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Chapter 62

Martingales

The centre of the theory of stochastic integration, since Itô 44, has been integrals $\int \boldsymbol{u}\,d\boldsymbol{v}$ where $\boldsymbol{v}$ is a martingale. In §621 I give a number of inequalities involving *finite* martingales which will make it possible to go straight to the general case in §622. In §622 we have to check some algebra concerning conditional expectations in order to make sense of the idea of 'fully adapted martingale', but the theorem that martingales are local integrators (622G) is a straightforward consequence of 621Hf.

It is not in general the case that an indefinite integral with respect to a martingale is again a martingale. For a full-strength theorem in this direction I think we need to turn to 'virtually local' martingales and do some hard work (623O). To use Itô's formula (619C) in its original form, in which the integrator was Brownian motion, we need of course to know the quadratic variation of Brownian motion, which I come to at last in 624F.

The next three sections are directed towards a structure theory for integrators in §627. This volume is devoted to structures based on probability algebras $(\mathfrak{A},\bar\mu)$. The concepts of Chapter 61 are generally law-independent in the sense that while the existence of the functional $\bar\mu$ is essential, its replacement by another functional $\bar\nu$ such that $(\mathfrak{A},\bar\nu)$ is still a probability algebra makes no difference. However nearly everything involving martingales is shaken up by a change in law. §625 examines such changes, and we find, remarkably, that we do not change the semi-martingales (625F). In §626 I introduce submartingales and previsible variations, with the Doob-Meyer theorem on the expression of submartingales as semi-martingales. In §627 I apply this to supermartingales, and show that local integrators are semi-martingales.

The essential inequality in 621Hf is proved by ordinary martingale methods in §621. There is an alternative route, incidentally yielding a better constant, which depends on a kind of interpolation; I present this in §628.

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621 Finite martingales

I have to justify my repeated assertion that martingales are integrators. This, together with the developments it leads to, will need some non-trivial facts about finite martingales which are most easily described in advance of their applications. However, the complexity of some of the lemmas below may be more bearable if you can see what they're for. So you may wish to treat this section as an appendix, and disentangle the ideas when you find them being called on in §§622, 624 and 626.

621A Notation $(\mathfrak{A},\bar\mu)$ will be a probability algebra, and for $1\le p<\infty$, $L^p_{\bar\mu}=L^p(\mathfrak{A},\bar\mu)\subseteq L^0(\mathfrak{A})$ will be the associated L^p-space, while $\mathbb{E}$ refers to the integral on $L^1_{\bar\mu}$.

621B Uniform integrability (a) Recall that a set $A\subseteq L^1_{\bar\mu}$ is **uniformly integrable** if for every $\epsilon>0$ there is an $M\ge 0$ such that $\mathbb{E}((|u|-M\chi 1)^+)\le\epsilon$ for every $u\in A$; equivalently, if A is $\|\;\|_1$-bounded and for every $\epsilon>0$ there is a $\delta>0$ such that $\mathbb{E}(|u|\times\chi a)\le\epsilon$ whenever $u\in A$, $a\in\mathfrak{A}$ and $\bar\mu a\le\delta$ (246Ca, 246G).

(b) A non-empty set $A\subseteq L^0=L^0(\mathfrak{A})$ is uniformly integrable iff

$$\lim_{\alpha\to\infty}\sup_{u\in A}\mathbb{E}(|u|\times\chi[\![|u|>\alpha]\!])=0$$

.

(c) Suppose that A, $B \subseteq L^1_{\bar\mu}$ are uniformly integrable.

(i) Every subset of A is uniformly integrable; αA is uniformly integrable for every $\alpha \in \mathbb{R}$; $A + B$ is uniformly integrable; the solid hull of A is uniformly integrable.

(ii) The $\mathfrak{T}$-closure $\overline{A}$ of A is uniformly integrable, where $\mathfrak{T}$ is the topology of convergence in measure on L^0, and $\mathfrak{T}$ agrees with the norm topology of $L^1_{\bar\mu}$ on $\overline{A}$.

(d) A subset of $L^1_{\bar\mu}$ is uniformly integrable iff it is relatively compact for the weak topology $\mathfrak{T}_s(L^1_{\bar\mu}, L^\infty(\mathfrak{A}))$.

(e) If $p > 1$, then any $\| \|_p$-bounded subset A of L^0 is uniformly integrable.

621C Conditional expectations (a) Following the definitions in §365, we find that if $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$ then $L^0(\mathfrak{B}) \cap L^1_{\bar\mu} = L^1(\mathfrak{B}, \bar\mu\restriction\mathfrak{B})$, and we have a unique positive linear operator $P_{\mathfrak{B}} : L^1_{\bar\mu} \to L^0(\mathfrak{B}) \cap L^1_{\bar\mu}$ such that $\mathbb{E}(P_{\mathfrak{B}}u \times \chi b) = \mathbb{E}(u \times \chi b)$ whenever $u \in L^1_{\bar\mu}$ and $b \in \mathfrak{B}$. Counting $\|u\|_p$ as ∞ if $u \in L^0(\mathfrak{A}) \setminus L^p(\mathfrak{A}, \bar\mu)$, $\|P_{\mathfrak{B}}u\|_p \le \|u\|_p$ for every $u \in L^1_{\bar\mu}$ and $p \in [1, \infty]$.

(b) If $\mathfrak{B}$ and $\mathfrak{C}$ are closed subalgebras of $\mathfrak{A}$ and $\mathfrak{B} \subseteq \mathfrak{C}$, then

$$P_{\mathfrak{B}}P_{\mathfrak{C}} = P_{\mathfrak{C}}P_{\mathfrak{B}} = P_{\mathfrak{B}}.$$

(c) If $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$, $u \in L^1_{\bar\mu}$, $v \in L^0(\mathfrak{B})$ and $u \times v \in L^1_{\bar\mu}$, then $P_{\mathfrak{B}}(u \times v) = P_{\mathfrak{B}}u \times v$. So if u, v, $u \times P_{\mathfrak{B}}v$ and $P_{\mathfrak{B}}u \times v$ all belong to $L^1_{\bar\mu}$,

$$P_{\mathfrak{B}}(u \times P_{\mathfrak{B}}v) = P_{\mathfrak{B}}u \times P_{\mathfrak{B}}v = P_{\mathfrak{B}}(P_{\mathfrak{B}}u \times v)$$

and $\mathbb{E}(u \times P_{\mathfrak{B}}v) = \mathbb{E}(P_{\mathfrak{B}}u \times v)$.

(d) Let $h : \mathbb{R} \to \mathbb{R}$ be a convex function and $\bar h : L^0 \to L^0$ the corresponding map. If u and $\bar h(u)$ both belong to $L^1_{\bar\mu}$, $\bar h(P_{\mathfrak{B}}u) \le P(\bar h_{\mathfrak{B}}(u))$ for every closed subalgebra $\mathfrak{B}$ of $\mathfrak{A}$.

(e) When $p = 2$, we have a sharper result: if $u \in L^2_{\bar\mu}$ and $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$, then $\|u\|_2^2 = \|P_{\mathfrak{B}}u\|_2^2 + \|u - P_{\mathfrak{B}}u\|^2$.

(f) If $A \subseteq L^1_{\bar\mu}$ is uniformly integrable, then $\{P_{\mathfrak{B}}u : u \in A, \mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}\}$ is uniformly integrable.

(g)(i) If $\mathbb{B}$ is a non-empty downwards-directed family of closed subalgebras of $\mathfrak{A}$ with intersection $\mathfrak{C}$, and $u \in L^1 = L^1_{\bar\mu}$, then $P_{\mathfrak{C}}u$ is the $\| \|_1$-limit $1\lim_{\mathfrak{B}\downarrow\mathbb{B}} P_{\mathfrak{B}}u$.

(ii) If $\mathbb{B}$ is a non-empty upwards-directed family of closed subalgebras of $\mathfrak{A}$, $\mathfrak{C}$ is the closed subalgebra generated by $\bigcup\mathbb{B}$ and $u \in L^1$, then $P_{\mathfrak{C}}u$ is the $\| \|_1$-limit $1\lim_{\mathfrak{B}\uparrow\mathbb{B}} P_{\mathfrak{B}}u$.

621D Definitions For the rest of this section, we shall be looking at a non-decreasing finite sequence $\langle\mathfrak{A}_i\rangle_{i\le n}$ of closed subalgebras of $\mathfrak{A}$. In this context, I will write $P_i : L^1_{\bar\mu} \to L^1_{\bar\mu}$ for the conditional expectation operator associated with $\mathfrak{A}_i$. $P_iP_j = P_{\min(i,j)}$ for all i, $j \le n$. Let $\boldsymbol{v} = \langle v_i\rangle_{i\le n}$ be a finite sequence in $L^1_{\bar\mu}$.

(a) $\boldsymbol{v}$ is a **martingale** adapted to $\langle\mathfrak{A}_i\rangle_{i\le n}$ if $v_i = P_iv_j$ whenever $i \le j \le n$.

(b) $\boldsymbol{v}$ is a **submartingale** adapted to $\langle\mathfrak{A}_i\rangle_{i\le n}$ if $v_i \in L^0(\mathfrak{A}_i)$ and $v_i \le P_iv_j$ whenever $i \le j \le n$.

621E Doob's maximal inequality If $\langle v_i\rangle_{i\le n}$ is a martingale adapted to $\langle\mathfrak{A}_i\rangle_{i\le n}$, and $\bar v = \sup_{i\le n}|v_i|$, then

$$t\bar\mu[\![\bar v > t]\!] \le \mathbb{E}(|v_n| \times \chi[\![\bar v > t]\!]) \le \|v_n\|_1,$$

$$t\bar\mu[\![\bar v \ge t]\!] \le \mathbb{E}(|v_n| \times \chi[\![\bar v \ge t]\!]) \le \|v_n\|_1$$

for every $t \ge 0$.

621F Lemma Suppose that $\langle u_i\rangle_{i<n}$ and $\langle v_i\rangle_{i\le n}$ are such that $u_i \in L^\infty(\mathfrak{A}_i)$ and $\|u_i\|_\infty \le 1$ for every $i<n$ and $\langle v_i\rangle_{i\le n}$ is a martingale adapted to $\langle\mathfrak{A}_i\rangle_{i\le n}$. Set $z=\sum_{i=0}^{n-1} u_i\times(v_{i+1}-v_i)$. Then $\|z\|_2\le\|v_n\|_2$.

