breakpoint string $(\tau_0, \dots, \tau_n)$ for $\boldsymbol{v}'$ starting with $\tau_0 = \tau$ and ending with $\tau_n = \tau'$ (see 612Kb), and let $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ be the simple process with domain $\mathcal{S}$ based on the starting value $u_\downarrow$, the breakpoint string $(\tau_0, \dots, \tau_n)$ and the values $(v'_{\tau_0}, \dots, v'_{\tau_n})$. If $\sigma \in \mathcal{S}$ then $|u_\sigma - v_\sigma| \le w + 2w' + 2w''$.
P

$$[\![\sigma < \tau]\!] \subseteq [\![v_\sigma = u_\downarrow]\!] \cap [\![u_\sigma = u_{\sigma \wedge \tau}]\!] \subseteq [\![|u_\sigma - v_\sigma| \le w]\!],$$

$$[\![\tau \le \sigma]\!] \cap [\![\sigma \le \tau']\!] \subseteq [\![u_\sigma = u_{\text{med}(\tau,\sigma,\tau')}]\!] \cap [\![v_\sigma = v'_{\text{med}(\tau,\sigma,\tau')}]\!] \subseteq [\![|u_\sigma - v_\sigma| \le w'']\!],$$

$$\begin{aligned} [\![\tau' \le \sigma]\!] &\subseteq [\![u_\sigma = u_{\sigma \vee \tau'}]\!] \cap [\![v_\sigma = v'_{\tau'}]\!] \\ &\subseteq [\![|u_\sigma - v_\sigma| \le |u_{\sigma \vee \tau'} - u_\uparrow| + |u_{\tau'} - u_\uparrow| + |u_{\tau'} - v'_{\tau'}|]\!] \\ &\subseteq [\![|u_\sigma - v_\sigma| \le 2w' + w'']\!]. \ \mathbf{Q} \end{aligned}$$

Accordingly $\theta(\sup |\boldsymbol{u} - \boldsymbol{v}|) \le \theta(w + 2w' + 2w'') \le 5\epsilon$. As ϵ is arbitrary. $\boldsymbol{u}$ is near-simple.

(ii) If $\boldsymbol{u}$ is locally near-simple and $\tau \le \tau'$ in $\mathcal{S}$, then $\boldsymbol{u} \restriction \mathcal{S} \wedge \tau'$ is near-simple so $\boldsymbol{u} \restriction \mathcal{S} \cap [\tau, \tau'] = \boldsymbol{u} \restriction (\mathcal{S} \wedge \tau') \vee \tau$ is near-simple ((a-iv) above). As τ and τ' are arbitrary, (i) here tells us that $\boldsymbol{u}$ is near-simple.

631G Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\hat{\mathcal{S}}$ its covered envelope, $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ a fully adapted process, and $\hat{\boldsymbol{u}}$ its fully adapted extension to $\hat{\mathcal{S}}$.

(a) $\boldsymbol{u}$ is near-simple iff $\hat{\boldsymbol{u}}$ is near-simple.

(b) $\boldsymbol{u}$ is locally near-simple iff $\hat{\boldsymbol{u}}$ is locally near-simple.

proof If $\mathcal{S}$ is empty both parts are trivial, so suppose otherwise.

(a)(i) Suppose that $\boldsymbol{u}$ is near-simple. Then it is order-bounded, so $\hat{\boldsymbol{u}}$ is order-bounded (614G(b-i)), Let $\epsilon > 0$. Then there is a simple process $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ such that $\theta(\sup |\boldsymbol{u} - \boldsymbol{v}|) \le \epsilon$. By 612Qf, the fully adapted extension $\hat{\boldsymbol{v}}$ of $\boldsymbol{v}$ to $\hat{\mathcal{S}}$ is simple, while $\sup |\hat{\boldsymbol{u}} - \hat{\boldsymbol{v}}| = \sup |\boldsymbol{u} - \boldsymbol{v}|$ (612Qb, 614G(b-i)). So $\theta(\sup |\hat{\boldsymbol{u}} - \hat{\boldsymbol{v}}|) \le \epsilon$; as ϵ is arbitrary, $\hat{\boldsymbol{u}}$ is near-simple.

(ii) Suppose that $\hat{\boldsymbol{u}}$ is near-simple.

(α) $\hat{\boldsymbol{u}}$ is order-bounded, so $\boldsymbol{u} = \hat{\boldsymbol{u}} \restriction \mathcal{S}$ is order-bounded. Let $\epsilon > 0$. Then there is a simple process $\boldsymbol{w} = \langle w_\tau \rangle_{\tau \in \hat{\mathcal{S}}}$ such that $\theta(\sup |\hat{\boldsymbol{u}} - \boldsymbol{w}|) \le \epsilon$. Let $w_\downarrow$ be the starting value of $\boldsymbol{w}$ and $(\tau_0, \dots, \tau_n)$ a breakpoint sequence for $\boldsymbol{w}$. For each $i \le n$, $\sup_{\sigma \in \mathcal{S}} [\![\tau_i = \sigma]\!] = 1$, so there is a finite set $A_i \subseteq \mathcal{S}$ such that $\bar{\mu}(\sup_{\sigma \in A_i} [\![\tau_i = \sigma]\!]) \ge 1 - \frac{\epsilon}{n+1}$. Let I be a finite sublattice of $\mathcal{S}$ including $\bigcup_{i \le n} A_i$, and take $(\rho_0, \dots, \rho_m)$ linearly generating the I-cells. If $i \le n$,

$$\sup_{\sigma \in A_i} [\![\tau_i = \sigma]\!] \subseteq \sup_{\sigma \in I} [\![\tau_i = \sigma]\!] = \sup_{\sigma \in I, j \le m} [\![\tau_i = \sigma]\!] \cap [\![\sigma = \rho_j]\!]$$

(611L)

$$\subseteq \sup_{j\le m} [\![\tau_i = \rho_j]\!],$$

so

$$a = \inf_{i\le n} \sup_{j\le m} [\![\tau_i = \rho_j]\!]$$

has measure at least $1-\epsilon$. Note that

$$a \subseteq \sup_{j\le m} [\![\rho_j \le \tau_0]\!] \cap \sup_{j\le m} [\![\tau_n \le \rho_j]\!] \subseteq [\![\rho_0 \le \tau_0]\!] \cap [\![\tau_n \le \rho_m]\!].$$

In between,

$$a \cap [\![\tau_i < \rho_{j+1}]\!] \subseteq [\![\tau_i \le \rho_j]\!]$$

whenever $i \le n$ and $j < m$. **P**

$$\begin{aligned} a \cap [\![\tau_i < \rho_{j+1}]\!] &\subseteq \sup_{k\le m} [\![\tau_i < \rho_{j+1}]\!] \cap [\![\tau_i = \rho_k]\!] = \sup_{k\le m} [\![\rho_k < \rho_{j+1}]\!] \cap [\![\tau_i = \rho_k]\!] \\ &\subseteq \sup_{k\le j} [\![\tau_i = \rho_k]\!] \subseteq [\![\tau_i \le \rho_j]\!]. \textbf{ Q} \end{aligned}$$

(β) Now let $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ be the simple process with starting value $w_\downarrow$, breakpoint string $(\rho_0,\dots,\rho_m)$ and values $v_{\rho_j} = w_{\rho_j}$ for $j \le m$. Take $\sigma \in \mathcal{S}$. Let $\mathfrak{B}$ be the finite subalgebra of $\mathfrak{A}$ generated by

$$\{[\![\tau_i = \rho_j]\!] : i \le n,\, j \le m\} \cup \{[\![\tau_i \le \sigma]\!] : i \le n\} \cup \{[\![\rho_j \le \sigma]\!],\, j \le m\};$$

note that $a \in \mathfrak{B}$.

(γ) Let b be a atom of $\mathfrak{B}$ included in a. Then $b \subseteq [\![v_\sigma = w_\sigma]\!]$. **P** Since

$$([\![\sigma < \rho_0]\!], [\![\rho_0 \le \sigma]\!] \cap [\![\sigma < \rho_1]\!], \dots, [\![\rho_{m-1} \le \sigma]\!] \cap [\![\sigma < \rho_m]\!], [\![\rho_m \le \sigma]\!])$$

is a partition of unity in $\mathfrak{A}$, one of these includes b; and similarly either $b \subseteq [\![\sigma < \tau_0]\!]$ or $b \subseteq [\![\tau_n \le \sigma]\!]$ or there is an $i < n$ such that $b \subseteq [\![\tau_i \le \sigma]\!] \cap [\![\sigma \le \tau_{i+1}]\!]$. Now $b \subseteq a \subseteq [\![\rho_0 \le \tau_0]\!]$ so if $b \subseteq [\![\sigma < \rho_0]\!]$ then $b \subseteq [\![\sigma < \tau_0]\!]$ and

$$b \subseteq [\![v_\sigma = w_\downarrow]\!] \cap [\![w_\sigma = w_\downarrow]\!] \subseteq [\![v_\sigma = w_\sigma]\!].$$

Similarly, $b \subseteq a \subseteq [\![\tau_n \le \rho_m]\!]$ so if $b \subseteq [\![\rho_m \le \sigma]\!]$ then $b \subseteq [\![\tau_n \le \sigma]\!]$ and

$$b \subseteq [\![v_\sigma = w_{\rho_m}]\!] \cap [\![w_{\rho_m} = w_{\tau_n}]\!] \cap [\![w_\sigma = w_{\tau_n}]\!] \subseteq [\![v_\sigma = w_\sigma]\!].$$

Otherwise, let $j < m$ be such that

$$b \subseteq [\![\rho_j \le \sigma]\!] \cap [\![\sigma < \rho_{j+1}]\!] \subseteq [\![v_\sigma = w_{\rho_j}]\!].$$

If $b \subseteq [\![\sigma < \tau_0]\!]$ then

$$\begin{aligned} b &\subseteq [\![v_\sigma = w_{\rho_j}]\!] \cap [\![\rho_j < \tau_0]\!] \cap [\![w_\sigma = w_\downarrow]\!] \\ &\subseteq [\![v_\sigma = w_{\rho_j}]\!] \cap [\![w_{\rho_j} = w_\sigma]\!] \subseteq [\![v_\sigma = w_\sigma]\!]. \end{aligned}$$

If $b \subseteq [\![\tau_n \le \sigma]\!]$ then

$$\begin{aligned} b &\subseteq a \cap [\![v_\sigma = w_{\rho_j}]\!] \cap [\![\tau_n < \rho_{j+1}]\!] \cap [\![w_\sigma = w_{\tau_n}]\!] \\ &\subseteq [\![v_\sigma = w_{\rho_j}]\!] \cap [\![\tau_n \le \rho_j]\!] \cap [\![w_\sigma = w_{\tau_n}]\!] \end{aligned}$$

(because $a \cap [\![\tau_n < \rho_{j+1}]\!] \subseteq [\![\tau_n \le \rho_j]\!]$)

$$\subseteq [\![v_\sigma = w_{\rho_j}]\!] \cap [\![w_{\rho_j} = w_{\tau_n}]\!] \cap [\![w_\sigma = w_{\tau_n}]\!] \subseteq [\![v_\sigma = w_\sigma]\!].$$

And in between, if $i < n$ and $b \subseteq [\![\tau_i \le \sigma]\!] \cap [\![\sigma < \tau_{i+1}]\!]$, then

$$b \subseteq a \cap [\![\tau_i < \rho_{j+1}]\!] \cap [\![\rho_j < \tau_{i+1}]\!] \subseteq [\![\tau_i \le \rho_j]\!] \cap [\![\rho_j < \tau_{i+1}]\!] \subseteq [\![w_{\rho_j} = w_{\tau_i}]\!]$$

so

$$b \subseteq [\![v_\sigma = w_{\rho_j}]\!] \cap [\![w_\sigma = w_{\tau_i}]\!] \cap [\![w_{\rho_j} = w_{\tau_i}]\!] \subseteq [\![v_\sigma = w_\sigma]\!]$$

in this case also, and every possibility is covered. **Q**

(δ) As b is arbitrary, $[\![v_\sigma = w_\sigma]\!] \supseteq a$, and this is true for every $\sigma \in \mathcal{S}$. But now we see that

$$|u_\sigma - v_\sigma| \times \chi a = |u_\sigma - w_\sigma| \times \chi a \le \sup|\hat{\boldsymbol{u}} - \boldsymbol{w}|$$

for every $\sigma \in \mathcal{S}$. So

$$\sup|\boldsymbol{u} - \boldsymbol{v}| \times \chi a \le \sup|\hat{\boldsymbol{u}} - \boldsymbol{w}|, \quad \theta(\sup|\boldsymbol{u} - \boldsymbol{v}|) \le \theta(\sup|\hat{\boldsymbol{u}} - \boldsymbol{w}|) + \bar{\mu}(1 \setminus a) \le 2\epsilon.$$

As ϵ is arbitrary, $\boldsymbol{u}$ is near-simple, as claimed.

(b)(i) Now suppose that $\boldsymbol{u}$ is locally near-simple. If $\sigma \in \mathcal{S}$, then $\hat{\mathcal{S}} \wedge \sigma$ is the covered envelope of $\mathcal{S} \wedge \sigma$ (611M(e-i)) so $\restriction\hat{\mathcal{S}} \wedge \sigma$ is the fully adapted extension of $\boldsymbol{u}\restriction\mathcal{S} \wedge \sigma$ to its covered envelope and is near-simple, by (a). As $\mathcal{S}$ covers $\hat{\mathcal{S}}$, 631F(b-v) tells us that $\hat{\boldsymbol{u}}$ is locally near-simple.

(ii) Finally, suppose that $\hat{\boldsymbol{u}}$ is locally near-simple and that $\sigma \in \mathcal{S}$. Again because the fully adapted extension of $\boldsymbol{u}\restriction\mathcal{S} \wedge \sigma$ to the covered envelope of $\mathcal{S} \wedge \sigma$ is $\hat{\boldsymbol{u}}\restriction\hat{\mathcal{S}} \wedge \sigma$, which is near-simple; so (a) in the other direction tells us that $\boldsymbol{u}\restriction\mathcal{S} \wedge \sigma$ is near-simple. As σ is arbitrary, $\boldsymbol{u}$ is locally near-simple.

631H Many of the arguments of §§614-617 are substantially simplified if we restrict attention to near-simple processes, and some curious new patterns emerge. A striking one, which will be useful in §641, is the following.

Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u}$ a process of bounded variation with domain $\mathcal{S}$.

(a)(i) $\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}$ is defined for every $\boldsymbol{v} \in M_{\text{n-s}}(\mathcal{S})$.

(ii) $\boldsymbol{v} \mapsto \int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v} : M_{\text{n-s}}(\mathcal{S}) \to L^0$ is continuous for the ucp topology on $M_{\text{n-s}}(\mathcal{S})$ and the topology of convergence in measure on L^0.

(b)(i) The indefinite integral $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is near-simple for every $\boldsymbol{v} \in M_{\text{n-s}}(\mathcal{S})$.

(ii) $\boldsymbol{v} \mapsto ii_{\boldsymbol{v}}(\boldsymbol{u}) : M_{\text{n-s}}(\mathcal{S}) \to M_{\text{n-s}}(\mathcal{S})$ is continuous for the ucp topology.

proof (a)(i)(α) Of course $\boldsymbol{u}$ is order-bounded (614La). Write $\bar{u}$ for $2\sup|\boldsymbol{u}| + \int_{\mathcal{S}} |d\boldsymbol{u}|$. The key is the following fact: for any $\boldsymbol{v} \in M_{\text{o-b}}(\mathcal{S})$,

$$|S_I(\boldsymbol{u}, d\boldsymbol{v})| \le \bar{u} \times \sup|\boldsymbol{v}|$$

for every $I \in \mathcal{I}(\mathcal{S})$. **P** If I is empty, this is trivial. Otherwise, take a sequence $(\tau_0, \dots, \tau_n)$ linearly generating the I-cells. Expressing $\boldsymbol{u}$ as $\langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ and $\boldsymbol{v}$ as $\langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$,

$$\begin{aligned} S_I(\boldsymbol{u}, d\boldsymbol{v}) &= \sum_{i=0}^{n-1} u_{\tau_i} \times (v_{\tau_{i+1}} - v_{\tau_i}) \\ &= \sum_{i=0}^{n-1} (u_{\tau_i} - u_{\tau_{i+1}}) \times v_{\tau_{i+1}} - u_{\tau_0} \times v_{\tau_0} + u_{\tau_n} \times v_{\tau_n} \end{aligned}$$

so

$$\begin{aligned} |S_I(\boldsymbol{u}, d\boldsymbol{v})| &\le \sum_{i=0}^{n-1} |u_{\tau_i} - u_{\tau_{i+1}}| \times |v_{\tau_{i+1}}| + |u_{\tau_0}| \times |v_{\tau_0}| + |u_{\tau_n}| \times |v_{\tau_n}| \\ &\le \bar{u} \times \sup|\boldsymbol{v}| \end{aligned}$$

(see 614J). **Q**

(β) If $\boldsymbol{v} \in M_{\text{simp}}(\mathcal{S})$ then $\boldsymbol{v}$ is of bounded variation (614Q(a-iii)), therefore an integrator (616Ra), while of course $\boldsymbol{u}$ is also an integrator and therefore moderately oscillatory. So in this case $\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}$ is defined. Now suppose that $\boldsymbol{v} \in M_{\text{n-s}}(\mathcal{S})$. Let $\epsilon > 0$. Then there is a $\delta > 0$ such that $\theta(\bar{u} \times x) \le \epsilon$ whenever $x \in L^0(\mathfrak{A})$ and $\theta(x) \le \delta$. Take $\boldsymbol{v}' \in M_{\text{simp}}(\mathcal{S})$ such that $\theta(\sup|\boldsymbol{v}' - \boldsymbol{v}|) \le \delta$ and $J \in \mathcal{I}(\mathcal{S})$ such that $\theta(S_I(\boldsymbol{u}, d\boldsymbol{v}') - \int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}') \le \epsilon$ whenever $J \subseteq I \in \mathcal{I}(\mathcal{S})$. In this case, if $J \subseteq I \in \mathcal{I}(\mathcal{S})$,

$$\theta(S_I(\boldsymbol{u},d\boldsymbol{v})-\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}')\le\theta(S_I(\boldsymbol{u},d\boldsymbol{v})-S_I(\boldsymbol{u},d\boldsymbol{v}'))+\theta(S_I(\boldsymbol{u},d\boldsymbol{v}')-\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}')$$
$$\le\theta(S_I(\boldsymbol{u},d(\boldsymbol{v}-\boldsymbol{v}')))+\epsilon\le\theta(\bar u\times\sup|\boldsymbol{v}-\boldsymbol{v}'|)+\epsilon\le 2\epsilon.$$

As ϵ is arbitrary, $\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}=\lim_{I\uparrow\mathcal{I}(\mathcal{S})}S_I(\boldsymbol{u},d\boldsymbol{v})$ is defined.

(ii) If $\boldsymbol{v}\in M_{\text{n-s}}(\mathcal{S})$ then

$$\theta(\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v})\le\sup_{I\in\mathcal{I}(\mathcal{S})}\theta(S_I(\boldsymbol{u},d\boldsymbol{v}))\le\theta(\bar u\times\sup|\boldsymbol{v}|),$$

so the linear operator $\boldsymbol{v}\mapsto\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}:M_{\text{n-s}}(\mathcal{S})\to L^0$ is continuous at 0, therefore continuous.

(b)(i) If $\boldsymbol{v}\in M_{\text{simp}}(\mathcal{S})$ then $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is simple, by 614D. Now suppose that $\boldsymbol{v}\in M_{\text{n-s}}(\mathcal{S})$ and $\epsilon>0$. Again take $\delta>0$ such that $\theta(\bar u\times x)\le\epsilon$ whenever $\theta(x)\le\delta$, and $\boldsymbol{v}'\in M_{\text{simp}}(\mathcal{S})$ such that $\theta(\sup|\boldsymbol{v}'-\boldsymbol{v}|)\le\delta$. For any $\tau\in\mathcal{S}$,

$$|\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\,d(\boldsymbol{v}-\boldsymbol{v}')|\le\sup_{I\in\mathcal{I}(\mathcal{S}\wedge\tau)}|S_I(\boldsymbol{u},d(\boldsymbol{v}-\boldsymbol{v}'))|\le\bar u\times\sup|\boldsymbol{v}-\boldsymbol{v}'|,$$

so

$$\theta(\sup|ii_{\boldsymbol{v}}(\boldsymbol{u})-ii_{\boldsymbol{v}'}(\boldsymbol{u})|)\le\theta(\bar u\times\sup|\boldsymbol{v}-\boldsymbol{v}'|)\le\epsilon.$$

As ϵ is arbitrary, $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is approximated in the ucp topology by simple processes and is near-simple.

(ii) Finally, the same arguments tell us that for $\boldsymbol{v}$, $\boldsymbol{v}'\in M_{\text{n-s}}(\mathcal{S})$,

$$\theta(\sup|ii_{\boldsymbol{v}}(\boldsymbol{u})-ii_{\boldsymbol{v}'}(\boldsymbol{u})|)\le\theta(\bar u\times\sup|\boldsymbol{v}-\boldsymbol{v}'|)$$

so that $\boldsymbol{v}\mapsto ii_{\boldsymbol{v}}(\boldsymbol{u})$ is continuous.

631I We have another result showing that an indefinite integral will share a property with the corresponding integrator, as in 614T, 615Rb, 616J, 616Q(c-i) and 618Q-618R.

Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u}$ a moderately oscillatory process and $\boldsymbol{v}$ a near-simple integrator, both with domain $\mathcal{S}$. Then $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is near-simple.

proof (a) Suppose to begin with that $\mathcal{S}$ is full and has a greatest element. Take $\epsilon>0$. Let $\delta>0$ be such that $\theta(\sup|ii_{\boldsymbol{v}}(\boldsymbol{w})|)\le\epsilon$ whenever $\boldsymbol{w}$ is a moderately oscillatory process with domain $\mathcal{S}$ such that $\theta(\sup|\boldsymbol{w}|)\le\delta$ (616J). Let $\boldsymbol{u}'$ be a process of bounded variation such that $\theta(\sup|\boldsymbol{u}-\boldsymbol{u}'|)\le\delta$. Then $ii_{\boldsymbol{v}}(\boldsymbol{u}')$ is near-simple, by 631H(b-i), while

$$\theta(\sup|ii_{\boldsymbol{v}}(\boldsymbol{u})-ii_{\boldsymbol{v}}(\tilde{\boldsymbol{u}})|)=\theta(\sup|ii_{\boldsymbol{v}}(\boldsymbol{u}-\tilde{\boldsymbol{u}})|)\le\epsilon.$$

As ϵ is arbitrary, $ii_{\boldsymbol{v}}(\boldsymbol{u})$ belongs to the closure of $M_{\text{n-s}}(\mathcal{S})$ in $M_{\text{o-b}}(\mathcal{S})$ and is near-simple, by the definition in 631B.

(b) Now suppose just that $\mathcal{S}$ has a greatest member. Let $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{v}}$ be the fully adapted extensions of $\boldsymbol{u}$ and $\boldsymbol{v}$ to the covered envelope $\hat{\mathcal{S}}$ of $\mathcal{S}$. Then $\hat{\boldsymbol{u}}$ is moderately oscillatory (615F(a-vi)) and $\hat{\boldsymbol{v}}$ is a near-simple integrator (631Ga, 616Ia) so $ii_{\hat{\boldsymbol{v}}}(\hat{\boldsymbol{u}})$ is near-simple, by (a). But this is just the fully adapted extension of $ii_{\boldsymbol{v}}(\boldsymbol{u})$ to $\hat{\mathcal{S}}$ (616Q(c-ii)). So $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is near-simple, by 631Ga in the other direction.

(c) For the general case, we can apply (b) to $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ and $\boldsymbol{v}\restriction\mathcal{S}\wedge\tau$ to see that $ii_{\boldsymbol{v}}(\boldsymbol{u})\restriction\mathcal{S}\wedge\tau$ is near-simple for every $\tau\in\mathcal{S}$, that is. $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is locally near-simple. We know also that $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is an integrator (616J), therefore moderately oscillatory (616Ib). By 631F(c-ii), $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is near-simple.

631J Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.

(a) If $\boldsymbol{v}$, $\boldsymbol{w}$ are near-simple integrators with domain $\mathcal{S}$, then $[\boldsymbol{v}\overset{*}{|}\boldsymbol{w}]$ and $\boldsymbol{v}^*$ are near-simple.

(b) If $\boldsymbol{v}$, $\boldsymbol{w}$ are locally near-simple local integrators with domain $\mathcal{S}$, then $[\boldsymbol{v}\overset{*}{|}\boldsymbol{w}]$ and $\boldsymbol{v}^*$ are locally near-simple.

proof (a) By 617I, applied in $\mathcal{S}\wedge\tau$ with $\boldsymbol{u}=\mathbf{1}$ for $\tau\in\mathcal{S}$,

$$[\boldsymbol{v}\overset{*}{|}\boldsymbol{w}]=\boldsymbol{v}\times\boldsymbol{w}-(v_\downarrow\times w_\downarrow)\mathbf{1}-ii_{\boldsymbol{v}}(\boldsymbol{w})-ii_{\boldsymbol{w}}(\boldsymbol{v})$$

where $v_\downarrow$ and $w_\downarrow$ are the starting values of $\boldsymbol{v}$ and $\boldsymbol{w}$. Now $ii_{\boldsymbol{v}}(\boldsymbol{w})$ and $ii_{\boldsymbol{w}}(\boldsymbol{v})$ are near-simple by 631I, and $(v_\downarrow \times w_\downarrow)\mathbf{1}$ is simple; so 631F(a-ii) tells us that $[\boldsymbol{v} \overset{*}{|} \boldsymbol{w}]$ is near-simple. Of course it follows at once that $\boldsymbol{v}^* = [\boldsymbol{v} \overset{*}{|} \boldsymbol{v}]$ is near-simple.

(b) Apply (a) to $\mathcal{S} \wedge \tau$ for $\tau \in \mathcal{S}$.

631K Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v}$ a near-simple process of bounded variation. Then its cumulative variation $\boldsymbol{v}^\uparrow$ (614O) is near-simple.

proof If $\mathcal{S}$ is empty, this is trivial; so suppose otherwise.

(a)(i) Let $\epsilon > 0$. Let $\boldsymbol{u}' = \langle u'_\sigma \rangle_{\sigma \in \mathcal{S}}$ be a simple process such that $\theta(\bar{w}_0) \le \epsilon$, where $\bar{w}_0 = \sup |\boldsymbol{v} - \boldsymbol{u}'|$. Let I_0 be a non-empty finite sublattice of $\mathcal{S}$ including a breakpoint string for $\boldsymbol{u}'$. Take $I \in \mathcal{I}(\mathcal{S})$ such that $I_0 \subseteq I$ and $\theta(\bar{w}_1) \le \epsilon$, where $\bar{w}_1 = \int_{\mathcal{S}} |d\boldsymbol{v}| - S_I(\mathbf{1}, |d\boldsymbol{v}|)$. Take $(\tau_0, \dots, \tau_n)$ linearly generating the I-cells; note that $(\tau_0, \dots, \tau_n)$ is a breakpoint string for $\boldsymbol{u}'$ (612Kb). Let $u'_\downarrow$ be the starting value of $\boldsymbol{u}'$, and $v_\downarrow$ the starting value of $\boldsymbol{v}$. Of course

$$|v_\downarrow - u'_\downarrow| = \lim_{\sigma \downarrow \mathcal{S}} |v_\sigma - u'_\sigma| \le \bar{w}_0.$$

(ii) Express $\boldsymbol{v}$ as $\langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$. Note that if $\sigma \in \mathcal{S}$ then

$$[\![\sigma < \tau_0]\!] \subseteq [\![u'_\sigma = u'_\downarrow]\!] \subseteq [\![|v_\sigma - u'_\downarrow| \le \bar{w}_0]\!] \subseteq [\![|v_\sigma - v_\downarrow| \le 2\bar{w}_0]\!],$$

$$\begin{aligned}[\![\tau_i \le \sigma]\!] \cap [\![\sigma < \tau_{i+1}]\!] &\subseteq [\![u'_\sigma = u'_{\tau_i}]\!] \subseteq [\![|v_\sigma - u'_{\tau_i}| \le \bar{w}_0]\!] \cap [\![|v_{\tau_i} - u'_{\tau_i}| \le \bar{w}_0]\!] \\ &\subseteq [\![|v_\sigma - v_{\tau_i}| \le 2\bar{w}_0]\!]\end{aligned}$$

for $i < n$,

$$[\![\tau_n \le \sigma]\!] \subseteq [\![u'_\sigma = u'_{\tau_n}]\!] \subseteq [\![|v_\sigma - v_{\tau_n}| \le 2\bar{w}_0]\!].$$

(b) Set $\bar{w} = \bar{w}_1 + 2\bar{w}_0$. Take any $\tau \in \mathcal{S}$.

(i) $[\![\tau < \tau_0]\!] \subseteq [\![|v^\uparrow_\tau| \le \bar{w}]\!]$. **P** Set $\sigma = \tau \wedge \tau_0$. Then $0 \le v^\uparrow_\sigma \le |v_\sigma - v_\downarrow| + \bar{w}_1$ (614P(c-ii-α)) and

$$\begin{aligned}[\![\tau < \tau_0]\!] &\subseteq [\![\tau = \sigma]\!] \cap [\![\sigma < \tau_0]\!] \subseteq [\![v^\uparrow_\tau = v^\uparrow_\sigma]\!] \cap [\![|v_\sigma - v_\downarrow| \le 2\bar{w}_0]\!] \\ &\subseteq [\![|v^\uparrow_\tau| \le \bar{w}_1 + 2\bar{w}_0]\!] = [\![|v^\uparrow_\tau| \le \bar{w}]\!]. \text{ **Q**}\end{aligned}$$

(ii) If $i < n$ then $[\![\tau_i \le \tau]\!] \cap [\![\tau < \tau_{i+1}]\!] \subseteq [\![|v^\uparrow_\tau - v^\uparrow_{\tau_i}| \le \bar{w}]\!]$. **P** Set $\sigma = \operatorname{med}(\tau_i, \tau, \tau_{i+1})$. Then $0 \le v^\uparrow_\sigma - v^\uparrow_{\tau_i} \le |v_\sigma - v_{\tau_i}| + \bar{w}_1$ (614P(c-ii-β)). Now

$$\begin{aligned}[\![\tau_i \le \tau]\!] \cap [\![\tau < \tau_{i+1}]\!] &\subseteq [\![\tau = \sigma]\!] \cap [\![\sigma < \tau_{i+1}]\!] \\ &\subseteq [\![|v^\uparrow_\tau - v^\uparrow_{\tau_i}| \le |v_\sigma - v_{\tau_i}| + \bar{w}_1]\!] \cap [\![|v_\sigma - v_{\tau_i}| \le 2\bar{w}_0]\!] \\ &\subseteq [\![|v^\uparrow_\tau - v^\uparrow_{\tau_i}| \le \bar{w}_1 + 2\bar{w}_0]\!]. \text{ **Q**}\end{aligned}$$

(iii) $[\![\tau_n \le \tau]\!] \subseteq [\![|v^\uparrow_\tau - v^\uparrow_{\tau_n}| \le \bar{w}]\!]$. **P** Set $\sigma = \tau \vee \tau_n$. Then $0 \le v^\uparrow_\sigma - v^\uparrow_{\tau_n} \le |v_\sigma - v_{\tau_n}| + \bar{w}_1$ (614P(c-ii-γ)). So

$$[\![\tau_n \le \tau]\!] \subseteq [\![\tau = \sigma]\!] \cap [\![|v_\sigma - v_{\tau_n}| \le 2\bar{w}_0]\!] \subseteq [\![|v^\uparrow_\tau - v^\uparrow_{\tau_n}| \le 2\bar{w}_0 + \bar{w}_1]\!]. \text{ **Q**}$$

(c) So if we take $\boldsymbol{v}' = \langle v'_\tau \rangle_{\tau \in \mathcal{S}}$ to be the simple process with breakpoint string $(\tau_0, \dots, \tau_n)$, $v'_{\tau_i} = v^\uparrow_{\tau_i}$ for $i \le n$ and $[\![\tau < \tau_0]\!] \subseteq [\![v'_\tau = 0]\!]$ for every $\tau \in \mathcal{S}$, we shall have, for any $\tau \in \mathcal{S}$,

$$[\![\tau < \tau_0]\!] \subseteq [\![|v^\uparrow_\tau| \le \bar{w}]\!] \cap [\![v'_\tau = 0]\!] \subseteq [\![|v^\uparrow_\tau - v'_\tau| \le \bar{w}]\!],$$

$$[\![\tau_i \le \tau]\!] \cap [\![\tau < \tau_{i+1}]\!] \subseteq [\![|v^\uparrow_\tau - v^\uparrow_{\tau_i}| \le \bar{w}]\!] \cap [\![v'_\tau = v^\uparrow_{\tau_i}]\!] \subseteq [\![|v^\uparrow_\tau - v'_\tau| \le \bar{w}]\!]$$

for $i < n$,

$$[\![\tau_n \le \tau]\!] \subseteq [\![|v^\uparrow_\tau - v^\uparrow_{\tau_n}| \le \bar{w}]\!] \cap [\![v'_\tau = v^\uparrow_{\tau_n}]\!] \subseteq [\![|v^\uparrow_\tau - v'_\tau| \le \bar{w}]\!];$$

assembling these, we see that $|v^\uparrow_\tau - v'_\tau| \le \bar w$.

Thus $\sup|\boldsymbol{v}^\uparrow - \boldsymbol{v}'| \le \bar w$, while $\theta(\bar w) \le 2\theta(w_0) + \theta(w_1) \le 3\epsilon$ and $\boldsymbol{v}'$ is simple. As ϵ was arbitrary, $\boldsymbol{v}^\uparrow$ is near-simple.

631L Corollary Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v}$ a fully adapted process with domain $\mathcal{S}$. Then $\boldsymbol{v}$ is near-simple and of bounded variation iff it is expressible as the difference of two non-negative non-decreasing near-simple processes.

proof Given that $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ is of bounded variation, let $\boldsymbol{v}^\uparrow$ be its cumulative variation and set $v_\downarrow = \lim_{\sigma\downarrow\mathcal{S}} v_\sigma$; then $\boldsymbol{v}^\uparrow + |v_\downarrow|\mathbf{1}$ and $\boldsymbol{v}^\uparrow - \boldsymbol{v} + |v_\downarrow|\mathbf{1}$ are order-bounded non-negative non-decreasing processes with difference $\boldsymbol{v}$, and are near-simple if $\boldsymbol{v}$ is (631K). In the other direction, given that $\boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{v}_2$ where $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are non-negative non-decreasing near-simple processes, these are order-bounded (by the definition 631Ba) so $\boldsymbol{v}$ is of bounded variation by 614J and near-simple by 631F(a-ii).

631M Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\mathcal{S}'$ a sublattice of $\mathcal{S}$ which is coinitial with $\mathcal{S}$.

(a)(i) There is a unique function $\Phi : M_{\rm simp}(\mathcal{S}') \to M_{\rm simp}(\mathcal{S})$ such that, for every $\boldsymbol{u} \in M_{\rm simp}(\mathcal{S}')$, $\Phi(\boldsymbol{u})$ extends $\boldsymbol{u}$ and has a breakpoint string in $\mathcal{S}'$.

(ii) $\Phi(\bar h\boldsymbol{u}) = \bar h\Phi(\boldsymbol{u})$ for every Borel measurable $h : \mathbb{R} \to \mathbb{R}$ and every $\boldsymbol{u} \in M_{\rm simp}(\mathcal{S}')$.

(iii) Φ is a multiplicative Riesz homomorphism.

(iv) If $\mathcal{S}' \ne \emptyset$ then $\boldsymbol{u}$ and $\Phi(\boldsymbol{u})$ have the same starting value for every $\boldsymbol{u} \in M_{\rm simp}(\mathcal{S}')$.

(v) $\sup|\Phi(\boldsymbol{u})| = \sup|\boldsymbol{u}|$ for every $\boldsymbol{u} \in M_{\rm simp}(\mathcal{S}')$.

(b)(i) There is a unique function $\Psi : M_{\text{n-s}}(\mathcal{S}') \to M_{\text{n-s}}(\mathcal{S})$ such that Ψ extends Φ and is continuous with respect to the ucp topologies on $M_{\text{n-s}}(\mathcal{S}')$ and $M_{\text{n-s}}(\mathcal{S})$.

(ii) $\Psi(\boldsymbol{u})\restriction\mathcal{S}' = \boldsymbol{u}$ and $\sup|\Psi(\boldsymbol{u})| = \sup|\boldsymbol{u}|$ for every $\boldsymbol{u} \in M_{\text{n-s}}(\mathcal{S}')$.

(iii) Ψ is a multiplicative Riesz homomorphism and $\Psi(\bar h\boldsymbol{u}) = \bar h\Psi(\boldsymbol{u})$ for every continuous $h : \mathbb{R} \to \mathbb{R}$ and every $\boldsymbol{u} \in M_{\text{n-s}}(\mathcal{S}')$.

(iv) For $\boldsymbol{u} \in M_{\text{n-s}}(\mathcal{S}')$, $\Psi(\boldsymbol{u})$ is non-decreasing iff $\boldsymbol{u}$ is non-decreasing.

(v) For $\boldsymbol{u} \in M_{\text{n-s}}(\mathcal{S}')$, $\sup|\Psi(\boldsymbol{u})| = \sup|\boldsymbol{u}|$, so $[\![\Psi(\boldsymbol{u}) \ne \mathbf{0}]\!] = [\![\boldsymbol{u} \ne \mathbf{0}]\!]$.

(c) Now suppose that $\sup_{\tau\in\mathcal{S}'}[\![\sigma \le \tau]\!] = 1$ for every $\sigma \in \mathcal{S}$.

(i) If $\boldsymbol{v}$ is an integrator with domain $\mathcal{S}$, then $\int_{\mathcal{S}'}\boldsymbol{u}\,d\boldsymbol{v} = \int_{\mathcal{S}}\Psi(\boldsymbol{u})\,d\boldsymbol{v}$ for every $\boldsymbol{u} \in M_{\text{n-s}}(\mathcal{S}')$.

(ii) There is a unique function $\Psi^* : M_{\text{ln-s}}(\mathcal{S}') \to M_{\text{ln-s}}(\mathcal{S})$ extending the map $\Phi : M_{\rm simp}(\mathcal{S}') \to M_{\rm simp}(\mathcal{S})$ and such that $\sup|\Psi^*(\boldsymbol{u})\restriction\mathcal{S}\wedge\tau| = \sup|\boldsymbol{u}\restriction\mathcal{S}'\wedge\tau|$ whenever $\boldsymbol{u} \in M_{\text{ln-s}}(\mathcal{S})$ and $\tau \in \mathcal{S}'$.

(iii) $\Psi^*(\boldsymbol{u})\restriction\mathcal{S}' = \boldsymbol{u}$ for every $\boldsymbol{u} \in M_{\text{ln-s}}(\mathcal{S}')$, Ψ^* is a multiplicative Riesz homomorphism, $\Psi^*(\bar h\boldsymbol{u}) = \bar h\Psi^*(\boldsymbol{u})$ for every continuous $h : \mathbb{R} \to \mathbb{R}$ and every $\boldsymbol{u} \in M_{\text{ln-s}}(\mathcal{S}')$, and $\Psi^*(\boldsymbol{u})$ is non-decreasing whenever $\boldsymbol{u} \in M_{\text{ln-s}}(\mathcal{S}')$ is non-decreasing.

proof If $\mathcal{S}'$ is empty so is $\mathcal{S}$ and everything is trivial, so suppose otherwise.

(a)(i)($\boldsymbol{\alpha}$) If $\boldsymbol{u} \in M_{\rm simp}(\mathcal{S}')$, there is a $\boldsymbol{v} \in M_{\rm simp}(\mathcal{S})$ such that $\boldsymbol{v}$ extends $\boldsymbol{u}$ and $\boldsymbol{v}$ has a breakpoint string consisting of members of $\mathcal{S}'$. **P** Let $(\tau_0,\dots,\tau_n)$ be a breakpoint string for $\boldsymbol{u}$, and $u_\downarrow$ its starting value. Then $u_\downarrow \in L^0(\bigcap_{\sigma\in\mathcal{S}'}\mathfrak{A}_\sigma)$; because $\mathcal{S}'$ is coinitial with $\mathcal{S}$, $\bigcap_{\sigma\in\mathcal{S}'}\mathfrak{A}_\sigma = \bigcap_{\sigma\in\mathcal{S}}\mathfrak{A}_\sigma$. We therefore have a simple process $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ defined from $\tau_0,\dots,\tau_n,u_\downarrow,u_{\tau_0},\dots,u_{\tau_n}$ as in 612Ka. Take $\sigma \in \mathcal{S}'$. Then $[\![u_\sigma = v_\sigma]\!]$ includes

$$[\![u_\sigma = u_\downarrow]\!] \cap [\![v_\sigma = u_\downarrow]\!] \supseteq [\![\sigma < \tau_0]\!],$$

$$[\![u_\sigma = u_{\tau_i}]\!] \cap [\![v_\sigma = u_{\tau_i}]\!] \supseteq [\![\tau_i \le \sigma]\!] \cap [\![\sigma < \tau_{i+1}]\!] \text{ for } i < n,$$

and

$$[\![u_\sigma = u_{\tau_n}]\!] \cap [\![v_\sigma = u_{\tau_n}]\!] \supseteq [\![\tau_n \le \sigma]\!],$$

so $u_\sigma = v_\sigma$. As σ is arbitrary, $\boldsymbol{v}$ extends $\boldsymbol{u}$, and of course it has a breakpoint string $(\tau_0,\dots,\tau_n)$ in $\mathcal{S}'$. **Q**

($\boldsymbol{\beta}$) If $\boldsymbol{u} \in M_{\rm simp}(\mathcal{S}')$, there is at most one $\boldsymbol{v} \in M_{\rm simp}(\mathcal{S})$ such that $\boldsymbol{v}$ extends $\boldsymbol{u}$ and $\boldsymbol{v}$ has a breakpoint string consisting of members of $\mathcal{S}'$. **P** Suppose that $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ and $\boldsymbol{v}' = \langle v'_\sigma\rangle_{\sigma\in\mathcal{S}}$ are two such processes. Then there is a finite sublattice of $\mathcal{S}'$ which includes breakpoint strings for both $\boldsymbol{v}$ and $\boldsymbol{v}'$, and now a string $(\tau_0,\dots,\tau_n)$ which linearly generates the I-cells will be a breakpoint string for both $\boldsymbol{v}$ and $\boldsymbol{v}'$ (612Kb). Observe also that $\boldsymbol{v}$, $\boldsymbol{v}'$ can be defined from $v_\downarrow,v_{\tau_0},\dots,v_{\tau_n}$ and $v'_\downarrow,v'_{\tau_0},\dots,v'_{\tau_n}$ where

$$v_\downarrow = \lim_{\sigma\downarrow\mathcal{S}} v_\sigma, \quad v'_\downarrow = \lim_{\sigma\downarrow\mathcal{S}} v'_\sigma$$

by 614Ba. But as $\mathcal{S}'$ is coinitial with $\mathcal{S}$,

$$v_\downarrow = \lim_{\sigma\downarrow\mathcal{S}} v_\sigma = \lim_{\sigma\downarrow\mathcal{S}'} v_\sigma = \lim_{\sigma\downarrow\mathcal{S}'} u_\sigma = u_\downarrow,$$

and similarly $v'_\downarrow = u_\downarrow$. And of course $v'_{\tau_i} = u_{\tau_i} = v_{\tau_i}$ for every $i \le n$. So $\boldsymbol{v} = \boldsymbol{v}'$. **Q**

We can therefore define $\Phi : M_{\mathrm{simp}}(\mathcal{S}') \to M_{\mathrm{simp}}(\mathcal{S})$ in the way claimed.

(ii) If $\boldsymbol{u} \in M_{\mathrm{simp}}(\mathcal{S}')$ and $h : \mathbb{R} \to \mathbb{R}$ is Borel measurable, then $\bar{h}\boldsymbol{u}$ is simple so $\bar{h}\Phi(\boldsymbol{u})$ is simple and extends $\bar{h}\boldsymbol{u}$; moreover, any breakpoint string for $\boldsymbol{u}$ is a breakpoint string for $\Phi(\boldsymbol{u})$ and $\bar{h}\Phi(\boldsymbol{u})$ (612La). So $\bar{h}\Phi(\boldsymbol{u}) = \Phi(\bar{h}\boldsymbol{u})$.

(iii) If $\boldsymbol{u}, \boldsymbol{u}' \in M_{\mathrm{simp}}(\mathcal{S}')$, then $\Phi(\boldsymbol{u}+\boldsymbol{u}') = \Phi(\boldsymbol{u})+\Phi(\boldsymbol{u}')$. **P** $\Phi(\boldsymbol{u})+\Phi(\boldsymbol{u}')$ belongs to $M_{\mathrm{simp}}(\mathcal{S})$, extends $\boldsymbol{u}+\boldsymbol{u}'$ and has a breakpoint string in $\mathcal{S}'$ (see the argument in (i-β) above), so must be equal to $\Phi(\boldsymbol{u}+\boldsymbol{u}')$. **Q**

It follows from (ii) just above that

$$\Phi(\alpha\boldsymbol{u}) = \alpha\Phi(\boldsymbol{u}), \quad \Phi(\boldsymbol{u}^2) = (\Phi(\boldsymbol{u}))^2, \quad \Phi|\boldsymbol{u}| = |\Phi(\boldsymbol{u})|$$

whenever $\boldsymbol{u} \in M_{\mathrm{simp}}(\mathcal{S}')$ and $\alpha \in \mathbb{R}$, so that Φ is a multiplicative Riesz homomorphism (cf. 612Bc).

(iv) I noted this in (i-β) above.

(v) Since $\boldsymbol{u}$ and $\Phi(\boldsymbol{u})$ have the same starting value $u_\downarrow$ and agree on a common breakpoint string $(\tau_0, \dots, \tau_n)$, 614Ec tells us that

$$\sup|\boldsymbol{u}| = |u_\downarrow \vee \sup_{i\le n} |u_{\tau_i}| = \sup|\Phi(\boldsymbol{u})|.$$

(b)(i) In the notation of 615B,

$$\widehat{\theta}\Phi(\boldsymbol{u}) = \theta(\sup|\Phi(\boldsymbol{u})|) = \widehat{\theta}(\boldsymbol{u})$$

for every $\boldsymbol{u} \in M_{\mathrm{simp}}(\mathcal{S}')$, and Φ, regarded as a linear operator from $M_{\mathrm{simp}}(\mathcal{S}')$ to $M_{\text{o-b}}(\mathcal{S})$ is continuous for the ucp topologies. Because $M_{\text{n-s}}(\mathcal{S}')$ is the closure of $M_{\mathrm{simp}}(\mathcal{S}')$ for the ucp topology on $M_{\text{o-b}}(\mathcal{S}')$ (631Ba), and $M_{\text{o-b}}(\mathcal{S})$ is complete as a linear topological space (615Cc), Φ has a unique extension to a continuous function $\Psi : M_{\text{n-s}}(\mathcal{S}') \to M_{\text{o-b}}(\mathcal{S})$ (use 3A4Cf[3] and 3A4G).

(ii) If $\tau \in \mathcal{S}'$ and we set

$$\pi'_\tau(\langle u_\sigma\rangle_{\sigma\in\mathcal{S}'}) = u_\tau, \quad \pi_\tau(\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}) = v_\tau,$$

we have continuous maps $\pi'_\tau : M_{\text{n-s}}(\mathcal{S}') \to L^0$ and $\pi_\tau : M_{\text{n-s}}(\mathcal{S}) \to L^0$; as $\pi_\tau\Psi$ and π'_τ agree on $M_{\mathrm{simp}}(\mathcal{S}')$ they agree on $M_{\text{n-s}}(\mathcal{S}')$; as τ is arbitrary, this means that $\Psi(\boldsymbol{u})\restriction\mathcal{S}' = \boldsymbol{u}$ for every $\boldsymbol{u} \in M_{\text{n-s}}(\mathcal{S}')$, that is, that $\Psi(\boldsymbol{u})$ always extends $\boldsymbol{u}$. Similarly, the functions $\boldsymbol{u} \mapsto \sup|\boldsymbol{u}| : M_{\text{o-b}}(\mathcal{S}') \to L^0(\mathfrak{A})$ and $\boldsymbol{u} \mapsto \sup|\Psi(\boldsymbol{u})| : M_{\text{n-s}}(\mathcal{S}') \to L^0(\mathfrak{A})$ are continuous (615C(b-ii)) and agree on $M_{\mathrm{simp}}(\mathcal{S}')$, so agree on $M_{\text{n-s}}(\mathcal{S}')$.

(iii) If $h : \mathbb{R} \to \mathbb{R}$ is continuous, then $\boldsymbol{u} \mapsto \bar{h}\boldsymbol{u} : M_{\text{n-s}}(\mathcal{S}') \to M_{\text{n-s}}(\mathcal{S}')$ and $\boldsymbol{v} \mapsto \bar{h}\boldsymbol{v} : M_{\text{n-s}}(\mathcal{S}) \to M_{\text{n-s}}(\mathcal{S})$ are continuous (615Ca). Since $\bar{h}\Psi(\boldsymbol{u}) = \Psi(\bar{h}\boldsymbol{u})$ for $\boldsymbol{u} \in M_{\mathrm{simp}}(\mathcal{S})$, $\bar{h}\Psi(\boldsymbol{u}) = \Psi(\bar{h}\boldsymbol{u})$ for $\boldsymbol{u} \in M_{\text{n-s}}(\mathcal{S})$. Now we see that

$$\Psi(\boldsymbol{u}^2) = (\Phi(\boldsymbol{u}))^2, \quad \Psi|\boldsymbol{u}| = |\Phi(\boldsymbol{u})|$$

whenever $\boldsymbol{u} \in M_{\text{n-s}}(\mathcal{S}')$, so that Ψ is a multiplicative Riesz homomorphism, as in (a-iii).

(iv) If $\boldsymbol{u} \in M_{\text{n-s}}(\mathcal{S}')$ and $\Psi(\boldsymbol{u})$ is non-decreasing then of course $\boldsymbol{u} = \Psi(\boldsymbol{u})\restriction\mathcal{S}'$ is non-decreasing. Conversely, looking at the method in (a-i) above, we see that if $\boldsymbol{u} \in M_{\mathrm{simp}}(\mathcal{S}')$ is non-decreasing then $\Phi(\boldsymbol{u})$ is non-decreasing. Next, if $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}'}$ is a non-decreasing near-simple process, then for any $\epsilon > 0$ there is a non-decreasing simple process $\boldsymbol{v}$ such that $\theta(\sup|\boldsymbol{u}-\boldsymbol{v}|) \le \epsilon$. **P** There is certainly a simple process $\boldsymbol{v}'$ such that $\theta(\sup|\boldsymbol{u}-\boldsymbol{v}'|) \le \frac{1}{2}\epsilon$. Let $(\tau_0, \dots, \tau_n)$ be a breakpoint sequence for $\boldsymbol{v}'$. Consider the simple process $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}'}$ with starting value equal to the starting value of $\boldsymbol{u}$, with breakpoint string $(\tau_0, \dots, \tau_n)$, and with $v_{\tau_i} = u_{\tau_i}$ for $i \le n$. Then $\boldsymbol{v}$ is non-decreasing and $\theta(\sup|\boldsymbol{v}-\boldsymbol{v}'|) \le \frac{1}{2}\epsilon$, so $\theta(\sup|\boldsymbol{v}-\boldsymbol{u}|) \le \epsilon$. **Q**

Now $\Psi(\boldsymbol{v}) = \Phi(\boldsymbol{v})$ is non-decreasing and

[3]Later editions only.

$$\theta(\sup|\Psi(\boldsymbol{u})-\Psi(\boldsymbol{v})|)=\theta(\sup|\Psi(\boldsymbol{u}-\boldsymbol{v})|)=\theta(\sup|\boldsymbol{u}-\boldsymbol{v}|)\le\epsilon.$$

As ϵ is arbitrary, Ψu must be non-decreasing.

(v) As $\boldsymbol{u}\mapsto\sup|\boldsymbol{u}|:M_{\text{o-b}}(\mathcal{S}')\to L^0(\mathfrak{A})$ and $\boldsymbol{u}\mapsto\sup|\Psi(\boldsymbol{u})|:M_{\text{n-s}}(\mathcal{S}')\to L^0(\mathfrak{A})$ are continuous (615C(b-ii)), it follows from (a-v) that $\sup|\Psi(\boldsymbol{u})|=\sup|\boldsymbol{u}|$ for every $\boldsymbol{u}\in M_{\text{n-s}}(\mathcal{S}')$. Now

$$[\![\Psi(\boldsymbol{u})\ne0]\!]=[\![\sup|\Psi(\boldsymbol{u})|>0]\!]=[\![\sup|\boldsymbol{u}|>0]\!]=[\![\boldsymbol{u}\ne0]\!].$$

(c) Write $\mathcal{S}'_1$ for $\bigcup_{\tau\in\mathcal{S}'}\mathcal{S}\wedge\tau$, the ideal of $\mathcal{S}$ generated by $\mathcal{S}'$.

(i)($\boldsymbol{\alpha}$) We know that $\boldsymbol{v}$ is moderately oscillatory (616Ib again), so $v_\uparrow=\lim_{\tau\uparrow\mathcal{S}}v_\tau$ and $v_\downarrow=\lim_{\tau\downarrow\mathcal{S}}v_\tau$ are defined (615G). Now if $\epsilon>0$ there are a $\tau\in\mathcal{S}$ such that $\theta(v_{\tau'}-v_\uparrow)\le\frac12\epsilon$ whenever $\tau'\in\mathcal{S}\vee\tau$, and a $\sigma\in\mathcal{S}'$ such that $\bar\mu[\![\sigma<\tau]\!]\le\frac12\epsilon$. Now if $\sigma'\in\mathcal{S}'\vee\sigma$,

$$\begin{aligned}\theta(v_{\sigma'\vee\tau}-v_\uparrow)&\le\bar\mu[\![v_{\sigma'}\ne v_{\sigma'\vee\tau}]\!]\le\bar\mu[\![\sigma'\ne\sigma'\vee\tau]\!]\\&=\bar\mu[\![\sigma'<\tau]\!]\le\bar\mu[\![\sigma<\tau]\!]\le\frac12\epsilon,\end{aligned}$$

so

$$\theta(v_{\sigma'}-v_\uparrow)\le\theta(v_{\sigma'}-v_{\sigma'\vee\tau})+\theta(v_{\sigma'\vee\tau}-v_\uparrow)\le\tfrac12\epsilon+\tfrac12\epsilon=\epsilon.$$

As ϵ is arbitrary, $v_\uparrow=\lim_{\sigma\uparrow\mathcal{S}'}v_\sigma$.

($\boldsymbol{\beta}$) Because $\mathcal{S}'$ is coinitial with $\mathcal{S}$, we see that $v_\downarrow=\lim_{\sigma\downarrow\mathcal{S}}v_\sigma$ is also $\lim_{\sigma\downarrow\mathcal{S}'}v_\sigma$. We also know that $\boldsymbol{v}'=\boldsymbol{v}\restriction\mathcal{S}'$ is an integrator (616P(b-ii)). Now $\int_{\mathcal{S}'}\boldsymbol{u}\,d\boldsymbol{v}'=\int_{\mathcal{S}}\Phi(\boldsymbol{u})\,d\boldsymbol{v}$ for every $\boldsymbol{u}\in M_{\text{simp}}(\mathcal{S}')$. **P** Let $(\tau_0,\dots,\tau_n)$ be a breakpoint string for $\boldsymbol{u}$ and $u_\downarrow$ its starting value. Then these are also a breakpoint string and starting value for $\Phi(\boldsymbol{u})$ as described in the proof of (a). So

$$\begin{aligned}\int_{\mathcal{S}'}\boldsymbol{u}\,d\boldsymbol{v}'&=u_\downarrow\times(v_{\tau_0}-v_\downarrow)+\sum_{i=0}^{n-1}u_{\tau_i}\times(v_{\tau_{i+1}}-v_{\tau_i})+u_{\tau_n}\times(v_\uparrow-v_{\tau_n})\\&=\int_{\mathcal{S}}\Phi(\boldsymbol{u})\,d\boldsymbol{v}\end{aligned}$$

by 614C. **Q**

($\boldsymbol{\gamma}$) Now the operators

$$\boldsymbol{u}\mapsto\int_{\mathcal{S}'}\boldsymbol{u}\,d\boldsymbol{v}',\quad\boldsymbol{u}\mapsto\int_{\mathcal{S}}\Psi(\boldsymbol{u})\,d\boldsymbol{v}$$

from $M_{\text{n-s}}(\mathcal{S}')$ to L^0 are continuous for the ucp topology on $M_{\text{n-s}}(\mathcal{S}')$ and the topology of convergence in measure on L^0, by 616J and (b) above. By (β), they agree on $M_{\text{simp}}(\mathcal{S}')$, which is dense in $M_{\text{n-s}}(\mathcal{S}')$, so agree everywhere on $M_{\text{n-s}}(\mathcal{S}')$, as claimed.

(ii)($\boldsymbol{\alpha}$) For $\tau\in\mathcal{S}'$, (a)-(b) here tell us that we have unique functions $\Phi_\tau:M_{\text{simp}}(\mathcal{S}'\wedge\tau)\to M_{\text{simp}}(\mathcal{S}\wedge\tau)$ and $\Psi_\tau:M_{\text{n-s}}(\mathcal{S}'\wedge\tau)\to M_{\text{n-s}}(\mathcal{S}\wedge\tau)$ such that $\Phi_\tau(\boldsymbol{u})$ is a simple process extending $\boldsymbol{u}$ and with a breakpoint string in $\mathcal{S}'\wedge\tau$ for every $\boldsymbol{u}\in M_{\text{simp}}(\mathcal{S}'\wedge\tau)$, while Ψ_τ extends Φ_τ and is continuous with respect to the ucp topologies. If $\tau\le\tau'$ in $\mathcal{S}'$ and $\boldsymbol{u}\in M_{\text{n-s}}(\mathcal{S}'\wedge\tau')$, then

$$\boldsymbol{u}\restriction\mathcal{S}'\wedge\tau=\boldsymbol{u}\restriction(\mathcal{S}'\wedge\tau')\wedge\tau\in M_{\text{n-s}}((\mathcal{S}'\wedge\tau')\wedge\tau)=M_{\text{n-s}}(\mathcal{S}'\wedge\tau)$$

(using 631F(a-iv)), while similarly $\Psi_{\tau'}(\boldsymbol{u})\restriction\mathcal{S}\wedge\tau\in M_{\text{n-s}}(\mathcal{S}\wedge\tau)$. If $\boldsymbol{u}\in M_{\text{simp}}(\mathcal{S}'\wedge\tau')$, then $\Psi_{\tau'}(\boldsymbol{u})\in M_{\text{simp}}(\mathcal{S}\wedge\tau')$ has a breakpoint string in $\mathcal{S}'\wedge\tau'$, so $\Psi_{\tau'}(\boldsymbol{u})\restriction\mathcal{S}\wedge\tau\in M_{\text{simp}}(\mathcal{S}\wedge\tau)$ has a breakpoint string in $\mathcal{S}'\wedge\tau$ (612K(d-ii)) and extends $\boldsymbol{u}\restriction\mathcal{S}'\wedge\tau$, so must be equal to $\Psi_\tau(\boldsymbol{u}\restriction\mathcal{S}'\wedge\tau)$, by (a-i) here.

If now $\boldsymbol{u}\in M_{\text{ln-s}}(\mathcal{S}')$, $\Psi_\tau(\boldsymbol{u}\restriction\mathcal{S}'\wedge\tau)=\Psi_{\tau'}(\boldsymbol{u}\restriction\mathcal{S}'\wedge\tau')\restriction\mathcal{S}\wedge\tau$ whenever $\tau\le\tau'$ in $\mathcal{S}'$. We therefore have a unique process $\boldsymbol{v}=\Psi_0^*(\boldsymbol{u})$ with domain $\mathcal{S}'_1$ such that $\boldsymbol{v}\restriction\mathcal{S}\wedge\tau=\Psi_\tau(\boldsymbol{u}\restriction\mathcal{S}'\wedge\tau)$ for every $\tau\in\mathcal{S}'$, and $\boldsymbol{v}\in M_{\text{ln-s}}(\mathcal{S}'_1)$. Now $\boldsymbol{v}$ has a unique fully adapted extension $\hat{\boldsymbol{v}}$ to the covered envelope $\hat{\mathcal{S}}'_1$ of $\mathcal{S}'_1$.

