Chapter 61

The Riemann-sum integral

I begin with an attempt to give a coherent and complete description of the principal form of stochastic integration which will be investigated in this volume.

As elsewhere in probability theory, it is customary to set this material out in terms of ordinary random variables, that is, measurable functions defined on probability spaces. We find immediately, however, that while integrands and integrators may well present themselves most naturally in this form, the integrals we construct are defined, in the cases for which this theory has been developed, in terms of convergence in $\| \|_1$ or $\| \|_2$ or in measure, and therefore correspond not to explicit functions, but to equivalence classes of functions. Moreover, integrands and integrators can be changed on negligible sets without affecting the values of the corresponding integrals. I believe that the theory becomes clearer and cleaner if we move directly to operations on evolving families in L^0. While this demands an initial investment by the reader in a more abstract framework for the ideas of elementary probability theory, the translation is not difficult, and a full exposition can be found in Chapter 36.

Again, stochastic processes are usually expressed as families $\langle X_t\rangle_{t\in T}$ of random variables, indexed by a set T of 'times'. There are very good reasons for this. However, to describe the stochastic integral in reasonable generality we need, as a first step, to discuss the random variable X_τ for a stopping time τ. The measure theory to make this possible (the notion of 'progressively measurable' process) is well understood and has been described in §455. When we come, following my principle above, to look at $\langle X_t^\bullet\rangle_{t\in T}$, we find that we can have $X_t^\bullet = Y_t^\bullet$, that is, $X_t =_{\text{a.e.}} Y_t$, for every t, while $X_\tau^\bullet \ne Y_\tau^\bullet$. This is just a nuisance. For our purposes here, it makes better sense to start from a family $\langle u_\tau\rangle_{\tau\in\mathcal{S}}$ where $\mathcal{S}$ is a set of stopping times and $u_\tau \in L^0$ for every $\tau\in\mathcal{S}$. The construction of such families from processes $\langle X_t\rangle_{t\in T}$ is important and interesting, but has nothing to do with the very substantial difficulties of the basic theory of stochastic integration.

Of course I now have to look at filtrations and stopping times, and these too are not best described in terms of σ-algebras of sets and real-valued functions. In the formulation I wish to use here, we don't even have a probability space for the functions to be defined on. Instead of thinking of a filtration as a family $\langle\Sigma_t\rangle_{t\in T}$ of σ-subalgebras of the domain Σ of a probability measure μ, I look at the corresponding family of subalgebras of the measure algebra $\mathfrak{A}$ of μ. This is easy (at least, if you have read Chapter 32; and this is my last apology for insisting that you know something of Volume 3). A stopping time τ now becomes defined in terms of elements $[\![\tau > t]\!] \in \mathfrak{A}$, 'the region where $\tau > t$'. We need to develop a theory of regions $[\![\sigma < \tau]\!]$, $[\![\sigma = \tau]\!]$ in $\mathfrak{A}$, and subalgebras $\mathfrak{A}_\tau$ of $\mathfrak{A}$, for stopping times σ, τ; and now the processes $\langle u_\tau\rangle_{\tau\in\mathcal{S}}$ we work with must be such that '$u_\sigma = u_\tau$ whenever $\sigma=\tau$', that is, $[\![\sigma=\tau]\!] \subseteq [\![u_\sigma = u_\tau]\!]$. Setting up these structures takes the greater part of §§611-612, which come to about a quarter of the chapter. It happens that nearly everything in these two sections can be done without mentioning 'measure' at all.

I say again that none of this is difficult, but it does take quite a long time; there are some new kinds of algebra to get a solid basis in, particularly the theory of stopping-time intervals (611E, 611J-611K) and fully adapted processes (612D). With this established, however, we are within reach of a direct definition of a stochastic integral as a limit of Riemann sums (§613). As long as we do not enquire about when the integral is actually defined, this is very straightforward and can be done in great generality. The next three sections are devoted to finding the basic cases of processes $\boldsymbol{u}$, $\boldsymbol{v}$ for which we shall have a well-defined integral $\int \boldsymbol{u}\,d\boldsymbol{v}$. Concerning $\boldsymbol{u}$, we have 'simple' and 'moderately oscillatory' processes (612J, 615E). Concerning $\boldsymbol{v}$, we have the concept of 'integrator' (616Fc), which is well adapted to the basic theorem 616K, but is otherwise obscure. It is easy enough to find a definition of 'bounded variation' for stochastic processes (614J) and to show that processes of bounded variation are integrators (616R), but this is not what the stochastic integral is for; in this case we have much more direct methods available.

Now we are ready, at least in a formal sense, for some proper stochastic calculus in §§617 and 619. Here I set out useful general manipulations. Some of them reproduce patterns familiar from the ordinary Riemann integral (616J), but others are radically different (617I, 619C). On the way to the latter ('Itô's formula') we need to understand 'jump-free' processes, corresponding to processes with continuous sample paths (§618).

The theory here involves a large number of constructions. Many of these have no short descriptions in terms of the concepts developed in Volumes 1-4, and correspondingly require new terminology and notation. I have tried to arrange the material in such a way that, within any individual section, substantial parts of the basic framework can be taken to be constant. From §614 on, these are indicated in introductory paragraphs headed 'Notation'. These paragraphs are highly repetitive. But until you are very familiar with my language, it is likely that opening at a random page, and scanning for the next 'Theorem', will lead you to something totally mysterious. Sometimes a check in the index for terminology will help. But sometimes there will be a baffling symbol, and then it will be worth while turning to the beginning of the section to see if the symbol appears there. It seems to me that while this expands the volume by several pages in total, it is kinder than referring you each time to a complete list of the terminological quirks of this presentation.

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611 Stopping times

The first step is to describe the structures within which the work of this volume will proceed. While everything really important will have to be based on probability algebras, I start with ideas which can be applied to arbitrary Dedekind complete Boolean algebras. This section introduces filtrations of subalgebras, the lattice of stopping times, the algebras associated with stopping times, stopping-time intervals and covered envelopes.

611A Filtrations Throughout this volume, $\mathfrak{A}$ will denote a Dedekind complete Boolean algebra (definitions: 311A, 314A), with Boolean operations $\triangle$, $\cap$, $\cup$ and $\setminus$, zero 0 and multiplicative identity 1.

(a) Let T be a non-empty totally ordered set. A **filtration of order-closed subalgebras of** $\mathfrak{A}$ will be a non-decreasing family $\langle\mathfrak{A}_t\rangle_{t\in T}$ of order-closed subalgebras of $\mathfrak{A}$ (definition: 313D).

(b)(i) A **stopping time** τ **adapted to** $\langle\mathfrak{A}_t\rangle_{t\in T}$ is a family $\langle[\![\tau>t]\!]\rangle_{t\in T}$ such that

$[\![\tau>t]\!]\in\mathfrak{A}_t$ for every $t\in T$,

if $s\le t$ in T then $[\![\tau>t]\!]\subseteq[\![\tau>s]\!]$,

if $t\in T$ is not isolated on the right, that is, t is neither the greatest element of T nor the lower endpoint of a gap in T, that is, $\{s:s>t\}$ is non-empty and has infimum t, then $[\![\tau>t]\!]=\sup_{s>t}[\![\tau>s]\!]$.

(Compare 364A.)

(ii) It will be worth checking each concept against the constant stopping times, where for $t\in T$ the **constant stopping time at** t, $\check{t}$, is given by setting

$$\begin{aligned}[\![\check{t}>s]\!]&=1\text{ if }s<t,\\&=0\text{ if }s\ge t.\end{aligned}$$

(iii) I will say that a stopping time τ is

— **finite-valued** if $\inf_{t\in T}[\![\tau>t]\!]=0$,

— **bounded** if there is a $t\in T$ such that $[\![\tau>t]\!]=0$.

Constant stopping times are bounded, and bounded stopping times are finite-valued.

(iv) I will write $\mathcal{T}$ for the set of stopping times adapted to $\langle\mathfrak{A}_t\rangle_{t\in T}$, $\mathcal{T}_f\subseteq\mathcal{T}$ for the set of finite-valued stopping times, and $\mathcal{T}_b\subseteq\mathcal{T}_f$ for the set of bounded stopping times.

(c) It is convenient to think of a stopping time $\tau\in\mathcal{T}$ as the element $\langle[\![\tau>t]\!]\rangle_{t\in T}$ of the simple product algebra $\prod_{t\in T}\mathfrak{A}_t$ (definition: 315A). But a warning! while this represents $\mathcal{T}$ as a sublattice of $\prod_{t\in T}\mathfrak{A}_t$ (611Ca-611Cc) below), $\mathcal{T}$ is not as a rule order-closed (see 611F and 632C(a-i)).

(d) We are going to have to think of $\mathcal{T}$ as disjoint from T (see, for instance, 611G below). Subject to the Axiom of Foundation (Kunen 80, §III.4), and on any ordinary formalization of the concepts of 'ordered pair' and 'function', T and $\prod_{t\in T}\mathfrak{A}_t$ will automatically be disjoint. I will therefore suppose without further comment that this is always the case.

611B The partial ordering of stopping times If σ, $\tau\in\mathcal{T}$, say that $\sigma\le\tau$ if $[\![\sigma>t]\!]\subseteq[\![\tau>t]\!]$ for every $t\in T$, that is, $\sigma\subseteq\tau$ in $\prod_{t\in T}\mathfrak{A}_t$. This defines a partial order on $\mathcal{T}$.

611C Proposition (a) $\mathcal{T}$ is a Dedekind complete distributive lattice. Consequently any finite subset of $\mathcal{T}$ is included in a finite sublattice of $\mathcal{T}$.

(b) If $C\subseteq\mathcal{T}$ is non-empty, then $\sup C$ is defined by saying that

$$[\![\sup C>t]\!]=\sup_{\tau\in C}[\![\tau>t]\!]$$

for every $t\in T$, that is, the supremum of C in $\mathcal{T}$ is the same as the supremum of C in $\prod_{t\in T}\mathfrak{A}_t$.

(c) If σ, $\tau\in\mathcal{T}$, then $\sigma\wedge\tau$ is defined by saying that

$$[\![\sigma\wedge\tau>t]\!]=[\![\sigma>t]\!]\cap[\![\tau>t]\!]$$

for every $t\in T$, that is, $\sigma\wedge\tau$ in $\mathcal{T}$ corresponds to $\sigma\cap\tau$ in $\prod_{t\in T}\mathfrak{A}_t$.

(d) If C, $C'\subseteq\mathcal{T}$ are non-empty, then $\sup C\wedge\sup C'=\sup\{\sigma\wedge\sigma':\sigma\in C,\,\sigma'\in C'\}$.

(e) Writing $\check t$ for the constant stopping time at t, the map $t\mapsto\check t:T\to\mathcal{T}$ is an order-continuous lattice homomorphism, which is injective if $\mathfrak{A}\ne\{0\}$.

(f) $\mathcal{T}$ has greatest and least elements defined by saying that

$$[\![\max\mathcal{T}>t]\!]=1,\qquad[\![\min\mathcal{T}>t]\!]=0$$

for every $t\in T$, that is, they correspond to the greatest and least elements 1 and 0 of $\prod_{t\in T}\mathfrak{A}_t$. If T has a least element $\min T$, then $\min\mathcal{T}$ is the constant stopping time at $\min T$.

(g) $\mathcal{T}_f$ and $\mathcal{T}_b$ are ideals[1] in $\mathcal{T}$.

(h) The function $\sigma\mapsto\sigma\wedge\tau:\mathcal{T}\to\mathcal{T}$ is order-continuous (definition: 313Ha) for every $\tau\in\mathcal{T}$.

proof (a)(i) I start with a direct verification of (b). Setting $a_t=\sup_{\tau\in C}[\![\tau>t]\!]$ for $t\in T$, we see that

$a_t\in\mathfrak{A}_t$ for every $t\in T$, because $\mathfrak{A}_t$ is order-closed in $\mathfrak{A}$,

if $s\le t$ in T then $a_t\subseteq a_s$,

if $t\in T$ is not isolated on the right then

$$a_t=\sup_{\tau\in C}[\![\tau>t]\!]=\sup_{\tau\in C,s>t}[\![\tau>s]\!]=\sup_{s>t}a_s.$$

So we have a stopping time σ defined by writing $[\![\sigma>t]\!]=a_t$ for every t. Now it is easy to see that $\tau\le\sigma$ for every $\tau\in C$ and that σ is the least such stopping time. Thus our formula defines $\sup C$ in the partially ordered set $\mathcal{T}$.

(ii) This is enough to show that $\mathcal{T}$ is a Dedekind complete lattice (314Aa, 314Bb).

(iii) Similarly, we can check the formula in (c). Set $b_t=[\![\sigma>t]\!]\cap[\![\tau>t]\!]$ for $t\in T$. Then

$b_t\in\mathfrak{A}_t$ for every $t\in T$,

if $s\le t$ then $b_t\subseteq b_s$,

if $t\in T$ is not isolated on the right, then

$$\begin{aligned}\sup_{s>t}b_s&=\sup_{s>t}[\![\sigma>s]\!]\cap[\![\tau>s]\!]=\sup_{s,s'>t}[\![\sigma>s]\!]\cap[\![\tau>s']\!]\\&=\sup_{s>t}[\![\sigma>s]\!]\cap\sup_{s'>t}[\![\tau>s']\!]\end{aligned}$$

(313Bc)

$$=[\![\sigma>t]\!]\cap[\![\tau>t]\!]=b_t.$$

[1] If P is a lattice, an **ideal** of P is a set $Q\subseteq P$ such that $p\vee q\in Q$ for all p, $q\in Q$ and $p\in Q$ whenever $q\in Q$ and $p\le q$ in P. In this context I do not insist that Q should be non-empty.

So we have a stopping time υ such that $[\![\upsilon > t]\!] = [\![\sigma > t]\!] \cap [\![\tau > t]\!]$ for every t. But now it is easy to see that $\upsilon = \sigma \wedge \tau$.

(iv) Putting these formulae together, we see that $\sigma \mapsto [\![\sigma > t]\!] : \mathcal{T} \to \mathfrak{A}$ is a lattice homomorphism for every $t \in T$, so $\sigma \mapsto \langle [\![\sigma > t]\!] \rangle_{t \in T}$ is an injective lattice homomorphism from $\mathcal{T}$ into the distributive lattice $\prod_{t \in T} \mathfrak{A}_t$, and identifies $\mathcal{T}$ with a sublattice of $\prod_{t \in T} \mathfrak{A}_t$, which must be distributive. And it is true in any distributive lattice that finitely-generated sublattices are finite (3A1I(c-iii)).

(b)-(c) have been dealt with in (a-i) and (a-iii) above.

(d) Translated into $\prod_{t \in T} \mathfrak{A}_t$, as in (b)-(c) just above, this becomes

$$\sup C \cap \sup C' = \sup\{\sigma \cap \sigma' : \sigma \in C,\ \sigma' \in C'\},$$

which is 313Bc.

(e)(i) Concerning constant stopping times, we see at once from the formulae in 611B-611B that $\check{s} \le \check{t}$ when $s \le t$, so that $t \mapsto \check{t}$ is order-preserving; because T is totally ordered, it is a lattice homomorphism.

(ii) If $A \subseteq T$ is a non-empty set with supremum t in T, then

$$[\![\check{t} > t']\!] = 0 = \sup_{s \in A} [\![\check{s} > t']\!]$$

whenever $t' \ge t$, while if $t' < t$ there is an $s' \in A$ such that $t' < s'$, so that

$$[\![\check{t} > t']\!] = 1 = [\![\check{s}' > t']\!] = \sup_{s \in A} [\![\check{s} > t']\!].$$

By (b), $\check{t}$ is the supremum $\sup_{s \in A} \check{s}$ in $\mathcal{T}$.

(iii) If $\emptyset \ne A \subseteq T$ and $t = \inf A$, then $\check{t} \le \check{s}$ for every $s \in A$, as noted in (i). Now suppose that $\tau \in \mathcal{T}$ and $\check{t} \le \tau \le \check{s}$ for every $s \in A$. If t is isolated on the right in T, then $t \in A$ so surely $\tau = \check{t}$. Otherwise,

$$[\![\tau > t]\!] = \sup_{s > t} [\![\tau > s]\!] = \sup_{s \in A} [\![\tau > s]\!] \subseteq \sup_{s \in A} [\![\check{s} > s]\!] = 0.$$

Of course we now have

$$[\![\tau > s]\!] \subseteq [\![\tau > t]\!] = 0 = [\![\check{t} > s]\!]$$

for $s \ge t$, while

$$[\![\tau > s]\!] \supseteq [\![\check{t} > s]\!] = 1$$

for $s < t$, so $\tau = \check{t}$. As τ is arbitrary, $\check{t} = \inf_{s \in A} \check{s}$.

Thus $t \mapsto \check{t}$ is order-continuous.

(iv) If $\mathfrak{A} \ne \{0\}$, that is, $0 \ne 1$ in $\mathfrak{A}$, and $s < t$ in T, then

$$[\![\check{s} > s]\!] = 0 \ne 1 = [\![\check{t} > s]\!]$$

so $\check{s} \ne \check{t}$; thus $t \mapsto \check{t}$ is injective.

(f) The formulae offered for $\max \mathcal{T}$ and $\min \mathcal{T}$ describe stopping times corresponding to 1 and 0 in $\prod_{t \in T} \mathfrak{A}_t$, so give us the greatest and least elements of $\mathcal{T}$. If $\min T$ is defined, the formula for $\min \mathcal{T}$ agrees with that for the constant stopping time $(\min T)\check{}$.

(g)(i) If $\sigma, \tau \in \mathcal{T}^f$,

$$\begin{aligned}\inf_{t \in T} [\![\sigma \vee \tau > t]\!] &= \inf_{t \in T} [\![\sigma > t]\!] \cup [\![\tau > t]\!] = \inf_{s, t \in T} [\![\sigma > s]\!] \cup [\![\tau > t]\!] \\ &= \inf_{s \in T} [\![\sigma > s]\!] \cup \inf_{t \in T} [\![\tau > t]\!]\end{aligned}$$

(313Bd)

$$= 0.$$

So $\sigma \vee \tau \in \mathcal{T}_f$. If $\sigma \le \tau$ in $\mathcal{T}$ and $\tau \in \mathcal{T}_f$, then

$$\inf_{t \in T} [\![\sigma > t]\!] \subseteq \inf_{t \in T} [\![\tau > t]\!] = 0,$$

so $\sigma \in \mathcal{T}_f$. Thus $\mathcal{T}_f$ is an ideal in $\mathcal{T}$.

(ii) As for $\mathcal{T}_b$, observe that $\tau \in \mathcal{T}$ is bounded iff there is a constant stopping time $\check{t}$ such that $\tau \le \check{t}$. Now we have just seen that the set of constant stopping times is totally ordered, so $\mathcal{T}_b$ is an ideal in $\mathcal{T}$.

(h)(i) If $A \subseteq \mathcal{T}$ is a non-empty downwards-directed set, then

$$\inf_{\sigma\in A} \tau \wedge \sigma = \inf(\{\tau\} \cup A) = \tau \wedge \inf A.$$

(ii) If $A \subseteq \mathcal{T}$ is a non-empty upwards-directed with supremum σ^*, and $\sigma_1^* = \sup_{\sigma\in A} \sigma \wedge \tau$, then, for any $t \in T$,

$$[\![\sigma_1^* > t]\!] = \sup_{\sigma\in A} [\![\tau \wedge \sigma > t]\!]$$

(by (b))

$$= \sup_{\sigma\in A} [\![\tau > t]\!] \cap [\![\sigma > t]\!]$$

(by (c))

$$= [\![\tau > t]\!] \cap \sup_{\sigma\in A} [\![\sigma > t]\!]$$

(313Bc again)

$$= [\![\tau > t]\!] \cap [\![\sigma^* > t]\!] = [\![\tau \wedge \sigma^* > t]\!],$$

so $\sigma_1^* = \tau \wedge \sigma^*$.

Remark (The following applies in any lattice.) If $A \subseteq \mathcal{T}$ and $\tau \in \mathcal{T}$, I will write $A \vee \tau$ for $\{\sigma \vee \tau : \sigma \in A\}$ and $A \wedge \tau$ for $\{\sigma \wedge \tau : \sigma \in A\}$. Note that if $\mathcal{S}$ is a sublattice of $\mathcal{T}$ and $\tau \in \mathcal{S}$, then

$$\mathcal{S} \vee \tau = \{\sigma : \sigma \in \mathcal{S},\, \tau \le \sigma\}, \quad \mathcal{S} \wedge \tau = \{\sigma : \sigma \in \mathcal{S},\, \sigma \le \tau\}.$$

So if $\mathcal{S}$ is a sublattice of $\mathcal{T}$, τ, $\tau' \in \mathcal{S}$ and $\tau \le \tau'$,

$$\mathcal{S} \cap [\tau, \tau'] = \{\sigma : \sigma \in \mathcal{S},\, \tau \le \sigma \le \tau'\} = \{\sigma : \sigma \in \mathcal{S} \vee \tau,\, \sigma \le \tau'\} = (\mathcal{S} \vee \tau) \wedge \tau'$$

because $\mathcal{S} \vee \tau = \{\sigma : \sigma \in \mathcal{S},\, \tau \le \sigma\}$ is a sublattice of $\mathcal{T}$.

611D The region where $\sigma < \tau$ If σ, $\tau \in \mathcal{T}$ set

$$[\![\sigma < \tau]\!] = \sup_{t\in T}([\![\tau > t]\!] \setminus [\![\sigma > t]\!]),$$

$$[\![\sigma \le \tau]\!] = 1 \setminus [\![\tau < \sigma]\!] = \inf_{t\in T}([\![\tau > t]\!] \cup (1 \setminus [\![\sigma > t]\!])),$$

$$[\![\sigma = \tau]\!] = [\![\sigma \le \tau]\!] \cap [\![\tau \le \sigma]\!] = 1 \setminus \sup_{t\in T}([\![\sigma > t]\!] \triangle [\![\tau > t]\!]).$$

611E Analysts commonly think of algebra as trivial, and so it is. But the algebra of stopping times and regions $[\![\sigma < \tau]\!]$ is a very rich structure, with a large number of elementary identities; and as the definition of $[\![\sigma < \tau]\!]$ includes the supremum of an infinite set, there is room for surprises. Consequently the fluency necessary for effective use of these ideas requires a good deal of practice. In the next theorem I have collected a more or less comprehensive list of facts which will be useful in one way or another. It is a very long list, and correspondingly few readers will be inclined to work through it systematically. I recommend rather that you treat it as a running buffet, to be sampled from time to time. I hope that there are enough cross-references later for you to know when you have to return for another fragment.

Theorem (a) Let σ, $\tau \in \mathcal{T}$.

(i)(α) $([\![\sigma < \tau]\!], [\![\sigma = \tau]\!], [\![\tau < \sigma]\!])$ is a partition of unity in $\mathfrak{A}$.

(β) $[\![\sigma > t]\!] \cap [\![\sigma = \tau]\!] = [\![\tau > t]\!] \cap [\![\sigma = \tau]\!]$ for every $t \in T$.

(γ) $[\![\sigma < \tau]\!] = 0$ iff $[\![\tau \le \sigma]\!] = 1$ iff $\tau \le \sigma$; $[\![\sigma = \tau]\!] = 1$ iff $\sigma = \tau$.

(δ) Writing $\check{t}$ for the constant stopping time at t, $[\![\check{t} < \tau]\!] = [\![\tau > t]\!]$ for every $t \in T$.

(ϵ) $[\![\min\mathcal{T} < \max\mathcal{T}]\!] = 1$.

(ζ) If $s < t$ in T, then $[\![\check{s} < \check{t}]\!] = 1$; $[\![\check{s} < \max\mathcal{T}]\!] = 1$ for every $s \in T$.

(ii)(α) $[\![\sigma < \tau]\!] = [\![\sigma \wedge \tau < \tau]\!] = [\![\sigma < \sigma \vee \tau]\!]$.

(β) $[\![\sigma \le \tau]\!] = [\![\sigma = \sigma \wedge \tau]\!] = [\![\tau = \sigma \vee \tau]\!]$.

(γ) $[\![\sigma \wedge \tau = \sigma]\!] \cup [\![\sigma \wedge \tau = \tau]\!] = [\![\sigma \vee \tau = \sigma]\!] \cup [\![\sigma \vee \tau = \tau]\!] = 1$.

(b) If $\sigma \in \mathcal{T}$ and $C \subseteq \mathcal{T}$ is non-empty then $[\![\sigma < \sup C]\!] = \sup_{\tau\in C} [\![\sigma < \tau]\!]$ and $[\![\sup C \le \sigma]\!] = \inf_{\tau\in C} [\![\tau \le \sigma]\!]$.

(c) Let σ, τ, $\upsilon \in \mathcal{T}$.

(i)(α) $[\![\sigma \wedge \tau < \upsilon]\!] = [\![\sigma < \upsilon]\!] \cup [\![\tau < \upsilon]\!]$, $[\![\upsilon \le \sigma \wedge \tau]\!] = [\![\upsilon \le \sigma]\!] \cap [\![\upsilon \le \tau]\!]$.

(β) $[\![\upsilon < \sigma \wedge \tau]\!] = [\![\upsilon < \sigma]\!] \cap [\![\upsilon < \tau]\!]$, $[\![\sigma \wedge \tau \le \upsilon]\!] = [\![\sigma \le \upsilon]\!] \cup [\![\tau \le \upsilon]\!]$.

(ii)(α) $[\![\sigma \vee \tau < \upsilon]\!] = [\![\sigma < \upsilon]\!] \cap [\![\tau < \upsilon]\!]$, $[\![\upsilon \le \sigma \vee \tau]\!] = [\![\upsilon \le \sigma]\!] \cup [\![\upsilon \le \tau]\!]$.

(β) $[\![\upsilon < \sigma \vee \tau]\!] = [\![\upsilon < \sigma]\!] \cup [\![\upsilon < \tau]\!]$, $[\![\sigma \vee \tau \le \upsilon]\!] = [\![\sigma \le \upsilon]\!] \cap [\![\tau \le \upsilon]\!]$.

(iii)(α) $[\![\sigma < \upsilon]\!] \subseteq [\![\sigma < \tau]\!] \cup [\![\sigma \vee \tau < \upsilon]\!] \subseteq [\![\sigma < \tau]\!] \cup [\![\tau < \upsilon]\!]$.

(β) $[\![\sigma \le \upsilon]\!] \subseteq [\![\sigma \le \tau]\!] \cup [\![\tau < \upsilon]\!]$.

(γ) $[\![\sigma < \upsilon]\!] \cap [\![\upsilon \le \tau]\!] \subseteq [\![\sigma < \tau]\!]$, $[\![\sigma \le \upsilon]\!] \cap [\![\upsilon < \tau]\!] \subseteq [\![\sigma < \tau]\!]$.

(iv)(α) $[\![\sigma \le \tau]\!] \cap [\![\tau \le \upsilon]\!] \subseteq [\![\sigma \le \upsilon]\!]$.

(β) $[\![\sigma \le \tau]\!] \cap [\![\tau < \upsilon]\!] \subseteq [\![\sigma < \upsilon]\!]$.

(γ) $[\![\sigma = \tau]\!] \cap [\![\tau = \upsilon]\!] = [\![\sigma = \tau]\!] \cap [\![\sigma = \upsilon]\!] \subseteq [\![\sigma = \upsilon]\!]$.

(v)(α) $[\![\sigma \wedge \upsilon = \tau \wedge \upsilon]\!] \supseteq [\![\sigma = \tau]\!]$.

(β) $[\![\sigma \vee \upsilon = \tau \vee \upsilon]\!] \supseteq [\![\sigma = \tau]\!]$.

(vi) If $\sigma \le \tau \le \upsilon$, then $[\![\sigma < \upsilon]\!] = [\![\sigma < \tau]\!] \cup [\![\tau < \upsilon]\!]$, $[\![\sigma = \upsilon]\!] = [\![\sigma = \tau]\!] \cap [\![\tau = \upsilon]\!]$.

(d) If $\tau_0 \le \ldots \le \tau_n$ in $\mathcal{T}$ and $\sigma \in \mathcal{T}$, then

$$([\![\sigma < \tau_0]\!], [\![\tau_0 \le \sigma]\!] \cap [\![\sigma < \tau_1]\!], \ldots, [\![\tau_{n-1} \le \sigma]\!] \cap [\![\sigma < \tau_n]\!], [\![\tau_n \le \sigma]\!])$$

is a partition of unity in $\mathfrak{A}$.

proof (a)(i)(α) We have only to check that $[\![\sigma < \tau]\!] \cap [\![\tau < \sigma]\!] = 0$. But

$$\begin{aligned}([\![\tau > t]\!] \setminus [\![\sigma > t]\!]) \cap ([\![\sigma > s]\!] \setminus [\![\tau > s]\!]) &\subseteq [\![\tau > t]\!] \setminus [\![\tau > s]\!] = 0 \text{ if } s \le t,\\ &\subseteq [\![\sigma > s]\!] \setminus [\![\sigma > t]\!] = 0 \text{ if } t \le s,\end{aligned}$$

so

$$\begin{aligned}[\![\sigma < \tau]\!] \cap [\![\tau < \sigma]\!] &= \sup_{t\in T}([\![\tau > t]\!] \setminus [\![\sigma > t]\!]) \cap \sup_{s\in T}([\![\sigma > s]\!] \setminus [\![\tau > s]\!])\\ &= \sup_{s,t\in T}([\![\tau > t]\!] \setminus [\![\sigma > t]\!]) \cap ([\![\sigma > s]\!] \setminus [\![\tau > s]\!]) = 0.\end{aligned}$$

(β)-(γ) are immediate from the definitions in 611B and 611D.

(δ) $[\![\check{t} < \tau]\!] = \sup_{s\in T} [\![\tau > s]\!] \setminus [\![\check{t} > s]\!] = \sup_{s\ge t} [\![\tau > s]\!] = [\![\tau > t]\!]$.

(ϵ) Recall that we are assuming that T is not empty. And $[\![\max\mathcal{T} > t]\!] \setminus [\![\min\mathcal{T} > t]\!] = 1$ for any $t \in T$.

(ζ) $[\![\check{s} < \check{t}]\!] \supseteq [\![\check{t} > s]\!] \setminus [\![\check{s} > s]\!] = 1$; $[\![\check{s} < \max\mathcal{T}]\!] \supseteq [\![\max\mathcal{T} > s]\!] \setminus [\![\check{s} > s]\!] = 1$.

(ii)(α)

$$\begin{aligned}[\![\sigma \wedge \tau < \tau]\!] &= \sup_{t\in T} [\![\tau > t]\!] \setminus [\![\sigma \wedge \tau > t]\!]\\ &= \sup_{t\in T} [\![\tau > t]\!] \setminus ([\![\sigma > t]\!] \cap [\![\tau > t]\!])\end{aligned}$$

(611Cc)

$$= \sup_{t\in T} [\![\tau > t]\!] \setminus [\![\sigma > t]\!] = [\![\sigma < \tau]\!],$$

$$[\![\sigma < \sigma \vee \tau]\!] = \sup_{t\in T} [\![\sigma \vee \tau > t]\!] \setminus [\![\sigma > t]\!]$$

$$= \sup_{t\in T}([\![\sigma > t]\!] \cup [\![\tau > t]\!]) \setminus [\![\sigma > t]\!]$$

(611Cb)

$$= \sup_{t\in T} [\![\tau > t]\!] \setminus [\![\sigma > t]\!] = [\![\sigma < \tau]\!].$$

(β)

$$[\![\sigma = \sigma \wedge \tau]\!] = [\![\sigma \le \sigma \wedge \tau]\!] \cap [\![\sigma \wedge \tau \le \sigma]\!] = (1 \setminus [\![\sigma \wedge \tau < \sigma]\!]) \cap 1$$

(by (i-γ))

$$= 1 \setminus [\![\tau < \sigma]\!]$$

(by (α) just above)

$$= [\![\sigma \le \tau]\!],$$

and similarly

$$[\![\tau = \sigma \vee \tau]\!] = [\![\tau \le \sigma \vee \tau]\!] \cap [\![\sigma \vee \tau \le \tau]\!] = 1 \setminus [\![\tau < \sigma \vee \tau]\!] = 1 \setminus [\![\tau < \sigma]\!] = [\![\sigma \le \tau]\!].$$

(γ) Using both parts of (β), we have

$$[\![\sigma \wedge \tau = \sigma]\!] \cup [\![\sigma \wedge \tau = \tau]\!] = [\![\tau \le \sigma]\!] \cup [\![\sigma \le \tau]\!] = 1$$

$$= [\![\sigma \vee \tau = \sigma]\!] \cup [\![\sigma \vee \tau = \tau]\!].$$

(b)

$$\sup_{t\in T} [\![\sup C > t]\!] \setminus [\![\sigma > t]\!] = \sup_{t\in T}(\sup_{\tau\in C} [\![\tau > t]\!]) \setminus [\![\sigma > t]\!]$$

(611Cb again)

$$= \sup_{t\in T} \sup_{\tau\in C}([\![\tau > t]\!] \setminus [\![\sigma > t]\!]) = \sup_{\tau\in C} [\![\sigma < \tau]\!].$$

Taking complements,

$$[\![\sup C \le \sigma]\!] = \inf_{\tau\in C} [\![\tau \le \sigma]\!].$$

(c)(i)(α) $[\![\sigma \wedge \tau > t]\!] = [\![\sigma > t]\!] \cap [\![\tau > t]\!]$ for every t, so

$$[\![\sigma \wedge \tau < \upsilon]\!] = \sup_{t\in T} [\![\upsilon > t]\!] \setminus ([\![\sigma > t]\!] \cap [\![\tau > t]\!])$$

$$= \sup_{t\in T}([\![\upsilon > t]\!] \setminus [\![\sigma > t]\!]) \cup ([\![\upsilon > t]\!] \setminus [\![\tau > t]\!]) = [\![\sigma < \upsilon]\!] \cup [\![\tau < \upsilon]\!].$$

Taking complements,

$$[\![\upsilon \le \sigma \wedge \tau]\!] = [\![\upsilon \le \sigma]\!] \cap [\![\upsilon \le \tau]\!].$$

(β)

$$\begin{aligned}
[\![\upsilon<\sigma]\!]\cap[\![\upsilon<\tau]\!] &= (\sup_{s\in T}[\![\sigma>s]\!]\setminus[\![\upsilon>s]\!])\cap(\sup_{t\in T}[\![\tau>t]\!]\setminus[\![\upsilon>t]\!])\\
&= \sup_{s,t\in T}([\![\sigma>s]\!]\setminus[\![\upsilon>s]\!])\cap([\![\tau>t]\!]\setminus[\![\upsilon>t]\!])\\
&= \sup_{s,t\in T}[\![\sigma>s]\!]\cap[\![\tau>t]\!]\setminus([\![\upsilon>s]\!]\cup[\![\upsilon>t]\!])\\
&= \sup_{s,t\in T}[\![\sigma>s]\!]\cap[\![\tau>t]\!]\setminus[\![\upsilon>\min(s,t)]\!]\\
&= \sup_{t'\in T}\sup_{s\ge t',t\ge t'}[\![\sigma>s]\!]\cap[\![\tau>t]\!]\setminus[\![\upsilon>t']\!]\\
&= \sup_{t'\in T}[\![\sigma>t']\!]\cap[\![\tau>t']\!]\setminus[\![\upsilon>t']\!]\\
&= \sup_{t'\in T}[\![\sigma\wedge\tau>t']\!]\setminus[\![\upsilon>t']\!]=[\![\upsilon<\sigma\wedge\tau]\!].
\end{aligned}$$

Taking complements,

$$[\![\sigma\wedge\tau\le\upsilon]\!]=[\![\sigma\le\upsilon]\!]\cup[\![\tau\le\upsilon]\!].$$

(ii)

$$\begin{aligned}
[\![\sigma<\upsilon]\!]\cap[\![\tau<\upsilon]\!] &= (\sup_{s\in T}[\![\upsilon>s]\!]\setminus[\![\sigma>s]\!])\cap(\sup_{t\in T}[\![\upsilon>t]\!]\setminus[\![\tau>t]\!])\\
&= \sup_{s,t\in T}([\![\upsilon>s]\!]\setminus[\![\sigma>s]\!])\cap([\![\upsilon>t]\!]\setminus[\![\tau>t]\!])\\
&= \sup_{s,t\in T}[\![\upsilon>s]\!]\cap[\![\upsilon>t]\!]\setminus([\![\sigma>s]\!]\cup[\![\tau>t]\!])\\
&= \sup_{s,t\in T}[\![\upsilon>\max(s,t)]\!]\setminus([\![\sigma>s]\!]\cup[\![\tau>t]\!])\\
&= \sup_{t'\in T}\sup_{s\le t',t\le t'}[\![\upsilon>t']\!]\setminus([\![\sigma>s]\!]\cup[\![\tau>t]\!])\\
&= \sup_{t'\in T}[\![\upsilon>t']\!]\setminus([\![\sigma>t']\!]\cup[\![\tau>t']\!])\\
&= \sup_{t'\in T}[\![\upsilon>t']\!]\setminus[\![\sigma\vee\tau>t']\!]=[\![\sigma\vee\tau<\upsilon]\!].
\end{aligned}$$

Taking complements,

$$[\![\upsilon\le\sigma\vee\tau]\!]=[\![\upsilon\le\sigma]\!]\cup[\![\upsilon\le\tau]\!].$$

(iii)(*α*) For every $t\in T$,

$$[\![\upsilon>t]\!]\setminus[\![\sigma>t]\!]\subseteq([\![\upsilon>t]\!]\setminus[\![\tau>t]\!])\cup([\![\tau>t]\!]\setminus[\![\sigma>t]\!]).$$

So $[\![\sigma<\upsilon]\!]\subseteq[\![\sigma<\tau]\!]\cup[\![\tau<\upsilon]\!]$. But equally we must also have

$$[\![\sigma<\upsilon]\!]\subseteq[\![\sigma<\sigma\vee\tau]\!]\cup[\![\sigma\vee\tau<\upsilon]\!]=[\![\sigma<\tau]\!]\cup[\![\sigma\vee\tau<\upsilon]\!]$$

by (a-ii-α). And $[\![\sigma<\tau]\!]\cup[\![\sigma\vee\tau<\upsilon]\!]\subseteq[\![\sigma<\tau]\!]\cup[\![\tau<\upsilon]\!]$ by (ii-α) just above.

(*β*) Now

$$1=[\![\sigma\le\tau]\!]\cup[\![\tau<\sigma]\!]\subseteq[\![\sigma\le\tau]\!]\cup[\![\tau<\upsilon]\!]\cup[\![\upsilon<\sigma]\!]$$

by (α), so

$$[\![\sigma\le\upsilon]\!]=1\setminus[\![\upsilon<\sigma]\!]\subseteq[\![\sigma\le\tau]\!]\cup[\![\tau<\upsilon]\!].$$

(*γ*) Using (α) twice,

$$[\![\sigma<\upsilon]\!]\cap[\![\upsilon\le\tau]\!]\subseteq([\![\sigma<\tau]\!]\cup[\![\tau<\upsilon]\!])\cap[\![\upsilon\le\tau]\!]\subseteq[\![\sigma<\tau]\!],$$

$$[\![\sigma \le \upsilon]\!] \cap [\![\upsilon < \tau]\!] \subseteq [\![\sigma \le \upsilon]\!] \cap ([\![\upsilon < \sigma]\!] \cup [\![\sigma < \tau]\!]) \subseteq [\![\sigma < \tau]\!].$$

(iv)(*α*) Take complements in (iii-α) and exchange the names σ, υ.

(*β*) Take complements in (iii-β) and exchange the names σ, υ.

(*γ*) Use (α) twice to prove the first equality; the second is trivial.

(v)(*α*)

$$[\![\sigma \wedge \upsilon < \tau \wedge \upsilon]\!] = [\![\sigma < \tau \wedge \upsilon]\!] \cup [\![\upsilon < \tau \wedge \upsilon]\!] \subseteq [\![\sigma < \tau]\!]$$

by (i) above. Similarly, $[\![\tau \wedge \upsilon < \sigma \wedge \upsilon]\!] \subseteq [\![\tau < \sigma]\!]$, and both are disjoint from $[\![\sigma = \tau]\!]$.

(*β*) As (α), but starting from

$$[\![\sigma \vee \upsilon < \tau \vee \upsilon]\!] = [\![\sigma \vee \upsilon < \tau]\!] \cap [\![\sigma \vee \upsilon < \upsilon]\!] \subseteq [\![\sigma < \tau]\!].$$

(vi) In this case, for every $t \in T$,

$$[\![\sigma > t]\!] \subseteq [\![\tau > t]\!] \subseteq [\![\upsilon > t]\!],$$

so $[\![\upsilon > t]\!] \setminus [\![\sigma > t]\!] = ([\![\upsilon > t]\!] \setminus [\![\tau > t]\!]) \cup ([\![\tau > t]\!] \setminus [\![\sigma > t]\!])$; taking the supremum over t, $[\![\sigma < \upsilon]\!] = [\![\sigma < \tau]\!] \cup [\![\tau < \upsilon]\!]$. Taking complements,

$$[\![\sigma = \upsilon]\!] = 1 \setminus [\![\sigma < \upsilon]\!]$$

(because $[\![\upsilon < \sigma]\!] = 0$, by (a-i-$\gamma$))

$$= (1 \setminus [\![\sigma < \tau]\!]) \cap (1 \setminus [\![\tau < \upsilon]\!]) = [\![\sigma = \tau]\!] \cap [\![\tau = \upsilon]\!].$$

(d) Induce on n. If $n = 0$ the list reduces to

$$([\![\sigma < \tau_0]\!],\ [\![\tau_0 \le \sigma]\!])$$

which is a partition of unity by the definition in 611D.

For the inductive step to $n + 1 > 1$, we know that

$$([\![\sigma < \tau_0]\!],\ [\![\tau_0 \le \sigma]\!] \cap [\![\sigma < \tau_1]\!], \dots , [\![\tau_{n-1} \le \sigma]\!] \cap [\![\sigma < \tau_n]\!],\ [\![\tau_n \le \sigma]\!])$$

and

$$([\![\sigma < \tau_{n+1}]\!],\ [\![\tau_{n+1} \le \sigma]\!])$$

are partitions of unity. So

$$\begin{gathered}([\![\sigma < \tau_0]\!], [\![\tau_0 \le \sigma]\!] \cap [\![\sigma < \tau_1]\!], \dots , [\![\tau_{n-1} \le \sigma]\!] \cap [\![\sigma < \tau_n]\!],\\ [\![\tau_n \le \sigma]\!] \cap [\![\sigma < \tau_{n+1}]\!],\ [\![\tau_n \le \sigma]\!] \cap [\![\tau_{n+1} \le \sigma]\!])\end{gathered}$$

is a partition of unity. And

$$[\![\tau_n \le \sigma]\!] \cap [\![\tau_{n+1} \le \sigma]\!] = [\![\tau_n \vee \tau_{n+1} \le \sigma]\!] = [\![\tau_{n+1} \le \sigma]\!]$$

by (c-ii-γ). So our partition of unity reduces to

$$\begin{gathered}([\![\sigma < \tau_0]\!], [\![\tau_0 \le \sigma]\!] \cap [\![\sigma < \tau_1]\!], \dots , [\![\tau_{n-1} \le \sigma]\!] \cap [\![\sigma < \tau_n]\!],\\ [\![\tau_n \le \sigma]\!] \cap [\![\sigma < \tau_{n+1}]\!],\ [\![\tau_{n+1} \le \sigma]\!])\end{gathered}$$

as required for the inductive step.

Remark I have taken the trouble to spell out direct proofs based on the definitions in 611A, 611B and 611D. But you will observe that every formula here corresponds to the case in which σ, τ and υ are real-valued functions defined on $[0, 1]$, with $\mathfrak{A} = \mathcal{P}[0, 1]$,

$$(\sigma \wedge \tau)(x) = \min(\sigma(x), \tau(x)), \quad (\sigma \vee \tau)(x) = \max(\sigma(x), \tau(x))$$

for $x \in [0, 1]$, and

$$[\![\sigma<\tau]\!]=\{x:\sigma(x)<\tau(x)\},$$

$$[\![\sigma=\tau]\!]=\{x:\sigma(x)=\tau(x)\},\quad [\![\sigma\le\tau]\!]=\{x:\sigma(x)\le\tau(x)\}.$$

In (b) we have a bit of luck – the formulae for infima are more complicated (see 632C(a-ii)). Elsewhere, with only finitely many stopping times involved, we are perfectly safe, though I do not attempt to state and prove an appropriate metatheorem.

611F Infima in $\mathcal{T}$: Proposition Let $A\subseteq\mathcal{T}$ be a non-empty set such that

$$\sup_{s>t}\inf_{\sigma\in A}[\![\sigma>s]\!]$$

belongs to $\mathfrak{A}_t$ whenever $t\in T$ is not isolated on the right.

(a)

$$\begin{aligned}[\![\inf A>t]\!]&=\inf_{\sigma\in A}[\![\sigma>t]\!]\text{ if }t\in T\text{ is isolated on the right}\\&=\sup_{s>t}\inf_{\sigma\in A}[\![\sigma>s]\!]\text{ for other }t\in T.\end{aligned}$$

(b) $[\![\inf A<\tau]\!]=\sup_{\sigma\in A}[\![\sigma<\tau]\!]$ for every $\tau\in\mathcal{T}$.

proof (a) For $t\in T$, set

$$a_t=\inf_{\sigma\in A}[\![\sigma>t]\!]\in\mathfrak{A}_t,$$

$$\begin{aligned}b_t&=a_t\text{ if }t\in T\text{ is isolated on the right,}\\&=\inf_{s>t}a_s\text{ for other }t\in T.\end{aligned}$$

By hypothesis, $b_t\in\mathfrak{A}_t$ for every $t\in T$. Now $a_s\subseteq a_t$ whenever $t\le s$, so $b_s\subseteq b_t$ whenever $t\le s$. If $t\in T$ is not isolated on the right,

$$\begin{aligned}\inf\{b_s:s>t\}&\inf\{a_{s'}:\text{ there is an }s\text{ such that }s'>s>t\}\\&=\inf\{a_{s'}:s'>t\}=b_t.\end{aligned}$$

Accordingly there is a $\tau_0\in\mathcal{T}$ such that $[\![\tau_0>t]\!]=b_t$ for every $t\in T$.

If $\sigma\in A$,

$$[\![\tau_0>t]\!]=b_t\subseteq a_t\subseteq[\![\sigma>t]\!]$$

for every $t\in T$, so $\tau_0\le\sigma$; as σ is arbitrary, $\tau_0\le\inf A$. On the other hand, for any $t\in T$, $[\![\inf A>t]\!]\subseteq[\![\sigma>t]\!]$ for every $\sigma\in A$, so $[\![\inf A>t]\!]\subseteq a_t$; and if t is not isolated on the right,

$$[\![\inf A>t]\!]=\sup_{s>t}[\![\inf A>s]\!]\subseteq\sup_{s>t}a_s=b_t.$$

Thus $[\![\inf A>t]\!]\subseteq b_t=[\![\tau_0>t]\!]$ for every t, and $\inf A\le\tau_0$.

Accordingly $\inf A=\tau_0$ satisfies the formula claimed.

(b) If $\sigma\in A$, $[\![\sigma<\tau]\!]\subseteq[\![\inf A<\tau]\!]$ by 611E(c-iv-β); so $\sup_{\sigma\in A}[\![\sigma<\tau]\!]\subseteq[\![\inf A<\tau]\!]$. In the other direction, take any $t\in T$. If t is isolated on the right, then

$$\begin{aligned}[\![\tau>t]\!]\setminus[\![\inf A>t]\!]&=[\![\tau>t]\!]\setminus\inf_{\sigma\in A}[\![\sigma>t]\!]\\&=\sup_{\sigma\in A}[\![\tau>t]\!]\setminus[\![\sigma>t]\!]\subseteq\sup_{\sigma\in A}[\![\sigma<\tau]\!];\end{aligned}$$

otherwise,

$$\begin{aligned}[\![\tau>t]\!]\setminus[\![\inf A>t]\!]&=\sup_{s>t}([\![\tau>s]\!]\setminus\sup_{s'>t}\inf_{\sigma\in A}[\![\sigma>s']\!])\\&\subseteq\sup_{s>t}([\![\tau>s]\!]\setminus\inf_{\sigma\in A}[\![\sigma>s]\!])\\&=\sup_{s>t}\sup_{\sigma\in A}([\![\tau>s]\!]\setminus[\![\sigma>s]\!])\subseteq\sup_{\sigma\in A}[\![\sigma<\tau]\!].\end{aligned}$$

So

$$[\![\inf A < \tau]\!] = \sup_{t\in T} [\![\tau > t]\!] \setminus [\![\inf A > t]\!] \subseteq \sup_{\sigma\in C} [\![\sigma < \tau]\!]$$

and we have equality.

611G The algebra defined by a stopping time: Definition If $\tau \in \mathcal{T}$, write $\mathfrak{A}_\tau$ for

$$\{a : a \in \mathfrak{A},\ a \setminus [\![\tau > t]\!] \in \mathfrak{A}_t \text{ for every } t \in T\}.$$

Then $\mathfrak{A}_\tau$ is an intersection of order-closed subalgebras, so is itself an order-closed subalgebra of $\mathfrak{A}$.

611H Proposition (a) Suppose that $\tau \in \mathcal{T}$ and $t \in T$.

(i) If $b \in \bigcap_{s>t} \mathfrak{A}_s$ and $b \subseteq [\![\tau > t]\!]$, then $b \in \mathfrak{A}_\tau$. In particular, $[\![\tau > t]\!]$ and therefore $1 \setminus [\![\tau > t]\!]$ belong to $\mathfrak{A}_\tau$.

(ii) If $b \in \mathfrak{A}_t$ and $b \subseteq [\![\tau > s]\!]$ for every $s < t$, then $b \in \mathfrak{A}_\tau$.

(iii) If $b \in \mathfrak{A}_\tau$ and $b \cap [\![\tau > t]\!] = 0$, then $b \in \mathfrak{A}_t$.

(b) If $\check t$ is the constant stopping time at t, then $\mathfrak{A}_{\check t} = \mathfrak{A}_t$.

(c) Suppose that σ, $\tau \in \mathcal{T}$.

(i) $[\![\sigma < \tau]\!]$, $[\![\sigma = \tau]\!]$ and $[\![\tau < \sigma]\!]$ belong to $\mathfrak{A}_\sigma \cap \mathfrak{A}_\tau$.

(ii) $\mathfrak{A}_{\sigma\wedge\tau} = \mathfrak{A}_\sigma \cap \mathfrak{A}_\tau$; in particular, $\mathfrak{A}_\sigma \subseteq \mathfrak{A}_\tau$ if $\sigma \le \tau$.

(iii) If $a \in \mathfrak{A}_\tau$ then $a \cap [\![\tau \le \sigma]\!] = a \setminus [\![\sigma < \tau]\!] \in \mathfrak{A}_{\sigma\wedge\tau}$.

(iv) $\mathfrak{A}_{\sigma\vee\tau}$ is the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_\sigma \cup \mathfrak{A}_\tau$.

proof (a)(i) If $s \le t$,

$$b \setminus [\![\tau > s]\!] \subseteq [\![\tau > t]\!] \setminus [\![\tau > s]\!] = 0$$

and $b \setminus [\![\tau > s]\!] \in \mathfrak{A}_s$. If $s > t$ then b and $[\![\tau > s]\!]$ both belong to $\mathfrak{A}_s$, so $b \setminus [\![\tau > s]\!] \in \mathfrak{A}_s$.

(ii) If $s < t$, $b \setminus [\![\tau > s]\!] = 0 \in \mathfrak{A}_s$. If $s \ge t$ then b and $[\![\tau > s]\!]$ both belong to $\mathfrak{A}_s$, so $b \setminus [\![\tau > s]\!] \in \mathfrak{A}_s$.

(iii) $b = b \setminus [\![\tau > t]\!]$.

(b) Use (a-ii) and (a-iii).

(c)(i) For every $t \in T$, $[\![\tau > t]\!] \setminus [\![\sigma > t]\!]$ belongs to $\mathfrak{A}_t$ and therefore to $\mathfrak{A}_\tau$, by (a-i) or (a-ii); now the supremum of these, $[\![\sigma < \tau]\!]$, will also belong to $\mathfrak{A}_\tau$. Moreover, for every t,

$$\begin{aligned}
[\![\sigma < \tau]\!] \setminus [\![\sigma > t]\!] &= \sup_{s\in T}([\![\tau > s]\!] \setminus [\![\sigma > s]\!]) \setminus [\![\sigma > t]\!] \\
&= \sup_{s\ge t}([\![\tau > s]\!] \setminus [\![\sigma > t]\!]) \cup \sup_{s\le t}([\![\tau > s]\!] \setminus [\![\sigma > s]\!]) \\
&= ([\![\tau > t]\!] \setminus [\![\sigma > t]\!]) \cup \sup_{s\le t}([\![\tau > s]\!] \setminus [\![\sigma > s]\!]) \\
&= \sup_{s\le t} [\![\tau > s]\!] \setminus [\![\sigma > s]\!] \in \mathfrak{A}_t.
\end{aligned}$$

So $[\![\sigma < \tau]\!]$ also belongs to $\mathfrak{A}_\sigma$.

In the same way, $[\![\tau < \sigma]\!] \in \mathfrak{A}_\sigma \cap \mathfrak{A}_\tau$; by 611E(a-i-$\alpha$), $[\![\sigma = \tau]\!] \in \mathfrak{A}_\sigma \cap \mathfrak{A}_\tau$.

(ii)($\boldsymbol{\alpha}$) If $\sigma \le \tau$, $a \in \mathfrak{A}_\sigma$ and $t \in T$, then

$$a \setminus [\![\tau > t]\!] = (a \setminus [\![\sigma > t]\!]) \setminus [\![\tau > t]\!] \in \mathfrak{A}_t$$

because $a \setminus [\![\sigma > t]\!]$ and $[\![\tau > t]\!]$ both belong to $\mathfrak{A}_t$. As a and t are arbitrary, $\mathfrak{A}_\sigma \subseteq \mathfrak{A}_\tau$.

($\boldsymbol{\beta}$) So for any σ, $\tau \in \mathcal{T}$ we shall have $\mathfrak{A}_{\sigma\wedge\tau} \subseteq \mathfrak{A}_\sigma \cap \mathfrak{A}_\tau$. Conversely, if $a \in \mathfrak{A}_\sigma \cap \mathfrak{A}_\tau$ and $t \in T$, then

$$a \setminus [\![\sigma \wedge \tau > t]\!] = a \setminus ([\![\sigma > t]\!] \cap [\![\tau > t]\!])$$

(611Cc)

$$= (a \setminus [\![\sigma > t]\!]) \cup (a \setminus [\![\tau > t]\!]) \in \mathfrak{A}_t$$

because both $a\setminus[\![\sigma>t]\!]$ and $a\setminus[\![\tau>t]\!]$ belong to $\mathfrak{A}_t$. As a and t are arbitrary, $\mathfrak{A}_\sigma\cap\mathfrak{A}_\tau\supseteq\mathfrak{A}_{\sigma\wedge\tau}$ and we have equality.

(iii) If $t\in T$, then $[\![\tau>t]\!]\setminus[\![\sigma>t]\!]\subseteq[\![\sigma<\tau]\!]$, so $[\![\tau>t]\!]\subseteq[\![\sigma>t]\!]\cup[\![\sigma<\tau]\!]$. Since $a\setminus[\![\tau>t]\!]$, $[\![\sigma>t]\!]$, $[\![\tau>t]\!]$ and $[\![\sigma<\tau]\!]\setminus[\![\tau>t]\!]$ all belong to $\mathfrak{A}_t$ (put (a-i), (a-iii) and (i) just above together for the last), so do $[\![\sigma<\tau]\!]\cup[\![\tau>t]\!]$, $[\![\sigma<\tau]\!]\cup[\![\tau>t]\!]\cup[\![\sigma>t]\!]$ and

$$\begin{aligned}(a\setminus[\![\tau>t]\!])\setminus([\![\sigma<\tau]\!]\cup[\![\sigma>t]\!]\cup[\![\tau>t]\!])\\ &=a\setminus([\![\sigma<\tau]\!]\cup[\![\tau>t]\!]\cup[\![\sigma>t]\!])\\ &=a\setminus([\![\sigma<\tau]\!]\cup[\![\sigma>t]\!])=(a\setminus[\![\sigma<\tau]\!])\setminus[\![\sigma>t]\!].\end{aligned}$$

As t is arbitrary, $a\setminus[\![\sigma<\tau]\!]\in\mathfrak{A}_\sigma$. But also a and $[\![\sigma<\tau]\!]$ belong to $\mathfrak{A}_\tau$, so

$$a\setminus[\![\sigma<\tau]\!]\in\mathfrak{A}_\sigma\cap\mathfrak{A}_\tau=\mathfrak{A}_{\sigma\wedge\tau}$$

by (i) again.

(iv) Let $\mathfrak{B}$ be the subalgebra generated by $\mathfrak{A}_\sigma\cup\mathfrak{A}_\tau$. By (i), $\mathfrak{B}\subseteq\mathfrak{A}_{\sigma\vee\tau}$. On the other hand, suppose that $a\in\mathfrak{A}_{\sigma\vee\tau}$. Then

$$a\setminus[\![\sigma<\tau]\!]=a\setminus[\![\sigma<\sigma\vee\tau]\!]\in\mathfrak{A}_\sigma\subseteq\mathfrak{B}$$

by 611E(a-ii-α) and (iii). Similarly $a\setminus[\![\tau<\sigma]\!]\in\mathfrak{B}$. As $[\![\sigma<\tau]\!]$ and $[\![\tau<\sigma]\!]$ are disjoint, $a\in\mathfrak{B}$. As a is arbitrary, $\mathfrak{A}_{\sigma\vee\tau}\subseteq\mathfrak{B}$ and we have equality.

611I Lemma Suppose that $\langle\tau_i\rangle_{i\in I}$ is a family in $\mathcal{T}$ and $\langle a_i\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$ such that $a_i\in\mathfrak{A}_{\tau_i}$ for every $i\in I$. Then there is a unique $\sigma\in\mathcal{T}$ such that $[\![\sigma=\tau_i]\!]\supseteq a_i$ for every $i\in I$, and $\inf_{i\in I}\tau_i\le\sigma\le\sup_{i\in I}\tau_i$.

proof For $t\in T$, set $b_t=\sup_{i\in I}a_i\cap[\![\tau_i>t]\!]$. Because $\langle a_i\rangle_{i\in I}$ is a partition of unity,

$$1\setminus b_t=\sup_{i\in I}a_i\setminus b_t=\sup_{i\in I}a_i\setminus[\![\tau_i>t]\!]$$

belongs to $\mathfrak{A}_t$, so $b_t\in\mathfrak{A}_t$. If t is not isolated on the right, then

$$\sup_{s>t}b_s=\sup_{i\in I,s>t}a_i\cap[\![\tau_i>s]\!]=\sup_{i\in I}a_i\cap[\![\tau_i>t]\!]=b_t.$$

So $\langle b_t\rangle_{t\in T}$ satisfies the conditions of 611A(b-i) and we have a stopping time $\sigma\in\mathcal{T}$ such that $[\![\sigma>t]\!]=b_t$ for every t. Now, for $i\in I$ and $t\in T$, $[\![\sigma>t]\!]\cap a_i=[\![\tau_i>t]\!]\cap a_i$, so $[\![\sigma<\tau_i]\!]$ and $[\![\tau_i<\sigma]\!]$ are both disjoint from a_i, and $a_i\subseteq[\![\sigma=\tau_i]\!]$.

To see that σ is unique, suppose that $\sigma'\in\mathcal{T}$ has the same property; then

$$[\![\sigma>t]\!]\cap a_i=[\![\tau_i>t]\!]\cap a_i=[\![\sigma'>t]\!]\cap a_i$$

for every $i\in I$, so $[\![\sigma>t]\!]=[\![\sigma'>t]\!]$ for every $t\in T$, and $\sigma'=\sigma$.

To see that $\inf_{i\in I}\tau_i\le\sigma$, observe that

$$[\![\inf_{i\in I}\tau_i\le\sigma]\!]\supseteq[\![\inf_{i\in I}\tau_i\le\tau_j]\!]\cap[\![\tau_j\le\sigma]\!]\supseteq a_j$$

for every $j\in I$, so $[\![\inf_{i\in I}\tau_i\le\sigma]\!]=1$ and $\inf_{i\in I}\tau_i\le\sigma$. Similarly, $\sigma\le\sup_{i\in I}\tau_i$.

611J Dissections by stopping times (a) Recall from 611B-611C that if we regard a stopping time $\tau=\langle[\![\tau>t]\!]\rangle_{t\in T}$ as a member of the algebra $\prod_{t\in T}\mathfrak{A}_t$, then the partial order $\le$ and the lattice operations $\vee$, $\wedge$ on $\mathcal{T}$ correspond to the Boolean relation and operations $\subseteq$, $\cup$, $\cap$ on $\prod_{t\in T}\mathfrak{A}_t$, and moreover that arbitrary suprema in $\mathcal{T}$ correspond to suprema in $\prod_{t\in T}\mathfrak{A}_t$ (611Cb), though there can be complications for general infima (632C(a-i)).

In view of this representation it is natural to consider other Boolean operations on members of $\mathcal{T}$, in particular, set difference. I will in fact prefer the notation

$$c(\sigma,\tau)=\langle[\![\tau>t]\!]\setminus[\![\sigma>t]\!]\rangle_{t\in T},$$

rather than writing $\tau\setminus\sigma$, as perhaps leaving less scope for confusion, and carrying the notion of an 'interval' from σ to τ. I will say that $c(\sigma,\tau)$ is the **stopping time interval** with **endpoints** σ, τ. (Warning! the endpoints are not uniquely defined; but see (d) here, and also 613Cc below.)

If $\mathcal{S}$ is a sublattice of $\mathcal{T}$, $\mathrm{Sti}(\mathcal{S})$ will be the set of stopping-time intervals expressible as $c(\sigma,\tau)$ where $\sigma\le\tau$ in $\mathcal{S}$.

(b) However, the Boolean interpretation of $c(\sigma,\tau)$, combined with the formulae in 611C and the distributive laws of Boolean algebra, leads us directly to such elementary facts as

$$c(\sigma,\tau)\cap c(\sigma',\tau')=c(\sigma\vee\sigma',\tau\wedge\tau')$$

for all σ, σ', τ, $\tau'\in\mathcal{T}$, corresponding to the formula

$$(b\setminus a)\cap(b'\setminus a')=(b\cap b')\setminus(a\cup a')$$

of Boolean algebra, and

$$c(\sigma\wedge\tau,\sigma\wedge\tau')\subseteq c(\tau,\tau'),$$

corresponding to $(a\cap b')\setminus(a\cap b)\subseteq b'\setminus b$. Similarly,

$$c(\sigma,\sup C)=\sup_{\tau\in C}c(\sigma,\tau),\quad c(\sigma\wedge\sigma',\tau)=c(\sigma,\tau)\cup c(\sigma',\tau)$$

for σ, σ', $\tau\in\mathcal{T}$ and $C\subseteq\mathcal{T}$, and if $\sigma\le\upsilon\le\tau$, then

$$c(\sigma,\upsilon)\cup c(\upsilon,\tau)=c(\sigma,\tau),\quad c(\sigma,\upsilon)\cap c(\upsilon,\tau)=0.$$

Of course $c(\sigma,\tau)=0$ iff $\tau\le\sigma$.

(c) We can now interpret $[\![\sigma<\tau]\!]$, as defined in 611D, as a kind of projection of $c(\sigma,\tau)$, so that, for instance, $[\![\sigma<\tau]\!]\subseteq[\![\sigma'<\tau']\!]$ whenever $c(\sigma,\tau)\subseteq c(\sigma',\tau')$. More precisely, if σ, τ, σ', $\tau'\in\mathcal{T}$ then $c(\sigma,\tau)\subseteq c(\sigma',\tau')$ iff

$$[\![\sigma<\tau]\!]\subseteq[\![\sigma'\le\sigma]\!]\cap[\![\tau\le\tau']\!].$$

P We have

$$[\![\sigma<\tau]\!]\subseteq[\![\sigma'\le\sigma]\!]\iff 0=[\![\sigma<\tau]\!]\cap[\![\sigma<\sigma']\!]=[\![\sigma<\tau\wedge\sigma']\!]$$

(611E(c-i-β))

$$\iff\tau\wedge\sigma'\le\sigma,$$

$$[\![\sigma<\tau]\!]\subseteq[\![\tau\le\tau']\!]\iff 0=[\![\sigma<\tau]\!]\cap[\![\tau'<\tau]\!]=[\![\sigma\vee\tau'<\tau]\!]$$

(611E(c-ii))

$$\iff\tau\le\sigma\vee\tau'.$$

Next, for elements a, b, a', b' of any Boolean algebra,

$$\begin{aligned}b\setminus a\subseteq b'\setminus a'&\iff b\setminus a\subseteq b'\text{ and }(b\setminus a)\cap b'\cap a'=0\\&\iff b\setminus a\subseteq b'\text{ and }(b\setminus a)\cap a'=0\\&\iff b\subseteq a\cup b'\text{ and }b\cap a'\subseteq a.\end{aligned}$$

Translating this into terms of $c(\sigma,\tau)=\tau\setminus\sigma$ and $c(\sigma',\tau')=\tau'\setminus\sigma'$ in $\prod_{t\in T}\mathfrak{A}_t$,

$$\begin{aligned}c(\sigma,\tau)\subseteq c(\sigma',\tau')&\iff\tau\subseteq\sigma\cup\tau'\text{ and }\tau\cap\sigma'\subseteq\sigma\\&\iff\tau\le\sigma\vee\tau'\text{ and }\tau\wedge\sigma'\le\sigma\end{aligned}$$

(translating into terms of $\le$, $\wedge$ and $\vee$ in the lattice $\mathcal{T}$)

$$\begin{aligned}&\iff[\![\sigma<\tau]\!]\subseteq[\![\tau\le\tau']\!]\text{ and }[\![\sigma<\tau]\!]\subseteq[\![\sigma'\le\sigma]\!]\\&\iff[\![\sigma<\tau]\!]\subseteq[\![\sigma'\le\sigma]\!]\cap[\![\tau\le\tau']\!].\ \mathbf{Q}\end{aligned}$$

(d) Similarly, if σ, τ, σ', $\tau' \in \mathcal{T}$ then $c(\sigma,\tau) = c(\sigma',\tau')$ iff

$$[\![\sigma < \tau]\!] = [\![\sigma' < \tau']\!] \subseteq [\![\sigma' = \sigma]\!] \cap [\![\tau = \tau']\!].$$

P If the condition is satisfied, then (c) shows at once that $c(\sigma,\tau) = c(\sigma',\tau')$. Conversely, if $c(\sigma,\tau) = c(\sigma',\tau')$ then the first remark in (c) tells us that $[\![\sigma < \tau]\!] = [\![\sigma' < \tau']\!]$, and now both are included in

$$[\![\sigma' \le \sigma]\!] \cap [\![\tau' \le \tau]\!] \cap [\![\sigma \le \sigma']\!] \cap [\![\tau \le \tau']\!] = [\![\sigma' = \sigma]\!] \cap [\![\tau = \tau']\!]. \ \mathbf{Q}$$

(e)(i) For a finite sublattice I of $\mathcal{T}$, an I**-cell** will be a minimal non-zero stopping time interval of the form $c(\sigma,\tau)$ where σ, $\tau \in I$. (If I is non-empty and we think of it as a sublattice of $\prod_{t\in T}\mathfrak{A}_t$, then an I-cell is an atom of the subalgebra of $\prod_{t\in T}\mathfrak{A}_t$ generated by I which is included in $c(\min I, \max I)$.)