621G Proposition Suppose that $\boldsymbol{v}=\langle v_i\rangle_{i\le n}$ is a submartingale adapted to $\langle\mathfrak{A}_i\rangle_{i\le n}$. Then there are a non-decreasing process $\boldsymbol{v}^\#=\langle v_i^\#\rangle_{i\le n}$ and a martingale $\hat{\boldsymbol{v}}$ adapted to $\langle\mathfrak{A}_i\rangle_{i\le n}$ such that $\boldsymbol{v}=\boldsymbol{v}^\#+\hat{\boldsymbol{v}}$ and $v_0^\#=0$. If $-\chi 1\le v_i\le 0$ for every $i\le n$, $\|\hat v_n\|_2^2\le\|v_n\|_1+2\|v_0\|_1$.

621H Lemma Let $\boldsymbol{v}=\langle v_i\rangle_{i\le n}$ be a finite sequence in $L^1_{\bar\mu}$ such that $v_i\in L^0(\mathfrak{A}_i)$ for $i\le n$. Suppose that $\langle\alpha_j\rangle_{j\le m}$, $\langle u_{ji}\rangle_{j\le m,i<n}$ are such that

$$\alpha_j\ge 0 \text{ for } j\le m,\quad \sum_{j=0}^m\alpha_j=1,$$

$$u_{ji}\in L^0(\mathfrak{A}_i),\quad \|u_{ji}\|_\infty\le 1 \text{ for } i<n,\ j\le m.$$

Set $z=\sum_{j=0}^m\alpha_j|\sum_{i=0}^{n-1}u_{ji}\times(v_{i+1}-v_i)|$.

(a) If $\boldsymbol{v}$ is non-negative and non-decreasing, then $\bar\mu[\![z>1]\!]\le\|v_n\|_1$.

(b) If $\boldsymbol{v}$ is a martingale adapted to $\langle\mathfrak{A}_i\rangle_{i\le n}$ then $\bar\mu[\![z>1]\!]\le\|v_n\|_2^2$.

(c) If $\boldsymbol{v}$ is a submartingale adapted to $\langle\mathfrak{A}_i\rangle_{i\le n}$ and $-\chi 1\le v_i\le 0$ for every $i\le n$, then $\bar\mu[\![z>2]\!]\le 3\|v_0\|_1$.

(d) If $\boldsymbol{v}$ is a non-negative martingale adapted to $\langle\mathfrak{A}_i\rangle_{i\le n}$, then $\bar\mu[\![z>2]\!]\le 4\mathbb{E}(v_n)$.

(e) If $\boldsymbol{v}$ is a martingale adapted to $\langle\mathfrak{A}_i\rangle_{i\le n}$, then $\bar\mu[\![z>4]\!]\le 4\|v_n\|_1$.

(f)(cf. Burkholder 66 and 628D below) If $\boldsymbol{v}$ is a martingale adapted to $\langle\mathfrak{A}_i\rangle_{i\le n}$, then $\bar\mu[\![z>\gamma]\!]\le\frac{16}{\gamma}\|v_n\|_1$ for every $\gamma>0$.

(g) If $\boldsymbol{v}$ is a submartingale adapted to $\langle\mathfrak{A}_i\rangle_{i\le n}$, then $\bar\mu[\![z>\gamma]\!]\le\frac{66}{\gamma}\|v_n\|_1-\frac{34}{\gamma}\mathbb{E}(v_0)$ for every $\gamma>0$.

621I Lemma Suppose that $\langle v_i\rangle_{i\le n}$ is a non-negative martingale adapted to $\langle\mathfrak{A}_i\rangle_{i\le n}$, and that $M\ge 0$ is such that $[\![v_i>M]\!]\subseteq[\![v_j=v_n]\!]$ whenever $i\le j\le n$. Suppose that $u_i\in L^\infty(\mathfrak{A}_i)$ and $\|u_i\|_\infty\le 1$ for $i<n$, and set $z=\sum_{i=0}^{n-1}u_i\times(v_{i+1}-v_i)$. Take any $\delta>0$.

(a) z is expressible as $z'+z''$ where z', $z''\in L^0(\mathfrak{A}_n)$ and

$$\|z'\|_1\le(2+\frac{M}{\delta})\|(v_n-M\chi 1)^+\|_1,$$

$$\|z''\|_2^2\le\delta^2+\|v_n\wedge M\chi 1\|_2^2.$$

(b) $\|z\|_1\le\delta+(2+\frac{M}{\delta})\|v_n\|_1+\sqrt{M\|v_n\|_1}$.

621J Lemma Suppose that $\langle v_i\rangle_{i\le n}$ is a non-negative submartingale adapted to $\langle\mathfrak{A}_i\rangle_{i\le n}$ and $z=\sum_{i=0}^{n-1}P_iv_{i+1}-v_i$. Then $\alpha\mathbb{E}(z\times\chi[\![z>2\alpha]\!])\le 3(\beta\mathbb{E}(v_n)+\alpha\mathbb{E}((v_n-\beta\chi 1)^+))$ whenever α, $\beta\ge 0$.

Version of 29.7.20

622 Fully adapted martingales

I come now to the promised central fact of the theory: martingales are local integrators. The first step is to establish a concept of 'martingale' for fully adapted processes (622C), which involves us in the properties of conditional expectations with respect to stopping-time algebras (622B). Elementary facts about martingales are in 622D-622F. The theorem that every martingale is a local integrator is now easy (622H); of course it depends on non-trivial ideas from §621. In 622L I check that Brownian motion is a local martingale. The rest of the section is a miscellany of results which will be needed later.

622A Notation As in Chapter 61, $(\mathfrak{A},\bar\mu,T,\langle\mathfrak{A}_t\rangle_{t\in T},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ will be a stochastic integration structure. For $\tau\in\mathcal{T}$, $P_\tau:L^1_{\bar\mu}\to L^0(\mathfrak{A}_\tau)\cap L^1_{\bar\mu}$, where $L^1_{\bar\mu}=L^1(\mathfrak{A},\bar\mu)$, will be the conditional expectation operator associated with the closed subalgebra $\mathfrak{A}_\tau$ of $\mathfrak{A}$.

622B Proposition Suppose that σ, $\tau \in \mathcal{T}$.
(a) $P_\sigma P_\tau = P_{\sigma\wedge\tau}$.
(b) $[\![\sigma = \tau]\!] \subseteq [\![P_\sigma u = P_\tau u]\!]$ for every $u \in L^1_{\bar\mu}$.

622C Fully adapted martingales Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a fully adapted process.

(a) $\boldsymbol{v}$ is an L^p**-process**, for $1 \le p < \infty$, if $v_\sigma \in L^p(\mathfrak{A}, \bar\mu)$ for every $\sigma \in \mathcal{S}$, and an L^∞**-process** if $v_\sigma \in L^\infty(\mathfrak{A})$ for every $\sigma \in \mathcal{S}$.

For $1 \le p \le \infty$, $\boldsymbol{v}$ is $\| \|_p$**-bounded** if $\sup_{\sigma\in\mathcal{S}} \|v_\sigma\|_p$ is finite (counting the supremum as 0 if $\mathcal{S}$ is empty). $\boldsymbol{v}$ is $\| \|_\infty$-bounded iff it is order-bounded and $\sup|\boldsymbol{v}|$ is in $L^\infty(\mathfrak{A})$.

(b) $\boldsymbol{v}$ is a **martingale** if it is an L^1-process and $v_\sigma = P_\sigma v_\tau$ whenever $\sigma \le \tau$ in $\mathcal{S}$.

(c) $\boldsymbol{v}$ is a **local martingale** if there is a covering ideal $\mathcal{S}'$ of $\mathcal{S}$ such that $\boldsymbol{v}\restriction\mathcal{S}'$ is a martingale.

(e)(i) $\boldsymbol{v}$ is **uniformly integrable** if it is an L^1-process and $\{v_\sigma : \sigma \in \mathcal{S}\}$ is uniformly integrable.

(ii) It will be convenient to use the phrase 'L^p-martingale' to mean a martingale which is also an L^p-process.

622D Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $M_{\text{fa}} = M_{\text{fa}}(\mathcal{S})$ the Riesz space of fully adapted processes with domain $\mathcal{S}$. Let $\mathcal{S}'$ be a sublattice of $\mathcal{S}$.
(a) For any $p \in [1, \infty]$, the set of L^p-processes with domain $\mathcal{S}$ is a solid linear subspace of M_{fa}, and $\boldsymbol{v}\restriction\mathcal{S}'$ is an L^p-process whenever $\boldsymbol{v} \in M_{\text{fa}}$ is an L^p-process.
(b)(i) The set of martingales with domain $\mathcal{S}$ is a linear subspace of M_{fa}.
(ii) If $\boldsymbol{v} \in M_{\text{fa}}$ is a martingale then $\boldsymbol{v}\restriction\mathcal{S}'$ is a martingale.
(c) The set of local martingales with domain $\mathcal{S}$ is a linear subspace of M_{fa}. If $\mathcal{S}'$ is an ideal of $\mathcal{S}$, then $\boldsymbol{v}\restriction\mathcal{S}'$ is a local martingale for every local martingale $\boldsymbol{v} \in M_{\text{fa}}$.
(d) The set of uniformly integrable processes with domain $\mathcal{S}$ is a solid linear subspace of M_{fa}, and $\boldsymbol{v}\restriction\mathcal{S}'$ is uniformly integrable whenever $\boldsymbol{v} \in M_{\text{fa}}$ is uniformly integrable.

622E Elementary facts Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a fully adapted process.

(a) If $\boldsymbol{u}$ is constant with a value in $L^1_{\bar\mu}$, then $\boldsymbol{u}$ is a uniformly integrable martingale.

(b)(i) If $\tau \in \mathcal{S}$, then $\boldsymbol{u}$ is a martingale iff $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ and $\boldsymbol{u}\restriction\mathcal{S}\vee\tau$ are martingales.

(ii) If $\tau \in \mathcal{S}$, $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ is a martingale and $\boldsymbol{u}\restriction\mathcal{S}\vee\tau$ is constant, then $\boldsymbol{u}$ is a martingale.

