Chapter 61

The Riemann-sum integral

I begin with an attempt to give a coherent and complete description of the principal form of stochastic integration which will be investigated in this volume.

As elsewhere in probability theory, it is customary to set this material out in terms of ordinary random variables, that is, measurable functions defined on probability spaces. We find immediately, however, that while integrands and integrators may well present themselves most naturally in this form, the integrals we construct are defined, in the cases for which this theory has been developed, in terms of convergence in $\| \|_1$ or $\| \|_2$ or in measure, and therefore correspond not to explicit functions, but to equivalence classes of functions. Moreover, integrands and integrators can be changed on negligible sets without affecting the values of the corresponding integrals. I believe that the theory becomes clearer and cleaner if we move directly to operations on evolving families in L^0. While this demands an initial investment by the reader in a more abstract framework for the ideas of elementary probability theory, the translation is not difficult, and a full exposition can be found in Chapter 36.

Again, stochastic processes are usually expressed as families $\langle X_t\rangle_{t\in T}$ of random variables, indexed by a set T of 'times'. There are very good reasons for this. However, to describe the stochastic integral in reasonable generality we need, as a first step, to discuss the random variable X_τ for a stopping time τ. The measure theory to make this possible (the notion of 'progressively measurable' process) is well understood and has been described in §455. When we come, following my principle above, to look at $\langle X_t^\bullet\rangle_{t\in T}$, we find that we can have $X_t^\bullet = Y_t^\bullet$, that is, $X_t =_{\text{a.e.}} Y_t$, for every t, while $X_\tau^\bullet \ne Y_\tau^\bullet$. This is just a nuisance. For our purposes here, it makes better sense to start from a family $\langle u_\tau\rangle_{\tau\in\mathcal{S}}$ where $\mathcal{S}$ is a set of stopping times and $u_\tau \in L^0$ for every $\tau\in\mathcal{S}$. The construction of such families from processes $\langle X_t\rangle_{t\in T}$ is important and interesting, but has nothing to do with the very substantial difficulties of the basic theory of stochastic integration.

Of course I now have to look at filtrations and stopping times, and these too are not best described in terms of σ-algebras of sets and real-valued functions. In the formulation I wish to use here, we don't even have a probability space for the functions to be defined on. Instead of thinking of a filtration as a family $\langle \Sigma_t\rangle_{t\in T}$ of σ-subalgebras of the domain Σ of a probability measure μ, I look at the corresponding family of subalgebras of the measure algebra $\mathfrak{A}$ of μ. This is easy (at least, if you have read Chapter 32; and this is my last apology for insisting that you know something of Volume 3). A stopping time τ now becomes defined in terms of elements $[\![\tau > t]\!] \in \mathfrak{A}$, 'the region where $\tau > t$'. We need to develop a theory of regions $[\![\sigma < \tau]\!]$, $[\![\sigma = \tau]\!]$ in $\mathfrak{A}$, and subalgebras $\mathfrak{A}_\tau$ of $\mathfrak{A}$, for stopping times σ, τ; and now the processes $\langle u_\tau\rangle_{\tau\in\mathcal{S}}$ we work with must be such that '$u_\sigma = u_\tau$ whenever $\sigma = \tau$', that is, $[\![\sigma = \tau]\!] \subseteq [\![u_\sigma = u_\tau]\!]$. Setting up these structures takes the greater part of §§611-612, which come to about a quarter of the chapter. It happens that nearly everything in these two sections can be done without mentioning 'measure' at all.

I say again that none of this is difficult, but it does take quite a long time; there are some new kinds of algebra to get a solid basis in, particularly the theory of stopping-time intervals (611E, 611J-611K) and fully adapted processes (612D). With this established, however, we are within reach of a direct definition of a stochastic integral as a limit of Riemann sums (§613). As long as we do not enquire about when the integral is actually defined, this is very straightforward and can be done in great generality. The next three sections are devoted to finding the basic cases of processes $\boldsymbol{u}$, $\boldsymbol{v}$ for which we shall have a well-defined integral $\int \boldsymbol{u}\,d\boldsymbol{v}$. Concerning $\boldsymbol{u}$, we have 'simple' and 'moderately oscillatory' processes (612J, 615E). Concerning $\boldsymbol{v}$, we have the concept of 'integrator' (616Fc), which is well adapted to the basic theorem 616K, but is otherwise obscure. It is easy enough to find a definition of 'bounded variation' for stochastic processes (614J) and to show that processes of bounded variation are integrators (616R), but this is not what the stochastic integral is for; in this case we have much more direct methods available.

Now we are ready, at least in a formal sense, for some proper stochastic calculus in §§617 and 619. Here I set out useful general manipulations. Some of them reproduce patterns familiar from the ordinary Riemann integral (616J), but others are radically different (617I, 619C). On the way to the latter ('Itô's formula') we need to understand 'jump-free' processes, corresponding to processes with continuous sample paths (§618).

The theory here involves a large number of constructions. Many of these have no short descriptions in terms of the concepts developed in Volumes 1-4, and correspondingly require new terminology and notation. I have tried to arrange the material in such a way that, within any individual section, substantial parts of the basic framework can be taken to be constant. From §614 on, these are indicated in introductory paragraphs headed 'Notation'. These paragraphs are highly repetitive. But until you are very familiar with my language, it is likely that opening at a random page, and scanning for the next 'Theorem', will lead you to something totally mysterious. Sometimes a check in the index for terminology will help. But sometimes there will be a baffling symbol, and then it will be worth while turning to the beginning of the section to see if the symbol appears there. It seems to me that while this expands the volume by several pages in total, it is kinder than referring you each time to a complete list of the terminological quirks of this presentation.

Version of 11.12.17/16.1.25

611 Stopping times

The first step is to describe the structures within which the work of this volume will proceed. While everything really important will have to be based on probability algebras, I start with ideas which can be applied to arbitrary Dedekind complete Boolean algebras. This section introduces filtrations of subalgebras, the lattice of stopping times, the algebras associated with stopping times, stopping-time intervals and covered envelopes.

611A Filtrations Throughout this volume, $\mathfrak{A}$ will denote a Dedekind complete Boolean algebra, with Boolean operations $\triangle$, $\cap$, $\cup$ and $\setminus$, zero 0 and multiplicative identity 1.

(a) Let T be a non-empty totally ordered set. A **filtration of order-closed subalgebras of** $\mathfrak{A}$ will be a non-decreasing family $\langle\mathfrak{A}_t\rangle_{t\in T}$ of order-closed subalgebras of $\mathfrak{A}$.

(b)(i) A **stopping time** τ **adapted to** $\langle\mathfrak{A}_t\rangle_{t\in T}$ is a family $\langle[\![\tau>t]\!]\rangle_{t\in T}$ such that

$[\![\tau>t]\!]\in\mathfrak{A}_t$ for every $t\in T$,

if $s\le t$ in T then $[\![\tau>t]\!]\subseteq[\![\tau>s]\!]$,

if $t\in T$ is not isolated on the right then $[\![\tau>t]\!]=\sup_{s>t}[\![\tau>s]\!]$.

(ii) It will be worth checking each concept against the constant stopping times, where for $t\in T$ the **constant stopping time at** t, $\check{t}$, is given by setting

$$\begin{aligned}[\![\check{t}>s]\!]&=1 \text{ if } s<t,\\ &=0 \text{ if } s\ge t.\end{aligned}$$

(iii) I will say that a stopping time τ is

— **finite-valued** if $\inf_{t\in T}[\![\tau>t]\!]=0$,

— **bounded** if there is a $t\in T$ such that $[\![\tau>t]\!]=0$.

Constant stopping times are bounded, and bounded stopping times are finite-valued.

(iv) I will write $\mathcal{T}$ for the set of stopping times adapted to $\langle\mathfrak{A}_t\rangle_{t\in T}$, $\mathcal{T}_f\subseteq\mathcal{T}$ for the set of finite-valued stopping times, and $\mathcal{T}_b\subseteq\mathcal{T}_f$ for the set of bounded stopping times.

(c) It is convenient to think of a stopping time $\tau\in\mathcal{T}$ as the element $\langle[\![\tau>t]\!]\rangle_{t\in T}$ of the simple product algebra $\prod_{t\in T}\mathfrak{A}_t$.

611B The partial ordering of stopping times If σ, $\tau\in\mathcal{T}$, say that $\sigma\le\tau$ if $[\![\sigma>t]\!]\subseteq[\![\tau>t]\!]$ for every $t\in T$, that is, $\sigma\subseteq\tau$ in $\prod_{t\in T}\mathfrak{A}_t$. This defines a partial order on $\mathcal{T}$.

611C Proposition (a) $\mathcal{T}$ is a Dedekind complete distributive lattice. Consequently any finite subset of $\mathcal{T}$ is included in a finite sublattice of $\mathcal{T}$.

(b) If $C \subseteq \mathcal{T}$ is non-empty, then $\sup C$ is defined by saying that

$$[\![\sup C > t]\!] = \sup_{\tau\in C} [\![\tau > t]\!]$$

for every $t \in T$, that is, the supremum of C in $\mathcal{T}$ is the same as the supremum of C in $\prod_{t\in T} \mathfrak{A}_t$.

(c) If σ, $\tau \in \mathcal{T}$, then $\sigma \wedge \tau$ is defined by saying that

$$[\![\sigma \wedge \tau > t]\!] = [\![\sigma > t]\!] \cap [\![\tau > t]\!]$$

for every $t \in T$, that is, $\sigma \wedge \tau$ in $\mathcal{T}$ corresponds to $\sigma \cap \tau$ in $\prod_{t\in T} \mathfrak{A}_t$.

(d) If C, $C' \subseteq \mathcal{T}$ are non-empty, then $\sup C \wedge \sup C' = \sup\{\sigma \wedge \sigma' : \sigma \in C,\, \sigma' \in C'\}$.

(e) Writing $\check{t}$ for the constant stopping time at t, the map $t \mapsto \check{t} : T \to \mathcal{T}$ is an order-continuous lattice homomorphism, which is injective if $\mathfrak{A} \neq \{0\}$.

(f) $\mathcal{T}$ has greatest and least elements defined by saying that

$$[\![\max \mathcal{T} > t]\!] = 1, \quad [\![\min \mathcal{T} > t]\!] = 0$$

for every $t \in T$, that is, they correspond to the greatest and least elements 1 and 0 of $\prod_{t\in T} \mathfrak{A}_t$. If T has a least element $\min T$, then $\min \mathcal{T}$ is the constant stopping time at $\min T$.

(g) $\mathcal{T}_f$ and $\mathcal{T}_b$ are ideals[1] in $\mathcal{T}$.

(h) The function $\sigma \mapsto \sigma \wedge \tau : \mathcal{T} \to \mathcal{T}$ is order-continuous for every $\tau \in \mathcal{T}$.

Remark If $A \subseteq \mathcal{T}$ and $\tau \in \mathcal{T}$, I will write $A \vee \tau$ for $\{\sigma \vee \tau : \sigma \in A\}$ and $A \wedge \tau$ for $\{\sigma \wedge \tau : \sigma \in A\}$. Note that if $\mathcal{S}$ is a sublattice of $\mathcal{T}$ and $\tau \in \mathcal{S}$, then

$$\mathcal{S} \vee \tau = \{\sigma : \sigma \in \mathcal{S},\, \tau \le \sigma\}, \quad \mathcal{S} \wedge \tau = \{\sigma : \sigma \in \mathcal{S},\, \sigma \le \tau\}.$$

So if $\mathcal{S}$ is a sublattice of $\mathcal{T}$, τ, $\tau' \in \mathcal{S}$ and $\tau \le \tau'$,

$$\mathcal{S} \cap [\tau, \tau'] = \{\sigma : \sigma \in \mathcal{S},\, \tau \le \sigma \le \tau'\} = \{\sigma : \sigma \in \mathcal{S} \vee \tau,\, \sigma \le \tau'\} = (\mathcal{S} \vee \tau) \wedge \tau'$$

because $\mathcal{S} \vee \tau = \{\sigma : \sigma \in \mathcal{S},\, \tau \le \sigma\}$ is a sublattice of $\mathcal{T}$.

611D The region where $\sigma < \tau$ If σ, $\tau \in \mathcal{T}$ set

$$[\![\sigma < \tau]\!] = \sup_{t\in T}([\![\tau > t]\!] \setminus [\![\sigma > t]\!]),$$

$$[\![\sigma \le \tau]\!] = 1 \setminus [\![\tau < \sigma]\!] = \inf_{t\in T}([\![\tau > t]\!] \cup (1 \setminus [\![\sigma > t]\!])),$$

$$[\![\sigma = \tau]\!] = [\![\sigma \le \tau]\!] \cap [\![\tau \le \sigma]\!] = 1 \setminus \sup_{t\in T}([\![\sigma > t]\!] \triangle [\![\tau > t]\!]).$$

611E Theorem (a) Let σ, $\tau \in \mathcal{T}$.

(i)(α) $([\![\sigma < \tau]\!], [\![\sigma = \tau]\!], [\![\tau < \sigma]\!])$ is a partition of unity in $\mathfrak{A}$.

(β) $[\![\sigma > t]\!] \cap [\![\sigma = \tau]\!] = [\![\tau > t]\!] \cap [\![\sigma = \tau]\!]$ for every $t \in T$.

(γ) $[\![\sigma < \tau]\!] = 0$ iff $[\![\tau \le \sigma]\!] = 1$ iff $\tau \le \sigma$; $[\![\sigma = \tau]\!] = 1$ iff $\sigma = \tau$.

(δ) Writing $\check{t}$ for the constant stopping time at t, $[\![\check{t} < \tau]\!] = [\![\tau > t]\!]$ for every $t \in T$.

(ϵ) $[\![\min \mathcal{T} < \max \mathcal{T}]\!] = 1$.

(ζ) If $s < t$ in T, then $[\![\check{s} < \check{t}]\!] = 1$; $[\![\check{s} < \max \mathcal{T}]\!] = 1$ for every $s \in T$.

(ii)(α) $[\![\sigma < \tau]\!] = [\![\sigma \wedge \tau < \tau]\!] = [\![\sigma < \sigma \vee \tau]\!]$.

(β) $[\![\sigma \le \tau]\!] = [\![\sigma = \sigma \wedge \tau]\!] = [\![\tau = \sigma \vee \tau]\!]$.

(γ) $[\![\sigma \wedge \tau = \sigma]\!] \cup [\![\sigma \wedge \tau = \tau]\!] = [\![\sigma \vee \tau = \sigma]\!] \cup [\![\sigma \vee \tau = \tau]\!] = 1$.

(b) If $\sigma \in \mathcal{T}$ and $C \subseteq \mathcal{T}$ is non-empty then $[\![\sigma < \sup C]\!] = \sup_{\tau\in C} [\![\sigma < \tau]\!]$ and $[\![\sup C \le \sigma]\!] = \inf_{\tau\in C} [\![\tau \le \sigma]\!]$.

(c) Let σ, τ, $\upsilon \in \mathcal{T}$.

(i)(α) $[\![\sigma \wedge \tau < \upsilon]\!] = [\![\sigma < \upsilon]\!] \cup [\![\tau < \upsilon]\!]$, $[\![\upsilon \le \sigma \wedge \tau]\!] = [\![\upsilon \le \sigma]\!] \cap [\![\upsilon \le \tau]\!]$.

(β) $[\![\upsilon < \sigma \wedge \tau]\!] = [\![\upsilon < \sigma]\!] \cap [\![\upsilon < \tau]\!]$, $[\![\sigma \wedge \tau \le \upsilon]\!] = [\![\sigma \le \upsilon]\!] \cup [\![\tau \le \upsilon]\!]$.

(ii)(α) $[\![\sigma \vee \tau < \upsilon]\!] = [\![\sigma < \upsilon]\!] \cap [\![\tau < \upsilon]\!]$, $[\![\upsilon \le \sigma \vee \tau]\!] = [\![\upsilon \le \sigma]\!] \cup [\![\upsilon \le \tau]\!]$.

---

[1] If P is a lattice, an **ideal** of P is a set $Q \subseteq P$ such that $p \vee q \in Q$ for all p, $q \in Q$ and $p \in Q$ whenever $q \in Q$ and $p \le q$ in P. In this context I do not insist that Q should be non-empty.

(β) $[\![\upsilon < \sigma \vee \tau]\!] = [\![\upsilon < \sigma]\!] \cup [\![\upsilon < \tau]\!]$, $[\![\sigma \vee \tau \le \upsilon]\!] = [\![\sigma \le \upsilon]\!] \cap [\![\tau \le \upsilon]\!]$.

(iii)(α) $[\![\sigma < \upsilon]\!] \subseteq [\![\sigma < \tau]\!] \cup [\![\sigma \vee \tau < \upsilon]\!] \subseteq [\![\sigma < \tau]\!] \cup [\![\tau < \upsilon]\!]$.

(β) $[\![\sigma \le \upsilon]\!] \subseteq [\![\sigma \le \tau]\!] \cup [\![\tau < \upsilon]\!]$.

(γ) $[\![\sigma < \upsilon]\!] \cap [\![\upsilon \le \tau]\!] \subseteq [\![\sigma < \tau]\!]$, $[\![\sigma \le \upsilon]\!] \cap [\![\upsilon < \tau]\!] \subseteq [\![\sigma < \tau]\!]$.

(iv)(α) $[\![\sigma \le \tau]\!] \cap [\![\tau \le \upsilon]\!] \subseteq [\![\sigma \le \upsilon]\!]$.

(β) $[\![\sigma \le \tau]\!] \cap [\![\tau < \upsilon]\!] \subseteq [\![\sigma < \upsilon]\!]$.

(γ) $[\![\sigma = \tau]\!] \cap [\![\tau = \upsilon]\!] = [\![\sigma = \tau]\!] \cap [\![\sigma = \upsilon]\!] \subseteq [\![\sigma = \upsilon]\!]$.

(v)(α) $[\![\sigma \wedge \upsilon = \tau \wedge \upsilon]\!] \supseteq [\![\sigma = \tau]\!]$.

(β) $[\![\sigma \vee \upsilon = \tau \vee \upsilon]\!] \supseteq [\![\sigma = \tau]\!]$.

(vi) If $\sigma \le \tau \le \upsilon$, then $[\![\sigma < \upsilon]\!] = [\![\sigma < \tau]\!] \cup [\![\tau < \upsilon]\!]$, $[\![\sigma = \upsilon]\!] = [\![\sigma = \tau]\!] \cap [\![\tau = \upsilon]\!]$.

(d) If $\tau_0 \le \ldots \le \tau_n$ in $\mathcal{T}$ and $\sigma \in \mathcal{T}$, then

$$([\![\sigma < \tau_0]\!], [\![\tau_0 \le \sigma]\!] \cap [\![\sigma < \tau_1]\!], \ldots, [\![\tau_{n-1} \le \sigma]\!] \cap [\![\sigma < \tau_n]\!], [\![\tau_n \le \sigma]\!])$$

is a partition of unity in $\mathfrak{A}$.

611F Infima in $\mathcal{T}$: Proposition Let $A \subseteq \mathcal{T}$ be a non-empty set such that

$$\sup_{s>t} \inf_{\sigma\in A} [\![\sigma > s]\!]$$

belongs to $\mathfrak{A}_t$ whenever $t \in T$ is not isolated on the right.

(a)

$$\begin{aligned}[\![\inf A > t]\!] &= \inf_{\sigma\in A} [\![\sigma > t]\!] \text{ if } t \in T \text{ is isolated on the right} \\ &= \sup_{s>t} \inf_{\sigma\in A} [\![\sigma > s]\!] \text{ for other } t \in T.\end{aligned}$$

(b) $[\![\inf A < \tau]\!] = \sup_{\sigma\in A} [\![\sigma < \tau]\!]$ for every $\tau \in \mathcal{T}$.

611G The algebra defined by a stopping time: Definition If $\tau \in \mathcal{T}$, write $\mathfrak{A}_\tau$ for

$$\{a : a \in \mathfrak{A},\ a \setminus [\![\tau > t]\!] \in \mathfrak{A}_t \text{ for every } t \in T\}.$$

Then $\mathfrak{A}_\tau$ is an order-closed subalgebra of $\mathfrak{A}$.

611H Proposition (a) Suppose that $\tau \in \mathcal{T}$ and $t \in T$.

(i) If $b \in \bigcap_{s>t} \mathfrak{A}_s$ and $b \subseteq [\![\tau > t]\!]$, then $b \in \mathfrak{A}_\tau$. $[\![\tau > t]\!]$ and $1 \setminus [\![\tau > t]\!]$ belong to $\mathfrak{A}_\tau$.

(ii) If $b \in \mathfrak{A}_t$ and $b \subseteq [\![\tau > s]\!]$ for every $s < t$, then $b \in \mathfrak{A}_\tau$.

(iii) If $b \in \mathfrak{A}_\tau$ and $b \cap [\![\tau > t]\!] = 0$, then $b \in \mathfrak{A}_t$.

(b) If $\check{t}$ is the constant stopping time at t, then $\mathfrak{A}_{\check{t}} = \mathfrak{A}_t$.

(c) Suppose that σ, $\tau \in \mathcal{T}$.

(i) $[\![\sigma < \tau]\!]$, $[\![\sigma = \tau]\!]$ and $[\![\tau < \sigma]\!]$ belong to $\mathfrak{A}_\sigma \cap \mathfrak{A}_\tau$.

(ii) $\mathfrak{A}_{\sigma\wedge\tau} = \mathfrak{A}_\sigma \cap \mathfrak{A}_\tau$; in particular, $\mathfrak{A}_\sigma \subseteq \mathfrak{A}_\tau$ if $\sigma \le \tau$.

(iii) If $a \in \mathfrak{A}_\tau$ then $a \cap [\![\tau \le \sigma]\!] = a \setminus [\![\sigma < \tau]\!] \in \mathfrak{A}_{\sigma\wedge\tau}$.

(iv) $\mathfrak{A}_{\sigma\vee\tau}$ is the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_\sigma \cup \mathfrak{A}_\tau$.

611I Lemma Suppose that $\langle\tau_i\rangle_{i\in I}$ is a family in $\mathcal{T}$ and $\langle a_i\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$ such that $a_i \in \mathfrak{A}_{\tau_i}$ for every $i \in I$. Then there is a unique $\sigma \in \mathcal{T}$ such that $[\![\sigma = \tau_i]\!] \supseteq a_i$ for every $i \in I$, and $\inf_{i\in I} \tau_i \le \sigma \le \sup_{i\in I} \tau_i$.

611J Dissections by stopping times (a) Recall that if we regard a stopping time $\tau = \langle[\![\tau > t]\!]\rangle_{t\in T}$ as a member of the algebra $\prod_{t\in T} \mathfrak{A}_t$, then the partial order $\le$ and the lattice operations $\vee$, $\wedge$ on $\mathcal{T}$ correspond to the Boolean relation and operations $\subseteq$, $\cup$, $\cap$ on $\prod_{t\in T} \mathfrak{A}_t$, and moreover that arbitrary suprema in $\mathcal{T}$ correspond to suprema in $\prod_{t\in T} \mathfrak{A}_t$.

In view of this representation it is natural to consider set difference. I will in fact prefer the notation

$$c(\sigma, \tau) = \langle[\![\tau > t]\!] \setminus [\![\sigma > t]\!]\rangle_{t\in T},$$

rather than writing $\tau \setminus \sigma$. I will say that $c(\sigma, \tau)$ is the **stopping time interval** with **endpoints** σ, τ.

If $\mathcal{S}$ is a sublattice of $\mathcal{T}$, $\mathrm{Sti}(\mathcal{S})$ will be the set of stopping-time intervals expressible as $c(\sigma, \tau)$ where $\sigma \le \tau$ in $\mathcal{S}$.

(b)

$$c(\sigma,\tau)\cap c(\sigma',\tau')=c(\sigma\vee\sigma',\tau\wedge\tau')$$

for all σ, σ', τ, $\tau'\in\mathcal{T}$, and

$$c(\sigma\wedge\tau,\sigma\wedge\tau')\subseteq c(\tau,\tau').$$

Similarly,

$$c(\sigma,\sup C)=\sup_{\tau\in C}c(\sigma,\tau),\quad c(\sigma\wedge\sigma',\tau)=c(\sigma,\tau)\cup c(\sigma',\tau)$$

for σ, σ', $\tau\in\mathcal{T}$ and $C\subseteq\mathcal{T}$, and if $\sigma\le\upsilon\le\tau$, then

$$c(\sigma,\upsilon)\cup c(\upsilon,\tau)=c(\sigma,\tau),\quad c(\sigma,\upsilon)\cap c(\upsilon,\tau)=0.$$

$c(\sigma,\tau)=0$ iff $\tau\le\sigma$.

(c) $[\![\sigma<\tau]\!]\subseteq[\![\sigma'<\tau']\!]$ whenever $c(\sigma,\tau)\subseteq c(\sigma',\tau')$. More precisely, if σ, τ, σ', $\tau'\in\mathcal{T}$ then $c(\sigma,\tau)\subseteq c(\sigma',\tau')$ iff

$$[\![\sigma<\tau]\!]\subseteq[\![\sigma'\le\sigma]\!]\cap[\![\tau\le\tau']\!].$$

(d) Similarly, if σ, τ, σ', $\tau'\in\mathcal{T}$ then $c(\sigma,\tau)=c(\sigma',\tau')$ iff

$$[\![\sigma<\tau]\!]=[\![\sigma'<\tau']\!]\subseteq[\![\sigma'=\sigma]\!]\cap[\![\tau=\tau']\!].$$

(e)(i) For a finite sublattice I of $\mathcal{T}$, an ***I*-cell** will be a minimal non-zero stopping time interval of the form $c(\sigma,\tau)$ where σ, $\tau\in I$.

(ii) Let I be a finite sublattice of $\mathcal{T}$, $\mathrm{Sti}_0(I)$ the set of I-cells, and $\tau\in I$. If we write

$$I\wedge\tau=\{\sigma\wedge\tau:\sigma\in I\},\quad I\vee\tau=\{\sigma\vee\tau:\sigma\in I\},$$

then $\mathrm{Sti}_0(I\wedge\tau)$, $\mathrm{Sti}_0(I\vee\tau)$ are disjoint sets with union $\mathrm{Sti}_0(I)$.

(iii) More generally, if I is a non-empty finite sublattice of $\mathcal{T}$ and $\tau_0\le\ldots\le\tau_n$ in I, then setting

$$I_{-1}=I\wedge\tau_0,\quad I_j=I\cap[\tau_j,\tau_{j+1}]\text{ for }j<n,\quad I_n=I\vee\tau_n,$$

$\langle\mathrm{Sti}_0(I_j)\rangle_{-1\le j\le n}$ is a partition of $\mathrm{Sti}_0(I)$.

611K Lemma Let $I\subseteq\mathcal{T}$ be a non-empty finite sublattice, and $\mathrm{Sti}_0(I)$ the set of I-cells. Let I_0 be a maximal totally ordered subset of I, and $\langle\tau_i\rangle_{i\le n}$ the increasing enumeration of I_0.

(a) $\tau_0=\min I$, $\tau_1=\max I$.
(b) If $i<n$ then $I\cap[\tau_i,\tau_{i+1}]=\{\tau_i,\tau_{i+1}\}$.
(c) $\mathrm{Sti}_0(I)=\{c(\tau_i,\tau_{i+1}):i<n\}$.
(d) $[\![\tau_i<\tau]\!]\cap[\![\tau<\tau_{i+1}]\!]=0$ whenever $i<n$ and $\tau\in I$.
(e) $\sup_{i\le n}[\![\tau=\tau_i]\!]=1$ for every $\tau\in I$.
*(f) If $\sigma\in\mathcal{T}$ then

$$J_0=\{\sigma\wedge\tau_0,\tau_0,\mathrm{med}(\tau_0,\sigma,\tau_1),\tau_1,\mathrm{med}(\tau_1,\sigma,\tau_2),\ldots,\tau_{n-1},\mathrm{med}(\tau_{n-1},\sigma,\tau_n),\tau_n,\sigma\vee\tau_n\}$$

is a maximal totally ordered subset of the sublattice $I\sqcup\{\sigma\}$ of $\mathcal{T}$ generated by $I\cup\{\sigma\}$.[2]

*(g) If $\sigma\in\mathcal{T}$, then $I\wedge\sigma=\{\tau\wedge\sigma:\tau\in I\}$ is a sublattice of $\mathcal{T}$, and $\{\tau_0\wedge\sigma,\ldots,\tau_n\wedge\sigma\}$ is a maximal totally ordered subset of $I\wedge\sigma$.

*(h) If $\tau_0\le\sigma_0\le\ldots\le\sigma_m\le\tau_n$ in $\mathcal{T}$, and K is the sublattice of $\mathcal{T}$ generated by $I\cup\{\sigma_0,\ldots,\sigma_m\}$, then $J_j=\{\mathrm{med}(\sigma_j,\tau_i,\sigma_{j+1}):i\le n\}$ is a maximal totally ordered subset of $K\cap[\sigma_j,\sigma_{j+1}]$, for every $j<m$.

[2] In a distributive lattice, $\mathrm{med}(p,q,r)=(p\wedge q)\vee(p\wedge r)\vee(q\wedge r)$; see 3A1Ic.