At this point, observe that our hypothesis '$\sup_{\tau\in\mathcal{S}'}[\![\sigma\le\tau]\!]=1$ for every $\sigma\in\mathcal{S}$' ensures that $\mathcal{S}'_1$ covers $\mathcal{S}$ in the sense of 611M, that is, that $\mathcal{S}\subseteq\hat{\mathcal{S}}'_1$. As $\mathcal{S}'_1\subseteq\mathcal{S}\subseteq\hat{\mathcal{S}}'_1$, $\hat{\mathcal{S}}'_1$ is also the covered envelope of $\mathcal{S}$. Using 631Gb in both directions, we see that $\hat{\boldsymbol{v}}$ and $\hat{\boldsymbol{v}}\restriction\mathcal{S}$ are both locally near-simple. I will say that $\Psi^*(\boldsymbol{u})=\hat{\boldsymbol{v}}\restriction\mathcal{S}$, the unique fully adapted extension of $\Psi_0(\boldsymbol{u})$ to $\mathcal{S}$.

(β) Just as $\Phi_\tau(\boldsymbol{u}\restriction\mathcal{S}'\wedge\tau)=\Phi_{\tau'}(\boldsymbol{u}\restriction\mathcal{S}'\wedge\tau')\restriction\mathcal{S}\wedge\tau$ whenever $\boldsymbol{u}\in M_{\mathrm{simp}}(\mathcal{S}')$ and $\tau\le\tau'$ in $\mathcal{S}'$, we have $\Phi_\tau(\boldsymbol{u}\restriction\mathcal{S}'\wedge\tau)=\Phi(\boldsymbol{u})\restriction\mathcal{S}\wedge\tau$ whenever $\boldsymbol{u}\in M_{\mathrm{simp}}(\mathcal{S}')$ and $\tau\in\mathcal{S}'$. Consequently $\Psi_0^*(\boldsymbol{u})=\Phi(\boldsymbol{u})\restriction\mathcal{S}_1'$ and $\Psi^*(\boldsymbol{u})=\Phi(\boldsymbol{u})$ whenever $\boldsymbol{u}\in M_{\mathrm{simp}}(\mathcal{S}')$.

(γ) I noted in (a-v) that $\sup|\Phi_\tau(\boldsymbol{u})|=\sup|\boldsymbol{u}|$ whenever $\tau\in\mathcal{S}'$ and $\boldsymbol{u}\in M_{\mathrm{simp}}(\mathcal{S}'\wedge\tau)$; by 615C(b-ii), $\sup|\Psi_\tau(\boldsymbol{u})|=\sup|\boldsymbol{u}|$ whenever $\tau\in\mathcal{S}'$ and $\boldsymbol{u}\in M_{\text{n-s}}(\mathcal{S}'\wedge\tau)$, so

$$\begin{aligned}\sup|\Psi^*(\boldsymbol{u})\restriction\mathcal{S}\wedge\tau|&=\sup|\Psi_0^*(\boldsymbol{u})\restriction\mathcal{S}\wedge\tau|\\&=\sup|\Psi_\tau(\boldsymbol{u}\restriction\mathcal{S}'\wedge\tau)|=\sup|\boldsymbol{u}\restriction\mathcal{S}'\wedge\tau|\end{aligned}$$

for every $\tau\in\mathcal{S}'$ and $\boldsymbol{u}\in M_{\text{ln-s}}(\mathcal{S}')$.

(δ) As for the uniqueness of the function Ψ^*, suppose that $\Psi_1^*:M_{\text{ln-s}}(\mathcal{S}')\to M_{\text{ln-s}}(\mathcal{S})$ extends Φ and is such that $\sup|\Psi_1^*(\boldsymbol{u})\restriction\mathcal{S}\wedge\tau|=\sup|\boldsymbol{u}\restriction\mathcal{S}'\wedge\tau|$ whenever $\boldsymbol{u}\in M_{\text{ln-s}}(\mathcal{S})$ and $\tau\in\mathcal{S}'$. Then $\Psi_1^*(\boldsymbol{u})\restriction\mathcal{S}\wedge\tau=\Phi_\tau(\boldsymbol{u}\restriction\mathcal{S}'\wedge\tau)$ whenever $\boldsymbol{u}\in M_{\mathrm{simp}}(\mathcal{S}')$ and $\tau\in\mathcal{S}'$, so $\Psi_1^*(\boldsymbol{u})\restriction\mathcal{S}\wedge\tau=\Psi_\tau(\boldsymbol{u}\restriction\mathcal{S}'\wedge\tau)$ whenever $\boldsymbol{u}\in M_{\text{ln-s}}(\mathcal{S}')$ and $\tau\in\mathcal{S}'$, $\Psi_1^*(\boldsymbol{u})\restriction\mathcal{S}_1'=\Psi_0^*(\boldsymbol{u})$ whenever $\boldsymbol{u}\in M_{\text{ln-s}}(\mathcal{S}')$ and $\Psi_1^*(\boldsymbol{u})=\Psi^*(\boldsymbol{u})$ whenever $\boldsymbol{u}\in M_{\text{ln-s}}(\mathcal{S}')$.

(iii) If $\tau\in\mathcal{S}'$ and $\boldsymbol{u}\in M_{\text{ln-s}}(\mathcal{S}')$,

$$\Psi^*(\boldsymbol{u})\restriction\mathcal{S}'\wedge\tau=\Psi_0^*(\boldsymbol{u})\restriction\mathcal{S}'\wedge\tau=\Psi_\tau(\boldsymbol{u}\restriction\mathcal{S}\wedge\tau)\restriction\mathcal{S}'\wedge\tau=\boldsymbol{u}\restriction\mathcal{S}'\wedge\tau$$

by (b-ii); as τ is arbitrary, $\Psi^*(\boldsymbol{u})$ extends $\boldsymbol{u}$. Next, it follows at once from (b-iii) that Ψ_0^* is a multiplicative Riesz homomorphism and $\Psi_0^*(\bar h\boldsymbol{u})=\bar h\Psi_0^*(\boldsymbol{u})$ for every continuous $h:\mathbb{R}\to\mathbb{R}$ and every $\boldsymbol{u}\in M_{\text{ln-s}}(\mathcal{S}')$. By 612Qb, we have the same result for Ψ^*. Finally, if $\boldsymbol{u}\in M_{\text{ln-s}}(\mathcal{S})$ is non-decreasing, then $\Psi_\tau(\boldsymbol{u}\restriction\mathcal{S}'\wedge\tau)$ is non-decreasing for every $\tau\in\mathcal{S}'$, by (b-iv), so $\Psi_0^*(\boldsymbol{u})$ is non-decreasing and $\Psi^*(\boldsymbol{u})$ is non-decreasing by 612Qg.

631N Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a locally near-simple process, and $A\subseteq\mathcal{S}$ a non-empty set such that $\inf A\in\mathcal{S}$. If $u_\rho=0$ for every $\rho\in A$, then $u_{\inf A}=0$.

proof Write τ for $\inf A$.

(a) To begin with, suppose that $\mathcal{S}=[\tau,\max\mathcal{S}]$ is a closed interval in $\mathcal{T}$ with least element τ, $u_{\max\mathcal{S}}=0$ and $\boldsymbol{u}\ge\mathbf{0}$. In this case, $\boldsymbol{u}$ is near-simple.

(i) If $\sigma\in\mathcal{S}$, $a=[\![u_\sigma>0]\!]$ and $\epsilon>0$, there are a $\sigma_1\in\mathcal{S}$ and a $c\in\mathfrak{A}$ such that $\sigma_1\ge\sigma$, $a\subseteq[\![\sigma<\sigma_1]\!]$, $\bar\mu c\le\epsilon$ and

$$a\subseteq c\cup[\![\rho<\sigma]\!]\cup[\![\sigma_1\le\rho]\!]\cup[\![u_\rho>0]\!]$$

whenever $\rho\in\mathcal{S}$. **P** Let $\eta>0$ be such that $c_0=a\cap[\![u_\sigma\le2\eta]\!])$ is at most $\frac12\epsilon$. Then there is a simple process $\boldsymbol{v}$ with domain $\mathcal{S}$ such that, writing $\bar v$ for $\sup|\boldsymbol{v}-\boldsymbol{u}|$, $\theta(\bar v)\le\frac12\eta\epsilon$, so that $c_1=[\![\bar v>\eta]\!]$ has measure at most $\frac12\epsilon$. Set $c=c_0\cup c_1$. As $\boldsymbol{v}\restriction\mathcal{S}\vee\sigma$ is simple (612K(d-iii)), it has a breakpoint string $(\sigma_0,\dots,\sigma_n)$ such that $\sigma_0=\sigma$, $\sigma_n=\max\mathcal{S}$ and $[\![\sigma<\sigma_1]\!]=[\![\sigma<\max\mathcal{S}]\!]$ (612M). Since $u_{\max\mathcal{S}}=0$, $a\subseteq[\![\sigma<\sigma_1]\!]$. Next, if $\rho\in\mathcal{S}$, then

$$[\![\sigma\le\rho]\!]\cap[\![\rho<\sigma_1]\!]\subseteq[\![v_\rho=v_\sigma]\!]\subseteq[\![|u_\rho-u_\sigma|\le2\bar v]\!]$$

and

$$\begin{aligned}a\cap[\![u_\rho=0]\!]\cap[\![u_\sigma>2\eta]\!]\cap[\![\sigma\le\rho]\!]\cap[\![\rho<\sigma_1]\!]&\\\subseteq a\cap[\![|u_\rho-u_\sigma|>2\eta]\!]\cap[\![|u_\rho-u_\sigma|\le2\bar v]\!]\subseteq[\![\bar v>\eta]\!]=c_1.&\end{aligned}$$

So

$$a\subseteq c_0\cup(a\cap[\![u_\sigma>2\eta]\!])\subseteq c_0\cup c_1\cup[\![\rho<\sigma]\!]\cup[\![\sigma_1\le\rho]\!]\cup[\![u_\rho>0]\!],$$

as required. **Q**

(ii) For $t\in T$, set $b_t=\inf\{[\![u_\sigma>0]\!]\cup[\![\sigma>t]\!]:\sigma\in\mathcal{S}\}$.

(α) $b_t\in\mathfrak{A}_t$ for every $t\in T$, because $[\![u_\sigma>0]\!]\in\mathfrak{A}_\sigma$ so

$$[\![u_\sigma>0]\!]\cup[\![\sigma>t]\!]=([\![u_\sigma>0]\!]\cap[\![\sigma\le\check t]\!])\cup[\![\sigma>t]\!]\in\mathfrak{A}_{\check t}=\mathfrak{A}_t$$

for every $\sigma\in\mathcal{T}$, using 611H(c-iii).

(β) If $s \le t$ then $b_t \subseteq b_s$ because $[\![\sigma > t]\!] \subseteq [\![\sigma > s]\!]$ for every σ.

(γ) If $t \in T$ is not isolated on the right, $b_t = \sup_{s>t} b_s$. **P** Take $\epsilon > 0$. Write υ for $\operatorname{med}(\tau, \check{t}, \max \mathcal{S}) \in \mathcal{S}$. By (i) above, there are a $\sigma_1 \in \mathcal{S}$ and a $c \in \mathfrak{A}$ such that

$$\upsilon \le \sigma_1, \quad [\![u_\upsilon > 0]\!] \subseteq [\![\upsilon < \sigma_1]\!], \quad \bar{\mu}c \le \epsilon,$$

and

$$[\![u_\upsilon > 0]\!] \subseteq c \cup [\![\sigma < \upsilon]\!] \cup [\![\sigma_1 \le \sigma]\!] \cup [\![u_\sigma > 0]\!]$$

for every $\sigma \in \mathcal{S}$. Let $s > t$ be such that $\bar{\mu}d \le \epsilon$ where $d = ([\![\tau > t]\!] \setminus [\![\tau > s]\!]) \cup ([\![\sigma_1 > t]\!] \setminus [\![\sigma_1 > s]\!])$.

Take any $\sigma \in \mathcal{S}$. We have

$$b_t \subseteq [\![u_\upsilon > 0]\!] \cup [\![\upsilon > t]\!] = [\![u_\upsilon > 0]\!] \cup [\![\tau > t]\!],$$

$$b_t \cap [\![\sigma < \upsilon]\!] \subseteq ([\![u_\sigma > 0]\!] \cup [\![\sigma > t]\!]) \cap [\![\sigma < \check{t}]\!]$$

(because $[\![\sigma < \tau]\!] = 0$)

$$\subseteq [\![u_\sigma > 0]\!], \tag{$*$}$$

$$[\![u_\upsilon > 0]\!] \cap [\![\upsilon \le \sigma]\!] \cap [\![\sigma < \sigma_1]\!] \subseteq c \cup [\![u_\sigma > 0]\!],$$

$$[\![\tau > t]\!] \subseteq d \cup [\![\tau > s]\!] \subseteq d \cup [\![\sigma > s]\!],$$

$$b_t \cap [\![\upsilon \le \sigma]\!] \cap [\![\sigma < \sigma_1]\!] \subseteq c \cup d \cup [\![u_\sigma > 0]\!] \cup [\![\sigma > s]\!], \tag{$*$}$$

$$\begin{aligned}[\![\sigma_1 \le \sigma]\!] &\subseteq [\![\sigma > s]\!] \cup [\![\sigma_1 \le \check{s}]\!] \\ &\subseteq d \cup [\![\sigma > s]\!] \cup [\![\sigma_1 \le \check{t}]\!] \\ &\subseteq d \cup [\![\sigma > s]\!] \cup ([\![\tau \le \check{t}]\!] \cap [\![u_\upsilon = 0]\!])\end{aligned}$$

(because $[\![\sigma_1 \le \upsilon]\!]$ is disjoint from $[\![u_\upsilon > 0]\!]$),

$$b_t \cap [\![\sigma_1 \le \sigma]\!] \subseteq d \cup [\![\sigma > s]\!]. \tag{$*$}$$

Collecting the three lines marked ($*$), we see that $b_t \subseteq c \cup d \cup [\![u_\sigma > 0]\!] \cup [\![\sigma > s]\!]$. As σ is arbitrary, $b_t \subseteq c \cup d \cup b_s$ and $\bar{\mu}(b_t \setminus b_s) \le \bar{\mu}c + \bar{\mu}d \le 2\epsilon$. As ϵ is arbitrary, $b_t = \sup_{s>t} b_s$. **Q**

(iii) Accordingly there is a $\tau' \in \mathcal{T}$ such that $[\![\tau' > t]\!] = b_t$ for every $t \in T$. If $t \in T$, $[\![\tau > t]\!] \subseteq [\![\sigma > t]\!]$ for every $\sigma \in \mathcal{S}$, so $[\![\tau > t]\!] \subseteq b_t$; thus $\tau \le \tau'$. On the other hand, if $\sigma \in A$, then $b_t \subseteq [\![\sigma > t]\!]$ for every t, so $\tau' \le \sigma$; as $\tau = \inf A$, $\tau' = \tau$. Thus $b_t = [\![\tau > t]\!]$ for every $t \in T$.

(iv) ? If $u_\tau \ne 0$, set $a = [\![u_\tau > 0]\!]$. By (i), we have a $\sigma_1 \in \mathcal{S}$ and a $c \in \mathfrak{A}$ such that $a \subseteq [\![\tau < \sigma_1]\!]$, $\bar{\mu}c \le \frac{1}{2}\bar{\mu}a$ and

$$a \subseteq c \cup [\![\sigma_1 \le \rho]\!] \cup [\![u_\rho > 0]\!]$$

for every $\rho \in \mathcal{S}$. Since

$$0 \ne a \setminus c \subseteq [\![\tau < \sigma_1]\!] = \sup_{t \in T}([\![\sigma_1 > t]\!] \setminus [\![\tau > t]\!]),$$

there is a $t \in T$ such that

$$0 \ne (a \setminus c) \cap [\![\sigma_1 > t]\!] \setminus [\![\tau > t]\!] = (a \setminus c) \cap [\![\sigma_1 > t]\!] \cap [\![\tau \le \check{t}]\!].$$

Next, we know that

$$[\![\tau > t]\!] = b_t = \inf_{\rho \in \mathcal{S}}([\![u_\rho > 0]\!] \cup [\![\rho > t]\!]);$$

taking complements,

$$[\![\tau \le \check{t}]\!] = \sup_{\rho \in \mathcal{S}}([\![u_\rho = 0]\!] \cap [\![\rho \le \check{t}]\!]).$$

There is therefore a $\rho \in \mathcal{S}$ such that

$$\begin{aligned}0 &\ne (a \setminus c) \cap [\![\sigma_1 > t]\!] \cap [\![u_\rho = 0]\!] \cap [\![\rho \le \check{t}]\!] \\ &\subseteq (a \setminus c) \cap [\![u_\rho = 0]\!] \cap [\![\rho < \sigma_1]\!] = (a \setminus c) \setminus ([\![\sigma_1 \le \rho]\!] \cup [\![u_\rho \ne 0]\!])\end{aligned}$$

and

$$a \not\subseteq c \cup [\![\sigma_1 \le \rho]\!] \cup [\![u_\rho > 0]\!]$$

contrary to the choice of σ_1 and c. $\mathbf{X}$

(v) So $u_\tau = 0$, and we have the result in the case when $\mathcal{S} = [\tau, \max \mathcal{S}]$, $u_{\max \mathcal{S}} = 0$ and $\boldsymbol{u} \ge \mathbf{0}$.

(b) For the general case, take any $\rho_0 \in A$. We know that $|\boldsymbol{u}|$ is locally near-simple (631F(b-ii)) so $|\boldsymbol{u}|{\restriction}\mathcal{S} \wedge \rho_0$ and $|\boldsymbol{u}|{\restriction}\mathcal{S} \cap [\tau, \rho_0]$ are near-simple (631F(a-iv)). By 631Mb, there is an extension of $|\boldsymbol{u}|{\restriction}\mathcal{S} \cap [\tau, \rho_0]$ to a non-negative near-simple process $\boldsymbol{u}' = \langle u'_\sigma \rangle_{\sigma \in [\tau, \rho_0]}$. Now $A \wedge \rho_0 = \{\rho \wedge \rho_0 : \rho \in A\}$ is a subset of $[\tau, \rho_0]$, and if $\rho \in A$ then

$$[\![\rho \le \rho_0]\!] \subseteq [\![u_{\rho \wedge \rho_0} = u_\rho]\!] \subseteq [\![u_{\rho \wedge \rho_0} = 0]\!], \quad [\![\rho_0 \le \rho]\!] \subseteq [\![u_{\rho \wedge \rho_0} = u_{\rho_0}]\!] \subseteq [\![u_{\rho \wedge \rho_0} = 0]\!]$$

so $u_{\rho \wedge \rho_0} = 0$. Thus $u'_\rho = |u_\rho| = 0$ for every $\rho \in A \wedge \rho_0$. Also, of course, $\tau = \inf(A \wedge \rho_0)$. So (a) tells us that $|u_\tau| = u'_\tau = 0$ and $u_\tau = 0$. Thus the result is true in the general case too.

631O Witnessing sequences Both 'near-simple' and 'jump-free' processes can be thought of as processes approximated in the right way by simple processes with finite breakpoint strings. In appropriate circumstances they can be characterized by the existence of suitable (infinite) non-decreasing sequences of stopping times, as follows. Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ with greatest and least elements, and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ a fully adapted process.

(a) $\mathrm{SL}_1(\boldsymbol{u})$ is the statement

for every $\delta > 0$ there is a non-decreasing sequence $\langle \tau_i \rangle_{i \in \mathbb{N}}$ in $\mathcal{S}$ such that

(α) $\tau_0 = \min \mathcal{S}$,

(β) $[\![|u_{\tau_{i+1}} - u_{\tau_i}| < \delta]\!] \subseteq [\![\tau_{i+1} = \max \mathcal{S}]\!]$ for every $i \in \mathbb{N}$,

(γ) $\inf_{i \in \mathbb{N}} [\![\tau_i < \max \mathcal{S}]\!] = 0$,

(δ) $[\![\sigma < \tau_{i+1}]\!] \subseteq [\![|u_\sigma - u_{\tau_i}| < \delta]\!]$ whenever $i \in \mathbb{N}$ and $\sigma \in \mathcal{S} \cap [\tau_i, \tau_{i+1}]$.

(b) $\mathrm{SL}_2(\boldsymbol{u})$ is the statement

for every $\delta > 0$ there is a non-decreasing sequence $\langle \tau_i \rangle_{i \in \mathbb{N}}$ in $\mathcal{S}$ such that

(α) $\tau_0 = \min \mathcal{S}$,

(β) $[\![|u_{\tau_{i+1}} - u_{\tau_i}| < \delta]\!] \subseteq [\![\tau_{i+1} = \max \mathcal{S}]\!]$ for every $i \in \mathbb{N}$,

(γ) $\inf_{i \in \mathbb{N}} [\![\tau_i < \max \mathcal{S}]\!] = 0$,

(δ) $[\![\sigma < \tau_{i+1}]\!] \subseteq [\![|u_\sigma - u_{\tau_i}| < \delta]\!]$ whenever $i \in \mathbb{N}$ and $\sigma \in \mathcal{S} \cap [\tau_i, \tau_{i+1}]$,

(ϵ) $|u_{\tau_{i+1}} - u_{\tau_i}| \le \delta$ for every $i \in \mathbb{N}$.

631P Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ with greatest and least elements and $\boldsymbol{u}$ a fully adapted process with domain $\mathcal{S}$.

(a) If $\mathrm{SL}_1(\boldsymbol{u})$ is true, then $\boldsymbol{u}$ is near-simple.

(b) If $\mathrm{SL}_2(\boldsymbol{u})$ is true, then $\boldsymbol{u}$ is jump-free.

proof Express $\boldsymbol{u}$ as $\langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$.

(a)(i) For every $\delta > 0$ there is a simple process $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ such that

$$\bar{\mu}(\sup_{\sigma \in \mathcal{S}} [\![|u_\sigma - v_\sigma| > \delta]\!]) \le \delta.$$

$\mathbf{P}$ Let $\langle \tau_i \rangle_{i \in \mathbb{N}}$ be as in $\mathrm{SL}_1(\boldsymbol{u})$. As $\langle [\![\tau_i < \max \mathcal{S}]\!] \rangle_{i \in \mathbb{N}}$ is a non-increasing sequence with infimum 0, there is an $n \in \mathbb{N}$ such that $\bar{\mu} d \le \delta$, where $d = [\![\tau_n < \max \mathcal{S}]\!]$. Let $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ be the simple process such that

$$[\![\tau_i \le \sigma]\!] \cap [\![\sigma < \tau_{i+1}]\!] \subseteq [\![v_\sigma = u_{\tau_i}]\!] \text{ for } i < n, \quad [\![\tau_n \le \sigma]\!] \subseteq [\![v_\sigma = u_{\tau_n}]\!]$$

for every $\sigma \in \mathcal{S}$. Then, for any $\sigma \in \mathcal{S}$,

$$1 \setminus d \subseteq [\![\tau_n = \sigma]\!] \cup \sup_{i<n} [\![\tau_i \le \sigma]\!] \cap [\![\sigma < \tau_{i+1}]\!]$$

(because $\tau_0 = \min \mathcal{S}$ and $[\![\tau_n \le \sigma]\!] \setminus d = [\![\tau_n = \max \mathcal{S}]\!] \cap [\![\sigma = \max \mathcal{S}]\!]$)

$$\subseteq (\llbracket v_\sigma = u_{\tau_n} \rrbracket \cap \llbracket u_\sigma = u_{\tau_n} \rrbracket) \cup \sup_{i<n} (\llbracket v_\sigma = u_{\tau_i} \rrbracket \cap \llbracket |u_\sigma - u_{\tau_i}| < \delta \rrbracket)$$
$$\subseteq \llbracket |u_\sigma - v_\sigma| < \delta \rrbracket.$$

So $\sup_{\sigma\in\mathcal{S}} \llbracket |u_\sigma - v_\sigma| > \delta \rrbracket \subseteq d$ has measure at most δ. **Q**

(ii) It follows that $\boldsymbol{u}$ is order-bounded. **P** For any $\delta > 0$ there is a simple process $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma\in\mathcal{S}}$ such that $\bar{\mu}a \geq 1 - \delta$ and

$$a \subseteq \llbracket |u_\sigma - v_\sigma| \leq \delta \rrbracket \subseteq \llbracket |u_\sigma| \leq \sup |\boldsymbol{v}| + \delta\chi 1 \rrbracket$$

for every $\sigma \in \mathcal{S}$. But this means that $\{|u_\sigma \times \chi a : \sigma \in \mathcal{S}\}$ is bounded above in L^0. As δ is arbitrary, 613C(p-i) tells us that $\{|u_\sigma| : \sigma \in \mathcal{S}\}$ is bounded above in L^0, that is, $\boldsymbol{u}$ is order-bounded. **Q**

(iii) Now we can re-phrase (i) as saying that

'For every $\delta > 0$ there is a simple process $\boldsymbol{v}$ such that $\bar{\mu}(\llbracket \sup |\boldsymbol{v} - \boldsymbol{u}|) > \delta \rrbracket) \leq \delta$'

so that for every $\delta > 0$ there is a simple process $\boldsymbol{v}$ such that $\theta(\sup |\boldsymbol{v} - \boldsymbol{u}|) \leq 2\delta$, and $\boldsymbol{u}$ is near-simple.

(b) Of course $\mathrm{SL}_2(\boldsymbol{u})$ implies $\mathrm{SL}_1(\boldsymbol{u})$, so (a) tells us that $\boldsymbol{u}$ is near-simple, therefore order-bounded; set $\bar{u} = \sup |\boldsymbol{u}|$.

Let $\delta > 0$, and take $\langle \tau_i \rangle_{i\in\mathbb{N}}$ as in the statement of $\mathrm{SL}_2(\boldsymbol{u})$. Let $n \geq 1$ be such that $\bar{\mu}\llbracket \tau_n < \max \mathcal{S} \rrbracket \leq \delta$; set $a = \llbracket \tau_n < \max \mathcal{S} \rrbracket$. Consider $I = \{\tau_0, \ldots , \tau_n\}$. If $i < n$ and σ, $\sigma' \in [\tau_i, \tau_{i+1}]$ then

$$\llbracket \sigma < \tau_{i+1} \rrbracket \subseteq \llbracket |u_\sigma - u_{\tau_i}| < \delta \rrbracket, \quad \llbracket \sigma = \tau_{i+1} \rrbracket \subseteq \llbracket |u_\sigma - u_{\tau_i}| \leq \delta \rrbracket,$$

so $|u_\sigma - u_{\tau_i}| \leq \delta\chi 1$. Similarly, $|u_{\sigma'} - u_{\tau_i}| \leq \delta\chi 1$ and $|u_\sigma - u_{\sigma'}| \leq 2\delta\chi 1$.

If σ, $\sigma' \in [\tau_n, \max \mathcal{S}]$ then $\llbracket u_\sigma \neq u_{\sigma'} \rrbracket \subseteq \llbracket \tau_n < \max \mathcal{S} \rrbracket = a$, while $|u_\sigma - u_{\sigma'}| \leq 2\bar{u}$, so $|u_\sigma - u_{\sigma'}| \leq 2\bar{u} \times \chi a$. By 618Ca,

$$\mathrm{Osclln}(\boldsymbol{u}) \leq \mathrm{Osclln}^*_I(\boldsymbol{u}) \leq 2\delta\chi 1 + 2\bar{u} \times \chi a$$

and

$$\theta(\mathrm{Osclln}(\boldsymbol{u})) \leq \theta(2\delta\chi 1 + 2\bar{u} \times \chi a) \leq 2\delta + \bar{\mu}a \leq 3\delta.$$

As δ is arbitrary, $\mathrm{Osclln}(\boldsymbol{u}) = 0$ and $\boldsymbol{u}$ is jump-free.

631Q Lemma Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ with a greatest member such that $\inf A \in \mathcal{S}$ for every non-empty $A \subseteq \mathcal{S}$, and $\boldsymbol{u}$ a near-simple process with domain $\mathcal{S}$. Take $\delta > 0$ and construct $\langle D_i \rangle_{i\in\mathbb{N}}$ and $\langle y_i \rangle_{i\in\mathbb{N}}$ from $\boldsymbol{u}$ and δ as in 615M; this is possible because $\boldsymbol{u}$ is moderately oscillatory (631Ca). Then $\inf D_i \in D_i$ and $u_{\inf D_i} = y_i$ for every $i \in \mathbb{N}$.

proof Induce on i. At the bottom, $\min \mathcal{S} = \min D_0 \in D_0$ and $y_0 = u_{\min D_0}$. For the inductive step to $i+1$, write τ for $\inf D_i$ and set $\mathcal{S}' = \mathcal{S} \vee \tau$. Define a process $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma\in\mathcal{S}'}$ by setting

$$v_\sigma = ((|u_\sigma - y_i| \wedge \delta\chi 1) \vee \delta w_\sigma) - \delta\chi 1$$

for $\sigma \in \mathcal{S}'$, where $w_\sigma = \chi\llbracket \sigma = \max \mathcal{S} \rrbracket$ for $\sigma \in \mathcal{S}'$. Because $M_{\text{n-s}}(\mathcal{S}')$ is closed under linear and lattice operations and contains $\boldsymbol{u}\restriction\mathcal{S}'$ (631Fa) and the simple processes $y_i\mathbf{1}\restriction\mathcal{S}'$ and $\boldsymbol{w} = \langle w_\sigma \rangle_{\sigma\in\mathcal{S}'}$ (612Jb), $\boldsymbol{v}$ is near-simple, with $v_{\max \mathcal{S}} = 0$.

Because $\tau \in D_i$, $\tau = \min D_i$ and

$$D_{i+1} = \{\sigma : \sigma \in \mathcal{S}', \, \llbracket \sigma < \max \mathcal{S} \rrbracket \subseteq \llbracket |u_\sigma - y_i| \geq \delta \rrbracket\} = \{\sigma : \sigma \in \mathcal{S}', \, v_\sigma = 0\}.$$

Write τ' for $\inf D_{i+1}$. Then 631N tells us that $v_{\tau'} = 0$, that is, that $\tau' \in D_{i+1}$. It follows at once that $y_{i+1} = \lim_{\sigma\downarrow D_{i+1}} u_\sigma = u_{\tau'}$, and the induction continues.

631R Stopping Lemmas: Theorem Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ with greatest and least members such that $\inf A \in \mathcal{S}$ for every non-empty $A \subseteq \mathcal{S}$, and $\boldsymbol{u}$ a moderately oscillatory process with domain $\mathcal{S}$.

(a) $\boldsymbol{u}$ is near-simple iff $\mathrm{SL}_1(\boldsymbol{u})$ is true.

(b) $\boldsymbol{u}$ is jump-free iff $\mathrm{SL}_2(\boldsymbol{u})$ is true.

proof Express $\boldsymbol{u}$ as $\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$.

(a) We know from 631Pa that if $\mathrm{SL}_1(\boldsymbol{u})$ is true then $\boldsymbol{u}$ is near-simple, so it will be enough to show the converse. Suppose that $\boldsymbol{u}$ is near-simple, and take any $\delta>0$.

(i) Construct $\langle D_i\rangle_{i\in\mathbb{N}}$, $\langle y_i\rangle_{i\in\mathbb{N}}$, $\langle d_i\rangle_{i\in\mathbb{N}}$ and $\langle c_{i\sigma}\rangle_{i\in\mathbb{N},\sigma\in\mathcal{S}}$ from $\boldsymbol{u}$ and δ as in 615M. Set $\tau_i=\inf D_i$ for each i. By 631Q, $\tau_i=\min D_i$ and $u_{\tau_i}=y_i$ for every i. Because every member of D_{i+1} is greater than or equal to some member of D_i, $\tau_i\le\tau_{i+1}$ for every i. Of course $\tau_0=\min D_0=\min\mathcal{S}$.

(ii) Now $d_i=[\![\tau_i<\max\mathcal{S}]\!]$ for each i, so $\inf_{i\in\mathbb{N}}[\![\tau_i<\max\mathcal{S}]\!]=0$. Since $\tau_{i+1}\in D_{i+1}$,

$$[\![|u_{\tau_{i+1}}-u_{\tau_i}|<\delta]\!]=[\![|u_{\tau_{i+1}}-y_i|<\delta]\!]\subseteq[\![\tau_{i+1}=\max\mathcal{S}]\!]$$

for every $i\in\mathbb{N}$.

If $i\in\mathbb{N}$ and $\sigma\in\mathcal{S}\cap[\tau_i,\tau_{i+1}]$ then

$$c_{i\sigma}=[\![\tau_i\le\sigma]\!]=1,\quad c_{i+1,\sigma}=[\![\tau_{i+1}\le\sigma]\!]=1\setminus[\![\sigma<\tau_{i+1}]\!].$$

So

$$[\![\sigma<\tau_{i+1}]\!]=c_{i\sigma}\setminus c_{i+1,\sigma}\subseteq[\![|u_\sigma-y_i|<\delta]\!]$$

(615M(d-v))

$$=[\![|u_\sigma-u_{\tau_i}|<\delta]\!].$$

Thus $\langle\tau_i\rangle_{i\in\mathbb{N}}$ satisfies (α)-(δ) of $\mathrm{SL}_1(\boldsymbol{u})$. As δ was arbitrary, $\mathrm{SL}_1(\boldsymbol{u})$ is true.

(b) Here we know from 631Pb that if $\mathrm{SL}_2(\boldsymbol{u})$ is true then $\boldsymbol{u}$ is jump-free, so again it will be enough to show the converse. Suppose that $\boldsymbol{u}$ is jump-free. By 631Cb, $\boldsymbol{u}$ is near-simple. By (a) just above, $\mathrm{SL}_1(\boldsymbol{u})$ is true. Take $\delta>0$, and let $\langle\tau_i\rangle_{i\in\mathbb{N}}$ be a sequence as in the statement of $\mathrm{SL}_1(\boldsymbol{u})$.

Take any $i\in\mathbb{N}$. **?** Suppose that $[\![u_{\tau_{i+1}}-u_{\tau_i}>\delta]\!]$ is non-zero. Then there is an $\eta>0$ such that $a=[\![u_{\tau_{i+1}}-u_{\tau_i}>\delta+\eta]\!]\ne0$. Let $I\in\mathcal{I}(\mathcal{S})$ be such that $\theta(\mathrm{Oscll n}^*_I(\boldsymbol{u}))<\eta\bar\mu a$; we can suppose that τ_i, $\tau_{i+1}\in I$. Set $b=[\![\mathrm{Oscll n}^*_I(\boldsymbol{u})\le\eta]\!]$, so that $a\cap b\ne0$. Let $\sigma_0\le\ldots\le\sigma_m$ linearly generate the $I\cap[\tau_i,\tau_{i+1}]$-cells. Then

$$\begin{aligned}a\cap b\cap[\![\sigma_j<\tau_{i+1}]\!]&\subseteq a\cap b\cap[\![u_{\sigma_j}\le u_{\tau_i}+\delta]\!]\\&\subseteq[\![u_{\sigma_{j+1}}\le u_{\sigma_j}+\eta]\!]\cap[\![u_{\sigma_j}<u_{\tau_{i+1}}-\eta]\!]\subseteq[\![\sigma_{j+1}<\tau_{i+1}]\!]\end{aligned}$$

for every $j<m$. Inducing on j, we see that $a\cap b\subseteq[\![\sigma_j<\tau_{i+1}]\!]$ for every $j\le m$. But $\sigma_m=\tau_{i+1}$, so this is impossible. **X**

Similarly, $[\![u_{\tau_{i+1}}-u_{\tau_i}<-\delta]\!]$ cannot be non-zero, and $|u_{\tau_{i+1}}-u_{\tau_i}|\le\delta\chi1$, for every i. Thus $\langle\tau_i\rangle_{i\in\mathbb{N}}$ has the property (ϵ) in the statement of $\mathrm{SL}_2(\boldsymbol{u})$ as well as the properties (α)-(δ) there. As δ is arbitrary, $\mathrm{SL}_2(\boldsymbol{u})$ is true.

631S Proposition Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ with greatest and least members and $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a moderately oscillatory process. If $\inf A\in\mathcal{S}$ and $u_{\inf A}=\lim_{\sigma\downarrow A}u_\sigma$ for every non-empty downwards-directed $A\subseteq\mathcal{S}$, then $\boldsymbol{u}$ is near-simple.

proof Take any $\delta>0$.

(a) Construct $\langle D_i\rangle_{i\in\mathbb{N}}$, $\langle y_i\rangle_{i\in\mathbb{N}}$, $\langle d_i\rangle_{i\in\mathbb{N}}$ and $\langle c_{i\sigma}\rangle_{i\in\mathbb{N},\sigma\in\mathcal{S}}$ from $\boldsymbol{u}$ and δ as in 615M. Set $\tau_i=\inf D_i$ for each i; then $\tau_i\in\mathcal{S}$. Since every member of D_{i+1} is greater than or equal to some member of D_i, $\tau_i\le\tau_{i+1}$ for every i.

(b) $\tau_i\in D_i$ for every $i\in\mathbb{N}$. **P** Induce on i. At the bottom, $\tau_0=\min\mathcal{S}=\min D_0\in D_0$. For the inductive step to $i+1$, we have

$$y_i=\lim_{\sigma\downarrow D_i}u_\sigma=u_{\tau_i}\in L^0(\mathfrak{A}_{\tau_i}).$$

Set

$$a=[\![|u_{\tau_{i+1}}-y_i|\ge\delta]\!]\in\mathfrak{A}_{\tau_{i+1}}\subseteq\mathfrak{A}_{\max\mathcal{S}}.$$

Then there is a $\tau\in\mathcal{T}$ such that $a\subseteq[\![\tau=\tau_{i+1}]\!]$ and $1\setminus a\subseteq[\![\tau=\max\mathcal{S}]\!]$. Because $\mathcal{S}$ is finitely full, $\tau\in\mathcal{S}\vee\tau_{i+1}$ and

$$[\![\tau<\max\mathcal{S}]\!]\subseteq a=[\![\tau=\tau_{i+1}]\!]\cap[\![|u_{\tau_{i+1}}-y_i|\ge\delta]\!]\subseteq[\![|u_\tau-y_i|\ge\delta]\!];$$

as also $\tau_i\in D_i$ and $\tau_i\le\tau$, $\tau\in D_{i+1}$.

If σ, $\sigma'\in D_{i+1}$ and $\sigma\le\sigma'$, then

$$[\![\sigma'<\max\mathcal{S}]\!]\subseteq[\![\sigma<\max\mathcal{S}]\!]\subseteq[\![|u_\sigma-y_i|\ge\delta]\!]$$

and $|u_\sigma-y_i|\ge\delta\chi[\![\sigma'<\max\mathcal{S}]\!]$. Taking the limit as $\sigma\downarrow D_{i+1}$,

$$|u_{\tau_{i+1}}-y_i|\ge\delta\chi[\![\sigma'<\max\mathcal{S}]\!],\quad [\![\sigma'<\max\mathcal{S}]\!]\subseteq a.$$

But now we have

$$a\subseteq[\![\tau=\tau_{i+1}]\!]\subseteq[\![\tau\le\sigma']\!],\quad 1\setminus a\subseteq[\![\sigma'=\max\mathcal{S}]\!]\subseteq[\![\tau\le\sigma']\!],$$

so $\tau\le\sigma'$. As σ' is arbitrary, $\tau\le\inf D_{i+1}=\tau_{i+1}$ and $\tau_{i+1}=\tau\in D_{i+1}$. **Q**

(c) So $\tau_i=\min D_i$ and $y_i=u_{\tau_i}\in L^0(\mathfrak{A}_{\tau_i})$ for each i. Since $[\![\tau_i<\max\mathcal{S}]\!]=d_i$ for each i, $\inf_{i\in\mathbb{N}}[\![\tau_i<\max\mathcal{S}]\!]=0$. Since $\tau_{i+1}\in D_{i+1}$,

$$[\![|u_{\tau_{i+1}}-u_{\tau_i}|<\delta]\!]=[\![|u_{\tau_{i+1}}-y_i|<\delta]\!]\subseteq[\![\tau_{i+1}=\max\mathcal{S}]\!]$$

for every $i\in\mathbb{N}$.

If $i\in\mathbb{N}$ and $\sigma\in\mathcal{S}\cap[\tau_i,\tau_{i+1}]$ then

$$c_{i\sigma}=[\![\tau_i\le\sigma]\!]=1,\quad c_{i+1,\sigma}=[\![\tau_{i+1}\le\sigma]\!]=1\setminus[\![\sigma<\tau_{i+1}]\!].$$

So

$$[\![\sigma<\tau_{i+1}]\!]=c_{i\sigma}\setminus c_{i+1,\sigma}\subseteq[\![|u_\sigma-y_i|<\delta]\!]$$

(615M(d-v) again)

$$=[\![|u_\sigma-u_{\tau_i}|<\delta]\!].$$

Thus $\langle\tau_i\rangle_{i\in\mathbb{N}}$ satisfies (α)-(δ) of $\mathrm{SL}_1(\boldsymbol{u})$. As δ was arbitrary, $\mathrm{SL}_1(\boldsymbol{u})$ is true; by 631Pa, $\boldsymbol{u}$ is near-simple.

631T So far we have been looking at questions arising naturally from the ideas of Chapter 61. I come now to something which will be important in §644.

Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and C the set of non-negative non-decreasing order-bounded near-simple processes with domain $\mathcal{S}$. If $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ belongs to C and $\mathrm{Osclln}(\boldsymbol{u})\ne0$, there is a non-zero simple process $\boldsymbol{v}\in C$ such that $\boldsymbol{u}-\boldsymbol{v}\in C$.

proof (a) Since we are supposing that $\mathrm{Osclln}(\boldsymbol{u})\ne0$, we can be sure that $\mathcal{S}$ is non-empty. Set

$$u_\downarrow=\lim_{\sigma\downarrow\mathcal{S}}u_\sigma=\inf_{\sigma\in\mathcal{S}}u_\sigma,\quad u_\uparrow=\lim_{\sigma\uparrow\mathcal{S}}u_\sigma=\sup_{\sigma\in\mathcal{S}}u_\sigma$$

(631C, 613Ba).

(b) Set $\epsilon=\frac{1}{9}\theta(\mathrm{Osclln}(\boldsymbol{u}))$. Let $\boldsymbol{w}=\langle w_\sigma\rangle_{\sigma\in\mathcal{S}}$ be a simple process such that $\theta(\bar w)\le\epsilon^2$, where $\bar w=\sup|\boldsymbol{u}-\boldsymbol{w}|$, and τ_*, $\tau^*\in\mathcal{S}$ such that $\theta(u_{\tau_*}-u_\downarrow)\le\epsilon$ and $\theta(u_\uparrow-u_{\tau^*})\le\epsilon$; let I be a finite sublattice of $\mathcal{S}$ containing τ_* and τ^* and including a breakpoint string for $\boldsymbol{w}$. Take $(\tau_0,\dots,\tau_n)$ linearly generating the I-cells; then $(\tau_0,\dots,\tau_n)$ is a breakpoint string for $\boldsymbol{w}$ (612Kb again), while $\tau_0\le\tau_*$ and $\tau^*\le\tau_n$. Now

$$\mathrm{Osclln}(\boldsymbol{u})\le\mathrm{Osclln}^*_I(\boldsymbol{u})=(u_{\tau_0}-u_\downarrow)\vee\sup_{i<n}(u_{\tau_{i+1}}-u_{\tau_i})\vee(u_\uparrow-u_{\tau_n})$$

(618Cb, because $\boldsymbol{u}$ is non-decreasing), so

$$\begin{aligned}\theta(\sup_{i<n}(u_{\tau_{i+1}}-u_{\tau_i}))&\ge\theta(\mathrm{Osclln}(\boldsymbol{u}))-\theta(u_{\tau_0}-u_\downarrow)-\theta(u_\uparrow-u_{\tau_n})\\&\ge\theta(\mathrm{Osclln}(\boldsymbol{u}))-\theta(u_{\tau_0}-u_{\tau_*})-\theta(u_\uparrow-u_{\tau^*})\ge7\epsilon.\end{aligned}$$

Consequently

$$a=[\![\sup_{i<n}u_{\tau_{i+1}}-u_{\tau_i}>4\epsilon]\!]=\sup_{i<n}[\![u_{\tau_{i+1}}-u_{\tau_i}>4\epsilon]\!]$$

has measure at least 2ϵ. On the other hand, $[\![\bar{w}\ge\epsilon]\!]$ has measure at most ϵ, so $a\cap[\![\bar{w}\le\epsilon]\!]$ is non-zero. There is therefore an $i<n$ such that

$$b=[\![u_{\tau_{i+1}}-u_{\tau_i}>4\epsilon]\!]\cap[\![\bar{w}\le\epsilon]\!]$$

is non-zero.

(c) Consider

$$c=[\![u_{\tau_{i+1}}\le\epsilon]\!]\cup\sup_{\sigma\in\mathcal{S}}([\![\sigma<\tau_{i+1}]\!]\cap[\![u_{\tau_{i+1}}-u_\sigma\le\epsilon]\!]).$$

Then $c\in\mathfrak{A}_{\tau_{i+1}}$. **P** For any $\sigma\in\mathcal{T}$, $[\![\sigma<\tau_{i+1}]\!]\in\mathfrak{A}_{\tau_{i+1}}$ (611H(c-i)) while $[\![u_{\tau_{i+1}}-u_\sigma\le\epsilon]\!]\in\mathfrak{A}_{\tau_{i+1}\vee\sigma}$, so

$$[\![u_{\tau_{i+1}}-u_\sigma\le\epsilon]\!]\cap[\![\sigma\le\tau_{i+1}]\!]=[\![u_{\tau_{i+1}}-u_\sigma\le\epsilon]\!]\cap[\![\sigma\vee\tau_{i+1}=\tau_{i+1}]\!]$$

(611E(a-ii-β) again)

$$=[\![u_{\tau_{i+1}}-u_\sigma\le\epsilon]\!]\cap[\![\sigma\vee\tau_{i+1}\le\tau_{i+1}]\!]\in\mathfrak{A}_{\tau_{i+1}}$$

(611H(c-iii)),

$$[\![\sigma<\tau_{i+1}]\!]\cap[\![u_{\tau_{i+1}}-u_\sigma\le\epsilon]\!]=[\![\sigma<\tau_{i+1}]\!]\cap[\![\sigma\le\tau_{i+1}]\!]\cap[\![u_{\tau_{i+1}}-u_\sigma\le\epsilon]\!]\in\mathfrak{A}_{\tau_{i+1}}.$$

Taking the supremum over σ,

$$\sup_{\sigma\in\mathcal{S}}([\![\sigma<\tau_{i+1}]\!]\cap[\![u_{\tau_{i+1}}-u_\sigma\le\epsilon]\!])\in\mathfrak{A}_{\tau_{i+1}};$$

as surely $[\![u_{\tau_{i+1}}\le\epsilon]\!]\in\mathfrak{A}_{\tau_{i+1}}$, $c\in\mathfrak{A}_{\tau_{i+1}}$. **Q**

Next, $c\cap b=0$. **P** Of course

$$[\![u_{\tau_{i+1}}\le\epsilon]\!]\cap b\subseteq[\![u_{\tau_{i+1}}-u_{\tau_i}\le4\epsilon]\!]\cap b=0.$$

If $\sigma\in\mathcal{S}$ then

$$\begin{aligned}[\![\sigma<\tau_{i+1}]\!]&=[\![\sigma<\tau_0]\!]\cup\sup_{j\le i}[\![\tau_j\le\sigma]\!]\cap[\![\sigma<\tau_{j+1}]\!]\\&\subseteq[\![u_\sigma=u_{\sigma\wedge\tau_0}]\!]\cup\sup_{j\le i}[\![w_\sigma=w_{\tau_j}]\!]\\&\subseteq[\![u_\sigma\le u_{\tau_0}]\!]\cup\sup_{j\le i}[\![u_\sigma\le u_{\tau_j}+2\bar{w}]\!]\subseteq[\![u_\sigma\le u_{\tau_i}+2\bar{w}]\!].\end{aligned}$$

So

$$\begin{aligned}&b\cap[\![\sigma<\tau_{i+1}]\!]\cap[\![u_{\tau_{i+1}}-u_\sigma\le\epsilon]\!]\\&\quad\subseteq[\![u_\sigma\le u_{\tau_i}+2\bar{w}]\!]\cap[\![u_{\tau_{i+1}}-u_\sigma\le\epsilon]\!]\cap[\![\bar{w}\le\epsilon]\!]\cap[\![u_{\tau_{i+1}}-u_{\tau_i}\ge4\epsilon]\!]\\&\quad\subseteq[\![u_{\tau_{i+1}}-u_{\tau_i}\le3\epsilon]\!]\cap[\![u_{\tau_{i+1}}-u_{\tau_i}\ge4\epsilon]\!]=0.\end{aligned}$$

Taking the supremum over σ, $b\cap c=0$. **Q**

(d) Since $c\in\mathfrak{A}_{\tau_{i+1}}$, we have a simple process $\boldsymbol{v}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$, with singleton breakpoint string (τ_{i+1}), defined by saying that

$$[\![\sigma<\tau_{i+1}]\!]\subseteq[\![v_\sigma=0]\!],\quad[\![\tau_{i+1}\le\sigma]\!]\subseteq[\![v_\sigma=\epsilon\chi(1\setminus c)]\!]$$

for every $\sigma\in\mathcal{S}$. Evidently $\boldsymbol{v}$ is near-simple and non-negative, while if $\sigma\le\sigma'$ in $\mathcal{S}$,

$$\begin{aligned}[\![\sigma'<\tau_{i+1}]\!]\cup[\![\tau_{i+1}\le\sigma]\!]\cup c&\subseteq[\![v_\sigma=v_{\sigma'}]\!]\\&\subseteq[\![v_\sigma\le v_{\sigma'}]\!]\cap[\![u_\sigma-v_\sigma\le u_{\sigma'}-v_{\sigma'}]\!],\end{aligned}$$

$$[\![\sigma<\tau_{i+1}]\!]\cup c\subseteq[\![v_\sigma=0]\!]\subseteq[\![u_\sigma-v_\sigma\ge0]\!],$$

$$\begin{aligned}&[\![\sigma < \tau_{i+1}]\!] \cap [\![\tau_{i+1} \le \sigma']\!] \setminus c \\ &\qquad\qquad \subseteq [\![v_\sigma = 0]\!] \cap [\![v_{\sigma'} = \epsilon]\!] \cap [\![u_{\tau_{i+1}} \le u_{\sigma'}]\!] \setminus [\![u_{\tau_{i+1}} - u_\sigma \le \epsilon]\!] \\ &\qquad\qquad \subseteq [\![v_\sigma \le v_{\sigma'}]\!] \cap [\![v_{\sigma'} - v_\sigma \le \epsilon]\!] \cap [\![u_{\sigma'} - u_\sigma \ge \epsilon]\!] \\ &\qquad\qquad \subseteq [\![v_\sigma \le v_{\sigma'}]\!] \cap [\![u_\sigma - v_\sigma \le u_{\sigma'} - v_{\sigma'}]\!],\end{aligned}$$

$$[\![\tau_{i+1} \le \sigma]\!] \setminus c \subseteq [\![u_{\tau_{i+1}} \le u_\sigma]\!] \cap [\![u_{\tau_{i+1}} \ge \epsilon]\!] \cap [\![v_\sigma \le \epsilon]\!] \subseteq [\![u_\sigma - v_\sigma \ge 0]\!].$$

So $\boldsymbol{v}$ and $\boldsymbol{u}-\boldsymbol{v}$ are both non-decreasing, while $\boldsymbol{u}-\boldsymbol{v}$ is non-negative. Finally,

$$v_{\tau_{i+1}} = \epsilon\chi(1 \setminus c) \ge \epsilon\chi b \ne 0$$

and $\boldsymbol{v}$ is non-zero. Thus we have an appropriate $\boldsymbol{v}$.

631U Theorem Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$ and $\boldsymbol{u}$ a non-negative, order-bounded, non-decreasing near-simple process with domain $\mathcal{S}$. Then for any $\epsilon > 0$ there are non-negative, non-decreasing processes $\boldsymbol{v}$, $\boldsymbol{w}$ with domain $\mathcal{S}$ such that $\boldsymbol{v}$ is simple, $\boldsymbol{w}$ is jump-free, $\boldsymbol{u}-\boldsymbol{v}-\boldsymbol{w}$ is non-negative and non-decreasing, and $\theta(\sup|\boldsymbol{u}-\boldsymbol{v}-\boldsymbol{w}|) \le \epsilon$.

proof (a) As in 631T, let C be the set of non-negative non-decreasing near-simple order-bounded processes with domain $\mathcal{S}$. Note that if $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a non-negative non-decreasing order-bounded process,

$$\sup|\boldsymbol{v}| = \sup_{\sigma\in\mathcal{S}} v_\sigma = \lim_{\sigma\uparrow\mathcal{S}} v_\sigma.$$

(I think the formulae will look cleaner if I write $\sup\boldsymbol{v}$ rather than $\sup|\boldsymbol{v}|$ in this case.) It follows that if $\boldsymbol{v}$, $\boldsymbol{v}' \in C$ then $\sup(\boldsymbol{v}+\boldsymbol{v}') = \sup\boldsymbol{v} + \sup\boldsymbol{v}'$.

(b) Write C' for the set of simple processes $\boldsymbol{v} \in C$ such that $\boldsymbol{u}-\boldsymbol{v} \in C$. For $\boldsymbol{v} \in C'$, set

$$q(\boldsymbol{v}) = \sup\{\theta(\sup\boldsymbol{v}') : \boldsymbol{v}' \in C,\ \boldsymbol{v}+\boldsymbol{v}' \in C'\}.$$

Choose $\langle\boldsymbol{v}_n\rangle_{n\in\mathbb{N}}$ inductively such that

$$\boldsymbol{v}_n \in C', \quad \boldsymbol{v}_{n+1}-\boldsymbol{v}_n \in C,$$

$$\theta(\sup(\boldsymbol{v}_{n+1}-\boldsymbol{v}_n)) \ge q(\boldsymbol{v}_n) - 2^{-n}$$

for every $n \in \mathbb{N}$. Then $\langle\boldsymbol{v}_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of processes bounded above by $\boldsymbol{u}$. Set $\boldsymbol{v}' = \sup_{n\in\mathbb{N}}\boldsymbol{v}_n$; by 612Ia, $\boldsymbol{v}'$ is a fully adapted process. Evidently $\boldsymbol{v}'$ is non-decreasing and non-negative; moreover, $\boldsymbol{v}_m - \boldsymbol{v}_n \in C$ whenever $m \ge n$, so $\boldsymbol{v}'-\boldsymbol{v}_n$ is non-decreasing and non-negative for every n.

(c) $\lim_{n\to\infty}\theta(\sup(\boldsymbol{v}_{n+1}-\boldsymbol{v}_n)) = 0$. **P** $\langle\sup\boldsymbol{v}_n\rangle_{n\in\mathbb{N}}$ is non-decreasing and bounded above by $\sup\boldsymbol{u}$, so is convergent for the topology of measure (613Ba); accordingly, using (a),

$$\lim_{n\to\infty}\theta(\sup(\boldsymbol{v}_{n+1}-\boldsymbol{v}_n)) = \lim_{n\to\infty}\theta(\sup\boldsymbol{v}_{n+1} - \sup\boldsymbol{v}_n) = 0. \ \mathbf{Q}$$

It follows that $\lim_{n\to\infty} q(\boldsymbol{v}_n) = 0$. Note also that $\sup\boldsymbol{v}' = \sup_{n\in\mathbb{N}}\sup\boldsymbol{v}_n$, so that

$$\lim_{n\to\infty}\theta(\sup(\boldsymbol{v}'-\boldsymbol{v}_n)) = \lim_{n\to\infty}\theta(\sup\boldsymbol{v}' - \sup\boldsymbol{v}_n) = 0;$$

as every $\boldsymbol{v}_n$ is simple, $\boldsymbol{v}'$ is near-simple and belongs to C.

(d) Set $\boldsymbol{w} = \boldsymbol{u}-\boldsymbol{v}'$. By (c), $\boldsymbol{w}$ is near-simple; moreover, as $\boldsymbol{u}-\boldsymbol{v}_n$ is non-decreasing and non-negative for every n, so is $\boldsymbol{w}$. Now $\boldsymbol{w}$ is jump-free. **P?** Otherwise, 631T tells us that there is a non-zero simple process $\boldsymbol{v} \in C$ such that $\boldsymbol{w}-\boldsymbol{v} \in C$. Consequently

$$\boldsymbol{v}_n + \boldsymbol{v}, \quad \boldsymbol{u}-\boldsymbol{v}_n-\boldsymbol{v} = (\boldsymbol{u}-\boldsymbol{v}'-\boldsymbol{v}) + (\boldsymbol{v}'-\boldsymbol{v}_n)$$

belong to C for every n, and $q(\boldsymbol{v}_n) \ge \theta(\sup\boldsymbol{v}) > 0$ for every n; which is impossible. **XQ**

(e) Now observe that

$$\theta(\sup(\boldsymbol{u}-\boldsymbol{v}_n-\boldsymbol{w})) = \theta(\sup\boldsymbol{v}' - \sup\boldsymbol{v}_n) \to 0$$

as $n \to \infty$. So $\boldsymbol{v}_n$ will serve for $\boldsymbol{v}$ for any sufficiently large n.

631V Corollary Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$ and $\boldsymbol{u}$ a near-simple process of bounded variation with domain $\mathcal{S}$. Then for any $\epsilon > 0$ there are processes $\boldsymbol{v}$, $\boldsymbol{w}$ with domain $\mathcal{S}$ such that $\boldsymbol{v}$ is simple, $\boldsymbol{w}$ is jump-free and of bounded variation and $\theta(\int_{\mathcal{S}} |d(\boldsymbol{u}-\boldsymbol{v}-\boldsymbol{w})|) \le \epsilon$.

proof By 631L, $\boldsymbol{u}$ is expressible as $\boldsymbol{u}_0 - \boldsymbol{u}_1$ where $\boldsymbol{u}_0$, $\boldsymbol{u}_1$ are non-negative non-decreasing near-simple order-bounded processes. By 631U, we have simple processes $\boldsymbol{v}_0$, $\boldsymbol{v}_1$ and jump-free processes of bounded variation $\boldsymbol{w}_0$, $\boldsymbol{w}_1$ such that $\boldsymbol{z}_0 = \boldsymbol{u}_0 - \boldsymbol{v}_0 - \boldsymbol{w}_0$ and $\boldsymbol{z}_1 = \boldsymbol{u}_1 - \boldsymbol{v}_1 - \boldsymbol{w}_1$ are non-decreasing and non-negative and $\theta(\sup|\boldsymbol{z}_0|)$, $\theta(\sup|\boldsymbol{z}_1|)$ are both at most $\frac{1}{2}\epsilon$.

Set $\boldsymbol{v} = \boldsymbol{v}_0 - \boldsymbol{v}_1$, $\boldsymbol{w} = \boldsymbol{w}_0 - \boldsymbol{w}_1$ and $\boldsymbol{z} = \boldsymbol{z}_0 - \boldsymbol{z}_1$, so that $\boldsymbol{v}$ is simple, $\boldsymbol{w}$ is jump-free and of bounded variation, and $\boldsymbol{u} - \boldsymbol{v} - \boldsymbol{w} = \boldsymbol{z}$. Now $\int_{\mathcal{S}} |d\boldsymbol{z}| \le \sup|\boldsymbol{z}_0| + \sup|\boldsymbol{z}_1|$. **P** Expressing $\boldsymbol{z}$, $\boldsymbol{z}_0$ and $\boldsymbol{z}_1$ as $\langle z_\sigma\rangle_{\sigma\in\mathcal{S}}$, $\langle z_{0\sigma}\rangle_{\sigma\in\mathcal{S}}$ and $\langle z_{1\sigma}\rangle_{\sigma\in\mathcal{S}}$, we see that

$$|z_\tau - z_\sigma| \le |z_{0\tau} - z_{0\sigma}| + |z_{1\tau} - z_{1\sigma}| = (z_{0\tau} - z_{0\sigma}) + (z_{1\tau} - z_{1\sigma})$$

whenever $\sigma \le \tau$ in $\mathcal{S}$. Consequently

$$S_I(\mathbf{1}, |d\boldsymbol{z}|) \le S_I(\mathbf{1}, d\boldsymbol{z}_0) + S_I(\mathbf{1}, d\boldsymbol{z}_1)$$

for every $I \in \mathcal{I}(\mathcal{S})$, and

$$\int_{\mathcal{S}} |d\boldsymbol{z}| \le \int_{\mathcal{S}} d\boldsymbol{z}_0 + \int_{\mathcal{S}} d\boldsymbol{z}_1 = \lim_{\sigma\uparrow\mathcal{S}}(z_{0\sigma} + z_{1\sigma}) - \lim_{\sigma\downarrow\mathcal{S}}(z_{0\sigma} + z_{1\sigma})$$

(613N)

$$\le \lim_{\sigma\uparrow\mathcal{S}} z_{0\sigma} + \lim_{\sigma\uparrow\mathcal{S}} z_{1\sigma} = \sup|\boldsymbol{z}_0| + \sup|\boldsymbol{z}_1|. \quad \mathbf{Q}$$

So we have

$$\theta(\int_{\mathcal{S}} |d\boldsymbol{z}|) \le \theta(\sup|\boldsymbol{z}_0|) + \theta(\sup|\boldsymbol{z}_1|) \le \epsilon$$

as required.