(ii) Let I be a finite sublattice of $\mathcal{T}$, $\mathrm{Sti}_0(I)$ the set of I-cells, and $\tau \in I$. If we write

$$I \wedge \tau = \{\sigma \wedge \tau : \sigma \in I\}, \quad I \vee \tau = \{\sigma \vee \tau : \sigma \in I\},$$

then $\mathrm{Sti}_0(I \wedge \tau)$, $\mathrm{Sti}_0(I \vee \tau)$ are disjoint sets with union $\mathrm{Sti}_0(I)$. **P** We can think of τ and $1 \setminus \tau$ as complementary elements of $\prod_{t\in T}\mathfrak{A}_t$, and if σ, $\sigma' \in \mathcal{T}$, then

$$c(\sigma,\sigma') \cap \tau = c(\sigma \wedge \tau, \sigma' \wedge \tau), \quad c(\sigma,\sigma') \setminus \tau = c(\sigma \vee \tau, \sigma' \vee \tau)$$

are stopping time intervals determined by endpoints in I. So if $c(\sigma,\sigma') \in \mathrm{Sti}_0(I)$, it must be equal either to $c(\sigma \wedge \tau, \sigma' \wedge \tau)$ or $c(\sigma \vee \tau, \sigma \vee \tau')$, and belong to $\mathrm{Sti}_0(I \wedge \tau)$ or $\mathrm{Sti}_0(I \vee \tau)$ accordingly. **Q**

(iii) More generally, if I is a non-empty finite sublattice of $\mathcal{T}$ and $\tau_0 \le \ldots \le \tau_n$ in I, then setting

$$I_{-1} = I \wedge \tau_0, \quad I_j = I \cap [\tau_j, \tau_{j+1}] \text{ for } j < n, \quad I_n = I \vee \tau_n,$$

$\langle \mathrm{Sti}_0(I_j)\rangle_{-1\le j\le n}$ is a partition of $\mathrm{Sti}_0(I)$. (Induce on n, noting that $(I \vee \tau_{n-1}) \wedge \tau_n = I \cap [\tau_{n-1}, \tau_n]$, $(I \vee \tau_{n-1}) \vee \tau_n = I \vee \tau_n$ if $n > 0$.)

611K The following facts will be extremely useful.

Lemma Let $I \subseteq \mathcal{T}$ be a non-empty finite sublattice, and $\mathrm{Sti}_0(I)$ the set of I-cells. Let I_0 be a maximal totally ordered subset of I, and $\langle \tau_i\rangle_{i\le n}$ the increasing enumeration of I_0.

(a) $\tau_0 = \min I$, $\tau_1 = \max I$.

(b) If $i < n$ then $I \cap [\tau_i, \tau_{i+1}] = \{\tau_i, \tau_{i+1}\}$.

(c) $\mathrm{Sti}_0(I) = \{c(\tau_i, \tau_{i+1}) : i < n\}$.

(d) $[\![\tau_i < \tau]\!] \cap [\![\tau < \tau_{i+1}]\!] = 0$ whenever $i < n$ and $\tau \in I$.

(e) $\sup_{i\le n} [\![\tau = \tau_i]\!] = 1$ for every $\tau \in I$.

*(f) If $\sigma \in \mathcal{T}$ then

$$J_0 = \{\sigma \wedge \tau_0, \tau_0, \mathrm{med}(\tau_0, \sigma, \tau_1), \tau_1, \mathrm{med}(\tau_1, \sigma, \tau_2), \\ \ldots, \tau_{n-1}, \mathrm{med}(\tau_{n-1}, \sigma, \tau_n), \tau_n, \sigma \vee \tau_n\}$$

is a maximal totally ordered subset of the sublattice $I \sqcup \{\sigma\}$ of $\mathcal{T}$ generated by $I \cup \{\sigma\}$.[2]

*(g) If $\sigma \in \mathcal{T}$, then $I \wedge \sigma = \{\tau \wedge \sigma : \tau \in I\}$ is a sublattice of $\mathcal{T}$, and $\{\tau_0 \wedge \sigma, \ldots, \tau_n \wedge \sigma\}$ is a maximal totally ordered subset of $I \wedge \sigma$.

*(h) If $\tau_0 \le \sigma_0 \le \ldots \le \sigma_m \le \tau_n$ in $\mathcal{T}$, and K is the sublattice of $\mathcal{T}$ generated by $I \cup \{\sigma_0, \ldots, \sigma_m\}$, then $J_j = \{\mathrm{med}(\sigma_j, \tau_i, \sigma_{j+1}) : i \le n\}$ is a maximal totally ordered subset of $K \cap [\sigma_j, \sigma_{j+1}]$, for every $j < m$.

proof (a) $I_0 \cup \{\min I, \max I\}$ is a totally ordered subset of I so must be equal to I_0.

(b) If $i < n$, $\tau \in I$ and $\tau_i \le \tau \le \tau_{i+1}$, then $I_0 \cup \{\tau\}$ is totally ordered and $\tau \in I_0$; thus $I \cap [\tau_i, \tau_{i+1}] = \{\tau_i, \tau_{i+1}\}$ for every $i < n$.

(c) Writing

$$I \wedge \tau_i = \{\sigma \wedge \tau_i : \sigma \in I\} = \{\sigma : \sigma \in I, \ \sigma \le \tau_i\}$$

as in 611Je above, we see that

[2] In a distributive lattice, $\mathrm{med}(p,q,r) = (p \wedge q) \vee (p \wedge r) \vee (q \wedge r)$; see 3A1Ic.

$$\mathrm{Sti}_0(I \wedge \tau_{i+1}) = \mathrm{Sti}_0(I \wedge \tau_i) \cup \mathrm{Sti}_0((I \wedge \tau_{i+1}) \vee \tau_i) = \mathrm{Sti}_0(I \wedge \tau_i) \cup \mathrm{Sti}_0(I \cap [\tau_i, \tau_{i+1}])$$

for each $i < n$. Since $I \wedge \tau_0$ is the singleton $\{\tau_0\}$, $\mathrm{Sti}_0(I \wedge \tau_0) = \emptyset$; since $I \cap [\tau_i, \tau_{i+1}] = \{\tau_i, \tau_{i+1}\}$, $\mathrm{Sti}_0(I \cap [\tau_i, \tau_{i+1}]) = \{c(\tau_i, \tau_{i+1})\}$ for each $i < n$. Inducing on m, $\mathrm{Sti}_0(I \wedge \tau_m) = \{c(\tau_i, \tau_{i+1}) : i < m\}$ for each $m \le n$, and

$$\mathrm{Sti}_0(I) = \mathrm{Sti}_0(I \wedge \tau_n) = \{c(\tau_i, \tau_{i+1}) : i < n\}.$$

(d) Set

$$\tau' = \mathrm{med}(\tau_i, \tau, \tau_{i+1}) = \tau_i \vee (\tau \wedge \tau_{i+1}) = (\tau_i \vee \tau) \wedge \tau_{i+1}.$$

Then $\tau_i \le \tau' \le \tau_{i+1}$ and $I_0 \cup \{\tau'\}$ is a totally ordered subset of I, so either $\tau_i = \tau'$ or $\tau' = \tau_{i+1}$. Accordingly

$$0 = [\![\tau_i < \tau']\!] \cap [\![\tau' < \tau_{i+1}]\!] = [\![\tau_i < \tau_i \vee (\tau \wedge \tau_{i+1})]\!] \cap [\![(\tau_i \vee \tau) \wedge \tau_{i+1} < \tau_{i+1}]\!]$$
$$= ([\![\tau_i < \tau_i]\!] \cup ([\![\tau_i < \tau]\!] \cap [\![\tau_i < \tau_{i+1}]\!])) \cap (([\![\tau_i < \tau_{i+1}]\!] \cap [\![\tau < \tau_{i+1}]\!]) \cup [\![\tau_{i+1} < \tau_{i+1}]\!])$$

(611E(c-i) and (c-ii))

$$= [\![\tau_i < \tau]\!] \cap [\![\tau_i < \tau_{i+1}]\!] \cap [\![\tau < \tau_{i+1}]\!] = [\![\tau_i < \tau]\!] \cap [\![\tau < \tau_{i+1}]\!]$$

(611C(c-iii-γ)), as required.

(e) If $\tau \in I$ then $[\![\tau \le \tau_m]\!] = \sup_{i \le m} [\![\tau = \tau_i]\!]$ for every $m \le n$. **P** Induce on m. If $m = 0$ we have $\tau_0 \le \tau$ so $[\![\tau \le \tau_0]\!] = [\![\tau \wedge \tau_0 = \tau_0]\!] = [\![\tau = \tau_0]\!]$. For the inductive step to $m + 1 \le n$,

$$[\![\tau \le \tau_{m+1}]\!] = ([\![\tau \le \tau_m]\!] \cap [\![\tau \le \tau_{m+1}]\!]) \cup ([\![\tau_m < \tau]\!] \cap [\![\tau \le \tau_{m+1}]\!])$$

(611E(a-i-α))

$$= [\![\tau \le \tau_m \wedge \tau_{m+1}]\!] \cup ([\![\tau_m < \tau]\!] \cap [\![\tau < \tau_{m+1}]\!]) \cup ([\![\tau_m < \tau]\!] \cap [\![\tau = \tau_{m+1}]\!])$$

(611E(c-i-α))

$$\subseteq [\![\tau \le \tau_m]\!] \cup [\![\tau = \tau_{m+1}]\!]$$

((d) above)

$$\subseteq \sup_{i \le m} [\![\tau = \tau_i]\!] \cup [\![\tau = \tau_{m+1}]\!]$$

(by the inductive hypothesis)

$$= \sup_{i \le m+1} [\![\tau = \tau_i]\!]. \textbf{ Q}$$

At the end of the induction,

$$1 = [\![\tau \le \tau_n]\!] = \sup_{i \le n} [\![\tau = \tau_i]\!]$$

for every $\tau \in I$.

(f) The set

$$\{\tau : \tau \in \mathcal{T}, \tau \wedge \tau_0 \in \{\sigma \wedge \tau_0, \tau_0\},\ \tau \vee \tau_n \in \{\tau_n, \sigma \vee \tau_n\},$$
$$\mathrm{med}(\tau_i, \tau, \tau_{i+1}) \in \{\tau_i, \mathrm{med}(\tau_i, \sigma, \tau_i), \tau_{i+1}\} \text{ for every } i < n\}$$

is a sublattice of $\mathcal{T}$ (because all the operations $\tau \mapsto \tau \wedge \tau_0$, $\tau \mapsto \mathrm{med}(\tau_i, \tau, \tau_{i+1})$, $\tau \mapsto \tau \vee \tau_n$ are lattice homomorphisms), containing σ (obviously) and including I (because $I \wedge \tau_0 = \{\tau_0\}$, $I \vee \tau_n = \{\tau_n\}$ and $I \cap [\tau_i, \tau_{i+1}] = \{\tau_i, \tau_{i+1}\}$ for $i < n$); so it includes $I \sqcup \{\sigma\}$. But this means that there is no member of $I \sqcup \{\sigma\}$ lying strictly between any two terms of the string defining J_0, while $\min(I \sqcup \{\sigma\}) = \sigma \wedge \tau_0$ is the first member of J_0 and $\max(I \sqcup \{\sigma\}) = \sigma \vee \tau_n$ is the last. Thus J_0 is a maximal totally ordered subset of $I \sqcup \{\sigma\}$, as claimed.

(g) $I \wedge \sigma$ is a sublattice of $\mathcal{T}$ because $\mathcal{T}$ is a distributive lattice. Since $\tau_0 = \min I$ and $\tau_n = \max I$, $\tau_0 \wedge \sigma$ and $\tau_n \wedge \sigma$ are the least and greatest members of $I \wedge \sigma$. Suppose that $\tau \in I$ and $i < n$ are such that $\tau_i \wedge \sigma \le \tau \wedge \sigma \le \tau_{i+1} \wedge \sigma$. Then $\tau' = \mathrm{med}(\tau_i, \tau, \tau_{i+1}) \in I$ and

$$\tau\wedge\sigma=((\tau\vee\tau_i)\wedge\tau_{i+1})\wedge\sigma=((\tau\wedge\sigma)\vee(\tau_i\wedge\sigma))\wedge\tau_{i+1})\wedge\sigma=\tau'\wedge\sigma.$$

But as $\tau'\in I\cap[\tau_i,\tau_{i+1}]$ and I_0 is a maximal totally ordered subset of I, $\tau'\in\{\tau_i,\tau_{i+1}\}$ and $\tau\wedge\sigma\in\{\tau_i\wedge\sigma,\tau_{i+1}\wedge\sigma\}$. So $\{\tau_i\wedge\sigma:i\le n\}$ is a maximal totally ordered subset of $I\wedge\sigma$.

(h) As $J_j\subseteq K$ and

$$\sigma_j=\mathrm{med}(\sigma_j,\tau_0,\sigma_{j+1})\le\mathrm{med}(\sigma_j,\tau_1,\sigma_{j+1})\le\ldots\le\mathrm{med}(\sigma_j,\tau_n,\sigma_{j+1})=\sigma_{j+1},$$

J_j is a totally ordered subset of $K\cap[\sigma_j,\sigma_{j+1}]$ containing σ_j and σ_{j+1}. As in (f), the set

$$\{\rho:\rho\in\mathcal{T},\,\mathrm{med}(\sigma_j,\rho,\sigma_{j+1})\in\{\mathrm{med}(\sigma_j,\tau,\sigma_{j+1}):\tau\in I\}\}$$

is a sublattice of $\mathcal{T}$ including $\{\sigma_0,\ldots,\sigma_m\}\cup I$ and therefore includes K. Now suppose that $\rho\in K\cap[\sigma_j,\sigma_{j+1}]$ and $J_j\cup\{\rho\}$ is totally ordered. Then there is an $i<n$ such that

$$\mathrm{med}(\sigma_j,\tau_i,\sigma_{j+1})\le\rho\le\mathrm{med}(\sigma_j,\tau_{i+1},\sigma_{j+1}).$$

Let $\tau\in I$ be such that $\mathrm{med}(\sigma_j,\rho,\sigma_{j+1})=\mathrm{med}(\sigma_j,\tau,\sigma_{j+1})$ and consider $\tau'=\mathrm{med}(\tau_i,\tau,\tau_{i+1})$. Then

$$\mathrm{med}(\sigma_j,\tau',\sigma_{j+1})=\mathrm{med}(\mathrm{med}(\sigma_j,\tau_i,\sigma_{j+1}),\mathrm{med}(\sigma_j,\tau,\sigma_{j+1}),\mathrm{med}(\sigma_j,\tau_{i+1},\sigma_{j+1}))$$

(because $\sigma\mapsto\mathrm{med}(\sigma_j,\sigma,\sigma_{j+1})$ is a lattice homomorphism)

$$=\mathrm{med}(\mathrm{med}(\sigma_j,\tau_i,\sigma_{j+1}),\mathrm{med}(\sigma_j,\rho,\sigma_{j+1}),\mathrm{med}(\sigma_j,\tau_{i+1},\sigma_{j+1}))$$

$$=\mathrm{med}(\mathrm{med}(\sigma_j,\tau_i,\sigma_{j+1}),\rho,\mathrm{med}(\sigma_j,\tau_{i+1},\sigma_{j+1}))$$

(because $\sigma_j\le\mathrm{med}(\sigma_j,\tau_i,\sigma_{j+1})\le\rho\le\mathrm{med}(\sigma_j,\tau_{i+1},\sigma_{j+1})\le\sigma_{j+1}$)

$$=\rho.$$

But as $\tau'\in I$ and $\{\tau_k:k\le n\}$ is a maximal totally ordered subset of I, either $\tau'=\tau_i$ and $\rho=\mathrm{med}(\sigma_j,\tau_i,\sigma_{j+1})$ or $\tau'=\tau_{i+1}$ and $\rho=\mathrm{med}(\sigma_j,\tau_{i+1},\sigma_{j+1})$; in either case $\rho\in J_j$. Thus J_j is a maximal totally ordered subset of $K\cap[\sigma_j,\sigma_{j+1}]$, as claimed.

611L Definition If I is a finite sublattice of $\mathcal{T}$, I will say that a sequence $\langle\tau_i\rangle_{i\le n}$ in I **linearly generates the I-cells** if it is non-decreasing and $\{\tau_i:i\le n\}$ is a maximal totally ordered subset of I. (It will be convenient not to insist that the sequence be strictly increasing. But we shall always have $\tau_0=\min I$, $\tau_n=\max I$, $I\cap[\tau_i,\tau_{i+1}]=\{\tau_i,\tau_{i+1}\}$ for every $i<n$, $\mathrm{Stio}_0(I)=\{c(\tau_i,\tau_{i+1}):i<n,\,\tau_i\ne\tau_{i+1}\}$, $\sup_{i\le n}[\![\sigma=\tau_i]\!]=1$ for every $\sigma\in I$, $[\![\tau_i<\sigma]\!]\cap[\![\sigma<\tau_{i+1}]\!]=0$ whenever $\sigma\in I$ and $i<n$, and

$$(\sigma\wedge\tau_0,\tau_0,\mathrm{med}(\tau_0,\sigma,\tau_1),\tau_1,\ldots,\tau_{n-1},\mathrm{med}(\tau_{n-1},\sigma,\tau_n),\tau_n,\sigma\vee\tau_n)$$

will linearly generate the $(I\sqcup\{\sigma\})$-cells for every $\sigma\in\mathcal{T}$.)

611M Covering and full sublattices (a)(i) If A, $B\subseteq\mathcal{T}$, I will say that A **covers** B if $\sup_{\sigma\in A}[\![\tau=\sigma]\!]=1$ for every $\tau\in B$. (The formula is to be interpreted as including a promise that, except in the trivial cases $\mathfrak{A}=\{0\}$ and $B=\emptyset$, A is non-empty, following the rule that $\sup\emptyset=0$ in any Boolean algebra.)

(ii) If $A\subseteq\mathcal{T}$, the **covered envelope** of A will be the set $\{\tau:\tau\in\mathcal{T},\,\sup_{\sigma\in A}[\![\tau=\sigma]\!]=1\}$, that is, the largest subset of $\mathcal{T}$ covered by A. Of course A covers itself, that is, the covered envelope of A includes A.

(b)(i) If $A\subseteq\mathcal{T}$ and $a\in\mathfrak{A}$, the set

$$\mathcal{S}=\{\tau:\tau\in\mathcal{T},\,a\subseteq\,\sup_{\sigma\in A}[\![\sigma=\tau]\!]\}$$

is a sublattice of $\mathcal{T}$. **P** If τ, $\tau'\in\mathcal{S}$, then

$$\sup_{\sigma\in A}[\![\tau\vee\tau'=\sigma]\!]\supseteq\sup_{\sigma\in A}([\![\tau\vee\tau'=\tau]\!]\cap[\![\tau=\sigma]\!])\cup\sup_{\sigma\in A}([\![\tau\vee\tau'=\tau]\!]\cap[\![\tau'=\sigma]\!])$$

(611E(c-iv-γ))

$$\supseteq(a\cap[\![\tau'\le\tau]\!])\cup(a\cap[\![\tau\le\tau']\!])$$

(611E(a-ii-β))

$$= a$$

(611D). Similarly

$$\sup_{\sigma\in A}[\![\tau\wedge\tau'=\sigma]\!]\supseteq\sup_{\sigma\in A}([\![\tau\wedge\tau'=\tau]\!]\cap[\![\tau=\sigma]\!])\cup\sup_{\sigma\in A}([\![\tau\wedge\tau'=\tau]\!]\cap[\![\tau'=\sigma]\!])$$
$$\supseteq(a\cap[\![\tau\le\tau']\!])\cup(a\cap[\![\tau'\le\tau]\!])=a.$$

So $\tau\vee\tau'$ and $\tau\wedge\tau'$ belong to $\mathcal{S}$. As τ and τ' are arbitrary, $\mathcal{S}$ is a sublattice of $\mathcal{T}$. **Q**

In particular, the covered envelope $\hat{A}$ of A is a sublattice of $\mathcal{T}$.

(ii) If ρ is an upper bound for A in $\mathcal{T}$, then ρ is an upper bound for $\hat{A}$. **P** If $\tau\in\hat{A}$, then

$$[\![\tau\le\rho]\!]\supseteq\sup_{\sigma\in A}[\![\tau=\sigma]\!]\cap[\![\sigma\le\rho]\!]\supseteq\sup_{\sigma\in A}[\![\tau=\sigma]\!]=1$$

and $\tau\le\rho$. **Q** Similarly, if ρ is a lower bound for A, it is a lower bound for $\hat{A}$.

(iii) Since $A\subseteq\hat{A}$, it follows that if A has a greatest member then this is also the greatest member of $\hat{A}$, and that if A has a least member then this is also the least member of $\hat{A}$.

(iv) Note that if σ, $\tau\in\mathcal{T}$ then $\{\sigma,\tau\}$ covers $\{\sigma\wedge\tau,\sigma\vee\tau\}$ (by (i) above) and also $\{\sigma\wedge\tau,\sigma\vee\tau\}$ covers $\{\sigma,\tau\}$ (because

$$[\![\sigma\le\tau]\!]\subseteq[\![\sigma=\sigma\wedge\tau]\!]\cap[\![\tau=\sigma\vee\tau]\!],\quad[\![\tau\le\sigma]\!]\subseteq[\![\sigma=\sigma\vee\tau]\!]\cap[\![\tau=\sigma\wedge\tau]\!]).$$

(c) I will say that a sublattice of $\mathcal{T}$ is **full** if it is equal to its covered envelope.

(i) The intersection of any non-empty family of full sublattices of $\mathcal{T}$ is full. **P** If $\mathbb{S}$ is a non-empty family of full sublattices of $\mathcal{T}$, and τ belongs to the covered envelope of $\mathcal{S}^*=\bigcap\mathbb{S}$, then for any $\mathcal{S}\in\mathbb{S}$ we have

$$\sup_{\sigma\in\mathcal{S}}[\![\tau=\sigma]\!]\supseteq\sup_{\sigma\in\mathcal{S}^*}[\![\tau=\sigma]\!]=1$$

so $\tau\in\mathcal{S}$. As $\mathcal{S}$ is arbitrary, $\tau\in\mathcal{S}^*$; as τ is arbitrary, $\mathcal{S}^*$ is full. **Q**

(ii) If $A\subseteq\mathcal{T}$, its covered envelope $\hat{A}$ is full. **P** If ρ belongs to the covered envelope of $\hat{A}$ then

$$\sup_{\sigma\in A}[\![\rho=\sigma]\!]\supseteq\sup_{\sigma\in A,\tau\in\mathcal{S}}[\![\rho=\tau]\!]\cap[\![\tau=\sigma]\!]=\sup_{\tau\in\mathcal{S}}[\![\rho=\tau]\!]=1$$

and $\rho\in\hat{A}$. **Q**

(d) For any $\rho\in\mathcal{T}$, $\mathcal{T}\wedge\rho$ is full. **P** If τ belongs to the covered envelope of $\mathcal{T}\wedge\rho$, then

$$[\![\tau\le\rho]\!]\supseteq\sup_{\sigma\in\mathcal{T}\wedge\rho}[\![\tau=\sigma]\!]\cap[\![\sigma\le\rho]\!]$$

(611E(c-iv-α))

$$=\sup_{\sigma\in\mathcal{T}\wedge\rho}[\![\tau=\sigma]\!]=1,$$

so $\tau\in\mathcal{T}\wedge\rho$ (611E(a-i-γ)). **Q** Similarly, $\mathcal{T}\vee\rho$ is full. Putting these together, $[\rho,\rho']=(\mathcal{T}\wedge\rho')\cap(\mathcal{T}\vee\rho)$ is full whenever $\rho\le\rho'$ in $\mathcal{T}$.

(e)(i) If $\mathcal{S}$ is a sublattice of $\mathcal{T}$ with covered envelope $\hat{\mathcal{S}}$, and $\rho\in\mathcal{S}$, then $\hat{\mathcal{S}}\wedge\rho$ is the covered envelope of $\mathcal{S}\wedge\rho$ and $\hat{\mathcal{S}}\vee\rho$ is the covered envelope of $\mathcal{S}\vee\rho$. **P** Since $\hat{\mathcal{S}}$ is a sublattice of $\mathcal{T}$ ((b-i) above) and $\rho\in\hat{\mathcal{S}}$,

$$\hat{\mathcal{S}}\wedge\rho=\{\tau:\tau\in\hat{\mathcal{S}},\,\tau\le\rho\}=\hat{\mathcal{S}}\cap(\mathcal{T}\wedge\rho)$$

(see the remark following 611C) and is full (putting (c) and (d) together). As it includes $\mathcal{S}\wedge\rho$, it includes the covered envelope of $\mathcal{S}\wedge\rho$. In the other direction, if $\tau\in\hat{\mathcal{S}}\wedge\rho$, then

$$\sup_{\sigma\in\mathcal{S}\wedge\rho}[\![\tau=\sigma]\!]=\sup_{\sigma\in\mathcal{S}}[\![\tau=\sigma\wedge\rho]\!]$$

(because $\mathcal{S}\wedge\rho=\{\sigma\wedge\rho:\sigma\in\mathcal{S}\}$)

$$=\sup_{\sigma\in\mathcal{S}}[\![\tau\wedge\rho=\sigma\wedge\rho]\!]$$

(because $\tau=\tau\wedge\rho$)

$$\supseteq\sup_{\sigma\in\mathcal{S}}[\![\tau=\sigma]\!]$$

(611E(c-v-α))

$$=1$$

because $\rho\in\hat{\mathcal{S}}$ and $\hat{\mathcal{S}}$ is a sublattice of $\mathcal{T}$, so $\tau\in\hat{\mathcal{S}}$. Thus τ belongs to the covered envelope of $\mathcal{S}\wedge\rho$, and this is the whole of $\hat{\mathcal{S}}\wedge\rho$.

Replacing every $\wedge$ above by $\vee$, we see that $\hat{\mathcal{S}}\vee\rho$ is the covered envelope of $\mathcal{S}\vee\rho$. **Q**

(ii) If $\mathcal{S}$ is a sublattice of $\mathcal{T}$, ρ, $\rho'\in\mathcal{S}$ and $\rho\le\rho'$, then the covered envelope of

$$\mathcal{S}\cap[\rho,\rho']=(\mathcal{S}\vee\rho)\wedge\rho'=\{\mathrm{med}(\rho,\sigma,\rho'):\sigma\in\mathcal{S}\}$$

is $(\hat{\mathcal{S}}\vee\rho)\wedge\rho'=\hat{\mathcal{S}}\cap[\rho,\rho']$. (Because $\mathcal{T}$ is a distributive lattice, $\mathcal{S}\vee\rho$ is a sublattice of $\mathcal{T}$, so we can apply (i) twice.)

(f) If $\mathcal{S}$ is a sublattice of $\mathcal{T}$ with covered envelope $\hat{\mathcal{S}}$, then $\bigcap_{\tau\in\hat{\mathcal{S}}}\mathfrak{A}_\tau=\bigcap_{\sigma\in\mathcal{S}}\mathfrak{A}_\sigma$. **P** Write $\mathfrak{B}$ for $\bigcap_{\sigma\in\mathcal{S}}\mathfrak{A}_\sigma$. Then $\bigcap_{\tau\in\hat{\mathcal{S}}}\mathfrak{A}_\tau\subseteq\mathfrak{B}$ just because $\mathcal{S}\subseteq\hat{\mathcal{S}}$. In the other direction, if $b\in\mathfrak{B}$ and $\tau\in\hat{\mathcal{S}}$, then for any $\sigma\in\mathcal{S}$ we have $b\in\mathfrak{a}_\sigma$ so $b\cap[\![\tau=\sigma]\!]\in\mathfrak{A}_\tau$ (611H(c-iii)); accordingly $b=\sup_{\sigma\in\mathcal{S}}b\cap[\![\tau=\sigma]\!]$ belongs to $\mathfrak{A}_\tau$. As b and τ are arbitrary, $\mathfrak{B}\subseteq\bigcap_{\tau\in\hat{\mathcal{S}}}\mathfrak{A}_\tau$ and we have equality. **Q**

(g) Suppose that A, $B\subseteq\mathcal{T}$ and A covers B.

(i) A covers the covered envelope of B, because A covers its own covered envelope which is a full sublattice including B.

(ii) If $\tau\in\mathcal{T}$, then $A\wedge\tau=\{\sigma\wedge\tau:\sigma\in A\}$ covers $B\wedge\tau=\{\sigma\wedge\tau:\sigma\in B\}$. **P** If $\sigma\in B$, then

$$1=\sup_{\sigma'\in A}[\![\sigma=\sigma']\!]\subseteq\sup_{\sigma'\in A}[\![\sigma\wedge\tau=\sigma'\wedge\tau]\!]$$

by 611E(c-v-α). **Q**

611N Covering ideals Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.

(a) Definition I will say that a **covering ideal** of $\mathcal{S}$ is an ideal $\mathcal{S}'$ of $\mathcal{S}$ which covers $\mathcal{S}$ in the sense of 611M.

(b)(i) If $\tau\in\mathcal{S}$ and $\mathcal{S}'$ is an ideal of $\mathcal{S}$, then $\{[\![\sigma=\tau]\!]:\sigma\in\mathcal{S}'\}$ is upwards-directed. **P** If σ, $\sigma'\in\mathcal{S}'$ then $\upsilon=(\sigma\vee\sigma')\wedge\tau$ belongs to $\mathcal{S}'$, and $[\![\upsilon=\tau]\!]\supseteq[\![\sigma=\tau]\!]\cup[\![\sigma'=\tau]\!]$. **Q**

(ii) If $\tau\in\mathcal{S}$ and $\mathcal{S}'$ is an ideal of $\mathcal{S}$, then $\sup_{\sigma\in\mathcal{S}'}[\![\sigma=\tau]\!]=\sup_{\sigma\in\mathcal{S}'}[\![\tau\le\sigma]\!]$. **P** For any $\sigma\in\mathcal{S}'$, $\sigma\wedge\tau\in\mathcal{S}'$ and

$$[\![\sigma=\tau]\!]\subseteq[\![\tau\le\sigma]\!]=[\![\tau=\sigma\wedge\tau]\!].\ \mathbf{Q}$$

(c) If $\mathcal{S}$ is a sublattice of $\mathcal{T}$ and $\mathcal{S}_1$, $\mathcal{S}_2$ are two covering ideals of $\mathcal{S}$, then $\mathcal{S}_0=\mathcal{S}_1\cap\mathcal{S}_2$ is a covering ideal of $\mathcal{S}$.

P Certainly $\mathcal{S}_0$ is an ideal of $\mathcal{S}$. Take $\tau\in\mathcal{S}$ and $a\in\mathfrak{A}\setminus\{0\}$. Then there is a $\sigma_1\in\mathcal{S}_1$ such that $a_1=a\cap[\![\tau=\sigma_1]\!]\ne0$. Next, there is a $\sigma_2\in\mathcal{S}_2$ such that $a_2=a_1\cap[\![\tau=\sigma_2]\!]\ne0$. In this case, $\sigma=\sigma_1\wedge\sigma_2$ belongs to $\mathcal{S}_0$, a

$$[\![\tau=\sigma]\!]\supseteq[\![\tau=\sigma_1]\!]\cap[\![\tau=\sigma_2]\!]\supseteq a_2$$

meets a. As a and τ are arbitrary, $\mathcal{S}_0$ is a covering ideal of $\mathcal{S}$. **Q**

(d) If $\mathcal{S}'$ is a covering ideal of $\mathcal{S}$ and $\mathcal{S}''$ is a covering ideal of $\mathcal{S}'$, then $\mathcal{S}''$ is a covering ideal of $\mathcal{S}$. **P** It is elementary to check that $\mathcal{S}''$ is an ideal of $\mathcal{S}$. If $\tau \in \mathcal{S}$ and $a \in \mathfrak{A} \setminus \{0\}$, there is a $\sigma \in \mathcal{S}'$ such that $b = a \cap [\![\sigma = \tau]\!]$ is not 0. Now there is an $\upsilon \in \mathcal{S}''$ such that $c = b \cap [\![\upsilon = \sigma]\!]$ is non-zero. But now $c \subseteq [\![\upsilon = \tau]\!]$, so $a \cap [\![\upsilon = \tau]\!] \neq 0$. As a is arbitrary, $\sup_{\upsilon \in \mathcal{S}''} [\![\upsilon = \tau]\!] = 0$; as τ is arbitrary, $\mathcal{S}''$ is covering in $\mathcal{S}$. **Q**

(e)(i) $\mathcal{T}_f$ is full. **P** If $\tau \in \mathcal{T}$ and $\sup_{\sigma \in \mathcal{T}_f} [\![\tau = \sigma]\!] = 1$, take any non-zero $a \in \mathfrak{A}$. Then there are a $\sigma \in \mathcal{T}_f$ such that $a \cap [\![\tau = \sigma]\!] \neq 0$ and a $t \in T$ such that

$$a \cap [\![\tau = \sigma]\!] \not\subseteq [\![\sigma > t]\!] \cap [\![\tau = \sigma]\!] = [\![\tau > t]\!] \cap [\![\tau = \sigma]\!]$$

(611E(i-β)), so $a \not\subseteq [\![\tau > t]\!]$. As a is arbitrary, $\inf_{t \in T} [\![\tau > t]\!] = 0$ and $\tau \in \mathcal{T}_f$. Thus $\mathcal{T}_f$ is its own covered envelope and is full. **Q**

(ii) $\mathcal{T}_b$ is a covering ideal of $\mathcal{T}_f$. **P** I observed in 611Cg that $\mathcal{T}_b$ is an ideal in $\mathcal{T}$ and therefore in $\mathcal{T}_f$. If $\tau \in \mathcal{T}_f$ and $t \in T$, then $\check{t}$ and $\tau \wedge \check{t}$ belong to $\mathcal{T}_b$, while

$$\begin{aligned}[\![\tau = \tau \wedge \check{t}]\!] &= 1 \setminus [\![\check{t} < \tau]\!] \\ &= 1 \setminus \sup_{s \in T} [\![\tau > s]\!] \setminus [\![\check{t} > s]\!] = 1 \setminus \sup_{s \geq t} [\![\tau > s]\!] = 1 \setminus [\![\tau > t]\!].\end{aligned}$$

(611E(a-ii-α))

So

$$\sup_{\sigma \in \mathcal{T}_f} [\![\tau = \sigma]\!] \supseteq \sup_{t \in T} [\![\tau = \tau \wedge \check{t}]\!] = 1 \setminus \inf_{t \in T} [\![\tau > t]\!] = 1$$

because $\tau \in \mathcal{T}_f$. As τ is arbitrary, $\mathcal{T}_b$ is covering in $\mathcal{T}_f$. **Q**

***611O Definitions** A variation on the concept of 'full' sublattice will be relevant in §615, and important in §626.

(a) If A, $B \subseteq \mathcal{T}$, I will say that A **finitely covers** B if for every $\tau \in B$ there is a finite $J \subseteq A$ such that $\sup_{\sigma \in J} [\![\tau = \sigma]\!] = 1$.

(b) If $A \subseteq \mathcal{T}$, the **finitely-covered envelope** of A is the set of those $\tau \in \mathcal{T}$ for which there is a finite subset $J \subseteq A$ such that $\sup_{\sigma \in J} [\![\tau = \sigma]\!] = 1$, that is, the largest subset of $\mathcal{T}$ finitely covered by A. Of course A finitely covers itself, so is included in its finitely-covered envelope.

(c) A subset of $\mathcal{T}$ is **finitely full** if it is equal to its finitely-covered envelope.

***611P Lemma** Suppose that $\mathfrak{A} \neq \{0\}$.

(a) Let A be a subset of $\mathcal{T}$ and $\hat{A}_f$ its finitely-covered envelope.

(i) $\hat{A}_f$ is finitely full.

(ii) $\hat{A}_f$ is a sublattice of the covered envelope $\hat{A}$ of A.

(iii) $\hat{A}_f$ is the intersection of all the finitely full subsets of $\mathcal{T}$ including A.

(b) The intersection of any non-empty family $\mathbb{S}$ of finitely full sublattices of $\mathcal{T}$ is finitely full.

(c) If $\mathcal{S}$ is a sublattice of $\mathcal{T}$ which is order-convex (that is, $\tau \in \mathcal{S}$ whenever $\sigma \leq \tau \leq \sigma'$ in $\mathcal{T}$ and σ, $\sigma' \in \mathcal{S}$), then $\mathcal{S}$ is finitely full.

(d) If $\mathcal{S}$ is a sublattice of $\mathcal{T}$ and $\tau \in \hat{\mathcal{S}}_f$, there are $\sigma_0 \leq \ldots \leq \sigma_n$ in $\mathcal{S}$ such that $\sup_{i \leq n} [\![\tau = \sigma_i]\!] = 1$.

(e) If $\mathcal{S}$ is a sublattice of $\mathcal{T}$ then $\mathcal{S}$ is both coinitial and cofinal with $\hat{\mathcal{S}}_f$.

proof (a)(i) If $\tau \in \mathcal{T}$ and $\{\tau\}$ is covered by a finite subset I of $\hat{A}_f$, then for each $\sigma \in I$ there is a finite subset J_σ of A covering $\{\sigma\}$; now $J = \bigcup_{\sigma \in I} J_\sigma$ is a finite subset of A, and

$$1 = \sup_{\sigma \in I} [\![\sigma = \tau]\!] = \sup_{\sigma \in I} \sup_{\rho \in J_\sigma} [\![\sigma = \tau]\!] \cap [\![\rho = \sigma]\!] \subseteq \sup_{\rho \in J} [\![\rho = \sigma]\!]$$

(using 611E(c-iv-γ) so $\tau \in \hat{A}_f$.

(ii) From the definitions in 611Ma and 611O we see at once that $\hat{A}_f \subseteq \hat{A}$. If σ and τ belong to $\hat{A}_f$, there are finite sets J, $K \subseteq A$ such that σ is covered by J and τ is covered by K; now $\{\sigma \wedge \tau, \sigma \vee \tau\}$ is covered by $\{\sigma, \tau\}$, by 611M(b-i), and therefore by $J \cup K$. So $\hat{A}_f$ is a sublattice of $\hat{A}$.

(iii) If B is a finitely full subset of $\mathcal{T}$ including A, then $\hat{A}_f \subseteq \hat{B}_f = B$; since $\hat{A}_f$ itself is finitely full, it is the intersection of all the finitely full sets including it.

(b) (Cf. 611M(c-i).) If I is a finite subset of $\bigcap \mathbb{S}$ then its covered envelope is included in every member of $\mathbb{S}$ so is included in $\bigcap \mathbb{S}$.

(c) If $\tau \in \mathcal{T}$ and there is a finite set $I \subseteq \mathcal{S}$ such that $\sup_{\sigma \in I} [\![\tau = \sigma]\!] = 1$, then (as $\mathfrak{A} \neq \{0\}$) I is non-empty, so we can speak of $\inf I$ and $\sup I$, which both belong to $\mathcal{S}$. Now

$$[\![\inf I \le \tau]\!] \subseteq \sup_{\sigma \in I} [\![\inf I \le \sigma]\!] \cap [\![\tau = \sigma]\!] = \sup_{\sigma \in I} [\![\tau = \sigma]\!] = 1$$

so $\inf I \le \tau$; similarly, $\tau \le \sup I$. As $\mathcal{S}$ is order-convex, $\tau \in \mathcal{S}$.

(d) There is a finite subset J of $\mathcal{S}$ covering $\{\tau\}$, and as $\mathfrak{A}$ is not $\{0\}$, J cannot be empty. Now the sublattice I of $\mathcal{T}$ generated by J is finite (611Ca) and not empty, so there is a finite sequence $(\sigma_0, \dots, \sigma_n)$ linearly generating the I-cells (611K-611L). Now $\{\sigma_0, \dots, \sigma_n\}$ covers I and therefore covers $\{\tau\}$.

(e) If $\tau \in \hat{\mathcal{S}}_f$, there is a finite set $J \subseteq \mathcal{S}$ such that $\sup_{\sigma \in J} [\![\sigma = \tau]\!] = 1$. As $\mathfrak{A} \neq \{0\}$, J is not empty, and we can speak of $\min J$ and $\max J$, which both belong to $\mathcal{S}$. Now

$$1 = \sup_{\sigma \in J} [\![\sigma = \tau]\!] \subseteq \sup_{\sigma \in J} [\![\sigma \le \tau]\!] \cap [\![\min J \le \sigma]\!] \subseteq [\![\min J \le \tau]\!]$$

(611E(c-iv-α)), so $\min J \le \tau$. Similarly, $\tau \le \max J$.

611X Basic exercises >(a) Suppose that $T = [0, \infty[$. (i) Show that $\mathcal{T}_f$ can be identified with a sublattice of $L^0(\mathfrak{A})^+$ which is closed under addition and multiplication by scalars greater than or equal to 1. (See §364 or 612A for the space $L^0 = L^0(\mathfrak{A})$.) (ii) Show that $\mathcal{T}_b$ becomes identified with $\mathcal{T}_f \cap L^\infty(\mathfrak{A})$. (iii) Show that a set $C \subseteq \mathcal{T}_f$ is bounded above in $\mathcal{T}_f$ iff it is bounded above in L^0, and that in this case its supremum taken in $\mathcal{T}_f$ is the same as its supremum taken in L^0.

>(b) Show that $\mathfrak{A}_{\max \mathcal{T}} = \mathfrak{A}$ and that $\mathfrak{A}_{\min \mathcal{T}} = \bigcap_{t \in T} \mathfrak{A}_t$.

>(c) In 611J, let Z be the Stone space of $\mathfrak{A}$, so that $\mathfrak{A}$ can be identified with the algebra $\mathcal{E}$ of open-and-closed subsets of Z, and $\mathfrak{A}^T$ with the family

$$\{W : W \subseteq Z \times T,\, W^{-1}[\{t\}] \in \mathcal{E} \text{ for every } t \in T\}.$$

(i) Show that a stopping time $\tau \in T$ corresponds to an ordinate set $W_\tau \subseteq Z \times T$ such that $(z, t) \in W_\tau$ whenever $(z, s) \in W_\tau$ and $t \le s$. (ii) Show that if σ, $\tau \in T$, then $c(\sigma, \tau)$ corresponds to a subset W of $Z \times T$ in which all vertical sections are intervals, and $[\![\sigma < \tau]\!]$ is now the interior of the closure of the projection $W^{-1}[T]$ of W onto Z.

(d) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. Show that $\mathcal{S}$ is order-convex (4A2A) iff $\mathrm{med}(\tau, \sigma, \tau')$ belongs to $\mathcal{S}$ whenever τ, $\tau' \in \mathcal{S}$ and $\sigma \in \mathcal{T}$.

(e) Suppose that I is a non-empty sublattice of $\mathcal{T}$, and that $(\tau_0, \dots, \tau_n)$ linearly generates the I-cells. Show that if $\sigma \in \mathcal{T}$ then

$$(\tau_0 \wedge \sigma, \dots, \tau_n \wedge \sigma, \sigma, \tau_0 \vee \sigma, \dots, \tau_n \vee \sigma)$$

linearly generates the $(I \sqcup \{\sigma\})$-cells, where $I \sqcup \{\sigma\}$ is the sublattice of $\mathcal{T}$ generated by $I \cup \{\sigma\}$.

(h) Suppose that $T = \mathbb{N}$. (i) Show that if $\sigma \in \mathcal{T}$ and $n \in \mathbb{N}$ then $[\![\sigma > n]\!]$ cannot meet $[\![(n+1)\check{} > \sigma]\!]$, so that $[\![\sigma > \check{n}]\!] = [\![\sigma \ge (n+1)\check{}]\!]$. (ii) Show that $\mathcal{T}$ is the covered envelope of $\{\check{n} : n \in \mathbb{N}\} \cup \{\max \mathcal{T}\}$.

(g) In 611M(b-i), show that $I = \{\tau : a \subseteq \sup_{\sigma \in A} [\![\tau = \sigma]\!]\}$ is a full sublattice of $\mathcal{T}$.

(f) Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ and $\mathcal{S}_0$ an ideal of $\mathcal{S}$. Show that $\mathcal{S}_0$ is finitely full.

611Y Further exercises (a) Suppose that $\mathfrak{A}$ is ccc (definition: 316A). Show that if $A \subseteq \mathcal{T}$ is non-empty, there is a countable $B \subseteq A$ such that $\sup B = \sup A$ in $\mathcal{T}$. (*Hint*: remember to cover the case $T = \omega_1$.)

(b) Let $\mathfrak{A} = \{0, a, 1 \setminus a, 1\}$ be a four-element Boolean algebra, and take $T = [0, \infty[$. Set $\mathfrak{A}_0 = \{0, 1\}$ and $\mathfrak{A}_t = \mathfrak{A}$ for $t > 0$. Show that for $s > 0$ there is a stopping time τ_s, adapted to $\langle \mathfrak{A}_t \rangle_{t \ge 0}$, defined by saying that $[\![\tau_s > t]\!] = a$ if $t \ge s$, 1 if $t < s$. (i) Show that $\inf_{s>0} \tau_s$ is the constant stopping time $\check{0} = \min \mathcal{T}$, and is not defined by the formula in 611F. (ii) Show that $\bigcap_{s>0} \mathfrak{A}_{\tau_s} \ne \mathfrak{A}_{\check{0}}$. (iii) Show that $\sup_{s>0} [\![\tau_s < \max \mathcal{T}]\!] \ne [\![\check{0} < \max \mathcal{T}]\!]$.

(c) Show that if $A \subseteq \mathcal{T}$ is finite then the covered envelope of A is order-closed in $\mathcal{T}$.

(d) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and write $\mathcal{I}$ for the set of totally ordered finite subsets of $\mathcal{S}$. For I, $J \in \mathcal{I}$ say that $I \sqsubseteq J$ if J covers I. Show that $\sqsubseteq$ is a pre-order on $\mathcal{I}$ (511A) under which $\mathcal{I}$ is upwards-directed.

611 Notes and comments Stochastic calculus is ordinarily presented in terms of probability spaces and random variables. I followed this line myself in the brief introduction to stochastic processes in §455. To go farther, however, I believe that (as with the ergodic theory of Chapter 38) the essential ideas can be expressed more clearly in terms of probability algebras $(\mathfrak{A}, \bar{\mu})$ and processes in the associated spaces $L^0(\mathfrak{A})$, $L^1(\mathfrak{A}, \bar{\mu})$ and $L^2(\mathfrak{A}, \bar{\mu})$. The machinery for this has already been developed in Chapter 36, so we can go directly to the new ideas in 611B, 611G and 612D. I give a translation from the standard framework in 612H.

In this context, let me recall what I wrote in the introduction to Chapter 27. The primary concept of probability theory is not a measure space of measure 1. Rather, it is 'random variable' with its associated distribution. 'Probability spaces', as I use the phrase in this treatise, can be regarded as models for a theory of random variables. But in any statistical question, it is the variable itself which we try to measure and make predictions for. If we knew where it came from, we'd study that. Now 'probability algebras', in my terminology, provide a perfectly adequate model for distributions, including joint distributions (see 653B and 653Xc below), while evading some of the technical problems associated with the arbitrary nature of any choice of probability space. I ought of course to admit that they simultaneously obscure some important sources of intuition.

Again, it is normal to think of filtrations and stochastic processes as based on real-valued times, so that the totally ordered set T of 611A is $[0, \infty[$ or something very like it. (See 611Xa for a note on how to look at this case.) I have nothing interesting to say about other cases, but there is very little extra work involved in the shift to an arbitrary totally ordered set, and this enables me to avoid an occasional shuffle when $]0, \infty[$ or $[0, 1]$ seems a more appropriate parameter space. In fact the extra steps needed in the general case, dealing with gaps in the parameter space (see the last clause in the definition 611B(b-i)), are already needed in the relatively elementary case $T = \mathbb{N}$. But it will become plain in Chapter 63 that some of the most important ideas of the theory apply only in contexts essentially excluding or erasing gaps in time.

I have deliberately cast the principal definitions in forms which make them applicable in such cases as $T = \mathbb{Z}$ or $T = \mathbb{R}$. But I note that the phrases 'finite-valued' and 'bounded' in 611A(b-iii) could be misleading when T has no least member. Of course we always have the option of adding a least member $-\infty$ to T and setting $\mathfrak{A}_{-\infty} = \bigcap_{t \in T} \mathfrak{A}_t$. And then we might be tempted to add yet another element $-\infty' < -\infty$ to T and set $\mathfrak{A}_{-\infty'} = \{0, 1\}$. Such manoeuvres can make no difference to the mathematical content of the work here, but they will sometimes smooth the task of adapting general results to specific applications.

The calculations in 611C, 611E and 611H are elaborate but fundamentally elementary. I ought to offer a word on the intuition behind the concept of 'stopping time'. The requirement '$[\![\tau > t]\!] \in \mathfrak{A}_t$' (or, if you prefer, '$[\![\tau \le t]\!] \in \mathfrak{A}_t$') is a declaration that the decision whether to continue beyond time t must be based on what can be observed at that time, the potential observations being those which can be represented by members of $\mathfrak{A}_t$. In 611I we have a direct expression of this idea: the stopping time σ corresponds to waiting until we reach a stopping time $\tau \in A$, checking whether we are in the region a_τ, and if so halting. This can be done only if the decision is based on something which will be observable when we reach τ, that is, if $a_\tau \in \mathfrak{A}_\tau$.

The point at which I think an imaginative effort is required is in 611J-611K. Here I expect that most people will find the expression in 611Xc useful in visualizing the sets $\mathrm{Sti}_0(I)$ of I-cells for finite sublattices I; they look like patchwork quilts. The same formulation can be used to help with 611C-611E.

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612 Fully adapted processes

The next step is to introduce the processes which this volume is devoted to studying. These are an abstract version of the real-valued stochastic processes $\langle X_t\rangle_{t\ge0}$ of §§455 and 477. Instead of starting from Σ_t-measurable functions $X_t:\Omega\to\mathbb{R}$ and then showing that it is possible to define Σ_h-measurable functions X_h for stopping times $h:\Omega\to[0,\infty[$, I move directly to families $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ of equivalence classes of measurable functions where $\mathcal{S}$ is a sublattice of the lattice $\mathcal{T}$ of stopping times discussed in §611. A 'fully adapted process' is one satisfying the essential measurability and consistency requirements of 612D. Among these, the 'simple' processes (612J), those which are constant between finitely many break points, are particularly important. I end with descriptions of Brownian motion (612T) and the standard Poisson process (612U) in this language.

612A $\mathfrak{A}$ and $L^0(\mathfrak{A})$ (a) Given a Dedekind complete Boolean algebra, we have a Dedekind complete f-algebra $L^0=L^0(\mathfrak{A})$ as described in §364. (For exact statements of the algebraic relationships between linear structure, lattice structure and multiplication which go to make an 'f-algebra', see 351A, 352A, 352D and 352W.) §364 was dedicated to setting up a coherent description of $L^0(\mathfrak{A})$ from $\mathfrak{A}$ in logically primitive terms, so that, in particular, it would be visibly free of any dependence on the axiom of choice. But if you are willing to relax this discipline, then I think that the easiest way to approach these formulae is to think of $\mathfrak{A}$ as a quotient $\Sigma/\mathcal{I}$, where Σ is a σ-algebra of subsets of a set Ω and $\mathcal{I}$ is a σ-ideal of Σ (314M), and to recall that L^0 can now be regarded as a space of equivalence classes of Σ-measurable functions (364C), with the natural definitions of addition, multiplication and lattice operations.

(b) In §364 I introduced the formulae $[\![u>\alpha]\!]$, $[\![u\in E]\!]$, where $u\in L^0$, $\alpha\in\mathbb{R}$ and $E\subseteq\mathbb{R}$ is a Borel set, to represent 'the region where u is greater than α' or 'the region where u lies in E' (364A, 364G). I mentioned formulae $[\![u\ge\alpha]\!]$, $[\![u<0]\!]$ and $[\![u\ne0]\!]$, and in the exercise 364Yb, I suggested a way of interpreting $[\![(u_1,\dots,u_n)\in E]\!]$ when E is a Borel subset of $\mathbb{R}^n$. Here it will be convenient to extend the notation to such formulae as $[\![u\ne v]\!]$, meaning, if you like, $[\![|u-v|>0]\!]$. In terms of the representation of L^0 as a space of equivalence classes of functions, we have

$$[\![(f_1^\bullet,\dots,f_n^\bullet)\in E]\!]=\{\omega:(f_1(\omega),\dots,f_n(\omega))\in E\}^\bullet$$

for all Σ-measurable functions $f_1,\dots,f_n:\Omega\to\mathbb{R}$. Similarly, $[\![f_1^\bullet\ne f_2^\bullet]\!]$ can be interpreted as $\{x:f_1(\omega)\ne f_2(\omega)\}^\bullet$, and while this interpretation skates over some technical issues, it gives clear signposts to such basic identities as

$$[\![u=v]\!]\cap[\![v=w]\!]\subseteq[\![u=w]\!]$$

without any real danger of your being led astray.

(c) Let $E\subseteq\mathbb{R}$ be a Borel set and $h:E\to\mathbb{R}$ a Borel measurable function. Set

$$Q_E=\{u:u\in L^0,\ [\![u\in E]\!]=1\}=\{f^\bullet:f:\Omega\to E\text{ is measurable}\}.$$

If $u\in Q_E$,we have an $\bar h(u)\in L^0$ defined by saying that $[\![\bar h(u)\in F]\!]=[\![u\in h^{-1}[F]]\!]$ for every Borel set $F\subseteq\mathbb{R}$ (364H). If u, $u'\in Q_E$ then $[\![u=u']\!]\subseteq[\![\bar h(u)=\bar h(u')]\!]$. Observe that if h_1, $h_2:\mathbb{R}\to\mathbb{R}$ are both Borel measurable, we now have $\overline{h_1h_2}(u)=\bar h_1(\bar h_2(u))$ for all h_1, h_2 and $u\in L^0$, because

$$[\![u\in(h_1h_2)^{-1}[F]]\!]=[\![u\in h_2^{-1}[h_1^{-1}[F]]]\!]=[\![\bar h_2(u)\in h_1^{-1}[F]]\!]=[\![\bar h_1(\bar h_2(u))\in F]\!]$$

for every Borel set F. Also, of course, $\bar h(u)=u$ if $E=\mathbb{R}$ and $h(\alpha)=\alpha$ for every $\alpha\in\mathbb{R}$, that is, h is the identity of the semigroup H of Borel measurable functions from $\mathbb{R}$ to itself under the operation of composition of functions. So we have a semigroup action $\bullet$ of H on L^0 defined by saying that $h\bullet u=\bar h(u)$ for $h\in H$ and $u\in L^0$.

(d)(i) The following elementary facts are easy to check.

(α) If $\gamma\in\mathbb{R}$ and $h(\alpha)=\gamma\alpha$ for $\alpha\in\mathbb{R}$, then $\bar h(u)=\gamma u$ for every $u\in L^0$.

(β) If $h(\alpha)=|\alpha|$ for $\alpha\in\mathbb{R}$, then $\bar h(u)=|u|$ for every $u\in L^0$.

(γ) If $h(\alpha)=\alpha^2$ for $\alpha\in\mathbb{R}$, then $\bar h(u)=u\times u=u^2$ for every $u\in L^0$.

(δ) If $h(\alpha)=1$ for $\alpha\in\mathbb{R}$, then $\bar h(u)=\chi 1$ is the multiplicative identity of L^0 for every $u\in L^0$.

(ϵ) If $h:\mathbb{R}\to\mathbb{R}$ is non-decreasing, then $\bar h(u)\le\bar h(v)$ whenever $u\le v$ in L^0.

(ii) It follows that if $V\subseteq L^0$ is such that $u+v\in V$ for all u, $v\in V$ and $\bar h(u)\in V$ for every convex function $h:\mathbb{R}\to\mathbb{R}$ such that $h(0)=0$, then V is an ***f*-subalgebra** of L^0, that is, a Riesz subspace closed under multiplication. **P** By (i-α), it is closed under scalar multiplication so is a linear subspace. By (i-β), $|u|\in V$ for every $u\in V$, so V is a Riesz subspace (352Ic). If u, $v\in V$ then $u\times v=\frac{1}{2}((u+v)^2-u^2-v^2)$ belongs to V by (i-γ), so V is closed under multiplication and is an f-subalgebra. **Q** *A fortiori*, if V is such that $u+v\in V$ for all u, $v\in V$ and $\bar h(u)\in V$ for every continuous function $h:\mathbb{R}\to\mathbb{R}$ such that $h(0)=0$, then V is an f-subalgebra of L^0.

(iii) Continuing from (c) above, it will be important also to note that, for any u, $v\in Q_E$, $[\![u\le v]\!]\subseteq[\![\bar h(u)\le\bar h(v)]\!]$. **P** Identifying Q_E with a set of equivalence classes of real-valued measurable functions from Ω to E, we find that $\bar h(f^\bullet)=(hf)^\bullet$ for all such functions f (364Ib), so that

$$\begin{aligned}[\![\bar h(f^\bullet)\le\bar h(g^\bullet)]\!]&=[\![(hf)^\bullet\le(hg)^\bullet]\!]=\{\omega:hf(\omega)\le hg(\omega)\}^\bullet\\&\supseteq\{\omega:f(\omega)\le g(\omega)\}^\bullet=[\![f^\bullet\le g^\bullet]\!]\end{aligned}$$

for all measurable f, $g:\Omega\to\mathbb{R}$. **Q** It follows at once that $[\![u=v]\!]\subseteq[\![\bar h(u)=\bar h(v)]\!]$.

(iv) Take any $u\in L^0$. Again writing H for the space of Borel measurable functions from $\mathbb{R}$ to itself, H is an f-subalgebra of the f-algebra $\mathbb{R}^{\mathbb{R}}$ as well as a sub-semigroup under composition. Treating H as an f-algebra, the map $h\mapsto\bar h(u):H\to L^0$ is a multiplicative Riesz homomorphism. **P** It is not especially hard to prove this directly from the formula in (c), but you may prefer to use the alternative description of $\bar h$ in 364Ib: expressing $\mathfrak{A}$ with $\Sigma/\mathcal{I}$, where Σ is a σ-algebra of subsets of a set Ω and $\mathcal{I}$ is a σ-ideal of Σ, as in (a), so that u can be thought of as the equivalence class of a Σ-measurable function $f:\Omega\to\mathbb{R}$ and $\bar h(u)$ becomes the equivalence class of the composition hf, then $h\mapsto hf$ is a multiplicative Riesz homomorphism, so $h\mapsto(hf)^\bullet=\bar h(u)$ also is. **Q**

(v) It will happen more than once that we have two Dedekind complete Boolean algebras $\mathfrak{A}$ and $\mathfrak{B}$, f-subalgebras V, W of $L^0(\mathfrak{A})$ and $L^0(\mathfrak{B})$ respectively, and a linear operator $Q:V\to W$ such that $Q|v|=|Qv|$ and $Q(v^2)=(Qv)^2$ for all $v\in V$. In this case, Q will be an ***f*-algebra homomorphism**, that is, a multiplicative Riesz homomorphism. (Use the ideas of (ii).)

(e)(i) Now suppose that $\mathfrak{B}$ is an order-closed subalgebra of $\mathfrak{A}$. In this case we can think of $L^0(\mathfrak{B})$ as being the subspace

$$\{u:u\in L^0(\mathfrak{A}),\ [\![u>\alpha]\!]\in\mathfrak{B}\text{ for every }\alpha\in\mathbb{R}\}.$$

The arguments of 364F show that this is equal to

$$\{u:u\in L^0(\mathfrak{A}),\ [\![u\in E]\!]\in\mathfrak{B}\text{ for every Borel set }E\subseteq\mathbb{R}\}$$

and therefore that $\bar h(u)\in L^0(\mathfrak{B})$ whenever $h\in H$ and $u\in L^0(\mathfrak{B})$. Looking at this a little more deeply, we see that if $h\in H$ we have two different functions $\bar h_{\mathfrak{A}}:L^0(\mathfrak{A})\to L^0(\mathfrak{A})$ and $\bar h_{\mathfrak{B}}:L^0(\mathfrak{B})\to L^0(\mathfrak{B})$, but that $\bar h_{\mathfrak{B}}=\bar h_{\mathfrak{A}}\restriction L^0(\mathfrak{B})$, so that we can fairly safely use the same symbol $\bar h$ for either.

Note also that if $A\subseteq L^0(\mathfrak{B})$ is non-empty and has a supremum v in $L^0(\mathfrak{A})$, then $v\in L^0(\mathfrak{B})$. **P** For any $\alpha\in\mathbb{R}$, $[\![v>\alpha]\!]=\sup_{u\in A}[\![u>\alpha]\!]$, by 364L(a-ii), and this belongs to $\mathfrak{B}$ because $\mathfrak{B}$ is order-closed in $\mathfrak{A}$. **Q** It follows that if $A\subseteq L^0(\mathfrak{B})$ is non-empty and has an infimum v in $L^0(\mathfrak{A})$, then $v\in L^0(\mathfrak{B})$ (because $-v=\sup(-A)$). So $L^0(\mathfrak{B})$ is an order-closed sublattice of $L^0(\mathfrak{A})$.

(ii) If $\langle\mathfrak{B}_i\rangle_{i\in I}$ is a non-empty family of order-closed subalgebras of $\mathfrak{A}$ with intersection $\mathfrak{B}$, then $\mathfrak{B}$ is an order-closed subalgebra of $\mathfrak{A}$ and

$$\begin{aligned}L^0(\mathfrak{B})&=\{u\in L^0(\mathfrak{A}),\ [\![u>\alpha]\!]\in\mathfrak{B}\text{ for every }\alpha\in\mathbb{R}\}\\&=\{u\in L^0(\mathfrak{A}),\ [\![u>\alpha]\!]\in\mathfrak{B}_i\text{ for every }\alpha\in\mathbb{R}\text{ and }i\in I\}=\bigcap_{i\in I}L^0(\mathfrak{B}_i).\end{aligned}$$

(iii) For any $u\in L^0(\mathfrak{A})$, the set $\{[\![u\in E]\!]:E\subseteq\mathbb{R}\text{ is Borel}\}$ is a σ-subalgebra of $\mathfrak{A}$, the smallest σ-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ such that $u\in L^0(\mathfrak{B})$; it is the σ-subalgebra generated by $\{[\![u>\alpha]\!]:\alpha\in\mathbb{R}\}$. Following

272C, I will say that it is the σ-subalgebra of $\mathfrak{A}$ defined by $\boldsymbol{u}$. Similarly, if $A \subseteq L^0(\mathfrak{A})$, I will say that the σ-subalgebra of $\mathfrak{A}$ generated by $\{[\![u > \alpha]\!] : u \in A,\ \alpha \in \mathbb{R}\}$ is the σ-subalgebra defined by A.

(f) Let $\mathfrak{C}$ be another Dedekind complete Boolean algebra, and $\phi : \mathfrak{A} \to \mathfrak{C}$ an order-continuous Boolean homomorphism. Then we have a unique order-continuous f-algebra homomorphism $T_\phi : L^0(\mathfrak{A}) \to L^0(\mathfrak{C})$ such that

$[\![T_\phi u > \alpha]\!] = \phi[\![u > \alpha]\!]$ for every $u \in L^0(\mathfrak{A})$ and $\alpha \in \mathbb{R}$,
$T_\phi(\chi a) = \chi(\phi a)$ for every $a \in \mathfrak{A}$,
$[\![T_\phi u \in E]\!] = \phi[\![u \in E]\!]$ for every Borel set $E \subseteq \mathbb{R}$,
$T_\phi \bar{h}_{\mathfrak{A}} = \bar{h}_{\mathfrak{C}} T_\phi$ for every Borel measurable $h : \mathbb{R} \to \mathbb{R}$,
T_ϕ is injective or surjective iff ϕ is

(364P).

612B Products and processes For the rest of this section, and indeed for nearly all the rest of the volume, $\mathfrak{A}$ will be a Dedekind complete Boolean algebra, T a totally ordered set, $\langle \mathfrak{A}_t \rangle_{t \in T}$ a filtration of closed subalgebras of $\mathfrak{A}$, $\mathcal{T}$ the associated lattice of stopping times, and $\langle \mathfrak{A}_\tau \rangle_{\tau \in \mathcal{T}}$ the corresponding family of order-closed subalgebras (611B-611C, 611G). For σ, $\tau \in \mathcal{T}$, $[\![\sigma < \tau]\!]$, $[\![\sigma \leq \tau]\!]$ $[\![\sigma = \tau]\!]$ will be the regions defined in 611D.

(a) If $\mathcal{S}$ is a sublattice of $\mathcal{T}$, we can form the family $\langle L^0(\mathfrak{A}_\sigma) \rangle_{\sigma \in \mathcal{S}}$. If we take the natural product linear space, lattice and multiplicative structures, we get an f-algebra $\prod_{\sigma \in \mathcal{S}} L^0(\mathfrak{A}_\sigma)$ (364R). Moreover, writing H for the semigroup of Borel measurable functions from $\mathbb{R}$ to itself as in 612Ac, we have a natural action of H on $\prod_{\sigma \in \mathcal{S}} L^0(\mathfrak{A}_\sigma)$ defined by setting

$$h\bullet\langle u_\sigma \rangle_{\sigma \in \mathcal{S}} = \langle h\bullet u_\sigma \rangle_{\sigma \in \mathcal{S}}$$

whenever $h \in H$ and $u_\sigma \in L^0(\mathfrak{A}_\sigma)$ for every $\sigma \in \mathcal{S}$.

Writing $\bar{h}(u)$ for $h\bullet u$, as in 612Ac, and thinking of $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ as a function from $\mathcal{S}$ to L^0, we find ourselves with a composition $\bar{h}\boldsymbol{u} = \bar{h} \circ \boldsymbol{u} : \mathcal{S} \to L^0$.

(b) Another way of looking at $\prod_{\sigma \in \mathcal{S}} L^0(\mathfrak{A}_\sigma)$ is to identify it with $L^0(\mathfrak{C})$, where $\mathfrak{C}$ is the simple Boolean algebra product $\prod_{\sigma \in \mathcal{S}} \mathfrak{A}_\sigma$ (315A, 364R). Once again, it is easy to see that if $h \in H$ then $\bar{h}_{\mathfrak{C}} : L^0(\mathfrak{C}) \to L^0(\mathfrak{C})$ matches the function $\boldsymbol{u} \mapsto \bar{h}\boldsymbol{u} : \prod_{\sigma \in \mathcal{S}} L^0(\mathfrak{A}_\sigma) \to \prod_{\sigma \in \mathcal{S}} L^0(\mathfrak{A}_\sigma)$ described in (a).

(c) From (b) and 612A(d-ii), or otherwise, we now see that if V is a subset of $\prod_{\sigma \in \mathcal{S}} L^0(\mathfrak{A}_\sigma)$ such that $\boldsymbol{u} + \boldsymbol{v} \in V$ and $\bar{h}\boldsymbol{u} \in V$ whenever $\boldsymbol{u}$, $\boldsymbol{v} \in V$, $h : \mathbb{R} \to \mathbb{R}$ is convex and $h(0) = 0$, then V is an f-subalgebra of $\prod_{\sigma \in \mathcal{S}} L^0(\mathfrak{A}_\sigma)$.

612C Before going farther, I give the following fragment complementing the results of §611.