611L Definition If I is a finite sublattice of $\mathcal{T}$, I will say that a sequence $\langle\tau_i\rangle_{i\le n}$ in I **linearly generates the I-cells** if it is non-decreasing and $\{\tau_i : i \le n\}$ is a maximal totally ordered subset of I.

611M Covering and full sublattices (a)(i) If A, $B \subseteq \mathcal{T}$, A **covers** B if $\sup_{\sigma\in A}[\![\tau=\sigma]\!] = 1$ for every $\tau \in B$.

(ii) If $A \subseteq \mathcal{T}$, the **covered envelope** of A will be the set $\{\tau : \tau \in \mathcal{T}, \sup_{\sigma\in A}[\![\tau=\sigma]\!] = 1\}$. Of course A covers itself, that is, the covered envelope of A includes A.

(b)(i) If $A \subseteq \mathcal{T}$ and $a \in \mathfrak{A}$, the set

$$\mathcal{S} = \{\tau : \tau \in \mathcal{T}, a \subseteq \sup_{\sigma\in A}[\![\sigma=\tau]\!]\}$$

is a sublattice of $\mathcal{T}$.

In particular, the covered envelope $\hat{A}$ of A is a sublattice of $\mathcal{T}$.

(ii) If ρ is an upper bound for A in $\mathcal{T}$, then ρ is an upper bound for $\hat{A}$. Similarly, if ρ is a lower bound for A, it is a lower bound for $\hat{A}$.

(iii) Since $A \subseteq \hat{A}$, it follows that if A has a greatest member then this is also the greatest member of $\hat{A}$, and that if A has a least member then this is also the least member of $\hat{A}$.

(iv) Note that if σ, $\tau \in \mathcal{T}$ then $\{\sigma,\tau\}$ covers $\{\sigma\wedge\tau,\sigma\vee\tau\}$ and also $\{\sigma\wedge\tau,\sigma\vee\tau\}$ covers $\{\sigma,\tau\}$.

(c) I will say that a sublattice of $\mathcal{T}$ is **full** if it is equal to its covered envelope.

(i) The intersection of any non-empty family of full sublattices of $\mathcal{T}$ is full.

(ii) If $A \subseteq \mathcal{T}$, its covered envelope $\hat{A}$ is full.

(d) For any $\rho \in \mathcal{T}$, $\mathcal{T}\wedge\rho$ is full. Similarly, $\mathcal{T}\vee\rho$ is full. Putting these together, $[\rho,\rho'] = (\mathcal{T}\wedge\rho')\cap(\mathcal{T}\vee\rho)$ is full whenever $\rho \le \rho'$ in $\mathcal{T}$.

(e)(i) If $\mathcal{S}$ is a sublattice of $\mathcal{T}$ with covered envelope $\hat{\mathcal{S}}$, and $\rho \in \mathcal{S}$, then $\hat{\mathcal{S}}\wedge\rho$ is the covered envelope of $\mathcal{S}\wedge\rho$ and $\hat{\mathcal{S}}\vee\rho$ is the covered envelope of $\mathcal{S}\vee\rho$.

(ii) If $\mathcal{S}$ is a sublattice of $\mathcal{T}$, ρ, $\rho' \in \mathcal{S}$ and $\rho \le \rho'$, then the covered envelope of

$$\mathcal{S}\cap[\rho,\rho'] = (\mathcal{S}\vee\rho)\wedge\rho' = \{\mathrm{med}(\rho,\sigma,\rho') : \sigma \in \mathcal{S}\}$$

is $(\hat{\mathcal{S}}\vee\rho)\wedge\rho' = \hat{\mathcal{S}}\cap[\rho,\rho']$.

(f) If $\mathcal{S}$ is a sublattice of $\mathcal{T}$ with covered envelope $\hat{\mathcal{S}}$, then $\bigcap_{\tau\in\hat{\mathcal{S}}}\mathfrak{A}_\tau = \bigcap_{\sigma\in\mathcal{S}}\mathfrak{A}_\sigma$.

(g) Suppose that A, $B \subseteq \mathcal{T}$ and A covers B.

(i) A covers the covered envelope of B.

(ii) If $\tau \in \mathcal{T}$, then $A\wedge\tau = \{\sigma\wedge\tau : \sigma \in A\}$ covers $B\wedge\tau = \{\sigma\wedge\tau : \sigma \in B\}$.

611N Covering ideals Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.

(a) Definition I will say that a **covering ideal** of $\mathcal{S}$ is an ideal $\mathcal{S}'$ of $\mathcal{S}$ which covers $\mathcal{S}$.

(b)(i) If $\tau \in \mathcal{S}$ and $\mathcal{S}'$ is an ideal of $\mathcal{S}$, then $\{[\![\sigma=\tau]\!] : \sigma \in \mathcal{S}'\}$ is upwards-directed.

(ii) If $\tau \in \mathcal{S}$ and $\mathcal{S}'$ is an ideal of $\mathcal{S}$, then $\sup_{\sigma\in\mathcal{S}'}[\![\sigma=\tau]\!] = \sup_{\sigma\in\mathcal{S}'}[\![\tau\le\sigma]\!]$.

(c) If $\mathcal{S}$ is a sublattice of $\mathcal{T}$ and $\mathcal{S}_1$, $\mathcal{S}_2$ are two covering ideals of $\mathcal{S}$, then $\mathcal{S}_0 = \mathcal{S}_1\cap\mathcal{S}_2$ is a covering ideal of $\mathcal{S}$.

(d) If $\mathcal{S}'$ is a covering ideal of $\mathcal{S}$ and $\mathcal{S}''$ is a covering ideal of $\mathcal{S}'$, then $\mathcal{S}''$ is a covering ideal of $\mathcal{S}$.

(e)(i) $\mathcal{T}_f$ is full.

(ii) $\mathcal{T}_b$ is a covering ideal of $\mathcal{T}_f$.

***611O Definitions**

(a) If A, $B \subseteq \mathcal{T}$, I will say that A **finitely covers** B if for every $\tau \in B$ there is a finite $J \subseteq A$ such that $\sup_{\sigma\in J} [\![\tau = \sigma]\!] = 1$.

(b) If $A \subseteq \mathcal{T}$, the **finitely-covered envelope** of A is the set of those $\tau \in \mathcal{T}$ for which there is a finite subset $J \subseteq A$ such that $\sup_{\sigma\in J} [\![\tau = \sigma]\!] = 1$.

(c) A subset of $\mathcal{T}$ is **finitely full** if it is equal to its finitely-covered envelope.

***611P Lemma** Suppose that $\mathfrak{A} \ne \{0\}$.
(a) Let A be a subset of $\mathcal{T}$ and $\hat A_f$ its finitely-covered envelope.
(i) $\hat A_f$ is finitely full.
(ii) $\hat A_f$ is a sublattice of the covered envelope $\hat A$ of A.
(iii) $\hat A_f$ is the intersection of all the finitely full subsets of $\mathcal{T}$ including A.
(b) The intersection of any non-empty family $\mathbb{S}$ of finitely full sublattices of $\mathcal{T}$ is finitely full.
(c) If $\mathcal{S}$ is a sublattice of $\mathcal{T}$ which is order-convex, then $\mathcal{S}$ is finitely full.
(d) If $\mathcal{S}$ is a sublattice of $\mathcal{T}$ and $\tau \in \hat{\mathcal{S}}_f$, there are $\sigma_0 \le \ldots \le \sigma_n$ in $\mathcal{S}$ such that $\sup_{i\le n} [\![\tau = \sigma_i]\!] = 1$.
(e) If $\mathcal{S}$ is a sublattice of $\mathcal{T}$ then $\mathcal{S}$ is both coinitial and cofinal with $\hat{\mathcal{S}}_f$.

Version of 24.12.16/21.10.22

612 Fully adapted processes

The next step is to introduce the processes which this volume is devoted to studying. These are an abstract version of the real-valued stochastic processes $\langle X_t\rangle_{t\ge0}$ of §§455 and 477. Instead of starting from Σ_t-measurable functions $X_t : \Omega \to \mathbb{R}$ and then showing that it is possible to define Σ_h-measurable functions X_h for stopping times $h : \Omega \to [0, \infty[$, I move directly to families $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ of equivalence classes of measurable functions where $\mathcal{S}$ is a sublattice of the lattice $\mathcal{T}$ of stopping times discussed in §611. A 'fully adapted process' is one satisfying the essential measurability and consistency requirements of 612D. Among these, the 'simple' processes (612J), those which are constant between finitely many break points, are particularly important. I end with descriptions of Brownian motion (612T) and the standard Poisson process (612U) in this language.

612A $\mathfrak{A}$ and $L^0(\mathfrak{A})$ (a) Given a Dedekind complete Boolean algebra, we have a Dedekind complete f-algebra $L^0 = L^0(\mathfrak{A})$ as described in §364.

(b) In §364 I introduced the formulae $[\![u > \alpha]\!]$, $[\![u \in E]\!]$, where $u \in L^0$, $\alpha \in \mathbb{R}$ and $E \subseteq \mathbb{R}$ is a Borel set. I mentioned formulae $[\![u \ge \alpha]\!]$, $[\![u < 0]\!]$ and $[\![u \ne 0]\!]$, and $[\![(u_1, \ldots , u_n) \in E]\!]$ when E is a Borel subset of $\mathbb{R}^n$. Here it will be convenient to extend the notation to such formulae as $[\![u \ne v]\!]$, meaning $[\![|u - v| > 0]\!]$. In terms of the representation of L^0 as a space of equivalence classes of functions, we have

$$[\![(f_1^\bullet, \ldots , f_n^\bullet) \in E]\!] = \{\omega : (f_1(\omega), \ldots , f_n(\omega)) \in E\}^\bullet$$

for all Σ-measurable functions $f_1, \ldots , f_n : \Omega \to \mathbb{R}$. $[\![f_1^\bullet \ne f_2^\bullet]\!]$ can be interpreted as $\{x : f_1(\omega) \ne f_2(\omega)\}^\bullet$.

(c) Let $E \subseteq \mathbb{R}$ be a Borel set and $h : E \to \mathbb{R}$ a Borel measurable function. Set

$$Q_E = \{u : u \in L^0,\ [\![u \in E]\!] = 1\} = \{f^\bullet : f : \Omega \to E \text{ is measurable}\}.$$

If $u \in Q_E$,we have an $\bar h(u) \in L^0$ defined by saying that $[\![\bar h(u) \in F]\!] = [\![u \in h^{-1}[F]]\!]$ for every Borel set $F \subseteq \mathbb{R}$. If u, $u' \in Q_E$ then $[\![u = u']\!] \subseteq [\![\bar h(u) = \bar h(u')]\!]$. Observe that if h_1, $h_2 : \mathbb{R} \to \mathbb{R}$ are both Borel measurable, we now have $\overline{h_1h_2}(u) = \bar h_1(\bar h_2(u))$ for all h_1, h_2 and $u \in L^0$. $\bar h(u) = u$ if $E = \mathbb{R}$ and $h(\alpha) = \alpha$ for every $\alpha \in \mathbb{R}$. So we have a semigroup action $\bullet$ of H on L^0 defined by saying that $h\bullet u = \bar h(u)$ for $h \in H$ and $u \in L^0$.

(d)(i)

(α) If $\gamma \in \mathbb{R}$ and $h(\alpha) = \gamma\alpha$ for $\alpha \in \mathbb{R}$, then $\bar{h}(u) = \gamma u$ for every $u \in L^0$.

(β) If $h(\alpha) = |\alpha|$ for $\alpha \in \mathbb{R}$, then $\bar{h}(u) = |u|$ for every $u \in L^0$.

(γ) If $h(\alpha) = \alpha^2$ for $\alpha \in \mathbb{R}$, then $\bar{h}(u) = u \times u = u^2$ for every $u \in L^0$.

(δ) If $h(\alpha) = 1$ for $\alpha \in \mathbb{R}$, then $\bar{h}(u) = \chi 1$ is the multiplicative identity of L^0 for every $u \in L^0$.

(ϵ) If $h : \mathbb{R} \to \mathbb{R}$ is non-decreasing, then $\bar{h}(u) \le \bar{h}(v)$ whenever $u \le v$ in L^0.

(ii) It follows that if $V \subseteq L^0$ is such that $u + v \in V$ for all u, $v \in V$ and $\bar{h}(u) \in V$ for every convex function $h : \mathbb{R} \to \mathbb{R}$ such that $h(0) = 0$, then V is an ***f*-subalgebra** of L^0, that is, a Riesz subspace closed under multiplication. *A fortiori*, if V is such that $u + v \in V$ for all u, $v \in V$ and $\bar{h}(u) \in V$ for every continuous function $h : \mathbb{R} \to \mathbb{R}$ such that $h(0) = 0$, then V is an f-subalgebra of L^0.

(iii) Continuing from (c) above, it will be important also to note that, for any u, $v \in Q_E$, $[\![u \le v]\!] \subseteq [\![\bar{h}(u) \le \bar{h}(v)]\!]$. $[\![u = v]\!] \subseteq [\![\bar{h}(u) = \bar{h}(v)]\!]$.

(iv) Take any $u \in L^0$. Again writing H for the space of Borel measurable functions from $\mathbb{R}$ to itself, H is an f-subalgebra of the f-algebra $\mathbb{R}^{\mathbb{R}}$ as well as a sub-semigroup under composition. Treating H as an f-algebra, the map $h \mapsto \bar{h}(u) : H \to L^0$ is a multiplicative Riesz homomorphism.

(v) It will happen more than once that we have two Dedekind complete Boolean algebras $\mathfrak{A}$ and $\mathfrak{B}$, f-subalgebras V, W of $L^0(\mathfrak{A})$ and $L^0(\mathfrak{B})$ respectively, and a linear operator $Q : V \to W$ such that $Q|v| = |Qv|$ and $Q(v^2) = (Qv)^2$ for all $v \in V$. In this case, Q will be an ***f*-algebra homomorphism**, that is, a multiplicative Riesz homomorphism.

(e)(i) Now suppose that $\mathfrak{B}$ is an order-closed subalgebra of $\mathfrak{A}$. In this case we can think of $L^0(\mathfrak{B})$ as being the subspace

$$\{u : u \in L^0(\mathfrak{A}),\ [\![u > \alpha]\!] \in \mathfrak{B} \text{ for every } \alpha \in \mathbb{R}\}.$$

The arguments of 364F show that this is equal to

$$\{u : u \in L^0(\mathfrak{A}),\ [\![u \in E]\!] \in \mathfrak{B} \text{ for every Borel set } E \subseteq \mathbb{R}\}$$

and that $\bar{h}(u) \in L^0(\mathfrak{B})$ whenever $h \in H$ and $u \in L^0(\mathfrak{B})$. Looking at this a little more deeply, we see that if $h \in H$ we have two different functions $\bar{h}_{\mathfrak{A}} : L^0(\mathfrak{A}) \to L^0(\mathfrak{A})$ and $\bar{h}_{\mathfrak{B}} : L^0(\mathfrak{B}) \to L^0(\mathfrak{B})$, but that $\bar{h}_{\mathfrak{B}} = \bar{h}_{\mathfrak{A}} \restriction L^0(\mathfrak{B})$, so that we can use the same symbol $\bar{h}$ for either.

$L^0(\mathfrak{B})$ is an order-closed sublattice of $L^0(\mathfrak{A})$.

(ii) If $\langle \mathfrak{B}_i \rangle_{i \in I}$ is a non-empty family of order-closed subalgebras of $\mathfrak{A}$ with intersection $\mathfrak{B}$, then $\mathfrak{B}$ is an order-closed subalgebra of $\mathfrak{A}$ and $L^0(\mathfrak{B}) = \bigcap_{i \in I} L^0(\mathfrak{B}_i)$.

(iii) For any $u \in L^0(\mathfrak{A})$, the set $\{[\![u \in E]\!] : E \subseteq \mathbb{R} \text{ is Borel}\}$ is a σ-subalgebra of $\mathfrak{A}$, the smallest σ-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ such that $u \in L^0(\mathfrak{B})$; it is the σ-subalgebra generated by $\{[\![u > \alpha]\!] : \alpha \in \mathbb{R}\}$. I will say that it is the σ-subalgebra of $\mathfrak{A}$ defined by $\boldsymbol{u}$. Similarly, if $A \subseteq L^0(\mathfrak{A})$, I will say that the σ-subalgebra of $\mathfrak{A}$ generated by $\{[\![u > \alpha]\!] : u \in A, \alpha \in \mathbb{R}\}$ is the σ-subalgebra defined by A.

(f) Let $\mathfrak{C}$ be another Dedekind complete Boolean algebra, and $\phi : \mathfrak{A} \to \mathfrak{C}$ an order-continuous Boolean homomorphism. Then we have a unique order-continuous f-algebra homomorphism $T_\phi : L^0(\mathfrak{A}) \to L^0(\mathfrak{C})$ such that

$[\![T_\phi u > \alpha]\!] = \phi[\![u > \alpha]\!]$ for every $u \in L^0(\mathfrak{A})$ and $\alpha \in \mathbb{R}$,

$T_\phi(\chi a) = \chi(\phi a)$ for every $a \in \mathfrak{A}$,

$[\![T_\phi u \in E]\!] = \phi[\![u \in E]\!]$ for every Borel set $E \subseteq \mathbb{R}$,

$T_\phi \bar{h}_{\mathfrak{A}} = \bar{h}_{\mathfrak{C}} T_\phi$ for every Borel measurable $h : \mathbb{R} \to \mathbb{R}$,

T_ϕ is injective or surjective iff ϕ is.

612B Products and processes For the rest of this section $\mathfrak{A}$ will be a Dedekind complete Boolean algebra, T a totally ordered set, $\langle \mathfrak{A}_t \rangle_{t \in T}$ a filtration of closed subalgebras of $\mathfrak{A}$, $\mathcal{T}$ the associated lattice of stopping times, and $\langle \mathfrak{A}_\tau \rangle_{\tau \in \mathcal{T}}$ the corresponding family of order-closed subalgebras. For σ, $\tau \in \mathcal{T}$, $[\![\sigma < \tau]\!]$, $[\![\sigma \le \tau]\!]$ $[\![\sigma = \tau]\!]$ will be the regions defined in 611D.

(a) If $\mathcal{S}$ is a sublattice of $\mathcal{T}$, we can form the family $\langle L^0(\mathfrak{A}_\sigma)\rangle_{\sigma\in\mathcal{S}}$. If we take the natural product linear space, lattice and multiplicative structures, we get an f-algebra $\prod_{\sigma\in\mathcal{S}} L^0(\mathfrak{A}_\sigma)$. Moreover, writing H for the semigroup of Borel measurable functions from $\mathbb{R}$ to itself, we have a natural action of H on $\prod_{\sigma\in\mathcal{S}} L^0(\mathfrak{A}_\sigma)$ defined by setting

$$h\bullet\langle u_\sigma\rangle_{\sigma\in\mathcal{S}} = \langle h\bullet u_\sigma\rangle_{\sigma\in\mathcal{S}}$$

whenever $h \in H$ and $u_\sigma \in L^0(\mathfrak{A}_\sigma)$ for every $\sigma \in \mathcal{S}$.

Writing $\bar{h}(u)$ for $h\bullet u$, as in 612Ac, and thinking of $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ as a function from $\mathcal{S}$ to L^0, we find ourselves with a composition $\bar{h}\boldsymbol{u} = \bar{h}\circ\boldsymbol{u} : \mathcal{S} \to L^0$.

(b) Another way of looking at $\prod_{\sigma\in\mathcal{S}} L^0(\mathfrak{A}_\sigma)$ is to identify it with $L^0(\mathfrak{C})$, where $\mathfrak{C}$ is the simple Boolean algebra product $\prod_{\sigma\in\mathcal{S}} \mathfrak{A}_\sigma$. Once again, it is easy to see that if $h \in H$ then $\bar{h}_{\mathfrak{C}} : L^0(\mathfrak{C}) \to L^0(\mathfrak{C})$ matches the function $\boldsymbol{u} \mapsto \bar{h}\boldsymbol{u} : \prod_{\sigma\in\mathcal{S}} L^0(\mathfrak{A}_\sigma) \to \prod_{\sigma\in\mathcal{S}} L^0(\mathfrak{A}_\sigma)$ described in (a).

(c) From (b) and 612A(d-ii) we now see that if V is a subset of $\prod_{\sigma\in\mathcal{S}} L^0(\mathfrak{A}_\sigma)$ such that $\boldsymbol{u}+\boldsymbol{v} \in V$ and $\bar{h}\boldsymbol{u} \in V$ whenever $\boldsymbol{u}$, $\boldsymbol{v} \in V$, $h : \mathbb{R} \to \mathbb{R}$ is convex and $h(0) = 0$, then V is an f-subalgebra of $\prod_{\sigma\in\mathcal{S}} L^0(\mathfrak{A}_\sigma)$.

612C Lemma If σ, $\tau \in \mathcal{T}$ and $u \in L^0(\mathfrak{A}_\tau)$ then $u \times \chi[\![\tau \le \sigma]\!]$ and $u \times \chi[\![\tau = \sigma]\!]$ and $u \times \chi[\![\tau < \sigma]\!]$ belong to $L^0(\mathfrak{A}_{\sigma\wedge\tau})$.

612D Fully adapted processes To continue the real work of this section, let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a family in $L^0(\mathfrak{A})$.

(a) Definition I will say that $\boldsymbol{u}$ is **fully adapted** to $\langle\mathfrak{A}_t\rangle_{t\in T}$ if $u_\sigma \in L^0(\mathfrak{A}_\sigma)$ and $[\![\sigma = \tau]\!] \subseteq [\![u_\sigma = u_\tau]\!]$ whenever σ, $\tau \in \mathcal{S}$.

(b) Note that if $u_\tau \in L^0(\mathfrak{A}_\tau)$ and $[\![\sigma = \tau]\!] \subseteq [\![u_\sigma = u_\tau]\!]$ whenever $\sigma \le \tau \in \mathcal{S}$, then $\boldsymbol{u}$ is fully adapted.

(c) If $\boldsymbol{u}$ is fully adapted and $\mathcal{S}'$ is a sublattice of $\mathcal{S}$, then $\boldsymbol{u}\restriction\mathcal{S}'$ is still a fully adapted process.

(d) If $\boldsymbol{u}$ is fully adapted, I is a finite sublattice of $\mathcal{S}$, and $(\tau_0\ \dots, \tau_n)$ linearly generates the I-cells, then for any $\sigma \in I$ we have

$$\sup_{i\le n} [\![u_\sigma = u_{\tau_i}]\!] \supseteq \sup_{i\le n} [\![\sigma = \tau_i]\!] = 1$$

So if $\bar{u} = \sup_{i\le n} u_{\tau_i}$, $\bar{u} = \sup_{\sigma\in I} u_\sigma$. $\sup_{\sigma\in I} |u_\sigma| = \sup_{i\le n} |u_{\tau_i}|$.

(e)(i) Note that if $\boldsymbol{u}$ is constant, say $u_\sigma = z$ for every $\sigma \in \mathcal{S}$, then $\boldsymbol{u}$ is fully adapted iff $z \in \bigcap_{\sigma\in\mathcal{S}} L^0(\mathfrak{A}_\sigma)$; if $\mathcal{S}$ has a least element, this will be so iff $z \in L^0(\mathfrak{A}_{\min\mathcal{S}})$. For any $z \in L^0(\mathfrak{A})$, I will write $z\mathbf{1}$ for the fully adapted process $\langle z\rangle_{\sigma\in\mathcal{S}}$ where $\mathcal{S}$ is the sublattice $\{\sigma : \sigma \in \mathcal{T},\ z \in L^0(\mathfrak{A}_\sigma)\}$. When $z = \chi 1$ and $\mathcal{S} = \mathcal{T}$ I will write just $\mathbf{1}$; similarly, $\mathbf{0}$ will be the constant process with value $0 \in L^0(\mathfrak{A})$.

(ii) Generally, if $z \in L^0(\mathfrak{A})$, I will write $z\boldsymbol{u}$ for the process $z\mathbf{1}\times\boldsymbol{u} = \langle z\times u_\sigma\rangle_{\sigma\in\mathcal{S}'}$, where $\mathcal{S}' = \{\sigma : \sigma \in \mathcal{S},\ z \in L^0(\mathfrak{A}_\sigma)\}$. Then $\mathcal{S}'$ is a sublattice of $\mathcal{S}$ and $z\boldsymbol{u}$ is fully adapted.

(f) Suppose that $\boldsymbol{u}$ is fully adapted.

(i) $u_{\sigma\wedge\tau} + u_{\sigma\vee\tau} = u_\sigma + u_\tau$ and $u_{\sigma\wedge\tau} \vee u_{\sigma\vee\tau} = u_\sigma \vee u_\tau$ for all σ, $\tau \in \mathcal{S}$.

(ii) $|u_\tau - u_\sigma| = |u_{\sigma\vee\tau} - u_{\sigma\wedge\tau}|$ for all σ, $\tau \in \mathcal{S}$.

(iii) $|u_\sigma - u_\rho| \le |u_{\sigma\wedge\tau} - u_{\rho\wedge\tau}| + |u_{\sigma\vee\tau} - u_{\rho\vee\tau}|$ for all ρ, σ, $\tau \in \mathcal{S}$.

612E Where fully adapted processes come from In applications, one commonly starts from a family $\langle X_t\rangle_{t\in T}$ of random variables, corresponding to a family $\langle u_t\rangle_{t\in T} \in \prod_{t\in T} L^0(\mathfrak{A}_t)$.

(a) If T is finite and not empty, with least value $\min T$, then for $\tau \in \mathcal{T}$ and $t \in T$ set

$$a_{\tau t} = (\inf_{s<t} [\![\tau > s]\!]) \setminus [\![\tau > t]\!]$$

(counting $\inf\emptyset$ as 1, as usual, so that $a_{\tau,\min T} = 1\setminus[\![\tau > \min T]\!]$). Then $\langle a_{\tau t}\rangle_{t\in T}$ is a partition of unity in $\mathfrak{A}$, and $a_{\tau t}\in\mathfrak{A}_t$ for every t. Now set

$$u'_\tau = \sum_{t\in T} u_t \times \chi a_{\tau t}.$$

(b) If T is well-ordered and not empty, we can use essentially the same formulae for $\tau\in\mathcal{T}_f$.

612F The identity process (a) Suppose that $T=[0,\infty[$. For $\tau\in\mathcal{T}_f$, we can define $\iota_\tau\in L^0(\mathfrak{A})$ by saying that, for $t\in\mathbb{R}$,

$$[\![\iota_\tau > t]\!] = [\![\tau > t]\!] \text{ if } t\ge 0,$$
$$= 1 \text{ if } t<0.$$

(b) $\boldsymbol{\iota} = \langle\iota_\tau\rangle_{\tau\in\mathcal{T}_f}$ is a fully adapted process.

(c) $\iota_{\check t} = t\chi 1$ for every $t\ge 0$.

I will call $\boldsymbol{\iota}$ the **identity process** for the structure $(\mathfrak{A},\langle\mathfrak{A}_t\rangle_{t\ge 0})$.

612H Theorem Let (Ω,Σ,μ) be a complete probability space, and $\langle\Sigma_t\rangle_{t\ge 0}$ a filtration of σ-subalgebras of Σ such that every μ-negligible set belongs to every Σ_t. Let $(\mathfrak{A},\bar\mu)$ be the measure algebra of μ and set $\mathfrak{A}_t=\{E^\bullet : E\in\Sigma_t\}$ for each $t\ge 0$; then $\langle\mathfrak{A}_t\rangle_{t\ge 0}$ is a filtration. Let $\mathcal{T}$ be the associated family of stopping times.

(a)(i) If $h:\Omega\to[0,\infty]$ is a stopping time, we have a stopping time $\tau\in\mathcal{T}$ defined by saying that $[\![\tau>t]\!]=\{\omega : h(\omega)>t\}^\bullet$ for every $t\ge 0$; I will say that h **represents** τ.

(ii) Conversely, if $\tau\in\mathcal{T}$, there is a stopping time $h:\Omega\to[0,\infty]$ representing τ.

(iii) If h represents τ, then $\Sigma_h=\{E : E\in\Sigma,\ E^\bullet\in\mathfrak{A}_\tau\}$ and $\mathfrak{A}_\tau=\{E^\bullet : E\in\Sigma_h\}$.

(iv) If g, $h:\Omega\to[0,\infty]$ are stopping times representing σ, $\tau\in\mathcal{T}$, then

$$[\![\sigma<\tau]\!]=\{\omega : g(\omega)<h(\omega)\}^\bullet,$$

$$[\![\sigma\le\tau]\!]=\{\omega : g(\omega)\le h(\omega)\}^\bullet,\quad [\![\sigma=\tau]\!]=\{\omega : g(\omega)=h(\omega)\}^\bullet.$$

g and h represent the same member of $\mathcal{T}$ iff they are equal almost everywhere.

(v) If h represents τ, then $\tau\in\mathcal{T}_f$ iff $h(\omega)<\infty$ for almost every ω; $\tau\in\mathcal{T}_f$ iff it can be represented by a stopping time $h:\Omega\to[0,\infty[$.

(vi) If $t\ge 0$, then the constant function with value t represents the constant stopping time at t.