631X Basic exercises (a)

(j) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and give $(L^0)^{\mathcal{S}}$ its product topology when each factor is given the topology of convergence in measure. Let $M_{\mathrm{simp}} \subseteq (L^0)^{\mathcal{S}}$ be the set of simple processes with domain $\mathcal{S}$. Show that the topological closure of M_{simp} in $(L^0)^{\mathcal{S}}$ is the set of all fully adapted processes with domain $\mathcal{S}$.

Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u}$ a near-simple process with domain $\mathcal{S}$. Show that $\boldsymbol{u}\restriction\mathcal{S}\cap[\tau,\tau']$ is near-simple whenever $\tau \le \tau'$ in $\mathcal{S}$.

(b) Give an example of a simple process $\boldsymbol{u}$ and a sublattice $\mathcal{S}'$ of $\operatorname{dom}\boldsymbol{u}$ such that $\boldsymbol{u}\restriction\mathcal{S}'$ is not near-simple. (*Hint*: take $\mathcal{S}'$ the lattice of constant stopping times.)

(c) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{u}$ a near-simple process with domain $\mathcal{S}$. Show that for any $\epsilon > 0$ there is a simple process $\boldsymbol{u}'$ with domain $\mathcal{S}$ such that $\theta(\sup|\boldsymbol{u}-\boldsymbol{u}'|) \le \epsilon$ and $\sup|\boldsymbol{u}'| \le \sup|\boldsymbol{u}|$.

(d) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a near-simple process. Set $v_\tau = \sup_{\sigma\in\mathcal{S}\wedge\tau} u_\sigma$ for $\tau \in \mathcal{S}$ (cf. 614Fb). Show that $\langle v_\tau\rangle_{\tau\in\mathcal{S}}$ is a near-simple process.

(e) In 631Mb, show that $\boldsymbol{v} \in M_{\text{n-s}}(\mathcal{S}')$ is of bounded variation iff $\Psi(\boldsymbol{v}) \in M_{\text{n-s}}(\mathcal{S})$ is of bounded variation, and in this case $\Psi(\boldsymbol{v}^\uparrow) = \Psi(\boldsymbol{v}^\uparrow)$.

(f) In 631Mb, show that if $\boldsymbol{v}$ is a local integrator with domain $\mathcal{S}$, then $ii_{\boldsymbol{v}\restriction\mathcal{S}'}(\boldsymbol{u}) = ii_{\boldsymbol{v}}(\Psi(\boldsymbol{u}))\restriction\mathcal{S}'$ for every $\boldsymbol{u} \in M_{\text{n-s}}(\mathcal{S}')$.

(h) Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{S}$ such that $\inf A \in \mathcal{S}$ for every non-empty $A \subseteq \mathcal{S}$. Show that $\mathcal{S}\wedge\tau$ is full for every $\tau \in \mathcal{S}$.

(i) Let $\boldsymbol{v}$ be a non-negative, non-decreasing near-simple process. (i) Show that there are a non-negative, non-decreasing jump-free process $\boldsymbol{w}$ and a sequence $\langle\boldsymbol{u}_n\rangle_{n\in\mathbb{N}}$ of non-negative non-decreasing simple processes such that $\boldsymbol{v}=\boldsymbol{w}+\sum_{n=0}^{\infty}\boldsymbol{u}_n$. (ii) Show that for every $\epsilon>0$ there is a non-decreasing simple process $\boldsymbol{u}$ such that $\boldsymbol{v}-\boldsymbol{u}$ is non-decreasing and $\theta(\mathrm{Osclln}(\boldsymbol{v}-\boldsymbol{u}))\le\epsilon$.

(k) In 631Mb, show that $\mathrm{Osclln}(\Psi(\boldsymbol{u}))=\mathrm{Osclln}(\boldsymbol{u})$ for every $\boldsymbol{u}\in M_{\text{n-s}}(\mathcal{S}')$, so that $\Psi(\boldsymbol{u})$ is jump-free iff $\boldsymbol{u}$ is jump-free.

(l) Let $\mathcal{S}$ be a non-empty finitely full sublattice of $\mathcal{T}$, and $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a fully adapted process. Show that $\boldsymbol{u}$ is near-simple iff $\boldsymbol{u}$ is locally near-simple and moderately oscillatory iff $\boldsymbol{u}$ is locally near-simple and order-bounded and $\lim_{\sigma\uparrow\mathcal{S}}u_\sigma$ is defined in $L^0(\mathfrak{A})$.

631Y Further exercises (a) Let $\boldsymbol{w}=\langle w_\sigma\rangle_{\sigma\in\mathcal{T}_f}$ be Brownian motion based on the real-time stochastic integration structure $(\mathfrak{C},\bar\nu,[0,\infty[\,,\langle\mathfrak{C}_t\rangle_{t\ge0})$, as described in 612T. Set $\mathcal{S}=\mathcal{T}\wedge\check{1}$ where $\check{1}$ is the constant stopping time with value 1. Show that there is a near-simple process $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ of bounded variation such that $ii_{\boldsymbol{w}}(\boldsymbol{u})$ is not a martingale.

(b) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a near-simple process. Show that if $\mathcal{S}'$ is a sublattice of $\mathcal{S}$ which is order-convex in $\mathcal{S}$ (that is, $\mathcal{S}\cap[\sigma,\tau]\subseteq\mathcal{S}'$ whenever $\sigma\le\tau$ in $\mathcal{S}'$), then $\boldsymbol{u}\restriction\mathcal{S}'$ is near-simple.

(c) Let $\boldsymbol{u}_1,\dots,\boldsymbol{u}_k$ be (locally) near-simple processes all with the same domain $\mathcal{S}$, and $h:\mathbb{R}^k\to\mathbb{R}$ a continuous function. Write $\boldsymbol{U}$ for $(\boldsymbol{u}_1,\dots,\boldsymbol{u}_k)$. Show that $\bar h\boldsymbol{U}$ (619G) is (locally) near-simple.

(e) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. Show that $M_{\text{ln-s}}(\mathcal{S})$ is closed in $M_{\text{lo-b}}(\mathcal{S})$ for the local ucp topology of 615Xb.

(d) Suppose that $\mathcal{S}=[\min\mathcal{S},\max\mathcal{S}]$ is a closed interval in $\mathcal{T}$ and that $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a jump-free process. Show that for every $z\in L^0(\mathfrak{A}_{\min\mathcal{S}})$ there is a $\tau\in\mathcal{S}$ such that $[\![u_{\min\mathcal{S}}\le z]\!]\cap[\![z\le u_{\max\mathcal{S}}]\!]\subseteq[\![u_\tau=z]\!]$.

(f) Let $(\mathfrak{A},\bar\mu)$ be the Lebesgue measure algebra and $\langle e_n\rangle_{n\in\mathbb{N}}$ a stochastically independent sequence of members of $\mathfrak{A}$ of measure $\frac{1}{2}$. For $n\in\mathbb{N}$ let $\mathfrak{A}_{2n}$ be the subalgebra of $\mathfrak{A}$ generated by $\{e_i:i<n\}$ and set $\mathfrak{A}_{2n+1}=\mathfrak{A}_{2n+2}$ for each n. In the stochastic integration structure $(\mathfrak{A},\bar\mu,\mathbb{N},\langle\mathfrak{A}_n\rangle_{n\in\mathbb{N}},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$, take $\mathcal{S}=\{\check n:n\in\mathbb{N}\}$ and $\mathcal{S}'=\{\sigma_n:n\in\mathbb{N}\}$ where $\sigma_n=\check{2n}$ for each n. Set $v_{\sigma_n}=\sum_{0\le i<n}\frac{1}{i+1}(2\chi e_i-\chi 1)$ for $n\in\mathbb{N}$. Show that $\boldsymbol{v}=\langle v_{\sigma_n}\rangle_{n\in\mathbb{N}}$ is a $\|\;\|_2$-bounded martingale and a near-simple integrator with domain $\mathcal{S}'$, but that if we define $\Psi(\boldsymbol{v})\in M_{\text{n-s}}(\mathcal{S})$ as in 631Mb, $\Psi(\boldsymbol{v})$ is not an integrator.

631 Notes and comments

From the beginning, the theory of continuous-time stochastic processes has given special prominence to those which have continuous or càdlàg sample paths; see §455. Just as the jump-free processes of §618 correspond to continuous sample paths (618H), near-simple processes correspond to càdlàg sample paths (631D). In comparison, the definition I offer of 'near-simple' process (631B) is agreeably more straightforward than the definition of 'jump-free' process in 618B. The only thing to remember is that, as with simple processes and jump-free processes, the restriction of a near-simple process to a sublattice need not be near-simple. So we have to take care when specifying the domain of any near-simple process we want to think about. But the journey through the elementary properties of the space $M_{\text{n-s}}$ of near-simple processes (631C, 631D-631J) gives no difficulties. 631K calls for a little ingenuity, but all the ingredients are in §614. Perhaps this is a good place to remark that near-simple processes are defined in terms of the ucp topology, so remain near-simple under any change of law.

In 631M I give a wide-ranging result on the extension of near-simple processes to larger sublattices. Like the previous extension theorem 612P, it is (in essence) too facile to offer much insight. But in the proofs of 612Qa and 631N these general propositions give us a helpful lift over potentially awkward obstacles.

The point of the stopping lemma SL_1 in 631O-631S is that we often have a description of (locally) near-simple processes which is of a quite different kind from that in 631B. Furthermore, this alternative description, when applicable, leads us to a strikingly stronger approximation by simple processes than that promised by 631B. If we take a sequence $\langle\tau_n\rangle_{n\in\mathbb{N}}$ as constructed in the proof of 631Ra, and truncate it appropriately, we find ourselves with a breakpoint string $(\tau_0,\ldots,\tau_n)$ from which we can define a simple process $\boldsymbol{u}'$ ϵ-approximating the near-simple process $\boldsymbol{u}$ except in the small region $[\![\tau_n < \max\mathcal{S}]\!]$. Now we knew that we had an approximation which would be ϵ-close to $\boldsymbol{u}$ except perhaps in a region of measure at most ϵ. But the Stopping Lemma gives us a progressive description of the approximation, each successive breakpoint being chosen in terms of a criterion observable at the time of the jump. Once again, a simple modification of SL_1 gives us a matching description SL_2 of jump-free processes.

I have tried to set these out in a way which exhibits one of the essential difficulties in the theory of stochastic processes. If a process is significant at all, stopping times defined by that process are likely to be important. (See §§477-479, for instance, or the jump-times of the Poisson process in 612Uf.) For jump-free processes like Brownian motion, especially if we have a representation with continuous sample paths, we expect hitting times to be easy to investigate, though there can be real surprises (see 652M-652N below). In fact we can have a kind of Darboux continuity, as in 631Yd. But with càdlàg paths we never know quite what will happen next; as we watch the process evolve, it might at any moment flip into a new state unforeseeable from anything we have seen before. I will go into this further in §643, with the notion of 'accessibility' of a stopping time.

As usual, my presentation is ahistorical. Near-simple processes, as I have defined them, are close enough to the classical concept of càdlàg process for the classical theory (see 633R below) to have results corresponding to 631R. The 'moderately oscillatory' processes of §614, and the fundamental facts in 615M, are my own invention. But you will have no difficulty in seeing how 615M could be inspired by a proof of 631Ra.

There are other ways in which a process may be expressible as the limit of a more or less special sequence of simple processes. I give 631U as a 'theorem' because this is the form I shall want to quote in §644. But 631V is an alternative form of the same idea. In the version of 631Xi(i), we see that $\boldsymbol{v}$ is expressed as the sum of a jump-free part and a 'saltus' part, recalling the Lebesgue decomposition of a function of bounded variation into a continuous part and a saltus part, as in 226Ca. In 631V I suggest a different measure of approximation, using $\theta(\int_{\mathcal{S}}|d\boldsymbol{u}|)$ in place of $\theta(\sup|\boldsymbol{u}|)$. And in 643M there will be a (much deeper) result on martingales which also seeks to express a given process as the sum of a simpler process and a process with small residual oscillation.

Version of 14.8.20/6.12.21

632 Right-continuous filtrations

Up to this point, we have been able (with some effort) to work in the full generality of stochastic integration structures $(\mathfrak{A},\bar\mu,T,\langle\mathfrak{A}_t\rangle_{t\in T})$ as described in §§611-613. We are now approaching territory in which we shall need to have filtrations which are 'right-continuous' in the sense of 632B. These include the standard examples (632D). The results I present here are a quick run through new features of the structures developed in §§611-612 (632C) and an important characterization of near-simple processes on infimum-closed full sublattices (632F). With this in hand, we see that in the most familiar contexts local martingales will be locally near-simple (632I) and we have a useful test for being a martingale (632J). In 632N I describe a classic example of a local martingale.

632A Notation As usual, $(\mathfrak{A},\bar\mu,T,\langle\mathfrak{A}_t\rangle_{t\in T},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ will be a stochastic integration structure. If $A\subseteq\mathcal{T}$ and $\tau\in\mathcal{T}$, I write $A\wedge\tau$ for $\{\sigma\wedge\tau:\sigma\in A\}$. The f-algebra $L^0=L^0(\mathfrak{A})$ will be given its linear space topology of convergence in measure. $\mathcal{T}_b$, $\mathcal{T}_f\subseteq\mathcal{T}$ will be the ideals of bounded and finite-valued stopping times, as in 611A(b-iii). For $\sigma\in\mathcal{T}$, P_σ will be the conditional expectation operator from $L^1_{\bar\mu}=L^1(\mathfrak{A},\bar\mu)$ to itself associated with the closed subalgebra $\mathfrak{A}_\sigma$; for $z\in L^1_{\bar\mu}$, $\boldsymbol{P}z$ will be the martingale $\langle P_\sigma z\rangle_{\sigma\in\mathcal{T}}$. I will write 1lim for $\|\,\|_1$-limits in $L^1_{\bar\mu}$.

632B Definition I will say that $\langle\mathfrak{A}_t\rangle_{t\in T}$ or $(\mathfrak{A},\bar\mu,T,\langle\mathfrak{A}_t\rangle_{t\in T})$ or $(\mathfrak{A},\bar\mu,T,\langle\mathfrak{A}_t\rangle_{t\in T},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ is **right-continuous** if $\mathfrak{A}_t=\bigcap_{s>t}\mathfrak{A}_s$ whenever $t\in T$ is not isolated on the right.

The notion of 'right-continuity' is of course a phenomenon which can appear anywhere in the theory of ordered sets, and will recur intermittently in this volume in a variety of contexts. I have therefore chosen this phrase for general use, as follows. If P and Q are partially ordered sets, an order-preserving function

$f : P \to Q$ is **right-continuous** if $\inf f[C]$ is defined and equal to $f(\inf C)$ whenever $C \subseteq P$ is non-empty and downwards-directed and has an infimum in P. When P is totally ordered, this will be the case iff $\inf_{q>p} f(q) = f(p)$ whenever $p \in P$ is not isolated on the right. So a filtration $\langle \mathfrak{A}_t \rangle_{t\in T}$ of closed subalgebras of $\mathfrak{A}$ is right-continuous if it is right-continuous when considered as an order-preserving function from T to the set of closed subalgebras of $\mathfrak{A}$ (or, of course, when considered as a function from T to $\mathcal{P}\mathfrak{A}$). Similarly, a filtration $\langle \Sigma_t \rangle_{t\in[0,\infty[}$ of σ-algebras is right-continuous if $\Sigma_t = \bigcap_{s>t} \Sigma_s$ for every $t \ge 0$. See 632C(a-iii) for another example, and 632C(a-ii) for a kind of inverted example.

632C Proposition Suppose that $\langle \mathfrak{A}_t \rangle_{t\in T}$ is right-continuous.

(a) Suppose that $C \subseteq \mathcal{T}$ is non-empty.

(i)

$$\begin{aligned}[\![\inf C > t]\!] &= \inf_{\tau\in C} [\![\tau > t]\!] \text{ if } t \text{ is isolated on the right,} \\ &= \sup_{s>t} \inf_{\tau\in C} [\![\tau > s]\!] \text{ otherwise.}\end{aligned}$$

(ii) $[\![\inf C < \tau]\!] = \sup_{\upsilon\in C} [\![\upsilon < \tau]\!]$ for every $\tau \in \mathcal{T}$.

(iii) $\mathfrak{A}_{\inf C} = \bigcap_{\tau\in C} \mathfrak{A}_\tau$.

(b) (Compare 611Cd.) If C, $D \subseteq \mathcal{T}$ are non-empty, then $\inf C \vee \inf D = \inf\{\sigma \vee \tau : \sigma \in C, \tau \in D\}$.

(c) If $\mathcal{S}$ is a sublattice of $\mathcal{T}$, $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$ is a fully adapted process and $A \subseteq \mathcal{S}$ is a non-empty downwards-directed set such that $u = \lim_{\sigma\downarrow A} u_\sigma$ is defined in L^0, then $u \in L^0(\mathfrak{A}_{\inf A})$.

proof (a) Write σ for $\inf C$.

(i) If $s' > t$ in T, then

$$\sup_{s>t} \inf_{\tau\in C} [\![\tau > s]\!] = \sup_{s'\ge s>t} \inf_{\tau\in C} [\![\tau > s]\!] \in \mathfrak{A}_{s'},$$

so

$$\sup_{s>t} \inf_{\tau\in C} [\![\tau > s]\!] \in \bigcap_{s>t} \mathfrak{A}_s = \mathfrak{A}_t$$

whenever $t \in T$ is not isolated on the right, and the result follows immediately from 611F.

(ii) By 611E(c-iv-β), $[\![\sigma < \tau]\!] \supseteq \sup_{\upsilon\in C} [\![\upsilon < \tau]\!]$. Let $t \in T$. If t is isolated on the right, then

$$[\![\tau > t]\!] \setminus [\![\sigma > t]\!] = [\![\tau > t]\!] \setminus \inf_{\upsilon\in C} [\![\upsilon > t]\!]$$

((i) above)

$$= \sup_{\upsilon\in C} [\![\tau > t]\!] \setminus [\![\upsilon > t]\!] \subseteq \sup_{\upsilon\in C} [\![\upsilon < \tau]\!];$$

otherwise, again using (i),

$$\begin{aligned}[\![\tau > t]\!] \setminus [\![\sigma > t]\!] &= \sup_{s>t} ([\![\tau > s]\!] \setminus \sup_{s'>t} \inf_{\upsilon\in C} [\![\upsilon > s']\!]) \\ &\subseteq \sup_{s>t} [\![\tau > s]\!] \setminus \inf_{\upsilon\in C} [\![\upsilon > s]\!] \\ &= \sup_{s>t} \sup_{\upsilon\in C} ([\![\tau > s]\!] \setminus [\![\upsilon > s]\!]) \subseteq \sup_{\upsilon\in C} [\![\upsilon < \tau]\!].\end{aligned}$$

So

$$[\![\sigma < \tau]\!] = \sup_{t\in T} [\![\tau > t]\!] \setminus [\![\sigma > t]\!] \subseteq \sup_{\upsilon\in C} [\![\upsilon < \tau]\!]$$

and we have equality.

(iii) Now suppose that $\sigma = \inf C$ in $\mathcal{T}$. By 611H(c-ii), $\mathfrak{A}_\sigma \subseteq \bigcap_{\tau\in C} \mathfrak{A}_\tau$. Conversely, suppose that $a \in \bigcap_{\tau\in C} \mathfrak{A}_\tau$ and that $t \in T$. For $s \in T$ set $a_s = \inf_{\tau\in C} [\![\tau > s]\!]$; then $a \setminus a_s = \sup_{\tau\in C} a \setminus [\![\tau > s]\!] \in \mathfrak{A}_s$ (611G). By (a), $[\![\sigma > t]\!] = a_t$ if t is isolated on the right, and is $\sup_{s>t} a_s$ otherwise. So

$$a \setminus [\![\sigma > t]\!] = a \setminus a_t \in \mathfrak{A}_t \text{ if } t \text{ is isolated on the right,}$$
$$= a \setminus \sup_{s>t} a_s = \inf_{s>t} a \setminus a_s \in \bigcap_{t'>t} \mathfrak{A}_{t'} = \mathfrak{A}_t \text{ otherwise}$$

because

$$\inf_{s>t} a \setminus a_s = \inf_{t' \ge s > t} a \setminus a_s \in \mathfrak{A}_{t'}$$

whenever $t' > t$. As t is arbitrary, $a \in \mathfrak{A}_\sigma$. As a is arbitrary, $\mathfrak{A}_\sigma \supseteq \bigcap_{\tau \in C} \mathfrak{A}_\tau$ and we have equality.

(b) Set $E = \{\sigma \vee \tau : \sigma \in C,\ \tau \in D\}$, $\phi_C(t) = \inf_{\sigma \in C} [\![\sigma > t]\!]$, $\phi_D(t) = \inf_{\sigma \in D} [\![\sigma > t]\!]$ and $\phi_E(t) = \inf_{\sigma \in E} [\![\sigma > t]\!]$ for $t \in T$. Then

$$\phi_E(t) = \inf_{\sigma \in C, \tau \in D} [\![\sigma \vee \tau > t]\!] = \inf_{\sigma \in C, \tau \in D} [\![\sigma > t]\!] \cup [\![\tau > t]\!] = \inf_{\sigma \in C} [\![\sigma > t]\!] \cup \inf_{\tau \in D} [\![\tau > t]\!]$$

(313Bd)

$$= \phi_C(t) \cup \phi_D(t)$$

for every $t \in T$. If t is isolated on the right, we have

$$[\![\inf E > t]\!] = \phi_E(t) = \phi_C(t) \cup \phi_D(t) = [\![\inf C > t]\!] \cup [\![\inf D > t]\!].$$

If t is not isolated on the right, then

$$[\![\inf E > t]\!] = \sup_{s>t} \phi_E(s) = \sup_{s>t} \phi_C(s) \cup \phi_D(s) = \sup_{s,s'>t} \phi_C(s) \cup \phi_D(s')$$
$$= \sup_{s>t} \phi_C(s) \cup \sup_{s'>t} \phi_D(s') = [\![\inf C > t]\!] \cup [\![\inf D > t]\!].$$

Thus

$$[\![\inf E > t]\!] = [\![\inf C > t]\!] \cup [\![\inf D > t]\!] = [\![\inf C \vee \inf D > t]\!]$$

for every t, and $\inf E = \inf C \vee \inf D$, as claimed.

(c) Put 613Bj and (a-iii) here together.

632D Examples (a) In the construction of Brownian motion in 612T, $\langle \mathfrak{C}_t \rangle_{t \ge 0}$ is right-continuous.

P Suppose that $t \ge 0$ and that $c \in \bigcap_{s>t} \mathfrak{C}_s$. For each $k \in \mathbb{N}$ let $E_k \in \Sigma_{t+2^{-k}}$ be such that $E_k^\bullet = c$. Since $E_j \triangle E_k$ is negligible for all j and k, $c = E^\bullet$ where $E = \bigcap_{k \in \mathbb{N}} \bigcup_{j \ge k} E_j$. But now $E \in \bigcap_{s>t} \Sigma_s$; as 477Hc tells us that $\bigcap_{s>t} \hat{\Sigma}_s = \hat{\Sigma}_t$, $E \in \hat{\Sigma}_t$ and $c \in \mathfrak{C}_t$. **Q**

(b) In the construction of the standard Poisson process in 612U, $\langle \mathfrak{A}_t \rangle_{t \ge 0}$ is right-continuous. **P** From the formulae

$$\Sigma_t = \{F : F \subseteq \Omega,\ F \in \hat{\hat{\Sigma}}_t\}$$

and

$$\hat{\hat{\Sigma}}_t = \bigcap_{s>t} \hat{\hat{\Sigma}}_s$$

in 612Uc, it is clear that $\Sigma_t = \bigcap_{s>t} \Sigma_s$ and therefore $\mathfrak{A}_t = \bigcap_{s>t} \mathfrak{A}_t$ for every $t \ge 0$. **Q**

632E Lemma (Compare 631N.) Suppose that $\langle \mathfrak{A}_t \rangle_{t \in T}$ is right-continuous. Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ a locally near-simple process, and $A \subseteq \mathcal{S}$ a non-empty downwards-directed set such that $\tau = \inf A$ belongs to $\mathcal{S}$. Then $u_\tau = \lim_{\sigma \downarrow A} u_\sigma$, and in fact for every $\epsilon > 0$ there is a $\sigma \in A$ such that $\theta(\sup_{\rho \in \mathcal{S} \cap [\tau, \sigma]} |u_\rho - u_\tau|) \le \epsilon$.

proof (a) Take any $\tau' \in A$. Then $\boldsymbol{u} \restriction \mathcal{S} \cap [\tau, \tau']$ is near-simple (631F(iv)), so there is a simple process $\boldsymbol{v} = \langle v_\rho \rangle_{\rho \in \mathcal{S} \cap [\tau, \tau']}$ such that $\theta(\sup |\boldsymbol{v} - \boldsymbol{u} \restriction \mathcal{S} \cap [\tau, \tau']) \le \frac{1}{3}\epsilon$. $\boldsymbol{v}$ has starting value v_τ and a breakpoint string $(\sigma_0,\ldots,\sigma_n)$ say. For each $i\le n$, $[\![\tau<\sigma_i]\!]=\sup_{\sigma\in A}[\![\sigma<\sigma_i]\!]$ (632C(a-ii)); as A is downwards-directed, there is a $\sigma\in A$ such that $\sigma\le\tau'$ and $\bar\mu([\![\tau<\sigma_i]\!]\setminus[\![\sigma<\sigma_i]\!])\le\frac{\epsilon}{3(n+1)}$ for every i. Set $b=\sup_{i\le n}[\![\tau<\sigma_i]\!]\setminus[\![\sigma<\sigma_i]\!]$, so that $\bar\mu b\le\frac{1}{3}\epsilon$ and

$$[\![v_\rho\ne v_\tau]\!]\subseteq\sup_{i\le n}[\![\sigma_i\le\rho]\!]\cap[\![\tau<\sigma_i]\!]\subseteq b,$$

$$1\setminus b\subseteq[\![v_\rho=v_\tau]\!]\subseteq[\![|u_\rho-u_\tau|\le\tfrac{2}{3}\epsilon]\!]$$

for every $\rho\in\mathcal{S}\cap[\tau,\sigma]$. Now $\boldsymbol{u}\restriction\mathcal{S}\cap[\tau,\tau']$ is order-bounded so $\bar u=\sup|\boldsymbol{u}\restriction\mathcal{S}\cap[\tau,\tau']|$ is defined in L^0, and now $|u_\rho-u_\tau|\le\dfrac{2}{3}\epsilon\chi1+\bar u\times\chi b$ for every $\rho\in\mathcal{S}\cap[\tau,\sigma]$, so

$$\theta(\sup_{\rho\in\mathcal{S}\cap[\tau,\sigma]}|u_\rho-u_\tau|)\le\theta(\frac{2}{3}\epsilon\chi1)+\theta(\bar u\times\chi b)\le\frac{2}{3}\epsilon+\bar\mu b\le\epsilon.$$

(b) Of course it follows at once that $u_\tau=\lim_{\sigma\downarrow A}u_\sigma$.

632F Theorem Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ such that $\inf A\in\mathcal{S}$ whenever $A\subseteq\mathcal{S}$ is non-empty and has a lower bound in $\mathcal{S}$. If $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a fully adapted process, then $\boldsymbol{u}$ is locally near-simple iff it is locally moderately oscillatory and

$$(\dagger)\ u_{\inf A}=\lim_{\sigma\downarrow A}u_\sigma \text{ for every non-empty downwards-directed } A\subseteq\mathcal{S}$$
$$\text{with a lower bound in } \mathcal{S}.$$

proof (a) Suppose that $\boldsymbol{u}$ satisfies the conditions.

(i) If $\mathcal{S}$ has a least element, we can apply 631S to see that $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ is near-simple for every $\tau\in\mathcal{S}$, that is, that $\boldsymbol{u}$ is locally near-simple.

(ii) Generally, we can apply (i) to see that $\boldsymbol{u}\restriction\mathcal{S}\cap[\tau,\tau']$ is near-simple whenever $\tau\le\tau'$ in $\mathcal{S}$. Now if $\tau'\in\mathcal{S}$, $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau'$ is moderately oscillatory and 631F(c-i) assures us that $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau'$ is near-simple. Accordingly $\boldsymbol{u}$ is locally near-simple.

(b) Now suppose that $\boldsymbol{u}$ is locally near-simple. By 631Ca, $\boldsymbol{u}$ is surely locally moderately oscillatory, so it will be enough to show that it satisfies the condition (γ). Take a non-empty downwards-directed set $A\subseteq\mathcal{S}$ with a lower bound in $\mathcal{S}$; by hypothesis, $\inf A\in\mathcal{S}$; by 632E, $u_{\inf A}=\lim_{\sigma\downarrow A}u_\sigma$; so $\boldsymbol{u}$ satisfies $(\dagger)$, and we're done.

632G Corollary Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ such that $\inf A\in\mathcal{S}$ for every non-empty $A\subseteq\mathcal{S}$ with a lower bound in $\mathcal{S}$, and $a\in\mathfrak{A}$. Set $u_\sigma=\chi(\mathrm{upr}(a,\mathfrak{A}_\sigma))$ for $\sigma\in\mathcal{S}$, and $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$. Then $\boldsymbol{u}$ is near-simple.

proof For $\sigma\in\mathcal{S}$, write a_σ for $\mathrm{upr}(a,\mathfrak{A}_\sigma)$. If $\sigma\le\tau$ in A, $\mathfrak{A}_\sigma\subseteq\mathfrak{A}_\tau$ so $a_\tau\subseteq a_\sigma$ and $\chi1\ge u_\sigma\ge u_\tau\ge0$. Accordingly $\boldsymbol{u}$ is of bounded variation and moderately oscillatory. If $A\subseteq\mathcal{S}$ is non-empty and downwards-directed and $\tau=\inf A$, then $a_\tau=\sup_{\sigma\in A}a_\sigma$. **P** Set $c=\sup_{\sigma\in A}a_\sigma$. If $\sigma\in A$, $c=\sup_{\sigma'\in A,\sigma'\le\sigma}a_{\sigma'}$ belongs to $\mathfrak{A}_\sigma$. So $c\in\bigcap_{\sigma\in A}\mathfrak{A}_\sigma=\mathfrak{A}_\tau$, by 632C(a-iii). As $a\subseteq c$, $a_\tau\subseteq c$. But of course $c\subseteq a_\tau$, so we have equality. **Q** This means that $a_\tau=\lim_{\sigma\downarrow A}a_\sigma$ for the measure-algebra topology on $\mathfrak{A}$ (323D(a-ii)) and

$$u_\tau=\chi(a_\tau)=\lim_{\sigma\downarrow A}\chi(a_\sigma)=\lim_{\sigma\downarrow A}u_\sigma$$

(367Ra). As A is arbitrary, 632F tells us that $\boldsymbol{u}$ is locally near-simple, and by 631F(c-ii) it is actually near-simple.

632H Corollary Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ such that $\inf A\in\mathcal{S}$ for every non-empty $A\subseteq\mathcal{S}$ with a lower bound in $\mathcal{S}$, and $\boldsymbol{v}=\langle v_\tau\rangle_{\tau\in\mathcal{S}}$ a locally jump-free non-decreasing process. Then $\boldsymbol{v}:\mathcal{S}\to L^0$ is an order-continuous lattice homomorphism.

proof By 614Ia, $\boldsymbol{v}$ is a lattice homomorphism. If $A\subseteq\mathcal{S}$ is upwards-directed and $\sup A\in\mathcal{S}$, then

$$v_{\sup A} = \lim_{\tau\uparrow A} v_\tau = \sup_{\tau\in A} v_\tau$$

by 618I and 613Ba. On the other side, $\boldsymbol{v}$ is locally near-simple (631Cb), so if $A \subseteq \mathcal{S}$ is downwards-directed and $\inf A \in \mathcal{S}$, then

$$v_{\inf A} = \lim_{\tau\downarrow A} v_\tau = \inf_{\tau\in A} v_\tau$$

by 632F. Accordingly $\boldsymbol{v}$ is order-continuous in the sense of 313H.

632I Theorem Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ such that $\inf A \in \mathcal{S}$ for every non-empty $A \subseteq \mathcal{S}$. Then a virtually local martingale with domain $\mathcal{S}$ is a locally near-simple local martingale.

proof (a) Let $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ be a virtually local martingale with domain $\mathcal{S}$. By 623K(b-iii), it is an approximately local martingale. Write $\mathcal{S}'$ for $\{\sigma : \sigma \in \mathcal{S},\ \boldsymbol{u}\restriction\mathcal{S}\wedge\sigma \text{ is a martingale}\}$, so that $\mathcal{S}'$ is an ideal of $\mathcal{S}$ (622M). Take $\tau \in \mathcal{S}$ and $\epsilon > 0$. Then there is a non-empty downwards-directed $A \subseteq \mathcal{S}\wedge\tau$ such that $\sup_{\rho\in A}\bar\mu[\![\rho<\tau]\!] \le \epsilon$ and $R_A(\boldsymbol{u})$, as defined in 623B, is a martingale (623J). Now $\inf A \in \mathcal{S}$, while $[\![\inf A<\tau]\!] = \sup_{\rho\in A}[\![\rho<\tau]\!]$, by 632C(a-ii). So $\bar\mu[\![\inf A<\tau]\!] = \sup_{\rho\in A}\bar\mu[\![\rho<\tau]\!]$ is at most ϵ. Moreover, for $\sigma \in \mathcal{S}\wedge\inf A$, $\lim_{\rho\downarrow A} u_{\sigma\wedge\rho} = u_\sigma$, so $\boldsymbol{u}\restriction\mathcal{S}\wedge\inf A = R_A(\boldsymbol{u})\restriction\mathcal{S}\wedge\inf A$ is a martingale, and $\inf A \in \mathcal{S}'$, while $[\![\tau=\inf A]\!] \ge 1-\epsilon$. As τ and ϵ are arbitrary, $\mathcal{S}'$ covers $\mathcal{S}$ and $\boldsymbol{u}$ is a local martingale.

(b) We know that $\boldsymbol{u}$ is locally moderately oscillatory (622H). If $A \subseteq \mathcal{S}$ is non-empty and downwards-directed, take any $\upsilon \in A$ and $\epsilon > 0$. Then there is a $\rho \in \mathcal{S}$ such that $\bar\mu[\![\rho<\upsilon]\!] \le \epsilon$ and $\boldsymbol{u}\restriction\mathcal{S}\wedge\rho$ is a martingale. Now $A\wedge\rho$ is downwards-directed and has infimum $\rho\wedge\inf A$ (611Ch), so $\mathfrak{A}_{\rho\wedge\inf A} = \bigcap_{\sigma\in A}\mathfrak{A}_{\rho\wedge\sigma}$ (632C(a-iii)) and

$$u_{\rho\wedge\inf A} = P_{\rho\wedge\inf A}u_{\rho\wedge\upsilon} = \operatorname{1lim}_{\sigma\downarrow A} P_{\rho\wedge\sigma}u_{\rho\wedge\upsilon}$$

(621C(g-i))

$$= \lim_{\sigma\downarrow A} P_{\rho\wedge\sigma}u_{\rho\wedge\upsilon}$$

(613B(d-i))

$$= \lim_{\sigma\downarrow A} u_{\rho\wedge\sigma}.$$

There is therefore a $\sigma_0 \in A\wedge\upsilon$ such that $\theta(u_{\rho\wedge\sigma} - u_{\rho\wedge\inf A}) \le \epsilon$ whenever $\sigma \in A$ and $\sigma \le \sigma_0$. But now we have, whenever $\sigma \in A$ and $\sigma \le \sigma_0$,

$$\theta(u_{\rho\wedge\sigma} - u_\sigma) \le \bar\mu[\![\rho<\sigma]\!] \le \bar\mu[\![\rho<\upsilon]\!] \le \epsilon$$

and similarly $\theta(u_{\rho\wedge\inf A} - u_{\inf A}) \le \epsilon$, so $\theta(u_\sigma - u_{\inf A}) \le 3\epsilon$. As ϵ is arbitrary, $u_{\inf A} = \lim_{\sigma\downarrow A} u_\sigma$. Thus condition (†) of 632F is satisfied and $\boldsymbol{u}$ is locally near-simple.

632J Where martingales come from Coming back to the basic theory of martingales, we have the following.

Proposition Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\langle u_t\rangle_{t\in T}$ be a martingale in the sense that $u_t \in L^1_{\bar\mu}$ and u_s is the conditional expectation of u_t on $\mathfrak{A}_s$ whenever $s \le t$ in T. If $\boldsymbol{w} = \langle w_\tau\rangle_{\tau\in\mathcal{T}_b}$ is a locally near-simple process such that $w_{\check t} = u_t$ for every $t \in T$, then $\boldsymbol{w}$ is a martingale.

proof (a) Fix $t \in T$ for the moment. Set $C = \{\sigma : \sigma \in \mathcal{T}_b,\ w_\sigma = P_\sigma u_t\}$.

(i) If $s \le t$, then $\check s \in C$, just because $\mathfrak{A}_s = \mathfrak{A}_{\check s}$, so $w_{\check s} = u_s = P_{\check s}u_t$.

(ii) If σ_0, $\sigma_1 \in C$ and $\tau \in \mathcal{T}_b$ is such that $[\![\tau=\sigma_0]\!] \cup [\![\tau=\sigma_1]\!] = 1$, then $\tau \in C$. **P**

$$\begin{aligned}[\![P_\tau u_t = w_\tau]\!] &\supseteq [\![\sigma_0=\tau]\!] \cap [\![P_{\sigma_0}u_t = w_{\sigma_0}]\!] \cap [\![P_\tau u_t = P_{\sigma_0}u_t]\!] \cap [\![w_\tau = w_{\sigma_0}]\!]\\ &= [\![\sigma_0=\tau]\!] \cap [\![P_\tau u_t = P_{\sigma_0}u_t]\!]\end{aligned}$$

(because $\sigma_0 \in C$ and $\boldsymbol{w}$ is fully adapted)

$$= [\![\sigma_0 = \tau]\!]$$

by 622Bb. Similarly, $[\![P_\tau u_t = w_\tau]\!] \supseteq [\![\tau = \sigma_1]\!]$ and

$$[\![P_\tau u_t = w_\tau]\!] \supseteq [\![\tau = \sigma_0]\!] \cup [\![\tau = \sigma_1]\!] = 1,$$

so $P_\tau u_t = w_\tau$, that is, $\tau \in C$. **Q**

(iii) In particular, putting (ii) here and 611E(a-ii-γ) together, $\sigma \wedge \tau \in C$ whenever σ, $\tau \in C$.

(iv) If $A \subseteq C$ is non-empty and downwards-directed, with infimum τ, then $\tau \in C$. **P** $\boldsymbol{P}u_t = \langle P_\sigma u_t \rangle_{\sigma \in \mathcal{T}}$ is a martingale (622Fa), so

$$P_\tau u_t = \lim_{\sigma \downarrow A} P_\sigma u_t$$

(put 632I and 632F together, or look at (a) of the proof of 632I)

$$= \lim_{\sigma \downarrow A} w_\sigma = w_\tau$$

by 632F again. **Q**

(v) If $\tau \le \check{t}$ then $\tau \in C$. **P** Set $A = \{\sigma : \sigma \in C, \tau \le \sigma\}$. Then A is downwards-directed, by (iii), and non-empty, because it contains $\check{t}$. Because $\mathcal{T}$ is Dedekind complete, $\tau^* = \inf A$ is defined, and $\tau^* \in C$ by (iv). Of course $\tau \le \tau^*$. **?** If $\tau \ne \tau^*$, then $[\![\tau < \tau^*]\!] \ne 0$ and there is an $s \in T$ such that $[\![\tau^* > s]\!] \setminus [\![\tau > s]\!] \ne 0$. As $[\![\tau > s]\!] \ne 0$, s must be less than or equal to t, and $\check{s} \in C$ by (i). Now consider $c = [\![\tau \le \check{s}]\!]$. We have

$$c \in \mathfrak{A}_{\check{s}} \cap \mathfrak{A}_\tau \subseteq \mathfrak{A}_{\check{s}} \cap \mathfrak{A}_{\tau^*}$$

by 611H(c-ii). So there is a $\tau' \in \mathcal{T}$ such that $[\![\tau' = \check{s}]\!] \supseteq c$ and $[\![\tau' = \tau^*]\!] \supseteq 1 \setminus c$ (611I), and $\tau' \in C$ by (ii).

Consider

$$[\![\tau \le \tau']\!] \supseteq ([\![\tau \le \check{s}]\!] \cap [\![\tau' = \check{s}]\!]) \cup ([\![\tau \le \tau^*]\!] \cap [\![\tau' = \tau^*]\!])$$
$$\supseteq ([\![\tau \le \check{s}]\!] \cap c) \cup ([\![\tau \le \tau^*]\!] \setminus c) = c \cup (1 \setminus c) = 1.$$

Thus $\tau \le \tau'$, so $\tau' \in A$ and $\tau^* \le \tau'$. On the other hand,

$$0 \ne [\![\tau^* > s]\!] \setminus [\![\tau > s]\!] = [\![\check{s} < \tau^*]\!] \cap [\![\tau \le \check{s}]\!]$$

(611E(a-i-δ))

$$\subseteq [\![\check{s} < \tau^*]\!] \cap [\![\tau' = \check{s}]\!] \subseteq [\![\tau' < \tau^*]\!]$$

(611E(c-iii-γ), so this is impossible. **X**

Accordingly $\tau = \tau^*$ belongs to C, as claimed. **Q**

(b) Thus $w_\tau = P_\tau u_t$ whenever $t \in T$ and $\tau \le \check{t}$. Now, of course, if $\sigma \le \tau$ in $\mathcal{T}_b$, there is a $t \in T$ such that $\tau \le \check{t}$, and

$$P_\sigma w_\tau = P_\sigma P_\tau u_t = P_\sigma u_t = w_\sigma.$$

As σ and τ are arbitrary, $\boldsymbol{w}$ is a martingale.

632K Lemma Let (Ω, Σ, μ) be a probability space with measure algebra $(\mathfrak{A}, \bar{\mu})$, and $\langle \Sigma_t \rangle_{t \ge 0}$ a right-continuous filtration of σ-subalgebras of Σ. If we set $\mathfrak{A}_t = \{E^\bullet : E \in \Sigma_t\}$ for $t \ge 0$, then $\langle \mathfrak{A}_t \rangle_{t \ge 0}$ is a right-continuous filtration.

proof Take $t \ge 0$ and $a \in \bigcap_{s>t} \mathfrak{A}_s$. Then for each $n \in \mathbb{N}$ there is an $E_n \in \Sigma_{t+2^{-n}}$ such that $E_n^\bullet = a$. Set $E = \bigcup_{n \in \mathbb{N}} \bigcap_{i \ge n} E_i$; then $E \in \bigcap_{s>t} \Sigma_s = \Sigma_t$ and $a = E^\bullet \in \mathfrak{A}_t$.

632L Proposition Let (Ω, Σ, μ) be a complete probability space and $\langle \Sigma_t \rangle_{t \in [0,\infty[}$ a right-continuous filtration of σ-subalgebras of Σ, all containing every negligible subset of Ω. Suppose that we are given a family $\langle X_t\rangle_{t\ge 0}$ of measurable real-valued functions on Ω such that $t\mapsto X_t(\omega)$ is càdlàg for every ω and X_s is a conditional expectation of X_t on Σ_s whenever $0\le s\le t$. Define $(\mathfrak{A},\bar\mu,\langle\mathfrak{A}_t\rangle_{t\ge 0})$ and $\boldsymbol{u}=\langle u_\tau\rangle_{\tau\in\mathcal{T}_f}$ as in 612H and 631D. Then $\boldsymbol{u}\restriction\mathcal{T}_b$ is a martingale and $\boldsymbol{u}$ is a local martingale.

proof By 631D, $\boldsymbol{u}$ is well-defined and is a locally near-simple fully adapted process. We have $u_{\check t}=X_t^\bullet$ for $t\in T$, so $u_{\check s}$ is the conditional expectation of $u_{\check t}$ on $\mathfrak{A}_s=\{E^\bullet:E\in\Sigma_s\}$ whenever $s\le t$. Moreover, $\langle\mathfrak{A}_t\rangle_{t\in[0,\infty[}$ is right-continuous, by 632K. But now 632J tells us that $\boldsymbol{u}\restriction\mathcal{T}_b$ is a martingale, so that $\boldsymbol{u}$ is a local martingale.

632M Proposition Let $\boldsymbol{v}=\langle v_\tau\rangle_{\tau\in\mathcal{T}_f}$ be the Poisson process as defined in 612U.

(a) If $\boldsymbol{\iota}$ is the identity process, $(\boldsymbol{v}-\boldsymbol{\iota})\restriction\mathcal{T}_b$ is a martingale, so that $\boldsymbol{v}-\boldsymbol{\iota}$ is a local martingale.

(b) The previsible variation of $\boldsymbol{v}\restriction\mathcal{T}_b$ is $\boldsymbol{\iota}\restriction\mathcal{T}_b$.

proof (a)(i) In order to use results from §455, I will in fact take a step back from the structure $(\Omega,\Sigma,\mu,\langle\Sigma_t\rangle_{t\in[0,\infty[})$ introduced in 612Uc, and for most of the argument work with the structure $(C_{\rm dlg},\ddot\Sigma,\ddot\mu,\langle\ddot\Sigma_t\rangle_{t\in[0,\infty[})$ from which it was defined, with the associated σ-algebras $\ddot\Sigma_t^+=\bigcap_{s>t}\ddot\Sigma_s$; here $C_{\rm dlg}$ is the space of all càdlàg functions from $[0,\infty[$ to $\mathbb{R}$. For $t\ge 0$ and $\omega\in C_{\rm dlg}$, set $Y_t(\omega)=\omega(t)-t$. If $0\le s\le t$, then Y_s is a conditional expectation of Y_t on $\ddot\Sigma_s^+$. **P** This is trivial if $s=t$, just because Y_s is $\ddot\Sigma_s$-measurable. If $s<t$, then we can use 455O and 455S, as follows. Let $h:C_{\rm dlg}\to[0,\infty[$ be the constant stopping time with value s. Then the σ-algebra $\ddot\Sigma_h^+$ of 455Ob is

$$\{F:F\in\ddot\Sigma,\ F\cap\{\omega:h(\omega)\le s'\}\in\ddot\Sigma_{s'}^+\text{ for every }s'\ge 0\}=\ddot\Sigma_s^+.$$

So if we define $\ddot\mu_{\omega s}$, for $\omega\in C_{\rm dlg}$, as in 455O, and set

$$g(\omega)=\int_{C_{\rm dlg}}Y_t(\omega')\,\ddot\mu_{\omega s}(d\omega')$$

whenever this is defined, then g will be a conditional expectation of Y_t on $\ddot\Sigma_s^+$, by 455Ob. Next, $\phi_{\omega s}:C_{\rm dlg}\to C_{\rm dlg}$ is inverse-measure-preserving for $\ddot\mu$ and $\ddot\mu_{\omega s}$, where

$$\begin{aligned}\phi_{\omega s}(\omega')(s')&=\omega(s')\text{ if }s'<s,\\ &=\omega(s)+\omega'(s'-s)\text{ if }s\le s'\end{aligned}$$

(455Sc). So

$$\begin{aligned}g(\omega)&=\int_{C_{\rm dlg}}Y_t(\phi_{\omega s}(\omega'))\ddot\mu(d\omega')=\int_{C_{\rm dlg}}\phi_{\omega s}(\omega')(t)-t\,\ddot\mu(d\omega')\\ &=\int_{C_{\rm dlg}}\omega(s)+\omega'(t-s)-t\,\ddot\mu(d\omega')\\ &=\omega(s)-t+\int_{C_{\rm dlg}}\omega'(t-s)\ddot\mu(d\omega')=\omega(s)-t+\mathbb{E}(\lambda_{t-s})\end{aligned}$$

(where λ_{t-s} is the distribution of the random variable $\omega'\mapsto\omega'(t-s)$)

$$=\omega(s)-s=Y_s(\omega)$$

because λ_{t-s} is the Poisson distribution with expectation $t-s$, as chosen in 612U.

This shows that Y_s is a conditional expectation of Y_t on $\ddot\Sigma_s^+$. **Q**

(ii) We know also from 455T that $\ddot\Sigma_s^+$ is included in

$$\hat{\ddot\Sigma}_s=\{F\triangle A:F\in\ddot\Sigma_s,\ A\text{ is }\ddot\mu\text{-negligible}\}.$$

It follows at once that Y_s is a conditional expectation of Y_t on the σ-algebra $\hat{\ddot\Sigma}_s$, and therefore that $Y_s\restriction\Omega$ is a conditional expectation of $Y_t\restriction\Omega$ on Σ_s, because Ω is a conegligible subset of $C_{\rm dlg}$. And this is true whenever $s\le t$ in $[0,\infty[$.

(iii) As noted in 632Db, $\langle\Sigma_t\rangle_{t\in[0,\infty[}$ is a right-continuous filtration, and it follows at once that $\langle\mathfrak{A}_t\rangle_{t\in[0,\infty[}$ is right-continuous. And we know that $\boldsymbol{v}$ and $\boldsymbol{\iota}$ are locally near-simple processes (631E), so (ii) and 632Jc tell us that $(\boldsymbol{v}-\boldsymbol{\iota})\restriction\mathcal{T}_b$ is a martingale. Of course this means that $\boldsymbol{v}-\boldsymbol{\iota}$ will be a local martingale on $\mathcal{T}_f$.

(b) Since $(\boldsymbol{v}-\boldsymbol{\iota})\restriction\mathcal{T}_b$ is a martingale, its previsible variation is zero (626Kc), so the previsible variation of $\boldsymbol{v}\restriction\mathcal{T}_b$ is the same as the previsible variation of $\boldsymbol{\iota}\restriction\mathcal{T}_b$ (using 626Kb), which is $\boldsymbol{\iota}\restriction\mathcal{T}_b$ (626Q).

***632N** I described local martingales which are not martingales in 622Xe and 622Xj, and plenty of other examples are provided by 632L; for instance, 'Brownian motion', as described in 612T, is a martingale if you take it to be defined on $\mathcal{T}_b$ (622L), but only a local martingale if you take it to be defined everywhere on $\mathcal{T}_f$ (632Ye). There is a classic example which is more interesting, because it arises naturally in the context of §§477-479.

Example (a) Let $\mu=\mu_W$ be three-dimensional Wiener measure (477D) on $\Omega=C([0,\infty[\,;\mathbb{R}^3)_0$, Σ its domain, and $(\mathfrak{A},\bar\mu)$ its measure algebra. For $t\ge 0$ set

$$\Sigma'_t=\{F:F\in\Sigma,\,\omega'\in F\text{ whenever }\omega\in F,\,\omega'\in\Omega\text{ and }\omega'\restriction[0,t]=\omega\restriction[0,t]\},$$

$$\Sigma_t=\{F\triangle H:F\in\Sigma'_t,\,\mu H=0\},$$

$$\mathfrak{A}_t=\{F^\bullet:F\in\Sigma_t\}.$$

By 477Hc in three dimensions, $\Sigma_t=\bigcap_{s>t}\Sigma_s$, so $\langle\mathfrak{A}_t\rangle_{t\ge0}$ is right-continuous.

(b) Let e be a unit vector in $\mathbb{R}^3$. Set $\Omega'=\{\omega:\omega\in\Omega,\,e\text{ is not a value of }\omega\}$; by 478Mc, Ω' is conegligible, so belongs to Σ_t for every t. For $t\ge0$ and $\omega\in\Omega$ set

$$\begin{aligned}Y_t(\omega)&=\tfrac{1}{\|\omega(t)-e\|}\text{ if }\omega\in\Omega',\\&=0\text{ otherwise }.\end{aligned}$$

Then Y_t is Σ_t-measurable for every t and $t\mapsto Y_t(\omega)$ is continuous for every ω, so we have a corresponding locally jump-free process $\boldsymbol{v}=\langle v_\tau\rangle_{\tau\in\mathcal{T}_f}$ (618H).

(c) $\lim_{t\to\infty}\mathbb{E}(Y_t)=0$. **P** For t, $R>0$, the probability density function of the random variable $\omega\mapsto\omega(t)$ is $x\mapsto\frac{1}{(\sqrt{2\pi t})^3}e^{-\|x\|^2/2t}$ (put 274Ad and 272I together), so by 271Ic

$$\mathbb{E}(Y_t)=\int_{\mathbb{R}^3}\frac{1}{(\sqrt{2\pi t})^3}e^{-\|x\|^2/2t}\frac{1}{\|x-e\|}dx$$

(where the integral is taken with respect to Lebesgue measure on $\mathbb{R}^3$)

$$\begin{aligned}&\le\frac1R+\int_{B(e,R)}\frac{1}{(\sqrt{2\pi t})^3}e^{-\|x\|^2/2t}\frac{1}{\|x-e\|}dx\\&\le\frac1R+\frac{1}{(\sqrt{2\pi t})^3}\int_{B(e,R)}\frac{1}{\|x-e\|}dx=\frac1R+\frac{1}{(\sqrt{2\pi t})^3}\int_{B(0,R)}\frac{1}{\|x\|}dx\\&=\frac1R+\frac{1}{(\sqrt{2\pi t})^3}\int_0^R4\pi s^2\cdot\frac1s\,ds=\frac1R+\frac{2\pi R^2}{(\sqrt{2\pi t})^3}.\end{aligned}$$

So $\limsup_{t\to\infty}\mathbb{E}(Y_t)\le\frac1R$ for every $R>0$ and must be 0. **Q**

Since $\mathbb{E}(Y_0)=1$, $\langle Y_t\rangle_{t\ge0}$ is not a martingale and $\boldsymbol{v}$ is not a martingale.

(d) If $n\in\mathbb{N}$ and h_n is the Brownian hitting time to the ball $B(e,2^{-n})$ (477I), then h_n is adapted to $\langle\Sigma_t\rangle_{t\ge0}$ (477I(c-iii)), so represents a stopping time τ_n adapted to $\langle\mathfrak{A}_t\rangle_{t\ge0}$ (612Ha). Set $\mathcal{S}=\bigcup_{n\in\mathbb{N}}\{\tau:\tau\in\mathcal{T}_f,\ \tau\le\tau_n\}$. Because $\langle\tau_n\rangle_{n\in\mathbb{N}}$ is non-decreasing, $\mathcal{S}$ is an ideal of $\mathcal{T}_f$. Now $\mathcal{S}$ is a covering ideal in the sense of 611N. **P** (This is where we need to be in three or more dimensions.) For each $n\in\mathbb{N}$ and $t\ge0$,

$$\begin{aligned}\bar\mu[\![\tau_n > t]\!] &= \mu_W\{\omega : h_n(\omega) > t\}\\ &\ge \mu_W\{\omega : B(e, 2^{-n}) \cap \omega[\,[0,\infty[\,] = \emptyset\} = 1 - \text{hp}(B(e, 2^{-n}))\end{aligned}$$

(where hp is 'Brownian hitting probability', see 477Ia)

$$= 1 - \tilde W_{B(e,2^{-n})}(0)$$

(where $\tilde W$ is 'equilibrium potential', 479Cb and 479Pb)

$$= 1 - 2^{-n}$$

by 479Da. But this means that if $\tau \in \mathcal{T}_f$ and $n \in \mathbb{N}$ there is a $t \ge 0$ such that $\bar\mu[\![\tau > t]\!] \le 2^{-n}$, and now

$$[\![\tau > \tau_n]\!] \subseteq [\![\tau > t]\!] \cup [\![\tau_n \le t]\!]$$

has measure at most 2^{-n+1}. So

$$\sup_{\sigma\in\mathcal{S}} \bar\mu[\![\tau = \sigma]\!] \ge \bar\mu[\![\tau = \tau \wedge \tau_n]\!] = 1 - \bar\mu[\![\tau > \tau_n]\!] \ge 1 - 2^{-n+1}.$$

As n is arbitrary, $\sup_{\sigma\in\mathcal{S}} [\![\tau = \sigma]\!] = 1$; as τ is arbitrary, $\mathcal{S}$ is a covering ideal of $\mathcal{T}_f$. **Q**

(e) At the same time, for each $n \in \mathbb{N}$, $\langle v_\sigma\rangle_{\sigma\le\tau_n}$ is a martingale. **P** Apply 478V with $G = \{x : \|x - e\| > 2^{-n}\}$ and $f(x) = \frac{1}{\|x-e\|}$, so that h_n is the Brownian exit time from G. Again because we are in at least three dimensions, G has few wandering paths (478N), so 478Vb tells us that if $g \le h_n$ is a stopping time representing $\sigma \le \tau_n$ then Y_g is the conditional expectation of Y_{h_n} on Σ_g, that is, $v_\sigma = P_\sigma v_{\tau_n}$. **Q**

Thus $\boldsymbol{v}$ is a local martingale.

632X Basic exercises (a) Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is not right-continuous. Show that in this case there are a simple process $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{T}}$ and a non-empty downwards-directed set $A \subseteq \mathcal{T}$ such that $\lim_{\sigma\downarrow A} u_\sigma \ne u_{\inf A}$. (*Hint*: reduce to the case in which $a \in \bigcap_{s>t}\mathfrak{A}_s \setminus \mathfrak{A}_t$ does not include any non-zero member of $\mathfrak{A}_t$. Try $A = \{\sigma : \sigma \ge \check t, [\![\sigma = \max\mathcal{T}]\!] \supseteq a\}$, $u_{\check t} = 0$, $u_{\max\mathcal{T}} = \chi 1$.)

(b) Suppose that $T = [0,\infty[$. Show that if $\langle\mathfrak{A}_t\rangle_{t\ge0}$ is right-continuous, and we identify $\mathcal{T}_f$ with a subset of L^0 as in 611Xa, then $\mathcal{T}_f$ is order-closed in L^0 (definition: 313Da).

(c) Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Show that the function $\sigma \mapsto \sigma \vee \tau : \mathcal{T} \to \mathcal{T}$ is order-continuous for every $\tau \in \mathcal{T}$.

(d) Suppose that $(\mathfrak{A}, \bar\mu, T, \langle\mathfrak{A}_t\rangle_{t\in T}, \mathcal{T}, \langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ is the structure described in 612U, and $\boldsymbol{v}$ the standard Poisson process. Show that $(\boldsymbol{v} - \boldsymbol{\iota})^2 - \boldsymbol{\iota}$ is a local martingale.

(e) Let $\boldsymbol{w}$ be Brownian motion (612T, 622L) and $\boldsymbol{\iota}$ the corresponding identity process. Use the method of 632M to show that $\boldsymbol{w}\restriction\mathcal{T}_b$ and $\boldsymbol{w}^2 - \boldsymbol{\iota}\restriction\mathcal{T}_b$ are martingales, without appealing to Dynkin's formula.

632Y Further exercises (a) Suppose that $\mathfrak{A}$ has countable Maharam type. Show that $\{t : t \in T, \mathfrak{A}_t \ne \mathfrak{A} \cap \bigcap_{s>t}\mathfrak{A}_s\}$ is countable.

(b) Define a family $\langle\mathfrak{A}_t^+\rangle_{t\in T}$ of closed subalgebras of $\mathfrak{A}$ by setting

$$\begin{aligned}\mathfrak{A}_t^+ &= \mathfrak{A}_t \text{ if } t \in T \text{ is isolated on the right,}\\ &= \bigcap_{s>t}\mathfrak{A}_s \text{ otherwise.}\end{aligned}$$

(i) Show that $\langle\mathfrak{A}_t^+\rangle_{t\in T}$ is a right-continuous filtration. (ii) Let $\mathcal{T}^+$ be the set of stopping times with respect to $\langle\mathfrak{A}_t^+\rangle_{t\in T}$. Show that if we think of $\mathcal{T}$ and $\mathcal{T}^+$ as subsets of $\mathfrak{A}^T$, as suggested in 611Ac, then $\mathcal{T}$ is a sublattice of $\mathcal{T}^+$. (iii) Show that every member of $\mathcal{T}^+$ is the infimum in $\mathcal{T}^+$ of a subset of $\mathcal{T}$. (iv) Show that if A is a non-empty subset of $\mathcal{T}$, its supremum in $\mathcal{T}^+$ belongs to $\mathcal{T}$. (v) Show that for every $\tau \in \mathcal{T}^+$ there is a greatest $\sigma \in \mathcal{T}$ such that $\sigma \le \tau$ in $\mathcal{T}^+$.