(c) If for every $\epsilon > 0$ there is a martingale $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ such that $\|u_\sigma - v_\sigma\|_1 \le \epsilon$ for every $\sigma \in \mathcal{S}$, then $\boldsymbol{u}$ is a martingale.

(d) If $\boldsymbol{u}$ is a martingale and $A \subseteq \mathcal{S}$ is non-empty and downwards-directed, then the $\| \|_1$-limit $z = 1\!\lim_{\sigma\downarrow A} u_\sigma$ is defined and is the limit $\lim_{\sigma\downarrow A} u_\sigma$ for the topology of convergence in measure; and if $\tau \in A$ then z is the conditional expectation of u_τ on $\bigcap_{\sigma\in A}\mathfrak{A}_\sigma$.

In particular, if $\mathcal{S}$ is non-empty, then the starting value $\lim_{\sigma\downarrow\mathcal{S}} u_\sigma$ is defined and belongs to $L^1_{\bar\mu}$.

622F Proposition Take any $u \in L^1_{\bar\mu}$.
(a) $\boldsymbol{P}u = \langle P_\tau u\rangle_{\tau\in\mathcal{T}}$ is a uniformly integrable martingale.
(b) Suppose that σ, $\tau \in \mathcal{T}$ and $[\![u \ne 0]\!] \subseteq [\![\sigma = \tau]\!]$. Then $P_\sigma u = P_\tau u$.

622G Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a $\| \|_1$-bounded martingale. Then $\boldsymbol{v}$ is an integrator, therefore moderately oscillatory and order-bounded.

622H Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. If $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a local martingale, then it is a local integrator, therefore locally moderately oscillatory.

622I Doob's maximal inequality (second form) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a martingale. Then $\boldsymbol{v}$ is locally order-bounded, and

$$\bar\mu(\sup_{\sigma\in\mathcal{S}\wedge\tau} [\![|v_\sigma| > \gamma]\!]) = \bar\mu([\![\sup_{\sigma\in\mathcal{S}\wedge\tau} |v_\sigma| > \gamma]\!]) \le \frac{1}{\gamma}\mathbb{E}(|v_\tau|)$$

for every $\tau \in \mathcal{S}$ and $\gamma > 0$.

622J Proposition Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a uniformly integrable martingale.

(a) The $\| \|_1$-limit $v = 1\lim_{\sigma\uparrow\mathcal{S}} v_\sigma$ is defined in $L^1_{\bar\mu}$, and v is also the limit $\lim_{\sigma\uparrow\mathcal{S}} v_\sigma$ for the topology of convergence in measure.

(b) $\boldsymbol{v} = \boldsymbol{P}v{\restriction}\mathcal{S}$ is order-bounded, and $\inf_{\tau\in\mathcal{S}} \sup_{\sigma\in\mathcal{S}\vee\tau} |v - v_\sigma| = 0$.

622K Lemma Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$, and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ an L^1-process such that $\mathbb{E}(u_\sigma) = \mathbb{E}(u_\tau)$ for all σ, $\tau \in \mathcal{S}$. Then $\boldsymbol{u}$ is a martingale.

622L Brownian motion: Theorem Let $\boldsymbol{w}$ be Brownian motion, and $\boldsymbol{\iota}$ the corresponding identity process. Then $\boldsymbol{w}$ and $\boldsymbol{w}^2 - \boldsymbol{\iota}$ are local martingales, and $\boldsymbol{w}{\restriction}\mathcal{T}_b$ is a martingale.

622M Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a fully adapted process. Then $\mathcal{S}' = \{\tau : \tau \in \mathcal{S}, \boldsymbol{v}{\restriction}\mathcal{S}\wedge\tau$ is a martingale$\}$ is an ideal in $\mathcal{S}$, and $\boldsymbol{v}{\restriction}\mathcal{S}'$ is a martingale.

622N Extensions to covered envelopes: Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ with covered envelope $\hat{\mathcal{S}}$, and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a fully adapted process with fully adapted extension $\hat{\boldsymbol{u}} = \langle \hat u_\sigma\rangle_{\sigma\in\hat{\mathcal{S}}}$.

(a) If $\hat{\boldsymbol{u}}$ is a martingale then $\boldsymbol{u}$ is a martingale.

(b) If $\boldsymbol{u}$ is a local martingale then $\hat{\boldsymbol{u}}$ is a local martingale.

(c) $\hat{\boldsymbol{u}}$ is a uniformly integrable martingale iff $\boldsymbol{u}$ is a uniformly integrable martingale.

622O Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a martingale.

(a) If $\mathcal{S}_1$ is the ideal of $\mathcal{T}$ generated by $\mathcal{S}$, there is a unique martingale $\boldsymbol{v} = \langle v_\tau\rangle_{\tau\in\mathcal{S}_1}$ extending $\boldsymbol{u}$.

(b) If $\mathcal{S}_2$ is the full ideal of $\mathcal{T}$ generated by $\mathcal{S}$, there is a local martingale $\hat{\boldsymbol{v}} = \langle \hat v_\tau\rangle_{\tau\in\mathcal{S}_2}$ extending $\boldsymbol{u}$.

622P Proposition Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$, $\boldsymbol{u}$ a moderately oscillatory process and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a martingale. Then $\| \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}\|_2 \le \|\boldsymbol{u}\|_\infty \sup_{\tau\in\mathcal{S}} \|v_\sigma\|_2$.

622Q Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$, $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ fully adapted processes such that $\boldsymbol{u}$ is locally moderately oscillatory and $\boldsymbol{v}$ is a martingale. Then $\| \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\boldsymbol{v}\|_2 \le \|\boldsymbol{u}{\restriction}\mathcal{S}\wedge\tau\|_\infty \|v_\tau\|_2$ for every $\tau \in \mathcal{S}$, and if the right-hand side is always finite, the indefinite integral $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is a martingale.

Version of 10.12.21/21.8.23

623 Approximately and virtually local martingales

I have presented a number of contexts in which an indefinite integral $ii_{\boldsymbol{v}}(\boldsymbol{u})$ can be expected to share properties with the integrator $\boldsymbol{v}$ (614D, 614T, 616J, 618Q). In contrast with this pattern, we can have a martingale with a corresponding indefinite integral which is not a martingale (622Xj), and this occurs in some of the central examples of the theory (631Ya). However the indefinite integral is often 'almost' a martingale in some sense. In this section I give what I think is the most important result in this direction for the Riemann-sum indefinite integral (623O). In the generality here, we need to go a good deal deeper than in §622, with what I call 'virtually local' martingales (623J). These depend, in turn, on a special class of operators on spaces of locally moderately oscillatory processes (623B).

623B The operators R_A: Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $A \subseteq \mathcal{S}$ a non-empty downwards-directed set.

(a) We have an f-algebra homomorphism $R_A : M_{\mathrm{lmo}}(\mathcal{S}) \to M_{\mathrm{lmo}}(\mathcal{S})$ defined by setting

$$R_A(\langle u_\sigma \rangle_{\sigma\in\mathcal{S}}) = \langle \lim_{\rho\downarrow A} u_{\sigma\wedge\rho} \rangle_{\sigma\in\mathcal{S}}$$

whenever $\langle u_\sigma \rangle_{\sigma\in\mathcal{S}} \in M_{\mathrm{lmo}}(\mathcal{S})$, and if $\boldsymbol{u} \in M_{\mathrm{mo}}(\mathcal{S})$ then $R_A(\boldsymbol{u}) \in M_{\mathrm{mo}}(\mathcal{S})$.

(b) $\bar{h}R_A = R_A\bar{h} : M_{\mathrm{lmo}}(\mathcal{S}) \to M_{\mathrm{lmo}}(\mathcal{S})$ for every continuous function $h : \mathbb{R} \to \mathbb{R}$.

(c) Take $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}} \in M_{\mathrm{lmo}}(\mathcal{S})$ and express $\boldsymbol{u}' = R_A(\boldsymbol{u})$ as $\langle u'_\sigma \rangle_{\sigma\in\mathcal{S}}$.

(i) The starting values $\lim_{\sigma\downarrow\mathcal{S}} u'_\sigma$ and $\lim_{\sigma\downarrow\mathcal{S}} u_\sigma$ are defined and equal.

(ii) If $\boldsymbol{u}$ is $\| \|_1$-bounded then $\boldsymbol{u}'$ is $\| \|_1$-bounded and $\sup_{\sigma\in\mathcal{S}} \|u'_\sigma\|_1 \le \sup_{\sigma\in\mathcal{S}} \|u_\sigma\|_1$.

(d) Write $\hat{\mathcal{S}}$ for the covered envelope of $\mathcal{S}$. If $\boldsymbol{u} \in M_{\mathrm{lmo}}(\mathcal{S})$ has fully adapted extension $\hat{\boldsymbol{u}}$ to $\hat{\mathcal{S}}$, then $R_A(\hat{\boldsymbol{u}})$ is the fully adapted extension of $R_A(\boldsymbol{u})$ to $\hat{\mathcal{S}}$.

623C Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. For a non-empty downwards-directed set $A \subseteq \mathcal{S}$ let $R_A : M_{\mathrm{lmo}}(\mathcal{S}) \to M_{\mathrm{lmo}}(\mathcal{S})$ be the operator described in 623B. Let A, $B \subseteq \mathcal{S}$ be non-empty downwards-directed sets.

(a) Setting $A\vee B = \{\rho\vee\rho' : \rho \in A, \rho' \in B\}$ and $A\wedge B = \{\rho\wedge\rho' : \rho \in A, \rho' \in B\}$, $R_{A\vee B}+R_{A\wedge B} = R_A+R_B$.

(b) $R_{A\wedge B} = R_A R_B = R_B R_A$.

(c) If $B \subseteq A$, then $R_A R_B = R_A$; in particular, $R_A^2 = R_A$.

(d) If B is a coinitial subset of A, then $R_B = R_A$.

623D Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $A \subseteq \mathcal{S}$ a non-empty downwards-directed set. Let $R_A : M_{\mathrm{lmo}}(\mathcal{S}) \to M_{\mathrm{lmo}}(\mathcal{S})$ be the operator described in 623B. If $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma\in\mathcal{S}}$ is a (local) integrator, $R_A(\boldsymbol{v})$ is a (local) integrator.

