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Chapter 62

Martingales

The centre of the theory of stochastic integration, since Itô 44, has been integrals $\int \boldsymbol{u}\,d\boldsymbol{v}$ where $\boldsymbol{v}$ is a martingale. In §621 I give a number of inequalities involving *finite* martingales which will make it possible to go straight to the general case in §622. In §622 we have to check some algebra concerning conditional expectations in order to make sense of the idea of 'fully adapted martingale', but the theorem that martingales are local integrators (622G) is a straightforward consequence of 621Hf.

It is not in general the case that an indefinite integral with respect to a martingale is again a martingale. For a full-strength theorem in this direction I think we need to turn to 'virtually local' martingales and do some hard work (623O). To use Itô's formula (619C) in its original form, in which the integrator was Brownian motion, we need of course to know the quadratic variation of Brownian motion, which I come to at last in 624F.

The next three sections are directed towards a structure theory for integrators in §627. This volume is devoted to structures based on probability algebras $(\mathfrak{A},\bar\mu)$. The concepts of Chapter 61 are generally law-independent in the sense that while the existence of the functional $\bar\mu$ is essential, its replacement by another functional $\bar\nu$ such that $(\mathfrak{A},\bar\nu)$ is still a probability algebra makes no difference. However nearly everything involving martingales is shaken up by a change in law. §625 examines such changes, and we find, remarkably, that we do not change the semi-martingales (625F). In §626 I introduce submartingales and previsible variations, with the Doob-Meyer theorem on the expression of submartingales as semi-martingales. In §627 I apply this to supermartingales, and show that local integrators are semi-martingales.

The essential inequality in 621Hf is proved by ordinary martingale methods in §621. There is an alternative route, incidentally yielding a better constant, which depends on a kind of interpolation; I present this in §628.

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621 Finite martingales

I have to justify my repeated assertion that martingales are integrators. This, together with the developments it leads to, will need some non-trivial facts about finite martingales which are most easily described in advance of their applications. However, the complexity of some of the lemmas below may be more bearable if you can see what they're for. So you may wish to treat this section as an appendix, and disentangle the ideas when you find them being called on in §§622, 624 and 626.

621A Notation This section will be almost independent of the work in Chapter 61, and will be based rather on the ideas of §275, interpreted as always in the language of Chapter 36. Once again, $(\mathfrak{A},\bar\mu)$ will be a probability algebra, and for $1\le p<\infty$, $L^p_{\bar\mu}=L^p(\mathfrak{A},\bar\mu)\subseteq L^0(\mathfrak{A})$ will be the associated L^p-space $\{w : w\in L^0(\mathfrak{A}),\ \|w\|_p<\infty\}$ (§366), while $\mathbb{E}$ refers to the integral on $L^1_{\bar\mu}$ (613Aa).

621B Uniform integrability We are going to need the following results from Volumes 2 and 3.

(a) Recall that a set $A\subseteq L^1_{\bar\mu}$ is **uniformly integrable** if for every $\epsilon>0$ there is an $M\ge 0$ such that $\mathbb{E}((|u|-M\chi 1)^+)\le\epsilon$ for every $u\in A$ (246Ab, 354P, 365T[1]); equivalently, if A is $\|\,\|_1$-bounded and for every $\epsilon>0$ there is a $\delta>0$ such that $\mathbb{E}(|u|\times\chi a)\le\epsilon$ whenever $u\in A$, $a\in\mathfrak{A}$ and $\bar\mu a\le\delta$ (246Ca, 246G).

[1] Formerly 365U.

(b) A non-empty set $A \subseteq L^0 = L^0(\mathfrak{A})$ is uniformly integrable iff

$$\lim_{\alpha\to\infty} \sup_{u\in A} \mathbb{E}(|u| \times \chi[\![|u| > \alpha]\!]) = 0$$

(246I).

(c) Suppose that A, $B \subseteq L^1_{\bar\mu}$ are uniformly integrable.

(i) Every subset of A is uniformly integrable; αA is uniformly integrable for every $\alpha \in \mathbb{R}$; $A + B$ is uniformly integrable; the solid hull of A is uniformly integrable (246C, 354Ra).

(ii) The $\mathfrak{T}$-closure $\overline{A}$ of A is uniformly integrable, where $\mathfrak{T}$ is the topology of convergence in measure on L^0 (613B), and $\mathfrak{T}$ agrees with the norm topology of $L^1_{\bar\mu}$ on $\overline{A}$ (246J).

(d) A subset of $L^1_{\bar\mu}$ is uniformly integrable iff it is relatively compact for the weak topology $\mathfrak{T}_s(L^1_{\bar\mu}, L^\infty(\mathfrak{A}))$ (243Gb and 247C, or 365T(a-v)).

(e) The following useful fact was left in the exercises for §246. If $p > 1$, then any $\| \|_p$-bounded subset A of L^0 is uniformly integrable. **P** Suppose that $\|u\|_p \le \gamma$ for $u \in A$. Given $\epsilon > 0$, there is an $M > 0$ such that $\gamma^p \le \epsilon M^{p-1}$. Now for any $u \in A$,

$$(|u| - M\chi 1)^+ \le \frac{1}{M^{p-1}}|u|^p, \quad \mathbb{E}((|u| - M\chi 1)^+) \le \frac{\gamma^p}{M^{p-1}} \le \epsilon.$$

As ϵ is arbitrary, A is uniformly integrable. **Q**

621C Conditional expectations Of course we cannot talk about martingales without speaking of conditional expectations, and this volume will call on the full resources developed in Volumes 2 and 3, which I now recapitulate.

(a) Following the definitions in §365, we find that if $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$ then $L^0(\mathfrak{B}) \cap L^1_{\bar\mu} = L^1(\mathfrak{B}, \bar\mu\restriction\mathfrak{B})$ (365Qa[2]), and we have a unique positive linear operator $P_{\mathfrak{B}} : L^1_{\bar\mu} \to L^0(\mathfrak{B}) \cap L^1_{\bar\mu}$ such that $\mathbb{E}(P_{\mathfrak{B}}u \times \chi b) = \mathbb{E}(u \times \chi b)$ whenever $u \in L^1_{\bar\mu}$ and $b \in \mathfrak{B}$ (365Q). Counting $\|u\|_p$ as ∞ if $u \in L^0(\mathfrak{A}) \setminus L^p(\mathfrak{A}, \bar\mu)$, $\|P_{\mathfrak{B}}u\|_p \le \|u\|_p$ for every $u \in L^1_{\bar\mu}$ and $p \in [1, \infty]$ (366J).

(b) If $\mathfrak{B}$ and $\mathfrak{C}$ are closed subalgebras of $\mathfrak{A}$ and $\mathfrak{B} \subseteq \mathfrak{C}$, then

$$P_{\mathfrak{B}}P_{\mathfrak{C}} = P_{\mathfrak{C}}P_{\mathfrak{B}} = P_{\mathfrak{B}}.$$

P (Cf. 458M.) Take any $u \in L^1_{\bar\mu}$. (i) $P_{\mathfrak{B}}P_{\mathfrak{C}}u \in L^0(\mathfrak{B})$ and

$$\mathbb{E}(P_{\mathfrak{B}}P_{\mathfrak{C}}u \times \chi b) = \mathbb{E}(P_{\mathfrak{C}}u \times \chi b) = \mathbb{E}(u \times \chi b) = \mathbb{E}(P_{\mathfrak{B}}u \times \chi b)$$

for every $b \in \mathfrak{B} \subseteq \mathfrak{C}$, so $P_{\mathfrak{B}}P_{\mathfrak{C}}u = P_{\mathfrak{B}}u$. (ii) $P_{\mathfrak{B}}u \in L^0(\mathfrak{B}) \subseteq L^0(\mathfrak{C})$ so $P_{\mathfrak{C}}P_{\mathfrak{B}}u = P_{\mathfrak{B}}u$. **Q**

(c) If $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$, $u \in L^1_{\bar\mu}$, $v \in L^0(\mathfrak{B})$ and $u \times v \in L^1_{\bar\mu}$, then $P_{\mathfrak{B}}(u \times v) = P_{\mathfrak{B}}u \times v$ (233K, 365Qa). So if u, v, $u \times P_{\mathfrak{B}}v$ and $P_{\mathfrak{B}}u \times v$ all belong to $L^1_{\bar\mu}$,

$$P_{\mathfrak{B}}(u \times P_{\mathfrak{B}}v) = P_{\mathfrak{B}}u \times P_{\mathfrak{B}}v = P_{\mathfrak{B}}(P_{\mathfrak{B}}u \times v)$$

and $\mathbb{E}(u \times P_{\mathfrak{B}}v) = \mathbb{E}(P_{\mathfrak{B}}u \times v)$.

(d) ('Jensen's inequality') Let $h : \mathbb{R} \to \mathbb{R}$ be a convex function and $\bar h : L^0 \to L^0$ the corresponding map (612Ac). If u and $\bar h(u)$ both belong to $L^1_{\bar\mu}$, $\bar h(P_{\mathfrak{B}}u) \le P(\bar h_{\mathfrak{B}}(u))$ for every closed subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ (365Qb).

(e) When $p = 2$, we have a sharper result: if $u \in L^2_{\bar\mu}$ and $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$, then $\|u\|_2^2 = \|P_{\mathfrak{B}}u\|_2^2 + \|u - P_{\mathfrak{B}}u\|^2$. **P** In the Hilbert space $L^2_{\bar\mu}$,

$$(P_{\mathfrak{B}}u|u) = \mathbb{E}(u \times P_{\mathfrak{B}}u) = \mathbb{E}(P_{\mathfrak{B}}(u \times P_{\mathfrak{B}}u)) = \mathbb{E}(P_{\mathfrak{B}}u \times P_{\mathfrak{B}}u)$$

so

$$\|u - P_{\mathfrak{B}}u\|_2^2 = \|u\|_2^2 - 2(u|P_{\mathfrak{B}}u) + \|P_{\mathfrak{B}}u\|_2^2 = \|u\|_2^2 - \|P_{\mathfrak{B}}u\|_2^2. \text{ **Q**}$$

[2] Formerly 365Ra.

(f) If $A \subseteq L^1_{\bar\mu}$ is uniformly integrable, then $\{P_{\mathfrak{B}}u : u \in A,\ \mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}\}$ is uniformly integrable (246D, 365Tb).

(g)(i) If $\mathbb{B}$ is a non-empty downwards-directed family of closed subalgebras of $\mathfrak{A}$ with intersection $\mathfrak{C}$, and $u \in L^1 = L^1_{\bar\mu}$, then $P_{\mathfrak{C}}u$ is the $\| \|_1$-limit $1\lim_{\mathfrak{B}\downarrow\mathbb{B}} P_{\mathfrak{B}}u$ (367Qa).

(ii) If $\mathbb{B}$ is a non-empty upwards-directed family of closed subalgebras of $\mathfrak{A}$, $\mathfrak{C}$ is the closed subalgebra generated by $\bigcup\mathbb{B}$ and $u \in L^1$, then $P_{\mathfrak{C}}u$ is the $\| \|_1$-limit $1\lim_{\mathfrak{B}\uparrow\mathbb{B}} P_{\mathfrak{B}}u$ (367Qb).

621D Definitions For the rest of this section, we shall be looking at a non-decreasing finite sequence $\langle\mathfrak{A}_i\rangle_{i\le n}$ of closed subalgebras of $\mathfrak{A}$; that is, a filtration in the sense of 611A in which the totally ordered set T is $\{0,\dots,n\}$ for some integer n. In this context, I will write $P_i : L^1_{\bar\mu} \to L^1_{\bar\mu}$ for the conditional expectation operator associated with $\mathfrak{A}_i$. Note that $P_iP_j = P_{\min(i,j)}$ for all i, $j \le n$, by 621Cb. Let $\boldsymbol{v} = \langle v_i\rangle_{i\le n}$ be a finite sequence in $L^1_{\bar\mu}$.

(a) $\boldsymbol{v}$ is a **martingale** adapted to $\langle\mathfrak{A}_i\rangle_{i\le n}$ if $v_i = P_iv_j$ whenever $i \le j \le n$; equivalently, if $v_n \in L^0(\mathfrak{A}_n)$ and $v_i = P_iv_{i+1}$ for every $i < n$.

(b) $\boldsymbol{v}$ is a **submartingale** adapted to $\langle\mathfrak{A}_i\rangle_{i\le n}$ if $v_i \in L^0(\mathfrak{A}_i)$ and $v_i \le P_iv_j$ whenever $i \le j \le n$; equivalently, if $v_i \in L^0(\mathfrak{A}_i)$ for every $i \le n$ and $v_i \le P_iv_{i+1}$ for every $i < n$.

(Cf. 275A, 275Yg, 626B.)

621E Doob's maximal inequality If $\langle v_i\rangle_{i\le n}$ is a martingale adapted to $\langle\mathfrak{A}_i\rangle_{i\le n}$, and $\bar v = \sup_{i\le n}|v_i|$, then

$$t\bar\mu[\![\bar v > t]\!] \le \mathbb{E}(|v_n| \times \chi[\![\bar v > t]\!]) \le \|v_n\|_1,$$

$$t\bar\mu[\![\bar v \ge t]\!] \le \mathbb{E}(|v_n| \times \chi[\![\bar v \ge t]\!]) \le \|v_n\|_1$$

for every $t \ge 0$.

proof (This is a small modification of 275D.) Set $a = [\![\bar v > t]\!]$, $b_i^+ = [\![v_i > t]\!]$, $c_i^+ = b_i^+ \setminus \sup_{j<i} b_j^+$ for $i \le n$ and $a^+ = \sup_{i\le n} b_i^+ = [\![\sup_{i\le n} v_i > t]\!]$ (364La). Then

$$t\bar\mu a^+ = \sum_{i=0}^n t\bar\mu c_i^+ \le \sum_{i=0}^n \mathbb{E}(v_i \times \chi c_i^+) = \sum_{i=0}^n \mathbb{E}(v_n \times \chi c_i^+)$$

(because $c_i^+ \in \mathfrak{A}_i$ and v_i is the conditional expectation of v_n on v_i)

$$= \mathbb{E}(v_n \times \chi a^+) \le \mathbb{E}((v_n \vee 0) \times \chi a).$$

Similarly, setting $a^- = \sup_{i\le n}[\![-v_{\tau_i} > t]\!] = [\![\inf_{i\le n} v_i < t]\!]$, we have $t\bar\mu a^- \le \mathbb{E}(((-v_n) \vee 0) \times \chi a)$. Since $a^+ \cup a^-$,

$$t\bar\mu a \le \mathbb{E}(((v_n \vee 0) + ((-v_n) \vee 0)) \times \chi a) = \mathbb{E}(|v_n| \times \chi a) \le \mathbb{E}(|v_n|) = \|v_n\|_1,$$

as claimed.

For the second version, the result is trivial if $t = 0$, and for $t > 0$ we have $[\![\bar v \ge t]\!] = \inf_{0\le s<t}[\![\bar v > s]\!]$, so

$$t\bar\mu[\![\bar v \ge t]\!] = \lim_{s\uparrow t} s\bar\mu[\![\bar v > s]\!]$$

$$\le \lim_{s\uparrow t} \mathbb{E}(|v_n| \times \chi[\![\bar v > s]\!]) = \mathbb{E}(|v_n| \times \chi[\![\bar v \ge t]\!]).$$

621F Lemma Suppose that $\langle u_i\rangle_{i<n}$ and $\langle v_i\rangle_{i\le n}$ are such that $u_i \in L^\infty(\mathfrak{A}_i)$ and $\|u_i\|_\infty \le 1$ for every $i < n$ and $\langle v_i\rangle_{i\le n}$ is a martingale adapted to $\langle\mathfrak{A}_i\rangle_{i\le n}$. Set $z = \sum_{i=0}^{n-1} u_i \times (v_{i+1} - v_i)$. Then $\|z\|_2 \le \|v_n\|_2$.

proof If v_n is not square-integrable, that is, $\|v_n\|_2 = \infty$, this is trivial. So let us suppose that $v_n \in L^2_{\bar\mu}$. In this case v_i is square-integrable for every i (621Cd), so $u_i \times (v_{i+1} - v_i)$ is square-integrable for every $i < n$. Now if $i < j < n$,

$$\begin{aligned}\mathbb{E}(u_i \times (v_{i+1} - v_i) \times u_j \times (v_{j+1} - v_j))& \\ &= \mathbb{E}(P_j(u_i \times (v_{i+1} - v_i) \times u_j \times (v_{j+1} - v_j))) \\ &= \mathbb{E}(u_i \times (v_{i+1} - v_i) \times u_j \times P_j(v_{j+1} - v_j)) = 0\end{aligned}$$

by 621Cc. At the same time, of course, $\mathbb{E}((v_{i+1} - v_i) \times (v_{j+1} - v_j)) = 0$. So

$$\mathbb{E}(z^2) = \sum_{i=0}^{n-1} \mathbb{E}(u_i^2 \times (v_{i+1} - v_i)^2) \le \sum_{i=0}^{n-1} \mathbb{E}((v_{i+1} - v_i)^2) = \sum_{i=0}^{n-1} \mathbb{E}(v_{i+1}^2 - v_i^2)$$

(621Ce)

$$\le \mathbb{E}(v_n^2)$$

and $\|z\|_2 \le \|v_n\|_2$.

621G Proposition Suppose that $\boldsymbol{v} = \langle v_i \rangle_{i \le n}$ is a submartingale adapted to $\langle \mathfrak{A}_i \rangle_{i \le n}$. Then there are a non-decreasing process $\boldsymbol{v}^\# = \langle v_i^\# \rangle_{i \le n}$ and a martingale $\hat{\boldsymbol{v}}$ adapted to $\langle \mathfrak{A}_i \rangle_{i \le n}$ such that $\boldsymbol{v} = \boldsymbol{v}^\# + \hat{\boldsymbol{v}}$ and $v_0^\# = 0$. If $-\chi 1 \le v_i \le 0$ for every $i \le n$, $\|\hat v_n\|_2^2 \le \|v_n\|_1 + 2\|v_0\|_1$.

proof (a) Set $w_j = P_j v_{j+1} - v_j$, so that $w_j \in L^0(\mathfrak{A}_j)$ and $w_j \ge 0$ for $j < n$; set $v_j^\# = \sum_{i=0}^{j-1} w_i$ for $j \le n$, so that $v_j^\# \in L^0(\mathfrak{A}_j)$ for each j, $v_0^\# = 0$ and $\boldsymbol{v}^\# = \langle v_i^\# \rangle_{i \le n}$ is non-decreasing. Set $\hat{\boldsymbol{v}} = \langle \hat v_i \rangle_{i \le n}$ where $\hat v_i = v_i - v_i^\#$ for each i, so that $\boldsymbol{v} = \boldsymbol{v}^\# + \hat{\boldsymbol{v}}$ and $\hat v_i \in L^0(\mathfrak{A}_i)$ for each i. Also, of course, $v_i^\#$ and $\hat v_i$ belong to $L^1_{\bar\mu}$ for every i.

(b) For $i < n$,

$$\hat v_{i+1} - \hat v_i = v_{i+1} - v_i - v_{i+1}^\# + v_i^\# = v_{i+1} - v_i - P_i v_{i+1} + v_i = v_{i+1} - P_i v_{i+1},$$

so

$$P_i \hat v_{i+1} - \hat v_i = P_i(\hat v_{i+1} - \hat v_i) = P_i(v_{i+1} - P_i v_{i+1}) = 0;$$

thus $\hat{\boldsymbol{v}}$ is a martingale adapted to $\langle \mathfrak{A}_i \rangle_{i \le n}$.

(c) Now suppose that $-\chi 1 \le v_i \le 0$ for $i \le n$. In this case all the v_i, $P_i v_j$, $v_i^\#$ and $\hat v_i$ belong to $L^\infty(\mathfrak{A}) \subseteq L^2_{\bar\mu}$. If $i < n$ then

$$\mathbb{E}(\hat v_{i+1}^2 - \hat v_i^2) = \mathbb{E}(\hat v_{i+1}^2 - (P_i \hat v_{i+1})^2) = \mathbb{E}((\hat v_{i+1} - P_i \hat v_{i+1})^2)$$

(621Ce)

$$= \mathbb{E}((\hat v_{i+1} - \hat v_i)^2) = \mathbb{E}((v_{i+1} - P_i v_{i+1})^2)$$

(by (b))

$$= \mathbb{E}(v_{i+1}^2 - (P_i v_{i+1})^2)$$

(621Ce again)

$$\begin{aligned}&= \mathbb{E}(v_{i+1}^2 - v_i^2) + \mathbb{E}(v_i^2 - P_i v_{i+1}^2) \\ &= \mathbb{E}(v_{i+1}^2 - v_i^2) + \mathbb{E}((v_i - P_i v_{i+1}) \times (v_i + P_i v_{i+1}))) \\ &\le \mathbb{E}(v_{i+1}^2 - v_i^2) + 2\mathbb{E}(|v_i - P_i v_{i+1}|)\end{aligned}$$

(because v_i and $P_i v_{i+1}$ both lie between $-\chi 1$ and 0)

$$= \mathbb{E}(v_{i+1}^2 - v_i^2) + 2\mathbb{E}(P_i v_{i+1} - v_i) = \mathbb{E}(v_{i+1}^2 - v_i^2) + 2\mathbb{E}(v_{i+1} - v_i).$$

Summing over i,

$$\begin{aligned}\mathbb{E}(\hat{v}_n^2) &\le \mathbb{E}(\hat{v}_0^2) + \mathbb{E}(v_n^2) - \mathbb{E}(v_0^2) + 2\mathbb{E}(v_n) - 2\mathbb{E}(v_0)\\ &\le \mathbb{E}(|v_n|) - 2\mathbb{E}(v_0)\end{aligned}$$

(because $v^\# = 0$ so $\hat{v}_0 = v_0$, while $v_n^2 \le |v_n|$ and $v_n \le 0$)

$$= \|v_n\|_1 + 2\|v_0\|_1.$$

621H Lemma Let $\boldsymbol{v} = \langle v_i\rangle_{i\le n}$ be a finite sequence in $L^1_{\bar\mu}$ such that $v_i \in L^0(\mathfrak{A}_i)$ for $i \le n$. Suppose that $\langle\alpha_j\rangle_{j\le m}$, $\langle u_{ji}\rangle_{j\le m, i<n}$ are such that

$$\alpha_j \ge 0 \text{ for } j \le m, \quad \sum_{j=0}^m \alpha_j = 1,$$

$$u_{ji} \in L^0(\mathfrak{A}_i), \quad \|u_{ji}\|_\infty \le 1 \text{ for } i < n,\ j \le m.$$

Set $z = \sum_{j=0}^m \alpha_j |\sum_{i=0}^{n-1} u_{ji} \times (v_{i+1} - v_i)|$.

(a) If $\boldsymbol{v}$ is non-negative and non-decreasing, then $\bar\mu[\![z > 1]\!] \le \|v_n\|_1$.

(b) If $\boldsymbol{v}$ is a martingale adapted to $\langle\mathfrak{A}_i\rangle_{i\le n}$ then $\bar\mu[\![z > 1]\!] \le \|v_n\|_2^2$.

(c) If $\boldsymbol{v}$ is a submartingale adapted to $\langle\mathfrak{A}_i\rangle_{i\le n}$ and $-\chi 1 \le v_i \le 0$ for every $i \le n$, then $\bar\mu[\![z > 2]\!] \le 3\|v_0\|_1$.

(d) If $\boldsymbol{v}$ is a non-negative martingale adapted to $\langle\mathfrak{A}_i\rangle_{i\le n}$, then $\bar\mu[\![z > 2]\!] \le 4\mathbb{E}(v_n)$.

(e) If $\boldsymbol{v}$ is a martingale adapted to $\langle\mathfrak{A}_i\rangle_{i\le n}$, then $\bar\mu[\![z > 4]\!] \le 4\|v_n\|_1$.

(f)(cf. Burkholder 66 and 628D below) If $\boldsymbol{v}$ is a martingale adapted to $\langle\mathfrak{A}_i\rangle_{i\le n}$, then $\bar\mu[\![z > \gamma]\!] \le \frac{16}{\gamma}\|v_n\|_1$ for every $\gamma > 0$.

(g) If $\boldsymbol{v}$ is a submartingale adapted to $\langle\mathfrak{A}_i\rangle_{i\le n}$, then $\bar\mu[\![z > \gamma]\!] \le \frac{66}{\gamma}\|v_n\|_1 - \frac{34}{\gamma}\mathbb{E}(v_0)$ for every $\gamma > 0$.

Remark For most applications (there is an important exception in 627M) it will be enough to consider the case $m = 0$, so that we are looking at $z = |\sum_{i=0}^{n-1} u_i \times (v_{i+1} - v_i)|$ where $u_i \in L^0(\mathfrak{A}_i)$ and $\|u_i\|_\infty \le 1$ for every i; this simplifies the formulae, but seems to make no difference to the ideas required.

proof (a)

$$\begin{aligned}\bar\mu[\![z > 1]\!] \le \mathbb{E}(z) &= \sum_{j=0}^m \alpha_j \sum_{i=0}^{n-1} \mathbb{E}(|u_{ji}| \times (v_{i+1} - v_i))\\ &\le \sum_{j=0}^m \alpha_j \sum_{i=0}^{n-1} \mathbb{E}(v_{i+1} - v_i) = \sum_{j=0}^m \alpha_j \mathbb{E}(v_n - v_0)\\ &= \mathbb{E}(v_n - v_0) \le \mathbb{E}(v_n) = \|v_n\|_1.\end{aligned}$$

(b) For $j \le n$, set $z_j = |\sum_{i=0}^n u_{ji} \times (v_{i+1} - v_i)|$. Then $\|z_j\|_2 \le \|v_n\|_2$, by 621F. Accordingly

$$\|z\|_2 = \|\sum_{j=0}^m \alpha_j z_j\|_2 \le \sum_{j=0}^m \alpha_j \|z_j\|_2 \le \|v_n\|_2,$$

and

$$\bar\mu[\![z > 1]\!] \le \mathbb{E}(z^2) = \|z\|_2^2 \le \|v_n\|_2^2.$$

(c) By 621G, we can express $\boldsymbol{v}$ as $\boldsymbol{v}^\# + \hat{\boldsymbol{v}}$ where $\boldsymbol{v}^\# = \langle v_i^\#\rangle_{i\le n}$ is non-decreasing, $\hat{\boldsymbol{v}} = \langle\hat{v}_i\rangle_{i\le n}$ is a martingale, $v_0^\# = 0$ and $\|\hat{v}_n\|_2^2 \le \|v_n\|_1 + 2\|v_0\|_1$.

Set

$$z^\# = \sum_{j=0}^m \alpha_j |\sum_{i=0}^{n-1} u_{ji} \times (v_{i+1}^\# - v_i^\#)|, \quad \hat{z} = \sum_{j=0}^m \alpha_j |\sum_{i=0}^{n-1} u_{ji} \times (\hat{v}_{i+1} - \hat{v}_i)|.$$

Then $z \le z^\# + \hat{z}$, so $[\![z > 2]\!] \subseteq [\![z^\# > 1]\!] \cup [\![\hat{z} > 1]\!]$ and

$$\bar{\mu}[\![z>2]\!] \le \bar{\mu}[\![z^\# > 1]\!] + \bar{\mu}[\![\hat{z} > 1]\!] \le \|v_n^\#\|_1 + \|\hat{v}_n\|_2^2$$

(by (a) and (b) above)

$$\le \mathbb{E}(v_n^\# - v_0^\#) + 2\|v_0\|_1 + \|v_n\|_1 = \mathbb{E}(v_n - v_0) + 2\|v_0\|_1 + \|v_n\|_1$$

(because $\boldsymbol{v} - \boldsymbol{v}^\#$ is a martingale)

$$= 3\|v_0\|_1.$$

(d) Set $\tilde{\boldsymbol{v}} = \langle \tilde{v}_i \rangle_{i\le n}$ where $\tilde{v}_i = -(v_i \wedge \chi 1)$ for each i. Then $\tilde{\boldsymbol{v}}$ is a submartingale adapted to $\langle \mathfrak{A}_i \rangle_{i\le n}$. **P** For each i, $\tilde{v}_i$ belongs to $L^0(\mathfrak{A}_i)$ because v_i does. If $i < n$, then

$$\tilde{v}_i = -(v_i \wedge \chi 1) = -(P_i v_{i+1} \wedge \chi 1) \le P_i(-(v_{i+1} \wedge \chi 1))$$

(621Cd, with $h(\alpha) = -\min(\alpha, 1) = \max(-\alpha, -1)$ for $\alpha \in \mathbb{R}$)

$$= P_i \tilde{v}_{i+1}. \quad \mathbf{Q}$$

Of course $-\chi 1 \le \tilde{v}_i \le 0$ for each i because $v_i \ge 0$. Set

$$\tilde{z} = \sum_{j=0}^m \alpha_j |\sum_{i=0}^{n-1} u_{ji} \times (\tilde{v}_{i+1} - \tilde{v}_i)| = \sum_{j=0}^m \alpha_j |\sum_{i=0}^{n-1} u_{ji} \times (-\tilde{v}_{i+1} + \tilde{v}_i)|.$$

Then

$$\bar{\mu}[\![\tilde{z} > 2]\!] \le 3\|\tilde{v}_0\|_1 \qquad \text{((c) above)}$$

$$\le 3\|v_0\|_1.$$

On the other hand,

$$[\![z \ne \tilde{z}]\!] \subseteq \sup_{i\le n} [\![v_i \ne -\tilde{v}_i]\!] = \sup_{i\le n} [\![v_i > 1]\!] = [\![\sup_{i\le n} |v_i| > 1]\!],$$

so

$$\bar{\mu}[\![z > 2]\!] \le \bar{\mu}[\![\tilde{z} > 2]\!] + \bar{\mu}[\![z \ne -\tilde{z}']\!] \le 3\|v_0\|_1 + \bar{\mu}[\![\sup_{i\le n} |v_i| > 1]\!]$$

$$\le \|v_n\|_1 + 3\|v_0\|_1 = \mathbb{E}(v_n + 3v_0) = 4\mathbb{E}(v_n)$$

by Doob's maximal inequality (621E).

(e) This time, set

$$v_i' = \frac{1}{2} P_i(|v_n| + v_n), \qquad v_i'' = \frac{1}{2} P_i(|v_n| - v_n)$$

for $i \le n$, and

$$z' = \sum_{j=0}^m \alpha_j |\sum_{i=0}^{n-1} u_i \times (v_{i+1}' - v_i')|, \qquad z'' = \sum_{j=0}^m \alpha_j |\sum_{i=0}^{n-1} u_i \times (v_{i+1}'' - v_i'')|.$$

Then $\langle v_i' \rangle_{i\le n}$ is a martingale adapted to $\langle \mathfrak{A}_i \rangle_{i\le n}$, because

$$P_i v_{i+1}' = \frac{1}{2} P_i P_{i+1}(|v_n| + v_n) = \frac{1}{2} P_i(|v_n| + v_n) = v_i'$$

for every $i < n$, and of course $v_i' \ge 0$ for every $i \le n$. Similarly, $\langle v_i'' \rangle_{i\le n}$ is a non-negative martingale adapted to $\langle \mathfrak{A}_i \rangle_{i\le n}$. Since $v_i' - v_i'' = P_i v_n = v_i$ for every i, $z \le z' + z''$. So $[\![z > 4]\!] \subseteq [\![z' > 2]\!] \cup [\![z'' > 2]\!]$ and

$$\bar{\mu}[\![z > 4]\!] \le \bar{\mu}[\![z' > 2]\!] + \bar{\mu}[\![z'' > 2]\!] \le 4\mathbb{E}(v_n' + v_n'') \qquad \text{((d) above)}$$

$$= 4\mathbb{E}(v_n) = 4\|v_n\|_1,$$

as claimed.

(f) We now have

$$\bar{\mu}[\![z>\gamma]\!]=\bar{\mu}[\![\frac{4}{\gamma}z>4]\!]\le 4\|\frac{4}{\gamma}v_n\|_1=\frac{16}{\gamma}\|v_n\|_1$$

for every $\gamma>0$.

(g) Again using 621G, express $\boldsymbol{v}$ as $\boldsymbol{v}^{\#}+\hat{\boldsymbol{v}}$ where $\boldsymbol{v}^{\#}=\langle v_i^{\#}\rangle_{i\le n}$ is non-decreasing, $\hat{\boldsymbol{v}}=\langle\hat{v}_i\rangle_{i\le n}$ is a martingale and $v_0^{\#}=0$. Set

$$\hat{z}=\sum_{j=0}^m\alpha_j|\sum_{i=0}^{n-1}u_i\times(\hat{v}_{i+1}-\hat{v}_i)|,\quad z^{\#}=\sum_{j=0}^m\alpha_j|\sum_{i=0}^{n-1}u_i\times(v_{i+1}^{\#}-v_i^{\#})|.$$

Applying (a) to $\frac{2}{\gamma}\boldsymbol{v}^{\#}$ and $\frac{2}{\gamma}z^{\#}$, we see that $\bar{\mu}[\![z^{\#}>\frac{1}{2}\gamma]\!]\le\frac{2}{\gamma}\|v_n^{\#}\|_1$; while (f) tells us that $\bar{\mu}[\![\hat{z}>\frac{1}{2}\gamma]\!]\le\frac{32}{\gamma}\|\hat{v}_n\|_1$. Since $z\le z^{\#}+\hat{z}$,

$$\bar{\mu}[\![z>\gamma]\!]\le\bar{\mu}[\![z^{\#}>\tfrac{1}{2}\gamma]\!]+\bar{\mu}[\![\hat{z}>\tfrac{1}{2}\gamma]\!]\le\frac{2}{\gamma}\|v_n^{\#}\|_1+\frac{32}{\gamma}\|\hat{v}_n\|_1.$$

Now we know that

$$\|v_n^{\#}\|_1=\mathbb{E}(v_n^{\#})=\mathbb{E}(v_n)-\mathbb{E}(\hat{v}_n)=\mathbb{E}(v_n)-\mathbb{E}(v_0)\le\|v_n\|_1-\mathbb{E}(v_0)$$

while

$$\|\hat{v}_n\|_1\le\|v_n\|_1+\|v_n^{\#}\|_1\le 2\|v_n\|_1-\mathbb{E}(v_0).$$

So we get

$$\bar{\mu}[\![z>\gamma]\!]\le\frac{66}{\gamma}\|v_n\|_1-\frac{34}{\gamma}\mathbb{E}(v_0).$$

621I Lemma Suppose that $\langle v_i\rangle_{i\le n}$ is a non-negative martingale adapted to $\langle\mathfrak{A}_i\rangle_{i\le n}$, and that $M\ge 0$ is such that $[\![v_i>M]\!]\subseteq[\![v_j=v_n]\!]$ whenever $i\le j\le n$. Suppose that $u_i\in L^\infty(\mathfrak{A}_i)$ and $\|u_i\|_\infty\le 1$ for $i<n$, and set $z=\sum_{i=0}^{n-1}u_i\times(v_{i+1}-v_i)$. Take any $\delta>0$.

(a) z is expressible as $z'+z''$ where z', $z''\in L^0(\mathfrak{A}_n)$ and

$$\|z'\|_1\le(2+\frac{M}{\delta})\|(v_n-M\chi 1)^+\|_1,$$

$$\|z''\|_2^2\le\delta^2+\|v_n\wedge M\chi 1\|_2^2.$$

(b) $\|z\|_1\le\delta+(2+\frac{M}{\delta})\|v_n\|_1+\sqrt{M\|v_n\|_1}$.

proof (a) Induce on n.

(i) The case $n=0$ is trivial. For the inductive step to $n\ge 1$, set $\beta=2+\frac{M}{\delta}$ and $z_0=\sum_{i=0}^{n-2}u_i\times(v_{i+1}-v_i)$; since

$$[\![v_i>M]\!]\subseteq[\![v_j=v_n]\!]\cap[\![v_{n-1}=v_n]\!]\subseteq[\![v_j=v_{n-1}]\!]$$

whenever $i\le j\le n-1$, the inductive hypothesis tells us that we can express z_0 as $z_0'+z_0''$ where z_0', $z_0''\in L^0(\mathfrak{A}_{n-1})$ and

$$\|z_0'\|_1\le\beta\|(v_{n-1}-M\chi 1)^+\|_1,$$

$$\|z_0''\|_2^2\le\delta^2+\|v_{n-1}\wedge M\chi 1\|_2^2.$$

Write P for P_{n-1} and $\hat{v}$ for $(v_n-M\chi 1)^+-(v_{n-1}-M\chi 1)^+$. Because

$$[\![(v_{n-1}-M\chi 1)^+\ne 0]\!]=[\![v_{n-1}>M]\!]\subseteq[\![v_{n-1}=v_n]\!]\subseteq[\![\hat{v}=0]\!],$$

$\hat{v}=(v_n-M\chi 1)^+\times\chi[\![v_{n-1}\le M]\!]\ge 0$, and

$$\begin{aligned}v_{n-1}\wedge M\chi 1-P(v_n\wedge M\chi 1)&=v_{n-1}-(v_{n-1}-M\chi 1)^+-Pv_n+P(v_n-M\chi 1)^+\\&=P(v_n-M\chi 1)^+-(v_{n-1}-M\chi 1)^+=P\hat v\ge 0.\end{aligned}$$

(ii) There is a $\tilde z\in L^0(\mathfrak{A}_{n-1})$ such that $\|\tilde z\|_2^2\le\delta^2+\|P(v_n\wedge M\chi 1)\|_2^2$ and $\|z_0''-\tilde z\|_1\le\frac{M}{\delta}\|\hat v\|_1$. **P** If $\|z_0''\|_2^2\le\delta^2+\|P(v_n\wedge M\chi 1)\|_2^2$, set $\tilde z=z_0''$. Otherwise, the function

$$\gamma\mapsto\|\operatorname{med}(-\gamma\chi 1,z_0'',\gamma\chi 1)\|_2^2$$

is continuous, so there is a $\gamma\ge\delta$ such that $\|\tilde z\|_2^2=\delta^2+\|P(v_n\wedge M\chi 1)\|_2^2$ where $\tilde z=\operatorname{med}(-\gamma\chi 1,z_0'',\gamma\chi 1)$. In this case,

$$\|z_0''\|_2^2-\|\tilde z\|_2^2=\mathbb{E}((|z_0''|+|\tilde z|)\times(|z_0''|-|\tilde z|))\ge 2\delta\mathbb{E}(|z_0''|-|\tilde z|)=2\delta\|z_0''-\tilde z\|_1$$

because $|z_0''|=|\tilde z|+|z_0''-\tilde z|$ and

$$[\![z_0''\ne\tilde z]\!]=[\![|z_0''|>\gamma]\!]\subseteq[\![|z_0''|\ge\delta]\!]\cap[\![|\tilde z|\ge\delta]\!],$$

and similarly

$$\begin{aligned}&\|v_{n-1}\wedge M\chi 1\|_2^2-\|P(v_n\wedge M\chi 1)\|_2^2\\&\quad=\mathbb{E}\big((v_{n-1}\wedge M\chi 1+P(v_n\wedge M\chi 1))\times(v_{n-1}\wedge M\chi 1-P(v_n\wedge M\chi 1))\big)\\&\quad\le 2M\|v_{n-1}\wedge M\chi 1-P(v_n\wedge M\chi 1)\|_1=2M\|P\hat v\|_1=2M\|\hat v\|_1\end{aligned}$$

because $\hat v\ge 0$. So

$$\begin{aligned}\|z_0''-\tilde z\|_1&\le\frac{1}{2\delta}(\|z_0''\|_2^2-\|\tilde z\|_2^2)\\&\le\frac{1}{2\delta}(\delta^2+\|v_{n-1}\wedge M\chi 1\|_2^2-\delta^2-\|P(v_n\wedge M\chi 1)\|_2^2)\le\frac{M}{\delta}\|\hat v\|_1.\ \mathbf{Q}\end{aligned}$$

(iii) Set $z''=\tilde z+u_{n-1}\times(v_n\wedge M\chi 1-P(v_n\wedge M\chi 1))$. Since $\tilde z\times u_{n-1}\in L^0(\mathfrak{A}_{n-1})\cap L^2_{\bar\mu}$ and $v_n\wedge M\chi 1-P(v_n\wedge M\chi 1)\in L^\infty(\mathfrak{A})$,

$$\begin{aligned}0&=\mathbb{E}(\tilde z\times u_{n-1}\times(v_n\wedge M\chi 1-P(v_n\wedge M\chi 1)))\\&=\mathbb{E}(P(v_n\wedge M\chi 1)\times(v_n\wedge M\chi 1-P(v_n\wedge M\chi 1)))\end{aligned}$$

and

$$\begin{aligned}\|z''\|_2^2&=\|\tilde z\|_2^2+\|u_{n-1}\times(v_n\wedge M\chi 1-P(v_n\wedge M\chi 1))\|_2^2\\&\le\delta^2+\|P(v_n\wedge M\chi 1)\|_2^2+\|v_n\wedge M\chi 1-P(v_n\wedge M\chi 1)\|_2^2=\delta^2+\|v_n\wedge M\chi 1\|_2^2.\end{aligned}$$

(iv) Set

$$\begin{aligned}z'&=z-z''\\&=z_0+u_{n-1}\times(v_n-v_{n-1})-\tilde z-u_{n-1}\times(v_n\wedge M\chi 1-P(v_n\wedge M\chi 1))\\&=z_0'+z_0''-\tilde z\\&\qquad+u_{n-1}\times((v_n-M\chi 1)^+-(v_{n-1}-M\chi 1)^+-v_{n-1}\wedge M\chi 1+P(v_n\wedge M\chi 1))\\&=z_0'+z_0''-\tilde z+u_{n-1}\times(\hat v-P\hat v),\end{aligned}$$

so that

$$\begin{aligned}\|z'\|_1&\le\|z_0'\|_1+\|z_0''-\tilde z\|_1+\|\hat v\|_1+\|P\hat v\|_1\\&\le\beta\|(v_{n-1}-M\chi 1)^+\|_1+\frac{M}{\delta}\|\hat v\|_1+2\|\hat v\|_1\\&=\beta(\|(v_{n-1}-M\chi 1)^+\|_1+\|\hat v\|_1)=\beta\|(v_n-M\chi 1)^+\|_1,\end{aligned}$$

and the induction proceeds.

(b) follows at once, because

$$\|z''\|_1^2 \le \|z''\|_2^2 \le \delta^2 + \|v_n \wedge M\chi 1\|_2^2 \le \delta^2 + M\|v_n\|_1$$

and $\|z''\|_1 \le \delta + \sqrt{M\|v_n\|_1}$.

621J Lemma Suppose that $\langle v_i\rangle_{i\le n}$ is a non-negative submartingale adapted to $\langle\mathfrak{A}_i\rangle_{i\le n}$ and $z = \sum_{i=0}^{n-1} P_i v_{i+1} - v_i$. Then $\alpha\mathbb{E}(z \times \chi[\![z > 2\alpha]\!]) \le 3(\beta\mathbb{E}(v_n) + \alpha\mathbb{E}((v_n - \beta\chi 1)^+))$ whenever α, $\beta \ge 0$.

proof (see Karatzas & Shreve 91, 1.4.10) **(a)** For $k \le n$ and $\gamma \ge 0$, set

$$u_k = P_k v_n - v_k, \quad z_k = \textstyle\sum_{i=0}^{k-1} P_i v_{i+1} - v_i,$$

(note that $z_{k+1} \in L^0(\mathfrak{A}_k)$ if $k < n$),

$$b_\gamma = [\![z > \gamma]\!],$$

$$a_{\gamma k} = [\![z_{k+1} > \gamma]\!] \setminus [\![z_k > \gamma]\!] \in \mathfrak{A}_k \text{ if } k < n, \quad a_{\gamma n} = 1 \setminus [\![z > \gamma]\!] \in \mathfrak{A}_n.$$

Because $\boldsymbol{v}$ is a submartingale, $0 = z_0 \le \ldots \le z_n = z$, $\langle a_{\gamma k}\rangle_{k\le n}$ is a partition of unity in $\mathfrak{A}$ and $b_\gamma = \sup_{k<n} a_{\gamma k}$.

Observe that if $k \le j$ then

$$P_k(P_j v_{j+1} - v_j) = P_k P_j(v_{j+1} - v_j) = P_k(v_{j+1} - v_j),$$

so if $k < n$ then

$$\begin{aligned} P_k z - z_k &= P_k(\sum_{j=k}^{n-1} P_j v_{j+1} - v_i) = P_k(\sum_{j=k}^{n-1} v_{j+1} - v_i) \\ &= P_k(v_n - v_k) = u_k. \end{aligned}$$

(b) For any $\gamma \ge 0$ and $k < n$.

$$\mathbb{E}(\chi a_{\gamma k} \times z) - \gamma\bar\mu a_{\gamma k} \le \mathbb{E}(\chi a_{\gamma k} \times u_k) \le \beta\bar\mu a_{\gamma k} + \mathbb{E}(\chi a_{\gamma k} \times (v_n - \beta\chi 1)^+).$$

P

$$\chi a_{\gamma k} \times u_k = \chi a_{\gamma k} \times (P_k z - z_k) \ge \chi a_{\gamma k} \times (P_k z - \gamma\chi 1)$$

because $a_{\gamma k} \cap [\![z_k > \gamma]\!] = 0$. So

$$\mathbb{E}(\chi a_{\gamma k} \times z) - \gamma\bar\mu a_{\gamma k} = \mathbb{E}(\chi a_{\gamma k} \times P_k z) - \gamma\bar\mu a_{\gamma k}$$

(because $a_{\gamma k} \in \mathfrak{A}_k$)

$$\begin{aligned} &= \mathbb{E}(\chi a_{\gamma k} \times P_k(z - \gamma\chi 1)) \le \mathbb{E}(\chi a_{\gamma k} \times u_k) \\ &\le \mathbb{E}(\chi a_{\gamma k} \times P_k v_n) = \mathbb{E}(\chi a_{\gamma k} \times v_n) \\ &= \mathbb{E}(\chi a_{\gamma k} \times (v_n \wedge \beta\chi 1)) + \mathbb{E}(\chi a_{\gamma k} \times (v_n - \beta\chi 1)^+) \\ &\le \mathbb{E}(\chi a_{\gamma k} \times \beta\chi 1) + \mathbb{E}(\chi a_{\gamma k} \times (v_n - \beta\chi 1)^+) \\ &= \beta\bar\mu a_{\gamma k} + \mathbb{E}(\chi a_{\gamma k} \times (v_n - \beta\chi 1)^+). \ \mathbf{Q} \end{aligned}$$

(c) Now we see that

$$\begin{aligned} \mathbb{E}(\chi b_\gamma \times z) - \gamma\bar\mu b_\gamma &= \sum_{k=0}^{n-1} \mathbb{E}(\chi a_{\gamma k} \times z) - \gamma\bar\mu a_{\gamma k} \\ &\le \sum_{k=0}^{n-1} \beta\bar\mu a_{\gamma k} + \mathbb{E}(\chi a_{\gamma k} \times (v_n - \beta\chi 1)^+) \\ &= \beta\bar\mu b_\gamma + \mathbb{E}(\chi b_\gamma \times (v_n - \beta\chi 1)^+) \end{aligned}$$

for every $\gamma \ge 0$. Next,

$$\alpha\bar{\mu}b_{2\alpha} \le \mathbb{E}(\chi b_{2\alpha} \times (z - \alpha\chi 1)) \le \mathbb{E}(\chi b_\alpha \times (z - \alpha\chi 1)) = \mathbb{E}(\chi b_\alpha \times z) - \alpha\bar{\mu}b_\alpha$$

because $b_{2\alpha} \subseteq [\![z - \alpha\chi 1 > \alpha]\!] \subseteq b_\alpha$. It follows that

$$\begin{aligned}\mathbb{E}(\chi b_{2\alpha} \times z) &\le (2\alpha + \beta)\bar{\mu}b_{2\alpha} + \mathbb{E}(\chi b_{2\alpha} \times (v_n - \beta\chi 1)^+)\\ &\le 2(\mathbb{E}(\chi b_\alpha \times z) - \alpha\bar{\mu}b_\alpha) + \beta\bar{\mu}b_\alpha + \mathbb{E}(\chi b_\alpha \times (v_n - \beta\chi 1)^+)\\ &\le 3(\beta\bar{\mu}b_\alpha + \mathbb{E}(\chi b_\alpha \times (v_n - \beta\chi 1)^+)).\end{aligned}$$

(d) On the other hand,

$$\begin{aligned}\alpha\bar{\mu}b_\alpha = \alpha\bar{\mu}[\![z > \alpha]\!] \le \mathbb{E}(z) &= \sum_{i=0}^{n-1} \mathbb{E}(P_i v_{i+1}) - \mathbb{E}(v_i)\\ &= \sum_{i=0}^{n-1} \mathbb{E}(v_{i+1}) - \mathbb{E}(v_i) = \mathbb{E}(v_n) - \mathbb{E}(v_{\sigma_0}) \le \mathbb{E}(v_n)\end{aligned}$$

and

$$\begin{aligned}\alpha\mathbb{E}(z \times \chi[\![z > 2\alpha]\!]) = \alpha\mathbb{E}(\chi b_{2\alpha} \times z) &\le 3(\beta\mathbb{E}(v_n) + \alpha\mathbb{E}((v_n - \beta\chi 1)^+))\\ &= 3(\beta\mathbb{E}(v_{\max I}) + \alpha\mathbb{E}((v_{\max I} - \beta\chi 1)^+)),\end{aligned}$$

as required.

621X Basic exercises (a) Suppose that we think of 621D-621J as applying to a stochastic integration structure $(\mathfrak{A}, \bar{\mu}, T, \langle\mathfrak{A}_t\rangle_{t\in T}, \mathcal{T}, \langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ such that $T = \{0, \dots, n\}$. Show that if $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{T}}$ is any fully adapted process, then

$$Q_{\mathcal{T}}(d\boldsymbol{v}) = \{\sum_{i=0}^{n-1} u_i \times (v_{(i+1)\check{}} - v_{\check{\imath}}) : u_i \in L^0(\mathfrak{A}_i),\ \|u_i\|_\infty \le 1 \text{ for every } i < n\}.$$

(b) In the context of Lemma 621H, show that, given $M \ge 0$, there are n, $\boldsymbol{v}$ and $\langle u_i\rangle_{i<n}$ such that $\boldsymbol{v}$ is a martingale and $\|z\|_1 \ge M\|v_n\|_1$. (*Hint*: $v_i = \gamma^i\chi a_i$, $u_i = (-1)^i\chi 1$ where $1 = a_0 \supseteq \dots \supseteq a_n$.)

621 Notes and comments There are some curious formulae here, which is why I think readers may wish to look ahead to see what they are supposed to do. But 621H, 621I and 621J, as well as 621E, are of the same kind; they seek to bound quantities calculated from sequences $\langle u_i\rangle_{i\le n}$ and $\langle v_i\rangle_{i\le n}$ in terms of a quantity determined by the final term v_n alone. The same is true of the fundamental inequalities 275D and 275F.

The constant 16 in 621Hf is far from best possible; in fact it is the case there that $\bar{\mu}[\![|z| > \gamma]\!] \le \frac{2}{\gamma}\|v_n\|_1$ (628D). (I do not know whether the inequality can be improved further.) However the proof in §628 is substantially longer than that in 621G-621H and demands techniques from measure theory which I do not think we shall need elsewhere in this volume. The argument above is a better preparation for what will come later. For instance, the idea in 621G will reappear, in much more general form, in the 'previsible variations' of §626.

The formulae in 621I-621J are bound to seem odd. In 621I we are presented with M but anticipate that δ and $\|v_n\|_1$ can be forced to be small, so that $\|z\|_1$ is small; this will be used in one of the main results of the chapter (623O). In 621J we see that $\lim_{\alpha\to\infty} f(\frac{1}{\alpha}\mathbb{E}(v_n)) = 0$, while a bound on $\mathbb{E}(z \times \chi[\![z > \alpha]\!])$ is something we look for if we want to prove that a set is uniformly integrable. See part (b) of the proof of 626M.

The notation of this section is a little clumsier than it might be, because from 621F onwards I repeatedly speak of elements $z = \sum_{i=0}^{n-1} u_i \times (v_{i+1} - v_i)$ without making the obvious association with $Q_{\mathcal{S}}(d\boldsymbol{v})$ (621Xa). I am in fact avoiding any appeal to the ideas introduced in Chapter 61; the material here can be regarded as a development of §275 in the language of Chapter 36, quite apart from its applications to stochastic integration, even though it is manifestly directed to those applications.

Version of 29.7.20

622 Fully adapted martingales

I come now to the promised central fact of the theory: martingales are local integrators. The first step is to establish a concept of 'martingale' for fully adapted processes (622C), which involves us in the properties of conditional expectations with respect to stopping-time algebras (622B). Elementary facts about martingales are in 622D-622F. The theorem that every martingale is a local integrator is now easy (622H); of course it depends on non-trivial ideas from §621. In 622L I check that Brownian motion, as defined in 612T, is a local martingale. The rest of the section is a miscellany of results which will be needed later.

622A Notation As in Chapter 61, $(\mathfrak{A},\bar{\mu},T,\langle\mathfrak{A}_t\rangle_{t\in T},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ will be a stochastic integration structure, with $\mathcal{T}_f$ and $\mathcal{T}_b$ the corresponding lattices of finite and bounded stopping times. For $t\in T$, $\check{t}$ is the constant stopping time at t. As before, I write $\theta(w)=\mathbb{E}(|w|\wedge\chi 1)$ for $w\in L^0(\mathfrak{A})$. For a sublattice $\mathcal{S}$ of $\mathcal{T}$, $\mathcal{I}(\mathcal{S})$ will be the set of finite sublattices of $\mathcal{S}$.

For $\tau\in\mathcal{T}$, $P_\tau:L^1_{\bar{\mu}}\to L^0(\mathfrak{A}_\tau)\cap L^1_{\bar{\mu}}$, where $L^1_{\bar{\mu}}=L^1(\mathfrak{A},\bar{\mu})$, will be the conditional expectation operator associated with the closed subalgebra $\mathfrak{A}_\tau$ of $\mathfrak{A}$ (621C).

622B We need something to match the familiar rule on composition of conditional expectation operators in 621Cb.

Proposition Suppose that σ, $\tau\in\mathcal{T}$.

(a) $P_\sigma P_\tau=P_{\sigma\wedge\tau}$.

(b) $[\![\sigma=\tau]\!]\subseteq[\![P_\sigma u=P_\tau u]\!]$ for every $u\in L^1_{\bar{\mu}}$.

proof (a)(i) Set $a=[\![\sigma\le\tau]\!]$, so that $a\in\mathfrak{A}_{\sigma\wedge\tau}=\mathfrak{A}_\sigma\cap\mathfrak{A}_\tau$ (611H(c-ii)), and $u\in L^0(\mathfrak{A}_{\sigma\wedge\tau})$ whenever $u\in L^0(\mathfrak{A}_\sigma)$ and $u=u\times\chi a$ (612C). Now if $u\in L^1_{\bar{\mu}}$ and $u=u\times\chi a$, we see that

$$(P_\sigma P_\tau u)\times\chi a=P_\sigma(P_\tau u\times\chi a)=P_\sigma P_\tau(u\times\chi a)=P_\sigma P_\tau u\in L^0(\mathfrak{A}_\sigma)$$

so that $P_\sigma P_\tau u\in L^0(\mathfrak{A}_{\sigma\wedge\tau})$. And if $b\in\mathfrak{A}_{\sigma\wedge\tau}$ we surely have

$$\mathbb{E}(P_\sigma P_\tau u\times\chi b)=\mathbb{E}(P_\sigma(P_\tau u\times\chi b))=\mathbb{E}(P_\tau u\times\chi b)=\mathbb{E}(u\times\chi b).$$

So $P_\sigma P_\tau u=P_{\sigma\wedge\tau}u$.

(ii) Next, setting $a'=[\![\tau\le\sigma]\!]$, we have $u\in L^0(\mathfrak{A}_{\sigma\wedge\tau})$ whenever $u\in L^0(\mathfrak{A}_\tau)$ and $u=u\times\chi a'$. So now, if $u\in L^1_{\bar{\mu}}$ and $u=u\times\chi a'$, we have $P_\tau u=P_\tau u\times\chi a'\in L^0(\mathfrak{A}_{\sigma\wedge\tau})$ and $P_\sigma P_\tau u=P_\tau u\in L^0(\mathfrak{A}_{\sigma\wedge\tau})$. As in (i),

$$\mathbb{E}(P_\sigma P_\tau u\times\chi b)=\mathbb{E}(P_\tau u\times\chi b)=\mathbb{E}(u\times\chi b)$$

whenever $b\in\mathfrak{A}_{\sigma\wedge\tau}$. So $P_\sigma P_\tau u=P_{\sigma\wedge\tau}u$ in this case also.

(iii) Assembling these, and noting that $1\setminus a\subseteq a'$,

$$\begin{aligned}P_\sigma P_\tau u&=P_\sigma P_\tau(u\times\chi a)+P_\sigma P_\tau(u\times\chi(1\setminus a))\\&=P_{\sigma\wedge\tau}(u\times\chi a)+P_{\sigma\wedge\tau}(u\times\chi(1\setminus a))=P_{\sigma\wedge\tau}u\end{aligned}$$

for every $u\in L^1_{\bar{\mu}}$, and $P_\sigma P_\tau=P_{\sigma\wedge\tau}$.

(b) Set $c=a\cap a'=[\![\sigma=\tau]\!]$. Then $c\in\mathfrak{A}_{\sigma\wedge\tau}$ and $(u\times\chi c)\times\chi a'=u\times\chi c$, so

$$P_\tau u\times\chi c=P_\tau(u\times\chi c)=P_\sigma P_\tau(u\times\chi c)$$

(see (a-ii) above)

$$=P_{\sigma\wedge\tau}(u\times\chi c).$$

Similarly,

$$P_\sigma u\times\chi c=P_{\tau\wedge\sigma}(u\times\chi c)=P_{\sigma\wedge\tau}(u\times\chi c)=P_\tau u\times\chi c.$$

So $[\![P_\sigma u=P_\tau u]\!]\supseteq c$.

622C Fully adapted martingales Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ a fully adapted process.

(a) $\boldsymbol{v}$ is an L^1**-process** if $v_\sigma \in L^1_{\bar{\mu}}$ for every $\sigma \in \mathcal{S}$. Generally, $\boldsymbol{v}$ is an L^p**-process**, for $1 \le p < \infty$, if $v_\sigma \in L^p(\mathfrak{A}, \bar{\mu})$ (366A) for every $\sigma \in \mathcal{S}$, and an L^∞**-process** if $v_\sigma \in L^\infty(\mathfrak{A})$ (363A, 364J) for every $\sigma \in \mathcal{S}$.

For $1 \le p \le \infty$, $\boldsymbol{v}$ is $\| \ \|_p$**-bounded** if $\sup_{\sigma \in \mathcal{S}} \|v_\sigma\|_p$ is finite (counting the supremum as 0 if $\mathcal{S}$ is empty). Note that $\boldsymbol{v}$ is $\| \ \|_\infty$-bounded iff it is order-bounded and $\sup |\boldsymbol{v}|$, as defined in 614Ea, is in $L^\infty(\mathfrak{A})$.

(b) $\boldsymbol{v}$ is a **martingale** if it is an L^1-process and $v_\sigma = P_\sigma v_\tau$ whenever $\sigma \le \tau$ in $\mathcal{S}$.

(c) $\boldsymbol{v}$ is a **local martingale** if there is a covering ideal $\mathcal{S}'$ of $\mathcal{S}$ (611N) such that $\boldsymbol{v} \restriction \mathcal{S}'$ is a martingale.

Taking $\mathcal{S}' = \mathcal{S}$ we see that every martingale is a local martingale. For classic examples of local martingales which are not martingales, see 622Xe and 632N.

Note that I do *not* say '$\boldsymbol{v}$ is a local martingale if $\boldsymbol{v} \restriction \mathcal{S} \wedge \tau$ is a martingale for every $\tau \in \mathcal{S}$'; the 'local' in 'local martingale' is not the same as the 'local' in 'local integrator' or 'locally order-bounded' or 'locally of bounded variation' or 'locally moderately oscillatory'.

(e)(i) $\boldsymbol{v}$ is **uniformly integrable** if it is an L^1-process and $\{v_\sigma : \sigma \in \mathcal{S}\}$ is uniformly integrable.

(ii) It will be convenient to use the phrase 'L^p-martingale' to mean a martingale which is also an L^p-process.

622D Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $M_{\mathrm{fa}} = M_{\mathrm{fa}}(\mathcal{S})$ the Riesz space of fully adapted processes with domain $\mathcal{S}$. Let $\mathcal{S}'$ be a sublattice of $\mathcal{S}$.

(a) For any $p \in [1, \infty]$, the set of L^p-processes with domain $\mathcal{S}$ is a solid linear subspace of M_{fa}, and $\boldsymbol{v} \restriction \mathcal{S}'$ is an L^p-process whenever $\boldsymbol{v} \in M_{\mathrm{fa}}$ is an L^p-process.

(b)(i) The set of martingales with domain $\mathcal{S}$ is a linear subspace of M_{fa}.

(ii) If $\boldsymbol{v} \in M_{\mathrm{fa}}$ is a martingale then $\boldsymbol{v} \restriction \mathcal{S}'$ is a martingale.

(c) The set of local martingales with domain $\mathcal{S}$ is a linear subspace of M_{fa}. If $\mathcal{S}'$ is an ideal of $\mathcal{S}$, then $\boldsymbol{v} \restriction \mathcal{S}'$ is a local martingale for every local martingale $\boldsymbol{v} \in M_{\mathrm{fa}}$.

(d) The set of uniformly integrable processes with domain $\mathcal{S}$ is a solid linear subspace of M_{fa}, and $\boldsymbol{v} \restriction \mathcal{S}'$ is uniformly integrable whenever $\boldsymbol{v} \in M_{\mathrm{fa}}$ is uniformly integrable.

proof (a) This is immediate from the definitions (622Ca).

(b)(i)-(ii) These too are immediate from the definitions.

(c) Applying (b-i) to appropriate ideals $\mathcal{S} \wedge \tau$, we see that the set of local martingales with domain $\mathcal{S}$ is a linear subspace of M_{fa}. Concerning ideals of $\mathcal{S}$, we need to know that if $\mathcal{S}'$ is an ideal of $\mathcal{S}$ and $\mathcal{S}_1$ is a covering ideal of $\mathcal{S}$, then $\mathcal{S}_1 \cap \mathcal{S}'$ is a covering ideal of $\mathcal{S}'$. **P** If $\tau \in \mathcal{S}'$, then

$$\sup_{\sigma \in \mathcal{S}_1 \cap \mathcal{S}'} [\![\tau = \sigma]\!] \supseteq \sup_{\sigma \in \mathcal{S}_1} [\![\tau = \tau \wedge \sigma]\!] \supseteq \sup_{\sigma \in \mathcal{S}_1} [\![\tau = \sigma]\!] = 1. \text{ } \mathbf{Q}$$

(d) All we need to know is that subsets, sums and scalar multiples and solid hulls of uniformly integrable sets are uniformly integrable, as declared in 621B(c-i).

622E Elementary facts Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ a fully adapted process.

(a) If $\boldsymbol{u}$ is constant with a value in $L^1_{\bar{\mu}}$, then $\boldsymbol{u}$ is a uniformly integrable martingale.

(b)(i) If $\tau \in \mathcal{S}$, then $\boldsymbol{u}$ is a martingale iff $\boldsymbol{u} \restriction \mathcal{S} \wedge \tau$ and $\boldsymbol{u} \restriction \mathcal{S} \vee \tau$ are martingales. **P** If $\boldsymbol{u}$ is a martingale then $\boldsymbol{u} \restriction \mathcal{S} \wedge \tau$ and $\boldsymbol{u} \restriction \mathcal{S} \vee \tau$ are martingales by 622Db. If $\boldsymbol{u} \restriction \mathcal{S} \wedge \tau$ and $\boldsymbol{u} \restriction \mathcal{S} \vee \tau$ are martingales, then we have

$$u_\sigma = u_{\sigma \vee \tau} + u_{\sigma \wedge \tau} - u_\tau \in L^1_{\bar{\mu}}$$

for every $\sigma \in \mathcal{S}$ (612Df). So $\boldsymbol{u}$ is an L^1-process. If $\sigma \le \sigma'$ in $\mathcal{S}$ then

$$P_{\sigma \wedge \tau} u_{\sigma' \vee \tau} = P_\sigma P_\tau u_{\sigma' \vee \tau} = P_\sigma u_\tau$$

(using 622Ba for the first equality), so

$$\begin{aligned}[\![\sigma\le\tau]\!] &= [\![\sigma=\sigma\wedge\tau]\!]\cap[\![\sigma=\sigma\wedge\sigma'\wedge\tau]\!]\\ &\qquad\cap[\![P_\sigma u_{\sigma'\vee\tau}=P_{\sigma\wedge\tau}u_{\sigma'\vee\tau}]\!]\cap[\![P_\sigma u_{\sigma'\wedge\tau}=P_{\sigma\wedge\sigma'\wedge\tau}u_{\sigma'\wedge\tau}]\!]\end{aligned}$$

(611E(a-ii-β, 622Bb)

$$\begin{aligned}&\subseteq[\![\sigma=\sigma\wedge\sigma'\wedge\tau]\!]\cap[\![P_\sigma u_{\sigma'\vee\tau}=P_\sigma u_\tau]\!]\\ &\qquad\cap[\![P_\sigma u_{\sigma'\wedge\tau}=u_{\sigma\wedge\sigma'\wedge\tau}]\!]\cap[\![u_\sigma=u_{\sigma\wedge\sigma'}\wedge\tau]\!]\\ &\subseteq[\![P_\sigma u_{\sigma'\vee\tau}=P_\sigma u_\tau]\!]\cap[\![P_\sigma u_{\sigma'\wedge\tau}=u_\sigma]\!]\\ &\subseteq[\![P_\sigma(u_{\sigma'\wedge\tau}+u_{\sigma'\vee\tau}-u_\tau)=u_\sigma]\!]=[\![P_\sigma u_{\sigma'}=u_\sigma]\!].\end{aligned}$$

On the other hand,

$$\begin{aligned}[\![\tau\le\sigma]\!] &\subseteq[\![\sigma=\sigma\vee\tau]\!]\cap[\![\sigma'\wedge\tau=\tau]\!]\\ &\subseteq[\![\sigma=\sigma\vee\tau]\!]\cap[\![\sigma'\wedge\tau=\tau]\!]\cap[\![P_\sigma u_\tau=P_{\sigma\vee\tau}u_\tau]\!]\\ &\qquad\cap[\![P_\sigma u_{\sigma'\vee\tau}=P_{\sigma\vee\tau}u_{\sigma'\vee\tau}]\!]\cap[\![P_\sigma u_{\sigma'\wedge\tau}=P_{\sigma\vee\tau}u_{\sigma'\wedge\tau}]\!]\\ &\subseteq[\![\sigma=\sigma\vee\tau]\!]\cap[\![\sigma'\wedge\tau=\tau]\!]\cap[\![P_\sigma u_\tau=u_\tau]\!]\\ &\qquad\cap[\![P_\sigma u_{\sigma'\vee\tau}=u_{\sigma\vee\tau}]\!]\cap[\![P_\sigma u_{\sigma'\wedge\tau}=u_{\sigma'\wedge\tau}]\!]\\ &\subseteq[\![P_\sigma u_\tau=u_\tau]\!]\cap[\![P_\sigma u_{\sigma'\vee\tau}=u_\sigma]\!]\cap[\![P_\sigma u_{\sigma'\wedge\tau}=u_\tau]\!]\subseteq[\![P_\sigma u_{\sigma'}=u_\sigma]\!].\end{aligned}$$

So in fact $[\![P_\sigma u_{\sigma'}=u_\sigma]\!]=1$ and $P_\sigma u_{\sigma'}=u_\sigma$. As σ and σ' are arbitrary, $\boldsymbol{u}$ is a martingale. **Q**

(ii) If $\tau\in\mathcal{S}$, $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ is a martingale and $\boldsymbol{u}\restriction\mathcal{S}\vee\tau$ is constant, then $\boldsymbol{u}$ is a martingale. (For the constant value of $\boldsymbol{u}\restriction\mathcal{S}\vee\tau$ must be u_τ, which is a value of $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ so is in $L^1_{\bar\mu}$.)

(c) If for every $\epsilon>0$ there is a martingale $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ such that $\|u_\sigma-v_\sigma\|_1\le\epsilon$ for every $\sigma\in\mathcal{S}$, then $\boldsymbol{u}$ is a martingale. **P** Note first that as there is an L^1-process $\boldsymbol{v}$ such that $\boldsymbol{u}-\boldsymbol{v}$ is an L^1-process, $\boldsymbol{u}$ also is an L^1-process. Now suppose that $\tau\le\tau'$ in $\mathcal{S}$ and $\epsilon>0$. Let $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ be a martingale such that $\|u_\sigma-v_\sigma\|_1\le\frac{1}{2}\epsilon$ for every $\sigma\in\mathcal{S}$. Then

$$\|P_\tau u_{\tau'}-u_\tau\|_1\le\|P_\tau u_{\tau'}-P_\tau v_{\tau'}\|_1+\|P_\tau v_{\tau'}-v_\tau\|_1+\|v_\tau-u_\tau\|_1\le\epsilon$$

(using 621Ca to see that $\|P_\tau(u_{\tau'}-v_{\tau'})\|_1\le\|u_{\tau'}-v_{\tau'}\|_1$). As ϵ is arbitrary, $P_\tau u_{\tau'}=u_\tau$; as τ and τ' are arbitrary, $\boldsymbol{u}$ is a martingale. **Q**

(d) If $\boldsymbol{u}$ is a martingale and $A\subseteq\mathcal{S}$ is non-empty and downwards-directed, then the $\|\;\|_1$-limit $z=1\text{lim}_{\sigma\downarrow A}u_\sigma$ is defined and is the limit $\lim_{\sigma\downarrow A}u_\sigma$ for the topology of convergence in measure; and if $\tau\in A$ then z is the conditional expectation of u_τ on $\bigcap_{\sigma\in A}\mathfrak{A}_\sigma$. **P** By 367Qa, $1\text{lim}_{\sigma\downarrow A}u_\sigma=1\text{lim}_{\sigma\downarrow A}P_\sigma u_\tau$ is defined and is the conditional expectation of u_τ on $\bigcap_{\sigma\in A}\mathfrak{A}_\sigma$. By 613B(d-i), this is also the limit $\lim_{\sigma\downarrow A}u_\sigma$. **Q**

In particular, if $\mathcal{S}$ is non-empty, then the starting value $\lim_{\sigma\downarrow\mathcal{S}}u_\sigma$ is defined and belongs to $L^1_{\bar\mu}$.

622F Proposition Take any $u\in L^1_{\bar\mu}$.

(a) $\boldsymbol{P}u=\langle P_\tau u\rangle_{\tau\in\mathcal{T}}$ is a uniformly integrable martingale.

(b) Suppose that σ, $\tau\in\mathcal{T}$ and $[\![u\ne0]\!]\subseteq[\![\sigma=\tau]\!]$. Then $P_\sigma u=P_\tau u$.

proof (a) By 622Bb, $\boldsymbol{P}u$ is fully adapted; by 622Ba, it is a martingale; by 621Cf, it is uniformly integrable.

(b) As in the proof of 622Ba, set $a=[\![\sigma\le\tau]\!]$. Then $u=u\times\chi a$ and $a\in\mathfrak{A}_\sigma$ so

$$P_\sigma u=P_\sigma(u\times\chi a)=P_\sigma u\times\chi a;$$

since $P_\sigma u\in L^0(\mathfrak{A}_\sigma)$, $P_\sigma u\in L^0(\mathfrak{A}_\tau)$ (612C again) and

$$P_\sigma u=P_\tau P_\sigma u=P_{\tau\wedge\sigma}u.$$

Similarly,

$$P_\tau u=P_{\sigma\wedge\tau}u=P_{\tau\wedge\sigma}u=P_\sigma u,$$

as claimed.

622G I have a great deal more to say about both martingales and local martingales. But I will move directly to the most important result in this section.

Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ a $\| \, \|_1$-bounded martingale. Then $\boldsymbol{v}$ is an integrator, therefore moderately oscillatory and order-bounded.

proof If $\mathcal{S}$ is empty, this is trivial, so let us suppose that $\mathcal{S} \ne \emptyset$. Then $Q_{\mathcal{S}}(d\boldsymbol{v})$ is topologically bounded. **P** Let $\epsilon > 0$. Let $\delta > 0$ be such that $\delta \sup_{\sigma \in \mathcal{S}} \|v_\sigma\|_1 \le \epsilon^2$. Take $z \in Q_{\mathcal{S}}(d\boldsymbol{v})$. Then there are a non-empty finite sublattice I of $\mathcal{S}$ and a fully adapted process $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in I}$ with $\|\boldsymbol{u}\|_\infty \le 1$ such that $z = S_I(\boldsymbol{u}, d\boldsymbol{v})$. Let $(\tau_0, \dots, \tau_n)$ linearly generate the I-cells (611L). Then $z = \sum_{i=0}^{n-1} u_{\tau_i} \times (v_{\tau_{i+1}} - v_{\tau_i})$. Now $\langle v_{\tau_i} \rangle_{i \le n}$ is a martingale in the classical sense of 621Da. So

$$\theta(\delta z) \le \epsilon + \bar{\mu}[\![\delta|z| > \epsilon]\!] \le \epsilon + \bar{\mu}[\![|z| > \frac{\epsilon}{\delta}]\!] \le \epsilon + \frac{16\delta}{\epsilon}\|v_{\tau_n}\|_1$$

(621Hf)

$$\le 17\epsilon.$$

As ϵ is arbitrary, $Q_{\mathcal{S}}(d\boldsymbol{v})$ is topologically bounded. **Q**

Thus $\boldsymbol{v}$ is an integrator. By 616Ib, it is moderately oscillatory and order-bounded.

622H Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. If $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ is a local martingale, then it is a local integrator, therefore locally moderately oscillatory.

proof Let $\mathcal{S}'$ be a covering ideal of $\mathcal{S}$ such that $\boldsymbol{v} \restriction \mathcal{S}'$ is a martingale. Suppose that $\tau \in \mathcal{S}$ and $\epsilon > 0$. Then there is a $\tau_1 \in \mathcal{S}'$ such that $c = [\![\tau \le \tau_1]\!]$ has measure at least $1 - \epsilon$ (611Mh). Set $\boldsymbol{v}' = \boldsymbol{P}v_{\tau_1} \restriction \mathcal{S} \wedge \tau$. As $\boldsymbol{P}v_{\tau_1}$ is a uniformly integrable martingale (622Fa), it is an integrator (622G), so $\boldsymbol{v}'$ is an integrator (616P(b-ii)).

Now if $\sigma \in \mathcal{S} \wedge \tau$,

$$[\![v_\sigma = v'_\sigma]\!] = [\![v_\sigma = P_\sigma v_{\tau_1}]\!] = [\![v_\sigma = P_\sigma P_{\tau_1} v_{\tau_1}]\!] = [\![v_\sigma = P_{\sigma \wedge \tau_1} v_{\tau_1}]\!] = [\![v_\sigma = v_{\sigma \wedge \tau_1}]\!]$$

(because $\sigma \wedge \tau_1$ belongs to $\mathcal{S}'$ and $\boldsymbol{v} \restriction \mathcal{S}'$ is a martingale)

$$\supseteq [\![\sigma \wedge \tau_1 = \sigma]\!] = [\![\sigma \le \tau_1]\!] \supseteq [\![\tau \le \tau_1]\!] = c.$$

So

$$[\![\boldsymbol{v}' \ne \boldsymbol{v} \restriction \mathcal{S} \wedge \tau]\!] = \sup_{\sigma \in \mathcal{S} \wedge \tau} [\![v_\sigma \ne v'_\sigma]\!]$$

is disjoint from c and has measure at most ϵ. As ϵ is arbitrary, $\boldsymbol{v} \restriction \mathcal{S} \wedge \tau$ is an integrator (616P(b-iii)). As τ is arbitrary, $\boldsymbol{v}$ is a local integrator. **Q**

622I The principal martingale theorems (see §275) take slightly different forms in the present context, so I take the space to spell one of them out.