(b) Now suppose that $\langle X_t\rangle_{t\ge 0}$ is a progressively measurable process on Ω.

(i) For every $\tau\in\mathcal{T}_f$ we have an $x_\tau\in L^0(\mathfrak{A})\cong L^0(\mu)$ defined by saying that x_τ is the equivalence class of the function X_h, where $X_h(\omega)=X_{h(\omega)}(\omega)$ for $\omega\in h^{-1}[[0,\infty[]$, whenever h represents τ.

(ii) The family $\langle x_\tau\rangle_{\tau\in\mathcal{T}_f}$ is fully adapted to $\langle\mathfrak{A}_t\rangle_{t\ge 0}$.

612I Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $M_{\mathrm{fa}}(\mathcal{S})\subseteq\prod_{\sigma\in\mathcal{S}}L^0(\mathfrak{A}_\sigma)$ the set of fully adapted processes with domain $\mathcal{S}$.

(a) $M_{\mathrm{fa}}(\mathcal{S})$ is an order-closed f-subalgebra of the f-algebra $L^0(\mathfrak{A})^{\mathcal{S}}$, and if $h:\mathbb{R}\to\mathbb{R}$ is a Borel measurable function then $\bar h\boldsymbol{u}\in M_{\mathrm{fa}}(\mathcal{S})$ for every $\boldsymbol{u}\in M_{\mathrm{fa}}(\mathcal{S})$. $M_{\mathrm{fa}}(\mathcal{S})$ is Dedekind complete.

(b) Suppose that $\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}\in M_{\mathrm{fa}}(\mathcal{S})$ and $\tau\in\mathcal{T}$. Set $\mathcal{S}'=\{\sigma : \sigma\in\mathcal{T},\ \sigma\wedge\tau\in\mathcal{S}\}$. Then $\langle u_{\sigma\wedge\tau}\rangle_{\sigma\in\mathcal{S}'}\in M_{\mathrm{fa}}(\mathcal{S}')$.

612J Simple processes (a) Definition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. A fully adapted process $\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is **simple** if either $\mathcal{S}$ is empty or there are $\tau_0\le\ldots\le\tau_n$ in $\mathcal{S}$ and $u_*\in L^0(\bigcap_{\sigma\in\mathcal{S}}\mathfrak{A}_\sigma)$ such that for every $\sigma\in\mathcal{S}$

$$[\![\sigma<\tau_0]\!]\subseteq[\![u_\sigma=u_*]\!],\quad [\![\tau_n\le\sigma]\!]\subseteq[\![u_\sigma=u_{\tau_n}]\!],$$

$$[\![\tau_i\le\sigma]\!]\cap[\![\sigma<\tau_{i+1}]\!]\subseteq[\![u_\sigma=u_{\tau_i}]\!] \text{ for every } i<n.$$

In this case I will say that $(\tau_0,\ldots,\tau_n)$ is a **breakpoint string** for $\boldsymbol{u}$.

(b) As a particularly elementary example, if $\mathcal{S}$ is a sublattice of $\mathcal{T}$ and $\tau \in \mathcal{S}$, then $\boldsymbol{u} = \langle \chi[\![\tau \le \sigma]\!]\rangle_{\sigma\in\mathcal{S}}$ is a simple process.

612K Lemma Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$. Write $\mathfrak{B}$ for $\bigcap_{\sigma\in\mathcal{S}}\mathfrak{A}_\sigma$.

(a) Suppose that $\tau_0 \le \ldots \le \tau_n$ in $\mathcal{S}$, $u_i \in L^0(\mathfrak{A}_{\tau_i})$ for $i \le n$ and $u_* \in L^0(\mathfrak{B})$. Then there is a unique simple fully adapted process $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ such that whenever $\sigma \in \mathcal{S}$ then

$$[\![v_\sigma = u_i]\!] \supseteq [\![\tau_i \le \sigma]\!] \cap [\![\sigma < \tau_{i+1}]\!],$$

for $i < n$, while

$$[\![v_\sigma = u_*]\!] \supseteq [\![\sigma < \tau_0]\!], \quad [\![v_\sigma = u_n]\!] \supseteq [\![\tau_n \le \sigma]\!];$$

and $(\tau_0, \ldots, \tau_n)$ is a breakpoint string for $\boldsymbol{v}$.

(b) Suppose that I is a non-empty finite sublattice of $\mathcal{S}$ and $(\tau_0, \ldots, \tau_n)$ linearly generates the I-cells. If a simple process $\boldsymbol{u}$ with domain $\mathcal{S}$ has a breakpoint string in I, then $(\tau_0, \ldots, \tau_n)$ is a breakpoint string for $\boldsymbol{u}$.

(c) Suppose that K is a finite set and $\boldsymbol{u}_k$ is a simple process with domain $\mathcal{S}$ for each $k \in K$. Then there is a single string $(\tau_0, \ldots, \tau_n)$ in $\mathcal{S}$ which is a breakpoint string for every $\boldsymbol{u}_k$.

(d) Suppose that $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a simple process with breakpoint string $(\tau_0, \ldots, \tau_n)$ in $\mathcal{S}$, and $\tau \in \mathcal{S}$.

(i) $(\tau_0 \wedge \tau, \ldots, \tau_0 \wedge \tau_n, \tau, \tau_0 \vee \tau, \ldots, \tau_n \vee \tau)$ is a breakpoint string for $\boldsymbol{u}$.

(ii) Writing $\mathcal{S} \wedge \tau$ for $\{\sigma \wedge \tau : \sigma \in \mathcal{S}\} = \mathcal{S} \cap [\min\mathcal{T}, \tau]$, $\boldsymbol{u}\restriction\mathcal{S} \wedge \tau$ is simple, with breakpoint string $(\tau_0 \wedge \tau, \ldots, \tau_n \wedge \tau, \tau)$.

(iii) Writing $\mathcal{S} \vee \tau$ for $\{\sigma \vee \tau : \sigma \in \mathcal{S}\} = \mathcal{S} \cap [\tau, \max\mathcal{T}]$, $\boldsymbol{u}\restriction\mathcal{S} \vee \tau$ is simple, with breakpoint string $(\tau_0 \vee \tau, \ldots, \tau_n \vee \tau)$.

(e) Suppose that $\boldsymbol{u}$ is a fully adapted process with domain $\mathcal{S}$, and that $\tau \in \mathcal{S}$. If $\boldsymbol{u}\restriction\mathcal{S} \wedge \tau$ and $\boldsymbol{u}\restriction\mathcal{S} \vee \tau$ are simple processes with breakpoint strings $(\tau_0, \ldots, \tau_m)$ and $(\tau'_0, \ldots, \tau'_n)$ respectively, then $\boldsymbol{u}$ is simple, with breakpoint string $(\tau_0, \ldots, \tau_m, \tau, \tau'_0, \ldots, \tau'_n)$.

612L Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. Write $M_{\mathrm{simp}} = M_{\mathrm{simp}}(\mathcal{S})$ for the set of simple processes with domain $\mathcal{S}$.

(a) If $h : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function and $\boldsymbol{u} \in M_{\mathrm{simp}}$, then $\bar h\boldsymbol{u} \in M_{\mathrm{simp}}$ and any breakpoint string for $\boldsymbol{u}$ is a breakpoint string for $\bar h\boldsymbol{u}$.

(b) M_{simp} is an f-subalgebra of $\prod_{\sigma\in\mathcal{S}} L^0(\mathfrak{A}_\sigma)$.

(c) If $z \in L^0(\mathfrak{A} \cap \bigcap_{\sigma\in\mathcal{S}}\mathfrak{A}_\sigma)$ and $\boldsymbol{u} \in M_{\mathrm{simp}}$, then $z\boldsymbol{u} \in M_{\mathrm{simp}}$.

612M Lemma Let $\mathcal{S} = [\min\mathcal{S}, \max\mathcal{S}]$ be a closed interval in $\mathcal{T}$, and $\boldsymbol{u}$ a simple process with domain $\mathcal{S}$. Then there is a breakpoint string $(\tau_0, \ldots, \tau_n)$ for $\boldsymbol{u}$ such that $\tau_0 = \min\mathcal{S}$, $\tau_n = \max\mathcal{S}$ and $[\![\tau_i < \tau_{i+1}]\!] = [\![\tau_i < \max\mathcal{S}]\!]$ for every $i < n$.

612P Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a fully adapted process. Then there is a fully adapted process $\boldsymbol{v} = \langle v_\tau\rangle_{\tau\in\mathcal{T}}$, extending $\boldsymbol{u}$, such that

$$[\![v_\tau \ne 0]\!] \subseteq \sup_{\sigma\in\mathcal{S}} [\![\sigma = \tau]\!]$$

for every $\tau \in \mathcal{T}$.

612Q Proposition Suppose that $\mathcal{S}$ is a sublattice of $\mathcal{T}$, $\hat{\mathcal{S}}$ its covered envelope and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a fully adapted process.

(a) $\boldsymbol{u}$ has a unique extension to a fully adapted process $\hat{\boldsymbol{u}} = \langle \hat u_\sigma\rangle_{\sigma\in\mathcal{S}}$ with domain $\hat{\mathcal{S}}$.

(b) The map $\boldsymbol{u} \mapsto \hat{\boldsymbol{u}}$ is an isomorphism from the f-algebra $M_{\mathrm{fa}}(\mathcal{S})$ of fully adapted processes with domain $\mathcal{S}$ to the f-algebra $M_{\mathrm{fa}}(\hat{\mathcal{S}})$, and $\bar h\hat{\boldsymbol{u}} = (\bar h\boldsymbol{u})\hat{}$ whenever $\boldsymbol{u} \in M_{\mathrm{fa}}(\mathcal{S})$ and $h : \mathbb{R} \to \mathbb{R}$ is Borel measurable.

(c) If $\tau \in \mathcal{S}$, then $\hat{\boldsymbol{u}}\restriction\hat{\mathcal{S}} \wedge \tau$ is the fully adapted extension of $\boldsymbol{u}\restriction\mathcal{S} \wedge \tau$ to the covered envelope of $\mathcal{S} \wedge \tau$.

(e) If $z \in L^0(\mathfrak{A} \cap \bigcap_{\sigma\in\mathcal{S}}\mathfrak{A}_\sigma)$ then $z\hat{\boldsymbol{u}}$ is the fully adapted extension of $z\boldsymbol{u}$.

(f) If $\boldsymbol{u}$ is simple, with a witnessing string $(u_*, \tau_0, \ldots, \tau_n)$ as in 612Ja, and $\mathcal{S}'$ is a sublattice of $\hat{\mathcal{S}}$ including $\mathcal{S}$, then $\hat{\boldsymbol{u}}\restriction\mathcal{S}'$ is simple, with the same witnessing string.

(g) If $\boldsymbol{u}$ is non-decreasing, so is $\hat{\boldsymbol{u}}$.

612R Corollary Suppose that $\mathcal{S}$ is a sublattice of $\mathcal{T}$ and $\mathcal{S}'$ is a sublattice of $\mathcal{S}$ covering $\mathcal{S}$. Then any fully adapted process $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}'}$ has a unique extension to a fully adapted process $\boldsymbol{v} = \langle v_\tau \rangle_{\tau \in \mathcal{S}}$.

612S Two more definitions Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.

(a) For a fully adapted process $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$, write $\|\boldsymbol{u}\|_\infty = \sup_{\sigma \in \mathcal{S}} \|u_\sigma\|_\infty$.

(b) For fully adapted processes $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$, write $[\![\boldsymbol{u} \neq \boldsymbol{v}]\!]$ for $\sup_{\sigma \in \mathcal{S}} [\![u_\sigma \neq v_\sigma]\!]$, and $[\![\boldsymbol{u} \neq \boldsymbol{0}]\!] = \sup_{\sigma \in \mathcal{S}} [\![u_\sigma \neq 0]\!]$.

(c) Suppose that $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ are fully adapted processes.

(i) If $h : \mathbb{R} \to \mathbb{R}$ is Borel measurable, then $[\![\bar{h}\boldsymbol{u} \neq \bar{h}\boldsymbol{v}]\!] \subseteq [\![\boldsymbol{u} \neq \boldsymbol{v}]\!]$.

(ii) If $\hat{\mathcal{S}}$ is the covered envelope of $\mathcal{S}$ and $\hat{\boldsymbol{u}} = \langle \hat{u}_\tau \rangle_{\tau \in \hat{\mathcal{S}}}$, $\hat{\boldsymbol{v}} = \langle \hat{v}_\tau \rangle_{\tau \in \hat{\mathcal{S}}}$ are the fully adapted extensions of $\boldsymbol{u}$, $\boldsymbol{v}$ to $\hat{\mathcal{S}}$, then $[\![\hat{\boldsymbol{u}} \neq \hat{\boldsymbol{v}}]\!] = [\![\boldsymbol{u} \neq \boldsymbol{v}]\!]$.

612T Example: Brownian motion (a) Let $\Omega = C([0,\infty[)_0$ be the set of continuous functions $\omega : [0,\infty[\to \mathbb{R}$ such that $\omega(0) = 0$, and ν one-dimensional Wiener measure on Ω, with Σ its domain. Recall that ν is a Radon measure with respect to the topology $\mathfrak{T}_c$ of uniform convergence on compact sets. Let $(\mathfrak{C}, \bar{\nu})$ be the measure algebra of ν. For $t \geq 0$, write Σ_t for

$$\{F : F \in \Sigma,\ \omega' \in F \text{ whenever } \omega \in F,\ \omega' \in \Omega \text{ and } \omega'\restriction[0,t] = \omega\restriction[0,t]\},$$

and let $\hat{\Sigma}_t$ be $\{F \triangle A : F \in \Sigma_t,\ \nu A = 0\}$; set $\mathfrak{C}_t = \{F^\bullet : F \in \hat{\Sigma}_t\} = \{F^\bullet : F \in \Sigma_t\}$ and $X_t(\omega) = \omega(t)$ for $t \geq 0$ and $\omega \in \Omega$. Then $(s,\omega) \mapsto X_s(\omega) : [0,t] \times \Omega \to \mathbb{R}$ is continuous for every $t \geq 0$, and $\langle X_t \rangle_{t \geq 0}$ is progressively measurable with respect to $\langle \hat{\Sigma}_t \rangle_{t \geq 0}$. We have a process $\boldsymbol{w} = \langle w_\tau \rangle_{\tau \in \mathcal{T}_f}$ fully adapted to $\langle \mathfrak{C}_t \rangle_{t \geq 0}$. In this volume I will use the phrase **Brownian motion** to mean the process $\boldsymbol{w}$.

(d) $\boldsymbol{w}$ determines $\mathfrak{C}$ and $\langle \mathfrak{C}_t \rangle_{t \geq 0}$, in that

(i) $\mathfrak{C}$ is the closed subalgebra $\mathfrak{D}$ of itself generated by $\{[\![w_{\check{t}} > \alpha]\!] : t \geq 0,\ \alpha \in \mathbb{R}\}$,

(ii) $\mathfrak{C}_t$ is the closed subalgebra generated by $\{[\![w_{\check{s}} > \alpha]\!] : s \in [0,t],\ \alpha \in \mathbb{R}\}$ for every $t \geq 0$.

(e) Every member of $\mathcal{T}_f$ can be represented by a stopping time adapted to $\langle \Sigma_t^+ \rangle_{t \geq 0}$, where $\Sigma_t^+ = \bigcap_{s > t} \Sigma_s$ for $t \geq 0$.

612U Example: the Poisson process (a) For $t > 0$ let λ_t be the Poisson distribution with expectation t, that is, the Radon probability measure on $\mathbb{R}$ such that $\lambda_t\{n\} = e^{-t}t^n/n!$ for every $n \in \mathbb{N}$. Then the convolution $\lambda_s * \lambda_t$ is equal to λ_{s+t} whenever $s, t > 0$, and $\lim_{t \downarrow 0} \lambda_t G = 1$ for every open set $G \subseteq \mathbb{R}$ including 0. So we have an associated probability measure $\ddot{\mu}$ on the space C_{dlg} of càdlàg real-valued functions defined on $[0,\infty[$. This measure is the subspace measure on C_{dlg} induced by a complete measure on $\mathbb{R}^{[0,\infty[}$ defined in terms of transitional probabilities. The formula of 455E tells us that if $0 = t_0 < \ldots < t_n$ in $\mathbb{R}$ and $0 = k_0 \leq \ldots \leq k_n$ in $\mathbb{N}$, then the measure of $\{\omega : \omega(t_i) = k_i \text{ for } i \leq n\}$ is

$$\prod_{i=1}^n \lambda_{t_i - t_{i-1}}\{k_i - k_{i-1}\} = e^{-t_n} \prod_{i=1}^n \frac{(t_i - t_{i-1})^{k_i - k_{i-1}}}{(k_i - k_{i-1})!}.$$

(b) $\ddot{\mu}$ is a completion regular quasi-Radon measure on C_{dlg} if we give C_{dlg} the topology of pointwise convergent inherited from $\mathbb{R}^{[0,\infty[}$.

$$\Omega = \{\omega : \omega \in C_{\text{dlg}} \text{ is non-decreasing, } \omega(t) \in \mathbb{N} \text{ for every } t \text{ and } \omega(0) = 0\}$$

is the support of $\ddot{\mu}$.

(c) Let μ be the subspace measure on Ω induced by $\ddot{\mu}$ and Σ its domain, so that μ is a quasi-Radon probability measure on Ω. For $t \geq 0$, set

$$\ddot{\Sigma}_t = \{F : F \in \operatorname{dom}\ddot{\mu},\ \omega' \in F \text{ whenever } \omega' \in C_{\text{dlg}},\ \omega \in F \text{ and } \omega'\restriction[0,t] = \omega\restriction[0,t]\}$$

(see 455O) and

$$\dot{\ddot{\Sigma}}_t = \{F \triangle A : F \in \ddot{\Sigma}_t,\ \ddot{\mu}A = 0\}.$$

Then $\dot{\ddot{\Sigma}}_t = \bigcap_{s>t} \dot{\ddot{\Sigma}}_s$ for every t. So if

$$\Sigma_t = \{F : F \subseteq \Omega,\ F \in \dot{\ddot{\Sigma}}_t\}$$

for $t \geq 0$, $\langle \Sigma_t \rangle_{t \geq 0}$ will be a filtration of σ-algebras. Consequently, if we take $(\mathfrak{A}, \bar{\mu})$ to be the measure algebra of μ, and set $\mathfrak{A}_t = \{F^\bullet : F \in \Sigma_t\}$ for each t, $\langle \mathfrak{A}_t \rangle_{t \geq 0}$ will be a filtration of closed subalgebras of $\mathfrak{A}$.

(d) For $\omega \in \Omega$ and $t \geq 0$ set $X_t(\omega) = \omega(t)$. Then X_t has a Poisson distribution with expectation t. Now $\langle X_t \rangle_{t \geq 0}$ is progressively measurable. We have a corresponding fully adapted process $\boldsymbol{v} = \langle v_\tau \rangle_{\tau \in \mathcal{T}_f}$; in this volume I will call this the **standard Poisson process**.

(e)(i) For each $n \in \mathbb{N}$ and $\omega \in \Omega$, set

$$g_n(\omega) = \inf\{t : t \in [0, \infty[,\ \omega(t) \geq n\},$$

counting $\inf \emptyset$ as ∞. Then $g_0(\omega) = 0$ for every ω. If $g_n(\omega)$ is finite, then $\omega(g_n(\omega)) \geq n$. g_n is finite a.e. $g_n \leq g_{n+1}$ for every n. In fact, for almost every ω, $\langle g_n(\omega) \rangle_{n \in \mathbb{N}}$ is strictly increasing.

Observe that, for any $n \in \mathbb{N}$, $\omega(g_n(\omega)) = n$ for almost every ω.

(ii) For $n \in \mathbb{N}$, g_n is a stopping time adapted to $\langle \Sigma_t \rangle_{t \geq 0}$. Let $\tau_n = g_n^\bullet$ be the corresponding stopping time in $\mathcal{T}_f$. $\sup_{n \in \mathbb{N}} \tau_n = \max \mathcal{T}$. $v_{\tau_n} = n\chi 1$, for every $n \in \mathbb{N}$.

I will call $\langle \tau_n \rangle_{n \in \mathbb{N}}$ the sequence of **jump times** for the process $\boldsymbol{v}$.

(f) If $\tau \in \mathcal{T}_f$, then

$$[\![v_\tau \in \mathbb{N}]\!] = 1, \quad [\![v_\tau = v_{\tau_n}]\!] = [\![v_\tau = n]\!] = [\![\tau_n \leq \tau]\!] \cap [\![\tau < \tau_{n+1}]\!] \text{ for every } n \in \mathbb{N}.$$

(g) $\boldsymbol{v}$ is locally order-bounded.

Version of 15.7.20/16.1.25

613 Definition of the integral

I come now to the definition of a stochastic integral which will be used for the next three chapters. We are looking for an effective way to interpret the formula $\int_\tau^{\tau'} \boldsymbol{u}\, d\boldsymbol{v}$ where $\tau \leq \tau'$ are stopping times and $\boldsymbol{u}$, $\boldsymbol{v}$ are fully adapted processes defined on an interval $[\tau, \tau']$ in $\mathcal{T}$. I will define this as a kind of Riemann-Stieltjes integral, a limit of 'Riemann sums' of the form $\sum_{i=0}^n u_{\tau_i} \times (v_{\tau_{i+1}} - v_{\tau_i})$ where $\tau = \tau_0 \leq \ldots \leq \tau_n = \tau'$. For this we need a notion of convergence, for which 'convergence in measure' turns out to be suitable, and a particular limiting process, to be described in 613Hb. Because our processes are defined on a lattice $\mathcal{T}$ of stopping times, rather than a totally ordered set, there are some technical obstacles to clear out of the way; I aim to do this in 613C-613G. The rest of the section is devoted to elementary properties of this new integral.

613A Probability algebras (a) For the rest of this volume, $(\mathfrak{A}, \bar{\mu})$ will denote a probability algebra. $L^1(\mathfrak{A}, \bar{\mu})$ or $L^1_{\bar{\mu}}$ will be its L^1 space. For w in $L^0 = L^0(\mathfrak{A})$, I will write $\mathbb{E}(w) = \mathbb{E}_{\bar{\mu}}(w) = \mathbb{E}(w^+) - \mathbb{E}(w^-)$ for its integral with respect to $\bar{\mu}$, provided that at most one of $\mathbb{E}(w^+)$, $\mathbb{E}(w^-)$ is infinite.

(b) T will be a totally ordered set and $\langle \mathfrak{A}_t \rangle_{t \in T}$ a filtration of order-closed subalgebras of $\mathfrak{A}$. $\mathcal{T}$ will be the set of stopping times adapted to $\langle \mathfrak{A}_t \rangle_{t \in T}$. For $\tau \in \mathcal{T}$, $\mathfrak{A}_\tau$ will be the closed subalgebra corresponding to τ. When I say that a process is 'fully adapted' I shall always mean that it is 'fully adapted to $\langle \mathfrak{A}_t \rangle_{t \in T}$'.

613B Convergence in measure (a) L^0 now has a topology $\mathfrak{T}$ of **convergence in measure** which can be defined by the F-norm θ where

$$\theta(w) = \mathbb{E}(|w| \wedge \chi 1) \text{ for every } w \in L^0.$$

This is a complete Hausdorff linear space topology for which multiplication and the lattice operations $\vee$, $\wedge$ and $|\,|$ are continuous. In particular, the positive cone $(L^0)^+$ is closed.

$\theta(\alpha w) \le \alpha\theta(w)$ if $w \in L^0$ and $\alpha \ge 1$. $\theta(v) \le \theta(w)$ whenever $|v| \le |w|$. $\lim_{w\downarrow A}\theta(w) = 0$ whenever $A \subseteq L^0$ is a non-empty downwards-directed family with infimum 0, so that $\sup A \in \overline{A}$ and $\lim_{w\uparrow A}\theta(w) = \theta(\sup A)$ whenever $A \subseteq L^0$ is a non-empty upwards-directed set with an upper bound in L^0; similarly, if $A \subseteq L^0$ is a non-empty downwards-directed set with a lower bound in L^0, $\lim_{w\downarrow A}\theta(w) = \theta(\inf A)$.

(b) If $E \subseteq \mathbb{R}$ is a Borel set and $Q_E = \{u : u \in L^0, [\![u \in E]\!] = 1\}$, then for any continuous $h : E \to \mathbb{R}$ the corresponding function $\bar{h} : Q_E \to L^0$ is continuous.

(c) If $1 \le p \le \infty$, all the $\|\,\|_p$-balls $\{u : u \in L^0, \|u\|_p \le \alpha\}$ are $\mathfrak{T}$-closed. Consequently the $\mathfrak{T}$-closure of a $\|\,\|_p$-bounded set is again $\|\,\|_p$-bounded, and $\|\,\|_p : L^0 \to [0,\infty]$ is lower semi-continuous.

(d)(i) For any $p \in [1,\infty]$, the embedding $L^p_{\bar\mu} \subsetneqq L^0$ is continuous for the norm topology of $L^p_{\bar\mu}$ and $\mathfrak{T}$.

(ii) If $A \subseteq L^1_{\bar\mu}$ is non-empty and downwards-directed and $\inf A = 0$ in $L^1_{\bar\mu}$, then $\inf_{u\in A}\|u\|_1 = \lim_{u\downarrow A}\|u\|_1 = 0$.

(iii) If $A \subseteq (L^1_{\bar\mu})^+$ is non-empty and upwards-directed and $\gamma = \sup_{u\in A}\|u\|_1$ is finite, then A is bounded above in $L^1_{\bar\mu}$, $\sup A$ belongs to the $\|\,\|_1$-closure of A and $\|\sup A\|_1 = \gamma$.

(iv) If $u \in L^1_{\bar\mu}$ and $\epsilon > 0$, there is a $\delta > 0$ such that $\|u - v\|_1 \le \epsilon$ whenever $v \in L^1_{\bar\mu}$, $\|v\|_1 \le \|u\|_1 + \delta$ and $\theta(u - v) \le \delta$.

(e) If $A \subseteq L^0$ and $v \in \overline{A}$ then $[\![v > \alpha]\!] \subseteq \sup_{u\in A}[\![u > \alpha]\!]$ for every $\alpha \in \mathbb{R}$.

(f)(i) I will say that a set $A \subseteq L^0$ is **topologically bounded** if for every neighbourhood G of 0 in L^0 there is an $n \in \mathbb{N}$ such that $A \subseteq nG$; equivalently, if for every $\epsilon > 0$ there is a $\delta > 0$ such that $\theta(\delta u) \le \epsilon$ for every $u \in A$.

(ii) If $A \subseteq L^0$ is non-empty, then A is topologically bounded iff $\inf_{\gamma>0}\sup_{u\in A}\bar\mu[\![|u| > \gamma]\!] = 0$.

(iii) If A, $B \subseteq L^0$ are topologically bounded, so are $A + B$ and αA for any $\alpha \in \mathbb{R}$, the closure $\overline{A}$ of A for the topology of convergence in measure, and any subset of A.

(iv) If $A \subseteq L^0$ is topologically bounded, so is its **solid hull** $\{u : u \in L^0, \exists\, v \in A, |u| \le |v|\}$. In particular, an order-bounded subset of L^0 is topologically bounded.

(v) An upwards-directed topologically bounded set is bounded above.

(vi) If $A \subseteq L^0$ is solid, so is $\overline{A}$.

(g) If $\bar\nu : \mathfrak{A} \to [0,1]$ is any functional such that $(\mathfrak{A}, \bar\nu)$ is a probability algebra, then $\bar\mu$ and $\bar\nu$ are mutually absolutely continuous, that is,

— for every $\epsilon > 0$ there is a $\delta > 0$ such that $\max(\bar\mu a, \bar\nu a) \le \epsilon$ whenever $a \in \mathfrak{A}$ and $\min(\bar\mu a, \bar\nu a) \le \delta$.

$\mathfrak{T}$ is still the topology of convergence in measure on L^0 if we apply the formulae of (a) with the integral $\mathbb{E}_{\bar\nu}$ defined from $\bar\nu$ in place of $\mathbb{E} = \mathbb{E}_{\bar\mu}$, and if we set $\theta_{\bar\nu}(w) = \mathbb{E}_{\bar\nu}(|w| \wedge \chi 1)$ for $w \in L^0$, then

— for every $\epsilon > 0$ there is a $\delta > 0$ such that $\max(\theta_{\bar\mu}(w), \theta_{\bar\nu}(w)) \le \epsilon$ whenever $\min(\theta_{\bar\mu}(w), \theta_{\bar\nu}(w)) \le \delta$.

I introduce a code phrase: the topology of convergence in measure is **law-independent**, since replacing the 'law' $\bar\mu$ by the law $\bar\nu$ leaves it unchanged.

(h) (L^0, θ) is a complete metric space; that is, L^0 is complete when regarded as a linear topological space.

(i) Now suppose that $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$.

(i) $L^0(\mathfrak{B})$, regarded as a subset of $L^0(\mathfrak{A})$, is closed for the topology of convergence in measure.

(ii) $L^1_{\bar\mu} \cap L^0(\mathfrak{B})$ is $\|\,\|_1$-closed in $L^1_{\bar\mu}$; it is also closed for the weak topology of $L^1_{\bar\mu}$.