623E Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $A \subseteq \mathcal{S}$ a non-empty downwards-directed set. Let $R_A : M_{\mathrm{lmo}}(\mathcal{S}) \to M_{\mathrm{lmo}}(\mathcal{S})$ be the operator described in 623B. If $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$ is a martingale, $R_A(\boldsymbol{u})$ is a martingale.

623F Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $A \subseteq \mathcal{S}$ a non-empty downwards-directed set and $R_A : M_{\mathrm{lmo}}(\mathcal{S}) \to M_{\mathrm{lmo}}(\mathcal{S})$ the operator described in 623B. Suppose that $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$ is locally moderately oscillatory.

(a) If $\boldsymbol{u}$ is order-bounded, the residual oscillation $\mathrm{Osclln}(R_A(\boldsymbol{u}))$ is at most $\mathrm{Osclln}(\boldsymbol{u})$.

(b) If $\boldsymbol{u}$ is (locally) jump-free, then $R_A(\boldsymbol{u})$ is (locally) jump-free.

623G Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $A \subseteq \mathcal{S}$ a non-empty downwards-directed set. Let $R_A : M_{\mathrm{lmo}}(\mathcal{S}) \to M_{\mathrm{lmo}}(\mathcal{S})$ be the operator described in 623B. If $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$ is locally moderately oscillatory and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma\in\mathcal{S}}$ is a local integrator,

$$R_A(ii_{\boldsymbol{v}}(\boldsymbol{u})) = ii_{R_A(\boldsymbol{v})}(\boldsymbol{u}) = ii_{R_A(\boldsymbol{v})}(R_A(\boldsymbol{u})).$$

623H Corollary Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $A \subseteq \mathcal{S}$ a non-empty downwards-directed set and $R_A : M_{\mathrm{lmo}}(\mathcal{S}) \to M_{\mathrm{lmo}}(\mathcal{S})$ the operator described in 623B. If $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma\in\mathcal{S}}$ is a local integrator with quadratic variation $\boldsymbol{v}^*$, then $R_A(\boldsymbol{v}^*)$ is the quadratic variation of $R_A(\boldsymbol{v})$.

623I Lemma Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$. Suppose that $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$ is a moderately oscillatory process, $\tau^* \in \mathcal{S}$ and $M \ge 0$.

(a) Set

$$A = \{\rho : \rho \in \mathcal{S}, [\![\rho < \tau^*]\!] \subseteq [\![|u_\rho| \ge M]\!]\}.$$

Then $\tau^* \in A$ and $\rho \wedge \rho' \in A$ whenever ρ, $\rho' \in A$.

(b) Let $R_A : M_{\mathrm{lmo}}(\mathcal{S}) \to M_{\mathrm{lmo}}(\mathcal{S})$ be the operator described in 623B. Suppose that $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma\in\mathcal{S}}$ is a moderately oscillatory process such that $R_A(\boldsymbol{v}) = \boldsymbol{v}$.

(i) $[\![|u_\sigma| \ge M]\!] \subseteq [\![v_\sigma = v_\tau]\!]$ whenever $\sigma \le \tau$ in $\mathcal{S} \wedge \tau^*$.

(ii) Expressing $R_A(\boldsymbol{u})$ as $\langle u'_\sigma \rangle_{\sigma\in\mathcal{S}}$, $[\![|u'_\sigma| > M]\!] \subseteq [\![v_\sigma = v_{\tau^*}]\!]$ for every $\sigma \in \mathcal{S}$. In particular, $[\![|u'_\sigma| > M]\!] \subseteq [\![u'_\sigma = u'_{\tau^*}]\!]$.

623J Definition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u}$ a locally moderately oscillatory process. Let $\hat{\mathcal{S}}$ be the covered envelope of $\mathcal{S}$ and $\hat{\boldsymbol{u}} = \langle \hat{u}_\sigma \rangle_{\sigma \in \hat{\mathcal{S}}}$ the fully adapted extension of $\boldsymbol{u}$ to $\hat{\mathcal{S}}$. Recall that $\hat{\boldsymbol{u}}$ is locally moderately oscillatory. I will say that $\boldsymbol{u}$ is an **approximately local martingale** if for every $\sigma \in \mathcal{S}$ and $\epsilon > 0$ there is a non-empty downwards-directed set $A \subseteq \mathcal{S}$ such that $\sup_{\rho \in A} \bar{\mu}[\![\rho < \sigma]\!] \le \epsilon$ and $R_A(\boldsymbol{u})$, as defined in 623B, is a martingale; while $\boldsymbol{u}$ is a **virtually local martingale** if $\hat{\boldsymbol{u}}$ is an approximately local martingale.

623K Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.

(a)(i) The space $M_{\mathrm{alm}}(\mathcal{S})$ of approximately local martingales on $\mathcal{S}$ is a linear subspace of $M_{\mathrm{lmo}}(\mathcal{S})$.

(ii) The space $M_{\mathrm{vlm}}(\mathcal{S})$ of virtually local martingales on $\mathcal{S}$ is a linear subspace of $M_{\mathrm{lmo}}(\mathcal{S})$.

(b)(i) A local martingale on $\mathcal{S}$ is an approximately local martingale.

(ii) An approximately local martingale on $\mathcal{S}$ is a virtually local martingale.

(iii) If $\mathcal{S}$ is finitely full, a virtually local martingale on $\mathcal{S}$ is an approximately local martingale.

(c) If $\boldsymbol{u} \in M_{\mathrm{vlm}}(\mathcal{S})$ and $A \subseteq \mathcal{S}$ is a non-empty downwards-directed set, $R_A(\boldsymbol{u})$, as defined in 623B, is a virtually local martingale.

(d) Every virtually local martingale on $\mathcal{S}$ is a local integrator, therefore locally moderately oscillatory.

(e)(i) If $\boldsymbol{u} \in M_{\mathrm{fa}}(\mathcal{S})$, then $\boldsymbol{u}$ is an approximately local martingale iff $\boldsymbol{u}\restriction\mathcal{S} \wedge \tau$ is an approximately local martingale for every $\tau \in \mathcal{S}$.

(ii) If $\boldsymbol{u} \in M_{\mathrm{fa}}(\mathcal{S})$, then $\boldsymbol{u}$ is a virtually local martingale iff $\boldsymbol{u}\restriction\mathcal{S} \wedge \tau$ is a virtually local martingale for every $\tau \in \mathcal{S}$.

(f) A uniformly integrable approximately local martingale on $\mathcal{S}$ is a martingale.

(g) If $\mathcal{S} \ne \emptyset$ and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ is a virtually local martingale, then $\lim_{\sigma \downarrow \mathcal{S}} u_\sigma$ is defined and belongs to L^1.

(h) If $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}} \in M_{\mathrm{vlm}}(\mathcal{S})$ and $\tau \in \mathcal{S}$ then $(\boldsymbol{u} - u_\tau \mathbf{1})\restriction\mathcal{S} \vee \tau$ is a virtually local martingale.

623L Theorem Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$, and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ a $\| \|_1$-bounded approximately local martingale. Write γ for $\sup_{\sigma \in \mathcal{S}} \|v_\sigma\|_1$.

(a) $\boldsymbol{v}$ is an integrator, therefore moderately oscillatory, and $\lim_{\sigma \uparrow \mathcal{S}} v_\sigma$ is defined.

(b) $\bar{v} = \sup_{\sigma \in \mathcal{S}} |v_\sigma|$ is defined in $L^0(\mathfrak{A})$, and $\theta(\bar{v}) \le 2\sqrt{\gamma}$.

623M Doob's quadratic maximal inequality: Proposition If $\mathcal{S}$ is a non-empty sublattice of $\mathcal{T}$, $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ is an approximately local martingale, and $\gamma = \sup_{\sigma \in \mathcal{S}} \|v_\sigma\|_2$ is finite, then $\boldsymbol{v}$ is order-bounded and $\| \sup |\boldsymbol{v}| \|_2 \le 2\gamma$.

623N Theorem Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ an approximately local martingale.

(a) $\boldsymbol{v}$ is a martingale iff $\boldsymbol{v}\restriction\mathcal{S} \wedge \tau$ is uniformly integrable for every $\tau \in \mathcal{S}$.

(b) The following are equiveridical:

(i) $\boldsymbol{v}$ is uniformly integrable;

(ii) there is a $z \in L^1$ such that $\boldsymbol{v} = \boldsymbol{P}z\restriction\mathcal{S}$;

(iii) $\{v_\sigma : \sigma \in \mathcal{S}\}$ is $\| \|_1$-bounded and $\|v_\uparrow\|_1 \ge \sup_{\sigma \in \mathcal{S}} \|v_\sigma\|_1$, where $v_\uparrow = \lim_{\sigma \uparrow \mathcal{S}} v_\sigma$;

(iv) $\boldsymbol{v}$ is a martingale and the limit $1\lim_{\sigma \uparrow \mathcal{S}} v_\sigma$ is defined in L^1.

623O Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ a locally moderately oscillatory process and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ a virtually local martingale. Then $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is a virtually local martingale.

Version of 21.2.23/22.8.23

624 Quadratic variation

We are at last ready to determine the quadratic variation of Brownian motion (624F). I take the opportunity to tidy up some simple consequences of results in §§617 and 623 (624B-624E), and to give useful facts about L^2-martingales (624G-624I).

624B Theorem Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$, $\boldsymbol{w} = \langle w_\sigma \rangle_{\sigma \in \mathcal{S}}$ virtually local martingales such that $v_\downarrow \times w_\downarrow \in L^1$ where $v_\downarrow = \lim_{\sigma \downarrow \mathcal{S}} v_\sigma$ and $w_\downarrow = \lim_{\sigma \downarrow \mathcal{S}} w_\sigma$. Then $\boldsymbol{v} \times \boldsymbol{w} - [\boldsymbol{v} \overset{*}{|} \boldsymbol{w}]$ is a virtually local martingale.

624C Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma\in\mathcal{S}}$, $\boldsymbol{w} = \langle w_\sigma \rangle_{\sigma\in\mathcal{S}}$ local integrators. If one of them is locally jump-free and one is locally of bounded variation, then $[\boldsymbol{v} \overset{*}{|} \boldsymbol{w}] = 0$. In particular, if $\boldsymbol{v}$ is locally jump-free and locally of bounded variation, then $\boldsymbol{v}^* = 0$.