Lemma If σ, $\tau \in \mathcal{T}$ and $u \in L^0(\mathfrak{A}_\tau)$ then $u \times \chi[\![\tau \leq \sigma]\!]$ and $u \times \chi[\![\tau = \sigma]\!]$ and $u \times \chi[\![\tau < \sigma]\!]$ belong to $L^0(\mathfrak{A}_{\sigma \wedge \tau})$.

proof It is enough to consider the case $u \geq 0$. In this case, for $\alpha \in \mathbb{R}$,

$$[\![u \times \chi[\![\tau \leq \sigma]\!] > \alpha]\!] = [\![u > \alpha]\!] \cap [\![\tau \leq \sigma]\!] = [\![u > \alpha]\!] \setminus [\![\sigma < \tau]\!] \in \mathfrak{A}_{\sigma \wedge \tau} \text{ if } \alpha \geq 0$$

(611H(c-iii)))

$$= 1 \in \mathfrak{A}_{\sigma \wedge \tau} \text{ if } \alpha < 0,$$

so $u \times \chi[\![\tau \leq \sigma]\!] \in L^0(\mathfrak{A}_{\sigma \wedge \tau})$. As for the other parts, $[\![\tau = \sigma]\!]$ belongs to $\mathfrak{A}_{\sigma \wedge \tau}$ so

$$u \times \chi[\![\tau = \sigma]\!] = u \times \chi[\![\tau \leq \sigma]\!] \times \chi[\![\tau = \sigma]\!]$$

belongs to $L^0(\mathfrak{A}_{\sigma \wedge \tau})$; while $u \times \chi[\![\tau < \sigma]\!]$ is the difference of the other two.

612D Fully adapted processes To continue the real work of this section, let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ a family in $L^0(\mathfrak{A})$.

(a) Definition I will say that $\boldsymbol{u}$ is **fully adapted** to $\langle\mathfrak{A}_t\rangle_{t\in T}$ if $u_\sigma \in L^0(\mathfrak{A}_\sigma)$ and $[\![\sigma=\tau]\!] \subseteq [\![u_\sigma = u_\tau]\!]$ whenever σ, $\tau \in \mathcal{S}$.

(b) Note that if $u_\tau \in L^0(\mathfrak{A}_\tau)$ and $[\![\sigma=\tau]\!] \subseteq [\![u_\sigma = u_\tau]\!]$ whenever $\sigma \le \tau \in \mathcal{S}$, then $\boldsymbol{u}$ is fully adapted. **P** For general σ, $\tau \in \mathcal{S}$ we now have

$$[\![\sigma=\tau]\!] \subseteq [\![\sigma=\sigma\wedge\tau]\!] \cap [\![\tau=\sigma\wedge\tau]\!]$$

(611E(a-ii-β))

$$\subseteq [\![u_\sigma=u_{\sigma\wedge\tau}]\!] \cap [\![u_\tau=u_{\sigma\wedge\tau}]\!] \subseteq [\![u_\sigma=u_\tau]\!]. \text{ **Q**}$$

(c) If $\boldsymbol{u}$ is fully adapted and $\mathcal{S}'$ is a sublattice of $\mathcal{S}$, then of course $\boldsymbol{u}{\restriction}\mathcal{S}'$ is still a fully adapted process.

(d) If $\boldsymbol{u}$ is fully adapted, I is a finite sublattice of $\mathcal{S}$, and $(\tau_0 \ldots, \tau_n)$ linearly generates the I-cells as in 611K-611L, then for any $\sigma \in I$ we have

$$\sup_{i\le n} [\![u_\sigma = u_{\tau_i}]\!] \supseteq \sup_{i\le n} [\![\sigma=\tau_i]\!] = 1$$

by 611Ke. So, for instance, if $\bar u = \sup_{i\le n} u_{\tau_i}$,

$$[\![u_\sigma \le \bar u]\!] \supseteq \sup_{i\le n} [\![u_\sigma = u_{\tau_i}]\!] = 1$$

and $u_\sigma \le \bar u$; thus $\bar u = \sup_{\sigma\in I} u_\sigma$. Similarly, $\sup_{\sigma\in I} |u_\sigma| = \sup_{i\le n} |u_{\tau_i}|$.

(e)(i) Note that if $\boldsymbol{u}$ is constant, say $u_\sigma = z$ for every $\sigma \in \mathcal{S}$, then $\boldsymbol{u}$ is fully adapted iff $z \in \bigcap_{\sigma\in\mathcal{S}} L^0(\mathfrak{A}_\sigma)$; if $\mathcal{S}$ has a least element, this will be so iff $z \in L^0(\mathfrak{A}_{\min\mathcal{S}})$. For any $z \in L^0(\mathfrak{A})$, I will write $z\mathbf{1}$ for the fully adapted process $\langle z\rangle_{\sigma\in\mathcal{S}}$ where $\mathcal{S}$ is the sublattice $\{\sigma : \sigma \in \mathcal{T},\ z \in L^0(\mathfrak{A}_\sigma)\}$. When $z = \chi 1$ and $\mathcal{S} = \mathcal{T}$ I will write just $\mathbf{1}$; similarly, $\mathbf{0}$ will be the constant process with value $0 \in L^0(\mathfrak{A})$.

(ii) Generally, if $z \in L^0(\mathfrak{A})$, I will write $z\boldsymbol{u}$ for the process $z\mathbf{1}\times\boldsymbol{u} = \langle z\times u_\sigma\rangle_{\sigma\in\mathcal{S}'}$, where $\mathcal{S}' = \{\sigma : \sigma \in \mathcal{S},\ z \in L^0(\mathfrak{A}_\sigma)\}$. Then $\mathcal{S}'$ is a sublattice of $\mathcal{S}$ and $z\boldsymbol{u}$ is fully adapted. **P** If σ, $\tau \in \mathcal{S}'$ then $\mathfrak{A}_{\sigma\wedge\tau} = \mathfrak{A}_\sigma \cap \mathfrak{A}_\tau$ and $\mathfrak{A}_{\sigma\vee\tau} \supseteq \mathfrak{A}_\sigma$ (611H(c-ii)), so $L^0(\mathfrak{A}_{\sigma\wedge\tau}) = L^0(\mathfrak{A}_\sigma) \cap L^0(\mathfrak{A}_\tau)$ and $L^0(\mathfrak{A}_{\sigma\vee\tau}) \supseteq L^0(\mathfrak{A}_\sigma)$ (612A(e-ii)) both contain z, and $\sigma\wedge\tau$ and $\sigma\vee\tau$ belong to $\mathcal{S}'$; thus $\mathcal{S}'$ is a sublattice of $\mathcal{S}$. If $\sigma \in \mathcal{S}'$, then z and u_σ both belong to $L^0(\mathfrak{A}_\sigma)$, so $z \times u_\sigma \in L^0(\mathfrak{A}_\sigma)$. If σ, $\tau \in \mathcal{S}'$, then

$$[\![z\times u_\sigma = z\times u_\tau]\!] = [\![z=0]\!] \cup [\![u_\sigma = u_\tau]\!] \supseteq [\![\sigma=\tau]\!],$$

so $z\boldsymbol{u}$ is fully adapted. **Q**

(f) Suppose that $\boldsymbol{u}$ is fully adapted.

(i) $u_{\sigma\wedge\tau} + u_{\sigma\vee\tau} = u_\sigma + u_\tau$ and $u_{\sigma\wedge\tau} \vee u_{\sigma\vee\tau} = u_\sigma \vee u_\tau$ for all σ, $\tau \in \mathcal{S}$. **P**

$$[\![\sigma\le\tau]\!] = [\![\sigma\wedge\tau=\sigma]\!] \cap [\![\sigma\vee\tau=\tau]\!]$$

(611E(a-ii-β))

$$\subseteq [\![u_{\sigma\wedge\tau}=u_\sigma]\!] \cap [\![u_{\sigma\vee\tau}=u_\tau]\!]$$
$$\subseteq [\![u_{\sigma\wedge\tau}+u_{\sigma\vee\tau}=u_\sigma+u_\tau]\!] \cap [\![u_{\sigma\wedge\tau}\vee u_{\sigma\vee\tau}=u_\sigma\vee u_\tau]\!],$$

and similarly

$$[\![\tau\le\sigma]\!] \subseteq [\![u_{\sigma\wedge\tau}+u_{\sigma\vee\tau}=u_\sigma+u_\tau]\!] \cap [\![u_{\sigma\wedge\tau}\vee u_{\sigma\vee\tau}=u_\sigma\vee u_\tau]\!];$$

accordingly

$$[\![u_{\sigma\wedge\tau}+u_{\sigma\vee\tau}=u_\sigma+u_\tau]\!] \cap [\![u_{\sigma\wedge\tau}\vee u_{\sigma\vee\tau}=u_\sigma\vee u_\tau]\!] \supseteq [\![\sigma\le\tau]\!] \cup [\![\tau\le\sigma]\!] = 1$$

and $u_{\sigma\wedge\tau}+u_{\sigma\vee\tau}=u_\sigma+u_\tau$, $u_{\sigma\wedge\tau}\vee u_{\sigma\vee\tau}=u_\sigma\vee u_\tau$. **Q**

(ii) $|u_\tau-u_\sigma|=|u_{\sigma\vee\tau}-u_{\sigma\wedge\tau}|$ for all σ, $\tau\in\mathcal{S}$. **P**

$$\begin{aligned}[\![\sigma\le\tau]\!]&\subseteq[\![u_{\sigma\wedge\tau}=u_\sigma]\!]\cap[\![u_{\sigma\vee\tau}=u_\tau]\!]\\&\subseteq[\![u_{\sigma\vee\tau}-u_{\sigma\wedge\tau}=u_\tau-u_\sigma]\!]\subseteq[\![|u_{\sigma\vee\tau}-u_{\sigma\wedge\tau}|=|u_\tau-u_\sigma|]\!]\end{aligned}$$

and similarly

$$[\![\tau\le\sigma]\!]\subseteq[\![|u_{\sigma\vee\tau}-u_{\sigma\wedge\tau}|=|u_\sigma-u_\tau|]\!]=[\![|u_{\sigma\vee\tau}-u_{\sigma\wedge\tau}|=|u_\tau-u_\sigma|]\!]$$

so $[\![|u_{\sigma\vee\tau}-u_{\sigma\wedge\tau}|=|u_\tau-u_\sigma|]\!]=1$ and $|u_\tau-u_\sigma|=|u_{\sigma\vee\tau}-u_{\sigma\wedge\tau}|$. **Q**

(iii) $|u_\sigma-u_\rho|\le|u_{\sigma\wedge\tau}-u_{\rho\wedge\tau}|+|u_{\sigma\vee\tau}-u_{\rho\vee\tau}|$ for all ρ, σ, $\tau\in\mathcal{S}$. **P**

$$\begin{aligned}[\![\rho\le\tau]\!]\cap[\![\sigma\le\tau]\!]&\subseteq[\![|u_\sigma-u_\rho|=|u_{\sigma\wedge\tau}-u_{\rho\wedge\tau}|]\!]\\&\subseteq[\![|u_\sigma-u_\rho|\le|u_{\sigma\wedge\tau}-u_{\rho\wedge\tau}|+|u_{\sigma\vee\tau}-u_{\rho\vee\tau}|]\!],\end{aligned}$$

$$\begin{aligned}[\![\tau\le\rho]\!]\cap[\![\tau\le\sigma]\!]&\subseteq[\![|u_\sigma-u_\rho|=|u_{\sigma\vee\tau}-u_{\rho\vee\tau}|]\!]\\&\subseteq[\![|u_\sigma-u_\rho|\le|u_{\sigma\wedge\tau}-u_{\rho\wedge\tau}|+|u_{\sigma\vee\tau}-u_{\rho\vee\tau}|]\!],\end{aligned}$$

$$\begin{aligned}[\![\rho\le\tau]\!]\cap[\![\tau\le\sigma]\!]&\subseteq[\![|u_\tau-u_\rho|+|u_\sigma-u_\tau|=|u_{\sigma\wedge\tau}-u_{\rho\wedge\tau}|+|u_{\sigma\vee\tau}-u_{\rho\vee\tau}|]\!]\\&\subseteq[\![|u_\sigma-u_\rho|\le|u_{\sigma\wedge\tau}-u_{\rho\wedge\tau}|+|u_{\sigma\vee\tau}-u_{\rho\vee\tau}|]\!],\end{aligned}$$

and similarly

$$[\![\sigma\le\tau]\!]\cap[\![\rho\le\sigma]\!]\subseteq[\![|u_\sigma-u_\rho|\le|u_{\sigma\wedge\tau}-u_{\rho\wedge\tau}|+|u_{\sigma\vee\tau}-u_{\rho\vee\tau}|]\!].$$

Assembling these,

$$[\![|u_\sigma-u_\rho|\le|u_{\sigma\wedge\tau}-u_{\rho\wedge\tau}|+|u_{\sigma\vee\tau}-u_{\rho\vee\tau}|]\!]\supseteq1,$$

that is, $|u_\sigma-u_\rho|\le|u_{\sigma\wedge\tau}-u_{\rho\wedge\tau}|+|u_{\sigma\vee\tau}-u_{\rho\vee\tau}|$. **Q**

612E Where fully adapted processes come from In applications, one commonly starts from a family $\langle X_t\rangle_{t\in T}$ of random variables, corresponding to a family $\langle u_t\rangle_{t\in T}\in\prod_{t\in T}L^0(\mathfrak{A}_t)$. In Chapter 63 I will look at general rules for converting such families into fully adapted processes. For the moment, I describe a couple of special cases. Another of the same kind is in 612R.

(a) If T is finite and not empty, with least value $\min T$, then for $\tau\in\mathcal{T}$ and $t\in T$ set

$$a_{\tau t}=(\inf\nolimits_{s<t}[\![\tau>s]\!])\setminus[\![\tau>t]\!]$$

(counting $\inf\emptyset$ as 1, as usual, so that $a_{\tau,\min T}=1\setminus[\![\tau>\min T]\!]$). Then $\langle a_{\tau t}\rangle_{t\in T}$ is a partition of unity in $\mathfrak{A}$, and $a_{\tau t}\in\mathfrak{A}_t$ for every t. Now set

$$u'_\tau=\sum\nolimits_{t\in T}u_t\times\chi a_{\tau t}.$$

If $t\in T$ and $\alpha\in\mathbb{R}$ then

$$[\![u'_\tau>\alpha]\!]\setminus[\![\tau>t]\!]=\sup_{s\in T}[\![u_s>\alpha]\!]\cap a_{\tau s}\setminus[\![\tau>t]\!]=\sup_{s\le t}[\![u_s>\alpha]\!]\cap a_{\tau s}\setminus[\![\tau>t]\!]$$

(because $a_{\tau s}\subseteq[\![\tau>t]\!]$ if $s>t$)

$$\in\mathfrak{A}_t$$

because $[\![\tau>t]\!]\in\mathfrak{A}_t$ and $[\![u_s>\alpha]\!]\cap a_{\tau s}\in\mathfrak{A}_s\subseteq\mathfrak{A}_t$ for $s\le t$. As t is arbitrary, $[\![u'_\tau>\alpha]\!]\in\mathfrak{A}_\tau$; as α is arbitrary, $u'_\tau\in L^0(\mathfrak{A}_\tau)$.

If σ, $\tau\in\mathcal{T}$ and $[\![\sigma=\tau]\!]=a$, then $a\cap[\![\sigma>t]\!]=a\cap[\![\tau>t]\!]$ for every t (611E(a-i-β), $a\cap a_{\sigma t}=a\cap a_{\tau t}$ for every t, $u'_\sigma\times\chi a=u'_\tau\times\chi a$ and $a\subseteq[\![u'_\sigma=u'_\tau]\!]$. So we have a fully adapted family.

If $\tau=\check t$ then $a_{\check t,t}=1$ so $u'_{\check t}=u_t$, and we have the required correspondence between $\langle u'_\tau\rangle_{\tau\in\mathcal{T}}$ and $\langle u_t\rangle_{t\in T}$.

(b) If T is well-ordered and not empty, we can use essentially the same formulae for $\tau \in \mathcal{T}_f$. We need to check that $\langle a_{\tau t}\rangle_{t\in T}$ is a partition of unity. **P** Certainly it is disjoint. If $a \in \mathfrak{A} \setminus \{0\}$, then there is a $t \in T$ such that $a' = a \setminus [\![\tau > t]\!]$ is non-zero, because τ is finite-valued; because T is well-ordered, we may suppose that t is minimal; now $0 \ne a' \subseteq a \cap a_{\tau t}$. **Q**

In defining u'_τ, if you do not like an infinite sum, simply declare that $[\![u'_\tau > \alpha]\!] = \sup_{t\in T} [\![u_t > \alpha]\!] \cap a_{\tau t}$ for every $\alpha \in \mathbb{R}$; then the rest of the argument proceeds as before.

612F The identity process In the leading special cases of this theory, in which $T = [0,\infty[$, we have special processes based on the similarity between the formula defining 'stopping time' in 611A and that defining L^0 in 364A.

(a) Suppose that $T = [0,\infty[$. For $\tau \in \mathcal{T}_f$, we can define $\iota_\tau \in L^0(\mathfrak{A})$ by saying that, for $t \in \mathbb{R}$,

$$\begin{aligned}[\![\iota_\tau > t]\!] &= [\![\tau > t]\!] \text{ if } t \ge 0,\\ &= 1 \text{ if } t < 0.\end{aligned}$$

P Since no member of T is isolated on the right, the conditions of 611A(b-i) imply that

$[\![\iota_\tau > t]\!] \in \mathfrak{A}$ for every $t \in \mathbb{R}$,

$[\![\iota_\tau > t]\!] = \sup_{s>t} [\![\iota_\tau > s]\!]$ for every $t \in \mathbb{R}$,

$\sup_{t\in\mathbb{R}} [\![\iota_\tau > t]\!] = 1$.

Since $\tau \in \mathcal{T}_f$, we also have

$\inf_{t\in\mathbb{R}} [\![\iota_\tau > t]\!] = 0$.

So all the conditions (α)-(γ) of 364Aa are satisfied, and $\iota_\tau \in L^0(\mathfrak{A})$. **Q**

(b) $\boldsymbol{\iota} = \langle \iota_\tau\rangle_{\tau\in\mathcal{T}_f}$ is a fully adapted process. **P** If $\tau \in \mathcal{T}_f$,

$$\begin{aligned}[\![\iota_\tau > s]\!] &= 1 \in \mathfrak{A}_\tau \text{ if } s < 0,\\ &= [\![\tau > s]\!] \in \mathfrak{A}_\tau \text{ if } s \ge 0,\end{aligned}$$

so $\iota_\tau \in L^0(\mathfrak{A}_\tau)$. If σ, $\tau \in \mathcal{T}_f$ and $a = [\![\sigma = \tau]\!]$, then

$$\begin{aligned}[\![\chi a \times \iota_\sigma > t]\!] &= a \cap [\![\iota_\tau > t]\!] = a \cap [\![\tau > t]\!] = a \cap [\![\sigma > t]\!] = [\![\chi a \times \iota_\sigma > t]\!] \text{ if } t \ge 0,\\ &= 1 = [\![\chi a \times \iota_\sigma > t]\!] \text{ if } t < 0,\end{aligned}$$

so $\chi a \times \iota_\sigma = \chi a \times \iota_\tau$ and $a \subseteq [\![\iota_\sigma = \iota_\tau]\!]$. **Q**

(c) $\iota_{\check t} = t\chi 1$ for every $t \ge 0$. **P** If $s \ge 0$,

$$\begin{aligned}[\![\iota_{\check t} > s]\!] = [\![\check t > s]\!] &= \chi 1 = [\![t\chi 1 > s]\!] \text{ if } t > s,\\ &= 0 = [\![t\chi 1 > s]\!] \text{ otherwise. } \mathbf{Q}\end{aligned}$$

I will call $\boldsymbol{\iota}$ the **identity process** for the structure $(\mathfrak{A}, \langle\mathfrak{A}_t\rangle_{t\ge0})$.

612G However, the fully adapted families of greatest importance to us will be those which can be constructed in the following way. I repeat some definitions from 455L.

Definitions Let (Ω, Σ, μ) be a probability space.

(a) A family $\langle\Sigma_t\rangle_{t\ge0}$ of σ-subalgebras of Σ is a **filtration** if $\Sigma_s \subseteq \Sigma_t$ whenever $0 \le s \le t$.

(b) A function $h : \Omega \to [0,\infty]$ is a **stopping time** (adapted to the filtration $\langle\Sigma_t\rangle_{t\ge0}$) if $\{\omega : h(\omega) \le t\} \in \Sigma_t$ for every $t \ge 0$. In this case, I will write Σ_h for the σ-algebra

$$\{E : E \in \Sigma,\ E \cap \{\omega : h(\omega) \le t\} \in \Sigma_t \text{ for every } t \ge 0\}$$

(see 455L(c-iii)).

(c) A family $\langle X_t\rangle_{t\ge0}$ of real-valued functions on Ω is a **progressively measurable process** (with respect to the filtration $\langle\Sigma_t\rangle_{t\ge0}$) if $(s,\omega) \mapsto X_s(\omega) : [0,t] \times \Omega \to Y$ is $\mathcal{B}([0,t])\widehat{\otimes}\Sigma_t$-measurable for every $t \ge 0$, where $\mathcal{B}([0,t])$ is the Borel σ-algebra of $[0,t]$ for each t.

612H Theorem Let (Ω,Σ,μ) be a complete probability space, and $\langle\Sigma_t\rangle_{t\ge 0}$ a filtration of σ-subalgebras of Σ such that every μ-negligible set belongs to every Σ_t. Let $(\mathfrak{A},\bar\mu)$ be the measure algebra of μ and set $\mathfrak{A}_t=\{E^\bullet:E\in\Sigma_t\}$ for each $t\ge 0$; then $\langle\mathfrak{A}_t\rangle_{t\ge 0}$ is a filtration in the sense of 611Aa. Let $\mathcal{T}$ be the associated family of stopping times.

(a)(i) If $h:\Omega\to[0,\infty]$ is a stopping time, we have a stopping time $\tau\in\mathcal{T}$ defined by saying that $[\![\tau>t]\!]=\{\omega:h(\omega)>t\}^\bullet$ for every $t\ge 0$; in this case, I will say that h **represents** τ.

(ii) Conversely, if $\tau\in\mathcal{T}$, there is a stopping time $h:\Omega\to[0,\infty]$ representing τ.

(iii) If h represents τ, then $\Sigma_h=\{E:E\in\Sigma,\,E^\bullet\in\mathfrak{A}_\tau\}$ and $\mathfrak{A}_\tau=\{E^\bullet:E\in\Sigma_h\}$.

(iv) If g, $h:\Omega\to[0,\infty]$ are stopping times representing σ, $\tau\in\mathcal{T}$, then

$$[\![\sigma<\tau]\!]=\{\omega:g(\omega)<h(\omega)\}^\bullet,$$

$$[\![\sigma\le\tau]\!]=\{\omega:g(\omega)\le h(\omega)\}^\bullet,\quad [\![\sigma=\tau]\!]=\{\omega:g(\omega)=h(\omega)\}^\bullet.$$

So g and h represent the same member of $\mathcal{T}$ iff they are equal almost everywhere.

(v) If h represents τ, then $\tau\in\mathcal{T}_f$ iff $h(\omega)<\infty$ for almost every ω; so $\tau\in\mathcal{T}_f$ iff it can be represented by a stopping time $h:\Omega\to[0,\infty[$.

(vi) If $t\ge 0$, then the constant function with value t represents the constant stopping time at t.

(b) Now suppose that $\langle X_t\rangle_{t\ge 0}$ is a progressively measurable process on Ω.

(i) For every $\tau\in\mathcal{T}_f$ we have an $x_\tau\in L^0(\mathfrak{A})\cong L^0(\mu)$ defined by saying that x_τ is the equivalence class of the function X_h, where $X_h(\omega)=X_{h(\omega)}(\omega)$ for $\omega\in h^{-1}[\,[0,\infty[\,]$, whenever h represents τ in the sense of (a-ii).

(ii) The family $\langle x_\tau\rangle_{\tau\in\mathcal{T}_f}$ is fully adapted to $\langle\mathfrak{A}_t\rangle_{t\ge 0}$.

proof (a)(i) If $h:\Omega\mapsto[0,\infty]$ is a stopping time, and $t\ge 0$, then $\{\omega:h(\omega)>t\}^\bullet$ belongs to $\mathfrak{A}_t$ and is equal to

$$\begin{aligned}(\bigcup_{s\in\mathbb{Q},s>t}\{\omega:h(\omega)>s\})^\bullet&=\sup_{s\in\mathbb{Q},s>t}\{\omega:h(\omega)>s\}^\bullet\\&=\sup_{t'>t}\{\omega:h(\omega)>t'\}^\bullet.\end{aligned}$$

So $t\mapsto\{\omega:h(\omega)>t\}^\bullet$ satisfies the conditions of 611A(b-i), and defines a member of $\mathcal{T}$.

(ii) Write $\mathbb{Q}^+$ for $\mathbb{Q}\cap[0,\infty[$. Take any $\tau\in\mathcal{T}$. For each $s\in\mathbb{Q}^+$, $[\![\tau>s]\!]\in\mathfrak{A}_s$, so we can choose $E_s\in\Sigma_s$ such that $E_s^\bullet=[\![\tau>s]\!]$. Next, for $s\in\mathbb{Q}^+$ set $F_s=\bigcap_{s'\in\mathbb{Q}\cap[0,s]}E_{s'}$; because $\mathbb{Q}$ is countable, $F_s^\bullet=\inf_{s'\in\mathbb{Q}\cap[0,s]}[\![\tau>s']\!]=[\![\tau>s]\!]$, while $\langle F_s\rangle_{s\in\mathbb{Q}^+}$ is non-increasing. Now, for $\omega\in\Omega$, set $h(\omega)=\sup\{s:s\in\mathbb{Q}^+,\,\omega\in F_s\}$, counting $\sup\emptyset$ as 0. Then, for any $t\ge 0$,

$$\begin{aligned}\{\omega:h(\omega)>t\}^\bullet&=(\bigcup_{s\in\mathbb{Q}^+,s>t}F_s)^\bullet=\sup_{s\in\mathbb{Q}^+,s>t}F_s^\bullet\\&=\sup_{s\in\mathbb{Q}^+,s>t}[\![\tau>s]\!]=\sup_{s>t}[\![\tau>s]\!]=[\![\tau>t]\!]\in\mathfrak{A}_t.\end{aligned}$$

So there is an $E\in\Sigma_t$ such that $\{\omega:h(\omega)>t\}\triangle E$ is negligible; as Σ_t contains every negligible set, $\{\omega:h(\omega)>t\}$ and $\{\omega:h(\omega)\le t\}$ belong to Σ_t. This is true for every $t\ge 0$, so h is a stopping time in the conventional sense; and as $\{\omega:h(\omega)>t\}^\bullet=[\![\tau>t]\!]$ for every t, h represents τ in the sense here.

(iii) If $E\in\Sigma_h$, then

$$E\setminus\{\omega:h(\omega)>t\}=E\cap\{\omega:h(\omega)\le t\}\in\Sigma_t,$$

$$E^\bullet\setminus[\![\tau>t]\!]=(E\setminus\{\omega:h(\omega)>t\})^\bullet\in\mathfrak{A}_t$$

for every $t\ge 0$, and $E^\bullet\in\mathfrak{A}_\tau$. Conversely, if $a\in\mathfrak{A}_\tau$, take $E\in\Sigma$ such that $E^\bullet=a$. Then, for any $t\ge 0$,

$$(E\cap\{\omega:h(\omega)\le t\})^\bullet=E^\bullet\setminus[\![\tau>t]\!]\in\mathfrak{A}_t$$

so there is an $F\in\Sigma_t$ such that $\mu((E\cap\{\omega:h(\omega)>t\})\triangle F)=0$; again because Σ_t contains all negligible sets, $E\cap\{\omega:h(\omega)>t\}$ belongs to Σ_t. As t is arbitrary, $E\in\Sigma_h$ and we have a suitable representative of a.

(iv) The point is that $[\![\sigma<\tau]\!]\subseteq\sup_{s\in\mathbb{Q}^+}[\![\tau>s]\!]\setminus[\![\sigma>s]\!]$. **P** For any $t\ge 0$, $t=\inf\{s:s\in\mathbb{Q},\,s\ge t\}$, so

$$[\![\tau>t]\!]\setminus[\![\sigma>t]\!]\subseteq\sup_{s\in\mathbb{Q},s\ge t}[\![\tau>s]\!]\setminus[\![\sigma>t]\!]\subseteq\sup_{s\in\mathbb{Q}^+}[\![\tau>s]\!]\setminus[\![\sigma>s]\!].\ \mathbf{Q}$$

So

$$\begin{aligned}[\![\sigma<\tau]\!]&=\sup_{t\ge 0}[\![\tau>t]\!]\setminus[\![\sigma>t]\!]=\sup_{s\in\mathbb{Q}^+}[\![\tau>s]\!]\setminus[\![\sigma>s]\!]\\&=\sup_{s\in\mathbb{Q}^+}(\{\omega:h(\omega)>s\}\setminus\{\omega:g(\omega)>s\})^\bullet\\&=(\bigcup_{s\in\mathbb{Q}^+}\{\omega:h(\omega)>s\}\setminus\{\omega:g(\omega)>s\})^\bullet=\{\omega:g(\omega)<h(\omega)\}^\bullet.\end{aligned}$$

It follows at once that

$$[\![\sigma\le\tau]\!]=1\setminus[\![\tau<\sigma]\!]=(\{\omega:g(\omega)\le h(\omega)\})^\bullet$$

and therefore that

$$[\![\sigma=\tau]\!]=(\{\omega:g(\omega)=h(\omega)\})^\bullet,$$

so that

$$\sigma=\tau\iff[\![\sigma=\tau]\!]=1\iff g=_{\text{a.e.}}h.$$

(v)

$$\begin{aligned}\tau\in\mathcal{T}_f&\iff\inf_{t\ge 0}[\![\tau>t]\!]=0\\&\iff\inf_{s\in\mathbb{Q}^+}[\![\tau>s]\!]=0\\&\iff\bigcap_{s\in\mathbb{Q}^+}\{\omega:h(\omega)>s\}\text{ is negligible,}\\&\iff\{\omega:h(\omega)=\infty\}\text{ is negligible.}\end{aligned}$$

In this case, because we are supposing that negligible sets belong to Σ_t for every t, we can adjust h on the negligible set $\Omega\setminus h^{-1}[\,[0,\infty[\,]$, if necessary, to get a stopping time with finite values which represents τ.

(vi) Immediate from the definition of 'constant stopping time' in 611A(b-ii).

(b) For the identification of $L^0(\mathfrak{A})$ with $L^0(\mu)$, see 364Ic. Note that as we are assuming that every negligible set belongs to every Σ_t, $(\Omega,\Sigma_t,\mu\restriction\Sigma_t)$ is a complete probability space and Σ_t is closed under Souslin's operation (431A), for every $t\ge 0$.

(i) By (a-v), every $\tau\in\mathcal{T}_f$ can be represented by a stopping time which takes finite values. If $h:\Omega\to[0,\infty[$ represents τ, then X_h is defined everywhere; because the process is progressively measurable, X_h is Σ_h-measurable (455Le), and $X_h^\bullet$ is defined in $L^0(\mu\restriction\Sigma_h)\cong L^0(\mathfrak{A}_\tau)$ ((a-iii) above).

We note also from (a-iv) that if g, h are stopping times both representing τ, then $g=_{\text{a.e.}}h$ so $X_g=_{\text{a.e.}}X_h$ and $X_g^\bullet=X_h^\bullet$. So we have a well-defined member x_τ of $L^0(\mathfrak{A})$.

(ii) We have already seen that $x_\tau\in L^0(\mathfrak{A}_\tau)$ for every $\tau\in\mathcal{T}_f$. If σ, $\tau\in\mathcal{T}_f$ are represented by stopping times g and h, then by (a-iv)

$$[\![\sigma=\tau]\!]=\{\omega:g(\omega)=h(\omega)\}^\bullet\subseteq\{\omega:X_g(\omega)=X_h(\omega)\}^\bullet=[\![X_g^\bullet=X_h^\bullet]\!]=[\![x_\sigma=x_\tau]\!].$$

So both conditions of 612Da are satisfied and $\langle x_\tau\rangle_{\tau\in\mathcal{T}_f}$ is fully adapted to $\langle\mathfrak{A}_t\rangle_{t\ge 0}$.

Remarks There will be a variation on this result in 649H.

Note that the representation of stopping times in (a) above corresponds to an identification of $\mathcal{T}_f$ with the set

$$\{u:u\ge 0,\ [\![u>t]\!]\in\mathfrak{A}_t\text{ for every }t\ge 0\}\subseteq L^0(\mathfrak{A}),$$

and that the lattice operations $\vee$, $\wedge$ on $\mathcal{T}_f$ agree with those on $L^0(\mathfrak{A})$, as do arbitrary suprema (611C(b)-(c), 364L(a-ii) and 364L(b-i)).

612I We do not need to spend much time on general fully adapted processes, but the following elementary facts are fundamental.

Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $M_{\text{fa}}(\mathcal{S}) \subseteq \prod_{\sigma\in\mathcal{S}} L^0(\mathfrak{A}_\sigma)$ the set of fully adapted processes with domain $\mathcal{S}$.

(a) $M_{\text{fa}}(\mathcal{S})$ is an order-closed f-subalgebra of the f-algebra $L^0(\mathfrak{A})^{\mathcal{S}}$, and if $h : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function then $\bar{h}\boldsymbol{u} \in M_{\text{fa}}(\mathcal{S})$ for every $\boldsymbol{u} \in M_{\text{fa}}(\mathcal{S})$. Regarded as a Riesz space in its own right, $M_{\text{fa}}(\mathcal{S})$ is Dedekind complete.

(b) Suppose that $\langle u_\sigma\rangle_{\sigma\in\mathcal{S}} \in M_{\text{fa}}(\mathcal{S})$ and $\tau \in \mathcal{T}$. Set $\mathcal{S}' = \{\sigma : \sigma \in \mathcal{T}, \sigma \wedge \tau \in \mathcal{S}\}$. Then $\langle u_{\sigma\wedge\tau}\rangle_{\sigma\in\mathcal{S}'} \in M_{\text{fa}}(\mathcal{S}')$.

proof (a)(i) If $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ belong to $M_{\text{fa}}(\mathcal{S})$, then

— $u_\sigma + v_\sigma \in L^0(\mathfrak{A}_\sigma)$ for every $\sigma \in \mathcal{S}$,

— if $\sigma, \tau \in \mathcal{S}$, then

$$[\![u_\sigma + v_\sigma = u_\tau + v_\tau]\!] \supseteq [\![u_\sigma = v_\sigma]\!] \cap [\![u_\tau = v_\tau]\!] \supseteq [\![\sigma = \tau]\!].$$

So $\boldsymbol{u} + \boldsymbol{v} = \langle u_\sigma + v_\sigma\rangle_{\sigma\in\mathcal{S}} \in M_{\text{fa}}(\mathcal{S})$.

Similarly, if $h : \mathbb{R} \to \mathbb{R}$ Borel measurable,

$$[\![\bar{h}(u_\sigma) = \bar{h}(u_\tau)]\!] \supseteq [\![\sigma = \tau]\!]$$

for all $\sigma, \tau \in \mathcal{S}$. **P** Set $c = [\![u_\sigma = u_\tau]\!]$, and write $\mathfrak{A}_c$ for the principal ideal of $\mathfrak{A}$ generated by c. Setting $\phi a = a \cap c$ for $a \in \mathfrak{A}$, we have an order-continuous Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{A}_c$, and the corresponding Riesz homomorphism T_ϕ from $L^0(\mathfrak{A})$ to $L^0(\mathfrak{A}_c)$ is defined by saying that $[\![T_\phi w > \alpha]\!] = [\![w > \alpha]\!] \cap c$ for every $w \in L^0(\mathfrak{A})$ (364P). But this means that $T_\phi u_\sigma = T_\phi u_\tau$, so

$$T_\phi \bar{h}(u_\sigma) = \bar{h}(T_\phi u_\sigma) = \bar{h}(T_\phi u_\tau) = T_\phi \bar{h}(u_\tau)$$

(612Af), and

$$[\![\bar{h}(u_\sigma) = \bar{h}(u_\tau)]\!] \supseteq c \supseteq [\![\sigma = \tau]\!].\ \mathbf{Q}$$

Thus $\bar{h}\boldsymbol{u} \in M_{\text{fa}}(\mathcal{S})$ for every $\boldsymbol{u} \in M_{\text{fa}}(\mathcal{S})$.

Putting these together, $M_{\text{fa}}(\mathcal{S})$ is an f-subalgebra of $\prod_{\sigma\in\mathcal{S}} L^0(\mathfrak{A}_\sigma)$, by 612Bc.

(ii) To see that $M_{\text{fa}}(\mathcal{S})$ is order-closed in $L^0(\mathfrak{A})^{\mathcal{S}}$, take a non-empty set $A \subseteq M_{\text{fa}}(\mathcal{S})$ with a supremum $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ in $L^0(\mathfrak{A})^{\mathcal{S}}$. For each $\sigma \in \mathcal{S}$, $v_\sigma = \sup_{\boldsymbol{u}\in A} u_\sigma$ belongs to $L^0(\mathfrak{A}_\sigma)$ (612Ae). If $\sigma, \tau \in \mathcal{S}$, set $a = [\![\sigma = \tau]\!]$; then $u_\sigma \times \chi a = u_\tau \times \chi a$ for every $\boldsymbol{u} \in A$ (taking $\boldsymbol{u}$ to be $\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ for every $\boldsymbol{u}$). Because multiplication by χa is order-continuous in $L^0(\mathfrak{A})$ (353Pa),

$$v_\sigma \times \chi a = \sup_{\boldsymbol{u}\in A} u_\sigma \times \chi a = \sup_{\boldsymbol{u}\in A} u_\tau \times \chi a = v_\tau \times \chi a,$$

and $[\![v_\sigma = v_\tau]\!] \supseteq a$. As σ and τ are arbitrary, $\boldsymbol{v} \in M_{\text{fa}}(\mathcal{S})$.

Similarly, or applying the argument above to $\{-\boldsymbol{u} : \boldsymbol{u} \in A\}$, $\inf A \in M_{\text{fa}}(\mathcal{S})$ whenever A is a non-empty subset of $M_{\text{fa}}(\mathcal{S})$ with an infimum in $L^0(\mathfrak{A})^{\mathcal{S}}$. So $M_{\text{fa}}(\mathcal{S})$ is order-closed in $L^0(\mathfrak{A})^{\mathcal{S}}$ in the sense of 313Da.

(iii) We know that $L^0(\mathfrak{A})$ is Dedekind complete. For every $\sigma \in \mathcal{S}$, $L^0(\mathfrak{A}_\sigma)$ is order-closed in $L^0(\mathfrak{A})$ (612Ae), so in itself is Dedekind complete (353K(b-ii)). So the product $\prod_{\sigma\in\mathcal{S}} L^0(\mathfrak{A}_\sigma)$ is Dedekind complete (315D(e-i)) and its order-closed subspace $M_{\text{fa}}(\mathcal{S})$ is Dedekind complete.

(b) Because $\mathcal{T}$ is a distributive lattice, $\mathcal{S}'$ is a sublattice of $\mathcal{T}$. For any $\sigma \in \mathcal{S}'$, $u_{\sigma\wedge\tau} \in L^0(\mathfrak{A}_{\sigma\wedge\tau}) \subseteq L^0(\mathfrak{A}_\sigma)$ (611H(c-ii)). If $\sigma, \sigma' \in \mathcal{S}'$, then by 611E(c-v-$\alpha$)

$$[\![\sigma = \sigma']\!] \subseteq [\![\sigma \wedge \tau = \sigma' \wedge \tau]\!] \subseteq [\![u_{\sigma\wedge\tau} = u_{\sigma'\wedge\tau}]\!].$$

So $\langle u_{\sigma\wedge\tau}\rangle_{\sigma\in\mathcal{S}'}$ is fully adapted.

612J Simple processes (a) Definition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. A fully adapted process $\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is **simple** if either $\mathcal{S}$ is empty or there are $\tau_0 \le \ldots \le \tau_n$ in $\mathcal{S}$ and $u_* \in L^0(\bigcap_{\sigma\in\mathcal{S}} \mathfrak{A}_\sigma)$ such that for every $\sigma \in \mathcal{S}$

$$[\![\sigma < \tau_0]\!] \subseteq [\![u_\sigma = u_*]\!], \quad [\![\tau_n \le \sigma]\!] \subseteq [\![u_\sigma = u_{\tau_n}]\!],$$

$$[\![\tau_i \le \sigma]\!] \cap [\![\sigma < \tau_{i+1}]\!] \subseteq [\![u_\sigma = u_{\tau_i}]\!] \text{ for every } i < n.$$

In this case I will say that $(\tau_0, \ldots, \tau_n)$ is a **breakpoint string** for $\boldsymbol{u}$.

(b) As a particularly elementary example, if $\mathcal{S}$ is a sublattice of $\mathcal{T}$ and $\tau \in \mathcal{S}$, then $\boldsymbol{u} = \langle \chi[\![\tau \le \sigma]\!]\rangle_{\sigma\in\mathcal{S}}$ is a simple process. **P** Setting $u_\sigma = \chi[\![\tau \le \sigma]\!]$ for $\sigma \in \mathcal{S}$ and $u_* = 0$, we have $u_\tau = \chi 1$ and

$$[\![\sigma < \tau]\!] = [\![u_\sigma = 0]\!] = [\![u_\sigma = u_*]\!], \quad [\![\tau \le \sigma]\!] = [\![u_\sigma = \chi 1]\!] = [\![u_\sigma = u_\tau]\!]$$

for every $\sigma \in \mathcal{S}$, so (τ) is a breakpoint string for $\boldsymbol{u}$. **Q**

Warning! most authors use the phrase 'simple process' for what I call a 'previsibly simple process'; see 612Ye. As we shall see shortly, a simple process will normally have many breakpoint strings. The following ideas will be useful.

612K Lemma Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$. Write $\mathfrak{B}$ for $\bigcap_{\sigma\in\mathcal{S}}\mathfrak{A}_\sigma$.

(a) Suppose that $\tau_0 \le \ldots \le \tau_n$ in $\mathcal{S}$, $u_i \in L^0(\mathfrak{A}_{\tau_i})$ for $i \le n$ and $u_* \in L^0(\mathfrak{B})$. Then there is a unique simple fully adapted process $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ such that whenever $\sigma \in \mathcal{S}$ then

$$[\![v_\sigma = u_i]\!] \supseteq [\![\tau_i \le \sigma]\!] \cap [\![\sigma < \tau_{i+1}]\!],$$

for $i < n$, while

$$[\![v_\sigma = u_*]\!] \supseteq [\![\sigma < \tau_0]\!], \quad [\![v_\sigma = u_n]\!] \supseteq [\![\tau_n \le \sigma]\!];$$

and $(\tau_0, \ldots, \tau_n)$ is a breakpoint string for $\boldsymbol{v}$.

(b) Suppose that I is a non-empty finite sublattice of $\mathcal{S}$ and $(\tau_0, \ldots, \tau_n)$ linearly generates the I-cells. If a simple process $\boldsymbol{u}$ with domain $\mathcal{S}$ has a breakpoint string in I, then $(\tau_0, \ldots, \tau_n)$ is a breakpoint string for $\boldsymbol{u}$.

(c) Suppose that K is a finite set and $\boldsymbol{u}_k$ is a simple process with domain $\mathcal{S}$ for each $k \in K$. Then there is a single string $(\tau_0, \ldots, \tau_n)$ in $\mathcal{S}$ which is a breakpoint string for every $\boldsymbol{u}_k$.

(d) Suppose that $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a simple process with breakpoint string $(\tau_0, \ldots, \tau_n)$ in $\mathcal{S}$, and $\tau \in \mathcal{S}$.

(i) $(\tau_0 \wedge \tau, \ldots, \tau_0 \wedge \tau_n, \tau, \tau_0 \vee \tau, \ldots, \tau_n \vee \tau)$ is a breakpoint string for $\boldsymbol{u}$.

(ii) Writing $\mathcal{S} \wedge \tau$ for $\{\sigma \wedge \tau : \sigma \in \mathcal{S}\} = \mathcal{S} \cap [\min\mathcal{T}, \tau]$, $\boldsymbol{u}\restriction\mathcal{S} \wedge \tau$ is simple, with breakpoint string $(\tau_0 \wedge \tau, \ldots, \tau_n \wedge \tau, \tau)$.

(iii) Writing $\mathcal{S} \vee \tau$ for $\{\sigma \vee \tau : \sigma \in \mathcal{S}\} = \mathcal{S} \cap [\tau, \max\mathcal{T}]$, $\boldsymbol{u}\restriction\mathcal{S} \vee \tau$ is simple, with breakpoint string $(\tau_0 \vee \tau, \ldots, \tau_n \vee \tau)$.

(e) Suppose that $\boldsymbol{u}$ is a fully adapted process with domain $\mathcal{S}$, and that $\tau \in \mathcal{S}$. If $\boldsymbol{u}\restriction\mathcal{S} \wedge \tau$ and $\boldsymbol{u}\restriction\mathcal{S} \vee \tau$ are simple processes with breakpoint strings $(\tau_0, \ldots, \tau_m)$ and $(\tau'_0, \ldots, \tau'_n)$ respectively, then $\boldsymbol{u}$ is simple, with breakpoint string $(\tau_0, \ldots, \tau_m, \tau, \tau'_0, \ldots, \tau'_n)$.

proof (a) Set $\tau_{-1} = \min\mathcal{T}$, $u_{-1} = u_*$. For $\sigma \in \mathcal{S}$, set $b_{\sigma i} = [\![\tau_i \le \sigma]\!] \cap [\![\sigma < \tau_{i+1}]\!]$ for $-1 \le i < n$, and $b_{\sigma n} = [\![\tau_n \le \sigma]\!]$; by 611H(c-i), $b_{\sigma i} \in \mathfrak{A}_\sigma$ for each i. Set

$$v_\sigma = \sum_{i=-1}^n u_i \times \chi b_{\sigma i}.$$

Then $v_\sigma \in L^0(\mathfrak{A}_\sigma)$. **P** For each i, $u_i \times \chi[\![\tau_i \le \sigma]\!] \in L^0(\mathfrak{A}_\sigma)$ (612C), so $u_i \times \chi([\![\tau_i \le \sigma]\!] \cap [\![\sigma < \tau_{i+1}]\!])$ belongs to $L^0(\mathfrak{A}_\sigma)$ if $i < n$, while also $u_n \times \chi[\![\tau_n \le \sigma]\!]$ belongs to $L^0(\mathfrak{A}_\sigma)$. **Q**

If σ, $\sigma' \in \mathcal{S}$ and $a = [\![\sigma = \sigma']\!]$, then $a \cap b_{\sigma i} = a \cap b_{\sigma' i}$ for each i (611E(c-iv-α) and (c-iv-β) twice), so

$$v_\sigma \times \chi a = \sum_{i=-1}^n u_i \times \chi(a \cap b_{\sigma i}) = v_{\sigma'} \times \chi a.$$

Thus $a \subseteq [\![v_\sigma = v_{\sigma'}]\!]$. As σ and σ' are arbitrary, $\boldsymbol{v}$ is fully adapted.

The definition of $\boldsymbol{v}$ makes it plain that, for any $\sigma \in \mathcal{S}$,

$$[\![\tau_n \le \sigma]\!] = b_{\sigma_n} \subseteq [\![v_\sigma = u_n]\!],$$

while for $-1 \le i < n$,

$$[\![\tau_i \le \sigma]\!] \cap [\![\sigma < \tau_{i+1}]\!] = b_{\sigma i} \subseteq [\![v_\sigma = u_i]\!].$$

Now we see that $[\![\tau_i < \tau_{i+1}]\!] = b_{\tau_i,i} \subseteq [\![v_{\tau_i} = u_i]\!]$ for $-1 \le i < n$, while $b_{\tau_n,n} = 1$ and $v_{\tau_n} = u_n$. So for any $\sigma \in \mathcal{S}$,

$$[\![\tau_i \le \sigma]\!] \cap [\![\sigma < \tau_{i+1}]\!] \subseteq [\![v_\sigma = u_i]\!] \cap [\![\tau_i < \tau_{i+1}]\!]$$

(611E(c-iii-γ))

$$\subseteq [\![v_\sigma = u_i]\!] \cap [\![v_{\tau_i} = u_i]\!] \subseteq [\![v_\sigma = v_{\tau_i}]\!]$$

whenever $-1 \le i < n$, while

$$[\![\tau_n \le \sigma]\!] \subseteq [\![v_\sigma = u_n]\!] = [\![v_\sigma = v_{\tau_n}]\!].$$

So $\boldsymbol{v}$ is a simple process and $(\tau_0, \dots, \tau_n)$ is a breakpoint string for $\boldsymbol{v}$.

Concerning the uniqueness of $\boldsymbol{v}$, we just have to note that for every $\sigma \in \mathcal{S}$, $\sup_{-1\le i\le n} b_{\sigma i} = 1$, so that the given formula defines v_σ uniquely.

(b) Express $\boldsymbol{u}$ as $\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$. Let $(\sigma_0, \dots, \sigma_m)$ be a breakpoint string for $\boldsymbol{u}$ in I, and take $u_* \in L^0(\mathfrak{B})$ as in 612J.

Set $\tau_{-1} = \sigma_{-1} = \min\mathcal{T}$, $\tau_{n+1} = \sigma_{m+1} = \max\mathcal{T}$. Take any $\sigma \in \mathcal{S}$. Since $\tau_0 \le \sigma_0$ and $\sigma_m \le \tau_n$,

$$[\![u_\sigma = u_*]\!] \supseteq [\![\sigma < \sigma_0]\!] \supseteq [\![\sigma < \tau_0]\!],$$

$$\begin{aligned}[\![u_\sigma = u_{\tau_n}]\!] &\supseteq [\![u_\sigma = u_{\sigma_m}]\!] \cap [\![u_{\tau_n} = u_{\sigma_m}]\!] \supseteq [\![\sigma_m \le \sigma]\!] \cap [\![\sigma_m \le \tau_n]\!]\\ &= [\![\sigma_m \le \sigma]\!] \supseteq [\![\tau_n \le \sigma]\!].\end{aligned}$$

Next, suppose that $i < n$. Then the stopping-time interval $c(\tau_i, \tau_{i+1})$ is either 0 or an I-cell; taking J to be $\{\sigma_i : -1 \le i \le m+1\}$, $c(\tau_i, \tau_{i+1})$ must be included in a J-cell, which must be of the form $c(\sigma_j, \sigma_{j+1})$ where $-1 \le j \le m$. In this case $[\![\tau_i < \tau_{i+1}]\!] \subseteq [\![\sigma_j \le \tau_i]\!] \cap [\![\tau_{i+1} \le \sigma_{j+1}]\!]$ (611Jc). So

$$[\![\tau_i \le \sigma]\!] \cap [\![\sigma < \tau_{i+1}]\!] = [\![\tau_i \le \sigma]\!] \cap [\![\sigma < \tau_{i+1}]\!] \cap [\![\tau_i < \tau_{i+1}]\!]$$

(611E(c-iii-γ))

$$\begin{aligned}&= [\![\tau_i \le \sigma]\!] \cap [\![\sigma < \tau_{i+1}]\!] \cap [\![\tau_i < \tau_{i+1}]\!] \cap [\![\sigma_j \le \tau_i]\!] \cap [\![\tau_{i+1} \le \sigma_{j+1}]\!]\\ &\subseteq [\![\sigma_j \le \sigma]\!] \cap [\![\sigma < \sigma_{j+1}]\!] \cap [\![\sigma_j \le \tau_i]\!] \cap [\![\tau_i < \sigma_{j+1}]\!]\end{aligned}$$

(611E(c-iv-α), 611E(c-iii-γ))

$$\subseteq [\![u_\sigma = u_{\sigma_j}]\!] \cap [\![u_{\tau_i} = u_{\sigma_j}]\!] \subseteq [\![u_\sigma = u_{\tau_i}]\!].$$

So $u_*, \tau_0, \dots, \tau_n$ satisfy the definition in 612J and $\tau_0, \dots, \tau_n$ is a breakpoint string for $\boldsymbol{u}$.

(c) Because breakpoint strings are finite, there is a finite set $A \subseteq \mathcal{S}$ such that every $\boldsymbol{u}_k$ has a breakpoint string in A. Let I be the sublattice of $\mathcal{S}$ generated by A, so that I is finite (611Ca) and there is a sequence $(\tau_0, \dots, \tau_n)$ linearly generating the I-cells. By (b), $(\tau_0, \dots, \tau_n)$ is a breakpoint string for $\boldsymbol{u}_k$, for any $k \in K$.

(d)(i) Of course $\tau_0 \wedge \tau \le \dots \le \tau_n \wedge \tau \le \tau \le \tau_0 \vee \tau \le \dots \le \tau_n \vee \tau$. Let $u_* \in L^0(\mathfrak{B})$ be such that $[\![\sigma < \tau_0]\!] \subseteq [\![u_\sigma = u_*]\!]$ for every $\sigma \in \mathcal{S}$.

Take any $\sigma \in \mathcal{S}$. Then

$$[\![\sigma < \tau_0 \wedge \tau]\!] \subseteq [\![\sigma < \tau_0]\!] \subseteq [\![u_\sigma = u_*]\!].$$

If $i < n$,

$$[\![\tau_i \wedge \tau \le \sigma]\!] \cap [\![\sigma < \tau_{i+1} \wedge \tau]\!] = ([\![\tau_i \le \sigma]\!] \cup [\![\tau \le \sigma]\!]) \cap [\![\sigma < \tau_{i+1}]\!] \cap [\![\sigma < \tau]\!]$$

(611E(c-i-β))

$$\begin{aligned}&= [\![\tau_i \le \sigma]\!] \cap [\![\sigma < \tau_{i+1}]\!] \cap [\![\sigma < \tau]\!]\\ &\subseteq [\![u_\sigma = u_{\tau_i}]\!] \cap [\![\tau_i < \tau]\!] \cap [\![\tau_i < \tau_{i+1}]\!]\end{aligned}$$

(611E(c-iii-γ))

$$\subseteq [\![u_\sigma = u_{\tau_i}]\!] \cap [\![\tau_i = \tau_i \wedge \tau]\!] \cap [\![\tau_i < \tau_{i+1} \wedge \tau]\!]$$

(611E(a-ii-α))

$$\begin{aligned}&\subseteq [\![u_\sigma = u_{\tau_i}]\!] \cap [\![\tau_i = \tau_i \wedge \tau]\!] \cap [\![\tau_i \wedge \tau < \tau_{i+1}]\!]\\ &\subseteq [\![u_\sigma = u_{\tau_i}]\!] \cap [\![u_{tau_i \wedge \tau} = u_{\tau_i}]\!] \subseteq [\![u_\sigma = u_{\tau_i \wedge \tau}]\!].\end{aligned}$$

Next,

$$\begin{aligned}\llbracket\tau_n\wedge\tau\le\sigma\rrbracket\cap\llbracket\sigma<\tau\rrbracket&=(\llbracket\tau_n\le\sigma\rrbracket\cup\llbracket\tau\le\sigma\rrbracket)\cap\llbracket\sigma<\tau\rrbracket\\&=\llbracket\tau_n\le\sigma\rrbracket\cap\llbracket\sigma<\tau\rrbracket\\&\subseteq\llbracket u_\sigma=u_{\tau_n}\rrbracket\cap\llbracket\tau_n\le\tau\rrbracket\\&=\llbracket u_\sigma=u_{\tau_n}\rrbracket\cap\llbracket\tau_n\le\tau_n\wedge\tau\rrbracket\\&\subseteq\llbracket u_\sigma=u_{\tau_n}\rrbracket\cap\llbracket u_{\tau_n\wedge\tau}=u_{\tau_n}\rrbracket\subseteq\llbracket u_\sigma=u_{\tau_n\wedge\tau}\rrbracket\end{aligned}$$

and

$$\begin{aligned}\llbracket\tau\le\sigma\rrbracket\cap\llbracket\sigma<\tau_0\vee\tau\rrbracket&=\llbracket\tau\le\sigma\rrbracket\cap(\llbracket\sigma<\tau_0\rrbracket\cup\llbracket\sigma<\tau\rrbracket)\\&=\llbracket\tau\le\sigma\rrbracket\cap\llbracket\sigma<\tau_0\rrbracket\\&\subseteq\llbracket\tau<\tau_0\rrbracket\cap\llbracket\sigma<\tau_0\rrbracket\\&\subseteq\llbracket u_\sigma=u_*\rrbracket\cap\llbracket u_\tau=u_*\rrbracket\subseteq\llbracket u_\sigma=u_\tau\rrbracket.\end{aligned}$$

(611Eb)

Continuing, if $i<n$ then

$$\begin{aligned}\llbracket\tau_i\vee\tau\le\sigma\rrbracket\cap\llbracket\sigma<\tau_{i+1}\vee\tau\rrbracket&=\llbracket\tau_i\le\sigma\rrbracket\cap\llbracket\tau\le\sigma\rrbracket\cap(\llbracket\sigma<\tau_{i+1}\rrbracket\cup\llbracket\sigma<\tau\rrbracket)\\&=\llbracket\tau_i\le\sigma\rrbracket\cap\llbracket\tau\le\sigma\rrbracket\cap\llbracket\sigma<\tau_{i+1}\rrbracket\\&\subseteq\llbracket u_\sigma=u_{\tau_i}\rrbracket\cap\llbracket\tau<\tau_{i+1}\rrbracket\cap\llbracket\tau_i<\tau_{i+1}\rrbracket\\&=\llbracket u_\sigma=u_{\tau_i}\rrbracket\cap\llbracket\tau_i\vee\tau<\tau_{i+1}\rrbracket\\&\subseteq\llbracket u_\sigma=u_{\tau_i}\rrbracket\cap\llbracket u_{\tau_i\vee\tau}=u_{\tau_i}\rrbracket\subseteq\llbracket u_\sigma=u_{\tau_i\vee\tau}\rrbracket.\end{aligned}$$

And finally

$$\llbracket\tau_n\vee\tau\le\sigma\rrbracket\subseteq\llbracket u_\sigma=u_{\tau_n}\rrbracket\cap\llbracket u_{\tau_n\vee\tau}=u_{\tau_n}\rrbracket\subseteq\llbracket u_\sigma=u_{\tau_n\vee\tau}\rrbracket.$$

So the string $(\tau_0\wedge\tau,\dots,\tau_0\wedge\tau_n,\tau,\tau_0\vee\tau,\dots,\tau_n\vee\tau)$ passes the test.

(ii) The formulae in (i) tell us that, for any $\sigma\in\mathcal{S}\wedge\tau$,

$$\llbracket\sigma<\tau_0\rrbracket\subseteq\llbracket u_\sigma=u_*\rrbracket,\quad\llbracket\tau_n\wedge\tau\le\sigma\rrbracket\cap\llbracket\sigma<\tau\rrbracket\subseteq\llbracket u_\sigma=u_{\tau_n\wedge\tau}\rrbracket,$$

$$\llbracket\tau_i\wedge\tau\le\sigma\rrbracket\cap\llbracket\sigma<\tau_{i+1}\wedge\tau\rrbracket\subseteq\llbracket u_\sigma=u_{\tau_i\wedge\tau}\rrbracket\text{ for every }i<n;$$

and of course

$$\llbracket\tau\le\sigma\rrbracket=\llbracket\tau=\sigma\rrbracket\subseteq\llbracket u_\sigma=u_\tau\rrbracket.$$

Also $u_*\in L^0(\bigcap_{\sigma\in\mathcal{S}\wedge\tau}\mathfrak{A}_\sigma)$. So $u_*,\tau_0\wedge\tau,\dots,\tau_n\wedge\tau,\tau)$ witness that $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ is simple, with breakpoint string $(\tau_0\wedge\tau,\dots,\tau_n\wedge\tau,\tau)$.

(iii) Similarly, for any $\sigma\in\mathcal{S}\vee\tau$,

$$\llbracket\sigma<\tau_0\vee\tau\rrbracket\subseteq\llbracket u_\sigma=u_\tau\rrbracket,\quad\llbracket\tau_n\vee\tau\le\sigma\rrbracket\subseteq\llbracket u_\sigma=u_{\tau_n\vee\tau}\rrbracket,$$

$$\llbracket\tau_i\vee\tau\le\sigma\rrbracket\cap\llbracket\sigma<\tau_{i+1}\vee\tau\rrbracket\subseteq\llbracket u_\sigma=u_{\tau_i\vee\tau}\rrbracket\text{ for every }i<n.$$

So $\boldsymbol{u}\restriction\mathcal{S}\vee\tau$ is simple, with breakpoint string $(\tau_0\vee\tau,\dots,\tau_n\vee\tau)$.

(e) Of course

$$\tau_0\le\dots\le\tau_m\le\tau\le\tau'_0\le\dots\le\tau'_m.$$

Let $u_*\in L^0(\bigcap_{\sigma\in\mathcal{S}\wedge\tau}\mathfrak{A}_\sigma)$ be such that $\llbracket\sigma<\tau_0\rrbracket\subseteq\llbracket u_\sigma=u_*\rrbracket$ for every $\sigma\in\mathcal{S}\wedge\tau$. Then $u_*\in L^0(\bigcap_{\sigma\in\mathcal{S}}\mathfrak{A}_{\sigma\wedge\tau})\subseteq L^0(\bigcap_{\sigma\in\mathcal{S}}\mathfrak{A}_\sigma)$. Now, for $\sigma\in\mathcal{S}$, $i<m$ and $j<n$,

$$\begin{aligned}
[\![\sigma<\tau_0]\!] &\subseteq [\![\sigma\wedge\tau=\sigma]\!]\cap[\![\sigma\wedge\tau<\tau_0]\!]\\
&\subseteq [\![u_\sigma=u_{\sigma\wedge\tau}]\!]\cap[\![u_{\sigma\wedge\tau}=u_*]\!]\subseteq[\![u_\sigma=u_*]\!],\\
[\![\tau_i\le\sigma]\!]\cap[\![\sigma<\tau_{i+1}]\!] &\subseteq [\![\sigma\wedge\tau=\sigma]\!]\cap[\![\tau_i\le\sigma\wedge\tau]\!]\cap[\![\sigma\wedge\tau<\tau_{i+1}]\!]\\
&\subseteq [\![u_\sigma=u_{\sigma\wedge\tau}]\!]\cap[\![u_{\sigma\wedge\tau}=u_{\tau_i}]\!]\subseteq[\![u_\sigma=u_{\tau_i}]\!],\\
[\![\tau_m\le\sigma]\!]\cap[\![\sigma<\tau]\!] &\subseteq [\![\sigma\wedge\tau=\sigma]\!]\cap[\![\tau_m\le\sigma\wedge\tau]\!]\\
&\subseteq [\![u_{\sigma\wedge\tau}=u_\sigma]\!]\cap[\![u_{\sigma\wedge\tau}=u_{\tau_m}]\!]\subseteq[\![u_\sigma=u_{\tau_m}]\!],\\
[\![\tau\le\sigma]\!]\cap[\![\sigma<\tau'_0]\!] &\subseteq [\![\tau\vee\sigma=\sigma]\!]\cap[\![\tau\vee\sigma<\tau'_0]\!]\cap[\![\tau<\tau'_0]\!]\\
&\subseteq [\![u_{\tau\vee\sigma}=u_\sigma]\!]\cap[\![u_{\tau\vee\sigma}=u_\tau]\!]\subseteq[\![u_\sigma=u_\tau]\!],\\
[\![\tau'_j\le\sigma]\!]\cap[\![\sigma<\tau'_{j+1}]\!] &\subseteq [\![\sigma\vee\tau=\sigma]\!]\cap[\![\tau'_j\le\sigma\vee\tau]\!]\cap[\![\sigma\vee\tau<\tau'_{j+1}]\!]\\
&\subseteq [\![u_\sigma=u_{\sigma\vee\tau}]\!]\cap[\![u_{\sigma\vee\tau}=u_{\tau'_j}]\!]\subseteq[\![u_\sigma=u_{\tau'_j}]\!],\\
[\![\tau'_n\le\sigma]\!] &\subseteq [\![\sigma\vee\tau=\sigma]\!]\cap[\![\tau'_n\le\sigma\vee\tau]\!]\\
&\subseteq [\![u_{\sigma\vee\tau}=u_\sigma]\!]\cap[\![u_{\sigma\vee\tau}=u_{\tau'_n}]\!]\subseteq[\![u_\sigma=u_{\tau'_n}]\!].
\end{aligned}$$

So $(\tau_0,\dots,\tau_m,\tau,\tau'_0,\dots,\tau'_m)$ is indeed a breakpoint string for $\boldsymbol{u}$ and witnesses that $\boldsymbol{u}$ is simple.

612L Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. Write $M_{\mathrm{simp}}=M_{\mathrm{simp}}(\mathcal{S})$ for the set of simple processes with domain $\mathcal{S}$.

(a) If $h:\mathbb{R}\to\mathbb{R}$ is a Borel measurable function and $\boldsymbol{u}\in M_{\mathrm{simp}}$, then $\bar{h}\boldsymbol{u}\in M_{\mathrm{simp}}$ and any breakpoint string for $\boldsymbol{u}$ is a breakpoint string for $\bar{h}\boldsymbol{u}$.

(b) M_{simp} is an f-subalgebra of $\prod_{\sigma\in\mathcal{S}}L^0(\mathfrak{A}_\sigma)$.

(c) If $z\in L^0(\mathfrak{A}\cap\bigcap_{\sigma\in\mathcal{S}}\mathfrak{A}_\sigma)$ and $\boldsymbol{u}\in M_{\mathrm{simp}}$, then $z\boldsymbol{u}\in M_{\mathrm{simp}}$.

proof If $\mathcal{S}$ is empty, this is trivial; suppose otherwise. Set $\mathfrak{B}=\bigcap_{\sigma\in\mathcal{S}}\mathfrak{A}_\sigma$.

(a) Express $\boldsymbol{u}$ as $\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$; let $(\tau_0,\dots,\tau_n)$ be a breakpoint string for $\boldsymbol{u}$, and take $u_*\in L^0(\mathfrak{B})$ such that $[\![\sigma<\tau_0]\!]\subseteq[\![u_\sigma=u_*]\!]$ for every $\sigma\in\mathcal{S}$. Then $\bar{h}(u_*)\in L^0(\mathfrak{B})$. If $\sigma\in\mathcal{S}$ then

$$[\![\sigma<\tau_0]\!]\subseteq[\![u_\sigma=u_*]\!]\subseteq[\![\bar{h}(u_\sigma)=\bar{h}(u_*)]\!],\quad [\![\tau_n\le\sigma]\!]\subseteq[\![u_\sigma=u_{\tau_n}]\!]\subseteq[\![\bar{h}(u_\sigma)=\bar{h}(u_{\tau_n})]\!],$$

and for $i<n$

$$[\![\tau_i\le\sigma]\!]\cap[\![\sigma<\tau_{i+1}]\!]\subseteq[\![u_\sigma=u_{\tau_i}]\!]\subseteq[\![\bar{h}(u_\sigma)=\bar{h}(u_{\tau_i})]\!].$$

So $\bar{h}(u_*),\tau_0,\dots,\tau_n$ witness that $\bar{h}\boldsymbol{u}$ is simple.