(j) Suppose that $\mathcal{S}$ is a sublattice of $\mathcal{T}$, $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$ is a fully adapted process, and $A \subseteq \mathcal{S}$ is a non-empty downwards-directed set such that the limit $u_* = \lim_{\sigma\downarrow A} u_\sigma$ is defined. Then $u_* \in \bigcap_{\sigma\in A} L^0(\mathfrak{A}_\sigma) = L^0(\bigcap_{\sigma\in A} \mathfrak{A}_\sigma)$.

(k) If $\mathcal{S}$ is non-empty and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$ is a fully adapted process such that the topological limit $u_\downarrow = \lim_{\sigma\downarrow\mathcal{S}} u_\sigma$ is defined in $L^0(\mathfrak{A})$, I will call $u_\downarrow$ the **starting value** of $\boldsymbol{u}$.

(l) If $\mathcal{S}$ is a sublattice of $\mathcal{T}$, then we can give $(L^0)^{\mathcal{S}}$ its product topology, under which it is a linear topological space. Now the space of fully adapted processes with domain $\mathcal{S}$ is a closed subspace of $(L^0)^{\mathcal{S}}$.

(m) Because the lattice operations on L^0 are continuous, and the topology is Hausdorff, sets of the form $\{|u| : u \le \bar u\} = \{u : |u| \vee \bar u = \bar u\}$ are closed for any $\bar u \in L^0$. Consequently, in a product space $(L^0)^{\mathcal{S}}$, sets of the form $\{\boldsymbol{u} : \boldsymbol{u} \in (L^0)^{\mathcal{S}},\ |\boldsymbol{u}| \le \bar{\boldsymbol{u}}\}$, where $\bar{\boldsymbol{u}} \in (L^0)^{\mathcal{S}}$, are closed for the product topology.

(n) Now suppose that $(\mathfrak{B}, \bar\nu)$ is another probability algebra, and $\phi : \mathfrak{A} \to \mathfrak{B}$ is a measure-preserving Boolean homomorphism. Then we have a corresponding injective f-algebra homomorphism $T_\phi : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$. Writing $\mathbb{E}_{\bar\mu}$, $\mathbb{E}_{\bar\nu}$ for expectations in $L^1_{\bar\mu}$, $L^1_{\bar\nu}$ respectively, and $\theta_{\bar\mu}$, $\theta_{\bar\nu}$ for the corresponding functionals on $L^0(\mathfrak{A})$ and $L^0(\mathfrak{B})$, $\mathbb{E}_{\bar\nu}(T_\phi u) = \mathbb{E}_{\bar\mu}(u)$ for every $u \in L^1_{\bar\mu}$; $\theta_{\bar\nu}(T_\phi u) = \theta_{\bar\mu}(u)$ for every $u \in L^0(\mathfrak{A})$, and T_ϕ is continuous for the topologies of convergence in measure.

(o) For any $\alpha \in \mathbb{R}$, the function $u \mapsto \bar\mu[\![u > \alpha]\!] : L^0 \to [0,1]$ is lower semi-continuous.

(p)(i) Suppose that $A \subseteq L^0$ and that for every $\epsilon > 0$ there is an $a \in \mathfrak{A}$ such that $\{u \times \chi a : u \in A\}$ is order-bounded in L^0 and $\bar\mu a \ge 1 - \epsilon$. Then A is order-bounded in L^0.

(ii) If $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$ is a fully adapted process and for every $\epsilon > 0$ there is an order-bounded process $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma\in\mathcal{S}}$ such that $\bar\mu[\![\boldsymbol{u} \ne \boldsymbol{v}]\!] \le \epsilon$, then $\boldsymbol{u}$ is order-bounded.

(q)(i) If $A \subseteq \mathcal{T}$, τ belongs to the covered envelope $\hat A$ of A and $\epsilon > 0$, there is a τ' in the finitely-covered envelope $\hat A_f$ of A such that $\bar\mu[\![\tau = \tau']\!] \ge 1 - \epsilon$.

(ii) If $\mathcal{S}$ is a sublattice of $\mathcal{T}$ with covered envelope $\hat{\mathcal{S}}$ and finitely covered envelope $\hat{\mathcal{S}}_f$, $\boldsymbol{u} = \langle u_\tau \rangle_{\tau\in\hat{\mathcal{S}}}$ is a fully adapted process and $\tau \in \hat{\mathcal{S}}$, then u_τ belongs to the closure of $\{u_\sigma : \sigma \in \hat{\mathcal{S}}_f\}$ for the topology of convergence in measure.

613C Interval functions (a) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. I will write $\mathcal{S}^{2\uparrow}$ for $\{(\sigma,\tau) : \sigma, \tau \in \mathcal{S}, \sigma \le \tau\}$.

(i) I say that a function $\psi : \mathcal{S}^{2\uparrow} \to L^0(\mathfrak{A})$ is an **adapted interval function** on $\mathcal{S}$ if

$$\psi(\sigma,\tau) \in L^0(\mathfrak{A}_\tau), \quad \psi(\sigma,\sigma) = 0, \quad b \subseteq [\![\psi(\sigma,\tau) = \psi(\sigma',\tau')]\!]$$

whenever $\sigma \le \sigma' \le \tau' \le \tau$ in $\mathcal{S}$, $b \in \mathfrak{A}_\sigma$ and $b \subseteq [\![\sigma = \sigma']\!] \cap [\![\tau' = \tau]\!]$.

(ii) In this case, I say that ψ is a **strictly adapted interval function** if

$$[\![\sigma = \sigma']\!] \cap [\![\tau' = \tau]\!] \subseteq [\![\psi(\sigma,\tau) = \psi(\sigma',\tau')]\!]$$

whenever $\sigma \le \sigma' \le \tau' \le \tau$ in $\mathcal{S}$.

(b) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\psi : \mathcal{S}^{2\uparrow} \to L^0(\mathfrak{A})$ an adapted interval function.

(i) $[\![\sigma = \tau]\!] \subseteq [\![\psi(\sigma,\tau) = 0]\!]$ whenever $\sigma \le \tau$ in $\mathcal{S}$.
$[\![\psi(\sigma,\tau) \ne 0]\!] \subseteq [\![\sigma < \tau]\!]$ whenever $\sigma \le \tau$ in $\mathcal{S}$.

(ii) $\psi(\sigma, \sigma \vee \tau) = \psi(\sigma \wedge \tau, \tau)$ for all σ, $\tau \in \mathcal{S}$.

(iii) If $\mathcal{S}_0$ is any sublattice of $\mathcal{S}$, then $\psi\restriction\mathcal{S}_0^{2\uparrow}$ is an adapted interval function on $\mathcal{S}_0$, and is strictly adapted if ψ is.

(iv) If ψ is strictly adapted then $[\![\sigma = \sigma']\!] \cap [\![\tau' = \tau]\!] \subseteq [\![\psi(\sigma,\tau) = \psi(\sigma',\tau')]\!]$ whenever $\sigma \le \tau$ and $\sigma' \le \tau'$ in $\mathcal{S}$.

(c) If $\mathcal{S} \subseteq \mathcal{T}$ is a sublattice and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ is a fully adapted process, we have a function $\Delta\boldsymbol{v} : \mathcal{S}^{2\uparrow} \to L^0(\mathfrak{A})$ defined by saying that

$$(\Delta\boldsymbol{v})(\sigma, \tau) = v_\tau - v_\sigma$$

whenever $\sigma \le \tau$ in $\mathcal{S}$, and $\Delta\boldsymbol{v}$ is a strictly adapted interval function on $\mathcal{S}$.

613D Constructions for interval functions Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and ψ, ψ' (strictly) adapted interval functions on $\mathcal{S}$.

(a) $\psi + \psi'$ and $\psi \times \psi'$ are (strictly) adapted interval functions.

(b) If $h : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function, then the composition $\bar{h}\psi$ is a (strictly) adapted interval function.

(c) ψ^2 and $|\psi|$ and $\alpha\psi$, for any $\alpha \in \mathbb{R}$, are (strictly) adapted interval functions; the space of (strictly) adapted interval functions on $\mathcal{S}$ is an f-subalgebra of $L^0(\mathfrak{A})^{\mathcal{S}^{2\uparrow}}$.

(d) If $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ is a fully adapted process, then we have a (strictly) adapted interval function $\boldsymbol{u}\psi$ on $\mathcal{S}$ defined by setting $(\boldsymbol{u}\psi)(\sigma, \tau) = u_\sigma \times \psi(\sigma, \tau)$ whenever $\sigma \le \tau$ in $\mathcal{S}$.

613E Riemann sums Let $\mathcal{S} \subseteq \mathcal{T}$ be a sublattice, ψ an adapted interval function on $\mathcal{S}$, and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ a fully adapted process.

(a) For a stopping-time interval $e \in \mathrm{Sti}(\mathcal{S})$, we can define $\Delta_e(\boldsymbol{u}, d\psi)$ by saying that $\Delta_e(\boldsymbol{u}, d\psi) = u_\sigma \times \psi(\sigma, \tau)$ whenever $e = c(\sigma, \tau)$ with $\sigma \le \tau$ in $\mathcal{S}$.

(b) If $I \subseteq \mathcal{S}$ is a finite sublattice and $\mathrm{Sti}_0(I) \subseteq \mathrm{Sti}(\mathcal{S})$ is the set of I-cells, write

$$S_I(\boldsymbol{u}, d\psi) = \textstyle\sum_{e \in \mathrm{Sti}_0(I)} \Delta_e(\boldsymbol{u}, d\psi).$$

(c) If $I \subseteq \mathcal{S}$ is a non-empty finite sublattice, then there is a string $(\tau_0, \dots, \tau_n)$ in I linearly generating the I-cells. $S_I(\boldsymbol{u}, d\psi)$ will be $\sum_{i=0}^{n-1} u_{\tau_i} \times \psi(\tau_i, \tau_{i+1})$.

(d) Now suppose that $\psi = \Delta\boldsymbol{v}$ for some fully adapted process $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$. If $I \subseteq \mathcal{S}$ is any non-empty sublattice, then $S_I(\mathbf{1}, d(\Delta\boldsymbol{v})) = v_{\max I} - v_{\min I}$.

(e) If $\psi = \boldsymbol{u}\psi'$, where ψ' is another adapted interval function with domain $\mathcal{S}^{2\uparrow}$ and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ is a fully adapted process, then $S_I(\mathbf{1}, d\psi) = S_I(\boldsymbol{u}, d\psi')$ for every finite sublattice I of $\mathcal{S}$.

613F Notation Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$, $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ and $\boldsymbol{w} = \langle w_\sigma \rangle_{\sigma \in \mathcal{S}}$ fully adapted processes.

(a) If $\sigma \le \tau$ in $\mathcal{S}$ and $e = c(\sigma, \tau)$, then I write

$$\Delta_e(\boldsymbol{u}, d\boldsymbol{v}) = \Delta_e(\boldsymbol{u}, d(\Delta\boldsymbol{v})) = u_\sigma \times (v_\tau - v_\sigma),$$

$$\Delta_e(\boldsymbol{u}, d\boldsymbol{v}d\boldsymbol{w}) = \Delta_e(\boldsymbol{u}, d(\Delta\boldsymbol{v} \times \Delta\boldsymbol{w})) = u_\sigma \times (v_\tau - v_\sigma) \times (w_\tau - w_\sigma),$$

$$\Delta_e(\boldsymbol{u}, |d\boldsymbol{v}|) = \Delta_e(\boldsymbol{u}, d|\Delta\boldsymbol{v}|) = u_\sigma \times |v_\tau - v_\sigma|.$$

(b) Now if $I \subseteq \mathcal{S}$ is a finite sublattice and $\mathrm{Sti}_0(I)$ is the set of I-cells, write

$$S_I(\boldsymbol{u}, d\boldsymbol{v}) = S_I(\boldsymbol{u}, d(\Delta\boldsymbol{v})) = \textstyle\sum_{e \in \mathrm{Sti}_0(I)} \Delta_e(\boldsymbol{u}, d\boldsymbol{v}),$$

$$S_I(\boldsymbol{u}, d\boldsymbol{v}d\boldsymbol{w}) = S_I(\boldsymbol{u}, d(\Delta\boldsymbol{v} \times \Delta\boldsymbol{w})) = \textstyle\sum_{e \in \mathrm{Sti}_0(I)} \Delta_e(\boldsymbol{u}, d\boldsymbol{v}d\boldsymbol{w}),$$

$$S_I(\boldsymbol{u}, |d\boldsymbol{v}|) = S_I(\boldsymbol{u}, d|\Delta\boldsymbol{v}|) = \textstyle\sum_{e \in \mathrm{Sti}_0(I)} \Delta_e(\boldsymbol{u}, |d\boldsymbol{v}|).$$

613G Proposition Suppose that I is a finite sublattice of $\mathcal{T}$, $\psi : I^{2\uparrow} \to L^0(\mathfrak{A})$ is an adapted interval function and $\boldsymbol{u} = \langle u_\tau \rangle_{\tau \in I}$ is a fully adapted process.

(a)(i) If $\tau \in I$ then $S_I(\boldsymbol{u}, d\psi) = S_{I\wedge\tau}(\boldsymbol{u}, d\psi) + S_{I\vee\tau}(\boldsymbol{u}, d\psi)$.

(ii) If $\tau_0, \dots, \tau_n \in I$ and $\min I = \tau_0 \le \tau_1 \le \dots \le \tau_n = \max I$, then $S_I(\boldsymbol{u}, d\psi) = \sum_{i=0}^{n-1} S_{I\cap[\tau_i,\tau_{i+1}]}(\boldsymbol{u}, d\psi)$.

(b) For $\tau \in I$ set $z_\tau = S_{I\wedge\tau}(\boldsymbol{u}, d\psi)$. Then $\langle z_\tau\rangle_{\tau\in I}$ is a fully adapted process.

(c) If τ, $\tau' \in I$ then $S_{I\wedge\tau}(\boldsymbol{u}, d\psi) + S_{I\wedge\tau'}(\boldsymbol{u}, d\psi) = S_{I\wedge(\tau\vee\tau')}(\boldsymbol{u}, d\psi) + S_{I\wedge(\tau\wedge\tau')}(\boldsymbol{u}, d\psi)$.

(d) $[\![S_I(\boldsymbol{u}, d\psi) \ne 0]\!] \subseteq [\![\boldsymbol{u} \ne \boldsymbol{0}]\!] \cap [\![\min I < \max I]\!]$.

(e) If $\boldsymbol{v} = \langle v_\tau\rangle_{\tau\in I}$ is another fully adapted process, then $S_I(\boldsymbol{u}, d(\boldsymbol{v}\psi)) = S_I(\boldsymbol{u}\times\boldsymbol{v}, d\psi)$.

613H Definitions (a) For a lattice $\mathcal{S}$, write $\mathcal{I}(\mathcal{S})$ for the family of finite sublattices of $\mathcal{S}$.

(b) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u}$ a fully adapted process with domain including $\mathcal{S}$ and ψ an adapted interval function defined on $\mathcal{S}^{2\uparrow}$. Then I define the **integral** of $\boldsymbol{u}$ over $\mathcal{S}$ with respect to ψ to be

$$\int_{\mathcal{S}} \boldsymbol{u}\, d\psi = \lim_{I\uparrow\mathcal{I}(\mathcal{S})} S_I(\boldsymbol{u}, d\psi)$$

if the limit is defined for the topology of convergence in measure.

(c) Note that if, in (b), we set $\psi' = \boldsymbol{u}\psi$, then

$$\int_{\mathcal{S}} d\psi' = \int_{\mathcal{S}} \mathbf{1}\, d\psi' = \int_{\mathcal{S}} \boldsymbol{u}\, d\psi$$

if either integral is defined.

613I Invariance under change of law The integral $\int_{\mathcal{S}} \boldsymbol{u}\, d\psi$ depends on the process $\boldsymbol{u}$, the interval function ψ and the lattice $\mathcal{S}$; behind these declared variables lie the undeclared structure $(\mathfrak{A}, T, \langle\mathfrak{A}_t\rangle_{t\in T})$ and the derived objects $L^0 = L^0(\mathfrak{A})$ and $\mathcal{T}$. But we do not really need the measure $\bar\mu$. What we use is the topology of convergence in measure on L^0. Now this topology can be defined in terms of the Boolean algebra structure of $\mathfrak{A}$.

So the Riemann-sum integral is law-independent, and we shall always be at liberty to replace the measure $\bar\mu$ by another strictly positive countably additive functional on $\mathfrak{A}$.

613J Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\operatorname{dom}\boldsymbol{u}}$ a fully adapted process with $\mathcal{S} \subseteq \operatorname{dom}\boldsymbol{u}$, and ψ an adapted interval function defined on $\mathcal{S}^{2\uparrow}$.

(a) Suppose that for every $\epsilon > 0$ there are a $z \in L^0(\mathfrak{A})$ and a $J \in \mathcal{I}(\mathcal{S})$ such that $\theta(S_I(\boldsymbol{u}, d\psi) - z) \le \epsilon$ whenever $J \subseteq I \in \mathcal{I}(\mathcal{S})$. Then $\int_{\mathcal{S}} \boldsymbol{u}\, d\psi$ is defined.

(b) If $\boldsymbol{u}'$ is another fully adapted process defined on $\mathcal{S}$, ψ' is another adapted interval function defined on $\mathcal{S}^{\uparrow 2}$, and $\int_{\mathcal{S}} \boldsymbol{u}\, d\psi$, $\int_{\mathcal{S}} \boldsymbol{u}'\, d\psi$ and $\int_{\mathcal{S}} \boldsymbol{u}\, d\psi'$ are all defined, then $\int_{\mathcal{S}} \boldsymbol{u} + \boldsymbol{u}'\, d\psi$ and $\int_{\mathcal{S}} \boldsymbol{u}\, d(\psi + \psi')$ are defined and

$$\int_{\mathcal{S}} \boldsymbol{u} + \boldsymbol{u}'\, d\psi = \int_{\mathcal{S}} \boldsymbol{u}\, d\psi + \int_{\mathcal{S}} \boldsymbol{u}'\, d\psi, \qquad \int_{\mathcal{S}} \boldsymbol{u}\, d(\psi + \psi') = \int_{\mathcal{S}} \boldsymbol{u}\, d\psi + \int_{\mathcal{S}} \boldsymbol{u}\, d\psi'.$$

Similarly, for any $\alpha \in \mathbb{R}$, $\int_{\mathcal{S}} \alpha\boldsymbol{u}\, d\psi$ and $\int_{\mathcal{S}} \boldsymbol{u}\, d(\alpha\psi)$ are defined and equal to $\alpha\int_{\mathcal{S}} \boldsymbol{u}\, d\psi$.

(c)(i) Suppose that $\tau \in \mathcal{S}$. Then

$$\int_{\mathcal{S}} \boldsymbol{u}\, d\psi = \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\psi + \int_{\mathcal{S}\vee\tau} \boldsymbol{u}\, d\psi$$

if either side is defined.

(ii) Suppose that $\tau_0 \le \dots \le \tau_n$ in $\mathcal{S}$. Then

$$\int_{\mathcal{S}} \boldsymbol{u}\, d\psi = \int_{\mathcal{S}\wedge\tau_0} \boldsymbol{u}\, d\psi + \sum_{i=0}^{n-1} \int_{\mathcal{S}\cap[\tau_i,\tau_{i+1}]} \boldsymbol{u}\, d\psi + \int_{\mathcal{S}\vee\tau_n} \boldsymbol{u}\, d\psi$$

if either side is defined.

(d) If $z = \int_{\mathcal{S}} \boldsymbol{u}\, d\psi$ is defined, then

$$\begin{aligned}[\![z \ne 0]\!] &\subseteq \sup_{\sigma\in\mathcal{S}} [\![u_\sigma \ne 0]\!] \cap \sup_{(\sigma,\tau)\in\mathcal{S}^{2\uparrow}} [\![\psi(\sigma,\tau) \ne 0]\!] \\ &\subseteq \sup_{\sigma,\tau\in\mathcal{S}} ([\![u_\sigma \ne 0]\!] \cap [\![\sigma < \tau]\!]) \subseteq [\![\boldsymbol{u} \ne \boldsymbol{0}]\!].\end{aligned}$$

(e) Set $\mathcal{S}' = \{\tau : \tau \in \mathcal{S}, \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\psi \text{ is defined}\}$.

(i) $\mathcal{S}'$ is an ideal of $\mathcal{S}$.

(ii) Setting $z_\tau = \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\psi$ for $\tau \in \mathcal{S}'$, $\langle z_\tau\rangle_{\tau\in\mathcal{S}'}$ is fully adapted.

(iii) If $\tau \in \mathcal{S}$ and $\sup_{\tau' \in \mathcal{S}'} [\![\tau' = \tau]\!] = 1$, then $\tau \in \mathcal{S}'$.

(f) Suppose that $\mathcal{S} \neq \emptyset$ and $z = \int_{\mathcal{S}} \boldsymbol{u}\, d\psi$ is defined. Set $z_\tau = \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\psi$ for $\tau \in \mathcal{S}$.

(i) The starting value $\lim_{\tau\downarrow\mathcal{S}} z_\tau$ is 0.

(ii) $\lim_{\tau\uparrow\mathcal{S}} \int_{\mathcal{S}\vee\tau} \boldsymbol{u}\, d\psi = 0$, $\lim_{\tau\uparrow\mathcal{S}} z_\tau = z$.

(g) Let $\boldsymbol{v}$ be another fully adapted process with domain $\mathcal{S}$. Then $\int_{\mathcal{S}} \boldsymbol{u}\, d(\boldsymbol{v}\psi) = \int_{\mathcal{S}} \boldsymbol{u} \times \boldsymbol{v}\, d\psi$ in the sense that if one is defined so is the other, and they are then equal.

613L More easy bits (a) If $\mathcal{S}$ is a sublattice of $\mathcal{T}$ and $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are fully adapted processes defined on $\mathcal{S}$, I will write

$$\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v} = \int_{\mathcal{S}} \boldsymbol{u}\, d(\Delta\boldsymbol{v}) = \lim_{I\uparrow\mathcal{I}(\mathcal{S})} S_I(\boldsymbol{u}, d\boldsymbol{v}),$$

$$\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}d\boldsymbol{w} = \int_{\mathcal{S}} \boldsymbol{u}\, d(\Delta\boldsymbol{v} \times \Delta\boldsymbol{w}) = \lim_{I\uparrow\mathcal{I}(\mathcal{S})} S_I(\boldsymbol{u}, d\boldsymbol{v}d\boldsymbol{w}),$$

$$\int_{\mathcal{S}} \boldsymbol{u}\, |d\boldsymbol{v}| = \int_{\mathcal{S}} \boldsymbol{u}\, d|\Delta\boldsymbol{v}| = \lim_{I\uparrow\mathcal{I}(\mathcal{S})} S_I(\boldsymbol{u}, |d\boldsymbol{v}|)$$

when the limits exist in $L^0(\mathfrak{A})$.

(b) Three trivial calculations: if $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ and $\boldsymbol{u}$ are fully adapted processes with domain a sublattice $\mathcal{S}$ of $\mathcal{T}$, then

(i) $S_I(\mathbf{1}, d\boldsymbol{v}) = v_{\max I} - v_{\min I}$ for every non-empty finite sublattice I of $\mathcal{S}$, so $\int_{\mathcal{S}\cap[\tau,\tau']} \mathbf{1}\, d\boldsymbol{v} = v_{\tau'} - v_\tau$ whenever $\tau \leq \tau'$ in $\mathcal{S}$;

(ii) if $\boldsymbol{v}$ is constant then $S_I(\boldsymbol{u}, d\boldsymbol{v}) = 0$ for every $I \in \mathcal{I}(\mathcal{S})$, so $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$ is defined and equal to 0;

(iii) if $z \in L^0(\bigcap_{\sigma\in\mathcal{S}} \mathfrak{A}_\sigma)$, then $S_I(z\boldsymbol{u}, d\boldsymbol{v}) = S_I(\boldsymbol{u}, d(z\boldsymbol{v})) = z \times S_I(\boldsymbol{u}, d\boldsymbol{v})$ for every $I \in \mathcal{I}(\mathcal{S})$, so $\int_{\mathcal{S}} z\boldsymbol{u}\, d\boldsymbol{v}$ and $\int_{\mathcal{S}} \boldsymbol{u}\, d(z\boldsymbol{v})$ are defined and equal to $z \times \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$ if the last integral is defined.

(c) Suppose that I is a finite sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in I}$, $\boldsymbol{u}' = \langle u'_\sigma\rangle_{\sigma\in I}$, $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in I}$ and $\boldsymbol{v}' = \langle v'_\sigma\rangle_{\sigma\in I}$ are fully adapted processes. Set $d = \sup_{\sigma\in I} [\![u_\sigma \neq u'_\sigma]\!] \cup [\![v_\sigma \neq v'_\sigma]\!]$. $[\![S_I(\boldsymbol{u}, d\boldsymbol{v}) \neq S_I(\boldsymbol{u}', d\boldsymbol{v}')]\!] \subseteq d$.

Similarly,

$$[\![S_I(\boldsymbol{u}, (d\boldsymbol{v})^2) \neq S_I(\boldsymbol{u}', (d\boldsymbol{v}')^2)]\!] \subseteq d, \quad [\![S_I(\boldsymbol{u}, |d\boldsymbol{v}|) \neq S_I(\boldsymbol{u}', |d\boldsymbol{v}'|)]\!] \subseteq d.$$

Indeed, if ψ, ψ' are any adapted interval functions defined on $\mathcal{I}^{2\uparrow}$, and we set

$$d = \sup_{\sigma\in I} [\![u_\sigma \neq u'_\sigma]\!] \cup \sup_{\sigma\leq\tau \text{ in } I} [\![\psi(\sigma,\tau) \neq \psi'(\sigma,\tau)]\!],$$

then $[\![S_I(\boldsymbol{u}, d\psi) \neq S_I(\boldsymbol{u}', d\psi')]\!] \subseteq d$.

(d) It follows that if $\mathcal{S}$ is any sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$, $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ are fully adapted processes such that $z = \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$ is defined, then $[\![z \neq 0]\!] \subseteq [\![\boldsymbol{u} \neq \mathbf{0}]\!] \cap [\![\boldsymbol{v} \neq \mathbf{0}]\!]$.

613M Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ fully adapted processes defined on $\mathcal{S}$. Then

$$\begin{aligned} S_I(\boldsymbol{u}, d\boldsymbol{v}d\boldsymbol{w}) &= S_I(\boldsymbol{u}, d(\boldsymbol{v}\times\boldsymbol{w})) - S_I(\boldsymbol{u}\times\boldsymbol{v}, d\boldsymbol{w}) - S_I(\boldsymbol{u}\times\boldsymbol{w}, d\boldsymbol{v}) \\ &= \frac{1}{2}\big(S_I(\boldsymbol{u}, (d(\boldsymbol{v}+\boldsymbol{w}))^2) - S_I(\boldsymbol{u}, (d\boldsymbol{v})^2) - S_I(\boldsymbol{u}, (d\boldsymbol{w})^2)\big) \end{aligned}$$

for every finite sublattice I of $\mathcal{S}$. Consequently

$$\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}d\boldsymbol{w} = \int_{\mathcal{S}} \boldsymbol{u}\, d(\boldsymbol{v}\times\boldsymbol{w}) - \int_{\mathcal{S}} \boldsymbol{u}\times\boldsymbol{v}\, d\boldsymbol{w} - \int_{\mathcal{S}} \boldsymbol{u}\times\boldsymbol{w}\, d\boldsymbol{v}$$

if any three of the four integrals are defined, and

$$\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}d\boldsymbol{w} = \frac{1}{2}\Big(\int_{\mathcal{S}} \boldsymbol{u}\, (d(\boldsymbol{v}+\boldsymbol{w}))^2 - \int_{\mathcal{S}} \boldsymbol{u}\, (d\boldsymbol{v})^2 - \int_{\mathcal{S}} \boldsymbol{u}\, (d\boldsymbol{w})^2\Big)$$

if any three of the integrals are defined.

613N Proposition Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a fully adapted process. Interpreting $\int_{\mathcal{S}} d\boldsymbol{v}$ as $\int_{\mathcal{S}} \mathbf{1}\, d\boldsymbol{v}$ where $\mathbf{1}$ is the constant process with value $\chi 1$, $\int_{\mathcal{S}} d\boldsymbol{v}$ is defined iff $v_\downarrow = \lim_{\sigma\downarrow\mathcal{S}} v_\sigma$ and $v_\uparrow = \lim_{\sigma\uparrow\mathcal{S}} v_\sigma$ are defined, and in this case $\int_{\mathcal{S}} d\boldsymbol{v} = v_\uparrow - v_\downarrow$.

613O Indefinite integrals (a) Definition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u}$ a fully adapted process with domain $\mathcal{S}$, and ψ an adapted interval function with domain $\mathcal{S}^{2\uparrow}$. Set $\mathcal{S}' = \{\tau : \tau \in \mathcal{S}, \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\,d\psi \text{ is defined}\}$; $\mathcal{S}'$ is an ideal of $\mathcal{S}$. The **indefinite integral** of $\boldsymbol{u}$ with respect to ψ is the process $ii_\psi(\boldsymbol{u}) = \langle\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\,d\psi\rangle_{\tau\in\mathcal{S}'}$; this is a fully adapted process.