(c) Dropping the measure $\bar{\mu}$, and supposing only that $\mathfrak{A}$ is a Dedekind complete Boolean algebra and $\langle\mathfrak{A}_t\rangle_{t\in T}$ is a filtration of order-closed subalgebras of $\mathfrak{A}$, as in 611A, show that parts (a)-(b) of 632C are still valid.

(d) Suppose that $\mathfrak{A}$ is a complete weakly (σ,∞)-distributive Boolean algebra (e.g., a probability algebra; see 316G and 322F) and $\langle\mathfrak{A}_t\rangle_{t\in T}$ is a right-continuous filtration of order-closed subalgebras of $\mathfrak{A}$. (i) Let $\langle C_n\rangle_{n\in\mathbb{N}}$ be a sequence of ideals of $\mathcal{T}$. Show that $\sup(\bigcap_{n\in\mathbb{N}} C_n) = \inf_{n\in\mathbb{N}}\sup C_n$. (ii) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ such that $\inf D \in \mathcal{S}$ for every non-empty countable set $D \subseteq \mathcal{S}$. Show that the intersection of a sequence of covering ideals of $\mathcal{S}$ is a covering ideal.

(e) Let $(\mathfrak{C},\bar{\nu},[0,\infty[\,,\langle\mathfrak{C}_t\rangle_{t\ge0},\mathcal{T},\langle\mathfrak{C}_\tau\rangle_{\tau\in\mathcal{T}})$ be the real-time stochastic integration structure of 612T, and $\boldsymbol{w} = \langle w_\sigma\rangle_{\sigma\in\mathcal{T}_f}$ Brownian motion. Show that for every $z \in L^0(\mathfrak{C})$ and $\sigma \in \mathcal{T}_f$ there is a $\tau \in \mathcal{T}_f$ such that $\sigma \le \tau$ and $w_\tau = z$; so that $\boldsymbol{w}$ is not a martingale.

(f) Let μ_L be Lebesgue measure on $[0,\infty[$, and ν the corresponding Poisson point process with intensity 1 as described in §495, so that ν is a complete probability measure on $\mathcal{P}\,[0,\infty[$. Let Ω be the set $\{\omega : \omega \subseteq\,]0,\infty[$, $\omega\cap[0,a]$ is finite for every $a \ge 0\}$. Show that $\nu\Omega = 1$; let μ be the subspace measure on Ω and Σ its domain. For $t \ge 0$ let Σ_t be the σ-algebra generated by sets of the form $\{\omega : \omega \in \Omega,\ \#(\omega)\cap[0,s] = n\}\triangle E$ where $n \in \mathbb{N}$, $s \le t$ and $\mu E = 0$; show that $\langle\Sigma_t\rangle_{t\ge0}$ is a right-continuous filtration of σ-subalgebras of Σ. For $t \ge 0$, $\omega \in \Omega$ set $X_t(\omega) = \#(\omega\cap[0,t])$; show that $\langle X_t\rangle_{t\ge0}$ is adapted to $\langle\Sigma_t\rangle_{t\ge0}$, and that $t \mapsto X_t(\omega)$ is càdlàg for every ω. Show that if we take the stochastic process defined from $(\Omega,\nu,\langle\Sigma_t\rangle_{t\ge0},\langle X_t\rangle_{t\ge0})$ by the method of 612H/631D, we obtain a structure $(\mathfrak{A},\bar{\mu},\langle\mathfrak{A}_t\rangle_{t\ge0},\langle v_\tau\rangle_{\tau\in\mathcal{T}_f})$ isomorphic to that of 612U.

(g) Give an example of near-simple processes $\boldsymbol{u}$, $\boldsymbol{v}$ defined on $\mathcal{T}$, in a real-time structure with a right-continuous filtration, such that $\int_{\check{T}}\boldsymbol{u}\,d\boldsymbol{v} = 0$, where $\check{T}$ is the set of constant stopping times, but $\int_{\mathcal{T}_s}\boldsymbol{u}\,d\boldsymbol{v}$ is undefined, where $\mathcal{T}_s$ is the finitely-covered envelope of $\check{T}$.

632 Notes and comments The aim of this section is to show the kind of simplification which is achieved by assuming right-continuity. After the basic list in 632C, the most important general facts are 632F and 632I.

As with 612U, we don't really need §455 in the proof of 632Ma. Instead of expressing the Poisson process in terms of a Lévy process as described in §455, we can start from a probability space better adapted to the problem (632Yf). But we still have some work to do, because the Markov property has got to come in somewhere, and the Poisson point processes of §495 aren't trivial. The method above also provides an alternative route to 622L, missing out the harmonic analysis, and giving a slightly stronger result (632Xe).

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633 Separating sublattices

At various points, I have looked at relations between a process $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ and its fully adapted extension to the covered envelope of $\mathcal{S}$; turning these round, we find connections between the properties of $\boldsymbol{u}$ and its restriction to a covering sublattice of $\mathcal{S}$. When the filtration is right-continuous and we have a near-simple process we can go much farther, and an effective concept is that of 'separating' sublattice (633B). Once again we have a useful result on equality of integrals (633K) and many correspondences between properties of $\boldsymbol{u}$ and $\boldsymbol{u}\restriction\mathcal{S}'$ (633O, 633P).

633A Notation Once again, $(\mathfrak{A},\bar{\mu},T,\langle\mathfrak{A}_t\rangle_{t\in T},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ will be a stochastic integration structure. If $\mathcal{S}$ is a sublattice of $\mathcal{T}$, $\mathcal{I}(\mathcal{S})$ is the set of finite sublattices of $\mathcal{S}$, and $M_{\text{ln-s}}(\mathcal{S})$ the space of locally near-simple processes with domain $\mathcal{S}$. For $t \in T$, $\check{t}$ will be the constant stopping time at t. $L^0 = L^0(\mathfrak{A})$ will be given the linear space topology of convergence in measure. For $w \in L^0$, $\theta(w)$ will be $\mathbb{E}(|w|\wedge\chi1)$, where $\mathbb{E}$ is integration with respect to $\bar{\mu}$; for an order-bounded fully adapted process $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$, $\sup|\boldsymbol{u}|$ will be $\sup(\{0\}\cup\{|u_\sigma| : \sigma \in \mathcal{S}\})$. If I, $J \subseteq \mathcal{T}$ are sublattices, I write $I \sqcup J$ for the sublattice of $\mathcal{T}$ generated by $I \cup J$.

633B Definition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and A, B subsets of $\mathcal{S}$.

(a) I will say that A **separates** B if $[\![\sigma < \tau]\!] = \sup_{\rho \in A}([\![\sigma \le \rho]\!] \cap [\![\sigma < \tau]\!])$ for all σ, $\tau \in B$.

(b) If $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ is a fully adapted process, I will say that A **$\boldsymbol{v}$-separates** B if whenever σ, $\tau \in B$ then $[\![\sigma < \tau]\!] \cap [\![v_\sigma \ne v_\tau]\!] \subseteq \sup_{\rho \in A}([\![\sigma \le \rho]\!] \cap [\![\rho < \tau]\!])$.

633C Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and A, B, C, D subsets of $\mathcal{S}$.
(a) If $A \subseteq B$, $C \subseteq D$ and A separates D then B separates C.
(b) A separates its covered envelope.
(c) If A separates B and B separates C then A separates C.
(d) If A separates B and τ^* is an upper bound of B in $\mathcal{T}$ then $A \wedge \tau^* = \{\sigma \wedge \tau^* : \sigma \in A\}$ separates B.
(e) A separates B iff A $\boldsymbol{v}$-separates B for every fully adapted process $\boldsymbol{v}$ with domain $\mathcal{S}$.

proof (a) Immediate from 633Ba.

(b) If σ, τ belong to the covered envelope of A,

$$\begin{aligned}
[\![\sigma < \tau]\!] &= \sup_{\sigma', \tau' \in A} [\![\sigma < \tau]\!] \cap [\![\sigma' = \sigma]\!] \cap [\![\tau' = \tau]\!] \\
&= \sup_{\sigma', \tau' \in A} [\![\sigma' = \sigma]\!] \cap [\![\tau' = \tau]\!] \cap [\![\sigma' < \tau']\!] \\
&= \sup_{\substack{\sigma', \tau' \in \mathcal{S} \\ \rho \in A}} [\![\sigma' = \sigma]\!] \cap [\![\tau' = \tau]\!] \cap [\![\sigma' \le \rho]\!] \cap [\![\rho < \tau']\!] \\
&\subseteq \sup_{\rho \in A} [\![\sigma \le \rho]\!] \cap [\![\rho < \tau]\!] \subseteq [\![\sigma < \tau]\!],
\end{aligned}$$

so $[\![\sigma < \tau]\!] = \sup_{\rho \in A} [\![\sigma \le \rho]\!] \cap [\![\rho < \tau]\!]$.

(c) If σ, $\tau \in C$, then

$$\begin{aligned}
[\![\sigma < \tau]\!] &\subseteq \sup_{\sigma' \in B} ([\![\sigma \le \sigma']\!] \cap [\![\sigma' < \tau]\!]) \\
&= \sup_{\sigma' \in B, \rho \in A} ([\![\sigma \le \sigma']\!] \cap [\![\sigma' < \tau]\!] \cap [\![\sigma' \le \rho]\!] \cap [\![\rho < \tau]\!] \\
&\subseteq \sup_{\rho \in A}([\![\sigma \le \rho]\!] \cap [\![\rho < \tau]\!] \subseteq [\![\sigma < \tau]\!].
\end{aligned}$$

(d) If σ, $\tau \in A$ then

$$\begin{aligned}
[\![\sigma \wedge \tau^* < \tau \wedge \tau^*]\!] = [\![\sigma < \tau]\!] \cap [\![\sigma < \tau^*]\!] &= \sup_{\rho \in A}([\![\sigma \le \rho]\!] \cap [\![\rho < \tau]\!] \cap [\![\sigma < \tau^*]\!]) \\
&\subseteq \sup_{\rho \in A}([\![\sigma \wedge \tau^* \le \rho \wedge \tau^*]\!] \cap [\![\rho \wedge \tau^* < \tau \wedge \tau^*]\!])
\end{aligned}$$

(e) Immediately from the definitions in 633B we see that if A separates B then A $\boldsymbol{v}$-separates B for every $\boldsymbol{v} \in M_{\mathrm{fa}}(\mathcal{S})$. Conversely, if A $\boldsymbol{v}$-separates B for every $\boldsymbol{v} \in M_{\mathrm{fa}}(\mathcal{S})$ and σ, $\tau \in B$, set $v_\rho = \chi[\![\rho < \tau]\!]$ for $\rho \in \mathcal{S}$. Then $\boldsymbol{v} = \langle v_\rho \rangle_{\rho \in \mathcal{S}}$ is fully adapted, being a simple process with breakpoint string (τ) (612J). And

$$[\![\sigma < \tau]\!] = [\![\sigma < \tau]\!] \cap [\![v_\sigma \ne v_\tau]\!] \subseteq \sup_{\rho \in A}([\![\sigma \le \rho]\!] \cap [\![\rho < \tau]\!]) \subseteq [\![\sigma < \tau]\!]$$

so $[\![\sigma < \tau]\!] = \sup_{\rho \in A}([\![\sigma \le \rho]\!] \cap [\![\rho < \tau]\!])$.

633D Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.
(a) If $T_0 \subseteq T$ is dense for the order topology of T (4A2R) and contains every point of T which is isolated on the right in T, then $\check{T}_0 = \{\check{t} : t \in T_0\}$ separates $\mathcal{S}$.
(b)(i) If $A \subseteq \mathcal{T}$ separates $\mathcal{S}$, $B \subseteq \mathcal{S}$ is coinitial with $\mathcal{S}$ and $C \subseteq \mathcal{S}$ is cofinal with $\mathcal{S}$, then $A' = \{\mathrm{med}(\tau, \sigma, \tau') : \tau \in B, \sigma \in A, \tau' \in C\}$ separates $\mathcal{S}$.

(ii) If $\boldsymbol{v}$ is a fully adapted process with domain $\mathcal{S}$, $A \subseteq \mathcal{T}$ $\boldsymbol{v}$-separates $\mathcal{S}$, $B \subseteq \mathcal{S}$ is coinitial with $\mathcal{S}$ and $C \subseteq \mathcal{S}$ is cofinal with $\mathcal{S}$, then $\mathcal{S}$ is $\boldsymbol{v}$-separated by $A' = \{\mathrm{med}(\tau,\sigma,\tau') : \tau \in B,\, \sigma \in A,\, \tau' \in C\}$.

proof (a) Suppose that τ, τ' in $\mathcal{S}$ and $t \in T$. Then $[\![\tau' > t]\!] \setminus [\![\tau > t]\!] \subseteq \sup_{s\in T_0}([\![\tau' > s]\!] \setminus [\![\tau > s]\!])$. **P** If t is isolated on the right in T, then $t \in T_0$ and the result is trivial. Otherwise,

$$[\![\tau' > t]\!] \setminus [\![\tau > t]\!] = \sup_{s'>t}([\![\tau' > s']\!] \setminus [\![\tau > t]\!]) \subseteq \sup_{s\in T_0}([\![\tau' > s]\!] \setminus [\![\tau > s]\!])$$

because if $s' > t$ then the open interval $]t, s'[$ is non-empty and meets T_0, and if $s \in T_0 \cap]t, s'[$ then $[\![\tau' > s']\!] \subseteq [\![\tau' > s]\!]$ and $[\![\tau > s]\!] \subseteq [\![\tau > t]\!]$. **Q**

Consequently

$$\begin{aligned}[\![\tau < \tau']\!] &= \sup_{t\in T}([\![\tau' > t]\!] \setminus [\![\tau > t]\!])\\ &= \sup_{s\in T_0}([\![\tau' > s]\!] \setminus [\![\tau > s]\!]) = \sup_{s\in T_0}([\![\tau \le \check{s}]\!] \cap [\![\check{s} < \tau]\!])\end{aligned}$$

by 611E(a-i-δ). As τ and τ' are arbitrary, $\check{T}_0$ separates $\mathcal{S}$.

(b)(i) Suppose that τ, τ' in $\mathcal{S}$ and $[\![\tau < \tau']\!] \ne 0$. Let $\tau_1 \in B$, $\tau_1' \in C$ be such that $\tau_1 \le \tau \le \tau' \le \tau_1'$. We know that $[\![\tau < \tau']\!] = \sup_{\sigma\in A} [\![\tau \le \sigma]\!] \cap [\![\sigma < \tau']\!]$. For $\sigma \in A$ set $\hat\sigma = \mathrm{med}(\tau_1, \sigma, \tau_1') \in A'$. We have

$$[\![\tau \le \hat\sigma]\!] \cap [\![\hat\sigma < \tau']\!] \supseteq [\![\tau \le \tau_1' \wedge \sigma]\!] \cap [\![\tau_1 \vee \sigma < \tau']\!]$$

(because $\tau_1' \wedge \sigma \le \hat\sigma \le \tau_1 \vee \sigma$)

$$= [\![\tau \le \tau_1']\!] \cap [\![\tau \le \sigma]\!] \cap [\![\tau_1 < \tau']\!] \cap [\![\sigma < \tau']\!]$$

(611E(c-i-α) and (c-ii-α))

$$\supseteq [\![\tau \le \sigma]\!] \cap [\![\tau < \tau']\!] \cap [\![\sigma < \tau']\!] = [\![\tau \le \sigma]\!] \cap [\![\sigma < \tau']\!]$$

(611E(c-iii-γ)). So

$$\begin{aligned}\sup_{\sigma\in A'}([\![\tau \le \sigma]\!] \cap [\![\sigma < \tau']\!]) &= \sup_{\sigma\in A}([\![\tau \le \hat\sigma]\!] \cap [\![\hat\sigma < \tau']\!])\\ &\supseteq \sup_{\sigma\in A}([\![\tau \le \sigma]\!] \cap [\![\sigma < \tau']\!]) = [\![\tau < \tau']\!]\end{aligned}$$

because A separates $\mathcal{S}$. As τ and τ' are arbitrary, A' separates $\mathcal{S}$.

(ii) Use the same argument, but looking only at the case $\tau \le \tau'$ and reading '$[\![v_\tau \ne v_{\tau'}]\!]$' for '$[\![\tau < \tau']\!]$'.

633E Lemma Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ such that $\inf A \in \mathcal{S}$ whenever $A \subseteq \mathcal{S}$ is non-empty and has a lower bound in $\mathcal{S}$, $\mathcal{S}'$ a sublattice of $\mathcal{S}$, $\hat{\mathcal{S}}'_f$ the finitely-covered envelope (611O) of $\mathcal{S}'$, and τ an element of $\bigcup_{\sigma\in\mathcal{S}'} \mathcal{S} \wedge \sigma$.

(a) $A = \{\sigma : \tau \le \sigma \in \hat{\mathcal{S}}'_f\}$ is non-empty and downwards-directed, and $\inf A \in \mathcal{S}$.

(b) If $\mathcal{S}'$ separates $\mathcal{S}$ then $\inf A = \tau$.

(c) If $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ is fully adapted and $\mathcal{S}'$ $\boldsymbol{v}$-separates $\mathcal{S}$, then $v_{\inf A} = v_\tau$.

proof (a) By the hypothesis on τ, A is non-empty, and of course it is downwards-directed. Because $\mathcal{S}$ is finitely full, $A \subseteq \mathcal{S}$, and τ is a lower bound for A. So $\tau^* = \inf A$ belongs to $\mathcal{S}$, and $\tau \le \tau^*$.

Note that if $\sigma \in \mathcal{S}'$ and $a = [\![\tau \le \sigma]\!] \cap [\![\sigma < \tau^*]\!]$, then $a = 0$. **P** Take any $\sigma_0 \in \mathcal{S}'$ such that $\tau \le \sigma_0$. Then we have a $\sigma^* \in \mathcal{T}$ such that $[\![\sigma^* = \sigma]\!] \supseteq a$, while $[\![\sigma^* = \sigma_0]\!] \supseteq 1 \setminus a$ (611I). In this case, $\sigma^* \in A$ so $[\![\sigma^* < \tau^*]\!] = 0$ and $a = 0$. **Q**

(b) It follows at once that if $\mathcal{S}'$ separates $\mathcal{S}$ then $\tau = \tau^*$, that is, $\inf A = \tau$.

(c) Similarly, if $\mathcal{S}'$ $\boldsymbol{v}$-separates $\mathcal{S}$, then $v_\tau = v_{\tau^*}$.

633F Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u}$, $\boldsymbol{v}$ locally near-simple processes with domain $\mathcal{S}$. Suppose that $C \subseteq \mathcal{S}$ separates $\mathcal{S}$ and that $\inf_{\sigma\in C} [\![\sigma < \tau]\!] = 0$ for every $\tau \in \mathcal{S}$. If $\boldsymbol{u}{\restriction}C = \boldsymbol{v}{\restriction}C$ then $\boldsymbol{u} = \boldsymbol{v}$.

proof (a) Set $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v}$, and express $\boldsymbol{w}$ as $\langle w_\sigma\rangle_{\sigma\in\mathcal{S}}$. Then $\boldsymbol{w}$ is near-simple (631F(a-ii)), $\boldsymbol{w}$ is zero on C and I need to show that $\boldsymbol{w}$ is zero on $\mathcal{S}$.

(a) Consider first the case in which $\mathcal{S}$ has a greatest element which belongs to C. If $\boldsymbol{z} = \langle z_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a simple process then $|\sup w| \le 2\sup|\boldsymbol{z}-\boldsymbol{w}|$. **P** Let $(\sigma_0, \dots, \sigma_n)$ be a breakpoint string for $\boldsymbol{z}$ with $\sigma_n = \max\mathcal{S}$, and write $z_\downarrow$ for the starting value of $\boldsymbol{z}$ and $\bar z$ for $\sup|\boldsymbol{z} - \boldsymbol{w}|$. Take any $\tau \in \mathcal{S}$. Then

$$\begin{aligned}[\![\tau < \tau_0]\!] &= \sup_{\sigma\in C}([\![\tau \le \sigma]\!] \cap [\![\sigma < \tau_0]\!]) \subseteq \sup_{\sigma\in C}([\![\tau > \tau_0]\!] \cap [\![\sigma < \tau_0]\!])\\ &\subseteq \sup_{\sigma\in C}([\![z_\tau = z_\downarrow]\!] \cap [\![z_\sigma = z_\downarrow]\!]) \subseteq \sup_{\sigma\in C}[\![z_\tau = z_\sigma]\!] \subseteq \sup_{\sigma\in C}[\![|z_\tau| = |z_\sigma - w_\sigma|]\!]\end{aligned}$$

(because $w_\sigma = 0$ for every $\sigma \in C$)

$$\subseteq [\![|z_\tau| \le \bar z]\!],$$

while if $i \le n$ then

$$\begin{aligned}[\![\sigma_i \le \tau]\!] \cap [\![\tau < \sigma_{i+1}]\!] &= \sup_{\sigma\in C}([\![\sigma_i \le \tau]\!] \cap [\![\tau < \sigma_{i+1}]\!] \cap [\![\tau \le \sigma]\!] \cap [\![\sigma < \sigma_{i+1}]\!])\\ &\subseteq \sup_{\sigma\in C}([\![\sigma_i \le \tau]\!] \cap [\![\tau < \sigma_{i+1}]\!] \cap [\![\sigma_i \le \sigma]\!] \cap [\![\sigma < \tau_{i+1}]\!])\\ &\subseteq \sup_{\sigma\in C}[\![z_\tau = z_\sigma]\!] \subseteq [\![|z_\tau| \le \bar z]\!]\end{aligned}$$

and

$$[\![\sigma_n \le \tau]\!] \subseteq [\![z_\tau = z_{\sigma_n} - w_{\sigma_n}]\!] \subseteq [\![|z_\tau| \le \bar z]\!].$$

Thus $|z_\tau| \le \bar z$; as τ is arbitrary, $\sup|\boldsymbol{z}| \le \bar z$. **Q**

Now we see that

$$\sup|\boldsymbol{w}| \le \sup|\boldsymbol{z}| + \sup|\boldsymbol{z} - \boldsymbol{w}| \le 2\sup|\boldsymbol{z} - \boldsymbol{w}|;$$

since $\boldsymbol{w}$ is near-simple, there is for any $\epsilon > 0$ a simple process $\boldsymbol{z}$ such that $\theta(\sup|\boldsymbol{z} - \boldsymbol{w}|) \le \epsilon$ and $\theta(\sup|\boldsymbol{w}|) \le 2\epsilon$. We conclude that $\boldsymbol{w} = \boldsymbol{0}$, as required.

(b) In the general case, take any $\tau \in \mathcal{S}$ and $\tau^* \in C$. Then C and $C \wedge \tau^*$ separate $\mathcal{S} \wedge \tau^*$ (633Ca, 633Cd), while $\{\sigma, \tau^*\}$ covere $\{\sigma \wedge \tau^*\}$ (611M(b-i)) so $|w_{\sigma\wedge\tau^*}| \le |w_\sigma| \vee |w_{\tau^*}| = 0$ (614Ga) for every $\sigma \in C$. By (a), applied to $\boldsymbol{w}{\restriction}\mathcal{S} \wedge \tau^*$, $w_{\tau\wedge\tau^*} = 0$ and $[\![\tau \le \tau^*]\!] \subseteq [\![w_\tau = 0]\!]$. As $\inf_{\tau^*\in C}[\![\tau^* < \tau]\!] = 0$, $w_\tau = 0$ and $\boldsymbol{w} = \boldsymbol{0}$ in this case also.

633G Lemma Let $\mathcal{S} \subseteq \mathcal{T}$ be a sublattice and $D \subseteq \mathcal{S}$ a cofinal finitely full set which separates $\mathcal{S}$ and is such that $\inf A \in D$ for every non-empty downwards-directed $A \subseteq D$ with a lower bound in $\mathcal{S}$. Then $D = \mathcal{S}$.

proof Take any $\tau \in \mathcal{S}$, and set $A = \{\sigma : \sigma \in D,\ \tau \le \sigma\}$. Because D is cofinal with $\mathcal{S}$, A is non-empty; because D is finitely full, it is closed under $\wedge$ (611P(a-ii)), so A also is closed under $\wedge$, and is downwards-directed, while τ is a lower bound of A belonging to $\mathcal{S}$. Set $\tau^* = \inf A$; then $\tau^* \in D$ and $\tau \le \tau^*$.

? If $\tau \ne \tau^*$, there is a $\sigma \in D$ such that $[\![\tau \le \sigma]\!] \cap [\![\sigma < \tau^*]\!] \ne 0$, because D separates $\mathcal{S}$. Set $c = [\![\tau \le \sigma]\!]$; then $c \in \mathfrak{A}_\sigma$ and $1 \setminus c \in \mathfrak{A}_\tau \subseteq \mathfrak{A}_{\tau^*}$. So there is a $\tau' \in \mathcal{T}$ such that $c \subseteq [\![\tau' = \sigma]\!]$ and $1 \setminus c \subseteq [\![\tau' = \tau^*]\!]$ (611I again). Since $[\![\tau' = \sigma]\!] \cup [\![\tau' = \tau^*]\!] = 1$, and D is finitely full, $\tau' \in D$; also

$$c = [\![\tau \le \sigma]\!] \cap [\![\tau' = \sigma]\!] \subseteq [\![\tau \le \tau']\!], \quad 1 \setminus c \subseteq [\![\tau' = \tau^*]\!] \subseteq [\![\tau \le \tau']\!]$$

so $\tau \le \tau'$. Accordingly $\tau' \in A$ and $\tau^* \le \tau'$. But

$$[\![\tau' < \tau^*]\!] \supseteq [\![\tau' = \sigma]\!] \cap [\![\sigma < \tau^*]\!] \supseteq [\![\tau \le \sigma]\!] \cap [\![\sigma < \tau^*]\!] \ne 0$$

so this is impossible. **X**

Thus $\tau = \tau^*$ belongs to D; as τ is arbitrary, $D = \mathcal{S}$.

633H Lemma Suppose that I, $J \in \mathcal{I}(\mathcal{T})$, with $J \subseteq I$. Then there are totally ordered sets $J_0 \subseteq J$ and $I_0 \subseteq I$ such that J_0 covers J, I_0 covers I and $J_0 \subseteq I_0$.

proof We know from 611Ke that there is a totally ordered set $J_0 \subseteq J$ covering J. If $J_0 = \emptyset$, all we have to do is to use 611Ke again to find a totally ordered set $I_0 \subseteq I$ covering I. Otherwise, enumerate J_0 in ascending order as $\langle \sigma_i \rangle_{i \le n}$. Set

$$K_0 = I \cap [\min \mathcal{T}, \sigma_0], \quad K_i = I \cap [\sigma_{i-1}, \sigma_i] \text{ for } 1 \le i \le n, \quad K_{n+1} = I \cap [\sigma_n, \max \mathcal{T}].$$

For each $i \le n+1$, let $K'_i \subseteq K_i$ be a totally ordered set covering the finite sublattice K_i; now set $I_0 = J_0 \cup \bigcup_{i \le n+1} K'_i$, so that I_0 is a totally ordered subset of I, including J_0, which covers $K = \bigcup_{i \le n+1} K_i$.

At the same time, K covers I. **P** Let $\tau \in I$. Set

$$\tau_0 = \tau \wedge \sigma_0, \quad \tau_i = \operatorname{med}(\sigma_{i-1}, \tau, \sigma_i) \text{ for } 1 \le i \le n, \quad \tau_{n+1} = \tau \vee \sigma_n.$$

Then $\tau_i \in K_i \subseteq K$ for every $i \le n+1$. Also $[\![\tau = \tau_0]\!] = [\![\tau \le \sigma_0]\!]$ and $[\![\tau = \tau_{n+1}]\!] = [\![\sigma_n \le \tau]\!]$, by 611E(a-ii-$\beta$). As for the middle terms, if $1 \le i \le n$ then

$$\begin{aligned}[\![\tau = \tau_i]\!] &= [\![\tau = (\tau \vee \sigma_{i-1}) \wedge \sigma_i]\!] \supseteq [\![\tau = \tau \vee \sigma_{i-1}]\!] \cap [\![\tau \vee \sigma_{i-1} = (\tau \vee \sigma_{i-1}) \wedge \sigma_i]\!] \\ &= [\![\sigma_{i-1} \le \tau]\!] \cap [\![\tau \vee \sigma_{i-1} \le \sigma_i]\!] = [\![\sigma_{i-1} \le \tau]\!] \cap [\![\tau \le \sigma_i]\!] \cap [\![\sigma_{i-1} \le \sigma_i]\!]\end{aligned}$$

(611Eb)

$$= [\![\sigma_{i-1} \le \tau]\!] \cap [\![\tau \le \sigma_i]\!].$$

So

$$\begin{aligned}\sup_{\sigma \in K} [\![\tau = \sigma]\!] &\supseteq \sup_{i \le n+1} [\![\tau = \tau_i]\!] \\ &\supseteq [\![\tau \le \sigma_0]\!] \cup \sup_{1 \le i \le n} ([\![\sigma_{i-1} \le \tau]\!] \cap [\![\tau \le \sigma_i]\!]) \cup [\![\sigma_n \le \tau]\!] = 1\end{aligned}$$

(611Ed), and K covers $\{\tau\}$. As τ is arbitrary, K covers I. **Q**

Consequently I_0 covers I, by 611M(g-i) as usual.

633I Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\mathcal{S}'$ a finitely full sublattice of $\mathcal{S}$ and ψ a strictly adapted interval function defined on $\mathcal{S}^{2\uparrow}$. Suppose that $J \in \mathcal{I}(\mathcal{S})$ and $\langle \sigma_\tau \rangle_{\tau \in J}$ are such that $\tau \le \sigma_\tau \in \mathcal{S}'$ and

$$\bar{u}_\tau = \sup_{\sigma \in \mathcal{S} \wedge \tau, \tau' \in \mathcal{S}' \cap [\tau, \sigma_\tau]} |\psi(\sigma, \tau') - \psi(\sigma, \tau)|$$

is defined in L^0 for each $\tau \in J$. Let $I \in \mathcal{I}(\mathcal{S}')$ be such that $\sigma_\tau \in I$ for every $\tau \in J$. Then

$$|S_{I \sqcup J}(\mathbf{1}, d\psi) - S_I(\mathbf{1}, d\psi)| \le 2 \textstyle\sum_{\tau \in J} \bar{u}_\tau.$$

proof (a) To begin with, suppose that $J = \{\tau\}$ is a singleton.

(i) Write $\hat{I}$ for the covered envelope of I. There is a $\tau' \in \hat{I}$ such that $\tau \le \tau'$ and $[\![\rho < \tau']\!] = [\![\rho < \tau]\!]$ for every $\rho \in \hat{I}$. **P** For each $\sigma \in I$, $[\![\tau \le \sigma]\!] \in \mathfrak{A}_\sigma \cap \mathfrak{A}_{\sigma_\tau}$ so there is a $\sigma' \in \mathcal{T}$ such that $[\![\tau_j \le \sigma]\!] \subseteq [\![\sigma' = \sigma]\!]$ and $[\![\sigma < \tau_j]\!] \subseteq [\![\sigma' = \sigma_\tau]\!]$. Now $\tau \le \sigma' \in \hat{I}$ for each $\sigma \in I$, so

$$\tau \le \tau' = \inf_{\sigma \in I} \sigma' \in \hat{I} \subseteq \mathcal{S}'.$$

Take $\rho \in \hat{I}$. Then certainly $[\![\rho < \tau]\!] \subseteq [\![\rho < \tau']\!]$. **?** If $a = [\![\rho < \tau']\!] \setminus [\![\rho < \tau]\!]$ is non-zero then there is a $\sigma \in I$ such that $a' = a \cap [\![\rho = \sigma]\!]$ is non-zero. In this case,

$$\begin{aligned}0 \ne a' &\subseteq [\![\sigma < \tau']\!] \setminus [\![\sigma < \tau]\!] \subseteq [\![\sigma < \tau']\!] \cap [\![\tau \le \sigma]\!] \\ &\subseteq [\![\sigma < \tau']\!] \cap [\![\sigma' = \sigma]\!] \subseteq [\![\sigma < \tau']\!] \cap [\![\tau' \le \sigma]\!]\end{aligned}$$

because $\tau' \le \sigma'$; but this is impossible. **X** So $[\![\rho < \tau]\!] = [\![\rho < \tau']\!]$, as required. **Q**

Of course $0 = [\![\sigma_\tau < \tau]\!] = [\![\sigma_\tau < \tau']\!]$ so $\tau' \le \sigma_\tau$.

(ii) Write $I' = I \sqcup \{\tau'\}$ and $K = I' \sqcup \{\tau\}$. Then $\{\rho : \rho \in \mathcal{T},\ \rho \vee \tau' \in I'\}$ is a sublattice including $I' \cup \{\tau\}$ so it includes K and $K \vee \tau' \subseteq I'$; consequently $K \vee \tau' = I' \vee \tau'$ and $S_{K \vee \tau'}(\mathbf{1}, d\psi) = S_{I' \vee \tau'}(\mathbf{1}, d\psi)$.

(iii) $|S_{K\wedge\tau'}(\mathbf{1},d\psi)-S_{I'\wedge\tau'}(\mathbf{1},d\psi)|\le 2\bar u_\tau$. **P** Take $(\sigma_0,\dots,\sigma_m)$ linearly generating the $(I'\wedge\tau')$-cells; note that every σ_i belongs to $\hat I$. Then $(\sigma_0\wedge\tau,\dots,\sigma_m\wedge\tau)$ linearly generates the $(K\wedge\tau')$-cells and $(\sigma_0\vee\tau,\dots,\sigma_m\vee\tau)$ linearly generates the $(K\cap[\tau,\tau'])$-cells. So

$$\begin{aligned}&S_{K\wedge\tau'}(\mathbf{1},d\psi)-S_{I'\wedge\tau'}(\mathbf{1},d\psi)\\&\qquad=\sum_{j=0}^{m-1}\psi(\sigma_i\wedge\tau,\sigma_{j+1}\wedge\tau)+\psi(\sigma_j\vee\tau,\sigma_{j+1}\vee\tau)-\psi(\sigma_j,\sigma_{j+1}).\end{aligned}$$

Set

$$\begin{aligned}b_i&=[\![\tau<\sigma_0]\!]\text{ if }i=-1,\\&=[\![\sigma_i\le\tau]\!]\cap[\![\tau<\sigma_{i+1}]\!]\text{ if }0\le i<m,\\&=[\![\tau=\sigma_m]\!]\text{ if }i=m;\end{aligned}$$

then $b_{-1},\dots,b_m$ is a partition of unity in $\mathfrak{A}$. Now observe that if $-1\le i<j<m$ then

$$\begin{aligned}b_i&\subseteq[\![\tau<\sigma_{i+1}]\!]\subseteq[\![\tau<\sigma_j]\!]\\&\subseteq[\![\sigma_j\wedge\tau=\sigma_{j+1}\wedge\tau]\!]\cap[\![\sigma_j\vee\tau=\sigma_j]\!]\cap[\![\sigma_{j+1}\vee\tau=\sigma_{j+1}]\!]\\&\subseteq[\![\psi(\sigma_j\wedge\tau,\sigma_{j+1}\wedge\tau)=0]\!]\cap[\![\psi(\sigma_j\vee\tau,\sigma_{j+1}\vee\tau)=\psi(\sigma_j,\sigma_{j+1})]\!]\\&\subseteq[\![\psi(\sigma_i\wedge\tau,\sigma_{j+1}\wedge\tau)+\psi(\sigma_j\vee\tau,\sigma_{j+1}\vee\tau)-\psi(\sigma_j,\sigma_{j+1})=0]\!],\end{aligned}$$

while if $0\le j<i\le m$ then

$$\begin{aligned}b_i&\subseteq[\![\sigma_i\le\tau]\!]\subseteq[\![\sigma_{j+1}\le\tau]\!]\subseteq[\![\sigma_j\vee\tau=\sigma_{j+1}\wedge\tau]\!]\cap[\![\sigma_j\wedge\tau=\sigma_j]\!]\cap[\![\sigma_{j+1}\wedge\tau=\sigma_{j+1}]\!]\\&\subseteq[\![\psi(\sigma_i\vee\tau,\sigma_{j+1}\vee\tau)=0]\!]\cap[\![\psi(\sigma_i\wedge\tau,\sigma_{j+1}\wedge\tau)=\psi(\sigma_i,\sigma_{j+1})]\!]\\&\subseteq[\![\psi(\sigma_j\wedge\tau,\sigma_{j+1}\wedge\tau)+\psi(\sigma_j\vee\tau,\sigma_{j+1}\vee\tau)-\psi(\sigma_j,\sigma_{j+1})=0]\!].\end{aligned}$$

So

$$\begin{aligned}&|S_{K\wedge\tau'}(\mathbf{1},d\psi)-S_{I'\wedge\tau'}(\mathbf{1},d\psi)|\\&\quad=|\sum_{j=0}^{m-1}\sum_{i=-1}^{m}\bigl(\psi(\sigma_j\wedge\tau,\sigma_{j+1}\wedge\tau)+\psi(\sigma_j\vee\tau,\sigma_{j+1}\vee\tau)-\psi(\sigma_j,\sigma_{j+1})\bigr)\times\chi b_i|\\&\quad=|\sum_{j=0}^{m-1}\bigl(\psi(\sigma_j\wedge\tau,\sigma_{j+1}\wedge\tau)+\psi(\sigma_j\vee\tau,\sigma_{j+1}\vee\tau)-\psi(\sigma_j,\sigma_{j+1})\bigr)\times\chi b_j|\\&\quad\le\sum_{j=0}^{m-1}|\psi(\sigma_j\wedge\tau,\tau)+\psi(\tau,\sigma_{j+1}\vee\tau)-\psi(\sigma_j\wedge\tau,\sigma_{j+1}\vee\tau)|\times\chi b_j\\&\quad\le\sum_{j=0}^{m-1}\bigl(|\psi(\sigma_j\wedge\tau,\sigma_{j+1}\vee\tau)-\psi(\sigma_j\wedge\tau,\tau)|+|\psi(\tau,\sigma_{j+1}\vee\tau)-\psi(\tau,\tau)|\bigr)\times\chi b_j\\&\quad=\sum_{j=0}^{m-1}\bigl(|\psi(\sigma_j\wedge\tau,\tau')-\psi(\sigma_j\wedge\tau,\tau)|+|\psi(\tau,\tau')-\psi(\tau,\tau)|\bigr)\times\chi b_j\end{aligned}$$

(because $b_j\subseteq[\![\tau\le\sigma_{j+1}]\!]=[\![\tau'\le\sigma_{j+1}]\!]=[\![\tau'=\sigma_{j+1}]\!]=[\![\tau'=\sigma_{j+1}\vee\tau]\!]$ for every $j<m$)

$$\le\sum_{j=0}^{m-1}2\bar u_\tau\times\chi b_j$$

(because $\tau'\in\mathcal{S}'$ and $\tau\le\tau'\le\sigma_\tau$)

$$\le 2\bar u_\tau$$

as claimed. **Q**

(iv) Putting (ii) and (iii) together, we see that $|S_K(\mathbf{1}, d\psi) - S_{I'}(\mathbf{1}, d\psi)| \le 2\bar{u}_\tau$. But $I \sqcup J$ is a sublattice of K covering I' and J, so it covers K, and $S_{I\sqcup J}(\mathbf{1}, d\psi) = S_K(\mathbf{1}, d\psi)$ by 613S. Similarly, $S_{I'}(\mathbf{1}, d\psi) = S_I(\mathbf{1}, d\psi)$, and $|S_{I\sqcup J}(\mathbf{1}, d\psi) - S_I(\mathbf{1}, d\psi)| \le 2\bar{u}_\tau$, as required.

(b) For the general case, induce on $\#(J)$. If J is empty, the result is trivial, and if $\#(J) = 1$ it is covered by (a) above. For the inductive step to $\#(J) = n \ge 2$, if J is not totally ordered then it has a totally ordered subset J' which covers J (611Ke), and

$$|S_{I\sqcup J}(\mathbf{1}, d\psi) - S_I(\mathbf{1}, d\psi)| = |S_{I\sqcup J'}(\mathbf{1}, d\psi) - S_I(\mathbf{1}, d\psi)| \le 2\sum_{\tau\in J'} \bar{u}_\tau$$

(by the inductive hypothesis)

$$\le 2\sum_{\tau\in J} \bar{u}_\tau.$$

If J is totally ordered, set $J' = J \setminus \{\max J\}$; then

$$\begin{aligned}|S_{I\sqcup J}(\mathbf{1}, d\psi) - S_I(\mathbf{1}, d\psi)| &\le |S_{I\sqcup J}(\mathbf{1}, d\psi) - S_{I\sqcup J'}(\mathbf{1}, d\psi)| + |S_{I\sqcup J'}(\mathbf{1}, d\psi) - S_I(\mathbf{1}, d\psi)| \\ &\le |S_{(I\sqcup J')\sqcup\{\max J\}})(\mathbf{1}, d\psi) - S_{I\sqcup J'}(\mathbf{1}, d\psi)| + 2\sum_{\tau\in J'} u_\tau\end{aligned}$$

(by the inductive hypothesis)

$$\le 2\sum_{\tau\in J} \bar{u}_\tau$$

by (a). Thus the induction continues in either case.

633J Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\mathcal{S}'$ a finitely full cofinal sublattice of $\mathcal{S}$ and ψ an order-bounded strictly adapted interval function defined on $\mathcal{S}^{2\uparrow}$. For $\tau \in \mathcal{S}$ set $A_\tau = \{\sigma : \tau \le \sigma \in \mathcal{S}'\}$, and for $\tau \in \mathcal{S}$, $\tau' \in A_\tau$ set

$$u_{\tau\tau'} = \sup_{\sigma\in\mathcal{S}\wedge\tau, \rho\in A_\tau\wedge\tau'} |\psi(\sigma, \rho) - \psi(\sigma, \tau)|.$$

Suppose that $z = \int_{\mathcal{S}} d\psi$ is defined and that $\inf_{\tau'\in A_\tau} u_{\tau\tau'} = 0$ for every $\tau \in \mathcal{S}$. Then $\int_{\mathcal{S}'} d\psi$ is defined and equal to z.

proof Let $\epsilon > 0$. Then there is a $J \in \mathcal{I}(\mathcal{S})$ such that $\theta(z - S_K(\mathbf{1}, d\psi)) \le \epsilon$ whenever $J \subseteq K \in \mathcal{I}\mathcal{S}$. For each $\tau \in J$, $\tau' \mapsto u_{\tau\tau'} : A_\tau \to (L^0)^+$ is order-preserving so $\lim_{\tau'\downarrow\tau} u_{\tau\tau'} = 0$ and there is a $\sigma_\tau \in A_\tau$ such that $\theta(u_{\tau\sigma_\tau}) \le \frac{\epsilon}{1+\#(J)}$. Set $I_0 = \{\sigma_\tau : \tau \in J\}$.

Suppose that $I_0 \subseteq I \in \mathcal{I}(\mathcal{S}')$. Then $|S_{I\sqcup J}(\mathbf{1}, d\psi) - S_I(\mathbf{1}, d\psi)| \le \sum_{\tau\in J} u_{\tau\sigma_\tau}$ by 633I. But now we see that

$$\theta(z - S_I(\mathbf{1}, d\psi)) \le \theta(z - S_{I\sqcup J}(\mathbf{1}, d\psi)) + \sum_{\tau\in J} \theta(u_{\tau\sigma_\tau}) \le \epsilon + 2\epsilon$$

by the choice of J and $\langle\sigma_J\rangle_{\tau\in J}$. As ϵ is arbitrary, $\int_{\mathcal{S}'} d\psi$ is defined and equal to z.

633K Theorem (a) Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ such that $\inf A \in \mathcal{S}$ for every non-empty subset A of $\mathcal{S}$ with a lower bound in $\mathcal{S}$, and $\boldsymbol{u}$, $\boldsymbol{v}$ fully adapted processes with domain $\mathcal{S}$ such that $\boldsymbol{u}$ is order-bounded and $\boldsymbol{v}$ is locally near-simple. Let $\mathcal{S}'$ be a sublattice of $\mathcal{S}$, cofinal with $\mathcal{S}$, which $\boldsymbol{v}$-separates $\mathcal{S}$. If $z = \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$ is defined then $\int_{\mathcal{S}'} \boldsymbol{u}\, d\boldsymbol{v}$ is defined and equal to z.

(b) Suppose that $\mathcal{S}$, $\mathcal{S}'$ are sublattices of $\mathcal{T}$ such that $\mathcal{S}'$ is finitely full and is included in $\mathcal{S}$. Let $\boldsymbol{u}$, $\boldsymbol{v}$ be fully adapted processes defined on $\mathcal{S}$. For $\tau \in \mathcal{S}$ set $A_\tau = \{\sigma : \tau \le \sigma \in \mathcal{S}'\}$. Suppose that A_τ is non-empty and

$$u_\tau = \lim_{\sigma\downarrow A_\tau} u_\sigma, \quad v_\tau = \lim_{\sigma\downarrow A_\tau} v_\sigma$$

for every $\tau \in \mathcal{S}$. If $z = \int_{\mathcal{S}'} \boldsymbol{u}\, d\boldsymbol{v}$ is defined then $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$ is defined and equal to z.

proof (a)(i) Suppose to begin with that $\mathcal{S}'$ is finitely full. For $\tau \in \mathcal{S}$, set $A_\tau = \{\sigma : \tau \le \sigma \in \hat{\mathcal{S}}'_f\}$. Because $\mathcal{S}'$ is cofinal with $\mathcal{S}$, $A_\tau \ne \emptyset$ and $\inf A_\tau \in \mathcal{S}$, while $v_\tau = v_{\inf A_\tau}$ by 633Ec. Because $\langle \mathfrak{A}_t \rangle_{t\in T}$ is right-continuous and $\boldsymbol{v}$ is locally near-simple,

$$\lim_{\sigma\downarrow A_\tau} \sup_{\rho\in A_\tau,\rho\le\sigma} |v_\rho - v_\tau| = \lim_{\sigma\downarrow A_\tau} \sup_{\rho\in A_\tau,\rho\le\sigma} |v_\rho - v_{\inf A}| = 0$$

(632E). Now set $\psi = \boldsymbol{u}\Delta\boldsymbol{v}$ (613Cc, 613Dd), so that $\psi(\sigma,\tau) = u_\sigma \times (v_\tau - v_\sigma)$ when $\sigma \le \tau$ in $\mathcal{S}$, and ψ is strictly adapted and order-bounded. Write $\bar u$ for $\sup|\boldsymbol{u}|$. Then whenever $\tau \in \mathcal{S}$ we have $v_\tau = v_{\inf A_\tau}$ (633Ec), so

$$\lim_{\tau'\downarrow A_\tau} \sup_{\sigma\in\mathcal{S}\wedge\tau,\rho\in A_\tau\wedge\tau'} |\psi(\sigma,\rho) - \psi(\sigma,\tau)| \le \lim_{\tau'\downarrow A_\tau} \sup_{\rho\in A_\tau\wedge\tau'} \bar u \times |v_\rho - v_\tau| = 0.$$

Thus ψ satisfies the conditions of 633J and

$$\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v} = \int_{\mathcal{S}} d\psi$$

(613Hc)

$$= \int_{\mathcal{S}'} d\psi$$

(633J)

$$= \int_{\mathcal{S}'} \boldsymbol{u}\, d\boldsymbol{v}$$

as required.

(ii) In general, the finitely-covered envelope $\hat{\mathcal{S}}'_f$ of $\mathcal{S}'$ is a cofinal finitely full sublattice of $\mathcal{S}$ which $\boldsymbol{v}$-separates $\mathcal{S}$, so

$$\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v} = \int_{\hat{\mathcal{S}}'_f} \boldsymbol{u}\, d\boldsymbol{v} = \int_{\mathcal{S}'} \boldsymbol{u}\, d\boldsymbol{v}$$

by 613T.

(b) Let $\epsilon > 0$. Let $J \in \mathcal{I}(\mathcal{S}')$ be such that $\theta(z - S_I(\boldsymbol{u}, d\boldsymbol{v})) \le \epsilon$ whenever $J \subseteq I \in \mathcal{I}(\mathcal{S}')$. Now suppose that I is any finite sublattice of $\mathcal{S}$ including J. Then there are a totally ordered subset J_0 of J covering J and a totally ordered subset I_0 of I, including J_0, which covers I (633H). Let $\langle \tau_i \rangle_{i\le n}$ be the increasing enumeration of I_0. Then we can find $\tau'_i \in A_{\tau_i}$, for $i \le n$, such that

— whenever $i, j \le n$, $\sigma \in A_{\tau_i}$, $\sigma' \in A_{\tau_j}$, $\sigma \le \tau'_i$ and $\sigma' \le \tau'_j$ then $\theta(u_\sigma \times v_{\sigma'} - u_{\tau_i} \times v_{\tau_j}) \le \frac{\epsilon}{n+1}$,

— if $i \le n$ and $\tau_i \in \mathcal{S}'$ then $\tau'_i = \tau_i$.

Set $\sigma_i = \inf_{n\ge j\ge i} \tau'_j$ for $i \le n$. Then $\sigma_0 \le \ldots \le \sigma_n$; for each $i \le n$, $\sigma_i \in \mathcal{S}'$ and $\tau_i \le \sigma_i \le \tau'_i$; and $\sigma_i = \tau_i$ if $\tau_i \in J_0$. So if we set $I'_0 = \{\sigma_i : i \le n\}$, we have

$$\begin{aligned}\theta(S_{I'_0}(\boldsymbol{u}, d\boldsymbol{v}) - S_I(\boldsymbol{u}, d\boldsymbol{v})) &= \theta\bigl(\sum_{i=0}^{n-1} u_{\sigma_i} \times v_{\sigma_{i+1}} - u_{\tau_i} \times v_{\tau_{i+1}} - u_{\sigma_i} \times v_{\sigma_i} + u_{\tau_i} \times v_{\tau_i}\bigr)\\ &\le 2\epsilon.\end{aligned}$$

Because J_0 covers J and $J_0 \subseteq I'_0$, the covered envelope of I'_0 includes $I' = I'_0 \sqcup J$ (611M(b-i)) and $S_{I'_0}(\boldsymbol{u}, d\boldsymbol{v}) = S_{I'}(\boldsymbol{u}, d\boldsymbol{v})$ (613T again). But $I' \supseteq J$ so $\theta(z - S_{I'}(\boldsymbol{u}, d\boldsymbol{v})) \le \epsilon$. Consequently $\theta(z - S_I(\boldsymbol{u}, d\boldsymbol{v})) \le 3\epsilon$; and this is true whenever $J \subseteq I \in \mathcal{I}(\mathcal{S})$.

As ϵ is arbitrary, $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$ is defined and equal to z.

633L Corollary Suppose that $\langle \mathfrak{A}_t \rangle_{t\in T}$ is right-continuous, and $\tau \le \tau'$ in $\mathcal{T}$. Let $\boldsymbol{u}$ be a near-simple process and $\boldsymbol{v}$ a near-simple integrator, both defined on $[\tau, \tau']$. Suppose that $T_0 \subseteq T$ is a dense set for the order topology containing every point of T which is isolated on the right. Set $\mathcal{S}' = \{\tau'\} \cup \{\mathrm{med}(\tau, \check{t}, \tau') : t \in T_0\}$. Then $\int_{\mathcal{S}'} \boldsymbol{u}\, d\boldsymbol{v}$ is defined and equal to $\int_{[\tau,\tau']} \boldsymbol{u}\, d\boldsymbol{v}$.

proof By 616K $\int_{[\tau,\tau']} \boldsymbol{u}\, d\boldsymbol{v}$ is defined. By 633D(b-i), $\mathcal{S}'$ separates $[\tau, \tau']$, and it is cofinal with $[\tau, \tau']$ just because it contains τ'. We can therefore apply 633Ka to see that $\int_{\mathcal{S}'} \boldsymbol{u}\, d\boldsymbol{v} = \int_{[\tau,\tau']} \boldsymbol{u}\, d\boldsymbol{v}$.

***633M** I star the next couple of paragraphs because they are here for a very special application in §652.

Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$ a moderately oscillatory process with domain $\mathcal{S}$.

(a) For every $\epsilon > 0$ and $\beta > 0$, there are a $b \in \mathfrak{A}$ and a $\gamma \ge 0$ such that $\bar\mu b \ge 1 - \epsilon$ and

$$\textstyle\sum_{i=0}^{n-1} \bar\mu(b \cap [\![|u_{\tau_{i+1}} - u_{\tau_i}| \ge \beta]\!]) \le \gamma$$

whenever $\tau_0 \le \ldots \le \tau_n$ in $\mathcal{S}$,

(b)(i) For $\sigma \le \tau$ in $\mathcal{S}$, set

$$\psi(\sigma, \tau) = \mathrm{med}(-\chi 1, u_\tau - u_\sigma, \chi 1), \quad \psi'(\sigma, \tau) = u_\tau - u_\sigma - \psi(\sigma, \tau).$$

Then ψ and ψ' are strictly adapted interval functions on $\mathcal{S}$.

(ii) If $\sigma \le \tau$ and $\sigma' \le \tau'$ in $\mathcal{S}$, then $|\psi(\sigma, \tau) - \psi(\sigma', \tau')|$ and $|\psi'(\sigma, \tau) - \psi'(\sigma', \tau')|$ are both at most $|u_\sigma - u_{\sigma'}| + |u_\tau - u_{\tau'}|$.

(c) $\int_{\mathcal{S}} d\psi'$ and $\int_{\mathcal{S}} d\psi$ are defined.

(d) $\boldsymbol{w}' = ii_{\psi'}(\boldsymbol{1})$ is of bounded variation and $\boldsymbol{w} = ii_\psi(\boldsymbol{1})$ is moderately oscillatory.

(e) $\mathrm{Osclln}(\boldsymbol{w}) \le \chi 1$.

(f) Express $\boldsymbol{w}$ as $\langle w_\tau \rangle_{\tau\in\mathcal{S}}$. If $\tau \le \tau'$ in $\mathcal{S}$, then $w_{\tau'} - w_\tau \in L^0(\mathfrak{D}_\tau)$, where $\mathfrak{D}_\tau$ is the closed subalgebra of $\mathfrak{A}$ generated by $\{u_{\sigma'} - u_\sigma : \sigma, \sigma' \in \mathcal{S} \cap [\tau, \tau'], \sigma \le \sigma'\}$.

(g) If $\boldsymbol{u}$ is near-simple, $\boldsymbol{w}$ is near-simple.

proof If $\mathcal{S} = \emptyset$ all of this is true for trivial reasons, so in the following arguments I will take it that $\mathcal{S}$ is non-empty.

(a) (Compare (i)⇒(iv) in the proof of 615N.) It is enough to deal with the case in which $\beta \le 1$. We know that there is a process $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma\in\mathcal{S}}$ of bounded variation such that $\theta(\sup |\boldsymbol{u} - \boldsymbol{v}|) < \frac{1}{6}\beta\epsilon$. Set $b_0 = [\![\sup |\boldsymbol{u} - \boldsymbol{v}| < \frac{1}{3}\beta]\!]$; then $\bar\mu b_0 \ge 1 - \frac{1}{2}\epsilon$. Writing $\bar v = \int_{\mathcal{S}} |d\boldsymbol{v}|$, we have a $\gamma \ge 0$ such that $\bar\mu b_1 \ge 1 - \frac{1}{2}\epsilon$ where $b_1 = [\![\bar v < \frac{1}{3}\gamma\beta]\!]$. Set $b = b_0 \cap b_1$; then $\bar\mu b \ge 1 - \epsilon$.

If $\sigma \le \tau$ in $\mathcal{S}$ then

$$[\![|u_\tau - u_\sigma| \ge \beta]\!] \subseteq [\![|u_\tau - v_\tau| \ge \tfrac{1}{3}\beta]\!] \cup [\![|v_\tau - v_\sigma| \ge \tfrac{1}{3}\beta]\!] \cup [\![|v_\tau - u_\tau| \ge \frac{1}{3}\beta]\!],$$

so $b \cap [\![|u_\tau - u_\sigma| \ge \beta]\!] \subseteq b \cap [\![|v_\tau - v_\sigma| \ge \frac{1}{3}\beta]\!]$. Now take any $\tau_0 \le \ldots \le \tau_n$ in $\mathcal{S}$. Then

$$\begin{aligned}
\sum_{i=0}^{n-1} \bar\mu(b \cap [\![|u_{\tau_{i+1}} - u_{\tau_i}| \ge \beta]\!]) &\le \sum_{i=0}^{n-1} \bar\mu(b_1 \cap [\![|v_{\tau_{i+1}} - v_{\tau_i}| \ge \tfrac{1}{3}\beta]\!]) \\
&\le \sum_{i=0}^{n-1} \frac{3}{\beta}\mathbb{E}(\chi b_1 \times |v_{\tau_{i+1}} - v_{\tau_i}|) \\
&\le \frac{3}{\beta}\mathbb{E}(\chi b_1 \times \bar v) \le \frac{3}{\beta} \cdot \frac{1}{3}\gamma\beta \le \gamma.
\end{aligned}$$

(b)(i) Since $\alpha \mapsto \mathrm{med}(-1, \alpha, 1) : \mathbb{R} \to \mathbb{R}$ is Borel measurable and $(\sigma, \tau) \mapsto u_\tau - u_\sigma$ is strictly adapted (613Cc), ψ and ψ' are strictly adapted (613Db-613Dc).

(ii) For x, y, z, x', y', $z' \in L^0$,

$$x' = x + (x' - x) \le (x \vee y) + (x' - x) \vee (y' - y),$$

$$x' \vee y' \le (x \vee y) + (x' - x) \vee (y' - y), \quad |x' \vee y' - x \vee y| \le |x' - x| \vee |y' - y|,$$

$$|x' \vee y' \vee z' - x \vee y \vee z| \le |x' \vee y' - x \vee y| \vee |z' - z| \le |x' - x| \vee |y' - y| \vee |z' - z|;$$

similarly, $|x' \wedge y' - x \wedge y| \le |x' - x| \vee |y' - y|$. Consequently

$$\begin{aligned} &|\operatorname{med}(x', y', z') - \operatorname{med}(x, y, z)| \\ &\qquad = |(x' \wedge y') \vee (x' \wedge z') \vee (y' \wedge z') - (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)| \\ &\qquad \le |x' \wedge y' - x \wedge y| \vee |x' \wedge z' - x \wedge z| \vee |y' \wedge z' - y \wedge z| \\ &\qquad \le |x' - x| \vee |y' - y| \vee |z' - z|. \end{aligned}$$

In particular, $|\operatorname{med}(-\chi 1, x', \chi 1) - \operatorname{med}(-\chi 1, x, \chi 1)| \le |x' - x|$, so

$$|\psi(\sigma, \tau) - \psi(\sigma', \tau')| \le |u_\tau - u_\sigma - u_{\tau'} + u_{\sigma'}| \le |u_\sigma - u_{\sigma'}| + |u_\tau - u_{\tau'}|.$$

Next, for any $x \in L^0$, $\operatorname{med}(-\chi 1, x, \chi 1) - x = \operatorname{med}(-\chi 1 - x, 0, \chi 1 - x)$, so

$$\begin{aligned} &|(\operatorname{med}(-\chi 1, y, \chi 1) - y) - (\operatorname{med}(-\chi 1, x, \chi 1) - x)| \\ &\qquad \le |-\chi 1 - x + \chi 1 + y| \vee |\chi 1 - y - \chi 1 + x| = |y - x| \end{aligned}$$

for all x, $y \in L^0$. And

$$|\psi'(\sigma, \tau) - \psi'(\sigma', \tau')| \le |u_\tau - u_\sigma - u_{\tau'} + u_{\sigma'}| \le |u_\sigma - u_{\sigma'}| + |u_\tau - u_{\tau'}|.$$

(c) To begin with (down to the end of (v) below), suppose that $\mathcal{S}$ is finitely full. Let $\epsilon > 0$.

(i) By 615Ga, $u_\uparrow = \lim_{\sigma\uparrow\mathcal{S}} u_\sigma$ is defined and there is a $\tau^* \in \mathcal{S}$ such that that $\theta(z^*) \le \epsilon$ where $z^* = \sup_{\tau \in \mathcal{S} \vee \tau^*} |u_\tau - u_\uparrow|$. Then $\bar\mu[\![z^* \ge \frac{1}{4}]\!] \le 4\epsilon$.