(b) If $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ and $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ are simple processes, then there is a string $(\tau_0,\dots,\tau_n)$ in $\mathcal{S}$ which is a breakpoint string for both, by 612Kc. As above, take u_*, $v_*\in L^0(\mathfrak{B})$ such that

$$[\![\sigma<\tau_0]\!]\subseteq[\![u_\sigma=u_*]\!],\quad [\![\sigma<\tau_0]\!]\subseteq[\![v_\sigma=v_*]\!]$$

for every $\sigma\in\mathcal{S}$. Now $u_*+v_*\in L^0(\mathfrak{B})$ and

$$[\![u_\sigma+v_\sigma=u_*+v_*]\!]\supseteq[\![u_\sigma=u_*]\!]\cap[\![v_\sigma=v_*]\!]\supseteq[\![\sigma<\tau_0]\!],$$

$$[\![u_\sigma+v_\sigma=u_{\tau_i}+v_{\tau_i}]\!]\supseteq[\![u_\sigma=u_{\tau_i}]\!]\cap[\![v_\sigma=v_{\tau_i}]\!]\supseteq[\![\tau_i\le\sigma]\!]\cap[\![\sigma<\tau_{i+1}]\!]$$

for $i<n$, and

$$[\![u_\sigma+v_\sigma=u_{\tau_n}+v_{\tau_n}]\!]\supseteq[\![u_\sigma=u_{\tau_n}]\!]\cap[\![v_\sigma=v_{\tau_n}]\!]\supseteq[\![\tau_n\le\sigma]\!].$$

So $\boldsymbol{u}+\boldsymbol{v}$ is a simple process with a breakpoint string $(\tau_0,\dots,\tau_n)$.

By 612Bc this, together with (a), implies that the set of simple processes with domain $\mathcal{S}$ is an f-subalgebra of $\prod_{\sigma\in\mathcal{S}}L^0(\mathfrak{A}_\sigma)$.

(c) Follow the argument of (a) above, but with $z\times u_*$, $z\times u_\sigma$ in place of $\bar{h}(u_*)$, $\bar{h}(u_\sigma)$, etc.

612M Lemma Let $\mathcal{S}=[\min\mathcal{S},\max\mathcal{S}]$ be a closed interval in $\mathcal{T}$, and $\boldsymbol{u}$ a simple process with domain $\mathcal{S}$. Then there is a breakpoint string $(\tau_0,\dots,\tau_n)$ for $\boldsymbol{u}$ such that $\tau_0=\min\mathcal{S}$, $\tau_n=\max\mathcal{S}$ and $[\![\tau_i<\tau_{i+1}]\!]=[\![\tau_i<\max\mathcal{S}]\!]$ for every $i<n$.

proof Applying 612K(d-i) twice, we see that $\boldsymbol{u}$ has a breakpoint string $(\sigma_0, \dots, \sigma_m)$ starting with $\sigma_0 = \min \mathcal{S}$ and ending with $\sigma_m = \max \mathcal{S}$. I seek to show by induction on m that the result is true in this case. If $m = 0$ then we can (and must) take $n = 0$ and $\tau_0 = \min \mathcal{S} = \max \mathcal{S}$.

For the inductive step to $m \ge 1$, set

$$a_i = [\![\min \mathcal{S} < \sigma_i]\!] \text{ for } i \le m,$$

so that

$$0 = a_0 \subseteq \dots \subseteq a_m = [\![\min \mathcal{S} < \max \mathcal{S}]\!]$$

and $a_i \in \mathfrak{A}_{\min \mathcal{S}}$ for every i. Now set $b_m = 1 \setminus a_m$ and

$$b_i = a_{i+1} \setminus a_i = [\![\min \mathcal{S} < \sigma_{i+1}]\!] \cap [\![\min \mathcal{S} = \sigma_i]\!] \text{ for } i < m,$$

so that $\langle b_i \rangle_{i \le m}$ is a partition of unity in $\mathfrak{A}$ and $b_i \in \mathfrak{A}_{\min \mathcal{S}}$ for $i \le m$. By 611I, there is a $\tau \in \mathcal{T}$ such that $b_i \subseteq [\![\tau = \sigma_{i+1}]\!]$ for $i < m$ and $b_m \subseteq [\![\tau = \sigma_m]\!]$, while $\sigma_1 \le \tau \le \max \mathcal{S}$, so $\tau \in \mathcal{S}$. Now

$$\begin{aligned}[\![\min \mathcal{S} < \tau]\!] &\supseteq \sup_{i<m} [\![\tau = \sigma_{i+1}]\!] \cap [\![\min \mathcal{S} < \sigma_{i+1}]\!] \\ &\supseteq \sup_{i<m} b_i \cap a_{i+1} = \sup_{i<m} b_i = a_m = [\![\min \mathcal{S} < \max \mathcal{S}]\!].\end{aligned}$$

Also, for any $\sigma \in \mathcal{S}$ and $i < m$,

$$\begin{aligned}b_i \cap [\![\sigma < \tau]\!] &\subseteq [\![\min \mathcal{S} = \sigma_i]\!] \cap [\![\tau = \sigma_{i+1}]\!] \cap [\![\sigma < \tau]\!] \\ &\subseteq [\![\min \mathcal{S} = \sigma_i]\!] \cap [\![\sigma_i \le \sigma]\!] \cap [\![\sigma < \sigma_{i+1}]\!] \\ &\subseteq [\![\min \mathcal{S} = \sigma_i]\!] \cap [\![u_\sigma = u_{\sigma_i}]\!] \subseteq [\![u_\sigma = u_{\min \mathcal{S}}]\!],\end{aligned}$$

while

$$b_m \cap [\![\sigma < \tau]\!] = [\![\min \mathcal{S} = \max \mathcal{S}]\!] \cap [\![\sigma < \tau]\!] = 0.$$

So

$$[\![\sigma < \tau]\!] = \sup_{i \le m} b_i \cap [\![\sigma < \tau]\!] \subseteq [\![u_\sigma = u_{\min \mathcal{S}}]\!].$$

Next, as $\sigma_1 \le \tau$, $(\sigma_1 \vee \tau, \dots, \sigma_m \vee \tau)$ is a breakpoint string for $\boldsymbol{u} \restriction \mathcal{S} \vee \tau$ (612K(d-iii)) of length m. By the inductive hypothesis, $\boldsymbol{u} \restriction \mathcal{S} \vee \tau = \boldsymbol{u} \restriction [\tau, \max \mathcal{S}]$ has a breakpoint string $(\tau_0, \dots, \tau_n)$ starting with $\tau_0 = \tau$, ending with $\tau_n = \max \mathcal{S}$, and such that $[\![\tau_i < \tau_{i+1}]\!] = [\![\tau_i < \max \mathcal{S}]\!]$ for every $i < n$.

We now find that $(\min \mathcal{S}, \tau_0, \dots, \tau_n)$ is a breakpoint sequence for $\boldsymbol{u}$. **P** Surely we have $\min \mathcal{S} \le \tau_0 \le \dots \le \tau_n$. If $\sigma \in \mathcal{S}$, then

$$[\![\min \mathcal{S} \le \sigma]\!] \cap [\![\sigma < \tau_0]\!] = [\![\sigma < \tau]\!] \subseteq [\![u_\sigma = u_{\min \mathcal{S}}]\!],$$

while

$$[\![\tau_i \le \sigma]\!] \cap [\![\sigma < \tau_{i+1}]\!] \subseteq [\![u_\sigma = u_{\tau_i}]\!]$$

for $i < n$, and

$$[\![\tau_n \le \sigma]\!] = [\![\tau_n = \sigma]\!] \subseteq [\![u_\sigma = u_{\tau_n}]\!]. \textbf{ Q}$$

And as $[\![\min \mathcal{S} < \tau]\!] = [\![\min \mathcal{S} < \max \mathcal{S}]\!]$, $(\min \mathcal{S}, \tau_0, \dots, \tau_n)$ is a breakpoint string of the right kind for $\boldsymbol{u}$, and the induction proceeds.

612P The next lemma is a bit of a sledgehammer, and in the form given here will be used only at the end of Chapter 22; but most of the ideas are required for the important result 612Qa.

Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ a fully adapted process. Then there is a fully adapted process $\boldsymbol{v} = \langle v_\tau \rangle_{\tau \in \mathcal{T}}$, extending $\boldsymbol{u}$, such that

$$[\![v_\tau \ne 0]\!] \subseteq \sup_{\sigma \in \mathcal{S}} [\![\sigma = \tau]\!]$$

for every $\tau \in \mathcal{T}$.

proof (a) For $\tau \in \mathcal{T}$, $\alpha \in \mathbb{R}$ set

$$a_\tau = 1 \setminus \sup_{\sigma\in\mathcal{S}} [\![\sigma = \tau]\!], \quad b_{\tau\alpha} = \sup_{\sigma\in\mathcal{S}} [\![\tau = \sigma]\!] \cap [\![u_\sigma > \alpha]\!].$$

Then $b_{\tau\alpha} = \sup_{\beta>\alpha} b_{\tau\beta}$ for every $\alpha \in \mathbb{R}$. **P**

$$\begin{aligned}\sup_{\beta>\alpha}\sup_{\sigma\in\mathcal{S}} [\![\tau = \sigma]\!] \cap [\![u_\sigma > \beta]\!] &= \sup_{\sigma\in\mathcal{S}}([\![\tau = \sigma]\!] \cap \sup_{\beta>\alpha} [\![u_\sigma > \beta]\!]) \\ &= \sup_{\sigma\in\mathcal{S}} [\![\tau = \sigma]\!] \cap [\![u_\sigma > \alpha]\!]. \text{ \textbf{Q}}\end{aligned}$$

Next, $\inf_{\alpha\ge0} b_{\tau\alpha} = 0$. **P** If $a \in \mathfrak{A} \setminus \{0\}$, either $a \cap a_\tau \ne 0$ and $a \not\subseteq b_{\tau0}$, or there are a $\sigma_0 \in \mathcal{S}$ such that $c = a \cap [\![\tau = \sigma_0]\!]$ is non-zero and an $\alpha \in \mathbb{R}$ such that $d = c \setminus [\![u_{\sigma_0} > \alpha]\!]$ is non-zero. Now

$$\begin{aligned}d \cap \sup_{\sigma\in\mathcal{S}} [\![\tau = \sigma]\!] \cap [\![u_\sigma > \alpha]\!] &\subseteq d \cap \sup_{\sigma\in\mathcal{S}} [\![\sigma_0 = \sigma]\!] \cap [\![u_\sigma > \alpha]\!] \\ &\subseteq d \cap [\![u_{\sigma_0} > \alpha]\!] = 0.\end{aligned}$$

Thus in either case $a \setminus \inf_{\alpha\ge0} b_{\tau\alpha}$ is non-zero. As a is arbitrary, $\inf_{\alpha\ge0} b_{\tau\alpha} = 0$. **Q**

On the other side,

$$\sup_{\alpha<0} b_{\tau\alpha} = \sup_{\sigma\in\mathcal{S}}([\![\tau = \sigma]\!] \cap \sup_{\alpha<0} [\![u_\sigma > \alpha]\!]) = \sup_{\sigma\in\mathcal{S}} [\![\tau = \sigma]\!] = 1 \setminus a_\tau.$$

(b) This means that if $\tau \in \mathcal{T}$ and we set

$$\begin{aligned}b'_{\tau\alpha} &= b_{\tau\alpha} \text{ if } \alpha \ge 0, \\ &= a_\tau \cup b_{\tau\alpha} \text{ if } \alpha < 0,\end{aligned}$$

we shall have

$$\begin{aligned}\sup_{\beta>\alpha} b'_{\tau\beta} &= \sup_{\beta>\alpha} b_{\tau\beta} = b_{\tau\alpha} = b'_{\tau\alpha} \text{ if } \alpha \ge 0, \\ &= \sup_{\beta>\alpha} a_\tau \cup b_{\tau\beta} = a_\tau \cup b_{\tau\alpha} = b'_{\tau\alpha} \text{ if } \alpha < 0,\end{aligned}$$

while

$$\inf_{\alpha\in\mathbb{R}} b'_{\tau\alpha} \subseteq \inf_{\alpha\ge0} b_{\tau\alpha} = 0,$$

$$\sup_{\alpha\in\mathbb{R}} b'_{\tau\alpha} \supseteq a_\tau \cup \sup_{\alpha<0} b_{\tau\alpha} = 1.$$

So $\alpha \mapsto b'_{\tau\alpha}$ satisfies all the conditions of 364Aa, and we have an element v_τ of L^0 defined by saying that $[\![v_\tau > \alpha]\!] = b'_{\tau\alpha}$ for every $\alpha \in \mathbb{R}$.

(c) $v_\tau \in L^0(\mathfrak{A}_\tau)$ for every $\tau \in \mathcal{T}$. **P** For any $\alpha \in \mathbb{R}$ and $\sigma \in \mathcal{S}$, $[\![u_\sigma > \alpha]\!] \in \mathfrak{A}_\sigma$ so $[\![\tau = \sigma]\!] \cap [\![u_\sigma > \alpha]\!] \in \mathfrak{A}_\tau$, by 612C. Accordingly $b_{\tau\alpha} = \sup_{\sigma\in\mathcal{S}} [\![\tau = \sigma]\!] \cap [\![u_\sigma > \alpha]\!]$ belongs to $\mathfrak{A}_\tau$. On the other hand, because $[\![\sigma = \tau]\!] \in \mathfrak{A}_\tau$ for every σ (611H(c-i) again), $a_\tau \in \mathfrak{A}_\tau$. So $[\![v_\tau > \alpha]\!] = b'_{\tau\alpha}$ belongs to $\mathfrak{A}_\tau$ for every α, and $v_\tau \in L^0(\mathfrak{A}_\tau)$. **Q**

(d) If $\tau \in \mathcal{T}$ and $\sigma \in \mathcal{S}$ then $[\![v_\tau = u_\sigma]\!] \supseteq [\![\tau = \sigma]\!]$. **P** If $\alpha \in \mathbb{R}$ then

$$[\![\tau = \sigma]\!] \cap [\![u_\sigma > \alpha]\!] \subseteq [\![\tau = \sigma]\!] \cap b_{\tau\alpha} = [\![\tau = \sigma]\!] \cap [\![v_\tau > \alpha]\!]$$

(because $a_\tau \cap [\![\tau = \sigma]\!] = 0$)

$$\begin{aligned}&= \sup_{\sigma'\in\mathcal{S}} [\![\tau = \sigma]\!] \cap [\![\tau = \sigma']\!] \cap [\![u_{\sigma'} > \alpha]\!] \\ &\subseteq \sup_{\sigma'\in\mathcal{S}} [\![\sigma = \sigma']\!] \cap [\![u_{\sigma'} > \alpha]\!] \\ &\subseteq \sup_{\sigma'\in\mathcal{S}} [\![u_\sigma = u_{\sigma'}]\!] \cap [\![u_{\sigma'} > \alpha]\!] \subseteq [\![u_\sigma > \alpha]\!],\end{aligned}$$

so in fact

$$[\![\tau = \sigma]\!] \cap [\![u_\sigma > \alpha]\!] = [\![\tau = \sigma]\!] \cap [\![v_\tau > \alpha]\!].$$

As α is arbitrary, $[\![\tau=\sigma]\!]\subseteq[\![v_\tau=u_\sigma]\!]$. **Q**

In particular, if $\sigma\in\mathcal{S}$ then $v_\sigma=u_\sigma$.

(e) If τ, $\tau'\in\mathcal{T}$ then $[\![\tau=\tau']\!]\subseteq[\![v_\tau=v_{\tau'}]\!]$. **P** For any $\alpha\in\mathbb{R}$,

$$\begin{aligned}[\![\tau=\tau']\!]\cap b_{\tau\alpha}&=\sup_{\sigma\in\mathcal{S}}[\![\tau=\tau']\!]\cap[\![\tau=\sigma]\!]\cap[\![u_\sigma>\alpha]\!]\\&=\sup_{\sigma\in\mathcal{S}}[\![\tau=\tau']\!]\cap[\![\tau'=\sigma]\!]\cap[\![u_\sigma>\alpha]\!]=[\![\tau=\tau']\!]\cap b_{\tau'\alpha}.\end{aligned}$$

At the same time,

$$[\![\tau=\tau']\!]\cap\sup_{\sigma\in\mathcal{S}}[\![\tau=\sigma]\!]=\sup_{\sigma\in\mathcal{S}}[\![\tau=\tau']\!]\cap[\![\tau=\sigma]\!]\subseteq\sup_{\sigma\in\mathcal{S}}[\![\tau'=\sigma]\!],$$

so in fact

$$[\![\tau=\tau']\!]\cap\sup_{\sigma\in\mathcal{S}}[\![\tau=\sigma]\!]=[\![\tau=\tau']\!]\cap\sup_{\sigma\in\mathcal{S}}[\![\tau'=\sigma]\!]$$

and $[\![\tau=\tau']\!]\cap a_\tau=[\![\tau=\tau']\!]\cap a_{\tau'}$. Accordingly

$$\begin{aligned}[\![\tau=\tau']\!]\cap[\![v_\tau>\alpha]\!]&=[\![\tau=\tau']\!]\cap b_{\tau\alpha}=[\![\tau=\tau']\!]\cap b_{\tau'\alpha}\\&\qquad=[\![\tau=\tau']\!]\cap[\![v_{\tau'}>\alpha]\!]\text{ if }\alpha\ge0,\\&=[\![\tau=\tau']\!]\cap(a_\tau\cup b_{\tau\alpha})=[\![\tau=\tau']\!]\cap(a_{\tau'}\cup b_{\tau'\alpha})\\&\qquad=[\![\tau=\tau']\!]\cap[\![v_{\tau'}>\alpha]\!]\text{ if }\alpha<0.\end{aligned}$$

As α is arbitrary, $[\![\tau=\tau']\!]\subseteq[\![v_\tau=v_{\tau'}]\!]$. **Q** As τ and τ' are arbitrary, $\boldsymbol{v}=\langle v_\tau\rangle_{\tau\in\mathcal{T}}$ is fully adapted, and we have already seen that it extends $\boldsymbol{u}$.

(f) Observe that, for any $\tau\in\mathcal{T}$,

$$[\![v_\tau>0]\!]=b_{\tau0}\subseteq\sup_{\sigma\in\mathcal{S}}[\![\sigma=\tau]\!]=1\setminus a_\tau,$$

while

$$[\![v_\tau>\alpha]\!]=b'_{\tau\alpha}\supseteq a_\tau$$

for every $\alpha<0$. But this means that $a_\tau\subseteq[\![v_\tau=0]\!]$, that is, that $[\![v_\tau\ne0]\!]\subseteq\sup_{\sigma\in\mathcal{S}}[\![\sigma=\tau]\!]$, as required.

612Q Proposition Suppose that $\mathcal{S}$ is a sublattice of $\mathcal{T}$, $\hat{\mathcal{S}}$ its covered envelope (611M) and $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a fully adapted process.

(a) $\boldsymbol{u}$ has a unique extension to a fully adapted process $\hat{\boldsymbol{u}}=\langle\hat u_\sigma\rangle_{\sigma\in\mathcal{S}}$ with domain $\hat{\mathcal{S}}$.

(b) The map $\boldsymbol{u}\mapsto\hat{\boldsymbol{u}}$ is an isomorphism from the f-algebra $M_{\text{fa}}(\mathcal{S})$ of fully adapted processes with domain $\mathcal{S}$ to the f-algebra $M_{\text{fa}}(\hat{\mathcal{S}})$, and $\bar h\hat{\boldsymbol{u}}=(\bar h\boldsymbol{u})\hat{}$ whenever $\boldsymbol{u}\in M_{\text{fa}}(\mathcal{S})$ and $h:\mathbb{R}\to\mathbb{R}$ is Borel measurable.

(c) If $\tau\in\mathcal{S}$, then $\hat{\boldsymbol{u}}\restriction\hat{\mathcal{S}}\wedge\tau$ is the fully adapted extension of $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ to the covered envelope of $\mathcal{S}\wedge\tau$.

(e) If $z\in L^0(\mathfrak{A}\cap\bigcap_{\sigma\in\mathcal{S}}\mathfrak{A}_\sigma)$ then $z\hat{\boldsymbol{u}}$ is the fully adapted extension of $z\boldsymbol{u}$.

(f) If $\boldsymbol{u}$ is simple, with a witnessing string $(u_*,\tau_0,\dots,\tau_n)$ as in 612Ja, and $\mathcal{S}'$ is a sublattice of $\hat{\mathcal{S}}$ including $\mathcal{S}$, then $\hat{\boldsymbol{u}}\restriction\mathcal{S}'$ is simple, with the same witnessing string.

(g) If $\boldsymbol{u}$ is non-decreasing, so is $\hat{\boldsymbol{u}}$.

proof (a) By 612P, there is a fully adapted process $\boldsymbol{v}=\langle v_\tau\rangle_{\tau\in\mathcal{T}}$ extending $\boldsymbol{u}$; set $\hat u_\tau=v_\tau$ for $\tau\in\hat{\mathcal{S}}$, so that $\hat{\boldsymbol{u}}=\langle\hat u_\tau\rangle_{\tau\in\hat{\mathcal{S}}}$ is a fully adapted process with domain $\hat{\mathcal{S}}$ extending $\boldsymbol{u}$. If $\langle w_\tau\rangle_{\tau\in\hat{\mathcal{S}}}$ is any fully adapted process extending $\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$, then

$$\begin{aligned}[\![\hat u_\tau=w_\tau]\!]&\supseteq[\![\hat u_\tau=u_\sigma]\!]\cap[\![w_\tau=u_\sigma]\!]\\&=[\![\hat u_\tau=\hat u_\sigma]\!]\cap[\![w_\tau=w_\sigma]\!]\supseteq[\![\tau=\sigma]\!]\end{aligned}$$

for every $\sigma\in\mathcal{S}$, so $\hat u_\tau=w_\tau$, for every $\tau\in\hat{\mathcal{S}}$. Thus the extension is unique.

(b) The point is just that $\boldsymbol{u}\mapsto\hat{\boldsymbol{u}}$ is the inverse of $\boldsymbol{v}\mapsto\boldsymbol{v}\restriction\mathcal{S}:M_{\text{fa}}(\hat{\mathcal{S}})\to M_{\text{fa}}(\mathcal{S})$, which has the declared properties.

(c) We saw in 611M(e-i) that $\hat{\mathcal{S}}\wedge\tau$ is the covered envelope of $\mathcal{S}\wedge\tau$. Now $\hat{\boldsymbol{u}}\restriction\hat{\mathcal{S}}\wedge\tau$ is fully adapted and extends $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$, so must be the fully adapted extension of $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$, by the uniqueness noted in (a) above.

(e) Because $\bigcap_{\tau\in\hat{\mathcal{S}}}\mathfrak{A}_\tau=\bigcap_{\sigma\in\hat{\mathcal{S}}}\mathfrak{A}_\sigma$ (611Mf), $z\hat{\boldsymbol{u}}$ is defined in $M_{\mathrm{fa}}(\hat{\mathcal{S}})$, and of course it extends $z\boldsymbol{u}$.

(f) Again because $\bigcap_{\tau\in\hat{\mathcal{S}}}\mathfrak{A}_\tau=\bigcap_{\sigma\in\mathcal{S}}\mathfrak{A}_\sigma$, $u_*\in L^0(\bigcap_{\tau\in\hat{\mathcal{S}}}\mathfrak{A}_\tau)$. Now if $\tau\in\hat{\mathcal{S}}$,

$$\begin{aligned}[\![\tau<\tau_0]\!]&=\sup_{\sigma\in\mathcal{S}}[\![\tau<\tau_0]\!]\cap\tau=\sigma=\sup_{\sigma\in\mathcal{S}}[\![\sigma<\tau_0]\!]\cap\tau=\sigma\\&\subseteq\sup_{\sigma\in\mathcal{S}}[\![u_\sigma=u_*]\!]\cap[\![\hat{u}_\tau=u_\sigma]\!]\subseteq[\![u_\tau=u_*]\!],\end{aligned}$$

$$\begin{aligned}[\![\tau_n\le\tau]\!]&=\sup_{\sigma\in\mathcal{S}}[\![\tau_n\le\tau]\!]\cap\tau=\sigma=\sup_{\sigma\in\mathcal{S}}[\![\tau_n\le\sigma]\!]\cap\tau=\sigma\\&\subseteq\sup_{\sigma\in\mathcal{S}}[\![u_\sigma=u_{\tau_n}]\!]\cap[\![\hat{u}_\tau=u_\sigma]\!]\subseteq[\![\hat{u}_\tau=u_{\tau_n}]\!]=[\![\hat{u}_\tau=\hat{u}_{\tau_n}]\!],\end{aligned}$$

and for $i<n$

$$\begin{aligned}[\![\tau_i\le\tau]\!]\cap[\![\tau<\tau_{i+1}]\!]&=\sup_{\sigma\in\mathcal{S}}[\![\tau_i\le\tau]\!]\cap[\![\tau<\tau_{i+1}]\!]\cap[\![\tau=\sigma]\!]\\&=\sup_{\sigma\in\mathcal{S}}[\![\tau_i\le\sigma]\!]\cap[\![\sigma<\tau_{i+1}]\!]\cap\tau=\sigma\\&\subseteq\sup_{\sigma\in\mathcal{S}}[\![u_\sigma=u_{\tau_i}]\!]\cap[\![\hat{u}_\tau=u_\sigma]\!]\subseteq[\![\hat{u}_\tau=\hat{u}_{\tau_i}]\!].\end{aligned}$$

So $\hat{\boldsymbol{u}}$ is simple, with the declared witnessing string.

As for $\hat{\boldsymbol{u}}\restriction\mathcal{S}'$, we have $\bigcap_{\tau\in\hat{\mathcal{S}}}\mathfrak{A}_\tau\subseteq\bigcap_{\sigma\in\mathcal{S}'}\mathfrak{A}_\sigma$, so $u_*\in L^0(\bigcap_{\tau\in\mathcal{S}'}\mathfrak{A}_\tau)$, while $\tau_0,\dots,\tau_n$ belong to $\mathcal{S}'$. Now the formulae of 612Ja show immediately that $(u_*,\tau_0,\dots,\tau_n)$ witnesses that $\hat{\boldsymbol{u}}\restriction\mathcal{S}'$ is simple.

(g) If $\tau\le\tau'$ in $\hat{\mathcal{S}}$ and $a\in\mathfrak{A}\setminus\{0\}$, there are σ, $\sigma'\in\mathcal{S}$ such that $b=a\cap[\![\tau=\sigma]\!]$ and $c=b\cap[\![\tau'=\sigma']\!]$ are non-zero. Now

$$\begin{aligned}c&\subseteq[\![\tau=\sigma]\!]\cap[\![\tau'=\sigma']\!]\cap[\![\tau\le\tau']\!]\subseteq[\![\hat{u}_\tau=u_\sigma]\!]\cap[\![\hat{u}_{\tau'}=u_{\sigma'}]\!]\cap[\![\sigma\le\sigma']\!]\\&\subseteq[\![\hat{u}_\tau=u_\sigma]\!]\cap[\![\hat{u}_{\tau'}=u_{\sigma'}]\!]\cap[\![\sigma=\sigma\wedge\sigma']\!]\cap[\![\sigma'=\sigma\vee\sigma']\!]\\&\subseteq[\![\hat{u}_\tau=u_{\sigma\wedge\sigma'}]\!]\cap[\![\hat{u}_{\tau'}=u_{\sigma\vee\sigma'}]\!]\subseteq[\![\hat{u}_\tau\le\hat{u}_{\tau'}]\!]\end{aligned}$$

because $u_{\sigma\wedge\sigma'}\le u_{\sigma\vee\sigma'}$, while $0\ne c\subseteq a$. Thus $a\cap[\![\hat{u}_\tau\le\hat{u}_{\tau'}]\!]\ne0$; as a is arbitrary, $[\![\hat{u}_\tau\le\hat{u}_{\tau'}]\!]=1$ and $\hat{u}_\tau\le\hat{u}_{\tau'}$; as τ and τ' are arbitrary, $\hat{\boldsymbol{u}}$ is non-decreasing.

612R Corollary Suppose that $\mathcal{S}$ is a sublattice of $\mathcal{T}$ and $\mathcal{S}'$ is a sublattice of $\mathcal{S}$ covering $\mathcal{S}$. Then any fully adapted process $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}'}$ has a unique extension to a fully adapted process $\boldsymbol{v}=\langle v_\tau\rangle_{\tau\in\mathcal{S}}$.

proof Let $\hat{\boldsymbol{u}}$ be the fully adapted extension of $\boldsymbol{u}$ to the covered envelope $\hat{\mathcal{S}}'$ of $\mathcal{S}'$; then $\boldsymbol{v}=\hat{\boldsymbol{u}}\restriction\mathcal{S}$ is fully adapted and extends $\boldsymbol{u}$. To see that $\boldsymbol{v}$ is unique, repeat the argument of part (a-iii) of the proof of 612Q; if $\langle v'_\tau\rangle_{\tau\in\mathcal{S}}$ is any fully adapted process extending $\langle u_\sigma\rangle_{\sigma\in\mathcal{S}'}$, then

$$\begin{aligned}[\![v_\tau=v'_\tau]\!]&\supseteq[\![v_\tau=u_\sigma]\!]\cap[\![v'_\tau=u_\sigma]\!]\\&=[\![v_\tau=v_\sigma]\!]\cap[\![v'_\tau=v'_\sigma]\!]\supseteq[\![\tau=\sigma]\!]\end{aligned}$$

for every $\sigma\in\mathcal{S}'$, so $v_\tau=v'_\tau$, for every $\tau\in\mathcal{S}$. Thus the extension is unique.

612S Two more definitions We shall have uses for the following ideas. Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.

(a) For a fully adapted process $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$, write $\|\boldsymbol{u}\|_\infty=\sup_{\sigma\in\mathcal{S}}\|u_\sigma\|_\infty$; counting the supremum as 0 if $\mathcal{S}$ is empty, and $\|u_\sigma\|_\infty$ as ∞ if u_σ does not belong to $L^\infty(\mathfrak{A})$ when this is identified as a subspace of $L^0(\mathfrak{A})$ as in 364J.

(b) For fully adapted processes $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ and $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$, write $[\![\boldsymbol{u}\ne\boldsymbol{v}]\!]$ for $\sup_{\sigma\in\mathcal{S}}[\![u_\sigma\ne v_\sigma]\!]$, and $[\![\boldsymbol{u}\ne\boldsymbol{0}]\!]=\sup_{\sigma\in\mathcal{S}}[\![u_\sigma\ne0]\!]$.

(c) Suppose that $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$ and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma\in\mathcal{S}}$ are fully adapted processes.

(i) If $h : \mathbb{R} \to \mathbb{R}$ is Borel measurable, then $[\![\bar{h}\boldsymbol{u} \neq \bar{h}\boldsymbol{v}]\!] \subseteq [\![\boldsymbol{u} \neq \boldsymbol{v}]\!]$. (Use 612A(d-iii).)

(ii) If $\hat{\mathcal{S}}$ is the covered envelope of $\mathcal{S}$ and $\hat{\boldsymbol{u}} = \langle \hat{u}_\tau \rangle_{\tau\in\hat{\mathcal{S}}}$, $\hat{\boldsymbol{v}} = \langle \hat{v}_\tau \rangle_{\tau\in\hat{\mathcal{S}}}$ are the fully adapted extensions of $\boldsymbol{u}$, $\boldsymbol{v}$ to $\hat{\mathcal{S}}$ (612Q), then $[\![\hat{\boldsymbol{u}} \neq \hat{\boldsymbol{v}}]\!] = [\![\boldsymbol{u} \neq \boldsymbol{v}]\!]$. **P** If $\tau \in \hat{\mathcal{S}}$ then

$$\begin{aligned}[\![\hat{u}_\tau \neq \hat{v}_\tau]\!] &= \sup_{\sigma\in\mathcal{S}} [\![\tau = \sigma]\!] \cap [\![\hat{u}_\tau \neq \hat{v}_\tau]\!] \\ &= \sup_{\sigma\in\mathcal{S}} [\![\hat{u}_\tau = \hat{u}_\sigma]\!] \cap [\![\hat{v}_\tau = \hat{v}_\sigma]\!] \cap [\![\hat{u}_\tau \neq \hat{v}_\tau]\!] \\ &= \subseteq \sup_{\sigma\in\mathcal{S}} [\![\hat{u}_\sigma \neq \hat{v}_\sigma]\!] = \sup_{\sigma\in\mathcal{S}} [\![u_\sigma \neq v_\sigma]\!] = [\![\boldsymbol{u} \neq \boldsymbol{v}]\!];\end{aligned}$$

as τ is arbitrary, $[\![\hat{\boldsymbol{u}} \neq \hat{\boldsymbol{v}}]\!] \subseteq [\![\boldsymbol{u} \neq \boldsymbol{v}]\!]$; and of course

$$[\![\boldsymbol{u} \neq \boldsymbol{v}]\!] = \sup_{\sigma\in\mathcal{S}} [\![\hat{u}_\sigma \neq \hat{v}_\sigma]\!] \subseteq [\![\hat{\boldsymbol{u}} \neq \hat{\boldsymbol{v}}]\!],$$

so we have equality. **Q**

612T The construction in 612H gives us direct routes to some of the leading examples of stochastic process, and most sections of this volume will introduce concepts which should be tested against these examples.

Example: Brownian motion (a) Let $\Omega = C([0,\infty[)_0$ be the set of continuous functions $\omega : [0,\infty[\to \mathbb{R}$ such that $\omega(0) = 0$, and ν one-dimensional Wiener measure on Ω (477D), with Σ its domain. Recall that ν is a Radon measure with respect to the topology $\mathfrak{T}_c$ of uniform convergence on compact sets (477B). Let $(\mathfrak{C}, \bar{\nu})$ be the measure algebra of ν. For $t \geq 0$, write Σ_t for

$$\{F : F \in \Sigma,\ \omega' \in F \text{ whenever } \omega \in F,\ \omega' \in \Omega \text{ and } \omega'{\restriction}[0,t] = \omega{\restriction}[0,t]\},$$

and let $\hat{\Sigma}_t$ be $\{F \triangle A : F \in \Sigma_t,\ \nu A = 0\}$ (cf. 477H); set $\mathfrak{C}_t = \{F^\bullet : F \in \hat{\Sigma}_t\} = \{F^\bullet : F \in \Sigma_t\}$ and $X_t(\omega) = \omega(t)$ for $t \geq 0$ and $\omega \in \Omega$. Then $(s,\omega) \mapsto X_s(\omega) : [0,t] \times \Omega \to \mathbb{R}$ is continuous, therefore $\mathcal{B}([0,t])\widehat{\otimes}\Sigma_t$-measurable and $\mathcal{B}([0,t])\widehat{\otimes}\hat{\Sigma}_t$-measurable (4A3Q(c-i)), for every $t \geq 0$, and $\langle X_t \rangle_{t\geq 0}$ is progressively measurable with respect to $\langle \hat{\Sigma}_t \rangle_{t\geq 0}$. We can therefore apply 612H to see that we have a process $\boldsymbol{w} = \langle w_\tau \rangle_{\tau\in\mathcal{T}_f}$ fully adapted to $\langle \mathfrak{C}_t \rangle_{t\geq 0}$. In this volume I will use the phrase **Brownian motion** to mean the process $\boldsymbol{w}$, rather than the process $\langle X_t \rangle_{t\geq 0}$ as in Chapter 47; I hope that this will not lead to any confusion.

(d) It will be important to know that $\boldsymbol{w}$ determines $\mathfrak{C}$ and $\langle \mathfrak{C}_t \rangle_{t\geq 0}$, in that

(i) $\mathfrak{C}$ is the closed subalgebra $\mathfrak{D}$ of itself generated by $\{[\![w_{\check{t}} > \alpha]\!] : t \geq 0,\ \alpha \in \mathbb{R}\}$,

(ii) $\mathfrak{C}_t$ is the closed subalgebra generated by $\{[\![w_{\check{s}} > \alpha]\!] : s \in [0,t],\ \alpha \in \mathbb{R}\}$ for every $t \geq 0$.

P(i) If $t \geq 0$ and $\alpha \in \mathbb{R}$, $[\![w_{\check{t}} > \alpha]\!] = \{\omega : \omega(t) > \alpha\}^\bullet$, so $\mathfrak{D} = \{E^\bullet : E \in \mathrm{T}_{[0,\infty[}\}$ where $\mathrm{T}_{[0,\infty[}$ is the σ-algebra of subsets of $\Omega = C([0,\infty[)_0$ generated by $\{\{\omega : \omega(t) > \alpha\} : t \geq 0,\ \alpha \in \mathbb{R}\}$.

Consider the family $\mathcal{V} = \mathfrak{T}_c \cap \mathrm{T}_{[0,\infty[}$. This is closed under union and intersection and the topology on Ω it generates is Hausdorff. It follows that if K, L are disjoint $\mathfrak{T}_c$-compact sets, then there are disjoint $U, V \in \mathcal{V}$ such that $K \subseteq U$ and $L \subseteq V$. But ν is inner regular with respect to the $\mathfrak{T}_c$-compact sets, so if $E \in \operatorname{dom}\nu$ and $\epsilon > 0$ there is a $V \in \mathcal{V}$ such that $\nu(E\triangle V) \leq \epsilon$. Consequently $\mathfrak{D}$ is dense in $\mathfrak{C}$ for the measure-algebra topology; as it is also closed, it is the whole of $\mathfrak{C}$, as claimed.

(ii) Now take any $t \geq 0$; write $\mathrm{T}_{[0,t]}$ for the σ-algebra of subsets of Ω generated by $\{\{\omega : \omega(s) > \alpha\} : s \in [0,t],\ \alpha \in \mathbb{R}\}$, and $\mathfrak{D}_t$ for the closed subalgebra generated by $\{[\![w_{\check{s}} > \alpha]\!] : s \in [0,t],\ \alpha \in \mathbb{R}\}$, so that $\mathfrak{D}_t = \{E^\bullet : E \in \mathrm{T}_{[0,t]}\}$. Consider Σ_t as defined in (a) above. Of course $\mathrm{T}_{[0,t]} \subseteq \Sigma_t$, so $\mathfrak{D}_t \subseteq \mathfrak{C}_t$. On the other hand, given $F \in \Sigma_t$ and $\epsilon > 0$, there are $\mathfrak{T}_c$-compact sets $K \subseteq F$ and $L \subseteq \Omega \setminus F$ such that $\nu(K \cup L) \geq \epsilon$. If we set $\mathcal{V}_t = \mathfrak{T}_c \cap \mathrm{T}_t$, then for any $\omega \in K$, $\omega' \in L$ there is an $s \leq t$ such that $\omega(s) \neq \omega'(s)$, so that there are disjoint U, $V \in \mathcal{V}_t$ such that $\omega \in U$ and $\omega' \in V$; because $\mathcal{V}_t$ is closed under finite unions and intersections, there are disjoint U, $V \in \mathcal{V}_t$ such that $K \subseteq U$ and $L \subseteq V$, so that $\nu(F\triangle U) \leq \epsilon$, while $U \in \mathrm{T}_t$. As ϵ is arbitrary, $F^\bullet \in \mathfrak{D}_t$; as F is arbitrary, $\mathfrak{C}_t \subseteq \mathfrak{D}_t$. **Q**

(e) In order to apply 612H directly, I have cast the discussion above in terms of stopping times adapted to the filtration $\langle\hat{\Sigma}_t\rangle_{t\ge 0}$. However it will make it easier to call on further results from §477 if I remark that every member of $\mathcal{T}_f$ can be represented by a stopping time adapted to $\langle\Sigma_t^+\rangle_{t\ge 0}$, where $\Sigma_t^+ = \bigcap_{s>t}\Sigma_s$ for $t \ge 0$. **P** Recall that $\langle\Sigma_t^+\rangle_{t\ge 0}$ is a filtration of σ-algebras (455L) and that

$$\Sigma_t^+ \subseteq \hat{\Sigma}_t = \hat{\Sigma}_t^+ = \bigcap_{s>t}\hat{\Sigma}_s$$

for every t (477Hc). If $\tau \in \mathcal{T}_f$ there is an $h : \Omega \to [0,\infty[$ representing τ which is adapted to $\langle\hat{\Sigma}_t\rangle_{t\ge 0} = \langle\hat{\Sigma}_t^+\rangle_{t\ge 0}$; by 455L(e-iii), there is a stopping time g, adapted to $\langle\Sigma_t^+\rangle_{t\ge 0}$, which is equal to h almost everywhere. Of course g is now adapted to $\langle\hat{\Sigma}_t\rangle_{t\ge 0}$ so represents a stopping time in $\mathcal{T}_f$ in the sense here, and by 612H(a-iv) g represents τ. **Q**

612U Example: the Poisson process (a) For $t > 0$ let λ_t be the Poisson distribution with expectation t, that is, the Radon probability measure on $\mathbb{R}$ such that $\lambda_t\{n\} = e^{-t}t^n/n!$ for every $n \in \mathbb{N}$ (495Aa). Then the convolution $\lambda_s * \lambda_t$ is equal to λ_{s+t} whenever s, $t > 0$ (495Ab), and $\lim_{t\downarrow 0}\lambda_t G = 1$ for every open set $G \subseteq \mathbb{R}$ including 0. So 455Pc tells us that we have an associated probability measure $\ddot{\mu}$ on the space C_{dlg} of càdlàg real-valued functions defined on $[0,\infty[$. This measure is described in 455P as the subspace measure on C_{dlg} induced by a complete measure on $\mathbb{R}^{[0,\infty[}$ defined in terms of transitional probabilities, following 455E. The formula of 455E tells us that if $0 = t_0 < \ldots < t_n$ in $\mathbb{R}$ and $0 = k_0 \le \ldots \le k_n$ in $\mathbb{N}$, then the measure of $\{\omega : \omega(t_i) = k_i \text{ for } i \le n\}$ is

$$\prod_{i=1}^n \lambda_{t_i - t_{i-1}}\{k_i - k_{i-1}\} = e^{-t_n}\prod_{i=1}^n \frac{(t_i-t_{i-1})^{k_i-k_{i-1}}}{(k_i-k_{i-1})!}.$$

(b) As in 455K, $\ddot{\mu}$ is a completion regular quasi-Radon measure on C_{dlg} if we give C_{dlg} the topology of pointwise convergent inherited from $\mathbb{R}^{[0,\infty[}$. Now the set

$$\Omega = \{\omega : \omega \in C_{\text{dlg}} \text{ is non-decreasing, } \omega(t) \in \mathbb{N} \text{ for every } t \text{ and } \omega(0) = 0\}$$

is the support of $\ddot{\mu}$. **P** Ω is a closed subset of C_{dlg}. The formula in (a) tells us that $\ddot{\mu}\{\omega : \omega(0) = 0\} = 1$ and

$$\ddot{\mu}\{\omega : \omega(t) \in \mathbb{N}\} = \sum_{k=0}^\infty e^{-t}\frac{t^k}{k!} = 1$$

for every $t > 0$, while

$$\ddot{\mu}\{\omega : \omega(s) \le \omega(t)\} = e^{-t}\sum_{k_0=0}^\infty\sum_{k_1=0}^\infty \frac{s^{k_0}}{k_0!}\frac{(t-s)^{k_1}}{k_1!} = 1$$

whenever $0 < s < t$. Thus Ω is expressible as the intersection of a family of conegligible closed sets and is itself a conegligible closed set. If $G \subseteq C_{\text{dlg}}$ is an open set meeting Ω, there are a $\tilde{\omega} \in G\cap\Omega$ and $t_0 < \ldots < t_n$ such that $t_0 = 0$ and $\{\omega : \omega(t_i) = \tilde{\omega}(t_i) \text{ for every } i \le n\} \subseteq G$; in this case $\ddot{\mu}(\Omega\cap G) > 0$. So Ω is the support of $\ddot{\mu}$. **Q**

(c) Let μ be the subspace measure on Ω induced by $\ddot{\mu}$ and Σ its domain, so that μ is a quasi-Radon probability measure on Ω (415B). For $t \ge 0$, set

$$\ddot{\Sigma}_t = \{F : F \in \operatorname{dom}\ddot{\mu},\ \omega' \in F \text{ whenever } \omega' \in C_{\text{dlg}},\ \omega \in F \text{ and } \omega'\restriction[0,t] = \omega\restriction[0,t]\}$$

(see 455O) and

$$\hat{\ddot{\Sigma}}_t = \{F\triangle A : F \in \ddot{\Sigma}_t,\ \ddot{\mu}A = 0\}.$$

Then $\hat{\ddot{\Sigma}}_t = \bigcap_{s>t}\hat{\ddot{\Sigma}}_s$ for every t (455T). So if we set

$$\Sigma_t = \{F : F \subseteq \Omega,\ F \in \hat{\ddot{\Sigma}}_t\}$$

for $t \ge 0$, $\langle\Sigma_t\rangle_{t\ge 0}$ will be a filtration of σ-algebras. Consequently, if we take $(\mathfrak{A}, \bar{\mu})$ to be the measure algebra of μ, and set $\mathfrak{A}_t = \{F^\bullet : F \in \Sigma_t\}$ for each t, $\langle\mathfrak{A}_t\rangle_{t\ge 0}$ will be a filtration of closed subalgebras of $\mathfrak{A}$.

(d) For $\omega \in \Omega$ and $t \ge 0$ set $X_t(\omega) = \omega(t)$. Then X_t has a Poisson distribution with expectation t. Now $\langle X_t\rangle_{t\ge 0}$ is progressively measurable (4A3Q(c-i) again). We therefore have a corresponding fully adapted process $\boldsymbol{v} = \langle v_\tau\rangle_{\tau\in\mathcal{T}_f}$ defined as in 612Hb; in this volume I will call this the **standard Poisson process**.

(e)(i) For each $n \in \mathbb{N}$ and $\omega \in \Omega$, set

$$g_n(\omega) = \inf\{t : t \in [0,\infty[, \, \omega(t) \geq n\},$$

counting $\inf \emptyset$ as ∞. Then $g_0(\omega) = 0$ for every ω. If $g_n(\omega)$ is finite, then $\omega(g_n(\omega)) \geq n$, because ω is càdlàg. Note that

$$\mu\{\omega : g_n(\omega) > t\} \leq \Pr(X_t < n) = e^{-t}\sum_{i=0}^{n-1} \frac{t^i}{i!} \to 0$$

as $t \to \infty$, so g_n is finite a.e. Of course $g_n \leq g_{n+1}$ for every n. In fact, for almost every ω, $\langle g_n(\omega)\rangle_{n\in\mathbb{N}}$ is strictly increasing. **P** If $0 \leq s < t$, then

$$\mu\{\omega : \omega(t) - \omega(s) \geq 2\} \leq 1 - e^{-(t-s)}(1 + t - s) \leq (t - s)^2.$$

Suppose that $\gamma > 0$, $n \in \mathbb{N}$, $m \geq 1$ and $\omega \in \Omega$ are such that $g_n(\omega) = g_{n+1}(\omega) \leq \gamma$. Set $t_i = \frac{i\gamma}{m}$ for $i \leq m$. Then there is a first $i \leq m$ such that $g_n(\omega) \leq t_i$, that is, $\omega(t_i) \geq n$. Since $g_{n+1}(\omega) \leq t_i$, $\omega(t_i) \geq n + 1$. By the definition of Ω, $\omega(t_0) = 0$, so $i > 0$ and we can speak of $\omega(t_{i-1})$, which must be less than n. But $\omega(t_{i-1})$ is an integer, so it is at most $n - 1$, and $\omega(t_i) \geq \omega(t_{i-1}) + 2$.

This shows that

$$\begin{aligned}\mu\{\omega : g_n(\omega) = g_{n+1}(\omega) \leq \gamma\} \\ &\leq \mu\{\omega : \text{there is an } i \text{ such that } 1 \leq i \leq m \text{ and } \omega(t_i) - \omega(t_{i-1}) \geq 2\} \\ &\leq m(\frac{\gamma}{m})^2 = \frac{\gamma^2}{m}.\end{aligned}$$

Letting $m \to \infty$, we see that $\{\omega : g_n(\omega) = g_{n+1}(\omega) \leq \gamma\}$ is negligible; letting $\gamma \to \infty$, $\{\omega : g_n(\omega) = g_{n+1}(\omega)\}$ is negligible and its complement is conegligible. **Q**

Observe that, for any $n \in \mathbb{N}$, $\omega(g_n(\omega)) = n$ for almost every ω. **P** If $g_n(\omega) < g_{n+1}(\omega)$, then $n \leq \omega(g_n(\omega)) < n + 1$ so $\omega(g_n(\omega)) = n$; and this is the case for almost every ω. **Q**

(ii) For $n \in \mathbb{N}$ and $t > 0$,

$$\{\omega : g_n(\omega) < t\} = \{\omega : \text{ there is a rational } q < t \text{ such that } \omega(q) \geq n\} \in \Sigma_t,$$

so g_n is a stopping time adapted to $\langle \Sigma_t^+\rangle_{t\geq 0}$ (455Lb), which here is just $\langle \Sigma_t\rangle_{t\geq 0}$. Let $\tau_n = g_n^\bullet$ be the corresponding stopping time in $\mathcal{T}_f$. Since $\lim_{n\to\infty} g_n(\omega) = \infty$ for every ω, $\sup_{n\in\mathbb{N}} [\![\tau_n > t]\!] = 1$ for every t, and $\sup_{n\in\mathbb{N}} \tau_n = \max \mathcal{T}$. Since $X_{g_n(\omega)}(\omega) = n$ for almost every ω, $v_{\tau_n} = n\chi 1$, for every $n \in \mathbb{N}$.

I will call $\langle \tau_n\rangle_{n\in\mathbb{N}}$ the sequence of **jump times** for the process $\boldsymbol{v}$.

(f) If $\tau \in \mathcal{T}_f$, then

$$[\![v_\tau \in \mathbb{N}]\!] = 1, \quad [\![v_\tau = v_{\tau_n}]\!] = [\![v_\tau = n]\!] = [\![\tau_n \leq \tau]\!] \cap [\![\tau < \tau_{n+1}]\!] \text{ for every } n \in \mathbb{N}.$$

P As in 612H, we have a stopping time $h : \Omega \to [0,\infty[$ representing τ, and $v_\tau = X_h^\bullet$. Now $X_{h(\omega)}(\omega) = \omega(h(\omega)) \in \mathbb{N}$ for every ω, so $[\![v_\tau \in \mathbb{N}]\!] = 1$. For any particular $n \in \mathbb{N}$,

$$\begin{aligned}[\![v_\tau = n]\!] &= \{\omega : \omega(h(\omega)) = n\}^\bullet \\ &= \{\omega : g_n(\omega) \leq h(\omega) < g_{n+1}(\omega)\}^\bullet = [\![\tau_n \leq \tau]\!] \cap [\![\tau < \tau_{n+1}]\!]. \textbf{ Q}\end{aligned}$$

(g) Because every ω is non-negative and non-decreasing, $0 \leq X_g \leq X_h$ whenever g, h are stopping times and $g \leq h$, and $0 \leq v_\sigma \leq v_\tau$ whenever $\sigma \leq \tau$ in $\mathcal{T}_f$. It follows immediately that $\boldsymbol{v}$ is locally order-bounded.

612X Basic exercises (a) In 612A(d-iv), show that if $u \geq 0$ then $h \mapsto \bar{h}(u) : H \to L^0$ is order-preserving and sequentially order-continuous.

(b) Suppose that $\mathcal{S}$ is a sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a fully adapted process. Show that

$$u_\sigma + u_\tau + u_\upsilon = u_{\sigma\wedge\tau\wedge\upsilon} + u_{\mathrm{med}(\sigma,\tau,\upsilon)} + u_{\sigma\vee\tau\vee\upsilon}$$

for all σ, τ, $\upsilon \in \mathcal{S}$.

(c) In 612Hb, show that we can have a progressively measurable family $\langle X_t\rangle_{t\ge 0}$ such that $X_t =_{\text{a.e.}} 0$ for every τ, but there is a $\tau \in \mathcal{T}_b$ such that $u_\tau \ne 0$.

(d) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u}$ a simple process with domain $\mathcal{S}$. Show that $\boldsymbol{u}\restriction\mathcal{S}\vee\tau = \boldsymbol{u}\restriction\mathcal{S}\cap[\tau, \max\mathcal{T}]$ and $\boldsymbol{u}\restriction\mathcal{S}\cap[\tau,\tau']$ are simple processes whenever $\tau \le \tau'$ in $\mathcal{S}$.

>(e) Let I be a finite sublattice of $\mathcal{T}$ and $\boldsymbol{u}$ a fully adapted process with domain I. Show that there is a simple process with domain $\mathcal{T}$ extending $\boldsymbol{u}$ and with a breakpoint string in I. How far is this extension unique?

(i) Let $A \subseteq \mathcal{T}$ be any set, and $\mathcal{S}$ its covered envelope (611Mb). (i) Show that if A is an ideal of $\mathcal{T}$ then $\mathcal{S}$ is an ideal of $\mathcal{T}$. (ii) Show that if $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in A} \in \prod_{\sigma\in A} L^0(\mathfrak{A}_\sigma)$ is such that $[\![u_\sigma = u_\tau]\!] \supseteq [\![\sigma = \tau]\!]$ for all σ, $\tau \in A$, then there is a unique fully adapted process $\hat{\boldsymbol{u}}$ with domain $\mathcal{S}$ extending $\boldsymbol{u}$. (iii) In (ii), show that $\|\hat{\boldsymbol{u}}\|_\infty = \sup_{\sigma\in A}\|u_\sigma\|_\infty$, $[\![\hat{\boldsymbol{u}} \ne 0]\!] = \sup_{\sigma\in A}[\![u_\sigma \ne 0]\!]$ and that $\{\hat u_\sigma : \sigma \in \mathcal{S}\}$ is and order-bounded subset of L^0 iff $\{u_\sigma : \sigma \in A\}$ is.

(j) Suppose that $\mathcal{S}$ is a sublattice of $\mathcal{T}$, $\hat{\mathcal{S}}$ is its covered envelope, and that $\boldsymbol{u}$ is a fully adapted process with domain $\mathcal{S}$ with corresponding extension $\hat{\boldsymbol{u}}$ to a fully adapted process with domain $\hat{\mathcal{S}}$. Show that if $\boldsymbol{u}$ is simple then $\hat{\boldsymbol{u}}$ is simple.

(k) Show that if $\tau \in \mathcal{T}$ then $\langle\chi[\![\sigma = \tau]\!]\rangle_{\sigma\in\mathcal{T}}$ is a fully adapted process.

(l) Show that if $\mathcal{S}$ is a sublattice of $\mathcal{T}$ with covered envelope $\hat{\mathcal{S}}$, and $\boldsymbol{u}$ is a non-decreasing fully adapted process with domain $\mathcal{S}$, then its fully adapted extension to $\hat{\mathcal{S}}$ is non-decreasing.

>(m) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and τ a member of $\mathcal{S}$. Suppose that $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}\wedge\tau}$ and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}\vee\tau}$ are fully adapted processes such that $u_\tau = v_\tau$. Show that there is a unique fully adapted process with domain $\mathcal{S}$ extending both $\boldsymbol{u}$ and $\boldsymbol{v}$.

612Y Further exercises (a) Show that the ideas of 612H can be applied to any totally ordered set T which is Polish in its order topology, in place of $[0,\infty[$. (First tackle the case $T = \mathbb{N}$ to establish rules when T has gaps. You may have to use T or $T \cup \{-\infty\}$ or $T \cup \{-\infty,\infty\}$ in place of $[0,\infty]$.)

(c) Let $\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ be a simple fully additive process, and $\tau \in \mathcal{S}$. Show that $\langle u_{\sigma\wedge\tau}\rangle_{\sigma\in\mathcal{S}}$ is a simple fully additive process.

(e) I will say that a fully adapted process $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ on a non-empty sublattice $\mathcal{S}$ of $\mathcal{T}$ is **previsibly simple** if there are $\tau_0 \le \ldots \le \tau_n$ in $\mathcal{S}$, $u_i \in L^0(\mathfrak{A}_{\tau_i})$ for $i \le n$ and a $u_* \in L^0(\bigcap_{\sigma\in\mathcal{S}}\mathfrak{A}_\sigma)$ such that, for every $\tau \in \mathcal{S}$,

$$\inf_{\sigma\in\mathcal{S}}[\![\tau \le \sigma]\!] \subseteq [\![u_\tau = 0]\!], \quad [\![\sigma < \tau]\!] \cap [\![\tau \le \tau_0]\!] \subseteq [\![u_\tau = u_*]\!] \text{ for every } \sigma \in \mathcal{S},$$

$$[\![\tau_i < \tau]\!] \cap [\![\tau \le \tau_{i+1}]\!] \subseteq [\![u_\tau = u_i]\!] \text{ for every } i < n, \quad [\![\tau_n < \tau]\!] \subseteq [\![u_\tau = u_n]\!].$$

Formulate and prove results for previsibly simple processes corresponding to the facts listed in 612K.

(f) Give an example in which $\mathcal{S}$ is a sublattice of $\mathcal{T}$ and $\boldsymbol{u}$ is a fully adapted process with domain $\mathcal{S}$ such that the fully adapted extension of $\boldsymbol{u}$ to the covered envelope of $\mathcal{S}$ is simple, but $\boldsymbol{u}$ is not simple.

612 Notes and comments In 612H, I look at filtrations of σ-algebras of measurable sets which are supposed all to contain every negligible set. In the most natural representations of the most important stochastic processes, the filtrations don't have this property; see 477H. An incidental advantage of working with measure algebras is that such questions disappear until we turn to specific examples.

Many special spaces of fully adapted processes will be important in the work below; here I mention only the simple processes (612J, 612Xd), as those which have descriptions accessible from our present position. More interesting is the use of 'covering ideals' (611N, 612R). We shall have many cases in which a restriction to an appropriate covering ideal will render a process more amenable – e.g., by making it a martingale.

You may have noticed that there is no mention of 'measure' or 'probability' in this section except in the construction described in 612H, and there is no '$\epsilon > 0$' anywhere. We are still working through the foothills, with essentially algebraic arguments. There are some suprema of infinite sets, but in so far as there is anything non-trivial here, it is a reflection of the work in §364 and §611. 'Simple' processes, however, demand a bit of attention. They are supposed to be a stochastic representation of step-functions on $\mathbb{R}$ (226Xb, 242O) and will play a similar role when we come to the theory of integration (see §614), but the extra complication of working on a lattice rather than on a totally ordered set makes some essential points (e.g., 612Kc) trickier.

If you have spent any time with Volumes 1-5 of this treatise, you will know that I consider a function to be inadequately defined if there is any doubt about its domain. This is a demanding discipline which is more important in some places than others. A point at which we can be relatively relaxed is in the definition of integration in 613Hb, where I shall insist only that $\operatorname{dom}\boldsymbol{u}$ and $\operatorname{dom}\psi$ should be large enough for $S_I(\boldsymbol{u}, d\psi)$ to be defined for every finite sublattice I of $\mathcal{S}$. A point at which we have to be more careful is in the definition of 'simple process' in 612J, where we have to know the exact domain of a process $\boldsymbol{u}$ before we can confirm that the proposed breakpoints belong to that domain. Manoeuvres like the proof of 612Qf will often be required.

Referring you to §455 in the course of 612U is unkind. It ought to be much easier than this, and indeed it is. You should have no real difficulty in finding your own way to a proof of the really important bit, which is that the formula in 612Ua defines a probability measure on the space Ω of non-decreasing càdlàg functions from $[0, \infty[$ to $\mathbb{N}$ starting at 0. But we are going to need the fact that the filtration $\langle\Sigma_t\rangle_{t\ge0}$ is right-continuous (632Da below), and this seems to demand thought. More generally, the processes considered in §455 furnish many other important examples for the theory here.

The construction $z\boldsymbol{u}$ of 612De will be one of the leitmotivs of this volume. For the theory here, we can expect $z\boldsymbol{u}$ to behave like a scalar multiple of $\boldsymbol{u}$; in effect, if $\mathcal{S}$ is a non-empty sublattice of $\mathcal{T}$, the f-algebra $\prod_{\sigma\in\mathcal{S}} L^0(\mathfrak{A}_\sigma)$ can be thought of as an $L^0(\bigcap_{\sigma\in\mathcal{S}}\mathfrak{A}_\sigma)$-module. The idea is that a fully adapted process with domain $\mathcal{S}$ is supposed to represent the evolution of a system over time, and that $\bigcap_{\sigma\in\mathcal{S}}\mathfrak{A}_\sigma$ is the algebra of events observable from the beginning of the process; so that if we think of z as a function rather than a member of L^0, its values are determinate scalars, and any feature of the process preserved by scalar multiplication ought to be preserved by multiplication by z. I shall give a fair bit of space, in total, to such calculations as 612Lc, but they will nearly always be elementary adaptations of ideas already indicated.

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613 Definition of the integral

I come now to the definition of a stochastic integral which will be used for the next three chapters. We are looking for an effective way to interpret the formula $\int_\tau^{\tau'}\boldsymbol{u}\,d\boldsymbol{v}$ where $\tau\le\tau'$ are stopping times and $\boldsymbol{u}$, $\boldsymbol{v}$ are fully adapted processes defined on an interval $[\tau,\tau']$ in $\mathcal{T}$. I will define this as a kind of Riemann-Stieltjes integral, a limit of 'Riemann sums' of the form $\sum_{i=0}^n u_{\tau_i}\times(v_{\tau_{i+1}}-v_{\tau_i})$ where $\tau=\tau_0\le\ldots\le\tau_n=\tau'$. For this we need a notion of convergence, for which 'convergence in measure' (§§245, 367) turns out to be suitable, and a particular limiting process, to be described in 613Hb. Because our processes are defined on a lattice $\mathcal{T}$ of stopping times, rather than a totally ordered set, there are some technical obstacles to clear out of the way; I aim to do this in 613C-613G. The rest of the section is devoted to elementary properties of this new integral.

613A Probability algebras (a) For the rest of this volume, $(\mathfrak{A},\bar\mu)$ will denote a probability algebra. $L^1(\mathfrak{A},\bar\mu)$ or $L^1_{\bar\mu}$ will be its L^1 space as described in §365, a linear subspace of $L^0(\mathfrak{A})$. For w in $L^0=L^0(\mathfrak{A})$, I will write $\mathbb{E}(w)=\mathbb{E}_{\bar\mu}(w)=\mathbb{E}(w^+)-\mathbb{E}(w^-)$ for its integral with respect to $\bar\mu$ as defined in 365D, provided that at most one of $\mathbb{E}(w^+)$, $\mathbb{E}(w^-)$ is infinite. (I am reserving the symbol $\int$ for the stochastic integral to be defined in 613H.)

(b) As in §§611-612, T will be a totally ordered set and $\langle\mathfrak{A}_t\rangle_{t\in T}$ a filtration of order-closed subalgebras of $\mathfrak{A}$. Recall from 316Fb and 323H that in this context a subalgebra of $\mathfrak{A}$ is order-closed iff it is a σ-subalgebra of $\mathfrak{A}$ iff it is topologically closed in the measure-algebra topology of $\mathfrak{A}$, which is that of the measure metric $(a,b)\mapsto\bar\mu(a\,\triangle\,b)$; so we can safely call such subalgebras 'closed' without specifying which aspect we have primarily in mind. $\mathcal{T}$ will be the set of stopping times adapted to $\langle\mathfrak{A}_t\rangle_{t\in T}$. For $\tau\in\mathcal{T}$, $\mathfrak{A}_\tau$ will be the closed subalgebra corresponding to τ (611G). When I say that a process is 'fully adapted' I shall always mean that it is 'fully adapted to $\langle\mathfrak{A}_t\rangle_{t\in T}$'.

613B Convergence in measure I begin with notes on two fundamental concepts from probability theory. The first is essential for the notion of 'integral' to be defined in this section.

(a) Recall that $L^0=L^0(\mathfrak{A})$ now has a topology $\mathfrak{T}$ of **convergence in measure** which can be defined by the F-norm θ where

$$\theta(w)=\mathbb{E}(|w|\wedge\chi 1)\text{ for every }w\in L^0$$

(245Da, 2A5B, 367L). This is a complete Hausdorff linear space topology for which multiplication and the lattice operations $\vee$, $\wedge$ and $|\;|$ are continuous (367M). Because $(\mathfrak{A},\bar\mu)$ is always isomorphic to the measure algebra of some probability space (321J), we can apply all the results proved in §§245-246 for the topology of convergence in measure on spaces $L^0(\mu)$, as well as those spelt out in §367. In particular, the positive cone $(L^0)^+$ is closed.

Concerning the functional θ, it is subadditive (that is,

$$\begin{aligned}\theta(w_1+w_2)&=\mathbb{E}(|w_1+w_2|\wedge\chi 1)\le\mathbb{E}((|w_1|+|w_2|)\wedge\chi 1)\\&\le\mathbb{E}(|w_1|\wedge\chi 1+|w_2|\wedge\chi 1)=\theta(w_1)+\theta(w_2)\end{aligned}$$

for all $w_1,\ w_2\in L^0$). It is not a seminorm except in trivial cases, but it does have the property that $\theta(\alpha w)\le\alpha\theta(w)$ if $w\in L^0$ and $\alpha\ge 1$. **P** $\mathbb{E}(\alpha|w|\wedge\chi 1)\le\mathbb{E}(\alpha|w|\wedge\alpha\chi 1)=\alpha\mathbb{E}(|w|\wedge\chi 1)$. **Q** Also, of course, $\theta(v)\le\theta(w)$ whenever $|v|\le|w|$ (cf. 354A). Finally, because $\mathbb{E}$ is order-continuous (365Da; cf. 354Dc), $\lim_{w\downarrow A}\theta(w)=0$ whenever $A\subseteq L^0$ is a non-empty downwards-directed family with infimum 0 (367Na), so that $\sup A\in\overline{A}$ and $\lim_{w\uparrow A}\theta(w)=\theta(\sup A)$ whenever $A\subseteq L^0$ is a non-empty upwards-directed set with an upper bound in L^0; similarly, if $A\subseteq L^0$ is a non-empty downwards-directed set with a lower bound in L^0, $\lim_{w\downarrow A}\theta(w)=\theta(\inf A)$.

(b) If $E\subseteq\mathbb{R}$ is a Borel set and $Q_E=\{u:u\in L^0,\ [\![u\in E]\!]=1\}$, then for any continuous $h:E\to\mathbb{R}$ the corresponding function $\bar h:Q_E\to L^0$ is continuous (367S).

(c) If $1\le p\le\infty$, all the $\|\;\|_p$-balls $\{u:u\in L^0,\ \|u\|_p\le\alpha\}$ are $\mathfrak{T}$-closed (245J(b-i), 245Xk). Consequently the $\mathfrak{T}$-closure of a $\|\;\|_p$-bounded set is again $\|\;\|_p$-bounded, and $\|\;\|_p:L^0\to[0,\infty]$ is lower semi-continuous (4A2A).

(d)(i) For any $p\in[1,\infty]$, the embedding $L^p_{\bar\mu}\subsetneq L^0$ is continuous for the norm topology of $L^p_{\bar\mu}$ and $\mathfrak{T}$ (245G).

(ii) If $A\subseteq L^1_{\bar\mu}$ is non-empty and downwards-directed and $\inf A=0$ in $L^1_{\bar\mu}$, then $\inf_{u\in A}\|u\|_1=\lim_{u\downarrow A}\|u\|_1=0$ (365C).

(iii) If $A\subseteq(L^1_{\bar\mu})^+$ is non-empty and upwards-directed and $\gamma=\sup_{u\in A}\|u\|_1$ is finite, then A is bounded above in $L^1_{\bar\mu}$, $\sup A$ belongs to the $\|\;\|_1$-closure of A and $\|\sup A\|_1=\gamma$ (365C again).

(iv) If $u\in L^1_{\bar\mu}$ and $\epsilon>0$, there is a $\delta>0$ such that $\|u-v\|_1\le\epsilon$ whenever $v\in L^1_{\bar\mu}$, $\|v\|_1\le\|u\|_1+\delta$ and $\theta(u-v)\le\delta$ (245H(b-i)).