When ψ is of the form $\Delta\boldsymbol{v}$ for a fully adapted process $\boldsymbol{v}$, I will write $ii_{\boldsymbol{v}}(\boldsymbol{u}) = \langle\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\,d\boldsymbol{v}\rangle_{\tau\in\mathcal{S}'}$ for the indefinite integral of $\boldsymbol{u}$ with respect to $\boldsymbol{v}$.

(b)(i) Note that if $\int_{\mathcal{S}} \boldsymbol{u}\,d\psi$ is defined, the domain $\mathcal{S}'$ of $ii_\psi(\boldsymbol{u})$ is the whole of $\mathcal{S}$.

(ii) It is obvious from the definition, but perhaps it is worth stating formally that if $\tau \in \mathcal{S}$ and $\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\,d\psi$ is defined then

$$\begin{aligned} ii_\psi(\boldsymbol{u})\restriction\mathcal{S}\wedge\tau &= \langle\int_{\mathcal{S}\wedge\sigma} \boldsymbol{u}\,d\psi\rangle_{\sigma\in\mathcal{S}\wedge\tau} = \langle\int_{\mathcal{S}\wedge\sigma} (\boldsymbol{u}\restriction\mathcal{S}\wedge\tau)d(\psi\restriction(\mathcal{S}\wedge\tau)^{2\uparrow})\rangle_{\sigma\in\mathcal{S}\wedge\tau} \\ &= ii_{\psi\restriction(\mathcal{S}\wedge\tau)^{2\uparrow}}(\boldsymbol{u}\restriction\mathcal{S}\wedge\tau). \end{aligned}$$

(iii) On the other side, if $\mathcal{S}' = \mathcal{S}$ and $\tau \in \mathcal{S}$, then $ii_{\psi\restriction(\mathcal{S}\vee\tau)^{2\uparrow}}(\boldsymbol{u}\restriction\mathcal{S}\vee\tau)$ is defined on the whole of $\mathcal{S}\vee\tau$ and is equal to $(ii_\psi(\boldsymbol{u})\restriction\mathcal{S}\vee\tau) - (\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\,d\psi)\mathbf{1}$.

613R Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u}$, $\boldsymbol{v}$ fully adapted processes with domain $\mathcal{S}$. Then $[\![ii_{\boldsymbol{v}}(\boldsymbol{u}) \ne \mathbf{0}]\!] \subseteq [\![\boldsymbol{u} \ne \mathbf{0}]\!] \cap [\![\boldsymbol{v} \ne \mathbf{0}]\!]$.

613S Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\psi : \mathcal{S}^{2\uparrow} \to L^0(\mathfrak{A})$ a strictly adapted interval function. Suppose that I, $J \in \mathcal{I}(\mathcal{S})$, $J \subseteq I$ and $a \subseteq \sup_{\sigma\in J} [\![\tau = \sigma]\!]$ for every $\tau \in I$. Then $a \subseteq [\![S_I(\mathbf{1}, d\psi) = S_J(\mathbf{1}, d\psi)]\!]$. In particular, if J covers I then $S_I(\mathbf{1}, d\psi) = S_J(\mathbf{1}, d\psi)$.

613T Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\mathcal{S}'$ a sublattice of $\mathcal{S}$ which covers $\mathcal{S}$, $\psi : \mathcal{S}^{2\uparrow} \to L^0(\mathfrak{A})$ a strictly adapted interval function and $\boldsymbol{u} : \mathcal{S} \to L^0(\mathfrak{A})$ a fully adapted process. If $\int_{\mathcal{S}} \boldsymbol{u}\,d\psi$ is defined, so is $\int_{\mathcal{S}'} \boldsymbol{u}\,d\psi$, and the integrals are equal.

613P Example If $T = [0, \infty[$, $(\mathfrak{A}, \bar\mu)$ is the measure algebra of Lebesgue measure on $[0, 1]$ and $\mathfrak{A}_t = \mathfrak{A}$ for every $t \ge 0$, then there are a sublattice $\mathcal{S}$ of $\mathcal{T}$ and fully adapted processes $\boldsymbol{u}$, $\boldsymbol{v}$ with domain $\mathcal{S}$ such that $\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}$ is defined but $\int_{\hat{\mathcal{S}}} \hat{\boldsymbol{u}}\,d\hat{\boldsymbol{v}}$ is not, where $\hat{\mathcal{S}}$ is the covered envelope of $\mathcal{S}$ and $\hat{\boldsymbol{u}}$, $\hat{\boldsymbol{v}}$ are the fully adapted extensions of $\boldsymbol{u}$ and $\boldsymbol{v}$ to $\hat{\mathcal{S}}$.

613U Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\hat{\mathcal{S}}$ its covered envelope.

(a) For every strictly adapted interval function $\psi : \mathcal{S}^{\uparrow 2} \to L^0$ there is a unique extension of ψ to a strictly adapted interval function $\hat\psi : \hat{\mathcal{S}}^{2\uparrow} \to L^0$.

(b)(i) The function $\psi \mapsto \hat\psi$ is an f-algebra homomorphism from the space of strictly adapted interval functions on $\mathcal{S}$ to the space of strictly adapted interval functions on $\hat{\mathcal{S}}$.

(ii) If ψ is a strictly adapted interval function on $\mathcal{S}$ and $h : \mathbb{R} \to \mathbb{R}$ is Borel measurable, then $(\bar h\psi)\hat{} = \bar h\hat\psi$.

(iii) If ψ is a strictly adapted interval function on $\mathcal{S}$ and $\boldsymbol{u}$ is a fully adapted process with domain $\mathcal{S}$, then $(\boldsymbol{u}\psi)\hat{} = \hat{\boldsymbol{u}}\hat\psi$, where $\hat{\boldsymbol{u}}$ is the fully adapted extension of $\boldsymbol{u}$ to $\hat{\mathcal{S}}$.

613V Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u}$ a fully adapted process with domain $\mathcal{S}$, and ψ an adapted interval function with domain $\mathcal{S}^{2\uparrow}$ such that $\int_{\mathcal{S}} \boldsymbol{u}\,d\psi$ is defined. Let $I \in \mathcal{I}(\mathcal{S})$ and $\epsilon > 0$ be such that $\theta(S_J(\boldsymbol{u}, d\psi) - S_K(\boldsymbol{u}, d\psi)) \le \epsilon$ whenever J, $K \in \mathcal{I}(\mathcal{S})$ include I.

(i) If $\tau_0 \le \tau_0' \le \tau_1 \le \tau_1' \le \ldots \le \tau_n \le \tau_n'$ in I, then

$$\theta(\textstyle\sum_{i=0}^n (S_{I\cap[\tau_i,\tau_i']}(\boldsymbol{u}, d\psi) - \int_{\mathcal{S}\cap[\tau_i,\tau_i']} \boldsymbol{u}\,d\psi)) \le \epsilon.$$

(ii)(α) If $\tau \in I$ then $\theta(S_{I\wedge\tau}(\boldsymbol{u}, d\psi) - \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\,d\psi) \le \epsilon$.

(β) For any $\tau \in \mathcal{S}$, $\theta(S_{I\wedge\tau}(\boldsymbol{u}, d\psi) - \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\,d\psi) \le 2\epsilon$.

613W The one-dimensional case (a) Suppose that $(\mathfrak{A}, \bar{\mu})$ is the trivial probability algebra in which $\mathfrak{A} = \{0, 1\}$. Then $L^0(\mathfrak{A}) = \{\alpha\chi 1 : \alpha \in \mathbb{R}\}$ can be identified, as f-algebra, with $\mathbb{R}$; of course we have $\theta(\alpha\chi 1) = \min(1, |\alpha|)$ for every α, so the topology of convergence in measure on $L^0(\mathfrak{A})$ corresponds to the usual topology of $\mathbb{R}$. Necessarily, $\mathfrak{A}_t = \mathfrak{A}$ for every $t \in T$, so the filtration is trivial. If it is also the case that T has no points isolated on the right, then every stopping time except $\max \mathcal{T}$ and possibly $\min \mathcal{T}$ will be a constant stopping time as described in 611A(b-ii), every subset of $\mathcal{T}$ is a sublattice, and every real-valued function f defined on a subset S of T corresponds to a fully adapted process $\{(\check{s}, f(s)\chi 1) : s \in S\}$.

(b) If also T has a least element, we can identify $\mathcal{T}_f$ with T and $M_{\mathrm{fa}}(\mathcal{T}_f) = (L^0)^{\mathcal{T}_f}$ with $\mathbb{R}^T$. Under this identification, if $f : T \to \mathbb{R}$ and $g : T \to \mathbb{R}$ represent processes $\boldsymbol{u}$, $\boldsymbol{v}$ with domain $\mathcal{T}_f$, and $I \subseteq \mathcal{T}_f$ is a non-empty finite set, there are $t_0 \le \ldots \le t_n$ in T such that $I = \{\check{t}_i : i \le n\}$, and

$$S_I(\boldsymbol{u}, d\boldsymbol{v}) = \big(\textstyle\sum_{i=0}^{n-1} f(t_i)(g(t_{i+1}) - g(t_i))\big)\chi 1.$$

(d) What this amounts to is that we have a kind of Riemann-Stieltjes integral on T, I spell this out in detail here partly because there are well-known Stieltjes integrals on the real line, of which the most important, from the point of view of my treatise as a whole, is integration with respect to Lebesgue-Stieltjes measures described in exercises from 114Xa onwards. Here we suppose that $g : \mathbb{R} \to \mathbb{R}$ is non-decreasing, so that there is a Radon measure ν_g on $\mathbb{R}$ with $\nu_g\,[a, b[= \lim_{x\uparrow b} g(x) - \lim_{x\uparrow a} g(x)$ whenever $a < b$ in $\mathbb{R}$. Now the point I need to make here is that if $\mathcal{S} = \{\check{s} : s \in \mathbb{R}\}$ then the integral $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$ is *not* the same as the Lebesgue-Stieltjes integral $\int f\, d\nu_g$. Consider the case in which $f = g = \chi\,[0, \infty[$. ν_g is the Dirac measure concentrated at 0, so that $\int f\, d\nu_g = f(0) = 1$. But when we look at sums $S_I(\boldsymbol{u}, d\boldsymbol{v})$ where $I = \{\check{t}_0, \ldots, \check{t}_n\}$ is a finite subset of $\mathcal{S}$, and supposing that $t_0 \le \ldots \le t_n$, we get

$$\begin{aligned} f(t_i)(g(t_{i+1}) - g(t_i)) &= 0(g(t_{i+1}) - g(t_i)) = 0 \text{ if } t_i < 0, \\ &= f(t_i)(1 - 1) = 0 \text{ if } t_i \ge 0, \end{aligned}$$

so $S_I(f, dg) = 0$; as this is true for every I, $\int_{\mathcal{S}} f\, dg = 0$. In the language of Lebesgue-Stieltjes integration, we are calculating $\int f_-\, d\nu_g$ where $f_-(x) = \lim_{y\uparrow x} f(y)$ for each x.

In my view, a theory of stochastic integration should insist on calculating integrals $\int \boldsymbol{u}\, d\boldsymbol{v}$ in terms of products $u_\sigma \times (v_\tau - v_\sigma)$ where $\sigma \le \tau$ (rather than $u_\tau \times (v_\tau - v_\sigma)$, for instance). We are going to have to return to this point from time to time, because it is one on which my presentation of the theory differs from that of most authors.

Version of 29.10.24

614 Simple and order-bounded processes and bounded variation

In §613 I gave a definition of an integral with no very useful indication of where it might be applicable. This section and the next two will be devoted to teasing out the basic case in which a Riemann-sum integral $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$ is defined: $\boldsymbol{u}$ should be 'moderately oscillatory' (615E) and $\boldsymbol{v}$ should be an 'integrator' (616K). Before we come to either of these notions, however, it will be helpful to have a firm grasp of three easier concepts: 'simple' processes (614B), 'order-bounded' processes (614E) and processes 'of bounded variation' (614J-614K).

614B Proposition Suppose that $\mathcal{S}$ is a non-empty sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$ a simple fully adapted process with a breakpoint string $(\tau_0, \ldots, \tau_n)$.

(a) The starting value $u_\downarrow = \lim_{\sigma\downarrow\mathcal{S}} u_\sigma$ is defined, and $[\![\sigma < \tau_0]\!] \subseteq [\![u_\sigma = u_\downarrow]\!]$ for every $\sigma \in \mathcal{S}$.

(b) Suppose that $\psi : \mathcal{S}^{2\uparrow} \to L^0$ is an adapted interval function such that $\int_{\mathcal{S}} d\psi$ is defined. Then $\int_{\mathcal{S}} \boldsymbol{u}\, d\psi$ is defined and equal to

$$u_\downarrow \times v_{\tau_0} + \textstyle\sum_{i=0}^{n-1} u_{\tau_i} \times (v_{\tau_{i+1}} - v_{\tau_i}) + u_{\tau_n} \times (v_\uparrow - v_{\tau_n})$$

where $v_\tau = \int_{\mathcal{S}\wedge\tau} d\psi$ for $\tau \in \mathcal{S}$, and $v_\uparrow = \int_{\mathcal{S}} d\psi$.

614C Corollary Suppose that $\mathcal{S}$ is a non-empty sublattice of $\mathcal{T}$, $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$ is a simple fully adapted process with starting value $u_\downarrow$ and a breakpoint string $(\tau_0, \dots, \tau_n)$, and $\boldsymbol{v} = \langle v_\tau \rangle_{\tau\in\mathcal{S}}$ is a fully adapted process such that $v_\uparrow = \lim_{\tau\uparrow\mathcal{S}} v_\tau$ and $v_\downarrow = \lim_{\sigma\downarrow\mathcal{S}} v_\sigma$ are defined. Then $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$ is defined and equal to

$$u_\downarrow \times (v_{\tau_0} - v_\downarrow) + \sum_{i=0}^{n-1} u_{\tau_i} \times (v_{\tau_{i+1}} - v_{\tau_i}) + u_{\tau_n} \times (v_\uparrow - v_{\tau_n}).$$

614D Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma\in\mathcal{S}}$ a simple process, and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$ a fully adapted process such that $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$ is defined. Then $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is simple.

614E Order-bounded processes: Definitions Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$ a fully adapted process.

(a) $\boldsymbol{u}$ is **order-bounded** if $\{u_\sigma : \sigma \in \mathcal{S}\}$ is bounded above and below in L^0. In this case, $\sup|\boldsymbol{u}| = \sup_{\sigma\in\mathcal{S}} |u_\sigma|$, taking the supremum in $(L^0)^+$, so that $\sup|\boldsymbol{u}| = 0$ if $\mathcal{S} = \operatorname{dom}\boldsymbol{u}$ is empty.

(b) $\boldsymbol{u}$ is **locally order-bounded** if $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau = \langle u_\sigma \rangle_{\sigma\in\mathcal{S},\sigma\le\tau}$ is order-bounded for every $\tau \in \mathcal{S}$.

(c) Suppose that $\mathcal{S}$ is non-empty and that $\boldsymbol{u}$ is simple, with breakpoint string $(\tau_0, \dots, \tau_n)$ and starting value $u_\downarrow$. Then $\boldsymbol{u}$ is order-bounded and $\sup|\boldsymbol{u}| = |u_\downarrow| \vee \sup_{i\le n} |u_{\tau_i}|$.

614F Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.

(a)(i) If $\boldsymbol{u}$ is an order-bounded process with domain $\mathcal{S}$, then $\boldsymbol{u}\restriction\mathcal{S}'$ is order-bounded for any sublattice $\mathcal{S}'$ of $\mathcal{S}$; in particular, $\boldsymbol{u}$ is locally order-bounded.

(ii) If $\boldsymbol{u}$ is a locally order-bounded process with domain $\mathcal{S}$, then $\boldsymbol{u}\restriction\mathcal{S}'$ is locally order-bounded for any sublattice $\mathcal{S}'$ of $\mathcal{S}$.

(b) Suppose that $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$ is a locally order-bounded process. Set $v_\tau = \sup_{\sigma\in\mathcal{S}\wedge\tau} |u_\sigma|$ for $\tau \in \mathcal{S}$. Then $\boldsymbol{v} = \langle v_\tau \rangle_{\tau\in\mathcal{S}}$ is a non-decreasing fully adapted process.

(c) Write $M_{\text{o-b}} = M_{\text{o-b}}(\mathcal{S})$ for the set of order-bounded fully adapted processes with domain $\mathcal{S}$.

(i) If $h : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function which is bounded on every bounded interval in $\mathbb{R}$, then $\bar{h}\boldsymbol{u} \in M_{\text{o-b}}$ for every $\boldsymbol{u} \in M_{\text{o-b}}$.

(ii) $M_{\text{o-b}}$ is an f-subalgebra of $\prod_{\sigma\in\mathcal{S}} L^0(\mathfrak{A}_\sigma)$.

(iii) If $z \in L^0(\mathfrak{A} \cap \bigcap_{\sigma\in\mathcal{S}} \mathfrak{A}_\sigma)$ then $z\boldsymbol{u} \in M_{\text{o-b}}$, with $\sup|z\boldsymbol{u}| = |z| \times \sup|\boldsymbol{u}|$, for every $\boldsymbol{u} \in M_{\text{o-b}}$.

(d) Write $M_{\text{lob}} = M_{\text{lob}}(\mathcal{S})$ for the set of locally order-bounded fully adapted processes with domain $\mathcal{S}$.

(i) If $h : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function which is bounded on every bounded interval in $\mathbb{R}$, then $\bar{h}\boldsymbol{u} \in M_{\text{lob}}$ for every $\boldsymbol{u} \in M_{\text{lob}}$.

(ii) M_{lob} is an f-subalgebra of $\prod_{\sigma\in\mathcal{S}} L^0(\mathfrak{A}_\sigma)$.

614G Proposition Suppose that $\mathcal{S}$ is a sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$ a fully adapted process.

(a) If A, $B \subseteq \mathcal{S}$, A covers B and $\{u_\sigma : \sigma \in A\}$ is order-bounded, then $\{u_\sigma : \sigma \in B\}$ is order-bounded and $\sup_{\sigma\in B} |u_\sigma| \le \sup_{\sigma\in A} |u_\sigma|$.

(b) If $\mathcal{S}'$ is a sublattice of $\mathcal{S}$ which covers $\mathcal{S}$

(i) $\boldsymbol{u}$ is order-bounded iff $\boldsymbol{u}\restriction\mathcal{S}'$ is order-bounded, and in this case $\sup|\boldsymbol{u}| = \sup|\boldsymbol{u}\restriction\mathcal{S}'|$,

(ii) $\boldsymbol{u}$ is locally order-bounded iff $\boldsymbol{u}\restriction\mathcal{S}'$ is locally order-bounded.

614H Proposition Brownian motion, as described in 612T, is locally order-bounded.

614I Non-decreasing processes Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma\in\mathcal{S}}$ a non-decreasing fully adapted process.

(a) $\boldsymbol{v}$ is a lattice homomorphism.

(b)

$$[\![\sigma \le \tau]\!] \subseteq [\![v_\sigma = v_{\sigma\wedge\tau}]\!] \cap [\![v_\tau = v_{\sigma\vee\tau}]\!] \subseteq [\![v_\sigma \le v_\tau]\!]$$

for all σ, $\tau \in \mathcal{S}$.

(c) If $\boldsymbol{v}$ is non-negative it is locally order-bounded.

(d) If $\mathcal{S} \neq \emptyset$ and $\boldsymbol{v}$ is order-bounded, then

$$\int_{\mathcal{S}} |d\boldsymbol{v}| = \int_{\mathcal{S}} d\boldsymbol{v} = v_\uparrow - v_\downarrow$$

where $v_\uparrow = \sup_{\sigma \in \mathcal{S}} v_\sigma = \lim_{\sigma\uparrow\mathcal{S}} v_\sigma$ and $v_\downarrow = \inf_{\sigma\in\mathcal{S}} v_\sigma = \lim_{\sigma\downarrow\mathcal{S}} v_\sigma$.

(e) Suppose that $w \in (L^0)^+$. For each $\sigma \in \mathcal{S}$, set $w_\sigma = \sup\{x : x \in L^0(\mathfrak{A}_\sigma,\, x \leq w\}$. Now $\boldsymbol{w} = \langle w_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a non-negative non-decreasing fully adapted process.

$|\boldsymbol{u}| \leq \boldsymbol{w}$ whenever $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is fully adapted and $|\sup \boldsymbol{u}| \leq w$.

(f) If $h : \mathbb{R} \to \mathbb{R}$ is non-decreasing, then $\bar{h}\boldsymbol{v}$ is non-decreasing.

(g) If $\boldsymbol{u}$ is non-negative and fully adapted and $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$ is defined, then $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v} \geq 0$ and $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is non-decreasing.

614J Bounded variation: Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a fully adapted process. Then the following are equiveridical:

(i) $\boldsymbol{v}$ is expressible as the difference of two order-bounded non-negative non-decreasing fully adapted processes,

(ii) $\{\sum_{i=0}^{n-1} |v_{\tau_{i+1}} - v_{\tau_i}| : \tau_0 \leq \ldots \leq \tau_n \text{ in } \mathcal{S}\}$ is bounded above in L^0,

(iii) $\int_{\mathcal{S}} |d\boldsymbol{v}|$ is defined;

and in this case

$$\int_{\mathcal{S}} |d\boldsymbol{v}| = \sup\{\textstyle\sum_{i=0}^{n-1} |v_{\tau_{i+1}} - v_{\tau_i}| : \tau_0 \leq \ldots \leq \tau_n \text{ in } \mathcal{S}\}$$

if we count $\sup\emptyset$ as 0.

614K Definitions Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v}$ a fully adapted process with domain $\mathcal{S}$.

(a) $\boldsymbol{v}$ is **of bounded variation** if it satisfies the conditions of Theorem 614J.

(b) $\boldsymbol{v}$ is **locally of bounded variation** if $\boldsymbol{v}{\restriction}\mathcal{S}\wedge\tau$ is of bounded variation for every $\tau \in \mathcal{S}$.

614L Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v}$ a fully adapted process with domain $\mathcal{S}$.

(a) If $\boldsymbol{v}$ is (locally) of bounded variation it is (locally) order-bounded.

(b)(i) If $\boldsymbol{v}$ is of bounded variation and $\mathcal{S}'$ is a sublattice of $\mathcal{S}$, then $\boldsymbol{v}{\restriction}\mathcal{S}'$ is of bounded variation and $\int_{\mathcal{S}'} |d\boldsymbol{v}| \leq \int_{\mathcal{S}} |d\boldsymbol{v}|$.

(ii) If $\boldsymbol{v}$ is locally of bounded variation and $\mathcal{S}'$ is a sublattice of $\mathcal{S}$, then $\boldsymbol{v}{\restriction}\mathcal{S}'$ is locally of bounded variation.

(c) If $\tau \in \mathcal{S}$, then $\boldsymbol{v}$ is (locally) of bounded variation iff $\boldsymbol{v}{\restriction}\mathcal{S}\wedge\tau$ and $\boldsymbol{v}{\restriction}\mathcal{S}\vee\tau$ are both (locally) of bounded variation.

614M Proposition The Poisson process, as described in 612U, is locally of bounded variation.

614N Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\bar{u} \in (L^0)^+$. Then $\{\boldsymbol{v} : \boldsymbol{v} \in M_{\mathrm{fa}}(\mathcal{S})$ is of bounded variation, $\int_{\mathcal{S}} |d\boldsymbol{v}| \leq \bar{u}\}$ is closed in $(L^0)^{\mathcal{S}}$ for its product topology.

614O Cumulative variation Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v}$ a process with domain $\mathcal{S}$ which is locally of bounded variation. Then $v^\uparrow_\tau = \int_{\mathcal{S}\wedge\tau} |d\boldsymbol{v}|$ is defined for every $\tau \in \mathcal{S}$, and $\boldsymbol{v}^\uparrow = \langle \int_{\mathcal{S}\wedge\tau} v^\uparrow_\tau\rangle_{\tau\in\mathcal{S}}$ is fully adapted. I will call $\boldsymbol{v}^\uparrow$ the **cumulative variation** of $\boldsymbol{v}$.

614P Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a process which is locally of bounded variation, and $\boldsymbol{v}^\uparrow = \langle v^\uparrow_\tau\rangle_{\tau\in\mathcal{S}}$ its cumulative variation.

(a)(i) If $\sigma \leq \tau$ in $\mathcal{S}$, then

$$v^{\uparrow}_{\tau}-v^{\uparrow}_{\sigma}=\int_{\mathcal{S}\cap[\sigma,\tau]}|d\boldsymbol{v}|\ge|v_{\tau}-v_{\sigma}|\ge 0.$$

(ii) $\boldsymbol{v}^{\uparrow}$ is non-negative and non-decreasing and has starting value 0 if $\mathcal{S}$ is not empty.

(iii) $\boldsymbol{v}^{\uparrow}+\boldsymbol{v}$ and $\boldsymbol{v}^{\uparrow}-\boldsymbol{v}$ are non-decreasing.

(iv) If $\mathcal{S}$ is non-empty, $\boldsymbol{v}$ has a starting value.

(v) $\boldsymbol{v}$ is of bounded variation iff $\lim_{\tau\uparrow\mathcal{S}}v^{\uparrow}_{\tau}=\sup|\boldsymbol{v}^{\uparrow}|$ is defined, and in this case the limit is $\int_{\mathcal{S}}|d\boldsymbol{v}|$.

(b) If $\tau\in\mathcal{S}$ then, writing $(\boldsymbol{v}\restriction\mathcal{S}\wedge\tau)^{\uparrow}$ and $(\boldsymbol{v}\restriction\mathcal{S}\vee\tau)^{\uparrow}$ for the cumulative variations of $\boldsymbol{v}\restriction\mathcal{S}\wedge\tau$ and $\boldsymbol{v}\restriction\mathcal{S}\vee\tau$,

$$(\boldsymbol{v}\restriction\mathcal{S}\wedge\tau)^{\uparrow}=\boldsymbol{v}^{\uparrow}\restriction\mathcal{S}\wedge\tau,\quad(\boldsymbol{v}\restriction\mathcal{S}\vee\tau)^{\uparrow}=\boldsymbol{v}^{\uparrow}\restriction\mathcal{S}\vee\tau-v^{\uparrow}_{\tau}\mathbf{1}.$$

(c) Suppose that $I\in\mathcal{I}(\mathcal{S})$ is not empty and $(\tau_0,\dots,\tau_n)$ linearly generates the I-cells.

(i)

$$v^{\uparrow}_{\tau_{i+1}}-v^{\uparrow}_{\tau_i}\le|v_{\tau_{i+1}}-v_{\tau_i}|+v^{\uparrow}_{\max I}-v^{\uparrow}_{\min I}-S_I(\mathbf{1},|d\boldsymbol{v}|)$$

for every $i<n$.

(ii) If $\boldsymbol{v}$ is of bounded variation, write w for $\int_{\mathcal{S}}|d\boldsymbol{v}|-S_I(\mathbf{1},|d\boldsymbol{v}|)$, and let $v_{\downarrow}$ be the starting value of $\boldsymbol{v}$.

(α) If $\tau\in\mathcal{S}\wedge\tau_0$, then $v^{\uparrow}_{\tau}\le|v_{\tau}-v_{\downarrow}|+w$.

(β) If $i<n$ and $\tau\in\mathcal{S}\cap[\tau_i,\tau_{i+1}]$ then $v^{\uparrow}_{\tau}-v^{\uparrow}_{\tau_i}\le|v_{\tau}-v_{\tau_i}|+w$.

(γ) If $\tau\in\mathcal{S}\vee\tau_n$ then $v^{\uparrow}_{\tau}-v^{\uparrow}_{\tau_n}\le|v_{\tau}-v_{\tau_n}|+w$.

614Q Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\hat{\mathcal{S}}$ its covered envelope.

(a) Write $M_{\mathrm{bv}}=M_{\mathrm{bv}}(\mathcal{S})$ for the set of fully adapted processes of bounded variation with domain $\mathcal{S}$.

(i) $\bar{h}\boldsymbol{v}\in M_{\mathrm{bv}}$ whenever $\boldsymbol{v}\in M_{\mathrm{bv}}$ and $h:\mathbb{R}\to\mathbb{R}$ is Lipschitz on every bounded interval.

(ii) M_{bv} is an f-subalgebra of $M_{\text{o-b}}(\mathcal{S})$.

(iii) The space M_{simp} of simple processes with domain $\mathcal{S}$ is an f-subalgebra of M_{bv} closed under $\bar{h}$ for every Borel measurable $h:\mathbb{R}\to\mathbb{R}$.

(iv) If $\boldsymbol{v}\in M_{\mathrm{fa}}(\mathcal{S})$ and $\hat{\boldsymbol{v}}$ is its fully adapted extension to $\hat{\mathcal{S}}$, then

(α) $\boldsymbol{v}$ is non-decreasing iff $\hat{\boldsymbol{v}}$ is non-decreasing,

(β) $\boldsymbol{v}$ is of bounded variation iff $\hat{\boldsymbol{v}}$ is of bounded variation, and in this case $\int_{\hat{\mathcal{S}}}|d\hat{\boldsymbol{v}}|=\int_{\mathcal{S}}|d\boldsymbol{v}|$ and the cumulative variation $\hat{\boldsymbol{v}}^{\uparrow}$ of $\hat{\boldsymbol{v}}$ is the fully adapted extension of the cumulative variation $\boldsymbol{v}^{\uparrow}$ of $\boldsymbol{v}$.