624D Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma\in\mathcal{S}}$ a virtually local martingale. Then the following are equiveridical:
(i) $\boldsymbol{v}$ is constant;
(ii) $\boldsymbol{v}$ is locally jump-free and locally of bounded variation;
(iii) the quadratic variation of $\boldsymbol{v}$ is zero.
[For 624E/653G, want $[\![\boldsymbol{v} \ne v_\downarrow \mathbf{1}]\!] \subseteq [\![\boldsymbol{v}^* \ne \mathbf{0}]\!]$]

624E Corollary Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma\in\mathcal{S}}$ a virtually local martingale with quadratic variation $\langle v^*_\sigma \rangle_{\sigma\in\mathcal{S}}$. If τ, $\tau' \in \mathcal{S}$ are such that $v^*_\tau = v^*_{\tau'}$, then $\boldsymbol{v}$ is constant on $\mathcal{S} \cap [\tau \wedge \tau', \tau \vee \tau']$.
we need $[\![v_\tau \ne v_{\tau'}]\!] \subseteq [\![v^*_\tau \ne v^*_{\tau'}]\!]$ for 653G

624F Theorem Let $\boldsymbol{w} = \langle w_\tau \rangle_{\tau\in\mathcal{T}_f}$ be Brownian motion. Then its quadratic variation $\boldsymbol{w}^*$ is the identity process $\boldsymbol{\iota}$.

624G Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma\in\mathcal{S}}$ an L^2-martingale and $\boldsymbol{v}^* = \langle v^*_\sigma \rangle_{\sigma\in\mathcal{S}}$ its quadratic variation. Then $\mathbb{E}(v^*_\tau) \le \mathbb{E}(v^2_\tau)$ for every $\tau \in \mathcal{S}$.

624H Proposition Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$, $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma\in\mathcal{S}}$ a virtually local martingale with starting value 0, and $\boldsymbol{v}^* = \langle v^*_\sigma \rangle_{\sigma\in\mathcal{S}}$ its quadratic variation.
(a) $\|v_\tau\|_2 \le \sqrt{\|v^*_\tau\|_1}$ for every $\tau \in \mathcal{S}$.
(b) If moreover $\boldsymbol{v}$ is an approximately local martingale and $\boldsymbol{v}^*$ is an L^1-process, then $\boldsymbol{v}$ and $ii_{\boldsymbol{v}}(\boldsymbol{v})$ are martingales, and $\|v_\tau\|_2 = \sqrt{\|v^*_\tau\|_1}$ for every $\tau \in \mathcal{S}$.

624I Corollary Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma\in\mathcal{S}}$ a $\| \|_2$-bounded martingale with quadratic variation $\boldsymbol{v}^*$, and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$ a $\| \|_\infty$-bounded moderately oscillatory process. Then $\mathbb{E}((\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v})^2)$ and $\mathbb{E}(\int_{\mathcal{S}} \boldsymbol{u}^2 d\boldsymbol{v}^*)$ are finite and equal.

Version of 6.3.24

625 Changing the measure

I give essential formulae for calculating the effect of replacing a given probability measure $\bar\mu$ with an equivalent probability measure $\bar\nu$ (625B-625C). Semi-martingales (625D) remain semi-martingales under any such change (625F).

625A Notation I continue in the framework developed in Chapter 61. $(\mathfrak{A}, \bar\mu, T, \langle\mathfrak{A}_t\rangle_{t\in T}, \mathcal{T}, \langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ will be a stochastic integration structure, and $\mathbb{E}_{\bar\mu}$ the integral corresponding to $\bar\mu$. For $\tau \in \mathcal{T}$, $P_\tau : L^1_{\bar\mu} \to L^1_{\bar\mu}$ will be the conditional expectation operator associated with $\mathfrak{A}_\tau$; if $z \in L^1_{\bar\mu}$, $\boldsymbol{P}z = \langle P_\tau z \rangle_{\tau\in\mathcal{T}}$ will be the martingale derived from z. If $\mathcal{S}$ is a sublattice of $\mathcal{T}$, $M_{\mathrm{fa}}(\mathcal{S})$ will be the space of fully adapted processes with domain $\mathcal{S}$, and $M_{\mathrm{lmo}}(\mathcal{S})$ the space of locally moderately oscillatory processes with domain $\mathcal{S}$.

625B Change of law: Theorem Let $\bar\nu$ be a second functional such that $(\mathfrak{A}, \bar\nu)$ is a probability algebra; write $\mathbb{E}_{\bar\nu}$ and $L^1_{\bar\nu}$ for the corresponding integral and L^1-space.
(a)(i) There is a unique $z \in L^1_{\bar\mu}$ such that $\bar\nu a = \mathbb{E}_{\bar\mu}(z \times \chi a)$ for every $a \in \mathfrak{A}$.
(ii) $[\![z > 0]\!] = 1$ and z has a multiplicative inverse $\frac{1}{z}$ in L^0.
(iii) For $w \in L^0$, $\mathbb{E}_{\bar\nu}(w) = \mathbb{E}_{\bar\mu}(w \times z)$ if either is defined in $[-\infty, \infty]$.
(iv) $\frac{1}{z} \in L^1_{\bar\nu}$ and $\bar\mu a = \mathbb{E}_{\bar\nu}(\frac{1}{z} \times \chi a)$ for every $a \in \mathfrak{A}$.

(v) For $w \in L^0$,

$$w \in L^1_{\bar\nu} \iff w \times z \in L^1_{\bar\mu}, \quad w \in L^1_{\bar\mu} \iff w \times \frac{1}{z} \in L^1_{\bar\nu}.$$

(vi) $[\![P_\tau z > 0]\!] = 1$ for every $\tau \in \mathcal{T}$.

(vii) If $\tau \in \mathcal{T}$ and $w \in L^0(\mathfrak{A}_\tau)$, then $w \in L^1_{\bar\nu}$ iff $w \times P_\tau z \in L^1_{\bar\mu}$.

(viii) We have a fully adapted process $\boldsymbol{u} = \langle u_\tau \rangle_{\tau\in\mathcal{T}}$ defined by saying that $u_\tau = \dfrac{1}{P_\tau z}$ is the multiplicative inverse of $P_\tau z$ for every $\tau \in \mathcal{T}$.

(b) For $\tau \in \mathcal{T}$, let $Q_\tau : L^1_{\bar\nu} \to L^1_{\bar\nu} \cap L^0(\mathfrak{A}_\tau)$ be the conditional expectation operator with respect to the closed subalgebra $\mathfrak{A}_\tau$ for the probability $\bar\nu$.

(i) If $w \in L^1_{\bar\nu}$, $P_\tau(w \times z) = Q_\tau w \times P_\tau z$.

(ii) If $w \in L^1_{\bar\mu}$, $Q_\tau(w \times \frac{1}{z}) = P_\tau w \times Q_\tau(\frac{1}{z})$.

(iii) $P_\tau(z) \times Q_\tau(\frac{1}{z}) = \chi 1$.

(c) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{w} = \langle w_\sigma \rangle_{\sigma\in\mathcal{S}}$ a fully adapted process. Write $\boldsymbol{Q}(\frac{1}{z})$ for the $\bar\nu$-martingale $\langle Q_\tau(\frac{1}{z}) \rangle_{\tau\in\mathcal{T}}$.

(i) $\boldsymbol{w}$ is is a $\bar\nu$-martingale iff $\boldsymbol{w} \times \boldsymbol{P}z$ is a $\bar\mu$-martingale. In particular, $\boldsymbol{u}$ in (a-viii) is a $\bar\nu$-martingale.

(ii) $\boldsymbol{w}$ is is a local $\bar\nu$-martingale iff $\boldsymbol{w} \times \boldsymbol{P}z$ is a local $\bar\mu$-martingale.

625C Proposition As in 625B, let $\bar\nu$ be such that $(\mathfrak{A}, \bar\nu)$ is a probability algebra; write $\mathbb{E}_{\bar\nu}$ and $L^1_{\bar\nu}$ for the corresponding integral and L^1-space, and let $z \in L^1_{\bar\mu}$ be such that $\bar\nu a = \mathbb{E}_{\bar\mu}(z \times \chi a)$ for every $a \in \mathfrak{A}$. Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{w} = \langle w_\sigma \rangle_{\sigma\in\mathcal{S}}$ a fully adapted process. Then $\boldsymbol{w}$ is an approximately local $\bar\nu$-martingale iff $\boldsymbol{w} \times \boldsymbol{P}z$ is an approximately local $\bar\mu$-martingale, and $\boldsymbol{w}$ is a virtually local $\bar\nu$-martingale iff $\boldsymbol{w} \times \boldsymbol{P}z$ is a virtually local $\bar\mu$-martingale.

625D Definition Let $\mathcal{S} \subseteq \mathcal{T}$ be a sublattice. A process with domain $\mathcal{S}$ is a **semi-martingale** if it is expressible as the sum of a virtually local martingale and a process which is locally of bounded variation.

625E Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. The set of semi-martingales with domain $\mathcal{S}$ is a linear subspace of the space of local integrators with domain $\mathcal{S}$. In particular, every semi-martingale is locally moderately oscillatory.

625F Theorem A semi-martingale remains a semi-martingale under any change of law.

Version of 21.1.25

626 Submartingales and previsible variations

Turning to submartingales, I start with the elementary theory (626B-626G). Serious work begins with what I call 'previsible variations' (626J-626K), based on a new adapted interval function $P\Delta\boldsymbol{v}$ (626H-626I). Now the final formula of §621 gives us the celebrated Doob-Meyer decomposition theorem (626M, 626O). The computation of previsible variations can be difficult, but I give some basic special cases (626Q, 626S and 626T).

626B Definition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma\in\mathcal{S}}$ a fully adapted process. $\boldsymbol{v}$ is a **submartingale** if it is an L^1-process and $v_\sigma \le P_\sigma v_\tau$ whenever $\sigma \le \tau$ in $\mathcal{S}$.

Clearly every martingale is a submartingale, and every non-decreasing L^1-process is a submartingale.

626C Elementary facts Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma\in\mathcal{S}}$ a submartingale.

(a) $\mathbb{E}(v_\sigma) \le \mathbb{E}(v_\tau)$ whenever $\sigma \le \tau$ in $\mathcal{S}$.

(b) If $\mathcal{S}'$ is a sublattice of $\mathcal{S}$, then $\boldsymbol{v}\restriction\mathcal{S}'$ is a submartingale.