(e) If $A\subseteq L^0$ and $v\in\overline{A}$ then $[\![v>\alpha]\!]\subseteq\sup_{u\in A}[\![u>\alpha]\!]$ for every $\alpha\in\mathbb{R}$. **P** Setting $a=1\setminus\sup_{u\in A}[\![u>\alpha]\!]$ we see that $u\times\chi a\le\alpha\chi a$ for every $u\in A$, so $v\times\chi a\le\alpha\chi a$ and $[\![v>\alpha]\!]$ does not meet a. **Q**

(f)(i) Because $\mathfrak{T}$ is a linear space topology, there is a corresponding notion of bounded set (3A5N). I will say that a set $A\subseteq L^0$ is **topologically bounded** if for every neighbourhood G of 0 in L^0 there is an $n\in\mathbb{N}$ such that $A\subseteq nG$; equivalently, if for every $\epsilon>0$ there is a $\delta>0$ such that $\theta(\delta u)\le\epsilon$ for every $u\in A$.

(ii) If $A\subseteq L^0$ is non-empty, then A is topologically bounded iff $\inf_{\gamma>0}\sup_{u\in A}\bar\mu[\![|u|>\gamma]\!]=0$. **P($\boldsymbol{\alpha}$)** If A is topologically bounded and $0<\epsilon\le 1$, let $\delta>0$ be such that $\theta(\delta u)\le\epsilon^2$ for every $u\in A$. If $u\in A$ then

$$\epsilon\bar{\mu}[\![|u| > \tfrac{1}{\delta}]\!] = \epsilon\bar{\mu}[\![|\delta u| > 1]\!] \le \theta(\delta u) \le \epsilon^2$$

so $\bar{\mu}[\![|u| > \frac{1}{\delta}]\!] \le \epsilon$. As ϵ is arbitrary, $\inf_{\gamma>0}\sup_{u\in A}\bar{\mu}[\![|u| > \gamma]\!] = 0$. **(β)** If $\inf_{\gamma>0}\sup_{u\in A}\bar{\mu}[\![|u| > \gamma]\!] = 0$, take $\epsilon > 0$. Let $\gamma > 0$ be such that $\bar{\mu}[\![|u| > \gamma]\!] \le \epsilon$ for every $u \in A$, and set $\delta = \dfrac{\epsilon}{\gamma}$. If $u \in A$, then

$$\theta(\delta u) \le \epsilon + \bar{\mu}[\![|\delta u| > \epsilon]\!] \le \epsilon + \bar{\mu}[\![|u| > \gamma]\!] \le 2\epsilon.$$

As ϵ is arbitrary, A is topologically bounded. **Q**

(iii) If A, $B \subseteq L^0$ are topologically bounded, so are $A + B$ and αA for any $\alpha \in \mathbb{R}$, the closure $\overline{A}$ of A for the topology of convergence in measure (3A5Nb), and any subset of A.

(iv) If $A \subseteq L^0$ is topologically bounded, so is its **solid hull** $\{u : u \in L^0, \exists\, v \in A, |u| \le |v|\}$. (For if $|u| \le |v|$, then $\theta(\delta u) \le \theta(\delta v)$.) In particular, an order-bounded subset of L^0 is topologically bounded.

(v) An upwards-directed topologically bounded set is bounded above. **P** If $A \subseteq L^0$ is an upwards-directed set which is not bounded above, then $c = \inf_{n\in\mathbb{N}}\sup_{u\in A}[\![u > n]\!]$ is non-zero (364La). If $n \in \mathbb{N}$, then $\{c \cap [\![u > n]\!] : u \in A\}$ is upwards-directed and has supremum c, so there is a $u_n \in A$ such that $\bar{\mu}(c \setminus [\![u_n > n]\!]) \le 2^{-n-2}\bar{\mu}c$. Set $d = c \setminus \sup_{n\in\mathbb{N}}[\![u_n > n]\!]$; then $\bar{\mu}d > 0$. But now observe that $\theta(\delta u_{n+1}) \ge \bar{\mu}d$ whenever $n \in \mathbb{N}$ and $\delta \ge \frac{1}{n+1}$, so A is not topologically bounded. **Q**

(vi) If $A \subseteq L^0$ is solid, so is $\overline{A}$. **P** If $v \in \overline{A}$, $|u| \le |v|$ and $\epsilon > 0$, there is a $v' \in A$ such that $\theta(v' - v) \le \epsilon$; now $u' = \mathrm{med}(-|v'|, u, |v'|)$ belongs to A, while $u = \mathrm{med}(-|v|, u, |v|)$, so $|u - u'| \le |v - v'|$ and $\theta(u - u') \le \epsilon$. As ϵ is arbitrary, $u \in \overline{A}$; as u and v are arbitrary, $\overline{A}$ is solid. **Q**

(g) If $\bar{\nu} : \mathfrak{A} \to [0, 1]$ is any functional such that $(\mathfrak{A}, \bar{\nu})$ is a probability algebra, then $\bar{\mu}$ and $\bar{\nu}$ are mutually absolutely continuous, that is,

— for every $\epsilon > 0$ there is a $\delta > 0$ such that $\max(\bar{\mu}a, \bar{\nu}a) \le \epsilon$ whenever $a \in \mathfrak{A}$ and $\min(\bar{\mu}a, \bar{\nu}a) \le \delta$ (393F).

(Compare 232Ba.) Consequently $\mathfrak{T}$ is still the topology of convergence in measure on L^0 if we apply the formulae of (a) with the integral $\mathbb{E}_{\bar{\nu}}$ defined from $\bar{\nu}$ in place of $\mathbb{E} = \mathbb{E}_{\bar{\mu}}$ (see 367T), and if we set $\theta_{\bar{\nu}}(w) = \mathbb{E}_{\bar{\nu}}(|w| \wedge \chi 1)$ for $w \in L^0$, then

— for every $\epsilon > 0$ there is a $\delta > 0$ such that $\max(\theta_{\bar{\mu}}(w), \theta_{\bar{\nu}}(w)) \le \epsilon$ whenever $\min(\theta_{\bar{\mu}}(w), \theta_{\bar{\nu}}(w)) \le \delta$.

This will be a recurring theme in the rest of this volume, so I introduce a code phrase here: the topology of convergence in measure is **law-independent**, since replacing the 'law' $\bar{\mu}$ by the law $\bar{\nu}$ leaves it unchanged.

(h) (L^0, θ) is a complete metric space; that is, L^0 is complete when regarded as a linear topological space (367Mc).

(i) Now suppose that $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$.

(i) $L^0(\mathfrak{B})$, regarded as a subset of $L^0(\mathfrak{A})$ (612Ae), is closed for the topology of convergence in measure (367Rc).

(ii) As the embedding $L^1_{\bar{\mu}} \subsetneqq L^0(\mathfrak{A})$ is continuous for the norm topology of $L^1_{\bar{\mu}}$ ((d-i) above), $L^1_{\bar{\mu}} \cap L^0(\mathfrak{B})$ is $\| \|_1$-closed in $L^1_{\bar{\mu}}$; being a linear subspace, it is also closed for the weak topology of $L^1_{\bar{\mu}}$ (4A4Ed).

(j) The following is a useful consequence of (i). Suppose that $\mathcal{S}$ is a sublattice of $\mathcal{T}$, $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a fully adapted process, and $A \subseteq \mathcal{S}$ is a non-empty downwards-directed set such that the limit $u_* = \lim_{\sigma\downarrow A} u_\sigma$ is defined. Then $u_* \in \bigcap_{\sigma\in A} L^0(\mathfrak{A}_\sigma) = L^0(\bigcap_{\sigma\in A}\mathfrak{A}_\sigma)$. **P** If $\sigma \in A$, then u_* belongs to the closure of $\{u_{\sigma'} : \sigma' \in A, \sigma' \le \sigma\} \subseteq L^0(\mathfrak{A}_\sigma)$; as $L^0(\mathfrak{A}_\sigma)$ is closed, $u_* \in L^0(\mathfrak{A}_\sigma)$. As σ is arbitrary, $u_* \in \bigcap_{\sigma\in A} L^0(\mathfrak{A}_\sigma)$. For the other expression, write $\mathfrak{B}$ for $\bigcap_{\sigma\in A}\mathfrak{A}_\sigma$; being the intersection of closed subalgebras of $\mathfrak{A}$, $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$. Now, for $v \in L^0$,

$$\begin{aligned} v \in L^0(\mathfrak{B}) &\iff [\![v > \alpha]\!] \in \mathfrak{B} \text{ for every } \alpha \in \mathbb{R} \\ &\iff [\![v > \alpha]\!] \in \mathfrak{A}_\sigma \text{ for every } \alpha \in \mathbb{R} \text{ and } \sigma \in A \\ &\iff v \in L^0(\mathfrak{A}_\sigma) \text{ for every } \sigma \in A, \end{aligned}$$

so $L^0(\mathfrak{B}) = \bigcap_{\sigma\in A} L^0(\mathfrak{A}_\sigma)$, and in particular contains u_*. **Q**

(k) A particular manifestation of the idea in (j) above appears often enough to be given a name. If $\mathcal{S}$ is non-empty and $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a fully adapted process such that the topological limit $u_\downarrow=\lim_{\sigma\downarrow\mathcal{S}}u_\sigma$ is defined in $L^0(\mathfrak{A})$, I will call $u_\downarrow\in L^0(\bigcap_{\sigma\in\mathcal{S}}\mathfrak{A}_\sigma)$ the **starting value** of $\boldsymbol{u}$.

(l) If $\mathcal{S}$ is a sublattice of $\mathcal{T}$, then we can give $(L^0)^{\mathcal{S}}$ its product topology, under which it is a linear topological space (4A4Bb). Now the space $M_{\mathrm{fa}}(\mathcal{S})$ of fully adapted processes with domain $\mathcal{S}$ is a closed subspace of $(L^0)^{\mathcal{S}}$. **P**

$$M_{\mathrm{fa}}(\mathcal{S})=(L^0)^{\mathcal{S}}\cap\bigcap_{\tau,\tau'\in\mathcal{S}}\{\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}:\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}\in(L^0)^{\mathcal{S}},$$

$$u_\tau\in L^0(\mathfrak{A}_\tau),\ [\![\tau=\tau']\!]\subseteq[\![u_\tau=u_{\tau'}]\!]\}.$$

Accordingly $L^0(\mathfrak{A}_\tau)$ is closed in L^0 for every τ, by (i-i) above. Moreover, if τ, $\tau'\in\mathcal{S}$ and $a\in\mathfrak{A}$, then

$$\{(u,v):u,\,v\in L^0,\,a\subseteq[\![u=v]\!]\}=\{(u,v):u,\,v\in L^0,\,u\times\chi a=v\times\chi a\}$$

is closed in $(L^0)^2$ because multiplication in L^0 is continuous and the topology of L^0 is Hausdorff. So $M_{\mathrm{fa}}(\mathcal{S})$ is an intersection of closed sets and is closed. **Q**

(m) Because the lattice operations on L^0 are continuous, and the topology is Hausdorff, sets of the form $\{|u|:u\le\bar u\}=\{u:|u|\vee\bar u=\bar u\}$ are closed for any $\bar u\in L^0$. Consequently, in a product space $(L^0)^{\mathcal{S}}$, sets of the form $\{\boldsymbol{u}:\boldsymbol{u}\in(L^0)^{\mathcal{S}},\,|\boldsymbol{u}|\le\bar{\boldsymbol{u}}\}$, where $\bar{\boldsymbol{u}}\in(L^0)^{\mathcal{S}}$, are closed for the product topology.

(n) Now suppose that $(\mathfrak{B},\bar\nu)$ is another probability algebra, and $\phi:\mathfrak{A}\to\mathfrak{B}$ is a measure-preserving Boolean homomorphism. Then ϕ is order-continuous (324Kb), so we have a corresponding injective f-algebra homomorphism $T_\phi:L^0(\mathfrak{A})\to L^0(\mathfrak{B})$ (612Af). Writing $\mathbb{E}_{\bar\mu}$, $\mathbb{E}_{\bar\nu}$ for expectations in $L^1_{\bar\mu}$, $L^1_{\bar\nu}$ respectively, and $\theta_{\bar\mu}$, $\theta_{\bar\nu}$ for the corresponding functionals on $L^0(\mathfrak{A})$ and $L^0(\mathfrak{B})$, $\mathbb{E}_{\bar\nu}(T_\phi u)=\mathbb{E}_{\bar\mu}(u)$ for every $u\in L^1_{\bar\mu}$ (365Nb[3]); as T_ϕ is a Riesz homomorphism and $T_\phi(\chi 1_{\mathfrak{A}})=\chi 1_{\mathfrak{B}}$, $\theta_{\bar\nu}(T_\phi u)=\theta_{\bar\mu}(u)$ for every $u\in L^0(\mathfrak{A})$, and T_ϕ is continuous for the topologies of convergence in measure.

(o) For any $\alpha\in\mathbb{R}$, the function $u\mapsto\bar\mu[\![u>\alpha]\!]:L^0\to[0,1]$ is lower semi-continuous. **P** Suppose that $u\in L^0=L^0(\mathfrak{A})$ and $\bar\mu[\![u>\alpha]\!]>t$. Then there is a $\delta\in\,]0,1]$ such that $\bar\mu[\![u>\alpha+\delta]\!]>t+\delta$. Suppose that $v\in L^0$ is such that $\theta(v-u)\le\delta^2$. Then $\bar\mu[\![|v-u|\ge\delta]\!]\le\delta$. Now

$$[\![v>\alpha]\!]\supseteq[\![u>\alpha+\delta]\!]\cap[\![v-u\ge-\delta]\!],$$

$$\bar\mu[\![v>\alpha]\!]\ge\bar\mu[\![u>\alpha+\delta]\!]-\bar\mu[\![|v-u|>\delta]\!]>t.$$

This shows that $\{u:\bar\mu[\![u>\alpha]\!]>t\}$ is open; as t is arbitrary, $u\mapsto\bar\mu[\![u>\alpha]\!]:L^0\to[0,1]$ is lower semi-continuous. **Q**

(p)(i) Suppose that $A\subseteq L^0$ and that for every $\epsilon>0$ there is an $a\in\mathfrak{A}$ such that $\{u\times\chi a:u\in A\}$ is order-bounded in L^0 and $\bar\mu a\ge 1-\epsilon$. Then A is order-bounded in L^0. **P** For each $n\in\mathbb{N}$ set $c_n=\sup_{u\in A}[\![|u|>n]\!]$. Then $\langle c_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$. Given $\epsilon>0$, there is an $a\in\mathfrak{A}$ such that $\bar\mu a\ge 1-\frac12\epsilon$ and $\bar u=\sup_{u\in A}|u\times\chi a|$ is defined. Let $n\in\mathbb{N}$ be such that $b=[\![\bar u>n]\!]$ has measure at most $\frac12\epsilon$. If $u\in A$,

$$[\![|u|>n]\!]\subseteq(1\setminus a)\cup[\![|u\times\chi a|>n]\!]\subseteq(1\setminus a)\cup b.$$

So if $m\ge n$,

$$c_m\subseteq c_n\subseteq(1\setminus a)\cup b,\quad \bar\mu c_m\le(1-\bar\mu a)+\bar\mu b\le\epsilon.$$

As ϵ is arbitrary, $\lim_{n\to\infty}\bar\mu c_n=0$.

Consequently $\inf_{n\in\mathbb{N}}c_n=0$ and $\{|u|:u\in A\}$ is bounded above in L^0 (364L(a-i)), that is, A is order-bounded in L^0. **Q**

(ii) If $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a fully adapted process and for every $\epsilon>0$ there is an order-bounded process $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ such that $\bar\mu[\![\boldsymbol{u}\ne\boldsymbol{v}]\!]\le\epsilon$, then $\boldsymbol{u}$ is order-bounded. **P** Set $A=\{u_\sigma:\sigma\in\mathcal{S}\}$. If $\epsilon>0$, there is an order-bounded process $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ such that $\bar\mu[\![\boldsymbol{u}\ne\boldsymbol{v}]\!]\le\epsilon$. Set $a=1\setminus[\![\boldsymbol{u}\ne\boldsymbol{v}]\!]$. Then $\bar\mu a\ge 1-\epsilon$ and

[3] Formerly 365Ob.

$$|u_\sigma \times \chi a| = |v_\sigma \times \chi a| \le \sup |\boldsymbol{v}|$$

for every $\sigma \in \mathcal{S}$; so $\{u \times \chi a : u \in A\}$ is order-bounded in L^0. By (i) just above, A is order-bounded in L^0, that is, $\boldsymbol{u}$ is order-bounded. **Q**

(q)(i) If $A \subseteq \mathcal{T}$, τ belongs to the covered envelope $\hat{A}$ of A (611Ma) and $\epsilon > 0$, there is a τ' in the finitely-covered envelope $\hat{A}_f$ of A (611Ob) such that $\bar{\mu}[\![\tau = \tau']\!] \ge 1-\epsilon$. **P** We know that $\sup_{\sigma\in A} [\![\sigma = \tau]\!] = 1$ in $\mathfrak{A}$, so there is a non-empty finite $J \subseteq A$ such that $\bar{\mu}(\sup_{\sigma\in J} [\![\sigma = \tau]\!]) \ge 1-\epsilon$. Let I be the sublattice of $\mathcal{T}$ generated by J; then I is finite (611Ca) and included in $\hat{A}_f$ (611Pa). Take a sequence $(\sigma_0, \dots, \sigma_n)$ linearly generating the I-cells (611K-611L). Set $a_k = [\![\sigma_k = \tau]\!]$ for $k \le n$, $a = \sup_{k\in\mathbb{N}} a_k$, $b_k = a_k \setminus \sup_{i<k} a_i$ for $k < n$ and $b_n = 1 \setminus \sup_{k<n} a_k$. Then a_k, $b_k \in \mathfrak{A}_{\sigma_k}$ for $k \le n$ and $\langle b_k\rangle_{k\le n}$ is a partition of unity in $\mathfrak{A}$.

Observe that if $\sigma \in J$ then $\sup_{k\le n} [\![\sigma = \sigma_k]\!] = 1$ (611Ke), so

$$[\![\sigma = \tau]\!] = \sup_{k\le n} [\![\sigma = \tau]\!] \cap [\![\sigma = \sigma_k]\!] \subseteq \sup_{k\le n} [\![\sigma_k = \tau]\!];$$

taking the supremum over σ, $a \subseteq \sup_{k\le n} [\![\sigma_k = \tau]\!]$.

By 611I there is a $\tau' \in \mathcal{T}$ such that $b_k \subseteq [\![\tau' = \sigma_k]\!]$ for $k \le n$. Next, for $k \le n$ let K_k be a finite subset of A such that $\sup_{\sigma\in K_k} [\![\sigma = \sigma_k]\!] = 1$, and set $K = \bigcup_{k\le n} K_k$. Then K is a finite subset of A,

$$\begin{aligned}\sup_{\sigma\in K} [\![\sigma = \tau']\!] &= \sup_{k\le n, \sigma\in K} b_k \cap [\![\sigma = \tau']\!] \\ &= \sup_{k\le n, \sigma\in K_k} b_k \cap [\![\sigma = \sigma_k]\!] = \sup_{k\le n} b_k = 1\end{aligned}$$

and $\tau' \in \hat{A}_f$.

Moreover,

$$\begin{aligned}[\![\tau = \tau']\!] &= \sup_{k\le n} b_k \cap [\![\tau = \tau']\!] = \sup_{k\le n} b_k \cap [\![\tau = \sigma_k]\!] \\ &= \sup_{k\le n} b_k \cap a_k = \sup_{k\le n}(a_k \setminus \sup_{i<k} a_i) = \sup_{k\le n} a_k \supseteq a\end{aligned}$$

has measure at least $1-\epsilon$, as required. **Q**

(ii) If $\mathcal{S}$ is a sublattice of $\mathcal{T}$ with covered envelope $\hat{\mathcal{S}}$ and finitely covered envelope $\hat{\mathcal{S}}_f$, $\boldsymbol{u} = \langle u_\tau\rangle_{\tau\in\hat{\mathcal{S}}}$ is a fully adapted process and $\tau \in \hat{\mathcal{S}}$, then u_τ belongs to the closure of $\{u_\sigma : \sigma \in \hat{\mathcal{S}}_f\}$ for the topology of convergence in measure. **P** For any $\epsilon > 0$ there is a $\sigma \in \hat{\mathcal{S}}_f$ such that $\bar{\mu}[\![\sigma - \tau]\!] \ge 1-\epsilon$, by (i) just above; now

$$\begin{aligned}\theta(u_\tau - u_\sigma) &= \mathbb{E}(|u_\tau - u_\sigma| \wedge \chi 1) \le \bar{\mu}[\![u_\tau \ne u_\sigma]\!] \\ &= 1 - \bar{\mu}[\![u_\tau = u_\sigma]\!] \le 1 - \bar{\mu}[\![\tau = \sigma]\!] \le \epsilon.\end{aligned}$$

As $\epsilon > 0$ is arbitrary, $u_\tau \in \overline{\{u_\sigma : \sigma \in \hat{\mathcal{S}}_f\}}$. **Q**

613C Interval functions Now for a new idea.

(a) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. I will write $\mathcal{S}^{2\uparrow}$ for $\{(\sigma, \tau) : \sigma, \tau \in \mathcal{S}, \sigma \le \tau\}$.

(i) I say that a function $\psi : \mathcal{S}^{2\uparrow} \to L^0(\mathfrak{A})$ is an **adapted interval function** on $\mathcal{S}$ if

$$\psi(\sigma, \tau) \in L^0(\mathfrak{A}_\tau), \quad \psi(\sigma, \sigma) = 0, \quad b \subseteq [\![\psi(\sigma, \tau) = \psi(\sigma', \tau')]\!]$$

whenever $\sigma \le \sigma' \le \tau' \le \tau$ in $\mathcal{S}$, $b \in \mathfrak{A}_\sigma$ and $b \subseteq [\![\sigma = \sigma']\!] \cap [\![\tau' = \tau]\!]$.

(ii) In this case, I say that ψ is a **strictly adapted interval function** if

$$[\![\sigma = \sigma']\!] \cap [\![\tau' = \tau]\!] \subseteq [\![\psi(\sigma, \tau) = \psi(\sigma', \tau')]\!]$$

whenever $\sigma \le \sigma' \le \tau' \le \tau$ in $\mathcal{S}$.

(b) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\psi : \mathcal{S}^{2\uparrow} \to L^0(\mathfrak{A})$ an adapted interval function.

(i) $[\![\sigma = \tau]\!] \subseteq [\![\psi(\sigma, \tau) = 0]\!]$ whenever $\sigma \le \tau$ in $\mathcal{S}$. **P** Setting $\sigma' = \tau' = \sigma$, we have

$$[\![\sigma = \tau]\!] = [\![\sigma = \sigma']\!] \cap [\![\tau' = \tau]\!] \in \mathfrak{A}_\sigma$$

so

$$[\![\sigma = \tau]\!] \subseteq [\![\psi(\sigma, \tau) = \psi(\sigma', \tau')]\!] \subseteq [\![\psi(\sigma, \tau) = 0]\!]. \textbf{ Q}$$

Taking complements, $[\![\psi(\sigma, \tau) \ne 0]\!] \subseteq [\![\sigma < \tau]\!]$ whenever $\sigma \le \tau$ in $\mathcal{S}$.

(ii) $\psi(\sigma, \sigma \vee \tau) = \psi(\sigma \wedge \tau, \tau)$ for all σ, $\tau \in \mathcal{S}$. **P** $[\![\sigma \le \tau]\!] \in \mathfrak{A}_{\sigma\wedge\tau}$ (611Hc) and

$$\begin{aligned}
[\![\sigma \le \tau]\!] &= [\![\sigma \wedge \tau = \sigma]\!] \cap [\![\tau = \sigma \vee \tau]\!] \\
&\subseteq [\![\psi(\sigma \wedge \tau, \sigma \vee \tau) = \psi(\sigma, \sigma \vee \tau)]\!] \cap [\![\psi(\sigma \wedge \tau, \sigma \vee \tau) = \psi(\sigma \wedge \tau, \tau)]\!] \\
&\subseteq [\![\psi(\sigma, \sigma \vee \tau) = \psi(\sigma \wedge \tau, \tau)]\!],
\end{aligned}$$

$$\begin{aligned}
[\![\tau \le \sigma]\!] &= [\![\sigma \wedge \tau = \tau]\!] \cap [\![\sigma = \sigma \vee \tau]\!] \\
&\subseteq [\![\psi(\sigma \wedge \tau, \tau) = 0]\!] \cap [\![\psi(\sigma, \sigma \vee \tau) = 0]\!] \subseteq [\![\psi(\sigma, \sigma \vee \tau) = \psi(\sigma \wedge \tau, \tau)]\!];
\end{aligned}$$

putting these together, $[\![\psi(\sigma, \sigma \vee \tau) = \psi(\sigma \wedge \tau, \tau)]\!] = 1$ and $\psi(\sigma, \sigma \vee \tau) = \psi(\sigma \wedge \tau, \tau)$. **Q**

(iii) If $\mathcal{S}_0$ is any sublattice of $\mathcal{S}$, then $\psi \restriction \mathcal{S}_0^{2\uparrow}$ is an adapted interval function on $\mathcal{S}_0$, and is strictly adapted if ψ is. (Immediate from the definitions in (a).)

(iv) If ψ is strictly adapted then $[\![\sigma = \sigma']\!] \cap [\![\tau' = \tau]\!] \subseteq [\![\psi(\sigma, \tau) = \psi(\sigma', \tau')]\!]$ whenever $\sigma \le \tau$ and $\sigma' \le \tau'$ in $\mathcal{S}$. **P**

$$\begin{aligned}
[\![\sigma = \sigma']\!] \cap [\![\tau' = \tau]\!] &= [\![\sigma \wedge \sigma' = \sigma]\!] \cap [\![\tau = \tau \vee \tau']\!] \cap [\![\sigma \wedge \sigma' = \sigma']\!] \cap [\![\tau' = \tau \vee \tau']\!] \\
&\subseteq [\![\psi(\sigma, \tau) = \psi(\sigma \wedge \sigma', \tau \vee \tau')]\!] \cap [\![\psi(\sigma', \tau') = \psi(\sigma \wedge \sigma', \tau \vee \tau')]\!] \\
&\subseteq [\![\psi(\sigma, \tau) = \psi(\sigma', \tau')]\!]. \textbf{ Q}
\end{aligned}$$

(c) Much the most important examples of such functions are the following. If $\mathcal{S} \subseteq \mathcal{T}$ is a sublattice and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ is a fully adapted process, we have a function $\Delta\boldsymbol{v} : \mathcal{S}^{2\uparrow} \to L^0(\mathfrak{A})$ defined by saying that

$$(\Delta\boldsymbol{v})(\sigma, \tau) = v_\tau - v_\sigma$$

whenever $\sigma \le \tau$ in $\mathcal{S}$, and $\Delta\boldsymbol{v}$ is a strictly adapted interval function on $\mathcal{S}$. **P** If $\sigma \le \sigma' \le \tau' \le \tau$ in $\mathcal{S}$, then $v_\tau - v_\sigma \in L^0(\mathfrak{A}_\tau)$ because $\sigma \le \tau$, $[\![v_\tau - v_\sigma \ne 0]\!] \subseteq [\![\sigma \ne \tau]\!] = [\![\sigma < \tau]\!]$, and

$$[\![\sigma = \sigma']\!] \cap [\![\tau = \tau']\!] \subseteq [\![v_\sigma = v_{\sigma'}]\!] \cap [\![v_{\tau'} = v_\tau]\!] \subseteq [\![v_\tau - v_\sigma = v_{\tau'} - v_{\sigma'}]\!]. \textbf{ Q}$$

613D Constructions for interval functions Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and ψ, ψ' (strictly) adapted interval functions on $\mathcal{S}$.

(a) $\psi + \psi'$ and $\psi \times \psi'$ are (strictly) adapted interval functions.

(b) If $h : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function, then the composition $\bar{h}\psi$ is a (strictly) adapted interval function.

(c) In particular, ψ^2 and $|\psi|$ and $\alpha\psi$, for any $\alpha \in \mathbb{R}$, are (strictly) adapted interval functions; the space of (strictly) adapted interval functions on $\mathcal{S}$ is an f-subalgebra of $L^0(\mathfrak{A})^{\mathcal{S}^{2\uparrow}}$.

(d) If $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ is a fully adapted process, then we have a (strictly) adapted interval function $\boldsymbol{u}\psi$ on $\mathcal{S}$ defined by setting $(\boldsymbol{u}\psi)(\sigma, \tau) = u_\sigma \times \psi(\sigma, \tau)$ whenever $\sigma \le \tau$ in $\mathcal{S}$.

proof (a)-(c) are immediate consequences of the definitions in 613Ca.

(d) Write $\psi'(\sigma, \tau) = u_\sigma \times \psi(\sigma, \tau)$. If $\sigma \le \tau$ in $\mathcal{S}$, then $\psi'(\sigma, \tau) \in L^0(\mathfrak{A}_\tau)$ because u_σ and $\psi(\sigma, \tau)$ both belong to $L^0(\mathfrak{A}_\tau)$; and

$$[\![\sigma = \tau]\!] \subseteq [\![\psi(\sigma, \tau) = 0]\!] \subseteq [\![\psi'(\sigma, \tau) = 0]\!].$$

Next, as in 613Cc, take σ, τ, σ', $\tau' \in \mathcal{S}$ such that $\sigma \le \sigma' \le \tau' \le \tau$ and $b \in \mathfrak{A}_\sigma$ such that $b \subseteq [\![\sigma = \sigma']\!] \cap [\![\tau' = \tau]\!]$. Then

$$b \subseteq [\![u_\sigma = u_{\sigma'}]\!] \cap [\![\psi(\sigma,\tau) = \psi(\sigma',\tau')]\!] \subseteq [\![\psi'(\sigma,\tau) = \psi'(\sigma',\tau')]\!].$$

If ψ is strictly adapted, then the same formula will be valid with $b = [\![\sigma = \sigma']\!] \cap [\![\tau' = \tau]\!]$

Remark Another class of adapted interval functions will be based on conditional expectations; see 626H.

613E Riemann sums Let $\mathcal{S} \subseteq \mathcal{T}$ be a sublattice, ψ an adapted interval function on $\mathcal{S}$, and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a fully adapted process. For $\sigma \le \tau$ in $\mathcal{S}$ set

$$c(\sigma,\tau) = \langle [\![\tau > t]\!] \setminus [\![\sigma > t]\!]\rangle_{t\in T}$$

as in 611J.

(a) For a stopping-time interval $e \in \mathrm{Sti}(\mathcal{S})$, we can define $\Delta_e(\boldsymbol{u}, d\psi)$ by saying that $\Delta_e(\boldsymbol{u}, d\psi) = u_\sigma \times \psi(\sigma,\tau)$ whenever $e = c(\sigma,\tau)$ with $\sigma \le \tau$ in $\mathcal{S}$. **P** I need to confirm that if $e = c(\sigma,\tau) = c(\sigma',\tau')$ then $u_\sigma \times \psi(\sigma,\tau) = u_{\sigma'} \times \psi(\sigma',\tau')$.

(i) Consider first the case in which $\sigma \le \sigma' \le \tau' \le \tau$. Set $b = [\![\sigma < \tau]\!] \in \mathfrak{A}_\sigma$. Then

$$b = [\![\sigma' < \tau']\!] \subseteq [\![\sigma = \sigma']\!] \cap [\![\tau = \tau']\!]$$

(611Jd), so

$$b \subseteq [\![u_\sigma = u_{\sigma'}]\!] \cap [\![\psi(\sigma,\tau) = \psi(\sigma',\tau')]\!] \subseteq [\![u_\sigma \times \psi(\sigma,\tau) = u_{\tau'} \times \psi(\sigma',\tau')]\!].$$

On the other hand,

$$\begin{aligned}1 \setminus b &\subseteq [\![\psi(\sigma,\tau) = 0]\!] \cap [\![\psi(\sigma',\tau') = 0]\!] \subseteq [\![u_\sigma \times \psi(\sigma,\tau) = 0]\!] \cap [\![u_{\tau'} \times \psi(\sigma',\tau') = 0]\!] \\ &\subseteq [\![u_\sigma \times \psi(\sigma,\tau) = u_{\tau'} \times \psi(\sigma',\tau')]\!].\end{aligned}$$

So $u_\sigma \times \psi(\sigma,\tau) = u_{\tau'} \times \psi(\sigma',\tau')$.

(ii) Generally, if $c(\sigma,\tau) = c(\sigma',\tau')$ then

$$[\![\sigma < \tau]\!] = [\![\sigma' < \tau']\!] \subseteq [\![\sigma = \sigma']\!] \cap [\![\tau = \tau']\!]$$

so

$$[\![\sigma \wedge \sigma' < \tau \vee \tau']\!] = [\![\sigma < \tau]\!] \subseteq [\![\sigma \wedge \sigma' = \sigma]\!] \cap [\![\tau = \tau \vee \tau']\!]$$

and $c(\sigma \wedge \sigma', \tau \vee \tau') = c(\sigma,\tau)$. By (i), $u_{\sigma'\wedge\tau'} \times \psi(\sigma \wedge \sigma', \tau \vee \tau') = u_\sigma \times \psi(\sigma,\tau)$. But the same argument shows that $u_{\sigma'\wedge\tau'} \times \psi(\sigma \wedge \sigma', \tau \vee \tau') = u_{\sigma'} \times \psi(\sigma',\tau')$, so $u_\sigma \times \psi(\sigma,\tau) = u_{\sigma'} \times \psi(\sigma',\tau')$, as required. **Q**

(b) If $I \subseteq \mathcal{S}$ is a finite sublattice and $\mathrm{Sti}_0(I) \subseteq \mathrm{Sti}(\mathcal{S})$ is the set of I-cells (611Je), write

$$S_I(\boldsymbol{u}, d\psi) = \textstyle\sum_{e\in\mathrm{Sti}_0(I)} \Delta_e(\boldsymbol{u}, d\psi).$$

(c) In this context, I will repeatedly use 611L, in the following way. If $I \subseteq \mathcal{S}$ is a non-empty finite sublattice, then there is a string $(\tau_0, \dots, \tau_n)$ in I linearly generating the I-cells. In this case $S_I(\boldsymbol{u}, d\psi)$ will be $\sum_{i=0}^{n-1} u_{\tau_i} \times \psi(\tau_i, \tau_{i+1})$. It will not be necessary to check that the sequence $\langle\tau_i\rangle_{i\le n}$ is strictly increasing, because if $\tau_i = \tau_{i+1}$ then $\psi(\tau_i, \tau_{i+1})$ will be 0.

(d) Now suppose that $\psi = \Delta\boldsymbol{v}$ for some fully adapted process $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$. If $I \subseteq \mathcal{S}$ is any non-empty sublattice, then $S_I(\mathbf{1}, d(\Delta\boldsymbol{v})) = v_{\max I} - v_{\min I}$. **P** (Recall from 612D that $\mathbf{1}$ is the constant process with value $\chi 1$.) Take $(\tau_0, \dots, \tau_n)$ linearly generating the I-cells. Then

$$\begin{aligned}S_I(\mathbf{1}, d(\Delta\boldsymbol{v})) &= \sum_{i=0}^{n-1} \chi 1 \times (\Delta\boldsymbol{v})(c(\tau_i,\tau_{i+1})) = \sum_{i=0}^{n-1} v_{\tau_{i+1}} - v_{\tau_i} \\ &= v_{\tau_n} - v_{\tau_0} = v_{\max I} - v_{\min I}.\ \mathbf{Q}\end{aligned}$$

(e) If $\psi = \boldsymbol{u}\psi'$, where ψ' is another adapted interval function with domain $\mathcal{S}^{2\uparrow}$ and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a fully adapted process (613Dd), then $S_I(\mathbf{1}, d\psi) = S_I(\boldsymbol{u}, d\psi')$ for every finite sublattice I of $\mathcal{S}$. **P**

$$\Delta_e(\mathbf{1}, d\psi) = \psi(\sigma,\tau) = u_\sigma \times \psi'(\sigma,\tau) = \Delta_e(\boldsymbol{u}, d\psi')$$

whenever $e = c(\sigma,\tau)$ is a stopping-time interval with endpoints σ, τ in $\mathcal{S}$. So

$$S_I(\mathbf{1}, d\psi) = \sum_{e\in\mathrm{Sti}_0(I)}\Delta_e(\mathbf{1}, d\psi) = \sum_{e\in\mathrm{Sti}_0(I)}\Delta_e(\boldsymbol{u}, d\psi') = S_I(\boldsymbol{u}, d\psi'). \text{ } \mathbf{Q}$$

613F Notation I will use abbreviations for some of the most important interval functions. Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$, $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ and $\boldsymbol{w} = \langle w_\sigma\rangle_{\sigma\in\mathcal{S}}$ fully adapted processes.

(a) If $\sigma \le \tau$ in $\mathcal{S}$ and $e = c(\sigma,\tau)$, then I write

$$\Delta_e(\boldsymbol{u}, d\boldsymbol{v}) = \Delta_e(\boldsymbol{u}, d(\Delta\boldsymbol{v})) = u_\sigma \times (v_\tau - v_\sigma),$$

$$\Delta_e(\boldsymbol{u}, d\boldsymbol{v}d\boldsymbol{w}) = \Delta_e(\boldsymbol{u}, d(\Delta\boldsymbol{v}\times\Delta\boldsymbol{w})) = u_\sigma \times (v_\tau - v_\sigma) \times (w_\tau - w_\sigma),$$

$$\Delta_e(\boldsymbol{u}, |d\boldsymbol{v}|) = \Delta_e(\boldsymbol{u}, d|\Delta\boldsymbol{v}|) = u_\sigma \times |v_\tau - v_\sigma|.$$

(b) Now if $I \subseteq \mathcal{S}$ is a finite sublattice and $\mathrm{Sti}_0(I)$ is the set of I-cells, write

$$S_I(\boldsymbol{u}, d\boldsymbol{v}) = S_I(\boldsymbol{u}, d(\Delta\boldsymbol{v})) = \sum_{e\in\mathrm{Sti}_0(I)}\Delta_e(\boldsymbol{u}, d\boldsymbol{v}),$$

$$S_I(\boldsymbol{u}, d\boldsymbol{v}d\boldsymbol{w}) = S_I(\boldsymbol{u}, d(\Delta\boldsymbol{v}\times\Delta\boldsymbol{w})) = \sum_{e\in\mathrm{Sti}_0(I)}\Delta_e(\boldsymbol{u}, d\boldsymbol{v}d\boldsymbol{w}),$$

$$S_I(\boldsymbol{u}, |d\boldsymbol{v}|) = S_I(\boldsymbol{u}, d|\Delta\boldsymbol{v}|) = \sum_{e\in\mathrm{Sti}_0(I)}\Delta_e(\boldsymbol{u}, |d\boldsymbol{v}|).$$

I hope that these expressions will make the formulae below more appealing, and perhaps offer some hints of the manipulations which will be possible.

613G Proposition Suppose that I is a finite sublattice of $\mathcal{T}$, $\psi : I^{2\uparrow} \to L^0(\mathfrak{A})$ is an adapted interval function and $\boldsymbol{u} = \langle u_\tau\rangle_{\tau\in I}$ is a fully adapted process.

(a)(i) If $\tau \in I$ and we set $I\vee\tau = \{\sigma\vee\tau : \sigma \in I\} = \{\sigma : \tau \le \sigma \in I\}$ and $I\wedge\tau = \{\sigma\wedge\tau : \sigma \in I\} = \{\sigma : \sigma \in I, \sigma \le \tau\}$, then $S_I(\boldsymbol{u}, d\psi) = S_{I\wedge\tau}(\boldsymbol{u}, d\psi) + S_{I\vee\tau}(\boldsymbol{u}, d\psi)$.

(ii) If $\tau_0, \dots, \tau_n \in I$ and $\min I = \tau_0 \le \tau_1 \le \ldots \le \tau_n = \max I$, then $S_I(\boldsymbol{u}, d\psi) = \sum_{i=0}^{n-1} S_{I\cap[\tau_i,\tau_{i+1}]}(\boldsymbol{u}, d\psi)$.

(b) For $\tau \in I$ set $z_\tau = S_{I\wedge\tau}(\boldsymbol{u}, d\psi)$. Then $\langle z_\tau\rangle_{\tau\in I}$ is a fully adapted process.

(c) If τ, $\tau' \in I$ then $S_{I\wedge\tau}(\boldsymbol{u}, d\psi) + S_{I\wedge\tau'}(\boldsymbol{u}, d\psi) = S_{I\wedge(\tau\vee\tau')}(\boldsymbol{u}, d\psi) + S_{I\wedge(\tau\wedge\tau')}(\boldsymbol{u}, d\psi)$.

(d) $[\![S_I(\boldsymbol{u}, d\psi) \ne 0]\!] \subseteq [\![\boldsymbol{u} \ne \mathbf{0}]\!] \cap [\![\min I < \max I]\!]$.

(e) If $\boldsymbol{v} = \langle v_\tau\rangle_{\tau\in I}$ is another fully adapted process, then $S_I(\boldsymbol{u}, d(\boldsymbol{v}\psi)) = S_I(\boldsymbol{u}\times\boldsymbol{v}, d\psi)$.

proof (a)(i) We have only to recall that $\mathrm{Sti}_0(I)$ is the disjoint union of $\mathrm{Sti}_0(I\wedge\tau)$ and $\mathrm{Sti}_0(I\vee\tau)$ (611J(e-ii)).

(ii) Use 611J(e-iii).

(b)(i) If $\tau \in I$ and $e \in \mathrm{Sti}_0(I\wedge\tau)$ then e is expressible as $c(\sigma,\sigma')$ where $\sigma \le \sigma' \le \tau$, in which case

$$u_\sigma \times \psi(\sigma,\sigma') \in L^0(\mathfrak{A}_{\sigma'}) \subseteq L^0(\mathfrak{A}_\tau).$$

So $z_\tau \in L^0(\mathfrak{A}_\tau)$.

(ii) Suppose that $\tau \le \tau' \in I$. Then

$$\begin{aligned} z_{\tau'} &= S_{I\wedge\tau'}(\boldsymbol{u}, d\psi) = S_{(I\wedge\tau')\wedge\tau}(\boldsymbol{u}, d\psi) + S_{(I\wedge\tau')\vee\tau}(\boldsymbol{u}, d\psi) \\ &= S_{I\wedge\tau}(\boldsymbol{u}, d\psi) + S_{I\cap[\tau,\tau']}(\boldsymbol{u}, d\psi) = z_\tau + S_{I\cap[\tau,\tau']}(\boldsymbol{u}, d\psi). \end{aligned}$$

Now if $e \in \mathrm{Sti}_0(I\cap[\tau,\tau'])$ there are σ, σ' such that $\tau \le \sigma \le \sigma' \le \tau'$ and $e = c(\sigma,\sigma')$. In this case

$$[\![\tau = \tau']\!] \subseteq [\![\sigma = \sigma']\!] \subseteq [\![\psi(\sigma,\sigma') = 0]\!] \subseteq [\![\Delta_e(\boldsymbol{u}, d\psi) = 0]\!]$$

(using 613C(b-i)). So

$$[\![\tau = \tau']\!] \subseteq [\![\sum_{e\in\mathrm{Sti}_0(I\cap[\tau,\tau'])}\Delta_e(\boldsymbol{u}, d\psi) = 0]\!] = [\![S_{I\cap[\tau,\tau']}(\boldsymbol{u}, d\psi) = 0]\!] = [\![z_\tau = z_{\tau'}]\!].$$

By 612Db, $\langle z_\tau\rangle_{\tau\in I}$ is fully adapted.

(c) Put (b) and 612D(f-i) together.

(d)(i) Setting $a=[\![\boldsymbol{u}\ne 0]\!]=\sup_{\sigma\in I}[\![u_\sigma\ne 0]\!]$, we have

$$\Delta_e(\boldsymbol{u},d\psi)=u_\sigma\times\psi(\sigma,\tau)=u_\sigma\times\chi a\times\psi(\sigma,\tau)=\Delta_e(\boldsymbol{u},d\psi)\times\chi a$$

whenever $e=c(\sigma,\tau)$ is an I-cell, so $S_I(\boldsymbol{u},d\psi)=S_I(\boldsymbol{u},d\psi)\times\chi a$ and $[\![S_I(\boldsymbol{u},d\psi)\ne 0]\!]\subseteq a$.

(ii) On the other hand, if $e=c(\sigma,\tau)$ is an I-cell,

$$[\![\Delta_e(\boldsymbol{u},d\psi)\ne 0]\!]\subseteq[\![\psi(\sigma,\tau)\ne 0]\!]\subseteq[\![\sigma<\tau]\!]\subseteq[\![\min I<\max I]\!],$$

by 613Cb, and

$$[\![S_I(\boldsymbol{u},d\psi)\ne 0]\!]\subseteq\sup_{e\in\mathrm{Stio}}[\![\Delta_e(\boldsymbol{u},d\psi)\ne 0]\!]\subseteq[\![\min I<\max I]\!].$$

(e) We have only to note that if $\sigma\le\tau$ in I and $e=c(\sigma,\tau)$, then

$$\Delta_e(\boldsymbol{u},d(\boldsymbol{v}\psi))=u_\sigma\times v_\sigma\times\psi(\sigma,\tau)=\Delta_e(\boldsymbol{u}\times\boldsymbol{v},d\psi);$$

summing over $e\in\mathrm{Stio}_0(I)$, we get the result.

613H Definitions (a) For a lattice $\mathcal{S}$, write $\mathcal{I}(\mathcal{S})$ for the family of finite sublattices of $\mathcal{S}$.

(b) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u}$ a fully adapted process with domain including $\mathcal{S}$ and ψ an adapted interval function defined (at least) on $\mathcal{S}^{2\uparrow}$. Then I define the **integral** of $\boldsymbol{u}$ over $\mathcal{S}$ with respect to ψ to be

$$\int_{\mathcal{S}}\boldsymbol{u}\,d\psi=\lim_{I\uparrow\mathcal{I}(\mathcal{S})}S_I(\boldsymbol{u},d\psi)$$

if the limit is defined for the topology of convergence in measure, that is, $z=\int_{\mathcal{S}}\boldsymbol{u}\,d\psi$ if for every $\epsilon>0$ there is a $J\in\mathcal{I}(\mathcal{S})$ such that $\theta(S_I(\boldsymbol{u},d\psi)-z)\le\epsilon$ whenever $I\in\mathcal{I}(\mathcal{S})$ includes J.

(c) Note that if, in (b), we set $\psi'=\boldsymbol{u}\psi$, as in 613Dd, then 613Ee tells us that

$$\int_{\mathcal{S}}d\psi'=\int_{\mathcal{S}}\mathbf{1}\,d\psi'=\lim_{I\uparrow\mathcal{I}(\mathcal{S})}S_I(\mathbf{1},d\psi')=\lim_{I\uparrow\mathcal{I}(\mathcal{S})}S_I(\boldsymbol{u},d\psi)=\int_{\mathcal{S}}\boldsymbol{u}\,d\psi$$

if either integral is defined.

(d) Remarks (i) I see that I am writing '$\lim_{I\uparrow\mathcal{I}(\mathcal{S})}$' where in Volume 3 I might have written '$\lim_{I\to\mathcal{F}(\mathcal{I}(\mathcal{S})\uparrow)}$'.

(ii) The notation $\int_{\mathcal{S}}\boldsymbol{u}\,d\psi$ is supposed to convey the fact that the existence and value of the integral depend only on $\boldsymbol{u}\restriction\mathcal{S}$ and $\psi\restriction\mathcal{S}^{2\uparrow}$. But a warning! regarded as a function of $\mathcal{S}$, it does not behave like an indefinite integral except in special circumstances. as in 613O below.

(iii) In this volume, the integral defined in (b) above will be the basic concept of 'stochastic integral', and $\int_{\mathcal{S}}$ should always be interpreted in the way described here unless there is some clear indication to the contrary. But in §§645-646 I will introduce a different integral, the 'S-integral', and it will then sometimes be helpful to be able to declare explicitly that I mean the integral I have just defined; so I will keep the phrase **Riemann-sum integral** in reserve to signify the integral of this chapter.

(iv) Of course a Riemann sum $S_I(\boldsymbol{u},d\psi)$ can also be thought of as $\int_I\boldsymbol{u}\,d\psi$.

613I Invariance under change of law The following point is perfectly elementary, but it is so important that I give it a numbered paragraph to itself. The integral $\int_{\mathcal{S}}\boldsymbol{u}\,d\psi$ depends, of course, on the process $\boldsymbol{u}$, the interval function ψ and the lattice $\mathcal{S}$; behind these declared variables lie the undeclared structure $(\mathfrak{A},T,\langle\mathfrak{A}_t\rangle_{t\in T})$ and the derived objects $L^0=L^0(\mathfrak{A})$ and $\mathcal{T}$. But we do not really need the measure $\bar\mu$. What we use is the topology of convergence in measure on L^0. Now this topology (though not the functionals $\mathbb{E}$ and θ of 613Aa) can be defined in terms of the Riesz space structure of L^0, which in turn depends only on the Boolean algebra structure of $\mathfrak{A}$ (613Bg). In place of introducing $(\mathfrak{A},\bar\mu)$ as a probability algebra, I could just as well have said that in this section $\mathfrak{A}$ would be a measurable algebra in the sense of 391Ba. (Indeed, the ideas so far would work perfectly if I asked only that $\mathfrak{A}$ should be a Maharam algebra in the sense of 393Ea. But this extension would fail when we came to the real meat of the theory in §622.)

So the Riemann-sum integral is law-independent, and we shall always be at liberty to replace the measure (or 'law') $\bar{\mu}$ by another strictly positive countably additive functional on $\mathfrak{A}$, if that seems to make calculations easier. As I have arranged my exposition, this will remain the case throughout this chapter. I will return to the issue in 622R.

613J Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\operatorname{dom}\boldsymbol{u}}$ a fully adapted process with $\mathcal{S} \subseteq \operatorname{dom}\boldsymbol{u}$, and ψ an adapted interval function defined (at least) on $\mathcal{S}^{2\uparrow}$.

(a) Suppose that for every $\epsilon > 0$ there are a $z \in L^0(\mathfrak{A})$ and a $J \in \mathcal{I}(\mathcal{S})$ such that $\theta(S_I(\boldsymbol{u}, d\psi) - z) \le \epsilon$ whenever $J \subseteq I \in \mathcal{I}(\mathcal{S})$. Then $\int_{\mathcal{S}} \boldsymbol{u}\, d\psi$ is defined.

(b) If $\boldsymbol{u}'$ is another fully adapted process defined on $\mathcal{S}$, ψ' is another adapted interval function defined on $\mathcal{S}^{\uparrow 2}$, and $\int_{\mathcal{S}} \boldsymbol{u}\, d\psi$, $\int_{\mathcal{S}} \boldsymbol{u}'\, d\psi$ and $\int_{\mathcal{S}} \boldsymbol{u}\, d\psi'$ are all defined, then $\int_{\mathcal{S}} \boldsymbol{u} + \boldsymbol{u}'\, d\psi$ and $\int_{\mathcal{S}} \boldsymbol{u}\, d(\psi + \psi')$ are defined and

$$\int_{\mathcal{S}} \boldsymbol{u} + \boldsymbol{u}'\, d\psi = \int_{\mathcal{S}} \boldsymbol{u}\, d\psi + \int_{\mathcal{S}} \boldsymbol{u}'\, d\psi, \quad \int_{\mathcal{S}} \boldsymbol{u}\, d(\psi + \psi') = \int_{\mathcal{S}} \boldsymbol{u}\, d\psi + \int_{\mathcal{S}} \boldsymbol{u}\, d\psi'.$$

Similarly, for any $\alpha \in \mathbb{R}$, $\int_{\mathcal{S}} \alpha\boldsymbol{u}\, d\psi$ and $\int_{\mathcal{S}} \boldsymbol{u}\, d(\alpha\psi)$ are defined and equal to $\alpha\int_{\mathcal{S}} \boldsymbol{u}\, d\psi$.

(c)(i) Suppose that $\tau \in \mathcal{S}$. Then

$$\int_{\mathcal{S}} \boldsymbol{u}\, d\psi = \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\psi + \int_{\mathcal{S}\vee\tau} \boldsymbol{u}\, d\psi$$

if either side is defined.

(ii) Suppose that $\tau_0 \le \ldots \le \tau_n$ in $\mathcal{S}$. Then

$$\int_{\mathcal{S}} \boldsymbol{u}\, d\psi = \int_{\mathcal{S}\wedge\tau_0} \boldsymbol{u}\, d\psi + \sum_{i=0}^{n-1} \int_{\mathcal{S}\cap[\tau_i,\tau_{i+1}]} \boldsymbol{u}\, d\psi + \int_{\mathcal{S}\vee\tau_n} \boldsymbol{u}\, d\psi$$

if either side is defined.

(d) If $z = \int_{\mathcal{S}} \boldsymbol{u}\, d\psi$ is defined, then

$$\begin{aligned}[\![z \ne 0]\!] &\subseteq \sup_{\sigma\in\mathcal{S}} [\![u_\sigma \ne 0]\!] \cap \sup_{(\sigma,\tau)\in\mathcal{S}^{2\uparrow}} [\![\psi(\sigma,\tau) \ne 0]\!] \\ &\subseteq \sup_{\sigma,\tau\in\mathcal{S}} ([\![u_\sigma \ne 0]\!] \cap [\![\sigma < \tau]\!]) \subseteq [\![\boldsymbol{u} \ne \mathbf{0}]\!].\end{aligned}$$

(e) Set $\mathcal{S}' = \{\tau : \tau \in \mathcal{S}, \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\psi \text{ is defined}\}$.

(i) $\mathcal{S}'$ is an ideal of $\mathcal{S}$.

(ii) Setting $z_\tau = \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\psi$ for $\tau \in \mathcal{S}'$, $\langle z_\tau \rangle_{\tau\in\mathcal{S}'}$ is fully adapted.

(iii) If $\tau \in \mathcal{S}$ and $\sup_{\tau'\in\mathcal{S}'} [\![\tau' = \tau]\!] = 1$, then $\tau \in \mathcal{S}'$.

(f) Suppose that $\mathcal{S} \ne \emptyset$ and $z = \int_{\mathcal{S}} \boldsymbol{u}\, d\psi$ is defined. Set $z_\tau = \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\psi$ for $\tau \in \mathcal{S}$.

(i) The starting value $\lim_{\tau\downarrow\mathcal{S}} z_\tau$ is 0.

(ii) $\lim_{\tau\uparrow\mathcal{S}} \int_{\mathcal{S}\vee\tau} \boldsymbol{u}\, d\psi = 0$, $\lim_{\tau\uparrow\mathcal{S}} z_\tau = z$.

(g) Let $\boldsymbol{v}$ be another fully adapted process with domain $\mathcal{S}$. Then $\int_{\mathcal{S}} \boldsymbol{u}\, d(\boldsymbol{v}\psi) = \int_{\mathcal{S}} \boldsymbol{u} \times \boldsymbol{v}\, d\psi$ in the sense that if one is defined so is the other, and they are then equal.

proof (a) For $J \in \mathcal{I}(\mathcal{S})$ set $C_J = \{S_I(\boldsymbol{u}, d\psi) : J \subseteq I \in \mathcal{I}(\mathcal{S})\}$. The hypothesis guarantees that the filter $\mathcal{F}$ on $L^0(\mathfrak{A})$ generated by $\{C_J : J \in \mathcal{I}(\mathcal{S})\}$ is Cauchy for the uniformity of convergence in measure; since $L^0(\mathfrak{A})$ is complete (367Mc again), $\mathcal{F}$ is convergent, and its limit is $\int_{\mathcal{S}} \boldsymbol{u}\, d\psi$.

(b) All we need to know is that the operators S_I are bilinear and that the linear space operations on $L^0(\mathfrak{A})$ are continuous (367Ma).

(c)(i)($\boldsymbol{\alpha}$) Suppose that $z_0 = \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\psi$ and $z_1 = \int_{\mathcal{S}\vee\tau} \boldsymbol{u}\, d\psi$ are defined. For any $\epsilon > 0$, there are $J_0 \in \mathcal{I}(\mathcal{S}\wedge\tau)$ and $J_1 \in \mathcal{I}(\mathcal{S}\vee\tau)$ such that

$$\theta(S_I(\boldsymbol{u}, d\psi) - z_0) \le \epsilon \text{ whenever } J_0 \subseteq I \in \mathcal{I}(\mathcal{S}\wedge\tau),$$

$$\theta(S_I(\boldsymbol{u}, d\psi) - z_1) \le \epsilon \text{ whenever } J_1 \subseteq I \in \mathcal{I}(\mathcal{S}\vee\tau).$$

Set $J = J_0 \cup \{\tau\} \cup J_1$, so that $J \in \mathcal{I}(\mathcal{S})$. Suppose that $J \subseteq I \in \mathcal{I}(\mathcal{S})$. Set

$$I \wedge \tau = \{\sigma \wedge \tau : \sigma \in I\} = I \cap [\min\mathcal{T}, \tau],$$

$$I \vee \tau = \{\sigma \vee \tau : \sigma \in I\} = I \cap [\tau, \max\mathcal{T}],$$

so that $J_0 \subseteq I \wedge \tau \in \mathcal{I}(\mathcal{S} \wedge \tau)$ and $J_1 \subseteq I \vee \tau \in \mathcal{I}(\mathcal{S} \vee \tau)$. By 613Ga, $S_I(\boldsymbol{u}, d\psi) = S_{I\wedge\tau}(\boldsymbol{u}, d\psi) + S_{I\vee\tau}(\boldsymbol{u}, d\psi)$, while $\theta(S_{I\wedge\tau}(\boldsymbol{u}, d\psi) - z_0) \le \epsilon$ and $\theta(S_{I\vee\tau}(\boldsymbol{u}, d\psi) - z_1) \le \epsilon$. Accordingly $\theta(S_I(\boldsymbol{u}, d\psi) - (z_0 + z_1)) \le 2\epsilon$; and this is true whenever $J \subseteq I \in \mathcal{I}(\mathcal{S})$. As ϵ is arbitrary, $\int_{\mathcal{S}} \boldsymbol{u}\, d\psi$ is defined and equal to $z_0 + z_1$.

(β) Now suppose that $z = \int_{\mathcal{S}} \boldsymbol{u}\, d\psi$ is defined. Let $\epsilon > 0$. Then there is a finite sublattice J of $\mathcal{S}$ such that $\theta(S_I(\boldsymbol{u}, d\psi) - z) \le \epsilon$ whenever $J \subseteq I \in \mathcal{I}(\mathcal{S})$. Of course we can suppose that $\tau \in J$. Take any $K \in \mathcal{I}(\mathcal{S} \wedge \tau)$ including $J \wedge \tau$. Let I be the (finite) sublattice of $\mathcal{S}$ generated by $J \cup K$. Since

$$\{\sigma : \sigma \in \mathcal{S},\, \sigma \wedge \tau \in K,\, \sigma \vee \tau \in J\}$$

is a sublattice of $\mathcal{S}$ including $J \cup K$, it includes I, so that $I \in \mathcal{I}(\mathcal{S})$, $I \wedge \tau = K$ and $I \vee \tau = J \vee \tau$. Now

$$\begin{aligned}\theta(S_K(\boldsymbol{u}, d\psi) - S_{J\wedge\tau}(\boldsymbol{u}, d\psi)) &= \theta(S_{I\wedge\tau}(\boldsymbol{u}, d\psi) - S_{J\wedge\tau}(\boldsymbol{u}, d\psi))\\ &= \theta(S_I(\boldsymbol{u}, d\psi) - S_{I\vee\tau}(\boldsymbol{u}, d\psi) - S_J(\boldsymbol{u}, d\psi) + S_{J\vee\tau}(\boldsymbol{u}, d\psi))\\ &= \theta(S_I(\boldsymbol{u}, d\psi) - S_J(\boldsymbol{u}, d\psi))\\ &\le \theta(S_I(\boldsymbol{u}, d\psi) - z) + \theta(z - S_J(\boldsymbol{u}, d\psi)) \le 2\epsilon.\end{aligned}$$

By (a) above, $\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\psi$ is defined.

(γ) For $\int_{\mathcal{S}\vee\tau} \boldsymbol{u}\, d\psi$ we can repeat the argument, inverted. Again suppose that $z = \int_{\mathcal{S}} \boldsymbol{u}\, d\psi$ is defined, and take $\epsilon > 0$ and $J \in \mathcal{I}(\mathcal{S})$ such that $\tau \in J$ and $\theta(S_I(\boldsymbol{u}, d\psi) - z) \le \epsilon$ whenever $J \subseteq I \in \mathcal{I}(\mathcal{S})$. This time, take any $K \in \mathcal{I}(\mathcal{S} \vee \tau)$ including $J \vee \tau$. Let I be the (finite) sublattice of $\mathcal{S}$ generated by $J \cup K$. Since

$$\{\sigma : \sigma \in \mathcal{S},\, \sigma \vee \tau \in K,\, \sigma \wedge \tau \in J\}$$

is a sublattice of $\mathcal{S}$ including $J \cup K$, it includes I, so that $I \in \mathcal{I}(\mathcal{S})$, $I \vee \tau = K$ and $I \wedge \tau = J \wedge \tau$. Now

$$\begin{aligned}\theta(S_K(\boldsymbol{u}, d\psi) - S_{J\vee\tau}(\boldsymbol{u}, d\psi)) &= \theta(S_{I\vee\tau}(\boldsymbol{u}, d\psi) - S_{J\vee\tau}(\boldsymbol{u}, d\psi))\\ &= \theta(S_I(\boldsymbol{u}, d\psi) - S_{I\wedge\tau}(\boldsymbol{u}, d\psi) - S_J(\boldsymbol{u}, d\psi) + S_{J\wedge\tau}(\boldsymbol{u}, d\psi))\\ &= \theta(S_I(\boldsymbol{u}, d\psi) - S_J(\boldsymbol{u}, d\psi)) \le 2\epsilon.\end{aligned}$$

So $\int_{\mathcal{S}\vee\tau} \boldsymbol{u}\, d\psi$ is defined.

(ii) Induce on n, using (i) for the inductive step.

$$[\![z \ne 0]\!] \subseteq \sup_{\sigma\in\mathcal{S}} [\![u_\sigma \ne 0]\!] \cap \sup_{(\sigma,\tau)\in\mathcal{S}^{2\uparrow}} [\![\psi(\sigma,\tau) \ne 0]\!] \subseteq \sup_{\sigma,\tau\in\mathcal{S}}([\![u_\sigma \ne 0]\!] \cap [\![\sigma < \tau]\!]).$$

(d) Setting

$$a = 1 \setminus (\sup_{\sigma\in\mathcal{S}} [\![u_\sigma \ne 0]\!] \cap \sup_{(\sigma,\tau)\in\mathcal{S}^{2\uparrow}} [\![\psi(\sigma,\tau) \ne 0]\!]),$$

$a \subseteq [\![\Delta_e(\boldsymbol{u}, d\psi) = 0]\!]$ for every stopping-time interval e with endpoints in $\mathcal{S}$, so $S_I(\boldsymbol{u}, d\psi) \times \chi a = 0$ for every $I \in \mathcal{I}(\mathcal{S})$; because multiplication in $L^0(\mathfrak{A})$ is continuous, $z \times \chi a = 0$, that is,

$$[\![z \ne 0]\!] \subseteq 1 \setminus a = \sup_{\sigma\in\mathcal{S}} [\![u_\sigma \ne 0]\!] \cap \sup_{(\sigma,\tau)\in\mathcal{S}^{2\uparrow}} [\![\psi(\sigma,\tau) \ne 0]\!].$$

For the second inequality, observe just that $[\![\psi(\sigma,\tau) \ne 0]\!] \subseteq [\![\sigma < \tau]\!]$ whenever $\sigma \le \tau$ in $\mathcal{S}$, by 613C(b-i); and the third inequality is immediate from the definition in 612Sb.

(e)(i)(α) If $\tau \in \mathcal{S}'$, $\sigma \in \mathcal{S}$ and $\sigma \le \tau$ then

$$\int_{\mathcal{S}\wedge\sigma} \boldsymbol{u}\, d\psi = \int_{(\mathcal{S}\wedge\tau)\wedge\sigma} \boldsymbol{u}\, d\psi$$

is defined, by (c).

(β) Suppose that τ, $\tau' \in \mathcal{S}'$. Then $z_\tau = \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\psi$ and $z_{\tau'} = \int_{\mathcal{S}\wedge\tau'} \boldsymbol{u}\, d\psi$ and $z_{\tau\wedge\tau'} = \int_{\mathcal{S}\wedge(\tau\wedge\tau')} \boldsymbol{u}\, d\psi$ are defined. We know also that if I is a sublattice of $\mathcal{S} \wedge (\tau \vee \tau')$ containing both τ and τ', then

$$S_I(\boldsymbol{u}, d\psi) = S_{I\wedge\tau}(\boldsymbol{u}, d\psi) + S_{I\wedge\tau'}(\boldsymbol{u}, d\psi) - S_{I\wedge\tau\wedge\tau'}(\boldsymbol{u}, d\psi)$$

by 613Gc. Looking at the limits as $I \uparrow \mathcal{I}(\mathcal{S} \wedge (\tau \vee \tau'))$, we see that

$$\lim_{I\uparrow\mathcal{I}(\mathcal{S}\wedge(\tau\vee\tau'))} S_{I\wedge\tau}(\boldsymbol{u}, d\psi) = \lim_{I\uparrow\mathcal{I}(\mathcal{S}\wedge\tau)} S_I(\boldsymbol{u}, d\psi) = z_\tau,$$

$$\lim_{I\uparrow\mathcal{I}(\mathcal{S}\wedge(\tau\vee\tau'))} S_{I\wedge\tau'}(\boldsymbol{u},d\psi)=\lim_{I\uparrow\mathcal{I}(\mathcal{S}\wedge\tau')} S_I(\boldsymbol{u},d\psi)=z_{\tau'},$$

$$\lim_{I\uparrow\mathcal{I}(\mathcal{S}\wedge(\tau\vee\tau')} S_{I\wedge(\tau\wedge\tau')}(\boldsymbol{u},d\psi)=\lim_{I\uparrow\mathcal{I}(\mathcal{S}\wedge(\tau\wedge\tau')} S_I(\boldsymbol{u},d\psi)=z_{\tau\wedge\tau'},$$

and therefore that

$$\int_{\mathcal{S}\wedge(\tau\vee\tau')}\boldsymbol{u}\,d\boldsymbol{v}=\lim_{I\uparrow\mathcal{I}(\mathcal{S}\wedge(\tau\vee\tau'))} S_I(\boldsymbol{u},d\psi)=z_\tau+z_{\tau'}-z_{\tau\wedge\tau'}$$

is defined. In particular, $\tau\vee\tau'\in\mathcal{S}'$. With (α), this shows that $\mathcal{S}'$ is an ideal of $\mathcal{S}$.

(ii) If $\sigma\le\sigma'$ in $\mathcal{S}$, then $u_\sigma\in L^0(\mathfrak{A}_\sigma)$ and $\psi(\sigma,\sigma')\in L^0(\mathfrak{A}_{\sigma'})$, so $\Delta_{c(\sigma,\sigma')}(\boldsymbol{u},d\psi)\in L^0(\mathfrak{A}_{\sigma'})$; consequently $S_I(\boldsymbol{u},d\psi)\in L^0(\mathfrak{A}_\tau)$ whenever $\tau\in\mathcal{S}$ and $I\in\mathcal{I}(\mathcal{S}\wedge\tau)$. Since $L^0(\mathfrak{A}_\tau)$ is closed in $L^0(\mathfrak{A})$ for the topology of convergence in measure (367Rc again), the limiting value z_τ belongs to $L^0(\mathfrak{A}_\tau)$, for every $\tau\in\mathcal{S}'$.