(b) Write $M_{\mathrm{lbv}}=M_{\mathrm{lbv}}(\mathcal{S})$ for the set of fully adapted processes with domain $\mathcal{S}$ which are locally of bounded variation.

(i) If $\boldsymbol{v}\in M_{\mathrm{lbv}}(\mathcal{S})$ then $\boldsymbol{v}\restriction\mathcal{S}'$ is locally of bounded variation for every sublattice $\mathcal{S}'$ of $\mathcal{S}$.

(ii) $\bar{h}\boldsymbol{v}\in M_{\mathrm{lbv}}$ whenever $\boldsymbol{v}\in M_{\mathrm{lbv}}$ and $h:\mathbb{R}\to\mathbb{R}$ is Lipschitz on every bounded interval.

(iii) M_{lbv} is an f-subalgebra of $M_{\mathrm{lob}}(\mathcal{S})$.

(iv) If $\boldsymbol{v}\in M_{\mathrm{fa}}(\mathcal{S})$ then $\boldsymbol{v}$ is locally of bounded variation iff it is expressible as the difference of two non-negative non-decreasing fully adapted processes.

(v) If $\boldsymbol{v}\in M_{\mathrm{fa}}(\mathcal{S})$, then $\boldsymbol{v}$ is locally of bounded variation iff its fully adapted extension to $\hat{\mathcal{S}}$ is locally of bounded variation, and in this case the cumulative variation of $\hat{\boldsymbol{v}}$ is the fully adapted extension of the cumulative variation of $\boldsymbol{v}$.

614R Lemma If $I\in\mathcal{I}(\mathcal{T})$ is non-empty and $\boldsymbol{u}=\langle u_{\sigma}\rangle_{\sigma\in I}$, $\boldsymbol{v}=\langle v_{\sigma}\rangle_{\sigma\in I}$ are fully adapted processes, then

$$|S_I(\boldsymbol{u},d\boldsymbol{v})|\le\min(\sup|\boldsymbol{u}|\times\int_I|d\boldsymbol{v}|,\sup|\boldsymbol{v}|\times(\int_I|d\boldsymbol{u}|+2\sup|\boldsymbol{u}|).$$

614S Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{u}=\langle u_{\sigma}\rangle_{\sigma\in\mathcal{S}}$, $\boldsymbol{v}=\langle v_{\sigma}\rangle_{\sigma\in\mathcal{S}}$ two processes of bounded variation with domain $\mathcal{S}$. Then $\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}$ is defined and

$$|\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}|\le\min(\sup|\boldsymbol{u}|\times\int_{\mathcal{S}}|d\boldsymbol{v}|,\sup|\boldsymbol{v}|\times(\int_{\mathcal{S}}|d\boldsymbol{u}|+2\sup|\boldsymbol{u}|)).$$

614T Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{u}$, $\boldsymbol{v}$ fully adapted processes with domain $\mathcal{S}$ such that $\boldsymbol{u}$ is order-bounded, $\boldsymbol{v}$ is of bounded variation and $\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}$ is defined. Then the indefinite integral $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is of bounded variation, and $\int_{\mathcal{S}}|d(ii_{\boldsymbol{v}}(\boldsymbol{u}))|\le\sup|\boldsymbol{u}|\times\int_{\mathcal{S}}|d\boldsymbol{v}|$.

614U Proposition Let (Ω,Σ,μ) be a complete probability space, and $\langle\Sigma_t\rangle_{t\ge0}$ a filtration of σ-subalgebras of Σ such that every μ-negligible set belongs to every Σ_t. Let $(\mathfrak{A},\bar\mu,[0,\infty[\,,\langle\mathfrak{A}_t\rangle_{t\ge0},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ be the associated stochastic integration structure as in 612H; for a stopping time $h:\Omega\to[0,\infty]$ let $h^\bullet$ be the corresponding member of $\mathcal{T}$. Suppose that $\langle X_t\rangle_{t\ge0}$ is a progressively measurable process on Ω, with associated fully adapted process $\boldsymbol{x}=\langle x_\tau\rangle_{\tau\in\mathcal{T}_f}$.

(a) If $\{X_s(\omega):s\ge0\}$ is bounded for almost every $\omega\in\Omega$, then $\boldsymbol{x}$ is order-bounded.

(b) If $s\mapsto X_s(\omega):[0,\infty[\,\to\mathbb{R}$ is of bounded variation for almost every $\omega\in\Omega$, then $\boldsymbol{x}$ is of bounded variation.

Version of 28.10.21/9.8.22

615 Moderately oscillatory processes

I come now to the class of integrands in the basic theorem 616K, the 'moderately oscillatory' processes. I have chosen a path which starts with a natural linear space topology on the space of order-bounded processes, the ucp topology (615B). This gives a straightforward definition of the space of moderately oscillatory processes (615E) with their elementary properties (615F-615H). When the domain is finitely full, we have an alternative definition in terms of convergence along monotonic sequences of stopping times (615I-615N). Classical stochastic processes with càdlàg sample paths give rise to locally moderately oscillatory processes (615P).

615B The ucp topology Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.

(a) For $\boldsymbol{u}\in M_{\text{o-b}}(\mathcal{S})$, set

$$\widehat{\theta}(\boldsymbol{u})=\theta(\sup|\boldsymbol{u}|).$$

(b) $\widehat{\theta}$ is an F-norm on $M_{\text{o-b}}(\mathcal{S})$.

(c) $\widehat{\theta}$ defines a metrizable linear space topology. I will call this the **ucp topology** on $M_{\text{o-b}}(\mathcal{S})$ and the associated uniformity the **ucp uniformity** on $M_{\text{o-b}}(\mathcal{S})$.

615C Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and give $M_{\text{o-b}}=M_{\text{o-b}}(\mathcal{S})$ its ucp topology.

(a) If $h:\mathbb{R}\to\mathbb{R}$ is continuous, then $\bar h\boldsymbol{u}\in M_{\text{o-b}}$ for every $\boldsymbol{u}\in M_{\text{o-b}}$, and $\boldsymbol{u}\mapsto\bar h\boldsymbol{u}:M_{\text{o-b}}\to M_{\text{o-b}}$ is continuous.

(b)(i) $(\boldsymbol{u},\boldsymbol{v})\mapsto\boldsymbol{u}\times\boldsymbol{v}:M_{\text{o-b}}\times M_{\text{o-b}}\to M_{\text{o-b}}$ is continuous.

(ii) $\boldsymbol{u}\mapsto\sup|\boldsymbol{u}|:M_{\text{o-b}}\to L^0$ is uniformly continuous.

(c) $M_{\text{o-b}}$ is complete as linear topological space.

615D Lemma Let $\mathcal{S}$ be a non-empty finitely full sublattice of $\mathcal{T}$, and $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a fully adapted process.

(a) If $\gamma>0$ then

$$\bar\mu(\sup_{\tau\in\mathcal{S}}[\![|u_\tau|>\gamma]\!])=\sup_{\tau\in\mathcal{S}}\bar\mu[\![|u_\tau|>\gamma]\!].$$

(b) If $\boldsymbol{u}$ is order-bounded, $\theta(\sup|\boldsymbol{u}|)\le2\sqrt{\sup_{\sigma\in\mathcal{S}}\theta(u_\sigma)}$.

615E Definition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.

(a) I will call a process with domain $\mathcal{S}$ **moderately oscillatory** if it is in the closure of $M_{\text{bv}}(\mathcal{S})$ in $M_{\text{o-b}}(\mathcal{S})$ for the ucp topology.

(b) A process $\boldsymbol{u}$ with domain $\mathcal{S}$ is **locally moderately oscillatory** if $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ is moderately oscillatory for every $\tau\in\mathcal{S}$.

Remark The definitions imply directly that (locally) moderately oscillatory processes are (locally) order-bounded. Of course processes of bounded variation (e.g., simple processes, 614Q(a-iii), and in particular constant processes) are moderately oscillatory, and processes which are locally of bounded variation are locally moderately oscillatory.

615F Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\hat{\mathcal{S}}$ its covered envelope.

(a) Write M_{mo} for the set of moderately oscillatory processes with domain $\mathcal{S}$.

(i) If $\mathcal{S}'$ is a sublattice of $\mathcal{S}$ then $\boldsymbol{u}\restriction\mathcal{S}'$ is moderately oscillatory for every $\boldsymbol{u} \in M_{\text{mo}}$.

(ii) If $h : \mathbb{R} \to \mathbb{R}$ is continuous, then $\bar{h}\boldsymbol{u} \in M_{\text{mo}}$ for every $\boldsymbol{u} \in M_{\text{mo}}$.

(iii) M_{mo} is an f-subalgebra of $M_{\text{o-b}} = M_{\text{o-b}}(\mathcal{S})$.

(iv) M_{mo} is closed in $M_{\text{o-b}}(\mathcal{S})$ for the ucp topology, so is complete for the ucp uniformity.

(v) If $\tau \in \mathcal{S}$, then a fully adapted process $\boldsymbol{u}$ with domain $\mathcal{S}$ is moderately oscillatory iff $\boldsymbol{u}\restriction\mathcal{S} \wedge \tau$ and $\boldsymbol{u}\restriction\mathcal{S} \vee \tau$ are both moderately oscillatory.

(vi) If $\boldsymbol{u} \in M_{\text{mo}}$, then its fully adapted extension to $\hat{\mathcal{S}}$ is moderately oscillatory.

(b) Write M_{lmo} for the set of locally moderately oscillatory processes with domain $\mathcal{S}$.

(i) $M_{\text{mo}} \subseteq M_{\text{lmo}}$.

(ii) If $h : \mathbb{R} \to \mathbb{R}$ is continuous, then $\bar{h}\boldsymbol{u} \in M_{\text{lmo}}$ for every $\boldsymbol{u} \in M_{\text{lmo}}$.

(iii) M_{lmo} is an f-subalgebra of the space $M_{\text{lob}} = M_{\text{lob}}(\mathcal{S})$ of locally order-bounded processes with domain $\mathcal{S}$.

(iv) If $\tau \in \mathcal{S}$, then a fully adapted process $\boldsymbol{u}$ with domain $\mathcal{S}$ is locally moderately oscillatory iff $\boldsymbol{u}\restriction\mathcal{S} \wedge \tau$ and $\boldsymbol{u}\restriction\mathcal{S} \vee \tau$ are both locally moderately oscillatory.

(v) If $\boldsymbol{u} \in M_{\text{lmo}}$, then its fully adapted extension $\hat{\boldsymbol{u}}$ to $\hat{\mathcal{S}}$ is locally moderately oscillatory.

615G Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ a fully adapted process.

(a) Suppose that $\boldsymbol{u}$ is moderately oscillatory and $A \subseteq \mathcal{S}$ is non-empty and upwards-directed. Then $w = \lim_{\sigma \uparrow A} u_\sigma$ is defined. Setting $A_* = \{\rho : \rho \in \mathcal{S}, \sup_{\sigma \in A} [\![\rho \le \sigma]\!] = 1\}$,

$$\lim_{\sigma \uparrow A} \sup_{\rho \in A_* \vee \sigma} |u_\rho - w| = 0.$$

(b) Suppose that $\boldsymbol{u}$ is locally moderately oscillatory and $A \subseteq \mathcal{S}$ is non-empty and downwards-directed. Then $w = \lim_{\sigma \downarrow A} u_\sigma$ is defined. Setting $A^* = \{\rho : \rho \in \mathcal{S}, \sup_{\sigma \in A} [\![\sigma \le \rho]\!] = 1\}$,

$$\lim_{\sigma \downarrow A} \sup_{\rho \in A^* \wedge \sigma} |u_\rho - w| = 0.$$

615H Corollary Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$, $\hat{\mathcal{S}}$ its covered envelope, $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ a locally moderately oscillatory process, and $\hat{\boldsymbol{u}} = \langle \hat{u}_\sigma \rangle_{\sigma \in \hat{\mathcal{S}}}$ its fully adapted extension to $\hat{\mathcal{S}}$. Then $\boldsymbol{u}$ and $\hat{\boldsymbol{u}}$ have starting values, which are the same.

615I Definition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ a fully adapted process. I will say that $\boldsymbol{u}$ is **⥮-convergent** if

(⥮) $\lim_{n \to \infty} u_{\sigma_n}$ is defined whenever $\langle \sigma_n \rangle_{n \in \mathbb{N}}$ is a sequence in $\mathcal{S}$ which is either non-increasing or non-decreasing.

615J Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ a moderately oscillatory process. Then $\boldsymbol{u}$ is ⥮-convergent.

615K Lemma Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ an ⥮-convergent process. Then $\boldsymbol{u}$ is order-bounded.

615L Lemma Let $\mathcal{S}$ be a non-empty finitely full sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ an ⥮-convergent process. Suppose that $A \subseteq \mathcal{S}$ is non-empty and downwards-directed. Then $w = \lim_{\sigma \downarrow A} u_\sigma$ is defined. Setting $A^* = \{\rho : \rho \in \mathcal{S}, \sup_{\sigma \in A} [\![\sigma \le \rho]\!] = 1\}$,

$$\lim_{\sigma \downarrow A} \sup_{\rho \in A^* \wedge \sigma} |u_\rho - w| = 0.$$

615M Construction Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ with a greatest member, $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ an ⥮-convergent process, and $\delta > 0$. Then there are sequences $\langle D_i \rangle_{i \in \mathbb{N}}$, $\langle y_i \rangle_{i \in \mathbb{N}}$, $\langle d_i \rangle_{i \in \mathbb{N}}$, a family $\langle c_{i\sigma} \rangle_{i \in \mathbb{N}, \sigma \in \mathcal{S}}$ and a process $\tilde{\boldsymbol{u}} = \langle \tilde{u}_\sigma \rangle_{\sigma \in \mathcal{S}}$ with the following properties.

(a) $D_0 = \mathcal{S}$; for every $i \in \mathbb{N}$, $\max \mathcal{S} \in D_i \subseteq \mathcal{S}$, D_i is closed under $\wedge$, $y_i = \lim_{\sigma \downarrow D_i} u_\sigma$ and

$$D_{i+1} = \{\sigma : \sigma \in \mathcal{S},\ [\![\sigma < \max \mathcal{S}]\!] \subseteq [\![|u_\sigma - y_i| \ge \delta]\!] \text{ and there is a } \sigma' \in D_i \text{ such that } \sigma' \le \sigma\}.$$

(b) $y_i \in \bigcap_{\sigma \in D_i} L^0(\mathfrak{A}_\sigma)$ for every $i \in \mathbb{N}$.

(c)(i) For every $i \in \mathbb{N}$,

$d_i = \sup_{\sigma \in D_i} [\![\sigma < \max \mathcal{S}]\!]$,
$d_i \in \bigcap_{\sigma \in D_i} \mathfrak{A}_\sigma$,
$d_{i+1} \subseteq d_i$,
$d_{i+1} \subseteq [\![|y_{i+1} - y_i| \ge \delta]\!]$,
$1 \setminus d_i \subseteq [\![y_i = u_{\max \mathcal{S}}]\!] \cap [\![y_i = y_{i+1}]\!]$.

(ii) $\inf_{i \in \mathbb{N}} d_i = 0$.

(d)(i) If $\sigma \in \mathcal{S}$ and $i \in \mathbb{N}$,

$$c_{i\sigma} = \sup_{\tau \in D_i} [\![\tau \le \sigma]\!], \quad c_{i+1,\sigma} \subseteq c_{i\sigma}, \quad [\![\sigma = \max \mathcal{S}]\!] \subseteq c_{i\sigma} \subseteq [\![\sigma = \max \mathcal{S}]\!] \cup d_i.$$

(ii) If $i \in \mathbb{N}$ and $\sigma \in D_i$ then $c_{i\sigma} = 1$; $c_{0\sigma} = 1$ for every $\sigma \in \mathcal{S}$.

(iii) If σ, σ' in $\mathcal{S}$ then $[\![\sigma \le \sigma']\!] \cap c_{i\sigma} \subseteq c_{i\sigma'}$ for every $i \in \mathbb{N}$.

(iv) $\inf_{i \in \mathbb{N}} c_{i\sigma} = [\![\sigma = \max \mathcal{S}]\!]$ for every $\sigma \in \mathcal{S}$.

(v) If $\sigma \in \mathcal{S}$ and $i \in \mathbb{N}$ then $c_{i\sigma} \setminus c_{i+1,\sigma} \subseteq [\![|u_\sigma - y_i| < \delta]\!]$.

(e) If $\sigma \in \mathcal{S}$ then

$$c_{i\sigma} \setminus c_{i+1,\sigma} \subseteq [\![\tilde{u}_\sigma = y_i]\!]$$

for every $i \in \mathbb{N}$, and $[\![\sigma = \max \mathcal{S}]\!] \subseteq [\![\tilde{u}_\sigma = u_{\max \mathcal{S}}]\!]$.

(f) $\tilde{\boldsymbol{u}}$ is fully adapted, $\sup |\tilde{\boldsymbol{u}}| \le \sup |\boldsymbol{u}|$ and $\sup |\boldsymbol{u} - \tilde{\boldsymbol{u}}| \le \delta \chi 1$.

(g) $\tilde{\boldsymbol{u}}$ is of bounded variation.

615N Theorem Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$, and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ a fully adapted process. Then the following are equiveridical:

(i) $\boldsymbol{u}$ is moderately oscillatory;

(ii) $\boldsymbol{u}$ is $\Vert$-convergent;

(iii) $\langle u_{\sigma_n} \rangle_{n \in \mathbb{N}}$ is Cauchy for every monotonic sequence $\langle \sigma_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{S}$;

(iv) for every $\epsilon > 0$ there is an $m \ge 1$ such that whenever $\sigma_0 \le \ldots \le \sigma_m$ in $\mathcal{S}$ there is a $j < m$ such that $\theta(u_{\sigma_j} - u_{\sigma_{j+1}}) \le \epsilon$.

615O Proposition Suppose that $\mathcal{S}$ is a sublattice of $\mathcal{T}$, $\boldsymbol{u} \in M_{\mathrm{mo}}(\mathcal{S})$ and $\epsilon > 0$. Then there is a $\boldsymbol{u}' \in M_{\mathrm{bv}}(\mathcal{S})$ such that $\theta(\sup |\boldsymbol{u}' - \boldsymbol{u}|) \le \epsilon$ and $\sup |\boldsymbol{u}'| \le \sup |\boldsymbol{u}|$.

615P Where moderately oscillatory processes come from There is an easy condition on the structure in 612H which will ensure that the process generated there is moderately oscillatory.

Proposition Let (Ω, Σ, μ) be a complete probability space, and $\langle \Sigma_t \rangle_{t \ge 0}$ a filtration of σ-subalgebras of Σ such that every μ-negligible set belongs to every Σ_t. Let $(\mathfrak{A}, \bar{\mu})$ be the measure algebra of μ and set $\mathfrak{A}_t = \{E^\bullet : E \in \Sigma_t\}$ for each $t \ge 0$; then we have a real-time stochastic integration structure $(\mathfrak{A}, \bar{\mu}, [0, \infty[, \langle \mathfrak{A}_r \rangle_{t \ge 0}, \mathcal{T}, \langle \mathfrak{A}_\tau \rangle_{\tau \in \mathcal{T}})$. Let $\langle X_t \rangle_{t \ge 0}$ be a progressively measurable process on Ω, and $\boldsymbol{x} = \langle x_\tau \rangle_{\tau \in \mathcal{T}_f}$ the corresponding fully adapted process as described in 612H. Suppose that $\lim_{n \to \infty} X_{t_n}(\omega)$ is defined in $\mathbb{R}$ for every bounded monotonic sequence $\langle t_n \rangle_{n \in \mathbb{N}}$ in $[0, \infty[$ and every $\omega \in \Omega$. Then $\boldsymbol{x}$ is locally moderately oscillatory.

615Q Proposition The identity process, Brownian motion and the Poisson process are all locally moderately oscillatory.

615R Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u}$ a process of bounded variation with domain $\mathcal{S}$.

(a) If $\boldsymbol{v} \in M_{\mathrm{mo}} = M_{\mathrm{mo}}(\mathcal{S})$ then $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$ is defined and $|\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}| \le (\int_{\mathcal{S}} |d\boldsymbol{u}| + 2 \sup |\boldsymbol{u}|) \times \sup |\boldsymbol{v}|$.

(b) $ii_{\boldsymbol{v}}(\boldsymbol{u}) \in M_{\mathrm{mo}}$ for every $\boldsymbol{v} \in M_{\mathrm{mo}}$, and $\boldsymbol{v} \mapsto ii_{\boldsymbol{v}}(\boldsymbol{u}) : M_{\mathrm{mo}} \to M_{\mathrm{mo}}$ is continuous for the ucp topology on M_{mo}.

Version of 31.1.23/3.2.23

616 Integrating interval functions

In this section I present a fundamental theorem on the existence of Riemann-sum integrals (616M), dealing with the case of moderately oscillatory integrands and integrating interval functions (616F). The most important integrating interval functions are those defined by integrators (616Fc, 616I). The integrators on a lattice $\mathcal{S}$ form an f-subalgebra of the space of moderately oscillatory processes with domain $\mathcal{S}$ (616P).

616B Definition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. If ψ is an adapted interval function defined on $\mathcal{S}^{2\uparrow}$, the **capped-stake variation set of** ψ **over** $\mathcal{S}$ is the set $Q_{\mathcal{S}}(d\psi)$ of Riemann sums $S_I(\boldsymbol{u}, d\psi)$ where $I \in \mathcal{I}(\mathcal{S})$, $\boldsymbol{u}$ is a fully adapted process with domain I and $\sup|\boldsymbol{u}| \le \chi 1$.

If $\boldsymbol{v}$, $\boldsymbol{w}$ are fully adapted processes defined on $\mathcal{S}$ then, corresponding to the basic interval functions of 613F, I will write $Q_{\mathcal{S}}(d\boldsymbol{v})$, $Q_{\mathcal{S}}(d\boldsymbol{v}d\boldsymbol{w})$, $Q_{\mathcal{S}}(|d\boldsymbol{v}|)$ for $Q_{\mathcal{S}}(d(\Delta\boldsymbol{v}))$, $Q_{\mathcal{S}}(d(\Delta\boldsymbol{v} \times \Delta\boldsymbol{w}))$ and $Q_{\mathcal{S}}(d|\Delta\boldsymbol{v}|)$.

616C Lemma Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$, ψ an adapted interval function defined on $\mathcal{S}^{2\uparrow}$, and z an element of $L^0(\mathfrak{A})$. Then the following are equiveridical:

(i) $z \in Q_{\mathcal{S}}(d\psi)$;

(ii) there are $\tau_0 \le \ldots \le \tau_n$ in $\mathcal{S}$ and $u_0, \ldots, u_{n-1}$ such that $u_i \in L^\infty(\mathfrak{A}_{\tau_i})$ and $|u_i| \le \chi 1$ for every $i < n$ and $z = \sum_{i=0}^{n-1} u_i \times \psi(\tau_i, \tau_{i+1})$;

(iii) there are $\tau_0 \le \ldots \le \tau_n$ in $\mathcal{S}$ and an order-bounded process $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ such that $\sup|\boldsymbol{u}| \le \chi 1$ and $z = \sum_{i=0}^{n-1} u_{\tau_i} \times \psi(\tau_i, \tau_{i+1})$.

616D Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and ψ, ψ' adapted interval functions defined on $\mathcal{S}^{2\uparrow}$.

(a) $Q_{\mathcal{S}}(d\psi) = \bigcup_{I \in \mathcal{I}(\mathcal{S})} Q_I(d\psi)$.

(b) $Q_{\mathcal{S}}(d(\alpha\psi)) = \alpha Q_{\mathcal{S}}(d\psi)$ for every $\alpha \in \mathbb{R}$.

(c) $Q_{\mathcal{S}}(d(\psi + \psi')) \subseteq Q_{\mathcal{S}}(d\psi) + Q_{\mathcal{S}}(d\psi')$.

(d) If $\mathcal{S}'$ is a sublattice of $\mathcal{S}$ then $Q_{\mathcal{S}'}(d\psi) \subseteq Q_{\mathcal{S}}(d\psi)$.

(e) If $w \in Q_{\mathcal{S}}(d\psi)$, $z \in L^0(\mathfrak{A} \cap \bigcap_{\sigma \in \mathcal{S}} \mathfrak{A}_\sigma)$ and $|z| \le \chi 1$, then $z \times w \in Q_{\mathcal{S}}(\psi)$.

(f) If $\tau \in \mathcal{S}$ then $Q_{\mathcal{S}\wedge\tau}(d\psi) + Q_{\mathcal{S}\vee\tau}(d\psi) \subseteq Q_{\mathcal{S}}(d\psi)$.

616E Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and ψ an adapted interval function on $\mathcal{S}$. Then the following are equiveridical:

(i) $Q_{\mathcal{S}}(d\psi)$ is topologically bounded;

(ii) for every $\epsilon > 0$ there is a $\delta > 0$ such that $\theta(S_I(\boldsymbol{u}, d\psi)) \le \epsilon$ whenever $I \in \mathcal{I}(\mathcal{S})$, $\boldsymbol{u} \in M_{\text{fa}}(I)$ and $\theta(\sup|\boldsymbol{u}|) \le \delta$;

(iii) for every $\epsilon > 0$ there is a $\delta > 0$ such that $\theta(S_I(\boldsymbol{u}, d\psi)) \le \epsilon$ whenever $I \in \mathcal{I}(\mathcal{S})$, $\boldsymbol{u} \in M_{\text{o-b}}(\mathcal{S})$ and $\theta(\sup|\boldsymbol{u}|) \le \delta$.

616F Definition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\psi : \mathcal{S}^{2\uparrow} \to L^0$ a function.

(a) ψ is an **integrating interval function** on $\mathcal{S}$ if

(α) ψ is a strictly adapted interval function;

(β) writing $\hat{\mathcal{S}}$ for the covered envelope of $\mathcal{S}$ and $\hat{\psi} : \hat{\mathcal{S}}^{2\uparrow} \to L^0$ for the strictly adapted extension of ψ, $\int_{\hat{\mathcal{S}}} d\hat{\psi} = \int_{\hat{\mathcal{S}}} \mathbf{1}\, d\hat{\psi}$ is defined;

(γ) $Q_{\hat{\mathcal{S}}}(d\hat{\psi})$ is topologically bounded in L^0.

(b) ψ is a **locally integrating interval function** if $\psi\restriction(\mathcal{S} \wedge \tau)^{2\uparrow}$ is an integrating interval function for every $\tau \in \mathcal{S}$.

(c) A fully adapted process $\boldsymbol{v}$ defined on $\mathcal{S}$ is an **integrator** if $Q_{\mathcal{S}}(d\boldsymbol{v})$ is topologically bounded in L^0, and a **local integrator** if $\boldsymbol{v}\restriction\mathcal{S} \wedge \tau$ is an integrator for every $\tau \in \mathcal{S}$.

I will write $M_{\mathrm{igtr}}(\mathcal{S})$ for the set of integrators with domain $\mathcal{S}$, and $M_{\mathrm{ligtr}}(\mathcal{S})$ for the set of local integrators with domain $\mathcal{S}$.

Remarks Evidently a strictly adapted interval function ψ on a sublattice $\mathcal{S}$ is an integrating interval function iff its adapted extension on the covered envelope of $\mathcal{S}$ is an integrating interval function.

616G Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and ψ, ψ' integrating interval functions on $\mathcal{S}$.
(a) $\psi+\psi'$ and $\alpha\psi$ are integrating interval functions on $\mathcal{S}$ for every $\alpha\in\mathbb{R}$.
(b) ψ is a locally integrating interval function.

616H Lemma Suppose that

$$\epsilon>0,\quad \gamma\ge 0,\quad m\ge 1,\quad m\epsilon\ge 2\gamma,$$

$$r\ge m,\quad 1-\frac{r!}{r^m(r-m)!}\le\tfrac12\epsilon^m,\quad k\ge 1,\quad 2k\epsilon^m\ge\epsilon,\quad n=rk.$$

Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and ψ an adapted interval function with domain $\mathcal{S}^{2\uparrow}$.

(a) Let $\langle a_i\rangle_{i<r}$ be a family in $\mathfrak{A}$ such that $\bar\mu a_i\ge\epsilon$ for every $i<r$. Then there is a $J\in[r]^m$ such that $\bar\mu(\inf_{i\in J}a_i)\ge\frac12\epsilon^m$.

(b) Let $\tau_0\le\ldots\le\tau_r$ in $\mathcal{S}$ be such that $\sup\{\theta(w):w\in Q_{\mathcal{S}\cap[\tau_i,\tau_{i+1}]}(d\psi)\}>4\epsilon$ for every $i<r$, while $z\in L^0(\mathfrak{A}_{\tau_0})$ is such that $\bar\mu[\![|z|\ge\gamma]\!]\le\epsilon$. Then there is a $w\in Q_{\mathcal{S}}(d\psi)$ such that $\bar\mu[\![|z+w|\ge\gamma]\!]\ge\bar\mu[\![|z|\ge\gamma]\!]+\frac12\epsilon^m$.