Doob's maximal inequality (second form) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ a martingale. Then $\boldsymbol{v}$ is locally order-bounded, and

$$\bar{\mu}(\sup_{\sigma \in \mathcal{S} \wedge \tau} [\![|v_\sigma| > \gamma]\!]) = \bar{\mu}([\![\sup_{\sigma \in \mathcal{S} \wedge \tau} |v_\sigma| > \gamma]\!]) \le \frac{1}{\gamma}\mathbb{E}(|v_\tau|)$$

for every $\tau \in \mathcal{S}$ and $\gamma > 0$.

proof If $A \subseteq \mathcal{S} \wedge \tau$ is a non-empty finite set, then

$$\bar{\mu}([\![\sup_{\sigma \in A} |v_\sigma| > \gamma]\!]) \le \frac{1}{\gamma}\mathbb{E}(|v_\tau|).$$

P Let I be the sublattice of $\mathcal{S}$ generated by A, and take $\tau_0 \le \ldots \le \tau_n$ linearly generating the I-cells (611L again). Then $(v_{\tau_0}, \dots, v_{\tau_n}, v_\tau)$ is a finite martingale adapted to $(\mathfrak{A}_{\tau_0}, \dots, \mathfrak{A}_{\tau_n}, \mathfrak{A}_\tau)$, so

$$\bar{\mu}([\![\sup_{i \le n} |v_{\tau_i}| > \gamma]\!]) \le \frac{1}{\gamma}\mathbb{E}(|v_\tau|)$$

by 621E. Write v for $\sup_{i\le n}|v_{\tau_i}|$. If $\sigma\in A$, then

$$[\![|v_\sigma|\le v]\!]\supseteq\sup_{i\le n}[\![v_\sigma=v_{\tau_i}]\!]\supseteq\sup_{i\le n}[\![\sigma=\tau_i]\!]=1$$

by 611Ke. So $v_\sigma\le v$. Thus $[\![\sup_{\sigma\in A}|v_\sigma|>\gamma]\!]\subseteq[\![v>\gamma]\!]$ has measure at most $\dfrac{1}{\gamma}\mathbb{E}(|v_\tau|)$. **Q**

Accordingly $c_\gamma=\sup_{\sigma\in\mathcal{S}\wedge\tau}[\![|v_\sigma|>\gamma]\!]$ has measure at most $\dfrac{1}{\gamma}\mathbb{E}(|v_\tau|)$, by 321D. Since this tends to 0 as γ increases to ∞, $\{|v_\sigma|:\sigma\in\mathcal{S}\wedge\tau\}$ is bounded above in $L^0(\mathfrak{A})$, and

$$\bar\mu([\![\sup_{\sigma\in\mathcal{S}\wedge\tau}|v_\sigma|>\gamma]\!])=\bar\mu c_\gamma\le\frac{1}{\gamma}\mathbb{E}(|v_\tau|)$$

for every $\gamma>0$ (364L(a-ii)), as required.

622J Proposition Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$ and $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a uniformly integrable martingale.

(a) The $\|\,\|_1$-limit $v=1\lim_{\sigma\uparrow\mathcal{S}}v_\sigma$ is defined in $L^1_{\bar\mu}$, and v is also the limit $\lim_{\sigma\uparrow\mathcal{S}}v_\sigma$ for the topology of convergence in measure.

(b) $\boldsymbol{v}=\boldsymbol{P}v\restriction\mathcal{S}$ is order-bounded, and $\inf_{\tau\in\mathcal{S}}\sup_{\sigma\in\mathcal{S}\vee\tau}|v-v_\sigma|=0$.

proof (a)(i) $\{v_\sigma:\sigma\in\mathcal{S}\}$ is relatively weakly compact in $L^1_{\bar\mu}$ (621Bd). Let $\mathcal{F}$ be an ultrafilter on $\mathcal{S}$ containing $\mathcal{S}\vee\sigma$ for every $\sigma\in\mathcal{S}$. Then the limit $u=\mathrm{w}1\lim_{\sigma\to\mathcal{F}}u_\sigma$ for the weak topology on $L^1_{\bar\mu}$ is defined (2A3R). If $\tau\in\mathcal{S}$, then $P_\tau:L^1_{\bar\mu}\to L^1_{\bar\mu}$ is a norm-continuous linear operator, so is weakly continuous (3A5Ec), and

$$P_\tau u=\mathrm{w}1\lim_{\sigma\to\mathcal{F}}P_\tau v_\sigma=v_\tau$$

because $\mathcal{S}\vee\tau\in\mathcal{F}$ and $P_\tau v_\sigma=v_\tau$ for every $\tau\in\mathcal{S}\vee\sigma$. So $\boldsymbol{v}=\boldsymbol{P}u\restriction\mathcal{S}$.

(ii) Let $\mathfrak{C}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{\sigma\in\mathcal{S}}\mathfrak{A}_\sigma$ and set $v=P_{\mathfrak{C}}u$. By 621C(g-ii),

$$v=1\lim_{\sigma\uparrow\mathcal{S}}P_\sigma u=1\lim_{\sigma\uparrow\mathcal{S}}v_\sigma.$$

By 613B(d-i) again, $v=\lim_{\sigma\uparrow\mathcal{S}}v_\sigma$.

(b) If $\tau\in\mathcal{S}$, then

$$P_\tau v=1\lim_{\sigma\uparrow\mathcal{S}}P_\tau v_\sigma=v_\tau;$$

accordingly $\boldsymbol{v}=\boldsymbol{P}v\restriction\mathcal{S}$. Being uniformly integrable, $\boldsymbol{v}$ is $\|\,\|_1$-bounded, so 622G tells us that it is moderately oscillatory; by 615Ga, $\inf_{\tau\in\mathcal{S}}\sup_{\sigma\in\mathcal{S}\vee\tau}|v-v_\sigma|=0$.

622K Lemma Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$, and $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ an L^1-process such that $\mathbb{E}(u_\sigma)=\mathbb{E}(u_\tau)$ for all σ, $\tau\in\mathcal{S}$. Then $\boldsymbol{u}$ is a martingale.

proof Take $\sigma\le\tau\in\mathcal{S}$ and $a\in\mathfrak{A}_\sigma\subseteq\mathfrak{A}_\tau$. Then there is a $\tau'\in\mathcal{T}$ such that $a\subseteq[\![\tau'=\tau]\!]$ and $1\setminus a\subseteq[\![\tau'=\sigma]\!]$ (611I). Now we have

$$\begin{aligned}\mathbb{E}(u_\sigma\times\chi a)+\mathbb{E}(u_\sigma\times\chi(1\setminus a))&=\mathbb{E}(u_\sigma)=\mathbb{E}(u_{\tau'})\\&=\mathbb{E}(u_{\tau'}\times\chi a)+\mathbb{E}(u_{\tau'}\times\chi(1\setminus a))\\&=\mathbb{E}(u_\tau\times\chi a)+\mathbb{E}(u_\sigma\times\chi(1\setminus a))\end{aligned}$$

so

$$\mathbb{E}(u_\sigma\times\chi a)=\mathbb{E}(u_\tau\times\chi a).$$

As a is arbitrary and $u_\sigma\in L^1_{\bar\mu}\cap L^0(\mathfrak{C}_\sigma)$, u_σ is the conditional expectation of u_τ on $\mathfrak{C}_\sigma$. As σ and τ are arbitrary, $\boldsymbol{u}$ is a martingale.

622L Brownian motion: Theorem Let $\boldsymbol{w}$ be Brownian motion, and $\boldsymbol{\iota}$ the corresponding identity process. Then $\boldsymbol{w}$ and $\boldsymbol{w}^2-\boldsymbol{\iota}$ are local martingales, and $\boldsymbol{w}\restriction\mathcal{T}_b$ is a martingale.

proof As in 612T and 612F, I regard $\boldsymbol{w} = \langle w_\sigma\rangle_{\sigma\in\mathcal{T}_f}$ and $\boldsymbol{\iota} = \langle \iota_\sigma\rangle_{\sigma\in\mathcal{T}_f}$ as processes on the real-time stochastic integration structure $(\mathfrak{C},\bar\nu,[0,\infty[\,,\langle\mathfrak{C}_t\rangle_{t\ge0},\mathcal{T},\langle\mathfrak{C}_\tau\rangle_{\tau\in\mathcal{T}})$, where $(\mathfrak{C},\bar\nu)$ is the measure algebra of (Ω,Σ,ν), $\Omega = C([0,\infty[)_0$, ν is one-dimensional Wiener measure and Σ is its domain.

(a) For $n\in\mathbb{N}$, let $h_n:\Omega\to[0,\infty]$ be the Brownian exit time from $]-n,n[$ (477I), so that h_n is a stopping time; write τ_n for the corresponding stopping time in $\mathcal{T}$ (612H). Because h is finite ν-almost everywhere (478Ma), $\tau_n\in\mathcal{T}_f$. Now $\mathbb{E}(w_\tau) = \mathbb{E}(w_\tau^2-\iota_\tau) = 0$ for every $\tau\in\mathcal{T}\wedge\tau_n$. **P** I use Dynkin's formula (478K). Express τ as $h^\bullet$ where $h\le h_n$ is a stopping time. By 478Jd[3], there is a smooth function $f:\mathbb{R}\to\mathbb{R}$ with compact support such that $f(x)=x$ for $x\in[-n-1,n+1]$. In the language of 612H and 478K,

$$\mathbb{E}(w_\tau) = \mathbb{E}(X_h) = \mathbb{E}(f(X_h))$$

(because $h(\omega)\le h_n(\omega)$, so $|X_h(\omega)| = |\omega(h(\omega))|\le n$ for almost every $\omega\in\Omega$)

$$= f(0)+\frac12\mathbb{E}(\int_0^h(\nabla^2 f)(X_s)ds = 0$$

because $\nabla^2 f(\xi)=0$ for $|\xi|\le n$, while $|X_s(\omega)| = |\omega(s)|\le n$ whenever $s\in h(\omega)$. Similarly,

$$\begin{aligned}\mathbb{E}(w_\tau^2-\iota_\tau) &= \mathbb{E}(X_h^2-h) = \mathbb{E}(f^2(X_h)-h)\\ &= f^2(0)+\frac12\mathbb{E}(\int_0^h(\nabla^2 f^2)(X_s)-2\,ds = 0\end{aligned}$$

because $(\nabla^2 f^2)(\xi) = 2$ for $|\xi|\le n$. **Q**

(b) By 622K, $\boldsymbol{w}\restriction\mathcal{T}\wedge\tau_n$ and $(\boldsymbol{w}^2-\boldsymbol{\iota})\restriction\mathcal{T}\wedge\tau_n$ are martingales for each $n\in\mathbb{N}$. Since $\langle\tau_n\rangle_{n\in\mathbb{N}}$ is non-decreasing, $\boldsymbol{w}\restriction\mathcal{S}$ and $(\boldsymbol{w}^2-\boldsymbol{\iota})\restriction\mathcal{S}$ are martingales, where $\mathcal{S} = \bigcup_{n\in\mathbb{N}}\mathcal{T}\wedge\tau_n$ is an ideal in $\mathcal{T}_f$. If τ is any member of $\mathcal{T}_f$, it can be represented as $h^\bullet$ where h is a finite-valued stopping time; now ω is bounded on $[0,h(\omega)]$ for every $\omega\in\Omega$, so $\Omega = \bigcup_{n\in\mathbb{N}}\{\omega : h(\omega)\le h_n(\omega)\}$ and $\sup_{n\in\mathbb{N}}[\![\tau\le\tau_n]\!] = 1$. Accordingly $\mathcal{S}$ is a covering ideal of $\mathcal{T}_f$ and $\boldsymbol{w}$ and $\boldsymbol{w}^2-\boldsymbol{\iota}$ are local martingales.

(c) To see that $\boldsymbol{w}\restriction\mathcal{T}_b$ is a martingale, take any $t\ge0$. If $\sigma\in\mathcal{S}\wedge\check t$,

$$\mathbb{E}(w_\sigma^2) = \mathbb{E}(\iota_\sigma) = \mathbb{E}(\sigma)\le t,$$

so $\boldsymbol{w}\restriction\mathcal{S}\wedge\check t$ is $\|\;\|_2$-bounded, therefore uniformly integrable, as well as being a martingale. By 622J, it is of the form $\boldsymbol{P}v\restriction\mathcal{S}\wedge\check t$ for some $v\in L^1_{\bar\mu}$. But $\mathcal{S}$ covers $\mathcal{T}_f$, so $\mathcal{S}\wedge\check t$ covers $\mathcal{T}_f\wedge\check t$ (611M(g-ii)) and $\boldsymbol{w}\restriction\mathcal{T}_f\wedge\check t = \boldsymbol{P}v\restriction\mathcal{T}_f\wedge\check t$ is a martingale. As t is arbitrary, $\boldsymbol{w}\restriction\mathcal{T}_b = \boldsymbol{w}\restriction\bigcup_{t\ge0}\mathcal{T}_f\wedge\check t$ is a martingale.

Remark It is also the case that $(\boldsymbol{w}^2-\boldsymbol{\iota})\restriction\mathcal{T}_b$ is a martingale; see 632Xe.

622M You will find various more or less elementary facts about martingales in the exercises. One which will be useful later in this chapter is the following.

Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a fully adapted process. Then $\mathcal{S}' = \{\tau:\tau\in\mathcal{S},\ \boldsymbol{v}\restriction\mathcal{S}\wedge\tau$ is a martingale$\}$ is an ideal in $\mathcal{S}$, and $\boldsymbol{v}\restriction\mathcal{S}'$ is a martingale.

proof (a) If $\tau\in\mathcal{S}$, then $\tau\in\mathcal{S}'$ iff $v_\sigma = P_\sigma v_\tau$ whenever $\sigma\in\mathcal{S}$ and $\sigma\le\tau$. **P**

$$\begin{aligned}\tau\in\mathcal{S}' &\iff v_\sigma = P_\sigma v_{\sigma'} \text{ whenever } \sigma,\,\sigma'\in\mathcal{S} \text{ and } \sigma\le\sigma'\le\tau\\ &\Longrightarrow v_\sigma = P_\sigma v_\tau \text{ whenever } \sigma\in\mathcal{S} \text{ and } \sigma\le\tau\\ &\Longrightarrow v_\sigma = P_\sigma v_\tau = P_\sigma P_{\sigma'}v_\tau = P_\sigma v_{\sigma'} \text{ whenever } \sigma,\,\sigma'\in\mathcal{S} \text{ and } \sigma\le\sigma'\le\tau.\ \mathbf{Q}\end{aligned}$$

(b) If $\tau\in\mathcal{S}$ and $\tau\le\tau'\in\mathcal{S}'$ then $\tau\in\mathcal{S}'$. **P** If $\sigma\in\mathcal{S}$ and $\sigma\le\tau$ then

$$v_\sigma = P_\sigma v_{\tau'} = P_\sigma P_\tau v_{\tau'} = P_\sigma v_\tau.\ \mathbf{Q}$$

[3]Later editions only.

(c) If τ, $\tau' \in \mathcal{S}'$, $\sigma \in \mathcal{S}$ and $\tau \le \sigma \le \tau \vee \tau'$ then $v_\sigma = P_\sigma v_{\tau\vee\tau'}$. **P** Set $a = [\![\sigma \le \tau']\!] \in \mathfrak{A}_\sigma$ (611H(c-i)). We have

$$a = [\![\tau \le \sigma]\!] \cap [\![\sigma \le \tau']\!] \subseteq [\![\tau \le \tau']\!] = [\![\tau' = \tau' \vee \tau]\!] \subseteq [\![v_{\tau'} = v_{\tau'\vee\tau}]\!]$$

(using 611E(c-iv-α) and (a-ii-β)) and

$$a = [\![\sigma = \sigma \wedge \tau']\!] \subseteq [\![P_\sigma v_{\tau'} = P_{\sigma\wedge\tau'} v_{\tau'}]\!] \cap [\![v_{\sigma\wedge\tau'} = v_\sigma]\!],$$

(611E(a-ii-β) again and 622Bb) so

$$\begin{aligned} P_\sigma(v_{\tau\vee\tau'} \times \chi a) &= P_\sigma(v_{\tau'} \times \chi a) = P_\sigma v_{\tau'} \times \chi a = P_{\sigma\wedge\tau'} v_{\tau'} \times \chi a \\ &= v_{\sigma\wedge\tau'} \times \chi a = v_\sigma \times \chi a. \end{aligned}$$

Next, setting

$$\begin{aligned} a' &= [\![\tau' < \sigma]\!] \subseteq [\![\tau' < \tau \vee \tau']\!] = [\![\tau' < \tau]\!] \cup [\![\tau' < \tau']\!] \\ &= [\![\tau' < \tau]\!] \subseteq [\![\tau' \le \tau]\!] = [\![\tau \vee \tau' = \tau]\!] \subseteq [\![\sigma \le \tau]\!] = [\![\sigma \vee \tau = \tau]\!] = [\![\sigma = \tau]\!] \\ &\subseteq [\![v_{\tau\vee\tau'} = v_\tau]\!] \cap [\![P_\sigma v_\tau = P_\tau v_\tau]\!] \cap [\![v_\tau = v_\sigma]\!], \end{aligned}$$

(611Eb)

we see that

$$P_\sigma(v_{\tau\vee\tau'} \times \chi a') = P_\sigma(v_\tau \times \chi a') = P_\sigma(v_\sigma \times \chi a') = v_\sigma \times \chi a'.$$

Adding, we have $P_\sigma v_{\tau\vee\tau'} = v_\sigma$, as claimed. **Q**

(d) If τ, $\tau' \in \mathcal{S}'$, $\sigma \in \mathcal{S}$ and $\sigma \le \tau \vee \tau'$ then $v_\sigma = P_\sigma v_{\sigma\vee\tau}$. **P** Set

$$\begin{aligned} a &= [\![\sigma \le \tau]\!] = [\![\sigma \vee \tau = \tau]\!] = [\![\sigma \wedge \tau = \sigma]\!] \\ &\subseteq [\![v_{\sigma\vee\tau} = v_\sigma]\!] \cap [\![P_{\sigma\wedge\tau} v_\tau = P_\sigma v_\tau]\!] \cap [\![v_{\sigma\wedge\tau} = v_\sigma]\!]. \end{aligned}$$

Then

$$\begin{aligned} P_\sigma(v_{\sigma\vee\tau} \times \chi a) &= P_\sigma(v_\tau \times \chi a) = P_\sigma v_\tau \times \chi a = P_\sigma P_\tau v_\tau \times \chi a \\ &= P_{\sigma\wedge\tau} v_\tau \times \chi a = v_{\sigma\wedge\tau} \times \chi a = v_\sigma \times \chi a. \end{aligned}$$

And setting

$$a' = [\![\tau < \sigma]\!] \subseteq [\![\sigma \vee \tau = \sigma]\!] \subseteq [\![v_{\sigma\vee\tau} = v_\sigma]\!],$$

we have

$$P_\sigma(v_{\sigma\vee\tau} \times \chi a') = P_\sigma(v_\sigma \times \chi a') = v_\sigma \times \chi a',$$

so $P_\sigma v_{\sigma\vee\tau} = v_\sigma$. **Q**

Now we have $\tau \le \sigma \vee \tau \le \tau \vee \tau'$, so (c) tells us that

$$v_\sigma = P_\sigma v_{\sigma\vee\tau} = P_\sigma P_{\sigma\vee\tau} v_{\tau\vee\tau'} = P_\sigma v_{\tau\vee\tau'}.$$

As σ is arbitrary, $\tau \vee \tau' \in \mathcal{S}'$. As τ and τ' are arbitrary, (a) tells us that $\mathcal{S}'$ is an ideal of $\mathcal{S}$.

(e) Finally, we have $v_\sigma = P_\sigma v_\tau$ whenever σ, $\tau \in \mathcal{S}'$ and $\sigma \le \tau$, so $\boldsymbol{v}\restriction\mathcal{S}'$ is a martingale.

622N Extensions to covered envelopes: Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ with covered envelope $\hat{\mathcal{S}}$, and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a fully adapted process with fully adapted extension $\hat{\boldsymbol{u}} = \langle \hat{u}_\sigma\rangle_{\sigma\in\hat{\mathcal{S}}}$.

(a) If $\hat{\boldsymbol{u}}$ is a martingale then $\boldsymbol{u}$ is a martingale.

(b) If $\boldsymbol{u}$ is a local martingale then $\hat{\boldsymbol{u}}$ is a local martingale.

(c) $\hat{\boldsymbol{u}}$ is a uniformly integrable martingale iff $\boldsymbol{u}$ is a uniformly integrable martingale.

proof (a) This is a special case of 622Db.

(b) Take $\tau \in \hat{\mathcal{S}}$ and $\epsilon > 0$. We know that $\sup_{\sigma\in\mathcal{S}} [\![\tau = \sigma]\!] = 1$, so there is a non-empty finite subset I of $\mathcal{S}$ such that $\bar{\mu}a \ge 1 - \epsilon$, where $a = \sup_{\sigma\in I} [\![\tau = \sigma]\!]$. Set $\tilde{\tau} = \sup I \in \mathcal{S}$. Then

$$[\![\tau \le \tilde{\tau}]\!] \supseteq \sup_{\sigma\in I} [\![\tau = \sigma]\!] \cap [\![\sigma \le \tilde{\tau}]\!] = a$$

so $\bar{\mu}[\![\tilde{\tau} < \tau]\!] \le \epsilon$. Because $\boldsymbol{u}$ is a local martingale, there is a $\tau' \in \mathcal{S}$ such that $\boldsymbol{u}\restriction\mathcal{S} \wedge \tau'$ is a martingale and $\bar{\mu}[\![\tau' < \tilde{\tau}]\!] \le \epsilon$. Consider the martingale $\boldsymbol{P}u_{\tau'}$ (622F). This is defined everywhere on $\mathcal{T}$ and agrees with $\boldsymbol{u}$ on $\mathcal{S}\wedge\tau'$. Now $\hat{\mathcal{S}}\wedge\tau'$ is the covered envelope of $\mathcal{S}\wedge\tau$ (611M(e-i)), while $\boldsymbol{P}u_{\tau'}\restriction\hat{\mathcal{S}}\wedge\tau'$ is fully adapted and extends $\boldsymbol{u}\restriction\mathcal{S} \wedge \tau'$, so must be equal to $\hat{\boldsymbol{u}}\restriction\hat{\mathcal{S}} \wedge \tau'$. Thus $\hat{\boldsymbol{u}}\restriction\hat{\mathcal{S}} \wedge \tau'$ is a martingale, while $[\![\tau' < \tau]\!] \subseteq [\![\tau' < \tilde{\tau}]\!] \cup [\![\tilde{\tau} < \tau]\!]$ has measure at most 2ϵ. As τ and ϵ are arbitrary, $\hat{\boldsymbol{u}}$ is a local martingale.

(c) If $\hat{\boldsymbol{u}}$ is a uniformly integrable martingale then $\boldsymbol{u} = \hat{\boldsymbol{u}}\restriction\mathcal{S}$ must be a uniformly integrable martingale. If $\boldsymbol{u}$ is a uniformly integrable martingale, it is of the form $\boldsymbol{P}u\restriction\mathcal{S}$ for some $u \in L^1_{\bar{\mu}}$ (622J), and now $\boldsymbol{P}u\restriction\hat{\mathcal{S}}$ is a uniformly integrable martingale. But $\boldsymbol{P}u\restriction\hat{\mathcal{S}}$ is a fully adapted process extending $\boldsymbol{u}$, so must be $\hat{\boldsymbol{u}}$, and $\hat{\boldsymbol{u}}$ is a uniformly integrable martingale.

622O Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a martingale.
(a) If $\mathcal{S}_1$ is the ideal of $\mathcal{T}$ generated by $\mathcal{S}$, there is a unique martingale $\boldsymbol{v} = \langle v_\tau\rangle_{\tau\in\mathcal{S}_1}$ extending $\boldsymbol{u}$.
(b) If $\mathcal{S}_2$ is the full ideal of $\mathcal{T}$ generated by $\mathcal{S}$, there is a local martingale $\hat{\boldsymbol{v}} = \langle \hat{v}_\tau\rangle_{\tau\in\mathcal{S}_2}$ extending $\boldsymbol{u}$.

proof (a) Since $\mathcal{S}$ is upwards-directed, $\bigcup_{\sigma\in\mathcal{S}} \mathcal{T}\wedge\sigma$ is an ideal of $\mathcal{T}$ and must be $\mathcal{S}_1$. If $\tau \in \mathcal{S}_1$ and σ, $\sigma' \in \mathcal{S}$ are such that $\tau \le \sigma$ and $\tau \le \sigma'$, then

$$P_\tau u_\sigma = P_\tau P_\sigma u_{\sigma\vee\sigma'} = P_\tau u_{\sigma\vee\sigma'} = P_\tau u_{\sigma'},$$

so we can define $v_\tau \in L^0(\mathfrak{A})$ by saying that $v_\tau = P_\tau v_\sigma$ whenever $\tau \le \sigma \in \mathcal{S}$. If $\tau \le \tau' \in \mathcal{S}_1$, there is a $\sigma \in \mathcal{S}$ such that $\tau' \le \sigma$ and

$$P_\tau v_{\tau'} = P_\tau P_{\tau'} u_\sigma = P_\tau v_\sigma = v_\tau.$$

In particular, $P_\tau v_\tau = v_\tau$ and $v_\tau \in L^0(\mathfrak{A}_\tau)$. Also

$$[\![\tau = \tau']\!] \subseteq [\![P_\tau v_{\tau'} = P_{\tau'} v_{\tau'}]\!] = [\![v_\tau = v_{\tau'}]\!]$$

by 622Bb. So $\boldsymbol{v}$ is fully adapted (612Db) and is a martingale. If $\sigma \in \mathcal{S}$ then $v_\sigma = P_\sigma u_\sigma = u_\sigma$, so $\boldsymbol{v}$ extends $\boldsymbol{u}$.

As for uniqueness, if $\boldsymbol{v}' = \langle v'_\tau\rangle_{\tau\in\mathcal{S}_1}$ is another martingale with domain $\mathcal{S}_1$ extending $\boldsymbol{u}$, then

$$v'_\tau = P_\tau v'_\sigma = P_\tau u_\sigma = v_\tau$$

whenever $\tau \le \sigma \in \mathcal{S}$, so $\boldsymbol{v}' = \boldsymbol{v}$.

(b) The covered envelope $\hat{\mathcal{S}}_1$ is an ideal of $\mathcal{T}$. **P** We know that it is a full sublattice (611M(b-i), 611M(c-ii)). If $\tau \in \mathcal{T}$ and $\tau \le \tau' \in \hat{\mathcal{S}}_1$, then

$$\sup_{\sigma\in\mathcal{S}_1} [\![\tau = \sigma]\!] \supseteq \sup_{\sigma\in\mathcal{S}_1} [\![\tau = \sigma \wedge \tau]\!]$$

(because $\sigma \wedge \tau \in \mathcal{S}_1$ for every $\sigma \in \mathcal{S}_1$)

$$= \sup_{\sigma\in\mathcal{S}_1} [\![\tau \le \sigma]\!] \supseteq \sup_{\sigma\in\mathcal{S}_1} [\![\tau \le \sigma]\!] \cap [\![\sigma = \tau']\!]$$

$$= \sup_{\sigma\in\mathcal{S}_1} [\![\tau \le \tau']\!] \cap [\![\sigma = \tau']\!] = \sup_{\sigma\in\mathcal{S}_1} [\![\sigma = \tau']\!] = 1$$

and $\tau \in \hat{\mathcal{S}}_1$. **Q**

So $\hat{\mathcal{S}}_1$ is a full ideal of $\mathcal{T}$, and of course it includes $\mathcal{S}$; while any full ideal including $\mathcal{S}$ must include $\mathcal{S}_1$ and $\hat{\mathcal{S}}_1$. Thus $\mathcal{S}_2 = \hat{\mathcal{S}}_1$. Now the fully adapted extension $\hat{\boldsymbol{v}}$ of $\boldsymbol{v}$ has domain $\mathcal{S}_2$, extends $\boldsymbol{u}$ and is a local martingale (622Nb).

622P Proposition Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$, $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a moderately oscillatory process and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a martingale. Then $\| \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}\|_2 \le \|\boldsymbol{u}\|_\infty \sup_{\tau\in\mathcal{S}} \|v_\sigma\|_2$.

proof The formulae here depend on my conventions for the use of ∞, so perhaps I should restate these. $0\cdot\infty=0$, so if either $\boldsymbol{u}$ or $\boldsymbol{v}$ is $\mathbf{0}$, the result is trivial. Otherwise, if $\boldsymbol{u}$ is not $\| \|_\infty$-bounded, then $\|\boldsymbol{u}\|_\infty=\infty$; if $\boldsymbol{v}$ is not $\| \|_2$-bounded, then $\sup_{\tau\in\mathcal{S}}\|v_\sigma\|_2=\infty$; both cases are trivial. So we can suppose that $M=\|\boldsymbol{u}\|_\infty$ and $M'=\sup_{\sigma\in\mathcal{S}}\|v_\sigma\|_2$ are both finite and non-zero.

As $\boldsymbol{v}$ is $\| \|_2$-bounded it is $\| \|_1$-bounded, therefore an integrator (622G), and $\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}$ is defined. The point is that if $I\in\mathcal{I}(\mathcal{S})$ then $\|S_I(\boldsymbol{u},d\boldsymbol{v})\|_2\le MM'$. **P** If $I=\emptyset$ or $M=0$ this is trivial. Otherwise, take $(\tau_0,\dots,\tau_n)$ linearly generating the I-cells. Then

$$\|S_I(\boldsymbol{u},d\boldsymbol{v})\|_2=M\|\sum_{i=0}^{n-1}\frac{1}{M}u_{\tau_i}\times(v_{\tau_{i+1}}-v_{\tau_i})\|_2\le M\|v_{\tau_n}\|_2$$

(by 621F, applied to the martingale $(v_{\tau_0},\dots,v_{\tau_n})$)

$$\le MM'. \text{ **Q**}$$

Since $\| \|_2$-balls are closed in $L^0(\mathfrak{A})$ (613Bc),

$$\|\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}\|_2=\|\lim_{I\uparrow\mathcal{I}(\mathcal{S})}S_I(\boldsymbol{u},d\boldsymbol{v})\|_2\le MM',$$

as claimed.

622Q Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$, $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ fully adapted processes such that $\boldsymbol{u}$ is locally moderately oscillatory and $\boldsymbol{v}$ is a martingale. Then $\|\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\,d\boldsymbol{v}\|_2\le\|\boldsymbol{u}{\restriction}\mathcal{S}\wedge\tau\|_\infty\|v_\tau\|_2$ for every $\tau\in\mathcal{S}$, and if the right-hand side is always finite, the indefinite integral $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is a martingale.

proof (a) We know from 622H that $\boldsymbol{v}$ is a local integrator so the process $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is defined everywhere on $\mathcal{S}$. For $\tau\in\mathcal{S}$ write z_τ for $\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\,d\boldsymbol{v}$. If $\tau\in\mathcal{S}$ and $\sigma\in\mathcal{S}\wedge\tau$, then $\|v_\sigma\|_2\le\|v_\tau\|_2$ by 366J, so 622P tells us that $\|z_\tau\|_2\le\|\boldsymbol{u}{\restriction}\mathcal{S}\wedge\tau\|_\infty\|v_\tau\|_2$. At the same time, we see from the argument in 622P that $\{S_I(\boldsymbol{u},d\boldsymbol{v}):I\in\mathcal{I}(\mathcal{S}\wedge\tau)\}$ is $\| \|_2$-bounded, therefore uniformly integrable (621Be).

(b) Now suppose that $\|\boldsymbol{u}{\restriction}\mathcal{S}\wedge\tau\|_\infty\|v_\tau\|_2$ is finite for every $\tau\in\mathcal{S}$. Take $\tau\le\tau'$ in $\mathcal{S}$. Because $\{S_I(\boldsymbol{u},d\boldsymbol{v}):I\in\mathcal{I}(\mathcal{S}\wedge\tau')\}\cup\{z_{\tau'}-z_\tau\}$ is uniformly integrable, and $z_{\tau'}-z_\tau=\int_{\mathcal{S}\cap[\tau,\tau']}\boldsymbol{u}\,d\boldsymbol{v}$ is the limit $\lim_{I\uparrow\mathcal{I}(\mathcal{S}\cap[\tau,\tau'])}S_I(\boldsymbol{u},d\boldsymbol{v})$ for the topology of convergence in measure, it is also the limit for $\| \|_1$ (621B(c-ii)). As P_τ is $\| \|_1$-continuous, this means that

$$P_\tau z_{\tau'}-z_\tau=P_\tau(z_{\tau'}-z_\tau)=\lim_{I\uparrow\mathcal{I}([\tau,\tau'])}P_\tau S_I(\boldsymbol{u},d\boldsymbol{v})=0$$

because if $\tau\le\sigma\le\sigma'\le\tau'$ then

$$\begin{aligned}P_\tau(u_\sigma\times(v_{\sigma'}-v_\sigma))&=P_\tau P_\sigma(u_\sigma\times(v_{\sigma'}-v_\sigma))=P_\tau(u_\sigma\times P_\sigma(v_{\sigma'}-v_\sigma))\\&=P_\tau(u_\sigma\times(P_\sigma v_{\sigma'}-v_\sigma))=0.\end{aligned}$$

So $P_\tau z_{\tau'}=z_\tau$; as τ and τ' are arbitrary, $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is a martingale.

622R Law-independence I remarked in 613I that the Riemann-sum integral, like the topology of convergence in measure on $L^0(\mathfrak{A})$, does not depend on the measure $\bar\mu$ assigned to $\mathfrak{A}$; if $\bar\nu$ is any strictly positive totaly finite countably additive functional on $\mathfrak{A}$, the stochastic integration structure $(\mathfrak{A},\bar\nu,T,\langle\mathfrak{A}_t\rangle_{t\in T})$ will behave exactly like the original structure $(\mathfrak{A},\bar\mu,T,\langle\mathfrak{A}_t\rangle_{t\in T})$. This is the case for most of the rest of Chapter 61. In a formal sense there are a great many definitions to check. Already in 615B I defined the ucp topology on a space $M_{\text{o-b}}(\mathcal{S})$ in terms of an F-norm $\widehat{\theta}$ defined in terms of the standard F-norm θ on $L^0(\mathfrak{A})$, and θ does depend on the measure. But if we take this into account, and speak of $\theta_{\bar\mu}$ and $\theta_{\bar\nu}$ giving rise to $\widehat{\theta}_{\bar\mu}$ and $\widehat{\theta}_{\bar\nu}$, then we know that

for every $\epsilon>0$ there is a $\delta>0$ such that $\bar\nu(a)\le\epsilon$ whenever $\bar\mu(b)\le\delta$

and consequently

for every $\epsilon>0$ there is a $\delta>0$ such that $\theta_{\bar\nu}(u)\le\epsilon$ whenever $\theta_{\bar\mu}(u)\le\delta$, for every $\epsilon > 0$ there is a $\delta > 0$ such that $\widehat{\theta}_{\bar\nu}(\boldsymbol{u}) \le \epsilon$ whenever $\widehat{\theta}_{\bar\mu}(\boldsymbol{u}) \le \delta$.

Since this works equally well with $\bar\mu$ and $\bar\nu$ exchanged, the $\bar\mu$-ucp topology and uniformity on $M_{\text{o-b}}(\mathcal{S})$ are the same as the $\bar\nu$-ucp topology and uniformity.

Continuing through 'moderately oscillatory' processes (615E), 'integrating interval functions' and 'integrators' (616F), these are defined in terms of the topology of $L^0(\mathfrak{A})$, so are the same in both structures. 'Bounded variation' (614K), 'cumulative variation' (614O) and '(residual) oscillation' (618B) can, with a little care, be defined in ways which do not call on the measure at all, but only on the Riesz space structure of $L^0(\mathfrak{A})$, so the same is true of 'jump-free process'. 'Covariations' and 'quadratic variations' (617H) are based on the Riemann-sum integral which doesn't change. So the way to Itô's formula (§619) is clear throughout.

Of course the examples, Brownian motion (612T, 622L) and Poisson processes (612U), are based on explicitly defined measures, and make no sense without them. But it is only in the present section that we have come to a general class of processes for which we really need to know which measure we are using. We do not expect anything to do with $L^p_{\bar\mu}$ or $\| \; \|_p$, for $p < \infty$, to be stable in the way that $L^0(\mathfrak{A})$, $L^\infty(\mathfrak{A})$ and $\| \; \|_\infty$ are. In particular, conditional expectations (621C) and martingales (622C) are dependent on the exact measure we have in hand. So in the present chapter our expectations are reversed. I will return to an occasion in which, surprisingly, we *do* have law-independence, in 625F.

622X Basic exercises >(a) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a martingale. Let $\tau \in \mathcal{T}$ and set $\mathcal{S}' = \{\sigma : \sigma \in \mathcal{T},\ \sigma \wedge \tau \in \mathcal{S}\}$. Show that $\sigma \mapsto v_{\sigma\wedge\tau} : \mathcal{S}' \to L^0(\mathfrak{A})$ is a martingale.

(b) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u}$ a process with domain $\mathcal{S}$, and z an element of $L^0(\bigcap_{\sigma\in\mathcal{S}}\mathfrak{A}_\sigma)$. Show that if $\boldsymbol{u}$ is a martingale and $z\boldsymbol{u}$ is an L^1-process then $z\boldsymbol{u}$ is a martingale.

(c) Let $\boldsymbol{v}$ be a local martingale defined on a sublattice $\mathcal{S}$ of $\mathcal{T}$. Show that $\boldsymbol{v}\restriction\mathcal{S}\vee\tau$ is a local martingale for every $\tau \in \mathcal{S}$.

(d) Suppose that $T = [0,\infty[$ and $\mathfrak{A} = \{0,1\}$, as in 613W, 616Xa and 615Xf. Let $f : [0,\infty[\to \mathbb{R}$ be a function. (i) Show that f corresponds to a martingale with domain $\mathcal{T}_f$ iff it corresponds to a local martingale iff it is constant. (ii) Show that f corresponds to a uniformly integrable process iff it is bounded.

>(e) Let $(\mathfrak{A}, \bar\mu)$ be the measure algebra of Lebesgue measure on $[0,1]$. (i) Set $T = [0,1]$ and for $t \in T$ set $a_t = [t,1]^\bullet \in \mathfrak{A}$, $\mathfrak{A}_t = \{a : a \cap a_t$ is either 0 or $a_t\}$; show that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is a filtration in the sense of 611B. (ii) Let $\mathcal{T}$ be the associated family of stopping times. Show that for any $\tau \in \mathcal{T}$ there is a least $s_\tau \in [0,1]$ such that $[\![\tau > s_\tau]\!] \cap a_{s_\tau} = 0$, and that $\tau \mapsto s_\tau : \mathcal{T} \to [0,1]$ is a lattice homomorphism. (iii) Show that there is a $\tau^* \in \mathcal{T}$ defined by saying that $[\![\tau^* > t]\!] = a_t$ for every $t \in [0,1]$, and that $s_{\tau^*} = 1$. (iv) Set $\mathcal{S} = \{\tau : \tau \le \tau^*\}$, $\mathcal{S}' = \{\tau : \tau \le \tau^*,\ s_\tau < 1\}$; show that $\mathcal{S}' = \mathcal{S}\setminus\{\tau^*\}$ is a covering ideal of $\mathcal{S}$. (v) Set $u_\tau = \frac{1}{1-s_\tau}\chi a_{s_\tau}$ for $\tau \in \mathcal{S}'$, $u_{\tau^*} = 0$; show that $\boldsymbol{u} = \langle u_\tau\rangle_{\tau\in\mathcal{S}}$ is jump-free, that $\boldsymbol{u}\restriction\mathcal{S}'$ is a martingale, and that $\boldsymbol{u}$ is a local martingale which is not a martingale. *(vi) Show that $\langle\mathfrak{A}_t\rangle_{t\in T}$ is right-continuous in the sense of 632B below.

(f) Let $\phi : [0,1] \to [0,\infty[$ be such that $\lim_{\delta\downarrow 0}\phi(\delta) = 0$. Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $M_{\text{mart},\phi}$ the set of martingales $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ such that $\mathbb{E}(|u_\sigma| \times \chi a) \le \phi(\bar\mu a)$ for every $\sigma \in \mathcal{S}$ and $a \in \mathfrak{A}$. Show that $M_{\text{mart},\phi}$ is a subset of the space $M_{\text{o-b}}(\mathcal{S})$ of order-bounded processes and is closed in the ucp topology.

>(g) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a family in $L^1_{\bar\mu}$ such that $P_\sigma v_\tau = v_\sigma$ whenever $\sigma, \tau \in \mathcal{S}$ and $\sigma \le \tau$. Show that $\boldsymbol{v}$ is fully adapted.

(h) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v}$ a local martingale with domain $\mathcal{S}$. Set $\mathcal{S}' = \{\tau : \tau \in \mathcal{T}$, $\inf_{\sigma\in\mathcal{S}}[\![\sigma < \tau]\!] = 0$. Show (i) that $\mathcal{S}'$ is the full ideal of $\mathcal{T}$ generated by $\mathcal{S}$ (ii) that there is a unique local martingale on $\mathcal{S}'$ extending $\boldsymbol{u}$.

(i) Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$, and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$, $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ fully adapted processes such that $\boldsymbol{v}$ is a martingale and $z = \int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}$ is defined. Show that $\|z\|_2 \le \|\boldsymbol{u}\|_\infty \sup_{\sigma\in\mathcal{S}}\|v_\sigma\|_2$.

>(j) Let (Ω, Σ, μ) be the interval $]0,1]$ with Lebesgue measure, and $T = [1,\infty[$. For $t \ge 1$, set $\Sigma_t = \{E : E \in \Sigma$, either $[0,\frac{1}{t}] \cap E$ or $[0,\frac{1}{t}] \setminus E$ is negligible$\}$. Set $X_t(\omega) = \frac{1}{\sqrt{\omega}}$ if $\omega t \ge 1$, $2\sqrt{t}$ if $\omega t < 1$, and

$X_\infty(\omega) = \frac{1}{\sqrt{\omega}}$ for every $\omega \in \Omega$. Set $Z_t(\omega) = -2$ if $\omega t \ge 1$, $2t-2$ if $\omega t < 1$, and $Z_\infty(\omega) = -2$ for every $\omega \in \Omega$. Let $(\mathfrak{A}, \bar\mu)$ be the measure algebra of μ, and $\langle \mathfrak{A}_t \rangle_{t \ge 1}$ the filtration corresponding to $\langle \Sigma_t \rangle_{t \ge 1}$. In $L^0(\mathfrak{A})$, set $v = X_\infty^\bullet$ and let $\boldsymbol{v}$ be the uniformly integrable martingale $\boldsymbol{P}v$; let $\boldsymbol{z} = \langle z_\tau \rangle_{\tau \in \mathcal{T}}$ be the indefinite integral $ii_{\boldsymbol{v}}(\boldsymbol{v})$. (i) Set $h(\omega) = \frac{1}{\omega}$ for $\omega \in]0,1]$; show that h is a stopping time adapted to $\langle \Sigma_t \rangle_{t \ge 1}$. (ii) Let $\tau \in \mathcal{T}$ be the stopping time associated with h. Show that $\mathcal{T} \wedge \tau = \{\check t \wedge \tau : t \ge 1\} \cup \{\tau\}$. (iii) Show that $P_{\check t} v = X_t^\bullet$ for every $t \ge 1$. (iv) Show that $z_{\check t} = Z_t^\bullet$ for every $t \ge 1$. (v) Show that $z_{\max \mathcal{T}} = Z_\infty^\bullet$. (vi) Show that $\boldsymbol{z}$ is a jump-free local martingale and an L^1-process but not a martingale. *(vii) Show that $\langle \mathfrak{A}_t \rangle_{t \ge 1}$ is right-continuous.

(k) Give an example of a stochastic integration structure $(\mathfrak{A}, \bar\mu, T, \langle \mathfrak{A}_t \rangle_{t \in T}, \mathcal{T}, \langle \mathfrak{A}_\tau \rangle_{\tau \in \mathcal{T}})$, a sublattice $\mathcal{S}$ of $\mathcal{T}$ and a uniformly integrable process with domain $\mathcal{S}$ such that its fully adapted extension to the covered envelope of $\mathcal{S}$ is not an L^1-process.

622Y Further exercises (a) Let $\boldsymbol{w} = \langle w_\sigma \rangle_{\sigma \in \mathcal{T}_f}$ be Brownian motion. Show that there is a sublattice $\mathcal{S}$ of $\mathcal{T}_f$, with covered envelope $\hat{\mathcal{S}}$, such that $\boldsymbol{w} \restriction \hat{\mathcal{S}}$ is a local martingale, but $\boldsymbol{w} \restriction \mathcal{S}$ is not.

622 Notes and comments Compared with the martingales of §275 and §621, the point of 'fully adapted' martingales is that they are defined on a lattice which is not totally ordered in any case in which the theory here is appropriate. So we have some new questions to ask. In particular, it is not quite obvious that $P_\sigma P_\tau$ will be $P_{\sigma \wedge \tau}$ (458M, 622Ba), or that every element of $L^1_{\bar\mu}$ will generate a fully adapted martingale (622Fa). 622M is another result which is easy in the totally ordered case, but demands finesse in the general context.

If you look at 612H and 632L below, you will see that some measure-theoretic considerations enter the argument, in particular the notion of progressive measurability, which have no direct parallel in the theory of stochastic processes in L^0. When eventually we come to applications of the theory here, they will generally be based on such processes as those examined in §455, where the measure theory is essential. But for the moment we can leave this to look after itself.

Innumerable variations on the concept of 'martingale' have been investigated. Here I have looked only at 'local' martingales; 'approximately local' and 'virtually local' martingales will come in the next section. Submartingales will reappear in §626, and supermartingales and quasimartingales in §627.

By far the most important martingale in mathematics is Brownian motion, which here appears in 622L. In the proof I appeal to Dynkin's formula, a fundamental result in the theory of harmonic functions in §478. I did warn you, in the introduction to this volume, that applications might depend on 'further non-trivial ideas'. But I ought to confess at once that there is an alternative proof, not dependent on anything in §478, using ideas in §632 below.

Once we have observed that martingales are local integrators (622H), it is clear that we should try to understand indefinite integrals with respect to martingales, which is indeed where stochastic integration began, with Itô's formula for integrals with respect to Brownian motion. Most of the rest of this chapter will be about integration with respect to martingales. Here I begin with a couple of baby steps, 622P-622Q, about L^2-martingales. Since jump-free processes will often (subject to an appropriate interpretation of the word 'locally') be locally $\| \|_\infty$-bounded, 622Q will take us a long way with jump-free martingales.

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623 Approximately and virtually local martingales

I have presented a number of contexts in which an indefinite integral $ii_{\boldsymbol{v}}(\boldsymbol{u})$ can be expected to share properties with the integrator $\boldsymbol{v}$ (614D, 614T, 616J, 618Q). In contrast with this pattern, we can have a martingale with a corresponding indefinite integral which is not a martingale (622Xj), and this occurs in some of the central examples of the theory (631Ya). However the indefinite integral is often 'almost' a martingale in some sense. In this section I give what I think is the most important result in this direction for the Riemann-sum indefinite integral (623O). In the generality here, we need to go a good deal deeper than in §622, with what I call 'virtually local' martingales (623J). These depend, in turn, on a special class of operators on spaces of locally moderately oscillatory processes (623B).

623A Notation $(\mathfrak{A},\bar{\mu},T,\langle\mathfrak{A}_t\rangle_{t\in T},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ will be a stochastic integration structure. If $\mathcal{S}$ is a sublattice of $\mathcal{T}$, $M_{\text{fa}}(\mathcal{S})$, $M_{\text{o-b}}(\mathcal{S})$, $M_{\text{mo}}(\mathcal{S})$ and $M_{\text{lmo}}(\mathcal{S})$ will be the spaces of fully adapted, order-bounded, moderately oscillatory and locally moderately oscillatory processes with domain $\mathcal{S}$, $M_{\text{bv}}(\mathcal{S})\subseteq M_{\text{fa}}(\mathcal{S})$ will be the space of processes with bounded variation, and $\mathcal{I}(\mathcal{S})$ will be the set of finite sublattices of $\mathcal{S}$. If $h:\mathbb{R}\to\mathbb{R}$ is a Borel function, $\bar{h}$ will denote either of the corresponding operators on $L^0(\mathfrak{A})$ or $M_{\text{lmo}}(\mathcal{S})$ (612Ac, 612Ia, 615Fb). L^1 will be $L^1(\mathfrak{A},\bar{\mu})$. For $\tau\in\mathcal{T}$, $P_\tau:L^1\to L^1\cap L^0(\mathfrak{A}_\tau)$ will be the conditional expectation associated with the closed subalgebra $\mathfrak{A}_\tau$, and if $z\in L^1$ $\boldsymbol{P}z$ will be the martingale $\langle P_\tau z\rangle_{\tau\in\mathcal{T}}$.

623B The operators R_A: Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $A\subseteq\mathcal{S}$ a non-empty downwards-directed set.

(a) We have an f-algebra homomorphism $R_A:M_{\text{lmo}}(\mathcal{S})\to M_{\text{lmo}}(\mathcal{S})$ defined by setting

$$R_A(\langle u_\sigma\rangle_{\sigma\in\mathcal{S}})=\langle\lim_{\rho\downarrow A}u_{\sigma\wedge\rho}\rangle_{\sigma\in\mathcal{S}}$$

whenever $\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}\in M_{\text{lmo}}(\mathcal{S})$, and if $\boldsymbol{u}\in M_{\text{mo}}(\mathcal{S})$ then $R_A(\boldsymbol{u})\in M_{\text{mo}}(\mathcal{S})$.

(b) $\bar{h}R_A=R_A\bar{h}:M_{\text{lmo}}(\mathcal{S})\to M_{\text{lmo}}(\mathcal{S})$ for every continuous function $h:\mathbb{R}\to\mathbb{R}$.

(c) Take $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}\in M_{\text{lmo}}(\mathcal{S})$ and express $\boldsymbol{u}'=R_A(\boldsymbol{u})$ as $\langle u'_\sigma\rangle_{\sigma\in\mathcal{S}}$.

(i) The starting values $\lim_{\sigma\downarrow\mathcal{S}}u'_\sigma$ and $\lim_{\sigma\downarrow\mathcal{S}}u_\sigma$ are defined and equal.

(ii) If $\boldsymbol{u}$ is $\|\,\|_1$-bounded then $\boldsymbol{u}'$ is $\|\,\|_1$-bounded and $\sup_{\sigma\in\mathcal{S}}\|u'_\sigma\|_1\le\sup_{\sigma\in\mathcal{S}}\|u_\sigma\|_1$.

(d) Write $\hat{\mathcal{S}}$ for the covered envelope of $\mathcal{S}$. If $\boldsymbol{u}\in M_{\text{lmo}}(\mathcal{S})$ has fully adapted extension $\hat{\boldsymbol{u}}$ to $\hat{\mathcal{S}}$, then $R_A(\hat{\boldsymbol{u}})$ is the fully adapted extension of $R_A(\boldsymbol{u})$ to $\hat{\mathcal{S}}$.

proof (a)(i) If $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is locally moderately oscillatory, $\lim_{\rho\downarrow A}u_{\sigma\wedge\rho}$ is defined (615Gb) and belongs to $L^0(\mathfrak{A}_\sigma)$ for every $\sigma\in\mathcal{S}$ (613Bj). If σ, $\tau\in\mathcal{S}$ then

$$[\![\sigma=\tau]\!]\subseteq\inf_{\rho\in A}[\![u_{\sigma\wedge\rho}=u_{\tau\wedge\rho}]\!]\subseteq[\![\lim_{\rho\downarrow A}u_{\sigma\wedge\rho}=\lim_{\rho\downarrow A}u_{\tau\wedge\rho}]\!],$$

so $R_A(\boldsymbol{u})\in M_{\text{fa}}(\mathcal{S})$.

(ii) Because addition, multiplication and modulus are continuous functions on $L^0(\mathfrak{A})$, $R_A:M_{\text{lmo}}(\mathcal{S})\to M_{\text{fa}}(\mathcal{S})$ is an f-algebra homomorphism.

(iii) If $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is moderately oscillatory, then $|u_{\sigma\wedge\rho}|\le\sup|\boldsymbol{u}|$ whenever $\sigma\in\mathcal{S}$ and $\rho\in A$, so $|\lim_{\rho\downarrow A}u_{\sigma\wedge\rho}|\le\sup|\boldsymbol{u}|$ whenever $\sigma\in\mathcal{S}$, and $R_A(\boldsymbol{u})$ is order-bounded, with $\sup|R_A(\boldsymbol{u})|\le\sup|\boldsymbol{u}|$. So we have an operator $R_A:M_{\text{mo}}(\mathcal{S})\to M_{\text{o-b}}(\mathcal{S})$ which is continuous for the ucp topology.

(iv) If $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is non-decreasing, non-negative and order-bounded, then whenever $\sigma\le\tau$ in $\mathcal{S}$ we shall have

$$0\le u_{\sigma\wedge\rho}\le u_{\tau\wedge\rho}\le\sup|\boldsymbol{u}|$$

for every $\rho\in A$, so

$$0\le\lim_{\rho\downarrow A}u_{\sigma\wedge\rho}\le\lim_{\rho\downarrow A}u_{\tau\wedge\rho}\le\sup|\boldsymbol{u}|.$$

Thus $R_A(\boldsymbol{u})$ is also non-decreasing, non-negative and order-bounded.

(v) It follows that if $\boldsymbol{u}\in M_{\text{bv}}(\mathcal{S})$, that is, $\boldsymbol{u}$ is expressible as the difference of two non-decreasing non-negative order-bounded processes. then $R_A(\boldsymbol{u})\in M_{\text{bv}}(\mathcal{S})$. Now as $R_A:M_{\text{mo}}(\mathcal{S})\to M_{\text{o-b}}(\mathcal{S})$ is continuous,

$$R_A[M_{\text{mo}}(\mathcal{S})]=R_A[\overline{M_{\text{bv}}(\mathcal{S})}]\subseteq\overline{R_A[M_{\text{bv}}(\mathcal{S})]}\subseteq\overline{M_{\text{bv}}(\mathcal{S})}=M_{\text{mo}}(\mathcal{S}),$$

and $R_A(\boldsymbol{u})$ is moderately oscillatory whenever $\boldsymbol{u}\in M_{\text{mo}}(\mathcal{S})$.

(vi) Generally, if $\boldsymbol{u}\in M_{\text{lmo}}(\mathcal{S})$, take any $\tau\in\mathcal{S}$. Then $A\wedge\tau=\{\rho\wedge\tau:\rho\in A\}$ is a non-empty downwards-directed subset of $\mathcal{S}\wedge\tau$, and $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ is moderately oscillatory, so $R_{A\wedge\tau}(\boldsymbol{u}\restriction\mathcal{S}\wedge\tau)$ is moderately oscillatory, by (i)-(v) above. And if $\sigma\in\mathcal{S}\wedge\tau$ then

$$\lim_{\rho\downarrow A}u_{\sigma\wedge\rho}=\lim_{\rho\downarrow A}u_{\sigma\wedge\tau\wedge\rho}=\lim_{\upsilon\downarrow A\wedge\tau}u_{\sigma\wedge\upsilon},$$

so $R_A(\boldsymbol{u})\restriction\mathcal{S}\wedge\tau=R_{A\wedge\tau}(\boldsymbol{u}\restriction\mathcal{S}\wedge\tau)$ is moderately oscillatory. As τ is arbitrary, $R_A(\boldsymbol{u})$ is locally moderately oscillatory, as claimed.

(b) If $h:\mathbb{R}\to\mathbb{R}$ is continuous, $\bar{h}:L^0(\mathfrak{A})\to L^0(\mathfrak{A})$ is continuous (613Bb) and $\bar{h}(\lim_{\rho\downarrow A}u_{\tau\wedge\rho})=\lim_{\rho\downarrow A}\bar{h}(u_{\tau\wedge\rho})$ for every locally moderately oscillatory $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ and $\tau\in\mathcal{S}$.

(c)(i) By 615Gb, $u_\downarrow = \lim_{\sigma\downarrow\mathcal{S}} u_\sigma$ is defined. Now if $\epsilon > 0$, there is a $\tau \in \mathcal{S}$ such that $\theta(u_\sigma - u_\downarrow) \le \epsilon$ for every $\sigma \in \mathcal{S} \wedge \tau$. In this case, if $\sigma \in \mathcal{S} \wedge \tau$, $\theta(u_{\sigma\wedge\rho} - u_\downarrow) \le \epsilon$ for every $\rho \in A$, so $\theta(u'_\sigma - u_\downarrow) \le \epsilon$. As ϵ is arbitrary, $\lim_{\sigma\downarrow\mathcal{S}} u'_\sigma = u_\downarrow$.

(ii) Writing γ for $\sup_{\sigma\in\mathcal{S}} \|u_\sigma\|_1$, $\{x : x \in L^0(\mathfrak{A}), \|x\|_1 \le \gamma\}$ is closed (613Bc), so contains $\lim_{\rho\downarrow A} u_{\sigma\wedge\rho}$ for every $\sigma \in \mathcal{S}$.

(d) We know that $\hat{\boldsymbol{u}}$ is locally moderately oscillatory (615F(a-vi)), while of course A is a non-empty downwards-directed subset of $\hat{\mathcal{S}}$, so we can speak of $R_A(\hat{\boldsymbol{u}})$. Looking at the formula in (a), we see that if we express $\boldsymbol{u}$ as $\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ and $\hat{\boldsymbol{u}}$ as $\langle \hat{u}_\sigma\rangle_{\sigma\in\hat{\mathcal{S}}}$, then we have $R_A(\boldsymbol{u}) = \langle u'_\sigma\rangle_{\sigma\in\mathcal{S}}$ and $R_A(\hat{\boldsymbol{u}}) = \langle \hat{u}'_\sigma\rangle_{\sigma\in\hat{\mathcal{S}}}$ where

$$u'_\sigma = \lim_{\rho\downarrow A} u_{\sigma\wedge\rho} \text{ for } \sigma \in \mathcal{S},$$

$$\hat{u}'_\sigma = \lim_{\rho\downarrow A} \hat{u}_{\sigma\wedge\rho} \text{ for } \sigma \in \hat{\mathcal{S}},$$

so that $\hat{u}'_\sigma = u'_\sigma$ for $\sigma \in \mathcal{S}$ and $R_A(\hat{\boldsymbol{u}})$ extends $R_A(\boldsymbol{u})$. Since $R_A(\hat{\boldsymbol{u}})$ is fully adapted ((a-i) above), it must be the fully adapted extension of $R_A(\boldsymbol{u})$

Remark The elementary case in which $A = \{\rho\}$ is a singleton, so that $R_A(\boldsymbol{u}) = \langle u_{\sigma\wedge\rho}\rangle_{\sigma\in\mathcal{S}}$ (612Ib), will be a useful guide. But I introduce the idea here primarily for the sake of applications based on the construction in 623I below.

623C Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. For a non-empty downwards-directed set $A \subseteq \mathcal{S}$ let $R_A : M_{\text{lmo}}(\mathcal{S}) \to M_{\text{lmo}}(\mathcal{S})$ be the operator described in 623B. Let A, $B \subseteq \mathcal{S}$ be non-empty downwards-directed sets.

(a) Setting $A\vee B = \{\rho\vee\rho' : \rho \in A, \rho' \in B\}$ and $A\wedge B = \{\rho\wedge\rho' : \rho \in A, \rho' \in B\}$, $R_{A\vee B}+R_{A\wedge B} = R_A+R_B$.
(b) $R_{A\wedge B} = R_A R_B = R_B R_A$.
(c) If $B \subseteq A$, then $R_A R_B = R_A$; in particular, $R_A^2 = R_A$.
(d) If B is a coinitial subset of A, then $R_B = R_A$.

proof (a) Of course both $A \vee B$ and $A \wedge B$ are non-emptu downwards-directed sets, so we can speak of $R_{A\vee B}$ and $R_{A\wedge B}$. Take a moderately oscillatory process $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$. If $\sigma \in \mathcal{S}$, then

$$\lim_{(\rho,\rho')\downarrow A\times B} u_{(\rho\vee\rho')\wedge\sigma} + \lim_{(\rho,\rho')\downarrow A\times B} u_{(\rho\wedge\rho')\wedge\sigma} = \lim_{(\rho,\rho')\downarrow A\times B} u_{(\rho\wedge\sigma)\vee(\rho'\wedge\sigma)} + u_{(\rho\wedge\sigma)\wedge(\rho'\wedge\sigma)}$$

(because $\mathcal{T}$ is a distributive lattice, by 611Ca)

$$= \lim_{(\rho,\rho')\downarrow A\times B} u_{\rho\wedge\sigma} + u_{\rho'\wedge\sigma}$$

(612D(f-i))

$$= \lim_{\rho\downarrow A} u_{\rho\wedge\sigma} + \lim_{\rho'\downarrow B} u_{\rho'\wedge\sigma}.$$

As σ is arbitrary, $R_{A\vee B}(\boldsymbol{u}) + R_{A\wedge B}(\boldsymbol{u}) = R_A(\boldsymbol{u}) + R_B(\boldsymbol{u})$; as $\boldsymbol{u}$ is arbitrary, $R_{A\vee B} + R_{A\wedge B} = R_A + R_B$.

(b) Take $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}} \in M_{\text{lmo}}(\mathcal{S})$ and $\sigma \in \mathcal{S}$, and set

$$v_\sigma = \lim_{\rho\downarrow A\wedge B} u_{\sigma\wedge\rho}, \quad w_\sigma = \lim_{\rho\downarrow A} \lim_{\rho'\downarrow B} u_{(\sigma\wedge\rho)\wedge\rho'}.$$

Let $\epsilon > 0$. Then there are $\tilde{\rho} \in A$, $\tilde{\rho}' \in B$ such that $\theta(v_\sigma - u_{\sigma\wedge\rho\wedge\rho'}) \le \epsilon$ whenever $\rho \in A$, $\rho' \in B$, $\rho \le \tilde{\rho}$ and $\rho' \le \tilde{\rho}'$. Next, there are $\tau \in A$, $\tau' \in B$ such that

$$\tau \le \tilde{\rho}, \quad \theta(w_\sigma - \lim_{\rho'\downarrow B} u_{\sigma\wedge\tau\wedge\rho'}) \le \epsilon,$$

$$\tau' \le \tilde{\rho}', \quad \theta(u_{\sigma\wedge\tau\wedge\tau'} - \lim_{\rho'\downarrow B} u_{\sigma\wedge\tau\wedge\rho'}) \le \epsilon.$$

Since we also have $\theta(v_\sigma - u_{\sigma\wedge\tau\wedge\tau'}) \le \epsilon$, we see that $\theta(v_\sigma - w_\sigma) \le 3\epsilon$. As ϵ is arbitrary, $v_\sigma = w_\sigma$. As σ is arbitrary,

$$R_{A\wedge B}(\boldsymbol{u}) = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}} = \langle w_\sigma\rangle_{\sigma\in\mathcal{S}} = R_A R_B(\boldsymbol{u}).$$

As $\boldsymbol{u}$ is arbitrary, $R_{A\wedge B} = R_A R_B$. Similarly, $R_B R_A = R_{B\wedge A} = R_{A\wedge B}$.

(c) If $B \subseteq A$, consider $A^* = \{\rho : \rho \in \mathcal{S},\ \rho' \le \rho$ for some $\rho' \in A\}$. Then A^* is closed under $\wedge$ and A is coinitial with A^*; as $B \subseteq A$, $A \wedge B \subseteq A^*$ is also coinitial with A^*. Now if $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ is locally moderately oscillatory, $\lim_{\rho\downarrow A} u_{\sigma\wedge\rho}$, $\lim_{\rho\downarrow A^*} u_{\sigma\wedge\rho}$ and $\lim_{\rho\downarrow A\wedge B} u_{\sigma\wedge\rho}$ are all defined and equal for every $\sigma \in \mathcal{S}$. As $\boldsymbol{u}$ and σ are arbitrary, $R_A = R_{A^*} = R_{A\wedge B}$.

(d) If B is a coinitial subset of A then of course $\lim_{\rho\downarrow B} u_{\sigma\wedge\rho} = \lim_{\rho\downarrow A} u_{\sigma\wedge\rho}$ whenever the latter is defined.

623D Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $A \subseteq \mathcal{S}$ a non-empty downwards-directed set. Let $R_A : M_{\mathrm{lmo}}(\mathcal{S}) \to M_{\mathrm{lmo}}(\mathcal{S})$ be the operator described in 623B. If $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma\in\mathcal{S}}$ is a (local) integrator, $R_A(\boldsymbol{v})$ is a (local) integrator.

proof Express $\boldsymbol{v}' = R_A(\boldsymbol{v})$ as $\langle v'_\sigma \rangle_{\sigma\in\mathcal{S}}$.

(a) Consider first the case in which $\boldsymbol{v}$ is an integrator.

(i) If $\rho \in A$ then $Q_{\mathcal{S}}(d(R_{\{\rho\}}(\boldsymbol{v}))) \subseteq Q_{\mathcal{S}}(d\boldsymbol{v})$. **P** Take $z \in Q_{\mathcal{S}}(dR_{\{\rho\}}(\boldsymbol{v}))$. If $\mathcal{S} = \emptyset$ then surely $z = 0 \in Q_{\mathcal{S}}(d\boldsymbol{v})$. Otherwise, 616C(iii) tells us that there are a fully adapted process $\tilde{\boldsymbol{u}} = \langle \tilde u_\sigma \rangle_{\sigma\in\mathcal{S}}$ such that $\|\tilde{\boldsymbol{u}}\|_\infty \le 1$ and $\tau_0 \le \ldots \le \tau_n$ such that

$$z = \sum_{i=0}^{n-1} \tilde u_{\tau_i} \times (v_{\tau_{i+1}\wedge\rho} - v_{\tau_i\wedge\rho}) = \sum_{i=0}^{n-1} \tilde u_{\tau_i\wedge\rho} \times (v_{\tau_{i+1}\wedge\rho} - v_{\tau_i\wedge\rho})$$

because if $i < n$ then

$$[\![\tilde u_{\tau_i\wedge\rho} \ne \tilde u_{\tau_i}]\!] \subseteq [\![\tau_i \wedge \rho < \tau_i]\!] \subseteq [\![\tau_i \wedge \rho = \tau_{i+1} \wedge \rho]\!] \subseteq [\![v_{\tau_{i+1}\wedge\rho} = v_{\tau_i\wedge\rho}]\!].$$

So $z \in Q_{\mathcal{S}}(\boldsymbol{v})$, by 616C(ii). **Q**

(ii) $Q_{\mathcal{S}}(d\boldsymbol{v}') \subseteq \overline{Q_{\mathcal{S}}(\boldsymbol{u}, d\boldsymbol{v})}$. **P** If $z \in Q_{\mathcal{S}}(d\boldsymbol{v}')\setminus\{0\}$, express it as $S_I(\boldsymbol{u}, d\boldsymbol{v}')$ where $I \in \mathcal{I}(\mathcal{S})$, $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in I} \in M_{\mathrm{fa}}(I)$ and $\|\boldsymbol{u}\|_\infty \le 1$. Let $(\tau_0, \ldots, \tau_n)$ linearly generate the I-cells. Then

$$\begin{aligned} z &= \sum_{i=0}^{n-1} u_{\tau_i} \times (v'_{\tau_{i+1}} - v'_{\tau_i}) = \lim_{\rho\downarrow A} \sum_{i=0}^{n-1} u_{\tau_i} \times (v_{\tau_{i+1}\wedge\rho} - v_{\tau_i\wedge\rho}) \\ &= \lim_{\rho\downarrow A} S_I(\boldsymbol{u}, dR_{\{\rho\}}(\boldsymbol{v})) \in \overline{Q_{\mathcal{S}}(\boldsymbol{u}, d\boldsymbol{v})} \end{aligned}$$

by (i). **Q**

(iii) Since $Q_{\mathcal{S}}(d\boldsymbol{v})$ is topologically bounded, so is its closure (613B(f-iii)), and $\boldsymbol{v}'$ is an integrator.

(b) Now suppose that $\boldsymbol{v}$ is a local integrator. Take $\tau \in \mathcal{S}$. Set $B = \{\tau \wedge \rho : \rho \in A\}$; then B is a non-empty downwards-directed subset of $\mathcal{S} \wedge \tau$, so we have a corresponding operator R_B on $M_{\mathrm{lmo}}(\mathcal{S} \wedge \tau)$. As $\boldsymbol{v}$ is a local integrator, $\boldsymbol{v}\restriction\mathcal{S}\wedge\tau$ is an integrator and $R_B(\boldsymbol{v}\restriction\mathcal{S}\wedge\tau)$ is an integrator, by (a). Express $R_B(\boldsymbol{v}\restriction\mathcal{S}\wedge\tau)$ as $\langle w'_\sigma \rangle_{\sigma\in\mathcal{S}\wedge\tau}$. If $\sigma \in \mathcal{S} \wedge \tau$,

$$v'_\sigma = \lim_{\rho\downarrow A} v_{\sigma\wedge\rho} = \lim_{\rho\downarrow A} v_{\sigma\wedge\tau\wedge\rho} = \lim_{\rho\downarrow B} v_{\sigma\wedge\rho} = w'_\sigma.$$

So $\boldsymbol{v}'\restriction\mathcal{S}\wedge\tau = R_B(\boldsymbol{v}\restriction\mathcal{S}\wedge\tau)$ is an integrator. As τ is arbitrary, $\boldsymbol{v}'$ is a local integrator.

623E Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $A \subseteq \mathcal{S}$ a non-empty downwards-directed set. Let $R_A : M_{\mathrm{lmo}}(\mathcal{S}) \to M_{\mathrm{lmo}}(\mathcal{S})$ be the operator described in 623B. If $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$ is a martingale, $R_A(\boldsymbol{u})$ is a martingale.

proof $\boldsymbol{u}$ is locally moderately oscillatory (622H) so $R_A(\boldsymbol{u})$ is defined; express it as $\langle u'_\sigma \rangle_{\sigma\in\mathcal{S}}$. The point is that if $\sigma \in \mathcal{S}$ then u'_σ is the limit $1\!\lim_{\rho\downarrow A} P_\rho u_\sigma$ for the norm topology of L^1. **P** For $\rho \in A$,

$$u_{\sigma\wedge\rho} = P_{\sigma\wedge\rho} u_\sigma = P_\rho P_\sigma u_\sigma = P_\rho u_\sigma$$

(622Ba). Next, $1\!\lim_{\rho\downarrow A} P_\rho z$ is defined for every $z \in L^0(\mathfrak{A})$, by 621Cg, and this must also be the limit $\lim_{\rho\downarrow A} P_\rho z$ for the topology of convergence in measure (613B(d-i)). So we have

$$u'_\sigma = \lim_{\rho\downarrow A} u_{\sigma\wedge\rho} = \lim_{\rho\downarrow A} P_\rho u_\sigma = 1\!\lim_{\rho\downarrow A} P_\rho u_\sigma,$$

as claimed. **Q**

If now $\sigma \le \tau$ in $\mathcal{S}$,

$$P_\sigma u'_\tau = P_\sigma(\operatorname{1lim}_{\rho\downarrow A} P_\rho u_\tau) = \operatorname{1lim}_{\rho\downarrow A} P_\sigma P_\rho u_\tau$$

(because $P_\sigma : L^1 \to L^1$ is $\| \|_1$-continuous)

$$= \operatorname{1lim}_{\rho\downarrow A} P_\rho P_\sigma u_\tau = \operatorname{1lim}_{\rho\downarrow A} P_\rho u_\sigma = u'_\sigma,$$

and $R_A(\boldsymbol{u}) = \langle u'_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a martingale.

623F Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $A \subseteq \mathcal{S}$ a non-empty downwards-directed set and $R_A : M_{\mathrm{lmo}}(\mathcal{S}) \to M_{\mathrm{lmo}}(\mathcal{S})$ the operator described in 623B. Suppose that $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is locally moderately oscillatory.

(a) If $\boldsymbol{u}$ is order-bounded, the residual oscillation $\operatorname{Osclln}(R_A(\boldsymbol{u}))$ is at most $\operatorname{Osclln}(\boldsymbol{u})$.

(b) If $\boldsymbol{u}$ is (locally) jump-free, then $R_A(\boldsymbol{u})$ is (locally) jump-free.

proof Express $\boldsymbol{u}' = R_A(\boldsymbol{u})$ as $\langle u'_\sigma\rangle_{\sigma\in\mathcal{S}}$.

(a)(i) As

$$|u'_\sigma| \le \sup_{\rho\in A} |u_{\sigma\wedge\rho}| \le \sup |\boldsymbol{u}|$$

for every $\sigma \in \mathcal{S}$, $\boldsymbol{u}'$ is order-bounded and $\operatorname{Osclln}(\boldsymbol{u}')$ is defined.