If $\sigma\le\tau$ in $\mathcal{S}'$, then

$$z_\tau-z_\sigma=\int_{(\mathcal{S}\wedge\tau)\vee\sigma}\boldsymbol{u}\,d\psi\int_{\mathcal{S}\cap[\sigma,\tau]}\boldsymbol{u}\,d\psi,$$

by (c). This time, for any finite sublattice I of $\mathcal{S}\cap[\sigma,\tau]$,

$$\begin{aligned}\llbracket S_I(\boldsymbol{u},d\psi)\ne 0\rrbracket&\subseteq\sup_{\rho\le\rho'\text{ in }\mathcal{S}\cap[\sigma,\tau]}\llbracket\psi(\rho,\rho')\ne 0\rrbracket\\&\subseteq\sup_{\rho\le\rho'\text{ in }\mathcal{S}\cap[\sigma,\tau]}\llbracket\rho<\rho'\rrbracket\subseteq\llbracket\sigma<\tau\rrbracket.\end{aligned}$$

Taking the limit as I increases, $\llbracket z_\sigma\ne z_\tau\rrbracket\subseteq\llbracket\sigma<\tau\rrbracket$, that is, $\llbracket z_\sigma=z_\tau\rrbracket\supseteq\llbracket\sigma=\tau\rrbracket$. By 612Db again, this shows that $\langle z_\sigma\rangle_{\sigma\in\mathcal{S}'}$ is fully adapted.

(iii) Let $\epsilon>0$. Then there is a $\tau'\in\mathcal{S}'$ such that $\bar\mu\llbracket\tau=\tau'\rrbracket\ge 1-\frac{1}{2}\epsilon$; as $\tau'\wedge\tau\in\mathcal{S}$ and $\llbracket\tau=\tau'\wedge\tau\rrbracket\supseteq\llbracket\tau=\tau'\rrbracket$, we may suppose that $\tau'\le\tau$. We have $\bar\mu\llbracket\tau'<\tau\rrbracket\le\frac{1}{2}\epsilon$. There is a $J'\in\mathcal{I}(\mathcal{S}\wedge\tau')$ such that $\theta(z_{\tau'}-S_{I'}(\boldsymbol{u},d\psi))\le\frac{1}{2}\epsilon$ whenever $J'\subseteq I'\in\mathcal{I}(\mathcal{S}\wedge\tau')$; we may suppose that $\tau'\in J'$. Now take any $I\in\mathcal{I}(\mathcal{S}\wedge\tau)$ such that $J'\subseteq I$. In this case $J'\subseteq I\wedge\tau'\in\mathcal{I}(\mathcal{S}\wedge\tau')$ and

$$\theta(z_{\tau'}-S_I(\boldsymbol{u},d\psi))=\theta(z_{\tau'}-S_{I\wedge\tau'}(\boldsymbol{u},d\psi)-S_{I\vee\tau'}(\boldsymbol{u},d\psi))$$

(613Ga)

$$\begin{aligned}&\le\theta(z_{\tau'}-S_{I\wedge\tau'}(\boldsymbol{u},d\psi))+\theta(S_{I\vee\tau'}(\boldsymbol{u},d\psi))\\&\le\frac{1}{2}\epsilon+\bar\mu\llbracket S_{I\vee\tau'}(\boldsymbol{u},d\psi)\ne 0\rrbracket\le\frac{1}{2}\epsilon+\bar\mu\llbracket\tau'<\sup I\rrbracket\end{aligned}$$

(613Gd)

$$\le\frac{1}{2}\epsilon+\bar\mu\llbracket\tau'<\tau\rrbracket\le\epsilon.$$

By (a) above, $\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\,d\psi$ is defined and $\tau\in\mathcal{S}'$, as claimed.

(f) Of course (c-i) assures us that z_τ is defined for every $\tau\in\mathcal{S}$.

(i) Let $\epsilon>0$. Then there is a non-empty $J\in\mathcal{I}(\mathcal{S})$ such that $\theta(S_I(\boldsymbol{u},d\psi)-z)\le\epsilon$ whenever $J\subseteq I\in\mathcal{S}$. Take any $\tau\in\mathcal{S}$ such that $\tau\le\min J$. Then $\theta(S_K(\boldsymbol{u},d\psi))\le 2\epsilon$ whenever $\tau\in K\in\mathcal{I}(\mathcal{S}\wedge\tau)$. **P** $I=K\cup J$ and $I'=\{\tau\}\cup J$ are both finite sublattices of $\mathcal{S}$ including J. So

$$\theta(S_K(\boldsymbol{u},d\psi))=\theta(S_{I\wedge\tau}(\boldsymbol{u},d\psi))=\theta(S_I(\boldsymbol{u},d\psi)-S_{I\vee\tau}(\boldsymbol{u},d\psi))$$

(613Ga)

$$=\theta(S_I(\boldsymbol{u},d\psi)-S_{I'}(\boldsymbol{u},d\psi))\le 2\epsilon.\ \mathbf{Q}$$

As K is arbitrary,

$$\theta(z_\tau)=\lim_{K\uparrow\mathcal{I}(\mathcal{S}\wedge\tau)}\theta(S_K(\boldsymbol{u},d\psi))\le 2\epsilon.$$

As ϵ is arbitrary, $\lim_{\tau\downarrow\mathcal{S}} z_\tau=0$.

(ii) The argument is essentially the same. Let $\epsilon > 0$. Again, there is a non-empty $J \in \mathcal{I}(\mathcal{S})$ such that $\theta(S_I(\boldsymbol{u}, d\psi) - z) \le \epsilon$ whenever $J \subseteq I \in \mathcal{S}$. This time, take $\tau \in \mathcal{S}$ such that $\max J \le \tau$. Then $\theta(S_K(\boldsymbol{u}, d\psi)) \le 2\epsilon$ whenever $\tau \in K \in \mathcal{I}(\mathcal{S} \vee \tau)$. **P** $I = K \cup J$ and $I' = \{\tau\} \cup J$ are both finite sublattices of $\mathcal{S}$ including J. So

$$\theta(S_K(\boldsymbol{u}, d\psi)) = \theta(S_{I\vee\tau}(\boldsymbol{u}, d\psi)) = \theta(S_I(\boldsymbol{u}, d\psi) - S_{I\wedge\tau}(\boldsymbol{u}, d\psi))$$

(613Ga)

$$= \theta(S_I(\boldsymbol{u}, d\psi) - S_{I'}(\boldsymbol{u}, d\psi)) \le 2\epsilon. \text{ } \mathbf{Q}$$

As K is arbitrary,

$$\int_{\mathcal{S}\vee\tau} \boldsymbol{u}\, d\psi = \lim_{K\uparrow\mathcal{I}(\mathcal{S}\vee\tau)} \theta(S_K(\boldsymbol{u}, d\psi)) \le 2\epsilon.$$

As ϵ is arbitrary, $\lim_{\tau\uparrow\mathcal{S}} \int_{\mathcal{S}\vee\tau} \boldsymbol{u}\, d\psi = 0$. By (c-i),

$$\lim_{\tau\uparrow\mathcal{S}} z_\tau = \lim_{\tau\uparrow\mathcal{S}}(z - \int_{\mathcal{S}\vee\tau} \boldsymbol{u}\, d\psi) = z.$$

(g) Immediate from 613Ge.

613K Remark The key step in the proof of 613Jc is that if $\tau^* \in J \in \mathcal{I}(\mathcal{S})$ and $J \wedge \tau^* \subseteq I \in \mathcal{I}(\mathcal{S} \wedge \tau^*)$, then there is a $K \in \mathcal{I}(\mathcal{S})$ such that $K \supseteq J$ and $K \wedge \tau^* = I \wedge \tau^*$. It follows that if ϕ is any function from $\mathcal{I}(\mathcal{S} \wedge \tau^*)$ to a Hausdorff topological space Y, $\lim_{I\uparrow\mathcal{I}(\mathcal{S}\wedge\tau^*)} \phi(I) = \lim_{I\uparrow\mathcal{I}(\mathcal{S})} \phi(I \wedge \tau^*)$ if either limit is defined. The same argument applies with $\vee$ in the place of $\wedge$, so that if ϕ is any function from $\mathcal{I}(\mathcal{S} \vee \tau^*)$ to a Hausdorff topological space Y, then $\lim_{I\uparrow\mathcal{I}(\mathcal{S}\vee\tau^*)} \phi(I) = \lim_{I\uparrow\mathcal{I}(\mathcal{S})} \phi(I \vee \tau^*)$ if either limit is defined. Mostly we shall be dealing with the case in which Y is L^0 with the topology of convergence in measure, but I shall also have an application in which $Y = L^1_{\bar\mu}$ with the weak topology (626J).

613L More easy bits (a) If $\mathcal{S}$ is a sublattice of $\mathcal{T}$ and $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are fully adapted processes defined on $\mathcal{S}$, I will write

$$\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v} = \int_{\mathcal{S}} \boldsymbol{u}\, d(\Delta\boldsymbol{v}) = \lim_{I\uparrow\mathcal{I}(\mathcal{S})} S_I(\boldsymbol{u}, d\boldsymbol{v}),$$

$$\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}d\boldsymbol{w} = \int_{\mathcal{S}} \boldsymbol{u}\, d(\Delta\boldsymbol{v} \times \Delta\boldsymbol{w}) = \lim_{I\uparrow\mathcal{I}(\mathcal{S})} S_I(\boldsymbol{u}, d\boldsymbol{v}d\boldsymbol{w}),$$

$$\int_{\mathcal{S}} \boldsymbol{u}\, |d\boldsymbol{v}| = \int_{\mathcal{S}} \boldsymbol{u}\, d|\Delta\boldsymbol{v}| = \lim_{I\uparrow\mathcal{I}(\mathcal{S})} S_I(\boldsymbol{u}, |d\boldsymbol{v}|)$$

when the limits exist in $L^0(\mathfrak{A})$.

(b) Three trivial calculations: if $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma\in\mathcal{S}}$ and $\boldsymbol{u}$ are fully adapted processes with domain a sublattice $\mathcal{S}$ of $\mathcal{T}$, then

(i) $S_I(\mathbf{1}, d\boldsymbol{v}) = v_{\max I} - v_{\min I}$ for every non-empty finite sublattice I of $\mathcal{S}$ (613Ed), so $\int_{\mathcal{S}\cap[\tau,\tau']} \mathbf{1}\, d\boldsymbol{v} = v_{\tau'} - v_\tau$ whenever $\tau \le \tau'$ in $\mathcal{S}$;

(ii) if $\boldsymbol{v}$ is constant then $S_I(\boldsymbol{u}, d\boldsymbol{v}) = 0$ for every $I \in \mathcal{I}(\mathcal{S})$, so $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$ is defined and equal to 0;

(iii) if $z \in L^0(\bigcap_{\sigma\in\mathcal{S}} \mathfrak{A}_\sigma)$, then (in the language of 612De) $S_I(z\boldsymbol{u}, d\boldsymbol{v}) = S_I(\boldsymbol{u}, d(z\boldsymbol{v})) = z \times S_I(\boldsymbol{u}, d\boldsymbol{v})$ for every $I \in \mathcal{I}(\mathcal{S})$, so $\int_{\mathcal{S}} z\boldsymbol{u}\, d\boldsymbol{v}$ and $\int_{\mathcal{S}} \boldsymbol{u}\, d(z\boldsymbol{v})$ are defined and equal to $z \times \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$ if the last integral is defined.

(c) The following straightforward fact, refining 613Jd, will frequently be useful. Suppose that I is a finite sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in I}$, $\boldsymbol{u}' = \langle u'_\sigma \rangle_{\sigma\in I}$, $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma\in I}$ and $\boldsymbol{v}' = \langle v'_\sigma \rangle_{\sigma\in I}$ are fully adapted processes. Set $d = \sup_{\sigma\in I} [\![u_\sigma \ne u'_\sigma]\!] \cup [\![v_\sigma \ne v'_\sigma]\!]$. For any stopping-time interval $e = c(\sigma, \tau)$ where $\sigma \le \tau$ in I,

$$[\![\Delta_e(\boldsymbol{u}, d\boldsymbol{v}) \ne \Delta_e(\boldsymbol{u}', d\boldsymbol{v}')]\!] \subseteq [\![u_\sigma \ne u'_\sigma]\!] \cup [\![v_\sigma \ne v'_\sigma]\!] \cup [\![v_\tau \ne v'_\tau]\!] \subseteq d,$$

so $[\![S_I(\boldsymbol{u}, d\boldsymbol{v}) \ne S_I(\boldsymbol{u}', d\boldsymbol{v}')]\!] \subseteq d$.

Similarly, of course,

$$[\![S_I(\boldsymbol{u}, (d\boldsymbol{v})^2) \ne S_I(\boldsymbol{u}', (d\boldsymbol{v}')^2)]\!] \subseteq d, \quad [\![S_I(\boldsymbol{u}, |d\boldsymbol{v}|) \ne S_I(\boldsymbol{u}', |d\boldsymbol{v}'|)]\!] \subseteq d.$$

Indeed, if ψ, ψ' are any adapted interval functions defined on $\mathcal{I}^{2\uparrow}$, and we set

$$d = \sup_{\sigma \in I} [\![u_\sigma \neq u'_\sigma]\!] \cup \sup_{\sigma \le \tau \text{ in } I} [\![\psi(\sigma,\tau) \neq \psi'(\sigma,\tau)]\!],$$

then $[\![S_I(\boldsymbol{u}, d\psi) \neq S_I(\boldsymbol{u}', d\psi')]\!] \subseteq d$.

(d) It follows that if $\mathcal{S}$ is any sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$, $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ are fully adapted processes such that $z = \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$ is defined, then

$$\begin{aligned}[\![z \neq 0]\!] &\subseteq \sup_{I \in \mathcal{I}(\mathcal{S})} [\![S_I(\boldsymbol{u}, d\boldsymbol{v}) \neq 0]\!] \subseteq \sup_{\sigma \in \mathcal{S}} [\![u_\sigma \neq 0]\!] \cap \sup_{\sigma,\tau \in \mathcal{S}} [\![v_\sigma \neq v_\tau]\!] \\ &\subseteq [\![\boldsymbol{u} \neq \boldsymbol{0}]\!] \cap [\![\boldsymbol{v} \neq \boldsymbol{0}]\!].\end{aligned}$$

613M The next calculation is perfectly elementary, but will be helpful more than once below, and is a useful exercise in the definitions here.

Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ fully adapted processes defined on $\mathcal{S}$. Then

$$\begin{aligned}S_I(\boldsymbol{u}, d\boldsymbol{v}d\boldsymbol{w}) &= S_I(\boldsymbol{u}, d(\boldsymbol{v} \times \boldsymbol{w})) - S_I(\boldsymbol{u} \times \boldsymbol{v}, d\boldsymbol{w}) - S_I(\boldsymbol{u} \times \boldsymbol{w}, d\boldsymbol{v}) \\ &= \frac{1}{2}\big(S_I(\boldsymbol{u}, (d(\boldsymbol{v} + \boldsymbol{w}))^2) - S_I(\boldsymbol{u}, (d\boldsymbol{v})^2) - S_I(\boldsymbol{u}, (d\boldsymbol{w})^2)\big)\end{aligned}$$

for every finite sublattice I of $\mathcal{S}$. Consequently

$$\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}d\boldsymbol{w} = \int_{\mathcal{S}} \boldsymbol{u}\, d(\boldsymbol{v} \times \boldsymbol{w}) - \int_{\mathcal{S}} \boldsymbol{u} \times \boldsymbol{v}\, d\boldsymbol{w} - \int_{\mathcal{S}} \boldsymbol{u} \times \boldsymbol{w}\, d\boldsymbol{v}$$

if any three of the four integrals are defined, and

$$\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}d\boldsymbol{w} = \frac{1}{2}\Big(\int_{\mathcal{S}} \boldsymbol{u}\, (d(\boldsymbol{v} + \boldsymbol{w}))^2 - \int_{\mathcal{S}} \boldsymbol{u}\, (d\boldsymbol{v})^2 - \int_{\mathcal{S}} \boldsymbol{u}\, (d\boldsymbol{w})^2\Big)$$

if any three of the integrals are defined.

proof (a) Express $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ as $\langle u_\sigma \rangle_{\sigma \in \mathrm{dom}\,\boldsymbol{u}}$, etc.

If $\sigma \le \tau$ in $\mathcal{S}$ then

$$\begin{aligned}u_\sigma \times (v_\tau - v_\sigma) \times (w_\tau - w_\sigma) &= u_\sigma \times (v_\tau \times w_\tau - v_\sigma \times w_\sigma) \\ &\qquad - u_\sigma \times v_\sigma \times (w_\tau - w_\sigma) - u_\sigma \times w_\sigma \times (v_\tau - v_\sigma) \\ &= \frac{1}{2}\big(u_\sigma \times (v_\tau + w_\tau - v_\sigma - w_\sigma)^2 \\ &\qquad - u_\sigma \times (v_\tau - v_\sigma)^2 - u_\sigma \times (w_\tau - w_\sigma)^2\big),\end{aligned}$$

that is,

$$\begin{aligned}\Delta_e(\boldsymbol{u}, d\boldsymbol{v}d\boldsymbol{w}) &= \Delta_e(\boldsymbol{u}, d(\boldsymbol{v} \times \boldsymbol{w})) - \Delta_e(\boldsymbol{u} \times \boldsymbol{v}, d\boldsymbol{w}) - \Delta_e(\boldsymbol{u} \times \boldsymbol{w}, d\boldsymbol{v}) \\ &= \frac{1}{2}\big(\Delta_e(\boldsymbol{u}, (d(\boldsymbol{v} + \boldsymbol{w}))^2) - \Delta_e(\boldsymbol{u}, (d\boldsymbol{v})^2) - \Delta_e(\boldsymbol{u}, (d\boldsymbol{w})^2)\big)\end{aligned}$$

for every $e \in \mathrm{Sti}(\mathcal{S})$. So, writing $\mathrm{Sti}_0(I)$ for the set of I-cells, as usual,

$$\begin{aligned}S_I(\boldsymbol{u},d\boldsymbol{v}d\boldsymbol{w})&=\sum_{e\in\mathrm{Sti}_0(I)}\Delta_e(\boldsymbol{u},d\boldsymbol{v}d\boldsymbol{w})\\&=\sum_{e\in\mathrm{Sti}_0(I)}\Delta_e(\boldsymbol{u},d(\boldsymbol{v}\times\boldsymbol{w}))-\sum_{e\in\mathrm{Sti}_0(I)}\Delta_e(\boldsymbol{u}\times\boldsymbol{v},d\boldsymbol{w})\\&\qquad-\sum_{e\in\mathrm{Sti}_0(I)}\Delta_e(\boldsymbol{u}\times\boldsymbol{w},d\boldsymbol{v})\\&=\frac{1}{2}\Bigl(\sum_{e\in\mathrm{Sti}_0(I)}\Delta_e(\boldsymbol{u},(d(\boldsymbol{v}+\boldsymbol{w}))^2)-\sum_{e\in\mathrm{Sti}_0(I)}\Delta_e(\boldsymbol{u},(d\boldsymbol{v})^2)\\&\qquad-\sum_{e\in\mathrm{Sti}_0(I)}\Delta_e(\boldsymbol{u},(d\boldsymbol{w})^2)\Bigr)\\&=\frac{1}{2}\bigl(S_I(\boldsymbol{u},(d(\boldsymbol{v}+\boldsymbol{w}))^2)-S_I(\boldsymbol{u},(d\boldsymbol{v})^2)-S_I(\boldsymbol{u},(d\boldsymbol{w})^2)\bigr).\end{aligned}$$

(b) Taking limits as $I\uparrow\mathcal{I}(\mathcal{S})$, we get the corresponding identities for the integrals, as long as all but one is defined.

613N Proposition Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$ and $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a fully adapted process. Interpreting $\int_{\mathcal{S}}d\boldsymbol{v}$ as $\int_{\mathcal{S}}\boldsymbol{1}\,d\boldsymbol{v}$ where $\boldsymbol{1}$ is the constant process with value $\chi1$, $\int_{\mathcal{S}}d\boldsymbol{v}$ is defined iff $v_\downarrow=\lim_{\sigma\downarrow\mathcal{S}}v_\sigma$ and $v_\uparrow=\lim_{\sigma\uparrow\mathcal{S}}v_\sigma$ are defined, and in this case $\int_{\mathcal{S}}d\boldsymbol{v}=v_\uparrow-v_\downarrow$.

proof (a) Suppose that $z=\int_{\mathcal{S}}d\boldsymbol{v}$ is defined. For $\tau\in\mathcal{S}$ set $z_\tau=\int_{\mathcal{S}\wedge\tau}d\boldsymbol{v}$; by 613Jc, this is defined. If $\tau\le\tau'$ in $\mathcal{S}$,

$$z_{\tau'}-z_\tau=\int_{\mathcal{S}\wedge\tau'}d\boldsymbol{v}-\int_{\mathcal{S}\wedge\tau'\wedge\tau}d\boldsymbol{v}=\int_{(\mathcal{S}\wedge\tau')\vee\tau}d\boldsymbol{v}$$

(613J(c-i) again)

$$=\int_{\mathcal{S}\cap[\tau,\tau']}d\boldsymbol{v}=v_{\tau'}-v_\tau$$

by 613Lb. Now we know that $\lim_{\tau\downarrow\mathcal{S}}z_\tau=0$ and $\lim_{\tau\uparrow\mathcal{S}}z_\tau=z$ (613Jf). So, starting from any $\tau^*\in\mathcal{S}$,

$$\begin{aligned}v_\downarrow=\lim_{\tau\downarrow\mathcal{S}}v_\tau&=v_{\tau^*}-\lim_{\tau\downarrow\mathcal{S}}(v_{\tau^*}-v_\tau)\\&=v_{\tau^*}-\lim_{\tau\downarrow\mathcal{S}}(z_{\tau^*}-z_\tau)=v_{\tau^*}-z_{\tau^*}\end{aligned}$$

and

$$\begin{aligned}v_\uparrow=\lim_{\tau'\uparrow\mathcal{S}}v_{\tau'}&=v_{\tau^*}+\lim_{\tau'\uparrow\mathcal{S}}v_{\tau'}-v_{\tau^*}\\&=v_{\tau^*}+\lim_{\tau'\uparrow\mathcal{S}}z_{\tau'}-z_{\tau^*}=v_{\tau^*}+z-z_{\tau^*}\end{aligned}$$

are defined, and $v_\uparrow-v_\downarrow=z$.

(b) Now suppose that the limits $v_\uparrow$ and $v_\downarrow$ are defined. Let $\epsilon>0$. Then there are τ_0, $\tau_1\in\mathcal{S}$ such that $\theta(v_{\tau'}-v_\uparrow)+\theta(v_{\tau'}-v_\downarrow)\le\epsilon$ whenever τ, $\tau'\in\mathcal{S}$, $\tau\le\tau_0$ and $\tau_1\le\tau'$. Now suppose that $I\in\mathcal{I}(\mathcal{S})$ includes $\{\tau_0\wedge\tau_1,\tau_0\vee\tau_1\}$. Then

$$\theta(v_\uparrow-v_\downarrow-S_I(\boldsymbol{1},d\boldsymbol{v}))=\theta(v_\uparrow-v_\downarrow-v_{\max I}+v_{\min I})$$

(613Lb again)

$$\le\theta(v_\uparrow-v_{\max I})+\theta(v_\downarrow-v_{\min I})\le\epsilon.$$

As ϵ is arbitrary, $\int_{\mathcal{S}}d\boldsymbol{v}=\int_{\mathcal{S}}\boldsymbol{1}\,d\boldsymbol{v}$ is defined and equal to $v_\uparrow-v_\downarrow$.

613O Indefinite integrals (a) Definition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u}$ a fully adapted process with domain $\mathcal{S}$, and ψ an adapted interval function with domain $\mathcal{S}^{2\uparrow}$. Set $\mathcal{S}' = \{\tau : \tau \in \mathcal{S}, \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\psi$ is defined$\}$; by 613J(e-i), $\mathcal{S}'$ is an ideal of $\mathcal{S}$. The **indefinite integral** of $\boldsymbol{u}$ with respect to ψ is the process $ii_\psi(\boldsymbol{u}) = \langle \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\psi\rangle_{\tau\in\mathcal{S}'}$; by 613J(e-ii), this is a fully adapted process.

When ψ is of the form $\Delta\boldsymbol{v}$ for a fully adapted process $\boldsymbol{v}$, I will write $ii_{\boldsymbol{v}}(\boldsymbol{u}) = \langle \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\boldsymbol{v}\rangle_{\tau\in\mathcal{S}'}$ for the indefinite integral of $\boldsymbol{u}$ with respect to $\boldsymbol{v}$.

(b)(i) Note that if $\int_{\mathcal{S}} \boldsymbol{u}\, d\psi$ is defined, the domain $\mathcal{S}'$ of $ii_\psi(\boldsymbol{u})$ is the whole of $\mathcal{S}$, by 613J(c-i).

(ii) It is obvious from the definition, but perhaps it is worth stating formally that if $\tau \in \mathcal{S}$ and $\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\psi$ is defined then

$$\begin{aligned} ii_\psi(\boldsymbol{u})\restriction\mathcal{S}\wedge\tau &= \langle \int_{\mathcal{S}\wedge\sigma} \boldsymbol{u}\, d\psi\rangle_{\sigma\in\mathcal{S}\wedge\tau} = \langle \int_{\mathcal{S}\wedge\sigma} (\boldsymbol{u}\restriction\mathcal{S}\wedge\tau) d(\psi\restriction(\mathcal{S}\wedge\tau)^{2\uparrow})\rangle_{\sigma\in\mathcal{S}\wedge\tau} \\ &= ii_{\psi\restriction(\mathcal{S}\wedge\tau)^{2\uparrow}}(\boldsymbol{u}\restriction\mathcal{S}\wedge\tau). \end{aligned}$$

(iii) On the other side, if $\mathcal{S}' = \mathcal{S}$ and $\tau \in \mathcal{S}$, then $ii_{\psi\restriction(\mathcal{S}\vee\tau)^{2\uparrow}}(\boldsymbol{u}\restriction\mathcal{S}\vee\tau)$ is defined on the whole of $\mathcal{S}\vee\tau$ and is equal to $(ii_\psi(\boldsymbol{u})\restriction\mathcal{S}\vee\tau) - (\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\psi)\mathbf{1}$. **P** Expanding the definitions, all this is saying is that if $\tau' \in \mathcal{S}\vee\tau$ then $\int_{(\mathcal{S}\vee\tau)\wedge\tau'} \boldsymbol{u}\, d\psi$ is defined and equal to $\int_{\mathcal{S}\wedge\tau'} \boldsymbol{u}\, d\psi - \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\psi$; since $(\mathcal{S}\vee\tau)\wedge\tau' = (\mathcal{S}\wedge\tau')\vee\tau$, this is immediate from 613J(c-i). **Q**

(c) I have put the definition in (a) in a form which can accommodate cases in which $\mathcal{S}' = \operatorname{dom} ii_\psi(\boldsymbol{u})$ is not the whole of $\mathcal{S}$, leaving open the possibility that $\mathcal{S}'$ is actually empty. But we know that $\mathcal{S}'$ is an ideal of $\mathcal{S}$, and from 613Jf that if $\mathcal{S}'$ is non-empty then

$$\lim_{\tau\downarrow\mathcal{S}} \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\psi = \lim_{\tau\downarrow\mathcal{S}'} \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\psi = 0.$$

In particular, if $\mathcal{S}$ has a least element $\min\mathcal{S}$, then $\min\mathcal{S} \in \mathcal{S}'$ and the value of $ii_\psi(\boldsymbol{u})$ there is equal to 0.

613R Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u}$, $\boldsymbol{v}$ fully adapted processes with domain $\mathcal{S}$. Then (in the notation of 612Sb) $[\![ii_{\boldsymbol{v}}(\boldsymbol{u}) \ne \mathbf{0}]\!] \subseteq [\![\boldsymbol{u} \ne \mathbf{0}]\!] \cap [\![\boldsymbol{v} \ne \mathbf{0}]\!]$.

proof Setting $c = 1 \setminus ([\![\boldsymbol{u} \ne \mathbf{0}]\!] \cap [\![\boldsymbol{v} \ne \mathbf{0}]\!])$, we see from 613Lc that $\chi c \times S_I(\boldsymbol{u}, d\boldsymbol{v}) = 0$ for every finite sublattice I of $\mathcal{S}$; because multiplication in $L^0(\mathfrak{A})$ is continuous, $\chi c \times \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\boldsymbol{v} = 0$ whenever $\tau \in \mathcal{S}$ and the integral is defined, that is,

$$[\![ii_{\boldsymbol{v}}(\boldsymbol{u}) \ne 0]\!] \subseteq 1 \setminus c \subseteq [\![\boldsymbol{u} \ne \mathbf{0}]\!] \cap [\![\boldsymbol{v} \ne \mathbf{0}]\!].$$

613S It is a striking and very convenient fact that we can often approach an integral $\int_{\mathcal{S}} \mathbf{1}\, d\psi$ over a general sublattice $\mathcal{S}$ through an integral over the covered envelope of $\mathcal{S}$, which is a full sublattice of $\mathcal{T}$.

Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\psi : \mathcal{S}^{2\uparrow} \to L^0(\mathfrak{A})$ a strictly adapted interval function. Suppose that I, $J \in \mathcal{I}(\mathcal{S})$, $J \subseteq I$ and $a \subseteq \sup_{\sigma\in J} [\![\tau = \sigma]\!]$ for every $\tau \in I$. Then $a \subseteq [\![S_I(\mathbf{1}, d\psi) = S_J(\mathbf{1}, d\psi)]\!]$. In particular, if J covers I (definition: 611Ma) then $S_I(\mathbf{1}, d\psi) = S_J(\mathbf{1}, d\psi)$.

proof (a) The case $a = 0$ is trivial; so is the case $a \ne 0$ and $J = \emptyset$, as then I also must be empty and

$$S_I(\mathbf{1}, d\psi) = S_J(\mathbf{1}, d\psi) = 0.$$

So suppose otherwise. Let $(\sigma_0, \dots, \sigma_n)$ linearly generate the J-cells. If $\tau \in I$ then

$$a \subseteq \sup_{\sigma\in J} [\![\tau = \sigma]\!] = \sup_{\sigma\in J} \sup_{j\le n} ([\![\tau = \sigma]\!] \cap [\![\sigma = \sigma_j]\!])$$

(611Ke)

$$\subseteq \sup_{j\le n} [\![\tau = \sigma_j]\!] \subseteq [\![\sigma_0 \le \tau]\!] \cap [\![\tau \le \sigma_n]\!].$$

Set $I_{-1} = I \wedge \sigma_0$, $I_j = I \cap [\sigma_j, \sigma_{j+1}]$ for $0 \le j < n$, $I_n = I \vee \sigma_n$.

(b) If $\tau \le \tau'$ in I_{-1}, then

$$a \cap [\![\psi(\tau,\tau') \ne 0]\!] \subseteq a \cap [\![\tau < \tau']\!]$$

(613C(b-i))

$$\subseteq a \cap [\![\tau < \sigma_0]\!] = 0,$$

so

$$a \cap [\![\Delta_e(\mathbf{1}, d\psi) \ne 0]\!] = 0$$

for every I_{-1}-cell e. Summing over e, $S_{I_{-1}}(\mathbf{1}, d\psi) \times \chi a = 0$.

(c) In the same way, if $\tau \le \tau'$ in I_n, then

$$a \cap [\![\psi(\tau,\tau') \ne 0]\!] \subseteq a \cap [\![\tau < \tau']\!] \subseteq a \cap [\![\sigma_n < \tau']\!] = 0,$$

so

$$S_{I_n}(\mathbf{1}, d\psi) \times \chi a = 0.$$

(d) If $0 \le j < n$, $a \subseteq [\![S_{I_j}(\mathbf{1}, d\psi) = \psi(\sigma_j, \sigma_{j+1})]\!]$. **P** For every $\tau \in I_j$,

$$\begin{aligned} a \cap [\![\sigma_j < \tau]\!] \cap [\![\tau < \sigma_{j+1}]\!] &\subseteq \sup_{k \le n}([\![\tau = \sigma_k]\!] \cap [\![\sigma_j < \tau]\!] \cap [\![\tau < \sigma_{j+1}]\!]) \\ &\subseteq \sup_{k \le n}([\![\sigma_j < \sigma_k]\!] \cap [\![\sigma_k < \sigma_{j+1}]\!]) \\ &\subseteq (\sup_{k \le j} [\![\sigma_j < \sigma_k]\!]) \cup (\sup_{j+1 \le k} [\![\sigma_k < \sigma_{j+1}]\!]) = 0. \end{aligned}$$

Take $(\tau_0, \dots, \tau_m)$ linearly generating the I_j-cells; then $\sigma_j = \tau_0 \le \dots \le \tau_m = \sigma_{j+1}$. For $i < m$, set $b_i = [\![\tau_i < \tau_{i+1}]\!]$, so that $\sup_{i<m} b_i = [\![\sigma_j < \sigma_{j+1}]\!]$. For $i < m$,

$$\begin{aligned} a \cap b_i = a \cap [\![\tau_i < \tau_{i+1}]\!] &= a \cap [\![\sigma_j < \tau_{i+1}]\!] \cap [\![\tau_i < \sigma_{j+1}]\!] \cap [\![\tau_i < \tau_{i+1}]\!] \\ &\subseteq a \cap [\![\tau_{i+1} = \sigma_{j+1}]\!] \cap [\![\tau_i = \sigma_j]\!] \cap [\![\tau_i < \tau_{i+1}]\!] \\ &\subseteq [\![\psi(\tau_i, \tau_{i+1}) = \psi(\sigma_j, \sigma_{j+1})]\!] \cap \inf_{k<i} [\![\tau_k = \tau_{k+1}]\!] \cap \inf_{i<k<m} [\![\tau_k = \tau_{k+1}]\!] \cap [\![u_{\tau_i} = u_{\sigma_j}]\!] \end{aligned}$$

(because ψ is strictly adapted)

$$\subseteq [\![\psi(\tau_i, \tau_{i+1}) = \psi(\sigma_j, \sigma_{j+1})]\!] \cap \inf_{\substack{k<m \\ k \ne i}} [\![\psi(\tau_k, \tau_{k+1}) = 0]\!] \cap [\![u_{\tau_i} = u_{\sigma_j}]\!]$$

(using 613C(b-i))

$$\begin{aligned} &\subseteq [\![\psi(\tau_i, \tau_{i+1}) = \psi(\sigma_j, \sigma_{j+1})]\!] \cap [\![S_{I_j}(\mathbf{1}, d\psi) = \psi(\tau_i, \tau_{i+1})]\!] \\ &\subseteq [\![S_{I_j}(\mathbf{1}, d\psi) = \psi(\sigma_j, \sigma_{j+1}]\!]. \end{aligned}$$

Taking the supremum over i,

$$a \cap [\![\sigma_j < \sigma_{j+1}]\!] \subseteq [\![S_{I_j}(\mathbf{1}, d\psi) = \psi(\sigma_j, \sigma_{j+1})]\!].$$

But

$$\begin{aligned} [\![\sigma_j = \sigma_{j+1}]\!] &= \inf_{k<m} [\![\tau_k = \tau_{k+1}]\!] \cap [\![\sigma_j = \sigma_{j+1}]\!] \\ &\subseteq [\![S_{I_j}(\mathbf{1}, d\psi) = 0]\!] \cap [\![\psi(\sigma_j, \sigma_{j+1}) = 0]\!] \\ &\subseteq [\![S_{I_j}(\mathbf{1}, d\psi) = \psi(\sigma_j, \sigma_{j+1})]\!], \end{aligned}$$

so $a \subseteq [\![S_{I_j}(\mathbf{1}, d\psi) = \psi(\sigma_j, \sigma_{j+1})]\!]$. **Q**

(e) Assembling these,

$$\begin{aligned} S_I(\mathbf{1}, d\psi) \times \chi a &= \sum_{j=-1}^{n} S_{I_j}(\mathbf{1}, d\psi) \times \chi a \\ &= \sum_{j=0}^{n-1} u_{\sigma_j} \times \psi(\sigma_j, \sigma_{j+1}) \times \chi a = S_J(\mathbf{1}, d\psi) \times \chi a \end{aligned}$$

and $a \subseteq [\![S_I(\mathbf{1}, d\psi) = S_J(\mathbf{1}, d\psi)]\!]$.

(f) If J actually covers I then we can take $a = 1$ and conclude that $S_I(\mathbf{1}, d\psi) = S_J(\mathbf{1}, d\psi)$.

613T Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\mathcal{S}'$ a sublattice of $\mathcal{S}$ which covers $\mathcal{S}$, $\psi : \mathcal{S}^{2\uparrow} \to L^0(\mathfrak{A})$ a strictly adapted interval function and $\boldsymbol{u} : \mathcal{S} \to L^0(\mathfrak{A})$ a fully adapted process. If $\int_{\mathcal{S}} \boldsymbol{u}\, d\psi$ is defined, so is $\int_{\mathcal{S}'} \boldsymbol{u}\, d\psi$, and the integrals are equal.

proof (a) Consider first the case in which $\boldsymbol{u} = \mathbf{1}$, so that $z = \int_{\mathcal{S}} d\hat{\psi}$ is defined. Let $\epsilon > 0$. Let $J \in \mathcal{I}(\mathcal{S})$ be such that $\theta(z - S_I(\mathbf{1}, d\hat{\psi})) \le \epsilon$ whenever $J \subseteq I \in \mathcal{I}(\mathcal{S})$. Let $A \subseteq \mathcal{S}'$ be a finite set such that $a = \inf_{\tau \in J} \sup_{\sigma \in A} [\![\sigma = \tau]\!]$ has measure at least $1 - \epsilon$. Let J_0 be the sublattice of $\mathcal{S}'$ generated by A, so that $a \subseteq \sup_{\sigma \in J_0} [\![\tau = \sigma]\!]$ for every $\tau \in J_0$. If $J_0 \subseteq K \in \mathcal{I}(\mathcal{S}')$, consider the sublattice $J \sqcup K$ of $\mathcal{S}$ generated by $J \cup K$. Since $a \subseteq \sup_{\sigma \in K} [\![\tau = \sigma]\!]$ for every $\tau \in J$, $a \subseteq \sup_{\sigma \in K} [\![\tau = \sigma]\!]$ for every $\tau \in J \cup K$ and therefore for every $\tau \in J \sqcup K$, by 611M(b-i). By 613S, $[\![S_K(\mathbf{1}, d\hat{\psi}) = S_{J \sqcup K}(\mathbf{1}, d\hat{\psi})]\!] \supseteq a$.

Consequently

$$\theta(z - S_K(\mathbf{1}, d\psi)) = \theta(z - S_K(\mathbf{1}, d\hat{\psi})) \le \theta(z - S_{J \sqcup K}(\mathbf{1}, d\hat{\psi})) + \bar{\mu}(1 \setminus a) \le 2\epsilon.$$

And this is true whenever $J_0 \subseteq K \in \mathcal{I}(\mathcal{S}')$.

As ϵ is arbitrary, $\int_{\mathcal{S}'} d\psi$ is defined and equal to z.

(b) For the general case, apply (i) to $\boldsymbol{u}\psi$, using 613Jg to see that

$$\int_{\mathcal{S}'} \boldsymbol{u}\, d\psi = \int_{\mathcal{S}'} d(\boldsymbol{u}\psi) = \int_{\mathcal{S}} d((\boldsymbol{u}\psi)\hat{\ }) = \int_{\mathcal{S}} \hat{\boldsymbol{u}}\, d\hat{\psi}.$$

613P Example If $T = [0, \infty[$, $(\mathfrak{A}, \bar{\mu})$ is the measure algebra of Lebesgue measure on $[0, 1]$ and $\mathfrak{A}_t = \mathfrak{A}$ for every $t \ge 0$, then there are a sublattice $\mathcal{S}$ of $\mathcal{T}$ and fully adapted processes $\boldsymbol{u}$, $\boldsymbol{v}$ with domain $\mathcal{S}$ such that $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$ is defined but $\int_{\hat{\mathcal{S}}} \hat{\boldsymbol{u}}\, d\hat{\boldsymbol{v}}$ is not, where $\hat{\mathcal{S}}$ is the covered envelope of $\mathcal{S}$ and $\hat{\boldsymbol{u}}$, $\hat{\boldsymbol{v}}$ are the fully adapted extensions of $\boldsymbol{u}$ and $\boldsymbol{v}$ to $\hat{\mathcal{S}}$.

Construction (a) For integers $k \in \mathbb{N}$ and $i, j < 2^k$ write Q_{kij} for the half-open square $\left[2^{-k}i, 2^{-k}(i+1)\right[\times \left[2^{-k}j, 2^{-k}(j+1)\right[$, so that $\{Q_{kij} : i, j < 2^k\}$ is a partition of $Q_{000} = [0, 1[^2$. For $k \ge 1$ set

$$\begin{aligned} \mathcal{Q}_k &= \{Q_{k,2i,2i+1} : i < 2^{k-1}\} \cup \{Q_{k,2i+1,2i} : i < 2k - 1\} \\ &\subseteq \bigcup_{i < 2^{k-1}} Q_{k-1,i,i} \setminus \bigcup_{i < 2^k} Q_{kii}. \end{aligned}$$

Then

if $Q, Q' \in \mathcal{Q}_k$ are distinct, their horizontal projections are disjoint and their vertical projections are disjoint,

the horizontal and vertical projections of $\bigcup \mathcal{Q}_k$ are both $[0, 1[$,

$\bigcup \mathcal{Q}_j \cap \bigcup \mathcal{Q}_k = \emptyset$ whenever $j \ne k$.

So $\mathcal{Q} = \bigcup_{k \ge 1} \mathcal{Q}_k$ is a disjoint family.

(b) Next, for $Q \in \mathcal{Q}$, take the $k \ge 1$ and $i, j < 2^k$ such that $Q = Q_{kij}$ and set

$$R(Q) = \left[2^{-k}i, 2^{-k}(i+1)\right[\times \left[2^{-k}(j + \tfrac{3}{4}, 2^{-k}(j+1)\right[,$$

$$\begin{aligned} R_l(Q) = &\left[2^{-k}(i + 2^{-k}l), 2^{-k}(i + 2^{-k}(l+1))\right[\\ &\times \left[2^{-k}(j + 2^{-k-2}l), 2^{-k}(j + 2^{-k-2}(l+1))\right[\end{aligned}$$

for $l < 2^k$, so that $R(Q)$ is the top quarter of Q and the $R_l(Q)$ are rectangles stepping up along the diagonal of the bottom quarter of Q. Now for $\omega \in [0,1[$ and $t \ge 0$ define

$$\begin{aligned} X_t(\omega) &= \frac{1}{\sqrt{k}} \text{ if } k \ge 1,\ Q \in \mathcal{Q}_k,\ l < 2^{-k} \text{ are such that } (\omega,t) \in R_l(Q), \\ &= 0 \text{ otherwise,} \\ Y_t(\omega) &= \frac{1}{\sqrt{k}} \text{ if } k \ge 1,\ Q \in \mathcal{Q}_k \text{ are such that } (\omega,t) \in R(Q), \\ &= 0 \text{ otherwise.} \end{aligned}$$

(c) Taking (Ω,Σ,μ) to be Lebesgue measure on $[0,1[$ and Σ_t to be Σ for every $t \ge 0$, $(\Omega,\mu,\langle\Sigma_t\rangle_{t\ge0})$ represents $(\mathfrak{A},\bar\mu,\langle\mathfrak{A}_t\rangle_{t\ge0}$ is the manner of Theorem 612H, while $\langle X_t\rangle_{t\ge0}$ and $\langle Y_t\rangle_{t\ge0}$ are progressively measurable, so we have corresponding fully adapted processes $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{T}_f}$ and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{T}_f}$ (612Hb).

(d)(i) Since $\mathcal{Q}$ is disjoint, $Y_t(\omega) = 0$ whenever $X_t(\omega) > 0$, and $S_I(\boldsymbol{u},d\boldsymbol{v}) \ge \Delta_e(\boldsymbol{u},d\boldsymbol{v}) \ge 0$ whenever I is a finite sublattice of $\mathcal{T}_f$ and e is an I-cell.

(ii) Suppose that $m \ge 1$, $Q \in \mathcal{Q}_m$, $(\omega,s) \in Q$, $0 \le s \le t$ and $X_s(\omega)(Y_t(\omega) - Y_s(\omega)) \ne 0$. Then we must have $X_s(\omega) = \frac{1}{\sqrt{m]}}$ and $Y_t(\omega) > 0$. In this case, there are i, j, $l < 2^m$ such that $Q = Q_{mij}$ and $(\omega,s) \in R_l(Q_{mij})$, that is,

$$2^{-m}(i + 2^{-m}l) \le \omega < 2^{-m}(i + 2^{-m}(l+1),$$

$$2^{-m}(j + 2^{-m-2}l) \le s < 2^{-m}(j + 2^{-m-2}(l+1)).$$

Now $Y_{t'}(\omega) = 0$ whenever (ω,t') belongs to the bottom three-quarters of Q, that is, whenever $2^{-m}j \le t' < 2^{-m}(j+\frac34)$, and t must be at least $2^{-m}(j+\frac34)$, while $s < 2^{-m}(j+\frac14)$. Accordingly $t - s > 2^{-m-1}$. Also, of course, $t < 1$.

(e)(i) Now suppose that $0 \le s_0 < \ldots < s_n$ and $k \ge 1$ is such that $s_{p+1} - s_p \le 2^{-k}$ for every $p < n$. For $m \ge 1$ and $Q \in \mathcal{Q}_m$, set

$$E_{mQ} = \bigcup_{p<n}\{\omega : (\omega,s_p) \in Q,\ X_{s_p}(\omega)Y_{s_{p+1}}(\omega) > 0\}.$$

(ii) If $m < k$ then $E_{mQ} = \emptyset$ for every $Q \in \mathcal{Q}_m$. **P** If $p < n$ then $s_{p+1} - s_p \le 2^{-m+1}$, so by (d-ii) there can be no (ω,s_p) such that $X_{s_p}(\omega)Y_{s_{p+1}}(\omega) > 0$. **Q**

(iii) If $m \ge k$ and $Q \in \mathcal{Q}_m$ then $\mu E_{mQ} \le 2^{-2m}$. **P** We have $Q = Q_{mij}$ where i, $j < 2^{-m}$. If $p < n$ and there is any ω such that $(\omega,s_p) \in Q$ and $X_{s_p}(\omega)Y_{s_{p+1}}(\omega) > 0$, then $s_p < 2^{-m}(j+\frac14)$ and $2^{-m}(j+\frac34) \le s_{p+1}$, by (d-ii). Now there can be at most one such p. So if $E_{mQ} \ne \emptyset$ there is a $p < n$ such that

$$E_{mQ} = \{\omega : (\omega,s_p) \in Q,\ X_{s_p}(\omega)Y_{s_{p+1}}(\omega) > 0\} \subseteq \{\omega : (\omega,s_p) \in R_l(Q)\}$$

where $l = \lfloor 2^{2m+1}(s_p - 2^{-m}j)\rfloor < 2^m$. Accordingly, in this case, $\mu E_{mQ} \le 2^{-2m}$, as required. **Q**

(iv) Putting (ii) and (iii) together,

$$\begin{aligned} &\{\omega : \sum_{p=0}^{n-1} X_{s_p(\omega)}(Y_{s_{p+1}}(\omega) - Y_{s_p}(\omega)) \ne 0\} \\ &\qquad = \bigcup_{p<n}\{\omega : X_{s_p}(\omega)Y_{s_{p+1}}(\omega) > 0\} \\ &\qquad = \bigcup_{m\ge1,Q\in\mathcal{Q}_m} E_{mQ} = \bigcup_{m\ge k,Q\in\mathcal{Q}_m} E_{mQ} \end{aligned}$$

has measure at most

$$\sum_{m=k}^\infty 2^{-2m}\#(\mathcal{Q}_m) = \sum_{m=k}^\infty 2^{-m} = 2^{-k+1}.$$

(f) Writing $\mathcal{T}_c$ for the set $\{\check{t} : t \ge 0\}$ of constant stopping times, $\int_{\mathcal{T}_c} \boldsymbol{u}\, d\boldsymbol{v}$ is defined and equal to 0. **P** Given $\epsilon > 0$, take $k \ge 1$ such that $2^{-k+1} \le \epsilon$, and let $I \in \mathcal{I}(\mathcal{T}_c)$ be such that $(2^{-k}i)\check{} \in I$ for every $i < 2^k$. Enumerate I in increasing order as $(\check{s}_0, \ldots, \check{s}_n)$. If $p < n$ and $s_p \ge 1$ then certainly $X_{s_p(\omega)}(Y_{s_{p+1}}(\omega) - Y_{s_p}(\omega)) = 0$ for every ω. So if n' is such that $s_{n'} = 1$,

$$\{\omega : \sum_{p=0}^{n-1} X_{s_p(\omega)}(Y_{s_{p+1}}(\omega) - Y_{s_p}(\omega)) \ne 0\}$$

$$= \{\omega : \sum_{p=0}^{n'-1} X_{s_p(\omega)}(Y_{s_{p+1}}(\omega) - Y_{s_p}(\omega)) \ne 0\}$$

has measure at most $2^{-k+1} \le \epsilon$, and

$$\theta(S_I(\boldsymbol{u}, \boldsymbol{v})) \le \bar{\mu}[\![S_I(\boldsymbol{u}, d\boldsymbol{v}) \ne 0]\!]$$

$$= \mu\{\omega : \sum_{p=0}^{n-1} X_{s_p(\omega)}(Y_{s_{p+1}}(\omega) - Y_{s_p}(\omega)) > 00\} \le \epsilon.$$

And this is true whenever $I \in \mathcal{I}(\mathcal{T}_c)$ includes $\{(2^{-k}i)\check{} : i < 2^k\}$. As ϵ is arbitrary, $\int_{\mathcal{T}_c} \boldsymbol{u}\, d\boldsymbol{v} = 0$.

(g) I turn now to the covered envelope $\hat{\mathcal{T}}_c$ of $\mathcal{T}_c$.

(i) For each $k \ge 1$ define stopping times h_k, $h'_k : [0,1[\to [0,\infty[$ by setting

$$h_k(\omega) = 2^{-k}(j + 2^{-k-2}l), \quad h'_k(\omega) = 2^{-k}(j + \tfrac{3}{4})$$

if i, j, $l < 2^k$ are such that $Q_{kij} \in \mathcal{Q}_k$ and $2^{-k}(i + 2^{-k}l) \le \omega < 2^{-k}(i + 2^{-k}(l+1))$. Then we have $h_k(\omega) < h'_k(\omega)$ and

$$X_{h_k}(\omega) = X_{h_k(\omega)}(\omega) = Y_{h'_k}(\omega) = \frac{1}{\sqrt{k}}, \quad X_{h_k}(\omega)(Y_{h'_k}(\omega) - Y_{h_k}(\omega)) = \frac{1}{k}$$

for every ω. If k, $m \ge 1$ are different, then for every ω there is a $Q \in \mathcal{Q}_k$ such that

$$[h_k(\omega), h'_k(\omega)] \subseteq \{s : (\omega, s) \in Q\} \subseteq \{s : (\omega, s) \in \bigcup \mathcal{Q}_k\};$$

as $\bigcup \mathcal{Q}_k$ is disjoint from $\bigcup \mathcal{Q}_m$, $[h_k(\omega), h'_k(\omega)] \cap [h_m(\omega), h'_m(\omega)] = \emptyset$.

(ii) For $k > 1$, write τ_k, τ'_k for the stopping times in $\mathcal{T}_f$ associated with h_k and h'_k. Since all the values of h_k and h'_k belong to the countable set $I = \{2^{-k}(j + 2^{-k-2}l) : j, l < 2^k\} \cup \{2^{-k}(j + \frac{3}{4}) : j < 2^k\}$,

$$\sup_{t \in I} [\![\tau_k = \check{t}]\!] = \sup_{t \in I}\{\omega : h(\omega) = t\}^\bullet = \{\omega : h_k(\omega) \in I\}^\bullet = 1$$

and $\tau_k \in \hat{\mathcal{T}}_c$; similarly, $\tau'_k \in \hat{\mathcal{T}}_c$;

Next, for each k, 612Hb tells us that

$$[\![\tau_k < \tau'_k]\!] = 1, \quad u_{\tau_k} = X^\bullet_{h_k} = \frac{1}{\sqrt{k}}\chi 1, \quad v_{\tau'_k} = Y^\bullet_{h'_k} = \frac{1}{\sqrt{k}}\chi 1, \quad v_{\tau_k} = Y^\bullet_{h_k} = 0,$$

so $u_{\tau_k} \times (v_{\tau'_k} - v_{\tau_k}) = \frac{1}{k}\chi 1$; moreover, if $k \ne m$, then for every $\omega \in [0,1[$ either $h'_k(\omega) < h_m(\omega)$ or $h'_m(\omega) < h_k(\omega)$, so $[\![\tau'_k < \tau_m]\!] \cup [\![\tau'_m < \tau_k]\!] = 1$.

It follows that if $\sigma \in \mathcal{T}_f$ then

$$([\![\tau_k < \sigma]\!] \cap [\![\sigma < \tau'_k]\!]) \cap ([\![\tau_m < \sigma]\!] \cap [\![\sigma < \tau'_m]\!]) = 0$$

whenever k, m are distinct.

(h)(i) Now suppose that $I \in \mathcal{I}(\hat{\mathcal{T}}_c)$. Let $n > \#(I) + 1$ be so large that $\sum_{k=\#(I)+2}^{n} \frac{1}{k} \ge 1$. Let J be the sublattice of $\hat{\mathcal{T}}_c$ generated by $I \cup \{\tau_k : 1 \le k \le n\} \cup \{\tau'_k : 1 \le k \le n\}$, and B the set of atoms of the subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ generated by $\{[\![\sigma < \tau]\!] : \sigma, \tau \in J\}$. Take any $b \in B$. If σ, $\tau \in J$ then b is either included in or disjoint from $[\![\sigma < \tau]\!]$. Now if $\sigma \in I$, there can be at most one $k \le n$ such that $b \subseteq [\![\tau_k < \sigma]\!] \cap [\![\sigma < \tau'_k]\!]$, so the set

$$K = \{k : \exists\, \sigma \in I,\, b \subseteq [\![\tau_k < \sigma]\!] \cap [\![\sigma < \tau'_k]\!]\}$$
$$= \{k : \exists\, \sigma \in I,\, b \cap [\![\tau_k < \sigma]\!] \cap [\![\sigma < \tau'_k]\!] \ne 0\}$$

has at most $\#(I)$ members. Set $K' = \{k : 1 \le k \le n,\, k \notin K\}$, so that $\sum_{k\in K'} \frac{1}{k} \ge 1$. Now

$$\{\sigma : b \cap [\![\tau_k < \sigma]\!] \cap [\![\sigma < \tau_k]\!] = 0 \text{ for every } k \in K'\}$$

is a sublattice of $\mathcal{T}$ including $I \cup \{\tau_k : 1 \le k \le n\} \cup \{\tau'_k : 1 \le k \le n\}$ and therefore including J.

(ii) Fix $k \in K'$ for the moment. If we think of intervals $c(\sigma,\tau)$, for $\sigma \le \tau$ in $\mathcal{T}$, as members of $\prod_{t\ge0} \mathfrak{A}_t = \mathfrak{A}^{[0,\infty[}$, as in 611J, then we see that

$$c(\tau_k, \tau'_k) = \sup\{e : e \in \mathrm{Sti}_0(J),\, e \subseteq c(\tau_k, \tau'_k)\}$$

where $\mathrm{Sti}_0(J)$ is the set of J-cells (611J(e-i)). Consequently

$$1 = [\![\tau_k, \tau'_k]\!] = \sup_{t\ge0} c(\tau_k,\tau'_k)(t) = \sup_{t\ge0} \sup_{\substack{e\in\mathrm{Sti}_0(J)\\ e\subseteq c(\tau_k,\tau'_k)}} e(t)$$

and there must be an $e_k \in \mathrm{Sti}_0(J)$ such that $e_k \subseteq c(\tau_k,\tau'_k)$ and $b \cap \sup_{t\ge0} e_k(t) \ne 0$. Expressing e_k as $c(\sigma,\tau)$ where $\sigma \le \tau$ in J, we have $\sup_{t\ge0} e_k(t) = [\![\sigma < \tau]\!]$. As this belongs to $\mathfrak{B}$ and meets b, it includes b, and $b \subseteq [\![\sigma < \tau]\!]$. At the same time, $[\![\sigma < \tau]\!] \subseteq [\![\tau_k \le \sigma]\!] \cap [\![\tau \le \tau'_k]\!]$ (611Jc) so $b \subseteq [\![\sigma \vee \tau_k < \tau'_k]\!]$; as $k \in K'$, $b \cap [\![\tau_k < \sigma \vee \tau_k]\!] = 0$ and $b \subseteq [\![\sigma = \tau_k]\!]$. Similarly, $b \subseteq [\![\tau = \tau_{k+1}]\!]$. But this means that

$$b \subseteq [\![u_\sigma \times (v_\tau - v_\sigma) = u_{\tau_k} \times (v_{\tau'_k} - v_{\tau_k})]\!] \subseteq [\![\Delta_{e_k}(\boldsymbol{u}, d\boldsymbol{v}) = \frac{1}{k}]\!].$$

(iii) This is true for every $k \in K'$. If k, $m \in K'$ are distinct, then either $\tau'_m \le \tau_k$ or $\tau'_k \le \tau_m$, and in either case $c(\tau_k,\tau'_k) \cap c(\tau_m,\tau'_m) = 0$ in $\mathfrak{A}^{[0,\infty[}$ and $e_k \ne e_m$. As we also know that $\Delta_e(\boldsymbol{u}, d\boldsymbol{v}) \ge 0$ for every stopping-time interval with endpoints in $\mathcal{T}_f$ ((d-i) above), we have

$$S_J(\boldsymbol{u}, d\boldsymbol{v}) = \sum_{e\in\mathrm{Sti}_0(J)} \Delta_e(\boldsymbol{u}, d\boldsymbol{v}) \ge \sum_{k\in K'} \Delta_{e_k}(\boldsymbol{u}, d\boldsymbol{v})$$

and

$$b \subseteq [\![S_J(\boldsymbol{u}, d\boldsymbol{v}) \ge \sum_{k\in K'} \frac{1}{k}]\!] \subseteq [\![S_J(\boldsymbol{u}, d\boldsymbol{v}) \ge 1]\!].$$

(iv) As b is arbitrary, $1 \subseteq [\![S_J(\boldsymbol{u}, d\boldsymbol{v}) \ge 1]\!]$ and $S_J(\boldsymbol{u}, d\boldsymbol{v}) \ge \chi 1$, while $I \subseteq J \in \mathcal{I}(\hat{\mathcal{T}}_c)$. As I is arbitrary, $\lim_{J\uparrow\mathcal{I}(\hat{\mathcal{T}}_c)} S_J(\boldsymbol{u}, d\boldsymbol{v})$ either does not exist or exists and is not equal to 0. But the latter is impossible, by 613T, because we saw in (f) that $\int_{\mathcal{T}_c} \boldsymbol{u}\, d\boldsymbol{v} = 0$. So we find that $\lim_{J\uparrow\mathcal{I}(\hat{\mathcal{T}}_c)} S_J(\boldsymbol{u}, d\boldsymbol{v})$ is undefined, that is, that $\int_{\hat{\mathcal{T}}_c} \boldsymbol{u}\, d\boldsymbol{v}$ is undefined.

Remark The processes $\boldsymbol{u}$ and $\boldsymbol{v}$ have a number of special properties which will be discussed in this volume; see in particular 615Yc and 618Yd.

613U Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\hat{\mathcal{S}}$ its covered envelope (611M).

(a) For every strictly adapted interval function $\psi : \mathcal{S}^{\uparrow2} \to L^0 = L^0(\mathfrak{A})$ there is a unique extension of ψ to a strictly adapted interval function $\hat{\psi} : \hat{\mathcal{S}}^{2\uparrow} \to L^0$.

(b)(i) The function $\psi \mapsto \hat{\psi}$ is an f-algebra homomorphism from the space of strictly adapted interval functions on $\mathcal{S}$ to the space of strictly adapted interval functions on $\hat{\mathcal{S}}$.

(ii) If ψ is a strictly adapted interval function on $\mathcal{S}$ and $h : \mathbb{R} \to \mathbb{R}$ is Borel measurable, then $(\bar{h}\psi)\hat{} = \bar{h}\hat{\psi}$.

(iii) If ψ is a strictly adapted interval function on $\mathcal{S}$ and $\boldsymbol{u}$ is a fully adapted process with domain $\mathcal{S}$, then $(\boldsymbol{u}\psi)\hat{} = \hat{\boldsymbol{u}}\hat{\psi}$, where $\hat{\boldsymbol{u}}$ is the fully adapted extension of $\boldsymbol{u}$ to $\hat{\mathcal{S}}$.

proof (Compare 612Q.)

(a)(i) The first thing to note is that if $(\sigma,\tau) \in \hat{\mathcal{S}}^{2\uparrow}$ and $c = \sup_{(\sigma',\tau')\in\mathcal{S}^{2\uparrow}}([\![\sigma = \sigma']\!] \cap [\![\tau = \tau']\!])$ then $c = 1$. **P** If $a \in \mathfrak{A}$ is non-zero, there is a $\sigma'' \in \mathcal{S}$ such that $a' = a \cap [\![\sigma = \sigma'']\!] \ne 0$, and now there is a $\tau' \in \mathcal{S}$ such that $a'' = a' \cap [\![\tau = \tau']\!] \ne 0$. Set $\sigma' = \sigma'' \wedge \tau'$; then $(\sigma',\tau') \in \mathcal{S}^{2\uparrow}$ and

$$a'' \subseteq [\![\sigma = \sigma'']\!] \cap [\![\tau = \tau']\!] \cap [\![\sigma \le \tau]\!] \subseteq [\![\sigma = \sigma'']\!] \cap [\![\tau = \tau']\!] \cap [\![\sigma'' \le \tau']\!]$$
$$\subseteq [\![\sigma = \sigma'']\!] \cap [\![\tau = \tau']\!] \cap [\![\sigma'' = \sigma']\!] \subseteq [\![\sigma = \sigma']\!] \cap [\![\tau = \tau']\!] \subseteq c.$$

So $a \cap c \supseteq a''$ is non-zero; as a is arbitrary, $c = 1$. **Q**

(ii) Take $(\sigma, \tau) \in \hat{\mathcal{S}}^{2\uparrow}$. For $\alpha \in \mathbb{R}$, set

$$A_{\sigma\tau\alpha} = \{[\![\psi(\sigma', \tau') > \alpha]\!] \cap [\![\sigma = \sigma']\!] \cap [\![\tau = \tau']\!] : \sigma' \le \tau' \text{ in } \mathcal{S}\}, \quad a_{\sigma\tau\alpha} = \sup A_{\sigma\tau\alpha}.$$

Then

$$a_{\sigma\tau\alpha} = \sup_{(\sigma',\tau')\in\mathcal{S}^{2\uparrow}} ([\![\psi(\sigma', \tau') > \alpha]\!] \cap [\![\sigma = \sigma']\!] \cap [\![\tau = \tau']\!])$$
$$= \sup_{\substack{(\sigma',\tau')\in\mathcal{S}^{2\uparrow} \\ \beta>\alpha}} ([\![\psi(\sigma', \tau') > \beta]\!] \cap [\![\sigma = \sigma']\!] \cap [\![\tau = \tau']\!]) = \sup_{\beta>\alpha} a_{\sigma\tau\beta}.$$

Moreover, if $a \in \mathfrak{A} \setminus \{0\}$, there are a pair $(\sigma', \tau') \in \mathcal{S}^{2\uparrow}$ such that $a' = a \cap [\![\sigma = \sigma']\!] \cap [\![\tau = \tau']\!]$ is non-zero, by (i), and α, $\beta \in \mathbb{R}$ such that $a'' = a' \cap [\![\psi(\sigma', \tau') > \alpha]\!] \cap [\![\psi(\sigma', \tau') \le \beta]\!]$ is non-zero; in which case, whenever $(\sigma'', \tau'') \in \mathcal{S}^{2\uparrow}$,

$$a'' \cap [\![\psi(\sigma'', \tau'') > \beta]\!] \cap [\![\sigma'' = \sigma]\!] \cap [\![\tau'' = \tau]\!]$$
$$\subseteq [\![\psi(\sigma'', \tau'') \ne \psi(\sigma', \tau']\!] \cap [\![\sigma'' = \sigma']\!] \cap [\![\tau'' = \tau']\!] = 0.$$

But this means that $a'' \cap a_{\sigma\tau\beta} = 0$, while on the other hand $a'' \subseteq a_{\sigma\tau\alpha}$. As a is arbitrary, $\inf_{\beta\in\mathbb{R}} a_{\sigma\tau\beta} = 0$ and $\sup_{\alpha\in\mathbb{R}} a_{\sigma\tau\alpha} = 1$.

Accordingly we have a $\hat{\psi}(\sigma, \tau) \in L^0(\mathfrak{A})$ such that $[\![\hat{\psi}(\sigma, \tau) > \alpha]\!] = a_{\sigma\tau\alpha}$ for every $\alpha \in \mathbb{R}$. Observe next that $A_{\sigma\tau\alpha} \subseteq \mathfrak{A}_\tau$ for every $\alpha \in \mathbb{R}$ (use 611H(c-iii)), so $a_{\sigma\tau\alpha} \in \mathfrak{A}_\tau$ for every α, and $\hat{\psi}(\sigma, \tau) \in L^0(\mathfrak{A}_\tau)$.

(iii) $\hat{\psi}$ extends ψ. **P** If $(\sigma, \tau) \in \mathcal{S}^{2\uparrow}$ and $\alpha \in \mathbb{R}$, then $[\![\psi(\sigma, \tau) > \alpha]\!] \in A_{\sigma\tau\alpha}$ so $[\![\psi(\sigma, \tau) > \alpha]\!] \subseteq a_{\sigma\tau\alpha}$. If $(\sigma', \tau') \in \mathcal{S}^{2\uparrow}$ then

$$[\![\psi(\sigma', \tau') > \alpha]\!] \cap [\![\sigma = \sigma']\!] \cap [\![\tau = \tau']\!]$$
$$= [\![\psi(\sigma, \tau) > \alpha]\!] \cap [\![\sigma = \sigma']\!] \cap [\![\tau = \tau']\!] \subseteq [\![\psi(\sigma, \tau) > \alpha]\!],$$

so $a_{\sigma\tau\alpha} \subseteq [\![\psi(\sigma, \tau) > \alpha]\!]$. Thus $[\![\hat{\psi}(\sigma, \tau) > \alpha]\!] = [\![\psi(\sigma, \tau) > \alpha]\!]$. As α is arbitrary, $\hat{\psi}(\sigma, \tau) = \psi(\sigma, \tau)$. **Q**

(iv) If (σ, τ), $(\tilde{\sigma}, \tilde{\tau}) \in \hat{\mathcal{S}}^{2\uparrow}$, $c = [\![\sigma = \tilde{\sigma}]\!] \cap [\![\tau = \tilde{\tau}]\!]$ and $\alpha \in \mathbb{R}$, then $c \cap a_{\sigma\tau\alpha} \subseteq a_{\tilde{\sigma}\tilde{\tau}\alpha}$. **P** If $a \in A_{\sigma\tau\alpha}$, express it as $[\![\psi(\sigma', \tau') > \alpha]\!] \cap [\![\sigma = \sigma']\!] \cap [\![\tau = \tau']\!]$ where $(\sigma', \tau') \in \mathcal{S}^{2\uparrow}$. Then $c \cap a \subseteq [\![\tilde{\sigma} = \sigma']\!] \cap [\![\tilde{\tau} = \tau']\!]$ (611E(c-iv-γ)), so $c \cap a \subseteq a_{\tilde{\sigma}\tilde{\tau}\alpha}$. It follows that

$$c \cap a_{\sigma\tau\alpha} = c \cap \sup A_{\sigma\tau\alpha} = \sup\nolimits_{a\in A_{\sigma\tau\alpha}} c \cap a \subseteq a_{\tilde{\sigma}\tilde{\tau}\alpha}. \text{ } \mathbf{Q}$$

Similarly, $c \cap a_{\tilde{\sigma},\tilde{\tau},\alpha} \subseteq a_{\sigma\tau\alpha}$, so $c \cap [\![\hat{\psi}(\sigma, \tau) > \alpha]\!] = c \cap [\![\hat{\psi}(\tilde{\sigma}, \tilde{\tau}) > \alpha]\!]$. As this is true for every α, $c \subseteq [\![\hat{\psi}(\sigma, \tau) = \hat{\psi}(\tilde{\sigma}, \tilde{\tau})]\!]$.