(ii) By (a) above, there are $b_0 \in \mathfrak{A}$ and $\gamma \ge 0$ such that $\bar\mu b_0 \ge 1 - \epsilon$ and

$$\sum_{i=0}^{n-1} \bar\mu(b_0 \cap [\![|u_{\tau_{i+1}} - u_{\tau_i}| \ge \tfrac{1}{4}]\!]) \le \gamma$$

whenever $\tau_0 \le \ldots \le \tau_n$ in $\mathcal{S}$. Set $\delta = \min(\dfrac{1}{8}, \dfrac{\epsilon}{\gamma})$.

(iii) By 615F(a-i) $\boldsymbol{u}\restriction\mathcal{S} \wedge \tau^*$ is moderately oscillatory, therefore $\Vdash$-convergent(615G/615J), while $\mathcal{S} \wedge \tau^*$ is full (611Me). Construct $\langle D_i\rangle_{i\in\mathbb{N}}$, $\langle y_i\rangle_{i\in\mathbb{N}}$ and $\langle d_i\rangle_{i\in\mathbb{N}}$ from $\boldsymbol{u}\restriction\mathcal{S} \wedge \tau^*$ and δ as in 615M. For $i \in \mathbb{N}$ set $D_i^* = \bigcup_{\sigma\in D_i} \mathcal{S} \vee \sigma$. There is an $m \in \mathbb{N}$ such that $\bar\mu d_m \le \epsilon$ (615M(c-ii)). Choose $\tau_m, \ldots, \tau_0$ in such a way that for each $i \le m$

$\tau_i \in D_i$,

$\theta(z_i) \le \frac{\epsilon\delta}{m+1}$ where $z_i = \sup\{|u_\sigma - y_i| : \sigma \in D_i^*, \sigma \le \tau_i\}$,

if $i < m$ then $\tau_i \le \tau_{i+1}$.

(By 615Gb, when we come to choose τ_k anything far enough down D_k will serve.) Set $z = \sup_{k\le m} z_k$; then $\theta(z) \le \epsilon\delta$ so $\bar\mu[\![z \ge \delta]\!] \le \epsilon$. Set $b = b_0 \cap [\![z < \delta]\!] \cap [\![z^* < \frac{1}{4}]\!] \setminus d_m$; then $\bar\mu b \ge 1 - 7\epsilon$.

(iv) Take $i < m$ and a finite sublattice K of $\mathcal{S} \cap [\tau_i, \tau_{i+1}]$ containing τ_i and τ_{i+1},

($\boldsymbol{\alpha}$) Let $\langle\sigma_j\rangle_{j\le n}$ be the increasing enumeration of a maximal totally ordered subset of K, so that $\tau_i = \sigma_0$ and $\tau_{i+1} = \sigma_n$. Define $\langle a_j\rangle_{j\le n}$ inductively by saying that

$$a_j = [\![|u_{\sigma_j} - y_i| \ge \delta]\!] \setminus \sup_{j'<j} a_{j'}$$

for each j and $a^* = 1 \setminus \sup_{j\le n} a_j$. Then the a_j are disjoint and $a_j \in \mathfrak{A}_{\sigma_j} \subseteq \mathfrak{A}_{\tau_{i+1}}$ for each j, so we have a $\tau_i' \in \mathcal{T}$ such that $a_j \subseteq [\![\tau_i' = \sigma_j]\!]$ for $j \le n$, while $a^* \subseteq [\![\tau_i' = \tau_i]\!]$. As $\mathcal{S}$ is supposed to be full, $\tau_i' \in \mathcal{S}$. Now $\sup_{j\le n} [\![\tau_i' = \sigma_j]\!] = 1$, $\tau_i \le \tau_i' \le \tau_{i+1}$ and $[\![|u_{\tau_i'} - y_i| < \delta]\!]$ is disjoint from every a_j and from $[\![|u_{\tau_{i+1}} - y_i| \ge \delta]\!]$ so is included in $[\![\tau_i' = \tau_{i+1}]\!] \cap [\![\tau_{i+1} = \tau^*]\!] \subseteq [\![\tau_i' = \tau^*]\!]$. So $\tau_i' \in D_{i+1}$.

($\boldsymbol{\beta}$) For $l \le n$,

$$|S_K(\mathbf{1}, d\psi') - \psi'(\tau_i, \tau_{i+1})| \times \chi(b \cap a_l) \le 4\delta\chi[\![|u_{\tau_{i+1}} - u_{\tau_i}| \ge \tfrac{1}{4}]\!].$$

P Concerning the case $l = 0$, observe that $a_0 = [\![|u_{\tau_i} - y_i| \ge \delta]\!] \subseteq [\![z_i \ge \delta]\!]$ is disjoint from b and the result is trivial. Otherwise, we know that $S_K(\mathbf{1}, d\psi') = \sum_{j=0}^{n-1} \psi'(\sigma_j, \sigma_{j+1})$ (613Ec). Now if $0 \le j \le l - 2$ then

$$a_l \subseteq [\![|u_{\sigma_j} - y_i| < \delta]\!] \cap [\![|u_{\sigma_{j+1}} - y_i| < \delta]\!] \subseteq [\![|u_{\sigma_{j+1}} - u_{\sigma_j}| < 2\delta]\!] \subseteq [\![\psi'(\sigma_j, \sigma_{j+1}) = 0]\!]$$

and if $l \le j < n$ then $\tau'_i \le \tau'_i \vee \sigma_j \le \tau'_i \vee \sigma_{j+1} \le \tau_{i+1}$ so $\tau'_i \vee \sigma_j$ and $\tau'_i \vee \sigma_{j+1}$ belong to D^*_{i+1} and

$$\begin{aligned} a_l \subseteq [\![\sigma_j = \tau'_i \vee \sigma_j]\!] &\cap [\![\sigma_{j+1} = \tau'_i \vee \sigma_{j+1}]\!] \\ &\cap [\![|u_{\tau'_i \vee \sigma_j} - u_{\tau_{i+1}}| \le z_{i+1}]\!] \cap [\![|u_{\tau'_i \vee \sigma_{j+1}} - u_{\tau_{i+1}}| \le z_{i+1}]\!] \\ \subseteq [\![|u_{\sigma_{j+1}} - u_{\sigma_j}| \le 2z]\!] \end{aligned}$$

and

$$a_l \cap b \subseteq [\![|u_{\sigma_{j+1}} - u_{\sigma_j}| \le \tfrac{1}{2}]\!] \subseteq [\![\psi'(\sigma_j, \sigma_{j+1}) = 0]\!].$$

So

$$|S_K(\mathbf{1}, d\psi') - \psi'(\tau_i, \tau_{i+1})| \times \chi(b \cap a_l) = |\psi'(\sigma_{l-1}, \sigma_l) - \psi'(\tau_i, \tau_{i+1})| \times \chi(b \cap a_l).$$

Now

$$a_l \subseteq [\![|u_{\sigma_{l-1}} - y_i| \le \delta]\!] \cap [\![|u_{\sigma_0} - y_i| \le \delta]\!] \subseteq [\![|u_{\sigma_{l-1}} - u_{\tau_i}| \le 2\delta]\!]$$

and also

$$a_l \subseteq [\![\sigma_l = \sigma_l \vee \tau'_i]\!] \subseteq [\![|u_{\sigma_l} - u_{\tau_{i+1}}| \le 2z_{i+1}]\!]$$

so

$$\begin{aligned} b \cap a_l &\subseteq [\![|u_{\sigma_{l-1}} - u_{\tau_i}| + |u_{\sigma_l} - u_{\tau_{i+1}}| \le 4\delta]\!] \\ &\subseteq [\![|\psi'(\sigma_{l-1}, \sigma_l) - \psi'(\tau_i, \tau_{i+1})| \le 4\delta]\!] \end{aligned}$$

((b-ii) above). So

$$|\psi'(\sigma_{l-1}, \sigma_l) - \psi'(\tau_i, \tau_{i+1})| \times \chi(b \cap a_l) \le 4\delta\chi 1.$$

At the same time,

$$\begin{aligned} b \cap a_l \cap [\![|u_{\tau_i} - u_{\tau_{i+1}}| \le \tfrac{1}{4}]\!] &\subseteq [\![\psi'(\tau_i, \tau_{i+1}) = 0]\!] \cap [\![|u_{\sigma_{l-1}} - u_{\sigma_l}| \le \tfrac{1}{4} + 4\delta]\!] \\ &\subseteq [\![\psi'(\tau_i, \tau_{i+1}) = 0]\!] \cap [\![\psi'(\sigma_{l-1}, \sigma_l) = 0]\!], \end{aligned}$$

and in fact

$$\begin{aligned} |S_K(\mathbf{1}, d\psi') - \psi'(\tau_i, \tau_{i+1})| \times \chi(b \cap a_l) &= |\psi'(\sigma_{l-1}, \sigma_l) - \psi'(\tau_i, \tau_{i+1})| \times \chi(b \cap a_l) \\ &\le 4\delta\chi[\![|u_{\tau_{i+1}} - u_{\tau_i}| \ge \tfrac{1}{4}]\!]. \text{ } \mathbf{Q} \end{aligned}$$

(γ) Concerning a^*, we have

$$\begin{aligned} a^* &\subseteq \inf_{j \le n} [\![|u_{\sigma_j} - y_i| < \delta]\!] \\ &\subseteq \inf_{j \le k \le n} [\![|u_{\sigma_k} - u_{\sigma_j}| \le 1]\!] \\ &\subseteq [\![\psi'(\tau_i, \tau_{i+1}) = 0]\!] \cap \inf_{j<n} [\![\psi'(\sigma_j, \sigma_{j+1}) = 0]\!] \\ &\subseteq [\![S_K(\mathbf{1}, d\psi') = \psi'(\tau_i, \tau_{i+1})]\!]. \end{aligned}$$

(δ) Since $a^* \cup \sup_{j \le n} a_j = 1$, we see that

$$|S_K(\mathbf{1}, d\psi') - \psi'(\tau_i, \tau_{i+1})| \times \chi b \le 4\delta\chi[\![|u_{\tau_{i+1}} - u_{\tau_i}| \ge \tfrac{1}{4}]\!].$$

And this is true whenever $i < m$ and K is a finite sublattice of $\mathcal{S} \cap [\tau_i, \tau_{i-1}]$ containing τ_i and τ_{i+1}.

(v) Now suppose that $J \in \mathcal{I}(\mathcal{S})$ includes $I = \{\tau_0, \dots, \tau_m\}$.

(α) $S_{J \wedge \tau_0}(\mathbf{1}, d\psi') \times \chi b = 0$. **P** If $\sigma \le \tau \le \tau_0$ in $\mathcal{S}$, then σ and τ both belong to $\mathcal{S} \wedge \tau^* = D_0$, so $|u_\tau - u_\sigma| \le 2z_i \le 2z$ and

$$b \subseteq [\![|u_\tau - u_\sigma| \le 2\delta]\!] \subseteq [\![\psi'(\sigma, \tau) = 0]\!],$$

that is, $\psi'(\sigma,\tau)\times\chi b=0$. As σ and τ are arbitrary, $S_{J\wedge\tau_0}(\mathbf{1},d\psi')\times\chi b=0$. **Q**

Similarly, $|u_\tau-u_\sigma|\le 2z^*$ and

$$b\subseteq[\![|u_\tau-u_\sigma|\le\tfrac12]\!]\subseteq[\![\psi'(\sigma,\tau)=0]\!]$$

whenever $\tau^*\le\sigma\le\tau$ in $\mathcal{S}$, so $S_{J\vee\tau^*}(\mathbf{1},d\psi')\times\chi b=0$. We see also that if $\tau_m\le\sigma\le\tau\le\tau^*$ then

$$[\![\psi'(\sigma,\tau)\ne0]\!]\subseteq[\![\sigma<\tau]\!]\subseteq[\![\tau_m<\tau^*]\!]\subseteq d_m$$

(615M(c-i)), so $\psi'(\sigma,\tau)\times\chi b=0$, and we conclude that $S_{J\cap[\tau_m,\tau^*]}(\mathbf{1},d\psi')\times\chi b=0$. So in fact we have

$$\begin{aligned}S_J(\mathbf{1},d\psi')\times\chi b&=\big(S_{J\wedge\tau_0}(\mathbf{1},d\psi')+\sum_{i=0}^{m-1}S_{J\cap[\tau_i,\tau_{i+1}]}\mathbf{1},d\psi')\\&\qquad+S_{J\cap[\tau_m,\tau^*]}(\mathbf{1},d\psi')+S_{J\vee\tau^*}(\mathbf{1},d\psi')\big)\times\chi b\\&=\sum_{i=0}^{m-1}S_{J\cap[\tau_i,\tau_{i+1}]}\mathbf{1},d\psi')\times\chi b,\end{aligned}$$

$$\begin{aligned}|S_J(\mathbf{1},d\psi')-S_I(\mathbf{1},d\psi')|\times\chi b&\le\sum_{i=0}^{m-1}|S_{J\cap[\tau_i,\tau_{i+1}]}(\mathbf{1},d\psi')\psi'(\tau_i,\tau_{i+1}|\times\chi b\\&\le\sum_{i=0}^{m-1}4\delta\chi[\![|u_{\tau_{i+1}}-u_{\tau_i}|\ge\tfrac14]\!]\end{aligned}$$

by (iv-δ) above, and

$$\begin{aligned}\theta(|S_J(\mathbf{1},d\psi')-S_I(\mathbf{1},d\psi')|\times\chi b)&\le\mathbb{E}(\sum_{i=0}^{m-1}4\delta\chi[\![|u_{\tau_{i+1}}-u_{\tau_i}|\ge\tfrac14]\!])\\&=4\delta\sum_{i=0}^{m-1}\bar\mu[\![|u_{\tau_{i+1}}-u_{\tau_i}|\ge\tfrac14]\!]\le4\delta\gamma\le4\epsilon\end{aligned}$$

by (ii) above. Since $\bar\mu b\ge1-7\epsilon$, $\theta(S_J(\mathbf{1},d\psi')-S_I(\mathbf{1},d\psi'))\le11\epsilon$, and this is true for every finite sublattice of $\mathcal{S}$ including I. As ϵ is arbitrary, $\int_{\mathcal{S}}d\psi'$ is defined.

(vi) All this has been on the assumption that $\mathcal{S}$ was full. But for the general case, we can take the covered envelope $\hat{\mathcal{S}}$ of $\mathcal{S}$ and the fully adapted extension $\hat{\boldsymbol{u}}=\langle\hat u_\sigma\rangle_{\sigma\in\hat{\mathcal{S}}}$ of $\boldsymbol{u}$ to $\hat{\mathcal{S}}$ (612Q). Setting $\hat\psi'(\sigma,\tau)=\hat u_\tau-\hat u_\sigma-\mathrm{med}(-\chi1,\hat u_\tau-\hat u_\sigma,\chi1)$ when $\sigma\le\tau$ in $\hat{\mathcal{S}}$, (i)-(v) tell us that $\int_{\hat{\mathcal{S}}}d\hat\psi'$ is defined. And now we see from 613T that $\int_{\mathcal{S}}d\psi'$ is defined.

(vii) In the notation of 613Cc, $\psi=\Delta\boldsymbol{u}-\psi'$. Because $\boldsymbol{u}$ is moderately oscillatory, $u_\downarrow=\lim_{\sigma\downarrow\mathcal{S}}u_\sigma$ is defined (615Gb) so $\int_{\mathcal{S}}d\boldsymbol{u}$ is defined (613N) and $\int_{\mathcal{S}}d\psi$ is defined.

(d)(i) As observed in 613O(b-i), $\boldsymbol{w}=\langle w_\sigma\rangle_{\sigma\in\mathcal{S}}=\langle\int_{\mathcal{S}\wedge\sigma}d\psi\rangle_{\sigma\in\mathcal{S}}$ and $\boldsymbol{w}'=\langle w'_\sigma\rangle_{\sigma\in\mathcal{S}}=\langle\int_{\mathcal{S}\wedge\sigma}d\psi'\rangle_{\sigma\in\mathcal{S}}$ are well-defined, while

$$\boldsymbol{u}=u_\downarrow\mathbf{1}+\langle\int_{\mathcal{S}\wedge\sigma}d\boldsymbol{u}\rangle_{\sigma\in\mathcal{S}}=u_\downarrow\mathbf{1}+\boldsymbol{w}+\boldsymbol{w}'.$$

Set $A=\{S_I(\mathbf{1},d|\psi'|):I\in\mathcal{I}(\mathcal{S})\}$.

(ii) Let $\epsilon>0$. Then there is a process $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ of bounded variation such that $\theta(\sup|\boldsymbol{u}-\boldsymbol{v}|)\le\dfrac13\epsilon$. Set $b=[\![\sup|\boldsymbol{v}-\boldsymbol{u}|\le\frac13]\!]$; then $\bar\mu b\ge1-\epsilon$. If $\sigma\le\tau$ in $\mathcal{S}$, then

$$\begin{aligned}|\psi'(\sigma,\tau)|\times\chi b&\le|u_\tau-u_\sigma|\times\chi(b\cap[\![|u_\tau-u_\sigma|>1]\!])\\&\le(|v_\tau-v_\sigma|+\frac23\chi1)\times\chi[\![|v_\tau-v_\sigma|>\tfrac13]\!]\le3|v_\tau-v_\sigma|.\end{aligned}$$

So

$$S_I(\mathbf{1},d|\psi'|)\times\chi b\le 3S_I(\mathbf{1},|d\boldsymbol{v}|)\le 3\int_{\mathcal{S}}|d\boldsymbol{v}|$$

for every $I\in\mathcal{I}(\mathcal{S})$, and $\{x\times\chi b:x\in A\}$ is bounded above in L^0. As ϵ is arbitrary, A is bounded above in L^0 (613B(p-i)). Set $\bar z=\sup A$.

(iii) If $\tau_0\le\ldots\le\tau_n$ in $\mathcal{S}$, $\sum_{i=0}^{n-1}|w'_{\tau_{i+1}}-w'_{\tau_i}|\le\bar z$. **P** Take $\epsilon>0$. Then for each $i<n$ there is a $J_i\in\mathcal{I}(\mathcal{S}\cap[\tau_i,\tau_{i+1}]$, containing τ_i and τ_{i+1}, such that $\theta(z_i)\le\frac{\epsilon}{n}$ where

$$\begin{aligned}z_i&=|\int_{\mathcal{S}\cap[\tau_i,\tau_{i+1}]}d\psi'-S_{J_i}(\mathbf{1},d\psi')|\\&=|w'_{\tau_{i+1}}-w'_{\tau_i}-S_{J_i}(\mathbf{1},d\psi')|\ge|w'_{\tau_{i+1}}-w'_{\tau_i}|-|S_{J_i}(\mathbf{1},d\psi')|\end{aligned}$$

Now

$$|w'_{\tau_{i+1}}-w'_{\tau_i}|\le z_i+|S_{J_i}(\mathbf{1},d\psi')|\le z_i+S_{J_i}(\mathbf{1},|d\psi'|)$$

for each i. Setting $I=\bigcup_{i<n}J_i$,

$$\textstyle\sum_{i=0}^{n-1}|w'_{\tau_{i+1}}-w'_{\tau_i}|\le S_I(\mathbf{1},|d\psi'|)+\sum_{i=0}^{n-1}z_i\le\bar z+\sum_{i=0}^{n-1}z_i$$

while $\theta(\sum_{i=0}^{n-1}z_i)\le\epsilon$. As ϵ is arbitrary, $\sum_{i=0}^{n-1}|w'_{\tau_{i+1}}-w'_{\tau_i}|\le\bar z$. **Q**

As $\tau_0,\dots,\tau_n$ are arbitrary, $\boldsymbol{w}'$ is of bounded variation.

(iv) In particular, $\boldsymbol{w}'$ is moderately oscillatory, so $\boldsymbol{w}=\boldsymbol{u}-u_\downarrow\mathbf{1}-\boldsymbol{w}'$ is moderately oscillatory.

(e) Now the arguments of (a)-(d) apply equally to the fully adapted extension $\hat{\boldsymbol{u}}$ of $\boldsymbol{u}$ to $\hat{\mathcal{S}}$ and the associated strictly adapted interval function $\hat\psi$, which must be the adapted extension of ψ to $\hat{\mathcal{S}}^{2\uparrow}$. The corresponding indefinite integral $\hat{\boldsymbol{w}}=ii_{\hat\psi}(\mathbf{1})=\langle\int_{\hat{\mathcal{S}}\wedge\tau}d\hat\psi\rangle_{\tau\in\hat{\mathcal{S}}}$ extends $\boldsymbol{w}$ (613T) so must be the fully adapted extension of $\boldsymbol{u}$ to $\hat{\mathcal{S}}$. Now $\mathrm{Osclln}(\boldsymbol{w})=\mathrm{Osclln}(\hat{\boldsymbol{w}})$ (618La) and $\mathrm{Osclln}(\hat{\boldsymbol{w}})$ is at most $\mathrm{Osclln}(\hat\psi)$ (618Pc), which is at most $\chi 1$, as can be seen directly from the formulae in 618O.

(f) We know that $w_{\tau'}-w_\tau=\int_{\mathcal{S}\cap[\tau,\tau']}d\psi$ (613J(c-i)). Now $\psi(\sigma,\sigma')\in L^0(\mathfrak{D}_\tau)$ for every $(\sigma,\sigma')\in(\mathcal{S}\vee\tau)^{2\uparrow}$, so $S_I(\mathbf{1},d\psi)\in L^0(\mathfrak{D}_\tau)$ for every $I\in\mathcal{I}(\mathcal{S}\vee\tau)$. As $L^0(\mathfrak{D}_\tau)$ is closed for the topology of convergence in measure (613B(i-i)), $w_{\tau'}-w_\tau=\lim_{I\uparrow\mathcal{I}(\mathcal{S}\cap[\tau,\tau'])}S_I(\mathbf{1},d\psi)$ belongs to $L^0(\mathfrak{D}_\tau)$.

(g) Let $\epsilon>0$. Then there is a simple process $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ such that $\theta(\sup|\boldsymbol{u}-\boldsymbol{v}|)\le\frac12\epsilon$. Set $d=[\![\sup|\boldsymbol{u}-\boldsymbol{v}|>\frac12]\!]$; then $\bar\mu d\le\epsilon$. Let $(\tau_0,\dots,\tau_n)$ be a breakpoint string for $\boldsymbol{v}$, and consider the simple process $\boldsymbol{z}=\langle z_\sigma\rangle_{\sigma\in\mathcal{S}}$ define by saying that if $\sigma\in\mathcal{S}$ then

$$[\![\sigma<\tau_0]\!]\subseteq[\![z_\sigma=0]\!],\quad [\![\tau_n\le\sigma]\!]\subseteq[\![z_\sigma=w_{\tau_n}]\!],$$

$$[\![\tau_i\le\sigma]\!]\cap[\![\sigma<\tau_{i+1}]\!]\subseteq[\![z_\sigma=w_{\tau_i}]\!]\text{ for }i<n.$$

Now $[\![\boldsymbol{w}\ne\boldsymbol{z}]\!]\subseteq d$. **P** Take $\tau\in\mathcal{S}$ and write a for $[\![w_\tau\ne z_\tau]\!]$. Then

$$\begin{aligned}a\cap[\![\tau<\tau_0]\!]=[\![w_\tau\ne0]\!]\cap[\![\tau<\tau_0]\!]&\subseteq\sup\{[\![\psi(\sigma,\sigma')\ne0]\!]\cap[\![\tau<\tau_0]\!]:\sigma\le\sigma'\le\tau\}\\&\subseteq\sup\{[\![\psi(\sigma,\sigma')\ne0]\!]\cap[\![\sigma<\tau_0]\!]\cap[\![\sigma'<\tau_0]\!]:\sigma\le\sigma'\}\\&\subseteq\sup\{[\![|u_{\sigma'}-u_\sigma|>1]\!]\cap[\![v_\sigma=v_{\sigma'}]\!]:\sigma,\,\sigma'\in\mathcal{S}\}\\&\subseteq\sup\{[\![|u_\sigma-v_\sigma|>\tfrac12]\!]:\sigma\in\mathcal{S}\}=d;\end{aligned}$$

$$\begin{aligned}a\cap[\![\tau_n\le\tau]\!]=[\![w_{\tau\vee\tau_n}\ne w_{\tau_n}]\!]&\subseteq\sup\{[\![\psi(\sigma,\sigma')\ne0]\!]:\tau_n\le\sigma\le\sigma'\le\tau\vee\tau_n\}\\&\subseteq\sup\{[\![|u_{\sigma'}-u_\sigma|>1]\!]\cap[\![\tau_n\le\sigma]\!]\cap[\![\tau_n\le\sigma']\!]:\sigma\le\sigma'\}\\&\subseteq\sup\{[\![|u_\sigma-v_\sigma|>\tfrac12]\!]:\sigma\in\mathcal{S}\}=d;\end{aligned}$$

and for $i<n$

$$\begin{aligned} a \cap [\![\tau_i \le \tau]\!] \cap [\![\tau < \tau_{i+1}]\!] &= [\![w_{\tau\vee\tau_i} \ne w_{\tau_i}]\!] \cap [\![\tau < \tau_{i+1}]\!] \\ &\subseteq \sup\{[\![\psi(\sigma,\sigma') \ne 0]\!] \cap [\![\sigma' < \tau_{i+1}]\!] : \tau_i \le \sigma \le \sigma'\} \\ &\subseteq \sup\{[\![|u_{\sigma'} - u_\sigma| > 1]\!] \cap [\![\sigma' < \tau_{i+1}]\!] : \tau_i \le \sigma \le \sigma'\} \\ &\subseteq \sup\{[\![|u_\sigma - v_\sigma| > \tfrac{1}{2}]\!] : \sigma \in \mathcal{S}\} = d. \end{aligned}$$

Thus in fact $a \subseteq d$; as τ is arbitrary, $[\![\boldsymbol{w} \ne \boldsymbol{z}]\!] \subseteq d$. **Q**

Now $\theta(\boldsymbol{w} - \boldsymbol{z}) \le \bar\mu d \le \epsilon$. As ϵ is arbitrary, $\boldsymbol{w}$ is near-simple.

***633N Lemma** Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ such that $\inf A \in \mathcal{S}$ whenever $A \subseteq \mathcal{S}$ is non-empty and has a lower bound in $\mathcal{S}$, and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a near-simple process. As in 633M, set

$$\psi(\sigma,\tau) = \mathrm{med}(-\chi 1, u_\tau - u_\sigma, \chi 1)$$

when $\sigma \le \tau$ in $\mathcal{S}$. Let $\mathcal{S}'$ be a cofinal sublattice of $\mathcal{S}$ which separates $\mathcal{S}$. Then $\int_{\mathcal{S}'} d\psi = \int_{\mathcal{S}} d\psi$.

proof We can use the arguments of 633I-633K.

(a) To begin with, suppose that $\mathcal{S}'$ is finitely full. The point is that if $\sigma \le \tau \le \tau'$ then $|\psi(\sigma,\tau') - \psi(\sigma,\tau)| \le |u_{\tau'} - u_\tau|$ (633M(b-ii)). Let $\mathcal{S}'$ be the finitely covered envelope of $\mathcal{S}_0$. If $\tau \in \mathcal{S}$, $A_\tau = \mathcal{S}' \cap (\mathcal{T} \vee \tau)$ is downwards-directed with infimum τ (633Eb) and $\lim_{\sigma\downarrow A_\tau} u_\sigma = u_\tau$ (632E); as $\mathcal{T} \vee \tau$ is order-convex, therefore finitely full (611Pc), A_τ is finitely full (611Pb), and

$$\lim_{\tau'\downarrow A_\tau} \sup_{\substack{\sigma\in\mathcal{S}\wedge\tau \\ \rho\in\mathcal{S}'\cap[\tau,\tau']}} |\psi(\sigma,\rho) - \psi(\sigma,\tau)| = \lim_{\tau'\downarrow A_\tau} \sup_{\rho\in A_\tau\wedge\tau'} |u_\rho - u_\tau| = 0$$

by 615Db.

Now we know from 633Mc that $z = \int_{\mathcal{S}} d\psi$ is defined. Let $\epsilon > 0$. Let $J \in \mathcal{I}(\mathcal{S})$ be such that $\theta(z - S_K(\mathbf{1}, d\psi)) \le \epsilon$ whenever $J \subseteq K \in \mathcal{I}(\mathcal{S})$. For each $\tau \in J$, let $\sigma_\tau \in A_\tau$ be such that $\theta(\bar u_\tau) \le \frac{\epsilon}{1+\#(J)}$ where

$$\bar u_\tau = \sup_{\sigma\in\mathcal{S}\wedge\tau, \rho\in\mathcal{S}'\cap[\tau,\tau']} |\psi(\sigma,\rho) - \psi(\sigma,\tau)|.$$

If $I \in \mathcal{I}(\mathcal{S}')$ contains σ_τ for every $\tau \in J$, consider $K = I \sqcup J$. By the choice of J, $\theta(S_K(\mathbf{1}, d\psi) - z) \le \epsilon$. By 633I, $|S_K(\mathbf{1}, d\psi) - S_I(\mathbf{1}, d\psi)| \le 2\sum_{\tau\in J} \bar u_\tau$ and $\theta(S_K(\mathbf{1}, d\psi) - S_I(\mathbf{1}, d\psi)) \le 2\epsilon$. But now

$$\begin{aligned} \theta(z - S_I(\mathbf{1}, d\psi)) &\le \theta(z - S_K(\mathbf{1}, d\psi)) + \theta(S_K(\mathbf{1}, d\psi) - S_I(\mathbf{1}, d\psi)) \\ &\le \epsilon + 2\epsilon = 3\epsilon. \end{aligned}$$

As ϵ is arbitrary, $\int_{\mathcal{S}'} d\psi$ is defined and equal to z.

(b) In general, writing $\hat{\mathcal{S}}'_f$ for the finitely-covered envelope of $\mathcal{S}$, $\hat{\mathcal{S}}'_f$ is a cofinal finitely full sublattice of $\mathcal{S}$ which separates $\mathcal{S}$, so $\int_{\hat{\mathcal{S}}'_f} d\psi = z$. But now 613T tells us that $\int_{\mathcal{S}'} d\psi = z$.

633O Proposition Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ such that $\inf A \in \mathcal{S}$ for every non-empty $A \subseteq \mathcal{S}$ with a lower bound in $\mathcal{S}$, $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ an order-bounded locally near-simple process, and $\mathcal{S}'$ a sublattice of $\mathcal{S}$ which is cofinal and coinitial with $\mathcal{S}$ and $\boldsymbol{u}$-separates $\mathcal{S}$. Write $\boldsymbol{u}'$ for $\boldsymbol{u}\restriction\mathcal{S}'$.

(a) $\mathrm{Osclln}(\boldsymbol{u}) = \mathrm{Osclln}(\boldsymbol{u}')$.

(b) $\boldsymbol{u}$ is jump-free iff $\boldsymbol{u}'$ is jump-free.

proof This is trivial if $\mathcal{S}$ is empty; suppose otherwise. Of course $\mathcal{S}'$ must also be non-empty.

(a)(i) If $\hat{\mathcal{S}}'_f$ is the finitely-covered envelope of $\mathcal{S}'$, then $\hat{\mathcal{S}}'_f \subseteq \mathcal{S}$, and of course $\hat{\mathcal{S}}'_f$ is cofinal with $\mathcal{S}$ and $\boldsymbol{u}$-separates $\mathcal{S}$. Since $\mathrm{Osclln}(\boldsymbol{u}') = \mathrm{Osclln}(\boldsymbol{u}\restriction\hat{\mathcal{S}}'_f)$ (618L), it is enough to consider the case in which $\mathcal{S}'$ itself is finitely full.

For $\tau \in \mathcal{S}$, set $A_\tau = \{\sigma : \tau \le \sigma \in \mathcal{S}'\}$ and $\tau^\# = \inf A_\tau$, so that $\tau^\# \in \mathcal{S}$, $u_{\tau^\#} = u_\tau$ (633Ec) and (because $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous and $\boldsymbol{u}$ is locally near-simple) $u_\tau = \lim_{\sigma\downarrow A_\tau} u_\sigma$. In fact, $\boldsymbol{u}$ is constant on $\mathcal{S}\cap[\tau,\tau^\#]$. **P** If $\tau' \in \mathcal{S}\cap[\tau,\tau^\#]$ then $A_{\tau'} = A_\tau$ so $u_{\tau'} = \lim_{\sigma\downarrow A_\tau} u_\sigma = u_\tau$. **Q**

(ii) If $I \in \mathcal{I}(\mathcal{S}')$ is non-empty then $\mathrm{Osclln}^*_I(\boldsymbol{u}) \le \mathrm{Osclln}^*_I(\boldsymbol{u}')$. **P** Let $\sigma_0 \le \ldots \le \sigma_n$ linearly generate the I-cells, and set $\sigma_{-1} = \min\mathcal{T}$, $\sigma_{n+1} = \max\mathcal{T}$. Write w for $\mathrm{Osclln}^*_I(\boldsymbol{u}')$. **?** Suppose that $-1 \le i \le n$, τ, $\tau' \in \mathcal{S}$ and $\sigma_i \le \tau \le \tau' \le \sigma_{i+1}$, $|u_\tau - u_{\tau'}| \not\le w$. Writing $A_{\tau'}\wedge\sigma_{i+1}$ for $\{\sigma\wedge\sigma_{i+1} : \sigma \in A_{\tau'}\}$, $A_{\tau'}\wedge\sigma_{i+1}$ is a coinitial subset of $A_{\tau'}$, so $u_{\tau'} = \lim_{\sigma\downarrow A_{\tau'}\wedge\sigma_{i+1}} u_\sigma$ and there is a $\sigma' \in \mathcal{S}'\cap[\tau',\sigma_{i+1}]$ such that $|u_{\sigma'} - u_\tau| \not\le w$. Similarly, there is a $\sigma \in \mathcal{S}'\cap[\tau,\sigma']$ such that $|u_{\sigma'} - u_\sigma| \not\le w$. But now $\sigma_i \le \sigma \le \sigma' \le \sigma_{i+1}$, so 618B(b-iv) and 618Ca tell us that

$$|u_{\sigma'} - u_\sigma| \le \mathrm{Osclln}^*_I(\boldsymbol{u}') = w. \ \mathbf{X}$$

Thus $|u_\tau - u_{\tau'}| \le w$ whenever $-1 \le i \le n$, τ, $\tau' \in \mathcal{S}$ and $\sigma_i \le \tau \le \tau' \le \sigma_{i+1}$. By 618Ca in the other direction, $\mathrm{Osclln}^*_I(\boldsymbol{u}) \le w$. **Q**

Accordingly

$$\begin{aligned}\mathrm{Osclln}(\boldsymbol{u}) &= \inf_{I\in\mathcal{I}(\mathcal{S})} \mathrm{Osclln}^*_I(\boldsymbol{u}) \le \inf_{I\in\mathcal{I}(\mathcal{S}')\setminus\{\emptyset\}} \mathrm{Osclln}^*_I(\boldsymbol{u})\\ &\le \inf_{I\in\mathcal{I}(\mathcal{S}')\setminus\{\emptyset\}} \mathrm{Osclln}^*_I(\boldsymbol{u}') = \lim_{I\uparrow\mathcal{I}(\mathcal{S}')} \mathrm{Osclln}^*_I(\boldsymbol{u}') = \mathrm{Osclln}(\boldsymbol{u}').\end{aligned}$$

(iii)(α) Take $\epsilon > 0$, a non-empty $I \in \mathcal{I}(\mathcal{S})$ and $(\tau_0,\ldots,\tau_n)$ linearly generating the I-cells. Set $\tau_{-1} = \min\mathcal{T}$ and $\tau_{n+1} = \max\mathcal{T}$.

(β) We know that $\boldsymbol{u}$ is locally moderately oscillatory (631Ca). So we can choose $\sigma_n \ge \ldots \ge \sigma_0$, in that order, such that $\sigma_i \in A_{\tau_i}$ and $\theta(\sup\{|u_{\sigma'} - u_{\tau_i}| : \sigma' \in A_{\tau_i}, \sigma' \le \sigma_i\}) \le \frac{\epsilon}{n+1}$ for each i (apply 615Gb to $\boldsymbol{u}{\restriction}\mathcal{S}\wedge(\sigma^*\wedge\inf_{i<j\le n}\sigma_j)$ where $\sigma^* \in A_{\tau_n}$). Set $w_i = \sup_{\sigma'\in A_{\tau_i},\sigma'\le\sigma_i} |u_{\sigma'} - u_{\tau_i}|$ for $0 \le i \le n$ and $w = \sup_{0\le i\le n} w_i$, so that $\theta(w) \le \epsilon$. Set $K = \{\sigma_0,\ldots,\sigma_n\} \in \mathcal{I}(\mathcal{S}')$.

Set $\sigma_{-1} = \min\mathcal{T}$ and $\sigma_{n+1} = \max\mathcal{T}$. If $0 \le i \le n+1$, $\sigma \in \mathcal{S}'$ and $\sigma \le \sigma_i$ then $|u_\sigma - u_{\sigma\wedge\tau_i}| \le w$. **P** If $i = n+1$ this is trivial. If $i \le n$, then $[\![\tau_i \le \sigma]\!] \in \mathfrak{A}_\sigma \subseteq \mathfrak{A}_{\sigma_i}$ so we can define $\sigma' \in \mathcal{T}$ by saying that

$$[\![\tau_i \le \sigma]\!] \subseteq [\![\sigma' = \sigma]\!], \quad [\![\sigma < \tau_i]\!] \subseteq [\![\sigma' = \sigma_i]\!]$$

(611I). Because $\mathcal{S}'$ is finitely full, $\sigma' \in \mathcal{S}'$, while of course $\tau_i \le \sigma'$, so $\sigma' \in A_{\tau_i}$. Now

$$|u_\sigma - u_{\sigma\wedge\tau_i}| = |u_{\sigma\vee\tau_i} - u_{\tau_i}|$$

(612D(f-i))

$$= |u_\sigma - u_{\tau_i}| \times \chi[\![\tau_i \le \sigma]\!] \le |u_{\sigma'} - u_{\tau_i}| \le w_i \le w. \ \mathbf{Q}$$

Now suppose that $-1 \le i \le n$ and σ, $\sigma' \in \mathcal{S}'$ are such that $\sigma_i \le \sigma \le \sigma' \le \sigma_{i+1}$. Then $\tau_i \le \sigma\wedge\tau_{i+1} \le \sigma'\wedge\tau_{i+1} \le \tau_{i+1}$. So

$$\begin{aligned}|u_{\sigma'} - u_\sigma| &\le |u_{\sigma'} - u_{\sigma'\wedge\tau_{i+1}}| + |u_{\sigma'\wedge\tau_{i+1}} - u_{\sigma\wedge\tau_{i+1}}| + |u_{\sigma\wedge\tau_{i+1}} - u_\sigma|\\ &\le \mathrm{Osclln}^*_I(\boldsymbol{u}) + 2w\end{aligned}$$

(using 618Ca again). Taking the supremum over i, σ and σ' and applying 618Ca in the other direction,

$$\mathrm{Osclln}(\boldsymbol{u}') \le \mathrm{Osclln}^*_K(\boldsymbol{u}') \le \mathrm{Osclln}^*_I(\boldsymbol{u}) + 2w$$

and

$$\theta((\mathrm{Osclln}(\boldsymbol{u}') - \mathrm{Osclln}^*_I(\boldsymbol{u}))^+) \le \theta(2w) \le 2\epsilon.$$

(γ) As ϵ is arbitrary, $\theta((\mathrm{Osclln}(\boldsymbol{u}') - \mathrm{Osclln}^*_I(\boldsymbol{u}))^+) = 0$ and $\mathrm{Osclln}(\boldsymbol{u}') \le \mathrm{Osclln}^*_I(\boldsymbol{u})$. As I is arbitrary, $\mathrm{Osclln}(\boldsymbol{u}') \le \mathrm{Osclln}(\boldsymbol{u})$. With (ii) above, this shows that $\mathrm{Osclln}(\boldsymbol{u}) = \mathrm{Osclln}(\boldsymbol{u}')$.

(b) In particular, $\mathrm{Osclln}(\boldsymbol{u}) = 0$ iff $\mathrm{Osclln}(\boldsymbol{u}') = 0$, that is, $\boldsymbol{u}$ is jump-free iff $\boldsymbol{u}'$ is jump-free.

633P 633K is an attempt to reproduce the success of 613T for sublattices which are separating rather than covering. We can try to follow other results about covering envelopes in the same way.

Theorem Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ such that $\inf A\in\mathcal{S}$ for every non-empty $A\subseteq\mathcal{S}$ with a lower bound in $\mathcal{S}$, $\mathcal{S}'$ a sublattice of $\mathcal{S}$ which is cofinal and coinitial with $\mathcal{S}$ and separates $\mathcal{S}$, and $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a locally near-simple fully adapted process. Write $\boldsymbol{u}'$ for $\boldsymbol{u}\restriction\mathcal{S}'$.

(a) If $\boldsymbol{u}'$ is simple then $\boldsymbol{u}$ is simple.

(b) If $\boldsymbol{u}'$ is near-simple then $\boldsymbol{u}$ is near-simple.

(c) $\boldsymbol{u}$ is order-bounded iff $\boldsymbol{u}'$ is order-bounded.

(d) $\boldsymbol{u}$ is (locally) of bounded variation iff $\boldsymbol{u}'$ is (locally) of bounded variation.

(e) $\boldsymbol{u}$ is an integrator iff $\boldsymbol{u}'$ is an integrator.

(f) $\boldsymbol{u}$ is a martingale iff $\boldsymbol{u}'$ is a martingale.

(g) If $\boldsymbol{u}'$ is a local martingale then $\boldsymbol{u}$ is a local martingale.

(h) Suppose that $\boldsymbol{u}'$ is a local integrator. Then $\boldsymbol{u}$ is a local integrator and the quadratic variation of $\boldsymbol{u}'$ is $\boldsymbol{u}^*\restriction\mathcal{S}'$, where $\boldsymbol{u}^*$ is the quadratic variation of $\boldsymbol{u}$.

proof Throughout the proof, write $\hat{\mathcal{S}}'_f$ for the finitely-covered envelope of $\mathcal{S}'$, so that $\hat{\mathcal{S}}'_f\subseteq\mathcal{S}$. For $\tau\in\mathcal{S}$, set $A_\tau=\{\sigma:\tau\le\sigma\in\hat{\mathcal{S}}'_f\}$, so that $\inf A_\tau=\tau$ (633Eb) and (because $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous and $\boldsymbol{u}$ is locally near-simple) $u_\tau=\lim_{\sigma\downarrow A_\tau}u_\sigma$ (632F once more). Everything is trivial if $\mathcal{S}'$ is empty, so I shall suppose otherwise.

(a) By 612Qf, $\boldsymbol{u}\restriction\hat{\mathcal{S}}'_f$ is simple. We have a starting value u_* and breakpoint string $(\tau_0,\dots,\tau_n)$ for $\boldsymbol{u}\restriction\hat{\mathcal{S}}'_f$. Because $\mathcal{S}'$ is coinitial with $\mathcal{S}$, $\bigcap_{\sigma\in\hat{\mathcal{S}}'_f}\mathfrak{A}_\sigma=\bigcap_{\sigma\in\mathcal{S}}\mathfrak{A}_\sigma$ and $u^*\in L^0(\bigcap_{\sigma\in\mathcal{S}}\mathfrak{A}_\sigma)$. By 612Ka, there is a simple fully adapted process $\boldsymbol{v}=\langle v_\tau\rangle_{\tau\in\mathcal{S}}$ defined by saying that

$$[\![v_\tau=u_*]\!]\supseteq[\![\tau<\tau_0]\!],\quad [\![v_\tau=u_{\tau_n}]\!]\supseteq[\![\tau_n\le\tau]\!],$$

$$[\![v_\tau=u_{\tau_i}]\!]\supseteq[\![\tau_i\le\tau]\!]\cap[\![\tau<\tau_{i+1}]\!]\text{ for }i<n$$

for every $\tau\in\mathcal{S}$. Evidently $\boldsymbol{v}$ extends $\boldsymbol{u}\restriction\hat{\mathcal{S}}'_f$. Because $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous,

$$v_\tau=\lim_{\sigma\downarrow A_\tau}v_\sigma=\lim_{\sigma\downarrow A_\tau}u_\sigma=u_\tau$$

for every $\tau\in\mathcal{S}$ (632F yet again), and $\boldsymbol{u}=\boldsymbol{v}$ is simple.

(b) By 631Ga, $\boldsymbol{u}\restriction\hat{\mathcal{S}}'_f$ is near-simple, because the covered envelopes of $\mathcal{S}'$ and $\hat{\mathcal{S}}'_f$ are the same. Let $\epsilon>0$. Then there is a simple process $\boldsymbol{u}'=\langle u'_\sigma\rangle_{\sigma\in\hat{\mathcal{S}}'_f}$ such that $\theta(\bar u)\le\epsilon$, where $\bar u=\sup_{\sigma\in\hat{\mathcal{S}}'_f}|u_\sigma-u'_\sigma|$. As in (a) just above, we have a simple process $\boldsymbol{v}=\langle v_\tau\rangle_{\tau\in\mathcal{S}}$ extending $\boldsymbol{u}'$. Now if $\tau\in\mathcal{S}$,

$$|u_\tau-v_\tau|=\lim_{\sigma\downarrow A_\tau}|u_\sigma-v_\sigma|=\lim_{\sigma\downarrow A_\tau}|u_\sigma-u'_\sigma|\le\bar u$$

because $A_\tau\subseteq\hat{\mathcal{S}}'_f$. Thus

$$\theta(\sup_{\tau\in\mathcal{S}}|u_\tau-v_\tau|)=\theta(\bar u)\le\epsilon.$$

As ϵ is arbitrary, $\boldsymbol{u}$ is near-simple.

(c) If $\boldsymbol{u}$ is order-bounded then $\boldsymbol{u}'$ is certainly order-bounded, by 614Fa. If $\boldsymbol{u}'$ is order-bounded, so is its fully adapted extension $\hat{\boldsymbol{u}}=\langle\hat u'_\sigma\rangle_{\sigma\in\hat{\mathcal{S}}'}$ to the covered envelope of $\mathcal{S}'$ (614G(b-i)). Writing $\bar u$ for $\sup|\hat{\boldsymbol{u}}'|$,

$$|u_\tau|=\lim_{\sigma\downarrow A_\tau}|u_\sigma|\le\bar u$$

for every $\tau\in\mathcal{S}$, so $\boldsymbol{u}$ is order-bounded.

(d)(i) If $\boldsymbol{u}$ is (locally) of bounded variation then $\boldsymbol{u}'$ is (locally) of bounded variation by 614Lb and 614Q(b-i).

(ii)($\boldsymbol{\alpha}$) If $\boldsymbol{u}'$ is of bounded variation, so is $\boldsymbol{u}\restriction\hat{\mathcal{S}}'_f$, by 614Lb; set $z=\int_{\hat{\mathcal{S}}'_f}|d\boldsymbol{u}|$. Suppose that $\tau_0\le\dots\le\tau_n$ in $\mathcal{S}$ and $\epsilon>0$. Then there are $\sigma_i\in A_{\tau_i}$, for $i\le n$, such that $\theta(u_\sigma-u_{\tau_i})\le\epsilon$ whenever $\sigma\in A_{\tau_i}$ and $\sigma\le\sigma_i$. Set $\sigma'_i=\inf_{i\le j\le n}\sigma_j$ for $i\le n$; then $\sigma'_i\in A_{\tau_i}$ and $\sigma'_i\le\sigma_i$ for each i, and $\sigma'_0\le\dots\le\sigma'_n$. Now

$$\theta((\sum_{i=0}^{n-1}|u_{\tau_{i+1}}-u_{\tau_i}|-z)^+)\le\theta((\sum_{i=0}^{n-1}|u_{\sigma'_{i+1}}-u_{\sigma'_i}|-z)^+)+2\sum_{i=0}^{n}\theta(u_{\sigma'_i}-u_{\tau_i})$$
$$\le 2(n+1)\epsilon.$$

As ϵ is arbitrary, $\sum_{i=0}^{n-1}|u_{\tau_{i+1}}-u_{\tau_i}|\le z$. As $\tau_0,\dots,\tau_n$ are arbitrary, $\boldsymbol{u}$ is of bounded variation.

(β) If $\boldsymbol{u}'$ is locally of bounded variation, and $\tau\in\mathcal{S}$, take $\sigma\in\mathcal{S}'$ such that $\tau\le\sigma$, and consider $\mathcal{S}\wedge\sigma$ and $\mathcal{S}'\wedge\sigma$. Evidently $\mathcal{S}\wedge\sigma$ is a finitely full sublattice of $\mathcal{T}$ such that $\inf A\in\mathcal{S}\wedge\sigma$ whenever $A\subseteq\mathcal{S}\wedge\sigma$ is non-empty and has a lower bound in $\mathcal{S}\wedge\sigma$, and $\mathcal{S}'\wedge\sigma$ separates $\mathcal{S}\wedge\sigma$ by 633C(b-iii), while $(\boldsymbol{u}\restriction\mathcal{S}\wedge\sigma)\restriction(\mathcal{S}'\wedge\sigma)=\boldsymbol{u}'\restriction\mathcal{S}'\wedge\sigma$ is of bounded variation. So (α) tells us that $\boldsymbol{u}\restriction\mathcal{S}\wedge\sigma$ is of bounded variation. It follows at once that $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ is of bounded variation; as τ is arbitrary, $\boldsymbol{u}$ is locally of bounded variation.

(e)(i) By 616P(b-ii), $\boldsymbol{u}'$ is surely an integrator if $\boldsymbol{u}$ is. So let us suppose that $\boldsymbol{u}'$ is an integrator and seek to show that $\boldsymbol{u}$ is.

(ii) By 616Ia, $\boldsymbol{u}\restriction\hat{\mathcal{S}}'_f$ is an integrator. Now $Q_{\mathcal{S}}(d\boldsymbol{u})\subseteq\overline{Q_{\hat{\mathcal{S}}'_f}(d\boldsymbol{u})}$. **P** Take $w\in Q_{\mathcal{S}}(d\boldsymbol{u})$ and $\epsilon>0$. There are $\tau_0\le\dots\le\tau_n$ in $\mathcal{S}$ and $v_i\in L^\infty(\mathfrak{A}_{\tau_i})$, for $i<n$, such that $\|v_i\|_\infty\le 1$ for every i and $w=\sum_{i=0}^{n-1}v_i\times(u_{\tau_{i+1}}-u_{\tau_i})$. For $i\le n$ there is a $\sigma_i\in A_{\tau_i}$ such that $\theta(u_\sigma-u_{\tau_i})\le\frac{\epsilon}{n+1}$ whenever $\sigma\in A_{\tau_i}$ and $\sigma\le\sigma_i$. Set $\sigma'_i=\inf_{i\le j\le n}\sigma_j$ for $i\le n$, so that $\sigma'_0\le\dots\le\sigma'_n$, while $\tau_i\le\sigma'_i\in\hat{\mathcal{S}}'_f$ and $\theta(u_{\sigma'_i}-u_{\tau_i})\le\frac{\epsilon}{n+1}$ for each i.

Because $\tau_i\le\sigma'_i$, $\mathfrak{A}_{\tau_i}\subseteq\mathfrak{A}_{\sigma'_i}$ and $v_i\in L^\infty(\mathfrak{A}_{\sigma'_i})$ for each i. By 616C(ii), $w'=\sum_{i=0}^{n}v_i\times(u_{\sigma'_{i+1}}-u_{\sigma'_i})$ belongs to $Q_{\hat{\mathcal{S}}'_f}(d\boldsymbol{u})$. But

$$\theta(w-w')=\theta(|w-w'|)\le\theta(\sum_{i=0}^{n-1}|v_i|\times(|u_{\tau_{i+1}}-u_{\sigma'_{i+1}}|+|u_{\tau_i}-u_{\sigma'_i}|))$$
$$\le\theta(2\sum_{i=0}^{n}|u_{\tau_i}-u_{\sigma'_i}|)\le 2\sum_{i=0}^{n}\theta(|u_{\tau_i}-u_{\sigma'_i}|)\le 2\epsilon.$$

As ϵ and w are arbitrary, $Q_{\mathcal{S}}(d\boldsymbol{u})\subseteq\overline{Q_{\hat{\mathcal{S}}'_f}(d\boldsymbol{u})}$. **Q**

As the closure of a topologically bounded set is topologically bounded, $Q_{\mathcal{S}}(d\boldsymbol{u})$ is topologically bounded.

(f)(i) If $\boldsymbol{u}$ is a martingale then of course $\boldsymbol{u}'$ is a martingale (622Db).

(ii) If $\boldsymbol{u}'$ is a martingale then there is a martingale $\boldsymbol{v}=\langle v_\tau\rangle_{\tau\in\mathcal{S}_1}$ extending $\boldsymbol{u}'$, where $\mathcal{S}_1$ is the ideal of $\mathcal{T}$ generated by $\mathcal{S}'$ (622Oa). Since $\mathcal{S}'$ is cofinal with $\mathcal{S}$, $\mathcal{S}\subseteq\mathcal{S}_1$ and $\boldsymbol{v}\restriction\mathcal{S}$ is a martingale, while $\boldsymbol{v}$ agrees with $\boldsymbol{u}$ on $\mathcal{S}'$ and therefore on $\hat{\mathcal{S}}'_f$.

If $\tau\in\mathcal{S}$, then $\mathcal{S}\vee\tau$ is a finitely full sublattice of $\mathcal{T}$ containing $\inf A$ whenever $A\subseteq\mathcal{S}\vee\tau$ is non-empty. so $\boldsymbol{v}\restriction\mathcal{S}\vee\tau$ is locally near-simple (632Ia). By 632F, as always,

$$v_\tau=\lim_{\sigma\downarrow A_\tau}v_\sigma=\lim_{\sigma\downarrow A_\tau}u_\sigma=u_\tau.$$

So $\boldsymbol{u}=\boldsymbol{v}\restriction\mathcal{S}$ is a martingale.

(g) If $\boldsymbol{u}'$ is a local martingale, there is a covering ideal $\mathcal{S}'_1$ of $\mathcal{S}'$ such that $\boldsymbol{u}'\restriction\mathcal{S}'_1$ is a martingale. Let $\mathcal{S}_1$ be the ideal of $\mathcal{S}$ generated by $\mathcal{S}'_1$. Then $\mathcal{S}_1$ covers $\mathcal{S}'$ which is cofinal with $\mathcal{S}$, so $\mathcal{S}_1$ covers $\mathcal{S}$. By 633C(b-ii), $\mathcal{S}'_1$ separates $\mathcal{S}_1$. Also $\mathcal{S}_1$ is a finitely full sublattice of $\mathcal{T}$ such that $\inf A\in\mathcal{S}_1$ for every non-empty $A\subseteq\mathcal{S}_1$ with a lower bound in $\mathcal{S}_1$, while $\boldsymbol{u}\restriction\mathcal{S}_1$ is locally near-simple (631Gb, since $\mathcal{S}_1$ and $\mathcal{S}$ have the same covered envelope). Moreover, $\mathcal{S}'_1$ is cofinal and coinitial with $\mathcal{S}_1$. Since $\boldsymbol{u}\restriction\mathcal{S}'_1=\boldsymbol{u}'\restriction\mathcal{S}'_1$ is a martingale, (f) tells us that $\boldsymbol{u}\restriction\mathcal{S}_1$ is a martingale, so that $\boldsymbol{u}$ is a local martingale.

(h) We have to check that (e) still works in a 'local' form. The point is that if $\tau\in\mathcal{S}'$ then $\mathcal{S}'\wedge\tau$ separates $\mathcal{S}\wedge\tau$ (633C(b-iii)). So if $\boldsymbol{u}'$ is a local integrator and $\tau\in\mathcal{S}$, there is a $\tau'\in\mathcal{S}'$ such that $\tau\le\tau'$, and now $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau'$ is an integrator because $\boldsymbol{u}'\restriction\mathcal{S}'\wedge\tau'$ is an integrator; by 616P(b-ii), $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau=(\boldsymbol{u}\restriction\mathcal{S}\wedge\tau')\restriction\mathcal{S}\wedge\tau$ is an integrator. As τ is arbitrary, $\boldsymbol{u}$ is a local integrator.

If $\tau \in \mathcal{S}'$, then $\mathcal{S}' \wedge \tau$ separates $\mathcal{S} \wedge \tau$ and therefore $\boldsymbol{u}$-separates $\mathcal{S} \wedge \tau$; and of course $\mathcal{S}' \wedge \tau$ is cofinal and coinitial with $\mathcal{S} \wedge \tau$. Setting $u_{\downarrow} = \lim_{\sigma \downarrow \mathcal{S}} u_\sigma$, this is also $\lim_{\sigma \downarrow \mathcal{S}'} u_\sigma$ because $\mathcal{S}'$ is coinitial with $\mathcal{S}$. So

$$u^*_\tau = u_\tau^2 - u_\downarrow^2 - 2\int_{\mathcal{S} \wedge \tau} \boldsymbol{u}\, d\boldsymbol{u} = u_\tau^2 - u_\downarrow^2 - 2\int_{\mathcal{S}' \wedge \tau} \boldsymbol{u}\, d\boldsymbol{u}$$

(633Ka). As τ is arbitrary, $\boldsymbol{u}^* \restriction \mathcal{S}'$ is the quadratic variation of $\boldsymbol{u}'$.

633S Proposition Suppose that $\langle \mathfrak{A}_t \rangle_{t \in T}$ is right-continuous. Let $\mathcal{S} \subseteq \mathcal{T}$ be a finitely full sublattice such that $\inf A \in \mathcal{S}$ whenever $A \subseteq \mathcal{S}$ is non-empty and has a lower bound in $\mathcal{S}$, and A a subset of $\mathcal{S}$.

(a) If $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ is fully adapted and whenever $\sigma \le \tau$ in $\mathcal{S}$ and $v_\sigma \ne v_\tau$ there is a $\rho \in A$ such that $[\![\sigma \le \rho]\!] \cap [\![\rho < \tau]\!] \ne 0$, then A $\boldsymbol{v}$-separates $\mathcal{S}$.

(b) If whenever $\sigma \le \tau$ in $\mathcal{S}$ and $[\![\sigma < \tau]\!] \ne 0$ there is a $\rho \in A$ such that $[\![\sigma \le \rho]\!] \cap [\![\rho < \tau]\!] \ne 0$, then A separates $\mathcal{S}$.

proof (a)(i) If A is empty then $\boldsymbol{v}$ is constant and the result is trivial, so suppose otherwise. For $\rho \in A$ set $b_\rho = [\![\sigma \le \rho]\!] \cap [\![\rho < \tau]\!]$. Then $b_\rho \subseteq [\![\sigma \wedge \tau \le \rho]\!] \cap [\![\rho \le \tau]\!]$ and $b_\rho \in \mathfrak{A}_\rho$ so $b_\rho \in \mathfrak{A}_\tau$. There is therefore a $\rho' \in \mathcal{T}$ such that $b_\rho \subseteq [\![\rho' = \rho]\!]$ and $1 \setminus b_\rho \subseteq [\![\rho' = \tau]\!]$. Because $\mathcal{S}$ is finitely full, $\rho' \in \mathcal{S}$. We see that $\sigma \wedge \tau \le \rho' \le \tau$, and that $b_\rho = [\![\rho' < \tau]\!]$.

(ii) Set $\tau' = \inf A$. By hypothesis, $\tau' \in \mathcal{S}$, and $\sigma \wedge \tau \le \tau' \le \tau$. By 632C(a-ii), $[\![\tau' < \tau]\!] = \sup_{\rho \in A} [\![\rho' < \tau]\!] = \sup_{\rho \in A} b_\rho$.

? If $v_{\tau'} \ne v_{\sigma \wedge \tau}$, there is a $\rho \in A$ such that $c = [\![\sigma \wedge \tau \le \rho]\!] \cap [\![\rho < \tau']\!]$ is non-zero. As $\tau' \le \tau$, $c \subseteq b_\rho \subseteq [\![\rho = \rho']\!] \subseteq [\![\tau' \le \rho]\!]$; but this is impossible. **X**

(iii) Thus $v_{\tau'} = v_{\sigma \wedge \tau}$ and

$$\begin{aligned}[\![v_\sigma \ne v_\tau]\!] \cap [\![\sigma < \tau]\!] &\subseteq [\![v_{\sigma \wedge \tau} \ne v_\tau]\!] \subseteq [\![v_{\tau'} \ne v_\tau]\!] \\ &\subseteq \tau' < \tau = \sup_{\rho \in A} b_\rho.\end{aligned}$$

As σ and τ are arbitrary, A separates $\mathcal{S}$.

(b) This now follows from 633Ce.

633Q Continuous time Using some of the ideas above, we can build stochastic integration structures in which no time is isolated on the right.