(c) If $h : \mathbb{R} \to \mathbb{R}$ is a convex function, $\bar h\boldsymbol{v} = \langle \bar h(v_\sigma) \rangle_{\sigma\in\mathcal{S}}$ is an L^1-process and *either* h is non-decreasing *or* $\boldsymbol{v}$ is a martingale, then $\bar h\boldsymbol{v}$ is a submartingale.

$\alpha\boldsymbol{v}$ is a submartingale for every $\alpha \ge 0$.

(d) If $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ is another submartingale, $\boldsymbol{u} + \boldsymbol{v}$ is a submartingale.

626D Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ a submartingale.
(a) If $\boldsymbol{v}$ is $\| \|_1$-bounded, it is an integrator.
(b) If $\mathcal{S}$ has a greatest element and $\{\mathbb{E}(v_\sigma) : \sigma \in \mathcal{S}\}$ is bounded below, $\boldsymbol{v}$ is $\| \|_1$-bounded.
(c) If $\{\mathbb{E}(v_\sigma) : \sigma \in \mathcal{S}\}$ is bounded below, $\boldsymbol{v}$ is a local integrator.

626E Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ a submartingale and $A \subseteq \mathcal{S}$ a non-empty downwards-directed set such that $\{\mathbb{E}(v_\sigma) : \sigma \in A\}$ is bounded below. Then the $\| \|_1$-limit $1\lim_{\sigma \downarrow A} v_\sigma$ is defined and equal to the limit $\lim_{\sigma \downarrow A} v_\sigma$ for the topology of convergence in measure.

626F Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ a submartingale such that $\{\mathbb{E}(v_\sigma) : \sigma \in \mathcal{S}\}$ is bounded below. Let $A \subseteq \mathcal{S}$ be a non-empty downwards-directed set and $R_A : M_{\mathrm{lmo}}(\mathcal{S}) \to M_{\mathrm{lmo}}(\mathcal{S})$ the corresponding operator as described in 623B. Then $R_A(\boldsymbol{v})$ is defined and is a submartingale.

626G Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ a submartingale. Let $\hat{\boldsymbol{v}} = \langle \hat{v}_\tau \rangle_{\tau \in \hat{\mathcal{S}}}$ be the fully adapted extension of $\boldsymbol{v}$ to the covered envelope $\hat{\mathcal{S}}$ of $\mathcal{S}$, and $\hat{\mathcal{S}}_f$ the finitely-covered envelope of $\mathcal{S}$.
(a) $\hat{\boldsymbol{v}} \restriction \hat{\mathcal{S}}_f$ is a submartingale.
(b) If $\boldsymbol{v}$ is $\| \|_1$-bounded then $\hat{\boldsymbol{v}}$ is $\| \|_1$-bounded.
(c) If $\mathcal{S}$ has a greatest element and $\{\mathbb{E}(v_\sigma) : \sigma \in \mathcal{S}\}$ is bounded below, then $\hat{\boldsymbol{v}}$ is a submartingale.

626H Proposition Suppose that $\mathcal{S}$ is a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ is an L^1-process. Then we have an adapted interval function $P\Delta\boldsymbol{v}$ defined by saying that $(P\Delta\boldsymbol{v})(\sigma, \tau) = P_\sigma v_\tau - v_\sigma$ whenever $\sigma \le \tau$ in $\mathcal{S}$.

626I Definitions Corresponding to the interval functions $P\Delta\boldsymbol{v}$ of 626H, write $\Delta_e(\boldsymbol{u}, Pd\boldsymbol{v})$, $S_I(\boldsymbol{u}, Pd\boldsymbol{v})$ and $Q_{\mathcal{S}}(Pd\boldsymbol{v})$ for $\Delta_e(\boldsymbol{u}, d(P\Delta\boldsymbol{v}))$ $S_I(\boldsymbol{u}, d(P\Delta\boldsymbol{v}))$ and $Q_{\mathcal{S}}(d(P\Delta\boldsymbol{v}))$ respectively. $\Delta_e(\boldsymbol{u}, |Pd\boldsymbol{v}|)$, $S_I(\boldsymbol{u}, |Pd\boldsymbol{v}|)$ and $Q_{\mathcal{S}}(|Pd\boldsymbol{v}|)$ will mean $\Delta_e(\boldsymbol{u}, d|P\Delta\boldsymbol{v}|)$, $S_I(\boldsymbol{u}, d|P\Delta\boldsymbol{v}|)$ and $Q_{\mathcal{S}}(d|P\Delta\boldsymbol{v}|)$.
Note that an L^1-process $\boldsymbol{v}$ is a submartingale iff $P\Delta\boldsymbol{v} \ge 0$.

626J Previsible variations Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ an L^1-process. If the weak limit

$$v^\#_\tau = \mathrm{w1lim}_{I \uparrow \mathcal{I}(\mathcal{S} \wedge \tau)} S_I(\mathbf{1}, Pd\boldsymbol{v})$$

is defined in $L^1_{\bar\mu}$ for every $\tau \in \mathcal{S}$, I will say that $\boldsymbol{v}^\# = \langle v^\#_\sigma \rangle_{\sigma \in \mathcal{S}}$ is the **previsible variation** of $\boldsymbol{v}$.

626K Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. Write $M_{\text{D-M}} = M_{\text{D-M}}(\mathcal{S})$ for the set of L^1-processes with domain $\mathcal{S}$ which have previsible variations.
(a) If $\boldsymbol{v} \in M_{\text{D-M}}$ then $\boldsymbol{v}^\#$ is an L^1-process and $\boldsymbol{v} - \boldsymbol{v}^\#$ is a martingale.
(b) $M_{\text{D-M}}$ is a linear subspace of $M_{\mathrm{fa}}(\mathcal{S})$, and the map $\boldsymbol{v} \mapsto \boldsymbol{v}^\# : M_{\text{D-M}} \to M_{\mathrm{fa}}(\mathcal{S})$ is linear.
(c) If $\boldsymbol{v}$ is a martingale with domain $\mathcal{S}$, then $\boldsymbol{v} \in M_{\text{D-M}}$ and $\boldsymbol{v}^\# = 0$.
(d) Suppose that $\boldsymbol{v} \in M_{\text{D-M}}$.
 (i) $\boldsymbol{v}$ is locally moderately oscillatory iff $\boldsymbol{v}^\#$ is locally moderately oscillatory.
 (ii) $\boldsymbol{v}$ is a local integrator iff $\boldsymbol{v}^\#$ is a local integrator.
 (iii) $\boldsymbol{v}$ is a submartingale iff $\boldsymbol{v}^\#$ is a submartingale.
 (iv) $\boldsymbol{v}$ is a martingale iff $\boldsymbol{v}^\#$ is a martingale.
(e) If $\boldsymbol{v} \in M_{\text{D-M}}$ then $P\Delta\boldsymbol{v}^\# = P\Delta\boldsymbol{v}$, $\boldsymbol{v}^\# \in M_{\text{D-M}}$ and $(\boldsymbol{v}^\#)^\# = \boldsymbol{v}^\#$.
(f) Suppose that $\boldsymbol{v} \in M_{\text{D-M}}$ and $\rho \in \mathcal{S}$. Express $\boldsymbol{v}^\#$ as $\langle v^\#_\sigma \rangle_{\sigma \in \mathcal{S}}$.
 (i) $\boldsymbol{v} \restriction \mathcal{S} \wedge \rho$ has a previsible variation, which is $\boldsymbol{v}^\# \restriction \mathcal{S} \wedge \rho$.
 (ii) $\boldsymbol{v} \restriction \mathcal{S} \vee \rho$ has a previsible variation, which is $\langle v^\#_\sigma - v^\#_\rho \rangle_{\sigma \in \mathcal{S} \vee \rho}$.
(g) If $\boldsymbol{v} \in M_{\mathrm{fa}}(\mathcal{S})$ is such that $\boldsymbol{v} \restriction \mathcal{S} \wedge \rho$ has a previsible variation for every $\rho \in \mathcal{S}$, then $\boldsymbol{v} \in M_{\text{D-M}}$.

626L Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ an L^1-process and z a member of $L^\infty(\mathfrak{A})$. Then

$$\mathbb{E}(z \times S_I(\mathbf{1}, Pd\boldsymbol{v})) = \mathbb{E}(S_I(\boldsymbol{P}z, d\boldsymbol{v}))$$

for every $I \in \mathcal{I}(\mathcal{S})$.

626M The Doob-Meyer theorem: first form Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ a non-negative submartingale. Then $\boldsymbol{v} \in M_{\text{D-M}}(\mathcal{S})$ and the previsible variation $\boldsymbol{v}^\#$ is non-negative and non-decreasing, with starting value 0.

626N Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $A \subseteq \mathcal{S}$ a non-empty downwards-directed set, and $R_A : M_{\text{lmo}}(\mathcal{S}) \to M_{\text{lmo}}(\mathcal{S})$ the associated operator. If $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ is a non-negative submartingale with domain $\mathcal{S}$ and previsible variation $\boldsymbol{v}^\#$, then the previsible variation $R_A(\boldsymbol{v})^\#$ of $R_A(\boldsymbol{v})$ is $R_A(\boldsymbol{v}^\#)$.

626O The Doob-Meyer theorem: second form Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ a submartingale such that $\{\mathbb{E}(v_\sigma) : \sigma \in \mathcal{S}\}$ is bounded below. Then $\boldsymbol{v}$ is expressible as the sum of a non-negative non-decreasing fully adapted process and a virtually local martingale.

626P Corollary If $\mathcal{S}$ is a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ is a submartingale such that $\{\mathbb{E}(v_\sigma) : \sigma \in \mathcal{S}\}$ is bounded below, then $\boldsymbol{v}$ is a semi-martingale.

626Q Proposition Suppose that $T = [0, \infty[$ and that $\boldsymbol{\iota} = \langle \iota_\tau \rangle_{\tau \in \mathcal{T}_f}$ is the identity process as described in 612F. Then the previsible variation of $\boldsymbol{\iota} \restriction \mathcal{T}_b$ is itself.