(v) If $\sigma \in \hat{\mathcal{S}}$ then $\hat{\psi}(\sigma, \sigma) = 0$. **P** If $\sigma' \in \mathcal{S}$ then

$$[\![\sigma = \sigma']\!] \subseteq [\![\hat{\psi}(\sigma, \sigma) = \psi(\sigma', \sigma')]\!] \subseteq [\![\hat{\psi}(\sigma, \sigma) = 0]\!];$$

since $\sigma \in \hat{\mathcal{S}}$, $\sup_{\sigma'\in\mathcal{S}} [\![\sigma = \sigma']\!] = 1$ and $\hat{\psi}(\sigma, \sigma) = 0$. **Q**

It follows that $\hat{\psi}$ is a strictly adapted interval function. **P** I have already checked that $\hat{\psi}(\sigma, \tau) \in L^0(\mathfrak{A}_\tau)$ whenever $(\sigma, \tau) \in \hat{\mathcal{S}}^{2\uparrow}$. Now (iv) tells us that $\hat{\psi}$ is a strictly adapted interval function. **Q**

(vi) Finally, if $\psi' : \hat{\mathcal{S}}^{2\uparrow} \to L^0(\mathfrak{A})$ is any strictly adapted interval function process extending ψ, and $(\sigma, \tau) \in \hat{\mathcal{S}}^{2\uparrow}$, we shall have

$$[\![\hat{\psi}(\sigma, \tau) = \psi'(\sigma, \tau)]\!] \supseteq \sup_{(\sigma',\tau')\in\mathcal{S}^{2\uparrow}} [\![\hat{\psi}(\sigma, \tau) = \hat{\psi}(\sigma', \tau')]\!] \cap [\![\psi'(\sigma, \tau) = \psi'(\sigma', \tau')]\!]$$
$$\supseteq \sup_{(\sigma',\tau')\in\mathcal{S}^{2\uparrow}} [\![\sigma = \sigma']\!] \cap [\![\tau = \tau']\!] = 1$$

by (i) again, so $\hat{\psi}(\sigma,\tau)=\psi'(\sigma,\tau)$. Thus $\hat{\psi}=\psi'$ and $\hat{\psi}$ is the unique strictly adapted interval function extending ψ.

(b) Really this is immediate from the definitions and the fact that $\hat{\psi}$ is always the unique strictly adapted extension of ψ, just as (in (iii)) $\hat{\boldsymbol{u}}$ is the unique fully adapted extension of $\boldsymbol{u}$. Of course we do need to know that $\hat{\psi}+\hat{\psi}'$, $\hat{\psi}\times\hat{\psi}'$, $\bar{h}\hat{\psi}$ and $\hat{\boldsymbol{u}}\hat{\psi}$ are always strictly adapted, as noted in 613D.

Remark Note that if $\boldsymbol{v}$ is a fully adapted process with domain $\mathcal{S}$ and fully adapted extension $\hat{\boldsymbol{v}}$ to $\hat{\mathcal{S}}$, then $(\Delta\boldsymbol{v})\hat{} = \Delta\hat{\boldsymbol{v}}$.

613V The definition of the integral means that we shall always be able to approximate an integral $\int_{\mathcal{S}}\boldsymbol{u}\,d\psi$ by Riemann sums $S_I(\boldsymbol{u},d\psi)$. There will be occasions when it is very useful to have simultaneous approximations for integrals over parts of $\mathcal{S}$.

Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u}$ a fully adapted process with domain $\mathcal{S}$, and ψ an adapted interval function with domain $\mathcal{S}^{2\uparrow}$ such that $\int_{\mathcal{S}}\boldsymbol{u}\,d\psi$ is defined. Let $I\in\mathcal{I}(\mathcal{S})$ and $\epsilon>0$ be such that $\theta(S_J(\boldsymbol{u},d\psi)-S_K(\boldsymbol{u},d\psi))\le\epsilon$ whenever J, $K\in\mathcal{I}(\mathcal{S})$ include I.

(i) If $\tau_0\le\tau_0'\le\tau_1\le\tau_1'\le\ldots\le\tau_n\le\tau_n'$ in I, then

$$\theta(\textstyle\sum_{i=0}^n(S_{I\cap[\tau_i,\tau_i']}(\boldsymbol{u},d\psi)-\int_{\mathcal{S}\cap[\tau_i,\tau_i']}\boldsymbol{u}\,d\psi))\le\epsilon.$$

(ii)(α) If $\tau\in I$ then $\theta(S_{I\wedge\tau}(\boldsymbol{u},d\psi)-\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\,d\psi)\le\epsilon$.

(β) For any $\tau\in\mathcal{S}$, $\theta(S_{I\wedge\tau}(\boldsymbol{u},d\psi)-\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\,d\psi)\le 2\epsilon$.

proof (a) Note straight away that by 613Jc all the integrals $\int_{\mathcal{S}\cap[\tau_i,\tau_i']}\boldsymbol{u}\,d\psi$, $\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\,d\psi$ will be defined. Observe also that

$$\theta(S_J(\boldsymbol{u},d\psi)-\textstyle\int_{\mathcal{S}}\boldsymbol{u}\,d\psi)=\lim_{K\uparrow\mathcal{I}(\mathcal{S})}\theta(S_J(\boldsymbol{u},d\psi)-S_K(\boldsymbol{u},d\psi))\le\epsilon$$

whenevever $I\subseteq J\in\mathcal{I}(\mathcal{S})$.

(b) For the time being, suppose that $\boldsymbol{u}$ is the constant process $\mathbf{1}$.

(i) Take any $\eta>0$. Then we have $J_0,\ldots,J_{n+1}$ such that

$$J_0\in\mathcal{I}(\mathcal{S}\wedge\tau_0),$$
$$\theta(S_K(\mathbf{1},d\psi)-\int_{\mathcal{S}\wedge\tau_0}d\psi)\le\eta\text{ whenever }J_0\subseteq K\in\mathcal{I}(\mathcal{S}\wedge\tau_0),$$
$$J_i\in\mathcal{I}(\mathcal{S}\cap[\tau_{i-1}',\tau_i]),$$
$$\theta(S_K(\mathbf{1},d\psi)-\int_{\mathcal{S}\cap[\tau_{i-1}',\tau_i]}d\psi)\le\eta\text{ whenever }J_i\subseteq K\in\mathcal{I}(\mathcal{S}\cap[\tau_{i-1}',\tau_i])$$

for $1\le i\le n$,

$$J_{n+1}\in\mathcal{I}(\mathcal{S}\vee\tau_n'),$$
$$\theta(S_K(\mathbf{1},d\psi)-\int_{\mathcal{S}\vee\tau_n'}d\psi)\le\eta\text{ whenever }J_{n+1}\subseteq K\in\mathcal{I}(\mathcal{S}\vee\tau_n).$$

Now take K to be the sublattice generated by $I\cup\bigcup_{i\le n+1}J_i$. Observe that if $i\le n$ then $\{\sigma:\sigma\in\mathcal{S},\ \mathrm{med}(\tau_i,\sigma,\tau_i')\in I\}$ is a sublattice of $\mathcal{S}$ including $I\cup\bigcup_{i\le n+1}J_i$ so includes K, and $K\cap[\tau_i,\tau_i']=I\cap[\tau_i,\tau_i']$. We see also that

$$\theta(S_K(\mathbf{1},d\psi)-\textstyle\int_{\mathcal{S}}d\psi)=\lim_{L\uparrow\mathcal{I}(\mathcal{S})}\theta(S_K(\mathbf{1},d\psi)-S_L(\mathbf{1},d\psi))\le\epsilon$$

because $K\supseteq I$. Now

$$S_K(\mathbf{1},d\psi)=S_{K\wedge\tau_0}(\mathbf{1},d\psi)+\sum_{i=0}^n S_{K\cap[\tau_i,\tau_i']}(\mathbf{1},d\psi)$$
$$+\sum_{i=1}^n S_{K\cap[\tau_{i-1}',\tau_i]}(\mathbf{1},d\psi)+S_{K\vee\tau_n'}(\mathbf{1},d\psi)$$

(613G(a-ii)),

$$\int_{\mathcal{S}} d\psi = \int_{\mathcal{S}\wedge\tau_0} d\psi + \sum_{i=0}^{n} \int_{\mathcal{S}\cap[\tau_i,\tau'_i]} d\psi + \sum_{i=1}^{n} \int_{\mathcal{S}\cap[\tau'_{i-1},\tau_i]} d\psi + \int_{\mathcal{S}\vee\tau'_n} d\psi$$

(613J(c-ii)). So

$$\begin{aligned}\theta(\sum_{i=0}^{n} S_{I\cap[\tau_i,\tau'_i]}(\mathbf{1},d\psi) - \int_{\mathcal{S}\cap[\tau_i,\tau'_i]} d\psi)& \\ &\le \theta(S_K(\mathbf{1},d\psi) - \int_{\mathcal{S}} d\psi) + \theta(S_{K\wedge\tau_0}(\mathbf{1},d\psi) - \int_{\mathcal{S}\wedge\tau_0} d\psi) \\ &\quad + \sum_{i=1}^{n} \theta(S_{K\cap[\tau'_{i-1},\tau_i]}(\mathbf{1},d\psi) - \int_{\mathcal{S}\cap[\tau'_{i-1},\tau_i]} d\psi) \\ &\quad + \theta(S_{K\vee\tau'_n}(\mathbf{1},d\psi) - \int_{\mathcal{S}\vee\tau'_n} d\psi) \\ &\le \epsilon + (n+2)\eta.\end{aligned}$$

As η is arbitrary,

$$\theta(\sum_{i=0}^{n} S_{I\cap[\tau_i,\tau'_i]}(\mathbf{1},d\psi) - \int_{\mathcal{S}\cap[\tau_i,\tau'_i]} d\psi) \le \epsilon,$$

as claimed.

(ii)(*α*) The argument is the same as that of (i), but simpler. For any $\eta > 0$, there is a $J \in \mathcal{I}(\mathcal{S}\vee\tau)$ such that $\theta(S_K(\mathbf{1},d\psi) - \int_{\mathcal{S}\vee\tau} d\psi) \le \eta$ whenever $K \in \mathcal{I}(\mathcal{S}\vee\tau)$ includes J. Let K be the sublattice generated by $I \cup J$. Since $I \cup J$ is included in the sublattice $\{\sigma : \sigma\wedge\tau \in I\}$, $K\wedge\tau = I\wedge\tau$, while $K\vee\tau \supseteq J$. So

$$\begin{aligned}\theta(S_{I\wedge\tau}(\mathbf{1},d\psi) - \int_{\mathcal{S}\wedge\tau} d\boldsymbol{v}) &= \theta(S_{K\wedge\tau}(\mathbf{1},d\boldsymbol{v}) - \int_{\mathcal{S}\wedge\tau} d\boldsymbol{v}) \\ &\le \theta(S_K(\mathbf{1},d\boldsymbol{v}) - \int_{\mathcal{S}} d\boldsymbol{v})) + \theta(S_{K\vee\tau}(\mathbf{1},d\boldsymbol{v}) - \int_{\mathcal{S}\vee\tau} d\boldsymbol{v}) \\ &\le \epsilon + \eta;\end{aligned}$$

as η is arbitrary, $\theta(S_{I\wedge\tau}(\mathbf{1},d\boldsymbol{v}) - \int_{\mathcal{S}\wedge\tau} d\boldsymbol{v}) \le \epsilon$.

(*β*) If I is empty, then $\theta(S_J(\mathbf{1},d\psi)) \le \epsilon$ for every $J \in \mathcal{I}(\mathcal{S})$, so

$$\theta(S_{I\wedge\tau}(\mathbf{1},d\psi) - \int_{\mathcal{S}\wedge\tau} d\psi) = \theta(\int_{\mathcal{S}\wedge\tau} d\psi) \le \epsilon.$$

Otherwise, let J be the sublattice of $\mathcal{S}$ generated by $I \cup \{\tau\}$, and write τ^* for $\max I$. Then

$$I\wedge\tau = I\wedge\tau\wedge\tau^* \subseteq J\wedge\tau\wedge\tau^*,$$

while $\{\sigma : \sigma\wedge\tau\wedge\tau^* \in I\wedge\tau\}$ is a sublattice of $\mathcal{S}$ including $I\cup\{\tau\}$, so includes J, and $I\wedge\tau = J\wedge\tau\wedge\tau^*$. Next, $\theta(S_L(\mathbf{1},d\psi) - S_{L'}(\mathbf{1},d\psi)) \le \epsilon$ whenever L, $L' \in \mathcal{I}(\mathcal{S})$ include J, so

$$\theta(S_{I\wedge\tau}(\mathbf{1},d\psi) - \int_{\mathcal{S}\wedge\tau\wedge\tau^*} d\psi) = \theta(S_{J\wedge\tau\wedge\tau^*}(\mathbf{1},d\psi) - \int_{\mathcal{S}\wedge\tau\wedge\tau^*} d\psi) \le \epsilon$$

by (α). Because $\sigma \mapsto \int_{\mathcal{S}\wedge\sigma} d\psi$ is fully adapted (613J(e-ii)),

$$\int_{\mathcal{S}\cap[\tau\wedge\tau^*,\tau]} d\psi = \int_{\mathcal{S}\wedge\tau} d\psi - \int_{\mathcal{S}\wedge\tau\wedge\tau^*} d\psi = \int_{\mathcal{S}\wedge(\tau\vee\tau^*)} d\psi - \int_{\mathcal{S}\wedge\tau^*} d\psi$$

(612D(f-i))

$$= \int_{\mathcal{S}\cap[\tau^*,\tau\vee\tau^*]} d\psi = \lim_{K\uparrow\mathcal{I}(\mathcal{S}\cap[\tau^*,\tau\vee\tau^*])} S_K(\mathbf{1},d\psi).$$

But if $K \in \mathcal{I}(\mathcal{S}\cap[\tau^*,\tau\vee\tau^*])$ contains τ^*, then $I \cup K \in \mathcal{I}(\mathcal{S})$,

$$\begin{aligned}S_K(\mathbf{1}, d\psi) =& S_{(I\cup K)\vee\tau^*}(\mathbf{1}, d\psi)\\ &= S_{I\cup K}(\mathbf{1}, d\psi) - S_{(I\cup K)\wedge\tau^*}(\mathbf{1}, d\psi) = S_{I\cup K}(\mathbf{1}, d\psi) - S_I(\mathbf{1}, d\psi)\end{aligned}$$

and

$$\theta(S_K(\mathbf{1}, d\psi)) = \theta(S_{I\cup K}(\mathbf{1}, d\psi) - S_I(\mathbf{1}, d\psi)) \le \epsilon;$$

as K is arbitrary, $\theta(\int_{\mathcal{S}\cap[\tau\wedge\tau^*,\tau]} d\psi) \le \epsilon$. Accordingly

$$\begin{aligned}\theta(S_{I\wedge\tau}(\mathbf{1}, d\psi) - \int_{\mathcal{S}\wedge\tau} d\psi) &= \theta(S_{I\wedge\tau}(\mathbf{1}, d\psi) - \int_{\mathcal{S}\wedge\tau\wedge\tau^*} d\psi - \int_{\mathcal{S}\cap[\tau\wedge\tau^*,\tau]} d\psi)\\ &\le \theta(S_{I\wedge\tau}(\mathbf{1}, d\psi) - \int_{\mathcal{S}\wedge\tau\wedge\tau^*} d\psi) + \theta(\int_{\mathcal{S}\cap[\tau\wedge\tau^*,\tau]} d\psi)\\ &\le 2\epsilon,\end{aligned}$$

as required.

(c) For the general case, recall that $\boldsymbol{u}\psi$, as defined in 613Dd, is an adapted interval function, and that every sum or integral $S_J(\boldsymbol{u}, d\psi)$ or $\int_{\mathcal{S}'} \boldsymbol{u}\, d\psi$ can be interpreted as $S_J(\mathbf{1}, d(\boldsymbol{u}\psi))$ or $\int_{\mathcal{S}'} d(\boldsymbol{u}\psi)$ (613Ee, 613Hc). So we get the general result at once.

613W The one-dimensional case (a) There are real difficulties in the theory of stochastic integration, and it will be a long time before you can expect it to feel natural and familiar. For the rest of this chapter we shall be teasing out more or less special cases in which the integral of 613H is defined. But there is a particular special case which we can approach immediately. Suppose that $(\mathfrak{A}, \bar\mu)$ is the trivial probability algebra in which $\mathfrak{A} = \{0, 1\}$. Then $L^0(\mathfrak{A}) = \{\alpha\chi 1 : \alpha \in \mathbb{R}\}$ can be identified, as f-algebra, with $\mathbb{R}$; of course we have $\theta(\alpha\chi 1) = \min(1, |\alpha|)$ for every α, so the topology of convergence in measure on $L^0(\mathfrak{A})$ corresponds to the usual topology of $\mathbb{R}$. Necessarily, $\mathfrak{A}_t = \mathfrak{A}$ for every $t \in T$, so the filtration is trivial. If it is also the case that T has no points isolated on the right, then every stopping time except $\max\mathcal{T}$ and possibly $\min\mathcal{T}$ will be a constant stopping time as described in 611A(b-ii), every subset of $\mathcal{T}$ is a sublattice, and every real-valued function f defined on a subset S of T corresponds to a fully adapted process $\{(\check s, f(s)\chi 1) : s \in S\}$.

(b) If also T has a least element, we can identify $\mathcal{T}_f$ with T and $M_{\mathrm{fa}}(\mathcal{T}_f) = (L^0)^{\mathcal{T}_f}$ with $\mathbb{R}^T$. Under this identification, if $f : T \to \mathbb{R}$ and $g : T \to \mathbb{R}$ represent processes $\boldsymbol{u}$, $\boldsymbol{v}$ with domain $\mathcal{T}_f$, and $I \subseteq \mathcal{T}_f$ is a non-empty finite set, there are $t_0 \le \ldots \le t_n$ in T such that $I = \{\check t_i : i \le n\}$, and

$$S_I(\boldsymbol{u}, d\boldsymbol{v}) = \big(\textstyle\sum_{i=0}^{n-1} f(t_i)(g(t_{i+1}) - g(t_i))\big)\chi 1.$$

(d) What this amounts to is that we have a kind of Riemann-Stieltjes integral on T, I spell this out in detail here partly because there are well-known Stieltjes integrals on the real line, of which the most important, from the point of view of my treatise as a whole, is integration with respect to Lebesgue-Stieltjes measures described in exercises from 114Xa onwards. Here we suppose that $g : \mathbb{R} \to \mathbb{R}$ is non-decreasing, so that there is a Radon measure ν_g on $\mathbb{R}$ with $\nu_g\,[a, b[\, = \lim_{x\uparrow b} g(x) - \lim_{x\uparrow a} g(x)$ whenever $a < b$ in $\mathbb{R}$. Now the point I need to make here is that if $\mathcal{S} = \{\check s : s \in \mathbb{R}\}$ then the integral $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$ is *not* the same as the Lebesgue-Stieltjes integral $\int f\, d\nu_g$, even in some of the most elementary situations. Consider, for instance, the case in which $f = g = \chi\,[0, \infty[$. In this case, ν_g is the Dirac measure concentrated at 0, so that $\int f\, d\nu_g = f(0) = 1$. But when we look at sums $S_I(\boldsymbol{u}, d\boldsymbol{v})$ where $I = \{\check t_0, \ldots, \check t_n\}$ is a finite subset of $\mathcal{S}$, and supposing that $t_0 \le \ldots \le t_n$, we get

$$\begin{aligned}f(t_i)(g(t_{i+1}) - g(t_i)) &= 0(g(t_{i+1}) - g(t_i)) = 0 \text{ if } t_i < 0,\\ &= f(t_i)(1 - 1) = 0 \text{ if } t_i \ge 0,\end{aligned}$$

so $S_I(f, dg) = 0$; as this is true for every I, $\int_{\mathcal{S}} f\, dg = 0$. In effect, the jump in g at the time 0 is necessarily applied to a value of f calculated before the time 0; in the language of Lebesgue-Stieltjes integration, we are calculating $\int f_-\, d\nu_g$ where $f_-(x) = \lim_{y\uparrow x} f(y)$ for each x.

In my view, there are excellent reasons (especially in view of its applications to financial mathematics) why a theory of stochastic integration should insist on calculating integrals $\int \boldsymbol{u}\,d\boldsymbol{v}$ in terms of products $u_\sigma \times (v_\tau - v_\sigma)$ where $\sigma \le \tau$ (rather than $u_\tau \times (v_\tau - v_\sigma)$, for instance). We are going to have to return to this point from time to time, because it is one on which my presentation of the theory differs from that of most authors.

613X Basic exercises (a) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v}$ a fully adapted process defined on $\mathcal{S}$. Show that $\int_{\mathcal{S}} \boldsymbol{v}\,d\boldsymbol{v}$ is defined whenever $\int_{\mathcal{S}} d\boldsymbol{v}$ and $\int_{\mathcal{S}} (d\boldsymbol{v})^2$ are defined, and that in this case

$$2\int_{\mathcal{S}} \boldsymbol{v}\,d\boldsymbol{v} + \int_{\mathcal{S}} (d\boldsymbol{v})^2 = \lim_{\tau\uparrow\mathcal{S}} v_\tau^2 - \lim_{\tau\downarrow\mathcal{S}} v_\tau^2.$$

(c) Let $\mathcal{S}$ be a full sublattice of $\mathcal{T}$, $\boldsymbol{u}$ a fully adapted process with domain $\mathcal{S}$, and ψ an adapted interval function with domain $\mathcal{S}^{2\uparrow}$. Write $\mathcal{S}'$ for the domain of the indefinite integral $ii_\psi(\boldsymbol{u})$. Show that $\mathcal{S}'$ is full.

(d) In 613V, show that if τ, $\tau' \in I$ and $I \subseteq J \in \mathcal{I}(\mathcal{S})$ then $\theta(S_{J\wedge\tau}(\boldsymbol{u}, d\psi) - \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\,d\psi)$, $\theta(S_{J\vee\tau'}(\boldsymbol{u}, d\psi) - \int_{\mathcal{S}\vee\tau'} \boldsymbol{u}\,d\psi)$ and $\theta(S_{J\cap[\tau,\tau']}(\boldsymbol{u}, d\psi) - \int_{\mathcal{S}\cap[\tau,\tau']} \boldsymbol{u}\,d\psi)$ are all at most ϵ.

(e) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\mathcal{I}_t$ the set of finite totally ordered subsets of $\mathcal{S}$. Suppose that $\boldsymbol{u}$ is a fully adapted process with domain $\mathcal{S}$, ψ is an adapted interval function with domain $\mathcal{S}^{2\uparrow}$ and z is a member of $L^0(\mathfrak{A})$. Show that $\int_{\mathcal{S}} \boldsymbol{u}\,d\psi = z$ iff for every $\epsilon > 0$ and $I \in \mathcal{I}_t$ there is a $J \in \mathcal{I}_t$, including I, such that $\theta(S_K(\boldsymbol{u}, d\psi) - z) \le \epsilon$ whenever $K \in \mathcal{I}_t$ includes J.

613Y Further exercises (a) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u}$ a fully adapted process defined on $\mathcal{S}$ and ψ an adapted interval function on $\mathcal{S}^{2\uparrow}$. Show that if τ, $\tau' \in \mathcal{S}$, then $\int_{\mathcal{S}\vee(\tau\wedge\tau')} \boldsymbol{u}\,d\psi$ is defined iff both $\int_{\mathcal{S}\vee\tau} \boldsymbol{u}\,d\psi$ and $\int_{\mathcal{S}\vee\tau'} \boldsymbol{u}\,d\psi$ are defined, and in this case

$$\int_{\mathcal{S}\vee(\tau\wedge\tau')} \boldsymbol{u}\,d\psi + \int_{\mathcal{S}\vee\tau\vee\tau'} \boldsymbol{u}\,d\psi = \int_{\mathcal{S}\vee\tau} \boldsymbol{u}\,d\psi + \int_{\mathcal{S}\vee\tau'} \boldsymbol{u}\,d\psi.$$

(b) Suppose that $\mathcal{S}$ is a sublattice of $\mathcal{T}$ and $\mathcal{S}'$ a covering ideal of $\mathcal{S}$. Let $\boldsymbol{u}$ be a fully adapted process defined on $\mathcal{S}$ and ψ an adapted interval function defined on $\mathcal{S}^{2\uparrow}$. Show that $\int_{\mathcal{S}} \boldsymbol{u}\,d\psi$ is defined iff $\int_{\mathcal{S}'} \boldsymbol{u}\,d\psi$ is defined, and the integrals are then equal.

(c) Let $\mathcal{C}$ be the set of all bounded intervals in $\mathbb{R}$, and T^* the straightforward set of tagged partitions generated by $Q = \{(a, C) : C \in \mathcal{C} \setminus \{\emptyset\},\ a = \inf C\}$ (see 481B). For a finite set $I \subseteq \mathbb{R}$, let δ_I be the set $\{\boldsymbol{t} : \boldsymbol{t} \in T^*,\ I \cap \operatorname{int} C = \emptyset$ whenever $(a, C) \in \boldsymbol{t}\}$; for $a \le b$ in $\mathbb{R}$, let $\mathcal{R}_{ab}$ be $\{\emptyset\} \cup \{\mathbb{R} \setminus [c, d] : c \le a,\ b \le d\}$. Set $\Delta = \{\delta_I : I \in [\mathbb{R}]^{<\omega}\}$ and $\mathfrak{R} = \{\mathcal{R}_{ab} : a \le b\}$ (see 481K). (i) Show that $(\mathbb{R}, T^*, \Delta, \mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$ (see 481G). (ii) For a function $g : [0, \infty[\to \mathbb{R}$ define $\nu_g : \mathcal{C} \to \mathbb{R}$ by saying that $\nu_g\emptyset = 0$, $\nu_g C = g(\sup C) - g(\inf C)$ for non-empty $C \in \mathcal{C}$. Let $\mathcal{F}(T^*, \Delta, \mathfrak{R})$ be the filter on T^* defined from Δ and $\mathfrak{R}$ as in 481F, and for $f : \mathbb{R} \to \mathbb{R}$ set $I_{\nu_g}(f) = \lim_{\boldsymbol{t}\to\mathcal{F}(T^*,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f, \nu_g)$ when this is defined, as in 481C. Next, for functions f, $g : \mathbb{R} \to \mathbb{R}$, define $\int_{\mathcal{S}} f\,dg$ as in 613W, taking $\mathcal{S}$ to be the set of constant stopping times when $\mathfrak{A} = \{0, 1\}$ and $T = \mathbb{R}$, and f and g are interpreted as functions from $\mathcal{S}$ to $L^0(\mathfrak{A})$. Show that $I_{\nu_g}(f) = \int_{\mathcal{S}} f\,dg$ when either is defined.

(d) In 613P, show that $\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}$ is undefined for any sublattice $\mathcal{S}$ of $\mathcal{T}_f$ including $\hat{\mathcal{T}}_c$.

(e) Give an example of a strictly adapted interval function ψ on a sublattice $\mathcal{S}$ such that $\int_{\mathcal{S}} d\psi$ is defined, but $\int_{\hat{\mathcal{S}}} d\hat{\psi}$ is not, where $\hat{\psi}$ is the strictly adapted extension of ψ on the covered envelope $\hat{\mathcal{S}}$ of $\mathcal{S}$.

(f) Show that for strictly adapted interval functions ψ we can re-work this section in terms of a definition of $S_I(\boldsymbol{u}, d\psi)$ restricted to finite totally ordered sets I, as in 613Ec, taking $\int_{\mathcal{S}} \boldsymbol{u}\,d\psi$ to be the limit $\lim_{I\uparrow\mathcal{I}_t(\mathcal{S})} S_I(\boldsymbol{u}, d\psi)$ where $\mathcal{I}_t(\mathcal{S})$ is the set of finite totally ordered subsets of $\mathcal{S}$ with the pre-order $\sqsubseteq$ of 611Yd. (Begin by showing that $S_I(\boldsymbol{u}, d\psi) = S_J(\boldsymbol{u}, d\psi)$ whenever $I \sqsubseteq J$ and $J \sqsubseteq I$.)

613 Notes and comments My objective in this section has been to reach a formally adequate definition of a stochastic integral as quickly as possible, with just enough of its properties to serve as a foundation for the rest of the volume.

The definition in 613H is a 'gauge integral' of the type examined in §§481-482. I cannot quote the results there because it is a vector-valued, rather than scalar-valued integral; and even on the basis of a vector-valued version of the material, which is easy to develop (there is a hint in 613Yc), the ideas of §482 do not directly illuminate the theory here. (Though there is an echo of the Saks-Henstock Lemma, 482B, in 613V.) The disadvantage of this headlong approach is that we have no real idea of which pairs $\boldsymbol{u}$, $\boldsymbol{v}$ will combine to give integrals $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$. The case of simple processes (614B-614D below) is plainly elementary, and 613C-613G don't belong to the topological theory at all, and could have been expressed in the context of §§611-612, with an arbitrary Dedekind complete algebra $\mathfrak{A}$ rather than a probability algebra. The integral $\int_{\mathcal{S}} |d\boldsymbol{v}|$ in 614J below can also be expressed in terms of a notion of convergence which does not involve the topology of convergence in measure, but is off the line of the main argument, as well as being elementary. The centre of the theory is really occupied by martingale integrators $\boldsymbol{v}$, and we have a fair amount of work to do before these become accessible in §622.

All the main work of this chapter will be done with strictly adapted interval functions, starting with the basic examples $\Delta\boldsymbol{v}$ as in 613Cc. I include the more general formulation of 'adapted interval function' in 613C(a-i) only because the language of 613E will be useful in §626 when talking about a quite different kind of integral.

While I have, I hope, given an exact definition of the integration process which will dominate the next three chapters, this section suffers from a singular lack of calculation of particular examples. In fact it is by no means trivial to show that even in the cases of our three leading examples (the identity process, Brownian motion and the Poisson process) we have a full set of indefinite integrals. There is an effective description of indefinite integrals of simple processes in 614Xb, but for other integrands, even when integrating with respect to simple processes as in 614D, we aren't yet in a position to get a formula. One will appear in 641J.

I ought to remark that the integral I have defined does not coincide with everyone's. I will return to this point in Chapter 64, in the course of defining what I call the 'S-integral'. Commonly the primary definition of the integral is based on what I call 'previsibly simple' processes, and the formula of 641Yd(ii) is used to define $\int_{[\tau,\tau']} \boldsymbol{u}\, d\boldsymbol{v}$. As will be noted in 641Yd, this in itself won't affect the principal cases.

I will present a large number of results showing that processes share properties with their fully adapted extensions to the covered envelopes of their domains. In those cases in which we have a good match between properties of $\boldsymbol{u}$ and $\hat{\boldsymbol{u}}$, starting with the isomorphism between $M_{\text{fa}}(\mathcal{S})$ and $M_{\text{fa}}(\hat{\mathcal{S}})$ (612Qb), we shall have a similar match between properties of $\boldsymbol{u}$ and of its alternative realisation $\hat{\boldsymbol{u}}\restriction\mathcal{S}'$ on any sublattice $\mathcal{S}'$ with the same covered envelope, that is, such that $\mathcal{S}$ covers $\mathcal{S}'$ and $\mathcal{S}'$ covers $\mathcal{S}$. These offer powerful methods for reducing problems to questions in which processes are defined on full sublattices of $\mathcal{T}$. It is correspondingly important to recognise cases in which the correspondence may not be exact. One is in 613T: if $\int_{\hat{\mathcal{S}}} \hat{\boldsymbol{u}}\, d\hat{\psi}$ is defined, then $\int_{\mathcal{S}} \boldsymbol{u}\, d\psi$ is defined, with the same value; but the converse is not universally true (613P). (In the opposite direction we have 612Xj and 612Yf.) I have starred 613P because the construction is hard work and the methods I use do not seem to contribute much to our toolkit. Ordinarily I would leave it as a 'further exercise'. But like a rocky outcrop in the bank of a river it affects the flow of the arguments in this volume, so I have taken three pages to include a detailed solution.

Nearly everything in the rest of this volume will be expressed in terms of structures $(\mathfrak{A}, \bar{\mu}, T, \langle\mathfrak{A}_t\rangle_{t\in T})$ where $(\mathfrak{A}, \bar{\mu})$ is a probability algebra, T is a non-empty totally ordered set, and $\langle\mathfrak{A}_t\rangle_{t\in T}$ is a filtration of closed subalgebras of $\mathfrak{A}$; for definiteness, it will often be useful to simultaneously declare names for the lattice of stopping times adapted to $\langle\mathfrak{A}_t\rangle_{t\in T}$ and the associated family of closed subalgebras of $\mathfrak{A}$ to get a sextuple $(\mathfrak{A}, \bar{\mu}, T, \langle\mathfrak{A}_t\rangle_{t\in T}, \mathcal{T}, \langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$. I will call such quadruples or sextuples **stochastic integration structures**. When $T = [0,\infty[$, as in 612F and 612H, so that we have a structure $(\mathfrak{A}, \bar{\mu}, [0,\infty[\,, \langle\mathfrak{A}_t\rangle_{t\ge0})$ or $(\mathfrak{A}, \bar{\mu}, [0,\infty[\,, \langle\mathfrak{A}_t\rangle_{t\ge0}, \mathcal{T}, \langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$, I will call it a **real-time** stochastic integration structure.

Perhaps I should note explicitly that a stochastic integration structure $(\mathfrak{A}, \bar{\mu}, T, \langle\mathfrak{A}_t\rangle_{t\in T}, \mathcal{T}, \langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ carries a lot of undeclared baggage. The symbol $\mathfrak{A}$ involves not just the set $\mathfrak{A}$ but its Boolean operations $\triangle$, $\cap$, $\cup$ and $\setminus$, with the induced relation $\subseteq$, and its greatest and least elements 1 and 0. The symbol T is accompanied by its total ordering $\le$ and the associated relation $<$. With $\mathcal{T}$ we have the functions $(\sigma,\tau) \mapsto [\![\sigma < \tau]\!]$, $[\![\sigma \le \tau]\!]$ and $[\![\sigma = \tau]\!]$. We are all well used to such things being omitted from hypotheses, but in the rest of this volume we shall have to be ready for a richer hidden substructure than is usual.

In the introduction to this volume I said I would try to explain why I am working with structures $(\mathfrak{A},\bar\mu)$ and $L^0(\mathfrak{A})$ rather than with probability spaces and measurable functions. The move from $(\Omega,\Sigma,\mu,\langle\Sigma_t\rangle_{t\ge0})$ to $(\mathfrak{A},\bar\mu,\langle\mathfrak{A}_t\rangle_{t\in T})$ is really no more than a change in notation. Every probability algebra, in the sense here, can be represented by a classical probability space (321J), and while the move from $[0,\infty[$ to an arbitrary non-empty totally ordered space is a generalisation, I shall have nothing significant to say about the new cases. But the move from real-valued measurable functions to their equivalence classes in L^0 is forced on us by the definition of the integral in 613H. The integral is defined as a limit in L^0 for the topology of convergence in measure; and as this is a Hausdorff topology, the limit, when it exists, is unique. There are variations of the theory using different topologies – in §626, for instance, there will be limits for the weak topology on L^1 – but none of them give functions as limits; they all provide members of L^0 or L^1 or L^2, corresponding to equivalence classes of functions. Starting in Chapter 62, conditional expectations will be enormously important, and if we want to speak of well-defined conditional expectation operators, we again need to work in L^1, not in an associated space of functions.

Of course, we could set up a theory in which we accepted that an integral $\int X_t\,dY_t$ would belong to L^0, while all the X_t, Y_t were real-valued functions. But when we get to stochastic calculus in §617 we shall want to look at integration with respect to indefinite integrals. So our theory needs to be able to deal with integrals $\int d\boldsymbol{v}$ where $\boldsymbol{v}$ is a process taking values in L^0.

Concerning stopping times, the argument takes a different form. Since the values of a process $\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ are going to be in L^0, and changing σ on a set of measure zero must not affect u_σ, it is to my mind more natural to define a stopping time $\sigma\in\mathcal{T}$ in such a way that it corresponds to an equivalence class of stopping times $h:X\to[0,\infty]$, even if this complicates the definition of the region $[\![\sigma<\tau]\!]$ in 611D and renders the algebra of 611E less transparent. And there is another issue. If you look at the leading examples in 612T-612U, or at the general formulations in 612H and 615P, you will see that I appeal to results in Volume 4 at several points. The intuitions behind Brownian motion and the Poisson process naturally lead to processes $\langle X_t\rangle_{t\ge0}$, and in order to go farther we need to look at the corresponding processes $\langle X_h\rangle_{h\text{ is a stopping time}}$. This step demands deeper ideas than anything else in the present chapter. By going directly to processes $\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$, where $\mathcal{S}$ may be any sublattice of $\mathcal{T}$, I can skate over these difficulties while still giving you something to cut your teeth on. Moreover, there are technical advantages; because $\mathcal{T}$ is Dedekind complete, I can speak uninhibitedly of infima, in such results as 611Ch and 618C, without needing any formula for calculating them.

Version of 29.10.24

614 Simple and order-bounded processes and bounded variation

In §613 I gave a definition of an integral with no very useful indication of where it might be applicable. This section and the next two will be devoted to teasing out the basic case in which a Riemann-sum integral $\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}$ is defined: $\boldsymbol{u}$ should be 'moderately oscillatory' (615E) and $\boldsymbol{v}$ should be an 'integrator' (616K). Before we come to either of these notions, however, it will be helpful to have a firm grasp of three easier concepts: 'simple' processes (614B), 'order-bounded' processes (614E) and processes 'of bounded variation' (614J-614K).

614A Notation $(\mathfrak{A},\bar\mu,T,\langle\mathfrak{A}_t\rangle_{t\in T},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ will be a stochastic integration structure, as described in the notes to §613, with regions $[\![\sigma<\tau]\!]$, $[\![\sigma\le\tau]\!]$ and $[\![\sigma=\tau]\!]$ (611D) for stopping times σ, $\tau\in\mathcal{T}$. If $\sigma\le\tau$ in $\mathcal{T}$, $c(\sigma,\tau)$ will be the corresponding stopping-time interval (611J). If $\mathcal{S}$ is a sublattice of $\mathcal{T}$, then $\mathrm{Sti}(\mathcal{S})$ will be the set of stopping-time intervals with endpoints in $\mathcal{S}$ and when I is a finite sublattice of $\mathcal{T}$ $\mathrm{Sti}_0(I)$ will be the set of I-cells (611Je). If $\mathcal{S}$ is a sublattice of $\mathcal{T}$ and $\tau\in\mathcal{S}$, then $\mathcal{S}\wedge\tau=\{\sigma\wedge\tau:\sigma\in\mathcal{S}\}$, $\mathcal{S}\vee\tau=\{\sigma\vee\tau:\sigma\in\mathcal{S}\}$, $\mathcal{I}(\mathcal{S})$ is the upwards-directed set of finite sublattices of $\mathcal{S}$ and $\mathcal{S}^{2\uparrow}=\{(\sigma,\tau):\sigma,\ \tau\in\mathcal{S},\ \sigma\le\tau\}$. $M_{\mathrm{fa}}(\mathcal{S})$ will be the space of fully adapted processes with domain $\mathcal{S}$ (612I). I write $\mathbf{1}$ for the constant process with value $\chi1$, and if $z\in L^0$ then $z\mathbf{1}$ will be the constant process with value z, defined on $\{\sigma:z\in L^0(\mathfrak{A}_\sigma)$ (612De).

$L^0=L^0(\mathfrak{A})$ (612A), and if $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}\in(L^0)^{\mathcal{S}}$, I write $\sup|\boldsymbol{u}|$ for $\sup_{\sigma\in\mathcal{S}}|u_\sigma|$ if this is defined in L^0. θ will be the standard F-seminorm defining the topology of convergence in measure on L^0 (613Ba), and limits will be taken with respect to this topology. If $\mathcal{S}$ is a sublattice of $\mathcal{T}$, $e \in \mathrm{Sti}(\mathcal{S})$, $I \in \mathcal{I}(\mathcal{S})$, $\boldsymbol{u}$ and $\boldsymbol{v}$ are processes defined on $\mathcal{S}$ and ψ is an interval function defined on $\mathcal{S}^{2\uparrow}$, we shall have $\Delta_e(\boldsymbol{u}, d\psi)$, $\Delta_e(\boldsymbol{u}, d\boldsymbol{v})$, $\Delta_e(\boldsymbol{u}, |d\boldsymbol{v}|)$ and the Riemann sums $S_I(\boldsymbol{u}, d\psi)$, $S_I(\boldsymbol{u}, d\boldsymbol{v})$ and $S_I(\boldsymbol{u}, |d\boldsymbol{v}|)$ as defined in 613E-613F, with the Riemann-sum integrals $\int_{\mathcal{S}} \boldsymbol{u}\, d\psi$, $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$ and $\int_{\mathcal{S}} \boldsymbol{u}\, |d\boldsymbol{v}|$ (when they are defined) as in 613H.

614B If either $\boldsymbol{u}$ or $\boldsymbol{v}$ is a simple process, then $\int \boldsymbol{u}\, d\boldsymbol{v}$ is particularly straightforward.

Proposition Suppose that $\mathcal{S}$ is a non-empty sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ a simple fully adapted process with a breakpoint string $(\tau_0, \dots, \tau_n)$ (612J).

(a) The starting value $u_\downarrow = \lim_{\sigma \downarrow \mathcal{S}} u_\sigma$ is defined, and $[\![\sigma < \tau_0]\!] \subseteq [\![u_\sigma = u_\downarrow]\!]$ for every $\sigma \in \mathcal{S}$.

(b) Suppose that $\psi : \mathcal{S}^{2\uparrow} \to L^0 = L^0(\mathfrak{A})$ is an adapted interval function such that $\int_{\mathcal{S}} d\psi = \int_{\mathcal{S}} \mathbf{1}\, d\psi$ is defined. Then $\int_{\mathcal{S}} \boldsymbol{u}\, d\psi$ is defined and equal to

$$u_\downarrow \times v_{\tau_0} + \sum_{i=0}^{n-1} u_{\tau_i} \times (v_{\tau_{i+1}} - v_{\tau_i}) + u_{\tau_n} \times (v_\uparrow - v_{\tau_n})$$

where $v_\tau = \int_{\mathcal{S} \wedge \tau} d\psi$ for $\tau \in \mathcal{S}$, and $v_\uparrow = \int_{\mathcal{S}} d\psi$.

proof (a) For $\sigma \in \mathcal{S}$, set $a_\sigma = [\![\sigma < \tau_0]\!]$; set $a = \sup_{\sigma \in \mathcal{S}} a_\sigma$. Note that $a_\sigma \supseteq a_\tau$ if $\sigma \le \tau$ in $\mathcal{S}$, so a is the limit $\lim_{\sigma \downarrow \mathcal{S}} a_\sigma$ for the measure-algebra topology of $\mathfrak{A}$ (323D(b-ii)), and $\chi a = \lim_{\sigma \downarrow \mathcal{S}} \chi a_\sigma$ for the topology of convergence in measure (367Ra). Next, we know that there is a $u_* \in L^0(\bigcap_{\sigma \in \mathcal{S}} \mathfrak{A}_\sigma)$ such that $a_\sigma \subseteq [\![u_\sigma = u_*]\!]$ for every $\sigma \in \mathcal{S}$. If $\sigma \in \mathcal{S}$ and $\sigma \le \tau_0$, then $1 \setminus a_\sigma = [\![\sigma = \tau_0]\!] \subseteq [\![u_\sigma = u_{\tau_0}]\!]$, so

$$u_\sigma = u_* \times \chi a_\sigma + u_{\tau_0} \times (\chi 1 - \chi a_\sigma).$$

But this means that

$$\begin{aligned} u_\downarrow = \lim_{\sigma \downarrow \mathcal{S}} u_\sigma &= \lim_{\sigma \downarrow \mathcal{S}} (u_* \times \chi a_\sigma + u_{\tau_0} \times (\chi 1 - \chi a_\sigma)) \\ &= u_* \times \chi a + u_{\tau_0} \times (\chi 1 - \chi a). \end{aligned}$$

Finally, for any $\sigma \in \mathcal{S}$,

$$[\![\sigma < \tau_0]\!] \subseteq a \cap [\![u_\sigma = u_*]\!] \subseteq [\![u_* = u_\downarrow]\!] \cap [\![u_\sigma = u_*]\!]$$

(because $u_* \times \chi a = u_\downarrow \times \chi a$)

$$\subseteq [\![u_\sigma = u_\downarrow]\!].$$

(b)(i) By 613J(c-i), v_τ is defined for every $\tau \in \mathcal{S}$.

(ii) $\int_{\mathcal{S} \wedge \tau_0} \boldsymbol{u}\, d\psi = u_\downarrow \times v_{\tau_0}$. **P** If σ, $\tau \in \mathcal{S}$ and $\sigma \le \tau \le \tau_0$, then

$$[\![\sigma = \tau_0]\!] \subseteq [\![\sigma = \tau]\!] \subseteq [\![\psi(\sigma, \tau) = 0]\!]$$

(611E(c-vi), 613C(b-i)), so $\psi(\sigma, \tau) = \chi[\![\sigma < \tau_0]\!] \times \psi(\sigma, \tau)$ and

$$\begin{aligned} u_\sigma \times \psi(\sigma, \tau) &= u_\sigma \times \chi[\![\sigma < \tau_0]\!] \times \psi(\sigma, \tau) \\ &= u_\downarrow \times \chi[\![\sigma < \tau_0]\!] \times \psi(\sigma, \tau) = u_\downarrow \times \psi(\sigma, \tau). \end{aligned}$$

So $\Delta_e(\boldsymbol{u}, d\psi) = u_\downarrow \times \Delta_e(\mathbf{1}, d\psi)$ for every stopping-time interval with endpoints in $\mathcal{S} \wedge \tau_0$, and $S_I(\boldsymbol{u}, d\psi) = u_\downarrow \times S_I(\mathbf{1}, d\psi)$ for every $I \in \mathcal{I}(\mathcal{S} \wedge \tau_0)$. Accordingly

$$\begin{aligned} \int_{\mathcal{S} \wedge \tau_0} \boldsymbol{u}\, d\psi &= \lim_{I \uparrow \mathcal{I}(\mathcal{S} \wedge \tau_0)} S_I(\boldsymbol{u}, d\psi) = \lim_{I \uparrow \mathcal{I}(\mathcal{S} \wedge \tau_0)} u_\downarrow \times S_I(\mathbf{1}, d\psi) \\ &= u_\downarrow \times \lim_{I \uparrow \mathcal{I}(\mathcal{S} \wedge \tau_0)} S_I(\mathbf{1}, d\psi) = u_\downarrow \times v_{\tau_0}. \ \mathbf{Q} \end{aligned}$$

(iii) If $i < n$, then $\int_{\mathcal{S} \cap [\tau_i, \tau_{i+1}]} \boldsymbol{u}\, d\psi = u_{\tau_i} \times (v_{\tau_{i+1}} - v_{\tau_i})$. **P** If σ, $\tau \in \mathcal{S}$ and $\tau_i \le \sigma \le \tau \le \tau_{i+1}$, then

$$[\![\sigma = \tau_{i+1}]\!] \subseteq [\![\sigma = \tau]\!] \subseteq [\![\psi(\sigma, \tau) = 0]\!],$$

$$\begin{aligned} u_\sigma \times \psi(\sigma,\tau) &= u_\sigma \times \chi[\![\sigma < \tau_{i+1}]\!] \times \psi(\sigma,\tau) \\ &= u_{\tau_i} \times \chi[\![\sigma < \tau_{i+1}]\!] \times \psi(\sigma,\tau) = u_{\tau_i} \times \psi(\sigma,\tau). \end{aligned}$$

So $\Delta_e(\boldsymbol{u}, d\psi) = u_{\tau_i} \times \Delta_e(\mathbf{1}, d\psi)$ for every stopping-time interval with endpoints in $\mathcal{S} \cap [\tau_i, \tau_{i+1}]$, and $S_I(\boldsymbol{u}, d\psi) = u_{\tau_i} \times S_I(\mathbf{1}, d\psi)$ for every $I \in \mathcal{I}(\mathcal{S} \cap [\tau_i, \tau_{i+1}])$. Accordingly

$$\begin{aligned} \int_{\mathcal{S}\cap[\tau_i,\tau_{i+1}]} \boldsymbol{u}\,d\psi &= \lim_{I\uparrow\mathcal{I}(\mathcal{S}\cap[\tau_i,\tau_{i+1}])} S_I(\boldsymbol{u}, d\psi) = \lim_{I\uparrow\mathcal{I}(\mathcal{S}\cap[\tau_i,\tau_{i+1}])} u_{\tau_i} \times S_I(\mathbf{1}, d\psi) \\ &= u_{\tau_i} \times \lim_{I\uparrow\mathcal{I}(\mathcal{S}\cap[\tau_i,\tau_{i+1}])} S_I(\mathbf{1}, d\psi) = u_{\tau_i} \times \int_{\mathcal{S}\cap[\tau_i,\tau_{i+1}]} d\psi \\ &= u_{\tau_i} \times (\int_{\mathcal{S}\wedge\tau_{i+1}} d\psi - \int_{\mathcal{S}\wedge\tau_i} d\psi) \end{aligned}$$

(using 613J(c-i) again)

$$= u_{\tau_i} \times (v_{\tau_{i+1}} - v_{\tau_i}). \quad \mathbf{Q}$$

(iv) $\int_{\mathcal{S}\vee\tau_n} \boldsymbol{u}\,d\psi = u_{\tau_n} \times (v_\uparrow - v_{\tau_n})$. **P** If σ, $\tau \in \mathcal{S}$ and $\tau_n \le \sigma \le \tau$, then $u_\sigma \times \psi(\sigma,\tau) = u_{\tau_n} \times \psi(\sigma,\tau)$. So $\Delta_e(\boldsymbol{u}, d\psi) = u_{\tau_n} \times \Delta_e(\mathbf{1}, d\psi)$ for every stopping-time interval with endpoints in $\mathcal{S} \vee \tau_n$, $S_I(\boldsymbol{u}, d\psi) = u_{\tau_n} \times S_I(\mathbf{1}, d\psi)$ for every $I \in \mathcal{I}(\mathcal{S} \vee \tau_n)$ and

$$\begin{aligned} \int_{\mathcal{S}\vee\tau_n} \boldsymbol{u}\,d\psi &= \lim_{I\uparrow\mathcal{I}(\mathcal{S}\vee\tau_n)} S_I(\boldsymbol{u}, d\psi) = \lim_{I\uparrow\mathcal{I}(\mathcal{S}\vee\tau_n)} u_{\tau_n} \times S_I(\mathbf{1}, d\psi) \\ &= u_{\tau_n} \times \lim_{I\uparrow\mathcal{I}(\mathcal{S}\vee\tau_n)} S_I(\mathbf{1}, d\psi) = u_{\tau_i} \times \int_{\mathcal{S}\vee\tau_n} d\psi \\ &= u_{\tau_i} \times (\int_{\mathcal{S}} d\psi - \int_{\mathcal{S}\wedge\tau_n} d\psi) = u_{\tau_i} \times (v_\uparrow - v_{\tau_n}). \quad \mathbf{Q} \end{aligned}$$

(v) Assembling these, as in 613J(c-ii), $\int_{\mathcal{S}} \boldsymbol{u}\,d\psi$ is defined and equal to

$$\begin{aligned} &\int_{\mathcal{S}\wedge\tau_0} \boldsymbol{u}\,d\psi + \sum_{i=0}^{n-1} \int_{\mathcal{S}\cap[\tau_i,\tau_{i+1}]} \boldsymbol{u}\,d\psi + \int_{\mathcal{S}\vee\tau_n} \boldsymbol{u}\,d\psi \\ &\qquad = u_\downarrow \times v_{\tau_0} + \sum_{i=0}^{n-1} u_{\tau_i} \times (v_{\tau_{i+1}} - v_{\tau_i}) + u_{\tau_n} \times (v_\uparrow - v_{\tau_n}), \end{aligned}$$

as claimed.

Remark I didn't have to say so in the course of the proof above, but of course $\int_{\mathcal{S}} d\psi = \lim_{\tau\uparrow\mathcal{S}} v_\tau$ (613J(f-ii)), so the formula $v_\uparrow$ here matches the usage in 613N.

614C Corollary Suppose that $\mathcal{S}$ is a non-empty sublattice of $\mathcal{T}$, $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$ is a simple fully adapted process with starting value $u_\downarrow$ and a breakpoint string $(\tau_0, \dots, \tau_n)$, and $\boldsymbol{v} = \langle v_\tau \rangle_{\tau\in\mathcal{S}}$ is a fully adapted process such that $v_\uparrow = \lim_{\tau\uparrow\mathcal{S}} v_\tau$ and $v_\downarrow = \lim_{\sigma\downarrow\mathcal{S}} v_\sigma$ are defined. Then $\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}$ is defined and equal to

$$u_\downarrow \times (v_{\tau_0} - v_\downarrow) + \textstyle\sum_{i=0}^{n-1} u_{\tau_i} \times (v_{\tau_{i+1}} - v_{\tau_i}) + u_{\tau_n} \times (v_\uparrow - v_{\tau_n}).$$

proof Apply 614B with $\psi = \Delta\boldsymbol{v}$; the point being just that $\int_{\mathcal{S}\wedge\tau} d\boldsymbol{v} = v_\tau - v_\downarrow$ for every $\tau \in \mathcal{S}$, by 613N applied to $\boldsymbol{v}{\restriction}\mathcal{S}\wedge\tau$, while $\int_{\mathcal{S}} d\boldsymbol{v} = v_\uparrow - v_\downarrow$.

614D Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma\in\mathcal{S}}$ a simple process, and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$ a fully adapted process such that $\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}$ is defined. Then $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is simple.

proof (a) If $\boldsymbol{v}$ is constant, this is trivial, as $\Delta\boldsymbol{v} = 0$ and $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is constant with value 0. Otherwise, let $(\tau_0, \dots, \tau_n)$ be a breakpoint string for $\boldsymbol{v}$, and $v_\downarrow$ the starting value of $\boldsymbol{v}$. For $\tau \in \mathcal{S}$ write w_τ for $\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\boldsymbol{v}$, so that $\langle w_\tau \rangle_{\tau\in\mathcal{S}} = ii_{\boldsymbol{v}}(\boldsymbol{u})$.

(b) If $\tau \in \mathcal{S} \wedge \tau_0$ then $[\![\tau < \tau_0]\!] \subseteq [\![w_\tau = 0]\!]$. **P** By 613Ld,

$$[\![w_\tau \ne 0]\!] \subseteq \sup_{\sigma,\sigma'\in\mathcal{S}\wedge\tau} [\![v_\sigma \ne v_{\sigma'}]\!] \subseteq \sup_{\sigma\in\mathcal{S}\wedge\tau} [\![v_\sigma \ne v_\downarrow]\!]$$
$$\subseteq \sup_{\sigma\in\mathcal{S}\wedge\tau} [\![\sigma = \tau_0]\!] \subseteq [\![\tau = \tau_0]\!]$$

so $[\![\tau < \tau_0]\!] \subseteq [\![w_\tau = 0]\!]$. **Q** Generally, if $\tau \in \mathcal{S}$,

$$[\![\tau < \tau_0]\!] \subseteq [\![\tau = \tau \wedge \tau_0]\!] \cap [\![\tau \wedge \tau_0 < \tau_0]\!]$$
$$\subseteq [\![w_\tau = w_{\tau\wedge\tau_0}]\!] \cap [\![w_{\tau\wedge\tau_0} = 0]\!] \subseteq [\![w_\tau = 0]\!]$$

because $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is fully adapted (613Oa).

(c) If $i \le n$ and $\tau \in \mathcal{S} \cap [\tau_i, \tau_{i+1}]$ then $[\![\tau < \tau_{i+1}]\!] \subseteq [\![w_\tau = w_{\tau_i}]\!]$. **P**

$$[\![w_\tau \ne w_{\tau_i}]\!] = [\![\textstyle\int_{\mathcal{S}\cap[\tau_i,\tau]} \boldsymbol{u}\, d\boldsymbol{v} \ne 0]\!]$$

(613Jc)

$$\subseteq \sup_{\sigma,\sigma'\in\mathcal{S}\cap[\tau_i,\tau]} [\![v_\sigma \ne v_{\sigma'}]\!] \subseteq \sup_{\sigma\in\mathcal{S}\cap[\tau_i,\tau]} [\![v_\sigma \ne v_{\tau_i}]\!]$$
$$\subseteq \sup_{\sigma\in\mathcal{S}\cap[\tau_i,\tau]} [\![\sigma = \tau_{i+1}]\!] \subseteq [\![\tau = \tau_{i+1}]\!],$$

so $[\![\tau < \tau_{i+1}]\!] \subseteq [\![w_\tau = w_{\tau_i}]\!]$. **Q** Generally, if $\tau \in \mathcal{S}$,

$$[\![\tau_i \le \tau]\!] \cap [\![\tau < \tau_{i+1}]\!] \subseteq [\![\tau = \operatorname{med}(\tau_i,\tau,\tau_{i+1})]\!] \cap [\![\operatorname{med}(\tau_i,\tau,\tau_{i+1}) < \tau_{i+1})]\!]$$
$$\subseteq [\![w_\tau = w_{\operatorname{med}(\tau_i,\tau,\tau_{i+1})}]\!] \cap [\![w_{\operatorname{med}(\tau_i,\tau,\tau_{i+1})} = w_{\tau_i}]\!]$$
$$\subseteq [\![w_\tau = w_{\tau_i}]\!].$$

(d) If $\tau \in \mathcal{S} \vee \tau_n$ then $w_\tau = w_{\tau_n}$. **P**

$$[\![w_\tau \ne w_{\tau_n}]\!] = [\![\textstyle\int_{\mathcal{S}\cap[\tau_n,\tau]} \boldsymbol{u}\, d\boldsymbol{v} \ne 0]\!]$$
$$\subseteq \sup_{\sigma,\sigma'\in\mathcal{S}\cap[\tau_n,\tau]} [\![v_\sigma \ne v_{\sigma'}]\!] \subseteq \sup_{\sigma\in\mathcal{S}\vee\tau_n} [\![v_\sigma \ne v_{\tau_n}]\!] = 0. \text{ \textbf{Q}}$$

Generally, if $\tau \in \mathcal{S}$,

$$[\![\tau_n \le \tau]\!] \subseteq [\![\tau = \tau \vee \tau_n]\!] \subseteq [\![w_\tau = w_{\tau\vee\tau_n}]\!] \subseteq [\![w_\tau = w_{\tau_n}]\!].$$

Thus $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is simple, with breakpoint string $(\tau_0, \dots, \tau_n)$.

614E Order-bounded processes Now for a much larger class of processes.

Definitions Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$ a fully adapted process.

(a) $\boldsymbol{u}$ is **order-bounded** if $\{u_\sigma : \sigma \in \mathcal{S}\}$ is bounded above and below in L^0. In this case, if $\mathcal{S} \ne \emptyset$, $\sup_{\sigma\in\mathcal{S}} |u_\sigma|$ is defined in L^0, because L^0 is Dedekind complete. It will be convenient to write $\sup|\boldsymbol{u}| = \sup_{\sigma\in\mathcal{S}} |u_\sigma|$, taking the supremum in $(L^0)^+$, so that $\sup|\boldsymbol{u}| = 0$ if $\mathcal{S} = \operatorname{dom}\boldsymbol{u}$ is empty.

(b) $\boldsymbol{u}$ is **locally order-bounded** if $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau = \langle u_\sigma \rangle_{\sigma\in\mathcal{S},\sigma\le\tau}$ is order-bounded for every $\tau \in \mathcal{S}$.

(c) Suppose that $\mathcal{S}$ is non-empty and that $\boldsymbol{u}$ is simple, with breakpoint string $(\tau_0, \dots, \tau_n)$ and starting value $u_\downarrow$. Then $\boldsymbol{u}$ is order-bounded and $\sup|\boldsymbol{u}| = |u_\downarrow| \vee \sup_{i\le n} |u_{\tau_i}|$. **P** Write $\bar{u}$ for $|u_\downarrow| \vee \sup_{i\le n} |u_{\tau_i}|$. If $\sigma \in \mathcal{S}$, then

$$\llbracket |u_\sigma| \le \bar u \rrbracket \supseteq \llbracket u_\sigma = u_\downarrow \rrbracket \cup \sup_{i\le n} \llbracket u_\sigma = u_{\tau_i} \rrbracket$$
$$\supseteq \llbracket \sigma < \tau_0 \rrbracket \cup \sup_{i<n} (\llbracket \tau_i \le \sigma \rrbracket \cap \llbracket \sigma < \tau_{i+1} \rrbracket) \cup \llbracket \tau_n \le \sigma \rrbracket = 1$$

and $|u_\sigma| \le \bar u$. As σ is arbitrary, $\boldsymbol{u}$ is order-bounded and $\sup |\boldsymbol{u}| \le \bar u$. On the other hand,

$$|u_\downarrow| = \lim_{\sigma\downarrow\mathcal{S}} |u_\sigma| \le \sup |\boldsymbol{u}|$$

so $\bar u \le \sup |\boldsymbol{u}|$ and we have equality. **Q**

614F Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.

(a)(i) If $\boldsymbol{u}$ is an order-bounded process with domain $\mathcal{S}$, then $\boldsymbol{u}\restriction\mathcal{S}'$ is order-bounded for any sublattice $\mathcal{S}'$ of $\mathcal{S}$; in particular, $\boldsymbol{u}$ is locally order-bounded.

(ii) If $\boldsymbol{u}$ is a locally order-bounded process with domain $\mathcal{S}$, then $\boldsymbol{u}\restriction\mathcal{S}'$ is locally order-bounded for any sublattice $\mathcal{S}'$ of $\mathcal{S}$.

(b) Suppose that $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a locally order-bounded process. Set $v_\tau = \sup_{\sigma\in\mathcal{S}\wedge\tau} |u_\sigma|$ for $\tau \in \mathcal{S}$. Then $\boldsymbol{v} = \langle v_\tau\rangle_{\tau\in\mathcal{S}}$ is a non-decreasing fully adapted process.

(c) Write $M_{\text{o-b}} = M_{\text{o-b}}(\mathcal{S})$ for the set of order-bounded fully adapted processes with domain $\mathcal{S}$.

(i) If $h : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function which is bounded on every bounded interval in $\mathbb{R}$, then $\bar h\boldsymbol{u} \in M_{\text{o-b}}$ for every $\boldsymbol{u} \in M_{\text{o-b}}$.

(ii) $M_{\text{o-b}}$ is an f-subalgebra of $\prod_{\sigma\in\mathcal{S}} L^0(\mathfrak{A}_\sigma)$.

(iii) If $z \in L^0(\mathfrak{A} \cap \bigcap_{\sigma\in\mathcal{S}} \mathfrak{A}_\sigma)$ then $z\boldsymbol{u} \in M_{\text{o-b}}$, with $\sup |z\boldsymbol{u}| = |z| \times \sup |\boldsymbol{u}|$, for every $\boldsymbol{u} \in M_{\text{o-b}}$.

(d) Write $M_{\text{lob}} = M_{\text{lob}}(\mathcal{S})$ for the set of locally order-bounded fully adapted processes with domain $\mathcal{S}$.

(i) If $h : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function which is bounded on every bounded interval in $\mathbb{R}$, then $\bar h\boldsymbol{u} \in M_{\text{lob}}$ for every $\boldsymbol{u} \in M_{\text{lob}}$.

(ii) M_{lob} is an f-subalgebra of $\prod_{\sigma\in\mathcal{S}} L^0(\mathfrak{A}_\sigma)$.

proof (a) Immediate from the definitions.

(b) For each $\tau \in \mathcal{S}$, $\{|u_\sigma| : \sigma \in \mathcal{S}\wedge\tau\}$ is bounded above in the Dedekind complete lattice $L^0(\mathfrak{A})$, so v_τ is defined. Moreover, $|u_\sigma| \in L^0(\mathfrak{A}_\sigma) \subseteq L^0(\mathfrak{A}_\tau)$ for every $\sigma \in \mathcal{S}\wedge\tau$; as $L^0(\mathfrak{A}_\tau)$ is an order-closed sublattice of $L^0(\mathfrak{A})$ (612A(e-i)), $v_\tau \in L^0(\mathfrak{A}_\tau)$. Of course $v_\tau \le v_{\tau'}$ whenever $\tau \le \tau'$ in $\mathcal{S}$, just because $\mathcal{S}\wedge\tau \subseteq \mathcal{S}\wedge\tau'$.

Now suppose that τ, $\tau' \in \mathcal{S}$ and $c = \llbracket \tau = \tau' \rrbracket$. If $\sigma \in \mathcal{S}\wedge\tau$, then $\sigma\wedge\tau' \in \mathcal{S}\wedge\tau'$ and

$$c \subseteq \llbracket \sigma \le \tau' \rrbracket = \llbracket \sigma = \sigma\wedge\tau' \rrbracket \subseteq \llbracket |u_\sigma| = |u_{\sigma\wedge\tau'}| \rrbracket \subseteq \llbracket |u_\sigma| \le v_{\tau'} \rrbracket.$$

So $|u_\sigma| \times \chi c \le v_{\tau'}$. Since $u \mapsto u \times \chi c : L^0(\mathfrak{A}) \to L^0(\mathfrak{A})$ is an order-continuous lattice homomorphism,

$$v_\tau \times \chi c = \sup_{\sigma\in\mathcal{S}\wedge\tau} |u_\sigma| \times \chi c \le v_{\tau'} \times \chi c,$$

and $c \subseteq \llbracket v_\tau \le v_{\tau'} \rrbracket$. Similarly, $c \subseteq \llbracket v_{\tau'} \le v_\tau \rrbracket$ and $\llbracket \tau = \tau' \rrbracket = c \subseteq \llbracket v_\tau = v_{\tau'} \rrbracket$. As τ and τ' are arbitrary, $\boldsymbol{v}$ is fully adapted.

(c)(i) For $x \in \mathbb{R}$, set $g(x) = \sup_{|y|\le|x|} |h(y)|$. Then g is monotonic on each of $]-\infty, 0]$ and $[0, \infty[$, so is Borel measurable, and $|\bar h(v)| \le \bar g(|u|)$ whenever u, $v \in L^0$ and $|v| \le u$. If $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is an order-bounded fully adapted process, and u is an upper bound of $\{|u_\sigma| : \sigma \in \mathcal{S}\}$, $\bar g(u)$ will be an upper bound of $\{|\bar h(u_\sigma)| : \sigma \in \mathcal{S}\}$, and $\bar h\boldsymbol{u}$ is order-bounded.

(ii) If $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ are order-bounded, let u, v be upper bounds of $\{|u_\sigma| : \sigma \in \mathcal{S}\}$, $\{|v_\sigma| : \sigma \in \mathcal{S}\}$ respectively; then $u + v$ is an upper bound of $\{|u_\sigma + v_\sigma| : \sigma \in \mathcal{S}\}$ and $\boldsymbol{u} + \boldsymbol{v}$ is order-bounded. Thus $M_{\text{o-b}}$ is closed under addition. By 612Bc as usual, $M_{\text{o-b}}$ is an f-subalgebra of $\prod_{\sigma\in\mathcal{S}} L^0(\mathfrak{A}_\sigma)$.