Proposition Let $(\mathfrak{A}, \bar\mu, T, \langle \mathfrak{A}_t \rangle_{t \in T}, \mathcal{T}, \langle \mathfrak{A}_\tau \rangle_{\tau \in \mathcal{T}})$ be a right-continuous stochastic integration structure. Then there are a right-continuous stochastic integration structure $(\mathfrak{A}, \bar\mu, T', \langle \mathfrak{A}_r \rangle_{r \in T'}, \mathcal{T}', \langle \mathfrak{A}_\rho \rangle_{\rho \in \mathcal{T}'})$, based on the same probability algebra $(\mathfrak{A}, \bar\mu)$, and a lattice homomorphism $\sigma \mapsto \sigma' : \mathcal{T} \to \mathcal{T}'$ such that

T' has no points isolated on the right,

$\mathfrak{A}_{\sigma'} = \mathfrak{A}_\sigma$ and $[\![\sigma' < \tau']\!] = [\![\sigma < \tau]\!]$ for all σ, $\tau \in \mathcal{T}$,

for every $\rho \in \mathcal{T}'$ there is a $\sigma \in \mathcal{T}$ such that $\sigma' \le \rho$ and $\mathfrak{A}_\sigma = \mathfrak{A}_\rho$.

Construction (a) It will be enough to deal with the case in which T is disjoint from $T' = T \times [0,1[$. Give T' its lexicographic ordering. Then T' is a non-empty totally ordered set with no points isolated on the right. For $t \in T$ and $\alpha \in [0,1[$ set $\mathfrak{A}_{(t,\alpha)} = \mathfrak{A}_t$. Then $\langle \mathfrak{A}_r \rangle_{r \in T'}$ is a right-continuous filtration of closed subalgebras. Let $(\mathfrak{A}, \bar\mu, T', \langle \mathfrak{A}_r \rangle_{r \in T'}, \mathcal{T}', \langle \mathfrak{A}_\rho \rangle_{\rho \in \mathcal{T}'})$ be the corresponding stochastic integration structure. Note that $\mathcal{T} \subseteq \mathfrak{A}^T$ and and $\mathcal{T}' \subseteq \mathfrak{A}^{T'}$ are disjoint.

(b) For $\sigma \in \mathcal{T}$, define $\sigma' \in \mathcal{T}'$ by setting

$$[\![\sigma' > (t,\alpha)]\!] = [\![\sigma > t]\!]$$

whenever $t \in T$ and $\alpha \in [0,1[$. Then $\sigma \mapsto \sigma' : \mathcal{T} \to \mathcal{T}'$ is a lattice homomorphism, and $(\min \mathcal{T})' = \min \mathcal{T}'$, $(\max \mathcal{T})' = \max \mathcal{T}'$ (use the formulae in 611C). If σ, $\tau \in \mathcal{T}$,

$$\begin{aligned}[\![\sigma' < \tau']\!] &= \sup_{t \in T, \alpha \in [0,1[} [\![\tau' > (t,\alpha)]\!] \setminus [\![\sigma' > (t,\alpha)]\!] \\ &= \sup_{t \in T} [\![\tau > t]\!] \setminus [\![\sigma > t]\!] = [\![\sigma < \tau]\!].\end{aligned}$$

Consequently $[\![\sigma' \le \tau']\!] = [\![\sigma \le \tau]\!]$ and $[\![\sigma' = \tau']\!] = [\![\sigma = \tau]\!]$ for all σ, $\tau \in \mathcal{T}$. Next, for $\sigma \in \mathcal{T}$,

$$\begin{aligned}\mathfrak{A}_{\sigma'} &= \{a : a \setminus [\![\sigma' > (t,\alpha)]\!] \in \mathfrak{A}_{(t,\alpha)} \text{ whenever } t \in T \text{ and } \alpha \in [0,1[\}\\ &= \{a : a \setminus [\![\sigma > t]\!] \in \mathfrak{A}_t \text{ whenever } t \in T\} = \mathfrak{A}_\sigma.\end{aligned}$$

If $t \in T$ and $\check{t} \in \mathcal{T}$ is the constant stopping time at t, and $(s,\beta) \in T'$, then

$$\begin{aligned}[\![\check{t}' > (s,\beta)]\!] &= [\![\check{t} > s]\!]\\ &= 1 \text{ if } s < t, \text{ that is, if } (s,\beta) < (t,0),\\ &= 0 \text{ if } s \ge t, \text{ that is, if } (t,0) \le (s,\beta).\end{aligned}$$

So $\check{t}' \in \mathcal{T}'$ is the constant stopping time at $(t,0)$.

(c) Consider the set

$$\mathcal{S} = \{\rho : \rho \in \mathcal{T}' \text{ and there is a } \sigma \in \mathcal{T} \text{ such that } \sigma' \le \rho \text{ and } \mathfrak{A}_\sigma = \mathfrak{A}_\rho\}.$$

(i) The constant stopping time $\check{r}$ belongs to $\mathcal{S}$ for every $r \in T'$. **P** If $r = (t,\alpha)$, then $r \ge (t,0)$ so $\check{r} \ge (t,0)\check{} = \check{t}'$, while $\mathfrak{A}_{\check{t}} = \mathfrak{A}_t = \mathfrak{A}_{(t,\alpha)} = \mathfrak{A}_r$. **Q** And of course $\max\mathcal{T}' = (\max\mathcal{T})'$ belongs to $\mathcal{S}$ (see 611Xb).

(ii) $\mathcal{S}$ is finitely full. **P** If $\rho \in \mathcal{T}'$ and there is a finite set $J \subseteq \mathcal{S}$ such that $\sup_{\rho' \in J} [\![\rho' = \rho]\!] = 1$, we may take it that J is a sublattice of $\mathcal{S}$. Taking $(\rho_0,\dots,\rho_n)$ linearly generating the J-cells, we have $\rho_0 \le \dots \le \rho_n$ and $\sup_{i \le n} a_i = 1$ where $a_i = [\![\rho_i = \rho]\!]$ for $i \le n$. Set $b_i = a_i \setminus \sup_{j<i} a_j$ for $i \le n$; then $b_i \in \mathfrak{A}_{\rho_i}$ and $b_i \subseteq [\![\rho_i = \rho]\!]$ for each i, while $\langle b_i\rangle_{i \le n}$ is a partition of unity in $\mathfrak{A}$.

For each $i \le n$, let $\sigma_i \in \mathcal{T}$ be such that $\sigma_i' \le \rho_i$ and $\mathfrak{A}_{\sigma_i} = \mathfrak{A}_{\rho_i}$. As $b_i \in \mathfrak{A}_{\sigma_i}$ for each i, there is a $\sigma \in \mathcal{T}$ such that $b_i \subseteq [\![\sigma = \sigma_i]\!]$ for each i (611I yet again). Now

$$b_i \subseteq [\![\sigma = \sigma_i]\!] \cap [\![\rho_i = \rho]\!] = [\![\sigma' = \sigma_i']\!] \cap [\![\sigma_i' \le \rho_i]\!] \cap [\![\rho_i = \rho]\!] \subseteq [\![\sigma' \le \rho]\!]$$

for each i, so $\sigma' \le \rho$. Thus $\rho \in \mathcal{S}$; as ρ is arbitrary, $\mathcal{S}$ is finitely full. **Q**

(iii) $\inf A \in \mathcal{S}$ for every non-empty $A \subseteq \mathcal{S}$. **P** Set

$$B = \{\sigma : \sigma \in \mathcal{T} \text{ and there is a } \rho \in A \text{ such that } \mathfrak{A}_\sigma = \mathfrak{A}_\rho\}, \quad \tau = \inf B.$$

Then for every $\rho \in A$ there is a $\sigma \in B$ such that $\sigma' \le \rho$ and $\mathfrak{A}_\sigma = \mathfrak{A}_\rho$, in which case $\tau' \le \sigma' \le \rho$. Accordingly $\tau' \le \inf A$. On the other hand, because $\langle\mathfrak{A}_t\rangle_{t \in T}$ and $\langle\mathfrak{A}_r\rangle_{r \in T'}$ are both right-continuous, 632C(a-iii) tells us that

$$\mathfrak{A}_\tau = \bigcap_{\sigma \in B} \mathfrak{A}_\sigma \supseteq \bigcap_{\rho \in A} \mathfrak{A}_\rho = \mathfrak{A}_{\inf A} \supseteq \mathfrak{A}_{\tau'} = \mathfrak{A}_\tau$$

as observed in (b). So τ witnesses that $\inf A \in \mathcal{S}$. **Q**

(iv) Consequently $\mathcal{S} = \mathcal{T}'$. **P** By (i), $\mathcal{S}$ separates $\mathcal{T}'$ and is cofinal with $\mathcal{T}'$; by (ii), $\mathcal{S}$ is equal to its finitely-covered envelope. So 633E tells us that if $\rho \in \mathcal{T}'$ there is a non-empty $A \subseteq \mathcal{S}$ such that $\inf A = \rho$ and $\rho \in \mathcal{S}$. **Q**

Thus we have a suitable stochastic integration structure $(\mathfrak{A},\bar{\mu},T',\langle\mathfrak{A}_r\rangle_{r \in T'},\mathcal{T}',\langle\mathfrak{A}_\rho\rangle_{\rho \in \mathcal{T}'})$.

633R Theorem If $(\mathfrak{A},\bar{\mu},[0,\infty[,\langle\mathfrak{A}_t\rangle_{t \ge 0},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau \in \mathcal{T}})$ is a real-time stochastic integration structure and $\boldsymbol{u} \in M_{\text{ln-s}}(\mathcal{T}_f)$, then $\boldsymbol{u}$ can be represented by a process with càdlàg sample paths as in Theorem 631D.

proof Express $\boldsymbol{u}$ as $\langle u_\sigma\rangle_{\sigma \in \mathcal{T}_f}$.

(a) By 321J, we can suppose that the probability algebra $(\mathfrak{A},\bar{\mu})$ is the measure algebra of a complete probability space $(\tilde{\Omega},\tilde{\Sigma},\tilde{\mu})$. Writing $\tilde{\Sigma}_t = \{E : E \in \tilde{\Sigma},\ E^\bullet \in \mathfrak{A}_t\}$ for $t \ge 0$, we have a filtration $\langle\tilde{\Sigma}_t\rangle_{t \ge 0}$ of σ-subalgebras of $\tilde{\Sigma}$, all containing every $\tilde{\mu}$-negligible set. Write Q for $\mathbb{Q} \cap [0,\infty[$, and for $q \in Q$ choose a $\tilde{\Sigma}$-measurable function $\tilde{X}_q : \tilde{\Omega} \to \mathbb{R}$ such that its equivalence class $\tilde{X}_q^\bullet$ in $L^0(\tilde{\mu}) \cong L^0(\mathfrak{A})$ is equal to $u_{\check{q}}$.

For each $n \in \mathbb{N}$, let $\boldsymbol{v}_n = \langle v_{n\sigma}\rangle_{\sigma \in \mathcal{T} \wedge \check{n}}$ be a simple process with domain $\mathcal{T} \wedge \check{n}$ such that $\theta(z_n) \le 4^{-n}$, where $z_n = \sup_{\sigma \in \mathcal{T} \wedge \check{n}} |u_\sigma - v_{n\sigma}|$, and let $(\tau_{n0},\dots,\tau_{nm_n})$ be a breakpoint sequence for $\boldsymbol{v}_n$ starting from $\tau_{n0} = \check{0}$ and ending with $\tau_{nm_n} = \check{n}$. Choose stopping times $\tilde{g}_{n0},\dots,\tilde{g}_{nm_n}$, adapted to $\langle\tilde{\Sigma}_t\rangle_{t \ge 0}$, representing

$\tau_{n0},\dots,\tau_{nm_n}$ respectively in the way described in 612Ha, so that $\{\omega:\tilde g_{ni}(\omega)>t\}^\bullet=[\![\tau_{ni}>t]\!]$ for every $t\ge 0$ and $i\le m_n$. Adjusting the $\tilde g_{ni}$ on a negligible set if necessary, we can arrange that $0=\tilde g_{n0}(\omega)\le\tilde g_{n1}(\omega)\le\ldots\le\tilde g_{nm_n}(\omega)=n$ for every $\omega\in\tilde\Omega$. Finally, choose a $\tilde\Sigma$-measurable function $\tilde h_n:\tilde\Omega\to[0,\infty[$ such that $\tilde h_n^\bullet=z_n$ in $L^0(\mathfrak A)$.

(b) As $\theta(z_n)\le 4^{-n}$, $\tilde\mu\{\omega:\omega\in\tilde\Omega,\,\tilde h_n(\omega)\ge 2^{-n}\}\le 2^{-n}$ for every $n\in\mathbb N$; consequently

$$F=\{\omega:\omega\in\tilde\Omega,\,\limsup_{n\to\infty}\tilde h_n(\omega)>0\}\subseteq\bigcap_{n\in\mathbb N}\bigcup_{m\ge n}\{\omega:\tilde h_m(\omega)\ge 2^{-m}\}$$

is negligible. Next, for $n\in\mathbb N$, $i<m_n$ and q, $q'\in Q$ such that $q\le q'\le n$, set

$$F_{niqq'}=\{\omega:\omega\in\tilde\Omega,\,\tilde g_{ni}\omega)\le q\le q'<\tilde g_{n,i+1}(\omega),\,|\tilde X_{q'}(\omega)-\tilde X_q(\omega)|>2\tilde h_n(\omega)\};$$

then

$$\begin{aligned}F^\bullet_{niqq'}&=[\![\tau_{ni}\le\check q]\!]\cap[\![\check q'<\tau_{n,i+1}]\!]\cap[\![|u_{\check q'}-u_{\check q}|>2z_n]\!]\\&\subseteq[\![|u_{\check q'}-v_{n\check q'}|>z_n]\!]\cup[\![|u_{\check q}-v_{n\check q}|>z_n]\!]\end{aligned}$$

(because $[\![\tau_{ni}\le\check q]\!]\cap[\![\check q'<\tau_{n,i+1}]\!]\subseteq[\![v_{n\check q'}=v_{n\tau_{ni}}]\!]\cap[\![v_{n\check q}=v_{n\tau_{ni}}]\!]$)

$$=0,$$

so $F_{niqq'}$ is negligible. Set

$$\Omega=\Omega'\setminus(F\cup\bigcup_{n\in\mathbb N,i<m_n,q,q'\in Q,q\le q'\le n}F_{niqq'};$$

then Ω is a conegligible subset of $\tilde\Omega$.

(c) Fix $\omega\in\Omega$ for the moment.

(i) For every $t\ge 0$, $\lim_{q\downarrow Q\cap[t,\infty[}\tilde X_q(\omega)$ is defined in $\mathbb R$. **P** If $\epsilon>0$, then (because $\omega\notin F$) there is an $n>t$ such that $\tilde h_n(\omega)\le\frac12\epsilon$. Since $0=\tilde g_{n0}(\omega)\le\ldots\le\tilde g_{nm_n}(\omega)=n$, there is an $i<m_n$ such that $\tilde g_{ni}(\omega)\le t<\tilde g_{n,i+1}(\omega)$. If now q, $q'\in Q$ and $t\le q\le q'<\tilde g_{n,i+1}(\omega)$, we know that $\omega\notin F_{niqq'}$ so $|\tilde X_{q'}(\omega)-\tilde X_q(\omega)|\le 2\tilde h_n(\omega)\le\epsilon$. As ϵ is arbitrary (and Q is dense in $[0,\infty[$) $\lim_{q\downarrow Q\cap[t,\infty[}\tilde X_q(\omega)$ is defined. **Q**

(ii) Set $X_t(\omega)=\lim_{q\downarrow Q\cap[t,\infty[}\tilde X_q(\omega)$ for $t\ge 0$. Note that $X_q(\omega)=\tilde X_q(\omega)$ for $q\in Q$.

(iii) $\langle X_t(\omega)\rangle_{t\ge 0}$ is càdlàg. **P** We just have to rerun the argument for (i) above. Given $t\ge 0$ and $\epsilon>0$, there is a $t'=\tilde g_{n,i+1}(\omega)$ such that $t<t'$ and $|\tilde X_{q'}(\omega)-\tilde X_q(\omega)|\le\epsilon$ whenever q, $q'\in Q$ and $t\le q\le q'<t'$. It follows at once that $|X_s(\omega)-X_t(\omega)|\le\epsilon$ whenever $t\le s<t'$. As t and ϵ are arbitrary, $\langle X_t(\omega)\rangle_{t\ge 0}$ is càdlàg. **Q**

(d) Because Ω is conegligible in $\tilde\Omega$ and μ is complete, the subspace σ-algebra $\Sigma=\{E\cap\Omega:E\in\Sigma\}$ is just $\Sigma\cap\mathcal P\Omega$, and the subspace measure $\mu=\tilde\mu\restriction\Sigma$ is a probability measure, with measure algebra isomorphic to $\mathfrak A$ (322Jb). We can identify $L^0(\mu)$ with $L^0(\tilde\mu)$ (indeed, in the formulations of Chapter 24 $\mathcal L^0(\mu)$ is actually a subset of $\mathcal L^0(\tilde\mu)$), and as every $\tilde\Sigma_t$ contains $\tilde\Omega\setminus\Omega$, we have a filtration $\langle\Sigma_t\rangle_{t\ge 0}$ of σ-subalgebras of Σ given by the formula

$$\Sigma_t=\{E\cap\Omega:E\in\Sigma_t\}=\Sigma_t\cap\mathcal P\Omega$$

for $t\ge 0$. If $\tilde g:\tilde\Omega\to[0,\infty[$ is a stopping time adapted to $\langle\tilde\Sigma_t\rangle_{t\ge 0}$, then $\tilde g\restriction\Omega$ is a stopping time adapted to $\langle\Sigma_t\rangle_{t\ge 0}$; and conversely, if $g:\Omega\to[0,\infty[$ is a stopping time adapted to $\langle\Sigma_t\rangle_{t\ge 0}$, then any extension of g to a real-valued function defined on $\tilde\Omega$ is a stopping time adapted to $\langle\tilde\Sigma_t\rangle_{t\ge 0}$. What this means is that $(\mathfrak A,\bar\mu,[0,\infty[\,,\langle\mathfrak A_t\rangle_{t\ge 0})$ is represented by $(\Omega,\Sigma,\mu,\langle\Sigma_t\rangle_{t\ge 0})$ just as well as by $(\tilde\Omega,\tilde\Sigma,\tilde\mu,\langle\tilde\Sigma_t\rangle_{t\ge 0})$.

(e) Now fix $t\ge 0$ for a bit.

(i) The first thing to note is that if $t<n\in\mathbb N$ then $v_{n\check t}=\lim_{q\downarrow Q\cap[t,\infty[}v_{n\check q}$ in L^0. **P** For any $s<n$,

$$v_{n\check{s}}=\sum_{i=0}^{m_n-1}v_{n\tau_{ni}}\times\chi([\![\tau_{ni}\le\check{s}]\!]\cap[\![\check{s}<\tau_{n,i+1}]\!])$$
$$=\sum_{i=0}^{m_n-1}v_{n\tau_{ni}}\times(\chi[\![\tau_{n,i+1}>s]\!]-\chi[\![\tau_{ni}>s]\!]).$$

But

$$\lim_{q\downarrow Q\cap[t,\infty[}\chi[\![\tau_{ni}>q]\!]=\chi[\![\tau_{ni}>t]\!]$$

for all $n\in\mathbb{N}$ and $i\le m_n$, just because $[\![\tau ni>t]\!]=\sup_{s>t}[\![\tau_{ni}>s]\!]$ (611A(b-i)). So $v_{n\check{t}}=\lim_{q\downarrow Q\cap[t,\infty[}v_{n\check{q}}$. **Q**

(ii) It follows that $u_{\check{t}}=\lim_{q\downarrow Q\cap[t,\infty[}u_{\check{q}}$. **P** If $n>t$ then we have

$$\limsup_{q\downarrow Q\cap[t,\infty[}|u_{\check{q}}-u_{\check{t}}|\le 2z_n+\limsup_{q\downarrow Q\cap[t,\infty[}|v_{n\check{q}}-v_{n\check{t}}|=2z_n,$$

so

$$\limsup_{q\downarrow Q\cap[t,\infty[}|u_{\check{q}}-u_{\check{t}}|\le 2\inf_{n>t}z_n=0$$

and $u_{\check{t}}=\lim_{q\downarrow Q\cap[t,\infty[}u_{\check{q}}$. **Q**

(iii) Since $Q\cap[t,\infty[$ is countable, $u_{\check{q}}=\tilde{X}_q^\bullet$ for every $q\in Q$ and $X_t=_{\text{a.e.}}\lim_{q\downarrow Q\cap[t,\infty[}\tilde{X}_q$,

$$X_t^\bullet=\lim_{q\downarrow Q\cap[t,\infty[}\tilde{X}_q^\bullet=\lim_{q\downarrow Q\cap[t,\infty[}u_{\check{q}}=u_{\check{t}}.$$

As $u_{\check{t}}\in L^0(\mathfrak{A}_t)$ and $\Sigma_t=\{E:E\subseteq\Omega,\,E^\bullet\in\mathfrak{A}_t\}$, X_t is Σ_t-measurable.

(e) Thus Ω, Σ, μ, $\langle\Sigma_t\rangle_{t\ge0}$ and $\langle X_t\rangle_{t\ge0}$) satisfy the conditions of 631D, and provide a locally near-simple process $\boldsymbol{x}=\langle x_\sigma\rangle_{\sigma\in\mathcal{T}_f}$ in the corresponding stochastic integration structure, which we are identifying with $(\mathfrak{A},\bar{\mu},[0,\infty[\,,\langle\mathfrak{A}_t\rangle_{t\ge0},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$. Now $\boldsymbol{u}$ and $\boldsymbol{x}$ are locally near-simple processes with domain $\mathcal{T}_f$ and $u_{\check{t}}=X_t^\bullet=x_{\check{t}}$ for every $t\ge0$. As $\{\check{t}:t\ge0\}$ separates $\mathcal{T}_f$ (633Da) and $\inf_{t\ge0}[\![\check{t}<\tau]\!]=\inf_{t\ge0}[\![\tau>t]\!]=0$ for every $\tau\in\mathcal{T}_f$, 633F tells us that $\boldsymbol{u}=\boldsymbol{x}$. So we have a representation of $\boldsymbol{u}$ of the kind we seek.

633X Basic exercises (a) Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ which separates its order-convex hull $\mathcal{S}^\sim$, $\boldsymbol{u}$ a near-simple process with domain $\mathcal{S}$, and $\boldsymbol{v}$ a martingale with domain $\mathcal{S}$. Show that $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is a local martingale. (*Hint*: reduce to the case in which $\mathcal{S}$ has a least member. Let $\boldsymbol{v}^\sim$ be the martingale with domain $\mathcal{S}^\sim$ extending $\boldsymbol{v}$, and $\boldsymbol{u}^\sim$ the near-simple process with domain $\mathcal{S}^\sim$ extending $\boldsymbol{u}$ (use 631Mb). Show that $ii_{\boldsymbol{v}}(\boldsymbol{u})=ii_{\boldsymbol{v}^\sim}(\boldsymbol{u}^\sim)\restriction\mathcal{S}$, and apply 623O, 632Ib and 622Dc.)

(b) Suppose that $T=[0,\infty[$, $(\mathfrak{A},\bar{\mu})$ is the measure algebra of Lebesgue measure μ on $[0,1]$, and $\mathfrak{A}_t=\mathfrak{A}$ for every $t\in T$. Define $X_t(\omega)$, for $t\in T$ and $\omega\in[0,1]$, by setting $X_t(\omega)=\frac{1}{\omega-t}$ if $\omega\ne t$, 0 if $\omega=t$. Let $\check{T}\subseteq\mathcal{T}$ be the set of constant stopping times, so that $\mathcal{S}=\check{T}\cup\{\max\mathcal{T}\}$ is a cofinal and coinitial sublattice which separates $\mathcal{T}$. Define $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ by setting $u_{\max\mathcal{T}}=0$ and $u_{\check{t}}=X_t^\bullet$ for $t\in T$. (i) Show that $u_{\inf A}=\lim_{\sigma\downarrow A}u_\sigma$ and $\lim_{\sigma\uparrow A}u_\sigma=u_{\sup A}$ for every non-empty $A\subseteq\mathcal{S}$. (ii) Show that if we take $\Sigma_t=\operatorname{dom}\mu$ for every $t\in T$, then $\langle X_t\rangle_{t\in T}$ is progressively measurable, so that we have a natural extension of $\boldsymbol{u}$ to a fully adapted process defined on $\mathcal{T}$, while $(\mathfrak{A},\bar{\mu},T,\langle\mathfrak{A}_t\rangle_{t\in T})$ is right-continuous. (iii) Show that $\boldsymbol{u}$ is not locally order-bounded.

(c) Show that every subset of $\mathcal{T}$ separates its covered envelope.

633Y Further exercises (a) Let $(\mathfrak{A},\bar{\mu},T,\langle\mathfrak{A}_t\rangle_{t\in T},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ be a stochastic integration structure. Write T_{ni} for $\{t:t\in T$ is not isolated on the right in $T\}$. Take an injective function $t\mapsto t^+$ from T to a set disjoint from T. Set $T^+=T\cup\{t^+:t\in T_{\text{ni}}\}$. On T^+ take the relation

$$\{(s,t):s,\,t\in T,\,s\le t\}\cup\{(s,t^+):s\in T,\,t\in T_{\text{ni}},\,s\le t\}$$
$$\cup\{(s^+,t):s\in T_{\text{ni}},\,t\in T,\,s<t\}\cup\{(s^+,t^+):s,\,t\in T_{\text{ni}},\,s\le t\}.$$

(i) Show that T^+ is totally ordered. (ii) Set $\mathfrak{A}_{t^+} = \bigcap_{s\in T,s>t} \mathfrak{A}_s$ for $t \in T_{ni}$; show that $\langle\mathfrak{A}_r\rangle_{r\in T^+}$ is a right-continuous filtration of closed subalgebras of $\mathfrak{A}$. Let $(\mathfrak{A}, \bar\mu, T^+, \langle\mathfrak{A}_r\rangle_{r\in T^+}, \mathcal{T}^+, \langle\mathfrak{A}_\rho\rangle_{\rho\in\mathcal{T}^+})$ be the corresponding stochastic integration structure. (iii) Show that we have a lattice homomorphism $\sigma \mapsto \sigma^+ : \mathcal{T} \to \mathcal{T}^+$ defined by saying that $[\![\sigma^+ > t]\!] = [\![\sigma > t]\!]$ for $t \in T$ and $[\![\sigma^+ > t^+]\!] = [\![\sigma > t]\!]$ for $t \in T_{ni}$. (iv) Show that $\mathfrak{A}_{\sigma^+} = \mathfrak{A}_\sigma$ and $[\![\sigma^+ < \tau^+]\!] = [\![\sigma < \tau]\!]$ for all σ, $\tau \in \mathcal{T}$. (v) Show that $\{\sigma^+ : \sigma \in \mathcal{T}\}$ separates $\mathcal{T}^+$. (vi) Suppose that $\boldsymbol{w} = \langle w_\rho\rangle_{\rho\in\mathcal{T}^+}$ is a fully adapted process. Show that $\boldsymbol{u} = \langle w_{\sigma^+}\rangle_{\sigma\in\mathcal{T}}$ is fully adapted, that $\boldsymbol{u}$ is a martingale if $\boldsymbol{w}$ is a martingale, and that $\operatorname{Osclln}(\boldsymbol{u}) \le 2\operatorname{Osclln}(\boldsymbol{w})$ if $\boldsymbol{w}$ is near-simple.

(b) Let $(\mathfrak{A}, \bar\mu, T^+, \langle\mathfrak{A}_r\rangle_{r\in T^+}, \mathcal{T}^+, \langle\mathfrak{A}_\rho\rangle_{\rho\in\mathcal{T}^+})$ be the right-continuous version of $(\mathfrak{A}, \bar\mu, T, \langle\mathfrak{A}_t\rangle_{t\in T}, \mathcal{T}, \langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$, as in 633Ya. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$ and $\mathcal{R} \subseteq \mathcal{T}^+$ the order-convex hull of $\mathcal{R}' = \{\sigma^+ : \sigma \in \mathcal{S}\}$. (i) Show that $\mathcal{R}'$ separates $\mathcal{R}$. (ii) Suppose that $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a near-simple process. Show that there is a near-simple process $\boldsymbol{v} = \langle v_\rho\rangle_{\rho\in\mathcal{R}}$ such that $v_{\sigma^+} = u_\sigma$ for every $\sigma \in \mathcal{S}$.

(c) Suppose that $(\mathfrak{A}, \bar\mu, [0,\infty[\,, \langle\mathfrak{A}_t\rangle_{t\ge0}, \mathcal{T}, \langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ is a real-time stochastic integration structure and $\boldsymbol{u}$ is a locally jump-free process with domain $\mathcal{T}_f$. Show that $\boldsymbol{u}$ can be represented by a process with continuous sample paths as in 618H.

(d) Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S} \subseteq \mathcal{T}$ be a finitely full sublattice such that $\inf A \in \mathcal{S}$ whenever $A \subseteq \mathcal{S}$ is non-empty and has a lower bound in $\mathcal{S}$, and $\mathcal{S}'$ a sublattice of $\mathcal{S}$ such that whenever τ, $\tau' \in \mathcal{S}$ and $[\![\tau < \tau']\!] \ne 0$ there is a $\sigma \in \mathcal{S}'$ such that $[\![\tau \le \sigma]\!] \cap [\![\sigma < \tau']\!] \ne 0$. Show that $\mathcal{S}'$ separates $\mathcal{S}$.

633 Notes and comments This volume began with two sections devoted to the algebra of lattices of stopping times and fully adapted processes. It would all have been much easier if we could have worked throughout with totally ordered sets of stopping times. Even for integrals of the form $\int_{[\check s,\check t]}$, however, this is inadequate (632Yg). The problem is not so much in the calculation of the integral as in its definition. In 633K-633L we see that if an integral $\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}$ is defined on the rules of §613, and subject to mild conditions on the processes $\boldsymbol{u}$ and $\boldsymbol{v}$, then we can hope to find a totally ordered sublattice $\mathcal{S}'$ of $\mathcal{S}$ such that $\int_{\mathcal{S}'} \boldsymbol{u}\,d\boldsymbol{v} = \int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}$. But the nearest thing to a converse result, in 633Kb, looks at finitely full sublattices $\mathcal{S}'$, which will not be totally ordered except in trivial cases.

It is true that we can reduce a Riemann sum S_I on an arbitrary finite sublattice I to the corresponding sum on a totally ordered sublattice I_0 of I (613Ec), and in the calculations so far this is what I have done more often than not. But whenever we want to look at sums S_I, S_J on two different sublattices, this method becomes problematic; there is no reason to suppose that there will be compatible totally ordered sublattices $I_0 \subseteq I$ and $J_0 \subseteq J$ which will be suitable. Even when $J \subseteq I$ we have to do some work (633H). What we want is a totally ordered sublattice of $\mathcal{T}$ which will deal with all integrals of interest simultaneously; and even in the most favourable case (633L), we should have to restrict ourselves to integrals between constant stopping times, which are inadequate for many of the most important applications of the theory.

I have given a number of results showing that classical stochastic processes, based on probability spaces and filtrations of σ-algebras, can be associated with processes dealt with in this volume (612H, 615P, 614U, 618H, 632L). I have not spent much time on converses. looking at cases in which a process defined by the properties considered here can be represented by one of the classical expressions. But in 633R we have a comfortingly straightforward result which seems entitled to a couple of pages.

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634 Changing the algebra

If $(\mathfrak{A}, \bar\mu)$ is a probability algebra with a filtration $\langle\mathfrak{A}_t\rangle_{t\in T}$ and $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$, then we have a probability algebra $(\mathfrak{B}, \bar\mu{\restriction}\mathfrak{B})$ with a filtration $\langle\mathfrak{B}\cap\mathfrak{A}_t\rangle_{t\in T}$. In this section I examine elementary connexions between stochastic calculus in the two structures, with notes on lattices of stopping times (634C) and stochastic processes (634E). The case in which $\mathfrak{B}$ and $\mathfrak{A}_t$ are relatively independent over their intersection for every t is particularly important (634F-634I). I end the section with a product construction adapted to the representation of families of independent stochastic processes (634K-634M) and a worked example on independent Poisson processes (634N).

634A Notation $\mathbb{A} = (\mathfrak{A}, \bar{\mu}, T, \langle\mathfrak{A}_t\rangle_{t\in T}, \mathcal{T}_{\mathbb{A}}, \langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}_{\mathbb{A}}})$ will be a stochastic integration structure, and $\theta = \theta_{\bar{\mu}}$ the usual functional defining the topology of convergence in measure on $L^0(\mathfrak{A})$, as in 613Ba. If $\mathcal{S}$ is a sublattice of $\mathcal{T}_{\mathbb{A}}$, then $\mathcal{I}(\mathcal{S})$ is the set of finite sublattices of $\mathcal{S}$.

If $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$, $P_{\mathfrak{B}} : L^1_{\bar{\mu}} \to L^1_{\bar{\mu}} \cap L^0(\mathfrak{B})$ will be the conditional expectation associated with $\mathfrak{B}$. For $\tau \in \mathcal{T}_{\mathbb{A}}$, I will write P_τ for $P_{\mathfrak{A}_\tau}$.

Recall that if $\mathfrak{B}$ is an order-closed subalgebra of $\mathfrak{A}$, we can identify $L^0(\mathfrak{B})$ with an order-closed f-subalgebra of $L^0(\mathfrak{A})$ (612Ae). In particular, a subset of $L^0(\mathfrak{B})$ is order-bounded in $L^0(\mathfrak{B})$ iff it is order-bounded in $L^0(\mathfrak{A})$. We have $L^1(\mathfrak{B}, \bar{\mu}\restriction\mathfrak{B}) = L^0(\mathfrak{B}) \cap L^1(\mathfrak{A}, \bar{\mu}) = L^0(\mathfrak{B}) \cap L^1_{\bar{\mu}}$, and the expectation $\mathbb{E}_{\bar{\mu}\restriction\mathfrak{B}}$ on $L^1(\mathfrak{B}, \bar{\mu}\restriction\mathfrak{B})$ is just the restriction of $\mathbb{E} = \mathbb{E}_{\bar{\mu}}$ to $L^1(\mathfrak{B}, \bar{\mu}\restriction\mathfrak{B})$ (see 365Oa[4], 365Qa[5]). Consequently $\theta_{\bar{\mu}\restriction\mathfrak{B}} = \theta_{\bar{\mu}}\restriction L^0(\mathfrak{B})$, and the topology of convergence in measure on $L^0(\mathfrak{B})$ is the subspace topology induced by the topology of convergence in measure on $L^0(\mathfrak{A})$. It will also be useful to note that a subset of $L^0(\mathfrak{B}) \cap L^1(\mathfrak{A}, \bar{\mu})$ is uniformly integrable in $L^1(\mathfrak{B}, \bar{\mu}\restriction\mathfrak{B})$ iff it is uniformly integrable in $L^1_{\bar{\mu}}$.

For a family $\langle\mathfrak{C}_k\rangle_{k\in K}$ of closed subalgebras of $\mathfrak{A}$, I write $\bigvee_{k\in K}\mathfrak{C}_k$ for the closed subalgebra generated by $\bigcup_{k\in K}\mathfrak{C}_k$. Similarly, for closed subalgebras $\mathfrak{C}$, $\mathfrak{C}'$ I will write $\mathfrak{C}\vee\mathfrak{C}'$ for the closed subalgebra generated by $\mathfrak{C}\cup\mathfrak{C}'$. (Cf. 458Ad.) Note that as $\mathfrak{A}$ is ccc (322G), a subalgebra of $\mathfrak{A}$ is closed iff it is a σ-subalgebra (331G).

634B I begin with a general result on morphisms of the structures here.

Proposition Let $\mathfrak{B}$ be a Dedekind complete algebra and $\langle\mathfrak{B}_t\rangle_{t\in T}$ a filtration of order-closed subalgebras of $\mathfrak{B}$; write $\mathcal{T}_{\mathbb{B}}$ and $\langle\mathfrak{B}_\sigma\rangle_{\sigma\in\mathcal{T}_{\mathbb{B}}}$ for the associated lattice of stopping times and family of closed subalgebras of $\mathfrak{B}$, as in §611. Suppose that $\phi : \mathfrak{B} \to \mathfrak{A}$ is an order-continuous Boolean homomorphism such that $\phi[\mathfrak{B}_t] \subseteq \mathfrak{A}_t$ for every $t \in T$.

(a) We have a lattice homomorphism $\hat{\phi} : \mathcal{T}_{\mathbb{B}} \to \mathcal{T}_{\mathbb{A}}$ defined by saying that $[\![\hat{\phi}(\sigma) > t]\!] = \phi[\![\sigma > t]\!]$ for every $\sigma \in \mathcal{T}_{\mathbb{B}}$.

(b)(i) $\hat{\phi}(\min\mathcal{T}_{\mathbb{B}}) = \min\mathcal{T}_{\mathbb{A}}$, $\hat{\phi}(\max\mathcal{T}_{\mathbb{B}}) = \max\mathcal{T}_{\mathbb{A}}$. If $t \in T$ and $\check{t}$ is the constant stopping time at t in $\mathcal{T}_{\mathbb{B}}$, then $\hat{\phi}(\check{t})$ is the constant stopping time at t in $\mathcal{T}_{\mathbb{A}}$.

(ii) If $C \subseteq \mathcal{T}_{\mathbb{B}}$ then $\hat{\phi}(\sup C) = \sup\hat{\phi}[C]$.

(iii) $\hat{\phi}[\mathcal{T}_{\mathbb{B}b}] \subseteq \mathcal{T}_{\mathbb{A}b}$ and $\hat{\phi}[\mathcal{T}_{\mathbb{B}f}] \subseteq \mathcal{T}_{\mathbb{A}f}$.

(c)(i) If σ, $\sigma' \in \mathcal{T}_{\mathbb{B}}$ then

$$[\![\hat{\phi}(\sigma) < \hat{\phi}(\sigma')]\!] = \phi[\![\sigma < \sigma']\!], \quad [\![\hat{\phi}(\sigma) \le \hat{\phi}(\sigma')]\!] = \phi[\![\sigma \le \sigma']\!],$$

$$[\![\hat{\phi}(\sigma) = \hat{\phi}(\sigma')]\!] = \phi[\![\sigma = \sigma']\!].$$

(ii) If ϕ is injective, so is $\hat{\phi}$.

(d) $\phi[\mathfrak{B}_\sigma] \subseteq \mathfrak{A}_{\hat{\phi}(\sigma)}$ for every $\sigma \in \mathcal{T}_{\mathbb{B}}$.

(e) If $\langle\mathfrak{B}_t\rangle_{t\in T}$ is right-continuous, then $\hat{\phi}$ is order-continuous,

(f) If $T = [0,\infty[$ and we define the identity processes $\langle\iota_\sigma\rangle_{\sigma\in\mathcal{T}_{\mathbb{B}f}}$, $\langle\iota_\tau\rangle_{\tau\in\mathcal{T}_{\mathbb{A}f}}$ as in 612F, then $\iota_{\hat{\phi}(\sigma)} = T_\phi\iota_\sigma$ for every $\sigma \in \mathcal{T}_{\mathbb{B}f}$, where $T_\phi : L^0(\mathfrak{B}) \to L^0(\mathfrak{A})$ is the f-algebra homomorphism associated with ϕ (612Af).

proof (a)(i) If $\sigma \in \mathcal{T}_{\mathbb{B}}$, then

$\phi[\![\sigma > t]\!] \in \phi[\mathfrak{B}_t] \subseteq \mathfrak{A}_t$ for every $t \in T$,

if $s \le t$ then $[\![\sigma > t]\!] \subseteq [\![\sigma > s]\!]$ so $\phi[\![\sigma > t]\!] \subseteq \phi[\![\sigma > s]\!]$,

if $t \in T$ is not isolated on the right then $[\![\sigma > t]\!] = \sup_{s>t}[\![\sigma > s]\!]$, and because ϕ is order-continuous, $\phi[\![\sigma > t]\!] = \sup_{s>t}\phi[\![\sigma > s]\!]$.

Thus the function $t \mapsto \phi[\![\sigma > t]\!]$ satisfies the conditions of 611A(b-i), and defines a member of $\mathcal{T}_{\mathbb{A}}$ which we may call $\hat{\phi}(\sigma)$.

(ii) If $C \subseteq \mathcal{T}_{\mathbb{B}}$ is non-empty, then

$$[\![\hat{\phi}(\sup C) > t]\!] = \phi[\![\sup C > t]\!] = \phi(\sup_{\sigma\in C}[\![\sigma > t]\!])$$

(611Cb)

[4]Formerly 365Pa.

[5]Formerly 365Ra.

$$= \sup_{\sigma\in C} \phi[\![\sigma > t]\!]$$

(again because ϕ is order-continuous)

$$= \sup_{\sigma\in C} [\![\hat{\phi}(\sigma) > t]\!] = [\![\sup_{\sigma\in C} \hat{\phi}(\sigma) > t]\!]$$

for every $t \in T$, so $\hat{\phi}(\sup C) = \sup \hat{\phi}[C]$. If σ, $\sigma' \in \mathcal{T}_{\mathbb{B}}$, then

$$[\![\hat{\phi}(\sigma \wedge \sigma') > t]\!] = \phi[\![\sigma \wedge \sigma' > t]\!] = \phi([\![\sigma > t]\!] \cap [\![\sigma' > t]\!])$$

(611Cc)

$$= \phi[\![\sigma > t]\!] \cap \phi[\![\sigma' > t]\!]$$
$$= [\![\hat{\phi}(\sigma) > t]\!] \cap [\![\hat{\phi}(\sigma') > t]\!] = [\![\hat{\phi}(\sigma) \wedge \hat{\phi}(\sigma') > t]\!]$$

for every $t \in T$, and $\hat{\phi}(\sigma\wedge\sigma') = \hat{\phi}(\sigma)\wedge\hat{\phi}(\sigma')$. Putting these together, we see that $\hat{\phi}$ is a lattice homomorphism.

(b)(i) Look at the descriptions of $\max \mathcal{T}$ and $\min \mathcal{T}$ in 611Cf, and remember that $\phi 0_{\mathfrak{B}} = 0_{\mathfrak{A}}$ and $\phi 1_{\mathfrak{B}} = 1_{\mathfrak{A}}$. Concerning constant stopping times, if $t \in T$ then

$$[\![\hat{\phi}(\check{t}) > s]\!] = \phi[\![\check{t} > s]\!] = \phi 1_{\mathfrak{B}} = 1_{\mathfrak{A}} \text{ if } s < t,$$
$$= \phi 0_{\mathfrak{B}} = 0_{\mathfrak{A}} \text{ if } s \geq t,$$

so $\hat{\phi}(\check{t})$ is the constant stopping time at t in the structure $(\mathfrak{A}, \langle\mathfrak{A}_t\rangle_{t\in T})$.

(ii) This is covered by (a-ii) above.

(iii) Because $\hat{\phi}$ is order-preserving and takes constant stopping times to constant stopping times, $\hat{\phi}[\mathcal{T}_{\mathbb{B}b}] \subseteq \mathcal{T}_{\mathbb{A}b}$. Now if $\sigma \in \mathcal{T}_{\mathbb{B}f}$, $\sigma = \sup_{t\in T} \sigma \wedge \check{t}$, so (ii) tells us that

$$\hat{\phi}(\sigma) = \sup_{t\in T} \hat{\phi}(\sigma) \wedge \check{t}$$

(now interpreting $\check{t}$ as a constant stopping time in $\mathcal{T}_{\mathbb{A}}$) and belongs to $\mathcal{T}_{\mathbb{A}f}$.

(c)(i) By the definition in 611D

$$\phi[\![\sigma < \sigma']\!] = \phi(\sup_{t\in T} [\![\sigma' > t]\!] \setminus [\![\sigma > t]\!]) = \sup_{t\in T}(\phi[\![\sigma' > t]\!] \setminus \phi[\![\sigma > t]\!])$$
$$= \sup_{t\in T}([\![\hat{\phi}(\sigma') > t]\!] \setminus [\![\hat{\phi}(\sigma) > t]\!]) = [\![\hat{\phi}(\sigma) < \hat{\phi}(\sigma')]\!].$$

The other equalities follow at once.

(ii) If ϕ is injective and σ, $\sigma' \in \mathcal{T}_{\mathbb{B}}$ are different, then $[\![\sigma = \sigma']\!] \neq 1_{\mathfrak{B}}$ so

$$[\![\hat{\phi}(\sigma) = \hat{\phi}(\sigma')]\!] = \phi[\![\sigma = \sigma']\!] \neq \phi(1_{\mathfrak{B}}) = 1_{\mathfrak{A}}$$

and $\hat{\phi}(\sigma) \neq \hat{\phi}(\sigma')$.

(d) Suppose that $b \in \mathfrak{B}_\sigma$. Then for any $t \in T$, $b \setminus [\![\sigma > t]\!] \in \mathfrak{B}_t$, so

$$\phi b \setminus [\![\hat{\phi}(\sigma) > t]\!] = \phi b \setminus \phi[\![\sigma > t]\!] = \phi(b \setminus [\![\sigma > t]\!]) \in \phi[\mathfrak{B}_t] \subseteq \mathfrak{A}_t.$$

Thus $\phi b \in \mathfrak{A}_{\hat{\phi}(\sigma)}$.

(e) If $C \subseteq \mathcal{T}_{\mathbb{B}}$ is a non-empty set with has infimum σ_*, set $\tau_* = \inf \hat{\phi}[C]$. As $\hat{\phi}$ is order-preserving, $\hat{\phi}(\sigma_*)$ is a lower bound of $\hat{\phi}[C]$ and $\hat{\phi}(\sigma_*) \leq \tau_*$.

If $t \in T$ is isolated on the right, then

$$[\![\tau_* > t]\!] \subseteq \inf_{\sigma\in C} [\![\hat{\phi}(\sigma) > t]\!] = \phi(\inf_{\sigma\in C} [\![\sigma > t]\!]) = \phi[\![\sigma_* > t]\!]$$

by 632C(a-i). If $t \in T$ is not isolated on the right, then

$$\llbracket\tau_* > t\rrbracket = \sup_{s>t} \llbracket\tau_* > s\rrbracket \subseteq \sup_{s>t} \inf_{\sigma\in C} \llbracket\hat{\phi}(\sigma) > s\rrbracket$$
$$= \phi(\sup_{s>t} \inf_{\sigma\in C} \llbracket\sigma > s\rrbracket) = \phi\llbracket\sigma_* > t\rrbracket$$

by the other formula in 632C(a-i). So

$$\llbracket\tau_* > t\rrbracket \subseteq \phi\llbracket\sigma_* > t\rrbracket = \llbracket\hat{\phi}(\sigma_*) > t\rrbracket$$

for every t, and $\tau_* \le \hat{\phi}(\sigma_*)$. Thus $\inf \hat{\phi}[C] = \hat{\phi}(\inf C)$. We know already from (b-ii) that $\sup \hat{\phi}[C] = \sup \phi(\inf C)$. As C is arbitrary, $\hat{\phi}$ is order-continuous in the sense of 313Ha.

(f) By (b-iii), $\hat{\phi}(\sigma) \in \mathcal{T}_{\mathbb{A}f}$. Now, for $t \in \mathbb{R}$,

$$\llbracket\iota_{\hat{\phi}(\sigma)} > t\rrbracket = \llbracket\hat{\phi}(\sigma) > t\rrbracket = \phi\llbracket\sigma > t\rrbracket = \phi\llbracket\iota_\sigma > t\rrbracket = \llbracket T_\phi\iota_\sigma > t\rrbracket \text{ if } t \ge 0,$$
$$= 1_{\mathfrak{A}} = \phi 1_{\mathfrak{B}} = \phi\llbracket\iota_\sigma > t\rrbracket = \llbracket T_\phi\iota_\sigma > t\rrbracket \text{ otherwise,}$$

so $\iota_{\hat{\phi}(\sigma)} = T_\phi\iota_\sigma$.

634C Proposition Suppose that $\mathfrak{B}$ is an (order-)closed subalgebra of $\mathfrak{A}$.

(a) Set $\mathfrak{B}_t = \mathfrak{B} \cap \mathfrak{A}_t$ for $t \in T$. Then $\langle\mathfrak{B}_t\rangle_{t\in T}$ is a filtration of closed subalgebras of $\mathfrak{B}$.

(b) Let $\mathcal{T}_{\mathbb{B}}$ be the set of stopping times defined from $(\mathfrak{B}, \langle\mathfrak{B}_t\rangle_{t\in T})$ by the formula of 611A(b-i).

(i) $\mathcal{T}_{\mathbb{B}}$ is a sublattice of $\mathcal{T}_{\mathbb{A}}$ containing $\min \mathcal{T}_{\mathbb{A}}$, $\max \mathcal{T}_{\mathbb{A}}$ and all constant stopping times.

(ii) If $C \subseteq \mathcal{T}_{\mathbb{B}}$ is non-empty then its supremum is the same whether calculated in $\mathcal{T}_{\mathbb{B}}$ or in $\mathcal{T}_{\mathbb{A}}$.

(iii) We can identify the order-ideals $\mathcal{T}_{\mathbb{B}b}$ and $\mathcal{T}_{\mathbb{B}f}$ of bounded and finite stopping times in $\mathcal{T}_{\mathbb{B}}$ with $\mathcal{T}_{\mathbb{B}} \cap \mathcal{T}_{\mathbb{A}b}$ and $\mathcal{T}_{\mathbb{B}} \cap \mathcal{T}_{\mathbb{A}f}$ respectively.

(c) If σ, $\tau \in \mathcal{T}_{\mathbb{B}}$ then the regions $\llbracket\sigma < \tau\rrbracket$, $\llbracket\sigma \le \tau\rrbracket$ and $\llbracket\sigma = \tau\rrbracket$, when defined by the formulae of 611D interpreted in either $(\mathfrak{A}, \langle\mathfrak{A}_t\rangle_{t\in T})$ or $(\mathfrak{B}, \langle\mathfrak{B}_t\rangle_{t\in T})$, are the same, and belong to $\mathfrak{B}$.

(d) If $\tau \in \mathcal{T}_{\mathbb{B}}$, and we define corresponding algebras $\mathfrak{A}_\tau$ and $\mathfrak{B}_\tau$ by the formula of 611G interpreted in $(\mathfrak{A}, \langle\mathfrak{A}_t\rangle_{t\in T})$, $(\mathfrak{B}, \langle\mathfrak{B}_t\rangle_{t\in T})$ respectively, then $\mathfrak{B}_\tau = \mathfrak{B} \cap \mathfrak{A}_\tau$.

(e) Suppose that $\mathcal{S}$ is a sublattice of $\mathcal{T}_{\mathbb{B}}$.

(i) If $\hat{\mathcal{S}}_{\mathbb{A}}$ is the covered envelope of $\mathcal{S}$ in $\mathcal{T}_{\mathbb{A}}$, then $\hat{\mathcal{S}}_{\mathbb{A}} \cap \mathcal{T}_{\mathbb{B}}$ is the covered envelope $\hat{\mathcal{S}}_{\mathbb{B}}$ of $\mathcal{S}$ in $\mathcal{T}_{\mathbb{B}}$.

(ii) A family $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ in $L^0(\mathfrak{B})$ is fully adapted to $\langle\mathfrak{B}_t\rangle_{t\in T}$ iff it is fully adapted to $\langle\mathfrak{A}_t\rangle_{t\in T}$.

(iii) If $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is fully adapted to $\langle\mathfrak{B}_t\rangle_{t\in T}$ and $\hat{\boldsymbol{u}}$ is the extension of $\boldsymbol{u}$ to $\hat{\mathcal{S}}_{\mathbb{A}}$ which is fully adapted to $\langle\mathfrak{A}_t\rangle_{t\in T}$, then $\hat{\boldsymbol{u}}\restriction\hat{\mathcal{S}}_{\mathbb{B}}$ is the extension of $\boldsymbol{u}$ to $\hat{\mathcal{S}}_{\mathbb{B}}$ which is fully adapted to $\langle\mathfrak{B}_t\rangle_{t\in T}$.

(f) Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous.

(i) $\langle\mathfrak{B}_t\rangle_{t\in T}$ is right-continuous.

(ii) $\mathcal{T}_{\mathbb{B}}$ is order-closed in $\mathcal{T}_{\mathbb{A}}$.

(g) If $T = [0, \infty[$ and we write $\boldsymbol{\iota} = \langle\iota_\sigma\rangle_{\sigma\in\mathcal{T}_{\mathbb{A}f}}$ for the identity process in the structure $(\mathfrak{A}, \langle\mathfrak{A}_t\rangle_{t\in T}, \mathcal{T}_{\mathbb{A}})$, then $\boldsymbol{\iota}\restriction\mathcal{T}_{\mathbb{B}f}$ is the identity process in the structure $(\mathfrak{B}, \langle\mathfrak{B}_t\rangle_{t\in T}, \mathcal{T}_{\mathbb{B}})$.

proof (a) Each $\mathfrak{B}_t$, being the intersection of order-closed subalgebras of $\mathfrak{A}$, is an order-closed subalgebra of $\mathfrak{A}$ and therefore an order-closed subalgebra of $\mathfrak{B}$. If $s \le t$ then $\mathfrak{A}_s \subseteq \mathfrak{A}_t$ so $\mathfrak{B}_s \subseteq \mathfrak{B}_t$.

We are now in the special case of 634B in which $\phi b = b$ for every $b \in \mathfrak{B}$. Consequently $\hat{\phi}(\sigma) = \sigma$ for every $\sigma \in \mathcal{T}_{\mathbb{B}}$, and $\mathcal{T}_{\mathbb{B}}$ is actually a subset of $\mathcal{T}_{\mathbb{A}}$. Because the identity map $\hat{\phi}$ is a lattice homomorphism (634B(b-i)), $\mathcal{T}_{\mathbb{B}}$ is a sublattice of $\mathcal{T}_{\mathbb{A}}$.

(b) We just have to adapt 634Bb to this special case, in which $\mathcal{T}_{\mathbb{B}} \subseteq \mathcal{T}_{\mathbb{A}}$ and $\hat{\phi}$ is the identity embedding.

(c) Similarly, this is just the form now taken by 634B(c-i).

(d) Going back to the formula in 611G,

$$\mathfrak{B}_\tau = \bigcap_{t\in T} \{b : b \in \mathfrak{B},\, b \setminus \llbracket\tau > t\rrbracket \in \mathfrak{B}_t\}$$
$$= \bigcap_{t\in T} \{b : b \in \mathfrak{B},\, b \setminus \llbracket\tau > t\rrbracket \in \mathfrak{B} \cap \mathfrak{A}_t\} = \bigcap_{t\in T} \{b : b \in \mathfrak{B},\, b \setminus \llbracket\tau > t\rrbracket \in \mathfrak{A}_t\}$$

(because $\llbracket\tau > t\rrbracket \in \mathfrak{B}$ for every $t \in T$)

$$= \mathfrak{B} \cap \bigcap_{t \in T} \{a : a \in \mathfrak{A},\ a \setminus [\![\tau > t]\!] \in \mathfrak{A}_t\} = \mathfrak{B} \cap \mathfrak{A}_\tau.$$

(e)(i) If $\tau \in \mathcal{T}_{\mathbb{B}}$, then

$$\tau \in \hat{\mathcal{S}}_{\mathbb{A}} \iff \sup_{\sigma \in \mathcal{S}} [\![\tau = \sigma]\!] = 1 \iff \tau \in \hat{\mathcal{S}}_{\mathbb{B}}.$$

So

$$\hat{\mathcal{S}}_{\mathbb{A}} \cap \mathcal{T}_{\mathbb{B}} = \hat{\mathcal{S}}_{\mathbb{B}} \cap \mathcal{T}_{\mathbb{B}} = \hat{\mathcal{S}}_{\mathbb{B}}.$$

(ii) Immediate from 612Da, (c) and (d), if we recall that

$$L^0(\mathfrak{B} \cap \mathfrak{A}_\tau) = L^0(\mathfrak{B}) \cap L^0(\mathfrak{A}_\tau)$$

for every τ (612A(e-i)).

(iii) The point is that if $\hat{\boldsymbol{u}} = \langle \hat{u}_\tau \rangle_{\tau \in \hat{\mathcal{S}}_{\mathbb{A}}}$, then $\hat{u}_\tau \in L^0(\mathfrak{B})$ for every $\tau \in \hat{\mathcal{S}}_{\mathbb{B}}$. **P** If $\tau \in \hat{\mathcal{S}}_{\mathbb{B}}$ and $\alpha \in \mathbb{R}$ then

$$[\![\hat{u}_\tau > \alpha]\!] = [\![\hat{u}_\tau > \alpha]\!] \cap \sup_{\sigma \in \mathcal{S}} [\![\tau = \sigma]\!] = \sup_{\sigma \in \mathcal{S}} ([\![\hat{u}_\tau > \alpha]\!] \cap [\![\tau = \sigma]\!])$$
$$= \sup_{\sigma \in \mathcal{S}} ([\![u_\sigma > \alpha]\!] \cap [\![\tau = \sigma]\!]) \in \mathfrak{B}.$$

(Of course I am relying on (c) for assurance that $[\![\tau = \sigma]\!]$ has a consistent interpretation.) **Q** Now (ii) just above tells us that $\hat{\boldsymbol{u}} \restriction \hat{\mathcal{S}}_{\mathfrak{B}}$ is fully adapted to $\langle \mathfrak{B}_t \rangle_{t \in T}$, while it surely extends $\boldsymbol{u}$.

(f)(i) Look at the definition in 632B: as $\langle \mathfrak{A}_t \rangle_{t \in T}$ is right-continuous,

$$\mathfrak{B}_t = \mathfrak{B} \cap \mathfrak{A}_t = \mathfrak{B} \cap \bigcap_{s > t} \mathfrak{A}_s = \bigcap_{s > t} \mathfrak{B} \cap \mathfrak{A}_s = \bigcap_{s > t} \mathfrak{B}_s$$

whenever $t \in T$ is not isolated on the right.

(ii) If $C \subseteq \mathcal{T}_{\mathbb{B}}$ is non-empty, then $\sup C$, calculated in $\mathcal{T}_{\mathbb{A}}$, belongs to $\mathcal{T}_{\mathbb{B}}$ by (b-ii) above. As for its infimum, this is given by the formula

$$[\![\inf C > t]\!] = \inf_{\tau \in C} [\![\tau > t]\!] \text{ if } t \text{ is isolated on the right,}$$
$$= \sup_{s > t} \inf_{\tau \in C} [\![\tau > s]\!] \text{ otherwise}$$

of 632C(a-i), which (because $\mathfrak{B}$ is order-closed in $\mathfrak{A}$) is the same whether calculated in $\mathfrak{B}$ or $\mathfrak{A}$. So $\sup C$ and $\inf C$, taken in $\mathcal{T}_{\mathbb{A}}$, both belong to $\mathcal{T}_{\mathbb{B}}$; as C is arbitrary, $\mathcal{T}_{\mathbb{B}}$ is an order-closed sublattice of $\mathcal{T}_{\mathbb{A}}$ as defined in 313Da.

(g) This is just the form taken by 634Bf in the present context.

Remark If we think of $\mathcal{T}_{\mathbb{A}}$ as a sublattice of $\prod_{t \in T} \mathfrak{A}_t$, as suggested in 611Ac, then $\mathcal{T}_{\mathbb{B}} = \mathcal{T}_{\mathbb{A}} \cap \prod_{t \in T} \mathfrak{B}_t$, while $\prod_{t \in T} \mathfrak{B}_t$ is an order-closed subalgebra of $\prod_{t \in T} \mathfrak{A}_t$ (315Xc).