(ii) If $I \in \mathcal{I}(\mathcal{S})$ then, in the language of 618B, $\operatorname{Osclln}_I(\boldsymbol{u}') \le \operatorname{Osclln}^*_I(\boldsymbol{u})$. **P** If I is empty, this is trivial. Otherwise, take $(\tau_0, \dots, \tau_n)$ linearly generating the I-cells. Take $\rho \in A$ and $i < n$. Then

$$\begin{aligned}[\![\rho \le \tau_i]\!] &\subseteq [\![\tau_i \wedge \rho = \rho]\!] \cap [\![\tau_{i+1} \wedge \rho = \rho]\!] \subseteq [\![u_{\tau_{i+1}\wedge\rho} - u_{\tau_i\wedge\rho} = 0]\!]\\ &\subseteq [\![|u_{\tau_{i+1}\wedge\rho} - u_{\tau_i\wedge\rho}| \le \operatorname{Osclln}^*_I(\boldsymbol{u})]\!],\end{aligned}$$

$$\begin{aligned}[\![\tau_i \le \rho]\!] \cap [\![\rho \le \tau_{i+1}]\!] &\subseteq [\![\tau_i \wedge \rho = \tau_i]\!] \cap [\![\tau_{i+1} \wedge \rho = \operatorname{med}(\tau_i, \rho, \tau_{i+1})]\!]\\ &\subseteq [\![u_{\tau_{i+1}\wedge\rho} - u_{\tau_i\wedge\rho} = u_{\operatorname{med}(\tau_i,\rho,\tau_{i+1})} - u_{\tau_i}]\!]\\ &\subseteq [\![|u_{\tau_{i+1}\wedge\rho} - u_{\tau_i\wedge\rho}| \le \operatorname{Osclln}^*_I(\boldsymbol{u})]\!]\end{aligned}$$

(618Ca),

$$\begin{aligned}[\![\tau_{i+1} \le \rho]\!] &\subseteq [\![\tau_i \wedge \rho = \tau_i]\!] \cap [\![\tau_{i+1} \wedge \rho = \tau_{i+1}]\!]\\ &\subseteq [\![u_{\tau_{i+1}\wedge\rho} - u_{\tau_i\wedge\rho} = u_{\tau_{i+1}} - u_{\tau_i}]\!] \subseteq [\![|u_{\tau_{i+1}\wedge\rho} - u_{\tau_i\wedge\rho}| \le \operatorname{Osclln}^*_I(\boldsymbol{u})]\!].\end{aligned}$$

So in fact $|u_{\tau_{i+1}\wedge\rho} - u_{\tau_i\wedge\rho}| \le \operatorname{Osclln}^*_I(\boldsymbol{u})$. Taking the limit as $\rho \downarrow A$, $|u'_{\tau_{i+1}} - u'_{\tau_i}| \le \operatorname{Osclln}^*_I(\boldsymbol{u})$. As i is arbitrary, $\operatorname{Osclln}_I(\boldsymbol{u}') \le \operatorname{Osclln}^*_I(\boldsymbol{u})$ (618Ba). **Q**

(iii) Now

$$\begin{aligned}\operatorname{Osclln}(\boldsymbol{u}') &= \inf_{I\in\mathcal{I}(\mathcal{S})} \sup_{I\subseteq J\in\mathcal{I}(\mathcal{S})} \operatorname{Osclln}_J(\boldsymbol{u}')\\ &\le \inf_{I\in\mathcal{I}(\mathcal{S})} \sup_{I\subseteq J\in\mathcal{I}(\mathcal{S})} \operatorname{Osclln}^*_J(\boldsymbol{u})\\ &= \inf_{I\in\mathcal{I}(\mathcal{S})} \sup_{I\subseteq J\subseteq K\in\mathcal{I}(\mathcal{S})} \operatorname{Osclln}_K(\boldsymbol{u})\\ &= \inf_{I\in\mathcal{I}(\mathcal{S})} \sup_{I\subseteq K\in\mathcal{I}(\mathcal{S})} \operatorname{Osclln}_K(\boldsymbol{u}) = \operatorname{Osclln}(\boldsymbol{u}).\end{aligned}$$

(b)(i) If $\boldsymbol{u}$ is jump-free, then it is order-bounded (618B(b-ii)) and $\operatorname{Osclln}(\boldsymbol{u}) = 0$, so $\operatorname{Osclln}(\boldsymbol{u}') = 0$ and $\boldsymbol{u}'$ is jump-free.

(ii) If $\boldsymbol{u}$ is locally jump-free, then we can apply (i) to $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ and $A \wedge \tau = \{\rho\wedge\tau : \rho \in A\}$ to see that $\boldsymbol{u}'\restriction\mathcal{S}\wedge\tau$ is jump-free for every $\tau \in \mathcal{S}$, so that $\boldsymbol{u}'$ is locally jump-free.

623G Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $A \subseteq \mathcal{S}$ a non-empty downwards-directed set. Let $R_A : M_{\mathrm{lmo}}(\mathcal{S}) \to M_{\mathrm{lmo}}(\mathcal{S})$ be the operator described in 623B. If $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ is locally moderately oscillatory and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ is a local integrator,

$$R_A(ii_{\boldsymbol{v}}(\boldsymbol{u})) = ii_{R_A(\boldsymbol{v})}(\boldsymbol{u}) = ii_{R_A(\boldsymbol{v})}(R_A(\boldsymbol{u})).$$

proof (a) Since $\boldsymbol{u}' = R_A(\boldsymbol{u})$ is locally moderately oscillatory (623Ba) and $\boldsymbol{v}' = R_A(\boldsymbol{v})$ is a local integrator (623D), all the indefinite integrals are defined everywhere on $\mathcal{S}$; while $ii_{\boldsymbol{v}}(\boldsymbol{u})$ also, being a local integrator (616Q(c-i)), is locally moderately oscillatory (616Ib), so we can speak of $\boldsymbol{w}' = R_A(ii_{\boldsymbol{v}}(\boldsymbol{u}))$.

For $I \in \mathcal{I}(\mathcal{S})$ and $\rho \in \mathcal{S}$ write $I \wedge \rho$ for $\{\sigma \wedge \rho : \sigma \in I\}$. Express $\boldsymbol{u}'$, $\boldsymbol{v}'$, $\boldsymbol{w}'$ and $\boldsymbol{z}' = ii_{\boldsymbol{v}'}(\boldsymbol{u}') = ii_{R_A(\boldsymbol{v})}(R_A(\boldsymbol{u}))$ as $\langle u'_\sigma \rangle_{\sigma \in \mathcal{S}}$, $\langle v'_\sigma \rangle_{\sigma \in \mathcal{S}}$, $\langle w'_\sigma \rangle_{\sigma \in \mathcal{S}}$ and $\langle z'_\sigma \rangle_{\sigma \in \mathcal{S}}$ respectively.

(b) If $\sigma \le \tau$ in $\mathcal{S}$ and $\rho \in \mathcal{S}$,

$$u_{\sigma \wedge \rho} \times (v_{\tau \wedge \rho} - v_{\sigma \wedge \rho}) = u_\sigma \times (v_{\tau \wedge \rho} - v_{\sigma \wedge \rho}).$$

P

$$[\![u_{\sigma \wedge \rho} \ne u_\sigma]\!] \subseteq [\![\rho < \sigma]\!] \subseteq [\![\sigma \wedge \rho = \tau \wedge \rho]\!] \subseteq [\![v_{\sigma \wedge \rho} = v_{\tau \wedge \rho}]\!],$$

so $(u_{\sigma \wedge \rho} - u_\sigma) \times (v_{\tau \wedge \rho} - v_{\sigma \wedge \rho}) = 0$. **Q**

Letting $\rho \downarrow A$, we see that

$$u'_\sigma \times (v'_\tau - v'_\sigma) = u_\sigma \times (v'_\tau - v'_\sigma).$$

It follows at once that

$$S_I(\boldsymbol{u}', d\boldsymbol{v}') = S_I(\boldsymbol{u}, d\boldsymbol{v}')$$

for every $I \in \mathcal{I}(\mathcal{S})$, and therefore that $ii_{\boldsymbol{v}'}(\boldsymbol{u}) = ii_{\boldsymbol{v}'}(\boldsymbol{u}') = \boldsymbol{z}'$.

(c) Take $\tau \in \mathcal{S}$ and $\epsilon > 0$. There is a $J_0 \in \mathcal{I}(\mathcal{S} \wedge \tau)$ such that $\theta(S_{I \wedge \tau \wedge \rho}(\boldsymbol{u}, d\boldsymbol{v}) - \int_{\mathcal{S} \wedge \tau \wedge \rho} \boldsymbol{u}\, d\boldsymbol{v}) \le \epsilon$ whenever $\rho \in \mathcal{S}$ and $I \in \mathcal{I}(\mathcal{S} \wedge \tau)$ includes J_0 (613V(ii-β)), and a $J_1 \in \mathcal{I}(\mathcal{S} \wedge \tau)$ such that $\theta(S_I(\boldsymbol{u}', d\boldsymbol{v}') - \int_{\mathcal{S} \wedge \tau} \boldsymbol{u}'\, d\boldsymbol{v}') \le \epsilon$ whenever $I \in \mathcal{I}(\mathcal{S} \wedge \tau)$ includes J_1. Let I be the sublattice generated by $J_0 \cup J_1 \cup \{\tau\}$. Let $(\tau_0, \dots, \tau_n)$ linearly generate the I-cells. If $\rho \in A$, then $(\tau_0 \wedge \rho, \dots, \tau_n \wedge \rho)$ linearly generates the $(I \wedge \rho)$-cells (611Kg), while of course $I \wedge \rho = I \wedge \tau \wedge \rho$ because $\tau = \max I$. Accordingly

$$S_{I \wedge \tau \wedge \rho}(\boldsymbol{u}, d\boldsymbol{v}) = \sum_{i=0}^{n-1} u_{\tau_i \wedge \rho} \times (v_{\tau_{i+1} \wedge \rho} - v_{\tau_i \wedge \rho})$$

so

$$\theta(\sum_{i=0}^{n-1} u_{\tau_i \wedge \rho} \times (v_{\tau_{i+1} \wedge \rho} - v_{\tau_i \wedge \rho}) - \int_{\mathcal{S} \wedge \tau \wedge \rho} \boldsymbol{u}\, d\boldsymbol{v}) \le \epsilon.$$

Taking the limit as $\rho \downarrow A$,

$$\theta(\sum_{i=0}^{n-1} u'_{\tau_i} \times (v'_{\tau_{i+1}} - v'_{\tau_i}) - w'_\tau) \le \epsilon,$$

that is,

$$\theta(S_I(\boldsymbol{u}', d\boldsymbol{v}') - w'_\tau) \le \epsilon.$$

Since we also have $J_1 \subseteq I \in \mathcal{I}(\mathcal{S} \wedge \tau)$, $\theta(S_I(\boldsymbol{u}', d\boldsymbol{v}') - z'_\tau) \le \epsilon$ and we conclude that $\theta(w'_\tau - z'_\tau) \le 2\epsilon$. As τ and ϵ are arbitrary,

$$\boldsymbol{w}' = \boldsymbol{z}' = ii_{\boldsymbol{v}'}(\boldsymbol{u}'),$$

and we know from (b) that the last is equal to $ii_{\boldsymbol{v}'}(\boldsymbol{u})$. So we have $R_A(ii_{\boldsymbol{v}}(\boldsymbol{u})) = ii_{R_A(\boldsymbol{v})}(R_A(\boldsymbol{u})) = ii_{R_A(\boldsymbol{v})}(\boldsymbol{u})$, as claimed.

623H Corollary Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $A \subseteq \mathcal{S}$ a non-empty downwards-directed set and $R_A : M_{\mathrm{lmo}}(\mathcal{S}) \to M_{\mathrm{lmo}}(\mathcal{S})$ the operator described in 623B. If $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ is a local integrator with quadratic variation $\boldsymbol{v}^*$, then $R_A(\boldsymbol{v}^*)$ is the quadratic variation of $R_A(\boldsymbol{v})$.

proof We know that $\boldsymbol{v}$ is locally moderately oscillatory (616Ib again). Writing $\boldsymbol{v}'$ for $R_A(\boldsymbol{v})$, 623D and 623G tell us that $\boldsymbol{v}'$ is a local integrator and $R_A(ii_{\boldsymbol{v}}(\boldsymbol{v})) = ii_{\boldsymbol{v}'}(\boldsymbol{v}')$. Now

$$(\boldsymbol{v}')^* = (\boldsymbol{v}')^2 - 2ii_{\boldsymbol{v}'}(\boldsymbol{v}') - (v'_\downarrow)^2\mathbf{1}$$

(where $v'_\downarrow$ is the starting value of $\boldsymbol{v}'$)

$$= R_A(\boldsymbol{v}^2) - 2R_A(ii_{\boldsymbol{v}}(\boldsymbol{v})) - v_\downarrow^2\mathbf{1}$$

(623Ba, 623B(c-i))

$$= R_A(\boldsymbol{v}^2 - 2ii_{\boldsymbol{v}}(\boldsymbol{v}) - v_\downarrow^2\mathbf{1})$$

(because $R_A(\mathbf{1}\restriction\mathcal{S}) = \mathbf{1}\restriction\mathcal{S}$)

$$= R_A(\boldsymbol{v})^*.$$

623I Lemma Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ (definition: 611O). Suppose that $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a moderately oscillatory process, $\tau^* \in \mathcal{S}$ and $M \ge 0$.

(a) Set

$$A = \{\rho : \rho \in \mathcal{S},\ [\![\rho < \tau^*]\!] \subseteq [\![|u_\rho| \ge M]\!]\}.$$

Then $\tau^* \in A$ and $\rho \wedge \rho' \in A$ whenever ρ, $\rho' \in A$.

(b) Let $R_A : M_{\mathrm{lmo}}(\mathcal{S}) \to M_{\mathrm{lmo}}(\mathcal{S})$ be the operator described in 623B. Suppose that $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a moderately oscillatory process such that $R_A(\boldsymbol{v}) = \boldsymbol{v}$.

(i) $[\![|u_\sigma| \ge M]\!] \subseteq [\![v_\sigma = v_\tau]\!]$ whenever $\sigma \le \tau$ in $\mathcal{S} \wedge \tau^*$.

(ii) Expressing $R_A(\boldsymbol{u})$ as $\langle u'_\sigma\rangle_{\sigma\in\mathcal{S}}$, $[\![|u'_\sigma| > M]\!] \subseteq [\![v_\sigma = v_{\tau^*}]\!]$ for every $\sigma \in \mathcal{S}$. In particular, $[\![|u'_\sigma| > M]\!] \subseteq [\![u'_\sigma = u'_{\tau^*}]\!]$.

proof (a) $\tau^* \in A$ just because $[\![\tau^* < \tau^*]\!] = 0$. To see that A is downwards-directed, repeat the formula in the proof of 615Ma: if ρ, $\rho' \in A$, then

$$\begin{aligned}[\![\rho \wedge \rho' < \tau^*]\!] &= ([\![\rho \le \rho']\!] \cap [\![\rho < \tau^*]\!]) \cup ([\![\rho' \le \rho]\!] \cap [\![\rho' < \tau^*]\!])\\ &\subseteq ([\![\rho \le \rho']\!] \cap [\![|u_\rho| \ge M]\!]) \cup ([\![\rho' \le \rho]\!] \cap [\![|u_{\rho'}| \ge M]\!])\\ &= ([\![\rho \le \rho']\!] \cap [\![|u_{\rho\wedge\rho'}| \ge M]\!]) \cup ([\![\rho' \le \rho]\!] \cap [\![|u_{\rho\wedge\rho'}| \ge M]\!])\\ &= [\![|u_{\rho\wedge\rho'}| \ge M]\!]\end{aligned}$$

and $\rho \wedge \rho' \in A$.

(b)(i) If $\sigma \le \tau$ in $\mathcal{S} \wedge \tau^*$, set $a = [\![|u_\sigma| \ge M]\!]$. Then $a \in \mathfrak{A}_\sigma \subseteq \mathfrak{A}_{\tau^*}$ so there is a $\rho \in \mathcal{T}$ such that $a \subseteq [\![\rho = \sigma]\!]$ and $1 \setminus a \subseteq [\![\rho = \tau^*]\!]$. As $\mathcal{S}$ is finitely full, $\rho \in \mathcal{S}$. Now

$$[\![\rho < \tau^*]\!] \subseteq a = [\![\rho = \sigma]\!] \cap [\![|u_\sigma| \ge M]\!] \subseteq [\![|u_\rho| \ge M]\!]$$

and $\rho \in A$.

If $\rho' \in A$ and $\rho' \le \rho$, then

$$a \subseteq [\![\rho = \sigma \wedge \rho]\!] \subseteq [\![\rho' = \sigma \wedge \rho']\!] \subseteq [\![\tau \wedge \rho' \le \sigma \wedge \rho']\!] = [\![\tau \wedge \rho' = \sigma \wedge \rho']\!]$$

(because we are supposing that $\sigma \le \tau$)

$$\subseteq [\![v_{\tau\wedge\rho'} = v_{\sigma\wedge\rho'}]\!].$$

As ρ is arbitrary,

$$a \subseteq [\![\lim_{\rho'\downarrow A} v_{\tau\wedge\rho'} = \lim_{\rho'\downarrow A} v_{\sigma\wedge\rho'}]\!] = [\![v_\tau = v_\sigma]\!]$$

because $R_A(\boldsymbol{v}) = \boldsymbol{v}$.

(ii) For any $\rho \in A \wedge \tau^*$, $\sigma \wedge \rho \le \tau^*$ so $[\![|u_{\sigma\wedge\rho}| \ge M]\!] \subseteq [\![v_{\sigma\wedge\rho} = v_{\tau^*}]\!]$. But we are supposing that

$$\boldsymbol{v} = R_A(\boldsymbol{v}) = R_{\{\rho\}}R_A(\boldsymbol{v})$$

(623Cc, with $B = \{\rho\}$)

$$= R_{\{\rho\}}(\boldsymbol{v}),$$

so $v_{\sigma\wedge\rho} = v_\sigma$ and $[\![|u_{\sigma\wedge\rho}| \ge M]\!] \subseteq [\![v_\sigma = v_{\tau^*}]\!]$. Now

$$[\![|u'_\sigma| > M]\!] = [\![|\lim_{\rho\downarrow A} u_{\sigma\wedge\rho}| > M]\!] \subseteq \sup_{\rho\in A\wedge\tau^*} [\![|u_{\sigma\wedge\rho}| > M]\!]$$

(because $\tau^* \in A$, so $A \wedge \tau^*$ is coinitial with A)

$$\subseteq [\![v_\sigma = v_{\tau^*}]\!].$$

As $R_A(R_A(\boldsymbol{u})) = R_A(\boldsymbol{u})$, we can apply this with $\boldsymbol{v} = R_A(\boldsymbol{u})$ and get $[\![|u'_\sigma| > M]\!] \subseteq [\![u'_\sigma = u'_{\tau^*}]\!]$.

623J Definition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u}$ a locally moderately oscillatory process. Let $\hat{\mathcal{S}}$ be the covered envelope of $\mathcal{S}$ and $\hat{\boldsymbol{u}} = \langle \hat{u}_\sigma \rangle_{\sigma\in\hat{\mathcal{S}}}$ the fully adapted extension of $\boldsymbol{u}$ to $\hat{\mathcal{S}}$. Recall that $\hat{\boldsymbol{u}}$ is locally moderately oscillatory (615F(b-v)). I will say that $\boldsymbol{u}$ is an **approximately local martingale** if for every $\sigma \in \mathcal{S}$ and $\epsilon > 0$ there is a non-empty downwards-directed set $A \subseteq \mathcal{S}$ such that $\sup_{\rho\in A} \bar{\mu}[\![\rho < \sigma]\!] \le \epsilon$ and $R_A(\boldsymbol{u})$, as defined in 623B, is a martingale; while $\boldsymbol{u}$ is a **virtually local martingale** if $\hat{\boldsymbol{u}}$ is an approximately local martingale.

Remarks Note that as the covered envelope of $\hat{\mathcal{S}}$ is itself, $\boldsymbol{u}$ is a virtually local martingale iff $\hat{\boldsymbol{u}}$ is. And if $\mathcal{S}$ has a greatest element we can drop the 'for every σ'; $\boldsymbol{u}$ will be an approximately local martingale iff for every $\epsilon > 0$ there is a non-empty downwards-directed set $A \subseteq \mathcal{S}$ such that $\sup_{\rho\in A} \bar{\mu}[\![\rho < \max\mathcal{S}]\!] \le \epsilon$ and $R_A(\boldsymbol{u})$ is a martingale.

In the context here, since A is downwards-directed, $\{[\![\rho < \tau]\!] : \rho \in A\}$ is upwards-directed, so that $\sup_{\rho\in A} \bar{\mu}[\![\rho < \sigma]\!]$ will always be $\bar{\mu}(\sup_{\rho\in A} [\![\rho < \sigma]\!])$ (321C).

623K Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.

(a)(i) The space $M_{\mathrm{alm}}(\mathcal{S})$ of approximately local martingales on $\mathcal{S}$ is a linear subspace of $M_{\mathrm{lmo}}(\mathcal{S})$.

(ii) The space $M_{\mathrm{vlm}}(\mathcal{S})$ of virtually local martingales on $\mathcal{S}$ is a linear subspace of $M_{\mathrm{lmo}}(\mathcal{S})$.

(b)(i) A local martingale on $\mathcal{S}$ is an approximately local martingale.

(ii) An approximately local martingale on $\mathcal{S}$ is a virtually local martingale.

(iii) If $\mathcal{S}$ is finitely full, a virtually local martingale on $\mathcal{S}$ is an approximately local martingale.

(c) If $\boldsymbol{u} \in M_{\mathrm{vlm}}(\mathcal{S})$ and $A \subseteq \mathcal{S}$ is a non-empty downwards-directed set, $R_A(\boldsymbol{u})$, as defined in 623B, is a virtually local martingale.

(d) Every virtually local martingale on $\mathcal{S}$ is a local integrator, therefore locally moderately oscillatory.

(e)(i) If $\boldsymbol{u} \in M_{\mathrm{fa}}(\mathcal{S})$, then $\boldsymbol{u}$ is an approximately local martingale iff $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ is an approximately local martingale for every $\tau \in \mathcal{S}$.

(ii) If $\boldsymbol{u} \in M_{\mathrm{fa}}(\mathcal{S})$, then $\boldsymbol{u}$ is a virtually local martingale iff $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ is a virtually local martingale for every $\tau \in \mathcal{S}$.

(f) A uniformly integrable approximately local martingale on $\mathcal{S}$ is a martingale.

(g) If $\mathcal{S} \ne \emptyset$ and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$ is a virtually local martingale, then $\lim_{\sigma\downarrow\mathcal{S}} u_\sigma$ is defined and belongs to L^1.

(h) If $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}} \in M_{\mathrm{vlm}}(\mathcal{S})$ and $\tau \in \mathcal{S}$ then $(\boldsymbol{u} - u_\tau\mathbf{1})\restriction\mathcal{S}\vee\tau$ is a virtually local martingale.

proof Let $\hat{\mathcal{S}}$ be the covered envelope of $\mathcal{S}$; for a fully adapted process $\boldsymbol{u}$ with domain $\mathcal{S}$, write $\hat{\boldsymbol{u}} = \langle \hat{u}_\sigma \rangle_{\sigma\in\hat{\mathcal{S}}}$ for its fully adapted extension to $\hat{\mathcal{S}}$.

(a)(i) It is built into the definition in 623J that a virtually local martingale is locally moderately oscillatory. If $\boldsymbol{u}$, $\boldsymbol{v} \in M_{\mathrm{alm}}(\mathcal{S})$, $\tau \in \mathcal{S}$ and $\epsilon > 0$, let A, $B \subseteq \mathcal{S}$ be non-empty downwards-directed sets such that $\sup_{\rho\in A} \bar{\mu}[\![\rho < \tau]\!] \le \frac{1}{2}\epsilon$, $R_A(\boldsymbol{u})$ is a martingale, $\sup_{\rho\in B} \bar{\mu}[\![\rho < \tau]\!] \le \frac{1}{2}\epsilon$ and $R_B(\boldsymbol{v})$ is a martingale. Then $A \wedge B$ is a non-empty downwards-directed subset of $\mathcal{S}$ and $R_{A\wedge B}(\boldsymbol{u}) = R_B R_A(\boldsymbol{u})$, $R_{A\wedge B}(\boldsymbol{v}) = R_A R_B(\boldsymbol{v})$ are martingales (623Cb, 623E). So $R_{A\wedge B}(\boldsymbol{u}+\boldsymbol{v})$ is a martingale (622Db), while

$$\sup_{\rho\in A\wedge B} \bar{\mu}[\![\rho < \tau]\!] = \sup_{\substack{\rho\in A\\ \rho'\in B}} \bar{\mu}[\![\rho\wedge\rho' < \tau]\!] = \sup_{\substack{\rho\in A\\ \rho'\in B}} \bar{\mu}([\![\rho < \tau]\!] \cup [\![\rho' < \tau]\!]) \le \epsilon.$$

As τ and ϵ are arbitrary, $\boldsymbol{u}+\boldsymbol{v}$ is an approximately local martingale.

If $\boldsymbol{u}\in M_{\text{alm}}(\mathcal{S})$ and $\alpha\in\mathbb{R}$, then for any $\tau\in\hat{\mathcal{S}}$ and $\epsilon>0$, we have a non-empty downwards-directed set $A\subseteq\mathcal{S}$ such that $\sup_{\rho\in A}\bar\mu[\![\rho<\tau]\!]\le\epsilon$ and $R_A(\boldsymbol{u})$ is a martingale. Now $R_A(\alpha\boldsymbol{u})=\alpha R_A(\boldsymbol{u})$ is a martingale. So $\alpha\boldsymbol{u}\in M_{\text{alm}}(\mathcal{S})$.

(ii) By definition, $M_{\text{vlm}}(\mathcal{S})=\{\boldsymbol{u}:\boldsymbol{u}\in M_{\text{lmo}}(\mathcal{S}),\,\hat{\boldsymbol{u}}\in M_{\text{alm}}(\hat{\mathcal{S}})\}$, where $\hat{\mathcal{S}}$ is the covered envelope of $\mathcal{S}$ and $\hat{\boldsymbol{u}}$ is the fully adapted extension of $\boldsymbol{u}$ to $\hat{\mathcal{S}}$. Since $M_{\text{alm}}(\hat{\mathcal{S}})$ is a linear subspace of $M_{\text{lmo}}(\hat{\mathcal{S}})$, by (i), and $\boldsymbol{u}\mapsto\hat{\boldsymbol{u}}:M_{\text{lmo}}(\mathcal{S})\to M_{\text{lmo}}(\hat{\mathcal{S}})$ is a linear operator (615F(b-v), 612Qb), $M_{\text{vlm}}(\mathcal{S})$ is a linear subspace of $M_{\text{lmo}}(\mathcal{S})$.

(b)(i) Suppose that $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a local martingale. Take any $\tau\in\hat{\mathcal{S}}$ and $\epsilon>0$. Then there is a $\tau'\in\mathcal{S}$ such that $\boldsymbol{u}\restriction\hat{\mathcal{S}}\wedge\tau'$ is a martingale and $\bar\mu[\![\tau'<\tau]\!]\le\epsilon$. Of course $A=\{\tau'\}$ is a non-empty downwards-directed subset of $\mathcal{S}$ and $\sup_{\rho\in A}[\![\rho<\tau]\!]=[\![\tau'<\tau]\!]$ has measure at most ϵ. Now

$$R_A(\hat{\boldsymbol{u}})=\langle\lim_{\rho\downarrow A}u_{\sigma\wedge\rho}\rangle_{\sigma\in\mathcal{S}}=\langle u_{\sigma\wedge\tau'}\rangle_{\sigma\in\mathcal{S}}$$

agrees with $\boldsymbol{u}$ on $\mathcal{S}\wedge\tau$ and is constant on $\mathcal{S}\vee\tau$, so is a martingale (622E(b-ii)). As ϵ is arbitrary, $\boldsymbol{u}$ is an approximately local martingale.

(ii) Now suppose that $\boldsymbol{u}$ is an approximately local martingale. Take $\tau\in\hat{\mathcal{S}}$ and $\epsilon>0$. As in part (b) of the proof of 622N, there is a $\tilde\tau\in\mathcal{S}$ such that $\bar\mu[\![\tilde\tau<\tau]\!]\le\epsilon$. Now there is a non-empty downwards-directed set $A\subseteq\mathcal{S}$ such that $\sup_{\rho\in A}\bar\mu[\![\rho<\tilde\tau]\!]\le\epsilon$ and $R_A(\boldsymbol{u})$ is a martingale. Of course $A\wedge\tilde\tau$ is now a non-empty downwards-directed subset of $\mathcal{S}$ and

$$\begin{aligned}\sup_{\rho\in A\wedge\tilde\tau}\bar\mu[\![\rho<\tau]\!]&=\sup_{\rho\in A}\bar\mu[\![\rho\wedge\tilde\tau<\tau]\!]\le\sup_{\rho\in A}\bar\mu([\![\rho<\tilde\tau]\!]\cup[\![\tilde\tau<\tau]\!])\\&\le\sup_{\rho\in A}(\bar\mu[\![\rho<\tilde\tau]\!]+\bar\mu[\![\tilde\tau<\tau]\!])\le 2\epsilon,\end{aligned}$$

while $R_{A\wedge\tilde\tau}(\boldsymbol{u})=R_{\{\tilde\tau\}}R_A(\boldsymbol{u})$ (623Cb) is a martingale (623E) and $R_{A\wedge\tilde\tau}(\hat{\boldsymbol{u}})$ is the fully adapted extension of $R_{A\wedge\tilde\tau}(\boldsymbol{u})$ (623Bd) and is therefore a local martingale (622Nb). Accordingly $R_{A\wedge\tilde\tau}(\hat{\boldsymbol{u}})\restriction\hat{\mathcal{S}}\wedge\tilde\tau$ is a martingale. On the other side of $\tilde\tau$,

$$R_{A\wedge\tilde\tau}(\hat{\boldsymbol{u}})\restriction\hat{\mathcal{S}}\vee\tilde\tau=R_AR_{\{\tilde\tau\}}(\hat{\boldsymbol{u}})\restriction\hat{\mathcal{S}}\vee\tilde\tau$$

is constant, so $R_{A\wedge\tilde\tau}(\hat{\boldsymbol{u}})$ is a martingale (622E(b-ii) again).

(iii) This time, suppose that $\boldsymbol{u}$ is a virtually local martingale, so that $\hat{\boldsymbol{u}}$ is an approximately local martingale, and that $\mathcal{S}$ is finitely full. Take $\tau\in\mathcal{S}$ and $\epsilon>0$. Then there is a non-empty downwards-directed set $A\subseteq\hat{\mathcal{S}}$ such that $R_A(\hat{\boldsymbol{u}})$ is a martingale and $\sup_{\rho\in A}\bar\mu[\![\rho<\tau]\!]\le\epsilon$. As we can replace A with $A\wedge\tau$ we can suppose that $A\subseteq\hat{\mathcal{S}}\wedge\tau$. Set $B=\bigcup_{\rho\in A}\{\sigma:\sigma\in\mathcal{S},\,\rho\le\sigma\}$. Then B is a downwards-directed subset of $\mathcal{S}$ and

$$\sup_{\sigma\in B}\bar\mu[\![\sigma<\tau]\!]\le\sup_{\rho\in A}\bar\mu[\![\rho<\tau]\!]\le\epsilon.$$

The point is that $R_B(\boldsymbol{u})=R_A(\hat{\boldsymbol{u}})\restriction\mathcal{S}$. **P** Express $\hat{\boldsymbol{u}}$, $R_A(\hat{\boldsymbol{u}})$ and $R_B(\boldsymbol{u})$ as $\langle\hat u_\sigma\rangle_{\sigma\in\hat{\mathcal{S}}}$, $\langle\hat u'_\sigma\rangle_{\sigma\in\hat{\mathcal{S}}}$ and $\langle\tilde u_\sigma\rangle_{\sigma\in\mathcal{S}}$ respectively.

Take any $\sigma\in\mathcal{S}$ and $\eta>0$; then $\hat u'_\sigma=\lim_{\rho\downarrow A}\hat u_{\sigma\wedge\rho}$ and $\tilde u_\sigma=\lim_{\rho\downarrow B}u_{\sigma\wedge\rho}$. Let $\tilde\rho\in B$ be such that $\theta(\tilde u_\sigma-u_{\sigma\wedge\rho})\le\eta$ whenever $\rho\in B$ and $\rho\le\tilde\rho$. As there is a $\rho\in A$ such that $\rho\le\tilde\rho$, we can find a $\hat\rho\in A$ such that $\hat\rho\le\tilde\rho$ and $\theta(\hat u'_\sigma-\hat u_{\sigma\wedge\hat\rho})\le\eta$. Because $\hat\rho\in\hat{\mathcal{S}}$, $\sup_{\upsilon\in\mathcal{S}}[\![\hat\rho=\upsilon]\!]=1$ and there is a finite set $I\subseteq\mathcal{S}$ such that $\bar\mu a\ge1-\eta$ where $a=\sup_{\upsilon\in I}[\![\hat\rho=\upsilon]\!]$. Now $a\in\mathfrak{A}_{\hat\rho}\subseteq\mathfrak{A}_\tau$ so there is a $\rho^*\in\mathcal{T}$ such that $a\subseteq[\![\rho^*=\hat\rho]\!]$ and $1\setminus a\subseteq[\![\rho^*=\tau]\!]$ (611I). We have

$$\sup_{\upsilon\in I\cup\{\tau\}}[\![\rho^*=\upsilon]\!]\supseteq\sup_{\upsilon\in I}([\![\rho^*=\hat\rho]\!]\cap[\![\hat\rho=\upsilon]\!])\cup[\![\rho^*=\tau]\!]\supseteq a\cup(1\setminus a)=1;$$

as $\mathcal{S}$ is finitely full, $\rho^*\in\mathcal{S}$. Consider $\rho=\rho^*\wedge\tilde\rho$. As $\hat\rho=\hat\rho\wedge\tau\le\rho^*$ (611I), $\hat\rho\le\rho$ and $\rho\in B$. Also

$$a\subseteq[\![\rho^*=\hat\rho]\!]=[\![\rho^*=\hat\rho\wedge\tilde\rho]\!]=[\![\rho^*=\rho]\!],$$

so

$$a\subseteq[\![\rho=\hat\rho]\!]\subseteq[\![\sigma\wedge\rho=\sigma\wedge\hat\rho]\!]\subseteq[\![u_{\sigma\wedge\rho}=\hat u_{\sigma\wedge\hat\rho}]\!]$$

and

$$\theta(u_{\sigma\wedge\rho}-\hat u_{\sigma\wedge\hat\rho})\le\bar\mu(1\setminus a)\le\eta.$$

But since also $\theta(\tilde u_\sigma-u_{\sigma\wedge\rho})\le\eta$ because $\rho\in B$ and $\rho\le\tilde\rho$, while $\theta(\hat u'_\sigma-\hat u_{\sigma\wedge\hat\rho})\le\eta$ by the choice of $\hat\rho$, we see that

$$\theta(\tilde u_\sigma-\hat u'_\sigma)\le\theta(\tilde u_\sigma-u_{\sigma\wedge\rho})+\theta(u_{\sigma\wedge\rho}-\hat u_{\sigma\wedge\hat\rho})+\theta(\hat u'_\sigma-\hat u_{\sigma\wedge\hat\rho})\le 3\eta.$$

As σ and η are arbitrary, $R_A(\hat{\boldsymbol{u}})\restriction\mathcal S=R_B(\boldsymbol{u})$. **Q**

Consequently $R_B(\boldsymbol{u})$ is a martingale (622D(b-ii)). As τ and ϵ are arbitrary, $\boldsymbol{u}$ is an approximately local martingale.

(c) As observed in (a-i) of the proof of 623B, we can regard R_A either as an operator on $M_{\text{lmo}}(\mathcal S)$ or as an operator on $M_{\text{lmo}}(\hat{\mathcal S})$. Again take $\tau\in\hat{\mathcal S}$ and $\epsilon>0$, and let $B\subseteq\hat{\mathcal S}$ be a non-empty downwards-directed set such that $\sup_{\rho\in B}\bar\mu[\![\rho<\tau]\!]\le\epsilon$ and $R_B(\hat{\boldsymbol{u}})$ is a martingale. Then $R_BR_A(\hat{\boldsymbol{u}})=R_AR_B(\hat{\boldsymbol{u}})$ (623Cb) is a martingale (623E). As τ and ϵ are arbitrary, $R_A(\boldsymbol{u})$ is a martingale.

(d) If $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal S}$ is a virtually local martingale and $\tau\in\hat{\mathcal S}$, let $\epsilon>0$. Then there is a non-empty downwards-directed set $A\subseteq\hat{\mathcal S}$ such that $R_A(\hat{\boldsymbol{u}})$ is a martingale and $\bar\mu a\le\epsilon$, where $a=\sup_{\rho\in A}[\![\rho<\tau]\!]$. Express the martingale $R_A(\hat{\boldsymbol{u}})$ as $\boldsymbol{u}'=\langle u'_\sigma\rangle_{\sigma\in\hat{\mathcal S}}$. This is a local integrator (622H), so $\boldsymbol{u}'\restriction\hat{\mathcal S}\wedge\tau$ is an integrator. If $\sigma\in\hat{\mathcal S}\wedge\tau$, then

$$1\setminus a\subseteq[\![\tau\le\rho]\!]\subseteq[\![\sigma\le\rho]\!]\subseteq[\![\sigma\wedge\rho=\sigma]\!]\subseteq[\![\hat u_{\sigma\wedge\rho}=\hat u_\sigma]\!]$$

for every $\rho\in A$, so

$$u'_\sigma\times\chi(1\setminus a)=\lim_{\rho\downarrow A}\hat u_{\sigma\wedge\rho}\times\chi(1\setminus a)=\lim_{\rho\downarrow A}\hat u_\sigma\times\chi(1\setminus a)=u_\sigma\times\chi(1\setminus a).$$

But this means that $[\![u'_\sigma\ne\hat u_\sigma]\!]\subseteq a$; as σ is arbitrary, $[\![\boldsymbol{u}'\restriction\hat{\mathcal S}\wedge\tau\ne\hat{\boldsymbol{u}}\restriction\mathcal S\wedge\tau]\!]\subseteq a$ has measure at most ϵ. As ϵ is arbitrary, $\hat{\boldsymbol{u}}\restriction\mathcal S\wedge\tau$ is an integrator (616P(b-iii)); as τ is arbitrary, $\hat{\boldsymbol{u}}$ is a local integrator. Now 616Q(b-i) tells us that $\boldsymbol{u}$ is a local integrator, therefore locally moderately oscillatory (616Ib once more).

(e)(i)(α) If $\boldsymbol{u}$ is an approximately local martingale and $\tau\in\mathcal S$, let $\epsilon>0$. Then there is a non-empty downwards-directed $A\subseteq\mathcal S$ such that $\bar\mu[\![\rho<\tau]\!]\le\epsilon$ for every $\rho\in A$ and $R_A(\hat{\boldsymbol{u}})$ is a martingale. We have

$$\bar\mu[\![\rho\wedge\tau<\tau]\!]=\bar\mu[\![\rho<\tau]\!]\le\epsilon$$

for every $\rho\in A$. Write $A\wedge\tau$ for $\{\rho\wedge\tau:\rho\in A\}$; then $A\wedge\tau$ is a non-empty downwards-directed subset of $\mathcal S\wedge\tau$ and $\bar\mu[\![\rho<\tau]\!]\le\epsilon$ for every $\rho\in A\wedge\tau$. Now

$$\begin{aligned}R_{A\wedge\tau}(\boldsymbol{u}\restriction\mathcal S\wedge\tau)&=\langle\lim_{\rho\downarrow A\wedge\tau}u_{\sigma\wedge\rho}\rangle_{\sigma\in\mathcal S\wedge\tau}=\langle\lim_{\rho\downarrow A}u_{\sigma\wedge\rho\wedge\tau}\rangle_{\sigma\in\mathcal S\wedge\tau}\\&=\langle\lim_{\rho\downarrow A}u_{\sigma\wedge\rho}\rangle_{\sigma\in\mathcal S\wedge\tau}=R_A(\boldsymbol{u})\restriction\mathcal S\wedge\tau\end{aligned}$$

is a martingale. As ϵ is arbitrary, $\boldsymbol{u}\restriction\mathcal S\wedge\tau$ is an approximately virtually local martingale (see the remarks in 623J).

(β) If $\boldsymbol{u}\restriction\mathcal S\wedge\tau$ is an approximately local martingale for every $\tau\in\mathcal S$, take $\tau\in\mathcal S$ and $\epsilon>0$. Then there is a non-empty downwards-directed set $A\subseteq\mathcal S\wedge\tau$ such that $\sup_{\rho\in A}\bar\mu[\![\rho<\tau]\!]\le\epsilon$ and $R_A(\boldsymbol{u}\restriction\mathcal S\wedge\tau)$ is a martingale. As τ and ϵ are arbitrary, $\boldsymbol{u}$ is an approximately local martingale.

(ii)(α) If $\boldsymbol{u}$ is a virtually local martingale and $\tau\in\mathcal S$, then $\hat{\boldsymbol{u}}$ is an approximately local martingale. By (i), $\hat{\boldsymbol{u}}\restriction\hat{\mathcal S}\wedge\tau$ is an approximately local martingale. But $\hat{\boldsymbol{u}}\restriction\hat{\mathcal S}\wedge\tau$ is the fully adapted extension of $\boldsymbol{u}\restriction\mathcal S\wedge\tau$ to the covered envelope of $\mathcal S\wedge\tau$, so $\boldsymbol{u}\restriction\mathcal S\wedge\tau$ is a virtually local martingale.

(β) If $\boldsymbol{u}\restriction\mathcal S\wedge\tau$ is a virtually local martingale for every $\tau\in\mathcal S$, that is, $\hat{\boldsymbol{u}}\restriction\hat{\mathcal S}\wedge\tau$ is an approximately local martingale for every $\tau\in\mathcal S$, take any $\sigma\in\hat{\mathcal S}$ and $\epsilon>0$. Let $\tau\in\mathcal S$ be such that $\bar\mu[\![\tau<\sigma]\!]\le\frac12\epsilon$ (611Mh). Then there is a non-empty downwards-directed set $A\subseteq\hat{\mathcal S}\wedge\tau$ such that $\sup_{\rho\in A}\bar\mu[\![\rho<\tau]\!]\le\frac12\epsilon$ and $R_A(\hat{\boldsymbol{u}}\restriction\hat{\mathcal S}\wedge\tau)$ is a martingale. By 623Cb, $R_{A\wedge\sigma\wedge\tau}(\hat{\boldsymbol{u}}\restriction\hat{\mathcal S}\wedge\tau)=R_{\{\sigma\wedge\tau\}}R_A(\hat{\boldsymbol{u}}\restriction\hat{\mathcal S}\wedge\tau)$ is a martingale and $R_{A\wedge\sigma\wedge\tau}(\hat{\boldsymbol{u}}\restriction\hat{\mathcal S}\wedge\sigma\wedge\tau)$ is a martingale; as $R_{A\wedge\sigma\wedge\tau}(\hat{\boldsymbol{u}}\restriction\hat{\mathcal S}\vee(\sigma\wedge\tau))$ is constant, $R_{A\wedge\sigma\wedge\tau}(\hat{\boldsymbol{u}})$ is a martingale (622E(b-ii) once more). Now

$$\bar\mu[\![\rho\wedge\sigma\wedge\tau<\sigma]\!]\le\bar\mu[\![\rho<\sigma\wedge\tau]\!]+\bar\mu[\![\tau<\sigma]\!]\le\epsilon$$

for every $\rho \in A$, that is, $\bar{\mu}[\![\rho < \sigma]\!] \le \epsilon$ for every $\rho \in A\wedge\sigma\wedge\tau$. As σ and ϵ are arbitrary, $\hat{\boldsymbol{u}}$ is an approximately local martingale and $\boldsymbol{u}$ is a virtually local martingale.

(f) Let $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ be a uniformly integrable approximately local martingale. Set $C = \{u_\sigma : \sigma \in \mathcal{S}\}$, so that C is uniformly integrable. Then its closure $\overline{C}$ (for the topology of convergence in measure) is also uniformly integrable (621B(c-ii)). Take $\tau \le \tau'$ in $\mathcal{S}$ and $\epsilon > 0$. Let $\delta > 0$ be such that $\|u\times\chi a\|_1 \le \epsilon$ whenever $u \in \overline{C}$ and $\bar{\mu}a \le \delta$ (621Ba). Then there is a downwards-directed set $A \subseteq \mathcal{S}$ such that $\sup_{\rho\in A}\bar{\mu}[\![\rho < \tau']\!] \le \delta$ and $R_A(\boldsymbol{u}) = \langle u'_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a martingale. For $\sigma \in \mathcal{S}$, we have $u'_\sigma = \lim_{\rho\downarrow A} u_{\sigma\wedge\rho} \in \overline{C}$ and if $\sigma \le \tau'$ then

$$a = [\![u'_\sigma \ne u_\sigma]\!] \subseteq \sup_{\rho\in A}[\![\sigma\wedge\rho \ne \sigma]\!] \subseteq \sup_{\rho\in A}[\![\rho < \tau']\!]$$

has measure at most δ, so

$$\|u'_\sigma - u_\sigma\|_1 = \|u'_\sigma \times \chi a - u_\sigma \times \chi a\|_1 \le 2\epsilon.$$

Consequently

$$\begin{aligned}\|P_\tau u_{\tau'} - u_\tau\|_1 &\le \|P_\tau u_{\tau'} - P_\tau u'_{\tau'}\|_1 + \|P_\tau u'_{\tau'} - u'_\tau\|_1 + \|u'_\tau - u_\tau\|_1\\ &\le \|u_{\tau'} - u'_{\tau'}\|_1 + 0 + 2\epsilon \le 4\epsilon.\end{aligned}$$

As ϵ is arbitrary, $P_\tau u_{\tau'} = u_\tau$; as τ and τ' are arbitrary, $\boldsymbol{u}$ is a martingale.

(g) By 615H, $\lim_{\sigma\downarrow\mathcal{S}} u_\sigma = \lim_{\sigma\downarrow\hat{\mathcal{S}}}\hat{u}_\sigma$. Because $\hat{\mathcal{S}}$ is non-empty. there is a non-empty downwards-directed $A \subseteq \hat{\mathcal{S}}$ such that $\langle u'_\sigma\rangle_{\sigma\in\hat{\mathcal{S}}} = R_A(\hat{\boldsymbol{u}})$ is a martingale. Take any $\tau \in \hat{\mathcal{S}}$. By 623B(c-i), $\lim_{\sigma\downarrow\hat{\mathcal{S}}} u'_\sigma$ is defined and equal to $\lim_{\sigma\downarrow\hat{\mathcal{S}}}\hat{u}_\sigma = \lim_{\sigma\downarrow\mathcal{S}} u_\sigma$; by 622Ed, this common value belongs to L^1.

(h)(i) To begin with, suppose that $\mathcal{S}$ is full. If $\tau' \in \mathcal{S}\vee\tau$ and $\epsilon > 0$, then there is a non-empty downwards-directed set $A \subseteq \mathcal{S}$ such that $\bar{\mu}[\![\rho < \tau']\!] \le \epsilon$ for every $\rho \in A$ and $R_A(\boldsymbol{u})$ is a martingale. Now $R_{\{\tau\}\wedge A}(\boldsymbol{u}) = R_{\{\tau\}}R_A(\boldsymbol{u})$ is a martingale, by 623Cc and 623D, so $R_{\{\tau\}\vee A}(\boldsymbol{u}) - R_{\{\tau\}}(\boldsymbol{u}) = R_A(\boldsymbol{u}) - R_{\{\tau\}\wedge A}(\boldsymbol{u})$ (623Ca) is a difference of martingales, therefor a martingale (622D(b-i)).

It follows at once that $R_{\{\tau\}\vee A}(\boldsymbol{u})\restriction\mathcal{S}\vee\tau - R_{\{\tau\}}(\boldsymbol{u})\restriction\mathcal{S}\vee\tau$ is a martingale. But as $\{\tau\}\vee A$ and $\{\tau\}$ are non-empty downwards-directed subsets of $\mathcal{S}\vee\tau$, we can speak of $R_{\{\tau\}\vee A}(\boldsymbol{u}\restriction\mathcal{S}\vee\tau)$ and $R_{\{\tau\}}(\boldsymbol{u}\restriction\mathcal{S}\vee\tau)$. Now $R_{\{\tau\}\vee A}(\boldsymbol{u}\restriction\mathcal{S}\vee\tau) = R_{\{\tau\}\vee A}(\boldsymbol{u})\restriction\mathcal{S}\vee\tau$ and

$$R_{\{\tau\}}(\boldsymbol{u}\restriction\mathcal{S}\vee\tau) = u_\tau\mathbf{1}\restriction\mathcal{S}\vee\tau = R_{\{\tau\}\vee A}(u_\tau\mathbf{1}\restriction\mathcal{S}\vee\tau).$$

So we see that $R_{\{\tau\}\vee A}((\boldsymbol{u} - u_\tau\mathbf{1})\restriction\mathcal{S}\vee\tau)$ is a martingale. And of course $\bar{\mu}[\![\rho < \tau']\!] \le \epsilon$ for every $\rho \in \{\tau\}\vee A$. As τ' and ϵ are arbitrary, $(\boldsymbol{u} - u_\tau\mathbf{1})\restriction\mathcal{S}\vee\tau$ is a virtually local martingale.

(ii) For the general case, given that $\boldsymbol{u} \in M_{\mathrm{vlm}}(\mathcal{S})$ and $\tau \in \mathcal{S}$, we know that $\hat{\boldsymbol{u}} \in M_{\mathrm{vlm}}(\hat{\mathcal{S}})$, so (i) tells us that $(\hat{\boldsymbol{u}} - \hat{u}_\tau\mathbf{1})\restriction\hat{\mathcal{S}}\vee\tau$ is a virtually local martingale; but this is just the fully adapted extension of $(\boldsymbol{u} - u_\tau\mathbf{1})\restriction\mathcal{S}\vee\tau$ to the covered envelope $\hat{\mathcal{S}}\vee\tau$ of $\mathcal{S}\vee\tau$ (611M(e-i)), so $(\boldsymbol{u} - u_\tau\mathbf{1})\restriction\mathcal{S}\vee\tau$ is a virtually local martingale.

623L Theorem Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$, and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a $\| \|_1$-bounded approximately local martingale. Write γ for $\sup_{\sigma\in\mathcal{S}}\|v_\sigma\|_1$.

(a) $\boldsymbol{v}$ is an integrator, therefore moderately oscillatory, and $\lim_{\sigma\uparrow\mathcal{S}} v_\sigma$ is defined.

(b) $\bar{v} = \sup_{\sigma\in\mathcal{S}}|v_\sigma|$ is defined in $L^0(\mathfrak{A})$, and $\theta(\bar{v}) \le 2\sqrt{\gamma}$.

proof (a) If $z \in Q_{\mathcal{S}}(d\boldsymbol{v})$, then $\theta(\delta z) \le \delta + 17\sqrt{\delta\gamma}$ for every $\delta > 0$. **P** Express z as $\sum_{i=0}^{n-1} u_i \times (v_{\tau_{i+1}} - v_{\tau_i})$ where $\tau_0 \le \ldots \le \tau_n$ in $\mathcal{S}$ and $u_{\tau_i} \in L^0(\mathfrak{A}_{\tau_i})$, $\|u_{\tau_i}\|_\infty \le 1$ for each $i \le n$. Then there is a non-empty, downwards-directed $A \subseteq \mathcal{S}$ such that $\sup_{\rho\in A}\bar{\mu}[\![\rho < \tau_n]\!] \le \delta$ and $\langle v'_\sigma\rangle_{\sigma\in\mathcal{S}} = R_A(\boldsymbol{v})$ is a martingale. Now $\|v'_\tau\|_1 \le \gamma$ for every $\tau \in \mathcal{S}$, just because the ball $\{z : \|z\|_1 \le \gamma\}$ is closed for the topology of convergence in measure (613Bc once more) and $v'_\tau = \lim_{\rho\downarrow A} v_{\tau\wedge\rho}$ belongs to $\overline{\{v_\sigma : \sigma \in \mathcal{S}\}}$.

Setting $z' = \sum_{i=0}^{n-1} u_i \times (v'_{\tau_{i+1}} - v'_{\tau_i})$, $[\![z' \ne z]\!] \subseteq \sup_{\rho\in A}[\![\rho < \tau_n]\!]$ has measure at most δ, so $\theta(\delta z) \le \theta(\delta z') + \delta$. But we see from 621Hf that

$$\theta(\delta z') \le \sqrt{\delta\gamma} + \bar{\mu}[\![\delta|z'| > \sqrt{\delta\gamma}]\!] \le \sqrt{\delta\gamma} + \frac{16\delta}{\sqrt{\delta\gamma}}\|v'_{\tau_n}\|_1 \le 17\sqrt{\delta\gamma}$$

and $\theta(\delta z) \le \delta + 17\sqrt{\delta\gamma}$. **Q**

As δ is arbitrary, $Q_{\mathcal{S}}(d\boldsymbol{v})$ is bounded and $\boldsymbol{v}$ is an integrator. Accordingly it is moderately oscillatory (616Ib, as always) and $\lim_{\sigma\uparrow\mathcal{S}} v_\sigma$ is defined (615Ga).

(b) If $B \subseteq \mathcal{S}$ is a finite set, then

$$\bar\mu([\![\sup_{\sigma\in B}|v_\sigma| > \beta]\!]) \le \frac{\gamma}{\beta}$$

for every $\beta > 0$. **P** If B is empty, this is trivial. Otherwise, let $\delta > 0$. Write I for the sublattice of $\mathcal{S}$ generated by B, and take $\tau_0 \le \ldots \le \tau_n$ linearly generating the I-cells (611L). Let $A \subseteq \mathcal{S}$ be a non-empty downwards-directed set such that $\sup_{\rho\in A}\bar\mu[\![\rho < \tau_n]\!] \le \delta$ and $\langle v'_\sigma\rangle_{\sigma\in\mathcal{S}} = R_A(\boldsymbol{v})$ is a martingale. Then $(v'_{\tau_0},\ldots,v'_{\tau_n})$ is a finite martingale adapted to $(\mathfrak{A}_{\tau_0},\ldots,\mathfrak{A}_{\tau_n})$, so

$$\bar\mu([\![\sup_{i\le n}|v'_{\tau_i}| > \beta]\!]) \le \frac{1}{\beta}\|v'_{\tau_n}\|_1$$

by 621E, while $\|v'_{\tau_n}\| \le \gamma$, as in (a) above. Again, $[\![v_{\tau_i} \ne v'_{\tau_i}]\!] \subseteq \sup_{\rho\in A}[\![\rho < \tau_n]\!]$ for each i, so $\bar\mu([\![\sup_{i\le n}|v_{\tau_i}| > \beta]\!]) \le \frac{\gamma}{\beta} + \delta$. But if we write v for $\sup_{i\le n}|v_{\tau_i}|$, then

$$[\![|v_\sigma| \le v]\!] \supseteq \sup_{i\le n}[\![v_\sigma = v_{\tau_i}]\!] \supseteq \sup_{i\le n}[\![\sigma = \tau_i]\!] = 1$$

for any $\sigma \in B$, by 611Ke. So $\sup_{\sigma\in B}|v_\sigma| \le v$ and $[\![\sup_{\sigma\in B}|v_\sigma| > \beta]\!] \subseteq [\![v > \gamma]\!]$ has measure at most $\frac{\gamma}{\beta} + \delta$. As δ is arbitrary, $\bar\mu[\![\sup_{\sigma\in B}|v_\sigma| > \beta]\!] \le \frac{\gamma}{\beta}$. **Q**

Accordingly $c_\gamma = \sup_{\sigma\in\mathcal{S}}[\![|v_\sigma| > \gamma]\!]$ has measure at most $\frac{\gamma}{\beta}$, by 321D. Since this tends to 0 as β increases to ∞, $\bar v = \sup_{\sigma\in\mathcal{S}}|v_\sigma|$ is defined in $L^0(\mathfrak{A})$, and

$$\bar\mu[\![\bar v > \beta]\!] = \bar\mu c_\gamma \le \frac{\gamma}{\beta}$$

for every $\beta > 0$ (364L(a-ii)). Consequently

$$\theta(\bar v) \le \sqrt{\gamma} + \bar\mu[\![\bar v > \sqrt{\gamma}]\!] \le 2\sqrt{\gamma}.$$

623M Doob's quadratic maximal inequality: Proposition If $\mathcal{S}$ is a non-empty sublattice of $\mathcal{T}$, $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ is an approximately local martingale, and $\gamma = \sup_{\sigma\in\mathcal{S}}\|v_\sigma\|_2$ is finite, then $\boldsymbol{v}$ is order-bounded and $\|\sup|\boldsymbol{v}|\|_2 \le 2\gamma$.

proof (a) It will simplify things if we note at once that as $\|\;\|_2$-bounded sets are uniformly integrable (621Be), $\boldsymbol{v}$ is a uniformly integrable approximately local martingale and is actually a martingale (623Kf).

(b)(i) Let us suppose to begin with that $\mathcal{S}$ is finite and totally ordered; let $\langle\tau_i\rangle_{i\le n}$ be its increasing enumeration. Write $\bar v$ for $\sup|\boldsymbol{v}|$ and v_n for v_{τ_n}; set $a_t = [\![\bar v > t]\!]$ for $t \ge 0$. Then $t\bar\mu a_t \le \mathbb{E}(|v_n| \times \chi a_t)$ for every t, by 621E.

(ii) We need to know that if $u \ge 0$ in $L^0(\mathfrak{A})$ then $\mathbb{E}(u \times \bar v) = \int_0^\infty \mathbb{E}(u \times \chi a_t)dt$, where $\int\ldots dt$ is integration with respect to Lebesgue measure. **P** If $u = \chi c$, then

$$\mathbb{E}(u \times \bar v) = \int_0^\infty \bar\mu[\![u \times \bar v > t]\!]dt$$

(by the definition of integration in $L^1(\mathfrak{A},\bar\mu)$, see 365A and 365Da)

$$= \int_0^\infty \bar\mu(c \cap a_t)dt = \int_0^\infty \mathbb{E}(u \times \chi a_t)dt.$$

Because $\mathbb{E}$ is a linear functional, $\mathbb{E}(u \times \bar v) = \int_0^\infty \mathbb{E}(u \times \chi a_t)dt$ for every u in $S(\mathfrak{A})$ as defined in §361. Generally, given $u \in L^0(\mathfrak{A})^+$, there is a non-decreasing sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in $S(\mathfrak{A})^+$ with supremum u, and now $\{u_n \times \bar v : w \in S(\mathfrak{A})^+, n \in \mathbb{N}\}$ is a non-decreasing sequence with supremum $u \times \bar v$ (353Pa[4]), so

[4]Formerly 353Oa.

$$\mathbb{E}(u\times\bar v)=\sup_{n\in\mathbb{N}}\mathbb{E}(u_n\times\bar v)=\sup_{n\in\mathbb{N}}\int_0^\infty\mathbb{E}(u_n\times\chi a_t)dt=\int_0^\infty\mathbb{E}(u\times\chi a_t)dt$$

because $\mathbb{E}(u\times\chi a_t)=\sup_{n\in\mathbb{N}}\mathbb{E}(u_n\times\chi a_t)$ for every t, so we can use B.Levi's theorem at the last step. **Q**

(iii) Note also that $v_{\tau_i}^2\le P_i(v_n^2)$ for every $i\le n$, where P_i is the conditional expectation associated with $\mathfrak{A}_{\tau_i}$, by Jensen's inequality (621Cd). So $\mathbb{E}(v_{\tau_i}^2)\le\mathbb{E}(v_n^2)$ is finite for every i.

(iv) We see now that

$$\|\bar v\|_2^2=\mathbb{E}(\bar v^2)=\int_0^\infty\bar\mu[\![\bar v^2>s]\!]ds=\int_0^\infty\bar\mu[\![\bar v>\sqrt{s}]\!]ds=\int_0^\infty 2t\bar\mu[\![\bar v>t]\!]dt$$
$$=2\int_0^\infty t\bar\mu a_t dt\le 2\int_0^\infty\mathbb{E}(|v_n|\times\chi a_t)dt=2\mathbb{E}(|v_n|\times\bar v)$$

(by (i) and (ii))

$$\le 2\|v_n\|_2\|\bar v\|_2$$

by Cauchy's inequality (244Eb). Since we know that

$$\|\bar v\|_2\le\sum_{i=0}^n\|v_{\tau_i}\|_2\le(n+1)\gamma$$

is finite, $\|\bar v\|_2\le 2\|v_n\|_2$.

(c) If $\mathcal{S}$ is any non-empty finite sublattice of $\mathcal{T}$, let $(\sigma_0,\dots,\sigma_n)$ linearly generate the $\mathcal{S}$-cells; then $\bar v=\sup_{i\le n}|v_{\sigma_i}|$ (612Dd) and we can apply (b) to see that $\|\bar v\|_2\le 2\|v_{\max\mathcal{S}}\|_2$. In general, setting $\bar v_I=\sup_{\sigma\in I}|v_\sigma|$ when $I\in\mathcal{I}(\mathcal{S})$, starting with $\bar v_\emptyset=0$, $\|\bar v_I\|_2\le 2\gamma$ for every I, while $\langle\bar v_I^2\rangle_{I\in\mathcal{I}(\mathcal{S})}$ is upwards-directed, so

$$\mathbb{E}(\bar v^2)=\mathbb{E}(\sup_{I\in\mathcal{I}(\mathcal{S})}\bar v_I^2)=\sup_{I\in\mathcal{I}(\mathcal{S})}\mathbb{E}(\bar v_I^2)$$

(365Df)

$$\le\sup_{I\in\mathcal{I}(\mathcal{S})}4\gamma^2;$$

taking square roots, we have the result in the general case.

Remark See 275Yd.

623N Theorem Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$ and $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ an approximately local martingale.

(a) $\boldsymbol{v}$ is a martingale iff $\boldsymbol{v}\restriction\mathcal{S}\wedge\tau$ is uniformly integrable for every $\tau\in\mathcal{S}$.

(b) The following are equiveridical:

(i) $\boldsymbol{v}$ is uniformly integrable;

(ii) there is a $z\in L^1$ such that $\boldsymbol{v}=\boldsymbol{P}z\restriction\mathcal{S}$;

(iii) $\{v_\sigma:\sigma\in\mathcal{S}\}$ is $\|\,\|_1$-bounded and $\|v_\uparrow\|_1\ge\sup_{\sigma\in\mathcal{S}}\|v_\sigma\|_1$, where $v_\uparrow=\lim_{\sigma\uparrow\mathcal{S}}v_\sigma$;

(iv) $\boldsymbol{v}$ is a martingale and the limit $1\text{lim}_{\sigma\uparrow\mathcal{S}}v_\sigma$ is defined in L^1.

proof (a)(i) If $\boldsymbol{v}$ is a martingale and $\tau\in\mathcal{S}$, then $\boldsymbol{v}\restriction\mathcal{S}\wedge\tau=\boldsymbol{P}v_\tau\restriction\mathcal{S}\wedge\tau$ is uniformly integrable by 622Fa.

(ii) Now suppose that $\boldsymbol{v}\restriction\mathcal{S}\wedge\tau$ is uniformly integrable for every $\tau\in\mathcal{S}$. If $\sigma\le\tau$ in $\mathcal{S}$, $\boldsymbol{v}\restriction\mathcal{S}\wedge\tau$ is a uniformly integrable approximately local martingale (623Ke) so is a martingale (623Kf) and $v_\sigma=P_\sigma v_\tau$. As σ and τ are arbitrary, $\boldsymbol{v}$ is a martingale.

(b)(i)⇒(iv) If (i) is true, then (a) tells us that $\boldsymbol{v}$ is a martingale. Of course $A=\{v_\sigma:\sigma\in\mathcal{S}\}$ is $\|\,\|_1$-bounded, so 623La tells us that $z=\lim_{\sigma\uparrow\mathcal{S}}v_\sigma$ is defined and belongs to L^1. Now $A\cup\{z\}$ is uniformly integrable, so the topology of convergence in measure and the topology defined by $\|\,\|_1$ agree on $A\cup\{z\}$, and $z=1\text{lim}_{\sigma\uparrow\mathcal{S}}v_\sigma$. Thus (iv) is true.

(iv)⇒(iii) Suppose that (iv) is true. Observe first that

$$\|v_\sigma\|_1 = \|P_\sigma v_\tau\|_1 \le \|v_\tau\|_1$$

whenever $\sigma \le \tau$ in $\mathcal{S}$. Now we are supposing that $z = 1\mathrm{lim}_{\sigma\uparrow\mathcal{S}}\, v_\sigma$. Because the embedding $L^1 \subsetneqq L^0$ is continuous (613B(d-i)), z is also $\lim_{\sigma\uparrow\mathcal{S}} v_\sigma$. Moreover,

$$\|z\|_1 = 1\mathrm{lim}_{\sigma\uparrow\mathcal{S}} \|v_\sigma\|_1 = \sup_{\sigma\in\mathcal{S}} \|v_\sigma\|_1$$

because $\sigma \mapsto \|v_\sigma\|_1 : \mathcal{S} \to \mathbb{R}$ is non-decreasing. So (iii) is true.

(iii)⇒(ii) Suppose that (iii) is true. Set $\gamma = \sup_{\sigma\in\mathcal{S}} \|v_\sigma\|_1$. By 623La, the limit $v_\uparrow$ is defined; we are supposing that $\|v_\uparrow\|_1 \ge \gamma$. In fact we must have equality, because $\| \,\|_1$-balls are closed; in particular, $v_\uparrow \in L^1_{\bar\mu}$.

($\boldsymbol{\alpha}$) $\boldsymbol{v}$ is a martingale. **P** Take $\sigma_0 \le \sigma_1$ in $\mathcal{S}$, and $\epsilon > 0$. Since $v_\uparrow \in \overline{\{v_\tau : \tau \in \mathcal{S} \vee \sigma_1\}}$, $\|v_\uparrow\|_1 \le \sup_{\tau\in\mathcal{S}\vee\sigma_1} \|v_\tau\|_1$ and there is a $\tau \in \mathcal{S}$ such that $\sigma_1 \le \tau$ and $\|v_\tau\|_1 \ge \|v_\uparrow\|_1 - \epsilon \ge \gamma - \epsilon$.

Let $\delta > 0$ be such that $\mathbb{E}(\|v_\sigma \times \chi a\|_1) \le \epsilon$ whenever $\sigma \in \{\sigma_0, \sigma_1, \tau\}$ and $\bar\mu a \le \delta$. Let $A \subseteq \mathcal{S}$ be a non-empty downwards-directed set such that $\sup_{\rho\in A} \bar\mu[\![\rho < \tau]\!] \le \delta$ and $\langle v'_\sigma\rangle_{\sigma\in\mathcal{S}} = R_A(\boldsymbol{v})$ is a martingale. Set $a = \sup_{\rho\in A} [\![\rho < \tau]\!]$, so that $\bar\mu a \le \delta$. Since $v'_\tau = \lim_{\rho\downarrow A} v_{\tau\wedge\rho}$,

$$\|v'_\tau\|_1 \le \sup_{\rho\in A} \|v_{\tau\wedge\rho}\|_1 \le \gamma.$$

On the other hand, $1 \setminus a \subseteq [\![v'_\tau = v_\tau]\!]$, so

$$\mathbb{E}(|v'_\tau \times \chi(1\setminus a)|) = \mathbb{E}(|v_\tau \times \chi(1\setminus a)|) = \|v_\tau\|_1 - \mathbb{E}(|v_\tau \times \chi a|) \ge \gamma - 2\epsilon.$$

So $\mathbb{E}(|v'_\tau \times \chi a|) \le 2\epsilon$. Next, for i either 0 or 1,

$$a_i = [\![v'_{\sigma_i} \ne v_{\sigma_i}]\!] \subseteq a, \quad \bar\mu a_i \le \delta$$

and $a_i \in \Sigma_{\sigma_i}$. So

$$\begin{aligned}\mathbb{E}(|v'_{\sigma_i}| \times \chi a_i) &= \mathbb{E}(|P_{\sigma_i} v'_\tau| \times \chi a_i) \le \mathbb{E}(P_{\sigma_i}|v'_\tau| \times \chi a_i)\\ &= \mathbb{E}(P_{\sigma_i}(|v'_\tau| \times \chi a_i)) = \mathbb{E}(|v'_\tau| \times \chi a_i) \le 2\epsilon,\end{aligned}$$

while also $\mathbb{E}(|v_{\sigma_i}| \times \chi a_i) \le \epsilon$, so $\|v_{\sigma_i} - v'_{\sigma_i}\|_1 \le 3\epsilon$.

Now we see that

$$\|v_{\sigma_0} - P_{\sigma_0} v_{\sigma_1}\|_1 \le 6\epsilon + \|v'_{\sigma_0} - P_{\sigma_0} v'_{\sigma_1}\|_1 = 6\epsilon.$$

As σ_0, σ_1 and ϵ are arbitrary, $\boldsymbol{v}$ is a martingale. **Q**

($\boldsymbol{\beta}$) Now note that $v_\uparrow = 1\mathrm{lim}_{\tau\uparrow\mathcal{S}}\, v_\tau$. **P** Let $\epsilon > 0$. By 613D(b-iv), there is a $\delta > 0$ such that $\|z - v_\uparrow\|_1 \le \epsilon$ whenever $\|z\|_1 \le \|v_\uparrow\|_1$ and $\theta(z - v_\uparrow) \le \delta$. But there is a $\tau \in \mathcal{S}$ such that $\theta(v_\sigma - v_\uparrow) \le \delta$ whenever $\sigma \in \mathcal{S} \vee \tau$, so $\|v_\sigma - v_\uparrow\|_1 \le \epsilon$ for every $\sigma \in \mathcal{S} \vee \tau$. **Q** But now we see that

$$v_\sigma = 1\mathrm{lim}_{\tau\uparrow\mathcal{S}}\, P_\sigma v_\tau = P_\sigma(1\mathrm{lim}_{\tau\uparrow\mathcal{S}}\, v_\tau) = P_\sigma v_\uparrow$$

for every $\sigma \in \mathcal{S}$. So (ii) is true.

(ii)⇒(i) As in (a-i), this is immediate from 622Fa.

623O Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a locally moderately oscillatory process and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a virtually local martingale. Then $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is a virtually local martingale.

proof It is worth noting straight away that because $\boldsymbol{u}$ is locally moderately oscillatory and $\boldsymbol{v}$ is a local integrator (623Kd), $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is certainly defined everywhere on $\mathcal{S}$. If $A \subseteq \mathcal{S}$ is non-empty and downwards-directed, $R_A : M_{\mathrm{lmo}}(\mathcal{S}) \to M_{\mathrm{lmo}}(\mathcal{S})$ will be the corresponding operator as described in 623B.

part A Suppose for the time being that $\mathcal{S}$ is full and has a greatest element, $\boldsymbol{v}$ is a martingale, and $\|\sup|\boldsymbol{u}|\|_\infty \le 1$. Then $\boldsymbol{v} = \boldsymbol{P}v_{\max\mathcal{S}}\restriction\mathcal{S}$ is an integrator and is order-bounded.

(a) Let $\epsilon > 0$. Consider the $\| \,\|_1$-bounded martingale $\tilde{\boldsymbol{v}} = \langle\tilde v_\sigma\rangle_{\sigma\in\mathcal{S}}$ where $\tilde v_\sigma = P_\sigma|v_{\max\mathcal{S}}|$ for $\sigma \in \mathcal{S}$. This is order-bounded (622G); let $M \ge 0$ be such that $\bar\mu a \le \epsilon$ where $a = [\![\sup|\tilde{\boldsymbol{v}}| \ge M]\!]$. Set

$$A = \{\rho : \rho \in \mathcal{S},\ [\![\rho < \max\mathcal{S}]\!] \subseteq [\![|\tilde v_\rho| \ge M]\!]\}$$

as in 623I. Note that

$$[\![\boldsymbol{w} \neq R_A(\boldsymbol{w})]\!] \subseteq \sup_{\rho\in A} [\![\rho < \max\mathcal{S}]\!] \subseteq a$$

for every $\boldsymbol{w} \in M_{\mathrm{lmo}}(\mathcal{S})$. Express $\boldsymbol{v}' = R_A(\boldsymbol{v})$ as $\langle v'_\sigma\rangle_{\sigma\in\mathcal{S}}$ and $\tilde{\boldsymbol{v}}' = R_A(\tilde{\boldsymbol{v}})$ as $\langle \tilde{v}'_\sigma\rangle_{\sigma\in\mathcal{S}}$.

(b) Now $ii_{\boldsymbol{v}'}(\boldsymbol{u})$ is a martingale.

P(i) Let $\delta > 0$. Then there is an $\eta > 0$ such that $(2+\frac{M}{\delta})\eta + \sqrt{M\eta} \le \delta$. Let $M' \ge M$ be such that $\|(|v_{\max\mathcal{S}}| - M'\chi 1)^+\|_1 \le \eta$. Set $w = \mathrm{med}(-M'\chi 1, v_{\max\mathcal{S}}, M'\chi 1)$, $w_1 = (v_{\max\mathcal{S}} - M'\chi 1)^+$ and $w_2 = (-v'_{\max\mathcal{S}} - M'\chi 1)^+$; then $v_{\max\mathcal{S}} = w + w_1 - w_2$, w is square-integrable, $w_1 \ge 0$, $w_2 \ge 0$, $\|w_1\|_1 \le \eta$ and $\|w_2\|_1 \le \eta$. Consider the martingales $\boldsymbol{w} = \langle P_\sigma w\rangle_{\sigma\in\mathcal{S}}$, $\boldsymbol{w}_1 = \langle P_\sigma w_1\rangle_{\sigma\in\mathcal{S}}$, $\boldsymbol{w}_2 = \langle P_\sigma w_2\rangle_{\sigma\in\mathcal{S}}$ and the associated martingales $\boldsymbol{w}' = R_A(\boldsymbol{w}) = \langle w'_\sigma\rangle_{\sigma\in\mathcal{S}}$, $\boldsymbol{w}'_1 = R_A(\boldsymbol{w}_1) = \langle w'_{1\sigma}\rangle_{\sigma\in\mathcal{S}}$ and $\boldsymbol{w}'_2 = R_A(\boldsymbol{w}_2) = \langle w'_{2\sigma}\rangle_{\sigma\in\mathcal{S}}$.

(ii) We have $\|P_\sigma w\|_2 \le \|w\|_2 \le M'$ for every σ (244M, 366H(b-iii)), so $\|w'_\sigma\|_2 = \|\lim_{\rho\downarrow A} P_{\sigma\wedge\rho} w\|_2 \le M'$ for every $\sigma \in \mathcal{S}$ (613Bc). Thus $\boldsymbol{w}'$ is a $\|\;\|_2$-bounded martingale, while we are supposing that $\boldsymbol{u}$ is moderately oscillatory, so $ii_{\boldsymbol{w}'}(\boldsymbol{u})$ is a martingale (622Q).

(iii) Observe that

$$\begin{aligned}0 \le w'_{1\sigma} &= \lim_{\rho\downarrow A} P_{\sigma\wedge\rho} w_1 \le \lim_{\rho\downarrow A} P_{\sigma\wedge\rho}|v_{\max\mathcal{S}}| \\ &= \lim_{\rho\downarrow A} \tilde{v}_{\sigma\wedge\rho} = \tilde{v}'_\sigma\end{aligned}$$

for every $\sigma \in \mathcal{S}$. Now $R_A(\boldsymbol{w}'_1) = \boldsymbol{w}'_1$ (623Cc), so if $\sigma \le \tau$ in $\mathcal{S}$,

$$[\![w'_{1\sigma} > M]\!] \subseteq [\![\tilde{v}'_\sigma > M]\!] \subseteq [\![w'_{1\sigma} = w'_{1\tau}]\!]$$

by 623I(b-i). It follows that if $\tau_0 \le \ldots \le \tau_n$ in $\mathcal{S}$, then $(w'_{1\tau_0}, \ldots, w'_{1\tau_n})$ is a non-negative martingale and

$$[\![w'_{1\tau_i} > M]\!] \subseteq [\![w'_{1\tau_i} = w'_{1\tau_j}]\!] \cap [\![w'_{1\tau_i} = w'_{1\tau_n}]\!] \subseteq [\![w'_{1\tau_j} = w'_{1\tau_n}]\!]$$

whenever $i \le j \le n$. So $(\mathfrak{A}_{\tau_0}, \ldots, \mathfrak{A}_{\tau_n})$, $(w'_{1\tau_0}, \ldots, w'_{1\tau_n})$ and $(u_{\tau_0}, \ldots, u_{\tau_n})$ satisfy the conditions of 621I. Moreover,

$$\|w'_{1\tau_n}\|_1 \le \sup_{\sigma\in\mathcal{S}} \|P_\sigma w_1\|_1$$

(623B(c-ii))

$$\le \|w_1\|_1 \le \eta.$$

So 621I tells us that

$$\begin{aligned}\|\sum_{i=0}^{n-1} u_{\tau_i} \times (w'_{1\tau_{i+1}} - w'_{1\tau_i})\|_1 &\le \delta + (2+\frac{M}{\delta})\|w'_{1\tau_n}\|_1 + \sqrt{M\|w'_{1\tau_n}\|_1} \\ &\le \delta + (2+\frac{M}{\delta})\eta + \sqrt{M\eta} \le 2\delta\end{aligned}$$

by the choice of η.