(c) Let $\tau_0\le\ldots\le\tau_n$ in $\mathcal{S}$ be such that $\sup\{\theta(w):w\in Q_{\mathcal{S}\cap[\tau_i,\tau_{i+1}]}(d\psi)\}>4\epsilon$ for every $i<n$. Then there is a $w\in Q_{\mathcal{S}}(d\psi)$ such that $\bar\mu[\![|w|\ge\gamma]\!]\ge\epsilon$.

(d) Let $\tau_0\le\ldots\le\tau_n$ in $\mathcal{S}$ be such that $\theta(\psi(\tau_i,\tau_{i+1}))>4\epsilon$ for every $i<n$. Then there is a $w\in Q_{\mathcal{S}}(d\psi)$ such that $\bar\mu[\![|w|\ge\gamma]\!]\ge\epsilon$.

616I Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a (local) integrator.
(a) The fully adapted extension of $\boldsymbol{v}$ to the covered envelope of $\mathcal{S}$ is a (local) integrator.
(b) $\boldsymbol{v}$ is (locally) moderately oscillatory, therefore (locally) order-bounded.
(c) $\Delta\boldsymbol{v}$ is a (locally) integrating interval function.

616J Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and ψ an integrating interval function with domain $\mathcal{S}^{2\uparrow}$. Set

$$M_\psi=\{\boldsymbol{u}:\boldsymbol{u}\in M_{\text{o-b}}(\mathcal{S}),\ \int_{\mathcal{S}}\boldsymbol{u}\,d\psi\text{ is defined}\}.$$

Then M_ψ is a closed linear subspace of $M_{\text{o-b}}(\mathcal{S})$ and we have an indefinite integral operator $ii_\psi:M_\psi\to M_{\mathrm{igtr}}(\mathcal{S})$ which is linear and continuous for the ucp topology on $M_{\text{o-b}}(\mathcal{S})$.

616K Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u}$ a moderately oscillatory process with domain $\mathcal{S}$, and $\boldsymbol{v}$ an integrator with domain $\mathcal{S}$. Then $\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}$ is defined, and $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is an integrator.

616L Corollary Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. If $\boldsymbol{u}$ is a locally moderately oscillatory process and $\boldsymbol{v}$ a fully adapted process which is locally of bounded variation, both with domain $\mathcal{S}$, then $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is locally of bounded variation.

616M Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and ψ an integrating interval function on $\mathcal{S}$. Write $\boldsymbol{v}$ for $ii_\psi(\mathbf{1})=\langle\int_{\mathcal{S}\wedge\tau}d\psi\rangle_{\tau\in\mathcal{S}}$. Then $\int_{\mathcal{S}}\boldsymbol{u}\,d\psi$ is defined and equal to $\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}$ whenever $\boldsymbol{u}$ is a moderately oscillatory process with domain $\mathcal{S}$.

616N Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a (local) integrator. If $f:\mathbb{R}\to\mathbb{R}$ is convex, then $\bar f\boldsymbol{v}$ is a (local) integrator.

616O Corollary If $\boldsymbol{v}$ is a (local) integrator and $f:\mathbb{R}\to\mathbb{R}$ is a function, absolutely continuous on every bounded interval in $\mathbb{R}$, such that its derivative f' has bounded variation on every bounded interval, then $\bar f\boldsymbol{v}$ is a (local) integrator.

616P Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.

(a) $M_{\mathrm{igtr}}(\mathcal{S})$ is an f-subalgebra of the space $M_{\mathrm{mo}}(\mathcal{S})$ of moderately oscillatory processes with domain $\mathcal{S}$.

(b)(i) Constant processes are integrators.

(ii) If $\boldsymbol{v} \in M_{\mathrm{igtr}}(\mathcal{S})$ then $\boldsymbol{v}\restriction\mathcal{S}' \in M_{\mathrm{igtr}}(\mathcal{S}')$ for any sublattice $\mathcal{S}'$ of $\mathcal{S}$. In particular, $\boldsymbol{v}$ is a local integrator.

(iii) Suppose that $\boldsymbol{v} \in M_{\mathrm{fa}}(\mathcal{S})$ and for every $\epsilon > 0$ there is a $\boldsymbol{v}' \in M_{\mathrm{igtr}}(\mathcal{S})$ such that $\bar{\mu}[\![\boldsymbol{v} \ne \boldsymbol{v}']\!] \le \epsilon$. Then $\boldsymbol{v} \in M_{\mathrm{igtr}}(\mathcal{S})$.

(iv) If $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}} \in M_{\mathrm{igtr}}(\mathcal{S})$ and $z \in L^0(\mathfrak{A} \cap \bigcap_{\sigma\in\mathcal{S}} \mathfrak{A}_\sigma)$, then $z\boldsymbol{v} \in M_{\mathrm{igtr}}(\mathcal{S})$.

(v) If $\boldsymbol{v} \in M_{\mathrm{fa}}(\mathcal{S})$ then $\boldsymbol{v} \in M_{\mathrm{igtr}}(\mathcal{S})$ iff $\boldsymbol{v}\restriction\mathcal{S}\wedge\tau \in M_{\mathrm{igtr}}(\mathcal{S}\wedge\tau)$ and $\boldsymbol{v}\restriction\mathcal{S}\vee\tau \in M_{\mathrm{igtr}}(\mathcal{S}\vee\tau)$.

616Q Corollary Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.

(a) $M_{\mathrm{ligtr}}(\mathcal{S})$ is an f-subalgebra of the space $M_{\mathrm{lmo}}(\mathcal{S})$ of locally moderately oscillatory processes with domain $\mathcal{S}$.

(b)(i) If $\boldsymbol{v} \in M_{\mathrm{ligtr}}(\mathcal{S})$ then $\boldsymbol{v}\restriction\mathcal{S}' \in M_{\mathrm{ligtr}}(\mathcal{S}')$ for any sublattice $\mathcal{S}'$ of $\mathcal{S}$.

(ii) If $\boldsymbol{v} \in M_{\mathrm{ligtr}}(\mathcal{S})$ and $z \in L^0(\mathfrak{A} \cap \bigcap_{\sigma\in\mathcal{S}} \mathfrak{A}_\sigma)$, then $z\boldsymbol{v} \in M_{\mathrm{ligtr}}(\mathcal{S})$.

(c) Suppose that $\boldsymbol{u} \in M_{\mathrm{lmo}}(\mathcal{S})$ and $\boldsymbol{v} \in M_{\mathrm{ligtr}}(\mathcal{S})$.

(i) The indefinite integral $ii_{\boldsymbol{v}}(\boldsymbol{u})$ belongs to $M_{\mathrm{ligtr}}(\mathcal{S})$.

(ii) Let $\hat{\mathcal{S}}$ be the covered envelope of $\mathcal{S}$, and $\hat{\boldsymbol{u}}$, $\hat{\boldsymbol{v}}$ the fully adapted extensions of $\boldsymbol{u}$, $\boldsymbol{v}$ to $\hat{\mathcal{S}}$. Then $ii_{\boldsymbol{v}}(\boldsymbol{u}) = ii_{\hat{\boldsymbol{v}}}(\hat{\boldsymbol{u}})\restriction\mathcal{S}$.

(d) Suppose that $\boldsymbol{v} \in M_{\mathrm{fa}}(\mathcal{S})$ and $\mathcal{S}'$ is a covering ideal of $\mathcal{S}$ such that $\boldsymbol{v}\restriction\mathcal{S}' \in M_{\mathrm{ligtr}}(\mathcal{S}')$. Then $\boldsymbol{v} \in M_{\mathrm{ligtr}}(\mathcal{S})$.

616R Proposition Suppose that $\mathcal{S}$ is a sublattice of $\mathcal{T}$, and that a fully adapted process $\boldsymbol{v}$ with domain $\mathcal{S}$ is (locally) of bounded variation.

(a) $\boldsymbol{v}$ is a (local) integrator.

(b) Now suppose that $\boldsymbol{v}$ is non-decreasing and that $\boldsymbol{u}$ is a non-negative moderately oscillatory process with domain $\mathcal{S}$.

(i) If $\boldsymbol{v}$ is of bounded variation then $\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v} \ge 0$.

(ii) If $\boldsymbol{v}$ is locally of bounded variation then $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is non-decreasing.

616S Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v}$ a process of bounded variation with domain $\mathcal{S}$. Then $|\Delta\boldsymbol{v}|$ is an integrating interval function.

616T Corollary Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{u}$, $\boldsymbol{v}$ fully adapted processes with domain $\mathcal{S}$.

(a) If $\boldsymbol{u}$ is moderately oscillatory and $\boldsymbol{v}$ is of bounded variation, then $\int_{\mathcal{S}} \boldsymbol{u}\,|d\boldsymbol{v}|$ is defined and equal to $\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}^\uparrow$, where $\boldsymbol{v}^\uparrow$ is the cumulative variation of $\boldsymbol{v}$.

(b) If $\boldsymbol{u}$ is locally moderately oscillatory and $\boldsymbol{v}$ is locally of bounded variation, then the indefinite integrals $ii_{|\Delta\boldsymbol{v}|}(\boldsymbol{u})$ and $ii_{\boldsymbol{v}^\uparrow}(\boldsymbol{u})$ are equal.

Mnemonic $|d\boldsymbol{v}| \sim d\boldsymbol{v}^\uparrow$.

Version of 10.11.21

617 Integral identities and quadratic variations

We come now to proper calculus, with change-of-variable theorems. 617D-617E is a stochastic-calculus version of the result that if $\nu = f\mu$ is an indefinite-integral measure, then $\int g\,d\nu = \int g \times f\,d\mu$ (235K). Similar formulae describe the cumulative variation of an indefinite integral with respect to a process of bounded variation (617G). The next theme is 'quadratic variation' (617H). Given two integrators $\boldsymbol{v}$ and $\boldsymbol{w}$, the interval function corresponding to $d\boldsymbol{v}d\boldsymbol{w}$ gives the same integrals as a process $[\boldsymbol{v}\overset{*}{\rceil}\boldsymbol{w}]$ (617I) which is locally of bounded variation. In particular, $(d\boldsymbol{v})^2$ mimics $d\boldsymbol{v}^*$ where the quadratic variation $\boldsymbol{v}^*$ is a non-decreasing process. Based on this, we have a second change-of-variable theorem (617P-617Q), using approximations of moderately oscillatory processes by simple processes (617B).

617B Lemma Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a moderately oscillatory process.

(a) For each non-empty $I \in \mathcal{I}(\mathcal{S})$ there is a unique simple process $\boldsymbol{u}_I = \langle u_{I\sigma}\rangle_{\sigma\in\mathcal{S}}$ such that $\boldsymbol{u}_I$ has a breakpoint string in I, $\boldsymbol{u}_I$ and $\boldsymbol{u}$ agree on I, and $[\![\sigma < \min I]\!] \subseteq [\![u_{I\sigma} = 0]\!]$ for every $\sigma \in \mathcal{S}$.

(b) Complete the definition in (a) by setting $u_{\emptyset\sigma} = 0$ for every $\sigma \in \mathcal{S}$. For every integrator $\boldsymbol{v}$ with domain $\mathcal{S}$,

(i) the indefinite integral $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is the limit $\lim_{I\uparrow\mathcal{I}(\mathcal{S})} ii_{\boldsymbol{v}}(\boldsymbol{u}_I)$ for the ucp topology,

(ii) $\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v} = \lim_{I\uparrow\mathcal{I}(\mathcal{S})} \int_{\mathcal{S}} \boldsymbol{u}_I d\boldsymbol{v}$ in L^0.

617D Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, ψ an integrating interval function on $\mathcal{S}$, and $\boldsymbol{u}$, $\boldsymbol{z}$ moderately oscillatory processes with domain $\mathcal{S}$.

(a) $\boldsymbol{z}\psi$, as defined in 613D, is an integrating interval function on $\mathcal{S}$.

(b) Set $\boldsymbol{w} = ii_\psi(\boldsymbol{z})$. Then $\boldsymbol{w}$ is an integrator and

$$\int_{\mathcal{S}} \boldsymbol{u}\,d(\boldsymbol{z}\psi) = \int_{\mathcal{S}} \boldsymbol{u}\times\boldsymbol{z}\,d\psi = \int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{w}.$$

617E Corollary Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u}$, $\boldsymbol{z}$ moderately oscillatory processes with domain $\mathcal{S}$, and $\boldsymbol{v}$ an integrator with domain $\mathcal{S}$. Set $\boldsymbol{w} = ii_{\boldsymbol{v}}(\boldsymbol{z})$. Then $\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{w} = \int_{\mathcal{S}} \boldsymbol{u}\times\boldsymbol{z}\,d\boldsymbol{v}$.

617F Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{z}$ a moderately oscillatory process and $\boldsymbol{v}$ a process of bounded variation, both with domain $\mathcal{S}$. Write $\boldsymbol{w}$ for $ii_{\boldsymbol{v}}(\boldsymbol{z})$. Then $\boldsymbol{w}$ is of bounded variation and $\int_{\mathcal{S}} |d\boldsymbol{w}| = \int_{\mathcal{S}} |\boldsymbol{z}||d\boldsymbol{v}|$.

617G Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u}$ and $\boldsymbol{z}$ moderately oscillatory processes and $\boldsymbol{v}$ a process of bounded variation, all with domain $\mathcal{S}$. Write $\boldsymbol{w}$ for $ii_{\boldsymbol{v}}(\boldsymbol{z})$, and $\boldsymbol{v}^\uparrow$, $\boldsymbol{w}^\uparrow$ for the cumulative variations of $\boldsymbol{v}$ and $\boldsymbol{w}$. Then

$$\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{w}^\uparrow = \int_{\mathcal{S}} \boldsymbol{u}\,|d\boldsymbol{w}| = \int_{\mathcal{S}} \boldsymbol{u}\times|\boldsymbol{z}|\,|d\boldsymbol{v}| = \int_{\mathcal{S}} \boldsymbol{u}\times|\boldsymbol{z}|\,d\boldsymbol{v}^\uparrow.$$

Mnemonic $d(ii_{\boldsymbol{v}}(\boldsymbol{z})^\uparrow) \sim |\boldsymbol{z}|d\boldsymbol{v}^\uparrow$.

617H Quadratic variation Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v}$, $\boldsymbol{w}$ local integrators with domain $\mathcal{S}$.

(a)(i) If $\boldsymbol{v}$ and $\boldsymbol{w}$ are integrators, then the strictly adapted interval function $\Delta\boldsymbol{v}\times\Delta\boldsymbol{w}$ on $\mathcal{S}$ is an integrating interval function.

(ii) In any case, $\Delta\boldsymbol{v}\times\Delta\boldsymbol{w}$ is a locally integrating interval function.

(b) The **covariation** of $\boldsymbol{v}$ and $\boldsymbol{w}$ is the indefinite integral

$$[\boldsymbol{v}\overset{*}{|}\boldsymbol{w}] = ii_{\Delta\boldsymbol{v}\times\Delta\boldsymbol{w}}(\mathbf{1}).$$

When $\boldsymbol{w} = \boldsymbol{v}$, $\boldsymbol{v}^* = [\boldsymbol{v}\overset{*}{|}\boldsymbol{v}] = ii_{(\Delta\boldsymbol{v})^2}(\mathbf{1})$ is the **quadratic variation** of $\boldsymbol{v}$.

(c) Note that as

$$(\boldsymbol{v},\boldsymbol{w}) \mapsto \Delta\boldsymbol{v}\times\Delta\boldsymbol{w}$$

is bilinear, so is $(\boldsymbol{v},\boldsymbol{w}) \mapsto [\boldsymbol{v}\overset{*}{|}\boldsymbol{w}]$.

617I Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{v}$, $\boldsymbol{w}$ two integrators and $\boldsymbol{u}$ a moderately oscillatory process, all with domain $\mathcal{S}$. Then $[\boldsymbol{v}\overset{*}{|}\boldsymbol{w}]$ is an integrator and

$$\int_{\mathcal{S}} \boldsymbol{u}\,d[\boldsymbol{v}\overset{*}{|}\boldsymbol{w}], \quad \int_{\mathcal{S}} \boldsymbol{u}\,d(\boldsymbol{v}\times\boldsymbol{w}) - \int_{\mathcal{S}} \boldsymbol{u}\times\boldsymbol{v}\,d\boldsymbol{w} - \int_{\mathcal{S}} \boldsymbol{u}\times\boldsymbol{w}\,d\boldsymbol{v}, \quad \int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}d\boldsymbol{w}$$

are defined and equal.

617J Corollary Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$ and $\boldsymbol{v}$ an integrator with domain $\mathcal{S}$. Let $\boldsymbol{v}^*$ be the quadratic variation of $\boldsymbol{v}$.

(a) $\boldsymbol{v}^*$ is an integrator, and if $\boldsymbol{u}$ is a moderately oscillatory process with domain $\mathcal{S}$ then

$$\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}^*, \quad \int_{\mathcal{S}} \boldsymbol{u}\, d(\boldsymbol{v}^2) - 2\int_{\mathcal{S}} \boldsymbol{u} \times \boldsymbol{v}\, d\boldsymbol{v}, \quad \int_{\mathcal{S}} \boldsymbol{u}\, (d\boldsymbol{v})^2$$

are defined and equal.

(b)(i) Expressing $\boldsymbol{v}^*$ as $\langle v^*_\tau\rangle_{\tau\in\mathcal{S}}$, $\lim_{\tau\downarrow\mathcal{S}} v^*_\tau = 0$.

(ii) $\boldsymbol{v}^*$ is non-negative, non-decreasing and order-bounded.

(c) If $\boldsymbol{w}$ is another integrator with domain $\mathcal{S}$, then $[\boldsymbol{v}\overset{*}{|}\boldsymbol{w}]$ is of bounded variation.

617K Remarks Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v}$, $\boldsymbol{w}$ local integrators with domain $\mathcal{S}$.

(a)

$$ii_{[\boldsymbol{v}\overset{*}{|}\boldsymbol{w}]}(\boldsymbol{u}) = ii_{\boldsymbol{v}\times\boldsymbol{w}}(\boldsymbol{u}) - ii_{\boldsymbol{w}}(\boldsymbol{u}\times\boldsymbol{v}) - ii_{\boldsymbol{v}}(\boldsymbol{u}\times\boldsymbol{w})$$

for every locally moderately oscillatory $\boldsymbol{u}$ with domain $\mathcal{S}$.

$$[\boldsymbol{v}\overset{*}{|}\boldsymbol{w}] = ii_{\boldsymbol{v}\times\boldsymbol{w}}(\mathbf{1}) - ii_{\boldsymbol{w}}(\boldsymbol{v}) - ii_{\boldsymbol{v}}(\boldsymbol{w});$$

$$\boldsymbol{v}^* = ii_{\boldsymbol{v}^2}(\mathbf{1}) - 2ii_{\boldsymbol{v}}(\boldsymbol{v}).$$

If $\mathcal{S}$ is not empty,

$$[\boldsymbol{v}\overset{*}{|}\boldsymbol{w}] = \boldsymbol{v}\times\boldsymbol{w} - (v_\downarrow \times w_\downarrow)\mathbf{1} - ii_{\boldsymbol{w}}(\boldsymbol{v}) - ii_{\boldsymbol{v}}(\boldsymbol{w}), \quad \boldsymbol{v}^* = \boldsymbol{v}^2 - v_\downarrow^2\mathbf{1} - 2ii_{\boldsymbol{v}}(\boldsymbol{v}),$$

where $v_\downarrow$ and $w_\downarrow$ are the starting values of $\boldsymbol{v}$ and $\boldsymbol{w}$.

(b) $[\boldsymbol{v}\restriction\mathcal{S}\wedge\tau\overset{*}{|}\boldsymbol{w}\restriction\mathcal{S}\wedge\tau] = [\boldsymbol{v}\overset{*}{|}\boldsymbol{w}]\restriction\mathcal{S}\wedge\tau$ for every $\tau\in\mathcal{S}$.

$$[\boldsymbol{v}\restriction\mathcal{S}\vee\tau\overset{*}{|}\boldsymbol{w}\restriction\mathcal{S}\vee\tau] = ([\boldsymbol{v}\overset{*}{|}\boldsymbol{w}]\restriction\mathcal{S}\vee\tau) - z\mathbf{1}$$

where $z = \int_{\mathcal{S}\wedge\tau} d\boldsymbol{v}d\boldsymbol{w} \in L^0(\mathfrak{A}_\tau)$.

$$\boldsymbol{v}^*\restriction\mathcal{S}\vee\tau = v^*_\tau\mathbf{1} + (\boldsymbol{v}\restriction\mathcal{S}\vee\tau)^*$$

where $v^*_\tau = \int_{\mathcal{S}\wedge\tau}(d\boldsymbol{v})^2$.

(c) A perfectly elementary fact is that if $\boldsymbol{v}-\boldsymbol{u}$ is constant then $\boldsymbol{u}^* = \boldsymbol{v}^*$.

617L Corollary Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v}$ a local integrator with domain $\mathcal{S}$. Let $\boldsymbol{v}^*$ be the quadratic variation of $\boldsymbol{v}$. Then $\boldsymbol{v}^*$ is non-negative, non-decreasing and locally of bounded variation. If $\boldsymbol{w}$ is another local integrator with domain $\mathcal{S}$, then $[\boldsymbol{v}\overset{*}{|}\boldsymbol{w}]$ is locally of bounded variation.

617M Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v}$, $\boldsymbol{w}$ local integrators with domain $\mathcal{S}$. Then $[\boldsymbol{v}\overset{*}{|}\boldsymbol{w}]^2 \le \boldsymbol{v}^*\times\boldsymbol{w}^*$.

617N Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v}$, $\boldsymbol{w}$ local integrators with domain $\mathcal{S}$. Let $\hat{\boldsymbol{v}}$, $\hat{\boldsymbol{w}}$ be their fully adapted extensions to the covered envelope $\hat{\mathcal{S}}$ of $\mathcal{S}$. Then $[\hat{\boldsymbol{v}}\overset{*}{|}\hat{\boldsymbol{w}}]$ is the fully adapted extension of $[\boldsymbol{v}\overset{*}{|}\boldsymbol{w}]$ to $\hat{\mathcal{S}}$. In particular, the quadratic variation of $\hat{\boldsymbol{v}}$ is the fully adapted extension to $\hat{\mathcal{S}}$ of the quadratic variation of $\boldsymbol{v}$.

617O Examples Suppose that $T = [0,\infty[$.

(a) Let $\boldsymbol{\iota}$ be the identity process. Then its quadratic variation $\boldsymbol{\iota}^*$ is zero.

(b) Let $\boldsymbol{v}$ be the standard Poisson process. Then $\boldsymbol{v}$ is equal to its quadratic variation $\boldsymbol{v}^*$.

617P Lemma Let $\mathcal{S}$ be a full sublattice of $\mathcal{T}$ with a greatest element, $\boldsymbol{z}$ a moderately oscillatory process and $\boldsymbol{v}$, $\boldsymbol{v}'$ integrators, all with domain $\mathcal{S}$. Set $\boldsymbol{w} = ii_{\boldsymbol{v}}(\boldsymbol{z})$. Then $\int_{\mathcal{S}} d\boldsymbol{w}\, d\boldsymbol{v}' = \int_{\mathcal{S}} \boldsymbol{z}\, d\boldsymbol{v}\, d\boldsymbol{v}'$.

617Q Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u}$, $\boldsymbol{z}$ and $\boldsymbol{z}'$ locally moderately oscillatory processes with domain $\mathcal{S}$, and $\boldsymbol{v}$, $\boldsymbol{v}'$ local integrators with domain $\mathcal{S}$. Set $\boldsymbol{w} = ii_{\boldsymbol{v}}(\boldsymbol{z})$, $\boldsymbol{w}' = ii_{\boldsymbol{v}'}(\boldsymbol{z}')$.

(a)(i) $[\boldsymbol{w} \overset{*}{|} \boldsymbol{v}'] = ii_{[\boldsymbol{v} \overset{*}{|} \boldsymbol{v}']}(\boldsymbol{z})$, $\quad ii_{[\boldsymbol{w} \overset{*}{|} \boldsymbol{v}']}(\boldsymbol{u}) = ii_{[\boldsymbol{v} \overset{*}{|} \boldsymbol{v}']}(\boldsymbol{u} \times \boldsymbol{z})$.

(ii) $[\boldsymbol{w} \overset{*}{|} \boldsymbol{w}'] = ii_{[\boldsymbol{v} \overset{*}{|} \boldsymbol{v}']}(\boldsymbol{z} \times \boldsymbol{z}')$, $\quad ii_{[\boldsymbol{w} \overset{*}{|} \boldsymbol{w}']}(\boldsymbol{u}) = ii_{[\boldsymbol{v} \overset{*}{|} \boldsymbol{v}']}(\boldsymbol{u} \times \boldsymbol{z} \times \boldsymbol{z}')$.

(iii) $\boldsymbol{w}^* = ii_{\boldsymbol{v}^*}(\boldsymbol{z}^2)$, $\quad ii_{\boldsymbol{w}^*}(\boldsymbol{u}) = ii_{\boldsymbol{v}^*}(\boldsymbol{u} \times \boldsymbol{z}^2)$.

(b) If $\boldsymbol{u}$, $\boldsymbol{z}$ and $\boldsymbol{z}'$ are moderately oscillatory and $\boldsymbol{v}$, $\boldsymbol{v}'$ are integrators,

$$\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{w}\, d\boldsymbol{w}' = \int_{\mathcal{S}} \boldsymbol{u} \times \boldsymbol{z} \times \boldsymbol{z}' d\boldsymbol{v}\, d\boldsymbol{v}', \quad \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{w}^* = \int_{\mathcal{S}} \boldsymbol{u} \times \boldsymbol{z}^2 d\boldsymbol{v}^*.$$

617R Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v}$ a process with domain $\mathcal{S}$ which is locally of bounded variation. Then $\boldsymbol{v}$ and its cumulative variation have the same quadratic variation.

Mnemonic $|d\boldsymbol{v}|^2 = d\boldsymbol{v}^2$.

Version of 8.9.12/26.8.22

618 Oscillations and jump-free processes

For the work so far, moderately oscillatory processes have been sufficiently regular for our needs. But for the next development (Itô's formula, 619C), we are going to need a new concept. In 618B I formulate a notion of 'jump-free' process corresponding to the idea of 'process with continuous sample paths' (618H).

618B Definitions (a) Let I be a finite sublattice of $\mathcal{T}$, and $\boldsymbol{u}$ a fully adapted process defined (at least) on I. The **I-oscillation of $\boldsymbol{u}$** is

$$\mathrm{Osclln}_I(\boldsymbol{u}) = \sup_{e \in \mathrm{Stio}_0(I)} \Delta_e(\mathbf{1}, |d\boldsymbol{u}|).$$

Note that if $(\tau_0, \dots, \tau_n)$ linearly generates the I-cells, then $\mathrm{Osclln}_I(\boldsymbol{u}) = \sup_{i<n} |u_{\tau_{i+1}} - u_{\tau_i}|$.

(b) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ an order-bounded process. Set $\bar{u} = \sup |\boldsymbol{u}|$.

(i) $\mathrm{Osclln}_J(\boldsymbol{u}) \le 2\bar{u}$ for every $J \in \mathcal{I}(\mathcal{S})$. We set

$$\mathrm{Osclln}^*_I(\boldsymbol{u}) = \sup_{J \in \mathcal{I}(\mathcal{S}), J \supseteq I} \mathrm{Osclln}_J(\boldsymbol{u}) \le 2\bar{u}$$

for every $I \in \mathcal{I}(\mathcal{S})$.

(ii) The **residual oscillation** $\mathrm{Osclln}(\boldsymbol{u})$ is $\inf_{I \in \mathcal{I}(\mathcal{S})} \mathrm{Osclln}^*_I(\boldsymbol{u}) \le 2\bar{u}$. $\boldsymbol{u}$ is **jump-free** if $\mathrm{Osclln}(\boldsymbol{u}) = 0$.

(iii) $\boldsymbol{u}$ is **locally jump-free** if $\boldsymbol{u}\restriction\mathcal{S} \wedge \tau$ is jump-free for every $\tau \in \mathcal{S}$.

(iv) $\mathrm{Osclln}^*_\emptyset(\boldsymbol{u}) = \sup\{|u_{\sigma'} - u_\sigma| : \sigma, \sigma' \in \mathcal{S}\}$.

(v) $\mathrm{Osclln}(\boldsymbol{u})$ is the limit $\lim_{I \uparrow \mathcal{I}(\mathcal{S})} \mathrm{Osclln}^*_I(\boldsymbol{u})$ and $\boldsymbol{u}$ is jump-free iff

$$\inf_{I \in \mathcal{I}(\mathcal{S})} \theta(\mathrm{Osclln}^*_I(\boldsymbol{u})) = \lim_{I \uparrow \mathcal{I}(\mathcal{S})} \theta(\mathrm{Osclln}^*_I(\boldsymbol{u})) = 0.$$

(c) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$, $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ order-bounded processes.