634D Notation In the context of 634Ce, we shall be able to regard a process $\boldsymbol{u} \in L^0(\mathfrak{B})^{\mathcal{S}}$ as being fully adapted either in the structure $\mathbb{A} = (\mathfrak{A}, \bar{\mu}, T, \langle \mathfrak{A}_t \rangle_{t \in T}, \mathcal{T}_{\mathbb{A}}, \langle \mathfrak{A}_\tau \rangle_{\tau \in \mathcal{T}_{\mathbb{A}}})$ $(\mathfrak{A}, \langle \mathfrak{A}_t \rangle_{t \in T})$ or in the structure $\mathbb{B} = (\mathfrak{B}, \bar{\mu} \restriction \mathfrak{B}, T, \langle \mathfrak{B}_t \rangle_{t \in T}, \mathcal{T}_{\mathbb{B}}, \langle \mathfrak{B}_\sigma \rangle_{\sigma \in \mathcal{T}_{\mathbb{B}}})$, where $\mathfrak{B}_t = \mathfrak{B} \cap \mathfrak{A}_t$ for $t \in T$. Consequently we shall be able to test $\boldsymbol{u}$ against the definitions in this volume in two different ways. We have to check which properties are 'absolute', in the sense that $\boldsymbol{u}$ will have them in one structure iff it has them in the other, and which are not. Most of the checks are very easy, just as 634C is. While working through the list, it will save a great many words if I use abbreviated expressions of the type '$\boldsymbol{u}$ is $\mathbb{A}$-simple', '$\boldsymbol{u}$ is a $\mathbb{B}$-integrator' to mean 'interpreted in the structure $\mathbb{A}$, $\boldsymbol{u}$ is a simple process', 'interpreted in the structure $\mathbb{B}$, $\boldsymbol{u}$ is an integrator', and so forth. But we have already seen in 634C(e-ii) that if $\boldsymbol{u}$ is a process such that its domain is a sublattice of $\mathcal{T}_{\mathbb{B}}$ and its values all belong to $L^0(\mathfrak{B})$ – these properties being intrinsic to $\boldsymbol{u}$ – then $\boldsymbol{u}$ is $\mathbb{A}$-fully-adapted iff it is $\mathbb{B}$-fully-adapted, so in this context I can say simply that '$\boldsymbol{u}$ is fully adapted' with little danger of confusion. A useful number of other concepts are equally easily handled, as in the next proposition.

634E Proposition Let $\mathfrak{B}$ be a closed subalgebra of $\mathfrak{A}$, and $\mathbb{B} = (\mathfrak{B}, \bar{\mu}\restriction\mathfrak{B}, T, \langle\mathfrak{B}_t\rangle_{t\in T}, \mathcal{T}_{\mathbb{B}}, \langle\mathfrak{B}_\sigma\rangle_{\sigma\in\mathcal{T}_{\mathbb{B}}})$ the corresponding stochastic integration structure, where $\mathfrak{B}_t = \mathfrak{B}\cap\mathfrak{A}_t$ for $t \in T$. Suppose that $\mathcal{S}$ is a sublattice of $\mathcal{T}_{\mathbb{B}}$ and that $\boldsymbol{u} \in L^0(\mathfrak{B})^{\mathcal{S}}$ is fully adapted.

(a) $\boldsymbol{u}$ is $\mathbb{B}$-simple iff it is $\mathbb{A}$-simple.

(b) $\boldsymbol{u}$ is $\mathbb{B}$-(locally)-near-simple iff it is $\mathbb{A}$-(locally)-near-simple.

(c) $\boldsymbol{u}$ is $\mathbb{B}$-order-bounded iff it is $\mathbb{A}$-order-bounded.

(d) $\boldsymbol{u}$ is of $\mathbb{B}$-bounded variation iff it is of $\mathbb{A}$-bounded variation.

(e) $\boldsymbol{u}$ is $\mathbb{B}$-(locally)-moderately-oscillatory iff it is $\mathbb{A}$-(locally)-moderately-oscillatory.

(f) $\boldsymbol{u}$ is $\mathbb{A}$-jump-free iff it is $\mathbb{B}$-jump-free.

(g) If $\boldsymbol{v} \in L^0(\mathfrak{B})^{\mathcal{S}}$ is another fully adapted process, then the integral $\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}$ is defined for $(\mathfrak{B}, \bar{\mu}\restriction\mathfrak{B}, \langle\mathfrak{B}_t\rangle_{t\in T})$ iff it is defined for $(\mathfrak{A}, \bar{\mu}, \langle\mathfrak{A}_t\rangle_{t\in T})$, with the same value; that is, $\mathbb{B}\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v} = \mathbb{A}\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}$ if either is defined.

proof Really all we have to do is to look at the definitions.

(a) Whether a process $\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is simple depends only on the ordering of $\mathcal{S}$ and the values of $[\![\sigma < \tau]\!]$, $[\![\sigma \le \tau]\!]$ for σ, $\tau \in \mathcal{S}$; given that $\mathcal{S} \subseteq \mathcal{T}_{\mathbb{B}}$. 634C(b-i) and 634Cc assure us that we can calculate these either in $\mathbb{A}$ or $\mathbb{B}$ and get the same results.

(b) The definition of 'near-simple' process calls on expressions of the form $\widehat{\theta}(\boldsymbol{u}-\boldsymbol{u}') = \theta(\sup_{\sigma\in\mathcal{S}} |u_\sigma - u'_\sigma|)$ (631Ba, 615B); but the supremum here will have the same value whether interpreted in $L^0(\mathfrak{A})$ or $L^0(\mathfrak{B})$, and I noted in 634A that there is no dispute about the value of $\theta(u)$ for $u \in L^0(\mathfrak{B})$. So $\boldsymbol{u}$ is $\mathbb{B}$-near-simple iff it is $\mathbb{A}$-near-simple; applying this to $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ for $\tau \in \mathcal{S}$, we see that $\boldsymbol{u}$ is $\mathbb{B}$-locally-near-simple iff it is $\mathbb{A}$-locally-near-simple.

(c) Here we just need to remember that a subset of $L^0(\mathfrak{B})$ is order-bounded in $L^0(\mathfrak{B})$ iff it is order-bounded in $L^0(\mathfrak{A})$, as noted in 634A.

(d) Satisfaction of any of the conditions (i)-(iii) in 614J is independent of the structure we are working in.

(e) Here it seems that we need a more sophisticated approach, using both characterizations of 'moderately oscillatory'.

(i)($\boldsymbol{\alpha}$) If $\boldsymbol{u}$ is $\mathbb{B}$-moderately-oscillatory, then for every $\epsilon > 0$ there is a fully adapted process $\boldsymbol{u}' \in (L^0(\mathfrak{B}))^{\mathcal{S}}$ of bounded variation (determined in either system, see (d)) such that $\theta(\sup|\boldsymbol{u}-\boldsymbol{u}'|) \le \epsilon$, so $\boldsymbol{u}$ is $\mathbb{A}$-moderately-oscillatory.

($\boldsymbol{\beta}$) If $\boldsymbol{u}$ is $\mathbb{A}$-moderately-oscillatory, write $\hat{\mathcal{S}}_{\mathbb{A}} \subseteq \mathcal{T}_{\mathbb{A}}$ for the $\mathbb{A}$-covered envelope of $\mathcal{S}$, and $\hat{\boldsymbol{u}} = \langle\hat{u}_\sigma\rangle_{\sigma\in\hat{\mathcal{S}}_{\mathbb{A}}}$ for the corresponding extension of $\boldsymbol{u}$. By 615N, $\langle u_{\sigma_n}\rangle_{n\in\mathbb{N}}$ is convergent in $L^0(\mathfrak{A})$ for every monotonic sequence $\langle\sigma\rangle_{n\in\mathbb{N}}$ in $\mathcal{S}$; but $L^0(\mathfrak{B})$ is a topologically closed set in $L^0(\mathfrak{A})$ (367Rc), so $\langle u_{\sigma_n}\rangle_{n\in\mathbb{N}}$ is convergent in $L^0(\mathfrak{B})$ for every monotonic sequence $\langle\sigma_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{S}$, and $\boldsymbol{u}$ is $\mathbb{B}$-moderately-oscillatory, by 615N in the other direction.

(ii) Now

$$\begin{aligned} &\boldsymbol{u} \text{ is } \mathbb{B}\text{-locally-moderately-oscillatory}\\ &\qquad\iff \boldsymbol{u}\restriction\mathcal{S}\wedge\tau \text{ is } \mathbb{B}\text{-moderately-oscillatory for every } \tau\in\mathcal{S}\\ &\qquad\iff \boldsymbol{u}\restriction\mathcal{S}\wedge\tau \text{ is } \mathbb{A}\text{-moderately-oscillatory for every } \tau\in\mathcal{S}\\ &\qquad\iff \boldsymbol{u} \text{ is } \mathbb{A}\text{-locally-moderately-oscillatory.} \end{aligned}$$

(f) See 618B. For $I \in \mathcal{I}(\mathcal{S})$, $\operatorname{Oscll}_I(\boldsymbol{u})$ and $\operatorname{Oscll}^*_I(\boldsymbol{u})$ are calculated in terms of suprema in L^0; again because $L^0(\mathfrak{B})$ is order-closed in $L^0(\mathfrak{A})$, and $\Delta_e(\mathbf{1}, |d\boldsymbol{u}|) \in L^0(\mathfrak{B})$ whenever e is a stopping-time interval with endpoints in $\mathcal{S}$, we will get the same values for $\operatorname{Oscll}_I(\boldsymbol{u})$, $\operatorname{Oscll}^*_I(\boldsymbol{u}) = \sup_{I\subseteq J\in\mathcal{I}(\mathcal{S})} \operatorname{Oscll}_J(\boldsymbol{u})$ and $\operatorname{Oscll}(\boldsymbol{u}) = \inf_{I\in\mathcal{I}(\mathcal{S})} \operatorname{Oscll}^*_I(\boldsymbol{u})$ on either interpretation.

(g) Again because $L^0(\mathfrak{B})$ is a topologically closed set in $L^0(\mathfrak{A})$, convergence in $L^0(\mathfrak{A})$ of a filter on $L^0(\mathfrak{B})$ implies convergence in $L^0(\mathfrak{B})$. And for $I \in \mathcal{I}(\mathcal{S})$, the evaluation of $S_I(\boldsymbol{u}, d\boldsymbol{v}) \in L^0(\mathfrak{B})$ is the same in either structure. So

$$\mathbb{B}\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v} = \lim_{I\uparrow\mathcal{I}(\mathcal{S})} S_I(\boldsymbol{u}, d\boldsymbol{v}) = \mathbb{A}\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$$

if either form of the integral is defined.

Remarks Thus I shall be able from now on, in the context of processes in $L^0(\mathfrak{B})^{\mathcal{S}}$ where $\mathcal{S} \subseteq \mathcal{T}_{\mathbb{B}}$, to use the phrases 'simple', 'near-simple', 'order-bounded', 'bounded variation', 'moderately oscillatory', 'jump-free', and the formula $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$, without specifying whether I am thinking in the structure $\mathbb{A}$ or the structure $\mathbb{B}$.

634F Relative independence You may have noticed an omission in 634E above. There is no mention of 'martingales'. In general, the shift from an algebra $\mathfrak{A}$ to a subalgebra $\mathfrak{B}$ need not respect conditional expectations in the right way. In many cases, however (for instance, if $\mathfrak{A}$ is based on a product probability measure, and $\mathfrak{B}$ on one of the factors), we do have strong connexions between martingales for $(\mathfrak{A}, \bar\mu, \langle\mathfrak{A}_t\rangle_{t\in T})$ and martingales for $(\mathfrak{B}, \bar\mu\restriction\mathfrak{B}, \langle\mathfrak{B}_t\rangle_{t\in T})$. The key concept is that of 'relative independence', already used in §628. In that section I was relying heavily on the work of §458, so did not separate the more elementary ideas; perhaps it will help if I set them out now.

Definitions (a) (see 458La) If $\mathfrak{B}$, $\mathfrak{C}$ and $\mathfrak{D}$ are closed subalgebras of $\mathfrak{A}$, I say that $\mathfrak{B}$ and $\mathfrak{C}$ are **relatively (stochastically) independent** over $\mathfrak{D}$ if $P_{\mathfrak{D}}\chi(b \cap c) = P_{\mathfrak{D}}\chi b \times P_{\mathfrak{D}}\chi c$ for all $b \in \mathfrak{B}$ and $c \in \mathfrak{C}$.

(b) I will say that a closed subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ is **coordinated** with the filtration $\langle\mathfrak{A}_t\rangle_{t\in T}$ if $\mathfrak{B}$ and $\mathfrak{A}_t$ are relatively independent over $\mathfrak{B}_t = \mathfrak{B} \cap \mathfrak{A}_t$ for every $t \in T$.

634G Proposition If $\mathfrak{B}$, $\mathfrak{C}$ are closed subalgebras of $\mathfrak{A}$, the following are equiveridical:
(i) $\mathfrak{B}$ and $\mathfrak{C}$ are relatively independent over $\mathfrak{B} \cap \mathfrak{C}$;
(ii) $P_{\mathfrak{B}\cap\mathfrak{C}}(u \times v) = P_{\mathfrak{B}\cap\mathfrak{C}}u \times P_{\mathfrak{B}\cap\mathfrak{C}}v$ for all $u \in L^\infty(\mathfrak{B})$, $v \in L^\infty(\mathfrak{C})$;
(iii) $P_{\mathfrak{B}}P_{\mathfrak{C}} = P_{\mathfrak{B}\cap\mathfrak{C}}$;
(iv) $P_{\mathfrak{B}}P_{\mathfrak{C}} = P_{\mathfrak{C}}P_{\mathfrak{B}}$;
(v) $P_{\mathfrak{B}}u \in L^0(\mathfrak{C})$ whenever $u \in L^1_{\bar\mu} \cap L^0(\mathfrak{C})$.

proof 458M.

634H Lemma Let $\mathfrak{B}$ be a closed subalgebra of $\mathfrak{A}$ which is coordinated with $\langle\mathfrak{A}_t\rangle_{t\in T}$, and $\mathbb{B} = (\mathfrak{B}, \bar\mu\restriction\mathfrak{B}, T, \langle\mathfrak{B}_t\rangle_{t\in T}, \mathcal{T}_{\mathbb{B}}, \langle\mathfrak{B}_\sigma\rangle_{\sigma\in\mathcal{T}_{\mathbb{B}}})$ the corresponding stochastic integration structure, where $\mathfrak{B}_t = \mathfrak{B} \cap \mathfrak{A}_t$ for $t \in T$. Then $\mathfrak{B}$ and $\mathfrak{A}_\sigma$ are relatively independent over $\mathfrak{B}_\sigma$ for every $\sigma \in \mathcal{T}_{\mathbb{B}}$.

proof (a) Writing $\check t$ for the constant stopping time at t (611A(b-ii)), we have $\mathfrak{A}_{\check t} = \mathfrak{A}_t$ (611Hb); by 634G(v), the hypothesis '$\mathfrak{B}$ is coordinated with $\langle\mathfrak{A}_t\rangle_{t\in T}$' amounts to saying that

$$P_{\check t}u \in L^0(\mathfrak{B}) \text{ for every } t \in T \text{ and } u \in L^1_{\bar\mu} \cap L^0(\mathfrak{B}).$$

(b) For $t \in T$ and $c \in \mathfrak{B}_t$, define $\check tc$ by saying that

$$[\![\check tc > s]\!] = 1 \text{ if } s < t,$$
$$= c \text{ if } s \ge t.$$

It is easy to check that (because $c \in \mathfrak{B}_t$) the conditions of 611B(b-i) are satisfied, so that $\check tc \in \mathcal{T}_{\mathbb{B}}$. We see that

$$\mathfrak{A}_{\check tc} = \{a : a \in \mathfrak{A},\ a \setminus [\![\check tc > s]\!] \in \mathfrak{A}_s \text{ for every } s \in T\}$$
$$= \{a : a \setminus c \in \mathfrak{A}_s \text{ for every } s \ge t\} = \{a : a \setminus c \in \mathfrak{A}_t\}.$$

Now $\check t \le \check tc$ and

$$[\![\check t < \check tc]\!] = \sup_{s\in T} [\![\check tc > s]\!] \setminus [\![\check t > s]\!] = c.$$

So $[\![\check tc = \check t]\!] = 1 \setminus c$. If $u \in L^1_{\bar\mu}$,

$$P_{\check tc}u = P_{\check tc}(u \times \chi c) + P_{\check tc}(u \times \chi(1 \setminus c)) = u \times \chi c + (P_{\check tc}u) \times \chi(1 \setminus c)$$

(because $u \times \chi c \in L^0(\mathfrak{A}_{\check tc})$ and $1 \setminus c \in \mathfrak{A}_{\check tc}$)

$$= u \times \chi c + (P_{\check{t}} u) \times \chi(1 \setminus c)$$

by 622Bb. It follows that if $u \in L^1_{\bar\mu} \cap L^0(\mathfrak{B})$, then $P_{\check{t}c} u \in L^0(\mathfrak{B})$.

(c) For $\sigma \in \mathcal{T}_{\mathbb{B}}$, set

$$C_\sigma = \{\check{t}c : t \in T,\, c = [\![\sigma > t]\!]\},$$

$$D_\sigma = \{\tau_0 \wedge \ldots \wedge \tau_n : \tau_i \in C_\sigma \text{ for } i \le n\}.$$

Then

$$\mathfrak{A}_\sigma = \bigcap_{t\in T}\{a : a \setminus [\![\sigma > t]\!] \in \mathfrak{A}_t\} = \bigcap_{\tau \in C_\sigma} \mathfrak{A}_\tau = \bigcap_{\tau\in D_\sigma} \mathfrak{A}_\tau$$

by 611Eb. We know from (b) that $P_\tau u \in L^0(\mathfrak{B}) \cap L^1_{\bar\mu}$ for every $\tau \in C_\sigma$; now if $\tau_0, \ldots, \tau_n \in C_\sigma$ and $\tau = \tau \wedge \ldots \wedge \tau_n$,

$$P_\tau u = P_{\tau_0} \ldots P_{\tau_n} u \in L^0(\mathfrak{B}) \cap L^1_{\bar\mu}$$

for every $u \in L^0(\mathfrak{B}) \cap L^1_{\bar\mu}$, by 622Ba. Thus $P_\tau u \in L^0(\mathfrak{B})$ whenever $\tau \in D_\sigma$ and $u \in L^0(\mathfrak{B}) \cap L^1_{\bar\mu}$.

Now observe that $\{\mathfrak{A}_\tau : \tau \in D_\sigma\}$ is downwards-directed and has intersection $\mathfrak{A}_\sigma$. So $P_\sigma u = \lim_{\tau\downarrow D_\sigma} P_\tau u$ for every $u \in L^1_{\bar\mu}$, by 621C(g-i). In 621Cg I called this a a $\| \|_1$-limit, but for our purposes here it is enough to think of it as a limit for the topology of convergence in measure, because $L^0(\mathfrak{B})$ is closed for the topology of convergence in measure, and all we need to know is that if $u \in L^0(\mathfrak{B}) \cap L^1_{\bar\mu}$ then $P_\sigma u \in L^0(\mathfrak{B})$. Thus $\mathfrak{A}_\sigma$ and $\mathfrak{B}$ satisfy 634G(v), and are relatively independent over $\mathfrak{A}_\sigma \cap \mathfrak{B} = \mathfrak{B}_\sigma$ (634Cd).

634I Theorem Let $\mathfrak{B}$ be a closed subalgebra of $\mathfrak{A}$ which is coordinated with $\langle\mathfrak{A}_t\rangle_{t\in T}$, and $\mathbb{B} = (\mathfrak{B}, \bar\mu{\restriction}\mathfrak{B}, T, \langle\mathfrak{B}_t\rangle_{t\in T}, \mathcal{T}_{\mathbb{B}}, \langle\mathfrak{B}_\sigma\rangle_{\sigma\in\mathcal{T}_{\mathbb{B}}})$ the corresponding stochastic integration structure, where $\mathfrak{B}_t = \mathfrak{B} \cap \mathfrak{A}_t$ for $t \in T$. Let $\mathcal{S}$ be a sublattice of $\mathcal{T}_{\mathbb{B}}$ and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a fully adapted process such that $u_\sigma \in L^0(\mathfrak{B})$ for every $\sigma \in \mathcal{S}$.

(a) $\boldsymbol{u}$ is a (local) $\mathbb{B}$-martingale iff it is a (local) $\mathbb{A}$-martingale.

(b) $\boldsymbol{u}$ is a (local) $\mathbb{B}$-integrator iff it is a (local) $\mathbb{A}$-integrator, and in this case its $\mathbb{B}$-quadratic variation is the same as its $\mathbb{A}$-quadratic variation.

proof (a) For $\sigma \in \mathcal{T}_{\mathbb{B}}$, write $Q_\sigma : L^1(\mathfrak{B}, \bar\mu{\restriction}\mathfrak{B}) \to L^1(\mathfrak{B}, \bar\mu{\restriction}\mathfrak{B})$ for the conditional expectation associated with the closed subalgebra $\mathfrak{B}_\sigma$. Suppose that $u \in L^0(\mathfrak{B}) \cap L^1_{\bar\mu} = L^1(\mathfrak{B}, \bar\mu{\restriction}\mathfrak{B})$. Then $\mathfrak{B}$ and $\mathfrak{A}_\sigma$ are relatively independent over $\mathfrak{B}_\sigma = \mathfrak{A} \cap \mathfrak{B}_\sigma$ (634H), so $P_\sigma u \in L^0(\mathfrak{B}) \cap L^0(\mathfrak{A}_\sigma) = L^0(\mathfrak{B}_\sigma)$, and we have $\mathbb{E}(P_\sigma u \times \chi b) = \mathbb{E}(u \times \chi b)$ for every $b \in \mathfrak{A}_\sigma$ and therefore for every $b \in \mathfrak{B}_\sigma$. But this means that $P_\sigma u$ satisfies the property defining $Q_\sigma u$, and $P_\sigma u = Q_\sigma u$.

Now we see that

$$\begin{aligned}
&\boldsymbol{u} \text{ is a } \mathfrak{B}\text{-martingale}\\
&\qquad\iff Q_\sigma u_\tau = u_\sigma \text{ whenever } \sigma \le \tau \text{ in } \mathcal{S}\\
&\qquad\iff P_\sigma u_\tau = u_\sigma \text{ whenever } \sigma \le \tau \text{ in } \mathcal{S}\\
&\qquad\iff \boldsymbol{u} \text{ is a } \mathfrak{A}\text{-martingale.}
\end{aligned}$$

Observe next that a sublattice $\mathcal{S}'$ of $\mathcal{S}$ is a $\mathbb{B}$-covering ideal of $\mathcal{S}$ iff it is an $\mathbb{A}$-covering ideal of $\mathcal{S}$, by 634Cc. So we have

$$\begin{aligned}
&\boldsymbol{u} \text{ is a } \mathfrak{B}\text{-local martingale}\\
&\qquad\iff \text{there is a covering ideal } \mathcal{S}' \text{ of } \mathcal{S}\\
&\qquad\qquad\qquad \text{such that } \boldsymbol{u}{\restriction}\mathcal{S}' \text{ is a } \mathfrak{B}\text{-martingale}\\
&\qquad\iff \text{there is a covering ideal } \mathcal{S}' \text{ of } \mathcal{S}\\
&\qquad\qquad\qquad \text{such that } \boldsymbol{u}{\restriction}\mathcal{S}' \text{ is a } \mathfrak{A}\text{-martingale}\\
&\qquad\iff \boldsymbol{u} \text{ is an } \mathfrak{A}\text{-local martingale.}
\end{aligned}$$

(b)(i)(α) Suppose that $\boldsymbol{u}$ is a $\mathbb{B}$-integrator. Then there is a $\bar\nu_0$ such that $(\mathfrak{B},\bar\nu_0)$ is a probability algebra and $\boldsymbol{u}$ is a $(\mathfrak{B},\bar\nu_0)$-strong integrator (627Ld). Let $z \in L^1_{\bar\mu} \cap L^0(\mathfrak{B})$ be such that $\bar\nu_0 b = \mathbb{E}_{\bar\mu}(z \times \chi b)$ for every $b \in \mathfrak{B}$ (625B; note that in this sentence we do not need to distinguish between $\mathbb{E}_{\bar\mu}$ and $\mathbb{E}_{\bar\mu\restriction\mathfrak{B}}$). Set $\bar\nu a = \mathbb{E}_{\bar\mu}(z \times \chi a)$ for every $a \in \mathfrak{A}$; because $[\![z > 0]\!] = 1$, $(\mathfrak{A},\bar\nu)$ is a probability algebra.

If $\tau \in \mathcal{T}_{\mathbb{B}}$, $\mathfrak{B}$ and $\mathfrak{A}_\tau$ are relatively independent over $\mathfrak{B} \cap \mathfrak{A}_\tau$ for $\bar\nu$ as well as for $\bar\mu$. **P** Let $Q_\tau : L^1_{\bar\nu} \to L^1_{\bar\nu} \cap L^0(\mathfrak{B}_\tau)$ be the conditional expectation associated with $\mathfrak{B}_\tau$ for $\bar\nu$. If $w \in L^1_{\bar\nu} \cap L^0(\mathfrak{B})$, then

$$Q_\tau w = P_\tau(w \times z) \times \frac{1}{P_\tau z}$$

(625B(b-i); of course $[\![P_\tau z > 0]\!] = 1$ by 625B(a-vi), so there is no problem with the reciprocal). But $w \times z \in L^1_{\bar\mu} \cap L^0(\mathfrak{B})$, because $z \in L^0(\mathfrak{B})$, so $P_\tau(w \times z) \in L^0(\mathfrak{B})$, using 634H. Similarly, $P_\tau z \in L^0(\mathfrak{B})$, so $Q_\tau w \in L^0(\mathfrak{B})$. By 634G(v), $\mathfrak{B}$ and $\mathfrak{A}_\tau$ are relatively independent over $\mathfrak{B} \cap \mathfrak{A}_\tau$ for $\bar\nu$. **Q**

Now suppose that $\epsilon > 0$. Then we have processes $\boldsymbol{w}$, $\boldsymbol{w}'$ with domain $\mathcal{S}$, $\mathbb{B}$-fully adapted, such that $\bar\nu[\![\boldsymbol{u} \ne \boldsymbol{w} + \boldsymbol{w}']\!] \ne 0$, $\boldsymbol{w}$ is a $(\mathfrak{B},\bar\nu_0)$-uniformly integrable martingale, and $\boldsymbol{w}'$ is of $\mathbb{B}$-bounded variation. By (a) above, $\boldsymbol{w}$ is an $(\mathfrak{A},\bar\nu)$-martingale, and $\boldsymbol{w}'$ is of $\mathbb{A}$-bounded variation, as noted in 634Ed. Also, of course, $\boldsymbol{w}$ is still uniformly integrable in $L^1_{\bar\nu}$. As ϵ is arbitrary, $\boldsymbol{u}$ is an $(\mathfrak{A},\bar\nu)$-strong integrator. It follows at once that $\boldsymbol{u}$ is an $\mathbb{A}$-integrator.

(β) If $\boldsymbol{u}$ is a $\mathbb{B}$-local-integrator, then we can apply (i) to see that $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ is an $\mathbb{A}$-integrator whenever $\tau \in \mathcal{S}$, so that $\boldsymbol{u}$ is an $\mathbb{A}$-local-integrator.

(ii) In the other direction, if $\boldsymbol{u}$ is an $\mathbb{A}$-integrator, then it is a $\mathbb{B}$-integrator. **P** Going back to the definition of 'integrator' in 616Fc, we see that $z \in \mathbb{B}Q_{\mathcal{S}}(\boldsymbol{u})$ there are $I \in \mathcal{I}(\mathcal{S})$ and $\boldsymbol{w} \in L^0(\mathfrak{B})^I$ such that $\boldsymbol{w}$ is $\mathbb{B}$-fully-adapted, $\|\boldsymbol{w}\|_\infty \le 1$ and $z = S_I(\boldsymbol{w}, d\boldsymbol{u})$. Now, of course, $\boldsymbol{w} \in L^0(\mathfrak{A})^I$ is $\mathbb{A}$-fully-adapted, so $z \in \mathbb{A}Q_{\mathcal{S}}(\boldsymbol{u})$. Thus $\mathbb{B}Q_{\mathcal{S}}(\boldsymbol{u}) \subseteq \mathbb{A}Q_{\mathcal{S}}(\boldsymbol{u}) \cap L^0(\mathfrak{B})$ is topologically bounded in $L^0(\mathfrak{A})$ and therefore also in $L^0(\mathfrak{B})$, since the linear space topology of $L^0(\mathfrak{B})$ is that induced by the linear space topology of $L^0(\mathfrak{A})$. So $\boldsymbol{u}$ is a $\mathbb{B}$-integrator. **Q**

Just as in (i-β) above, it follows at once that if $\boldsymbol{u}$ is an $\mathbb{A}$-local-integrator it is a $\mathbb{B}$-local-integrator.

(iii) If $\boldsymbol{u}$ is a local integrator (either in $\mathbb{A}$ or in $\mathbb{B}$), set $u_\downarrow = \lim_{\sigma\downarrow\mathcal{S}} u_\sigma$. For any $\tau \in \mathcal{S}$,

$$u_\tau^2 - u_\downarrow^2 - 2\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}$$

is the same whether calculated in $\mathbb{A}$ or $\mathbb{B}$, as noted in 634Eg. By 617Ka, the quadratic variation $\langle u^*_\tau\rangle_{\tau\in\mathcal{S}}$ of $\boldsymbol{u}$ is the same in either structure.

634J One of the most important operations in probability theory is the use of product measures to simultaneously represent independent random variables. In order to do the same for the stochastic processes we have been studying here, we have to pay attention to filtrations. The next lemma provides one of the tools we need.

Lemma Let (Ω,Σ,μ) and (Ω',Σ',μ') be probability spaces, and $(\Xi,\le)$ a non-empty downwards-directed partially ordered set. Let $\langle\Sigma_\xi\rangle_{\xi\in\Xi}$ and $\langle\Sigma'_\xi\rangle_{\xi\in\Xi}$ be families of σ-subalgebras of Σ, Σ' respectively such that $\Sigma_\xi \subseteq \Sigma_\eta$ and $\Sigma'_\xi \subseteq \Sigma'_\eta$ whenever $\xi \le \eta$, while every Σ_ξ contains every μ-negligible set and every Σ'_ξ contains every μ'-negligible set. Write $\mathrm{T} = \bigcap_{\xi\in\Xi}\Sigma_\xi$ and $\mathrm{T}' = \bigcap_{\xi\in\Xi}\Sigma'_\xi$. Let λ be the product probability measure on $\Omega\times\Omega'$, and Λ its domain.

(a) Suppose that $W \in \Lambda$ is such that for every $\xi \in \Xi$ there is a $W_\xi \in \Sigma_\xi\widehat{\otimes}\Sigma'$ such that $W\triangle W_\xi$ is λ-negligible. Then there is a $W' \in \mathrm{T}\widehat{\otimes}\Sigma'$ such that $W\triangle W'$ is λ-negligible.

(b) Suppose that $W \in \Lambda$ is such that for every $\xi \in \Xi$ there is a $W_\xi \in \Sigma_\xi\widehat{\otimes}\Sigma'_\xi$ such that $W\triangle W_\xi$ is λ-negligible. Then there is a $W' \in \mathrm{T}\widehat{\otimes}\mathrm{T}'$ such that $W\triangle W'$ is λ-negligible.

(c) Suppose that $W \in \bigcap_{\xi\in\Xi}\Sigma_\xi\widehat{\otimes}\Sigma'$, $E \in \Sigma$ and $\epsilon \ge 0$ are such that $\lambda(W\triangle(E\times\Omega')) \le \epsilon$. Then there is an $E_1 \in \mathrm{T}$ such that $\lambda(W\triangle(E_1\times\Omega')) \le 3\epsilon$.

proof (a) Since there is certainly a member of $\Sigma\widehat{\otimes}\Sigma'$ differing from W by a λ-negligible set (251Ib), we can suppose that W itself belongs to $\Sigma\widehat{\otimes}\Sigma'$. Let $\mathfrak{A}'$ be the measure algebra of (Ω',Σ',μ'). Then 418Ta tells us that we have a function $f : \Omega \to \mathfrak{A}'$ defined by setting $f(\omega) = W[\{\omega\}]^\bullet$ for $\omega \in \Omega$, and that $f[\Omega]$ is a separable subset of $\mathfrak{A}'$ for the measure-algebra topology of $\mathfrak{A}'$ and f is Σ-measurable for this topology.

Similarly, for any $\xi \in \Xi$, we have a Σ_ξ-measurable $f_\xi : \Omega \to \mathfrak{A}'$ such that $f_\xi(\omega) = W_\xi[\{\omega\}]^\bullet$ for every ω. But since $W \triangle W_\xi$ is λ-negligible, $W[\{\omega\}] \triangle W_\xi[\{\omega\}]$ is μ'-negligible for μ-almost every ω, by Fubini's theorem (252D). So $f =_{\text{a.e.}} f_\xi$. As we are supposing that every μ-negligible set belongs to Σ_ξ, f is Σ_ξ-measurable.

This is true for every $\xi \in \Xi$, so f is actually T-measurable. But now we can use 418Ta in the other direction to see that there is a $W' \in \mathrm{T} \triangle \Sigma'$ such that $W'[\{\omega\}]^\bullet = f(\omega)$ for every $\omega \in \Omega$. By Fubini's theorem again, $\lambda(W \triangle W') = 0$, as required.

(b) For each $\eta \in \Xi$, we can apply (a) to (Ω, Σ, μ), $(\Omega', \Sigma'_\eta, \mu'_\eta \restriction \Sigma'_\eta)$ and $\langle W_\xi \rangle_{\xi \in \Xi, \xi \le \eta}$ to see that there is a $W'_\eta \in \mathrm{T} \widehat{\otimes} \Sigma'_\eta$ such that $\lambda_\eta(W \triangle W'_\eta) = 0$, where λ_η is the product of μ and μ'_η. Of course it follows that $\lambda(W \triangle W'_\eta) = 0$ and also that $\tilde{\lambda}(W'_\eta \triangle W'_\zeta) = 0$ whenever η, $\zeta \in \Xi$, where $\tilde{\lambda}$ is the product of $\mu \restriction \mathrm{T}$ and μ'. Fixing $\zeta \in \Xi$, we can apply (a) again to (Ω', Σ', μ'), $(\Omega, \mathrm{T}, \mu \restriction \mathrm{T})$ and $\langle W'_\eta \rangle_{\eta \in \Xi}$ to see that there is a $W' \in \mathrm{T} \widehat{\otimes} \mathrm{T}'$ such that $\tilde{\lambda}(W' \triangle W'_\zeta) = 0$, in which case $\lambda(W' \triangle W)$ will be zero.

(c) Try $E_1 = \{\omega : \omega \in \Omega, \mu' W[\{\omega\}] \ge \frac{1}{2}\}$. By 252P, $E_1 \in \Sigma_\xi$ for every $\xi \in \Xi$, so $E_1 \in \mathrm{T}$. We have

$$\frac{1}{2}\mu(E_1 \setminus E) \le \int \mu'((W \setminus (E \times \Omega'))[\{\omega\}]\mu(d\omega) = \lambda(W \setminus (E \times \Omega')),$$

$$\frac{1}{2}\mu(E \setminus E_1) \le \int \mu'((E \times \Omega') \setminus W)[\{\omega\}]\mu(d\omega) = \lambda((E \times \Omega') \setminus W)$$

so

$$\frac{1}{2}\mu(E_1 \triangle E) \le \lambda(W \triangle (E \times \Omega')) \le \epsilon$$

and

$$\lambda(W \triangle (E_1 \times \Omega')) \le \mu(E_1 \triangle E) + \lambda(W \triangle (E \times \Omega')) \le 3\epsilon.$$

634K Theorem Let $\langle \mathfrak{A}_i \rangle_{i \in I}$ be a stochastically independent family of closed subalgebras of $\mathfrak{A}$ (see 325L). Suppose that $(\Xi, \le)$ is a non-empty downwards-directed partially ordered set and that for each $i \in I$ we have a non-decreasing family $\langle \mathfrak{B}_{i\xi} \rangle_{\xi \in \Xi}$ of closed subalgebras of $\mathfrak{A}_i$ with intersection $\mathfrak{B}_i$. Set $\mathfrak{D} = \bigvee_{i \in I} \mathfrak{B}_i$, and for $\xi \in \Xi$ set $\mathfrak{D}_\xi = \bigvee_{i \in I} \mathfrak{B}_{i\xi}$. Then $\mathfrak{D} = \bigcap_{\xi \in \Xi} \mathfrak{D}_\xi$.

proof (a) If I is empty or a singleton, the result is trivial. Note also that for any I we surely have $\mathfrak{D} \subseteq \bigcap_{\xi \in \Xi} \mathfrak{D}_\xi$, so it will be enough to show that $\bigcap_{\xi \in \Xi} \mathfrak{D}_\xi \subseteq \mathfrak{D}$.

(b) Suppose that $I = \{i, j\}$ has two members. Let (Ω, Σ, μ) and (Ω', Σ', μ') be probability spaces with measure algebras which can be identified with $(\mathfrak{A}_i, \bar{\mu} \restriction \mathfrak{A}_i)$, $(\mathfrak{A}_j, \bar{\mu} \restriction \mathfrak{A}_j)$ respectively (321J); set $\Sigma_\xi = \{E : E \in \Sigma, E^\bullet \in \mathfrak{B}_{i\xi}\}$, $\Sigma'_\xi = \{F : E \in \Sigma', F^\bullet \in \mathfrak{B}_{j\xi}\}$ for $\xi \in \Xi$. Note that every Σ_ξ and Σ'_ξ is a σ-algebra containing all negligible sets, and that $\langle \Sigma_\xi \rangle_{\xi \in \Xi}$ and $\langle \Sigma'_\xi \rangle_{\xi \in \Xi}$ are non-decreasing.

We can identify the measure algebra $(\mathfrak{C}, \bar{\lambda})$ of the product $(\Omega \times \Omega', \Lambda, \lambda)$ with the probability algebra free product of $(\mathfrak{A}_i, \bar{\mu} \restriction \mathfrak{A}_i)$ and $(\mathfrak{A}_j, \bar{\mu} \restriction \mathfrak{A}_j)$ (325D or 325I), which in turn embeds naturally in $\mathfrak{A}$ (325J, 325L); write $\phi : \mathfrak{C} \to \mathfrak{A}$ for the embedding. If $E \in \Sigma$ and $F \in \Sigma'$, then $\phi((E \times F)^\bullet) = E^\bullet \cap F^\bullet$. Now, for $\xi \in \Xi$, $\phi[\{W^\bullet : W \in \Sigma_\xi \widehat{\otimes} \Sigma'_\xi\}]$ is the σ-subalgebra of $\mathfrak{A}$ generated by $\mathfrak{B}_{i\xi} \cup \mathfrak{B}_{j\xi}$, that is, $\mathfrak{D}_\xi$. So if $d \in \bigcap_{\xi \in \Xi} \mathfrak{D}_\xi$, we have for each $\xi \in \Xi$ a $W_\xi \in \Sigma_\xi \widehat{\otimes} \Sigma'_\xi$ such that $\phi(W_\xi^\bullet) = d$. It follows that $\lambda(W_\xi \triangle W_\eta) = 0$ for all ξ, $\eta \in \Xi$. By 634Jb, there is a $W \in \mathrm{T} \widehat{\otimes} \mathrm{T}'$ such that $\lambda(W_\xi \triangle W) = 0$ for every $\xi \in \Xi$, that is, $\phi W^\bullet = d$, where $\mathrm{T} = \bigcap_{\xi \in \Xi} \Sigma_\xi$ and $\mathrm{T}' = \bigcap_{\xi \in \Xi} \Sigma'_\xi$. But now $\phi W^\bullet$ belongs to the closed subalgebra of $\mathfrak{A}$ generated by $\{E^\bullet : E \in \mathrm{T}\} \cup \{F^\bullet : F \in \mathrm{T}'\} = \mathfrak{B}_i \cup \mathfrak{B}_j$, which is $\mathfrak{D}$. As d is arbitrary, $\bigcap_{\xi \in \Xi} \mathfrak{D}_\xi \subseteq \mathfrak{D}$, as required.

(c) Suppose that I is finite. Then $\mathfrak{D} = \bigcap_{\xi \in \Xi} \mathfrak{D}_\xi$. **P** Induce on $\#(I)$. I have already dealt with the cases $\#(I) \le 2$. For the inductive step to $\#(I) \ge 3$, take any $j \in I$ and set $I' = I \setminus \{j\}$. Set $\mathfrak{A}' = \bigvee_{i \in I'} \mathfrak{A}_i$; then $\mathfrak{A}'$ and $\mathfrak{A}_j$ are independent, as in 272K or 458Le.

For $\xi \in \Xi$, set $\mathfrak{D}'_\xi = \bigvee_{i \in I'} \mathfrak{B}_{i\xi}$. By the inductive hypothesis, $\mathfrak{D}' = \bigvee_{i \in I'} \mathfrak{B}_i$ is $\bigcap_{\xi \in \Xi} \mathfrak{D}'_\xi$. Now, of course, $\langle \mathfrak{D}'_\xi \rangle_{\xi \in \Xi}$ is non-decreasing, and $\mathfrak{D}_\xi = \mathfrak{B}_{j\xi} \vee \mathfrak{D}'_\xi$ for each ξ, while $\mathfrak{D} = \mathfrak{B}_j \vee \mathfrak{D}'$. So (b) tells us that $\mathfrak{D} = \bigcap_{\xi \in \Xi} \mathfrak{D}_\xi$. Thus the induction proceeds. **Q**

(d) Now consider the case of arbitrary I. Suppose that $d \in \bigcap_{\xi\in\Xi}\mathfrak{D}_\xi$ and $0 < \epsilon \le \frac{1}{2}$. Because the topological closure of the subalgebra $\bigcup_{J\subseteq I \text{ is finite}}\bigvee_{i\in J}\mathfrak{A}_i$ is $\bigvee_{i\in I}\mathfrak{A}_i$ (323J), which contains d, there are a finite $J\subseteq I$ and an element e of $\bigvee_{i\in J}\mathfrak{A}_i$ such that $\bar\lambda(d\,\triangle\,e)\le\epsilon$. This time, observe that $\tilde{\mathfrak{A}} = \bigvee_{i\in J}\mathfrak{A}_i$ and $\tilde{\mathfrak{A}}' = \bigvee_{i\in I\setminus J}\mathfrak{A}_i$ are independent. Setting $\tilde{\mathfrak{B}}_\xi = \bigvee_{i\in J}\mathfrak{B}_{i\xi}$ for $\xi\in\Xi$, we see from (c) that $\bigcap_{\xi\in\Xi}\tilde{\mathfrak{B}}_\xi = \bigvee_{i\in J}\mathfrak{D}_i$.

Now, following the pattern of (b), take probability spaces (Ω,Σ,μ) and (Ω',Σ',μ') with measure algebras isomorphic to $\tilde{\mathfrak{A}}$ and $\tilde{\mathfrak{A}}'$ respectively, and set $\Sigma_\xi = \{E : E^\bullet\in\tilde{\mathfrak{B}}_\xi\}$ for $\xi\in\Xi$. Translating 634Jc into terms of the measure algebras, we see that there is an $e_1\in\bigvee_{i\in J}\mathfrak{D}_i$ such that $\bar\mu(d\,\triangle\,e_1)\le 3\epsilon$. Of course $e_1\in\mathfrak{D}$.

As ϵ is arbitrary, d belongs to the closure of $\mathfrak{D}$ for the measure-algebra topology; but $\mathfrak{D}$ is topologically closed, so $d\in\mathfrak{D}$. Thus we have the required result in this case also, and the proof is complete.

634L Theorem Let $\langle\mathfrak{B}_i\rangle_{i\in I}$ be a stochastically independent family of closed subalgebras of $\mathfrak{A}$. Suppose that for each $i\in I$ we have a filtration $\langle\mathfrak{B}_{it}\rangle_{t\in T}$ of closed subalgebras of $\mathfrak{B}_i$. For each $t\in T$ set $\mathfrak{C}_t = \bigvee_{i\in I}\mathfrak{B}_{it}$.

(a) $\langle\mathfrak{C}_t\rangle_{t\in T}$ is a filtration.

(b) For $i\in I$ and $t\in T$, $\mathfrak{B}_i\cap\mathfrak{C}_t = \mathfrak{B}_{it}$ and $\mathfrak{B}_i$ and $\mathfrak{C}_t$ are relatively independent over $\mathfrak{B}_{it}$.

(c) If $\langle\mathfrak{B}_{it}\rangle_{t\in T}$ is right-continuous for every $i\in I$, then $\langle\mathfrak{C}_t\rangle_{t\in T}$ is right-continuous.

proof (a) Immediate from the definitions of 'filtration' and '$\bigvee$'.

(b) Of course $\mathfrak{B}_i\cap\mathfrak{C}_t\supseteq\mathfrak{B}_{it}$. Next, $\mathfrak{B}_i$ and $\bigvee_{j\in I\setminus\{i\}}\mathfrak{B}_j$ are independent, that is, they are relatively independent over $\{0,1\}$; by 458Ld, they are relatively independent over $\mathfrak{B}_{it}\subseteq\mathfrak{B}_i$. *A fortiori*, $\mathfrak{B}_i$ and $\bigvee_{j\ne i}\mathfrak{B}_{jt}$ are relatively independent over $\mathfrak{B}_{it}$, and therefore $\mathfrak{B}_i$ and $\mathfrak{B}_{it}\vee\bigvee_{j\ne i}\mathfrak{B}_{jt} = \mathfrak{C}_t$ are relatively independent over $\mathfrak{B}_{it}$ (458Ld again). But this means that $\mathfrak{B}_i\cap\mathfrak{C}_t$ is relatively independent of itself over $\mathfrak{B}_{it}$, and must be equal to $\mathfrak{B}_{it}$.

(c) If $t\in T$ is not isolated on the right, apply Theorem 634K with $\Xi = \{s : s > t\}$ to see that $\mathfrak{C}_t = \bigcap_{s>t}\mathfrak{C}_s$.

634M Corollary Suppose that $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ is a family of probability algebras, and that $\langle\mathfrak{A}_{it}\rangle_{t\in T}$ is a filtration in $\mathfrak{A}_i$ for each i. Then there are a probability algebra $(\mathfrak{C},\bar\lambda)$ with a filtration $\langle\mathfrak{C}_t\rangle_{t\in T}$ and a stochastically independent family $\langle\mathfrak{B}_i\rangle_{i\in I}$ of closed subalgebras such that $\mathfrak{B}_i$ is coordinated with $\langle\mathfrak{C}_t\rangle_{t\in T}$ and $(\mathfrak{B}_i,\bar\lambda\restriction\mathfrak{B}_i,\langle\mathfrak{B}_i\cap\mathfrak{C}_t\rangle_{t\in T})$ is isomorphic to $(\mathfrak{A}_i,\bar\mu_i,\langle\mathfrak{A}_{it}\rangle_{t\in T})$ for every $i\in I$. If every $\langle\mathfrak{A}_{it}\rangle_{t\in T}$ is right-continuous, we can arrange that $\langle\mathfrak{C}_t\rangle_{t\in T}$ should be right-continuous.

proof Take $(\mathfrak{C},\bar\lambda,\langle\varepsilon_i\rangle_{i\in I})$ to be the probability algebra free product of $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ (325K), and set $\mathfrak{B}_i = \varepsilon_i[\mathfrak{A}_i]$ for $i\in I$, $\mathfrak{C}_t = \bigvee_{i\in I}\varepsilon_i[\mathfrak{A}_{it}]$ for $t\in T$. As noted in 325L, $\langle\mathfrak{B}_i\rangle_{i\in I}$ is an independent family of closed subalgebras of $\mathfrak{C}$, while $(\mathfrak{B}_i,\bar\lambda\restriction\mathfrak{B}_i)\cong(\mathfrak{A}_i,\bar\mu_i)$ for every i, so 634L gives the result.

634N Example: independent Poisson processes To show how these ideas may be used, I work through an important special case.

(a) Let $(\mathfrak{B},\bar\nu,\langle\mathfrak{B}_t\rangle_{t\ge0},\mathcal{T}_{\mathbb{B}},\langle u_\sigma\rangle_{\sigma\in\mathcal{T}_{\mathbb{B}f}})$ be the standard Poisson process of 612U in its measure-algebra form. Let $(\mathfrak{A},\bar\mu)$ be the probability algebra free product of $(\mathfrak{B},\bar\nu)$ with itself, with associated embeddings $\varepsilon_1:\mathfrak{B}\to\mathfrak{A}$, $\varepsilon_2:\mathfrak{B}\to\mathfrak{A}$; write $\mathfrak{B}^{(i)}$ for $\varepsilon_i[\mathfrak{B}]$ for each i. Set $\mathfrak{A}_t = \varepsilon_1[\mathfrak{B}_t]\vee\varepsilon_2[\mathfrak{B}_t]$ for $t\ge0$, so that $\langle\mathfrak{A}_t\rangle_{t\ge0}$ is a right-continuous filtration (612Uc, 632Db, 634Lc), while $\varepsilon_i[\mathfrak{B}_t] = \mathfrak{B}^{(i)}\cap\mathfrak{A}_t$ for both i and every t, and each $\mathfrak{B}^{(i)}$ is coordinated with $\langle\mathfrak{A}_t\rangle_{t\ge0}$ (634Lb).

(b) For each i, ε_i is an isomorphism between $(\mathfrak{B},\bar\nu,\langle\mathfrak{B}_t\rangle_{t\ge0})$ and $(\mathfrak{B}^{(i)},\bar\mu\restriction\mathfrak{B}^{(i)},\langle\mathfrak{B}^{(i)}\cap\mathfrak{A}_t\rangle_{t\in[0,\infty[})$, so matches $\mathcal{T}_{\mathbb{B}f}$ with $\mathcal{T}_{\mathbb{B}^{(i)}f} = \mathcal{T}_{\mathbb{B}^{(i)}}\cap\mathcal{T}_{\mathbb{A}f}$ (634Cb). Now $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{T}_{\mathbb{B}f}}$ is matched with $\boldsymbol{u}_i = \langle u_{i\sigma}\rangle_{\sigma\in\mathcal{T}_{\mathbb{B}^{(i)}f}}$, and $\boldsymbol{u}_i$ is locally near-simple because $\boldsymbol{u}$ is (634Eb). Because $\mathcal{T}_{\mathbb{B}^{(i)}f}$ contains all the constant processes, $\sup\{[\![\tau\le\sigma]\!] : \sigma\in\mathcal{T}_{\mathbb{B}^{(i)}f}\} = 1$ for every $\tau\in\mathcal{T}_{\mathbb{A}f}$, $\boldsymbol{u}_i$ has an extension to a locally near-simple process defined on $\mathcal{T}_{\mathbb{A}f}$ (631M(c-ii)). Because $\mathcal{T}_{\mathbb{B}^{(i)}f}$ contains the constant processes, it separates $\mathcal{T}_{\mathbb{A}f}$ (633D), so the extension is unique (633F); I will call it $\boldsymbol{v}_i$.

(c) For each i, $\boldsymbol{v}_i - \boldsymbol{\iota}_{\mathfrak{A}}$ is a local martingale. **P** By 632Ma, $\boldsymbol{u} - \boldsymbol{\iota}_{\mathfrak{B}}$ (where $\boldsymbol{\iota}_{\mathfrak{B}}$ as the identity process on $\mathcal{T}_{\mathbb{B}f}$) is a local martingale, so its copy $(\boldsymbol{v}_i - \boldsymbol{\iota}_{\mathfrak{B}_i^{(f)}})\restriction\mathcal{T}_{\mathbb{B}^{(i)}f}$ is a local martingale (634Ia) and $\boldsymbol{v}_i - \boldsymbol{\iota}_{\mathfrak{A}}$ is a local martingale (633Pg; of course $\mathcal{T}_{\mathbb{B}^{(i)}f}$ separates $\mathcal{T}_{\mathbb{A}f}$ because it contains all constant processes). **Q** It follows that $\boldsymbol{w} = \boldsymbol{v}_1 - \boldsymbol{v}_2$ is a local martingale.

(d)(i) $\boldsymbol{v}_i^* = \boldsymbol{v}_i$, because $\boldsymbol{u}^* = \boldsymbol{u}$ (617Ob) so $\boldsymbol{u}_i^* = \boldsymbol{u}_i$, and we can apply 633Ph to see that $\boldsymbol{v}_i^*$ and $\boldsymbol{v}_i$ agree on $\mathcal{T}_{\mathbb{B}f}$, and therefore on $\mathcal{T}_{\mathbb{A}f}$ (633F).

(ii) The covariation $[\boldsymbol{v}_1 \overset{*}{|} \boldsymbol{v}_2]$ is zero. **P** Express $\boldsymbol{v}_i$ as $\langle v_{i\sigma}\rangle_{\sigma\in\mathcal{T}_{\mathbb{A}f}}$ for each i. If $0 \le s \le t$, then, writing $\check{s}$ and $\check{t}$ for the constant stopping times at s and t respectively, $v_{i\check{t}} - v_{i\check{s}}$ has a Poisson distribution with mean $t-s$, so

$$\bar{\mu}[\![v_{i,\check{t}} \ne v_{i,\check{s}}]\!] = 1 - e^{-(t-s)} \le t - s$$

for each i. Now suppose that $t \ge 0$, $n \ge 1$ and that I is a sublattice of $[\check{0},\check{t}]$ containing $\check{s}_j$ for every $j \le n$, where $s_j = jt/n$. In this case

$$\begin{aligned}\theta(S_I(\mathbf{1}, d\boldsymbol{v}_1 d\boldsymbol{v}_2)) &\le \bar{\mu}[\![S_I(\mathbf{1}, d\boldsymbol{v}_1 d\boldsymbol{v}_2) \ne 0]\!]\\ &\le \bar{\mu}(\sup_{e\in\mathrm{Sti}_0(I)} [\![\Delta_e(\mathbf{1}, d\boldsymbol{v}_1 d\boldsymbol{v}_2) \ne 0]\!])\end{aligned}$$

(where $\mathrm{Sti}_0(I)$ is the set of I-cells)

$$\le \bar{\mu}(\sup_{j<n} [\![v_{1,\check{s}_{j+1}} \ne v_{1,\check{s}_j}]\!] \cap [\![v_{2,\check{s}_{j+1}} \ne v_{2,\check{s}_j}]\!])$$

(because both $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are non-decreasing)

$$\begin{aligned}&\le \sum_{j=0}^{n-1} \bar{\mu}([\![v_{1,\check{s}_{j+1}} \ne v_{1,\check{s}_j}]\!] \cap [\![v_{2,\check{s}_{j+1}} \ne v_{2,\check{s}_j}]\!])\\ &= \sum_{j=0}^{n-1} \bar{\mu}([\![v_{1,\check{s}_{j+1}} \ne v_{1,\check{s}_j}]\!]) \cdot \bar{\mu}([\![v_{2,\check{s}_{j+1}} \ne v_{2,\check{s}_j}]\!])\end{aligned}$$

(because $[\![v_{i,\check{s}_{j+1}} \ne v_{i,\check{s}_j}]\!] \in \mathfrak{B}^{(i)}$ for all i and j, and $\mathfrak{B}^{(1)}$ and $\mathfrak{B}^{(2)}$ are stochastically independent)

$$\le \sum_{j=0}^{n-1} (s_{j+1} - s_j)^2 = \frac{t^2}{n}.$$

Taking the limit as $I \uparrow \mathcal{S}([\check{0},\check{t}])$, $\int_{[\check{0},\check{t}]} d\boldsymbol{v}_1 d\boldsymbol{v}_2 = 0$. As t is arbitrary, the covariation $[\boldsymbol{v}_1 \overset{*}{|} \boldsymbol{v}_2]$ is zero at every constant stopping time. Because the constant stopping times separate $\mathcal{T}_{\mathbb{A}}$, and $[\boldsymbol{v}_1 \overset{*}{|} \boldsymbol{v}_2]$ is locally near-simple (631Jb), $[\boldsymbol{v}_1 \overset{*}{|} \boldsymbol{v}_2] = 0$, by 633F. **Q**

(iii) $\boldsymbol{w}^* = \boldsymbol{v}_1 + \boldsymbol{v}_2$.

$$\mathbf{P}\ \boldsymbol{w}^* = [\boldsymbol{v}_1 - \boldsymbol{v}_2 \overset{*}{|} \boldsymbol{v}_1 - \boldsymbol{v}_2] = [\boldsymbol{v}_1 \overset{*}{|} \boldsymbol{v}_1] + [\boldsymbol{v}_2 \overset{*}{|} \boldsymbol{v}_2] = \boldsymbol{v}_1^* + \boldsymbol{v}_2^* = \boldsymbol{v}_1 + \boldsymbol{v}_2.\ \mathbf{Q}$$

(iv) The previsible variation of $\boldsymbol{w}^2{\restriction}\mathcal{T}_{\mathbb{A}b}$ is $2\boldsymbol{\iota}{\restriction}\mathcal{T}_{\mathbb{A}b}$. **P** We know that $\boldsymbol{w}$ is a local martingale, so $\boldsymbol{w}^2 - \boldsymbol{w}^*$ is a virtually local martingale (624B), therefore a local martingale (632I); now (iii) just above, with (c), tells us that $\boldsymbol{w}^2 - 2\boldsymbol{\iota}_{\mathfrak{A}}$ is a local martingale, so $\boldsymbol{w}^2{\restriction}\mathcal{T}_{\mathbb{A}b}$ and $2\boldsymbol{\iota}{\restriction}\mathcal{T}_{\mathbb{A}b}$ have the same previsible variation (as in the proof of 632Mb), which is $2\boldsymbol{\iota}{\restriction}\mathcal{T}_{\mathbb{A}b}$, by 626Q. **Q**

(e) Note that $\boldsymbol{w}$ corresponds to a Lévy process (455Q, §652) derived from the family $\langle\lambda'_t\rangle_{t\ge0}$ where

$$\lambda'_t(\{n\}) = e^{-2t} t^n \sum_{k=\max(-n,0)}^{\infty} \frac{t^{2k}}{k!(k+n)!}$$

for $n \in \mathbb{Z}$, because $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ can be determined from $\boldsymbol{w}$ and $\boldsymbol{w}^*$.

634X Basic exercises (a) Let $\mathfrak{B}$ be a closed subalgebra of $\mathfrak{A}$, and $\mathbb{B} = (\mathfrak{B}, \bar{\mu}{\restriction}\mathfrak{B}, T, \langle\mathfrak{B}_t\rangle_{t\in T}, \mathcal{T}_{\mathbb{B}}, \langle\mathfrak{B}_\sigma\rangle_{\sigma\in\mathcal{T}_{\mathbb{B}}})$ the corresponding stochastic integration structure, where $\mathfrak{B}_t = \mathfrak{B} \cap \mathfrak{A}_t$ for $t \in T$. Suppose that $\mathcal{S}$ is a sublattice of $\mathcal{T}_{\mathbb{B}}$ and that $\boldsymbol{u} \in L^0(\mathfrak{B})^{\mathcal{S}}$ is fully adapted. Show that (i) if $\boldsymbol{u}$ is an $\mathbb{A}$-integrator it is a $\mathbb{B}$-integrator (ii) if $\boldsymbol{u}$ is a local $\mathbb{A}$-integrator it is a local $\mathbb{B}$-integrator and its $\mathbb{B}$-quadratic variation is the same as its $\mathbb{A}$-quadratic variation.

634Y Further exercises (a) Let $\mathfrak{A}$ be an eight-element Boolean algebra with atoms a, b, c, and $\mathfrak{B}$ the subalgebra generated by $\{b\}$. Let $\mathfrak{A}_0$ be the subalgebra generated by $\{a\}$ and for $t > 0$ let $\mathfrak{A}_t$ be the subalgebra generated by $\{a, b\}$. For $t > 0$ set $\tau_t = \check{t}(a \cup c)$ in the language of part (b) of the proof of 634H. Show that the infimum $\tau = \inf_{s>0} \tau_s$, taken in $\mathcal{T}_{\mathbb{A}}$, is such that $[\![\tau > 0]\!] = a$, so is not the infimum of $\{\tau_s : s > 0\}$ in $\mathcal{T}_{\mathbb{B}}$.

(b) Suppose that $\mathcal{S}$ is a sublattice of $\mathcal{T}_{\mathbb{A}}$ and that $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ is a fully adapted process which is not of bounded variation. (i) Show that there are $\mathfrak{C}$, $\bar{\lambda}$, $\langle \mathfrak{C}_t \rangle_{t \in T}$, $\mathfrak{B}$, $\langle \mathfrak{B}_t \rangle_{t \in T}$ and $\boldsymbol{v}'$ such that $(\mathfrak{C}, \bar{\lambda})$ is a probability algebra, $\langle \mathfrak{C}_t \rangle_{t \in T}$ is a filtration of closed subalgebras of $\mathfrak{C}$, $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{C}$, $\mathfrak{B}_t = \mathfrak{B} \cap \mathfrak{C}_t$ for every $t \in T$, $(\mathfrak{A}, \bar{\mu}, \langle \mathfrak{A}_t \rangle_{t \in T}, \boldsymbol{v})$ is isomorphic to $(\mathfrak{B}, \bar{\lambda} \restriction \mathfrak{B}, \langle \mathfrak{B}_t \rangle_{t \in T}, \boldsymbol{v}')$, but $\boldsymbol{v}'$ is not an integrator for the structure $(\mathfrak{C}, \langle \mathfrak{C}_t \rangle_{t \in T})$. (ii) Show that if $\langle \mathfrak{A}_t \rangle_{t \in T}$ is right-continuous we can arrange that $\langle \mathfrak{C}_t \rangle_{t \in T}$ is right-continuous.