Re-expressing this in the standard form I am using for Riemann sums, we have

$$\|S_I(\boldsymbol{u}, d\boldsymbol{w}'_1)\|_1 \le 2\delta$$

for every finite sublattice I of $\mathcal{S}$. Again because $\|\;\|_1$-balls are closed for the topology of convergence in measure, it follows that $\|\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\boldsymbol{w}'_1\|_1 \le 2\delta$ for every $\tau \in \mathcal{S}$.

(iv) Similarly. $\|\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\boldsymbol{w}'_2\|_1 \le 2\delta$ for every $\tau \in \mathcal{S}$. But now recall that $v_{\max\mathcal{S}} = w + w_1 - w_2$ so $\boldsymbol{P}v_{\max\mathcal{S}} = \boldsymbol{P}w + \boldsymbol{P}w_1 - \boldsymbol{P}w_2$, $\boldsymbol{v} = \boldsymbol{w} + \boldsymbol{w}_1 - \boldsymbol{w}_2$, $\boldsymbol{v}' = \boldsymbol{w}' + \boldsymbol{w}'_1 - \boldsymbol{w}'_2$ and

$$\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\boldsymbol{v}' = \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\boldsymbol{w}' + \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\boldsymbol{w}'_1 - \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\boldsymbol{w}'_2$$

for every $\tau \in \mathcal{S}$. So

$$\|\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\,d\boldsymbol{v}'-\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\,d\boldsymbol{w}'\|_1\le 4\delta$$

for every $\tau\in\mathcal{S}$. Since $ii_{\boldsymbol{w}'}(\boldsymbol{u})$ is a martingale, by (ii) above, and δ is arbitrary, $ii_{\boldsymbol{v}'}(\boldsymbol{u})$ is a martingale (622Ec). **Q**

(c) Recall now that

$$ii_{\boldsymbol{v}'}(\boldsymbol{u})=ii_{R_A(\boldsymbol{v})}(\boldsymbol{u})=R_A(ii_{\boldsymbol{v}}(\boldsymbol{u}))$$

by 623G. So $R_A(ii_{\boldsymbol{v}}(\boldsymbol{u}))$ is a martingale. Since ϵ was arbitrary, $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is a virtually local martingale (see the remarks in 623J).

part B I set out to strip away the extra hypotheses demanded in part A.

(a) Of course it will be enough to suppose that $\mathcal{S}$ is full and has a greatest element, $\boldsymbol{v}$ is a martingale and $\|\sup|\boldsymbol{u}|\|_\infty$ is finite, since then we can apply (A) to a non-zero multiple of $\boldsymbol{u}$.

(b) Now suppose that $\mathcal{S}$ is full and has a greatest element and that $\boldsymbol{v}$ is a martingale. In this case, $\boldsymbol{u}$ is still moderately oscillatory and order-bounded. Let $\epsilon>0$, and take $M\ge 0$ such that $\bar\mu[\![\sup|\boldsymbol{u}|\ge M]\!]\le\frac{1}{2}\epsilon$. Set

$$A=\{\rho:\rho\in\mathcal{S},\,[\![\rho<\max\mathcal{S}]\!]\subseteq[\![|u_\rho|\ge M]\!]\}$$

as in 623I. Then A is non-empty and downwards-directed and $\sup_{\rho\in A}\bar\mu[\![\rho<\max\mathcal{S}]\!]\le\frac{1}{2}\epsilon$. Set $\boldsymbol{u}'=\langle u'_\sigma\rangle_{\sigma\in\mathcal{S}}=R_A(\boldsymbol{u})$, $\boldsymbol{v}'=\langle v'_\sigma\rangle_{\sigma\in\mathcal{S}}=R_A(\boldsymbol{v})$ and $\tilde{\boldsymbol{u}}=\langle\tilde u_\sigma\rangle_{\sigma\in\mathcal{S}}=\mathrm{med}(-M\mathbf{1},\boldsymbol{u}',M\mathbf{1})$. Then $\boldsymbol{u}'$ and $\tilde{\boldsymbol{u}}$ are moderately oscillatory, $\boldsymbol{v}'$ is an integrator (623D) and $\|\sup|\tilde{\boldsymbol{u}}|\|_\infty\le M$ is finite. If $\sigma\le\tau$ in $\mathcal{S}$, then

$$[\![\tilde u_\sigma\ne u'_\sigma]\!]\subseteq[\![|u'_\sigma|>M]\!]\subseteq[\![v_\sigma=v_\tau]\!]$$

(623I(b-i) again), so $\tilde u_\sigma\times(v'_\tau-v'_\sigma)=u'_\sigma\times(v'_\tau-v'_\sigma)$. Accordingly $S_I(\tilde{\boldsymbol{u}},d\boldsymbol{v}')=S_I(\boldsymbol{u}',d\boldsymbol{v}')$ for every $I\in\mathcal{I}(\mathcal{S})$ and $ii_{\boldsymbol{v}'}(\boldsymbol{u}')=ii_{\boldsymbol{v}'}(\tilde{\boldsymbol{u}})$. But $\boldsymbol{v}'$ is a martingale (623E again), so (a) just above tells us that $ii_{\boldsymbol{v}'}(\tilde{\boldsymbol{u}})$ is a virtually local martingale and accordingly $ii_{\boldsymbol{v}'}(\boldsymbol{u}')$ is a virtually local martingale.

Let $B\subseteq\mathcal{S}$ be a non-empty downwards-directed set such that $\sup_{\rho\in B}\bar\mu[\![\rho<\max\mathcal{S}]\!]\le\frac{1}{2}\epsilon$ and $R_B(ii_{\boldsymbol{v}'}(\boldsymbol{u}'))$ is a martingale. Setting $\boldsymbol{v}''=R_B(\boldsymbol{v}')$ and $\boldsymbol{u}''=R_B(\boldsymbol{u}')$, 623G tells us that $ii_{\boldsymbol{v}''}(\boldsymbol{u}'')=R_B(ii_{\boldsymbol{v}'}(\boldsymbol{u}'))$ is a martingale. But $\boldsymbol{v}''=R_BR_A(\boldsymbol{v})=R_{A\wedge B}(\boldsymbol{v})$ (623Cb) and similarly $\boldsymbol{u}''=R_{A\wedge B}(\boldsymbol{u})$. Applying 623G again, we see that

$$R_{A\wedge B}(ii_{\boldsymbol{v}}(\boldsymbol{u}))=ii_{R_{A\wedge B}(\boldsymbol{v})}(R_{A\wedge B}(\boldsymbol{u}))=ii_{\boldsymbol{v}''}(\boldsymbol{u}'')$$

is a martingale. And for any $\sigma\in\mathcal{S}$ we have

$$\begin{aligned}\sup_{\rho\in A\wedge B}\bar\mu[\![\rho<\sigma]\!]\le\sup_{\rho\in A\wedge B}\bar\mu[\![\rho<\max\mathcal{S}]\!]&=\sup_{\substack{\rho\in A\\ \rho'\in B}}\bar\mu[\![\rho\wedge\rho'<\max\mathcal{S}]\!]\\&=\sup_{\substack{\rho\in A\\ \rho'\in B}}\bar\mu([\![\rho<\max\mathcal{S}]\!]\cup[\![\rho'<\max\mathcal{S}]\!])\\&\le\sup_{\substack{\rho\in A\\ \rho'\in B}}(\bar\mu[\![\rho<\max\mathcal{S}]\!]+\bar\mu[\![\rho'<\max\mathcal{S}]\!])\le\epsilon.\end{aligned}$$

As ϵ is arbitrary, $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is a virtually local martingale.

(c) Thirdly, consider the case in which $\mathcal{S}$ is full and has a greatest element but $\boldsymbol{v}$ is only a virtually local martingale. Take $\epsilon>0$. Then we have a non-empty downwards-directed set $A\subseteq\mathcal{S}$ such that $\bar\mu[\![\rho<\max\mathcal{S}]\!]\le\frac{1}{2}\epsilon$ for every $\rho\in A$ and $R_A(\boldsymbol{v})$ is a martingale. By (b), $ii_{R_A(\boldsymbol{v})}(\boldsymbol{u})$ is a virtually local martingale. Let $B\subseteq\mathcal{S}$ be a non-empty downwards-directed set such that $\sup_{\rho\in B}\bar\mu[\![\rho<\max\mathcal{S}]\!]\le\frac{1}{2}\epsilon$ and $R_B(ii_{R_A(\boldsymbol{v})}(\boldsymbol{u}))$ is a martingale. By 623G and 623Cb once more,

$$R_{A\wedge B}(ii_{\boldsymbol{v}}(\boldsymbol{u}))=ii_{R_{A\wedge B}(\boldsymbol{v})}(\boldsymbol{u})=ii_{R_BR_A(\boldsymbol{v})}(\boldsymbol{u})=R_B(ii_{R_A(\boldsymbol{v})}(\boldsymbol{u}))$$

is a martingale. As in (b), we have $\sup_{\rho\in A\wedge B}\bar\mu[\![\rho<\max\mathcal{S}]\!]\le\epsilon$. As ϵ is arbitrary, $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is a virtually local martingale.

(d) If we suppose only that $\mathcal{S}$ is full, then for each $\tau\in\mathcal{S}$ we have a moderately oscillatory process $\boldsymbol{u}{\restriction}\mathcal{S}\wedge\tau$ and a virtually local martingale $\boldsymbol{v}{\restriction}\mathcal{S}\wedge\tau$ (623Ke again), while $\mathcal{S}\wedge\tau$ is full and has a greatest element. By (c), $ii_{\boldsymbol{v}\restriction\mathcal{S}\wedge\tau}(\boldsymbol{u}\restriction\mathcal{S}\wedge\tau)$ is a virtually local martingale. But this is just $ii_{\boldsymbol{v}}(\boldsymbol{u})\restriction\mathcal{S}\wedge\tau$. So $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is a virtually local martingale, by 623Ke in the other direction.

(e) Finally, in the general case, the covered envelope $\hat{\mathcal{S}}$ of $\mathcal{S}$ is full, the fully adapted extension $\hat{\boldsymbol{u}}$ of $\boldsymbol{u}$ to $\hat{\mathcal{S}}$ is locally moderately oscillatory (615F(b-v)) and the fully adapted extension $\hat{\boldsymbol{v}}$ of $\boldsymbol{v}$ is a virtually local martingale (623J). So (d) tells us that $ii_{\hat{\boldsymbol{v}}}(\hat{\boldsymbol{u}})$ is a virtually local martingale. But $ii_{\hat{\boldsymbol{v}}}(\hat{\boldsymbol{u}})$ is the fully adapted extension of $ii_{\boldsymbol{v}}(\boldsymbol{u})$ (apply 613Uc to $\mathcal{S}\wedge\tau$ for $\tau\in\mathcal{S}$), so $ii_{\boldsymbol{v}}(\boldsymbol{u})$ itself is a virtually local martingale (623J again).

This completes the proof.

623X Basic exercises (a) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $A\subseteq\mathcal{S}$ a non-empty downwards-directed set and $\boldsymbol{u}$ a locally moderately oscillatory process with domain $\mathcal{S}$. (i) Show that if $\boldsymbol{u}$ is order-bounded then $R_A(\boldsymbol{u})$ is order-bounded and $\sup|R_A(\boldsymbol{u})|\le\sup|\boldsymbol{u}|$, (ii) Show that if $\boldsymbol{u}$ is locally order-bounded then $R_A(\boldsymbol{u})$ is locally order-bounded, (iii) Show that if $\boldsymbol{u}$ is of bounded variation, then $R_A(\boldsymbol{u})$ is of bounded variation and $\int_{\mathcal{S}}|dR_A(\boldsymbol{u})|\le\int_{\mathcal{S}}|d\boldsymbol{u}|$. (iv) Show that if $\boldsymbol{u}$ is locally of bounded variation, then $R_A(\boldsymbol{u})$ is locally of bounded variation. (v) Show that if $1\le p\le\infty$ and $\boldsymbol{u}$ is $\|\,\|_p$-bounded (622Ca), then $R_A(\boldsymbol{u})$ is $\|\,\|_p$-bounded.

(b) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $A\subseteq\mathcal{S}$ a non-empty downwards-directed set. Let $R_A:M_{\mathrm{lmo}}(\mathcal{S})\to M_{\mathrm{lmo}}(\mathcal{S})$ be the operator described in 623B. Show that if $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a local martingale then $R_A(\boldsymbol{u})$ is a local martingale.

(c) Show that Brownian motion $\boldsymbol{w}$ on $\mathcal{T}_f$, as described in 612T, is not an L^1-process, and that there is a $\tau\in\mathcal{T}_f$ such that $\boldsymbol{w}\restriction\mathcal{T}_f\vee\tau$ is not a virtually local martingale.

(e) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a virtually local martingale. Show that $\boldsymbol{u}\restriction\mathcal{S}'$ is a virtually local martingale for any ideal $\mathcal{S}'$ of $\mathcal{S}$.

(f) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a virtually local martingale. Show that $u_\sigma=u_\tau$ whenever σ, $\tau\in\mathcal{S}$ and $\mathfrak{A}_\sigma=\mathfrak{A}_\tau$.

(g) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a fully adapted process. Show that the following are equiveridical: (α) $\boldsymbol{u}$ is a virtually local martingale and $u_\tau\in L^1$; (β) $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ and $\boldsymbol{u}\restriction\mathcal{S}\vee\tau$ are virtually local martingales.

(h) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. Suppose that $\boldsymbol{u}\in M_{\mathrm{fa}}(\mathcal{S})$ and for every $\tau\in\mathcal{S}$ and $\epsilon>0$ there is a $\sigma\in\mathcal{S}$ such that $\bar\mu[\![\sigma<\tau]\!]\le\epsilon$ and $\boldsymbol{u}\restriction\mathcal{S}\wedge\sigma$ is a virtually local martingale. Show that $\boldsymbol{u}$ is a virtually local martingale.

623Y Further exercises (a) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a virtually local martingale. Suppose that $A\subseteq\mathcal{S}$ is non-empty and downwards-directed, $\inf A\in\mathcal{S}$ and $\mathfrak{A}_{\inf A}=\bigcap_{\sigma\in A}\mathfrak{A}_\sigma$. Show that $u_{\inf A}=\lim_{\sigma\downarrow A}u_\sigma$.

623 Notes and comments At the price of a substantial effort, we have a theorem on indefinite integrals with respect to virtually local martingales which matches the form of the corresponding results on processes of bounded variation (614T), integrators (616J) and jump-free integrators (618Q). But 623O is especially important because martingales, and in particular Brownian motion, are central to any theory of stochastic integration. And its difficulty lies largely in the fact that an indefinite integral with respect to a martingale is *not* necessarily a martingale (631Ya). In the framework I have settled on for this volume so far, 'virtually local martingale' is the best I can do. I ought to tell you that in the more conventional framework of right-continuous filtrations (§632), virtually local martingales on ideals of $\mathcal{T}$ are actually local martingales (632Ib), and that the corresponding special case of 623O is a good deal easier to prove, while being a sufficient foundation for the standard theory.

Observe that 623La covers Doob's martingale convergence theorem (275G, 367Ja). If we have a $\|\,\|_1$-bounded virtually local martingale $\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$, it is moderately oscillatory, so $\lim_{\sigma\uparrow\mathcal{S}}v_\sigma=v_\uparrow$ is defined, and

$\inf_{\tau\in\mathcal{S}}\sup_{\sigma\in\mathcal{S}\vee\tau}|v_\sigma - v_\uparrow| = 0$ (615Ga), corresponding to order*-convergence in L^0 as defined in 367A, that is, to almost-everywhere pointwise convergence of sequences of measurable functions (367F).

The main theorems of this section (623L-623O) involve awkward shifts between 'approximately local' and 'virtually local', which will have echoes later. As long as we restrict ourselves to finitely full lattices, there is no difference (623K(b-iii)). I am reluctant to impose such a restriction generally because the Riemann-sum integral does not insist on it (613T), and many of the ideas of this volume can be effectively expressed in terms of lattices of constant stopping times. Indeed applications often begin with processes defined on such a lattice, as in 612H. When we come to the structure theory of integrators, the concept of 'virtually local' martingale will provide a particularly striking formulation of the main theorem (627Q).

Version of 21.2.23/22.8.23

624 Quadratic variation

We are at last ready to determine the quadratic variation of Brownian motion (624F). I take the opportunity to tidy up some simple consequences of results in §§617 and 623 (624B-624E), and to give useful facts about L^2-martingales (624G-624I).

624A Notation $(\mathfrak{A},\bar\mu,T,\langle\mathfrak{A}_t\rangle_{t\in T},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ will be a stochastic integration structure. I write L^1, L^2, L^∞ for $L^1(\mathfrak{A},\bar\mu)$, $L^2(\mathfrak{A},\bar\mu)$ and $L^\infty(\mathfrak{A})$ respectively. $\mathbb{E}$ will be the integral on L^1 and θ will be the F-norm defining the topology of convergence in measure on $L^0(\mathfrak{A})$, as in 613B. If $\boldsymbol{v}$ and $\boldsymbol{w}$ are local integrators with the same domain, $[\boldsymbol{v}\overset{*}{|}\boldsymbol{w}]$ will be their covariation (617H); $\boldsymbol{v}^* = [\boldsymbol{v}\overset{*}{|}\boldsymbol{v}]$ is the quadratic variation of $\boldsymbol{v}$.

624B Theorem Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$, $\boldsymbol{w} = \langle w_\sigma\rangle_{\sigma\in\mathcal{S}}$ virtually local martingales such that $v_\downarrow\times w_\downarrow\in L^1$ where $v_\downarrow = \lim_{\sigma\downarrow\mathcal{S}} v_\sigma$ and $w_\downarrow = \lim_{\sigma\downarrow\mathcal{S}} w_\sigma$. Then $\boldsymbol{v}\times\boldsymbol{w} - [\boldsymbol{v}\overset{*}{|}\boldsymbol{w}]$ is a virtually local martingale.

proof We know that $\boldsymbol{v}$ and $\boldsymbol{w}$ are local integrators (623Kd), so the covariance $[\boldsymbol{v}\overset{*}{|}\boldsymbol{w}]$ is defined everywhere on $\mathcal{S}$ (617Hb). We have

$$\boldsymbol{v}\times\boldsymbol{w} - [\boldsymbol{v}\overset{*}{|}\boldsymbol{w}] = ii_{\boldsymbol{w}}(\boldsymbol{v}) + ii_{\boldsymbol{v}}(\boldsymbol{w}) + z\mathbf{1}$$

where $z = v_\downarrow\times w_\downarrow \in L^0(\bigcap_{\sigma\in\mathcal{S}}\mathfrak{A}_\sigma)$; as $z\in L^1$, $z\mathbf{1}$ is a martingale. But $ii_{\boldsymbol{w}}(\boldsymbol{v})$ and $ii_{\boldsymbol{v}}(\boldsymbol{w})$ are virtually local martingales, by 623O, so $\boldsymbol{v}\times\boldsymbol{w} - [\boldsymbol{v}\overset{*}{|}\boldsymbol{w}]$ is a virtually local martingale.

624C Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$, $\boldsymbol{w} = \langle w_\sigma\rangle_{\sigma\in\mathcal{S}}$ local integrators. If one of them is locally jump-free and one is locally of bounded variation, then $[\boldsymbol{v}\overset{*}{|}\boldsymbol{w}] = 0$. In particular, if $\boldsymbol{v}$ is locally jump-free and locally of bounded variation, then $\boldsymbol{v}^* = 0$.

proof (a) Suppose that $\boldsymbol{v}$ is locally jump-free and $\boldsymbol{w}$ is locally of bounded variation. Take any $\tau\in\mathcal{S}$. Set $\bar w = \int_{\mathcal{S}\wedge\tau}|d\boldsymbol{w}|$. Take any $\epsilon>0$. Let $\delta>0$ be such that $\theta(z\times\bar w)\le\epsilon$ whenever $\theta(z)\le\delta$. Let $I\in\mathcal{I}(\mathcal{S}\wedge\tau)$ be such that $\theta(\mathrm{Osclln}^*_I(\boldsymbol{v}\restriction\mathcal{S}\wedge\tau))\le\delta$ (618B). If $J\in\mathcal{I}(\mathcal{S}\wedge\tau)$, $J\supseteq I$ and e is a J-cell, express e as $c(\sigma,\tau)$ where $\sigma\le\tau$ in J; then

$$|\Delta_e(\mathbf{1}, d\boldsymbol{v}\,d\boldsymbol{w})| = |v_\tau - v_\sigma|\times|w_\tau - w_\sigma| \le \mathrm{Osclln}_J(\boldsymbol{v})\times\Delta_e(\mathbf{1},|d\boldsymbol{w}|).$$

Summing over the J-cells,

$$|S_J(\mathbf{1}, d\boldsymbol{v}\,d\boldsymbol{w})| \le \mathrm{Osclln}_J(\boldsymbol{v})\times S_J(\mathbf{1},|d\boldsymbol{w}|) \le \mathrm{Osclln}^*_I(\boldsymbol{v}\restriction\mathcal{S}\wedge\tau)\times\bar w$$

and $\theta(S_J(\mathbf{1}, d\boldsymbol{v}\,d\boldsymbol{w}))\le\epsilon$. As ϵ is arbitrary,

$$\int_{\mathcal{S}\wedge\tau} d\boldsymbol{v}\,d\boldsymbol{w} = \lim_{J\uparrow\mathcal{I}(\mathcal{S}\wedge\tau)} S_J(\mathbf{1}, d\boldsymbol{v}\,d\boldsymbol{w})) = 0.$$

This is true for every $\tau\in\mathcal{S}$, so $[\boldsymbol{v}\overset{*}{|}\boldsymbol{w}] = 0$.

Of course the same arguments will apply if $\boldsymbol{v}$ is locally of bounded variation and $\boldsymbol{w}$ is locally jump-free.

(b) If $\boldsymbol{v}$ is both locally jump-free and locally of bounded variation, then (a) tells us that $\boldsymbol{v}^* = [\boldsymbol{v}\overset{*}{|}\boldsymbol{v}] = 0$. By 617M, it follows that $[\boldsymbol{v}\overset{*}{|}\boldsymbol{w}] = 0$. Similarly, $[\boldsymbol{v}\overset{*}{|}\boldsymbol{w}] = 0$ if $\boldsymbol{w}$ is locally jump-free and locally of bounded variation.

624D Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma\in\mathcal{S}}$ a virtually local martingale. Then the following are equiveridical:

(i) $\boldsymbol{v}$ is constant;

(ii) $\boldsymbol{v}$ is locally jump-free and locally of bounded variation;

(iii) the quadratic variation of $\boldsymbol{v}$ is zero.

[For 624E/653G, want $[\![\boldsymbol{v} \ne v_\downarrow \mathbf{1}]\!] \subseteq [\![\boldsymbol{v}^* \ne \mathbf{0}]\!]$]

proof (i)⇒(ii) is trivial.

(ii)⇒(iii) is immediate from 624C.

(iii)⇒(i) The result is trivial if $\mathcal{S}$ is empty, so let us suppose otherwise. By 623Kd again, $\boldsymbol{v}$ is locally integrable, so its quadratic variation is defined everywhere on $\mathcal{S}$. By 623Kg, $v_\downarrow = \lim_{\sigma\downarrow\mathcal{S}} v_\sigma$ is defined and belongs to $L^1(\mathfrak{A}, \bar\mu)$.

($\boldsymbol{\alpha}$) Suppose to begin with that $\mathcal{S}$ is full and that $v_\downarrow = 0$. Because the quadratic variation of $\boldsymbol{v}$ is zero, $\boldsymbol{v}^2 = 2ii_{\boldsymbol{v}}(\boldsymbol{v})$ is a virtually local martingale (623O again). Take $\tau \in \mathcal{S}$ and $\epsilon > 0$. Let $A \subseteq \mathcal{S}$ be a non-empty downwards-directed set such that $\sup_{\rho\in A} \bar\mu[\![\rho < \tau]\!] \le \epsilon$ and $\boldsymbol{z} = R_A(\boldsymbol{v}^2)$, as defined in 623B, is a martingale. Express $\boldsymbol{z}$ as $\langle z_\sigma \rangle_{\sigma\in\mathcal{S}}$. Then $\lim_{\sigma\downarrow\mathcal{S}} z_\sigma = \lim_{\sigma\downarrow\mathcal{S}} v_\sigma^2 = 0$ (623B(c-i)). The set $\{z_\sigma : \sigma \in \mathcal{S} \wedge \tau\} = \{P_\sigma z_\tau : \sigma \in \mathcal{S} \wedge \tau\}$ is uniformly integrable (621Cf), so the $\| \|_1$-limit $1\lim_{\sigma\downarrow\mathcal{S}} z_\sigma$ is zero (621B(c-ii)) and

$$\mathbb{E}(z_\tau) = \lim_{\sigma\downarrow\mathcal{S}} \mathbb{E}(P_\sigma z_\tau) = \lim_{\sigma\downarrow\mathcal{S}} \mathbb{E}(z_\sigma) = 0.$$

As $z_\tau = \lim_{\rho\downarrow A} v_{\tau\wedge\rho}^2 \ge 0$, $z_\tau = 0$.

Now note that

$$[\![v_\tau \ne 0]\!] = [\![v_\tau^2 \ne z_\tau]\!] \subseteq \sup_{\rho\in A} [\![v_\tau^2 \ne v_{\tau\wedge\rho}^2]\!] \subseteq \sup_{\rho\in A} [\![\rho < \tau]\!]$$

has measure at most ϵ (because A is downwards-directed). As ϵ is arbitrary, $v_\tau = 0$; as τ is arbitrary, $\boldsymbol{v} = \mathbf{0}$ is constant.

($\boldsymbol{\beta}$) Next, suppose just that $\mathcal{S}$ is full. Set $w_\sigma = v_\sigma - v_\downarrow$ for $\sigma \in \mathcal{S}$ and $\boldsymbol{w} = \langle w_\sigma \rangle_{\sigma\in\mathcal{S}}$. Since $v_\downarrow \in L^1(\mathfrak{A}, \bar\mu)$, $v_\downarrow \mathbf{1}$ is a martingale and $\boldsymbol{w} = \boldsymbol{v} - v_\downarrow \mathbf{1}$ is a virtually local martingale. Next, $(w_\tau - w_\sigma)^2 = (v_\tau - v_\sigma)^2$ whenever $\sigma \le \tau$ in $\mathcal{S}$, so $\int_{\mathcal{S}\wedge\tau} (d\boldsymbol{w})^2 = \int_{\mathcal{S}\wedge\tau} (d\boldsymbol{v})^2$ for every $\tau \in \mathcal{S}$, and the quadratic variation of $\boldsymbol{w}$ is equal to $\boldsymbol{v}^* = \mathbf{0}$. By ($\alpha$), $\boldsymbol{w} = \mathbf{0}$ and $\boldsymbol{v}$ is constant.

($\boldsymbol{\gamma}$) Finally, for the general case, let $\hat{\mathcal{S}}$ be the covered envelope of $\mathcal{S}$ and $\hat{\boldsymbol{v}}$ the fully adapted extension of $\boldsymbol{v}$ to $\hat{\mathcal{S}}$. As noted in 623J, $\hat{\boldsymbol{v}}$ is a virtually local martingale. Its quadratic variation is the fully adapted extension of $\boldsymbol{v}^*$ (617N) so is zero. By (β), $\hat{\boldsymbol{v}}$ is constant, so $\boldsymbol{v}$ is.

624E Corollary Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma\in\mathcal{S}}$ a virtually local martingale with quadratic variation $\langle v_\sigma^* \rangle_{\sigma\in\mathcal{S}}$. If τ, $\tau' \in \mathcal{S}$ are such that $v_\tau^* = v_{\tau'}^*$, then $\boldsymbol{v}$ is constant on $\mathcal{S} \cap [\tau \wedge \tau', \tau \vee \tau']$.

we need $[\![v_\tau \ne v_{\tau'}]\!] \subseteq [\![v_\tau^* \ne v_{\tau'}^*]\!]$ for 653G

proof (a) To begin with, suppose that $\tau \le \tau'$. Write $\mathcal{S}_0$ for $\mathcal{S} \cap [\tau, \tau']$. As $\boldsymbol{v}^*$ is non-decreasing, $v_\sigma^* = v_\tau^*$, that is, $\int_{\mathcal{S}\wedge\sigma} (d\boldsymbol{v})^2 = \int_{\mathcal{S}\wedge\tau} (d\boldsymbol{v})^2$, for every $\sigma \in \mathcal{S} \cap [\tau, \tau']$. By 623Kh and 623K(e-ii), $(\boldsymbol{v} - v_\tau \mathbf{1}) \restriction \mathcal{S} \vee \tau$ and $\boldsymbol{w} = (\boldsymbol{v} - v_\tau \mathbf{1}) \restriction \mathcal{S}_0$ are virtually local martingales. Now $\Delta\boldsymbol{w} = \Delta\boldsymbol{v} \restriction \mathcal{S}_0^{2\uparrow}$, so

$$\int_{\mathcal{S}_0\wedge\sigma} (d\boldsymbol{w})^2 = \int_{\mathcal{S}_0\wedge\sigma} (d\boldsymbol{v})^2 = \int_{\mathcal{S}\wedge\sigma} (d\boldsymbol{v})^2 - \int_{\mathcal{S}'\wedge\tau} (d\boldsymbol{v})^2 = 0$$

for every $\sigma \in \mathcal{S}$, that is, the quadratic variation of $\boldsymbol{w}$ is zero. By 624D, $\boldsymbol{w}$ and $\boldsymbol{v} = \boldsymbol{w} + v_\tau \mathbf{1}$ are constant on $\mathcal{S}_0$.

(b) This deals with the case $\tau \le \tau'$. But for the general case, given that $v_\tau^* = v_{\tau'}^*$, we have

$$\begin{aligned} [\![\tau \le \tau']\!] &\subseteq [\![\tau \wedge \tau' = \tau]\!] \cap [\![\tau \vee \tau' = \tau']\!] \\ &\subseteq [\![v_{\tau\wedge\tau'}^* = v_\tau^*]\!] \cap [\![v_{\tau\vee\tau'}^* = v_{\tau'}^*]\!] \subseteq [\![v_{\tau\wedge\tau'}^* = v_{\tau\vee\tau'}^*]\!] \end{aligned}$$

and similarly $[\![\tau' \le \tau]\!] \subseteq [\![v_{\tau\wedge\tau'}^* = v_{\tau\vee\tau'}^*]\!]$. So $v_{\tau\wedge\tau'}^* = v_{\tau\vee\tau'}^*$ and (a) shows that $\boldsymbol{v}$ is constant on $\mathcal{S} \cap [\tau \wedge \tau', \tau \vee \tau']$.

624F Theorem Let $\boldsymbol{w} = \langle w_\tau \rangle_{\tau \in \mathcal{T}_f}$ be Brownian motion. Then its quadratic variation $\boldsymbol{w}^*$ is the identity process $\boldsymbol{\iota}$.

proof By 622L, $\boldsymbol{w}$ and $\boldsymbol{w}^2 - \boldsymbol{\iota}$ are local martingales. Next, $\boldsymbol{w}^2 - \boldsymbol{w}^* = 2ii_{\boldsymbol{w}}(\boldsymbol{w})$ is a virtually local martingale, by 624B, so $\boldsymbol{w}^* - \boldsymbol{\iota}$ is a virtually local martingale. Because both $\boldsymbol{w}^*$ and $\boldsymbol{\iota}$ are non-decreasing, $\boldsymbol{w}^* - \boldsymbol{\iota}$ is locally of bounded variation. Now recall that $\boldsymbol{w}$ is locally jump-free (618Jc), so $\boldsymbol{w}^*$ is locally jump-free (618T), while of course $\boldsymbol{\iota}$ is locally jump-free (618Ja). So $\boldsymbol{w}^* - \boldsymbol{\iota}$ is a virtually local martingale, locally jump-free and locally of bounded variation, and must be constant (624D). Since it starts from 0 at $\min \mathcal{T}$, it is zero everywhere and $\boldsymbol{w}^* = \boldsymbol{\iota}$, as claimed.

624G We saw in 622Q that indefinite integrals with respect to L^2-martingales are particularly easy to analyze. Here I give a characterization and a striking property of such martingales.

Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ an L^2-martingale and $\boldsymbol{v}^* = \langle v^*_\sigma \rangle_{\sigma \in \mathcal{S}}$ its quadratic variation. Then $\mathbb{E}(v^*_\tau) \le \mathbb{E}(v^2_\tau)$ for every $\tau \in \mathcal{S}$.

proof As before, $\boldsymbol{v}^*$ is defined everywhere on $\mathcal{S}$. Now $\mathbb{E}(S_I(\mathbf{1}, (d\boldsymbol{v})^2)) \le \mathbb{E}(v^2_{\max I})$ for every non-empty $I \in \mathcal{I}(\mathcal{S})$. **P** Take $(\tau_0, \dots, \tau_n)$ linearly generating the I-cells. If $i < n$, then

$$P_{\tau_i}(v_{\tau_i} \times (v_{\tau_{i+1}} - v_{\tau_i})) = v_{\tau_i} \times P_{\tau_i}(v_{\tau_{i+1}} - v_{\tau_i}) = 0,$$

so

$$\mathbb{E}(v_{\tau_i} \times (v_{\tau_{i+1}} - v_{\tau_i})) = \mathbb{E}(P_{\tau_i}(v_{\tau_i} \times (v_{\tau_{i+1}} - v_{\tau_i}))) = 0.$$

Consequently

$$\begin{aligned}\mathbb{E}(S_I(\mathbf{1}, (d\boldsymbol{v})^2)) &= \mathbb{E}(\sum_{i=0}^{n-1} (v_{\tau_{i+1}} - v_{\tau_i})^2) \\ &= \sum_{i=0}^{n-1} \mathbb{E}(v_{\tau_{i+1}}) - \mathbb{E}(v_{\tau_i})^2) - 2\mathbb{E}(v_{\tau_i} \times (v_{\tau_{i+1}} - v_{\tau_i})) \\ &= \mathbb{E}(v^2_{\tau_n}) - \mathbb{E}(v^2_{\tau_0}) \le \mathbb{E}(v^2_{\max I}). \ \mathbf{Q}\end{aligned}$$

Now if $\tau \in \mathcal{S}$,

$$v^*_\tau = \lim_{I \uparrow \mathcal{I}(\mathcal{S} \wedge \tau)} S_I(\mathbf{1}, (d\boldsymbol{v})^2) \in \overline{\{x : \|x\|_1 \le \mathbb{E}(v^2_\tau)\}}.$$

Bu $\| \|_1$-balls are closed for the topology of convergence in measure (613Bc), so $\mathbb{E}(v^*_\tau) = \|v^*_\tau\|_1 \le \mathbb{E}(v^2_\tau)$, as clsimed.

624H Proposition Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$, $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ a virtually local martingale with starting value 0, and $\boldsymbol{v}^* = \langle v^*_\sigma \rangle_{\sigma \in \mathcal{S}}$ its quadratic variation.

(a) $\|v_\tau\|_2 \le \sqrt{\|v^*_\tau\|_1}$ for every $\tau \in \mathcal{S}$.

(b) If moreover $\boldsymbol{v}$ is an approximately local martingale and $\boldsymbol{v}^*$ is an L^1-process, then $\boldsymbol{v}$ and $ii_{\boldsymbol{v}}(\boldsymbol{v})$ are martingales, and $\|v_\tau\|_2 = \sqrt{\|v^*_\tau\|_1}$ for every $\tau \in \mathcal{S}$.

Remark Recall that I count $\|v_\tau\|_2$ as ∞ if $v_\tau \in L^0$ is not square-integrable, and $\|v^*_\tau\|_1$ as ∞ if $v^*_\tau \in (L^0)^+$ does not have finite expectation; while of course $\sqrt{\infty}$ is to be interpreted as ∞.

proof (a)(i) To begin with, suppose that $\mathcal{S}$ is full. As in 624B $\boldsymbol{v}^*$ is well-defined and $\boldsymbol{v}^* - \boldsymbol{v}^2$ is a virtually local martingale. Take $\tau \in \mathcal{S}$ and $\epsilon > 0$. Then there is a non-empty downwards-directed subset A of $\mathcal{S}$ such that $a = \sup_{\rho \in A} [\![\rho < \tau]\!]$ has measure at most ϵ and $R_A(\boldsymbol{v}^* - \boldsymbol{v}^2)$ is a martingale, where $R_A : M_{\text{lmo}}(\mathcal{S}) \to M_{\text{lmo}}(\mathcal{S})$ is defined as in 623B. Now $R_A(\boldsymbol{v})^* = R_A(\boldsymbol{v}^*)$, by 623H, so $\boldsymbol{z} = R_A(\boldsymbol{v}^*) - R_A(\boldsymbol{v})^2$ is a martingale. The starting value of $R_A(\boldsymbol{v})$ is 0, by 623B(c-i), as is the starting value of $R_A(\boldsymbol{v}^*)$, so the starting value of $\boldsymbol{z}$ is 0; expressing $\boldsymbol{z}$ as $\langle z_\sigma \rangle_{\sigma \in \mathcal{S}}$, 0 is the conditional expectation of z_τ on $\bigcap_{\sigma \in \mathcal{S}} \mathfrak{A}_\sigma$ (622Ed), so $0 = \mathbb{E}(z_\tau) = \mathbb{E}(v^*_{A\tau} - v^2_{A\tau})$, where $v_{A\tau} = \lim_{\rho \downarrow A} v_{\rho \wedge \tau}$ and $v^*_{A\tau} = \lim_{\rho \downarrow A} v^*_{\rho \wedge \tau}$. Accordingly

$$\|v^2_{A\tau}\|_1 = \mathbb{E}(v^2_{A\tau}) = \mathbb{E}(v^*_{A\tau}) = \|v^*_{A\tau}\|_1 \le \|v^*_\tau\|_1$$

because $0 \le v^*_{A\tau} \le v^*_\tau$.

Now observe that

$$[\![v_{A\tau}^2 \ne v_\tau^2]\!] \subseteq [\![v_{A\tau} \ne v_\tau]\!] \subseteq \sup_{\rho\in A} [\![v_{\rho\wedge\tau} \ne v_\tau]\!] \subseteq \sup_{\rho\in A} [\![\rho < \tau]\!] = a$$

so $\theta(v_\tau^2 - v_{A\tau}^2) \le \epsilon$. As ϵ is arbitrary, v_τ^2 belongs to the closure of $\{x : \|x\|_1 \le \|v_\tau^*\|_1\}$. But this is a closed set for the topology of convergence in measure (613Bc again), so $\|v_\tau^2\|_1 \le \|v_\tau^*\|_1$, that is, $\|v_\tau\|_2 \le \sqrt{\|v_\tau^*\|_1}$.

(ii) For the general case, let $\hat{\mathcal{S}}$ be the covered envelope of $\mathcal{S}$ and $\hat{\boldsymbol{v}}$ the fully adapted extension of $\boldsymbol{v}$ to $\hat{\mathcal{S}}$, Then $\hat{\mathcal{S}}$ is full and $\hat{\boldsymbol{v}}$ is a virtually local martingale (623J) with starting value 0 (615H, as $\boldsymbol{v}$ and $\hat{\boldsymbol{v}}$ are locally moderately oscillatory). Express $\hat{\boldsymbol{v}}$ as $\langle\hat{v}_\sigma\rangle_{\sigma\in\hat{\mathcal{S}}}$ and its quadratic variation $\hat{\boldsymbol{v}}^*$ as $\langle\hat{v}_\sigma^*\rangle_{\sigma\in\hat{\mathcal{S}}}$. By 617N, $\hat{\boldsymbol{v}}^*$ extends $\boldsymbol{v}^*$. So (a) tells us that

$$\|v_\tau\|_2 = \|\hat{v}_\tau\|_2 \le \sqrt{\|\hat{v}_\tau^*\|_1} = \sqrt{\|v_\tau^*\|_1}$$

for every $\tau \in \mathcal{S}$.

(b) Now suppose that $\boldsymbol{v}$ is an approximately local martingale and $\boldsymbol{v}^*$ is an L^1-process, that is, $\|v_\tau^*\|_1$ and therefore $\|v_\tau\|_2$ are finite for every $\tau \in \mathcal{S}$.

(i) If $\tau \in \mathcal{S}$, then

$$\|v_\sigma\|_2 \le \sqrt{\|v_\sigma^*\|_1} \le \sqrt{\|v_\tau^*\|_1}$$

for every $\sigma \in \mathcal{S}\wedge\tau$, and $\{v_\sigma : \sigma \in \mathcal{S}\wedge\tau\}$ is $\|\,\|_2$-bounded, therefore uniformly integrable (621Be), that is, $\boldsymbol{v}\restriction\mathcal{S}\wedge\tau$ is uniformly integrable. By 623Na, $\boldsymbol{v}$ is a martingale.

As $\boldsymbol{v}$ is also an L^2-process, 624G tells us that $\sqrt{\|v_\tau^*\|_1} \le \|v_\tau\|_2$ for every $\tau \in \mathcal{S}$, so we have equality.

(ii) As for $ii_{\boldsymbol{v}}(\boldsymbol{v})$, take $\tau \le \tau'$ in $\mathcal{S}$ and $c \in \mathfrak{A}_\tau$. For $\sigma \in \mathcal{S}\vee\tau$ set

$$w_\sigma = P_\sigma((v_{\tau'} - v_\tau) \times \chi c)$$

so that $\boldsymbol{w} = \langle w_\sigma\rangle_{\sigma\in\mathcal{S}\vee\tau}$ is a martingale. Write $\boldsymbol{w}^* = \langle w_\sigma^*\rangle_{\sigma\in\mathcal{S}\vee\tau}$ for its quadratic variation. We see that if $\sigma \in \mathcal{S}\vee\tau$ then $c \in \mathfrak{A}_\sigma$ so

$$w_\sigma = P_\sigma(v_{\tau'} - v_\tau) \times \chi c = (v_\sigma - v_\tau) \times \chi c$$

because $\boldsymbol{v}$ is a martingale; in particular, the starting value w_τ of $\boldsymbol{w}$ is 0. As $(v_{\tau'} - v_\tau) \times \chi c \in L^2$, $\boldsymbol{w}$ is an L^2-martingale and $\mathbb{E}(w_\sigma^*) \le \mathbb{E}(w_\sigma^2)$ is finite for every $\sigma \in \mathcal{S}\vee\tau$, by 624G. But from (a) we know that $\mathbb{E}(w_\sigma^2) \le \mathbb{E}(w_\sigma^*)$, so we have equality.

Now $w_{\tau'}^* - w_\tau^* = (v_{\tau'}^* - v_\tau^*) \times \chi c$. **P** If $\tau_0 \le \ldots \le \tau_n$ in $\mathcal{S}\cap[\tau,\tau']$,

$$\sum_{i=0}^n (w_{\tau_{i+1}} - w_{\tau_i})^2 = \sum_{i=0}^n (v_{\tau_{i+1}} - v_{\tau_i})^2 \times \chi c.$$

So $S_I(\mathbf{1},(d\boldsymbol{w})^2) = S_I(\mathbf{1},(d\boldsymbol{v})^2) \times \chi c$ for every finite sublattice I of $\mathcal{S}\cap[\tau,\tau']$. Taking the limit as $I \uparrow \mathcal{I}(\mathcal{S}\cap[\tau,\tau'])$,

$$\begin{aligned} w_{\tau'}^* &= \int_{\mathcal{S}\cap[\tau,\tau']} (d\boldsymbol{w})^2 = \lim_{I\uparrow\mathcal{I}(\mathcal{S}\cap[\tau,\tau'])} S_I(\mathbf{1},(d\boldsymbol{w})^2) \\ &= \lim_{I\uparrow\mathcal{I}(\mathcal{S}\cap[\tau,\tau'])} S_I(\mathbf{1},(d\boldsymbol{v})^2) \times \chi c \\ &= \int_{\mathcal{S}\cap[\tau,\tau']} (d\boldsymbol{v})^2 \times \chi c = (v_{\tau'}^* - v_\tau^*) \times \chi c. \ \mathbf{Q} \end{aligned}$$

Turning to expectations, we have

$$\begin{aligned} \mathbb{E}((v_{\tau'}^* - v_\tau^*) \times \chi c) &= \mathbb{E}(w_{\tau'}^*) = \mathbb{E}(w_{\tau'}^2) = \mathbb{E}((v_{\tau'} - v_\tau)^2 \times \chi c) \\ &= \mathbb{E}((v_{\tau'}^2 - v_\tau^2) \times \chi c) + 2\mathbb{E}(v_\tau \times (v_{\tau'} - v_\tau) \times \chi c). \end{aligned}$$

But

$$\begin{aligned} \mathbb{E}(v_\tau \times (v_{\tau'} - v_\tau) \times \chi c) &= \mathbb{E}(P_\tau(v_\tau \times (v_{\tau'} - v_\tau) \times \chi c)) \\ &= \mathbb{E}(v_\tau \times P_\tau(v_{\tau'} - v_\tau) \times \chi c) = 0 \end{aligned}$$

so

$$\mathbb{E}((v^*_{\tau'} - v^*_\tau) \times \chi c) = \mathbb{E}((v^2_{\tau'} - v^2_\tau) \times \chi c),$$

that is to say,

$$\mathbb{E}((v^*_{\tau'} - v^2_{\tau'}) \times \chi c) = \mathbb{E}((v^*_\tau - v^2_\tau) \times \chi c).$$

As c is arbitrary, $P_\tau(v^*_{\tau'} - v^2_{\tau'}) = v^*_\tau - v^2_\tau$. As τ and τ' are arbitrary, $\boldsymbol{v}^* - \boldsymbol{v}^2$ is a martingale. But as $\boldsymbol{v}$ has starting value 0,

$$ii_{\boldsymbol{v}}(\boldsymbol{v}) = \frac{1}{2}(\boldsymbol{v}^2 - \boldsymbol{v}^*)$$

is a martingale, and the proof is complete.

624I Corollary Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma\in\mathcal{S}}$ a $\| \|_2$-bounded martingale with quadratic variation $\boldsymbol{v}^*$, and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$ a $\| \|_\infty$-bounded moderately oscillatory process. Then $\mathbb{E}((\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v})^2)$ and $\mathbb{E}(\int_{\mathcal{S}} \boldsymbol{u}^2 d\boldsymbol{v}^*)$ are finite and equal.

proof By 622Q, $\boldsymbol{z} = ii_{\boldsymbol{v}}(\boldsymbol{u})$ is a $\| \|_2$-bounded martingale, and of course its starting value is 0 (613J(f-i)). Writing $\boldsymbol{z}^* = \langle z^*_\sigma \rangle_{\sigma\in\mathcal{S}}$ for its quadratic variation, and expressing $\boldsymbol{z}$ as $\langle z_\sigma \rangle_{\sigma\in\mathcal{S}}$, we have $\mathbb{E}(z^*_\tau) = \mathbb{E}(z^2_\tau)$ for every $\tau \in \mathcal{S}$, by 624G and 624Hb. We know also that

$$z^*_\tau = \int_{\mathcal{S}\wedge\tau} d\boldsymbol{z}^* = \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}^2 d\boldsymbol{v}^*$$

for every $\tau \in \mathcal{S}$, by 617Qb.

If $\mathcal{S}$ has a greatest member, we just take $\tau = \max \mathcal{S}$. In general, we need to check the limits as $\tau \uparrow \mathcal{S}$. By 622Q again, $\mathbb{E}(z^2_\tau) \le \|\boldsymbol{u}\|^2_\infty \mathbb{E}(v^2_\tau)$ for every $\tau \in \mathcal{S}$. Since $\boldsymbol{v}$ is $\| \|_2$-bounded,

$$\beta = \sup_{\tau\in\mathcal{S}} \mathbb{E}(z^*_\tau) = \sup_{\tau\in\mathcal{S}} \mathbb{E}(z^2_\tau)$$

is finite. Now $\boldsymbol{z}^*$ is non-decreasing, so

$$\begin{aligned}\mathbb{E}(\int_{\mathcal{S}} \boldsymbol{u}^2 d\boldsymbol{v}^*) &= \mathbb{E}(\lim_{\tau\uparrow\mathcal{S}} \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}^2 d\boldsymbol{v}^*) = \mathbb{E}(\lim_{\tau\uparrow\mathcal{S}} z^*_\tau) \\ &= \mathbb{E}(\sup_{\tau\in\mathcal{S}} z^*_\tau) = \sup_{\tau\in\mathcal{S}} \mathbb{E}(z^*_\tau) = \beta\end{aligned}$$

(613B(d-iii)). On the other side, we know that if $\sigma \le \tau$ in $\mathcal{S}$ then

$$\mathbb{E}((z_\tau - z_\sigma)^2) = \mathbb{E}((z^2_\tau - z^2_\sigma - 2z_\sigma \times (z_\tau - z_\sigma)) = \mathbb{E}(z^2_\tau) - \mathbb{E}(z^2_\sigma)$$

as in (a-i) of the proof of 624H. Generally, for σ, $\tau \in \mathcal{S}$,

$$\|z_\tau - z_\sigma\|^2_2 = \mathbb{E}((z_\tau - z_\sigma)^2) = \mathbb{E}((z_{\sigma\vee\tau} - z_{\sigma\wedge\tau})^2)$$

(612D(f-ii))

$$= \mathbb{E}(z^2_{\sigma\vee\tau}) - \mathbb{E}(z^2_{\sigma\wedge\tau}) \le \beta - \mathbb{E}(z^2_{\sigma\wedge\tau}) \to 0$$

as σ, $\tau \uparrow \mathcal{S}$. So the $\| \|_2$-limit $2\mathrm{lim}_{\tau\uparrow\mathcal{S}} z_\tau$ is defined and must be equal to the limit $\lim_{\tau\uparrow\mathcal{S}} z_\tau = \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$ for the topology of convergence in measure (613B(d-i)); moreover,

$$\begin{aligned}\mathbb{E}((\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v})^2) &= \| \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}\|^2_2 = \| 2\mathrm{lim}_{\tau\uparrow\mathcal{S}} z_\tau\|^2_2 \\ &= \lim_{\tau\uparrow\mathcal{S}} \|z_\tau\|^2_2 = \beta = \mathbb{E}(\int_{\mathcal{S}} \boldsymbol{u}^2 d\boldsymbol{v}^*),\end{aligned}$$

as claimed.

624X Basic exercises (a) (i) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma\in\mathcal{S}}$ a martingale. Suppose that τ, $\tau' \in \mathcal{S}$ are such that $v_\tau = v_{\tau'}$. Show that $\boldsymbol{v}$ is constant on $\mathcal{S} \cap [\tau\wedge\tau', \tau\vee\tau']$. (ii) Give an example in which there is a local martingale $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma\in\mathcal{T}}$, with quadratic variation $\boldsymbol{v}^* = \langle v^*_\sigma \rangle_{\sigma\in\mathcal{T}}$, such that $v_{\max\mathcal{T}} = v_{\min\mathcal{T}}$ but $v^*_{\max\mathcal{T}} \ne v^*_{\min\mathcal{T}}$.

(b) Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ and $\boldsymbol{v}$, $\boldsymbol{w}$ L^2-martingales on $\mathcal{S}$. Show that $\boldsymbol{v}\times\boldsymbol{w}-[\boldsymbol{v}\,{}^*_{|}\,\boldsymbol{w}]$ is a martingale. (*Hint*: 624B, 624G, 623K(b-iii), 623Kf.)

(c) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{v}$ a $\|\;\|_2$-bounded martingale with domain $\mathcal{S}$, and $\boldsymbol{u}$ a locally moderately osciallatory process with domain $\mathcal{S}$. Write $\boldsymbol{v}^*$ for the quadratic variation of $\boldsymbol{v}$. Show that $\mathbb{E}((\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v})^2)=\mathbb{E}(\int_{\mathcal{S}}\boldsymbol{u}^2d\boldsymbol{v}^*)$ if either is finite.

624 Notes and comments Nothing in this section is surprising, but to get complete arguments I think a little care is needed. It is easy to believe that $\|v_\tau\|_2$ and $\|v^*_\tau\|_1$ in 624Ha are related, but not so simple to find exact conditions on the process $\boldsymbol{v}$ which will ensure this in the cases we might encounter. Of course there are many paths through the forest. In 624Xb, for instance, I sketch an alternative route to the result of 622L, not relying on Dynkin's formula.

Version of 6.3.24

625 Changing the measure

I give essential formulae for calculating the effect of replacing a given probability measure $\bar\mu$ with an equivalent probability measure $\bar\nu$ (625B-625C). Semi-martingales (625D) remain semi-martingales under any such change (625F).

625A Notation I continue in the framework developed in Chapter 61. $(\mathfrak{A},\bar\mu,T,\langle\mathfrak{A}_t\rangle_{t\in T},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ will be a stochastic integration structure, and $\mathbb{E}_{\bar\mu}$ the integral corresponding to $\bar\mu$. For $\tau\in\mathcal{T}$, $P_\tau:L^1_{\bar\mu}\to L^1_{\bar\mu}$ will be the conditional expectation operator associated with $\mathfrak{A}_\tau$; if $z\in L^1_{\bar\mu}$, $\boldsymbol{P}z=\langle P_\tau z\rangle_{\tau\in\mathcal{T}}$ will be the martingale derived from z (622F). If $\mathcal{S}$ is a sublattice of $\mathcal{T}$, $M_{\rm fa}(\mathcal{S})$ will be the space of fully adapted processes with domain $\mathcal{S}$, and $M_{\rm lmo}(\mathcal{S})$ the space of locally moderately oscillatory processes with domain $\mathcal{S}$.

625B Change of law: Theorem Let $\bar\nu$ be a second functional such that $(\mathfrak{A},\bar\nu)$ is a probability algebra; write $\mathbb{E}_{\bar\nu}$ and $L^1_{\bar\nu}$ for the corresponding integral and L^1-space.

(a)(i) There is a unique $z\in L^1_{\bar\mu}$ such that $\bar\nu a=\mathbb{E}_{\bar\mu}(z\times\chi a)$ for every $a\in\mathfrak{A}$.

(ii) $[\![z>0]\!]=1$ and z has a multiplicative inverse $\frac{1}{z}$ in L^0.

(iii) For $w\in L^0$, $\mathbb{E}_{\bar\nu}(w)=\mathbb{E}_{\bar\mu}(w\times z)$ if either is defined in $[-\infty,\infty]$.

(iv) $\frac{1}{z}\in L^1_{\bar\nu}$ and $\bar\mu a=\mathbb{E}_{\bar\nu}(\frac{1}{z}\times\chi a)$ for every $a\in\mathfrak{A}$.

(v) For $w\in L^0$,

$$w\in L^1_{\bar\nu}\iff w\times z\in L^1_{\bar\mu},\quad w\in L^1_{\bar\mu}\iff w\times\tfrac{1}{z}\in L^1_{\bar\nu}.$$

(vi) $[\![P_\tau z>0]\!]=1$ for every $\tau\in\mathcal{T}$.

(vii) If $\tau\in\mathcal{T}$ and $w\in L^0(\mathfrak{A}_\tau)$, then $w\in L^1_{\bar\nu}$ iff $w\times P_\tau z\in L^1_{\bar\mu}$.

(viii) We have a fully adapted process $\boldsymbol{u}=\langle u_\tau\rangle_{\tau\in\mathcal{T}}$ defined by saying that $u_\tau=\dfrac{1}{P_\tau z}$ is the multiplicative inverse of $P_\tau z$ for every $\tau\in\mathcal{T}$.

(b) For $\tau\in\mathcal{T}$, let $Q_\tau:L^1_{\bar\nu}\to L^1_{\bar\nu}\cap L^0(\mathfrak{A}_\tau)$ be the conditional expectation operator with respect to the closed subalgebra $\mathfrak{A}_\tau$ for the probability $\bar\nu$.

(i) If $w\in L^1_{\bar\nu}$, $P_\tau(w\times z)=Q_\tau w\times P_\tau z$.

(ii) If $w\in L^1_{\bar\mu}$, $Q_\tau(w\times\frac{1}{z})=P_\tau w\times Q_\tau(\frac{1}{z})$.

(iii) $P_\tau(z)\times Q_\tau(\frac{1}{z})=\chi 1$.

(c) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{w}=\langle w_\sigma\rangle_{\sigma\in\mathcal{S}}$ a fully adapted process. Write $\boldsymbol{Q}(\frac{1}{z})$ for the $\bar\nu$-martingale $\langle Q_\tau(\frac{1}{z})\rangle_{\tau\in\mathcal{T}}$.

(i) $\boldsymbol{w}$ is is a $\bar\nu$-martingale iff $\boldsymbol{w}\times\boldsymbol{P}z$ is a $\bar\mu$-martingale. In particular, $\boldsymbol{u}$ in (a-viii) is a $\bar\nu$-martingale.

(ii) $\boldsymbol{w}$ is is a local $\bar\nu$-martingale iff $\boldsymbol{w}\times\boldsymbol{P}z$ is a local $\bar\mu$-martingale.

proof (a) Parts (i)-(iii) are in 365S[5]. Now (iv) and (v) follow immediately.

[5]Formerly 365T.

For (vi), we just have to note that

$$\mathbb{E}_{\bar\nu}(P_\tau z \times \chi a) = \mathbb{E}_{\bar\nu}(z \times \chi a) = \bar\nu a > 0$$

for every non-zero $a \in \mathfrak{A}_\tau$, because $[\![z > 0]\!] = 1$. It follows at once that $u_\tau = \frac{1}{P_\tau z}$ is defined for every τ (364N). We see also that $P_\tau z$ is the Radon-Nikodým derivative of $\bar\nu{\restriction}\mathfrak{A}_\tau$ with respect to $\bar\mu{\restriction}\mathfrak{A}_\tau$. So, just as in (iii), given $w \in L^0(\mathfrak{A}_\tau)$, $\mathbb{E}_{\bar\nu}(w) = \mathbb{E}_{\bar\mu}(w \times P_\tau z)$ if either is defined, and $w \in L^1_{\bar\nu}$ iff $w \times P_\tau z \in L^1_{\bar\mu}$. This deals with (vii).

For any $\tau \in \mathcal{T}$, $u_\tau \in L^0(\mathfrak{A}_\tau)$ because $P_\tau z \in L^0(\mathfrak{A}_\tau)$. If $\sigma, \tau \in \mathcal{T}$ and $c = [\![\sigma = \tau]\!]$, then $P_\sigma z \times \chi c = P_\tau z \times \chi c$ (622Bb), so

$$u_\sigma \times \chi c = u_\sigma \times u_\tau \times P_\tau z \times \chi c = u_\sigma \times u_\tau \times P_\sigma z \times \chi c = u_\tau \times \chi c$$

and $c \subseteq [\![u_\sigma = u_\tau]\!]$. Thus $\boldsymbol{u}$ is fully adapted and (viii) is true.

(b)(i) We know from (a-v) that $w \times z \in L^1_{\bar\mu}$ so $P_\tau(w \times z)$ is defined. Take any $a \in \mathfrak{A}_\tau$. We know that $Q_\tau w \in L^1_{\bar\nu}$ so $Q_\tau w \times z \in L^1_{\bar\mu}$. Since $Q_\tau w \times \chi a \in L^0(\mathfrak{A}_\tau)$, 621Cc tells us that $P_\tau(Q_\tau w \times \chi a \times z) = Q_\tau w \times \chi a \times P_\tau z$. But this means that

$$\begin{aligned}\mathbb{E}_{\bar\mu}(P_\tau(w \times z) \times \chi a) &= \mathbb{E}_{\bar\mu}(w \times z \times \chi a) = \mathbb{E}_{\bar\nu}(w \times \chi a) = \mathbb{E}_{\bar\nu}(Q_\tau w \times \chi a)\\ &= \mathbb{E}_{\bar\mu}(Q_\tau w \times z \times \chi a) = \mathbb{E}_{\bar\mu}(P_\tau(Q_\tau w \times z \times \chi a))\\ &= \mathbb{E}_{\bar\mu}(Q_\tau w \times P_\tau z \times \chi a).\end{aligned}$$

As a is arbitrary, and $P_\tau(w \times z) \in L^1_{\bar\mu} \cap L^0(\mathfrak{A}_\tau)$, while $Q_\tau w \times P_\tau z \in L^0(\mathfrak{A}_\tau)$, $P_\tau(w \times z) = Q_\tau w \times P_\tau z$.

(ii) Exchange $\bar\mu$ and $\bar\nu$ in (i).

(iii) Set $w = z$ in (ii).

(c)(i) By (a-vii), $\boldsymbol{w}$ is an $L^1_{\bar\nu}$-process iff $\boldsymbol{w} \times \boldsymbol{P}z$ is an $L^1_{\bar\mu}$-process. So we can suppose that both of these are the case. Now, for $\sigma \le \tau$ in $\mathcal{S}$,

$$w_\sigma = Q_\sigma w_\tau \iff w_\sigma \times P_\sigma z = Q_\sigma w_\tau \times P_\sigma z$$

(because $[\![P_\sigma z > 0]\!] = 1$)

$$\iff w_\sigma \times P_\sigma z = P_\sigma(w_\tau \times z)$$

(by (b-i))

$$\iff w_\sigma \times P_\sigma z = P_\sigma P_\tau(w_\tau \times z)$$

(because $P_\sigma P_\tau = P_\sigma$, by 622Ba)

$$\iff w_\sigma \times P_\sigma z = P_\sigma(w_\tau \times P_\tau z)$$

by 621Cc again. So

$$\begin{aligned}\boldsymbol{w} \text{ is a } \bar\nu\text{-martingale} &\iff w_\sigma = Q_\sigma w_\tau \text{ whenever } \sigma \le \tau \text{ in } \mathcal{S}\\ &\iff w_\sigma \times P_\sigma z = P_\sigma(w_\tau \times P_\tau z) \text{ whenever } \sigma \le \tau \text{ in } \mathcal{S}\\ &\iff \boldsymbol{w} \times \boldsymbol{P}z \text{ is a } \bar\mu\text{-martingale.}\end{aligned}$$

(ii)

$$\begin{aligned}\boldsymbol{w} \text{ is a local } \bar\nu\text{-martingale} &\iff \text{there is a covering ideal } \mathcal{S}' \text{ of } \mathcal{S}\\ &\qquad\quad \text{such that } \boldsymbol{w}{\restriction}\mathcal{S}' \text{ is a } \bar\nu\text{-martingale}\\ &\iff \text{there is a covering ideal } \mathcal{S}' \text{ of } \mathcal{S}\\ &\qquad\quad \text{such that } (\boldsymbol{w}{\restriction}\mathcal{S}') \times \boldsymbol{P}z \text{ is a } \bar\mu\text{-martingale}\\ &\iff \text{there is a covering ideal } \mathcal{S}' \text{ of } \mathcal{S}\\ &\qquad\quad \text{such that } (\boldsymbol{w} \times \boldsymbol{P}z){\restriction}\mathcal{S}' \text{ is a } \bar\mu\text{-martingale}\\ &\iff \boldsymbol{w} \times \boldsymbol{P}z \text{ is a local } \bar\mu\text{-martingale.}\end{aligned}$$

625C The next fact belongs with 625Bc, but the proof demands new ideas, as well as being rather long, so I have separated it out.

Proposition As in 625B, let $\bar\nu$ be such that $(\mathfrak{A},\bar\nu)$ is a probability algebra; write $\mathbb{E}_{\bar\nu}$ and $L^1_{\bar\nu}$ for the corresponding integral and L^1-space, and let $z\in L^1_{\bar\mu}$ be such that $\bar\nu a=\mathbb{E}_{\bar\mu}(z\times\chi a)$ for every $a\in\mathfrak{A}$. Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{w}=\langle w_\sigma\rangle_{\sigma\in\mathcal{S}}$ a fully adapted process. Then $\boldsymbol{w}$ is an approximately local $\bar\nu$-martingale iff $\boldsymbol{w}\times\boldsymbol{P}z$ is an approximately local $\bar\mu$-martingale, and $\boldsymbol{w}$ is a virtually local $\bar\nu$-martingale iff $\boldsymbol{w}\times\boldsymbol{P}z$ is a virtually local $\bar\mu$-martingale.

proof (a)(i) For a non-empty downwards-directed set $A\subseteq\mathcal{S}$, write $\mathfrak{A}_A$ for $\bigcap_{\sigma\in A}\mathfrak{A}_\sigma$ and $P_A:L^1_{\bar\mu}\to L^1_{\bar\mu}\cap L^0(\mathfrak{A}_A)$ for the corresponding conditional expectation. If $w\in L^1_{\bar\mu}$, then $P_Aw=\lim_{\rho\downarrow A}P_\rho w=1\text{lim}_{\rho\downarrow A}P_\rho w$ is the limit for the norm topology of $L^1_{\bar\mu}$ (621C(g-i) again). Consequently

$$\begin{aligned}P_\tau P_Aw&=\operatorname*{1lim}_{\rho\downarrow A}P_\tau P_\rho w=\operatorname*{1lim}_{\rho\downarrow A}P_{\tau\wedge\rho}w=\operatorname*{1lim}_{\rho\downarrow A\wedge\tau}P_{\tau\wedge\rho}w=P_{A\wedge\tau}w\\&=\operatorname*{1lim}_{\rho\downarrow A}P_\rho P_\tau w=P_AP_\tau w\end{aligned}$$

for any $\tau\in\mathcal{S}$. Thus $P_\tau P_A=P_AP_\tau=P_{A\wedge\tau}$. Similarly, writing $Q_A:L^1_{\bar\nu}\to L^1_{\bar\nu}\cap L^0(\mathfrak{A}_A)$ for the conditional expectation corresponding to $\mathfrak{A}_A$ with respect to the probability measure $\bar\nu$, $Q_\tau Q_A=Q_AQ_\tau=Q_{A\wedge\tau}$ for every $\tau\in\mathcal{S}$.

(ii) $Q_Aw\times P_Az=P_A(w\times z)$ whenever $w\in L^1_{\bar\nu}$, that is, whenever $w\times z\in L^1_{\bar\mu}$. **P**

$$\begin{aligned}Q_A(w)\times P_Az&=\lim_{\rho\downarrow A}Q_\rho w\times P_\rho z=\lim_{\rho\downarrow A}P_\rho(w\times z)&&\text{(625B(b-i))}\\&=P_A(w\times z).\ \mathbf{Q}\end{aligned}$$

(b) Let $\mathfrak{B}$ be a closed subalgebra of $\mathfrak{A}$ and $P:L^1_{\bar\mu}\to L^1_{\bar\mu}\cap L^0(\mathfrak{B})=L^1_{\bar\mu\restriction\mathfrak{B}}$ the associated conditional expectation. Then

$$\bar\nu b=\mathbb{E}_{\bar\mu}(z\times\chi b)=\mathbb{E}_{\bar\mu}(P(z\times\chi b))=\mathbb{E}_{\bar\mu\restriction\mathfrak{B}}(Pz\times\chi b),$$

for every $b\in\mathfrak{B}$, that is, Pz is the conditional expectation of $\bar\nu\restriction\mathfrak{B}$ on $\bar\mu\restriction\mathfrak{B}$. So for $w\in L^0(\mathfrak{B})$

$$w\in L^1_{\bar\nu}\iff w\in L^1_{\bar\nu\restriction\mathfrak{B}}\iff w\times Pz\in L^1_{\bar\mu\restriction\mathfrak{B}}\iff w\times Pz\in L^1_{\bar\mu}.$$

Also $[\![Pz>0]\!]=1$, just as in 625B(a-ii).

(c) Suppose that $A\subseteq\mathcal{S}$ is a non-empty downwards-directed set. Let $R_A:M_{\mathrm{lmo}}(\mathcal{S})\to M_{\mathrm{lmo}}(\mathcal{S})$ be the corresponding operator defined in 623B. Then $R_A(\boldsymbol{w})$ is a $\bar\nu$-martingale iff $R_A(\boldsymbol{w}\times\boldsymbol{P}z)$ is a $\bar\mu$-martingale. **P** Express $R_A(\boldsymbol{w})$ as $\langle w'_\sigma\rangle_{\sigma\in\mathcal{S}}$. We have

$$R_A(\boldsymbol{P}z)=\langle\lim_{\rho\downarrow A}P_{\sigma\wedge\rho}z\rangle_{\sigma\in\mathcal{S}}=\langle P_{A\wedge\sigma}(z)\rangle_{\sigma\in\mathcal{S}}.$$

So

$$\begin{aligned}R_A(\boldsymbol{w}\times\boldsymbol{P}z)&=R_A(\boldsymbol{w})\times R_A(\boldsymbol{P}z)&&\text{(623Ba)}\\&=\langle w'_\sigma\times P_{A\wedge\sigma}z\rangle_{\sigma\in\mathcal{S}}.\end{aligned}$$

If $\tau\in\mathcal{S}$, then

$$w'_\tau=\lim_{\rho\downarrow A}w_{\tau\wedge\rho}=\lim_{\rho\downarrow A\wedge\tau}w_\rho\in L^0(\mathfrak{A}_{A\wedge\tau})$$

by 613Bj. So $w'_\tau\in L^1_{\bar\nu}$ iff $w'_\tau\times P_{A\wedge\tau}z\in L^1_{\bar\mu}$, by (b) above. Thus $R_A(\boldsymbol{w})$ is an $L^1_{\bar\nu}$-process iff $R_A(\boldsymbol{w}\times\boldsymbol{P}z)$ is an $L^1_{\bar\mu}$-process.

Suppose that this is the case. If $\sigma\le\tau$ in $\mathcal{S}$,

$$Q_\sigma(w'_\tau)\times P_{A\wedge\sigma}z=Q_\sigma Q_A(w'_\tau)\times P_{A\wedge\sigma}z$$

(because $w'_\tau\in L^0(\mathfrak{A}_{A\wedge\tau})\subseteq L^0(\mathfrak{A}_A)$)

$$= Q_{A\wedge\sigma}(w'_\tau) \times P_{A\wedge\sigma}z = P_{A\wedge\sigma}(w'_\tau \times z)$$

(by (a))

$$= P_{A\wedge\sigma\wedge\tau}(w'_\tau \times z) = P_\sigma P_{A\wedge\tau}(w'_\tau \times z) = P_\sigma(w'_\tau \times P_{A\wedge\tau}z).$$

But now we see that

$$R_A(\boldsymbol{w}) \text{ is a } \bar\nu\text{-martingale} \iff Q_\sigma w'_\tau = w'_\sigma \text{ whenever } \sigma \le \tau$$

$$\iff Q_\sigma w'_\tau \times P_{A\wedge\sigma}z = w'_\sigma \times P_{A\wedge\sigma}z \text{ whenever } \sigma \le \tau$$

(because $[\![P_{A\wedge\sigma}z > 0]\!] = 1$, as noted in (b))

$$\iff P_\sigma(w'_\tau \times P_{A\wedge\tau}z) = w'_\sigma \times P_{A\wedge\sigma}z \text{ whenever } \sigma \le \tau$$

$$\iff R_A(\boldsymbol{w} \times \boldsymbol{P}z) \text{ is a } \bar\mu\text{-martingale}$$

which is what I set out to prove. **Q**

(d)(i) If $\boldsymbol{w}$ is an approximately local $\bar\nu$-martingale then $\boldsymbol{w} \times \boldsymbol{P}z$ is an approximately local $\bar\mu$-martingale. **P** If $\tau \in \mathcal{S}$ and $\epsilon > 0$, there is a $\delta > 0$ such that $\bar\mu a \le \epsilon$ whenever $a \in \mathfrak{A}$ and $\bar\nu a \le \delta$. Now there is a non-empty downwards-directed set $A \subseteq \mathcal{S}$ such that $\sup_{\rho\in A} \bar\nu[\![\rho < \tau]\!] \le \delta$ and $R_A(\boldsymbol{w})$ is a $\bar\nu$-martingale (623J). By (c), $R_A(\boldsymbol{w} \times \boldsymbol{P}z)$ is a $\bar\mu$-martingale, while $\sup_{\rho\in A} \bar\mu[\![\rho < \tau]\!] \le \epsilon$; as τ and ϵ are arbitrary, $\boldsymbol{w} \times \boldsymbol{P}z$ is an approximately local $\bar\mu$-martingale. **Q**

(ii) If $\boldsymbol{w} \times \boldsymbol{P}z$ is an approximately local $\bar\mu$-martingale then $\boldsymbol{w} \times \boldsymbol{P}z \times \boldsymbol{Q}(\frac{1}{z})$ is an approximately local $\bar\nu$-martingale, where $\boldsymbol{Q}(\frac{1}{z}) = \langle Q_\tau(\frac{1}{z})\rangle_{\tau\in\mathcal{T}}$ as in 625Bc. **P** Apply (i) with $(\bar\mu, \bar\nu, \frac{1}{z}, \boldsymbol{w}\times\boldsymbol{P}z)$ in place of $(\bar\nu, \bar\mu, z, \boldsymbol{w})$. **Q** But $\boldsymbol{P}z \times \boldsymbol{Q}(\frac{1}{z}) = \mathbf{1}$, by 625B(b-iii), so $\boldsymbol{w}$ is an approximately local $\bar\nu$-martingale.

(iii) Thus $\boldsymbol{w}$ is an approximately local $\bar\nu$-martingale iff $\boldsymbol{w} \times \boldsymbol{P}z$ is an approximately local $\bar\mu$-martingale.

(e) As for virtually local martingales, let $\hat{\mathcal{S}}$ be the covered envelope of $\mathcal{S}$ and $\hat{\boldsymbol{w}}$ the fully adapted extension of $\boldsymbol{w}$ to $\hat{\mathcal{S}}$. Recalling that I interpret $\boldsymbol{w} \times \boldsymbol{P}z$ to be defined on $\operatorname{dom}\boldsymbol{w} \cap \operatorname{dom}\boldsymbol{P}z = \mathcal{S} \cap \mathcal{T} = \mathcal{S}$, so that it is $\boldsymbol{w} \times (\boldsymbol{P}z\restriction\mathcal{S})$, then it is clear that $\boldsymbol{P}z\restriction\hat{\mathcal{S}}$, being fully adapted, must be the fully adapted extension of $\boldsymbol{P}z\restriction\mathcal{S}$ to $\hat{\mathcal{S}}$, and that $\hat{\boldsymbol{w}} \times \boldsymbol{P}z$ is the extension of $\boldsymbol{w} \times \boldsymbol{P}z$ (612Qb). Now we have

$$\boldsymbol{w} \text{ is a virtually local } \bar\nu\text{-martingale}$$

$$\iff \hat{\boldsymbol{w}} \text{ is an approximately local } \bar\nu\text{-martingale} \qquad (623J)$$

$$\iff \hat{\boldsymbol{w}} \times \boldsymbol{P}z \text{ is an approximately local } \bar\mu\text{-martingale} \qquad ((d) \text{ above})$$

$$\iff \boldsymbol{w} \times \boldsymbol{P}z \text{ is a virtually local } \bar\mu\text{-martingale.}$$

This completes the proof.

625D Definition Let $\mathcal{S} \subseteq \mathcal{T}$ be a sublattice. A process with domain $\mathcal{S}$ is a **semi-martingale** if it is expressible as the sum of a virtually local martingale and a process which is locally of bounded variation (both, of course, with domain $\mathcal{S}$).

Warning! The standard definition of 'semimartingale' (no hyphen) is a process which is the sum of a local martingale, in the sense of 622Cc, and a process which is locally of bounded variation. Most presentations of the theory take it for granted that the conditions of 632Ib below will be satisfied, so that the distinction vanishes. Nevertheless, to limit the opportunities for confusion, I will try to be consistent in hyphenating 'semi-martingale' when I have the 'virtually local martingale' form in mind.