626R Lemma Let $\mathcal{S}$ be a full sublattice of $\mathcal{T}$ with greatest and least members, and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ a non-decreasing non-negative jump-free L^1-process. Then for every $\epsilon > 0$ there is an $I \in \mathcal{I}(\mathcal{S})$, containing $\min \mathcal{S}$ and $\max \mathcal{S}$, such that $\|S_I(\mathbf{1}, Pd\boldsymbol{u}) - u_{\max \mathcal{S}} + u_{\min \mathcal{S}}\|_1 \le \epsilon$.

626S Proposition Let $\mathcal{S}$ be a non-empty full sublattice of $\mathcal{T}$, and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ a non-decreasing locally jump-free L^1-process starting from 0. Then it is equal to its previsible variation.

626T Proposition Let $\mathcal{S}$ be a full sublattice of $\mathcal{T}$, and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ a locally jump-free L^2-martingale. Then the quadratic variation $\boldsymbol{v}^* = \langle v^*_\sigma \rangle_{\sigma \in \mathcal{S}}$ of $\boldsymbol{v}$ is the previsible variation $(\boldsymbol{v}^2)^\#$ of the submartingale $\boldsymbol{v}^2$.

Version of 27.3.21

627 Integrators and semi-martingales

This section is devoted to a kind of structure theory for integrators (627I-627J, 627L, 627Q); I take a route which passes some further important classes of stochastic process (627B) and ideas from the theory of linear topological spaces (627F-627G).

627B Definitions Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v}$ a fully adapted process with domain $\mathcal{S}$.

(a) $\boldsymbol{v}$ is a **supermartingale** if $-\boldsymbol{v}$ is a submartingale.

(b) $\boldsymbol{v}$ is a **quasimartingale** if $\boldsymbol{v}$ is an L^1-process and $\{\mathbb{E}(w) : w \in Q_{\mathcal{S}}(d\boldsymbol{v})\}$ is bounded in $\mathbb{R}$,

(c) I will say that $\boldsymbol{v}$ is a **strong integrator** if whenever $\epsilon > 0$ there are a uniformly integrable martingale $\boldsymbol{w}$ and a fully adapted process $\boldsymbol{w}'$ of bounded variation, both with domain $\mathcal{S}$, such that $\bar{\mu}[\![\boldsymbol{v} \ne \boldsymbol{w} + \boldsymbol{w}']\!] \le \epsilon$.

627C Elementary facts (a)(i) If $\boldsymbol{v}$ is a supermartingale, so is $\boldsymbol{v} \restriction \mathcal{S}'$ for any sublattice $\mathcal{S}'$ of $\operatorname{dom} \boldsymbol{v}$. If $\boldsymbol{v}$ and $\boldsymbol{w}$ are supermartingales, so is $\boldsymbol{v} + \boldsymbol{w}$.

(ii) If $h : \mathbb{R} \to \mathbb{R}$ is concave and non-decreasing, $\boldsymbol{v}$ is a supermartingale and $\bar{h}\boldsymbol{v}$ is an L^1-process, then $\bar{h}\boldsymbol{v}$ is a supermartingale.

(iii) A supermartingale $\langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ is a martingale iff $\mathbb{E}(v_\sigma) = \mathbb{E}(v_\tau)$ whenever $\sigma \le \tau$ in $\mathcal{S}$.

(b) Every martingale is a quasimartingale.

(c)(i) A strong integrator is an integrator.

(ii) A linear combination of strong integrators is a strong integrator.

(iii) If $\boldsymbol{v}$ is a strong integrator with domain $\mathcal{S}$ and $\mathcal{S}'$ is a sublattice of $\mathcal{S}$, then $\boldsymbol{v}\restriction\mathcal{S}'$ is a strong integrator.

(iv) If $\boldsymbol{v}$ is a fully adapted process with domain $\mathcal{S}$ and for every $\epsilon > 0$ there is a strong integrator $\boldsymbol{v}'$ with domain $\mathcal{S}$, such that $[\![\boldsymbol{v} \ne \boldsymbol{v}']\!]$ has measure at most ϵ, then $\boldsymbol{v}$ is a strong integrator.

627D Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v}$ a non-negative fully adapted process with domain $\mathcal{S}$.

(a) If $\boldsymbol{v}$ is a virtually local martingale, it is a $\| \|_1$-bounded supermartingale.

(b) If $\boldsymbol{v}$ is a $\| \|_1$-bounded supermartingale it is order-bounded.

627E Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma\in\mathcal{S}}$ a quasimartingale.

(a) There is a non-negative $\| \|_1$-bounded supermartingale $\boldsymbol{w}$ such that $\boldsymbol{v} + \boldsymbol{w}$ is a supermartingale.

(b) If $\mathcal{S}$ has a greatest element then $\boldsymbol{v}$ is expressible as the difference of two non-negative supermartingales.

(c) If $\boldsymbol{v}$ is $\| \|_1$-bounded then it is a semi-martingale.

627F Lemma Let U be a Banach space, C a convex subset of U and K a non-empty weak*-compact convex subset of the dual U^* of U. Suppose that $\gamma \ge 0$ is such that for every $u \in C$ there is an $f \in K$ such that $f(u) \le \gamma$. Then there is a $g \in K$ such that $g(u) \le \gamma$ for every $u \in C$.

627G Lemma Suppose that $C \subseteq L^1_{\bar\mu}$ is a non-empty topologically bounded convex set. Then there is a $w \in L^\infty(\mathfrak{A})$ such that $[\![w > 0]\!] = 1$ and $\sup_{u\in C} \mathbb{E}(u \times w)$ is finite.

627H Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v}$ a fully adapted process with domain $\mathcal{S}$.

(a) If $I \in \mathcal{I}(\mathcal{S})$, $\boldsymbol{u} \in M_{\mathrm{fa}}(I)$ and $\|\boldsymbol{u}\|_\infty \le 1$, then there is a $\boldsymbol{w} \in M_{\mathrm{fa}}(\mathcal{S})$ such that $\|\boldsymbol{w}\|_\infty \le 1$ and $S_J(\boldsymbol{w}, d\boldsymbol{v}) = S_I(\boldsymbol{u}, d\boldsymbol{v})$ whenever $I \subseteq J \in \mathcal{I}(\mathcal{S})$, so that $\int_{\mathcal{S}} \boldsymbol{w}\, d\boldsymbol{v}$ is defined and equal to $S_I(\boldsymbol{u}, d\boldsymbol{v})$.

(b) $Q_{\mathcal{S}}(d\boldsymbol{v})$ is convex.

627I Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v}$ an integrator with domain $\mathcal{S}$. Then there is a $\bar\nu$ such that $(\mathfrak{A}, \bar\nu)$ is a probability algebra and $\boldsymbol{v}$ is a uniformly integrable quasimartingale with respect to $\bar\nu$.

627J Corollary Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v}$ an integrator with domain $\mathcal{S}$. Then $\boldsymbol{v}$ is a semi-martingale.

627K Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v}$ an integrator with domain $\mathcal{S}$. Set $\mathcal{S}' = \mathcal{S} \cup \{\min \mathcal{T}, \max \mathcal{T}\}$. Then there is an integrator $\boldsymbol{v}'$ with domain $\mathcal{S}'$ extending $\boldsymbol{v}$.

627L Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v}$ a fully adapted process with domain $\mathcal{S}$.

(a) If $\mathcal{S}$ has a greatest element and $\boldsymbol{v}$ is a non-negative submartingale, $\boldsymbol{v}$ is a strong integrator.

(b) If $\mathcal{S}$ has greatest and least elements and $\boldsymbol{v}$ is a non-negative supermartingale, $\boldsymbol{v}$ is a strong integrator.

(c) If $\mathcal{S}$ has greatest and least elements and $\boldsymbol{v}$ is a quasimartingale, $\boldsymbol{v}$ is a strong integrator.

(d) The following are equiveridical:

(i) $\boldsymbol{v}$ is an integrator;

(ii) there is a functional $\bar\nu$ such that $(\mathfrak{A}, \bar\nu)$ is a probability algebra and $\boldsymbol{v}$ is a strong integrator with respect to $\bar\nu$.

627M Corollary Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v}$ an integrator with domain $\mathcal{S}$. Then the solid convex hull of $Q_{\mathcal{S}}(d\boldsymbol{v})$ is topologically bounded.

***627N Lemma** Let $\mathcal{S}$ be a non-empty finitely full sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$ a fully adapted process such that $\lim_{\sigma\uparrow A} u_\sigma$ is defined in $L^0(\mathfrak{A})$ for every non-empty upwards-directed set $A \subseteq \mathcal{S}$ with an upper bound in $\mathcal{S}$. Then there are a non-decreasing sequence $\langle\tau_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{S}$ and a non-decreasing sequence $\langle d_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that

$$d_n \in \mathfrak{A}_{\tau_n}, \quad d_n \subseteq [\![\tau_{n+1} = \tau_n]\!]$$

for every $n \in \mathbb{N}$, and

$$\sup_{n\in\mathbb{N}}(d_n \cup [\![\tau \le \tau_n]\!]) = 1, \quad u_\tau = \lim_{n\to\infty} u_{\tau\wedge\tau_n}$$

for every $\tau \in \mathcal{S}$.

***627O Lemma** Suppose that we are given a sublattice $\mathcal{S}$ of $\mathcal{T}$, a non-decreasing sequence $\langle\tau_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{S}$ and a non-decreasing sequence $\langle d_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that

$$d_n \in \mathfrak{A}_{\tau_n}, \quad d_n \subseteq [\![\tau_{n+1} = \tau_n]\!]$$

for every $n \in \mathbb{N}$, and

$$1 = \sup_{n\in\mathbb{N}} d_n \cup [\![\tau \le \tau_n]\!]$$

for every $\tau \in \mathcal{S}$. Set $\mathcal{S}_0 = \bigcup_{n\in\mathbb{N}} \mathcal{S} \wedge \tau_n$ and suppose that $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}_0}$ is a fully adapted process.

(a) There is a fully adapted process $\tilde{\boldsymbol{u}} = \langle\tilde{u}_\tau\rangle_{\tau\in\mathcal{S}}$ such that

(i) $\tilde{u}_\tau = \lim_{n\to\infty} u_{\tau\wedge\tau_n}$ for every $\tau \in \mathcal{S}$,

(ii) $d_n \cup [\![\tau \le \tau_n]\!] \subseteq [\![\tilde{u}_\tau = u_{\tau\wedge\tau_n}]\!]$ for every $\tau \in \mathcal{S}$ and $n \in \mathbb{N}$,

(iii) $\tilde{\boldsymbol{u}}$ extends $\boldsymbol{u}$.