(c) Give an example of a probability algebra $(\mathfrak{A}, \bar{\mu})$ with a closed subalgebra $\mathfrak{C}$ and a non-increasing sequence $\langle \mathfrak{B}_n \rangle_{n \in \mathbb{N}}$ of closed subalgebras such that $\bigcap_{n \in \mathbb{N}} \mathfrak{B}_n = \{0, 1\}$ but $\bigcap_{n \in \mathbb{N}} (\mathfrak{C} \vee \mathfrak{B}_n) \neq \mathfrak{C}$.

(d) ('Emery's example', see Protter 05, IV.2) In the structure defined in 634N, let $\boldsymbol{x} = \langle x_\tau \rangle_{\tau \in \mathcal{T}_{\mathbb{A}f}}$ be the fully adapted process defined by saying that $x_{\check{0}} = 0$ and $x_\tau = \dfrac{1}{\iota_\tau}$ if $\tau \in \mathcal{T}_{\mathbb{A}f}$ and $[\![\tau > 0]\!] = 1$, where $\boldsymbol{\iota}$ is the identity process. (i) Show that $\int_{[\check{0}, \tau]} \boldsymbol{x}\, d\boldsymbol{w}$ is defined for every $\tau \in \mathcal{T}_{\mathbb{A}f}$. (ii) Show that the indefinite integral $\boldsymbol{z} = ii_{\boldsymbol{w}}(\boldsymbol{x})$ is not a local martingale, but that for every $\epsilon > 0$ there is a local martingale $\boldsymbol{z}'$ with domain $\mathcal{T}_{\mathbb{A}f}$ such that $\bar{\mu}[\![\boldsymbol{z} \neq \boldsymbol{z}']\!] \leq \epsilon$.

634 Notes and comments Even by the standards of this volume, I have taken 634B-634E extremely slowly. The point is that nearly everything up to this point was written on the understanding that we are starting from a settled structure $(\mathfrak{A}, \bar{\mu}, \langle \mathfrak{A}_t \rangle_{t \in T})$. I have remarked many times that the topology of convergence in measure is more important than the measure itself, but this is fairly easy to incorporate into one's intuitive picture, because so many of the definitions and arguments refer directly to the topology. In this section I am exploring a much more radical change. Substituting $\mathfrak{B}$ for $\mathfrak{A}$, in the measure-algebra version of probability theory I am working in here, corresponds to a deliberate closing of the eyes to some of the randomness in the 'outer' model $(\mathfrak{A}, \bar{\mu})$. Probability theory would of course be impossible without this. We never suppose that we have grasped all the possible stochastic elements of a real-world situation, and must always be ready to extend our model by elaborating on our probability space or probability algebra. I have discussed this aspect of probability theory in the introduction to §275. But this means that we must at some point check every concept and theorem for the transformations which will be appropriate when the framework shifts.

Many of the checks are so elementary that they hardly need mentioning, and you may feel that I have laboured unnecessarily. If so, do feel at liberty to write 'obvious' in the margin of your copy. In particular, it is not clear that we need to spell 634B out explicitly, rather than proceed directly to the more intuitively appealing 634C. But I fear that the tempting intuitions here are not perfectly safe, and that pulling the algebras $\mathfrak{B}$, $\mathfrak{A}$ apart as in 634B forces a necessary extra clarity, if you can face the extra elaboration in the notation.

We come to some new phenomena when we investigate relative independence (634F-634G). The point is that the concepts of 'martingale' and 'integrator' are less absolute than 'fully adapted', 'bounded variation' and so on. Even Brownian motion, unsuitably embedded, can become a process which is not an integrator (634Yb). Without checking, we cannot be sure that the conditional expectation associated with $\mathfrak{B}_t = \mathfrak{B} \cap \mathfrak{A}_t$ is appropriately related to the conditional expectation associated with $\mathfrak{A}_t$. When it is, we have a much closer relationship between the structure $(\mathfrak{B}, \bar{\mu} \restriction \mathfrak{B}, \langle \mathfrak{B}_t \rangle_{t \in T}, \mathcal{T}_{\mathbb{B}})$ and its attendant space $M_{\mathrm{fa}\mathbb{B}}$ of fully adapted processes, regarded in isolation, and its alternative realization as a subspace of $M_{\mathrm{fa}\mathbb{A}}$ (634H, 634I)..

Product measures provide a tool for representing arbitrary independent families of random variables (272J). The corresponding construction for probability algebras is the probability algebra free product of §325. To use this, we need to be able to define a filtration on the product from given filtrations on the factors. To begin with, this is easy (634La-634Lb), at least if you know the right things about relative independence. But elsewhere in this section we repeatedly depend on right-continuity, and to be sure that the elementary construction I offer produces a right-continuous filtration on the product we have to look hard at product measures (634J).

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635 Changing the filtration

In this section I introduce the elementary theory of 'local times'. In the principal applications, we have a process which is easier to handle if we replace the standard clock T with a variable-speed clock $\langle\pi_r\rangle_{r\in R}$ where the clock-times are now a totally ordered family of stopping times. I will come to such applications in Chapter 65. Here I want to set up a language to discuss the transformation in which a process $\langle u_\tau\rangle_{\tau\in\mathcal{T}}$ is mapped to $\langle u_{\pi(\rho)}\rangle_{\rho\in\mathcal{R}}$, where $\mathcal{R}$ is the lattice of R-based stopping times and $\pi(\rho)\in\mathcal{T}$ corresponds to $\rho\in\mathcal{R}$. Starting from the construction in 635B, we have basic algebraic properties (corresponding to ideas in §611) in 635C and can then follow a programme along the same lines as elsewhere, looking at the usual kinds of process and Riemann-sum integrals.

635A Notation $(\mathfrak{A},\bar\mu,T,\langle\mathfrak{A}_t\rangle_{t\in T},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ will be a stochastic integration structure, and θ the usual F-norm on $L^0(\mathfrak{A})$ (613Ba). For $\tau\in\mathcal{T}$, P_τ will be the conditional expectation operator associated with $\mathfrak{A}_\tau$. If $\mathcal{S}$ is a sublattice of $\mathcal{T}$, $\mathcal{I}(\mathcal{S})$ will be the family of finite sublattices of $\mathcal{S}$.

Now for the new idea.

635B Construction For the whole of this section, $(R,\le)$ will be a new (non-empty) totally ordered set, and $\langle\pi_r\rangle_{r\in R}$ a non-decreasing family in $\mathcal{T}$. For $r\in R$, I will write $\mathfrak{B}_r$ for $\mathfrak{A}_{\pi_r}$, so that $\langle\mathfrak{B}_r\rangle_{r\in R}$ is a filtration of closed subalgebras of $\mathfrak{A}$, and we shall have a corresponding stochastic integration structure $(\mathfrak{A},\bar\mu,R,R,\langle\mathfrak{B}_r\rangle_{r\in R},\mathcal{R},\langle\mathfrak{B}_\rho\rangle_{\rho\in\mathcal{R}})$.

For $\rho\in\mathcal{R}$, $\pi(\rho)\in L^0(\mathfrak{A})^T$ will be defined by saying that

$$\begin{aligned}[\![\pi(\rho)>t]\!]&=\inf_{r\in R}([\![\rho>r]\!]\cup[\![\pi_r>t]\!])\text{ if }t\in T\text{ is isolated on the right in }T,\\&=\sup_{s>t}\inf_{r\in R}([\![\rho>r]\!]\cup[\![\pi_r>s]\!])\text{ for other }t\in T.\end{aligned}$$

635C Theorem Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous.

(a) For every $\rho\in\mathcal{R}$, $\pi(\rho)$, as defined in 635B, belongs to $\mathcal{T}$, and $\mathfrak{B}_\rho=\mathfrak{A}_{\pi(\rho)}$.

(b)(i) The map $\pi:\mathcal{R}\to\mathcal{T}$ is a lattice homomorphism.

(ii) $\pi(\min\mathcal{R})=\inf_{r\in R}\pi_r$ in $\mathcal{T}$, $\pi(\max\mathcal{R})=\max\mathcal{T}$.

(iii) If $r\in R$ and $\check r\in\mathcal{R}$ is the corresponding constant stopping time, then $\pi(\check r)=\pi_r$.

(iv) If $\rho\in\mathcal{R}_f$ then $\pi(\rho)\le\sup_{r\in R}\pi_r$.

(c) $[\![\pi(\rho)<\pi(\rho')]\!]\subseteq[\![\rho<\rho']\!]$, $[\![\pi(\rho)\le\pi(\rho')]\!]\supseteq[\![\rho\le\rho']\!]$ and $[\![\rho=\rho']\!]\subseteq[\![\pi(\rho)=\pi(\rho')]\!]$ for all ρ, $\rho'\in\mathcal{R}$.

(d) Suppose that $\langle\pi_r\rangle_{r\in R}$ is **right-continuous** in the sense that $\pi_r=\inf_{q\in R,q>r}\pi_q$ in $\mathcal{T}$ whenever $r\in R$ is not isolated on the right in R. Then

(i) $\langle\mathfrak{B}_r\rangle_{r\in R}$ is right-continuous;

(ii) π is right-continuous (see 632B).

proof (a)(i) For $\rho\in\mathcal{R}$ and $r\in R$ there is a $\pi(\rho,r)\in\mathcal{T}$ defined by saying that $[\![\pi(\rho,r)>t]\!]=[\![\rho>r]\!]\cup[\![\pi_r>t]\!]$ for $t\in T$. **P** $[\![\rho>r]\!]\in\mathfrak{B}_r=\mathfrak{A}_{\pi_r}$; looking at the definition of $\mathfrak{A}_{\pi_r}$ (611G), we see that $[\![\rho>r]\!]\cup[\![\pi_r>t]\!]\in\mathfrak{A}_t$ for every $t\in T$. If $s\le t$ in T then $[\![\pi_r>t]\!]\subseteq[\![\pi_r>s]\!]$ so $[\![\rho>r]\!]\cup[\![\pi_r>t]\!]\subseteq[\![\rho>r]\!]\cup[\![\pi_r>s]\!]$. If $t\in T$ is not isolated on the right, $[\![\pi_r>t]\!]=\sup_{s>t}[\![\pi_r>s]\!]$ so $[\![\rho>r]\!]\cup[\![\pi_r>t]\!]=\sup_{s>t}[\![\rho>r]\!]\cup[\![\pi_r>s]\!]$. So the function $t\mapsto[\![\rho>r]\!]\cup[\![\pi_r>t]\!]$ satisfies the conditions of 611A(b-i). **Q**

(ii) Comparing the definition of $\pi(\rho)$ in 635B with the formula in 632C(a-i) (which is applicable because $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous), we see that $\pi(\rho)$ is just $\inf_{r\in R}\pi(\rho,r)$ in $\mathcal{T}$, and certainly belongs to $\mathcal{T}$.

(iii) Take any $\rho\in\mathcal{R}$. For $a\in\mathfrak{A}$,

$$\begin{aligned}a\in\mathfrak{B}_\rho&\iff a\setminus[\![\rho>r]\!]\in\mathfrak{B}_r=\mathfrak{A}_{\pi_r}\text{ for every }r\in R\\&\iff a\setminus([\![\rho>r]\!]\cup[\![\pi_r>t]\!])\in\mathfrak{A}_t\text{ for every }r\in R\text{ and }t\in T\\&\iff a\setminus[\![\pi(\rho,r)>t]\!]\in\mathfrak{A}_t\text{ for every }r\in R\text{ and }t\in T\\&\iff a\in\mathfrak{A}_{\pi(\rho,r)}\text{ for every }r\in R\iff a\in\bigcap_{r\in R}\mathfrak{A}_{\pi(\rho,r)}=\mathfrak{A}_{\pi(\rho)}\end{aligned}$$

by 632C(a-iii), again using the general hypothesis that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. So $\mathfrak{B}_\rho = \mathfrak{A}_{\pi(\rho)}$.

(b)(i)(α) For $r \in R$, $\rho \mapsto \pi(\rho, r) : \mathcal{R} \to \mathcal{T}$ is a lattice homomorphism. **P** Use 611Cb and 611Cc; we just have to note that $\rho \mapsto [\![\rho > r]\!]$ is a lattice homomorphism, so that $\rho \mapsto [\![\pi(\rho, r) > t]\!]$ is a lattice homomorphism for every $t \in T$. **Q**

(β) $\pi(\rho \wedge \rho') = \pi(\rho) \wedge \pi(\rho')$ for ρ, $\rho' \in \mathcal{R}$. **P** By (α), both are equal to $\inf_{r\in R}(\pi(\rho, r) \wedge \pi(\rho', r))$. **Q**

(γ) $\pi(\rho) \vee \pi(\rho') \le \pi(\rho \vee \rho')$ for all ρ, $\rho \in \mathcal{R}$. (By (β), or otherwise, π is order-preserving.)

(δ) $\pi(\rho \vee \rho') \le \pi(\rho) \vee \pi(\rho')$ for all ρ, $\rho' \in \mathcal{R}$. **P** For any $t \in T$,

$$\begin{aligned}\inf_{r\in R}[\![\pi(\rho, r) > t]\!] \cup \inf_{r\in R}[\![\pi(\rho', r) > t]\!] &= \inf_{r,r'\in R}([\![\pi(\rho, r) > t]\!] \cup [\![\pi(\rho', r') > t]\!]) \\ &= \inf_{r,r'\in R}([\![\rho > r]\!] \cup [\![\pi_r > t]\!] \cup [\![\rho' > r']\!] \cup [\![\pi_{r'} > t]\!]) \\ &\supseteq \inf_{r,r'\in R}([\![\rho > \max(r, r')]\!] \cup [\![\rho' > \max(r, r')]\!] \cup [\![\pi_r \vee \pi_{r'} > t]\!]) \\ &= \inf_{r,r'\in R}([\![\rho \vee \rho' > \max(r, r')]\!] \cup [\![\pi_{\max(r,r')} > t]\!] = \inf_{r\in R}[\![\pi(\rho \vee \rho', r) > t]\!].\end{aligned}$$

So if t is isolated on the right,

$$[\![\pi(\rho \vee \rho') > t]\!] = \inf_{r\in R}[\![\pi(\rho \vee \rho', r) > t]\!]$$

(632C(a-i) again)

$$\begin{aligned}&\subseteq \inf_{r\in R}[\![\pi(\rho, r) > t]\!] \cup \inf_{r\in R}[\![\pi(\rho', r) > t]\!] \\ &= [\![\pi(\rho) > t]\!] \cup [\![\pi(\rho') > t]\!] = [\![\pi(\rho) \vee \pi(\rho') > t]\!],\end{aligned}$$

while if t is not isolated on the right,

$$\begin{aligned}[\![\pi(\rho \vee \rho') > t]\!] &= \sup_{s>t}\inf_{r\in R}[\![\pi(\rho \vee \rho', r) > s]\!] \subseteq \sup_{s>t}(\inf_{r\in R}[\![\pi(\rho, r) > s]\!] \cup \inf_{r\in R}[\![\pi(\rho', r) > s]\!]) \\ &= \sup_{s>t}\inf_{r\in R}[\![\pi(\rho', r) > s]\!] \cup \sup_{s>t}\inf_{r\in R}[\![\pi(\rho', r) > s]\!] \\ &= [\![\pi(\rho) > t]\!] \cup [\![\pi(\rho') > t]\!] = [\![\pi(\rho) \vee \pi(\rho') > t]\!].\end{aligned}$$

Accordingly $\pi(\rho \vee \rho') \le \pi(\rho) \vee \pi(\rho')$. **Q**

(ϵ) Thus $\pi : \mathcal{R} \to \mathcal{T}$ is a lattice homomorphism.

(ii) If $r \in R$, then

$$[\![\pi(\min\mathcal{R}, r) > t]\!] = [\![\pi_r > t]\!], \quad [\![\pi(\max\mathcal{R}, r) > t]\!] = 1$$

for every $t \in T$, so $\pi(\min\mathcal{R}, r) = \pi_r$ and $\pi(\max\mathcal{R}, r) = \max\mathcal{T}$. Accordingly

$$\pi(\min\mathcal{R}) = \inf_{r\in R}\pi(\min\mathcal{R}, r) = \inf_{r\in R}\pi_r,$$

$$\pi(\max\mathcal{R}) = \inf_{r\in R}\pi(\max\mathcal{R}, r) = \max\mathcal{T}.$$

(iii) For $q \in R$ and $t \in T$,

$$\begin{aligned}[\![\pi(\check{r}, q) > t]\!] &= [\![\pi_q > t]\!] \text{ if } r \le q, \\ &= 1 \text{ otherwise,}\end{aligned}$$

so

$$\begin{aligned}\pi(\check{r}, q) &= \pi_q \text{ if } r \le q, \\ &= \max\mathcal{T} \text{ otherwise.}\end{aligned}$$

Taking the infimum over r, $\pi(\check{r}) = \inf_{q\ge r}\pi_q = \pi_r$.

(iv) Write τ for the supremum $\sup_{r\in R}\pi_r$ taken in $\mathcal{T}$. If $\rho \in \mathcal{R}_f$ then

$$\begin{aligned}\inf_{r\in R}([\![\rho > r]\!] \cup [\![\pi_r > t]\!]) &\subseteq \inf_{r\in R}([\![\rho > r]\!] \cup [\![\tau > t]\!])\\ &= [\![\tau > t]\!] \cup \inf_{r\in R}[\![\rho > r]\!] = [\![\tau > t]\!].\end{aligned}$$

So

$$\begin{aligned}[\![\pi(\rho) > t]\!] &= \inf_{r\in R}([\![\rho > r]\!] \cup [\![\pi_r > t]\!]) \subseteq [\![\tau > t]\!] \text{ if } t \in T \text{ is isolated on the right in } T,\\ &= \sup_{s>t}\inf_{r\in R}([\![\rho > s]\!] \cup [\![\pi_s > t]\!]) \subseteq \sup_{s>t}[\![\tau > s]\!] = [\![\tau > t]\!] \text{ for other } t \in T,\end{aligned}$$

and $\pi(\rho) \le \tau$, as claimed.

(c)(i) Set $a = [\![\rho < \rho']\!]$. If $t \in T$ and $r \in R$, then

$$[\![\pi(\rho', r) > t]\!] = [\![\rho' > r]\!] \cup [\![\pi_r > t]\!] \subseteq a \cup [\![\rho > r]\!] \cup [\![\pi_r > t]\!] = a \cup [\![\pi(\rho, r) > t]\!].$$

If t is isolated on the right in T, then

$$\begin{aligned}[\![\pi(\rho') > t]\!] &= \inf_{r\in R}[\![\pi(\rho', r) > t]\!] \subseteq \inf_{r\in R}(a \cup [\![\pi(\rho, r) > t]\!])\\ &= a \cup \inf_{r\in R}[\![\pi(\rho, r) > t]\!] = a \cup [\![\pi(\rho) > t]\!],\end{aligned}$$

while if t is not isolated on the right,

$$\begin{aligned}[\![\pi(\rho') > t]\!] &= \sup_{s>t}\inf_{r\in R}[\![\pi(\rho', r) > s]\!] \subseteq \sup_{s>t}\inf_{r\in R} a \cup [\![\pi(\rho, r) > s]\!]\\ &= a \cup \sup_{s>t}\inf_{r\in R}[\![\pi(\rho, r) > s]\!] = a \cup [\![\pi(\rho) > t]\!].\end{aligned}$$

So

$$[\![\pi(\rho) < \pi(\rho')]\!] = \sup_{t\in T}([\![\pi(\rho') > t]\!] \setminus [\![\pi(\rho) > t]\!]) \subseteq a.$$

(ii) Similarly, $[\![\pi(\rho') < \pi(\rho)]\!] \subseteq [\![\rho' < \rho]\!]$; taking complements, $[\![\pi(\rho) \le \pi(\rho')]\!] \supseteq [\![\rho \le \rho']\!]$.

(iii) Now

$$\begin{aligned}[\![\pi(\rho) = \pi(\rho')]\!] &= [\![\pi(\rho) \le \pi(\rho')]\!] \cap [\![\pi(\rho') \le \pi(\rho)]\!]\\ &\supseteq [\![\rho \le \rho']\!] \cap [\![\rho' \le \rho]\!] = [\![\rho = \rho']\!].\end{aligned}$$

(d)(i) If $r \in R$ is not isolated on the right, then $\pi_r = \inf_{q>r}\pi_q$, so

$$\mathfrak{B}_r = \mathfrak{A}_{\pi_r} = \bigcap_{q>r}\mathfrak{A}_{\pi_q} = \bigcap_{q>r}\mathfrak{B}_q$$

(632C(a-iii) again). Thus $\langle\mathfrak{B}_r\rangle_{r\in R}$ is right-continuous.

(ii) Set

$$\tau = \inf_{\rho\in D}\pi(\rho) = \inf_{\rho\in D, r\in R}\pi(\rho, r),$$

taken in $\mathcal{T}$.

($\boldsymbol{\alpha}$) If $r \in R$ is isolated on the right, then $\tau \le \pi(\inf D, r)$. **P** For any $t \in T$,

$$[\![\pi(\inf D, r) > t]\!] = [\![\inf D > r]\!] \cup [\![\pi_r > t]\!] = (\inf_{\rho\in D}[\![\rho > r]\!]) \cup [\![\pi_r > t]\!]$$

(applying 632C(a-i) to $\langle\mathfrak{B}_r\rangle_{r\in\mathcal{R}}$)

$$= \inf_{\rho\in D}[\![\pi(\rho, r) > t]\!] \supseteq [\![\inf_{\rho\in D}\pi(\rho, r) > t]\!] \supseteq [\![\tau > t]\!]. \text{ } \mathbf{Q}$$

(β) If $r\in R$ is not isolated on the right, then $\tau\le\pi(\inf D,r)$. **P** We have $\pi_r=\inf_{q>r}\pi_q$. If $t\in T$ is isolated on the right,

$$\begin{aligned}[\![\pi(\inf D,r)>t]\!]&=[\![\inf D>r]\!]\cup[\![\pi_r>t]\!]=(\sup_{q>r}\inf_{\rho\in D}[\![\rho>q]\!])\cup[\![\inf_{q>r}\pi_q>t]\!]\\&=(\sup_{q>r}\inf_{\rho\in D}[\![\rho>q]\!])\cup(\inf_{q>r}[\![\pi_q>t]\!])\supseteq\inf_{q>r}([\![\pi_q>t]\!]\cup\inf_{\rho\in D}[\![\rho>q]\!])\\&=\inf_{q>r}\inf_{\rho\in D}([\![\rho>q]\!]\cup[\![\pi_q>t]\!])=\inf_{q>r}\inf_{\rho\in D}[\![\pi(\rho,q)>t]\!]\supseteq[\![\tau>t]\!].\end{aligned}$$

If $t\in T$ is not isolated on the right, then

$$\begin{aligned}[\![\pi(\inf D,r)>t]\!]&=[\![\inf D>r]\!]\cup[\![\pi_r>t]\!]=(\sup_{q>r}\inf_{\rho\in D}[\![\rho>q]\!])\cup(\sup_{s>t}\inf_{q>r}[\![\pi_q>s]\!])\\&=\sup_{s>t}((\sup_{q>r}\inf_{\rho\in D}[\![\rho>q]\!])\cup(\inf_{q>r}[\![\pi_q>s]\!]))\\&\supseteq\sup_{s>t}\inf_{q>r}(\inf_{\rho\in D}[\![\rho>q]\!]\cup[\![\pi_q>s]\!])\\&=\sup_{s>t}\inf_{q>r}\inf_{\rho\in D}[\![\pi(\rho,q)>s]\!]\supseteq\sup_{s>t}[\![\tau>s]\!]=[\![\tau>t]\!].\end{aligned}$$

Thus $[\![\tau>t]\!]\subseteq[\![\pi(\inf D,r)>t]\!]$ for every t and $\tau\le\pi(\inf D,r)$. **Q**

(γ) Putting these together, $\tau\le\pi(\inf D,r)$ for every r and $\tau\le\pi(\inf D)$. On the other hand,

$$\pi(\inf D)=\inf_{r\in R}\pi(\inf D,r)\le\inf_{\rho\in D,r\in R}\pi(\rho,r)=\tau.$$

So in fact $\pi(\inf D)=\tau=\inf_{\rho\in D}\pi(\rho)$, as required.

635D Proposition Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{Q}$ be a sublattice of $\mathcal{R}$.

(a)(i) $\pi[\mathcal{Q}]$ is a sublattice of $\mathcal{T}$.

(ii) If $\boldsymbol{u}=\langle u_\tau\rangle_{\tau\in\pi[\mathcal{Q}]}$ is a process fully adapted to $\langle\mathfrak{A}_t\rangle_{t\in T}$, then $\boldsymbol{u}\pi=\langle u_{\pi(\rho)}\rangle_{\rho\in\mathcal{Q}}$ is fully adapted to $\langle\mathfrak{B}_r\rangle_{r\in R}$.

(iii) Let $\psi:\pi[\mathcal{Q}]^{2\uparrow}\to L^0(\mathfrak{A})$ be an adapted interval function (613C), and set $\psi_\pi(\rho,\rho')=\psi(\pi(\rho),\pi(\rho'))$ when $\rho\le\rho'$ in $\mathcal{Q}$. Then ψ_π is an adapted interval function.

(iv) In (iii), if ψ is strictly adapted then ψ_π is strictly adapted.

(b) Now suppose that $\boldsymbol{u}=\langle u_\tau\rangle_{\tau\in\pi[\mathcal{Q}]}$ is fully adapted and that ψ is an adapted interval function on $\pi[\mathcal{Q}]$. Then $\int_{\mathcal{Q}}\boldsymbol{u}\pi\,d\psi_\pi=\int_{\pi[\mathcal{Q}]}\boldsymbol{u}\,d\psi$ if either is defined.

proof (a)(i) This is just because π is a lattice homomorphism (635C(b-i)).

(ii) If $\rho\in\mathcal{Q}$ then $u_{\pi(\rho)}\in L^0(\mathfrak{A}_{\pi(\rho)})=L^0(\mathfrak{B}_\rho)$ by 635Ca. If ρ, $\rho\in\mathcal{Q}$ then

$$[\![u_{\pi(\rho)}=u_{\pi(\rho)}]\!]\supseteq[\![\pi(\rho)=\pi(\rho)]\!]\supseteq[\![\rho=\rho]\!]$$

by 635Cc. So $\langle u_{\pi(\rho)}\rangle_{\rho\in\mathcal{Q}}$ satisfies both the conditions of 612Da.

(iii)(α) If $\rho\le\rho'$ in $\mathcal{Q}$ then $\pi(\rho)\le\pi(\rho')$ in $\pi[\mathcal{Q}]$ (by 635C(b-i) again), so $\psi(\pi(\rho),\pi(\rho'))$ is defined and belongs to $L^0(\mathfrak{A}_{\pi(\rho')})=L^0(\mathfrak{B}_{\rho'})$ (635Ca). Thus ψ_π is a well-defined function on $\mathcal{Q}^{2\uparrow}$. And of course $\psi_\pi(\rho,\rho)=\psi(\pi(\rho),\pi(\rho))=0$ for every $\rho\in\mathcal{Q}$.

(β) If ρ, ρ', σ, $\sigma'\in\mathcal{Q}$, $\rho\le\rho'\le\sigma'\le\sigma$, $b\in\mathfrak{B}_\rho$ and $b\subseteq[\![\rho=\rho']\!]\cap[\![\sigma'=\sigma]\!]$, then $\pi(\rho)\le\pi(\rho')\le\pi(\sigma')\le\pi(\sigma)$ in $\pi[\mathcal{Q}]$, $b\in\mathfrak{A}_{\pi(\rho)}$ and $b\subseteq[\![\pi(\rho)=\pi(\rho')]\!]\cap[\![\pi(\sigma')=\pi(\sigma)]\!]$ (635Cc), so $b\subseteq[\![\psi(\pi(\rho),\pi(\sigma))=\psi(\pi(\rho'),\pi(\sigma'))]\!]$. With (α), this shows that ψ_π is an adapted interval function.

(iv) Continuing from (iii), suppose that ψ is strictly adapted. Then

$$\begin{aligned}[\![\rho=\rho']\!]\cap[\![\sigma'=\sigma]\!]&\subseteq[\![\pi(\rho)=\pi(\rho')]\!]\cap[\![\pi(\sigma')=\pi(\sigma)]\!]\\&\subseteq[\![\psi(\pi(\rho),\pi(\sigma))=\psi(\pi(\rho'),\pi(\sigma'))]\!]\end{aligned}$$

so ψ_π is strictly adapted.

(b)(i) We know from (a) that $\boldsymbol{u}\pi$ is a fully adapted process and ψ_π is an adapted interval function. If $J \in \mathcal{I}(\mathcal{Q})$, then $\pi[J] \in \mathcal{I}(\pi[\mathcal{Q}])$ because π is a lattice homomorphism. Now $S_J(\boldsymbol{u}\pi, d\psi_\pi) = S_{\pi[J]}(\boldsymbol{u}, d\psi)$. **P** If J is empty, this is trivial. Otherwise, let $(\rho_0, \dots, \rho_n)$ linearly generate the J-cells (611L). If $\rho \in J$, then

$$\sup_{i\le n} [\![\pi(\rho) = \pi(\rho_i)]\!] \supseteq \sup_{i\le n} [\![\rho = \rho_i]\!] = 1.$$

Consider the string $(\tau_0, \dots, \tau_n)$ where $\tau_i = \pi(\rho_i)$ for $i \le n$. This is a non-decreasing sequence in $\pi[J]$. If $\tau \in \pi[J]$, there must be a $\rho \in J$ such that $\pi(\rho) = \tau$; now $\rho_0 \le \rho \le \rho_n$, so $\tau_0 \le \tau \le \tau_n$. Thus $\tau_0 = \min \pi[J]$ and $\tau_n = \max \pi[J]$. If $i < n$ and $\tau \in \pi[J] \cap [\tau_i, \tau_{i+1}]$, let $\rho \in J$ be such that $\tau = \pi(\rho)$; setting $\rho' = \operatorname{med}(\rho_i, \rho, \rho_{i+1})$, $\pi(\rho') = \operatorname{med}(\tau_i, \tau, \tau_{i+1}) = \tau$ while $\rho' \in \{\rho_i, \rho_{i+1}\}$, so that $\tau \in \{\tau_i, \tau_{i+1}\}$. But this means that $\{\tau_0, \dots, \tau_n\}$ is a maximal totally ordered subset of $\pi[J]$, so that $(\tau_0, \dots, \tau_n)$ linearly generates the $\pi[J]$-cells.

Consequently

$$S_{\pi[J]}(\boldsymbol{u}, d\psi) = \sum_{i=0}^{n-1} u_{\tau_i} \times \psi(\tau_i, \tau_{i+1}) = \sum_{i=0}^{n-1} u_{\pi(\rho_i)} \times \psi_\pi(\rho_i, \rho_{i+1}) = S_J(\boldsymbol{u}\pi, d\psi_\pi). \textbf{ Q}$$

(ii) Suppose that $z = \int_{\pi[\mathcal{Q}]} \boldsymbol{u}\, d\psi$ is defined. Let $\epsilon > 0$. Then there is an $I_0 \in \mathcal{I}(\pi[\mathcal{Q}])$ such that $\theta(z - S_I(\boldsymbol{u}, d\psi)) \le \epsilon$ whenever $I_0 \subseteq I \in \mathcal{I}(\pi[\mathcal{Q}])$. Let A be a finite subset of $\mathcal{Q}$ such that $I_0 = \pi[A]$, and $J_0 \in \mathcal{I}(\mathcal{Q})$ the sublattice generated by A. If $J_0 \subseteq J \in \mathcal{I}(\mathcal{Q})$, then $\pi[J] \supseteq \pi[J_0] \supseteq I_0$, so

$$\theta(z - S_J(\boldsymbol{u}\pi, d\psi_\pi)) = \theta(z - S_{\pi[J]}(\boldsymbol{u}, d\psi)) \le \epsilon.$$

As ϵ is arbitrary, $\int_{\mathcal{Q}} \boldsymbol{u}\pi\, d\psi_\pi$ is defined and equal to z.

(iii) Suppose that $z = \int_{\mathcal{Q}} \boldsymbol{u}\pi\, d\psi_\pi$ is defined. Let $\epsilon > 0$. Then there is a $J_0 \in \mathcal{I}(\mathcal{Q})$ such that $\theta(z - S_J(\boldsymbol{u}\pi, d\psi_\pi)) \le \epsilon$ whenever $J_0 \subseteq J \in \mathcal{I}(\mathcal{Q})$. If $\pi[J_0] \subseteq I \in \mathcal{I}(\pi[\mathcal{Q}])$, let A be a finite subset of $\mathcal{Q}$ such that $I = \pi[A]$, and let J be the sublattice of $\mathcal{Q}$ generated by $J_0 \cup A$; then $J_0 \subseteq J \in \mathcal{I}(\mathcal{Q})$ and $\pi[J] = I$, so

$$\theta(z - S_I(\boldsymbol{u}, d\psi)) = \theta(z - S_J(\boldsymbol{u}\pi, d\psi_\pi)) \le \epsilon.$$

As ϵ is arbitrary, $\int_{\pi[\mathcal{Q}]} \boldsymbol{u}\, d\psi$ is defined and equal to z. This completes the proof.

635E Proposition Suppose that $\langle \mathfrak{A}_t \rangle_{t\in T}$ is right-continuous. Let $\mathcal{S} \subseteq \mathcal{T}$ be a sublattice and $\boldsymbol{u} = \langle u_\tau \rangle_{\tau\in\mathcal{S}}$ a fully adapted process. Set $\mathcal{Q} = \pi^{-1}[\mathcal{S}] = \operatorname{dom}(\pi\boldsymbol{u})$.

(a) $\boldsymbol{u}\pi$ is order-bounded iff $\boldsymbol{u}\restriction\pi[\mathcal{Q}]$ is order-bounded.

(b) $\boldsymbol{u}\pi$ is of bounded variation iff $\boldsymbol{u}\restriction\pi[\mathcal{Q}]$ is of bounded variation.

(c) $\boldsymbol{u}\pi$ is an integrator for the structure $(\mathfrak{A}, \bar\mu, R, \langle \mathfrak{B}_r \rangle_{r\in R})$ iff $\boldsymbol{u}\restriction\pi[\mathcal{Q}]$ is an integrator for the structure $(\mathfrak{A}, \bar\mu, T, \langle \mathfrak{A}_t \rangle_{t\in T})$.

(d) $\boldsymbol{u}\pi$ is a martingale for the structure $(\mathfrak{A}, \bar\mu, R, \langle \mathfrak{B}_r \rangle_{r\in R})$ iff $\boldsymbol{u}\restriction\pi[\mathcal{Q}]$ is a martingale for the structure $(\mathfrak{A}, \bar\mu, T, \langle \mathfrak{A}_t \rangle_{t\in T})$.

(e) $\boldsymbol{u}\pi$ is jump-free iff $\boldsymbol{u}\restriction\pi[\mathcal{Q}]$ is jump-free.

(f) Let $\hat{\mathcal{S}}$ be the covered envelope of $\mathcal{S}$ in $\mathcal{T}$, $\hat{\mathcal{Q}}$ the covered envelope of $\mathcal{Q}$ in $\mathcal{R}$ and $\hat{\boldsymbol{u}}$ the fully adapted extension of $\boldsymbol{u}$ to $\hat{\mathcal{S}}$. Then $\pi[\hat{\mathcal{Q}}] \subseteq \hat{\mathcal{S}}$ and $\hat{\boldsymbol{u}}\pi\restriction\hat{\mathcal{Q}}$ is the fully adapted extension of $\boldsymbol{u}\pi$ to $\hat{\mathcal{Q}}$.

(g) If $\boldsymbol{u}$ is moderately oscillatory then $\boldsymbol{u}\pi$ is moderately oscillatory.

(h)(i) If $\mathcal{S}$ is order-convex in $\mathcal{T}$ then $\mathcal{Q}$ is order-convex in $\mathcal{R}$.

(ii) Suppose that $\langle \pi_r \rangle_{r\in R}$ is right-continuous (635Cd). If $\mathcal{S}$ is order-convex and $\boldsymbol{u}$ is near-simple, then $\boldsymbol{u}\pi$ is near-simple.

proof If $\mathcal{Q}$ is empty, then $\pi[\mathcal{Q}]$ will be empty and everything is trivial. So suppose that $\mathcal{Q}$ is non-empty. Because $\pi : \mathcal{R} \to \mathcal{T}$ is a lattice homomorphism (635C(b-i)), $\pi[\mathcal{Q}]$ is a sublattice of $\mathcal{T}$.

(a) We have only to note that $\{u_{\pi(\rho)} : \rho \in \mathcal{Q}\} = \{u_\rho : \rho \in \pi[\mathcal{Q}]\}$.

(b)(i) If $\boldsymbol{u}\restriction\pi[\mathcal{Q}]$ is of bounded variation and $\rho_0 \le \ldots \le \rho_n$ in $\mathcal{Q}$ then

$$\sum_{i=0}^{n-1} |u_{\pi(\rho_{i+1})} - u_{\pi(\rho_i)}| \le \int_{\pi[\mathcal{Q}]} |d\boldsymbol{u}|,$$

so $\boldsymbol{u}\pi$ is of bounded variation.

(ii) If $\boldsymbol{u}\pi$ is of bounded variation and $\tau_0 \le \ldots \le \tau_n$ in $\pi[\mathcal{Q}]$, take $\rho'_i \in \mathcal{Q}$ such that $\pi(\rho'_i) = \tau_i$ for each i. Set $\rho_i = \sup_{j\le i} \rho'_j$ for $i \le n$, so that $\rho_0 \le \ldots \le \rho_n$, $\pi(\rho_i) = \tau_i$ for $i \le n$, and

$$\sum_{i=0}^{n-1} |u_{\tau_{i+1}} - u_{\tau_i}| = \sum_{i=0}^{n-1} |u_{\pi(\rho_{i+1})} - u_{\pi(\rho_i)}| \le \int_{\mathcal{Q}} |d(\boldsymbol{u}\pi)|.$$

As $\tau_0, \ldots, \tau_n$ are arbitrary, $\boldsymbol{u}\restriction\pi[\mathcal{Q}]$ is of bounded variation.

(c)(i) $Q_{\pi[\mathcal{Q}]}(d\boldsymbol{u}) \subseteq Q_{\mathcal{Q}}(d(\boldsymbol{u}\pi))$. **P** Suppose that $I \in \mathcal{I}(\pi[\mathcal{Q}])$ is non-empty and that $\boldsymbol{w} = \langle w_\tau\rangle_{\tau\in I}$ is a fully adapted process with $\|\boldsymbol{w}\|_\infty \le 1$. Let $(\tau_0, \ldots, \tau_n)$ linearly generate the I-cells. As in (b-ii) above, we have $\rho_0 \le \ldots \le \rho_n$ in $\mathcal{Q}$ such that $\pi(\rho_i) = \tau_i$ for each i. Now $w_{\tau_i} \in L^0(\mathfrak{A}_{\pi(\rho_i)}) = L^0(\mathfrak{B}_{\rho_i})$ and $\|w_{\tau_i}\|_\infty \le 1$. So

$$\begin{aligned} S_I(\boldsymbol{w}, d\boldsymbol{u}) &= \sum_{i=0}^{n-1} w_{\tau_i} \times (u_{\tau_{i+1}} - u_{\tau_i}) \\ &= \sum_{i=0}^{n-1} w_{\tau_i} \times (u_{\pi(\rho_{i+1})} - u_{\pi(\rho_i)}) \in Q_{\mathcal{Q}}(d(\boldsymbol{u}\pi)) \end{aligned}$$

(616C(ii)). As I and $\boldsymbol{w}$ are arbitrary, $Q_{\pi[\mathcal{Q}]}(d\boldsymbol{u}) \subseteq Q_{\mathcal{Q}}(d(\boldsymbol{u}\pi))$. **Q**

So if $\boldsymbol{u}\pi$ is an integrator, $\boldsymbol{u}\restriction\pi[\mathcal{Q}]$ is an integrator (616Fc, 613B(f-iii)).

(ii) $Q_{\mathcal{Q}}(d(\boldsymbol{u}\pi)) \subseteq Q_{\pi[\mathcal{Q}]}(d\boldsymbol{u})$. **P** Suppose that $I \in \mathcal{I}(\mathcal{Q})$ is non-empty and that $\boldsymbol{w} = \langle w_\tau\rangle_{\tau\in I}$ is a fully adapted process with $\|\boldsymbol{w}\|_\infty \le 1$. Let $(\rho_0, \ldots, \rho_n)$ linearly generate the I-cells. Now $\pi(\rho_0) \le \ldots \le \pi(\rho_n)$ and $w_{\rho_i} \in L^0(\mathfrak{A}_{\pi(\rho_i)})$ and $\|w_{\rho_i}\|_\infty \le 1$ for each i. So

$$S_I(\boldsymbol{w}, d(\boldsymbol{u}\pi)) = \sum_{i=0}^{n-1} w_{\rho_i} \times (u_{\pi(\rho_{i+1})} - u_{\pi(\rho_i)}) \in Q_{\pi[\mathcal{Q}]}(d\boldsymbol{u}).$$

As I and $\boldsymbol{w}$ are arbitrary, $Q_{\mathcal{Q}}(d(\boldsymbol{u}\pi)) \subseteq Q_{\pi[\mathcal{Q}]}(d\boldsymbol{u})$. **Q**

So if $\boldsymbol{u}\restriction\pi[\mathcal{Q}]$ is an integrator, $\boldsymbol{u}\pi$ is an integrator.

(d)(i) Suppose that $\boldsymbol{u}\pi$ is a martingale. Take $\tau \le \tau'$ in $\pi[\mathcal{Q}]$. Again as in (b-ii) of this proof, there are $\rho \le \rho'$ in $\mathcal{Q}$ such that $\tau = \pi(\rho)$ and $\tau' = \pi(\rho')$. Now $\mathfrak{A}_\tau = \mathfrak{B}_\rho$, so

$$P_\tau u_{\tau'} = P_\tau u_{\pi(\rho')} = u_{\pi(\rho)} = u_\tau.$$

As τ and τ' are arbitrary, $\boldsymbol{u}\restriction\pi[\mathcal{Q}]$ is a martingale.

(ii) Suppose that $\boldsymbol{u}\restriction\pi[\mathcal{Q}]$ is a martingale. Take $\rho \le \rho'$ in $\mathcal{Q}$. Then $\pi(\rho) \le \pi(\rho')$ in $\pi[\mathcal{Q}]$, and $\mathfrak{B}_\rho = \mathfrak{A}_{\pi(\rho)}$. So the conditional expectation of $u_{\pi(\rho')}$ on $\mathfrak{B}_\rho$ is $P_{\pi(\rho)}u_{\pi(\rho')} = u_{\pi(\rho)}$. As ρ and ρ' are arbitrary, $\boldsymbol{u}\pi$ is a martingale.

(e) By (a), we know that if either $\boldsymbol{u}\pi$ or $\boldsymbol{u}\restriction\pi[\mathcal{Q}]$ is jump-free, then both are order-bounded. So we can suppose throughout that this is so.

(i) Suppose that $\rho_0 \le \ldots \le \rho_n$ in $\mathcal{Q}$. Set $\rho_{-1} = \min\mathcal{R}$, $\rho_{n+1} = \max\mathcal{R}$, $\tau_{-1} = \min\mathcal{T}$, $\tau_{n+1} = \max\mathcal{T}$ and $\tau_i = \pi(\rho_i)$ for $0 \le i \le n$. Then

$$\begin{aligned} \tilde{w} = \sup\{|u_{\pi(\rho')} - u_{\pi(\rho)}| : \rho,\ \rho' \in \mathcal{Q} \text{ and there is an } i \\ \text{such that } -1 \le i \le n \text{ and } \rho_i \le \rho \le \rho' \le \rho_{i+1}\}, \\ w = \sup\{|u_{\tau'} - u_\tau| : \tau,\ \tau' \in \pi[\mathcal{Q}] \text{ and there is an } i \\ \text{such that } -1 \le i \le n \text{ and } \tau_i \le \tau \le \tau' \le \tau_{i+1}\} \end{aligned}$$

are equal.

P$(\boldsymbol{\alpha})$ If τ, $\tau' \in \pi[\mathcal{Q}]$ are such that $\tau_i \le \tau \le \tau' \le \tau_{i+1}$ where $-1 \le i \le n+1$, then we can express τ, τ' as $\pi(\rho)$, $\pi(\rho')$ for ρ, $\rho' \in \mathcal{Q}$; replacing ρ by $\operatorname{med}(\rho_i, \rho, \rho_{i+1})$ if necessary, we can suppose that $\rho_i \le \rho \le \rho_{i+1}$; replacing ρ' by $\operatorname{med}(\rho, \rho', \rho_{i+1})$ if necessary, we can suppose that $\rho \le \rho' \le \rho_{i+1}$. Now

$$|u_{\tau'} - u_\tau| = |u_{\pi(\rho')} - u_{\pi(\rho)}| \le \tilde{w}.$$

As τ and τ' are arbitrary, $w \le \tilde{w}$.

(β) If w is defined and ρ, $\rho' \in \mathcal{Q}$ are such that $\rho_i \le \rho \le \rho' \le \rho_{i+1}$ where $-1 \le i \le n+1$, then $\tau_i \le \pi(\rho) \le \pi(\rho') \le \tau_{i+1}$, so $|u_{\pi(\rho')} - u_{\pi(\rho)}| \le w$. As ρ and ρ' are arbitrary, $\tilde{w} \le w$. **Q**

(ii) Suppose that $\rho_0 \le \ldots \le \rho_n$ in $\mathcal{Q}$ and that $I \in \mathcal{I}(Q)$, $I' \in \mathcal{I}(\pi[\mathcal{Q}])$ are such that $(\rho_0, \ldots, \rho_n)$ linearly generates the I-cells, while $(\pi(\rho_0), \ldots, \pi(\rho_n))$ linearly generates the I'-cells. Then $\mathrm{Osclln}^*_I(\boldsymbol{u}\pi) = \mathrm{Osclln}^*_{I'}(\boldsymbol{u}\restriction\pi[\mathcal{Q}])$. **P** Put (i) here together with 618Ca. **Q**

(iii) Suppose that $\boldsymbol{u}\pi$ is jump-free. Let $\epsilon > 0$. Then there is a non-empty $I \in \mathcal{I}(\mathcal{Q})$ such that $\mathrm{Osclln}^*_I(\boldsymbol{u}\pi) \le \epsilon$. Let $(\rho_0, \ldots, \rho_n)$ linearly generate the I-cells. Set $I' = \{\pi(\rho_0), \ldots, \pi(\rho_n)\} \in \mathcal{I}(\pi[\mathcal{Q}])$. Then $(\pi(\rho_0), \ldots, \pi(\rho_n))$ linearly generates the I'-cells. By (ii),

$$\theta(\mathrm{Osclln}^*_{I'}(\boldsymbol{u}\restriction\pi[\mathcal{Q}])) = \theta(\mathrm{Osclln}^*_I(\boldsymbol{u}\pi)) \le \epsilon.$$

As ϵ is arbitrary, $\boldsymbol{u}\restriction\pi[\mathcal{Q}]$ is jump-free.

(iv) Suppose that $\boldsymbol{u}\restriction\pi[\mathcal{Q}]$ is jump-free. Let $\epsilon > 0$. Then there is a non-empty $I' \in \mathcal{I}(\pi[\mathcal{Q}])$ such that $\mathrm{Osclln}^*_{I'}(\boldsymbol{u}\restriction\pi[\mathcal{Q}]) \le \epsilon$. Let $(\tau_0, \ldots, \tau_n)$ linearly generate the I'-cells. Take $\rho_0 \le \ldots \le \rho_n$ in $\mathcal{Q}$ such that $\pi(\rho_i) = \tau_i$ for each i. Set $I = \{\rho_0, \ldots, \rho_n\} \in \mathcal{I}(\mathcal{Q})$. Then $(\rho_0, \ldots, \rho_n)$ linearly generates the I-cells, so

$$\theta(\mathrm{Osclln}^*_I(\boldsymbol{u}\pi)) = \theta(\mathrm{Osclln}^*_{I'}(\boldsymbol{u}\restriction\pi[\mathcal{Q}])) \le \epsilon.$$

As ϵ is arbitrary, $\boldsymbol{u}\pi$ is jump-free.

(f) If $\rho \in \hat{\mathcal{Q}}$ then

$$\sup_{\tau\in\mathcal{S}} [\![\pi(\rho) = \tau]\!] \supseteq \sup_{\rho'\in\mathcal{Q}} [\![\pi(\rho) = \pi(\rho')]\!] \supseteq \sup_{\rho'\in\mathcal{Q}} [\![\rho = \rho']\!] = 1,$$

so $\pi(\rho) \in \hat{\mathcal{S}}$. Now $\hat{\boldsymbol{u}}\pi\restriction\hat{\mathcal{Q}}$ is fully adapted and extends $\boldsymbol{u}\pi$, so is the fully adapted extension of $\boldsymbol{u}\pi$.

(g) For any $\epsilon > 0$, there is a process $\boldsymbol{v} = \langle v_\tau\rangle_{\tau\in\mathcal{S}}$ of bounded variation such that $\theta(\sup|\boldsymbol{u} - \boldsymbol{v}|) \le \epsilon$. Now $\boldsymbol{v}\pi$ is of bounded variation ((b) above) and

$$\sup|\boldsymbol{u}\pi - \boldsymbol{v}\pi| = \sup_{\rho\in\mathcal{Q}} |u_{\pi(\rho)} - v_{\pi(\rho)}| \le \sup_{\tau\in\mathcal{S}} |u_\tau - v_\tau| = \sup|\boldsymbol{u} - \boldsymbol{v}|,$$

so $\theta(\sup|\boldsymbol{u}\pi - \boldsymbol{v}\pi|) \le \epsilon$. As ϵ is arbitrary, $\boldsymbol{u}\pi$ is moderately oscillatory.

(h)(i) If ρ_0, $\rho_1 \in \mathcal{Q}$, $\rho \in \mathcal{R}$ and $\rho_0 \le \rho \le \rho_1$, then $\pi(\rho_0) \le \pi(\rho) \le \pi(\rho_1)$, while $\pi(\rho_0)$ and $\pi(\rho_1)$ belong to $\mathcal{S}$; so $\pi(\rho) \in \mathcal{S}$ and $\rho \in \mathcal{Q}$. Thus $\mathcal{Q}$ is order-convex.

(ii) Suppose that $A \subseteq \mathcal{Q}$ is non-empty and downwards-directed and has a lower bound $\rho_* \in \mathcal{Q}$. Then $\inf A$, the infimum of A in $\mathcal{R}$, lies between ρ_* and any member of A, so belongs to $\mathcal{Q}$. Next, $B = \pi[A]$ is non-empty and downwards-directed and has a lower bound in $\mathcal{S}$, so $\inf B \in \mathcal{S}$, while $\inf B = \pi(\inf A)$ by 635C(d-ii). Given $\epsilon > 0$, there is a $\tau_0 \in B$ such that $\theta(u_\tau - u_{\inf B}) \le \epsilon$ whenever $\tau \in B$ and $\tau \le \tau_0$ (632E). Take $\rho_0 \in A$ such that $\tau_0 = \pi(\rho_0)$; then $\theta(u_{\pi(\rho)} - u_{\pi(\inf A)}) \le \epsilon$ whenever $\rho \in A$ and $\rho \le \rho_0$. As A and ϵ are arbitrary, $\boldsymbol{u}\pi$ satisfies (†) of 632F.

As $\boldsymbol{u}$ is near-simple, it is moderately oscillatory, so $\boldsymbol{u}\pi$ is moderately oscillatory. By 632F, $\boldsymbol{u}\pi$ is locally near-simple, and in fact near-simple, by 631F(c-ii).

635F Theorem Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$, and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a near-simple integrator; let $\mathcal{Q}$ be a sublattice of $\mathcal{R}$ such that $\pi[\mathcal{Q}]$ is a cofinal sublattice of $\mathcal{S}$ which $\boldsymbol{v}$-separates $\mathcal{S}$ (633Bb). If $\boldsymbol{u} = \langle u_\tau\rangle_{\tau\in\mathcal{S}}$ is a moderately oscillatory process, then $\int_{\mathcal{Q}} \boldsymbol{u}\pi\, d(\boldsymbol{v}\pi)$ is defined and equal to $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$.

proof By 633Ka, $\int_{\pi[\mathcal{Q}]} \boldsymbol{u}\, d\boldsymbol{v}$ is defined and equal to $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$. By 635Db, this is also $\int_{\mathcal{Q}} \boldsymbol{u}\pi\, d(\boldsymbol{v}\pi)$, since of course the adapted interval function $\Delta(\boldsymbol{v}\pi)$ (613Cc) is precisely $(\Delta\boldsymbol{v})_\pi$ as defined in 635Db.

635G Corollary Suppose that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$, and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a near-simple integrator, with quadratic variation $\boldsymbol{v}^*$. If $\mathcal{Q}$ is a sublattice of $\pi^{-1}[\mathcal{S}]$ such that $\pi[\mathcal{Q}]$ $\boldsymbol{v}$-separates $\mathcal{S}$, then the quadratic variation of $\boldsymbol{v}\pi\restriction\mathcal{Q}$ is $\boldsymbol{v}^*\pi\restriction\mathcal{Q}$.

proof If $\mathcal{Q}$ is empty, this is trivial; so suppose otherwise. By 635Ec, $\boldsymbol{v}\pi$ is an integrator, so $\boldsymbol{v}\pi\restriction\mathcal{Q}$ is an integrator and has a quadratic variation. Take any $\rho \in \mathcal{Q}$. Then $\pi[\mathcal{Q} \wedge \rho]$ $\boldsymbol{v}$-separates $\mathcal{S} \wedge \pi(\rho)$. **P** If τ, $\tau' \in \mathcal{S} \wedge \pi(\rho)$ and $\sigma \in \mathcal{Q}$, then

$$[\![\tau \le \pi(\sigma \wedge \rho)]\!] = [\![\tau \le \pi(\sigma) \wedge \pi(\rho)]\!] = [\![\tau \le \pi(\sigma)]\!] \cap [\![\tau \le \pi(\rho)]\!] = [\![\tau \le \pi(\sigma)]\!],$$

$$[\![\pi(\sigma \wedge \rho) < \tau']\!] = [\![\pi(\sigma) \wedge \pi(\rho) < \tau']\!] = [\![\pi(\sigma) < \tau']\!] \cup [\![\pi(\rho) < \tau']\!] = [\![\pi(\sigma) < \tau']\!],$$

$$[\![\tau \le \pi(\sigma \wedge \rho)]\!] \cap [\![\pi(\sigma \wedge \rho) < \tau']\!] = [\![\tau \le \pi(\sigma)]\!] \cap [\![\pi(\sigma) < \tau']\!].$$

So if $\tau \le \tau'$, then

$$\begin{aligned}[\![v_\tau \ne v_{\tau'}]\!] &\subseteq \sup_{\sigma \in \mathcal{Q}} ([\![\tau \le \pi(\sigma)]\!] \cap [\![\pi(\sigma) < \tau']\!]) \\ &= \sup_{\sigma \in \mathcal{Q}} ([\![\tau \le \pi(\sigma \wedge \rho)]\!] \cap [\![\pi(\sigma \wedge \rho) < \tau']\!]).\end{aligned}$$

As τ and τ' are arbitrary, $\pi[\mathcal{Q} \wedge \rho]$ $\boldsymbol{v}$-separates $\mathcal{S} \wedge \pi(\rho)$. **Q**

Of course $\mathcal{S} \wedge \pi(\rho)$ is still order-convex, and $\pi[\mathcal{Q} \wedge \rho]$ is cofinal with $\mathcal{S} \wedge \pi(\rho)$ because it contains $\pi(\rho)$. So 635F tells us that $\int_{\mathcal{Q} \wedge \rho} \boldsymbol{v}\pi \, d(\boldsymbol{v}\pi) = \int_{\mathcal{S} \wedge \pi(\rho)} \boldsymbol{v} \, d\boldsymbol{v}$. Next, $\boldsymbol{v}^2$ is also a near-simple integrator (631F(a-i), 616N), and clearly $\boldsymbol{v}^2\pi = (\boldsymbol{v}\pi)^2$, so

$$\int_{\mathcal{Q} \wedge \rho} d((\boldsymbol{v}\pi)^2) = \int_{\mathcal{Q} \wedge \rho} d(\boldsymbol{v}^2\pi) = \int_{\mathcal{S} \wedge \pi(\rho)} d(\boldsymbol{v}^2)$$

by 635F again. Accordingly

$$\int_{\mathcal{Q} \wedge \rho} d((\boldsymbol{v}\pi)^2) - 2\int_{\mathcal{Q} \wedge \rho} \boldsymbol{v}\pi \, d(\boldsymbol{v}\pi) = \int_{\mathcal{S} \wedge \pi(\rho)} d(\boldsymbol{v}^2) - 2\int_{\mathcal{S} \wedge \pi(\rho)} \boldsymbol{v} \, d\boldsymbol{v} = v^*_{\pi(\rho)}$$

(617Kb). As ρ is arbitrary, the quadratic variation of $\boldsymbol{v}\pi \restriction \mathcal{Q}$ is the function $\rho \mapsto v^*_{\pi(\rho)} : \mathcal{Q} \to L^0(\mathfrak{A})$, that is, $\boldsymbol{v}^*\pi \restriction \mathcal{Q}$.

635X Basic exercises (a) Suppose that $\langle \mathfrak{A}_t \rangle_{t \in T}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$, and $\boldsymbol{v}$, $\boldsymbol{w}$ local integrators with domain $\mathcal{S}$, with covariation $[\boldsymbol{v} \overset{*}{|} \boldsymbol{w}]$. Show that if $\mathcal{Q}$ is a sublattice of $\pi^{-1}[\mathcal{S}]$ such that $\pi[\mathcal{Q}]$ separates $\mathcal{S}$, then $[\boldsymbol{v}\pi \restriction \mathcal{Q} \overset{*}{|} \boldsymbol{w}\pi \restriction \mathcal{Q}]$ is defined and equal to $[\boldsymbol{v} \overset{*}{|} \boldsymbol{w}]\pi \restriction \mathcal{Q}$.

635Y Further exercises (a) Set $T^{\|} = \{t^- : t \in T\} \cup \{t^+ : t \in T\}$ with the total ordering defined by saying that, for s, $t \in T$,

$$s^+ \le t^+ \iff s^- \le t^- \iff s^- \le t^+ \iff s \le t, \quad s^+ \le t^- \iff s < t.$$

(i) Show that $T^{\|}$ is totally ordered. (ii) For $t \in T$, set $\mathfrak{B}_{t^-} = \mathfrak{A}_t$, $\mathfrak{B}_{t^+} = \mathfrak{A}$ if $t = \max T$, and $\mathfrak{B}_{t^+} = \bigcap_{s>t} \mathfrak{A}_s$ otherwise. Show that $\langle \mathfrak{B}_r \rangle_{r \in T^{\|}}$ is a right-continuous filtration. Write $\mathcal{T}^{\|}$ for the corresponding lattice of stopping times. (iii) For $t \in T$ set $\pi_t = \check{t}^- \in \mathcal{T}^{\|}$. For $\rho \in \mathcal{T}$ let $\pi(\rho) \in \mathcal{T}^{\|}$ be given by the formula in 635B. Show that

$$[\![\pi(\rho) > t^+]\!] = [\![\pi(\rho) > t^-]\!] = [\![\rho > t]\!]$$

for every $t \in T$. (iv) Show that $[\![\pi(\rho) < \pi(\tau)]\!] = [\![\rho < \tau]\!]$ for all ρ, $\tau \in \mathcal{T}$, and that $\pi(\rho) = \pi(\tau)$ iff $\rho = \tau$. (v) Show that if $\tau \in \mathcal{T}^{\|}$ then $\mathfrak{B}_\tau = \bigcap \{\mathfrak{A}_\rho : \rho \in \mathcal{T}, \pi(\rho) \ge \tau\}$.

635 Notes and comments In this section, right-continuity of the filtration $\langle \mathfrak{A}_t \rangle_{t \in T}$ is a generally ruling hypothesis; this is because it seems to be necessary for the basic algebra of the lattice homomorphism $\pi : \mathcal{R} \to \mathcal{T}$ (635Ca). With the elementary properties of this homomorphism in hand, the programme of 635D-635G is obvious, and we have just to make sure that it is watertight. In 653G-653J I will explain how the ideas behind 635F can sometimes be used to replace an integral with respect to an unfamiliar jump-free martingale by an integral with respect to Brownian motion.

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