625E Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. The set of semi-martingales with domain $\mathcal{S}$ is a linear subspace of the space of local integrators with domain $\mathcal{S}$. In particular, every semi-martingale is locally moderately oscillatory.

proof Since every virtually local martingale and every process of locally bounded variation is a local integrator (623Kd, 616Ra) and the sum of two local integrators is a local integrator (616Qa), every semi-martingale is a local integrator. Since sums and scalar multiples of virtually local martingales are virtually local martingales (623K(a-ii)) and sums and scalar multiples of processes which are locally of bounded variation are again locally of bounded variation (614Q(b-iii)), the set of semi-martingales is a linear subspace of $L^0(\mathfrak{A})^{\mathcal{S}}$.

625F Theorem A semi-martingale remains a semi-martingale under any change of law.

proof Let $\bar\nu$ be a functional such that $(\mathfrak{A},\bar\nu)$ is a probability algebra. Suppose that $\boldsymbol{v}$ is a $\bar\nu$-semi-martingale. Express it as $\boldsymbol{v}_1+\boldsymbol{v}_2$ where $\boldsymbol{v}_1$ is a virtually local $\bar\nu$-martingale and $\boldsymbol{v}_2$ is locally of bounded variation. Let $z\in L^1_{\bar\mu}$ be the Radon-Nikodým derivative of $\bar\nu$ with respect to $\bar\mu$, as in 625B. As in 625B, write $Q_\tau:L^1_{\bar\nu}\to L^1_{\bar\nu}\cap L^0(\mathfrak{A}_\tau)$ for the conditional expectation associated with $\mathfrak{A}_\tau$ with respect to $\bar\nu$, and $\boldsymbol{Q}(\frac{1}{z})$ for the $\bar\nu$-martingale $\langle Q_\tau(\frac{1}{z})\rangle_{\tau\in\mathcal{T}}$. As in (d-ii) of the proof of 625C, $\boldsymbol{Q}(\frac{1}{z})\times\boldsymbol{P}z=\mathbf{1}$.

Set $\boldsymbol{w}=\boldsymbol{v}_1\times\boldsymbol{Q}(\frac{1}{z})$. Applying 625C with $(\bar\mu,\bar\nu,z,\boldsymbol{P})$ replaced by $(\bar\nu,\bar\mu,\frac{1}{z},\boldsymbol{Q})$, we see that $\boldsymbol{w}$ is a virtually local $\bar\mu$-martingale and

$$\boldsymbol{v}_1=\boldsymbol{v}_1\times\boldsymbol{Q}(\tfrac{1}{z})\times\boldsymbol{P}z=\boldsymbol{w}\times\boldsymbol{P}z.$$

Writing $w_\downarrow$ and $P_\downarrow z$ for the starting values of the local integrators $\boldsymbol{w}$ (625E) and $\boldsymbol{P}z\restriction\mathcal{S}$, and $[\boldsymbol{w}\overset{*}{|}\boldsymbol{P}z]$ for the covariation of $\boldsymbol{w}$ and $\boldsymbol{P}z$,

$$\begin{aligned}\boldsymbol{v}&=\boldsymbol{w}\times\boldsymbol{P}z+\boldsymbol{v}_2\\&=[\boldsymbol{w}\overset{*}{|}\boldsymbol{P}z]+ii_{\boldsymbol{P}z}(\boldsymbol{w})+ii_{\boldsymbol{w}}(\boldsymbol{P}z)+(w_\downarrow\times P_\downarrow z)\mathbf{1}+\boldsymbol{v}_2\end{aligned}$$

by 617Ka. But $ii_{\boldsymbol{P}z}(\boldsymbol{w})$ and $ii_{\boldsymbol{w}}(\boldsymbol{P}z)$ are virtually local $\bar\mu$-martingales, by 623O, while $[\boldsymbol{w}\overset{*}{|}\boldsymbol{P}z]$ is locally of bounded variation (617L). Thus $\boldsymbol{v}$ is the sum of three processes which are locally of bounded variation and two virtually local $\bar\mu$-martingales, so is a $\bar\mu$-semi-martingale.

Similarly, any $\bar\mu$-semi-martingale is a $\bar\nu$-semi-martingale.

625X Basic exercises (a) Suppose that $T=[0,\infty[$ and $\mathfrak{A}=\{0,1\}$, as in 613W, 615Xf, 616Xa and 622Xd. Let $f:[0,\infty[\,\to\mathbb{R}$ be a function. Show that f corresponds to a semi-martingale iff $f\restriction[0,t]$ is of bounded variation for every $t\ge0$.

(b) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a semi-martingale. Show that $\boldsymbol{v}\restriction\mathcal{S}'$ is a semi-martingale for any ideal $\mathcal{S}'$ of $\mathcal{S}$.

(c) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u}$ a process with domain $\mathcal{S}$, and z an element of $L^0(\bigcap_{\sigma\in\mathcal{S}}\mathfrak{A}_\sigma)$. Show that if $\boldsymbol{u}$ is a semi-martingale then $z\boldsymbol{u}$ is a semi-martingale.

(d) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u}$ a locally moderately oscillatory process and $\boldsymbol{v}$ a semi-martingale, both with domain $\mathcal{S}$. Show that $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is a semi-martingale.

(e) Let $\boldsymbol{v}$, $\boldsymbol{w}$ be semi-martingales with the same domain. Show that $\boldsymbol{v}\times\boldsymbol{w}$ is a semi-martingale.

625 Notes and comments I have said repeatedly that stochastic integration is law-independent. At the same time it is intimately entwined with the theory of martingales, which are emphatically not law-independent. 625B is a brisk run through the formulae we need if we are to move freely between different probability measures on a fixed algebra. In 625F we find that the concept of 'semi-martingale' again turns out, remarkably, to be law-independent. The ideas on stochastic processes required to state the result are not trivial, but they do not mention any kind of integration; while the proof depends on an excursion through most of the theory of the Riemann-sum integral so far developed. I do not know whether there is anything one could call an 'elementary' proof of 625F.

There is a great deal more to be said about semi-martingales. In 625X I offer a handful of tasters. But these will be dramatically upstaged by 627Q below.

Version of 21.1.25

626 Submartingales and previsible variations

Turning to submartingales, I start with the elementary theory (626B-626G). Serious work begins with what I call 'previsible variations' (626J-626K), based on a new adapted interval function $P\Delta\boldsymbol{v}$ (626H-626I). Now the final formula of §621 gives us the celebrated Doob-Meyer decomposition theorem (626M, 626O). The computation of previsible variations can be difficult, but I give some basic special cases (626Q, 626S and 626T).

626A Notation $(\mathfrak{A}, \bar{\mu}, T, \langle\mathfrak{A}_t\rangle_{t\in T}, \mathcal{T}, \langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ is a stochastic integration structure. For $t \in T$, $\check{t}$ is the constant stopping time at t (611A(b-ii)). If $\mathcal{S}$ is a sublattice of $\mathcal{T}$, $\mathcal{I}(\mathcal{S})$ is the set of finite sublattices of $\mathcal{S}$. If $\sigma \le \tau$ in $\mathcal{T}$, $c(\sigma, \tau)$ is the corresponding stopping time interval (611J).

On the L-space $L^1_{\bar{\mu}} = L^1(\mathfrak{A}, \bar{\mu})$ we have the integral $\mathbb{E}$ defined by $\bar{\mu}$, giving rise to the functional $\theta(w) = \mathbb{E}(|w| \wedge \chi 1)$ for $w \in L^0(\mathfrak{A})$, so that θ defines the topology of convergence in measure on $L^0(\mathfrak{A})$. For $\tau \in \mathcal{T}$, P_τ is the conditional expectation associated with $\mathfrak{A}_\tau$. If $w \in L^1_{\bar{\mu}}$, $\boldsymbol{P}w$ is the martingale $\langle P_\tau w\rangle_{\tau\in\mathcal{T}}$.

If $\mathcal{S}$ is a sublattice of $\mathcal{T}$, $M_{\mathrm{fa}}(\mathcal{S})$ will be the space of fully adapted processes with domain $\mathcal{S}$, and $M_{\mathrm{lmo}}(\mathcal{S}) \subseteq M_{\mathrm{fa}}(\mathcal{S})$ the space of locally moderately oscillatory processes.

We shall need to look at the norm and weak topologies on $L^1_{\bar{\mu}}$ as well as the topology of convergence in measure on $L^0(\mathfrak{A})$. It will therefore be helpful to have a notation which distinguishes between the three corresponding notions of limit. I will use 'lim' for limits in $L^0(\mathfrak{A})$ for the topology of convergence in measure, '1lim' for limits in $L^1_{\bar{\mu}}$ for the norm topology defined by $\| \|_1$, and 'w1lim' for limits in $L^1_{\bar{\mu}}$ for the weak topology $\mathfrak{T}_s(L^1_{\bar{\mu}}, L^\infty(\mathfrak{A}))$ (365Lc[6]).

626B Definition (Compare 621Db.) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a fully adapted process. $\boldsymbol{v}$ is a **submartingale** if it is an L^1-process (definition: 622Ca) and $v_\sigma \le P_\sigma v_\tau$ whenever $\sigma \le \tau$ in $\mathcal{S}$.

Clearly every martingale is a submartingale, and every non-decreasing L^1-process is a submartingale (because if $\sigma \le \tau$ and $v_\sigma \le v_\tau$, then $P_\sigma v_\tau - v_\sigma = P_\sigma(v_\tau - v_\sigma) \ge 0$).

626C Elementary facts Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a submartingale.

(a) $\mathbb{E}(v_\sigma) \le \mathbb{E}(P_\sigma v_\tau) = \mathbb{E}(v_\tau)$ whenever $\sigma \le \tau$ in $\mathcal{S}$.

(b) If $\mathcal{S}'$ is a sublattice of $\mathcal{S}$, then $\boldsymbol{v}{\restriction}\mathcal{S}'$ is a submartingale. (Immediate from 626B.)

(c) If $h : \mathbb{R} \to \mathbb{R}$ is a convex function, $\bar{h}\boldsymbol{v} = \langle\bar{h}(v_\sigma)\rangle_{\sigma\in\mathcal{S}}$ (612B) is an L^1-process and *either* h is non-decreasing *or* $\boldsymbol{v}$ is a martingale, then $\bar{h}\boldsymbol{v}$ is a submartingale. **P** If $\sigma \le \tau$ in $\mathcal{S}$ then $\bar{h}(P_\sigma v_\tau) \le P_\sigma(\bar{h}(v_\tau))$ by Jensen's inequality (621Cd). If h is non-decreasing, then $\bar{h}(v_\sigma) \le \bar{h}(P_\sigma v_\tau)$ because $v_\sigma \le P_\sigma v_\tau$; if $\boldsymbol{v}$ is a martingale, then $\bar{h}(v_\sigma) = \bar{h}(P_\sigma v_\tau)$. So in either case we have $\bar{h}(v_\sigma) \le P_\sigma(\bar{h}(v_\tau))$. **Q**

In particular, $\alpha\boldsymbol{v}$ is a submartingale for every $\alpha \ge 0$.

(d) If $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is another submartingale, $\boldsymbol{u} + \boldsymbol{v}$ is a submartingale. (Immediate from 626B, because conditional expectations are linear operators.)

626D Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a submartingale.

(a) If $\boldsymbol{v}$ is $\| \|_1$-bounded, it is an integrator.

(b) If $\mathcal{S}$ has a greatest element and $\{\mathbb{E}(v_\sigma) : \sigma \in \mathcal{S}\}$ is bounded below, $\boldsymbol{v}$ is $\| \|_1$-bounded.

(c) If $\{\mathbb{E}(v_\sigma) : \sigma \in \mathcal{S}\}$ is bounded below, $\boldsymbol{v}$ is a local integrator.

proof (a) (Cf. 622G) If $\mathcal{S}$ is empty, this is trivial. Otherwise, set $\beta = \sup_{\sigma\in\mathcal{S}} \|v_\sigma\|_1$. If $z \in Q_{\mathcal{S}}(\boldsymbol{v})$, it is expressible as $\sum_{i=0}^{n-1} u_i \times (v_{\tau_{i+1}} - v_{\tau_i})$ where $\tau_0 \le \ldots \le \tau_n$ in $\mathcal{S}$ and $u_i \in L^0(\mathfrak{A}_{\tau_i})$ and $\|u_i\|_\infty \le 1$ for every $i < n$ (616C(ii)). By 621Hg,

$$\bar{\mu}[\![|z| > \gamma]\!] \le \frac{66}{\gamma}\|v_{\tau_n}\|_1 - \frac{34}{\gamma}\mathbb{E}(v_{\tau_0}) \le \frac{100}{\gamma}\beta.$$

[6] Formerly 365Mc.

So

$$\inf_{\gamma>0}\sup_{z\in Q_{\mathcal{S}}(\boldsymbol{v})}\bar{\mu}[\![|z|>\gamma]\!]=0$$

and $Q_{\mathcal{S}}(\boldsymbol{v})$ is topologically bounded (613B(f-ii)), that is, $\boldsymbol{v}$ is an integrator.

(b) If $\alpha=\inf_{\sigma\in\mathcal{S}}\mathbb{E}(v_\sigma)$, and $\sigma\in\mathcal{S}$, then

$$v_\sigma\le P_\sigma v_{\max\mathcal{S}},\quad v_\sigma^+\le(P_\sigma v_{\max\mathcal{S}})^+\le P_\sigma v_{\max\mathcal{S}}^+\le P_\sigma|v_{\max\mathcal{S}}|$$

and

$$\|v_\sigma\|_1=\mathbb{E}(v_\sigma^+)+\mathbb{E}(v_\sigma^-)=2\mathbb{E}(v_\sigma^+)-\mathbb{E}(v_\sigma)\le 2\mathbb{E}(P_\sigma|v_{\max\mathcal{S}}|)-\alpha=2\mathbb{E}(|v_{\max\mathcal{S}}|)-\alpha.$$

(c) Take $\tau\in\mathcal{S}$. Since of course $\boldsymbol{v}\restriction\mathcal{S}\wedge\tau$ is a submartingale (626Cb), (b) tells us that $\boldsymbol{v}\restriction\mathcal{S}\wedge\tau$ is $\|\ \|_1$-bounded and (a) tells us that $\boldsymbol{v}\restriction\mathcal{S}\wedge\tau$ is an integrator. As τ is arbitrary, $\boldsymbol{v}$ is a local integrator.

626E From 626D we see that a submartingale $\boldsymbol{v}$ will often be moderately oscillatory (616Ib), so will have limits along directed sets (615G). But for downwards-directed sets we can look for more.

Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a submartingale and $A\subseteq\mathcal{S}$ a non-empty downwards-directed set such that $\{\mathbb{E}(v_\sigma):\sigma\in A\}$ is bounded below. Then the $\|\ \|_1$-limit $1\lim_{\sigma\downarrow A}v_\sigma$ is defined and equal to the limit $\lim_{\sigma\downarrow A}v_\sigma$ for the topology of convergence in measure.

proof Let $\epsilon>0$. Then there is a $\tau\in A$ such that $\|v_\sigma-v_\tau\|_1\le 3\epsilon$ whenever $\sigma\in A$ and $\sigma\le\tau$. **P** Set $\gamma=\inf_{\sigma\in A}\mathbb{E}(v_\sigma)$. Let $\tau_0\in A$ be such that $\mathbb{E}(v_{\tau_0})\le\gamma+\epsilon$. Then $1\lim_{\sigma\downarrow A}P_\sigma v_{\tau_0}$ is defined (621C(g-i)); let $\tau\in A$ be such that $\tau\le\tau_0$ and $\|P_\sigma v_{\tau_0}-P_\tau v_{\tau_0}\|_1\le\epsilon$ whenever $\sigma\in A$ and $\sigma\le\tau$. In this case, if $\sigma\in A$ and $\sigma\le\tau$, we have

$$\begin{aligned}\|v_\sigma-v_\tau\|_1&\le\|v_\sigma-P_\sigma v_{\tau_0}\|_1+\|P_\sigma v_{\tau_0}-P_\tau v_{\tau_0}\|_1+\|P_\tau v_{\tau_0}-v_\tau\|_1\\&\le\mathbb{E}(P_\sigma v_{\tau_0}-v_\sigma)+\epsilon+\mathbb{E}(P_\tau v_{\tau_0}-v_\tau)\\&=\mathbb{E}(v_{\tau_0}-v_\sigma)+\epsilon+\mathbb{E}(v_{\tau_0}-v_\tau)\le 2(\mathbb{E}(v_{\tau_0})-\gamma)+\epsilon\le 3\epsilon.\ \mathbf{Q}\end{aligned}$$

As $L^1_{\bar{\mu}}$ is complete under $\|\ \|_1$, the limit $1\lim_{\sigma\downarrow A}v_\sigma$ is defined. As the embedding $L^1_{\bar{\mu}}\subsetneqq L^0(\mathfrak{A})$ is continuous, this is also the limit $\lim_{\sigma\downarrow A}v_\sigma$.

626F Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a submartingale such that $\{\mathbb{E}(v_\sigma):\sigma\in\mathcal{S}\}$ is bounded below. Let $A\subseteq\mathcal{S}$ be a non-empty downwards-directed set and $R_A:M_{\text{lmo}}(\mathcal{S})\to M_{\text{lmo}}(\mathcal{S})$ the corresponding operator as described in 623B. Then $R_A(\boldsymbol{v})$ is defined and is a submartingale.

proof Because $\boldsymbol{v}$ is a local integrator (626Dc), it is locally moderately oscillatory (616Ib) and $R_A(\boldsymbol{v})$ is defined; express it as $\langle v_{A\sigma}\rangle_{\sigma\in\mathcal{S}}$. If $\sigma\le\tau$ in $\mathcal{S}$ and $\rho\in A$, $\sigma\wedge\rho\le\tau\wedge\rho$ and

$$v_{\sigma\wedge\rho}\le P_{\sigma\wedge\rho}v_{\tau\wedge\rho}=P_\sigma P_\rho v_{\tau\wedge\rho}=P_\sigma v_{\tau\wedge\rho}.$$

Now (because $\{\mathbb{E}(v_\sigma):\sigma\in\mathcal{S}\}$ is bounded below) 626E tells us that

$$v_{A\sigma}=\lim_{\rho\downarrow A}v_{\sigma\wedge\rho}=1\lim_{\rho\downarrow A}v_{\sigma\wedge\rho},\quad v_{A\tau}=1\lim_{\rho\downarrow A}v_{\tau\wedge\rho}$$

and therefore

$$P_\sigma v_{A\tau}=1\lim_{\rho\downarrow A}P_\sigma v_{\tau\wedge\rho}.$$

Accordingly

$$P_\sigma v_{A\tau}-v_{A\sigma}=1\lim_{\rho\downarrow A}P_\sigma v_{\tau\wedge\rho}-v_{\sigma\wedge\rho}\ge 0$$

and $v_{A\sigma}\le P_\sigma v_{A\tau}$. As σ and τ are arbitrary, $R_A(\boldsymbol{v})$ is a submartingale.

626G Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a submartingale. Let $\hat{\boldsymbol{v}}=\langle\hat{v}_\tau\rangle_{\tau\in\hat{\mathcal{S}}}$ be the fully adapted extension of $\boldsymbol{v}$ to the covered envelope $\hat{\mathcal{S}}$ of $\mathcal{S}$, and $\hat{\mathcal{S}}_f$ the finitely-covered envelope of $\mathcal{S}$.

(a) $\hat{\boldsymbol{v}}\restriction\hat{\mathcal{S}}_f$ is a submartingale.

(b) If $\boldsymbol{v}$ is $\|\ \|_1$-bounded then $\hat{\boldsymbol{v}}$ is $\|\ \|_1$-bounded.

(c) If $\mathcal{S}$ has a greatest element and $\{\mathbb{E}(v_\sigma) : \sigma \in \mathcal{S}\}$ is bounded below, then $\hat{\boldsymbol{v}}$ is a submartingale.

proof (a)(i) $\hat{\boldsymbol{v}}\restriction\hat{\mathcal{S}}_f$ **is an** L^1-**process.** **P** If $\tau \in \hat{\mathcal{S}}_f$, there is a finite set $J \subseteq \mathcal{S}$ such that

$$1 = \sup_{\sigma\in J} [\![\tau = \sigma]\!] \subseteq \sup_{\sigma\in J} [\![\hat{v}_\tau = v_\sigma]\!] \subseteq [\![|\hat{v}_\tau| \le \sup_{\sigma\in J} |v_\sigma|]\!]$$

so $\|\hat{v}_\tau\|_1 \le \sum_{\sigma\in J} \|v_\sigma\|_1$ is finite. **Q**

(ii) If $\tau \in \hat{\mathcal{S}}_f$, $\sigma_0 \le \ldots \le \sigma_n$ in $\mathcal{S}$ and $\sup_{i\le n} [\![\tau = \sigma_i]\!] = 1$, then $\hat{v}_{\sigma\wedge\tau} \le P_{\sigma\wedge\tau}\hat{v}_\tau$ for every $\sigma \in \mathcal{S}$. **P** Induce on n. The induction starts with $n = 0$ and $\tau = \sigma_0$ and $v_{\sigma\wedge\sigma_0} \le P_{\sigma\wedge\sigma_0} v_{\sigma_0}$.

(α) For the inductive step to $n \ge 1$, set $\tau' = \tau \wedge \sigma_{n-1}$ and $d = [\![\sigma_{n-1} < \tau]\!] = [\![\tau' < \tau]\!]$. Then $d \in \mathfrak{A}_{\sigma_{n-1}}$ and $d \subseteq [\![\tau' = \sigma_{n-1}]\!] \cap [\![\tau = \sigma_n]\!]$. Now

$$\sup_{i<n} [\![\tau' = \sigma_i]\!] \supseteq \sup_{i<n} [\![\tau = \sigma_i]\!] \cup d = 1.$$

So the inductive hypothesis assures us that $\hat{v}_{\sigma\wedge\tau'} \le P_{\sigma\wedge\tau'}\hat{v}_{\tau'}$. Next,

$$\begin{aligned} P_{\sigma_{n-1}}\hat{v}_\tau - \hat{v}_{\tau'} &= P_{\sigma_{n-1}}(\hat{v}_\tau - \hat{v}_{\tau'}) = P_{\sigma_{n-1}}((\hat{v}_\tau - \hat{v}_{\tau'}) \times \chi d) \\ &= P_{\sigma_{n-1}}((v_{\sigma_n} - v_{\sigma_{n-1}}) \times \chi d) = P_{\sigma_{n-1}}(v_{\sigma_n} - v_{\sigma_{n-1}}) \times \chi d \\ &= (P_{\sigma_{n-1}} v_{\sigma_n} - v_{\sigma_{n-1}}) \times \chi d \ge 0 \end{aligned}$$

and

$$P_{\sigma\wedge\tau'}\hat{v}_\tau = P_{\sigma\wedge\tau'\wedge\sigma_{n-1}}\hat{v}_\tau = P_{\sigma\wedge\tau'}P_{\sigma_{n-1}}\hat{v}_\tau \ge P_{\sigma\wedge\tau'}\hat{v}_{\tau'} \ge \hat{v}_{\sigma\wedge\tau'}.$$

(β) Set $b = [\![\sigma \le \sigma_{n-1}]\!]$. Then

$$\begin{aligned} b &\subseteq [\![\sigma\wedge\tau = \sigma\wedge\sigma_{n-1}\wedge\tau]\!] = [\![\sigma\wedge\tau = \sigma\wedge\tau']\!] \\ &\subseteq [\![\hat{v}_{\sigma\wedge\tau} = \hat{v}_{\sigma\wedge\tau'}]\!] \cap [\![P_{\sigma\wedge\tau}\hat{v}_\tau = P_{\sigma\wedge\tau'}\hat{v}_\tau]\!] \subseteq [\![\hat{v}_{\sigma\wedge\tau} = \hat{v}_{\sigma\wedge\tau'}]\!] \cap [\![P_{\sigma\wedge\tau}\hat{v}_\tau \ge \hat{v}_{\sigma\wedge\tau'}]\!] \end{aligned}$$

(using the last formula in (α))

$$\subseteq [\![P_{\sigma\wedge\tau}\hat{v}_\tau \ge \hat{v}_{\sigma\wedge\tau}]\!].$$

(γ) Set

$$b' = [\![\sigma_{n-1} \le \sigma]\!] \cap [\![\sigma < \tau]\!] \in \mathfrak{A}_\sigma.$$

Then

$$P_{\sigma\wedge\tau}\hat{v}_\tau \times \chi b' = P_\sigma P_\tau \hat{v}_\tau \times \chi b' = P_\sigma \hat{v}_\tau \times \chi b' = P_\sigma(\hat{v}_\tau \times \chi b')$$

(because $b' \in \mathfrak{A}_\sigma$)

$$= P_\sigma(v_{\sigma_n} \times \chi b')$$

(because $b' \subseteq d \subseteq [\![\tau = \sigma_n]\!]$)

$$\begin{aligned} &= P_\sigma v_{\sigma_n} \times \chi b' = P_\sigma P_{\sigma_n} v_{\sigma_n} \times \chi b' \\ &= P_{\sigma\wedge\sigma_n} v_{\sigma_n} \times \chi b' \ge v_{\sigma\wedge\sigma_n} \times \chi b' = \hat{v}_{\sigma\wedge\tau} \times \chi b' \end{aligned}$$

because $b' \subseteq [\![\sigma\wedge\sigma_n = \sigma\wedge\tau]\!]$.

(δ) Set $b'' = [\![\tau \le \sigma]\!]$. Then

$$b'' = [\![\sigma\wedge\tau = \tau]\!] \subseteq [\![\hat{v}_{\sigma\wedge\tau} = \hat{v}_\tau]\!] \cap [\![P_{\sigma\wedge\tau}\hat{v}_\tau = P_\tau\hat{v}_\tau]\!] \subseteq [\![\hat{v}_{\sigma\wedge\tau} = P_{\sigma\wedge\tau}\hat{v}_\tau]\!].$$

(ϵ) Now observe that $b \cup b' \cup b'' = 1$. So $[\![\hat{v}_{\sigma\wedge\tau} \le P_{\sigma\wedge\tau}\hat{v}_\tau]\!] = 1$, $\hat{v}_{\sigma\wedge\tau} \le P_{\sigma\wedge\tau}\hat{v}_\tau$ and the induction proceeds. **Q**

(iii) With 611Pd, this tells us that $\hat{v}_{\sigma\wedge\tau} \le P_{\sigma\wedge\tau}\hat{v}_\tau$ whenever $\sigma \in \mathcal{S}$ and $\tau \in \hat{\mathcal{S}}_f$. Now suppose that $\tau' \le \tau$ in $\hat{\mathcal{S}}_f$ and $\sigma \in \mathcal{S}$. Then

$$[\![\tau' = \sigma]\!] \subseteq [\![\sigma \wedge \tau = \tau']\!]$$
$$\subseteq [\![\hat{v}_{\sigma\wedge\tau} = \hat{v}_{\tau'}]\!] \cap [\![P_{\sigma\wedge\tau}\hat{v}_\tau = P_{\tau'}\hat{v}_\tau]\!] \subseteq [\![\hat{v}_{\tau'} \le P_{\tau'}\hat{v}_\tau]\!].$$

Taking the supremum over σ, $[\![v_{\tau'} \le P_{\tau'}v_\tau]\!] = 1$ and $v_{\tau'} \le P_{\tau'}v_\tau$. As τ' and τ are arbitrary, $\hat{\boldsymbol{v}}\restriction\hat{\mathcal{S}}_f$ is a submartingale.

(b)(i) Set $\gamma = \sup_{\sigma\in\mathcal{S}}\|v_\sigma\|_1$. Then $\|\hat{v}_\tau\|_1 \le 3\gamma$ for every $\tau \in \hat{\mathcal{S}}_f$. **P** If $\tau \in \hat{\mathcal{S}}_f$, there are σ, $\sigma' \in \mathcal{S}$ such that $\sigma \le \tau \le \sigma'$, by 611Pe. Now $\hat{v}_\tau \le P_\tau v_{\sigma'}$, by (a) above, so

$$\mathbb{E}(\hat{v}_\tau^+) \le \|P_\tau v_{\sigma'}\|_1 \le \|v_{\sigma'}\|_1.$$

On the other side, $v_\sigma \le P_\sigma\hat{v}_\tau$ so

$$\mathbb{E}(\hat{v}_\tau) = \mathbb{E}(P_\sigma\hat{v}_\tau) \ge \mathbb{E}(v_\sigma)$$

and

$$\|\hat{v}_\tau\|_1 = 2\mathbb{E}(\hat{v}_\tau^+) - \mathbb{E}(v_\tau) \le 2\|v_{\sigma'}\|_1 - \mathbb{E}(v_\sigma) \le 2\|v_{\sigma'}\|_1 + \|v_\sigma\|_1 \le 3\gamma. \text{ **Q**}$$

(ii) Since $\{\hat{v}_\tau : \tau \in \hat{\mathcal{S}}\} \subseteq \overline{\{\hat{v}_\tau : \tau \in \hat{\mathcal{S}}_f\}}$ (613B(q-ii)), and $\|\;\|_1$-balls are closed in $L^0(\mathfrak{A})$ (613Bc), $\sup_{\tau\in\hat{\mathcal{S}}}\|\hat{v}_\tau\|_1 \le 3\gamma$ is finite and $\hat{\boldsymbol{v}}$ is $\|\;\|_1$-bounded.

(c)(i) We know from 626Db and (b) here that $\hat{\boldsymbol{v}}$ is an L^1-process. Suppose that $\tau \in \hat{\mathcal{S}}$ and $\sigma \in \mathcal{S}$. Then $\hat{v}_{\tau\wedge\sigma} \le P_{\tau\wedge\sigma}\hat{v}_\tau$. **P** Let $\epsilon > 0$. Then there is a $\delta \in \,]0,\epsilon]$ such that $\mathbb{E}(|v_{\max\mathcal{S}} - \hat{v}_\tau| \times \chi a) \le \epsilon$ whenever $\bar{\mu}a \le \delta$. As $\sup_{\rho\in\mathcal{S}}[\![\tau = \rho]\!] = 1$, there is a finite set $J \subseteq \mathcal{S}$ such that $a = 1 \setminus \sup_{\rho\in J}[\![\tau = \rho]\!]$ has measure at most δ. Since $a \in \mathfrak{A}_\tau \subseteq \mathfrak{A}_{\max\mathcal{S}}$, there is a $\tau' \in \mathcal{T}$ such that

$$1 \setminus a \subseteq [\![\tau' = \tau]\!], \quad a \subseteq [\![\tau' = \max\mathcal{S}]\!].$$

Now

$$\|v_{\tau'} - v_\tau\|_1 \le \mathbb{E}(\chi a \times |v_{\max\mathcal{S}} - v_\tau|) \le \epsilon$$

and $\tau' \in \hat{\mathcal{S}}_f$, so $\hat{v}_{\sigma\wedge\tau'} \le P_{\sigma\wedge\tau'}\hat{v}_{\tau'}$ and

$$1 \setminus a \subseteq [\![\hat{v}_{\sigma\wedge\tau} \le P_{\sigma\wedge\tau}\hat{v}_{\tau'}]\!], \quad \theta((\hat{v}_{\sigma\wedge\tau} - P_{\sigma\wedge\tau}\hat{v}_{\tau'})^+) \le \bar{\mu}a \le \epsilon.$$

On the other hand,

$$\theta(P_{\sigma\wedge\tau}\hat{v}_{\tau'} - P_{\sigma\wedge\tau}\hat{v}_\tau) \le \|P_{\sigma\wedge\tau}\hat{v}_{\tau'} - P_{\sigma\wedge\tau}\hat{v}_\tau\|_1 \le \|\hat{v}_{\tau'} - \hat{v}_\tau\|_1 \le \epsilon,$$

so

$$\theta((\hat{v}_{\sigma\wedge\tau} - P_{\sigma\wedge\tau}\hat{v}_\tau)^+) \le \theta((\hat{v}_{\sigma\wedge\tau} - P_{\sigma\wedge\tau}\hat{v}_{\tau'})^+ + |P_{\sigma\wedge\tau}\hat{v}_{\tau'} - P_{\sigma\wedge\tau}\hat{v}_\tau|) \le 2\epsilon.$$

As ϵ is arbitrary, $\hat{v}_{\sigma\wedge\tau} \le P_{\sigma\wedge\tau}\hat{v}_\tau$. **Q**

(ii) Repeating the argument of (a-iii), we now see that if $\tau' \in \hat{\mathcal{S}}$ and $\tau' \le \tau$ then $[\![\tau' = \sigma]\!] \subseteq [\![\hat{v}_{\tau'} \le P_{\tau'}\hat{v}_\tau]\!]$ for every $\sigma \in \mathcal{S}$ and $\hat{v}_{\tau'} \le P_{\tau'}\hat{v}_\tau$. So $\hat{\boldsymbol{v}}$ is a submartingale.

626H Proposition Suppose that $\mathcal{S}$ is a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ is an L^1-process. Then we have an adapted interval function (definition: 613C) $P\Delta\boldsymbol{v}$ defined by saying that $(P\Delta\boldsymbol{v})(\sigma,\tau) = P_\sigma v_\tau - v_\sigma$ whenever $\sigma \le \tau$ in $\mathcal{S}$.

proof Of course $P_\sigma v_\tau - v_\sigma \in L^0(\mathfrak{A}_\sigma) \subseteq \mathfrak{A}_\tau$ whenever $\sigma \le \tau$ in $\mathcal{S}$, and $P_\sigma v_\sigma - v_\sigma = 0$ for every $\sigma \in \mathcal{S}$. Suppose that σ, σ', τ, $\tau' \in \mathcal{S}$, $\sigma \le \sigma' \le \tau' \le \tau$ and $b \in \mathfrak{A}_\sigma$ is such that $b \subseteq [\![\sigma = \sigma']\!] \cap [\![\tau' = \tau]\!]$. Then $b \subseteq [\![P_\sigma v_{\tau'} = P_{\sigma'}v_{\tau'}]\!]$ by 622Bb. At the same time,

$$\chi b \times P_\sigma v_\tau = P_\sigma(\chi b \times v_\tau) = P_\sigma(\chi b \times v_{\tau'}) = \chi b \times P_\sigma v_{\tau'}$$

so $b \subseteq [\![P_\sigma v_\tau = P_\sigma v_{\tau'}]\!]$ and $b \subseteq [\![P_\sigma v_\tau = P_{\sigma'}v_{\tau'}]\!]$. Since we also have $b \subseteq [\![v_\sigma = v_{\sigma'}]\!] \cap [\![v_{\tau'} = v_\tau]\!]$, $b \subseteq [\![P_\sigma v_\tau - v_\sigma = P_{\sigma'}v_{\tau'} -$
So the conditions of 613C(a-i) are satisfied by $P\Delta\boldsymbol{v}$.

626I Definitions Corresponding to the interval functions $P\Delta\boldsymbol{v}$ of 626H, I will write $\Delta_e(\boldsymbol{u}, Pd\boldsymbol{v})$, $S_I(\boldsymbol{u}, Pd\boldsymbol{v})$ and $Q_{\mathcal{S}}(Pd\boldsymbol{v})$ for $\Delta_e(\boldsymbol{u}, d(P\Delta\boldsymbol{v}))$ $S_I(\boldsymbol{u}, d(P\Delta\boldsymbol{v}))$ and $Q_{\mathcal{S}}(d(P\Delta\boldsymbol{v}))$ respectively, as in 613F and 616B. Similarly, $\Delta_e(\boldsymbol{u}, |Pd\boldsymbol{v}|)$, $S_I(\boldsymbol{u}, |Pd\boldsymbol{v}|)$ and $Q_{\mathcal{S}}(|Pd\boldsymbol{v}|)$ will mean $\Delta_e(\boldsymbol{u}, d|P\Delta\boldsymbol{v}|)$, $S_I(\boldsymbol{u}, d|P\Delta\boldsymbol{v}|)$ and $Q_{\mathcal{S}}(d|P\Delta\boldsymbol{v}|)$.

Note that an L^1-process $\boldsymbol{v}$ is a submartingale iff $P\Delta\boldsymbol{v} \ge 0$.

626J Previsible variations Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ an L^1-process. If the weak limit

$$v^\#_\tau = \operatorname{w1lim}_{I\uparrow\mathcal{I}(\mathcal{S}\wedge\tau)} S_I(\mathbf{1}, Pd\boldsymbol{v})$$

is defined in $L^1_{\bar\mu}$ for every $\tau \in \mathcal{S}$, I will say that $\boldsymbol{v}^\# = \langle v^\#_\sigma\rangle_{\sigma\in\mathcal{S}}$ is the **previsible variation** of $\boldsymbol{v}$.

626K Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. Write $M_{\text{D-M}} = M_{\text{D-M}}(\mathcal{S})$ for the set of L^1-processes with domain $\mathcal{S}$ which have previsible variations.

(a) If $\boldsymbol{v} \in M_{\text{D-M}}$ then $\boldsymbol{v}^\#$ is an L^1-process (in particular, it is fully adapted) and $\boldsymbol{v} - \boldsymbol{v}^\#$ is a martingale.

(b) $M_{\text{D-M}}$ is a linear subspace of $M_{\text{fa}}(\mathcal{S})$, and the map $\boldsymbol{v} \mapsto \boldsymbol{v}^\# : M_{\text{D-M}} \to M_{\text{fa}}(\mathcal{S})$ is linear.

(c) If $\boldsymbol{v}$ is a martingale with domain $\mathcal{S}$, then $\boldsymbol{v} \in M_{\text{D-M}}$ and $\boldsymbol{v}^\# = 0$.

(d) Suppose that $\boldsymbol{v} \in M_{\text{D-M}}$.

(i) $\boldsymbol{v}$ is locally moderately oscillatory iff $\boldsymbol{v}^\#$ is locally moderately oscillatory.

(ii) $\boldsymbol{v}$ is a local integrator iff $\boldsymbol{v}^\#$ is a local integrator.

(iii) $\boldsymbol{v}$ is a submartingale iff $\boldsymbol{v}^\#$ is a submartingale.

(iv) $\boldsymbol{v}$ is a martingale iff $\boldsymbol{v}^\#$ is a martingale.

(e) If $\boldsymbol{v} \in M_{\text{D-M}}$ then $P\Delta\boldsymbol{v}^\# = P\Delta\boldsymbol{v}$, $\boldsymbol{v}^\# \in M_{\text{D-M}}$ and $(\boldsymbol{v}^\#)^\# = \boldsymbol{v}^\#$.

(f) Suppose that $\boldsymbol{v} \in M_{\text{D-M}}$ and $\rho \in \mathcal{S}$. Express $\boldsymbol{v}^\#$ as $\langle v^\#_\sigma\rangle_{\sigma\in\mathcal{S}}$.

(i) $\boldsymbol{v}\restriction\mathcal{S}\wedge\rho$ has a previsible variation, which is $\boldsymbol{v}^\#\restriction\mathcal{S}\wedge\rho$.

(ii) $\boldsymbol{v}\restriction\mathcal{S}\vee\rho$ has a previsible variation, which is $\langle v^\#_\sigma - v^\#_\rho\rangle_{\sigma\in\mathcal{S}\vee\rho}$.

(g) If $\boldsymbol{v} \in M_{\text{fa}}(\mathcal{S})$ is such that $\boldsymbol{v}\restriction\mathcal{S}\wedge\rho$ has a previsible variation for every $\rho \in \mathcal{S}$, then $\boldsymbol{v} \in M_{\text{D-M}}$.

proof (a) Express $\boldsymbol{v}^\#$ as $\langle v^\#_\sigma\rangle_{\sigma\in\mathcal{S}}$.

(i) If $\sigma \le \sigma' \le \tau$ in $\mathcal{S}$ then $P_\sigma v_{\sigma'} - v_\sigma \in L^0(\mathfrak{A}_\tau)$, so $S_I(\mathbf{1}, Pd\boldsymbol{v}) \in L^0(\mathfrak{A}_\tau)$ whenever $I \in \mathcal{I}(\mathcal{S}\wedge\tau)$. Since $L^0(\mathfrak{A}_\tau) \cap L^1_{\bar\mu} = L^1(\mathfrak{A}_\tau, \bar\mu\restriction\mathfrak{A}_\tau)$ is a norm-closed subspace of $L^1_{\bar\mu}$, therefore weakly closed, it contains $\operatorname{w1lim}_{I\uparrow\mathcal{I}(\mathcal{S}\wedge\tau)} S_I(\mathbf{1}, Pd\boldsymbol{v}) = v^\#_\tau$.

Suppose that $\tau \le \tau'$ in $\mathcal{S}$ and $a = [\![\tau = \tau']\!]$. Then $a \in \mathfrak{A}_\tau$ and $a \subseteq [\![\sigma = \sigma']\!]$ whenever $\tau \le \sigma \le \sigma' \le \tau'$. Accordingly

$$\chi a \times P_\tau(P_\sigma v_{\sigma'} - v_\sigma) = \chi a \times P_\tau(v_{\sigma'} - v_\sigma) = P_\tau(\chi a \times (v_{\sigma'} - v_\sigma)) = 0$$

whenever $\tau \le \sigma \le \sigma' \le \tau'$. But this means that $\chi a \times S_I(\mathbf{1}, Pd\boldsymbol{v}) = 0$ whenever $I \in \mathcal{I}(\mathcal{S}\cap[\tau,\tau'])$. Now if $I \in \mathcal{I}(\mathcal{S}\wedge\tau')$ contains τ, we shall have

$$\chi a \times (S_I(\mathbf{1}, Pd\boldsymbol{v}) - S_{I\wedge\tau}(\mathbf{1}, Pd\boldsymbol{v})) = \chi a \times S_{I\vee\tau}(\mathbf{1}, Pd\boldsymbol{v}) = 0$$

(613G(a-i)). As $u \mapsto \chi a \times u : L^1_{\bar\mu} \to L^1_{\bar\mu}$ is linear and norm-continuous, therefore weakly continuous,

$$\begin{aligned}\chi a \times (v^\#_{\tau'} - v_\tau) &= \chi a \times \operatorname*{w1lim}_{I\uparrow\mathcal{I}(\mathcal{S}\wedge\tau')} (S_I(\mathbf{1}, Pd\boldsymbol{v}) - S_{I\wedge\tau}(\mathbf{1}, Pd\boldsymbol{v})) \\ &= \operatorname*{w1lim}_{I\uparrow\mathcal{I}(\mathcal{S}\wedge\tau')} \chi a \times (S_I(\mathbf{1}, Pd\boldsymbol{v}) - S_{I\wedge\tau}(\mathbf{1}, Pd\boldsymbol{v})) = 0\end{aligned}$$

and $a \subseteq [\![v^\#_\tau = v^\#_{\tau'}]\!]$.

Thus $\boldsymbol{v}^\#$ is fully adapted. Since $v^\#_\tau$ is defined as a weak limit in $L^1_{\bar\mu}$ for every τ, $\boldsymbol{v}^\#$ is an L^1-process.

(ii) If $\tau \le \sigma \le \sigma' \le \tau'$ in $\mathcal{S}$, then $P_\tau(P_\sigma v_{\sigma'} - v_\sigma) = P_\tau(v_{\sigma'} - v_\tau)$; consequently $P_\tau S_I(\mathbf{1}, Pd\boldsymbol{v}) = P_\tau v_{\tau'} - v_\tau$ whenever $I \in \mathcal{I}(\mathcal{S}\cap[\tau,\tau'])$ contains τ and τ'. Since $P_\tau : L^1_{\bar\mu} \to L^1_{\bar\mu}$ is norm-continuous, therefore weakly continuous, $P_\tau(v^\#_{\tau'} - v^\#_\tau) = P_\tau(v_{\tau'} - v_\tau)$, that is, $P_\tau(v_{\tau'} - v^\#_{\tau'}) = v_\tau - v^\#_\tau$. As τ and τ' are arbitrary, $\boldsymbol{v} - \boldsymbol{v}^\#$ is a martingale.

(b) If $\boldsymbol{u}$, $\boldsymbol{v} \in M_{\text{D-M}}$ and $\alpha \in \mathbb{R}$, $\boldsymbol{u} + \boldsymbol{v}$ and $\alpha\boldsymbol{u}$ are L^1-processes. So if $I \in \mathcal{I}(\mathcal{S})$, $S_I(\mathbf{1}, Pd(\boldsymbol{u} + \boldsymbol{v}))$ and $S_I(\mathbf{1}, Pd(\alpha\boldsymbol{u}))$ belong to $L^1_{\bar\mu}$ and are equal to $S_I(\mathbf{1}, Pd\boldsymbol{u}) + S_I(\mathbf{1}, Pd\boldsymbol{v})$ and $\alpha S_I(\mathbf{1}, Pd\boldsymbol{u})$ respectively. As the weak topology on $L^1_{\bar\mu}$ is a linear space topology. the weak limits

$$\operatorname{w1lim}_{I\uparrow\mathcal{S}\wedge\tau} S_I(\mathbf{1}, Pd(\boldsymbol{u} + \boldsymbol{v})), \quad \operatorname{w1lim}_{I\uparrow\mathcal{S}\wedge\tau} S_I(\mathbf{1}, Pd(\alpha\boldsymbol{u}))$$

are defined and equal to

$$\operatorname{w1lim}_{I\uparrow\mathcal{S}\wedge\tau} S_I(\mathbf{1},Pd\boldsymbol{u})+\operatorname{w1lim}_{I\uparrow\mathcal{S}\wedge\tau} S_I(\mathbf{1},Pd\boldsymbol{v}),\quad \alpha\operatorname{w1lim}_{I\uparrow\mathcal{S}\wedge\tau} S_I(\mathbf{1},Pd\boldsymbol{u})$$

for every $\tau\in\mathcal{S}$.

(c) If $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a martingale, then $P_\sigma v_\tau-v_\sigma=0$ whenever $\sigma\le\tau$ in $\mathcal{S}$, so $S_I(\mathbf{1},Pd\boldsymbol{v})=0$ for every $I\in\mathcal{I}(\mathcal{S})$ and $v^\#_\tau$ is defined and zero for every $\tau\in\mathcal{S}$.

(d) We know that $\boldsymbol{v}^\#-\boldsymbol{v}$ is a martingale, therefore a local integrator and locally moderately oscillatory (622H), and $\boldsymbol{v}^\#-\boldsymbol{v}$ is a submartingale. Accordingly $\boldsymbol{v}^\#=\boldsymbol{v}+(\boldsymbol{v}^\#-\boldsymbol{v})$ will have any of these properties iff $\boldsymbol{v}$ does (615F(b-iii), 616Qa, 626Cd, 622Db).

(e) Take σ, $\tau\in\mathcal{S}$ such that $\sigma\le\tau$. As $\boldsymbol{v}-\boldsymbol{v}^\#$ is a martingale,

$$P_\sigma(v_\tau-v^\#_\tau)=v_\sigma-v^\#_\sigma,$$

and

$$(P\Delta\boldsymbol{v}^\#)(\sigma,\tau)=P_\sigma v^\#_\tau-v^\#_\sigma=P_\sigma v_\tau-v_\sigma=(P\Delta\boldsymbol{v})(\sigma,\tau).$$

So $P\Delta\boldsymbol{v}^\#=P\Delta\boldsymbol{v}$. Putting (b) and (c) together, $(\boldsymbol{v}^\#)^\#=\boldsymbol{v}^\#-(\boldsymbol{v}-\boldsymbol{v}^\#)^\#$ is defined and equal to $\boldsymbol{v}^\#$.

(f)(i) If $\sigma\in\mathcal{S}\wedge\rho$ then the calculation

$$v^\#_\sigma=\operatorname{w1lim}_{I\uparrow\mathcal{I}(\mathcal{S}\wedge\sigma)} S_I(\mathbf{1},Pd\boldsymbol{v})$$

yields the same result if we interpret the right-hand side as

$$\operatorname{w1lim}_{I\uparrow\mathcal{I}((\mathcal{S}\wedge\rho)\wedge\sigma)} S_I(\mathbf{1},Pd(\boldsymbol{v}\restriction\mathcal{S}\wedge\rho)).$$

(ii) If $\sigma\in\mathcal{S}\vee\rho$ then for any $I\in\mathcal{I}(\mathcal{S}\wedge\sigma)$ containing ρ we have

$$v^\#_\sigma=\operatorname*{w1lim}_{I\uparrow\mathcal{I}(\mathcal{S}\wedge\sigma)} S_I(\mathbf{1},Pd\boldsymbol{v})=\operatorname*{w1lim}_{I\uparrow\mathcal{I}(\mathcal{S}\wedge\sigma)} S_{I\wedge\rho}(\mathbf{1},Pd\boldsymbol{v})+\operatorname*{w1lim}_{I\uparrow\mathcal{I}(\mathcal{S}\wedge\sigma)} S_{I\vee\rho}(\mathbf{1},Pd\boldsymbol{v})$$

(613G(a-i) again)

$$=\operatorname*{w1lim}_{I\uparrow\mathcal{I}(\mathcal{S}\wedge\rho)} S_I(\mathbf{1},Pd\boldsymbol{v})+\operatorname*{w1lim}_{I\uparrow\mathcal{I}((\mathcal{S}\wedge\sigma)\vee\rho)} S_I(\mathbf{1},Pd\boldsymbol{v})$$

(613K)

$$=v^\#_\rho+\operatorname*{w1lim}_{I\uparrow\mathcal{I}((\mathcal{S}\vee\rho)\wedge\sigma)} S_I(\mathbf{1},Pd(\boldsymbol{v}\restriction\mathcal{S}\vee\rho)).$$

(g) As with (f-i), this is immediate from the definition in 626J.

626L In this context, it is worth having an elementary fact set out in quotable form.

Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ an L^1-process and z a member of $L^\infty(\mathfrak{A})$. Then

$$\mathbb{E}(z\times S_I(\mathbf{1},Pd\boldsymbol{v}))=\mathbb{E}(S_I(\boldsymbol{P}z,d\boldsymbol{v}))$$

for every $I\in\mathcal{I}(\mathcal{S})$.

proof If $\sigma\le\tau$ in $\mathcal{S}$,

$$\mathbb{E}(z\times(P_\sigma v_\tau-v_\sigma))=\mathbb{E}(z\times P_\sigma(v_\tau-v_\sigma))=\mathbb{E}(P_\sigma z\times(v_\tau-v_\sigma))$$

by 621Cb; that is, $\mathbb{E}(z\times\Delta_e(\mathbf{1},Pd\boldsymbol{v}))=\mathbb{E}(\Delta_e(\boldsymbol{P}z,d\boldsymbol{v}))$ for every stopping-time interval e with endpoints in $\mathcal{S}$. Summing over the I-cells, $\mathbb{E}(z\times S_I(\mathbf{1},Pd\boldsymbol{v}))=\mathbb{E}(S_I(\boldsymbol{P}z,d\boldsymbol{v}))$

626M The Doob-Meyer theorem: first form Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a non-negative submartingale. Then $\boldsymbol{v}\in M_{\text{D-M}}(\mathcal{S})$ and the previsible variation $\boldsymbol{v}^\#$ is non-negative and non-decreasing, with starting value 0.

proof (a) If $\sigma\le\sigma'$ in $\mathcal{S}$, then (in the language of 613C and 613E)

$$\Delta_{c(\sigma,\sigma')}(\mathbf{1}, P d\boldsymbol{v}) = P_\sigma v_{\sigma'} - v_\sigma \ge 0$$

so $S_I(\mathbf{1}, Pd\boldsymbol{v}) \ge 0$ for every $I \in \mathcal{I}(\mathcal{S})$. For $\tau \in \mathcal{S}$, set $A_\tau = \{S_I(\mathbf{1}, Pd\boldsymbol{v}) : \tau \in I \in \mathcal{I}(\mathcal{S} \wedge \tau)\}$.

(b) Take any $\tau \in \mathcal{S}$.

(i) $\mathbb{E}(z \times \chi[\![z > 2\beta^2]\!] \le 3(\frac{1}{\beta}\mathbb{E}(v_\tau) + \mathbb{E}((v_\tau - \beta\chi 1)^+))$ whenever $z \in A_\tau$ and $\beta > 0$. **P** Let $I \in \mathcal{I}(\mathcal{S} \wedge \tau)$ be such that $\tau \in I$ and $z = S_I(\mathbf{1}, Pd\boldsymbol{v})$. Take $\sigma_0 \le \ldots \le \sigma_n$ linearly generating the I-cells. Then $\langle v_{\sigma_i}\rangle_{i \le n}$ is a non-negative submartingale in the sense of 621Db adapted to $\langle \mathfrak{A}_{\sigma_i}\rangle_{i\le n}$, and $z = \sum_{i=0}^{n-1} P_{\sigma_i} v_{\sigma_{i+1}} - v_{\sigma_i}$. By Lemma 621J, $\beta^2\mathbb{E}(z \times \chi[\![z > 2\beta^2]\!] \le 3(\beta\mathbb{E}(v_{\sigma_n}) + \beta^2\mathbb{E}((v_{\sigma_n} - \beta\chi 1)^+))$, that is, $\mathbb{E}(z \times \chi[\![z > 2\beta^2]\!] \le 3(\frac{1}{\beta}\mathbb{E}(v_\tau) + \mathbb{E}((v_\tau - \beta\chi 1)^+))$.**Q**

(ii) The closure $\overline{A}_\tau$ of A_τ for the weak topology $\mathfrak{S}$ of $L^1_{\bar\mu}$ is compact for $\mathfrak{S}$. **P** We can estimate

$$\begin{aligned}\limsup_{\alpha\to\infty} \sup_{z\in A_\tau} \mathbb{E}(|z| \times \chi[\![|z| > \alpha]\!]) &= \limsup_{\alpha\to\infty} \sup_{z\in A_\tau} \mathbb{E}(z \times \chi[\![z > \alpha]\!]) \\ &= \limsup_{\beta\to\infty} \sup_{z\in A_\tau} \mathbb{E}(z \times \chi[\![z > 2\beta^2]\!]) \\ &\le \limsup_{\beta\to\infty} 3(\frac{1}{\beta}\mathbb{E}(v_\tau) + \mathbb{E}((v_\tau - \beta\chi 1)^+)) = 0\end{aligned}$$

so A_τ is uniformly integrable by 621Bb. By 247C, $\overline{A}_\tau$ is weakly compact. **Q**

(c) Let $\mathcal{F}$ be an ultrafilter on $\mathcal{I}(\mathcal{S})$ such that $\{I : \sigma \in I \in \mathcal{I}(\mathcal{S})\}$ belongs to $\mathcal{F}$ for every $\sigma \in \mathcal{S}$.

(i) For $\tau \in \mathcal{S}$ and $I \in \mathcal{I}(\mathcal{S})$ write $I \wedge \tau$ for $\{\sigma \wedge \tau : \sigma \in I\} \in \mathcal{I}(\mathcal{S} \wedge \tau)$. Because $\{S_I(\mathbf{1}, Pd\boldsymbol{v}) : \tau \in I \in \mathcal{I}(\mathcal{S} \wedge \tau)\}$ is relatively weakly compact in $L^1_{\bar\mu}$, the weak limit $v^\#_{\mathcal{F}\tau} = \text{w1lim}_{I\to\mathcal{F}} S_{I\wedge\tau}(\mathbf{1}, Pd\boldsymbol{v})$ is defined in $L^1_{\bar\mu}$ (3A3De).

(ii) The arguments of the proof of 626Ka now show that $\boldsymbol{v}^\#_{\mathcal{F}} = \langle v^\#_{\mathcal{F}\tau}\rangle_{\tau\in\mathcal{S}}$ is an L^1-process and that $\boldsymbol{v} - \boldsymbol{v}^\#_{\mathcal{F}}$ is a martingale.

(iii) If $\sigma \le \tau$ in $\mathcal{S}$ and $I \in \mathcal{I}(\mathcal{S})$ contains both σ and τ, then

$$S_{I\wedge\tau}(\mathbf{1}, Pd\boldsymbol{v}) = S_{I\wedge\sigma}(\mathbf{1}, Pd\boldsymbol{v}) + S_{I\cap[\sigma,\tau]}(\mathbf{1}, Pd\boldsymbol{v}) \ge S_{I\wedge\sigma}(\mathbf{1}, Pd\boldsymbol{v})$$

(613G(a-i) once more). Taking the limit,

$$v^\#_{\mathcal{F}\tau} = \text{w1lim}_{I\to\mathcal{F}} S_{I\wedge\tau}(\mathbf{1}, Pd\boldsymbol{v}) \ge \text{w1lim}_{I\to\mathcal{F}} S_{I\wedge\tau}(\mathbf{1}, Pd\boldsymbol{v}) = v^\#_{\mathcal{F}\sigma}$$

because the weak topology of $L^1_{\bar\mu}$ is a linear space topology for which the positive cone $\{u : u \ge 0\}$ is closed. Similarly, $v^\#_{\mathcal{F}\tau} \ge 0$ for every $\tau \in \mathcal{S}$. Thus $\boldsymbol{v}^\#_{\mathcal{F}}$ is non-negative and non-decreasing.

(d) Now take any $z \in L^\infty(\mathfrak{A})$ and consider the martingale $\boldsymbol{P}z = \langle P_\sigma z\rangle_{\sigma\in\mathcal{S}}$.

(i) If $I \in \mathcal{I}(\mathcal{S})$ then

$$\mathbb{E}(z \times S_I(\mathbf{1}, Pd\boldsymbol{v})) = \mathbb{E}(S_I(\boldsymbol{P}z, d\boldsymbol{v})) = \mathbb{E}(S_I(\boldsymbol{P}z, d\boldsymbol{v}^\#_{\mathcal{F}}))$$

P If $\sigma \le \tau$ in $\mathcal{S}$ then

$$P_\sigma(v_\tau - v^\#_{\mathcal{F}\tau}) = v_\sigma - v^\#_{\mathcal{F}\sigma}, \quad P_\sigma v_\tau - v_\sigma = P_\sigma v^\#_{\mathcal{F}\tau} - v^\#_{\mathcal{F}\sigma}$$

because $\boldsymbol{v} - \boldsymbol{v}^\#_{\mathcal{F}}$ is a martingale. So $P\Delta\boldsymbol{v} = P\Delta\boldsymbol{v}^\#_{\mathcal{F}}$ and

$$\mathbb{E}(S_I(\boldsymbol{P}z, d\boldsymbol{v})) = \mathbb{E}(z \times S_I(\mathbf{1}, Pd\boldsymbol{v})) = \mathbb{E}(z \times S_I(\mathbf{1}, Pd\boldsymbol{v}^\#_{\mathcal{F}})) = \mathbb{E}(S_I(\boldsymbol{P}z, d\boldsymbol{v}^\#_{\mathcal{F}}))$$

by 626L. **Q**

(ii) Suppose that $\tau \in \mathcal{S}$. Taking the limit as $I \to \mathcal{F}$,

$$\begin{aligned}\mathbb{E}(z \times v^\#_{\mathcal{F}\tau}) &= \mathbb{E}(z \times \operatorname*{w1lim}_{I\to\mathcal{F}} S_{I\wedge\tau}(\mathbf{1}, Pd\boldsymbol{v})) \\ &= \lim_{I\to\mathcal{F}} \mathbb{E}(z \times S_{I\wedge\tau}(\mathbf{1}, Pd\boldsymbol{v})) = \lim_{I\to\mathcal{F}} \mathbb{E}(S_{I\wedge\tau}(\boldsymbol{P}z, d\boldsymbol{v}^\#_{\mathcal{F}})).\end{aligned}$$

But if $\sigma_0 \le \ldots \le \sigma_n \le \tau$ in $\mathcal{S}$,

$$\begin{aligned}|\sum_{i=0}^{n-1} P_{\sigma_i}z \times (v^{\#}_{\mathcal{F}\sigma_{i+1}} - v^{\#}_{\mathcal{F}\sigma_i})| &\le \sum_{i=0}^{n-1} \|P_{\sigma_i}z\|_\infty |v^{\#}_{\mathcal{F}\sigma_{i+1}} - v^{\#}_{\mathcal{F}\sigma_i}| \\ &\le \sum_{i=0}^{n-1} \|z\|_\infty (v^{\#}_{\mathcal{F}\sigma_{i+1}} - v^{\#}_{\mathcal{F}\sigma_i})\end{aligned}$$

(because $\boldsymbol{v}^{\#}_{\mathcal{F}}$ is non-decreasing)

$$= \|z\|_\infty (v^{\#}_{\mathcal{F}\sigma_n} - v^{\#}_{\mathcal{F}\sigma_0}) \le \|z\|_\infty v^{\#}_{\mathcal{F}\tau}.$$

This shows that $\{S_{I\wedge\tau}(\boldsymbol{P}z, d\boldsymbol{v}^{\#}_{\mathcal{F}}) : I \in \mathcal{I}(\mathcal{S})\}$ is order-bounded in $L^1_{\bar\mu}$, therefore uniformly integrable. Now we know also that $\boldsymbol{P}z$ is moderately oscillatory (622G), while $\boldsymbol{v}^{\#}_{\mathcal{F}}$ is a local integrator (616Ra), so

$$\int_{\mathcal{S}\wedge\tau} \boldsymbol{P}z\, d\boldsymbol{v}^{\#}_{\mathcal{F}} = \lim_{I\uparrow\mathcal{I}(\mathcal{S}\wedge\tau)} S_I(\boldsymbol{P}z, d\boldsymbol{v}^{\#}_{\mathcal{F}}) = \lim_{I\uparrow\mathcal{I}(\mathcal{S})} S_{I\wedge\tau}(\boldsymbol{P}z, d\boldsymbol{v}^{\#}_{\mathcal{F}})$$

(see 613K) is defined, and must be equal to $1\text{lim}_{I\uparrow\mathcal{I}(\mathcal{S})} S_{I\wedge\tau}(\boldsymbol{P}z, d\boldsymbol{v}^{\#}_{\mathcal{F}})$, by 621B(c-ii). As $\mathcal{F}$ includes the filter generated by $\{\{I : \sigma \in I \in \mathcal{I}(\mathcal{S})\} : \sigma \in \mathcal{S}\}$, this is also $1\text{lim}_{I\to\mathcal{F}} S_{I\wedge\tau}(\boldsymbol{P}z, d\boldsymbol{v}^{\#}_{\mathcal{F}})$. Accordingly

$$\begin{aligned}\mathbb{E}(z \times v^{\#}_{\mathcal{F}\tau}) &= \lim_{I\to\mathcal{F}} \mathbb{E}(S_{I\wedge\tau}(\boldsymbol{P}z, d\boldsymbol{v}^{\#}_{\mathcal{F}})) = \mathbb{E}(1\text{lim}_{I\to\mathcal{F}} S_{I\wedge\tau}(\boldsymbol{P}z, d\boldsymbol{v}^{\#}_{\mathcal{F}})) \\ &= \mathbb{E}(1\text{lim}_{I\uparrow\mathcal{I}(\mathcal{S})} S_{I\wedge\tau}(\boldsymbol{P}z, d\boldsymbol{v}^{\#}_{\mathcal{F}})) = \lim_{I\uparrow\mathcal{I}(\mathcal{S})} \mathbb{E}(S_{I\wedge\tau}(\boldsymbol{P}z, d\boldsymbol{v}^{\#}_{\mathcal{F}})) \\ &= \lim_{I\uparrow\mathcal{I}(\mathcal{S})} \mathbb{E}(S_{I\wedge\tau}(\boldsymbol{P}z, d\boldsymbol{v})) = \lim_{I\uparrow\mathcal{I}(\mathcal{S})} \mathbb{E}(z \times S_{I\wedge\tau}(\mathbf{1}, Pd\boldsymbol{v})) \\ &= \lim_{I\uparrow\mathcal{I}(\mathcal{S}\wedge\tau} \mathbb{E}(S_I(\boldsymbol{P}z, d\boldsymbol{v})) = \lim_{I\uparrow\mathcal{I}(\mathcal{S}\wedge\tau)} \mathbb{E}(z \times S_I(\mathbf{1}, Pd\boldsymbol{v})).\end{aligned}$$

Since this is true for every $z \in L^\infty(\mathfrak{A}) \cong (L^1_{\bar\mu})'$,

$$v^{\#}_{\mathcal{F}\tau} = \text{w1lim}_{I\uparrow\mathcal{I}(\mathcal{S}\wedge\tau)} S_I(\mathbf{1}, Pd\boldsymbol{v}).$$

Thus $\boldsymbol{v}^{\#}_{\mathcal{F}}$ is the previsible variation of $\boldsymbol{v}$ as defined in 626J.

(e) To find the starting value $v^{\#}_{\downarrow} = \lim_{\sigma\downarrow\mathcal{S}} v^{\#}_{\mathcal{F}\sigma}$ (613Bk), consider the case $z = \chi 1$ in the formula of (c-i) above. We have $\boldsymbol{P}z = \mathbf{1}$ so $\mathbb{E}(S_I(\mathbf{1}, Pd\boldsymbol{v})) = \mathbb{E}(S_I(\mathbf{1}, d\boldsymbol{v}))$ for every $I \in \mathcal{I}(\mathcal{S})$. Taking the limit as in 626J,

$$\begin{aligned}\mathbb{E}(v^{\#}_{\mathcal{F}\tau}) &= \lim_{I\uparrow\mathcal{I}(\mathcal{S}\wedge\tau)} \mathbb{E}(S_I(\mathbf{1}, d\boldsymbol{v})) = \lim_{I\uparrow\mathcal{I}(\mathcal{S}\wedge\tau)} \mathbb{E}(v_{\max I} - v_{\min I}) \\ &= \lim_{I\uparrow\mathcal{I}(\mathcal{S}\wedge\tau)} \mathbb{E}(v_\tau - v_{\min I}) = \mathbb{E}(v_\tau) - \lim_{\sigma\downarrow\mathcal{S}} \mathbb{E}(v_\sigma)\end{aligned}$$

for every $\tau \in \mathcal{S}$. But this means that $\lim_{\tau\downarrow\mathcal{S}} \mathbb{E}(v^{\#}_{\mathcal{F}\tau}) = 0$, and as $v^{\#}_{\mathcal{F}\tau} \ge 0$ for every $\tau \in \mathcal{S}$ and $\boldsymbol{v}^{\#}_{\mathcal{F}}$ is non-decreasing, $\lim_{\tau\downarrow\mathcal{S}} v^{\#}_{\mathcal{F}\tau} = \inf_{\tau\in\mathcal{S}} v^{\#}_{\mathcal{F}\tau} = 0$.

626N Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $A \subseteq \mathcal{S}$ a non-empty downwards-directed set, and $R_A : M_{\text{lmo}}(\mathcal{S}) \to M_{\text{lmo}}(\mathcal{S})$ the associated operator. If $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma\in\mathcal{S}}$ is a non-negative submartingale with domain $\mathcal{S}$ and previsible variation $\boldsymbol{v}^{\#}$, then the previsible variation $R_A(\boldsymbol{v})^{\#}$ of $R_A(\boldsymbol{v})$ is $R_A(\boldsymbol{v}^{\#})$.

proof (a) By 626F, $R_A(\boldsymbol{v})$ is defined and a submartingale and by 623Ba it is non-negative. By 626M, $\boldsymbol{v}^{\#}$ and $R_A(\boldsymbol{v})^{\#}$ are defined and are non-negative, non-decreasing and start at 0. Express $\boldsymbol{v}$, $\boldsymbol{v}^{\#}$, $R_A(\boldsymbol{v})$, $R_A(\boldsymbol{v})^{\#}$ and $R_A(\boldsymbol{v}^{\#})$ as $\langle v_\sigma \rangle_{\sigma\in\mathcal{S}}$, $\langle v^{\#}_\sigma \rangle_{\sigma\in\mathcal{S}}$, $\langle v_{A\sigma} \rangle_{\sigma\in\mathcal{S}}$, $\langle v^{\#}_{A\sigma} \rangle_{\sigma\in\mathcal{S}}$ and $\langle z_\sigma \rangle_{\sigma\in\mathcal{S}}$. As $\boldsymbol{v}^{\#}$ is a non-negative non-decreasing L^1-process, it is a submartingale, and $z_\tau = \lim_{\rho\downarrow A\wedge\tau} v^{\#}_\rho = 1\text{lim}_{\rho\downarrow A\wedge\tau} v^{\#}_\rho$ for every $\tau \in \mathcal{S}$, as in 626E.

(b) Consider first the case in which $A = \{\rho\}$ is a singleton, so that $v_{A\sigma} = v_{\sigma\wedge\rho}$ and $z_\sigma = v^{\#}_{\sigma\wedge\rho}$ for every $\sigma \in \mathcal{S}$. In this case, $\boldsymbol{v}\restriction\mathcal{S}\wedge\rho = R_{\{\rho\}}(\boldsymbol{v})\restriction\mathcal{S}\wedge\rho$ and

$$R_{\{\rho\}}(\boldsymbol{v})^{\#}\restriction\mathcal{S}\wedge\rho = \boldsymbol{v}^{\#}\restriction\mathcal{S}\wedge\rho = R_{\{\rho\}}(\boldsymbol{v}^{\#})\restriction\mathcal{S}\wedge\rho$$

by 626K(f-i). In particular, $v^{\#}_{\{\rho\}\rho} = z_\rho$. On the other side, $R_{\{\rho\}}(\boldsymbol{v})\restriction\mathcal{S}\vee\rho$ is constant so $(R_{\{\rho\}}(\boldsymbol{v})\restriction\mathcal{S}\vee\rho)^{\#} = 0$ and $R_{\{\rho\}}(\boldsymbol{v})^{\#}\restriction\mathcal{S}\vee\rho$ is constant with value $v^{\#}_{\{\rho\}\rho}$ (626K(f-ii)). At the same time, $R_{\{\rho\}}(\boldsymbol{v}^{\#})$ is constant with value $z_\rho = v^{\#}_{\{\rho\}\rho}$ on $\mathcal{S}\vee\rho$. Thus $R_{\{\rho\}}(\boldsymbol{v})^{\#}$ and $R_{\{\rho\}}(\boldsymbol{v}^{\#})$ agree on both $\mathcal{S}\wedge\rho$ and $\mathcal{S}\vee\rho$; as they are fully adapted processes, they agree on $\mathcal{S}$ and are identical.

(c) Still supposing that $A = \{\rho\}$, take $\tau\in\mathcal{S}$ and $I\in\mathcal{I}(\mathcal{S}\wedge\tau)$ such that $\tau\wedge\rho\in I$. Then $\|S_I(\mathbf{1}, Pd\boldsymbol{v}^{\#}) - S_I(\mathbf{1}, Pd(R_{\{\rho\}}(\boldsymbol{v}^{\#}))\|_1 \le \|v^{\#}_\tau - v^{\#}_{\tau\wedge\rho}\|_1$. **P** Let $(\sigma_0,\dots,\sigma_n)$ be a sequence linearly generating the I-cells such that $\tau\wedge\rho = \sigma_k$ for some $k\le n$. For $i<k$, $\sigma_{i+1}\le\rho$ so

$$P_{\sigma_i}v^{\#}_{\sigma_{i+1}} - v^{\#}_{\sigma_i} = P_{\sigma_i}v^{\#}_{\sigma_{i+1}\wedge\rho} - v^{\#}_{\sigma_i\wedge\rho};$$

summing over $i<k$, $S_{I\wedge\tau\wedge\rho}(\mathbf{1}, Pd\boldsymbol{v}^{\#}) = S_{I\wedge\tau\wedge\rho}(\mathbf{1}, Pd(R_{\{\rho\}}(\boldsymbol{v}^{\#})))$. On the other side, if $k\le i\le n$, we have $\tau\wedge\rho\le\sigma_i\le\tau$ so $\sigma_i\wedge\rho = \tau\wedge\rho$. Now if $k\le i<n$,

$$P_{\sigma_i}v^{\#}_{\sigma_{i+1}\wedge\rho} = P_{\sigma_i}v^{\#}_{\tau\wedge\rho} = v^{\#}_{\tau\wedge\rho} = v^{\#}_{\sigma_i\wedge\rho};$$

it follows that $S_{I\vee(\tau\wedge\rho)}(\mathbf{1}, Pd(R_{\{\rho\}}(\boldsymbol{v}^{\#}))) = 0$. So

$$\begin{aligned}
&\|S_I(\mathbf{1}, Pd\boldsymbol{v}^{\#}) - S_I(\mathbf{1}, Pd(R_{\{\rho\}}(\boldsymbol{v}^{\#}))\|_1\\
&\qquad = \|S_{I\vee(\tau\wedge\rho)}(\mathbf{1}, Pd\boldsymbol{v}^{\#}) - S_{I\vee(\tau\wedge\rho)}(\mathbf{1}, Pd(R_{\{\rho\}}(\boldsymbol{v}^{\#}))\|_1\\
&\qquad = \|S_{I\vee(\tau\wedge\rho)}(\mathbf{1}, Pd\boldsymbol{v}^{\#})\|_1 = \|\sum_{i=k}^{n-1} P_{\sigma_i}v^{\#}_{\sigma_{i+1}} - v^{\#}_{\sigma_i})\|_1\\
&\qquad = \sum_{i=k}^{n-1}\mathbb{E}(P_{\sigma_i}v^{\#}_{\sigma_{i+1}} - v^{\#}_{\sigma_i}) = \sum_{i=k}^{n-1}\mathbb{E}(v^{\#}_{\sigma_{i+1}} - v^{\#}_{\sigma_i})\\
&\qquad = \mathbb{E}(v^{\#}_{\sigma_n} - v^{\#}_{\sigma_k}) \le \mathbb{E}(v^{\#}_\tau - v^{\#}_{\tau\wedge\rho}) = \|v^{\#}_\tau - v^{\#}_{\tau\wedge\rho}\|_1. \;\mathbf{Q}
\end{aligned}$$

(d) Returning to the case of general A, we have $\|v^{\#}_\tau - v^{\#}_{A\tau}\|_1 \le \|v^{\#}_\tau - z_\tau\|_1$ for every $\tau\in\mathcal{S}$. **P** Whenever $\sigma\le\sigma'$ in $\mathcal{S}\wedge\tau$,

$$P_\sigma v_{A\sigma'} = P_\sigma(\lim_{\rho\downarrow A} v_{\sigma'\wedge\rho}) = P_\sigma(1\lim_{\rho\downarrow A} v_{\sigma'\wedge\rho})$$

(626E)

$$= 1\lim_{\rho\downarrow A} P_\sigma v_{\sigma'\wedge\rho}$$

because $P_\sigma : L^1_{\bar\mu}\to L^1_{\bar\mu}$ is $\|\;\|_1$-continuous. Since $v_{A\sigma'} = 1\lim_{\rho\downarrow A} v_{\sigma'\wedge\rho}$, $P_\sigma v_{A\sigma'} - v_{A\sigma} = 1\lim_{\rho\downarrow A}(P_\sigma v_{\sigma'\wedge\rho} - v_{\sigma\wedge\rho})$. Another way of expressing this is to say that if e is a stopping time interval with endpoints in $\mathcal{S}\wedge\tau$,

$$\Delta_e(\mathbf{1}, PdR_A(\boldsymbol{v})) = 1\lim_{\rho\downarrow A}\Delta_e(\mathbf{1}, PdR_{\{\rho\}}(\boldsymbol{v})).$$

It follows that if $I\in\mathcal{I}(\mathcal{S}\wedge\tau)$ then

$$S_I(\mathbf{1}, PdR_A(\boldsymbol{v})) = 1\lim_{\rho\downarrow A} S_I(\mathbf{1}, PdR_{\{\rho\}}(\boldsymbol{v})) = 1\lim_{\rho\downarrow A} S_I(\mathbf{1}, Pd(R_{\{\rho\}}(\boldsymbol{v})^{\#}))$$

(626Kh)

$$= 1\lim_{\rho\downarrow A} S_I(\mathbf{1}, PdR_{\{\rho\}}(\boldsymbol{v}^{\#}))$$

by (b). Consequently

$$\begin{aligned}
\|S_I(\mathbf{1}, Pd\boldsymbol{v}^{\#}) - S_I(\mathbf{1}, PdR_A(\boldsymbol{v}))\|_1 &\le \sup_{\rho\in A}\|S_I(\mathbf{1}, Pd\boldsymbol{v}^{\#}) - S_I(\mathbf{1}, PdR_{\{\rho\}}(\boldsymbol{v}^{\#}))\|_1\\
&\le \sup_{\rho\in A}\|v^{\#}_\tau - v^{\#}_{\tau\wedge\rho}\|_1
\end{aligned}$$

(by (c))

$$= \|v^\#_\tau - z_\tau\|_1$$

because $\{v^\#_\tau - v^\#_{\tau\wedge\rho} : \rho \in A\}$ is non-negative and upwards-directed, as in 613B(d-iii), and $z_\tau = \lim_{\rho\downarrow A} v^\#_{\tau\wedge\rho} = 1\lim_{\rho\downarrow A} v^\#_{\tau\wedge\rho}$, by 626E again. **Q**

(e) Now take $\tau \in \mathcal{S}$ and $\epsilon > 0$. Then there is a $\rho \in A$ such that $\|v^\#_{\tau\wedge\rho} - z_\tau\|_1 \le \epsilon$. Write $\tilde{\boldsymbol{v}} = R_{\{\rho\}}(\boldsymbol{v})$; since $\tilde{\boldsymbol{v}}$ is a non-negative submartingale, everything above can be applied to $\tilde{\boldsymbol{v}}$. Expressing $\tilde{\boldsymbol{v}}^\#$, $R_A(\tilde{\boldsymbol{v}})^\#$ and $R_A(\tilde{\boldsymbol{v}}^\#)$ as $\langle \tilde{v}^\#_\sigma\rangle_{\sigma\in\mathcal{S}}$, $\langle \tilde{v}^\#_{A\sigma}\rangle_{\sigma\in\mathcal{S}}$ and $\langle \tilde{z}_\sigma\rangle_{\sigma\in\mathcal{S}}$, (d) just above tells us that

$$\|\tilde{v}^\#_\tau - \tilde{v}^\#_{A\tau}\|_1 \le \|\tilde{v}^\#_\tau - \tilde{z}_\tau\|_1. \qquad (*)$$

Since $R_A R_{\{\rho\}} = R_A$ (623Cc), $R_A(\tilde{\boldsymbol{v}}) = R_A(\boldsymbol{v})$, $R_A(\tilde{\boldsymbol{v}})^\# = R_A(\boldsymbol{v})^\#$ and $\tilde{v}^\#_{A\tau} = v^\#_{A\tau}$. From (b) we see that $\tilde{\boldsymbol{v}}^\# = R_{\{\rho\}}(\boldsymbol{v}^\#)$ and therefore $R_A(\tilde{\boldsymbol{v}}^\#)$ is equal to $R_A(\boldsymbol{v}^\#)$, so $\tilde{z}_\tau = z_\tau$ and $\tilde{v}^\#_\tau = v^\#_{\tau\wedge\rho}$. Translating the formula (*), we see that

$$\|\tilde{v}^\#_{\tau\wedge\rho} - v^\#_{A\tau}\|_1 \le \|v^\#_{\tau\wedge\rho} - z_\tau\|_1 \le \epsilon, \quad \|v^\#_{A\tau} - z_\tau\|_1 \le 2\epsilon.$$

As τ and ϵ are arbitrary, $R_A(\boldsymbol{v})^\# = R_A(\boldsymbol{v}^\#)$.