(i) For any $\alpha \in \mathbb{R}$,

$$\mathrm{Osclln}_I(\alpha\boldsymbol{u}) = |\alpha| \,\mathrm{Osclln}_I(\boldsymbol{u}) \text{ for every } I \in \mathcal{I}(\mathcal{S}),$$

$$\mathrm{Osclln}^*_I(\alpha\boldsymbol{u}) = |\alpha| \,\mathrm{Osclln}^*_I(\boldsymbol{u}) \text{ for every } I \in \mathcal{I}(\mathcal{S}),$$

$$\mathrm{Osclln}(\alpha\boldsymbol{u}) = |\alpha| \,\mathrm{Osclln}(\boldsymbol{u}).$$

(ii)

$$\mathrm{Osclln}_I(\boldsymbol{u} + \boldsymbol{v}) \le \mathrm{Osclln}_I(\boldsymbol{u}) + \mathrm{Osclln}_I(\boldsymbol{v}) \text{ for every } I \in \mathcal{I}(\mathcal{S}),$$

$$\mathrm{Osclln}^*_I(\boldsymbol{u} + \boldsymbol{v}) \le \mathrm{Osclln}^*_I(\boldsymbol{u}) + \mathrm{Osclln}^*_I(\boldsymbol{v}) \text{ for every } I \in \mathcal{I}(\mathcal{S}),$$

$$\mathrm{Osclln}(\boldsymbol{u} + \boldsymbol{v}) \le \mathrm{Osclln}(\boldsymbol{u}) + \mathrm{Osclln}(\boldsymbol{v}).$$

(iii) Writing $\bar{u}$, $\bar{v}$ for $\sup|\boldsymbol{u}|$ and $\sup|\boldsymbol{v}|$,

$$\mathrm{Osclln}_I(\boldsymbol{u}\times\boldsymbol{v}) \le \bar{v}\times\mathrm{Osclln}_I(\boldsymbol{u}) + \bar{u}\times\mathrm{Osclln}_I(\boldsymbol{v}) \text{ for every } I\in\mathcal{I}(\mathcal{S}),$$

$$\mathrm{Osclln}^*_I(\boldsymbol{u}\times\boldsymbol{v}) \le \bar{v}\times\mathrm{Osclln}^*_I(\boldsymbol{u}) + \bar{u}\times\mathrm{Osclln}^*_I(\boldsymbol{v}) \text{ for every } I\in\mathcal{I}(\mathcal{S}),$$

$$\mathrm{Osclln}(\boldsymbol{u}\times\boldsymbol{v}) \le \bar{v}\times\mathrm{Osclln}(\boldsymbol{u}) + \bar{u}\times\mathrm{Osclln}(\boldsymbol{v}).$$

618C Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ an order-bounded fully adapted process. Let I be a non-empty finite sublattice of $\mathcal{S}$; suppose that $(\tau_0,\dots,\tau_n)$ linearly generates the I-cells.

(a) Set $\tau_{-1} = \inf\mathcal{S}$ and $\tau_{n+1} = \sup\mathcal{S}$ and

$$\begin{aligned} w = \sup\{|u_{\sigma'} - u_\sigma| : \ &\sigma,\ \sigma'\in\mathcal{S} \text{ and there is an } i \\ &\text{such that } -1\le i\le n \text{ and } \tau_i\le\sigma\le\sigma'\le\tau_{i+1}\}, \\ w' = \sup\{|u_{\sigma'} - u_\sigma| : \ &\sigma,\ \sigma'\in\mathcal{S} \text{ and there is an } i \\ &\text{such that } -1\le i\le n \text{ and } \sigma,\ \sigma'\in[\tau_i,\tau_{i+1}]\}. \end{aligned}$$

Then $w = w' = \mathrm{Osclln}^*_I(\boldsymbol{u})$.

(b) Now suppose that $\boldsymbol{u}$ is non-decreasing. Set $u_\downarrow = \inf_{\sigma\in\mathcal{S}} u_\sigma$ and $u_\uparrow = \sup_{\sigma\in\mathcal{S}} u_\sigma$. Then

$$\mathrm{Osclln}^*_I(\boldsymbol{u}) = (u_{\tau_0} - u_\downarrow)\vee\sup_{i<n}(u_{\tau_{i+1}} - u_{\tau_i})\vee(u_\uparrow - u_{\tau_n}).$$

618D Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a locally order-bounded process.

(a) Set $v_\tau = \mathrm{Osclln}(\boldsymbol{u}\restriction\mathcal{S}\wedge\tau)$ for $\tau\in\mathcal{S}$. Then $\boldsymbol{v} = \langle v_\tau\rangle_{\tau\in\mathcal{S}}$ is a non-decreasing fully adapted process.

(b) If $\boldsymbol{u}$ is order-bounded, then

(i) $\mathrm{Osclln}(\boldsymbol{u}) = \mathrm{Osclln}(\boldsymbol{u}\restriction\mathcal{S}\wedge\tau)\vee\mathrm{Osclln}(\boldsymbol{u}\restriction\mathcal{S}\vee\tau)$ for every $\tau\in\mathcal{S}$,

(ii) $\mathrm{Osclln}(\boldsymbol{u}\restriction\mathcal{S}\cap[\tau,\tau']) \le \mathrm{Osclln}(\boldsymbol{u})$ whenever $\tau\le\tau'$ in $\mathcal{S}$.

***618E Lemma** Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ with a greatest element, $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a jump-free process, $\tau\in\mathcal{S}$ and $\epsilon>0$. Then there is a $\tau'\in\mathcal{S}\vee\tau$ such that $[\![\tau<\tau']\!] = [\![\tau<\max\mathcal{S}]\!]$ and $\theta(\sup_{\sigma\in\mathcal{S}\cap[\tau,\tau']}|u_\sigma - u_\tau|)\le\epsilon$.

618F Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.

(a) If $\boldsymbol{u}$, $\boldsymbol{v}$ are order-bounded processes with domain $\mathcal{S}$, then $|\mathrm{Osclln}(\boldsymbol{u}) - \mathrm{Osclln}(\boldsymbol{v})| \le 2\sup|\boldsymbol{u}-\boldsymbol{v}|$.

(b) $\mathrm{Osclln} : M_{\text{o-b}}(\mathcal{S})\to L^0(\mathfrak{A})$ is uniformly continuous if $M_{\text{o-b}}(\mathcal{S})$ is given its ucp uniformity.

618G Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. Write $M_{\text{j-f}}(\mathcal{S})$ for the set of jump-free fully adapted processes with domain $\mathcal{S}$.

(a) The set $M_{\text{j-f}}(\mathcal{S})$ of jump-free fully adapted processes with domain $\mathcal{S}$ is a topologically closed f-subalgebra of $M_{\text{o-b}}(\mathcal{S})$, and $\bar{h}\boldsymbol{v}\in M_{\text{j-f}}(\mathcal{S})$ whenever $\boldsymbol{v}\in M_{\text{j-f}}(\mathcal{S})$ and $h:\mathbb{R}\to\mathbb{R}$ is continuous.

(b) A (locally) jump-free fully adapted process on $\mathcal{S}$ is (locally) moderately oscillatory.

(c) If $\boldsymbol{v}\in M_{\text{j-f}}(\mathcal{S})$, then $\boldsymbol{v}\restriction\mathcal{S}\vee\tau$, $\boldsymbol{v}\restriction\mathcal{S}\wedge\tau'$ and $\boldsymbol{v}\restriction\mathcal{S}\cap[\tau,\tau']$ are jump-free whenever $\tau\le\tau'$ in $\mathcal{S}$. In particular, $\boldsymbol{v}$ is locally jump-free.

618H Where jump-free processes come from: Proposition Let (Ω,Σ,μ) be a complete probability space and $\langle\Sigma_t\rangle_{t\ge0}$ a family of σ-subalgebras of Σ, all containing every negligible subset of Ω. Suppose that we are given a family $\langle X_t\rangle_{t\ge0}$ of real-valued functions on Ω such that X_t is Σ_t-measurable for every $t\ge0$ and $t\mapsto X_t(\omega) : [0,\infty[\ \to\mathbb{R}$ is continuous for every $\omega\in\Omega$. Then $\langle X_t\rangle_{t\ge0}$ is progressively measurable, and if $(\mathfrak{A},\bar{\mu},\langle\mathfrak{A}_t\rangle_{t\ge0},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ and $\langle x_\sigma\rangle_{\sigma\in\mathcal{T}_f}$ are defined as in 612H, $\boldsymbol{x} = \langle x_\sigma\rangle_{\sigma\in\mathcal{T}_f}$ is locally jump-free.

618I Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a locally jump-free fully adapted process. If $A\subseteq\mathcal{S}$ is non-empty and upwards-directed and $\sup A\in\mathcal{S}$, then $u_{\sup A} = \lim_{\sigma\uparrow A} u_\sigma$.

618J Examples Take $T=[0,\infty[$.

(a) The identity process is locally jump-free.

(b) The standard Poisson process is not locally jump-free.

(c) Brownian motion is locally jump-free.

618K Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. If I, $J\in\mathcal{I}(\mathcal{S})$ and $a\in\mathfrak{A}$ are such that $J\subseteq I$ and $a\subseteq\sup_{\sigma\in J}[\![\tau=\sigma]\!]$ for every $\tau\in I$, then $a\subseteq[\![\mathrm{Osclln}_I(\boldsymbol{u})=\mathrm{Osclln}_J(\boldsymbol{u})]\!]$ for every fully adapted process $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$.

618L Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a fully adapted process, $\hat{\mathcal{S}}$ the covered envelope of $\mathcal{S}$, and $\hat{\boldsymbol{u}}=\langle\hat u_\sigma\rangle_{\sigma\in\hat{\mathcal{S}}}$ the fully adapted extension of $\boldsymbol{u}$ to $\hat{\mathcal{S}}$.

(a) If either $\boldsymbol{u}$ or $\hat{\boldsymbol{u}}$ is order-bounded, so is the other, and in this case $\mathrm{Osclln}(\hat{\boldsymbol{u}})=\mathrm{Osclln}(\boldsymbol{u})$.

(b) In particular, $\boldsymbol{u}$ is jump-free iff $\hat{\boldsymbol{u}}$ is jump-free.

(c) If either $\sup_{\sigma\in\mathcal{S}}\mathrm{Osclln}(\boldsymbol{u}\restriction\mathcal{S}\wedge\sigma)$ or $\sup_{\tau\in\hat{\mathcal{S}}}\mathrm{Osclln}(\hat{\boldsymbol{u}}\restriction\hat{\mathcal{S}}\wedge\tau)$ is defined in $L^0(\mathfrak{A})$, so is the other, and they are equal.

618M Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u}$ a moderately oscillatory process. Then $\mathrm{Osclln}(\boldsymbol{u})=\lim_{I\uparrow\mathcal{I}(\mathcal{S})}\mathrm{Osclln}_I(\boldsymbol{u})$.

618N Lemma Let $\mathcal{S}$ be a full sublattice of $\mathcal{T}$ with a greatest element, $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a moderately oscillatory process, and $\delta>0$. Let $\langle y_i\rangle_{i\in\mathbb{N}}$ be the sequence constructed from $\boldsymbol{u}$ and δ as in 615M. Then $|y_{i+1}-y_i|\le\mathrm{Osclln}(\boldsymbol{u})+\delta\chi 1$ for every $i\in\mathbb{N}$.

618O Definition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\psi:\mathcal{S}^{2\uparrow}\to L^0(\mathfrak{A})$ an adapted interval function which is order-bounded. Following 618B, set

$$\mathrm{Osclln}_I(\psi)=\sup_{e\in\mathrm{Stio}_0(I)}\Delta_e(\mathbf{1},|d\psi|)$$

for $I\in\mathcal{I}(\mathcal{S})$ (counting $\sup\emptyset$ as 0),

$$\mathrm{Osclln}_I^*(\psi)=\sup_{J\in\mathcal{I}(\mathcal{S}),J\supseteq I}\mathrm{Osclln}_J(\psi)$$

for $I\in\mathcal{I}(\mathcal{S})$, and

$$\mathrm{Osclln}(\psi)=\inf_{I\in\mathcal{I}(\mathcal{S})}\mathrm{Osclln}_I^*(\psi).$$

$\mathrm{Osclln}(\psi)=\lim_{I\uparrow\mathcal{I}(\mathcal{S})}\mathrm{Osclln}_I^*(\psi)$. Moreover, if $\boldsymbol{u}$ is an order-bounded fully adapted process and $\psi=\Delta\boldsymbol{u}$ the corresponding interval function, $\mathrm{Osclln}_I(\psi)=\mathrm{Osclln}_I(\boldsymbol{u})$ for every $I\in\mathcal{I}(\mathcal{S})$ and $\mathrm{Osclln}(\psi)=\mathrm{Osclln}(\boldsymbol{u})$.

618P Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\psi:\mathcal{S}^{2\uparrow}\to L^0(\mathfrak{A})$ a strictly adapted interval function. For $I\in\mathcal{I}(\mathcal{S})$, set $\boldsymbol{w}_I=\langle w_{I\tau}\rangle_{\tau\in\mathcal{S}}$ where $w_{I\tau}=S_{I\wedge\tau}(\mathbf{1},d\psi)$ for $\tau\in\mathcal{S}$.

(a) For any $I\in\mathcal{I}(\mathcal{S})$, $\boldsymbol{w}_I$ is fully adapted.

(b) Suppose that $\mathcal{S}$ is finitely full, ψ is order-bounded and $\int_{\mathcal{S}}d\psi$ is defined. Then $ii_\psi(\mathbf{1})$ is order-bounded and $\mathrm{Osclln}(ii_\psi(\mathbf{1}))\le 2\,\mathrm{Osclln}(\psi)$.

(c) Suppose tht ψ is order-bounded and $\int_{\hat{\mathcal{S}}}d\hat\psi$ is defined, where $\hat{\mathcal{S}}$ is the covered envelope of $\mathcal{S}$ and $\hat\psi$ is the adapted extension of ψ to $\hat{\mathcal{S}}^\uparrow$. If $ii_\psi(\mathbf{1})$ is moderately oscillatory, then $\mathrm{Osclln}(ii_\psi(\mathbf{1}))\le\mathrm{Osclln}(\psi)$.

(d) If ψ is an order-bounded integrating interval function, then $\mathrm{Osclln}(ii_\psi(\mathbf{1}))\le\mathrm{Osclln}(\psi)$.

618Q Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a moderately oscillatory process, and $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ an integrator. Then $\mathrm{Osclln}(ii_{\boldsymbol{v}}(\boldsymbol{u}))\le\sup|\boldsymbol{u}|\times\mathrm{Osclln}(\boldsymbol{v})$. $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is jump-free if $\boldsymbol{v}$ is.

618R Corollary Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a locally moderately oscillatory process, and $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a locally jump-free local integrator. Then $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is locally jump-free.

618S Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$, $\boldsymbol{w} = \langle w_\sigma \rangle_{\sigma \in \mathcal{S}}$ two integrators.
(a) $\operatorname{Osclln}([\boldsymbol{v} \overset{*}{|} \boldsymbol{w}]) \le \operatorname{Osclln}(\boldsymbol{v}) \times \operatorname{Osclln}(\boldsymbol{w})$.
(b) $\operatorname{Osclln}(\boldsymbol{v}^*) = (\operatorname{Osclln}(\boldsymbol{v}))^2$.

618T Corollary Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$, $\boldsymbol{w} = \langle w_\sigma \rangle_{\sigma \in \mathcal{S}}$ two (local) integrators with domain $\mathcal{S}$ of which $\boldsymbol{v}$ is (locally) jump-free. Then the covariation $[\boldsymbol{v} \overset{*}{|} \boldsymbol{w}]$ and the quadratic variation $\boldsymbol{v}^*$ are (locally) jump-free.

618U Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ a process of bounded variation. Let $\boldsymbol{v}^\uparrow$ be its cumulative variation. Then $\operatorname{Osclln}(\boldsymbol{v}^\uparrow)$ is equal to $\operatorname{Osclln}(\boldsymbol{v})$; $\boldsymbol{v}$ is jump-free iff $\boldsymbol{v}^\uparrow$ is jump-free.

618V Corollary Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v}$ a fully adapted process with domain $\mathcal{S}$. Then $\boldsymbol{v}$ is jump-free and of bounded variation iff it is expressible as the difference of two non-negative non-decreasing order-bounded jump-free processes.

618zO Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\psi : \mathcal{S}^{2\uparrow} \to L^0(\mathfrak{A})$ an order-bounded integrating interval function with indefinite integral $\boldsymbol{v} = ii_\psi(\mathbf{1})$. Then

$$\operatorname{Osclln}(\boldsymbol{v}) \le \operatorname{Osclln}(\psi).$$

Version of 13.3.17/29.7.19

619 Itô's formula

I give three versions of Itô's formula (619C, 619D and 619J). The last depends on elementary facts about the action of functions of more than one real variable on strings of processes (619E-619G).

619B Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ an integrator. Then for every $\epsilon > 0$ there is a $\delta > 0$ such that $\theta(S_I(\boldsymbol{u}, (d\boldsymbol{v})^2) \le \epsilon$ whenever $I \in \mathcal{I}(\mathcal{S})$, $\boldsymbol{u} \in M_{\mathrm{fa}}(\mathcal{I})$ and $\theta(\sup |\boldsymbol{u}|) \le \delta$.

619C Itô's Formula, first form Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{v} = \langle v_\tau \rangle_{\tau \in \mathcal{S}}$ a jump-free integrator, and $\boldsymbol{v}^*$ its quadratic variation. If $h : \mathbb{R} \to \mathbb{R}$ is a twice-differentiable function with continuous second derivative, then

$$\int_{\mathcal{S}} \bar{h}'\boldsymbol{v}\, d\boldsymbol{v} + \frac{1}{2}\int_{\mathcal{S}} \bar{h}''\boldsymbol{v}\, d\boldsymbol{v}^*$$

is defined and equal to $\bar{h}(v_\uparrow) - \bar{h}(v_\downarrow)$, where

$$v_\uparrow = \lim_{\sigma\uparrow\mathcal{S}} v_\sigma, \quad v_\downarrow = \lim_{\sigma\downarrow\mathcal{S}} v_\sigma.$$

Remark In the formula above, $\bar{h}' : L^0 \to L^0$ and $\bar{h}'' : L^0 \to L^0$ should be read as $\overline{h'}$ and $\overline{h''}$.

619D Itô's Formula, second form Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v}$ a jump-free integrator with domain $\mathcal{S}$ and quadratic variation $\boldsymbol{v}^*$. If $\boldsymbol{u}$ is a moderately oscillatory process with domain $\mathcal{S}$, and $h : \mathbb{R} \to \mathbb{R}$ is a twice-differentiable function with continuous second derivative, then

$$\int_{\mathcal{S}} \boldsymbol{u}\, d(\bar{h}\boldsymbol{v}) = \int_{\mathcal{S}} \boldsymbol{u} \times \bar{h}'\boldsymbol{v}\, d\boldsymbol{v} + \frac{1}{2}\int_{\mathcal{S}} \boldsymbol{u} \times \bar{h}''\boldsymbol{v}\, d\boldsymbol{v}^*.$$

619E Proposition Let $k \ge 1$ be an integer.
(a) Suppose that $u_1, \dots, u_k \in L^0$. Let $\mathcal{B}_k$ be the Borel σ-algebra of $\mathbb{R}^k$. Then there is a unique sequentially order-continuous Boolean homomorphism $\phi : \mathcal{B}_k \to \mathfrak{A}$ such that $\phi\{(\xi_1, \dots, \xi_k) : \xi_i > \alpha\} = [\![u_i > \alpha]\!]$ whenever $1 \le i \le k$ and $\alpha \in \mathbb{R}$.
In this context, write $[\![(u_1, \dots, u_k) \in E]\!]$ for ϕE, for every Borel set $E \subseteq \mathbb{R}^k$.

(b) Suppose that $h : \mathbb{R}^k \to \mathbb{R}$ is a Borel measurable function. Then there is a unique operator $\bar{h} : (L^0)^k \to L^0$ such that $[\![\bar{h}(u_1,\dots,u_k) \in F]\!] = [\![(u_1,\dots,u_k) \in h^{-1}[F]]\!]$ whenever $F \subseteq \mathbb{R}$ is a Borel set and $u_1,\dots,u_k \in L^0$.

(c) If $u_1,\dots,u_k,v_1,\dots,v_k \in L^0$, then

$$\inf_{1\le i\le k} [\![u_i = v_i]\!] \subseteq [\![\bar{h}(u_1,\dots,u_k) = \bar{h}(v_1,\dots,v_k)]\!].$$

(d) If $h : \mathbb{R}^k \to \mathbb{R}$ is continuous, then $\bar{h} : (L^0)^k \to L^0$ is continuous for the topology of convergence in measure.

(e) Suppose that Ω is a set, Σ is a σ-algebra of subsets of Ω, $\mathcal{N}$ is a σ-ideal of Σ, and $\mathfrak{A}$ is isomorphic to the quotient Boolean algebra $\Sigma/\mathcal{N}$. Write $\mathcal{L}^0$ for the f-algebra of real-valued Σ-measurable functions on Ω, and $\mathcal{W}$ for the ideal

$$\{f : f \in \mathcal{L}^0,\ \{\omega : f(\omega) \ne 0\} \in \mathcal{N}\},$$

so that L^0 can be identified with the f-algebra quotient $\mathcal{L}^0/\mathcal{W}$. Write $E \mapsto E^\bullet : \Sigma \to \mathfrak{A}$ and $f \mapsto f^\bullet : \mathcal{L}^0 \to L^0$ for the homomorphisms corresponding to the identifications $\mathfrak{A} \cong \Sigma/\mathcal{N}$ and $L^0 \cong \mathcal{L}^0/\mathcal{W}$. Then if $h : \mathbb{R}^k \to \mathbb{R}$ is a Borel measurable function,

$$\bar{h}(f_1^\bullet,\dots,f_k^\bullet) = (h(f_1,\dots,f_k))^\bullet$$

for all $f_1,\dots,f_k \in \mathcal{L}^0$, defining the composition $h(f_1,\dots,f_k)$ by setting $(h(f_1,\dots,f_k))(\omega) = h(f_1(\omega),\dots,f_k(\omega))$ for every $\omega \in \Omega$.

(f) Suppose that $\langle h_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of Borel measurable functions from $\mathbb{R}^k$ to $\mathbb{R}$, and that $h(x) = \sup_{n\in\mathbb{N}} h_n(x)$ is finite for every $x \in \mathbb{R}^k$. Then $\langle \bar{h}_n(u_1,\dots,u_k)\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in L^0 with supremum $\bar{h}(u_1,\dots,u_k)$, for all $u_1,\dots,u_k \in L^0$.

(g) Now suppose that $(\mathfrak{C},\nu)$ is another probability algebra and $\phi : \mathfrak{A} \to \mathfrak{C}$ is an order-continuous Boolean homomorphism. Let $T_\phi : L^0(\mathfrak{A}) \to L^0(\mathfrak{C})$ be the corresponding f-algebra homomorphism. Take $u_1,\dots,u_k \in L^0(\mathfrak{A})$.

(i) If $E \in \mathcal{B}_k$ is a Borel set, then $[\![(T_\phi u_1,\dots,T_\phi u_k) \in E]\!] = \phi[\![(u_1,\dots,u_k) \in E]\!]$.

(ii) If $h : \mathbb{R}^k \to \mathbb{R}$ is Borel measurable, then $\bar{h}(T_\phi u_1,\dots,T_\phi u_k) = T_\phi \bar{h}(u_1,\dots,u_k)$.

619F Definition Let $\boldsymbol{u}_1,\dots,\boldsymbol{u}_k$ be fully adapted processes defined on sublattices $\mathcal{S}_1,\dots,\mathcal{S}_k$ of $\mathcal{T}$ and $h : \mathbb{R}^k \to \mathbb{R}$ a Borel measurable function. Regarding $\boldsymbol{U} = (\boldsymbol{u}_1,\dots,\boldsymbol{u}_k)$ as the function $\sigma \mapsto (u_{1\sigma},\dots,u_{k\sigma}) : \mathcal{S} \to (L^0)^k$, where $\boldsymbol{u}_i = \langle u_{i\sigma}\rangle_{\sigma\in\mathcal{S}_i}$ for each i and $\mathcal{S} = \bigcap_{1\le i\le k} \mathcal{S}_i$, we have a composition

$$\bar{h}\boldsymbol{U} = \langle \bar{h}(u_{1\sigma},\dots,u_{k\sigma})\rangle_{\sigma\in\mathcal{S}}.$$

619G Proposition Let $\boldsymbol{u}_1,\dots,\boldsymbol{u}_k$ be fully adapted processes all with the same domain $\mathcal{S}$, and $h : \mathbb{R}^k \to \mathbb{R}$ a Borel measurable function. Write $\boldsymbol{U}$ for $(\boldsymbol{u}_1,\dots,\boldsymbol{u}_k)$.

(a) $\bar{h}\boldsymbol{U}$ is fully adapted.

(b) If every $\boldsymbol{u}_i$ is order-bounded and h is **locally bounded**, that is, bounded on bounded subsets of $\mathbb{R}^k$, then $\bar{h}\boldsymbol{U}$ is order-bounded.

(c) If every $\boldsymbol{u}_i$ is (locally) moderately oscillatory and h is continuous, then $\bar{h}\boldsymbol{U}$ is (locally) moderately oscillatory.

(d) If every $\boldsymbol{u}_i$ is (locally) jump-free and h is continuous, then $\bar{h}\boldsymbol{U}$ is (locally) jump-free.

*(e) If $\boldsymbol{z}$ is a fully adapted process with domain $\mathcal{S}$ and $\boldsymbol{z}^2 = \boldsymbol{z}$, then

(i) $\boldsymbol{z} \times \bar{h}(\boldsymbol{z} \times \boldsymbol{u}_1,\dots,\boldsymbol{z} \times \boldsymbol{u}_k) = \boldsymbol{z} \times \bar{h}\boldsymbol{U}$,

(ii) and if $h(0,\dots,0) = 0$, then $\bar{h}(\boldsymbol{z} \times \boldsymbol{u}_1,\dots,\boldsymbol{z} \times \boldsymbol{u}_k) = \boldsymbol{z} \times \bar{h}\boldsymbol{U}$.

619H Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $k \ge 1$ an integer and $h : \mathbb{R}^k \to \mathbb{R}$ a continuous function. Then $\bar{h} : M_{\text{o-b}}(\mathcal{S})^k \to M_{\text{o-b}}(\mathcal{S})$ is continuous when $M_{\text{o-b}}(\mathcal{S})$ is given the ucp topology and $M_{\text{o-b}}(\mathcal{S})^k$ the corresponding product topology.

619I Theorem Let $h : \mathbb{R}^k \to \mathbb{R}$ be a differentiable function; write $h_1,\dots,h_k$ for its partial derivatives. Suppose that every h_i is Lipschitz on every bounded set in $\mathbb{R}^k$. Let $\boldsymbol{v}_1,\dots,\boldsymbol{v}_k$ be integrators, all with the same domain $\mathcal{S}$. Then $\bar{h}(\boldsymbol{v}_1,\dots,\boldsymbol{v}_k)$ is an integrator.

619J Itô's Formula, third form Let $k \ge 1$ be an integer, and $h : \mathbb{R}^k \to \mathbb{R}$ a twice-differentiable function with continuous second derivative. Denote its first partial derivatives by $h_1, \dots , h_k$ and its second partial derivatives by $h_{11}, \dots , h_{kk}$. Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v}_1, \dots , \boldsymbol{v}_k$ jump-free integrators with domain $\mathcal{S}$; let $\boldsymbol{u}$ be a moderately oscillatory process with domain $\mathcal{S}$. Write $\boldsymbol{V} = (\boldsymbol{v}_1, \dots , \boldsymbol{v}_k)$. Then

$$\int_{\mathcal{S}} \boldsymbol{u}\, d(\bar{h}\boldsymbol{V}) = \sum_{i=1}^{k} \int_{\mathcal{S}} \boldsymbol{u} \times \bar{h}_i \boldsymbol{V}\, d\boldsymbol{v}_i + \frac{1}{2}\sum_{i=1}^{k} \sum_{j=1}^{k} \int_{\mathcal{S}} \boldsymbol{u} \times \bar{h}_{ij} \boldsymbol{V}\, d[\boldsymbol{v}_i \overset{*}{|} \boldsymbol{v}_j].$$

619K Corollary Let $k \ge 1$ be an integer, and $h : \mathbb{R}^k \to \mathbb{R}$ a twice-differentiable function with continuous second derivative. Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v}_1, \dots , \boldsymbol{v}_k$ locally jump-free local integrators with domain $\mathcal{S}$; let $\boldsymbol{u}$ be a locally moderately oscillatory process with domain $\mathcal{S}$. Write $\boldsymbol{V} = (\boldsymbol{v}_1, \dots , \boldsymbol{v}_k)$. Then

$$ii_{\bar{h}\boldsymbol{V}}(\boldsymbol{u}) = \sum_{i=1}^{k} ii_{\boldsymbol{v}_i}(\boldsymbol{u} \times \bar{h}_i \boldsymbol{V}) + \frac{1}{2}\sum_{i=1}^{k} \sum_{j=1}^{k} ii_{[\boldsymbol{v}_i \overset{*}{|} \boldsymbol{v}_j]}(\boldsymbol{u} \times \bar{h}_{ij} \boldsymbol{V}).$$