(b) Write $\hat{\mathcal{S}}$ for the covered envelope of $\mathcal{S}$, $\hat{\mathcal{S}}_0$ for $\bigcup_{n\in\mathbb{N}} \hat{\mathcal{S}} \wedge \tau_n$ and $\hat{\boldsymbol{u}} = \langle\hat{u}_\sigma\rangle_{\sigma\in\hat{\mathcal{S}}_0}$ for the fully adapted extension of $\boldsymbol{u}$ to $\hat{\mathcal{S}}_0$. Set $\tilde{\hat{u}}_\tau = \lim_{n\to\infty} \hat{u}_{\tau\wedge\tau_n}$ for every $\tau \in \hat{\mathcal{S}}$. Then $\tilde{\hat{\boldsymbol{u}}} = \langle\tilde{\hat{u}}_\tau\rangle_{\tau\in\hat{\mathcal{S}}}$ is the fully adapted extension of $\tilde{\boldsymbol{u}}$ to $\hat{\mathcal{S}}$.

(c) If $\boldsymbol{u}$ is locally moderately oscillatory, $\tilde{\boldsymbol{u}}$ is locally moderately oscillatory.

(d) If $\boldsymbol{u}$ is a virtually local martingale, $\tilde{\boldsymbol{u}}$ is a virtually local martingale.

(e) If $\boldsymbol{u}$ is locally of bounded variation, $\tilde{\boldsymbol{u}}$ is locally of bounded variation.

(f) If $\boldsymbol{u}$ is locally order-bounded and $\bar{w} = \sup_{n\in\mathbb{N}} \operatorname{Osclln}(\boldsymbol{u}\restriction\mathcal{S}_0\wedge\tau_n)$ is defined, then $\bar{w} \ge \sup_{\tau\in\mathcal{S}} \operatorname{Osclln}(\tilde{\boldsymbol{u}}\restriction\mathcal{S}\wedge\tau)$.

(g) If $\boldsymbol{u}$ is a semi-martingale, $\tilde{\boldsymbol{u}}$ is a semi-martingale.

627P Corollary Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, τ a member of $\mathcal{S}$, and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$, $\boldsymbol{w} = \langle w_\sigma\rangle_{\sigma\in\mathcal{S}\wedge\tau}$ two fully adapted processes. Set

$$v_\sigma = u_\sigma - u_{\sigma\wedge\tau} + w_{\sigma\wedge\tau}$$

for $\sigma \in \mathcal{S}$. Then $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ is fully adapted, and is locally of bounded variation, or a virtually local martingale, or locally moderately oscillatory, or a semi-martingale if $\boldsymbol{u}$ and $\boldsymbol{w}$ both are, while

$$v_\sigma = w_\sigma \text{ if } \sigma \in \mathcal{S} \wedge \tau, \quad v_\sigma = u_\sigma - u_\tau + w_\tau \text{ if } \sigma \in \mathcal{S} \vee \tau.$$

627Q Theorem A fully adapted process is a semi-martingale iff it is a local integrator.

627R Proposition Suppose that T is separable in its order topology. If $\mathcal{S}$ is any sublattice of $\mathcal{T}$, there is a non-decreasing sequence $\langle\tau_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{S}$ such that $\sup_{n\in\mathbb{N}} [\![\sigma \le \tau_n]\!] = 1$ for every $\sigma \in \mathcal{S}$.

Version of 31.12.17

*628 Refining a martingale inequality

I remarked in §621 that the constant 16 in the inequality 621Hf can be reduced to 2 if we are willing to use some rather more advanced measure theory. This treatise is not about finding best constants. But 2 is a much prettier number than 16 and the method I have devised passes through a construction (628C) which may have other uses, as in 628F-628G, so I present it here.

628A Lemma Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra, and $\mathfrak{A}_0$ a closed subalgebra of $\mathfrak{A}$; write $P_0 : L^1_{\bar{\mu}} \to L^1_{\bar{\mu}}$ for the corresponding conditional expectation operator. Suppose that $v \in L^1_{\bar{\mu}}$. Set $v_0 = P_0 v$. Then there are a probability algebra $(\mathfrak{B},\bar{\nu})$, closed subalgebras $\mathfrak{B}_0 \subseteq \mathfrak{B}_1 \subseteq \mathfrak{B}$, and a measure-preserving Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ such that $\pi[\mathfrak{A}_0] = \mathfrak{B}_0$ and if $T = T_\pi : L^1_{\bar{\mu}} \to L^1_{\bar{\nu}}$ is the associated embedding and w_i is the conditional expectation of Tv on $\mathfrak{B}_i$ for both i, then

$$w_0 = Tv_0,$$

$$\begin{aligned}[\![|w_1| = 1]\!] \cap [\![|w_0| < 1]\!] &= [\![|w_1| \ge 1]\!] \cap [\![|w_0| < 1]\!]\\ &\supseteq [\![|Tv| \ge 1]\!] \cap [\![|w_0| < 1]\!].\end{aligned}$$

628B Lemma Suppose that $(\mathfrak{A},\bar{\mu})$ is a probability algebra and $\langle u_i\rangle_{i\le n}$ is a martingale adapted to a non-decreasing finite sequence $\langle\mathfrak{A}_i\rangle_{i\le n}$ of closed subalgebras of $\mathfrak{A}$. Then there are a probability algebra $(\mathfrak{B},\bar{\nu})$, closed subalgebras $\mathfrak{B}_0 \subseteq \ldots \subseteq \mathfrak{B}_{2n}$ of $\mathfrak{B}$, a martingale $\langle w_j\rangle_{j\le 2n}$ adapted to $\langle\mathfrak{B}_j\rangle_{j\le 2n}$ and a measure-preserving Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ such that if $T = T_\pi : L^1_{\bar{\mu}} \to L^1_{\bar{\nu}}$ is the associated embedding then

$$\pi[\mathfrak{A}_i] \subseteq \mathfrak{B}_{2i},\ w_{2i} = Tu_i \text{ for } i \le n,$$

$$\begin{aligned}[\![|w_j| = 1]\!] \cap [\![|w_{j-1}| < 1]\!] &= [\![|w_j| \ge 1]\!] \cap [\![|w_{j-1}| < 1]\!]\\ &\supseteq [\![|w_{j+1}| \ge 1]\!] \cap [\![|w_{j-1}| < 1]\!]\end{aligned}$$

for odd $j < 2n$.

628C Corollary Suppose that $(\mathfrak{A},\bar{\mu})$ is a probability algebra and $\langle u_i\rangle_{i\le n}$ is a martingale adapted to a non-decreasing finite sequence $\langle\mathfrak{A}_i\rangle_{i\le n}$ of closed subalgebras of $\mathfrak{A}$. Then there are a probability algebra $(\mathfrak{B},\bar{\nu})$, closed subalgebras $\mathfrak{C}_0 \subseteq \ldots \subseteq \mathfrak{C}_n$ of $\mathfrak{B}$, a martingale $\langle v_i\rangle_{i\le n}$ adapted to $\langle\mathfrak{C}_i\rangle_{i\le n}$ and a measure-preserving Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ such that

$$\pi[\mathfrak{A}_i] \subseteq \mathfrak{C}_i, \quad \|v_i\|_\infty \le 1, \quad \|v_i\|_1 \le \|u_i\|_1$$

for every $i \le n$, and

$$\bar{\nu}(\sup_{i\le n} [\![v_i \ne T_\pi u_i]\!]) \le \|u_n\|_1.$$

628D Proposition Suppose that $(\mathfrak{A},\bar{\mu})$ is a probability algebra and $\langle v_i\rangle_{i\le n}$ is a martingale adapted to a non-decreasing finite sequence $\langle\mathfrak{A}_i\rangle_{i\le n}$ of closed subalgebras of $\mathfrak{A}$. Let $\langle\alpha_j\rangle_{j\le m}$, $\langle u_{ji}\rangle_{j\le m, i<n}$ be such that

$$\alpha_j \ge 0 \text{ for } j \le m, \quad \textstyle\sum_{j=0}^m \alpha_j = 1,$$

$$u_{ji} \in L^0(\mathfrak{A}_i), \quad \|u_{ji}\|_\infty \le 1 \text{ for } i < n,\ j \le m.$$

Set $z = \sum_{j=0}^m \alpha_j |\sum_{i=0}^{n-1} u_{ji} \times (v_{i+1} - v_i)|$. Then $\bar{\mu}[\![z > \gamma]\!] \le \frac{2}{\gamma}\|v_n\|_1$ for every $\gamma > 0$.

628E Corollary Let $(\mathfrak{A},\bar{\mu},T,\langle\mathfrak{A}_t\rangle_{t\in T},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ be a stochastic integration structure, $\mathcal{S}$ a non-empty sublattice of $\mathcal{T}$, and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a martingale. Then

$$\bar{\mu}[\![|z| > \gamma]\!] \le \frac{1}{\gamma}\sup_{\sigma\in\mathcal{S}} \|v_\sigma\|_1$$

whenever $z \in Q_{\mathcal{S}}(\boldsymbol{v})$ and $\gamma > 0$.

628F Proposition Suppose that $(\mathfrak{A},\bar{\mu})$ is a probability algebra and $\langle u_i\rangle_{i\le n}$ is a martingale adapted to a non-decreasing finite sequence $\langle\mathfrak{A}_i\rangle_{i\le n}$ of closed subalgebras of $\mathfrak{A}$. Set $u^* = \sum_{i=0}^{n-1}(u_{i+1} - u_i)^2$. Then $\bar{\mu}[\![u^* > \gamma^2]\!] \le \frac{2}{\gamma}\|u_n\|_1$ for every $\gamma > 0$.

628G Proposition Let $(\mathfrak{A}, \bar{\mu}, T, \langle\mathfrak{A}_t\rangle_{t\in T}, \mathcal{T}, \langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ be a stochastic integration structure, $\mathcal{S}$ a sublattice of $\mathcal{T}$, and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a martingale. Let $\boldsymbol{v}^* = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ be its quadratic variation. Then

$$\bar{\mu}[\![v_\tau^* > \gamma^2]\!] \le \frac{2}{\gamma}\|v_\tau\|_1$$

whenever $\gamma > 0$ and $\tau \in \mathcal{S}$.