626O The Doob-Meyer theorem: second form Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a submartingale such that $\{\mathbb{E}(v_\sigma) : \sigma \in \mathcal{S}\}$ is bounded below. Then $\boldsymbol{v}$ is expressible as the sum of a non-negative non-decreasing fully adapted process and a virtually local martingale.

proof (a) For the time being (down to the end of (d) below), suppose that $\mathcal{S}$ is finitely full.

(i) For $\tau \in \mathcal{S}$ and $M \ge 0$ set

$$A_{M\tau} = \{\rho : \rho \in \mathcal{S}, [\![\rho < \tau]\!] \subseteq [\![|v_\rho| \ge M]\!]\}.$$

Then $\tau \in A_{M\tau}$ and $A_{M\tau}$ is closed under $\wedge$ (see part (a) of the proof of 623I). Set $\mathcal{A} = \{A_{M\tau} : \tau \in \mathcal{S}, M \ge 0\}$.

(ii) If $\tau \le \tau'$ in $\mathcal{S}$ and $0 \le M \le M'$, then

$$A_{M'\tau'} = \{\rho : [\![\tau' \le \rho]\!] \cup [\![|v_\rho| \ge M']\!] = 1\} \subseteq \{\rho : [\![\tau \le \rho]\!] \cup [\![|v_\rho| \ge M]\!] = 1\} = A_{M\tau}$$

so $\mathcal{A}$ is downwards-directed.

(iii) By 626F, $R_A(\boldsymbol{v})$ is defined and is a submartingale for every $A \in \mathcal{A}$.

(iv) If $\tau \in \mathcal{S}$ and $\epsilon > 0$, there is an $A \in \mathcal{A}$ such that $\sup_{\rho\in A} \bar{\mu}[\![\rho < \tau]\!] \le \epsilon$. **P** Because $\boldsymbol{v}$ is a local integrator (626Dc), $\boldsymbol{v}\restriction\mathcal{S}\wedge\tau$ is order-bounded; set $\bar{v} = \sup_{\sigma\in\mathcal{S}\wedge\tau} |v_\sigma|$. Let $M \ge 0$ be such that $\bar{\mu}[\![\bar{v} \ge M]\!] \le \epsilon$, and set $A = A_{M\tau}$. If $\rho \in A$, then

$$[\![\rho < \tau]\!] \subseteq [\![|v_\rho| \ge M]\!] \cap [\![v_{\rho\wedge\tau} = v_\rho]\!] \subseteq [\![|v_{\rho\wedge\tau}| \ge M]\!] \subseteq [\![\bar{v} \ge M]\!]$$

has measure at most ϵ. **Q**

(b) Take any $A \in \mathcal{A}$, and express $R_A(\boldsymbol{v})$ as $\langle v_{A\sigma}\rangle_{\sigma\in\mathcal{S}}$.

(i) There is a martingale $\boldsymbol{w}$ such that $R_A(\boldsymbol{v}) + \boldsymbol{w} \ge 0$. **P** Express A as $A_{M\tau}$ where $M \ge 0$ and $\tau \in \mathcal{S}$. As noted in (a-iii), $R_A(\boldsymbol{v})$ is a submartingale, in particular an L^1-process, and $v_{A\tau} \in L^1_{\bar{\mu}}$. Set $w_\sigma = P_\sigma|v_{A\tau}| + M\chi 1$ for $\sigma \in \mathcal{S}$, so that $\boldsymbol{w} = \langle w_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a martingale.

If $\sigma \in \mathcal{S}$ and $a = [\![v_{A\sigma} < -M]\!]$, then $a \in \mathfrak{A}_\sigma$ and $a \subseteq [\![v_{A\sigma} = v_{A\tau}]\!]$ (623I(b-ii)), so

$$a \subseteq [\![P_\sigma v_{A\sigma} = P_\sigma v_{A\tau}]\!] \subseteq [\![v_{A\sigma} + P_\sigma|v_{A\tau}| \ge 0]\!] \subseteq [\![v_{A\sigma} + w_\sigma \ge 0]\!],$$

while of course $1 \setminus a$ also is included in $[\![v_{A\sigma} + M\chi 1 \ge 0]\!] \subseteq [\![v_{A\sigma} + w_\sigma \ge 0]\!]$. Thus $R_A(\boldsymbol{v})+\boldsymbol{w}$ is non-negative. **Q**

(ii) $R_A(\boldsymbol{v}) \in M_{\text{D-M}}(\mathcal{S})$ and its previsible variation $R_A(\boldsymbol{v})^\#$ is non-negative and non-decreasing, with starting value 0. **P** $R_A(\boldsymbol{v}) + \boldsymbol{w}$ is a submartingale (626Cd) and we have just seen that it is non-negative. By 626M, it has a previsible variation $(R_A(\boldsymbol{v}) + \boldsymbol{w})^\#$ which is non-negative and non-decreasing, therefore a submartingale, and starts at 0. But now $R_A(\boldsymbol{v})^\#$ is defined and equal to $(R_A(\boldsymbol{v}) + \boldsymbol{w})^\#$, by 626Kb-626Kc. **Q** Express $R_A(\boldsymbol{v})^\#$ as $\langle v^\#_{A\sigma}\rangle_{\sigma\in\mathcal{S}}$.

(iii) Writing γ for $\inf_{\sigma\in\mathcal{S}}\mathbb{E}(v_\sigma)$, $\mathbb{E}(v^{\#}_{A\tau})\le\mathbb{E}(v_\tau)-\gamma$ for every $\tau\in\mathcal{S}$. **P** For $\sigma\in\mathcal{S}$,

$$v_{A\sigma}=\lim_{\rho\downarrow A}v_{\sigma\wedge\rho}=1\lim_{\rho\downarrow A}v_{\sigma\wedge\rho}$$

(626E) so

$$\mathbb{E}(v_{A\sigma})=\lim_{\rho\downarrow A}\mathbb{E}(v_{\sigma\wedge\rho})=\inf_{\rho\in A}\mathbb{E}(v_{\sigma\wedge\rho})$$

(626Ca) and $\gamma\le\mathbb{E}(v_{A\sigma})\le\mathbb{E}(v_\sigma)$. Next,

$$1\lim_{\sigma\downarrow\mathcal{S}}v^{\#}_{A\sigma}=\lim_{\sigma\downarrow\mathcal{S}}v^{\#}_{A\sigma}=0$$

so $\lim_{\sigma\downarrow\mathcal{S}}\mathbb{E}(v^{\#}_{A\sigma})=0$. Now $R_A(\boldsymbol{v})^{\#}-R_A(\boldsymbol{v})$ is a martingale (626Ka), so

$$\mathbb{E}(v^{\#}_{A\tau})=\lim_{\sigma\downarrow\mathcal{S}}\mathbb{E}(v^{\#}_{A\tau}-v^{\#}_{A\sigma})=\lim_{\sigma\downarrow\mathcal{S}}\mathbb{E}(v_{A\tau}-v_{A\sigma})\le\mathbb{E}(v_\tau)-\gamma.\ \mathbf{Q}$$

(iv) If now $B\subseteq\mathcal{S}$ is non-empty and downwards-directed, $R_BR_A(\boldsymbol{v})\in M_{\text{D-M}}(\mathcal{S})$ and $(R_BR_A(\boldsymbol{v}))^{\#}=R_B(R_A(\boldsymbol{v})^{\#})$. **P** By 626N,

$$\begin{aligned}R_B(R_A(\boldsymbol{v})^{\#})&=R_B((R_A(\boldsymbol{v})+\boldsymbol{w})^{\#})=(R_B(R_A(\boldsymbol{v})+\boldsymbol{w}))^{\#}\\&=(R_BR_A(\boldsymbol{v})+R_B(\boldsymbol{w}))^{\#}=(R_BR_A(\boldsymbol{v}))^{\#}\end{aligned}$$

because $R_B(\boldsymbol{w})$ is a martingale (623E). **Q**

(c) Thus we have a family $\langle R_A(\boldsymbol{v})^{\#}\rangle_{A\in\mathcal{A}}$ of previsible variations, all non-decreasing and non-negative, and $R_B(\boldsymbol{v})^{\#}=(R_BR_A(\boldsymbol{v}))^{\#}=R_B(R_A(\boldsymbol{v})^{\#})$ whenever A, $B\in\mathcal{A}$ and $A\subseteq B$.

(i) If $\sigma\in\mathcal{S}$ and $A\subseteq B$ in $\mathcal{A}$ then

$$v^{\#}_{B\sigma}=\lim_{\rho\downarrow B}v^{\#}_{A,\sigma\wedge\rho}\le v^{\#}_{A\sigma}$$

for every σ. We know also that $v^{\#}_{A\sigma}\ge0$ and $\mathbb{E}(v^{\#}_{A\sigma})\le\mathbb{E}(v_\sigma)-\gamma$ for every $A\in\mathcal{A}$. As $\mathcal{A}$ is downwards-directed, $\{v^{\#}_{A\sigma}:A\in\mathcal{A}\}$ is upwards-directed and $\|\,\|_1$-bounded, so $1\lim_{A\downarrow\mathcal{A}}v^{\#}_{A\sigma}=\sup_{A\in\mathcal{A}}v^{\#}_{A\sigma}$ is defined for every $\sigma\in\mathcal{S}$; I will call it v''_σ.

(ii) $\boldsymbol{v}''=\langle v''_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a fully adapted process. **P** If $\sigma\in\mathcal{S}$, $v^{\#}_{A\sigma}\in L^0(\mathfrak{A}_\sigma)$ for every $A\in\mathcal{A}$, so $v''_\sigma=\sup_{A\in\mathcal{A}}v^{\#}_{A\sigma}$ belongs to $L^0(\mathfrak{A}_\sigma)$ (612A(e-i)). If σ, $\tau\in\mathcal{S}$ and $a=[\![\sigma=\tau]\!]$, then $\chi a\times v^{\#}_{A\sigma}=\chi a\times v^{\#}_{A\tau}$ for every $A\in\mathcal{A}$, so $\chi a\times v''_\sigma=\chi a\times v''_\tau$, that is, $a\subseteq[\![v''_\sigma=v''_\tau]\!]$. **Q**

(iii) I am *not* claiming that $\boldsymbol{v}''=\langle v''_\sigma\rangle_{\sigma\in\mathcal{S}}$ is actually a previsible variation in the sense of 626J, or even an L^1-process. But surely it is non-negative and non-decreasing, because $0\le v^{\#}_{A\sigma}\le v^{\#}_{A\tau}$ whenever $A\in\mathcal{A}$ and $\sigma\le\tau$, so $0\le v''_\sigma\le v''_\tau$ whenever $\sigma\le\tau$ in $\mathcal{S}$. Consequently it is locally moderately oscillatory (616Ra again).

(iv) If $\tau\in\mathcal{S}$ and $\epsilon>0$, there are an $a\in\mathfrak{A}$ and a $B\in\mathcal{A}$ such that $\bar\mu a\le\epsilon$ and $[\![v''_\sigma\ne v^{\#}_{A\sigma}]\!]\subseteq a$ whenever $\sigma\in\mathcal{S}\wedge\tau$, $A\in\mathcal{A}$ and $A\subseteq B$. **P** By (a-iv), there is a $B\in\mathcal{A}$ such that $a=\sup_{\rho\in B}[\![\rho<\tau]\!]$ has measure at most ϵ. If $A\in\mathcal{A}$, $A\subseteq B$ and $\sigma\le\tau$ in $\mathcal{S}$, then $R_B(\boldsymbol{v})^{\#}=R_B(R_A(\boldsymbol{v}^{\#}))$ so $v^{\#}_{B\sigma}=\lim_{\rho\downarrow B}v^{\#}_{A,\sigma\wedge\rho}$. But for each $\rho\in B$,

$$[\![v^{\#}_{A,\sigma\wedge\rho}\ne v^{\#}_{A\sigma}]\!]\subseteq[\![\rho<\sigma]\!]\subseteq[\![\rho<\tau]\!]\subseteq a,$$

so $[\![v^{\#}_{B\sigma}\ne v^{\#}_{A\sigma}]\!]\subseteq a$. Taking the limit as $A\downarrow\mathcal{A}$, $[\![v^{\#}_{B\sigma}\ne v''_\sigma]\!]\subseteq a$. But now returning to an arbitrary $A\in\mathcal{A}$ included in B, we also have $[\![v^{\#}_{A\sigma}\ne v''_\sigma]\!]\subseteq a$. **Q**

(v) If $B\in\mathcal{A}$, then $R_B(\boldsymbol{v}'')=R_B(\boldsymbol{v})^{\#}$. **P** Take $\tau\in\mathcal{S}$. For any $\epsilon>0$ there are an $a\in\mathfrak{A}$ and an $A^*\in\mathcal{A}$ such that $\bar\mu a\le\epsilon$ and $[\![v''_\sigma\ne v^{\#}_{A\sigma}]\!]\subseteq a$ whenever $\sigma\in\mathcal{S}\wedge\tau$, $A\in\mathcal{A}$ and $A\subseteq A^*$; take $A\in\mathcal{A}$ such that $A\subseteq B\cap A^*$. Since $R_B(\boldsymbol{v})^{\#}=R_B(R_A(\boldsymbol{v}''))$,

$$\begin{aligned}[\![\lim_{\rho\downarrow B}v^{\#}_{\tau\wedge\rho}\ne v^{\#}_{B\tau}]\!]&=\lim_{\rho\downarrow B}[\![v^{\#}_{\tau\wedge\rho}\ne\lim_{\rho\downarrow B}v^{\#}_{A,\tau\wedge\rho}]\!]\\&\subseteq\sup_{\rho\in B}[\![v^{\#}_{\tau\wedge\rho}\ne v^{\#}_{A,\tau\wedge\rho}]\!]\subseteq a\end{aligned}$$

has measure at most ϵ. As ϵ is arbitrary, $\lim_{\rho\downarrow B} v^{\#}_{\tau\wedge\rho} = v^{\#}_{B\tau}$; as τ is arbitrary, $R_B(\boldsymbol{v}'') = R_B(\boldsymbol{v})^{\#}$. **Q**

(d) Set $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{v}''$. Because $\boldsymbol{v}$ and $\boldsymbol{v}''$ are locally moderately oscillatory, so is $\boldsymbol{w}$. For every $A \in \mathcal{A}$,

$$R_A(\boldsymbol{w}) = R_A(\boldsymbol{v}) - R_A(\boldsymbol{v}'') = R_A(\boldsymbol{v}) - R_A(\boldsymbol{v})^{\#}$$

is a martingale. Note that $R_A(\boldsymbol{w})$ is actually a uniformly integrable martingale, because if $\rho \in A$ then $R_A(\boldsymbol{w})$ is constant on $\mathcal{S} \vee \rho$. It follows that $\boldsymbol{w}$ is a virtually local martingale. **P** The definition in 623J referred to the fully adapted extension $\hat{\boldsymbol{w}}$ of $\boldsymbol{w}$ to the covered envelope $\hat{\mathcal{S}}$ of $\mathcal{S}$, and our assumption here is only that $\mathcal{S}$ is finitely full, that is, that it is equal to its finitely-covered envelope $\hat{\mathcal{S}}_f$. But each $A \in \mathcal{A}$ is a non-empty downwards-directed subset of $\hat{\mathcal{S}}$, so we can speak of R_A as an operator from $M_{\text{lmo}}(\hat{\mathcal{S}})$ to itself, and $\hat{\boldsymbol{w}}$ is locally moderately oscillatory (615F(b-v)), so $R_A(\hat{\boldsymbol{w}})$ is defined as a fully adapted process on $\hat{\mathcal{S}}$. But $R_A(\hat{\boldsymbol{w}})$ extends $R_A(\boldsymbol{w})$, so must be the fully adapted extension of $R_A(\boldsymbol{w})$, and is again a uniformly integrable martingale (622Nc).

Suppose now that $\tau \in \hat{\mathcal{S}}$ and $\epsilon > 0$. Then there is a $\sigma \in \hat{\mathcal{S}}_f$ such that $\bar{\mu}[\![\sigma = \tau]\!] \ge 1 - \epsilon$ (613B(q-i)). There is an $A \in \mathcal{A}$ such that $\sup_{\rho\in A} \bar{\mu}[\![\rho < \sigma]\!] \le \epsilon$. But now $\sup_{\rho\in A} \bar{\mu}[\![\rho < \tau]\!] \le 2\epsilon$. As τ and ϵ are arbitrary, $\{R_A : A \in \mathcal{A}\}$ is a sufficient family of operators to ensure that $\boldsymbol{w}$ is a virtually local martingale. **Q**

Accordingly $\boldsymbol{v} = \boldsymbol{v}'' + \boldsymbol{w}$ is expressed as the sum of a non-negative non-decreasing fully adapted process and a virtually local martingale, as required.

(e) This deals with the case in which $\mathcal{S}$ is finitely full. For the general case, writing $\hat{\boldsymbol{v}}$ for the fully adapted extension of $\boldsymbol{v}$ to the covered envelope $\hat{\mathcal{S}}$ of $\mathcal{S}$, and $\hat{\mathcal{S}}_f$ for the finitely-covered envelope of $\mathcal{S}$, $\hat{\boldsymbol{v}}\restriction\hat{\mathcal{S}}_f$ is a submartingale (626Ga). By (a)-(d) above, $\hat{\boldsymbol{v}}\restriction\hat{\mathcal{S}}_f$ is expressible in the form $\boldsymbol{v}'' + \boldsymbol{w}$ where $\boldsymbol{v}''$ is non-negative and non-decreasing and $\boldsymbol{w}$ is a virtually local martingale. Now $\boldsymbol{v} = (\boldsymbol{v}''\restriction\mathcal{S}) + (\boldsymbol{w}\restriction\mathcal{S})$. $\boldsymbol{v}''\restriction\mathcal{S}$ is surely non-negative and non-decreasing. But $\boldsymbol{w}$ and $\boldsymbol{w}\restriction\mathcal{S}$ have the same fully adapted extension $\hat{\boldsymbol{w}}$ to $\hat{\mathcal{S}}$, and $\hat{\boldsymbol{w}}$ and $\boldsymbol{w}\restriction\mathcal{S}$ will be virtually local martingales, as noted in 623J. Thus we have a decomposition of $\boldsymbol{v}$ of the type claimed.

626P Corollary If $\mathcal{S}$ is a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a submartingale such that $\{\mathbb{E}(v_\sigma) : \sigma \in \mathcal{S}\}$ is bounded below, then $\boldsymbol{v}$ is a semi-martingale.

proof We just have to look at the definition in 625D and remember that non-negative non-decreasing processes are locally of bounded variation (614Ic-614Id).

626Q We know that the previsible variation of a martingale is always zero. Otherwise, it seems that the calculation of previsible variations is not a trivial matter, even in the most basic cases. I go through the argument for one of my leading examples.

Proposition Suppose that $T = [0, \infty[$ and that $\boldsymbol{\iota} = \langle\iota_\tau\rangle_{\tau\in\mathcal{T}_f}$ is the identity process as described in 612F. Then the previsible variation of $\boldsymbol{\iota}\restriction\mathcal{T}_b$ is itself.

proof (a) Take $\tau \in \mathcal{T}_b$ and $\epsilon > 0$. Let $m \in \mathbb{N}$ be such that τ is less than or equal to the constant stopping time $(m\epsilon)\check{}$ and let J be the finite sublattice of $\mathcal{T}_b$ generated by $\{\tau \wedge (k\epsilon)\check{} : k \le m\}$. Note that $\tau = \max J$ and $\check{0} = \min \mathcal{T}_b = \min J$. Suppose that $I \in \mathcal{I}(\mathcal{T}_b \wedge \tau)$ and $J \subseteq I$. If $e \in \text{Sti}_0(I \wedge \tau)$, there is a $k < m$ such that e is included in the stopping-time interval $c(\tau \wedge (k\epsilon)\check{}, \tau \wedge ((k+1)\epsilon)\check{})$ and $e = c(\sigma, \sigma')$ where $\tau \wedge (k\epsilon)\check{} \le \sigma \le \sigma' \le \tau \wedge ((k+1)\epsilon)\check{}$, so that $\iota_{\sigma'} - \iota_\sigma \le \epsilon\chi 1$.

(b) Let $\langle\sigma_i\rangle_{i\le n}$ linearly generate the $(I \wedge \tau)$-cells. Then $0 \le \iota_{\sigma_{i+1}} - \iota_{\sigma_i} \le \epsilon\chi 1$ for every $i < n$, $\sigma_0 = \check{0}$ and $\sigma_n = \tau$. Set

$$v_i = \sum_{j=0}^{i-1} P_{\sigma_j}\iota_{\sigma_{j+1}} - \iota_{\sigma_j}, \quad w_i = \iota_{\sigma_i} - v_i$$

for $i \le n$. Then $v_0 = 0$, $v_{i+1} \in L^0(\mathfrak{A}_{\sigma_i})$ and $v_i \le v_{i+1}$ for $i < n$. So, for $i < n$,

$$P_{\sigma_i}w_{i+1} = P_{\sigma_i}(\iota_{\sigma_{i+1}} - v_i - P_{\sigma_i}\iota_{\sigma_{i+1}} + P_{\sigma_i}\iota_{\sigma_i}) = \iota_{\sigma_i} - v_i = w_i;$$

thus $\langle w_i\rangle_{i\le n}$ is an L^∞-martingale adapted to $\langle\mathfrak{A}_{\sigma_i}\rangle_{i\le n}$, starting from $w_0 = \iota_{\sigma_0}$. We have

$$\begin{aligned}\mathbb{E}(\sum_{i=0}^{n-1}(v_{i+1}-v_i)^2) &= \mathbb{E}(\sum_{i=0}^{n-1}(v_{i+1}-v_i)\times P_{\sigma_i}(\iota_{\sigma_{i+1}}-\iota_{\sigma_i}))\\ &\le \mathbb{E}(\sum_{i=0}^{n-1}(v_{i+1}-v_i)\times\epsilon\chi 1) = \epsilon\mathbb{E}(\sum_{i=0}^{n-1}v_{i+1}-v_i)\\ &= \epsilon\mathbb{E}(\sum_{i=0}^{n-1}P_{\sigma_i}\iota_{\sigma_{i+1}}-\iota_{\sigma_i}) = \epsilon\mathbb{E}(\sum_{i=0}^{n-1}\iota_{\sigma_{i+1}}-\iota_{\sigma_i})\\ &= \epsilon\mathbb{E}(\iota_\tau-\iota_{\breve{0}}) = \epsilon\mathbb{E}(\iota_\tau),\end{aligned}$$

and similarly

$$\mathbb{E}(\textstyle\sum_{i=0}^{n-1}(\iota_{\sigma_{i+1}}-\iota_{\sigma_i})^2) \le \mathbb{E}(\sum_{i=0}^{n-1}(\iota_{\sigma_{i+1}}-\iota_{\sigma_i})\times\epsilon\chi 1) \le \epsilon\mathbb{E}(\iota_\tau).$$

So

$$\mathbb{E}((\iota_\tau - S_I(\mathbf{1},Pd\boldsymbol{\iota}))^2) = \mathbb{E}((\iota_\tau - v_n)^2) = \mathbb{E}(w_n^2) = \sum_{i=0}^{n-1}\mathbb{E}((w_{i+1}-w_i)^2)$$

(because $\langle w_i\rangle_{i\le n}$ is a martingale, so $\mathbb{E}((w_{i+1}-w_i)\times(w_{j+1}-w_j)) = 0$ when $i\ne j$)

$$\le 2\sum_{i=0}^{n-1}\mathbb{E}((\iota_{\sigma_{i+1}}-\iota_{\sigma_i})^2+(v_{i+1}-v_i)^2) \le 4\epsilon\mathbb{E}(\iota_\tau).$$

(c) This is true whenever $I\in\mathcal{I}(\mathcal{T}_b\wedge\tau)$ includes J. As ϵ is arbitrary, ι_τ is the limit $2\mathrm{lim}_{I\uparrow\mathcal{I}(\mathcal{T}_b\wedge\tau)} S_{I\wedge\tau}(\mathbf{1},Pd\boldsymbol{\iota})$ for the norm topology of $L^2_{\bar\mu}$. It is therefore also the limit $\mathrm{w1lim}_{I\uparrow\mathcal{I}(\mathcal{T}_b\wedge\tau)} S_I(\mathbf{1},Pd\boldsymbol{\iota})$ for the weak topology of $L^1_{\bar\mu}$.

(d) As τ is arbitrary, the previsible variation $(\boldsymbol{\iota}\restriction\mathcal{T}_b)^\#$ agrees with $\boldsymbol{\iota}$ on $\mathcal{T}_b$.

626R Lemma Let $\mathcal{S}$ be a full sublattice of $\mathcal{T}$ with greatest and least members, and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a non-decreasing non-negative jump-free L^1-process. Then for every $\epsilon>0$ there is an $I\in\mathcal{I}(\mathcal{S})$, containing $\min\mathcal{S}$ and $\max\mathcal{S}$, such that $\|S_I(\mathbf{1},Pd\boldsymbol{u}) - u_{\max\mathcal{S}} + u_{\min\mathcal{S}}\|_1 \le \epsilon$.

proof By 618Gb, $\boldsymbol{u}$ is moderately oscillatory. Since $0\le u_\sigma\le u_{\max\mathcal{S}}\in L^1_{\bar\mu}$ for every $\sigma\in\mathcal{S}$, $\boldsymbol{u}$ is uniformly integrable. Set $\delta = \frac{1}{7}\epsilon$. Construct $\langle D_i\rangle_{i\in\mathbb{N}}$, $\langle y_i\rangle_{i\in\mathbb{N}}$ and $\langle d_i\rangle_{i\in\mathbb{N}}$ from $\boldsymbol{u}$ and δ as in 615M. Note that $\min D_0 = \min\mathcal{S}$ so $y_0 = u_{\min\mathcal{S}}$ (615Ma). If $i\in\mathbb{N}$, then $D_i\subseteq\mathcal{S}$ is closed under $\wedge$ and $y_i = \lim_{\sigma\downarrow D_i} u_\sigma$, while $d_{i+1}\subseteq d_i\cap[\![|y_{i+1}-y_i|\ge\delta]\!]$ (615Mc) and $|y_{i+1}-y_i|\le\delta\chi 1$ (618N, because $\boldsymbol{u}$ is jump-free). Because $0\le y_i\le u_{\max\mathcal{S}}$, or otherwise, $y_i\in L^1_{\bar\mu}$. Set $\mathfrak{B}_i = \bigcap_{\sigma\in D_i}\mathfrak{A}_\sigma$ and write $Q_i : L^1_{\bar\mu}\to L^1_{\bar\mu}\cap L^0(\mathfrak{B}_i)$ for the associated conditional expectation. As $y_i\in L^0(\mathfrak{B}_i)$ (615Mb), $Q_iy_i = y_i$. Because $\boldsymbol{u}$ is uniformly integrable, $y_i = 1\mathrm{lim}_{\sigma\downarrow D_i} u_\sigma$ (621B(c-ii) again).

If $i\in\mathbb{N}$ and $\sigma\in D_{i+1}$, there is a $\sigma'\in D_i$ such that $\sigma'\le\sigma$ (615Ma); now $\sigma''\le\sigma'$ and $u_{\sigma''}\le u_{\sigma'}$ for every $\sigma''\in D_i\wedge\sigma'$, so $y_i\le u_\sigma$; as σ is arbitrary, $y_i\le y_{i+1}$. So in fact $d_{i+1}\subseteq d_i\cap[\![y_{i+1}-y_i=\delta]\!]$ and $y_i\le y_{i+1}\le y_i+\delta\chi d_i$. Now

$$y_i = Q_iy_i \le Q_iy_{i+1} \le Q_i(y_i+\delta\chi d_i) = y_i+\delta\chi d_i$$

(because $d_i\in\mathfrak{B}_i$), and $(Q_iy_{i+1}-y_{i+1})\times\chi(1\setminus d_i) = 0$. At the same time, $y_{i+1}\ge y_i+\delta\chi d_{i+1}$, so

$$(Q_iy_{i+1}-y_{i+1})^+ \le ((y_i+\delta\chi d_i)-(y_i+\delta\chi d_{i+1}))^+ = \delta\chi(d_i\setminus d_{i+1})$$

and

$$\|Q_iy_{i+1}-y_{i+1}\|_1 = 2\mathbb{E}((Q_iy_{i+1}-y_{i+1})^+) - \mathbb{E}(Q_iy_{i+1}-y_{i+1}) \le 2\delta\bar\mu(d_i\setminus d_{i+1}).$$

Summing over i,

$$\textstyle\sum_{i=0}^\infty \|Q_iy_{i+1}-y_{i+1}\|_1 \le 2\delta.$$

By 615M(c-ii), there is an $m \ge 1$ such that $\mathbb{E}(u_{\max\mathcal{S}} \times \chi d_m) \le \delta$. Set $\eta = \frac{1}{m}\delta$. Again because every member of D_{i+1} dominates some member of D_i, we can choose $\sigma_m, \sigma_{m-1}, \dots, \sigma_0 \in \mathcal{S}$ such that $\sigma_m \in D_m$, $\|u_{\sigma_m} - y_m\|_1 \le \eta$ and

$$\sigma_i \in D_i, \quad \sigma_i \le \sigma_{i+1}, \quad \|u_{\sigma_i} - y_i\|_1 \le \eta, \quad \|P_{\sigma_i} y_{i+1} - Q_i y_{i+1}\|_1 \le \eta$$

for $i < m$ (using 621C(g-i) for the last clause), while $\sigma_0 = \min\mathcal{S}$. Since $[\![\sigma_m < \max\mathcal{S}]\!] \subseteq d_m$ and $0 \le u_{\sigma_m} \le u_{\max\mathcal{S}}$,

$$\|u_{\max\mathcal{S}} - u_{\sigma_m}\|_1 \le \mathbb{E}(u_{\max\mathcal{S}} \times \chi d_m) \le \delta.$$

Set $I = \{\sigma_i : i \le m\} \cup \{\max\mathcal{S}\}$. Then

$$\begin{aligned}
&\|S_I(\mathbf{1}, P d\boldsymbol{u}) - u_{\max\mathcal{S}} + u_{\min\mathcal{S}}\|_1 \\
&\quad = \|P_{\sigma_m} u_{\max\mathcal{S}} - u_{\sigma_m} - u_{\max\mathcal{S}} + u_{\sigma_m} + \sum_{i=0}^{m-1} (P_{\sigma_i} u_{\sigma_{i+1}} - u_{\sigma_i} - u_{\sigma_{i+1}} + u_{\sigma_i})\|_1 \\
&\quad \le \|P_{\sigma_m}(u_{\max\mathcal{S}} - u_{\sigma_m})\|_1 + \|u_{\max\mathcal{S}} - u_{\sigma_m}\|_1 + \sum_{i=0}^{m-1} \|P_{\sigma_i} u_{\sigma_{i+1}} - u_{\sigma_{i+1}}\|_1 \\
&\quad \le 2\|u_{\max\mathcal{S}} - u_{\sigma_m}\|_1 + \sum_{i=0}^{m-1} \|Q_i y_{i+1} - y_{i+1}\|_1 \\
&\qquad + \sum_{i=0}^{m-1} \|P_{\sigma_i} u_{\sigma_{i+1}} - u_{\sigma_{i+1}} - Q_i y_{i+1} + y_{i+1}\|_1 \\
&\quad \le 2\delta + 2\delta + \sum_{i=0}^{m-1} \|P_{\sigma_i} u_{\sigma_{i+1}} - P_{\sigma_i} y_{i+1}\|_1 + \sum_{i=0}^{m-1} \|P_{\sigma_i} y_{i+1} - Q_i y_{i+1}\|_1 \\
&\qquad + \sum_{i=0}^{m-1} \|y_{i+1} - u_{\sigma_{i+1}}\|_1 \\
&\quad \le 4\delta + m\eta + m\eta + m\eta = \epsilon.
\end{aligned}$$

626S Proposition Let $\mathcal{S}$ be a non-empty full sublattice of $\mathcal{T}$, and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma\in\mathcal{S}}$ a non-decreasing locally jump-free L^1-process starting from 0. Then it is equal to its previsible variation.

proof (a) Because $\boldsymbol{v}$ is moderately oscillatory (618Gb), its starting value $v_\downarrow$ is well-defined (615Gb). We are supposing that $v_\downarrow$ is zero; as $\boldsymbol{v}$ is a non-decreasing L^1-process it is a non-negative submartingale and has a previsible variation $\boldsymbol{v}^\# = \langle v^\#_\sigma \rangle_{\sigma\in\mathcal{S}}$. I need to show that $v^\#_\tau = v_\tau$ for every $\tau \in \mathcal{S}$. Clearly it is enough to consider the case in which $\mathcal{S} = \mathcal{S} \wedge \tau$. Take $w \in L^\infty(\mathfrak{A})$ and $\epsilon > 0$.

(b) If $J \in \mathcal{I}(\mathcal{S})$ is non-empty, there is an $I \in \mathcal{I}(\mathcal{S})$ such that $J \subseteq I$ and $\|S_I(\mathbf{1}, Pd\boldsymbol{v}) - v_{\max J} + v_{\min J}\|_1 \le \epsilon$. **P** If J is a singleton, set $I = J$. Otherwise, take $(\tau_0, \dots, \tau_k)$ linearly generating the J-cells, so that $\tau_0 = \min J$ and $\tau_k = \max J$. If $j < k$, $\boldsymbol{v}\restriction\mathcal{S} \cap [\tau_j, \tau_{j+1}]$ is a non-decreasing non-negative L^1-process, and is jump-free by 618Gc, while $\mathcal{S} \cap [\tau_j, \tau_{j+1}]$ is full (see 611Md and 611Me). So 626R tells us that there is a finite sublattice I_j of $\mathcal{S} \cap [\tau_j, \tau_{j+1}]$, containing τ_j and τ_{j+1}, such that $\|S_{I_j}(\mathbf{1}, Pd\boldsymbol{v}) - v_{\tau_{j+1}} + v_{\tau_j}\|_1 \le \frac{1}{k}\epsilon$. Let I be the sublattice of $\mathcal{S}$ generated by $\bigcup_{j<k} I_j$. Then $\min I = \tau_0 = \min J$, $\max I = \tau_k = \max J$ and $I \cap [\tau_j, \tau_{j+1}] = I_j$ for each j. So

$$S_I(\mathbf{1}, Pd\boldsymbol{v}) = \sum_{j=0}^{k-1} S_{I_j}(\mathbf{1}, Pd\boldsymbol{v}), \quad v_{\max I} - v_{\min I} = \sum_{j=0}^{k} v_{\tau_{j+1}} - v_{\tau_j}$$

and

$$\|S_I(\mathbf{1}, Pd\boldsymbol{v}) - v_{\max J} + v_{\min J}\|_1 \le \sum_{j=0}^{k-1} \|S_{I_j}(\mathbf{1}, Pd\boldsymbol{v}) - v_{\tau_{j+1}} + v_{\tau_j}\|_1 \le \epsilon. \textbf{ Q}$$

(c) If $J \in \mathcal{I}(\mathcal{S})$ is non-empty, there is an $I \in \mathcal{I}(\mathcal{S})$ such that $J \subseteq I$ and $\|S_I(\mathbf{1}, Pd\boldsymbol{v}) - v_\tau\|_1 \le 2\epsilon$. **P** As in the proof of 626R, $\boldsymbol{v}$ here is uniformly integrable, so $0 = 1\lim_{\sigma\downarrow\mathcal{S}} v_\sigma$. Let $\sigma \in \mathcal{S}$ be such that $\sigma \le \min J$

and $\|v_\sigma\|_1 \le \epsilon$. By (b), there is an $I \in \mathcal{I}(\mathcal{S})$ such that $I \supseteq J \cup \{\sigma,\tau\}$ and $\|S_I(\mathbf{1}, Pd\boldsymbol{v}) - v_\tau + v_\sigma\|_1 \le \epsilon$, so that $\|S_I(\mathbf{1}, Pd\boldsymbol{v}) - v_\tau\|_1 \le 2\epsilon$. **Q**

(d) As $v_\tau^\# = \text{w1lim}_{I\uparrow\mathcal{S}} S_I(\mathbf{1}, Pd\boldsymbol{v})$, there is a non-empty $J \in \mathcal{I}(\mathcal{S})$ such that $|\mathbb{E}(w \times (S_I(\mathbf{1}, Pd\boldsymbol{v}) - v_\tau^\#))| \le \epsilon$ whenever $I \in \mathcal{I}(\mathcal{S})$ includes J. By (c), there is such an I with $\|S_I(\mathbf{1}, Pd\boldsymbol{v}) - v_\tau\|_1 \le 2\epsilon$, so that $|\mathbb{E}(w \times (S_I(\mathbf{1}, Pd\boldsymbol{v}) - v_\tau))| \le 2\epsilon\|w\|_\infty$ and $|\mathbb{E}(w \times (v_\tau^\# - v_\tau))| \le \epsilon(1 + 2\|w\|_\infty)$. As ϵ and w are arbitrary, $v_\tau^\# = v_\tau$.

626T Proposition Let $\mathcal{S}$ be a full sublattice of $\mathcal{T}$, and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a locally jump-free L^2-martingale. Then the quadratic variation $\boldsymbol{v}^* = \langle v_\sigma^*\rangle_{\sigma\in\mathcal{S}}$ of $\boldsymbol{v}$ is the previsible variation $(\boldsymbol{v}^2)^\#$ of the submartingale $\boldsymbol{v}^2$.

proof If $\mathcal{S}$ is empty, this is trivial; suppose otherwise. By 622H again, $\boldsymbol{v}$ is locally moderately oscillatory, so its starting value $v_\downarrow$ is defined (615Gb again). Note that $v_\downarrow \in L^2_{\bar\mu}$. **P** Take any $\tau \in \mathcal{S}$. Then $\{x : x \in L^0(\mathfrak{A}), \|x\|_2 \le \|v_\tau\|_2\}$ is closed (613Bc) and includes $\{v_\sigma : \sigma \in \mathcal{S} \wedge \tau\}$ (621Ce) so contains $v_\downarrow$. **Q**

(a) Suppose to begin with that $v_\downarrow = 0$. By Jensen's inequality (621Cd again), $\boldsymbol{v}^2$ is a submartingale and has a previsible variation (626M). Also $\boldsymbol{v}^*$ is an L^1-process (624G) so $\boldsymbol{v}^2 - \boldsymbol{v}^* = 2ii_{\boldsymbol{v}}(\boldsymbol{v})$ (617Ka) is a martingale (624Hb) and $(\boldsymbol{v}^2)^\# = (v^*)^\#$ (626Kc). Now $\boldsymbol{v}^*$ is an L^1-process, non-decreasing and starting from 0 (617Jb), while $\boldsymbol{v}^2$ and $ii_{\boldsymbol{v}}(\boldsymbol{v})$ are locally jump-free (618Ga, 618R). By 626S, $\boldsymbol{v}^* = (\boldsymbol{v}^*)^\# = (\boldsymbol{v}^2)^\#$.

(b) For the general case, set $\boldsymbol{w} = \boldsymbol{v} - v_\downarrow\mathbf{1}$. Then $\boldsymbol{w}$ is a locally jump-free martingale and an L^2-process starting at 0, so $(\boldsymbol{w}^2)^\# = \boldsymbol{w}^*$. Now, expressing $\boldsymbol{w}^*$ as $\langle w_\sigma^*\rangle_{\sigma\in\mathcal{S}}$,

$$w_\tau^* = \int_{\mathcal{S}\wedge\tau} d\boldsymbol{w}^* = \int_{\mathcal{S}\wedge\tau} (d\boldsymbol{w})^2$$

(617I)

$$= \int_{\mathcal{S}\wedge\tau} (d\boldsymbol{v})^2$$

(because the interval functions $\Delta\boldsymbol{w}$, $\Delta\boldsymbol{v}$ are equal)

$$= v_\tau^*$$

for every $\tau \in \mathcal{S}$. At the same time,

$$\boldsymbol{v}^2 - \boldsymbol{w}^2 = 2\boldsymbol{v} - v_\downarrow^2\mathbf{1}$$

is a martingale, so

$$(\boldsymbol{v}^2)^\# = (\boldsymbol{w}^2)^\# = \boldsymbol{w}^* = \boldsymbol{v}^*.$$

626X Basic exercises (a) Suppose that $T = [0,\infty[$ and $\mathfrak{A} = \{0,1\}$, as in 613W, 615Xf, 616Xa, 617Xb, 618Xa and 622Xd. Let $f : [0,\infty[\to \mathbb{R}$ be a function and $\boldsymbol{u}$ the corresponding process on $\mathcal{T}_f$. (i) Show that $\boldsymbol{u}$ is a submartingale iff f is non-decreasing. (ii) Show that this case the previsible variation of $\boldsymbol{u}$ corresponds to the function $t \mapsto f(t) - f(0)$.

(b) Let $\mathcal{S}$ be a sublattice, $\boldsymbol{v}$ a fully adapted process with domain $\mathcal{S}$ and $\tau \in \mathcal{S}$. Show that $\boldsymbol{v}$ is a submartingale iff $\boldsymbol{v}\restriction\mathcal{S}\wedge\tau$ and $\boldsymbol{v}\restriction\mathcal{S}\vee\tau$ are both submartingales.

(c) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a submartingale such that $\{\mathbb{E}(v_\sigma) : \sigma \in \mathcal{S}\}$ is bounded below. Show that $\boldsymbol{v}$ is locally moderately oscillatory. (*Hint*: 626Gc.)

(d) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{v}$ a non-negative submartingale with domain $\mathcal{S}$, and z a non-negative member of $L^0(\bigcap_{\sigma\in\mathcal{S}}\mathfrak{A}_\sigma)$. Show that if $z\boldsymbol{v}$ (612D(e-ii)) is an L^1-process then it is a submartingale, and that $z\boldsymbol{v}^\#$ is the previsible variation of $z\boldsymbol{v}$.

(e) Suppose that $T = \mathbb{N}$ and that $\mathfrak{A}$ is atomless. Let $u \in L^0(\mathfrak{A})$ be such that $\bar\mu[\![u > 1-\alpha]\!] = \alpha$ for $\alpha \in [0,1]$, and suppose that

$$\mathfrak{A}_n = \{a : a \in \mathfrak{A},\ a \cap [\![u > 1-2^{-n}]\!] \in \{0, [\![u > 1-2^{-n}]\!]\} \text{ for every } n \in \mathbb{N}\}.$$

Show that the martingale $\boldsymbol{P}u$ is of bounded variation. (*Hint*: 611Xh.)

626 Notes and comments The previsible variations of 626J-626K can be thought of as indefinite integrals of interval functions $P d\boldsymbol{v}$ based on the weak topology of $L^1_{\bar{\mu}}$ instead of the topology of convergence in measure on $L^0(\mathfrak{A})$. I do not really wish to go farther along this route. But there are obvious questions to ask about the applicability of the ideas of §613 in this context.

In both 626M and 626O, we start with a submartingale $\boldsymbol{v}$ and seek to express it as $\boldsymbol{v}^\# + \boldsymbol{w}$ where $\boldsymbol{v}^\#$ is non-decreasing and $\boldsymbol{w}$ is more or less a martingale. In 626M we have an explicit formula, with $\boldsymbol{v}^\#$ the 'previsible variation' as defined in 626J, so that we have picked out a particular solution. In 626O the statement of the theorem makes no claim that the solution is unique, and of course it is not, because a constant process can always be added to one term and subtracted from the other. Indeed it is not difficult to show that there are non-trivial martingales of bounded variation (626Xe) and therefore that there are many non-negative non-decreasing processes which are not their own previsible variations. The construction in the proof of 626O gives rise to a particular pair $\boldsymbol{v}^\#$, $\boldsymbol{w}$ but I have not seen a simple characterization of the processes $\boldsymbol{v}^\#$ which can arise in this way.

The calculation in 626Q is rather elaborate, but I do not know of an essentially more direct method. Concerning the other two examples in §612, Brownian motion has previsible variation 0 just because it is a martingale, and the Poisson process turns out to have previsible variation $\boldsymbol{\iota}$; but for a proof of this we shall have to wait for some more of the general theory (632Mb).

Version of 27.3.21

627 Integrators and semi-martingales

This section is devoted to a kind of structure theory for integrators (627I-627J, 627L, 627Q); I take a route which passes some further important classes of stochastic process (627B) and ideas from the theory of linear topological spaces (627F-627G).

627A Notation As always, $(\mathfrak{A}, \bar{\mu}, T, \langle\mathfrak{A}_t\rangle_{t\in T}, \mathcal{T}, \langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ is a stochastic integration structure. For a sublattice $\mathcal{S}$ of $\mathcal{T}$, $\mathcal{I}(\mathcal{S})$ is the set of finite sublattices of $\mathcal{S}$; $M_{\mathrm{fa}}(\mathcal{S})$ and $M_{\mathrm{lmo}}(\mathcal{S})$ are the spaces of fully adapted and locally moderately oscillatory processes with domain $\mathcal{S}$. If $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is order-bounded, $\sup|\boldsymbol{u}| = \sup(\{0\} \cup \{|u_\sigma| : \sigma \in \mathcal{S}\})$ (614E).

$\mathbf{1}$ is the constant process with value $\chi 1$. For $I \subseteq \mathcal{T}$ and $\tau \in \mathcal{T}$, $I \wedge \tau$ and $I \vee \tau$ are $\{\sigma \wedge \tau : \sigma \in I\}$, $\{\sigma \vee \tau : \sigma \in I\}$ respectively. $\mathbb{E}$ is the standard integral on $L^1_{\bar{\mu}} = L^1(\mathfrak{A}, \bar{\mu})$, and $\theta(w) = \mathbb{E}(|w| \wedge \chi 1)$ for $w \in L^0 = L^0(\mathfrak{A})$. For $\tau \in \mathcal{T}$, $P_\tau : L^1_{\bar{\mu}} \to L^1_{\bar{\mu}} \cap L^0(\mathfrak{A}_\tau)$ is the associated conditional expectation. If $z \in L^1_{\bar{\mu}}$, $\boldsymbol{P}z = \langle P_\tau z\rangle_{\tau\in\mathcal{T}}$.

627B Definitions Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v}$ a fully adapted process with domain $\mathcal{S}$.

(a) $\boldsymbol{v}$ is a **supermartingale** if $-\boldsymbol{v}$ is a submartingale (626B), that is, $\boldsymbol{v}$ is an L^1-process and $P_\sigma v_\tau \le v_\sigma$ whenever $\sigma \le \tau$ in $\mathcal{S}$. (Mnemonic: $P d\boldsymbol{v} \le 0$.)

(b) $\boldsymbol{v}$ is a **quasimartingale** if $\boldsymbol{v}$ is an L^1-process and $\{\mathbb{E}(w) : w \in Q_{\mathcal{S}}(d\boldsymbol{v})\}$ (definition: 616B) is bounded in $\mathbb{R}$,

(c) I will say that $\boldsymbol{v}$ is a **strong integrator** if whenever $\epsilon > 0$ there are a uniformly integrable martingale $\boldsymbol{w}$ and a fully adapted process $\boldsymbol{w}'$ of bounded variation, both with domain $\mathcal{S}$, such that $\bar{\mu}[\![\boldsymbol{v} \ne \boldsymbol{w} + \boldsymbol{w}']\!] \le \epsilon$.

627C Elementary facts (a)(i) If $\boldsymbol{v}$ is a supermartingale, so is $\boldsymbol{v}\restriction\mathcal{S}'$ for any sublattice $\mathcal{S}'$ of $\operatorname{dom}\boldsymbol{v}$. If $\boldsymbol{v}$ and $\boldsymbol{w}$ are supermartingales, so is $\boldsymbol{v} + \boldsymbol{w}$. (Immediate from the definition.)

(ii) If $h : \mathbb{R} \to \mathbb{R}$ is concave and non-decreasing, $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a supermartingale and $\bar{h}\boldsymbol{v}$ is an L^1-process, then $\bar{h}\boldsymbol{v}$ is a supermartingale. **P** Set $g(x) = -h(-x)$ for $x \in \mathbb{R}$. Then g is convex and non-decreasing, so $\bar{g}\circ(-\boldsymbol{v})$ is a submartingale (626Cc) and $\bar{h}\boldsymbol{v} = -\bar{g}\circ(-\boldsymbol{v})$ is a supermartingale. **Q**

(iii) A supermartingale $\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a martingale iff $\mathbb{E}(v_\sigma) = \mathbb{E}(v_\tau)$ whenever $\sigma \le \tau$ in $\mathcal{S}$. (For in this case we have $P_\sigma v_\tau \le v_\sigma$ while $\mathbb{E}(P_\sigma v_\tau) = \mathbb{E}(v_\tau) = \mathbb{E}(v_\sigma)$, so $P_\sigma v_\tau = v_\sigma$.)

(b) Every martingale $\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a quasimartingale. **P** If $\sigma\in\tau$ in $\mathcal{S}$ and $u\in L^\infty(\mathfrak{A}_\sigma)$, then

$$\mathbb{E}(u\times(v_\tau-v_\sigma))=\mathbb{E}(P_\sigma(u\times(v_\tau-v_\sigma)))=\mathbb{E}(u\times(P_\sigma v_\tau-v_\sigma))=0,$$

so $\mathbb{E}(w)=0$ for every $w\in Q_{\mathcal{S}}(d\boldsymbol{v})$. **Q**

(c)(i) A strong integrator is an integrator. **P** By 622G, 616Ra and 616Pa, the sum of a uniformly integrable martingale and a fully adapted process of bounded variation is an integrator. Now 616P(b-iii) shows that a strong integrator is an integrator. **Q**

(ii) A linear combination of strong integrators is a strong integrator. (We just have to recall that sums and scalar multiples of martingales are martingales, sums and scalar multiples of uniformly integrable sets are uniformly integrable and sums and scalar multiples of processes of bounded variation have bounded variation.)

(iii) If $\boldsymbol{v}$ is a strong integrator with domain $\mathcal{S}$ and $\mathcal{S}'$ is a sublattice of $\mathcal{S}$, then $\boldsymbol{v}\restriction\mathcal{S}'$ is a strong integrator. **P** Given $\epsilon>0$, there are a uniformly integrable martingale $\boldsymbol{w}$ and a process $\boldsymbol{w}'$ of bounded variation, both with domain $\mathcal{S}$, such that $[\![\boldsymbol{v}\ne\boldsymbol{w}+\boldsymbol{w}']\!]$ has measure at most ϵ. Now $\boldsymbol{w}\restriction\mathcal{S}'$ is a uniformly integrable martingale (622Dd), $\boldsymbol{w}\restriction\mathcal{S}'$ is of bounded variation (614Lb) and $[\![\boldsymbol{v}\restriction\mathcal{S}'\ne\boldsymbol{w}\restriction\mathcal{S}'+\boldsymbol{w}'\restriction\mathcal{S}']\!]$ is included in $[\![\boldsymbol{v}\ne\boldsymbol{w}+\boldsymbol{w}']\!]$, so has measure at most ϵ. **Q**

(iv) If $\boldsymbol{v}$ is a fully adapted process with domain $\mathcal{S}$ and for every $\epsilon>0$ there is a strong integrator $\boldsymbol{v}'$ with domain $\mathcal{S}$, such that $[\![\boldsymbol{v}\ne\boldsymbol{v}']\!]$ has measure at most ϵ, then $\boldsymbol{v}$ is a strong integrator. (Immediate from the definition of 'strong integrator' and from the fact that $[\![\boldsymbol{v}\ne\boldsymbol{v}'']\!]\subseteq[\![\boldsymbol{v}\ne\boldsymbol{v}']\!]\cup[\![\boldsymbol{v}'\ne\boldsymbol{v}'']\!]$ for any fully adapted processes $\boldsymbol{v}$, $\boldsymbol{v}'$ and $\boldsymbol{v}''$ with domain $\mathcal{S}$.)

627D Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v}$ a non-negative fully adapted process with domain $\mathcal{S}$.

(a) If $\boldsymbol{v}$ is a virtually local martingale, it is a $\|\;\|_1$-bounded supermartingale.

(b) If $\boldsymbol{v}$ is a $\|\;\|_1$-bounded supermartingale it is order-bounded.

proof If $\mathcal{S}$ is empty, both parts are trivial, so let us suppose that $\mathcal{S}\ne\emptyset$. Express $\boldsymbol{v}$ as $\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$.

(a)(i) The fully adapted extension $\hat{\boldsymbol{v}}=\langle\hat v_\sigma\rangle_{\sigma\in\hat{\mathcal{S}}}$ of $\boldsymbol{v}$ to the covered envelope $\hat{\mathcal{S}}$ of $\mathcal{S}$ is an approximately local martingale (623J). Let $\mathcal{A}$ be the family of non-empty downwards-directed subsets A of $\hat{\mathcal{S}}$ such that $R_A(\hat{\boldsymbol{v}})$, as defined in 623B, is a martingale, and for $A\in\mathcal{A}$ express $R_A(\hat{\boldsymbol{v}})$ as $\langle\hat v_{A\sigma}\rangle_{\sigma\in\hat{\mathcal{S}}}$.

(ii) If $A\in\mathcal{A}$, the starting values of $R_A(\hat{\boldsymbol{v}})$ and $\hat{\boldsymbol{v}}$ are the same (623B(c-i)); so we have a common starting value $v_\downarrow=\lim_{\sigma\downarrow\hat{\mathcal{S}}}v_{A\sigma}$ for all the $R_A(\hat{\boldsymbol{v}})$. Now $v_\downarrow$ is always the $\|\;\|_1$-limit $1\text{lim}_{\sigma\downarrow\hat{\mathcal{S}}}\hat v_{A\sigma}$ (626E, since $R_A(\hat{\boldsymbol{v}})$ is a submartingale, as noted in 626F). Since all the expectations $\mathbb{E}(\hat v_{A\sigma})$, for $\sigma\in\mathcal{S}$, must be the same, this is also $\mathbb{E}(v_\downarrow)$. This is so for every $A\in\mathcal{A}$, so we have $\mathbb{E}(\hat v_{A\sigma})=\mathbb{E}(v_\downarrow)$ for every $A\in\mathcal{A}$ and $\sigma\in\hat{\mathcal{S}}$.

(iii) Since $v_\sigma\ge0$ for every $\sigma\in\mathcal{S}$, $\hat v_\sigma\ge0$ for every $\sigma\in\hat{\mathcal{S}}$, $\hat v_{A\sigma}\ge0$ for every $A\in\mathcal{A}$ and $\sigma\in\hat{\mathcal{S}}$, and $v_\downarrow\ge0$. So in fact we have $\|\hat v_{A\sigma}\|_1=\|v_\downarrow\|_1$ for every $A\in\mathcal{A}$ and $\sigma\in\hat{\mathcal{S}}$.

Consequently $\|\hat v_\sigma\|_1\le\|v_\downarrow\|_1$ for every $\sigma\in\hat{\mathcal{S}}$. **P** Because $\hat{\boldsymbol{v}}$ is a virtually local martingale, there is for every $\epsilon>0$ an $A\in\mathcal{A}$ such that $\bar\mu[\![\hat v_{A\sigma}\ne\hat v_\sigma]\!]\le\epsilon$. So v_σ belongs to the closure $\overline{\{\hat v_{A\sigma}:A\in\mathcal{A}\}}$ for the topology of convergence in measure. Because $\|\;\|_1$-balls are closed for this topology (613Bc),

$$\|\hat v_\sigma\|_1\le\sup_{A\in\mathcal{A}}\|\hat v_{A\sigma}\|_1=\|v_\downarrow\|_1.\ \mathbf{Q}$$

In particular, $\hat{\boldsymbol{v}}$ is an L^1-process.

(iv) ? Suppose, if possible, that $\hat{\boldsymbol{v}}$ is not a supermartingale. Then there are $\sigma\le\tau$ in $\hat{\mathcal{S}}$ such that $P_\sigma\hat v_\tau\not\le\hat v_\sigma$, that is, $c=[\![\hat v_\sigma<P_\sigma\hat v_\tau]\!]$ is non-zero. For each $n\in\mathbb{N}$ we can find an $A_n\in\mathcal{A}$ such that $a_n=\sup_{\rho\in A}[\![\rho<\tau]\!]$ has measure at most $2^{-n-2}\bar\mu c$. Setting $b_n=\sup_{\rho\in A}[\![\rho<\sigma]\!]$ for each n, $b_n\in\mathfrak{A}_\sigma$ and

$$\sum_{n=0}^\infty\bar\mu b_n\le\sum_{n=0}^\infty\bar\mu a_n<\bar\mu c,$$

so $c'=c\setminus\sup_{n\in\mathbb{N}}b_n$ is a non-zero member of $\mathfrak{A}_\sigma$. Consider $\mathbb{E}(v_\tau\times\chi c')$. For each $n\in\mathbb{N}$,

$$\theta(v_\tau-v_{A_n\tau})\le\bar\mu[\![v_\tau\ne v_{A_n\tau}]\!]\le\bar\mu a_n\to0$$

as $n \to \infty$, so $v_\tau = \lim_{n\to\infty} v_{A_n\tau}$ and $v_\tau \times \chi c' = \lim_{n\to\infty} v_{A_n\tau} \times \chi v'$. Now

$$\mathbb{E}(v_\sigma \times \chi c') < \mathbb{E}(P_\sigma v_\tau \times \chi c')$$

(because $0 \ne c' \subseteq [\![v_\sigma < P_\sigma v_\tau]\!]$)

$$= \mathbb{E}(P_\sigma(v_\tau \times \chi c'))$$

(because $c' \in \mathfrak{A}_\sigma$)

$$= \mathbb{E}(v_\tau \times \chi c') = \|v_\tau \times \chi c'\|_1 \le \sup_{n\in\mathbb{N}} \|v_{A_n\tau} \times \chi c'\|_1$$

(because $\| \|_1$-balls are closed for the topology of convergence in measure)

$$= \sup_{n\in\mathbb{N}} \mathbb{E}(v_{A_n\tau} \times \chi c').$$

There is therefore an $n \in \mathbb{N}$ such that $\mathbb{E}(v_{A_n\tau} \times \chi c') > \mathbb{E}(v_\sigma \times \chi c')$. But c' is disjoint from b_n, so $v_\sigma \times \chi c' = v_{A_n\sigma} \times \chi c'$ and

$$\begin{aligned}\mathbb{E}(v_{A_n\sigma} \times \chi c') < \mathbb{E}(v_{A_n\tau} \times \chi c') &= \mathbb{E}(P_\sigma(v_{A_n\tau} \times \chi c')) \\ &= \mathbb{E}(P_\sigma v_{A_n\tau} \times \chi c') = \mathbb{E}(v_{A_n\sigma} \times \chi c')\end{aligned}$$

because $R_{A_n}(\boldsymbol{v})$ is a martingale. But this is absurd. **X**

(v) Thus $\hat{\boldsymbol{v}}$ is a supermartingale. But it follows at once that $\boldsymbol{v} = \hat{\boldsymbol{v}}\restriction\mathcal{S}$ is a supermartingale. And we saw in (iii) that $\hat{\boldsymbol{v}}$ is $\| \|_1$-bounded, so $\boldsymbol{v}$ also is $\| \|_1$-bounded.

(b)(i) Set $\gamma = \sup_{\sigma\in\mathcal{S}} \|v_\sigma\|_1$. If $\tau_0 \le \tau_1 \le \ldots \le \tau_n$ in $\mathcal{S}$ there is a $v \ge 0$ such that $P_{\tau_i} v \ge v_{\tau_i}$ for every $i \le n$ and $\mathbb{E}(v) \le \gamma$. **P** Induce on n. If $n = 0$ we can set $v = v_{\tau_0}$. For the inductive step to $n+1 > 0$, take $v' \ge 0$ such that $\mathbb{E}(v') \le \gamma$ and $P_{\tau_i} v' \ge v_{\tau_i}$ for $i \le n$. Then $P_{\tau_n} v_{\tau_{n+1}} \le v_{\tau_n} \le P_{\tau_n} v'$; set $v = v_{\tau_{n+1}} + P_{\tau_n} v' - P_{\tau_n} v_{\tau_{n+1}}$. In this case,

$$\mathbb{E}(v) = \mathbb{E}(v') \le \gamma,$$

while

$$P_{\tau_{n+1}} v = v \ge v_{\tau_{n+1}} \ge 0$$

and for $i \le n$

$$P_{\tau_i} v = P_{\tau_i} v_{\tau_{n+1}} + P_{\tau_i} v' - P_{\tau_i} v_{\tau_{n+1}} \ge v_{\tau_i}. \textbf{ Q}$$

So if $w = \sup_{i\le n} v_{\tau_i}$,

$$\bar{\mu}[\![w \ge M]\!] \le \bar{\mu}[\![\sup_{i\le n} P_{\tau_i} v \ge M]\!] \le \frac{1}{M}\|P_{\tau_n} v\|_1 \le \frac{\gamma}{M}$$

for every $M > 0$, by Doob's maximal inequality (621E).

(ii) If $A \subseteq \mathcal{S}$ is finite and not empty and $z_A = \sup_{\sigma\in A} v_\sigma$, then $\bar{\mu}[\![z_A \ge M]\!] \le \frac{\gamma}{M}$ for every $M \ge 0$. **P** Let I be the sublattice of $\mathcal{S}$ generated by A, and $\tau_0 \le \ldots \le \tau_n$ a sequence in I linearly generating the I-cells as in 611L. Set $z' = \sup_{i\le n} v_{\tau_i}$. Because $\sup_{i\le n} [\![\tau = \tau_i]\!] = 1$, $v_\tau \le z'$ for every $\tau \in I$, $z_A \le z'$ and

$$\bar{\mu}[\![z_A \ge M]\!] \le \bar{\mu}[\![z' \ge M]\!] \le \frac{\gamma}{M}$$

by (i). **Q**

(iii) For $n \in \mathbb{N}$, set $c_n = \sup\{[\![z_A \ge n]\!] : \emptyset \ne A \in [\mathcal{S}]^{<\omega}\}$. Then $\bar{\mu} c_n \le \frac{\gamma}{n}$ for every $n > 1$. By 364L(a-i), $\{z_A : \min\mathcal{S} \in A \in [\mathcal{S}]^{<\omega}\}$ is bounded above in L^0, that is, $\{v_\sigma : \sigma \in \mathcal{S}\}$ is bounded above. Since it is also bounded below, $\boldsymbol{v}$ is order-bounded.

627E Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a quasimartingale.
(a) There is a non-negative $\| \|_1$-bounded supermartingale $\boldsymbol{w}$ such that $\boldsymbol{v} + \boldsymbol{w}$ is a supermartingale.

(b) If $\mathcal{S}$ has a greatest element then $\boldsymbol{v}$ is expressible as the difference of two non-negative supermartingales.

(c) If $\boldsymbol{v}$ is $\| \|_1$-bounded then it is a semi-martingale (definition: 625D).

proof (a)(i) Set $\gamma = \sup\{\mathbb{E}(w) : w \in Q_{\mathcal{S}}(d\boldsymbol{v})\}$. Then $\mathbb{E}(w) \le \gamma$ for every $w \in Q_{\mathcal{S}}(Pd\boldsymbol{v})$. **P** If $w = 0$ this is trivial, as $0 \in Q_{\mathcal{S}}(d\boldsymbol{v})$, so $\gamma \ge 0$. Otherwise, there are a non-empty $I \in \mathcal{I}(\mathcal{S})$ and a fully adapted process $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in I}$ such that $\|\boldsymbol{u}\|_\infty \le 1$ and $w = S_I(\boldsymbol{u}, Pd\boldsymbol{v})$. Let $\langle \tau_i \rangle_{i \le n}$ be a sequence linearly generating the I-cells. Then

$$\begin{aligned}\mathbb{E}(w) &= \mathbb{E}(\sum_{i=0}^{n-1} u_{\tau_i} \times P_{\tau_i}(v_{\tau_{i+1}} - v_{\tau_i})) = \mathbb{E}(\sum_{i=0}^{n-1} P_{\tau_i}(u_{\tau_i} \times (v_{\tau_{i+1}} - v_{\tau_i}))) \\ &= \mathbb{E}(\sum_{i=0}^{n-1} u_{\tau_i} \times (v_{\tau_{i+1}} - v_{\tau_i})) = \mathbb{E}(S_I(\boldsymbol{u}, d\boldsymbol{v})) \le \gamma\end{aligned}$$

because $S_I(\boldsymbol{u}, d\boldsymbol{v}) \in Q_{\mathcal{S}}(d\boldsymbol{v})$. **Q**

(ii) If $I \in \mathcal{I}(\mathcal{S})$, then $S_I(\mathbf{1}, |Pd\boldsymbol{v}|) \in Q_{\mathcal{S}}(Pd\boldsymbol{v})$. **P** If I is empty this is trivial. Otherwise, let $\tau_0 \le \ldots \le \tau_n$ linearly generate the I-cells. For each $i < n$, set

$$a_i = [\![P_{\tau_i} v_{\tau_{i+1}} - v_{\tau_i} \ge 0]\!], \quad u_i = \chi a_i - \chi(1 \setminus a_i).$$

Then $a_i \in \mathfrak{A}_{\tau_i}$, $u_i \in L^\infty(\mathfrak{A}_{\tau_i})$ and $\|u_i\|_\infty \le 1$. So

$$S_I(\mathbf{1}, |Pd\boldsymbol{v}|) = \textstyle\sum_{i=0}^{n-1} |P_{\tau_i} v_{\tau_{i+1}} - v_{\tau_i}| = \sum_{i=0}^{n-1} u_i \times (P_{\tau_i} v_{\tau_{i+1}} - v_{\tau_i}) \in Q_{\mathcal{S}}(Pd\boldsymbol{v})$$

by 616C(ii). **Q**

(iii) For $\tau \in \mathcal{S}$ set

$$A_\tau = \{P_\tau S_I(\mathbf{1}, |Pd\boldsymbol{v}|) : I \in \mathcal{I}(\mathcal{S} \vee \tau)\}.$$

($\boldsymbol{\alpha}$) If $\tau \le \sigma_0 \le \sigma_1 \le \sigma_2$ in $\mathcal{S}$,

$$\begin{aligned}|P_{\sigma_0}(v_{\sigma_2} - v_{\sigma_0})| &= |P_{\sigma_0}(v_{\sigma_1} - v_{\sigma_0}) + P_{\sigma_0} P_{\sigma_1}(v_{\sigma_2} - v_{\sigma_1})| \\ &\le |P_{\sigma_0}(v_{\sigma_1} - v_{\sigma_0}| + P_{\sigma_0}|P_{\sigma_1}(v_{\sigma_2} - v_{\sigma_1})|, \\ P_\tau|P_{\sigma_0}(v_{\sigma_2} - v_{\sigma_0})| &\le P_\tau|P_{\sigma_0}(v_{\sigma_1} - v_{\sigma_0})| + P_\tau|P_{\sigma_1}(v_{\sigma_2} - v_{\sigma_1})|.\end{aligned}$$

So if $\tau \le \tau_0 \le \ldots \le \tau_n$ in $\mathcal{S}$,

$$P_\tau|P_{\tau_0}(v_{\tau_n} - v_{\tau_0})| \le P_\tau \textstyle\sum_{i=0}^{n-1} |P_{\tau_i}(v_{\tau_{i+1}} - v_{\tau_i})|.$$

P Induce on n. The case $n = 1$ is trivial. For the inductive step to $n + 1 > 1$, we have

$$\begin{aligned}P_\tau|P_{\tau_0}(v_{\tau_{n+1}} - v_{\tau_0})| &\le P_\tau|P_{\tau_0}(v_{\tau_n} - v_{\tau_0})| + P_\tau|P_{\tau_n}(v_{\tau_{n+1}} - v_{\tau_n})| \\ &\le P_\tau \sum_{i=0}^{n-1} |P_{\tau_i}(v_{\tau_{i+1}} - v_{\tau_i})| + P_\tau|P_{\tau_n}(v_{\tau_{n+1}} - v_{\tau_n})| \\ &= P_\tau \sum_{i=0}^{n} |P_{\tau_i}(v_{\tau_{i+1}} - v_{\tau_i})|. \ \mathbf{Q}\end{aligned}$$

($\boldsymbol{\beta}$) If $\tau \in \mathcal{S}$ and $I \in \mathcal{I}(\mathcal{S} \vee \tau)$ is non-empty, then $P_\tau|P_{\min I} v_{\max I} - v_{\min I}| \le P_\tau S_I(\mathbf{1}, |Pd\boldsymbol{v}|)$. **P** Apply ($\alpha$) with $\tau_0, \ldots, \tau_n$ a sequence linearly generating the I-cells. **Q**

($\boldsymbol{\gamma}$) If $\tau \in \mathcal{S}$, I, $J \in \mathcal{I}(\mathcal{S} \vee \tau)$ and $J \subseteq I$, then $P_\tau S_J(\mathbf{1}, |Pd\boldsymbol{v}|) \le P_\tau S_I(\mathbf{1}, |Pd\boldsymbol{v}|)$. **P** If J is empty this is trivial. Otherwise, let $\tau_1 \le \ldots \le \tau_n$ be a sequence linearly generating the J-cells; set $\tau_0 = \min I$ and $\tau_{n+1} = \max I$. For $j \le n$ set $I_j = I \cap [\tau_j, \tau_{j+1}]$. Then

$$
\begin{aligned}
P_\tau S_J(\mathbf{1},|Pd\boldsymbol{v}|) &= \sum_{j=1}^{n-1} P_\tau|P_{\tau_j}v_{\tau_{j+1}}-v_{\tau_j}| \le \sum_{j=0}^{n} P_\tau|P_{\tau_j}v_{\tau_{j+1}}-v_{\tau_j}| \\
&\le \sum_{j=0}^{n} P_\tau S_{I_j}(\mathbf{1},|Pd\boldsymbol{v}|) = P_\tau\sum_{j=0}^{n} S_{I_j}(\mathbf{1},|Pd\boldsymbol{v}|) = P_\tau S_I(\mathbf{1},|Pd\boldsymbol{v}|)
\end{aligned}
$$

by 613G(a-ii). **Q**

(δ) Thus A_τ is upwards-directed. If $w\in A_\tau$, then $w=P_\tau w'$ for some $w'\in Q_{\mathcal{S}}(Pd\boldsymbol{v})$, by (ii); in which case

$$\|w\|_1=\mathbb{E}(w)=\mathbb{E}(w')\le\gamma.$$

It follows that $w_\tau=\sup A_\tau$ is defined in $L^1_{\bar\mu}$, and belongs to the $\|\,\|_1$-closure of A_τ (613B(d-iii)). As $A_\tau\subseteq L^0(\mathfrak{A}_\tau)$, $w_\tau\in L^0(\mathfrak{A}_\tau)$. Also $\|w_\tau\|_1\le\sup_{w\in A_\tau}\|w\|_1\le\gamma$.

(ϵ) Suppose that τ, $\tau'\in\mathcal{S}$ and $[\![\tau=\tau']\!]=a$. If $w\in A_\tau$, then $w\times\chi a\le w_{\tau'}\times\chi a$. **P** Let $\tau_0\le\ldots\le\tau_n$ in $\mathcal{S}\vee\tau$ be such that $w=\sum_{i=0}^{n-1}P_\tau|P_{\tau_i}(v_{\tau_{i+1}}-v_{\tau_i})|$; set $\tau'_i=\tau_i\vee\tau'$ for $i\le n$, and $w'=\sum_{i=0}^{n-1}P_{\tau'}|P_{\tau'_i}(v_{\tau'_{i+1}}-v'_{\tau_i})|$, so that $w'\in A_{\tau'}$. Now, for $i\le n$,

$$[\![\tau'_i=\tau_i]\!]=[\![\tau_i\vee\tau'=\tau_i\vee\tau]\!]\supseteq a$$

by 611E(c-v-β), so $[\![v_{\tau'_i}=v_{\tau_i}]\!]\supseteq a$ and $[\![P_{\tau'_i}v=P_{\tau_i}v]\!]\supseteq a$ for every $v\in L^1_{\bar\mu}$ (622Bb); also, of course $[\![P_\tau v=P_{\tau'}v]\!]\supseteq a$ for every $v\in L^1_{\bar\mu}$. Moreover, $a\in\mathfrak{A}_{\tau\wedge\tau'}$ (611H(c-i)). Accordingly

$$
\begin{aligned}
w\times\chi a &= \sum_{i=0}^{n-1}\chi a\times P_\tau|P_{\tau_i}(v_{\tau_{i+1}}-v_{\tau_i})| = \sum_{i=0}^{n-1}\chi a\times P_{\tau'}|P_{\tau_i}(v_{\tau_{i+1}}-v_{\tau_i})| \\
&= \sum_{i=0}^{n-1}P_{\tau'}(\chi a\times|P_{\tau_i}(v_{\tau_{i+1}}-v_{\tau_i})|)
\end{aligned}
$$

(because $\chi a\in L^\infty(\mathfrak{A}_{\tau'})$)

$$
\begin{aligned}
&= \sum_{i=0}^{n-1}P_{\tau'}|\chi a\times P_{\tau_i}(v_{\tau_{i+1}}-v_{\tau_i})| = \sum_{i=0}^{n-1}P_{\tau'}|\chi a\times P_{\tau'_i}(v_{\tau_{i+1}}-v_{\tau_i})| \\
&= \sum_{i=0}^{n-1}P_{\tau'}|P_{\tau'_i}(\chi a\times(v_{\tau_{i+1}}-v_{\tau_i}))|
\end{aligned}
$$

(because $\chi a\in L^\infty(\mathfrak{A}_{\tau'_i})$ for every i)

$$= \sum_{i=0}^{n-1}P_{\tau'}|P_{\tau'_i}(\chi a\times(v_{\tau'_{i+1}}-v_{\tau'_i}))| = w'\times\chi a$$

(following a parallel path back)

$$\le w_{\tau'}\times\chi a. \ \mathbf{Q}$$

As w is arbitrary, $w_\tau\times\chi a\le w_{\tau'}\times\chi a$. Similarly, $w'_\tau\times\chi a\le w_\tau\times\chi a$ and $a\subseteq[\![w_\tau=w_{\tau'}]\!]$. As τ, τ' are arbitrary, $\boldsymbol{w}=\langle w_\tau\rangle_{\tau\in\mathcal{S}}$ is a fully adapted process. From (δ), we know that it is $\|\,\|_1$-bounded.

(iv)(α) If $\tau\le\tau'$ in $\mathcal{S}$ and $I\in\mathcal{I}(\mathcal{S}\vee\tau')$, then

$$P_\tau P_{\tau'}S_I(\mathbf{1},|Pd\boldsymbol{v}|)=P_\tau S_I(\mathbf{1},|Pd\boldsymbol{v}|)\le w_\tau;$$

that is, $P_\tau w\le w_\tau$ for every $w\in A_{\tau'}$. As P_τ is $\|\,\|_1$-continuous, and $w_{\tau'}$ belongs to the $\|\,\|_1$-closure of $A_{\tau'}$, $P_\tau w_{\tau'}$ also is less than or equal to w_τ. Thus $\boldsymbol{w}$ is a supermartingale and of course it is non-negative.

(β) If $\tau\le\tau'$ in $\mathcal{S}$ and $I\in\mathcal{I}(\mathcal{S}\vee\tau')$, then

$$P_\tau(v_{\tau'} + P_{\tau'}S_I(\mathbf{1}, |Pd\boldsymbol{v}|)) \le v_\tau + |P_\tau(v_{\tau'} - v_\tau)| + P_\tau S_I(\mathbf{1}, |Pd\boldsymbol{v}|))$$
$$= v_\tau + P_\tau S_{I\cup\{\tau\}}(\mathbf{1}, |Pd\boldsymbol{v}|)) \le v_\tau + w_\tau.$$

So $P_\tau(v_{\tau'} + w_{\tau'}) \le v_\tau + w_\tau$. This shows that $\boldsymbol{v} + \boldsymbol{w}$ is a supermartingale.

(b) If $\mathcal{S}$ has a greatest element, set

$$\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w} + \boldsymbol{P}|v_{\max\mathcal{S}}|, \quad \boldsymbol{u}' = \boldsymbol{w} + \boldsymbol{P}|v_{\max\mathcal{S}}|.$$

Then $\boldsymbol{u}$ and $\boldsymbol{u}'$ are sums of supermartingales, therefore themselves supermartingales. As their final values $v_{\max\mathcal{S}}+w_{\max\mathcal{S}}+|v_{\max\mathcal{S}}|$, $w_{\max\mathcal{S}}+|v_{\max\mathcal{S}}|$ are both greater than or equal to 0, $\boldsymbol{u}$ and $\boldsymbol{u}'$ are both non-negative, and their difference $\boldsymbol{v}$ is expressed in the required form.

(c) Since $\boldsymbol{w}$ is a $\| \|_1$-bounded supermartingale, $-\boldsymbol{w}$ is a $\| \|_1$-bounded submartingale, so is a semi-martingale, by 626P. Thus $\boldsymbol{w}$ is a semi-martingale. Also we are now supposing that $\boldsymbol{v}$ is $\| \|_1$-bounded, so $\boldsymbol{v} + \boldsymbol{w}$ is a $\| \|_1$-bounded supermartingale, and it too is a semi-martingale. Accordingly $\boldsymbol{v} = (\boldsymbol{v} + \boldsymbol{w}) - \boldsymbol{w}$ is a difference of semi-martingales and is a semi-martingale.

627F For the next step, we need a couple of facts from the theory of linear topological spaces.

Lemma Let U be a Banach space, C a convex subset of U and K a non-empty weak*-compact convex subset of the dual U^* of U. Suppose that $\gamma \ge 0$ is such that for every $u \in C$ there is an $f \in K$ such that $f(u) \le \gamma$. Then there is a $g \in K$ such that $g(u) \le \gamma$ for every $u \in C$.

proof (a) For each finite $I \subseteq C$ set $K_I = \{f : f \in K,\ f(u) \le \gamma$ for every $u \in I\}$. Then $K_I \ne \emptyset$. **P?** Otherwise, I is certainly non-empty. Set $Tf = \langle f(u)\rangle_{u\in I}$ for $f \in U^*$. Then $T : U^* \to \mathbb{R}^I$ is a linear operator which is continuous for the weak* topology on U^*, so $T[K]$ is a convex compact subset of $\mathbb{R}^I$; and as K_I is empty, $T[K]$ does not meet the closed convex set $F = \{v : v \in \mathbb{R}^I,\ v(u) \le \gamma$ for every $u \in I\}$.

The set $D = T[K] - F$ is convex (2A5Ea) and closed (4A5Ef) and does not contain 0. We therefore have a linear functional $h : \mathbb{R}^I \to \mathbb{R}$ such that $\inf_{v\in D} h(v) > 0$ (3A5Cb), that is, $\sup_{v\in F} h(v) < \inf_{v'\in T[K]} h(v')$. For $u \in I$ write e_u for the corresponding unit vector in $\mathbb{R}^I$, and set $\alpha_u = h(e_u)$, so that $h(v) = \sum_{u\in I} \alpha_u v(u)$ for every $v \in \mathbb{R}^I$. Because F contains βe_u for every $\beta \le 0$, and $\sup_{v\in F} h(v)$ is finite, α_u is at least 0, for every $u \in I$. Also h cannot be zero, so not every α_u is zero. Set $\alpha = \sum_{u\in I} \alpha_u$ and consider $\tilde{u} = \frac{1}{\alpha}\sum_{u\in I} \alpha_u u$, so that $\tilde{u} \in C$ and there is a $\tilde{f} \in K$ such that $\tilde{f}(\tilde{u}) \le \gamma$. In this case, $T\tilde{f} \in T[K]$ and

$$h(T\tilde{f}) = \textstyle\sum_{u\in I} \alpha_u \tilde{f}(u) = \tilde{f}(\sum_{u\in I} \alpha_u u) = \alpha\tilde{f}(\tilde{u}) \le \alpha\gamma = h(\sum_{u\in I} \gamma e_u).$$

But $\sum_{u\in I} \gamma e_u \in F$, so this is impossible, by the choice of h. **XQ**

(b) Now $\{K_I : I \in [C]^{<\omega}\}$ is a downwards-directed family of non-empty closed subsets of the compact set K, so has non-empty intersection, and any member of the intersection will serve for g.

627G Lemma Suppose that $C \subseteq L^1_{\bar{\mu}}$ is a non-empty topologically bounded convex set. Then there is a $w \in L^\infty = L^\infty(\mathfrak{A})$ such that $[\![w > 0]\!] = 1$ and $\sup_{u\in C} \mathbb{E}(u \times w)$ is finite.

proof (a) For any $\epsilon \in \,]0, 1]$ there is a $w \in L^\infty$ such that $0 \le w \le \chi 1$, $\bar{\mu}[\![w = 0]\!] \le \epsilon$ and $\sup_{u\in C} \mathbb{E}(u \times w)$ is finite. **P** Let K be the set $\{w : 0 \le w \le \chi 1,\ \mathbb{E}(w) \ge 1 - \epsilon\}$. Then K is convex and is closed for the topology $\mathfrak{T}_s(L^\infty, L^1_{\bar{\mu}})$, so can be regarded as a convex subset of $(L^1_{\bar{\mu}})^*$ which is compact for the weak* topology (365Lc[7]). Let $\gamma \ge 1$ be such that $\theta(\frac{1}{\gamma}u) \le \epsilon$ for every $u \in C$; then $\bar{\mu}[\![u \ge \gamma]\!] \le \epsilon$ for every $u \in C$. So for every $u \in C$ there is a $w \in K$ such that $\mathbb{E}(u \times w) \le \gamma$ (we can take $w = \chi[\![u \le \gamma]\!]$). By 627F, there is a $w \in K$ such that $\mathbb{E}(u \times w) \le \gamma$ for every $u \in C$. As $\mathbb{E}(w) \ge 1 - \epsilon$, $\bar{\mu}[\![w = 0]\!] \le \epsilon$. **Q**

(b) We can therefore find sequences $\langle w_n\rangle_{n\in\mathbb{N}}$ in L^∞ and $\langle \gamma_n\rangle_{n\in\mathbb{N}}$ in $[0, \infty[$ such that $0 \le w_n \le \chi 1$, $\bar{\mu}[\![w_n = 0]\!] \le 2^{-n}$ and $\mathbb{E}(u \times w_n) \le \gamma_n$ for every $n \in \mathbb{N}$ and $u \in C$. Now $w = \sum_{n=0}^\infty \frac{2^{-n}}{\gamma_n+1} w_n$ is defined in L^∞, $[\![w > 0]\!] = 1$ and $\mathbb{E}(u \times w) \le 2$ for every $u \in C$.

[7]Formerly 365Mc.

627H Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v}$ a fully adapted process with domain $\mathcal{S}$.

(a) If $I \in \mathcal{I}(\mathcal{S})$, $\boldsymbol{u} \in M_{\mathrm{fa}}(I)$ and $\|\boldsymbol{u}\|_\infty \le 1$, then there is a $\boldsymbol{w} \in M_{\mathrm{fa}}(\mathcal{S})$ such that $\|\boldsymbol{w}\|_\infty \le 1$ and $S_J(\boldsymbol{w}, d\boldsymbol{v}) = S_I(\boldsymbol{u}, d\boldsymbol{v})$ whenever $I \subseteq J \in \mathcal{I}(\mathcal{S})$, so that $\int_{\mathcal{S}} \boldsymbol{w}\, d\boldsymbol{v}$ is defined and equal to $S_I(\boldsymbol{u}, d\boldsymbol{v})$.

(b) $Q_{\mathcal{S}}(d\boldsymbol{v})$ is convex.

proof (a) (The key.) Express $\boldsymbol{u}$ as $\langle u_\sigma\rangle_{\sigma\in I}$. If I is empty then of course we can take $\boldsymbol{w}$ to be the zero process with domain $\mathcal{S}$. Otherwise, let $\tau_0 \le \ldots \le \tau_n$ linearly generate the I-cells. By 612Ka, applied to $(u_{\tau_0}, \ldots, u_{\tau_{n-1}}, 0)$ and $u_* = 0$, there is a simple fully adapted process $\boldsymbol{w} = \langle w_\sigma\rangle_{\sigma\in\mathcal{S}}$ such that

$$[\![w_\sigma = u_{\tau_i}]\!] \supseteq [\![\tau_i \le \sigma]\!] \cap [\![\sigma < \tau_{i+1}]\!],$$

for $i < n$, while

$$[\![w_\sigma = 0]\!] \supseteq [\![\sigma < \tau_0]\!] \cup [\![\tau_n \le \sigma]\!],$$

for every $\sigma \in \mathcal{S}$. Evidently $\|\boldsymbol{w}\|_\infty \le \sup_{i\le n} \|u_{\tau_i}\|_\infty \le 1$. Now suppose that $I \subseteq J \in \mathcal{I}(\mathcal{S})$. Then, expressing $\boldsymbol{v}$ as $\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$,

$$\begin{aligned} S_J(\boldsymbol{w}, d\boldsymbol{v}) &= \sum_{e\in\mathrm{Sti}_0(J)} \Delta_e(\boldsymbol{w}, d\boldsymbol{v}) \\ &= \sum_{e\in\mathrm{Sti}_0(J\wedge\tau_0)} \Delta_e(\boldsymbol{w}, d\boldsymbol{v}) + \sum_{e\in\mathrm{Sti}_0(J\vee\tau_n)} \Delta_e(\boldsymbol{w}, d\boldsymbol{v}) \\ &\qquad + \sum_{i=0}^{n-1} \sum_{e\in\mathrm{Sti}_0(J\cap[\tau_i,\tau_{i+1}])} \Delta_e(\boldsymbol{w}, d\boldsymbol{v}) \end{aligned}$$

(611J(e-iii))

$$= 0 + 0 + \sum_{i=0}^{n-1} \sum_{e\in\mathrm{Sti}_0(J\cap[\tau_i,\tau_{i+1}])} u_{\tau_i} \times \Delta_e(\mathbf{1}, d\boldsymbol{v})$$

(because for $\sigma \le \tau$ in J, if $\tau \le \tau_0$ then $[\![v_\sigma \ne v_\tau]\!] \subseteq [\![\sigma < \tau_0]\!] \subseteq [\![w_\sigma = 0]\!]$; if $\tau_n \le \sigma$ then $w_\sigma = 0$; and if $\tau_i \le \sigma \le \tau \le \tau_{i+1}$, then $[\![v_\sigma \ne v_\tau]\!] \subseteq [\![\tau_i \le \sigma]\!] \cap [\![\sigma < \tau_{i+1}]\!] \subseteq [\![w_\sigma = u_{\tau_i}]\!]$)

$$= \sum_{i=0}^{n-1} u_{\tau_i} \times (v_{\tau_{i+1}} - v_{\tau_i})$$

(613L(b-i))

$$= S_I(\boldsymbol{u}, d\boldsymbol{v})$$

(613Ec).

Taking the limit as J increases through $\mathcal{I}(\mathcal{S})$, $\int_{\mathcal{S}} \boldsymbol{w}\, d\boldsymbol{v} = S_I(\boldsymbol{u}, d\boldsymbol{v})$.

(b) Suppose that z, $z' \in Q_{\mathcal{S}}(d\boldsymbol{v})$ and $\alpha \in [0,1]$. Then there are I, $I' \in \mathcal{I}(\mathcal{S})$ and $\boldsymbol{u} \in M_{\mathrm{fa}}(I)$, $\boldsymbol{u}' \in M_{\mathrm{fa}}(I')$ such that $\|\boldsymbol{u}\|_\infty \le 1$, $\|\boldsymbol{u}'\|_\infty \le 1$, $z = S_I(\boldsymbol{u}, d\boldsymbol{v})$ and $z' = S_{I'}(\boldsymbol{u}', d\boldsymbol{v})$. Let $\boldsymbol{w}$, $\boldsymbol{w}' \in M_{\mathrm{fa}}(\mathcal{S})$ be as in (a), starting from $\boldsymbol{u}$, $\boldsymbol{u}'$ respectively. Set $\tilde{\boldsymbol{w}} = \alpha\boldsymbol{w} + (1-\alpha)\boldsymbol{w}'$; then $\|\tilde{\boldsymbol{w}}\|_\infty \le 1$. If $J = I \sqcup I'$ is the sublattice generated by $I \cup I'$,

$$\begin{aligned} \alpha z + (1-\alpha)z' &= \alpha S_I(\boldsymbol{u}, d\boldsymbol{v}) + (1-\alpha) S_{I'}(\boldsymbol{u}, d\boldsymbol{v}) \\ &= \alpha S_J(\boldsymbol{w}, d\boldsymbol{v}) + (1-\alpha) S_J(\boldsymbol{w}, d\boldsymbol{v}) = S_J(\tilde{\boldsymbol{w}}, d\boldsymbol{v}) \in Q_{\mathcal{S}}(d\boldsymbol{v}). \end{aligned}$$

As z, z' and α are arbitrary, $Q_{\mathcal{S}}(d\boldsymbol{v})$ is convex.

627I Theorem (Bichteler 79, Dellacherie & Meyer 82, §VIII.4) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v}$ an integrator with domain $\mathcal{S}$. Then there is a $\bar\nu$ such that $(\mathfrak{A}, \bar\nu)$ is a probability algebra and $\boldsymbol{v}$ is a uniformly integrable quasimartingale with respect to $\bar\nu$.

proof (a) Express $\boldsymbol{v}$ as $\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$. $\boldsymbol{v}$ is order-bounded, by 616Ib; set $\bar w = \sup|\boldsymbol{v}|$.

(b) Suppose to begin with that $\bar{w} \in L^1_{\bar{\mu}}$. Set $C = Q_{\mathcal{S}}(d\boldsymbol{v})$, so that C is topologically bounded in L^0. By 627H, C is convex. Of course $0 \in C$, and $C \subseteq L^1_{\bar{\mu}}$ because $u \times (v_\tau - v_\sigma) \in L^1_{\bar{\mu}}$ whenever $\sigma \le \tau$ in $\mathcal{S}$ and $u \in L^\infty(\mathfrak{A}_\sigma)$. Let C' be the linear sum $C + [-\bar{w}, \bar{w}]$; then C' is convex, non-empty, topologically bounded and included in $L^1_{\bar{\mu}}$. By 627G, there is a $w \in L^\infty(\mathfrak{A})$ such that $[\![w > 0]\!] = 1$ and $\sup_{z \in C'} \mathbb{E}_{\bar{\mu}}(z \times w)$ is finite. Adjusting w by a scalar factor if necessary, we can arrange that $\mathbb{E}_{\bar{\mu}}(w) = 1$. In this case, we have a strictly positive probability $\bar{\nu}$ on $\mathfrak{A}$ defined by saying that $\bar{\nu}a = \mathbb{E}_{\bar{\mu}}(w \times \chi a)$ for every $a \in \mathfrak{A}$, and now $\mathbb{E}_{\bar{\nu}}(z) = \mathbb{E}_{\bar{\mu}}(w \times z)$ for every $z \in L^0$ for which either expectation is defined (625B(a-iii)).

It follows that $\sup_{z \in C} \mathbb{E}_{\bar{\nu}}(z)$ is finite. Since $-z \in Q_{\mathcal{S}}(d\boldsymbol{v})$ whenever $z \in Q_{\mathcal{S}}(d\boldsymbol{v})$, $-z \in C$ whenever $z \in C$. So $\sup_{z \in C} |\mathbb{E}_{\bar{\nu}}(z)|$ is finite, and $\boldsymbol{v}$ is a quasimartingale with respect to $\bar{\nu}$. At the same time, $\mathbb{E}_{\bar{\nu}}(\bar{w}) = \mathbb{E}_{\bar{\mu}}(\bar{w} \times w)$ is finite, so $\boldsymbol{v}$ is actually order-bounded in $L^1_{\bar{\nu}}$ and is surely uniformly integrable.

(c) In general, let w' be a scalar multiple of $\frac{1}{\bar{w}+\chi 1}$ such that $\mathbb{E}_{\bar{\mu}}(w') = 1$, and set $\bar{\lambda}a = \mathbb{E}_{\bar{\mu}}(w' \times \chi a)$ for $a \in \mathfrak{A}$. Then $(\mathfrak{A}, \bar{\lambda})$ is a probability algebra, and $\bar{w} \in L^1_{\bar{\lambda}}$. By (b), we now have a $\bar{\nu}$ of the kind required.

627J Corollary Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v}$ an integrator with domain $\mathcal{S}$. Then $\boldsymbol{v}$ is a semi-martingale.

proof By 627I, there is a probability measure $\bar{\nu}$ such that $\boldsymbol{v}$ is a $\bar{\nu}$-uniformly integrable $\bar{\nu}$-quasimartingale. By 627Ec, $\boldsymbol{v}$ is a $\bar{\nu}$-semi-martingale. By 625F, $\boldsymbol{v}$ is a $\bar{\mu}$-semi-martingale.

627K Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v}$ an integrator with domain $\mathcal{S}$. Set $\mathcal{S}' = \mathcal{S} \cup \{\min \mathcal{T}, \max \mathcal{T}\}$. Then there is an integrator $\boldsymbol{v}'$ with domain $\mathcal{S}'$ extending $\boldsymbol{v}$.

proof (a) Express $\boldsymbol{v}$ as $\langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$, and set $\bar{v} = \sup |\boldsymbol{v}|$. Let $\boldsymbol{w} = \langle w_\tau \rangle_{\tau \in \mathcal{T}}$ be an extension of $\boldsymbol{v}$ to a fully adapted process defined on the whole of $\mathcal{T}$ as described in 612P. Then

$$\begin{aligned}[\![|w_\tau| \le \bar{v}]\!] &\supseteq [\![w_\tau = 0]\!] \cup \sup_{\sigma \in \mathcal{S}} ([\![w_\tau = w_\sigma]\!] \cap [\![w_\sigma = v_\sigma]\!] \cap [\![|v_\sigma| \le \bar{v}]\!]) \\ &\supseteq [\![w_\tau = 0]\!] \cup \sup_{\sigma \in \mathcal{S}} [\![\sigma = \tau]\!] = 1,\end{aligned}$$

so $|w_\tau| \le \bar{v}$, for every $\tau \in \mathcal{T}$.

Of course $\mathcal{S}'$ is a sublattice of $\mathcal{T}$. Set $\boldsymbol{v}' = \boldsymbol{w} \restriction \mathcal{S}'$; then $\boldsymbol{v}'$ is fully adapted and extends $\boldsymbol{v}$.

(b) Now $Q_{\mathcal{S}'}(d\boldsymbol{v}') \subseteq Q_{\mathcal{S}}(d\boldsymbol{v}) + [-4\bar{v}, 4\bar{v}]$. **P** Take $z \in Q_{\mathcal{S}'}(d\boldsymbol{v}')$. Express z as $S_I(\boldsymbol{u}, d\boldsymbol{v}')$ where $I \in \mathcal{I}(\mathcal{S}')$, $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in I}$ is fully adapted and $\|\boldsymbol{u}\|_\infty \le 1$. If $I = \emptyset$ then $z = 0$ certainly belongs to $Q_{\mathcal{S}}(d\boldsymbol{v}) + [-4\bar{v}, 4\bar{v}]$. Otherwise, let $\langle \tau_i \rangle_{i \le n}$ be the increasing enumeration of a maximal totally ordered subset of I. If $n = 0$ then again $z = 0$ belongs to $Q_{\mathcal{S}}(d\boldsymbol{v}) + [-4\bar{v}, 4\bar{v}]$. If $n = 1$ then

$$|z| = |u_{\tau_0} \times (w_{\tau_1} - w_{\tau_0})| \le 2\bar{v}$$

and $z \in Q_{\mathcal{S}}(d\boldsymbol{v}) + [-4\bar{v}, 4\bar{v}]$. If $n = 2$ then

$$|z| = |u_{\tau_0} \times (w_{\tau_1} - w_{\tau_0}) + u_{\tau_1} \times (w_{\tau_2} - w_{\tau_1})| \le 4\bar{v}$$

and $z \in Q_{\mathcal{S}}(d\boldsymbol{v}) + [-4\bar{v}, 4\bar{v}]$. If $n \ge 3$ then, because $\tau_0 \le \tau_1 \le \ldots \le \tau_{n-1} \le \tau_n$ are all different, $\tau_1, \ldots, \tau_{n-1}$ must all belong to $\mathcal{S}$, so $\sum_{i=1}^{n-2} u_{\tau_i} \times (w_{\tau_{i+1}} - w_{\tau_i}) = \sum_{i=1}^{n-2} u_{\tau_i} \times (v_{\tau_{i+1}} - v_{\tau_i})$ belongs to $Q_{\mathcal{S}}(d\boldsymbol{v})$, while

$$u_{\tau_0} \times (w_{\tau_1} - w_{\tau_0}) + u_{\tau_{n-1}} \times (w_{\tau_n} - w_{\tau_{n-1}}) \in [-4\bar{v}, 4\bar{v}],$$

so

$$\begin{aligned}z &= u_{\tau_0} \times (w_{\tau_1} - w_{\tau_0}) + \sum_{i=1}^{n-2} u_{\tau_i} \times (w_{\tau_{i+1}} - w_{\tau_i}) + u_{n-1} \times (w_{\tau_n} - w_{\tau_{n-1}}) \\ &\in Q_{\mathcal{S}}(d\boldsymbol{v}) + [-4\bar{v}, 4\bar{v}]. \ \mathbf{Q}\end{aligned}$$

Since $Q_{\mathcal{S}}(d\boldsymbol{v})$ and $[-4\bar{v}, 4\bar{v}]$ are topologically bounded and the sum of topologically bounded sets is topologically bounded, $Q_{\mathcal{S}'}(d\boldsymbol{v}')$ is topologically bounded and $\boldsymbol{v}'$ is an integrator.

627L Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v}$ a fully adapted process with domain $\mathcal{S}$.

(a) If $\mathcal{S}$ has a greatest element and $\boldsymbol{v}$ is a non-negative submartingale, $\boldsymbol{v}$ is a strong integrator.

(b) If $\mathcal{S}$ has greatest and least elements and $\boldsymbol{v}$ is a non-negative supermartingale, $\boldsymbol{v}$ is a strong integrator.

(c) If $\mathcal{S}$ has greatest and least elements and $\boldsymbol{v}$ is a quasimartingale, $\boldsymbol{v}$ is a strong integrator.

(d) The following are equiveridical:

(i) $\boldsymbol{v}$ is an integrator;

(ii) there is a functional $\bar\nu$ such that $(\mathfrak{A},\bar\nu)$ is a probability algebra and $\boldsymbol{v}$ is a strong integrator with respect to $\bar\nu$.

proof (a) By the Doob-Meyer theorem (626M), $\boldsymbol{v}$ has a previsible variation $\boldsymbol{v}^\#$ which is non-negative and non-decreasing, therefore of bounded variation (because $\mathcal{S}$ has a greatest element). Now $\boldsymbol{w}=\boldsymbol{v}-\boldsymbol{v}^\#$ is a martingale (626Ka); again because $\mathcal{S}$ has a greatest element, $\boldsymbol{w}$ is uniformly integrable. So $\boldsymbol{v}$ is a strong integrator.

(b) Express $\boldsymbol{v}$ as $\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$. For any $\sigma\in\mathcal{S}$,

$$\|v_\sigma\|_1=\mathbb{E}(v_\sigma)=\mathbb{E}(P_{\min\mathcal{S}}v_\sigma)\le\mathbb{E}(v_{\min\mathcal{S}}),$$

so $\boldsymbol{v}$ is $\|\;\|_1$-bounded, therefore order-bounded (627Db). Write $\bar v$ for $\sup|\boldsymbol{v}|$.

Take any $\epsilon>0$. Let $M\ge0$ be such that $\bar\mu[\![\bar v\ge M]\!]\le\epsilon$. Then $M\mathbf{1}\wedge\boldsymbol{v}$ is a supermartingale (627C(a-ii)) and $\boldsymbol{u}=M\mathbf{1}-M\mathbf{1}\wedge\boldsymbol{v}$ is a non-negative submartingale, therefore a strong integrator, by (a). It follows that $\boldsymbol{v}'=M\mathbf{1}-\boldsymbol{u}$ is a strong integrator. But

$$[\![\boldsymbol{v}'\ne\boldsymbol{v}]\!]\subseteq[\![\bar v\ge M]\!]$$

has measure at most ϵ. As ϵ is arbitrary, $\boldsymbol{v}$ is a strong integrator, by 627C(c-iv).

(c) follows at once from 627Eb, (b) here and 627C(c-ii).

(d)(ii)⇒(i) is immediate from 627C(c-i) and the definition 616Fc, which shows that the property of being an integrator depends only on the linear space topology of L^0, not on the measure.

(ii)⇒(i) Suppose that $\boldsymbol{v}$ is an integrator. By 627K, there is an integrator $\boldsymbol{v}'$ with domain $\mathcal{S}\cup\{\min\mathcal{T},\max\mathcal{T}\}$ extending $\boldsymbol{v}$. 627I tells us that there is a probability measure $\bar\nu$ such that $\boldsymbol{v}'$ is a $\bar\nu$-quasimartingale, therefore a $\bar\nu$-strong integrator, by (c) just above. Consequently $\boldsymbol{v}$ is a $\bar\nu$-strong integrator, by 627C(c-iii).

627M Corollary Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v}$ an integrator with domain $\mathcal{S}$. Then the solid convex hull of $Q_{\mathcal{S}}(d\boldsymbol{v})$ is topologically bounded.

proof If $\mathcal{S}$ is empty, this is trivial, so I suppose otherwise.

(a) I should begin by noting that the set of those $z\in L^0$ for which there are $\alpha_0,\dots,\alpha_m\ge0$ and $z_0,\dots,z_m\in Q_{\mathcal{S}}(d\boldsymbol{v})$ such that $\sum_{j=0}^m\alpha_j=1$ and $|z|\le\sum_{j=0}^m\alpha_j|z_j|$ is a solid convex set including $Q_{\mathcal{S}}(d\boldsymbol{v})$; moreover, as noted in 613B(f-iv), the solid hull of a topologically bounded set in L^0 is topologically bounded. It will therefore be enough to show that

$$C(d\boldsymbol{v})=\{\sum_{j=0}^m\alpha_j|z_j|:\alpha_j\ge0,\ z_j\in Q_{\mathcal{S}}(d\boldsymbol{v})\text{ for every }j\le m,\ \sum_{j=0}^m\alpha_j=1\}$$

is topologically bounded. Moreover, for this it will be enough to show that $\inf_{\gamma>0}\sup_{z\in C(d\boldsymbol{v})}\bar\mu[\![z>\gamma]\!]=0$ (613B(f-ii)).

Next, if $z\in C(d\boldsymbol{v})$, there are $\alpha_0,\dots,\alpha_m\ge0$, $\sigma_0\le\dots\le\sigma_n$ in $\mathcal{S}$ and a family $\langle u_{ji}\rangle_{j\le m,i<n}$ in L^0 such that $u_{ji}\in L^0(\mathfrak{A}_{\sigma_i})$ and $\|u_{ji}\|_\infty\le1$ for all j and i, $\sum_{j=1}^m\alpha_j=1$ and $z=\sum_{j=0}^m\alpha_j|\sum_{i=0}^{n-1}u_{ji}\times(v_{\sigma_{i+1}}-v_{\sigma_i})|$. **P** Let $\alpha_0,\dots,\alpha_m,z_0,\dots,z_m$ be such that $\alpha_j\ge0$ and $z_j\in Q_{\mathcal{S}}(d\boldsymbol{v})$ for every j, while $\sum_{j=0}^m\alpha_j=1$ and $z=\sum_{j=0}^m\alpha_j|z_j|$. For each $j\le m$, let $I_j\in\mathcal{I}(\mathcal{S})$ be such that $z_j\in Q_{I_j}(d\boldsymbol{v})$ (616Da). Let I be a non-empty finite sublattice of $\mathcal{S}$ including $\bigcup_{j\le m}I_j$. Then $z_j\in Q_I(d\boldsymbol{v})$ for every j (616Dd). Let $\langle\sigma_i\rangle_{i\le n}$ linearly generate the I-cells. Then for each j we can find $u_{j0},\dots,u_{j,n-1}$ such that $u_{ji}\in L^0(\mathfrak{A}_{\sigma_i})$ and $\|u_{ji}\|_\infty\le1$ for each $i<n$ and $z_j=\sum_{i=0}^{n-1}u_{ji}\times(v_{\sigma_{i+1}}-v_{\sigma_i})$ (616C(ii) again). Now $z=\sum_{j=0}^m\alpha_j|\sum_{i=0}^{n-1}u_{ji}\times(v_{\sigma_{i+1}}-v_{\sigma_i})|$, as required. **Q**

(b) Suppose that $\boldsymbol{v}$ is a uniformly integrable martingale. Then $C(d\boldsymbol{v})$ is topologically bounded. **P** By 621Ba, $\beta = \sup_{\sigma\in\mathcal{S}} \|v_\sigma\|_1$ is finite. Take $z \in C(d\boldsymbol{v})$ and $\gamma > 0$. Take $\alpha_0, \dots, \alpha_m \ge 0$, $\sigma_0 \le \dots \le \sigma_n$ in $\mathcal{S}$ and a family $\langle u_{ji}\rangle_{j\le m, i<n}$ in L^0 such that $u_{ji} \in L^0(\mathfrak{A}_{\sigma_i})$ and $\|u_{ji}\|_\infty \le 1$ for all j and i, $\sum_{j=1}^m \alpha_j = 1$ and $z = \sum_{j=0}^m \alpha_j |\sum_{i=0}^{n-1} u_{ji} \times (v_{\sigma_{i+1}} - v_{\sigma_i})|$. By 621Hf, in its full strength,

$$\bar\mu[\![|z| > \gamma]\!] = \bar\mu[\![z > \gamma]\!] \le \frac{16}{\gamma}\|v_{\sigma_n}\|_1 \le \frac{16\beta}{\gamma}.$$

As z is arbitrary,

$$\inf_{\gamma>0}\sup_{z\in C(d\boldsymbol{v})} \bar\mu[\![|z| > \gamma]\!] \le \inf_{\gamma>0}\frac{16\beta}{\gamma} = 0$$

and $C(d\boldsymbol{v})$ is topologically bounded. **Q**

(c) Suppose that $\boldsymbol{v}$ is order-bounded and non-decreasing. Then $C(d\boldsymbol{v})$ is topologically bounded. **P** Set $v_\uparrow = \sup_{\sigma\in\mathcal{S}} v_\sigma$ and $v_\downarrow = \inf_{\sigma\in\mathcal{S}} v_\sigma$. If $z \in C(d\boldsymbol{v})$, express it as $\sum_{j=0}^m \alpha_j |\sum_{i=0}^{n-1} u_{ji} \times (v_{\sigma_{i+1}} - v_{\sigma_i})|$ where $\alpha_0, \dots, \alpha_m \ge 0$, $\sigma_0 \le \dots \le \sigma_n$ in $\mathcal{S}$ and $\langle u_{ji}\rangle_{j\le m,i<n}$ are such that $\sum_{j=0}^m \alpha_j = 1$ and $u_{ji} \in L^0(\mathfrak{A}_{\sigma_i})$ and $\|u_{ji}\|_\infty \le 1$ for all j and i. Then

$$\begin{aligned} z &\le \sum_{j=0}^m \alpha_j \sum_{i=0}^{n-1} |u_{ji}| \times (v_{\sigma_{i+1}} - v_{\sigma_i}) \\ &\le \sum_{j=0}^m \alpha_j \sum_{i=0}^{n-1} v_{\sigma_{i+1}} - v_{\sigma_i} = v_{\sigma_n} - v_{\sigma_0} \le v_\uparrow - v_\downarrow. \end{aligned}$$

Thus $C(d\boldsymbol{v}) \subseteq [0, v_\uparrow - v_\downarrow]$ is order-bounded, therefore topologically bounded. **Q**

(d) Suppose that $\boldsymbol{v}$ is a strong integrator. Then $C(d\boldsymbol{v})$ is topologically bounded. **P** Let $\epsilon > 0$. Then there are a uniformly integrable martingale $\boldsymbol{w}$ and non-decreasing, order-bounded processes $\boldsymbol{w}'$, $\boldsymbol{w}''$, all with domain $\mathcal{S}$, such that $\bar\mu[\![\boldsymbol{v} \ne \boldsymbol{w} + \boldsymbol{w}' - \boldsymbol{w}'']\!] \le \epsilon$. By (b)-(c), $C(d\boldsymbol{w})$, $C(d\boldsymbol{w}')$ and $C(d\boldsymbol{w}'')$ are all topologically bounded, so their algebraic sum is topologically bounded and there is a $\gamma > 0$ such that $\bar\mu[\![z + z' + z'' > \gamma]\!] \le \epsilon$ whenever $z \in C(d\boldsymbol{w})$, $z' \in C(d\boldsymbol{w}')$ and $z'' \in C(d\boldsymbol{w}'')$. Now suppose that $z^* \in C(d\boldsymbol{v})$. Express z^* as $\sum_{j=0}^m \alpha_j |\sum_{i=0}^{n-1} u_{ji} \times (v_{\sigma_{i+1}} - v_{\sigma_i})|$ where $\alpha_0, \dots, \alpha_m \ge 0$, $\sigma_0 \le \dots \le \sigma_n$ in $\mathcal{S}$ and $\langle u_{ji}\rangle_{j\le m,i<n}$ are such that $\sum_{j=0}^m \alpha_j = 1$ and $u_{ji} \in L^0(\mathfrak{A}_{\sigma_i})$ and $\|u_{ji}\|_\infty \le 1$ for all j and i. Set $\tilde{\boldsymbol{v}} = \boldsymbol{w} + \boldsymbol{w}' - \boldsymbol{w}''$,

$$z = \textstyle\sum_{j=0}^m \alpha_j |\sum_{i=0}^{n-1} u_{ji} \times (w_{\sigma_{i+1}} - w_{\sigma_i})| \in C(d\boldsymbol{w}),$$

$$z' = \textstyle\sum_{j=0}^m \alpha_j |\sum_{i=0}^{n-1} u_{ji} \times (w'_{\sigma_{i+1}} - w'_{\sigma_i})| \in C(d\boldsymbol{w}'),$$

$$z'' = \textstyle\sum_{j=0}^m \alpha_j |\sum_{i=0}^{n-1} u_{ji} \times (w''_{\sigma_{i+1}} - w''_{\sigma_i})| \in C(d\boldsymbol{w}'')$$

and

$$\tilde z = \textstyle\sum_{j=0}^m \alpha_j |\sum_{i=0}^{n-1} u_{ji} \times (\tilde v_{\sigma_{i+1}} - \tilde v_{\sigma_i})|$$

where $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$, etc. Then

$$\tilde z \le z + z' + z'', \quad [\![z^* \ne \tilde z]\!] \subseteq [\![\boldsymbol{v} \ne \tilde{\boldsymbol{v}}]\!],$$

so

$$\begin{aligned} \bar\mu[\![|z^*| > \gamma]\!] = \bar\mu[\![z^* > \gamma]\!] &\le \bar\mu[\![\tilde z > \gamma]\!] + \bar\mu[\![\boldsymbol{v} \ne \tilde{\boldsymbol{v}}]\!] \\ &\le \bar\mu[\![z + z' + z'' > \gamma]\!] + \epsilon \le 2\epsilon. \end{aligned}$$

As ϵ is arbitrary, $C(d\boldsymbol{v})$ is topologically bounded. By (a), the solid convex hull of $Q_{\mathcal{S}}(d\boldsymbol{v})$ is topologically bounded. **Q**

(e) For the general case, we know from 627Ld that $\boldsymbol{v}$ is a strong integrator with respect to an alternative law, which gives the same topologically bounded sets in L^0, so the solid convex hull of $Q_{\mathcal{S}}(d\boldsymbol{v})$ is topologically bounded in this case also.

***627N** I star the next couple of lemmas because I am sure that most readers will be interested primarily in the case $T = [0, \infty[$, for which they are essentially irrelevant, in view of 627R.

Lemma Let $\mathcal{S}$ be a non-empty finitely full sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$ a fully adapted process such that $\lim_{\sigma\uparrow A} u_\sigma$ is defined in $L^0(\mathfrak{A})$ for every non-empty upwards-directed set $A \subseteq \mathcal{S}$ with an upper bound in $\mathcal{S}$. Then there are a non-decreasing sequence $\langle \tau_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{S}$ and a non-decreasing sequence $\langle d_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that

$$d_n \in \mathfrak{A}_{\tau_n}, \quad d_n \subseteq [\![\tau_{n+1} = \tau_n]\!]$$

for every $n \in \mathbb{N}$, and

$$\sup_{n\in\mathbb{N}}(d_n \cup [\![\tau \le \tau_n]\!]) = 1, \quad u_\tau = \lim_{n\to\infty} u_{\tau\wedge\tau_n}$$

for every $\tau \in \mathcal{S}$.

proof (a) For $\sigma \in \mathcal{S}$ and $k \in \mathbb{N}$, set

$$a_{\sigma k} = \sup_{\tau\in\mathcal{S}\vee\sigma} [\![|u_\tau - u_\sigma| \ge 2^{-k}]\!];$$

for $\sigma \in \mathcal{S}$ set

$$b_\sigma = \sup_{k\in\mathbb{N}} a_{\sigma k} = \sup_{\tau\in\mathcal{S}\vee\sigma} [\![u_\tau \ne u_\sigma]\!], \quad c_\sigma = 1 \setminus \operatorname{upr}(b_\sigma, \mathfrak{A}_\sigma).$$

(Here $\operatorname{upr}(b_\sigma, \mathfrak{A}_\sigma) = \inf\{a : a \in \mathfrak{A}_\sigma, b_\sigma \subseteq a\}$, the upper envelope of b_σ in $\mathfrak{A}_\sigma$, as in 313S.) Set $d = \sup_{\sigma\in\mathcal{S}} c_\sigma$.

(b) If $\sigma \in \mathcal{S}$ and $k \in \mathbb{N}$, then $\inf_{\tau\in\mathcal{S}\vee\sigma} \bar\mu(a_{\sigma k} \setminus [\![|u_\tau - u_\sigma| \ge 2^{-k}]\!]) = 0$. **P** For $\tau \in \mathcal{S} \vee \sigma$ set $e_\tau = [\![|u_\tau - u_\sigma| \ge 2^{-k}]\!]$, so that $a_{\sigma k} = \sup_{\tau\in\mathcal{S}\vee\sigma} e_\tau$.

(i) If τ, $\tau' \in \mathcal{S} \vee \sigma$ then

$$\begin{aligned} e_\tau \cap [\![\tau = \tau']\!] &= [\![|u_\tau - u_\sigma| \ge 2^{-k}]\!] \cap [\![\tau = \tau']\!] \\ &= [\![|u_{\tau'} - u_\sigma| \ge 2^{-k}]\!] \cap [\![\tau = \tau']\!] = e_{\tau'} \cap [\![\tau = \tau']\!], \end{aligned}$$

$$e_\tau \cap [\![\tau \le \tau']\!] = e_\tau \cap [\![\tau = \tau \wedge \tau']\!] \subseteq e_{\tau\wedge\tau'},$$

$$e_\tau \cap [\![\tau' \le \tau]\!] = e_\tau \cap [\![\tau = \tau \vee \tau']\!] \subseteq e_{\tau\vee\tau'},$$

and $e_\tau \subseteq e_{\tau\wedge\tau'} \cup e_{\tau\vee\tau'}$. Similarly, $e_{\tau'} \subseteq e_{\tau\wedge\tau'} \cup e_{\tau\vee\tau'}$.

Now $e_{\tau\wedge\tau'} \in \mathfrak{A}_{\tau\wedge\tau'} \subseteq \mathfrak{A}_{\tau\vee\tau'}$, so there is a $\tau'' \in \mathcal{T}$ such that

$$e_{\tau\wedge\tau'} \subseteq [\![\tau'' = \tau \wedge \tau']\!], \quad 1 \setminus e_{\tau\wedge\tau'} \subseteq [\![\tau'' = \tau \vee \tau']\!]$$

(611I). Because $\mathcal{S}$ is finitely full, $\tau'' \in \mathcal{S}$, and of course $\sigma \le \tau \wedge \tau' \le \tau''$. Now

$$e_{\tau''} \supseteq [\![\tau'' = \tau \wedge \tau']\!] \cap e_{\tau\wedge\tau'} = e_{\tau\wedge\tau'}$$

and also

$$e_{\tau''} \supseteq [\![\tau'' = \tau \vee \tau']\!] \cap e_{\tau\vee\tau'} \supseteq e_{\tau\vee\tau'} \setminus e_{\tau\wedge\tau'},$$

so

$$e_{\tau''} \supseteq e_{\tau\wedge\tau'} \cup e_{\tau\vee\tau'} \supseteq e_\tau \cup e_{\tau'}.$$

(ii) Thus $\{e_\tau : \tau \in \mathcal{S} \wedge \sigma\}$ is upwards-directed and

$$\sup_{\tau\in\mathcal{S}\wedge\sigma} \bar\mu e_\tau = \bar\mu(\sup_{\tau\in\mathcal{S}\wedge\sigma} e_\tau) = \bar\mu a_{\sigma k}.$$

Taking complements in $a_{\sigma k}$,

$$\inf_{\tau\in\mathcal{S}\wedge\sigma} \bar\mu(a_{\sigma_k} \setminus e_\tau) = 0,$$

as required. **Q**

(c)(i) For any $\sigma \in \mathcal{S}$, $c_\sigma \in \mathfrak{A}_\sigma$ and $c_\sigma \cap a_{\sigma k} = 0$ for every $k \in \mathbb{N}$, so $c_\sigma \subseteq [\![u_\tau = u_\sigma]\!]$ for every $\tau \in \mathcal{S} \vee \sigma$. Generally, for $\tau \in \mathcal{S}$,

$$\llbracket u_\tau = u_{\tau\wedge\sigma} \rrbracket \supseteq \llbracket \tau = \tau \wedge \sigma \rrbracket \cup (\llbracket \tau = \tau \vee \sigma \rrbracket \cap \llbracket u_\sigma = u_{\tau\vee\sigma} \rrbracket)$$
$$\supseteq \llbracket \tau \le \sigma \rrbracket \cup (\llbracket \sigma \le \tau \rrbracket \cap c_\sigma) \supseteq c_\sigma.$$

(ii) If $\sigma \le \sigma' \le \tau$ in $\mathcal{S}$,

$$\llbracket u_\tau \ne u_{\sigma'} \rrbracket \subseteq \llbracket u_\tau \ne u_\sigma \rrbracket \cup \llbracket u_{\sigma'} \ne u_\sigma \rrbracket \subseteq b_\sigma;$$

taking the supremum over τ,

$$b_{\sigma'} \subseteq b_\sigma \subseteq \operatorname{upr}(b_\sigma, \mathfrak{A}_\sigma) \in \mathfrak{A}_\sigma \subseteq \mathfrak{A}_{\sigma'}$$

and

$$\operatorname{upr}(b_{\sigma'}, \mathfrak{A}_{\sigma'}) \subseteq \operatorname{upr}(b_\sigma, \mathfrak{A}_\sigma),$$

that is, $c_\sigma \subseteq c_{\sigma'}$.

(d)(i) There is a non-empty countable subset $A_0 \subseteq \mathcal{S}$ such that $d = \sup_{\sigma\in A_0} c_\sigma$. Now there is a countable sublattice $\mathcal{S}_0$ of $\mathcal{S}$, including A_0, such that whenever $\sigma \in \mathcal{S}_0$ and $k \in \mathbb{N}$ then $\inf_{\tau\in\mathcal{S}_0\vee\sigma} \bar\mu(a_{\sigma k} \setminus \llbracket |u_\tau - u_\sigma| \ge 2^{-k} \rrbracket) = 0$. **P** For $k \in \mathbb{N}$ and $\sigma \in \mathcal{S}$ there is a sequence $\langle \tau_{\sigma ki} \rangle_{i\in\mathbb{N}}$ in $\mathcal{S}\vee\sigma$ such that $\inf_{i\in\mathbb{N}} \bar\mu(a_{\sigma k} \setminus \llbracket |u_{\tau_{\sigma ki}} - u_\sigma| \ge 2^{-k} \rrbracket) = 0$, by (b). Set $\mathcal{S}_0 = \bigcup_{n\in\mathbb{N}} A_n$ where

$$A_{n+1} = \{\sigma \wedge \sigma' : \sigma,\, \sigma' \in A_n\} \cup \{\sigma \vee \sigma' : \sigma,\, \sigma' \in A_n\} \cup \{\tau_{\sigma ki} : \sigma \in A_n,\, k,\, i \in \mathbb{N}\}$$

for each $n \in \mathbb{N}$. **Q**

(ii) Let $\langle \sigma'_n \rangle_{n\in\mathbb{N}}$ be a sequence running over $\mathcal{S}_0$ and set $\sigma_n = \sup_{i\le n} \sigma'_i$ for $n \in \mathbb{N}$; then $\langle \sigma_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathcal{S}_0$ and $\{\sigma_n : n \in \mathbb{N}\}$ is cofinal with $\mathcal{S}_0$. Set $d_n = c_{\sigma_n}$ for $n \in \mathbb{N}$; by (c),

$$d = \sup_{\sigma\in A_0} c_\sigma \subseteq \sup_{n\in\mathbb{N}} c_{\sigma'_n} \subseteq \sup_{n\in\mathbb{N}} c_{\sigma_n} = \sup_{n\in\mathbb{N}} d_n \subseteq d$$

and $d = \sup_{n\in\mathbb{N}} d_n$. Note that $d_n \subseteq \llbracket u_\tau = u_{\tau\wedge\sigma_n} \rrbracket$ for every $n \in \mathbb{N}$ and $\tau \in \mathcal{S}$, by (c-i) above, while $d_n \subseteq d_{n+1}$ for every n by (c-ii).

(e) If $\tau \in \mathcal{S}$ then $\inf_{\sigma\in\mathcal{S}_0} \llbracket \sigma < \tau \rrbracket \subseteq c_\tau$. **P?** Otherwise,

$$\operatorname{upr}(b_\tau, \mathfrak{A}_\tau) \cap \inf_{\sigma\in\mathcal{S}_0} \llbracket \sigma < \tau \rrbracket \ne 0.$$

Because $\inf_{\sigma\in\mathcal{S}_0} \llbracket \sigma < \tau \rrbracket \in \mathfrak{A}_\tau$,

$$b_\tau \cap \inf_{\sigma\in\mathcal{S}_0} \llbracket \sigma < \tau \rrbracket \ne 0$$

and there is a $k \ge 1$ such that

$$a_{\tau,k-1} \cap \inf_{\sigma\in\mathcal{S}_0} \llbracket \sigma < \tau \rrbracket \ne 0;$$

finally, there is a $\tau' \in \mathcal{S} \vee \tau$ such that

$$a = \llbracket |u_{\tau'} - u_\tau| \ge 2^{-k+1} \rrbracket \cap \inf_{\sigma\in\mathcal{S}_0} \llbracket \sigma < \tau \rrbracket$$

is non-zero.

If $\sigma \in \mathcal{S}_0$ then

$$a \subseteq (\llbracket \sigma < \tau \rrbracket \cap \llbracket |u_{\tau'} - u_\tau| \ge 2^{-k+1} \rrbracket)$$
$$\subseteq (\llbracket \sigma \le \tau' \rrbracket \cap \llbracket |u_{\tau'} - u_\sigma| \ge 2^{-k} \rrbracket) \cup (\llbracket \sigma \le \tau \rrbracket \cap \llbracket |u_\tau - u_\sigma| \ge 2^{-k} \rrbracket)$$
$$\subseteq \llbracket |u_{\sigma\vee\tau'} - u_\sigma| \ge 2^{-k} \rrbracket) \cup \llbracket |u_{\sigma\vee\tau} - u_\sigma| \ge 2^{-k} \rrbracket) \subseteq a_{\sigma k}.$$

Consequently

$$\inf_{\rho\in\mathcal{S}_0\vee\sigma} \bar\mu(a \setminus \llbracket |u_\rho - u_\sigma| \ge 2^{-k} \rrbracket) \le \inf_{\rho\in\mathcal{S}_0\vee\sigma} \bar\mu(a_{\sigma k} \setminus \llbracket |u_\rho - u_\sigma| \ge 2^{-k} \rrbracket) = 0.$$

We can therefore choose inductively a non-decreasing sequence $\langle \rho_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{S}_0$ such that

$$\bar\mu(a \setminus \llbracket |u_{\rho_{n+1}} - u_{\rho_n}| \ge 2^{-k} \rrbracket) \le \frac{1}{2}\bar\mu a$$

for every $n \in \mathbb{N}$. But this means that

$$\begin{aligned}\theta(u_{\tau\wedge\rho_{n+1}}-u_{\tau\wedge\rho_n}) &\ge 2^{-k}\bar\mu[\![|u_{\tau\wedge\rho_{n+1}}-u_{\tau\wedge\rho_n}|\ge 2^{-k}]\!]) \\ &\ge 2^{-k}\bar\mu([\![\rho_{n+1}<\tau]\!]\cap[\![|u_{\rho_{n+1}}-u_{\rho_n}|\ge 2^{-k}]\!]) \\ &\ge 2^{-k}\bar\mu(a\cap[\![|u_{\rho_{n+1}}-u_{\rho_n}|\ge 2^{-k}]\!])\ge 2^{-k-1}\bar\mu a\end{aligned}$$

for every n; which contradicts our hypothesis that $\lim_{\sigma\uparrow A}u_\sigma$ is defined in $L^0(\mathfrak{A})$ for every non-empty upwards-directed set $A\subseteq\mathcal{S}$ with an upper bound in $\mathcal{S}$. **XQ**

(f) So for any $\tau\in\mathcal{S}$

$$\sup_{n\in\mathbb{N}}(d_n\cup[\![\tau\le\sigma_n]\!])=d\cup(1\setminus\inf_{n\in\mathbb{N}}[\![\sigma_n<\tau]\!])=d\cup(1\setminus\inf_{\sigma\in\mathcal{S}_0}[\![\sigma<\tau]\!])$$

(because if $\sigma\in\mathcal{S}_0$ there is an $n\in\mathbb{N}$ such that $\sigma\le\sigma_n$ and $[\![\sigma_n<\tau]\!]\subseteq[\![\sigma<\tau]\!]$)

$$=1$$

because $\inf_{\sigma\in\mathcal{S}_0}[\![\sigma<\tau]\!]\subseteq c_\tau\subseteq d$.

(g) We have most of what we want. But as there is no reason why d_n should be included in $[\![\sigma_{n+1}=\sigma_n]\!]$, we have to make a further adjustment.

(i) We can define a non-decreasing sequence $\langle\tau_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{S}$ inductively by saying that $\tau_0=\sigma_0$ and that

$$d_n\subseteq[\![\tau_{n+1}=\tau_n]\!],\quad 1\setminus d_n\subseteq[\![\tau_{n+1}=\sigma_{n+1}]\!]$$

for $n\in\mathbb{N}$. **P** The point is that $\tau_n\le\sigma_n$ and $d_n\in\mathfrak{A}_{\tau_n}$ for every n. To see this, we know that we have $\tau_0\le\sigma_0$ and

$$d_0=c_{\sigma_0}\in\mathfrak{A}_{\sigma_0}=\mathfrak{A}_{\tau_0}.$$

At the inductive step, given that $\tau_n\le\sigma_n$ and $d_n\in\mathfrak{A}_{\tau_n}$, 611I tells us that τ_{n+1} is well-defined in $\mathcal{T}$ and that $\tau_n\le\tau_{n+1}\le\sigma_{n+1}$; $\tau_{n+1}\in\mathcal{S}$ because $\mathcal{S}$ is finitely full. To see that $d_{n+1}\in\mathfrak{A}_{\tau_{n+1}}$, note first that, because $\sigma_n\le\sigma_{n+1}$,

$$d_n=c_{\sigma_n}\subseteq c_{\sigma_{n+1}}=d_{n+1}$$

by (c-ii) again. We know that $d_n\in\mathfrak{A}_{\tau_n}\subseteq\mathfrak{A}_{\tau_{n+1}}$, while $d_{n+1}\setminus d_n$ belongs to $\mathfrak{A}_{\sigma_{n+1}}$ and is included in $[\![\tau_{n+1}=\sigma_{n+1}]\!]$, so belongs to $\mathfrak{A}_{\tau_{n+1}}$, by 611H(c-iii). Accordingly $d_{n+1}=d_n\cup(d_{n+1}\setminus d_n)$ belongs to $\mathfrak{A}_{\tau_{n+1}}$. So the induction proceeds. **Q**

(ii) We now certainly have $d_n\in\mathfrak{A}_{\tau_n}$ and $d_n\subseteq[\![\tau_{n+1}=\tau_n]\!]$ for every n. If $\tau\in\mathcal{S}$,

$$1=\sup_{n\in\mathbb{N}}(d_n\cup[\![\tau\le\sigma_n]\!])$$

(by (f))

$$\begin{aligned}&\subseteq\sup_{n\in\mathbb{N}}(d_n\cup[\![\tau\le\sigma_{n+1}]\!])=\sup_{n\in\mathbb{N}}(d_n\cup([\![\tau\le\sigma_{n+1}]\!]\setminus d_n)) \\ &\subseteq\sup_{n\in\mathbb{N}}(d_n\cup([\![\tau\le\sigma_{n+1}]\!]\cap[\![\sigma_{n+1}=\tau_{n+1}]\!])) \\ &\subseteq\sup_{n\in\mathbb{N}}(d_{n+1}\cup[\![\tau\le\tau_{n+1}]\!])\subseteq\sup_{n\in\mathbb{N}}(d_n\cup[\![\tau\le\tau_n]\!]).\end{aligned}$$

(iii) If $n\in\mathbb{N}$ and $\tau\in\mathcal{S}$, then $d_n\subseteq[\![u_\tau=u_{\tau\wedge\tau_n}]\!]$. **P** Induce on n. If $n=0$ we just have to recall from (d-ii) that $d_0\subseteq[\![u_\tau=u_{\tau\wedge\sigma_0}]\!]$. For the inductive step to $n+1\ge 1$, we have

$$d_n\subseteq[\![\tau_n=\tau_{n+1}]\!]\cap[\![u_\tau=u_{\tau\wedge\tau_n}]\!]\subseteq[\![\tau\wedge\tau_n=\tau\wedge\tau_{n+1}]\!]\cap[\![u_\tau=u_{\tau\wedge\tau_n}]\!]$$

(611E(c-v-α))

$$\subseteq[\![u_\tau=u_{\tau\wedge\tau_{n+1}}]\!],$$

while $d_{n+1} \subseteq [\![u_\tau = u_{\tau\wedge\sigma_{n+1}}]\!]$ by (d-ii), so

$$d_{n+1} \setminus d_n \subseteq [\![u_\tau = u_{\tau\wedge\sigma_{n+1}}]\!] \cap [\![\tau_{n+1} = \sigma_{n+1}]\!] \subseteq [\![u_\tau = u_{\tau\wedge\tau_{n+1}}]\!].$$

Putting these together, $d_{n+1} \subseteq [\![u_\tau = u_{\tau\wedge\tau_{n+1}}]\!]$, as required to continue the induction. **Q**

(iv) It follows that $u_\tau = \lim_{n\to\infty} u_{\tau\wedge\tau_n}$ for every $\tau \in \mathcal{S}$. **P** By (iii), we have

$$d_n \cup [\![\tau \le \tau_n]\!] = d_n \cup [\![\tau = \tau \wedge \tau_n]\!] \subseteq [\![u_\tau = u_{\tau\wedge\tau_n}]\!],$$

while $\langle d_n \cup [\![\tau \le \tau_n]\!]\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{A}$ with supremum 1, so

$$\theta(u_\tau - u_{\tau\wedge\tau_n}) \le \bar\mu(1 \setminus (d_n \cup [\![\tau \le \tau_n]\!])) \to 0$$

as $n \to \infty$. **Q** So $\langle d_n\rangle_{n\in\mathbb{N}}$ and $\langle \tau_n\rangle_{n\in\mathbb{N}}$ have all the listed properties.

***627O Lemma** Suppose that we are given a sublattice $\mathcal{S}$ of $\mathcal{T}$, a non-decreasing sequence $\langle \tau_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{S}$ and a non-decreasing sequence $\langle d_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that

$$d_n \in \mathfrak{A}_{\tau_n}, \quad d_n \subseteq [\![\tau_{n+1} = \tau_n]\!]$$

for every $n \in \mathbb{N}$, and

$$1 = \sup_{n\in\mathbb{N}} d_n \cup [\![\tau \le \tau_n]\!]$$

for every $\tau \in \mathcal{S}$. Set $\mathcal{S}_0 = \bigcup_{n\in\mathbb{N}} \mathcal{S} \wedge \tau_n$ and suppose that $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}_0}$ is a fully adapted process.

(a) There is a fully adapted process $\tilde{\boldsymbol{u}} = \langle \tilde u_\tau\rangle_{\tau\in\mathcal{S}}$ such that

(i) $\tilde u_\tau = \lim_{n\to\infty} u_{\tau\wedge\tau_n}$ for every $\tau \in \mathcal{S}$,

(ii) $d_n \cup [\![\tau \le \tau_n]\!] \subseteq [\![\tilde u_\tau = u_{\tau\wedge\tau_n}]\!]$ for every $\tau \in \mathcal{S}$ and $n \in \mathbb{N}$,

(iii) $\tilde{\boldsymbol{u}}$ extends $\boldsymbol{u}$.

(b) Write $\hat{\mathcal{S}}$ for the covered envelope of $\mathcal{S}$, $\hat{\mathcal{S}}_0$ for $\bigcup_{n\in\mathbb{N}} \hat{\mathcal{S}} \wedge \tau_n$ and $\hat{\boldsymbol{u}} = \langle \hat u_\sigma\rangle_{\sigma\in\hat{\mathcal{S}}_0}$ for the fully adapted extension of $\boldsymbol{u}$ to $\hat{\mathcal{S}}_0$. Set $\tilde{\hat u}_\tau = \lim_{n\to\infty} \hat u_{\tau\wedge\tau_n}$ for every $\tau \in \hat{\mathcal{S}}$. Then $\tilde{\hat{\boldsymbol{u}}} = \langle \tilde{\hat u}_\tau\rangle_{\tau\in\hat{\mathcal{S}}}$ is the fully adapted extension of $\tilde{\boldsymbol{u}}$ to $\hat{\mathcal{S}}$.

(c) If $\boldsymbol{u}$ is locally moderately oscillatory, $\tilde{\boldsymbol{u}}$ is locally moderately oscillatory.

(d) If $\boldsymbol{u}$ is a virtually local martingale, $\tilde{\boldsymbol{u}}$ is a virtually local martingale.

(e) If $\boldsymbol{u}$ is locally of bounded variation, $\tilde{\boldsymbol{u}}$ is locally of bounded variation.

(f) If $\boldsymbol{u}$ is locally order-bounded and $\bar w = \sup_{n\in\mathbb{N}} \mathrm{Osclln}(\boldsymbol{u}\restriction\mathcal{S}_0\wedge\tau_n)$ is defined, then $\bar w \ge \sup_{\tau\in\mathcal{S}} \mathrm{Osclln}(\tilde{\boldsymbol{u}}\restriction\mathcal{S}\wedge\tau)$.

(g) If $\boldsymbol{u}$ is a semi-martingale, $\tilde{\boldsymbol{u}}$ is a semi-martingale.

proof (a)(i)($\boldsymbol{\alpha}$) If $\tau \in \mathcal{S}$ and $k \le n \in \mathbb{N}$, then

$$d_k \subseteq [\![\tau_k = \tau_n]\!] \subseteq [\![u_{\tau\wedge\tau_k} = u_{\tau\wedge\tau_n}]\!]$$

while similarly

$$[\![\tau \le \tau_k]\!] \subseteq [\![\tau \wedge \tau_k = \tau \wedge \tau_n]\!] \subseteq [\![u_{\tau\wedge\tau_k} = u_{\tau\wedge\tau_n}]\!].$$

So if $k \le m \le n$ in $\mathbb{N}$, $d_k \cup [\![tau \le \tau_k]\!] \subseteq [\![u_{\tau\wedge\tau_m} = u_{\tau\wedge\tau_n}]\!]$ and

$$\theta(u_{\tau\wedge\tau_m} - u_{\tau\wedge\tau_n}) \le \bar\mu(1 \setminus (d_k \cup [\![\tau \le \tau_k]\!])) \to 0$$

as $k \to \infty$. Thus $\langle u_{\tau\wedge\tau_n}\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence and $\tilde u_\tau = \lim_{n\to\infty} u_{\tau\wedge\tau_n}$ is defined in $L^0(\mathfrak{A})$. Moreover, since $u_{\tau\wedge\tau_n} \in L^0(\mathfrak{A}_{\tau\wedge\tau_n}) \subseteq L^0(\mathfrak{A}_\tau)$ for every n, $\tilde u_\tau \in L^0(\mathfrak{A}_\tau)$.

($\boldsymbol{\beta}$) If τ, $\tau' \in \mathcal{S}$ and $c = [\![\tau = \tau']\!]$, then $c \subseteq [\![\tau \wedge \tau_n = \tau' \wedge \tau_n]\!]$ for every n, so

$$\tilde u_\tau \times \chi c = \lim_{n\to\infty} u_{\tau\wedge\tau_n} \times \chi c = \lim_{n\to\infty} u_{\tau'\wedge\tau_n} \times \chi c = \tilde u_{\tau'} \times \chi c$$

and $c \subseteq [\![\tilde u_\tau = \tilde u_{\tau'}]\!]$. As τ and τ' are arbitrary, $\tilde{\boldsymbol{u}}$ is fully adapted.

(ii) Set $d'_n = d_n \cup [\![\tau \le \tau_n]\!]$. For any $k \ge n$,

$$d'_n \subseteq ([\![\tau_k = \tau_n]\!] \cup ([\![\tau \le \tau_n]\!] \cap [\![\tau_n \le \tau_k]\!]) \subseteq [\![\tau \wedge \tau_k = \tau \wedge \tau_n]\!] \subseteq [\![u_{\tau\wedge\tau_k} = u_{\tau\wedge\tau_n}]\!].$$

So

$$\tilde u_\tau \times \chi d'_n = \lim_{k\to\infty} u_{\tau\wedge\tau_k} \times \chi d'_n = u_{\tau\wedge\tau_n} \times \chi d'_n$$

and $d'_n \subseteq [\![\tilde{u}_\tau = u_{\tau\wedge\tau_n}]\!]$.

(iii) If $\tau \in \mathcal{S}_0$ then $\tau \wedge \tau_n = \tau$ for all n large enough, so $\tilde{u}_\tau = \lim_{n\to\infty} u_{\tau\wedge\tau_n} = u_\tau$.

(b) We just have to note that $\tilde{\hat{\boldsymbol{u}}}$ is fully adapted, by (a-i) applied to $\hat{\mathcal{S}}$ and $\hat{\boldsymbol{u}}$, and extends $\tilde{\boldsymbol{u}}$.

(c)(i) Consider first the case in which $\mathcal{S}$ is full. Suppose that $\tau \in \mathcal{S}$ and $\langle\sigma_i\rangle_{i\in\mathbb{N}}$ is a monotonic sequence in $\mathcal{S} \wedge \tau$. For $n \in \mathbb{N}$ set $d'_n = d_n \cup [\![\tau \le \tau_n]\!]$. Then

$$d'_n \subseteq d_n \cup [\![\sigma_i \le \tau_n]\!] \subseteq [\![u_{\sigma_i} = u_{\sigma_i\wedge\tau_n}]\!]$$

for every $i \in \mathbb{N}$, and

$$\langle\tilde{u}_{\sigma_i} \times \chi d'_n\rangle_{i\in\mathbb{N}} = \langle u_{\sigma_i\wedge\tau_n} \times \chi d'_n\rangle_{i\in\mathbb{N}}$$

is convergent. Since $\theta(\tilde{u}_{\sigma_i} - \tilde{u}_{\sigma_i} \times \chi d'_n) \le \bar{\mu}(1 \setminus d'_n)$ for all i and n, and $\lim_{n\to\infty} \bar{\mu}(1 \setminus d'_n) = 0$, $\langle\tilde{u}_{\sigma_i}\rangle_{i\in\mathbb{N}}$ is convergent. As $\langle\sigma_i\rangle_{i\in\mathbb{N}}$ is arbitrary, $\tilde{\boldsymbol{u}}\restriction\mathcal{S}\wedge\tau$ is moderately oscillatory; as τ is arbitrary, $\tilde{\boldsymbol{u}}$ is locally moderately oscillatory.

(ii) In general, take $\hat{\mathcal{S}}$, $\hat{\mathcal{S}}_0$, $\hat{\boldsymbol{u}}$ and $\tilde{\hat{\boldsymbol{u}}}$ as in (b). For each $n \in \mathbb{N}$, $\hat{\mathcal{S}}_0 \wedge \tau_n = \hat{\mathcal{S}} \wedge \tau_n$ is the covered envelope of $\mathcal{S} \wedge \tau_n$ (611M(e-i)), and $\boldsymbol{u}\restriction\mathcal{S} \wedge \tau_n$ is moderately oscillatory, so $\hat{\boldsymbol{u}}\restriction\hat{\mathcal{S}} \wedge \tau_n$ is moderately oscillatory (615F(a-i)); as $\{\tau_n : n \in \mathbb{N}\}$ is cofinal with $\hat{\mathcal{S}}_0$, $\hat{\boldsymbol{u}}$ is locally moderately oscillatory. By (i), $\tilde{\hat{\boldsymbol{u}}}$ is locally moderately oscillatory, so $\tilde{\boldsymbol{u}}$ is locally moderately oscillatory (615F(b-v)).

(d) We know from (c) that $\tilde{\boldsymbol{u}}$ is locally moderately oscillatory.

(i) Consider first the case in which $\mathcal{S}$ is full.

($\boldsymbol{\alpha}$) Take $\tau \in \mathcal{S}$ and $\epsilon > 0$. Then there is an $n \in \mathbb{N}$ such that $\bar{\mu}(d_n \cup [\![\tau \le \tau_n]\!]) \ge 1 - \epsilon$. By 623Ke, $\boldsymbol{u}\restriction\mathcal{S} \wedge \tau_n = \boldsymbol{u}\restriction\mathcal{S}_0 \wedge \tau_n$ is a virtually local martingale. Let $A \subseteq \mathcal{S} \wedge \tau_n$ be a non-empty downwards-directed set such that $\sup_{\rho\in A} \bar{\mu}[\![\rho < \tau_n]\!] < \epsilon$ and $R_A(\boldsymbol{u}\restriction\mathcal{S} \wedge \tau_n)$, as defined in 623B, is a martingale. If $\rho \in A$, then $a_\rho = d_n \cap [\![\tau_n \le \rho \wedge \tau]\!]$ belongs to $\mathfrak{A}_{\rho\wedge\tau}$, by 611H(c-iii) again. So we have a $\rho' \in \mathcal{T}$ such that

$$1 \setminus a_\rho \subseteq [\![\rho' = \rho \wedge \tau]\!], \quad a_\rho \subseteq [\![\rho' = \tau]\!],$$

and $\rho' \in \mathcal{S}$. If $\rho_0 \le \rho_1$ in A then $\rho'_0 \le \rho'_1$. **P** $a_{\rho_0} \subseteq a_{\rho_1}$. Now

$$\begin{aligned} a_{\rho_0} &\subseteq [\![\rho'_0 = \tau]\!] \cap [\![\rho'_1 = \tau]\!] \subseteq [\![\rho'_0 \le \rho'_1]\!], \\ a_{\rho_1} \setminus a_{\rho_0} &\subseteq [\![\rho'_0 = \rho_0 \wedge \tau]\!] \cap [\![\rho'_1 = \tau]\!] \subseteq [\![\rho'_0 \le \rho'_1]\!], \\ 1 \setminus a_{\rho_1} &\subseteq [\![\rho'_0 = \rho_0 \wedge \tau]\!] \cap [\![\rho'_1 = \rho_1 \wedge \tau]\!] \subseteq [\![\rho'_0 \le \rho'_1]\!]. \end{aligned}$$

So $\rho'_0 \le \rho'_1$. **Q**

($\boldsymbol{\beta}$) If $\rho \in A$ and $\sigma \in \mathcal{S}$, $\tilde{u}_{\sigma\wedge\rho'} = u_{\sigma\wedge\tau\wedge\rho}$. **P** By (a-ii), $d_n \subseteq [\![\tilde{u}_\sigma = u_{\sigma\wedge\tau_n}]\!]$ for every $\sigma \in \mathcal{S}$. So

$$\begin{aligned} a_\rho &\subseteq d_n \cap [\![\tau_n \le \tau \wedge \rho]\!] \cap [\![\rho' = \tau]\!] \subseteq [\![\tilde{u}_{\sigma\wedge\rho'} = \tilde{u}_{\sigma\wedge\rho'\wedge\tau_n}]\!] \cap [\![\rho = \tau_n]\!] \cap [\![\rho' = \tau]\!] \\ &\subseteq [\![\tilde{u}_{\sigma\wedge\rho'} = \tilde{u}_{\sigma\wedge\tau\wedge\rho}]\!] = [\![\tilde{u}_{\sigma\wedge\rho'} = u_{\sigma\wedge\tau\wedge\rho}]\!] \end{aligned}$$

because $\sigma \wedge \tau \wedge \rho \in \mathcal{S} \wedge \tau_n \subseteq \mathcal{S}_0$. On the other hand,

$$1 \setminus a_\rho \subseteq [\![\rho' = \tau \wedge \rho]\!] \subseteq [\![\tilde{u}_{\sigma\wedge\rho'} = \tilde{u}_{\sigma\wedge\tau\wedge\rho}]\!] = [\![\tilde{u}_{\sigma\wedge\rho'} = u_{\sigma\wedge\tau\wedge\rho}]\!].$$

Putting these together, we have the result. **Q**

($\boldsymbol{\gamma}$) Set $A' = \{\rho' : \rho \in A\}$; then $A' \subseteq \mathcal{S}$ is downwards-directed and not empty. Now $\sup_{\rho\in A'} \bar{\mu}[\![\rho < \tau]\!] = \sup_{\rho\in A} \bar{\mu}[\![\rho' < \tau]\!]$ is at most 2ϵ. **P** If $\rho \in A$, then

$$\begin{aligned} [\![\rho' < \tau]\!] &= [\![\rho \wedge \tau < \tau]\!] \setminus a_\rho = [\![\rho < \tau]\!] \setminus (d_n \cap [\![\tau_n \le \rho \wedge \tau]\!]) \\ &\subseteq [\![\rho < \tau_n]\!] \cup ([\![\rho = \tau_n]\!] \cap [\![\tau_n < \tau]\!] \setminus (d_n \cap [\![\tau_n \le \rho \wedge \tau]\!])) \\ &= [\![\rho < \tau_n]\!] \cup ([\![\rho = \tau_n]\!] \cap [\![\tau_n < \tau]\!] \setminus d_n) \\ &\subseteq [\![\rho < \tau_n]\!] \cup (1 \setminus (d_n \cup [\![\tau \le \tau_n]\!])) \end{aligned}$$

has measure at most 2ϵ. **Q**

(δ) Because $\tilde{\boldsymbol{u}}$ is locally moderately oscillatory, $R_{A'}(\tilde{\boldsymbol{u}})$ is defined. Furthermore, it is a martingale. **P** Express the martingale $R_A(\boldsymbol{u}\restriction\mathcal{S}\wedge\tau_n)$ as $\langle u'_\sigma\rangle_{\sigma\in\mathcal{S}\wedge\tau_n}$, so that $u'_\sigma=\lim_{\rho\downarrow A}u_{\sigma\wedge\rho}$ belongs to $L^1_{\bar\mu}$ for $\sigma\in\mathcal{S}\wedge\tau_n$, and $P_\sigma u'_{\sigma'}=u'_\sigma$ whenever $\sigma\le\sigma'$ in $\mathcal{S}\wedge\tau_n$.

For $\sigma\in\mathcal{S}$, write v'_σ for

$$\lim_{\rho\downarrow A'}\tilde u_{\sigma\wedge\rho}=\lim_{\rho\downarrow A}\tilde u_{\sigma\wedge\rho'}=\lim_{\rho\downarrow A}u_{\sigma\wedge\tau\wedge\rho}=\lim_{\rho\downarrow A}u_{\sigma\wedge\tau_n\wedge\tau\wedge\rho}=u'_{\sigma\wedge\tau_n\wedge\tau};$$

this belongs to $L^1_{\bar\mu}$. And if $\sigma\le\sigma'$ in $\mathcal{S}$,

$$P_\sigma(v'_{\sigma'})=P_\sigma u'_{\sigma'\wedge\tau_n\wedge\tau}=P_{\sigma\wedge\tau_n\wedge\tau}u'_{\sigma'\wedge\tau_n\wedge\tau}=u'_{\sigma\wedge\tau_n\wedge\tau}=v'_\sigma.$$

So $\langle v'_\sigma\rangle_{\sigma\in\mathcal{S}}=R_{A'}(\tilde{\boldsymbol{u}})$ is a martingale. **Q**

(ϵ) Since τ and ϵ are arbitrary, $\tilde{\boldsymbol{u}}$ is a virtually local martingale.

(ii) Generally, take $\hat{\mathcal{S}}$, $\hat{\mathcal{S}}_0$, $\tilde{\hat{\boldsymbol{u}}}$ and $\hat{\boldsymbol{u}}$ as in (b) and (c-ii). Because $\hat{\boldsymbol{u}}\restriction\hat{\mathcal{S}}_0\wedge\tau_n$ is the fully adapted extension of $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau_n$, it is a virtually local martingale for every $n\in\mathbb{N}$; because $\{\tau_n:n\in\mathbb{N}\}$ is cofinal with $\hat{\mathcal{S}}_0$, $\hat{\boldsymbol{u}}$ is a virtually local martingale; by (i) here, $\tilde{\hat{\boldsymbol{u}}}$ is a virtually local martingale; it follows that $\tilde{\boldsymbol{u}}$ is a virtually local martingale.

(e) Take $\tau\in\mathcal{S}$. For $n\in\mathbb{N}$, $\bar z_n=\int_{\mathcal{S}\wedge\tau_n}|d\boldsymbol{u}|$ is defined in $L^0(\mathfrak{A})$. Set $d'_{-1}=0$, $d'_n=d_n\cup[\![\tau\le\tau_n]\!]$ for $n\ge0$ and

$$\bar z=\sum_{n=0}^\infty(\bar z_n\times\chi(d'_n\setminus d'_{n-1}))\in L^0.$$

Now suppose that $\sigma_0\le\ldots\le\sigma_k\le\tau$ in $\mathcal{S}$, and set $z=\sum_{i=0}^{k-1}|\tilde u_{\sigma_{i+1}}-\tilde u_{\sigma_i}|$. Then

$$\begin{aligned}d'_n&\subseteq\inf_{i\le k}[\![\tilde u_{\sigma_i}=u_{\sigma_i\wedge\tau_n}]\!]\\&\subseteq[\![z=\textstyle\sum_{i=0}^{k-1}|u_{\sigma_{i+1}\wedge\tau_n}-u_{\sigma_i\wedge\tau_n}|]\!]\subseteq[\![z\le\bar z_n]\!]\end{aligned}$$

so

$$d'_n\setminus d'_{n-1}\subseteq[\![z\le\bar z_n]\!]\cap[\![\bar z_n\le\bar z]\!]\subseteq[\![z\le\bar z]\!]$$

for every $n\in\mathbb{N}$. Since $\langle d'_n\rangle_{n\in\mathbb{N}}$ is non-decreasing and has supremum 1, $z\le\bar z$. As $\sigma_0,\ldots,\sigma_k$ are arbitrary, $\tilde{\boldsymbol{u}}\restriction\mathcal{S}\wedge\tau$ is of bounded variation.

(f) Take $\tau^*\in\mathcal{S}$. I need to show that $\mathrm{Osclln}(\tilde{\boldsymbol{u}}\restriction\mathcal{S}\wedge\tau^*)\le\bar w$. We know from (c) that $\tilde{\boldsymbol{u}}\restriction\mathcal{S}\wedge\tau^*$ is order-bounded; set $\bar w'=\mathrm{Osclln}(\tilde{\boldsymbol{u}}\restriction\mathcal{S}\wedge\tau^*)$. Take any $n\in\mathbb{N}$ and $\epsilon>0$. Set $\tilde\tau=\tau^*\wedge\tau_n$. Note that by 618Da,

$$\mathrm{Osclln}(\boldsymbol{u}\restriction\mathcal{S}\wedge\tilde\tau)\le\mathrm{Osclln}(\boldsymbol{u}\restriction\mathcal{S}\wedge\tau_n)\le\bar w.$$

Suppose that $I\subseteq J\in\mathcal{I}(\mathcal{S}\wedge\tau^*)$. If $\tilde\tau\le\tau\le\tau'\le\tau^*$ in $\mathcal{S}$, then $\tau\wedge\tau_n=\tilde\tau=\tau'\wedge\tau_n$, so

$$d_n\cup[\![\tau^*\le\tau_n]\!]\subseteq[\![\tilde u_\tau=u_{\tau\wedge\tau_n}]\!]\cap[\![\tilde u_{\tau'}=u_{\tau'\wedge\tau_n}]\!]\subseteq[\![\tilde u_\tau=\tilde u_{\tau'}]\!].$$

As τ and τ' are arbitrary,

$$\begin{aligned}d_n\cup[\![\tau^*\le\tau_n]\!]&\subseteq[\![\mathrm{Osclln}_{J\vee\tilde\tau}(\tilde{\boldsymbol{u}})=0]\!]\subseteq[\![\mathrm{Osclln}_J(\tilde{\boldsymbol{u}})=\mathrm{Osclln}_{J\wedge\tilde\tau}(\tilde{\boldsymbol{u}})]\!]\\&=[\![\mathrm{Osclln}_J(\tilde{\boldsymbol{u}})=\mathrm{Osclln}_{J\wedge\tilde\tau}(\boldsymbol{u})]\!]\end{aligned}$$

(because $\tilde{\boldsymbol{u}}\restriction\mathcal{S}\wedge\tilde\tau=\boldsymbol{u}\restriction\mathcal{S}\wedge\tilde\tau$)

$$\subseteq[\![\mathrm{Osclln}_J(\tilde{\boldsymbol{u}})\le\mathrm{Osclln}^*_I(\boldsymbol{u}\restriction\mathcal{S}\wedge\tilde\tau)]\!].$$

As J is arbitrary,

$$\begin{aligned}d_n\cup[\![\tau^*\le\tau_n]\!]&\subseteq[\![\mathrm{Osclln}^*_I(\tilde{\boldsymbol{u}}\restriction\mathcal{S}\wedge\tau^*)\le\mathrm{Osclln}^*_I(\boldsymbol{u}\restriction\mathcal{S}\wedge\tilde\tau)]\!]\\&\subseteq[\![\bar w'\le\mathrm{Osclln}^*_I(\boldsymbol{u}\restriction\mathcal{S}\wedge\tilde\tau)]\!].\end{aligned}$$

As I is arbitrary,

$$d_n\cup[\![\tau^*\le\tau_n]\!]\subseteq[\![\bar w'\le\mathrm{Osclln}(\boldsymbol{u}\restriction\mathcal{S}\wedge\tilde\tau)]\!]\subseteq[\![\bar w'\le\bar w]\!].$$

As n is arbitrary, $\operatorname{Osclln}(\tilde{\boldsymbol{u}}\restriction\mathcal{S}\wedge\tau^*) = \bar{w}' \le \bar{w}$, as required.

(g) Express $\boldsymbol{u}$ as $\boldsymbol{v}+\boldsymbol{w}$ where $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}_0}$ is a virtually local martingale and $\boldsymbol{w} = \langle w_\sigma\rangle_{\sigma\in\mathcal{S}_0}$ is locally of bounded variation. Let $\tilde{\boldsymbol{v}} = \langle\tilde{v}_\tau\rangle_{\tau\in\mathcal{S}}$ and $\tilde{\boldsymbol{w}} = \langle\tilde{w}_\tau\rangle_{\tau\in\mathcal{S}}$ be the corresponding extensions, so that $\tilde{\boldsymbol{v}}$ is a virtually local martingale and $\tilde{\boldsymbol{w}}$ is locally of bounded variation ((d) and (e) above). Then

$$\tilde{u}_\tau = \lim_{n\to\infty} u_{\tau\wedge\tau_n} = \lim_{n\to\infty} v_{\tau\wedge\tau_n} + \lim_{n\to\infty} w_{\tau\wedge\tau_n} = \tilde{v}_\tau + \tilde{w}_\tau$$

for every $\tau\in\mathcal{S}$, so that $\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{v}}+\tilde{\boldsymbol{w}}$ is a semi-martingale.

627P Corollary Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, τ a member of $\mathcal{S}$, and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$, $\boldsymbol{w} = \langle w_\sigma\rangle_{\sigma\in\mathcal{S}\wedge\tau}$ two fully adapted processes. Set

$$v_\sigma = u_\sigma - u_{\sigma\wedge\tau} + w_{\sigma\wedge\tau}$$

for $\sigma\in\mathcal{S}$. Then $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ is fully adapted, and is locally of bounded variation, or a virtually local martingale, or locally moderately oscillatory, or a semi-martingale if $\boldsymbol{u}$ and $\boldsymbol{w}$ both are, while

$$v_\sigma = w_\sigma \text{ if } \sigma\in\mathcal{S}\wedge\tau, \quad v_\sigma = u_\sigma - u_\tau + w_\tau \text{ if } \sigma\in\mathcal{S}\vee\tau.$$

proof Apply 627O with $d_n = 1$, $\tau_n = \tau$ for every $n\in\mathbb{N}$, so that $\mathcal{S}_0 = \mathcal{S}\wedge\tau$ and $\sigma\mapsto u_{\sigma\wedge\tau}$, $\sigma\mapsto w_{\sigma\wedge\tau}$ are the extensions $\tilde{\boldsymbol{u}}$, $\tilde{\boldsymbol{w}}$ of $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ and $\boldsymbol{w}$ as described in 627O. Now for each of the four properties listed, $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ will have the property if $\boldsymbol{u}$ does; see 623Ke for virtually local martingales, and the others are almost immediate. So 627O tells us that $\tilde{\boldsymbol{u}}$ and $\tilde{\boldsymbol{w}}$ also have the property considered, so that $\boldsymbol{v} = \boldsymbol{u}-\tilde{\boldsymbol{u}}+\tilde{\boldsymbol{w}}$ also does, using 614Q(b-iii), 615F(b-iii) or 623Ka.

627Q Theorem A fully adapted process is a semi-martingale iff it is a local integrator.

proof (a) We saw in 625E that a semi-martingale is a local integrator. So it will be enough to show that if $\mathcal{S}$ is a sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a local integrator, then it is a semi-martingale.

(b) To begin with, suppose that $\mathcal{S}$ is full. Since $\boldsymbol{u}$ is locally moderately oscillatory (616Ib), the hypotheses of 627N are satisfied; let $\langle d_n\rangle_{n\in\mathbb{N}}$ and $\langle\tau_n\rangle_{n\in\mathbb{N}}$ be as described there. For each $n\in\mathbb{N}$, $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau_n$ is an integrator, therefore a semi-martingale (627J); let $\boldsymbol{v}_n = \langle v_{n\sigma}\rangle_{\sigma\in\mathcal{S}\wedge\tau_n}$ be a virtually local martingale and $\boldsymbol{w}_n = \langle w_{n\sigma}\rangle_{\sigma\in\mathcal{S}\wedge\tau_n}$ a process of bounded variation such that $\boldsymbol{v}_n+\boldsymbol{w}_n = \boldsymbol{u}_n\restriction\mathcal{S}\wedge\tau_n$. Define $\boldsymbol{v}'_n = \langle v'_{n\sigma}\rangle_{\sigma\in\mathcal{S}\wedge\tau_n}$ and $\boldsymbol{w}'_n = \langle w'_{n\sigma}\rangle_{\sigma\in\mathcal{S}\wedge\tau_n}$ inductively by saying that

$$\boldsymbol{v}'_0 = \boldsymbol{v}_0, \quad \boldsymbol{w}'_0 = \boldsymbol{w}_0,$$

$$v'_{n+1,\sigma} = v_{n+1,\sigma} - v_{n+1,\sigma\wedge\tau_n} + v'_{n,\sigma\wedge\tau_n},$$

$$w'_{n+1,\sigma} = w_{n+1,\sigma} - w_{n+1,\sigma\wedge\tau_n} + w'_{n,\sigma\wedge\tau_n}$$

for $n\in\mathbb{N}$ and $\sigma\in\mathcal{S}\wedge\tau_{n+1}$. We see by induction, using 627P, that $\boldsymbol{v}'_n$ is a virtually local martingale and $\boldsymbol{w}'_n$ is of bounded variation, while $\boldsymbol{v}'_{n+1}$ extends $\boldsymbol{v}'_n$, $\boldsymbol{w}'_{n+1}$ extends $\boldsymbol{w}'_n$ and $\boldsymbol{v}'_n+\boldsymbol{w}'_n = \boldsymbol{u}_n$, for every $n\in\mathbb{N}$. We therefore have processes $\boldsymbol{v} = \bigcup_{n\in\mathbb{N}}\boldsymbol{v}'_n$ and $\boldsymbol{w} = \bigcup_{n\in\mathbb{N}}\boldsymbol{w}'_n$, both with domain $\mathcal{S}_0 = \bigcup_{n\in\mathbb{N}}\mathcal{S}\wedge\tau_n$, such that $\boldsymbol{v}$ is a virtually local martingale (623Ke again) and $\boldsymbol{w}$ is locally of bounded variation, while $\boldsymbol{v}+\boldsymbol{w} = \boldsymbol{u}\restriction\mathcal{S}_0$. Thus $\boldsymbol{u}\restriction\mathcal{S}_0$ is a semi-martingale. But as the construction in 627N arranges that $u_\tau = \lim_{n\to\infty} u_{\tau\wedge\tau_n}$ for every $\tau\in\mathcal{S}$, $\boldsymbol{u}$ itself is a semi-martingale, by 627Og.

(c) For general $\mathcal{S}$, let $\hat{\mathcal{S}}$ be the covered envelope of $\mathcal{S}$, and write $\hat{\boldsymbol{u}}$ for the fully adapted extension of $\boldsymbol{u}$ to $\hat{\mathcal{S}}$. By 616Ia, $\hat{\boldsymbol{u}}$ is a local integrator, By (b) here, $\hat{\boldsymbol{u}}$ is a semi-martingale, and is expressible as the sum of a virtually local martingale $\hat{\boldsymbol{v}}$ and a process $\hat{\boldsymbol{w}}$ which is locally of bounded variation. Setting $\boldsymbol{v} = \hat{\boldsymbol{v}}\restriction\mathcal{S}$ and $\boldsymbol{w} = \hat{\boldsymbol{w}}\restriction\mathcal{S}$, $\boldsymbol{u} = \boldsymbol{v}+\boldsymbol{w}$, while $\boldsymbol{v}$ is a virtually local martingale (623J) and $\boldsymbol{w}$ is locally of bounded variation (614L(b-ii)). So $\boldsymbol{u}$ is a semi-martingale, which is what we need to know.

627R In the cases we really care about we can escape most of the work in Lemmas 627N and 627O by using the following.

Proposition Suppose that T is separable in its order topology. If $\mathcal{S}$ is any sublattice of $\mathcal{T}$, there is a non-decreasing sequence $\langle\tau_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{S}$ such that $\sup_{n\in\mathbb{N}}[\![\sigma\le\tau_n]\!] = 1$ for every $\sigma\in\mathcal{S}$.

proof (a) Set $\tau^* = \sup \mathcal{S}$ in $\mathcal{T}$. Then there is a countable set $T_0 \subseteq T$ such that $[\![\tau^* > t]\!] = \sup_{s \in T_0, s \ge t} [\![\tau^* > s]\!]$ for every $t \in T$. **P** Take T_1 to be a countable dense subset of T and set

$$T_2 = \{t : [\![\tau^* > t]\!] \ne \sup_{s>t} [\![\tau^* > s]\!]\} = \{t : \bar{\mu}[\![\tau^* > t]\!] > \sup_{s>t} \bar{\mu}[\![\tau^* > s]\!]\},$$

$$T_3 = \{t : \inf\{s : s > t\} \in T_2\},$$

so that T_2 and T_3 are countable; then $T_0 = T_1 \cup T_2 \cup T_3$ is countable. If $t \in T_0$ then of course $[\![\tau^* > t]\!] = \sup_{s \in T_0, s \ge t} [\![\tau^* > s]\!]$. If $t \in T \setminus T_0$ and $\epsilon > 0$, then $t \notin T_2$ so there is an $s_0 > t$ such that $\bar{\mu}[\![\tau^* > t]\!] \le \bar{\mu}[\![\tau^* > s_0]\!] + \epsilon$; as $t \notin T_3$, we can suppose that $s_0 \notin T_2$, so that there is an $s_1 > s_0$ such that $\bar{\mu}[\![\tau^* > s_0]\!] \le \bar{\mu}[\![\tau^* > s_1]\!] + \epsilon$. Now the open interval $]\tau^*, s_1[$ is non-empty, so meets T_1, and there is an $s \in T_1$ such that $\bar{\mu}[\![\tau^* > t]\!] \le \bar{\mu}[\![\tau^* > s]\!] + 2\epsilon$. As ϵ is arbitrary, $[\![\tau^* > t]\!] = \sup_{s \in T_0, s \ge t} [\![\tau^* > s]\!]$ in this case also. **Q**

(b) Set $a = \sup_{\sigma \in \mathcal{S}} [\![\sigma = \tau^*]\!]$. Then we have a countable $C \subseteq \mathcal{S}$ such that $a = \sup_{\tau \in C} [\![\tau = \tau^*]\!]$. Next, for each $t \in T_0$, let D_t be a countable subset of $\mathcal{S}$ such that

$$\sup_{\tau \in D_t} [\![\tau > t]\!] = \sup_{\sigma \in \mathcal{S}} [\![\sigma > t]\!] = [\![\tau^* > t]\!]$$

(611Cb). Set $D = C \cup \bigcup_{t \in T_0} D_t \in [\mathcal{S}]^{\le \omega}$.

(c) $\sup D = \tau^*$. **P** For any $t \in T$,

$$[\![\tau^* > t]\!] = \sup_{\substack{s \in T_0 \\ s \ge t}} [\![\tau^* > s]\!] = \sup_{\substack{s \in T_0 \\ s \ge t \\ \tau \in D_s}} [\![\tau > s]\!] \subseteq \sup_{\substack{s \in T_0 \\ \tau \in D_s}} [\![\tau > t]\!] \subseteq [\![\sup D > t]\!]$$

so $\tau^* \le \sup D$; and of course $\sup D \le \tau^*$. **Q**

(d) Take any $\sigma \in \mathcal{S}$. We have

$$[\![\sigma < \tau^*]\!] = [\![\sigma < \sup D]\!] = \sup_{\tau \in D} [\![\sigma < \tau]\!],$$

while $[\![\sigma = \tau^*]\!] = \sup_{\tau \in D} [\![\sigma = \tau]\!]$ because $C \subseteq D$. Consequently

$$\sup_{\tau \in D} [\![\sigma \le \tau]\!] = [\![\sigma \le \tau^*]\!] = 1.$$

(e) Let $\langle \tau'_n \rangle_{n \in \mathbb{N}}$ be a sequence running over D and set $\tau_n = \sup_{i \le n} \tau'_i$ for $n \in \mathbb{N}$. Then $\sup_{n \in \mathbb{N}} [\![\sigma \le \tau_n]\!] = \sup_{\tau \in D} [\![\sigma \le \tau]\!] = 1$, as required.

Remark Accordingly we have the result of 627N in a much stronger form, with every d_n equal to 0, and with the sequence $\langle \tau_n \rangle_{n \in \mathbb{N}}$ independent of the process $\boldsymbol{u}$.

627X Basic exercises (a) Suppose that $T = [0, \infty[$ and $\mathfrak{A} = \{0, 1\}$, as in 613W, 615Xf, 616Xa, 617Xb, 618Xa, 622Xd and 626Xa, Let $f : [0, \infty[\to \mathbb{R}$ be a function and $\boldsymbol{u}$ the corresponding process on $\mathcal{T}_f$. (i) Show that $\boldsymbol{u}$ is a supermartingale iff f is non-increasing. (ii) Show that $\boldsymbol{u}$ is a quasimartingale iff it is a strong integrator iff f is of bounded variation.

(b) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ a supermartingale. Write $\hat{\boldsymbol{v}}$ for the fully adapted extension of $\boldsymbol{v}$ to the covered envelope $\hat{\mathcal{S}}$ of $\mathcal{S}$, and $\hat{\mathcal{S}}_f$ for the finitely-covered envelope of $\mathcal{S}$. Show that

(i) if $\boldsymbol{v}$ is $\| \|_1$-bounded, it is an integrator;

(ii) if $\mathcal{S}$ has a greatest element and $\{\mathbb{E}(v_\sigma) : \sigma \in \mathcal{S}\}$ is bounded above, $\boldsymbol{v}$ is $\| \|_1$-bounded:

(iii) if $A \subseteq \mathcal{S}$ is non-empty and downwards-directed and $\{\mathbb{E}(v_\sigma) : \sigma \in A\}$ is bounded above, then (α) $\lim_{\sigma \downarrow A} v_\sigma$ and $1\lim_{\sigma \downarrow A} v_\sigma$ are defined and equal (β) $R_A(\boldsymbol{v})$ is defined and is a supermartingale;

(iv)(α) $\hat{\boldsymbol{v}} \restriction \hat{\mathcal{S}}_f$ is a supermartingale (β) if $\boldsymbol{v}$ is $\| \|_1$-bounded then $\hat{\boldsymbol{v}}$ is $\| \|_1$-bounded (γ) if $\mathcal{S}$ has a greatest element and $\{\mathbb{E}(v_\sigma) : \sigma \in \mathcal{S}\}$ is bounded above, then $\hat{\boldsymbol{v}}$ is a supermartingale;

(v) if $\boldsymbol{v}$ has a previsible variation $\boldsymbol{v}^\#$ (definition: 626J), then $\boldsymbol{v}^\#$ is non-increasing, therefore a supermartingale;

(vi) if $\{\mathbb{E}(v_\sigma) : \sigma \in \mathcal{S}\}$ is bounded above, then $\boldsymbol{v}$ is expressible as the sum of a non-increasing fully adapted process and a virtually local martingale.

(c) Show that sums and scalar multiples of quasimartingales are quasimartingales.

(d) Let $\boldsymbol{v}$ be an L^1-process defined on a sublattice $\mathcal{S}$ of $\mathcal{T}$. (i) Show that $Q_{\mathcal{S}}(|Pd\boldsymbol{v}|) \subseteq Q_{\mathcal{S}}(Pd\boldsymbol{v})$. (ii) Show that the following are equiveridical: (α) $\boldsymbol{v}$ is a quasimartingale; (β) $Q_{\mathcal{S}}(Pd\boldsymbol{v})$ is $\| \|_1$-bounded; (γ) $Q_{\mathcal{S}}(|Pd\boldsymbol{v}|)$ is $\| \|_1$-bounded; (δ) $\{\mathbb{E}(S_I(\mathbf{1}, |Pd\boldsymbol{v}|)) : I \in \mathcal{I}(\mathcal{S})\}$ is bounded. (*Hint*: for $(\alpha) \Rightarrow (\beta)$, recall that weakly bounded sets in Banach spaces are bounded.)

(e) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ with greatest and least elements. Show that a submartingale with domain $\mathcal{S}$ is a quasimartingale.

627 Notes and comments The target of this section is Theorem 627Q. The traditional approach to stochastic integration has been integration with respect to semi-martingales, typically in contexts which ensure that they can be described in terms of local martingales rather than virtually local martingales. so that Protter 05, for instance, uses the word 'semimartingale' for what I call 'integrators', and the phrase 'classical semimartingale' for a sum of a process of bounded variation and a local martingale. Protter's Theorem 47 corresponds to my 627J. The first steps are to prove that a semi-martingale is a local integrator (625E) and that an integrator is a semi-martingale. To show that every local integrator is a semi-martingale we need some more technique, using 627N-627O or 627R.

Clearly supermartingales are going to be like submartingales in many ways. But it does not at all follow that non-negative supermartingales, as in 627D-627E, are going to behave like non-negative submartingales, as in 626M. The idea of quasimartingales is to get back to something symmetric, like integrators and semi-martingales.

The principal results here (627I, 627J, 627L) depend on the theory of convex sets in locally convex linear topological spaces (627F-627G), and our difficulty is that the topology of convergence in measure is not locally convex in the interesting cases. So we have to negotiate carefully to ensure that we have topologically bounded convex sets. If $\boldsymbol{v}$ is an integrator, $Q_{\mathcal{S}}(d\boldsymbol{v})$ is convex, by 627Hb, and its solid hull is topologically bounded by 613B(f-iv). But it does not seem to follow directly that the convex hull of its solid hull will be bounded. So for 627M we need to refer back to the methods used in 621H, and these work best on what I am calling 'strong integrators'. In fact 627L-627M will be used rarely in this volume. But I think we need them to get a proper idea of what an integrator is.

I remarked in the notes to §625 that the law-independence of the class of semi-martingales (625F) was surprising. Using that independence, we have come to a law-independent characterization of semi-martingales as local integrators (627Q). There are no coincidences in mathematics. But there are many deep structures with unexpected outcrops.

Version of 31.12.17

*628 Refining a martingale inequality

I remarked in §621 that the constant 16 in the inequality 621Hf can be reduced to 2 if we are willing to use some rather more advanced measure theory. This treatise is not about finding best constants. But 2 is a much prettier number than 16 and the method I have devised passes through a construction (628C) which may have other uses, as in 628F-628G, so I present it here.

628A Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, and $\mathfrak{A}_0$ a closed subalgebra of $\mathfrak{A}$; write $P_0 : L^1_{\bar{\mu}} \to L^1_{\bar{\mu}}$ for the corresponding conditional expectation operator. Suppose that $v \in L^1_{\bar{\mu}}$. Set $v_0 = P_0 v$. Then there are a probability algebra $(\mathfrak{B}, \bar{\nu})$, closed subalgebras $\mathfrak{B}_0 \subseteq \mathfrak{B}_1 \subseteq \mathfrak{B}$, and a measure-preserving Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ such that $\pi[\mathfrak{A}_0] = \mathfrak{B}_0$ and if $T = T_\pi : L^1_{\bar{\mu}} \to L^1_{\bar{\nu}}$ is the associated embedding (365N[8]) and w_i is the conditional expectation of Tv on $\mathfrak{B}_i$ for both i, then

$$w_0 = Tv_0,$$

$$\begin{aligned}[\![|w_1| = 1]\!] \cap [\![|w_0| < 1]\!] &= [\![|w_1| \ge 1]\!] \cap [\![|w_0| < 1]\!] \\ &\supseteq [\![|Tv| \ge 1]\!] \cap [\![|w_0| < 1]\!].\end{aligned}$$

[8]Formerly 365O.

proof (a) Let $(\mathfrak{A}',\bar{\mu}')$ be a homogeneous probability algebra of infinite Maharam type at least equal to the relative Maharam type of $\mathfrak{A}$ over $\mathfrak{A}_0$ (333Aa). Let $(\mathfrak{C},\bar{\lambda})$ be the probability algebra free product of $(\mathfrak{A}_0,\bar{\mu}\restriction\mathfrak{A}_0)$ and $(\mathfrak{A}',\bar{\mu}')$ (325K). Then there is a measure-preserving homomorphism $\pi_0:\mathfrak{A}\to\mathfrak{C}$ extending the canonical homomorphism $a\mapsto a\otimes 1:\mathfrak{A}_0\to\mathfrak{A}_0\widehat{\otimes}\mathfrak{A}'=\mathfrak{C}$ (333Fa). Set $\mathfrak{C}_0=\pi_0[\mathfrak{A}_0]=\{a\otimes 1:a\in\mathfrak{A}_0\}$; let $T_0:L^1_{\bar{\mu}}\to L^1_{\bar{\lambda}}$ be the embedding associated with π_0, and $Q_0:L^1_{\bar{\lambda}}\to L^1_{\bar{\lambda}}$ the conditional expectation operator corresponding to $\mathfrak{C}_0$. Then $Q_0T_0=T_0P_0$, by 365Qd[9].

(b) Let (Ω,Σ,μ) and (Ω',Σ',μ') be probability spaces with measure algebras isomorphic to $(\mathfrak{A}_0,\bar{\mu}\restriction\mathfrak{A}_0)$ and $(\mathfrak{A}',\bar{\mu}')$ respectively (321J). Let λ be the product measure on $\Omega\times\Omega'$. Then the measure algebra of λ is isomorphic to $(\mathfrak{C},\bar{\lambda})$ (325I), and if $E\in\Sigma$ then $(E\times\Omega')^\bullet$ corresponds to $E^\bullet\otimes 1$, so that $\pi_0\restriction\mathfrak{A}_0:\mathfrak{A}_0\to\mathfrak{C}$ corresponds to the map $E\mapsto E\times\Omega'$, and $\mathfrak{C}_0$ to the σ-algebra $\{E\times\Omega':E\in\Sigma\}$. Accordingly the conditional expectation operator Q_0 can be defined by saying that if $g\in\mathcal{L}^1(\lambda)$ then $Q_0(g^\bullet)=f^\bullet$, where $f(\omega,\tilde{\omega}')=\int g(\omega,\omega')d\omega'$ whenever this is defined (253H), the integral here being with respect to μ'.

(c) Let $g\in\mathcal{L}^1(\lambda)$ be such that $g^\bullet=T_0v$; we may suppose that g is defined everywhere on $\Omega\times\Omega'$ and is $\Sigma\widehat{\otimes}\Sigma'$-measurable (because λ is the completion of its restriction to $\Sigma\widehat{\otimes}\Sigma'$, by 251K). Moreover, adjusting g on a negligible set of the form $E\times\Omega'$ if necessary, we can arrange that $\int g(\omega,\omega')d\omega'$ is defined for every $\omega\in\Omega$ (252B). Set $E=\{\omega:-1<\int g(\omega,\omega')d\omega'<1\}$; by 252P, $E\in\Sigma$.

(d) For $\omega\in E$ set

$$F_\omega=\{\omega':g(\omega,\omega')\le -1\},\quad F'_\omega=\{\omega':g(\omega,\omega')\ge 1\}.$$

Let me explain where I think I am going. I seek to define measurable sets G_ω, $G'_\omega\subseteq\Omega'\times[0,1]$ with the properties that

$$G_\omega\cap G'_\omega=\emptyset,\quad G_\omega\cup G'_\omega\supseteq(F_\omega\cup F'_\omega)\times[0,1],$$

$$\iint_{G_\omega}g(\omega,\omega')d\omega'd\omega''=-(\mu'\times\mu'')(G_\omega),$$

$$\iint_{G'_\omega}g(\omega,\omega')d\omega'd\omega''=(\mu'\times\mu'')(G'_\omega),$$

where μ'' is Lebesgue measure on $[0,1]$ and the integrals $\int\ldots d\omega''$ are taken with respect to μ''; moreover, I wish to do this in such a way that

$$G=\{(\omega,\omega',\omega''):\omega\in E,\ (\omega',\omega'')\in G_\omega\},$$

$$G'=\{(\omega,\omega',\omega''):\omega\in E,\ (\omega',\omega'')\in G'_\omega\}$$

belong to $\Sigma\widehat{\otimes}\Sigma'\widehat{\otimes}\Sigma''$, where Σ'' is the Borel σ-algebra of $[0,1]$.

(e) Set

$$E_0=\{\omega:\omega\in E,\ \mu'F_\omega=\mu'F'_\omega=0\},$$

$$E_1=\{\omega:\omega\in E\setminus E_0,\ \int_{F_\omega\cup F'_\omega}g(\omega,\omega')d\omega'\le-\mu'(F_\omega\cup F'_\omega)\},$$

$$E_2=\{\omega:\omega\in E\setminus E_0,\ \int_{F_\omega\cup F'_\omega}g(\omega,\omega')d\omega'\ge\mu'(F_\omega\cup F'_\omega)\},$$

$$E_3=\{\omega:\omega\in E\setminus E_0,\ -\mu'(F_\omega\cup F'_\omega)<\int_{F_\omega\cup F'_\omega}g(\omega,\omega')d\omega'<\mu'(F_\omega\cup F'_\omega)\}.$$

Because $\omega\mapsto\mu'F_\omega$, $\omega\mapsto\mu'F'_\omega$, $\omega\mapsto\int_{F_\omega}g(\omega,\omega')d\omega'$ and $\omega\mapsto\int_{F'_\omega}g(\omega,\omega')d\omega'$ are Σ-measurable (252P again), (E_0,E_1,E_2,E_3) is a partition of E into members of Σ.

(f)(i) If $\omega\in E_0$, set $G_\omega=F_\omega\times[0,1]$ and $G'_\omega=F'_\omega\times[0,1]$.

(ii) If $\omega\in E_1$, then (because $\int_{\Omega'}g(\omega,\omega')d\omega'>-1$) there is exactly one $\alpha_\omega\in[0,1[$ such that

$$(1-\alpha_\omega)\int_{F_\omega\cup F'_\omega}g(\omega,\omega')d\omega'+\alpha_\omega\int_{\Omega'}g(\omega,\omega')d\omega'=-(1-\alpha_\omega)\mu'(F_\omega\cup F'_\omega)-\alpha_\omega;$$

[9]Formerly 365Rd.

set

$$G_\omega = ((F_\omega \cup F'_\omega) \times [0,1]) \cup (\Omega' \times [0,\alpha_\omega]), \quad G'_\omega = \emptyset.$$

Then

$$\begin{aligned}\iint_{G_\omega} g(\omega,\omega')d\omega' d\omega'' &= (1-\alpha_\omega)\int_{F_\omega \cup F'_\omega} g(\omega,\omega')d\omega' + \alpha_\omega \int_{\Omega'} g(\omega,\omega')d\omega' \\ &= -(1-\alpha_\omega)\mu'(F_\omega \cup F'_\omega) - \alpha_\omega = -(\mu' \times \mu'')(G_\omega),\end{aligned}$$

$$\iint_{G'_\omega} g(\omega,\omega')d\omega' d\omega'' = 0 = (\mu' \times \mu'')(G'_\omega).$$

Because $\omega \mapsto \alpha_\omega$ is expressible as a rational combination of measurable functions, it is Σ-measurable and

$$\{(\omega,\omega',\omega'') : \omega \in E_1,\ (\omega',\omega'') \in G_\omega\}$$

belongs to $\Sigma\widehat{\otimes}\Sigma'\widehat{\otimes}\Sigma''$, while of course

$$\{(\omega,\omega',\omega'') : \omega \in E_1,\ (\omega',\omega'') \in G'_\omega\} = \emptyset$$

also does.

(iii) Similarly, if $\omega \in E_2$, then there is exactly one $\alpha_\omega \in \,]0,1]$ such that

$$(1-\alpha_\omega)\int_{F_\omega \cup F'_\omega} g(\omega,\omega')d\omega' + \alpha_\omega \int_{\Omega'} g(\omega,\omega')d\omega' = (1-\alpha_\omega)\mu'(F_\omega \cup F'_\omega) + \alpha_\omega,$$

and we can set

$$G_\omega = \emptyset, \quad G'_\omega = ((F_\omega \cup F'_\omega) \times [0,1]) \cup (\Omega' \times [0,\alpha_\omega]).$$

(iv) Now consider the case in which $\omega \in E_3$. In this case, because

$$\int_{F_\omega} g(\omega,\omega')d\omega' \le -\mu' F_\omega, \quad \int_{F_\omega} g(\omega,\omega')d\omega' + \int_{F'_\omega} g(\omega,\omega')d\omega' > -\mu' F_\omega - \mu' F'_\omega,$$

there is a unique $\alpha_\omega \in [0,1[$ such that

$$\int_{F_\omega} g(\omega,\omega')d\omega' + \alpha_\omega \int_{F'_\omega} g(\omega,\omega')d\omega' = -\mu' F_\omega - \alpha_\omega \mu' F'_\omega.$$

Next, because

$$(1-\alpha_\omega)\int_{F'_\omega} g(\omega,\omega')d\omega' \ge (1-\alpha_\omega)\mu' F'_\omega,$$

$$\int_{F_\omega} g(\omega,\omega')d\omega' + \int_{F'_\omega} g(\omega,\omega')d\omega' < \mu' F_\omega + \mu' F'_\omega,$$

there is a unique $\beta_\omega \in \,]0,1]$ such that

$$\begin{aligned}(1-\beta_\omega)\int_{F_\omega} g(\omega,\omega')d\omega' + (1-\alpha_\omega\beta_\omega)\int_{F'_\omega} g(\omega,\omega')d\omega' \\ = (1-\beta_\omega)\mu' F_\omega + (1-\alpha_\omega\beta_\omega)\mu' F'_\omega.\end{aligned}$$

So if we set

$$G_\omega = (F_\omega \times [0,\beta_\omega]) \cup (F'_\omega \times [0,\alpha_\omega\beta_\omega]),$$

$$G'_\omega = (F_\omega \times \,]\beta_\omega,1]) \cup (F'_\omega \times \,]\alpha_\omega\beta_\omega,1]),$$

we shall have $G_\omega \cap G'_\omega = \emptyset$, $G_\omega \cup G'_\omega = (F_\omega \cup F'_\omega) \times [0,1]$,

$$\begin{aligned}\iint_{G_\omega} g(\omega,\omega')d\omega' d\omega'' &= \beta_\omega \int_{F_\omega} g(\omega,\omega')d\omega' + \alpha_\omega\beta_\omega \int_{F'_\omega} g(\omega,\omega')d\omega' \\ &= \beta_\omega\Big(\int_{F_\omega} g(\omega,\omega')d\omega' + \alpha_\omega \int_{F'_\omega} g(\omega,\omega')d\omega'\Big) \\ &= -\beta_\omega(\mu' F_\omega + \alpha_\omega \mu' F'_\omega) = -(\mu' \times \mu'')(G_\omega),\end{aligned}$$

and

$$\iint_{G'_\omega} g(\omega,\omega')d\omega' d\omega'' = (1-\beta_\omega)\int_{F_\omega} g(\omega,\omega')d\omega' + (1-\alpha_\omega\beta_\omega)\int_{F'_\omega} g(\omega,\omega')d\omega'$$
$$= (1-\beta_\omega)\mu' F_\omega + (1-\alpha_\omega\beta_\omega)\mu' F'_\omega = (\mu'\times\mu'')(G'_\omega).$$

Once again, $\omega\mapsto\alpha_\omega$ and $\omega\mapsto\beta_\omega$ are Σ-measurable, so $\{(\omega,\omega',\omega''):\omega\in E_3,\ (\omega',\omega'')\in G_\omega\}$ and $\{(\omega,\omega',\omega''):\omega\in E_3,\ (\omega',\omega'')\in G'_\omega\}$ belong to $\Sigma\widehat{\otimes}\Sigma'\widehat{\otimes}\Sigma''$.

(g) Thus the project set out in (d) has been accomplished. We need one more refinement: define $\tilde G_\omega$ and γ_ω, for $\omega\in E$, by setting

$$\tilde G_\omega = (\Omega'\times\Omega'')\setminus(G_\omega\cup G'_\omega),$$

$$\gamma_\omega = 0 \text{ if } (\mu'\times\mu'')(\tilde G_\omega)=0,$$
$$= \tfrac{1}{(\mu'\times\mu'')(\tilde G_\omega)}\iint_{\tilde G_\omega} g(\omega,\omega')d\omega' d\omega'' \text{ otherwise.}$$

Note that as $-1<g(\omega,\omega')<1$ whenever $(\omega',\omega'')\in\tilde G_\omega$, $\gamma_\omega\in\,]-1,1[$ for every ω; and by the same arguments as in (f), $\tilde G=\{(\omega,\omega',\omega''):\omega\in E,\ (\omega',\omega'')\in\tilde G_\omega\}$ belongs to $\Sigma\widehat{\otimes}\Sigma'\widehat{\otimes}\Sigma''$ and $\omega\mapsto\gamma_\omega$ is Σ-measurable. (Of course $\tilde G=(E\times\Omega'\times\Omega'')\setminus(G\cup G')$.)

(h) Now, in $\Omega\times\Omega'\times\Omega''$, consider the product measure $\nu=\mu\times\mu'\times\mu''$, the σ-algebras $\mathrm{T}=\operatorname{dom}\nu$ and $\mathrm{T}_0=\{E\times\Omega'\times[0,1]:E\in\Sigma\}$, and the σ-algebra T_1 generated by $\mathrm{T}_0\cup\{G,G'\}$. For $\omega\in\Omega$, $\omega'\in\Omega'$ and $\omega''\in[0,1]$ set

$$h(\omega,\omega',\omega'') = g(\omega,\omega'),$$
$$h_0(\omega,\omega',\omega'') = \int g(\omega,\omega')d\omega' = \iint h(\omega,\omega',\omega'')d\omega' d\omega'',$$

$$h_1(\omega,\omega',\omega'') = -1 \text{ if } (\omega,\omega',\omega'')\in G,$$
$$= 1 \text{ if } (\omega,\omega',\omega'')\in G',$$
$$= \gamma_\omega \text{ if } (\omega,\omega,\omega'')\in\tilde G,$$
$$= \int g(\omega,\omega')d\omega' \text{ if } \omega\in\Omega\setminus E.$$

Then h_0 is a conditional expectation of h on T_0, by 253H again. The point is that h_1 is a conditional expectation of h on T_1. **P** Of course h_1 is T_1-measurable because G, G' and $\tilde G$ belong to T_1 and $\mathrm{T}_0\subseteq\mathrm{T}_1$. Now any element of T_1 is of the form $W^*=W\cup W'\cup\tilde W\cup W_0$ where

$$W=(H\times\Omega'\times\Omega'')\cap G,\quad W'=(H'\times\Omega'\times\Omega'')\cap G',$$

$$\tilde W=(\tilde H\times\Omega'\times\Omega'')\cap\tilde G,\quad W_0=H_0\times\Omega'\times\Omega'',$$

H, H', $\tilde H$ and H_0 belong to Σ, and $H_0\cap E=\emptyset$. Now

$$\int_W h\,d\nu = \int_{H\cap E}\iint_{G_\omega} g(\omega,\omega')d\omega' d\omega'' d\omega$$
$$= -\int_{H\cap E}(\mu'\times\mu'')(G_\omega)d\omega = -\nu W = \int_W h_1 d\nu,$$
$$\int_{W'} h\,d\nu = \int_{H\cap E}\iint_{G'_\omega} g(\omega,\omega')d\omega' d\omega'' d\omega$$
$$= \int_{H\cap E}(\mu'\times\mu'')(G'_\omega)d\omega = \nu W' = \int_{W'} h_1 d\nu,$$
$$\int_{\tilde W} h\,d\nu = \int_{H\cap E}\iint_{\tilde G_\omega} g(\omega,\omega')d\omega' d\omega'' d\omega$$
$$= \int_{H\cap E}\gamma_\omega(\mu'\times\mu'')(\tilde G_\omega)d\omega = \int_W h_1 d\nu,$$
$$\int_{W_0} h\,d\nu = \int_{H_0}\int g(\omega,\omega')d\omega' d\omega = \int_{W_0} h_1 d\nu.$$

Adding, $\int_{W^*} h\,d\nu = \int_{W^*} h_1 d\nu$; as W^* is arbitrary, h_1 is a conditional expectation of h on T_1. **Q**

Just because $\mathrm{T}_0 \subseteq \mathrm{T}_1$, it follows at once that h_0 is a conditional expectation of h_1 on T_0.

(i) We are almost home. Let $(\mathfrak{B},\bar\nu)$ be the measure algebra of ν, $\mathfrak{B}_0 = \{W^\bullet : W \in \mathrm{T}_0\}$ and $\mathfrak{B}_1 = \{W^\bullet : W \in \mathrm{T}_1\}$, so that $\mathfrak{B}_0 \subseteq \mathfrak{B}_1$ are closed subalgebras of $\mathfrak{B}$. Set $w = h^\bullet$, $w_0 = h_0^\bullet$ and $w_1 = h_1^\bullet$; then w_0, w_1 are the conditional expectations of w on $\mathfrak{B}_0$, $\mathfrak{B}_1$ respectively. Next,

$$[\![|w_0| < 1]\!] = (E\times\Omega'\times\Omega'')^\bullet = (G\cup G'\cup\tilde G)^\bullet,$$

$$[\![|w_1| \ge 1]\!] = \{(\omega,\omega',\omega'') : |h_1(\omega,\omega',\omega'')| \ge 1\}^\bullet,$$

so

$$[\![|w_1| \ge 1]\!] \cap [\![|w_0| < 1]\!] = [\![|w_1| = 1]\!] \cap [\![|w_0| < 1]\!] = (G\cup G')^\bullet.$$

Moreover, if we set

$$F = \{(\omega,\omega',\omega'') : \omega\in E,\ h(\omega,\omega',\omega'') \le -1\},$$

$$F' = \{(\omega,\omega',\omega'') : \omega\in E,\ h(\omega,\omega',\omega'') \ge 1\},$$

then $F\cup F' \subseteq G\cup G'$ because $(F_\omega\cup F'_\omega)\times[0,1] \subseteq G_\omega\cup G'_\omega$ for every $\omega\in E$. So

$$[\![|w| \ge 1]\!] \cap [\![|w_0| < 1]\!] = (F\cup F')^\bullet \subseteq [\![|w_1| = 1]\!] \cap [\![|w_0| < 1]\!].$$

(j) To complete the pattern demanded in the statement of the lemma, I must describe the homomorphism π. Let $\varepsilon : \mathfrak{C} \to \mathfrak{B}$ be the canonical map corresponding to the inverse-measure-preserving function $(\omega,\omega',\omega'') \mapsto (\omega,\omega')$, and $\pi = \varepsilon\pi_0 : \mathfrak{A} \to \mathfrak{B}$, so that π is a measure-preserving Boolean homomorphism; let $T : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$ be the associated embedding, corresponding to the inverse-measure-preserving function $(\omega,\omega',\omega'') \to \omega$. Because $g^\bullet = T_0 v$, $w = h^\bullet = Tv$; and $\pi[\mathfrak{A}_0] = \varepsilon[\mathfrak{C}_0] = \mathfrak{B}_0$, while $w_0 = Tv_0$.

This ends the proof.

628B Lemma Suppose that $(\mathfrak{A},\bar\mu)$ is a probability algebra and $\langle u_i\rangle_{i\le n}$ is a martingale adapted to a non-decreasing finite sequence $\langle\mathfrak{A}_i\rangle_{i\le n}$ of closed subalgebras of $\mathfrak{A}$. Then there are a probability algebra $(\mathfrak{B},\bar\nu)$, closed subalgebras $\mathfrak{B}_0 \subseteq \ldots \subseteq \mathfrak{B}_{2n}$ of $\mathfrak{B}$, a martingale $\langle w_j\rangle_{j\le 2n}$ adapted to $\langle\mathfrak{B}_j\rangle_{j\le 2n}$ and a measure-preserving Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ such that if $T = T_\pi : L^1_{\bar\mu} \to L^1_{\bar\nu}$ is the associated embedding then

$$\pi[\mathfrak{A}_i] \subseteq \mathfrak{B}_{2i},\ w_{2i} = Tu_i \text{ for } i \le n,$$

$$[\![|w_j| = 1]\!] \cap [\![|w_{j-1}| < 1]\!] = [\![|w_j| \ge 1]\!] \cap [\![|w_{j-1}| < 1]\!]$$
$$\supseteq [\![|w_{j+1}| \ge 1]\!] \cap [\![|w_{j-1}| < 1]\!]$$

for odd $j < 2n$.

proof Induce on n. If $n = 0$ the result is trivial (we can take $\mathfrak{B}_0 = \mathfrak{A}_0$, $\mathfrak{B} = \mathfrak{A}$).

(a) For the inductive step to $n+1$, we suppose that we have a probability algebra $(\mathfrak{A}, \bar{\mu})$, closed subalgebras $\mathfrak{A}_0 \subseteq \ldots \subseteq \mathfrak{A}_{n+1}$, and a martingale $\langle u_i \rangle_{i \le n+1}$ adapted to $\langle \mathfrak{A}_i \rangle_{i \le n+1}$. The inductive hypothesis tells us that there are a probability algebra $(\mathfrak{C}, \bar{\lambda})$, closed subalgebras $\mathfrak{C}_0 \subseteq \ldots \subseteq \mathfrak{C}_{2n}$ of $\mathfrak{C}$, a martingale $\langle v_j \rangle_{j \le 2n}$ adapted to $\langle \mathfrak{C}_j \rangle_{j \le 2n}$, and a measure-preserving Boolean homomorphism $\phi : \mathfrak{A} \to \mathfrak{C}$ such that, writing T_ϕ for the associated embedding of $L^0(\mathfrak{A})$ into $L^0(\mathfrak{C})$,

$$\phi[\mathfrak{A}_i] \subseteq \mathfrak{C}_{2i},\ v_{2i} = T_\phi u_i \text{ for } i \le n,$$

$$\begin{aligned}[\![|v_j| = 1]\!] \cap [\![|v_{j-1}| < 1]\!] &= [\![|v_j| \ge 1]\!] \cap [\![|v_{j-1}| < 1]\!] \\ &\supseteq [\![|v_{j+1}| \ge 1]\!] \cap [\![|v_{j-1}| < 1]\!]\end{aligned}$$

for every odd $j < 2n$.

(b) Let $(\mathfrak{D}, \bar{\zeta}, \varepsilon_1, \varepsilon_2)$ be the relative free product of $(\mathfrak{A}, \bar{\mu}, \iota)$ and $(\mathfrak{C}, \bar{\lambda}, \phi)$ over $(\mathfrak{A}_n, \bar{\mu} \restriction \mathfrak{A}_n)$ in the sense of 458N-458O, where $\iota : \mathfrak{A}_n \subsetneqq \mathfrak{A}$ is the identity map; so that $(\mathfrak{D}, \bar{\zeta})$ is a probability algebra, $\varepsilon_1 : \mathfrak{A} \to \mathfrak{D}$ and $\varepsilon_2 : \mathfrak{C} \to \mathfrak{D}$ are measure-preserving homomorphisms, $\varepsilon_1 \restriction \mathfrak{A}_n = \varepsilon_2 \phi \restriction \mathfrak{A}_n$, and $\varepsilon_1[\mathfrak{A}]$, $\varepsilon_2[\mathfrak{C}]$ are relatively independent over

$$\mathfrak{D}' = \varepsilon_1[\mathfrak{A}_n] = \varepsilon_2[\phi[\mathfrak{A}_n]] \subseteq \varepsilon_2[\mathfrak{C}_{2n}].$$

It follows that $T_{\varepsilon_1} u_n$ is the conditional expectation of $T_{\varepsilon_1} u_{n+1}$ on $\varepsilon_2[\mathfrak{C}_{2n}]$. **P** $T_{\varepsilon_1} u_n$ is certainly the conditional expectation of $T_{\varepsilon_1} u_{n+1}$ on $\mathfrak{D}'$, by 365Qd again. By 458Fb or 458M, $T_{\varepsilon_1} u_n$ is the conditional expectation of $T_{\varepsilon_1} u_{n+1}$ on $\varepsilon_2[\mathfrak{C}_{2n}]$, just because $\varepsilon_1[\mathfrak{A}]$ and $\varepsilon_2[\mathfrak{C}_{2n}] \subseteq \varepsilon_2[\mathfrak{C}]$ are relatively independent over $\mathfrak{D}' \subseteq \varepsilon_2[\mathfrak{C}_{2n}]$. **Q**

Note that as ε_1 and $\varepsilon_2 \phi$ agree on $\mathfrak{A}_n$, $T_{\varepsilon_1} u = T_{\varepsilon_2 \phi} u = T_{\varepsilon_2} T_\phi u$ (364Pe) for every $u \in L^0(\mathfrak{A}_n)$; in particular, $T_{\varepsilon_1} u_i = T_{\varepsilon_2} v_{2i}$ for every $i \le n$.

(c) Apply 628A to $(\mathfrak{D}, \bar{\zeta})$, the closed subalgebra $\varepsilon_2[\mathfrak{C}_{2n}]$ of $\mathfrak{D}$ and the element $T_{\varepsilon_1} u_{n+1}$ of $L^1_{\bar{\zeta}}$ to find a probability algebra $(\mathfrak{B}, \bar{\nu})$, a measure-preserving Boolean homomorphism $\psi : \mathfrak{D} \to \mathfrak{B}$, and a closed subalgebra $\mathfrak{B}_{2n+1}$ of $\mathfrak{B}$ such that $\mathfrak{B}_{2n} = \psi[\varepsilon_2[\mathfrak{C}_{2n}]] \subseteq \mathfrak{B}_{2n+1}$ and if $w_{2n+2} = T_\psi T_{\varepsilon_1} u_{n+1}$, w_{2n} is the conditional expectation of w_{2n+2} on $\mathfrak{B}_{2n}$ and w_{2n+1} is the conditional expectation of w_{2n+2} on $\mathfrak{B}_{2n+1}$, then

$$w_{2n} = T_\psi T_{\varepsilon_1} u_n,$$

$$\begin{aligned}[\![|w_{2n+1}| = 1]\!] \cap [\![|w_{2n}| < 1]\!] &= [\![|w_{2n+1}| \ge 1]\!] \cap [\![|w_{2n}| < 1]\!] \\ &\supseteq [\![|w_{2n+2}| \ge 1]\!] \cap [\![|w_{2n}| < 1]\!].\end{aligned}$$

For $j < 2n$, set $\mathfrak{B}_j = \psi\varepsilon_2[\mathfrak{C}_j]$ and $w_j = T_{\psi\varepsilon_2} v_j$. By 365Qd once more, w_j is the conditional expectation of

$$T_{\psi\varepsilon_2} v_{2n} = T_\psi T_{\varepsilon_2} v_{2n} = T_\psi T_{\epsilon_1} u_n = w_{2n}$$

on $\mathfrak{B}_j$. So if we set $\mathfrak{B}_{2n+2} = \mathfrak{B}$, $\langle w_j \rangle_{j \le 2n+2}$ is a martingale adapted to $\langle \mathfrak{B}_j \rangle_{j \le 2n+2}$.

(d) Set $\pi = \psi\varepsilon_1 : \mathfrak{A} \to \mathfrak{B}$. Then

$$\pi[\mathfrak{A}_i] = \psi[\varepsilon_1[\mathfrak{A}_i]] = \psi[\varepsilon_2\phi[\mathfrak{A}_i]] \subseteq \psi\varepsilon_2[\mathfrak{C}_{2i}] = \mathfrak{B}_{2i},$$

$$T_\pi u_i = T_\psi T_{\varepsilon_1} u_i = T_\psi T_{\varepsilon_2} v_{2i} = w_{2i}$$

for $i \le n$. Next, for odd $j < 2n$,

$$\begin{aligned}[\![|w_j| = 1]\!] \cap [\![|w_{j-1}| < 1]\!] &= \psi\varepsilon_2([\![|v_j| = 1]\!] \cap [\![|v_{j-1}| < 1]\!]) \\ &= \psi\varepsilon_2([\![|v_j| \ge 1]\!] \cap [\![|v_{j-1}| < 1]\!]) \\ &= [\![|w_j| \ge 1]\!] \cap [\![|w_{j-1}| < 1]\!],\end{aligned}$$

$$\begin{aligned}[\![|w_j|=1]\!]\cap[\![|w_{j-1}|<1]\!]&\supseteq\psi\varepsilon_2([\![|v_{j+1}|\ge 1]\!]\cap[\![|v_{j-1}|<1]\!])\\&=[\![|w_{j+1}|\ge 1]\!]\cap[\![|w_{j-1}|<1]\!].\end{aligned}$$

At the last step,

$$\pi[\mathfrak{A}_{n+1}]\subseteq\mathfrak{B}=\mathfrak{B}_{2n+2},$$

$$w_{2n+2}=T_\psi T_{\varepsilon_1}u_{n+1}=T_\pi u_{n+1},$$

$$\begin{aligned}[\![|w_{2n+1}|=1]\!]\cap[\![|w_{2n}|<1]\!]&=[\![|w_{2n+1}|\ge 1]\!]\cap[\![|w_{2n}|<1]\!]\\&\supseteq[\![|w_{2n+2}|\ge 1]\!]\cap[\![|w_{2n}|<1]\!]\end{aligned}$$

by the choice of $\mathfrak{B}$, w_{2n}, w_{2n+1} and w_{2n+2}. So $\mathfrak{B}$, $\bar\nu$, $\mathfrak{B}_0,\dots,\mathfrak{B}_{2n+2}$ and π witness that the induction proceeds.

628C Corollary Suppose that $(\mathfrak{A},\bar\mu)$ is a probability algebra and $\langle u_i\rangle_{i\le n}$ is a martingale adapted to a non-decreasing finite sequence $\langle\mathfrak{A}_i\rangle_{i\le n}$ of closed subalgebras of $\mathfrak{A}$. Then there are a probability algebra $(\mathfrak{B},\bar\nu)$, closed subalgebras $\mathfrak{C}_0\subseteq\ldots\subseteq\mathfrak{C}_n$ of $\mathfrak{B}$, a martingale $\langle v_i\rangle_{i\le n}$ adapted to $\langle\mathfrak{C}_i\rangle_{i\le n}$ and a measure-preserving Boolean homomorphism $\pi:\mathfrak{A}\to\mathfrak{B}$ such that

$$\pi[\mathfrak{A}_i]\subseteq\mathfrak{C}_i,\quad\|v_i\|_\infty\le 1,\quad\|v_i\|_1\le\|u_i\|_1$$

for every $i\le n$, and

$$\bar\nu(\sup_{i\le n}[\![v_i\ne T_\pi u_i]\!])\le\|u_n\|_1.$$

proof (a) Let $\mathfrak{B}$, $\bar\nu$, $\mathfrak{B}_0,\dots,\mathfrak{B}_{2n}$, $w_0,\dots,w_{2n}$ and π be as in 628B. For $j\le 2n$ write Q_j for the conditional expectation operator associated with $\mathfrak{B}_j$, and set $c_j=\inf_{k\le j}[\![|w_k|<1]\!]\in\mathfrak{B}_j$; for $1\le j\le 2n$ set

$$\begin{aligned}b_j=c_{j-1}\setminus c_j&=[\![|w_j|\ge 1]\!]\cap\inf_{k<j}[\![|w_k|<1]\!]\\&=0\text{ if }0<j\le 2n\text{ and }j\text{ is even,}\\&\subseteq[\![|w_j|=1]\!]\text{ if }j<2n\text{ is odd.}\end{aligned}$$

Now set

$$\hat w_j=w_j\times\chi c_j+\sum_{k=1}^j w_k\times\chi b_k\in L^0(\mathfrak{B}_j).$$

Then $\langle\hat w_j\rangle_{j\le 2n}$ is a martingale adapted to $\langle\mathfrak{B}_j\rangle_{j\le 2n}$. **P** If $j<2n$,

$$Q_j(\hat w_{j+1}-\hat w_j)=Q_j(w_{j+1}\times\chi c_j-w_j\times\chi c_j)=Q_j(w_{j+1}-w_j)\times\chi c_j=0.\ \mathbf{Q}$$

Observe that because $b_j\subseteq[\![|w_j|=1]\!]$ for $0<j\le 2n$, while $c_j\subseteq[\![|w_j|<1]\!]$ for every j, we have $\|\hat w_j\|_\infty\le 1$ for every j. We also see that $\hat w_j\times\chi c_{2n}=w_j\times\chi c_{2n}$ for every j.

(b) This means that if we set $\mathfrak{C}_i=\mathfrak{B}_{2i}$ and $v_i=\hat w_{2i}$ for $i\le n$, we shall have a martingale $\langle v_i\rangle_{i\le n}$ adapted to $\langle\mathfrak{C}_i\rangle_{i\le n}$ with

$$\pi[\mathfrak{A}_i]\subseteq\mathfrak{C}_i,\quad\|v_i\|_\infty\le 1$$

for each $i\le n$, while

$$\sup_{i\le n}[\![T_\pi u_i\ne v_i]\!]=\sup_{i\le n}[\![w_{2i}\ne\hat w_{2i}]\!]\subseteq 1\setminus c_{2n}=\sup_{j\le 2n}[\![|w_j|\ge 1]\!]$$

has measure at most $\|w_{2n}\|_1=\|u_n\|_1$, by 621E.

(c) Note also that $\|\hat w_j\|_1\le\|w_j\|_1$ for $j\le 2n$. **P** Since $c_j\in\mathfrak{B}_j$ and $b_k\in\mathfrak{B}_k$ for $1\le k\le j$,

$$\mathbb{E}(|\hat{w}_j|) = \mathbb{E}(|w_j| \times \chi c_j) + \sum_{k=1}^{j} \mathbb{E}(|w_k| \times \chi b_k) = \mathbb{E}(|w_j| \times \chi c_j) + \sum_{k=1}^{j} \mathbb{E}(|Q_k w_j| \times \chi b_k)$$
$$\le \mathbb{E}(|w_j| \times \chi c_j) + \sum_{k=1}^{j} \mathbb{E}(Q_k |w_j| \times \chi b_k)$$
$$= \mathbb{E}(|w_j| \times \chi c_j) + \sum_{k=1}^{j} \mathbb{E}(|w_j| \times \chi b_k) \le \mathbb{E}(|w_j|). \text{ }\mathbf{Q}$$

So

$$\|v_i\|_1 = \|\hat{w}_{2i}\|_1 \le \|w_{2i}\|_1 = \|u_i\|_1$$

for every $i \le n$.

628D Proposition Suppose that $(\mathfrak{A}, \bar{\mu})$ is a probability algebra and $\langle v_i \rangle_{i \le n}$ is a martingale adapted to a non-decreasing finite sequence $\langle \mathfrak{A}_i \rangle_{i \le n}$ of closed subalgebras of $\mathfrak{A}$. Let $\langle \alpha_j \rangle_{j \le m}$, $\langle u_{ji} \rangle_{j \le m, i < n}$ be such that

$$\alpha_j \ge 0 \text{ for } j \le m, \quad \textstyle\sum_{j=0}^{m} \alpha_j = 1,$$

$$u_{ji} \in L^0(\mathfrak{A}_i), \quad \|u_{ji}\|_\infty \le 1 \text{ for } i < n,\ j \le m.$$

Set $z = \sum_{j=0}^{m} \alpha_j |\sum_{i=0}^{n-1} u_{ji} \times (v_{i+1} - v_i)|$. Then $\bar{\mu}[\![z > \gamma]\!] \le \frac{2}{\gamma}\|v_n\|_1$ for every $\gamma > 0$.

proof (a) By 628C, applied to the martingale $\langle v_i \rangle_{i \le n}$, there are a probability algebra $(\mathfrak{B}, \bar{\nu})$, closed subalgebras $\mathfrak{B}_0 \subseteq \ldots \subseteq \mathfrak{B}_n$ of $\mathfrak{B}$, a martingale $\langle w_i \rangle_{i \le n}$ adapted to $\langle \mathfrak{B}_i \rangle_{i \le n}$ and a measure-preserving Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ such that

$$\pi[\mathfrak{A}_i] \subseteq \mathfrak{B}_i, \quad \|w_i\|_\infty \le 1$$

for every $i \le n$, and

$$\bar{\nu}(\sup_{i \le n} [\![w_i \ne T_\pi v_i]\!]) \le \|v_n\|_1$$

where $T_\pi : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$ is the f-algebra homomorphism corresponding to π. Now

$$T_\pi z = \textstyle\sum_{j=0}^{m} \alpha_j |\sum_{i=0}^{n-1} T_\pi u_{ji} \times (v_{i+1} - v_i)|$$

and if we set

$$z' = \textstyle\sum_{j=0}^{m} \alpha_j |\sum_{i=0}^{n-1} T_\pi u_{ji} \times (w_{i+1} - w_i)|.$$

then $[\![z' \ne T_\pi z]\!] \subseteq \sup_{i \le n} [\![w_i \ne T_\pi v_i]\!]$ and $\bar{\nu}[\![z' \ne T_\pi z]\!] \le \|v_n\|_1$.

For $j \le m$, set $z'_j = \sum_{i=0}^{n-1} T_\pi u_{ji} \times (w_{i+1} - w_i)$. Because $\|T_\pi u_{ji}\|_\infty = \|u_{ji}\|_\infty \le 1$ and $\pi[\mathfrak{A}_i] \subseteq \mathfrak{B}_i$ so $T_\pi u_{ji} \in L^0(\mathfrak{B}_i)$ for every i, 621F tells us that $\|z'_j\|_2 \le \|w_n\|_2$. As $z' = \sum_{j=0}^{n} \alpha_j |z'_j|$,

$$\|z'\|_2 \le \textstyle\sum_{j=0}^{m} \alpha_j \|w_n\|_2 = \|w_n\|_2 \le \|w_n\|_\infty \|w_n\|_1 \le \|w_n\|_1 = \|v_n\|_1.$$

Now we see that

$$\bar{\mu}[\![z > 1]\!] = \bar{\nu}[\![T_\pi z > 1]\!] \le \bar{\nu}[\![z' > 1]\!] + \bar{\nu}[\![z' \ne T_\pi z]\!] \le \|v_n\|_1 + \|v_n\|_1 = 2\|v_n\|_1.$$

(b) For the general case, observe that

$$\frac{1}{\gamma} z = \textstyle\sum_{j=0}^{m} \alpha_j |\sum_{i=0}^{n-1} u_{ji} \times (\frac{1}{\gamma} v_{i+1} - \frac{1}{\gamma} v_i)|,$$

while $\langle \frac{1}{\gamma} v_i \rangle_{i \le n}$ is a martingale. So (a) tells us that

$$\bar{\nu}[\![z > \gamma]\!] = \bar{\nu}[\![\tfrac{1}{\gamma} z > 1]\!] \le 2\|\tfrac{1}{\gamma} v_n\|_1 = \frac{2}{\gamma}\|v_n\|_1.$$

628E Corollary Let $(\mathfrak{A},\bar{\mu},T,\langle\mathfrak{A}_t\rangle_{t\in T},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ be a stochastic integration structure, $\mathcal{S}$ a non-empty sublattice of $\mathcal{T}$, and $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a martingale. Then

$$\bar{\mu}[\![|z|>\gamma]\!]\le\frac{1}{\gamma}\sup_{\sigma\in\mathcal{S}}\|v_\sigma\|_1$$

whenever $z\in Q_{\mathcal{S}}(\boldsymbol{v})$ and $\gamma>0$.

proof z is expressible as $\sum_{i=0}^{n-1}u_i\times(v_{\tau_{i+1}}-v_{\tau_i})$ where $\tau_0\le\ldots\le\tau_n$ in $\mathcal{S}$ and $u_i\in\mathfrak{A}_{\tau_i}$ for every $i<n$. Now apply 628D to the martingale $\langle v_{\tau_i}\rangle_{i\le n}$ adapted to $\langle\mathfrak{A}_{\tau_i}\rangle_{i\le n}$.

628F An argument along the same lines as that in 628D gives a similar result for quadratic variations.

Proposition Suppose that $(\mathfrak{A},\bar{\mu})$ is a probability algebra and $\langle u_i\rangle_{i\le n}$ is a martingale adapted to a non-decreasing finite sequence $\langle\mathfrak{A}_i\rangle_{i\le n}$ of closed subalgebras of $\mathfrak{A}$. Set $u^*=\sum_{i=0}^{n-1}(u_{i+1}-u_i)^2$. Then $\bar{\mu}[\![u^*>\gamma^2]\!]\le\frac{2}{\gamma}\|u_n\|_1$ for every $\gamma>0$.

proof (a) If $\|u_n\|_\infty\le1$, then $\|u^*\|_1\le\|u_n\|_2^2$. **P** This is a greatly simplified version of 624G. For a direct argument, note that every u_i is square-integrable and $(u_{i+1}-u_i)^2=u_{i+1}^2-u_i^2-2(u_{i+1}-u_i)\times u_i$; take the expectation of both sides and sum over i. **Q**

(b) Take $\mathfrak{B}$, $\bar{\nu}$, $\mathfrak{C}_0,\ldots,\mathfrak{C}_n$, $\langle v_i\rangle_{i\le n}$, $\pi:\mathfrak{A}\to\mathfrak{B}$ and $T_\pi:L^0(\mathfrak{A})\to L^0(\mathfrak{B})$ as in 628C. Setting $v^*=\sum_{i=0}^{n-1}(v_{i+1}-v_i)^2$, we see that

$$\bar{\nu}[\![v^*\ne T_\pi u^*]\!]\le\bar{\nu}(\sup_{i\le n}[\![v_i\ne T_\pi]\!])\le\|u_n\|_1$$

because T_π is an f-algebra homomorphism. Now

$$\bar{\nu}[\![v^*>1]\!]\le\|v^*\|_1\le\|v_n\|^2\le\|v_n\|_1\le\|u_n\|_1.$$

Accordingly

$$\bar{\mu}[\![u^*>1]\!]=\bar{\nu}[\![T_\pi u^*>1]\!]\le\bar{\nu}[\![v^*>1]\!]+\bar{\nu}[\![v^*\ne T_\pi u^*]\!]\le2\|u_n\|_1.$$

(c) This deals with the case $\gamma=1$. For the general case, look at the martingale $\langle\hat{u}_i\rangle_{i\le n}$ where $\hat{u}_i=\frac{1}{\gamma}u_i$ for $i\le n$. Setting

$$\hat{u}^*=\sum_{i=0}^{n-1}(\hat{u}_{i+1}-\hat{u}_i)^2=\frac{1}{\gamma^2}u^*,$$

we have

$$\bar{\mu}[\![u^*>\gamma^2]\!]=\bar{\mu}[\![\hat{u}^*>1]\!]\le2\|\hat{u}_n\|_1=\frac{2}{\gamma}\|u_n\|_1,$$

as claimed.

628G Proposition Let $(\mathfrak{A},\bar{\mu},T,\langle\mathfrak{A}_t\rangle_{t\in T},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ be a stochastic integration structure, $\mathcal{S}$ a sublattice of $\mathcal{T}$, and $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a martingale. Let $\boldsymbol{v}^*=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ be its quadratic variation. Then

$$\bar{\mu}[\![v_\tau^*>\gamma^2]\!]\le\frac{2}{\gamma}\|v_\tau\|_1$$

whenever $\gamma>0$ and $\tau\in\mathcal{S}$.

proof If I belongs to $\mathcal{I}(\mathcal{S}\wedge\tau)$, the set of finite sublattices of $\mathcal{S}$ bounded above by τ, and $(\sigma_0,\ldots,\sigma_n)$ linearly generates the I-cells, then

$$\bar{\mu}[\![S_I(\mathbf{1},(d\boldsymbol{v})^2))>\gamma^2]\!]=\bar{\mu}[\![\sum_{i=0}^{n-1}(v_{\sigma_{i+1}}-v_{\sigma_i})^2>\gamma^2]\!]\le\frac{2}{\gamma}\|v_{\sigma_n}\|$$

(by 612F applied to the martingale $\langle v_{\sigma_i}\rangle_{i\le n}$ adapted to $\langle\mathfrak{A}_{\sigma_i}\rangle_{i\le n}$)

$$\le\frac{2}{\gamma}\|v_\tau\|_1.$$

Now $\{u : \bar{\mu}[\![u > \gamma^2]\!] \le \frac{2}{\gamma}\|v_\tau\|_1\}$ is closed for the topology of convergence in measure, by 613Bo, and

$$v_\tau^* = \int_{\mathcal{S}\wedge\tau}(d\boldsymbol{v})^2 = \lim_{I\uparrow\mathcal{I}(\mathcal{S}\wedge\tau)} S_I(\mathbf{1},(d\boldsymbol{v})^2),$$

so we have $\bar{\mu}[\![v_\tau^* > \gamma^2]\!] \le \frac{2}{\gamma}\|v_\tau\|_1$.

628X Basic exercises (a) Suppose that $\langle u_i\rangle_{i\le n}$ is a martingale, and $\alpha < \beta$ in $\mathbb{R}$. Show that there is a martingale $\langle w_i\rangle_{i\le 2n}$ (possibly on a different probability algebra) such that $(w_0, w_2, w_4, \ldots, w_{2n})$ has the same joint distribution as $(u_0, \ldots, u_n)$ (definition: 364Yo, 653B) and

$$\begin{aligned}[\![\alpha < w_{j-1} < \beta]\!] \setminus [\![\alpha < w_j < \beta]\!] &= [\![\alpha < w_{j-1} < \beta]\!] \cap ([\![w_j = \alpha]\!] \cup [\![w_j = \beta]\!]) \\ &\supseteq [\![\alpha < w_{j-1} < \beta]\!] \setminus [\![\alpha < w_{j+1} < \beta]\!]\end{aligned}$$

for odd $j < 2n$.

(b) Let $(\mathfrak{A}, \bar{\mu}, T, \langle\mathfrak{A}_t\rangle_{t\in T}, \mathcal{T}, \langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ be a stochastic integration structure, $\mathcal{S}$ a non-empty sublattice of $\mathcal{T}$, and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a virtually local martingale. Show that $\bar{\mu}[\![|z| > \gamma]\!] \le \frac{1}{\gamma}\sup_{\sigma\in\mathcal{S}}\|v_\sigma\|_1$ whenever $\gamma > 0$ and $z \in Q_{\mathcal{S}}(\boldsymbol{v})$.

628 Notes and comments Both 628C and 621I are addressed to the same question. We have a martingale $\langle u_i\rangle_{i\le n}$ and would very much rather it was $\|\;\|_\infty$-bounded, with $-\chi 1 \le u_i \le \chi 1$ for every i. The natural approach is to stop it as soon as it leaves the interval $]-1,1[$. In 621I this has been done, and we are looking at the stopped martingale. Here a crude, if complex, inequality is enough for the application in 623O. In 622G, however, we need something more like 621Hf or 628C. For the latter, I offer a method of approximating the given martingale by a martingale which really never leaves the interval $[-1,1]$. I have no application in mind for 628F-628G, but I include them as a further motive for mastering the technique of 628A-628C.

The proof I give of 628C depends on 628B, which is a kind of interpolation theorem. Given a martingale $\langle u_i\rangle_{i\le n}$ and the interval $]-1,1[$, we can interpolate terms to convert $\langle u_i\rangle_{i\le n}$ into the even terms of a martingale $\langle w_j\rangle_{j\le 2n}$ in which the first exit from this interval takes one of the end-point values. We shall now have a stopping time τ such that the martingale $\langle w_{\tau\wedge j}\rangle_{j\le 2n}$ either starts outside $]-1,1[$ or runs to full time inside $]-1,1[$ or stops at ± 1 precisely, which is something we expect of a continuous martingale but not of a discrete martingale. To achieve this, of course, we have to enlarge our probability algebra. You will see that the proof I give of 628B depends on the one-step case 628A, and that here I abandon the abstract formulation in terms of probability algebras and move to ordinary probability spaces, for which the key calculations in part (f) of the proof of 628A are elementary and reasonably natural.

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