(iii) The map $u \mapsto |z| \times u : L^0 \to L^0$ is an order-continuous Riesz homomorphism, so if $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ belongs to $M_{\text{o-b}}$ we shall have

$$|z| \times \sup |\boldsymbol{u}| = |z| \times \sup_{\sigma\in\mathcal{S}} |u_\sigma| = \sup_{\sigma\in\mathcal{S}} |z| \times |u_\sigma| = \sup_{\sigma\in\mathcal{S}} |z \times u_\sigma|$$

and $z\boldsymbol{u}$ is order-bounded, with $\sup |z\boldsymbol{u}| = |z| \times \sup |\boldsymbol{u}|$.

(d) Apply (c) to $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ for each $\tau \in \mathcal{S}$.

614G Proposition Suppose that $\mathcal{S}$ is a sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$ a fully adapted process.

(a) If A, $B \subseteq \mathcal{S}$, A covers B and $\{u_\sigma : \sigma \in A\}$ is order-bounded, then $\{u_\sigma : \sigma \in B\}$ is order-bounded and $\sup_{\sigma\in B} |u_\sigma| \le \sup_{\sigma\in A} |u_\sigma|$.

(b) If $\mathcal{S}'$ is a sublattice of $\mathcal{S}$ which covers $\mathcal{S}$

(i) $\boldsymbol{u}$ is order-bounded iff $\boldsymbol{u}\restriction\mathcal{S}'$ is order-bounded, and in this case $\sup|\boldsymbol{u}| = \sup|\boldsymbol{u}\restriction\mathcal{S}'|$,

(ii) $\boldsymbol{u}$ is locally order-bounded iff $\boldsymbol{u}\restriction\mathcal{S}'$ is locally order-bounded.

proof (a) Write $\bar u$ for $\sup_{\sigma\in A}|u_\sigma|$, counting the supremum of the empty set as 0. If $\tau \in B$ then

$$[\![|u_\tau| \le \bar u]\!] \supseteq \sup_{\sigma\in A} [\![u_\tau = u_\sigma]\!] \cap [\![|u_\sigma| \le \bar u]\!] \supseteq \sup_{\sigma\in A} [\![\tau = \sigma]\!] = 1,$$

so $|u_\tau| \le \bar u$. As τ is arbitrary, $\boldsymbol{u}\restriction B$ is order-bounded and $\sup|\boldsymbol{u}\restriction B| \le \bar u = \sup|\boldsymbol{u}\restriction A|$.

(b)(i) Immediate from (a), as each of $\mathcal{S}$, $\mathcal{S}'$ covers the other.

(ii) If $\boldsymbol{u}$ is locally order-bounded and $\sigma^* \in \mathcal{S}'$, $\{u_\sigma : \sigma \in \mathcal{S}' \wedge \sigma^*\} \subseteq \{u_\tau : \tau \in \mathcal{S} \wedge \sigma^*\}$ is order-bounded in $L^0(\mathfrak{A})$, so $\boldsymbol{u}\restriction\mathcal{S}'$ is locally order-bounded.

Now suppose that $\boldsymbol{u}$ is locally order-bounded, $\tau^* \in \mathcal{S}$ and $a \in \mathfrak{A}$ is non-zero. Since $\sup_{\sigma\in\mathcal{S}'} [\![\tau^* = \sigma]\!] = 1$, there is a $\sigma \in \mathcal{S}'$ such that $a' = a \cap [\![\tau^* = \sigma]\!]$ is non-zero. Now $\mathcal{S} \wedge \sigma$ is covered by $\mathcal{S}' \wedge \sigma$ (611M(e-i)) and $\boldsymbol{u}\restriction\mathcal{S}' \wedge \sigma$ is order-bounded, so $\boldsymbol{u}\restriction\mathcal{S} \wedge \sigma$ is order-bounded, by (i) above. There is therefore a $\alpha \ge 0$ such that $a' = a \setminus \sup_{\tau\in\mathcal{S}\wedge\sigma} [\![|u_\tau| > \alpha]\!]$ is non-zero. But in this case

$$\begin{aligned} a' \cap [\![|u_{\tau\wedge\tau^*}| > \alpha]\!] &= a' \cap [\![|u_{\tau\wedge\tau^*}| > \alpha]\!] \cap [\![\tau^* = \sigma]\!] \\ &\subseteq a' \cap [\![|u_{\tau\wedge\sigma}| > \alpha]\!] = 0 \end{aligned}$$

for every $\tau \in \mathcal{S}$, that is, $a' \cap [\![|u_\tau| > \alpha]\!] = 0$ for every $\tau \in \mathcal{S} \wedge \tau^*$. So $\sup_{\tau\in\mathcal{S}\wedge\tau^*} [\![|u_\tau| > \alpha]\!]$ does not include a; as a is arbitrary, $\inf_{\gamma>0}\sup_{\tau\in\mathcal{S}\wedge\tau^*} [\![|u_\tau| > \gamma]\!] = 0$ and $\{u_\tau : \tau \in \mathcal{S} \wedge \tau^*\}$ is bounded above in $L^0(\mathfrak{A})$ (364L(a-ii)). As τ^* is arbitrary, $\boldsymbol{u}$ is locally order-bounded.

614H Proposition Brownian motion, as described in 612T, is locally order-bounded.

proof I follow the notation of 612T. If $\tau \in \mathcal{T}_f$, let $h : \Omega \to [0, \infty[$ be a stopping time representing τ. Then $f(\omega) = \sup_{t\in[0,h(\omega)]} |\omega(t)|$ is finite for every $\omega \in \Omega$, because ω is continuous. Moreover, again because every ω is continuous, $f(\omega) = \sup_{q\in\mathbb{Q}} f_q(\omega)$ for every ω, where $f_q(\omega) = |\omega(q)|$ if $q \le h(\omega)$, 0 otherwise; as every f_q is measurable, so is f. Now we see that if $\sigma \le \tau$ in $\mathcal{T}_f$ there is a stopping time $g : \Omega \to [0, \infty[$ such that g represents σ and $g \le h$, so that $|X_g| \le f$ and $|w_\sigma| = |X_g^\bullet| \le f^\bullet$ in $L^0(\mathfrak{C})$. Thus $\{w_\sigma : \sigma \le \tau\}$ is order-bounded; as τ is arbitrary, $\boldsymbol{w}$ is locally order-bounded.

614I Non-decreasing processes I pause for some nearly trivial remarks. Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma\in\mathcal{S}}$ a non-decreasing fully adapted process.

(a) $\boldsymbol{v}$ is a lattice homomorphism. **P** If σ, $\tau \in \mathcal{S}$, then $v_{\sigma\wedge\tau} \le v_{\sigma\vee\tau}$ and

$$[\![\sigma \le \tau]\!] = [\![\sigma = \sigma \wedge \tau]\!] \cap [\![\tau = \sigma \vee \tau]\!]$$

(611E(a-ii-β))

$$\subseteq [\![v_\sigma = v_{\sigma\wedge\tau}]\!] \cap [\![v_\tau = v_{\sigma\vee\tau}]\!] \subseteq [\![v_\sigma \wedge v_\tau = v_{\sigma\wedge\tau}]\!] \cap [\![v_\sigma \vee v_\tau = v_{\sigma\vee\tau}]\!]$$

and similarly

$$[\![\tau \le \sigma]\!] \subseteq [\![v_\sigma \wedge v_\tau = v_{\sigma\wedge\tau}]\!] \cap [\![v_\sigma \vee v_\tau = v_{\sigma\vee\tau}]\!],$$

so $v_\sigma \wedge v_\tau = v_{\sigma\wedge\tau}$ and $v_\sigma \vee v_\tau = v_{\sigma\vee\tau}$. As σ and τ are arbitrary, $\boldsymbol{v}$ is a lattice homomorphism. **Q**

(b) As in the proof of (a) just above,

$$[\![\sigma \le \tau]\!] \subseteq [\![v_\sigma = v_{\sigma\wedge\tau}]\!] \cap [\![v_\tau = v_{\sigma\vee\tau}]\!] \subseteq [\![v_\sigma \le v_\tau]\!]$$

for all σ, $\tau \in \mathcal{S}$.

(c) If $\boldsymbol{v}$ is non-negative it is locally order-bounded. **P** If $\tau \in \mathcal{S}$ then $0 \le v_\sigma \le v_\tau$ for every $\sigma \in \mathcal{S} \wedge \tau$. **Q**

(d) If $\mathcal{S} \ne \emptyset$ and $\boldsymbol{v}$ is order-bounded, then

$$\int_{\mathcal{S}} |d\boldsymbol{v}| = \int_{\mathcal{S}} d\boldsymbol{v} = v_\uparrow - v_\downarrow$$

where $v_\uparrow = \sup_{\sigma\in\mathcal{S}} v_\sigma = \lim_{\sigma\uparrow\mathcal{S}} v_\sigma$ and $v_\downarrow = \inf_{\sigma\in\mathcal{S}} v_\sigma = \lim_{\sigma\downarrow\mathcal{S}} v_\sigma$. **P** To identify $\inf_{\sigma\in\mathcal{S}} v_\sigma$ with $\lim_{\sigma\downarrow\mathcal{S}} v_\sigma$ and $\sup_{\sigma\in\mathcal{S}} v_\sigma$ with $\lim_{\sigma\uparrow\mathcal{S}} v_\sigma$, use the last sentence of 613Ba. Now 613N tells us that $\int_{\mathcal{S}} d\boldsymbol{v} = v_\uparrow - v_\downarrow$, and since the interval functions $\Delta\boldsymbol{v}$ and $|\Delta\boldsymbol{v}|$ (613Cc) are equal, this is also $\int_{\mathcal{S}} |d\boldsymbol{v}|$. **Q**

(e) There is a special kind of non-decreasing process which it is sometimes useful to remember. Suppose that $w \in (L^0)^+$. For each $\sigma \in \mathcal{S}$, set $w_\sigma = \sup\{x : x \in L^0(\mathfrak{A}_\sigma, x \le w\}$. Now $\boldsymbol{w} = \langle w_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a non-negative non-decreasing fully adapted process. **P** If $\sigma \in \mathcal{S}$, then $L^0(\mathfrak{A}_\sigma)$ is order-closed in $L^0 = L^0(\mathfrak{A})$ (612A(e-i)), so $w_\sigma \in L^0(\mathfrak{A}_\sigma)$, and of course $0 \le w_\sigma$. If $\sigma \le \tau$ in $\mathcal{S}$ then $\mathfrak{A}_\sigma \subseteq \mathfrak{A}_\tau$ so $L^0(\mathfrak{A}_\sigma) \subseteq L^0(\mathfrak{A}_\tau)$ and $w_\sigma \le w_\tau$. Finally, if σ, $\tau \in \mathcal{S}$ then $w_\tau \times \chi[\![\sigma = \tau]\!] \in L^0(\mathfrak{A}_\sigma)$ (612C) and $w_\tau \times \chi[\![\sigma = \tau]\!] \le w$ so $w_\tau \times \chi[\![\sigma = \tau]\!] \le w_\sigma$; similarly, $w_\sigma \times \chi[\![\sigma = \tau]\!] \le w_\tau$, so $w_\tau \times \chi[\![\sigma = \tau]\!] = w_\sigma \times \chi[\![\sigma = \tau]\!]$, that is, $[\![\sigma = \tau]\!] \subseteq [\![w_\sigma = w_\tau]\!]$. Thus $\langle w_\sigma\rangle_{\sigma\in\mathcal{S}}$ is fully adapted. **Q**

Observe also that $|\boldsymbol{u}| \le \boldsymbol{w}$ whenever $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is fully adapted and $|\sup\boldsymbol{u}| \le w$, just because $|u_\sigma| \in L^0(\mathfrak{A}_\sigma)$ and $|u_\sigma| \le w$ for every $\sigma \in \mathcal{S}$.

(f) If $h : \mathbb{R} \to \mathbb{R}$ is non-decreasing, then $\bar h\boldsymbol{v}$ is non-decreasing. (Use 612A(d-i-ϵ).)

(g) If $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is non-negative and fully adapted and $\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}$ is defined, then $\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v} \ge 0$ and $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is non-decreasing. **P** For any finite sublattice I of $\mathcal{S}$, $S_I(\boldsymbol{u}, d\boldsymbol{v})$ is either 0 or expressible in the form $\sum_{i=0}^{n-1} u_{\tau_i} \times (v_{\tau_{i+1}} - v_{\tau_i})$ where $\tau_i \le \tau_{i+1}$ for every $i < n$; in the latter case u_{τ_i} and $v_{\tau_{i+1}} - v_{\tau_i}$ and $u_{\tau_i} \times (v_{\tau_{i+1}} - v_{\tau_i})$ are non-negative for every i, so $S_I(\boldsymbol{u}, d\boldsymbol{v}) \ge 0$. Now $\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v} = \lim_{I\uparrow\mathcal{I}(\mathcal{S})} S_I(\boldsymbol{u}, d\boldsymbol{v})$ must be positive because the cone $(L^0)^+$ is closed (613Ba).

As for the indefinite integral, we see now that if $\sigma \le \tau$ in $\mathcal{S}$ then $\boldsymbol{u}\restriction\mathcal{S}\cap[\sigma,\tau]$ is non-negative and $\boldsymbol{v}\restriction\mathcal{S}\cap[\sigma,\tau]$ is non-decreasing, so $\int_{\mathcal{S}\cap[\sigma,\tau]} \boldsymbol{u}\,d\boldsymbol{v} \ge 0$ and

$$\int_{\mathcal{S}\wedge\sigma} \boldsymbol{u}\,d\boldsymbol{v} \le \int_{\mathcal{S}\wedge\sigma} \boldsymbol{u}\,d\boldsymbol{v} + \int_{\mathcal{S}\cap[\sigma,\tau]} \boldsymbol{u}\,d\boldsymbol{v} = \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\,d\boldsymbol{v}$$

by 613K(c-i). **Q**

614J Bounded variation The third class of processes I wish to discuss is intermediate between the other two.

Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a fully adapted process. Then the following are equiveridical:

(i) $\boldsymbol{v}$ is expressible as the difference of two order-bounded non-negative non-decreasing fully adapted processes,

(ii) $\{\sum_{i=0}^{n-1} |v_{\tau_{i+1}} - v_{\tau_i}| : \tau_0 \le \ldots \le \tau_n \text{ in } \mathcal{S}\}$ is bounded above in L^0,

(iii) $\int_{\mathcal{S}} |d\boldsymbol{v}|$ is defined;

and in this case

$$\int_{\mathcal{S}} |d\boldsymbol{v}| = \sup\{\sum_{i=0}^{n-1} |v_{\tau_{i+1}} - v_{\tau_i}| : \tau_0 \le \ldots \le \tau_n \text{ in } \mathcal{S}\}$$

if we count $\sup\emptyset$ as 0.

proof If $\mathcal{S}$ is empty, then (interpreting $\sup\emptyset$ as 0 in the last clause) the result is true for trivial reasons, so let us suppose that $\mathcal{S} \ne \emptyset$.

(i)⇒(ii) If $\boldsymbol{v} = \boldsymbol{v}' - \boldsymbol{v}''$ where $\boldsymbol{v}' = \langle v'_\sigma\rangle_{\sigma\in\mathcal{S}}$ and $\boldsymbol{v}'' = \langle v''_\sigma\rangle_{\sigma\in\mathcal{S}}$ are non-decreasing, non-negative and order-bounded, set $\bar v = \sup_{\sigma\in\mathcal{S}} v'_\sigma + \sup_{\sigma\in\mathcal{S}} v''_\sigma$. Then

$$\begin{aligned}\sum_{i=0}^{n-1}|v_{\tau_{i+1}}-v_{\tau_i}| &\le \sum_{i=0}^{n-1}|v'_{\tau_{i+1}}-v'_{\tau_i}|+\sum_{i=0}^{n-1}|v''_{\tau_{i+1}}-v''_{\tau_i}|\\ &=\sum_{i=0}^{n-1}v'_{\tau_{i+1}}-v'_{\tau_i}+\sum_{i=0}^{n-1}v''_{\tau_{i+1}}-v''_{\tau_i}\\ &=v'_{\tau_n}-v'_{\tau_0}+v''_{\tau_n}-v'_{\tau_0}\le\bar v\end{aligned}$$

whenever $\tau_0\le\ldots\le\tau_n$ in $\mathcal{S}$. So (ii) is true.

(ii)⇒(iii)(α) The key fact is that $S_J(\mathbf{1},|d\boldsymbol{v}|)\le S_I(\mathbf{1},|d\boldsymbol{v}|)$ whenever $J\subseteq I$ in $\mathcal{I}(\mathcal{S})$. **P** If J is empty this is trivial. Otherwise, let $(\tau_0,\ldots,\tau_n)$ linearly generate the J-cells, so that

$$S_J(\mathbf{1},|d\boldsymbol{v}|)=\sum_{i=0}^{n-1}|v_{\tau_{i+1}}-v_{\tau_i}|$$

(613Ec),

$$S_I(\mathbf{1},|d\boldsymbol{v}|)=S_{I\cap[\min I,\tau_0]}(\mathbf{1},|d\boldsymbol{v}|)+\sum_{i=0}^{n-1}S_{I\cap[\tau_i,\tau_{i+1}]}(\mathbf{1},|d\boldsymbol{v}|)+S_{I\cap[\tau_n,\max I]}(\mathbf{1},|d\boldsymbol{v}|)$$

(613G(a-ii)). Now for any stopping-time interval $e=c(\sigma,\tau)$ with endpoints in $\mathcal{S}$,

$$\Delta_e(\mathbf{1},|d\boldsymbol{v}|)=|v_\tau-v_\sigma|=|\Delta_e(\mathbf{1},d\boldsymbol{v})|,$$

so for any $K\in\mathcal{I}(\mathcal{S})$

$$\begin{aligned}|S_K(\mathbf{1},d\boldsymbol{v})|&=|\sum_{e\in\mathrm{Stio}_0(K)}\Delta_e(\mathbf{1},d\boldsymbol{v})|\le\sum_{e\in\mathrm{Stio}_0(K)}|\Delta_e(\mathbf{1},d\boldsymbol{v})|\\ &=\sum_{e\in\mathrm{Stio}_0(K)}\Delta_e(\mathbf{1},|d\boldsymbol{v}|)=S_K(\mathbf{1},|d\boldsymbol{v}|).\end{aligned}$$

In particular, for $i<n$,

$$\begin{aligned}|v_{\tau_{i+1}}-v_{\tau_i}|&=|S_{I\cap[\tau_i,\tau_{i+1}]}(\mathbf{1},d\boldsymbol{v})|\\ &\le S_{I\cap[\tau_i,\tau_{i+1}]}(\mathbf{1},|d\boldsymbol{v}|).\end{aligned}$$

(613Ed)

Summing over i,

$$S_J(\mathbf{1},|d\boldsymbol{v}|)=\sum_{i=0}^{n-1}|v_{\tau_{i+1}}-v_{\tau_i}|\le\sum_{i=0}^{n-1}S_{I\cap[\tau_i,\tau_{i+1}]}(\mathbf{1},|d\boldsymbol{v}|)\le S_I(\mathbf{1},|d\boldsymbol{v}|).\ \mathbf{Q}$$

(β) Setting $A=\{\sum_{i=0}^{n-1}|v_{\tau_{i+1}}-v_{\tau_i}|:\tau_0\le\ldots\le\tau_n\text{ in }\mathcal{S}\}$, $A'=\{S_I(\mathbf{1},|d\boldsymbol{v}|):I\in\mathcal{I}(\mathcal{S})\}$, we have $A'\subseteq A$, and we know from (α) that A' is upwards-directed; moreover,

$$\int_{\mathcal{S}}|d\boldsymbol{v}|=\lim_{I\uparrow\mathcal{I}(\mathcal{S})}S_I(\mathbf{1},|d\boldsymbol{v}|)=\lim_{z\uparrow A'}z=\sup A'$$

if any of these is defined in L^0. If we assume that (ii) is true, so that A and A' are bounded above, then $\sup A'$ is defined and (iii) is true.

In fact, of course, $A\subseteq A'$, because if $\tau_0\le\ldots\le\tau_n$ in $\mathcal{S}$ then $\sum_{i=0}^{n-1}|v_{\tau_{i+1}}-v_{\tau_i}|=S_I(\mathbf{1},|d\boldsymbol{v}|)$ where $I=\{\tau_0,\ldots,\tau_n\}$. So we must have $\sup A=\int_{\mathcal{S}}|d\boldsymbol{v}|$, as claimed in the final clause of the statement of this theorem.

(iii)⇒(i) Suppose that $\int_{\mathcal{S}}|d\boldsymbol{v}|$ is defined.

(α) $v_\downarrow=\lim_{\sigma\downarrow\mathcal{S}}v_\sigma$ is defined. **P** Let $\epsilon>0$. Then there is a non-empty $J\in\mathcal{I}(\mathcal{S})$ such that $\theta(S_I(\mathbf{1},|d\boldsymbol{v}|)-S_J(\mathbf{1},|d\boldsymbol{v}|))\le\epsilon$ whenever $I\in\mathcal{I}(\mathcal{S})$ includes J. If $\sigma\in\mathcal{S}\wedge\min J$, set $I=J\cup\{\sigma\}$; then

$$\begin{aligned}\epsilon&\ge\theta(S_I(\mathbf{1},|d\boldsymbol{v}|)-S_J(\mathbf{1},|d\boldsymbol{v}|))=\theta(\Delta_{c(\sigma,\min J)}(\mathbf{1},|d\boldsymbol{v}|))\\ &=\theta(|v_{\min J}-v_\sigma|)=\theta(v_{\min J}-v_\sigma).\end{aligned}$$

Because L^0 is complete, this is enough to ensure that $v_\downarrow$ is defined. **Q**

(β) $v_\uparrow = \lim_{\sigma\uparrow\mathcal{S}} v_\sigma$ is defined. **P** Argue as in (α). Let $\epsilon > 0$. Then there is a non-empty $J \in \mathcal{I}(\mathcal{S})$ such that $\theta(S_I(\mathbf{1}, |d\boldsymbol{v}|) - S_J(\mathbf{1}, |d\boldsymbol{v}|)) \le \epsilon$ whenever $I \in \mathcal{I}(\mathcal{S})$ includes J. Now

$$\theta(v_\sigma - v_{\max J}) = \theta(S_{J\cup\{\sigma\}}(\mathbf{1}, |d\boldsymbol{v}|) - S_J(\mathbf{1}, |d\boldsymbol{v}|) \le \epsilon$$

whenever $\max J \le \sigma \in \mathcal{S}$. Because L^0 is complete, this is enough to ensure that $v_\uparrow$ is defined. **Q**

(γ) Set $v'_\tau = v_\downarrow^+ + \int_{\mathcal{S}\wedge\tau} |d\boldsymbol{v}|$ for $\tau \in \mathcal{S}$. Because $v_\downarrow \in L^0(\mathfrak{A}_\tau)$ for every $\tau \in \mathcal{S}$ (613Bj), $\boldsymbol{v}' = \langle v'_\tau\rangle_{\tau\in\mathcal{S}}$ is fully adapted. If $\sigma \le \tau$ in $\mathcal{S}$, $v'_\tau - v'_\sigma = \int_{\mathcal{S}\cap[\sigma,\tau]} |d\boldsymbol{v}| \ge 0$, so $\boldsymbol{v}'$ is non-decreasing. Because $v_\downarrow^+ \ge 0$, $\boldsymbol{v}'$ is non-negative. As $\lim_{\tau\uparrow\mathcal{S}} v'_\tau = v_\downarrow^+ + \int_{\mathcal{S}} |d\boldsymbol{v}|$ is defined (613J(f-ii)), this is $\sup_{\tau\in\mathcal{S}} v'_\tau$ and $\boldsymbol{v}'$ is order-bounded.

(δ) Set $v''_\tau = v'_\tau - v_\tau$ for $\tau \in \mathcal{S}$, so that $\boldsymbol{v}'' = \langle v''_\tau\rangle_{\tau\in\mathcal{S}}$ is fully adapted. If $\sigma \le \tau$ in $\mathcal{S}$,

$$v''_\tau - v''_\sigma = \int_{\mathcal{S}\wedge\tau} |d\boldsymbol{v}| - \int_{\mathcal{S}\wedge\sigma} |d\boldsymbol{v}| - v_\tau + v_\sigma$$

$$= \int_{\mathcal{S}\cap[\sigma,\tau]} |d\boldsymbol{v}| - v_\tau + v_\sigma \ge |v_\tau - v_\sigma| - v_\tau + v_\sigma$$

(as in (α) of (ii)$\Rightarrow$(iii) above)

$$\ge 0.$$

So $\boldsymbol{v}''$ is non-decreasing. We have

$$\lim_{\tau\downarrow\mathcal{S}} v''_\tau = v_\downarrow^+ + \lim_{\tau\downarrow\mathcal{S}} \int_{\mathcal{S}\wedge\tau} |d\boldsymbol{v}| - v_\downarrow = v_\downarrow^+ - v_\downarrow$$

(613J(f-i))

$$\ge 0,$$

so $\boldsymbol{v}''$ is non-negative. At the other end,

$$\lim_{\tau\uparrow\mathcal{S}} v''_\tau = v_\downarrow^+ + \lim_{\tau\uparrow\mathcal{S}} \int_{\mathcal{S}\wedge\tau} |d\boldsymbol{v}| - \lim_{\tau\uparrow\mathcal{S}} v_\tau = v_\downarrow^+ + \int_{\mathcal{S}} |d\boldsymbol{v}| - v_\uparrow$$

is defined, so this must be an upper bound of $\{v''_\tau : \tau \in \mathcal{S}\}$, and $\boldsymbol{v}''$ is order-bounded.

(ϵ) Accordingly $\boldsymbol{v}' - \boldsymbol{v}'' = \boldsymbol{v}$ witnesses that (i) is true.

614K Definitions Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v}$ a fully adapted process with domain $\mathcal{S}$.

(a) $\boldsymbol{v}$ is **of bounded variation** if it satisfies the conditions of Theorem 614J.

(b) $\boldsymbol{v}$ is **locally of bounded variation** if $\boldsymbol{v}\restriction\mathcal{S}\wedge\tau$ is of bounded variation for every $\tau \in \mathcal{S}$.

614L Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v}$ a fully adapted process with domain $\mathcal{S}$.

(a) If $\boldsymbol{v}$ is (locally) of bounded variation it is (locally) order-bounded.

(b)(i) If $\boldsymbol{v}$ is of bounded variation and $\mathcal{S}'$ is a sublattice of $\mathcal{S}$, then $\boldsymbol{v}\restriction\mathcal{S}'$ is of bounded variation and $\int_{\mathcal{S}'} |d\boldsymbol{v}| \le \int_{\mathcal{S}} |d\boldsymbol{v}|$.

(ii) If $\boldsymbol{v}$ is locally of bounded variation and $\mathcal{S}'$ is a sublattice of $\mathcal{S}$, then $\boldsymbol{v}\restriction\mathcal{S}'$ is locally of bounded variation.

(c) If $\tau \in \mathcal{S}$, then $\boldsymbol{v}$ is (locally) of bounded variation iff $\boldsymbol{v}\restriction\mathcal{S}\wedge\tau$ and $\boldsymbol{v}\restriction\mathcal{S}\vee\tau$ are both (locally) of bounded variation.

proof (a) Since $M_{\text{o-b}}(\mathcal{S})$ is closed under subtraction (614F(c-ii)), it follows at once from 614J(i) that if $\boldsymbol{v}$ is of bounded variation then it is order-bounded. Applying this to $\mathcal{S}\wedge\tau$ for $\tau \in \mathcal{S}$, we see that if $\boldsymbol{v}$ is locally of bounded variation it is locally order-bounded.

(b)(i) $\boldsymbol{v}$ satisfies condition (i) of 614J, so $\boldsymbol{v}\restriction\mathcal{S}'$ also does, and $\int_{\mathcal{S}'}|d\boldsymbol{v}| \le \int_{\mathcal{S}'}|d\boldsymbol{v}|$ by the last formula in 614J.

(ii) If $\boldsymbol{v}$ is locally of bounded variation and $\tau \in \mathcal{S}'$, then $\boldsymbol{v}\restriction\mathcal{S}\wedge\tau$ is of bounded variation and $\mathcal{S}'\wedge\tau$ is a sublattice of $\mathcal{S}\wedge\tau$, so $\boldsymbol{v}\restriction\mathcal{S}'\wedge\tau$ is of bounded variation; as τ is arbitrary, $\boldsymbol{v}\restriction\mathcal{S}'$ is locally of bounded variation.

(c) If $\boldsymbol{v}$ is (locally) of bounded variation, then $\boldsymbol{v}\restriction\mathcal{S}\wedge\tau$ and $\boldsymbol{v}\restriction\mathcal{S}\vee\tau$ are (locally) of bounded variation because $\mathcal{S}\wedge\tau$ and $\mathcal{S}\vee\tau$ are sublattices of $\mathcal{S}$.

If $\boldsymbol{v}\restriction\mathcal{S}\wedge\tau$ and $\boldsymbol{v}\restriction\mathcal{S}\vee\tau$ are of bounded variation then $\int_{\mathcal{S}\wedge\tau}|d\boldsymbol{v}|$ and $\int_{\mathcal{S}\vee\tau}|d\boldsymbol{v}|$ are defined, so $\int_{\mathcal{S}}|d\boldsymbol{v}|$ is defined (613J(c-i) once more) and $\boldsymbol{v}$ is of bounded variation.

If $\boldsymbol{v}\restriction\mathcal{S}\wedge\tau$ and $\boldsymbol{v}\restriction\mathcal{S}\vee\tau$ are locally of bounded variation and $\sigma\in\mathcal{S}$, then $\boldsymbol{v}\restriction(\mathcal{S}\wedge\sigma)\wedge\tau = \boldsymbol{v}\restriction(\mathcal{S}\wedge\tau)\wedge\sigma$ and $\boldsymbol{v}\restriction(\mathcal{S}\wedge\sigma)\vee\tau = \boldsymbol{v}\restriction(\mathcal{S}\vee\tau)\wedge(\sigma\vee\tau)$ are of bounded variation, so $\boldsymbol{v}\restriction\mathcal{S}\wedge\sigma$ is of bounded variation; accordingly $\boldsymbol{v}$ is locally of bounded variation.

614M Proposition The Poisson process, as described in 612U, is locally of bounded variation.

proof It is non-negative and non-decreasing, so 614Ic applies.

614N Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\bar{u}\in(L^0)^+$. Then $\{\boldsymbol{v} : \boldsymbol{v}\in M_{\mathrm{fa}}(\mathcal{S})$ is of bounded variation, $\int_{\mathcal{S}}|d\boldsymbol{v}|\le\bar{u}\}$ is closed in $(L^0)^{\mathcal{S}}$ for its product topology.

proof The point is just that if $I\in\mathcal{I}(\mathcal{S})$, then $\boldsymbol{v}\mapsto S_I(\mathbf{1},|d\boldsymbol{v}|):(L^0)^{\mathcal{S}}\to L^0$ is continuous, so

$$A=\{\boldsymbol{v}:\boldsymbol{v}\in(L^0)^{\mathcal{S}},\ S_I(\mathbf{1},|d\boldsymbol{v}|)\le\bar{u}\text{ for every }I\in\mathcal{I}(\mathcal{S})\}$$

is closed in $(L^0)^{\mathcal{S}}$. Now the set $M_{\mathrm{fa}}(\mathcal{S})$ of fully adapted processes with domain $\mathcal{S}$ is closed in $(L^0)^{\mathcal{S}}$ (613Bl), so $\{\boldsymbol{v}:\boldsymbol{v}\in M_{\mathrm{bv}}(\mathcal{S}),\ \int_{\mathcal{S}}|d\boldsymbol{v}|\le\bar{u}\}=A\cap M_{\mathrm{fa}}(\mathcal{S})$ is closed.

614O Cumulative variation Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v}$ a process with domain $\mathcal{S}$ which is locally of bounded variation. Then $v^{\uparrow}_{\tau}=\int_{\mathcal{S}\wedge\tau}|d\boldsymbol{v}|$ is defined for every $\tau\in\mathcal{S}$ (614J-614K), and $\boldsymbol{v}^{\uparrow}=\langle\int_{\mathcal{S}\wedge\tau}v^{\uparrow}_{\tau}\rangle_{\tau\in\mathcal{S}}$ is fully adapted (613J(e-ii)). I will call $\boldsymbol{v}^{\uparrow}$ the **cumulative variation** of $\boldsymbol{v}$.

614P Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a process which is locally of bounded variation, and $\boldsymbol{v}^{\uparrow}=\langle v^{\uparrow}_{\tau}\rangle_{\tau\in\mathcal{S}}$ its cumulative variation.

(a)(i) If $\sigma\le\tau$ in $\mathcal{S}$, then

$$v^{\uparrow}_{\tau}-v^{\uparrow}_{\sigma}=\int_{\mathcal{S}\cap[\sigma,\tau]}|d\boldsymbol{v}|\ge|v_\tau-v_\sigma|\ge 0.$$

(ii) $\boldsymbol{v}^{\uparrow}$ is non-negative and non-decreasing and has starting value 0 if $\mathcal{S}$ is not empty.

(iii) $\boldsymbol{v}^{\uparrow}+\boldsymbol{v}$ and $\boldsymbol{v}^{\uparrow}-\boldsymbol{v}$ are non-decreasing.

(iv) If $\mathcal{S}$ is non-empty, $\boldsymbol{v}$ has a starting value.

(v) $\boldsymbol{v}$ is of bounded variation iff $\lim_{\tau\uparrow\mathcal{S}}v^{\uparrow}_{\tau}=\sup|\boldsymbol{v}^{\uparrow}|$ is defined, and in this case the limit is $\int_{\mathcal{S}}|d\boldsymbol{v}|$.

(b) If $\tau\in\mathcal{S}$ then, writing $(\boldsymbol{v}\restriction\mathcal{S}\wedge\tau)^{\uparrow}$ and $(\boldsymbol{v}\restriction\mathcal{S}\vee\tau)^{\uparrow}$ for the cumulative variations of $\boldsymbol{v}\restriction\mathcal{S}\wedge\tau$ and $\boldsymbol{v}\restriction\mathcal{S}\vee\tau$,

$$(\boldsymbol{v}\restriction\mathcal{S}\wedge\tau)^{\uparrow}=\boldsymbol{v}^{\uparrow}\restriction\mathcal{S}\wedge\tau,\qquad(\boldsymbol{v}\restriction\mathcal{S}\vee\tau)^{\uparrow}=\boldsymbol{v}^{\uparrow}\restriction\mathcal{S}\vee\tau-v^{\uparrow}_{\tau}\mathbf{1}.$$

(c) Suppose that $I\in\mathcal{I}(\mathcal{S})$ is not empty and $(\tau_0,\dots,\tau_n)$ linearly generates the I-cells (611L).

(i)

$$v^{\uparrow}_{\tau_{i+1}}-v^{\uparrow}_{\tau_i}\le|v_{\tau_{i+1}}-v_{\tau_i}|+v^{\uparrow}_{\max I}-v^{\uparrow}_{\min I}-S_I(\mathbf{1},|d\boldsymbol{v}|)$$

for every $i<n$.

(ii) If $\boldsymbol{v}$ is of bounded variation, write w for $\int_{\mathcal{S}}|d\boldsymbol{v}|-S_I(\mathbf{1},|d\boldsymbol{v}|)$, and let $v_{\downarrow}$ be the starting value of $\boldsymbol{v}$.

(α) If $\tau\in\mathcal{S}\wedge\tau_0$, then $v^{\uparrow}_{\tau}\le|v_\tau-v_{\downarrow}|+w$.

(β) If $i<n$ and $\tau\in\mathcal{S}\cap[\tau_i,\tau_{i+1}]$ then $v^{\uparrow}_{\tau}-v^{\uparrow}_{\tau_i}\le|v_\tau-v_{\tau_i}|+w$.

(γ) If $\tau\in\mathcal{S}\vee\tau_n$ then $v^{\uparrow}_{\tau}-v^{\uparrow}_{\tau_n}\le|v_\tau-v_{\tau_n}|+w$.

proof (a)(i) Immediate from 613J(c-i) and 614J.

(ii) By 614J, $\boldsymbol{v}^{\uparrow}$ is non-negative; by (i) here, it is non-decreasing; by 613J(f-i) again its starting value is 0 if $\mathcal{S}\ne\emptyset$.

(iii) If $\sigma \le \tau$ in $\mathcal{S}$, then

$$(v^\uparrow_\tau \pm v_\tau) - (v^\uparrow_\sigma \pm v_\sigma) = v^\uparrow_\tau - v^\uparrow_\sigma \mp (v_\tau - v_\sigma) \ge v^\uparrow_\tau - v^\uparrow_\sigma - |v_\tau - v_\sigma| \ge 0$$

by (i) above.

(iv) Suppose to begin with that $\boldsymbol{v}$ is non-negative and non-decreasing. Take any $\tau \in \mathcal{S}$. Then $\boldsymbol{v}{\restriction}\mathcal{S} \wedge \tau$ is order-bounded (614Ic) so has a starting value, as noted in 614Id, and this will also be the starting value of $\boldsymbol{v}$.

Generally, $\boldsymbol{v}$ is expressible as $\boldsymbol{v}' - \boldsymbol{v}''$ where $\boldsymbol{v}' = \langle v'_\sigma\rangle_{\sigma\in\mathcal{S}}$ and $\boldsymbol{v}'' = \langle v''_\sigma\rangle_{\sigma\in\mathcal{S}}$ are non-negative and non-decreasing. Now

$$\lim_{\sigma\downarrow\mathcal{S}} v_\sigma = \lim_{\sigma\downarrow\mathcal{S}} v'_\sigma - v''_\sigma = \lim_{\sigma\downarrow\mathcal{S}} v'_\sigma - \lim_{\sigma\downarrow\mathcal{S}} v''_\sigma$$

is defined and is the starting value of $\boldsymbol{v}$.

(v) Because $\boldsymbol{v}^\uparrow$ is non-negative and non-decreasing, $\lim_{\tau\uparrow\mathcal{S}} v^\uparrow_\tau = \sup|\boldsymbol{v}^\uparrow|$ if either is defined in $L^0(\mathfrak{A})$. If $\boldsymbol{v}$ is of bounded variation,

$$\int_{\mathcal{S}} |d\boldsymbol{v}| = v^\uparrow_\tau + \int_{\mathcal{S}\vee\tau} |d\boldsymbol{v}| \ge v^\uparrow_\tau$$

for every $\tau \in \mathcal{S}$, so $\sup|\boldsymbol{v}^\uparrow|$ is defined and less than or equal to $\int_{\mathcal{S}} |d\boldsymbol{v}|$. If $\sup|\boldsymbol{v}^\uparrow|$ is defined, then whenever $\tau_0 \le \ldots \le \tau_n$ in $\mathcal{S}$,

$$\sum_{i=0}^{n-1} |v_{\tau_{i+1}} - v_{\tau_i}| \le \int_{\mathcal{S}\wedge\tau_n} |d\boldsymbol{v}| \le \sup|\boldsymbol{v}^\uparrow|,$$

so $\int_{\mathcal{S}} |d\boldsymbol{v}|$ is defined and is at most $\sup|\boldsymbol{v}^\uparrow|$. Thus in either case we have equality.

(b) Of course

$$(\boldsymbol{v}{\restriction}\mathcal{S}\wedge\tau)^\uparrow = \langle \int_{\mathcal{S}\wedge\sigma} |d\boldsymbol{v}|\rangle_{\sigma\in\mathcal{S}\wedge\tau} = \boldsymbol{v}^\uparrow{\restriction}\mathcal{S}\wedge\tau.$$

On the other hand, for $\sigma \in \mathcal{S}\vee\tau$,

$$v^\uparrow_\sigma = \int_{\mathcal{S}\wedge\sigma} |d\boldsymbol{v}| = \int_{(\mathcal{S}\wedge\sigma)\wedge\tau} |d\boldsymbol{v}| + \int_{(\mathcal{S}\wedge\sigma)\vee\tau} |d\boldsymbol{v}|$$

(613G(a-i))

$$= v^\uparrow_\tau + \int_{(\mathcal{S}\vee\tau)\wedge\sigma} |d\boldsymbol{v}|,$$

so

$$\boldsymbol{v}^\uparrow{\restriction}\mathcal{S}\vee\tau = v^\uparrow_\tau\mathbf{1} + (\boldsymbol{v}{\restriction}\mathcal{S}\vee\tau)^\uparrow.$$

(c)(i)

$$\begin{aligned} S_I(\mathbf{1}, |d\boldsymbol{v}|) &= |v_{\tau_{i+1}} - v_{\tau_i}| + \sum_{\substack{j<n\\ j\ne i}} |v_{\tau_{j+1}} - v_{\tau_j}| \le |v_{\tau_{i+1}} - v_{\tau_i}| + \sum_{\substack{j<n\\ j\ne i}} v^\uparrow_{\tau_{j+1}} - v^\uparrow_{\tau_j} \\ &= |v_{\tau_{i+1}} - v_{\tau_i}| + S_I(\mathbf{1}, d\boldsymbol{v}^\uparrow) - v^\uparrow_{\tau_{i+1}} + v^\uparrow_{\tau_i} \\ &= |v_{\tau_{i+1}} - v_{\tau_i}| + v^\uparrow_{\max I} - v^\uparrow_{\min I} - v^\uparrow_{\tau_{i+1}} + v^\uparrow_{\tau_i}; \end{aligned}$$

rearranging, we have the result.

(ii)(α) If $\sigma \in \mathcal{S}\wedge\tau$ then $v^\uparrow_\tau - v^\uparrow_\sigma \le |v_\tau - v_\sigma| + w$. **P** Set $J = \{\sigma, \tau, \tau_0, \ldots, \tau_n\}$. Then $J \in \mathcal{I}(\mathcal{S})$ and $(\sigma, \tau, \tau_0, \ldots, \tau_n)$ linearly generates the J-cells, so

$$\begin{aligned} v^\uparrow_\tau - v^\uparrow_\sigma &\le |v_\tau - v_\sigma| + v^\uparrow_{\tau_n} - v^\uparrow_\sigma - S_J(\mathbf{1}, |d\boldsymbol{v}|) \le |v_\tau - v_\sigma| + \int_{\mathcal{S}} |d\boldsymbol{v}| - \sum_{i=0}^{n-1} |v_{\tau_{i+1}} - v_{\tau_i}| \\ &= |v_\tau - v_\sigma| + \int_{\mathcal{S}} |d\boldsymbol{v}| - S_I(\mathbf{1}, |d\boldsymbol{v}|) = |v_\tau - v_\sigma| + w. \ \mathbf{Q} \end{aligned}$$

Now

$$v_\tau^\uparrow = \lim_{\sigma\downarrow\mathcal{S}} v_\tau^\uparrow - v_\sigma^\uparrow \le \lim_{\sigma\downarrow\mathcal{S}} |v_\tau - v_\sigma| + w = |v_\tau - v_\downarrow| + w.$$

(β) Set $J = \{\tau_0, \dots, \tau_i, \tau, \tau_{i+1}, \dots, \tau_n\}$. Then

$$v_\tau^\uparrow - v_{\tau_i}^\uparrow \le |v_\tau - v_{\tau_i}| + v_{\tau_n}^\uparrow - v_{\tau_0}^\uparrow - S_J(\mathbf{1}, |d\boldsymbol{v}|) \le |v_\tau - v_{\tau_i}| + w.$$

(γ) This time, set $J = \{\tau_0, \dots, \tau_n, \tau\}$, so that

$$v_\tau^\uparrow - v_{\tau_n}^\uparrow \le |v_\tau - v_{\tau_n}| + v_\tau^\uparrow - v_{\tau_0}^\uparrow - S_J(\mathbf{1}, |d\boldsymbol{v}|) \le |v_\tau - v_{\tau_n}| + w.$$

614Q Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\hat{\mathcal{S}}$ its covered envelope.

(a) Write $M_{\mathrm{bv}} = M_{\mathrm{bv}}(\mathcal{S})$ for the set of fully adapted processes of bounded variation with domain $\mathcal{S}$.

(i) $\bar{h}\boldsymbol{v} \in M_{\mathrm{bv}}$ whenever $\boldsymbol{v} \in M_{\mathrm{bv}}$ and $h : \mathbb{R} \to \mathbb{R}$ is Lipschitz on every bounded interval.

(ii) M_{bv} is an f-subalgebra of $M_{\text{o-b}}(\mathcal{S})$.

(iii) The space M_{simp} of simple processes with domain $\mathcal{S}$ is an f-subalgebra of M_{bv} closed under $\bar{h}$ for every Borel measurable $h : \mathbb{R} \to \mathbb{R}$.

(iv) If $\boldsymbol{v} \in M_{\mathrm{fa}}(\mathcal{S})$ and $\hat{\boldsymbol{v}}$ is its fully adapted extension to $\hat{\mathcal{S}}$, then

(α) $\boldsymbol{v}$ is non-decreasing iff $\hat{\boldsymbol{v}}$ is non-decreasing,

(β) $\boldsymbol{v}$ is of bounded variation iff $\hat{\boldsymbol{v}}$ is of bounded variation, and in this case $\int_{\hat{\mathcal{S}}} |d\hat{\boldsymbol{v}}| = \int_{\mathcal{S}} |d\boldsymbol{v}|$ and the cumulative variation $\hat{\boldsymbol{v}}^\uparrow$ of $\hat{\boldsymbol{v}}$ is the fully adapted extension of the cumulative variation $\boldsymbol{v}^\uparrow$ of $\boldsymbol{v}$.

(b) Write $M_{\mathrm{lbv}} = M_{\mathrm{lbv}}(\mathcal{S})$ for the set of fully adapted processes with domain $\mathcal{S}$ which are locally of bounded variation.

(i) If $\boldsymbol{v} \in M_{\mathrm{lbv}}(\mathcal{S})$ then $\boldsymbol{v}\restriction\mathcal{S}'$ is locally of bounded variation for every sublattice $\mathcal{S}'$ of $\mathcal{S}$.

(ii) $\bar{h}\boldsymbol{v} \in M_{\mathrm{lbv}}$ whenever $\boldsymbol{v} \in M_{\mathrm{lbv}}$ and $h : \mathbb{R} \to \mathbb{R}$ is Lipschitz on every bounded interval.

(iii) M_{lbv} is an f-subalgebra of $M_{\mathrm{lob}}(\mathcal{S})$.

(iv) If $\boldsymbol{v} \in M_{\mathrm{fa}}(\mathcal{S})$ then $\boldsymbol{v}$ is locally of bounded variation iff it is expressible as the difference of two non-negative non-decreasing fully adapted processes.

(v) If $\boldsymbol{v} \in M_{\mathrm{fa}}(\mathcal{S})$, then $\boldsymbol{v}$ is locally of bounded variation iff its fully adapted extension to $\hat{\mathcal{S}}$ is locally of bounded variation, and in this case the cumulative variation of $\hat{\boldsymbol{v}}$ is the fully adapted extension of the cumulative variation of $\boldsymbol{v}$.

proof If $\mathcal{S}$ is empty, this is trivial. So suppose otherwise.

(a)(i) Define $g : \mathbb{R} \to \mathbb{R}$ by setting $g(0) = 0$ and

$$g(x) = \sup_{-|x| \le y < y' \le |x|} \tfrac{|h(y') - h(y)|}{y' - y}$$

for $x \ne 0$. Then g is monotonic on both $]-\infty, 0]$ and $[0, \infty[$, so is Borel measurable. As

$$|h(y') - h(y)| \le g(x)|y' - y| \text{ whenever } |y|,\ |y'| \le x \text{ in } \mathbb{R},$$

we shall have

$$|\bar{h}(v') - \bar{h}(v)| \le \bar{g}(u) \times |v' - v| \text{ whenever } |v|,\ |v'| \le u \text{ in } L^0.$$

Now suppose that $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ is of bounded variation. By 614La, it is order-bounded; let u be an upper bound of $\{v_\sigma : \sigma \in \mathcal{S}\}$. If $\tau_0 \le \ldots \le \tau_n$ in $\mathcal{S}$, then

$$\begin{aligned}\sum_{i=0}^{n-1} |\bar{h}(v_{\tau_{i+1}}) - \bar{h}(v_{\tau_i})| &\le \sum_{i=0}^{n-1} \bar{g}(u) \times |v_{\tau_{i+1}} - v_{\tau_i}| \\ &= \bar{g}(u) \times \sum_{i=0}^{n-1} |v_{\tau_{i+1}} - v_{\tau_i}| \le \bar{g}(u) \times \int_{\mathcal{S}} |d\boldsymbol{v}|\end{aligned}$$

(614J). Thus $\bar{h}\boldsymbol{v}$ satisfies (ii) of 614J and is of bounded variation.

(ii) If $\boldsymbol{v}, \boldsymbol{v}'$ are non-negative non-decreasing processes defined on $\mathcal{S}$, then of course $\boldsymbol{v} + \boldsymbol{v}'$ is non-negative and non-decreasing. Condition (i) of 614J now makes it plain that the sum of processes of bounded variation is of bounded variation. From (i) we see that $\bar{h}\boldsymbol{v} \in M_{\mathrm{bv}}$ whenever $\boldsymbol{v} \in M_{\mathrm{bv}}$ and $h : \mathbb{R} \to \mathbb{R}$ is convex, so 612Bc tells us that M_{bv} is an f-subalgebra of $\prod_{\sigma\in\mathcal{S}} L^0(\mathfrak{A}_\sigma)$ and therefore of $M_{\text{o-b}}$.

(iii) Let $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ be a simple process. If $\boldsymbol{v}$ is constant, then surely it is of bounded variation. Otherwise, let $(\tau_0,\dots,\tau_n)$ be a breakpoint sequence for $\boldsymbol{v}$ and v_* its starting value. Consider the simple process $\boldsymbol{w} = \langle w_\sigma\rangle_{\sigma\in\mathcal{S}}$ with the same breakpoint sequence defined by saying that

$$[\![\sigma < \tau_0]\!] \subseteq [\![w_\sigma = v_*]\!], \quad [\![\tau_n \le \sigma]\!] \subseteq [\![w_\sigma = v_* + |v_{\tau_0} - v_*| + \sum_{i=0}^{n-1} |v_{\tau_{i+1}} - v_{\tau_i}|]\!],$$

$$[\![\tau_j \le \sigma]\!] \cap [\![\sigma < \tau_{j+1}]\!] \subseteq [\![w_\sigma = v_* + |v_{\tau_0} - v_*| + \sum_{i=0}^{j-1} |v_{\tau_{i+1}} - v_{\tau_i}|]\!]$$

for $0 \le j < n$. Since $(v_*, w_{\tau_0},\dots,w_{\tau_n})$ and $(0, w_{\tau_0} - v_{\tau_0},\dots,w_{\tau_n} - v_{\tau_n})$ are both non-decreasing, $\boldsymbol{w}$ and $\boldsymbol{w} - \boldsymbol{v}$ are both non-decreasing, so their difference $\boldsymbol{v}$ is of bounded variation.

Thus $M_{\mathrm{simp}} \subseteq M_{\mathrm{bv}}$. Now refer to 612L for its other properties.

(iv) Express $\boldsymbol{v}$ and $\hat{\boldsymbol{v}}$ as $\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ and $\langle \hat{v}_\sigma\rangle_{\sigma\in\hat{\mathcal{S}}}$.

($\boldsymbol{\alpha}$) If $\hat{\boldsymbol{v}}$ is non-decreasing then of course $\boldsymbol{v} = \hat{\boldsymbol{v}}\restriction\mathcal{S}$ is non-decreasing. If $\boldsymbol{v}$ is non-decreasing and $\tau_0 \le \tau_1$ in $\hat{\mathcal{S}}$, take any non-zero $a \in \mathfrak{A}$. Then there are a $\sigma_0 \in \mathcal{S}$ such that $a_0 = a \cap [\![\tau_0 = \sigma_0]\!]$ is non-zero, and a $\sigma_1 \in \mathcal{S}$ such that $a_1 = a_0 \cap [\![\tau_1 = \sigma_1]\!]$ is non-zero. Now

$$\begin{aligned} a_1 &\subseteq [\![\hat{v}_{\tau_0} = v_{\sigma_0}]\!] \cap [\![\hat{v}_{\tau_1} = v_{\sigma_1}]\!] \cap [\![\sigma_0 \le \sigma_1]\!] \\ &\subseteq [\![\hat{v}_{\tau_0} = v_{\sigma_0}]\!] \cap [\![\hat{v}_{\tau_1} = v_{\sigma_0\vee\sigma_1}]\!] \subseteq [\![\hat{v}_{\tau_0} \le \hat{v}_{\tau_1}]\!]. \end{aligned}$$

As a is arbitrary, $\hat{v}'_{\tau_0} \le \hat{v}'_{\tau_1}$. So $\hat{\boldsymbol{v}}$ is non-decreasing.

($\boldsymbol{\beta}$) If $\hat{\boldsymbol{v}}$ is of bounded variation then $\boldsymbol{v} = \hat{\boldsymbol{v}}\restriction\mathcal{S}$ is of bounded variation, by 614L(b-i) above. If $\boldsymbol{v}$ is of bounded variation, express it as $\boldsymbol{v}' - \boldsymbol{v}''$ where $\boldsymbol{v}' = \langle v'_\sigma\rangle_{\sigma\in\mathcal{S}}$, $\boldsymbol{v}''$ are order-bounded non-negative non-decreasing fully adapted processes. Let $\hat{\boldsymbol{v}}'$, $\hat{\boldsymbol{v}}''$ be their fully adapted extensions to $\hat{\mathcal{S}}$, so that $\hat{\boldsymbol{v}} = \hat{\boldsymbol{v}}' - \hat{\boldsymbol{v}}''$ (612Qb). By (α), $\hat{\boldsymbol{v}}'$ and $\hat{\boldsymbol{v}}''$ are non-decreasing; by 614G(b-i) they are order-bounded; and by 612Qb again they are non-negative. So they witness that $\hat{\boldsymbol{v}}$ is of bounded variation.

Now 613T, with $\boldsymbol{u} = \mathbf{1}$ and $\psi(\sigma,\tau) = |\hat{v}_\tau - \hat{v}_\sigma|$ for $\sigma \le \tau$ in $\hat{\mathcal{S}}$, tells us that $\int_{\mathcal{S}} |d\boldsymbol{v}| = \int_{\hat{\mathcal{S}}} |d\hat{\boldsymbol{v}}|$. And if $\tau \in \mathcal{S}$ then $\hat{\mathcal{S}} \wedge \tau$ is the covered envelope of $\mathcal{S} \wedge \tau$ (611M(e-i)) and $\int_{\mathcal{S}\wedge\tau} |d\boldsymbol{v}| = \int_{\hat{\mathcal{S}}\wedge\tau} |d\hat{\boldsymbol{v}}|$, so $\boldsymbol{v}^\uparrow = \hat{\boldsymbol{v}}^\uparrow\restriction\mathcal{S}$ and $\hat{\boldsymbol{v}}^\uparrow$ is the fully adapted extension of $\boldsymbol{v}^\uparrow$.

(b)(i)-(iii) These follow immediately from 614L and (a-i).

(iv)($\boldsymbol{\alpha}$) If $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ is non-negative and non-decreasing and $\tau \in \mathcal{S}$, then $\boldsymbol{v}\restriction\mathcal{S}\wedge\tau$ takes all its values in $[0, v_\tau]$ so is order-bounded and of bounded variation; as τ is arbitrary, $\boldsymbol{v}$ is locally of bounded variation. By (iii), if $\boldsymbol{v}$ is the difference of two non-negative non-decreasing processes then it is locally of bounded variation.

($\boldsymbol{\beta}$) Conversely, if $\boldsymbol{v}$ is locally of bounded variation, then it has a cumulative variation $\boldsymbol{v}^\uparrow = \langle v^\uparrow_\sigma\rangle_{\sigma\in\mathcal{S}}$ (614O), which is non-negative and non-decreasing and has starting value 0 (614P(a-ii)). By 614P(a-iv), $v_\downarrow = \lim_{\sigma\downarrow\mathcal{S}} v_\sigma$ is defined. We have

$$v^\uparrow_\tau = \lim_{\sigma\downarrow\mathcal{S}} v^\uparrow_\tau - v^\uparrow_\sigma \ge \lim_{\sigma\downarrow\mathcal{S}} |v_\tau - v_\sigma| = |v_\tau - v_\downarrow| \ge v_\tau - v_\downarrow$$

so $v^\uparrow_\tau + v^+_\downarrow \ge v_\tau$, for every $\tau \in \mathcal{S}$. Now if we set $\boldsymbol{w} = \boldsymbol{v}^\uparrow + v^+_\downarrow\mathbf{1}$, both $\boldsymbol{w}$ and $\boldsymbol{w} - \boldsymbol{v}$ will be non-negative and non-decreasing, and $\boldsymbol{v} = \boldsymbol{w} - (\boldsymbol{w} - \boldsymbol{v})$ is expressed as a difference of non-negative non-decreasing processes.

(v) We can follow the proof of (a-iv-β) to show that $\hat{\boldsymbol{v}}$ is locally of bounded variation iff $\boldsymbol{v}$ is. In this case, $\hat{v}^\uparrow_\tau = \int_{\hat{\mathcal{S}}\wedge\tau} |d\hat{\boldsymbol{v}}|$ is defined for every $\tau \in \hat{\mathcal{S}}$, and will be equal to $\int_{\mathcal{S}\wedge\tau} |d\boldsymbol{v}| = v^\uparrow_\tau$ if $\tau \in \mathcal{S}$, by 613T. So $\hat{\boldsymbol{v}}^\uparrow$ extends $\boldsymbol{v}^\uparrow$ and must be the fully adapted extension of $\boldsymbol{v}^\uparrow$.

614R Lemma If $I \in \mathcal{I}(\mathcal{T})$ is non-empty and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in I}$, $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in I}$ are fully adapted processes, then

$$|S_I(\boldsymbol{u}, d\boldsymbol{v})| \le \min(\sup|\boldsymbol{u}| \times \int_I |d\boldsymbol{v}|, \sup|\boldsymbol{v}| \times (\int_I |d\boldsymbol{u}| + 2\sup|\boldsymbol{u}|).$$

proof If $\#(I) \le 1$ this is trivial. Otherwise, take $(\tau_0,\dots,\tau_n)$ linearly generating the I-cells. Then

$$|S_I(\boldsymbol{u},d\boldsymbol{v})| = |\sum_{i=0}^{n-1} u_{\tau_i}\times(v_{\tau_{i+1}}-v_{\tau_i})| \le \sum_{i=0}^{n-1}|u_{\tau_i}\times(v_{\tau_{i+1}}-v_{\tau_i})|$$
$$\le \sup|\boldsymbol{u}|\times\sum_{i=0}^{n-1}|v_{\tau_{i+1}}-v_{\tau_i}| = \sup|\boldsymbol{u}|\times\int_I|d\boldsymbol{v}|$$

and also

$$|S_I(\boldsymbol{u},d\boldsymbol{v})| = |\sum_{i=0}^{n-1}(u_{\tau_i}-u_{\tau_{i+1}})\times v_{\tau_{i+1}} + u_{\tau_n}\times v_{\tau_n} - u_{\tau_0}\times v_{\tau_0}|$$
$$\le \sup|\boldsymbol{v}|\times(\int_I|d\boldsymbol{u}|+|u_{\tau_n}|+|u_{\tau_0}|) \le \sup|\boldsymbol{v}|\times(\int_I|d\boldsymbol{u}|+2\sup|\boldsymbol{u}|).$$

614S Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$, $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ two processes of bounded variation with domain $\mathcal{S}$. Then $\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}$ is defined and

$$|\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}| \le \min(\sup|\boldsymbol{u}|\times\int_{\mathcal{S}}|d\boldsymbol{v}|, \sup|\boldsymbol{v}|\times(\int_{\mathcal{S}}|d\boldsymbol{u}|+2\sup|\boldsymbol{u}|)).$$

proof Note first that $\boldsymbol{u}$ and $\boldsymbol{v}$ are order-bounded (614La), so $\sup|\boldsymbol{u}|$ and $\sup|\boldsymbol{v}|$ are defined.

(a) To begin with, consider the case in which both $\boldsymbol{u}$ and $\boldsymbol{v}$ are non-decreasing and non-negative.

(i) If $I\in\mathcal{I}(\mathcal{S})$ is non-empty and $(\tau_0,\dots,\tau_n)$ linearly generate the I-cells, then

$$0 \le u_{\tau_0}\times(v_{\tau_n}-v_{\tau_0}) = u_{\tau_0}\times\sum_{i=0}^{n-1}(v_{\tau_{i+1}}-v_{\tau_i}) \le \sum_{i=0}^{n-1}u_{\tau_i}\times(v_{\tau_{i+1}}-v_{\tau_i})$$
$$= S_I(\boldsymbol{u},d\boldsymbol{v}) \le \sum_{i=0}^{n-1}u_{\tau_n}\times(v_{\tau_{i+1}}-v_{\tau_i}) = u_{\tau_n}\times(v_{\tau_n}-v_{\tau_0}) \le \sup|\boldsymbol{u}|\times\sup|\boldsymbol{v}|.$$

(ii) If $I\subseteq J$ in $\mathcal{I}(\mathcal{S})$, then $S_I(\boldsymbol{u},d\boldsymbol{v})\le S_J(\boldsymbol{u},d\boldsymbol{v})$. **P** If I is empty this is trivial. Otherwise, take $(\tau_0,\dots,\tau_n)$ linearly generating the I-cells. Then

$$S_J(\boldsymbol{u},d\boldsymbol{v}) = S_{J\cap[\min J,\tau_0]}(\boldsymbol{u},d\boldsymbol{v}) + \sum_{i=0}^{n-1}S_{J\cap[\tau_i,\tau_{i+1}]}(\boldsymbol{u},d\boldsymbol{v}) + S_{J\cap[\tau_n,\max J]}(\boldsymbol{u},d\boldsymbol{v})$$

(613G(a-ii))

$$\ge 0+\sum_{i=0}^{n-1}u_{\tau_i}\times(v_{\tau_{i+1}}-v_{\tau_i})+0$$

(by (i) above)

$$= S_I(\boldsymbol{u},d\boldsymbol{v}).\ \mathbf{Q}$$

(iii) Thus $\{S_I(\boldsymbol{u},d\boldsymbol{v}):I\in\mathcal{I}(\mathcal{S})\}$ is upwards-directed. As it is also bounded above in L^0, by (i), it has a supremum, which is $\lim_{I\uparrow\mathcal{I}(\mathcal{S})}S_I(\boldsymbol{u},d\boldsymbol{v})=\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}$.

(c) Thus the integral is defined when both $\boldsymbol{u}$ and $\boldsymbol{v}$ are non-negative and non-decreasing; as integration is bilinear (613Jb), the integral is defined for all processes $\boldsymbol{u}$, $\boldsymbol{v}$ of bounded variation. Concerning the bound for $|\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}|$, we see from 614R that if $I\in\mathcal{I}(\mathcal{S})$ then

$$|S_I(\boldsymbol{u},d\boldsymbol{v})| \le \min\bigl(\sup|\boldsymbol{u}\restriction I|\times\int_I|d\boldsymbol{v}|,\ \sup|\boldsymbol{v}\restriction I|\times(\int_I|d\boldsymbol{u}|+2\sup|\boldsymbol{u}\restriction I|)\bigr)$$
$$\le \min\bigl(\sup|\boldsymbol{u}|\times\int_{\mathcal{S}}|d\boldsymbol{v}|,\ \sup|\boldsymbol{v}|\times(\int_{\mathcal{S}}|d\boldsymbol{u}|+2\sup|\boldsymbol{u}|)\bigr)$$

(using 614L(b-i)). Accordingly

$$|\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}| = |\lim_{I\uparrow\mathcal{S}} S_I(\boldsymbol{u},d\boldsymbol{v})| = \lim_{I\uparrow\mathcal{S}} |S_I(\boldsymbol{u},d\boldsymbol{v})|$$
$$\le \min\bigl(\sup|\boldsymbol{u}| \times \int_{\mathcal{S}} |d\boldsymbol{v}|,\ \sup|\boldsymbol{v}| \times (\int_{\mathcal{S}} |d\boldsymbol{u}| + 2\sup|\boldsymbol{u}|)\bigr).$$

614T Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{u}$, $\boldsymbol{v}$ fully adapted processes with domain $\mathcal{S}$ such that $\boldsymbol{u}$ is order-bounded, $\boldsymbol{v}$ is of bounded variation and $\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}$ is defined. Then the indefinite integral $ii_{\boldsymbol{v}}(\boldsymbol{u})$ (613O) is of bounded variation, and $\int_{\mathcal{S}} |d(ii_{\boldsymbol{v}}(\boldsymbol{u}))| \le \sup|\boldsymbol{u}| \times \int_{\mathcal{S}} |d\boldsymbol{v}|$.

proof Set $z_\tau = \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\,d\boldsymbol{v}$ for $\tau \in \mathcal{S}$. As noted in 613Oa and 613O(b-i), $ii_{\boldsymbol{v}}(\boldsymbol{u}) = \langle z_\tau\rangle_{\tau\in\mathcal{S}}$ is fully adapted and has domain $\mathcal{S}$. Now suppose that $\tau_0 \le \ldots \le \tau_n$ in $\mathcal{S}$. For $i < n$ write $\mathcal{S}_i$ for $\mathcal{S}\cap[\tau_i,\tau_{i+1}] = (\mathcal{S}\wedge\tau_{i+1})\vee\tau_i$. Then

$$\sum_{i=0}^{n-1} |z_{\tau_{i+1}} - z_{\tau_i}| = \sum_{i=0}^{n-1} |\int_{\mathcal{S}_i} \boldsymbol{u}\,d\boldsymbol{v}| \le \sum_{i=0}^{n-1} \sup|u\restriction\mathcal{S}_i| \times \int_{\mathcal{S}_i} |d\boldsymbol{v}|$$

(614S)

$$\le \sup|\boldsymbol{u}| \times (\int_{\mathcal{S}\wedge\tau_0} |d\boldsymbol{v}| + \sum_{i=0}^{n-1} \int_{\mathcal{S}_i} |d\boldsymbol{v}| + \int_{\mathcal{S}\vee\tau_n} |d\boldsymbol{v}|)$$
$$= \sup|\boldsymbol{u}| \times \int_{\mathcal{S}} |d\boldsymbol{v}|$$

(613J(c-ii)). Thus $ii_{\boldsymbol{v}}(\boldsymbol{u})$ satisfies the condition 614J(ii) and is of bounded variation, with $\int_{\mathcal{S}} |d(ii_{\boldsymbol{v}}(\boldsymbol{u}))| \le \sup|\boldsymbol{u}| \times \int_{\mathcal{S}} |d\boldsymbol{v}|$.

614U Following on from 612H, I give a result on the construction of order-bounded processes and processes of bounded variation.

Proposition Let (Ω,Σ,μ) be a complete probability space, and $\langle\Sigma_t\rangle_{t\ge0}$ a filtration of σ-subalgebras of Σ such that every μ-negligible set belongs to every Σ_t. Let $(\mathfrak{A},\bar\mu,[0,\infty[\,,\langle\mathfrak{A}_t\rangle_{t\ge0},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ be the associated stochastic integration structure as in 612H; for a stopping time $h:\Omega\to[0,\infty]$ let $h^\bullet$ be the corresponding member of $\mathcal{T}$. Suppose that $\langle X_t\rangle_{t\ge0}$ is a progressively measurable process on Ω, with associated fully adapted process $\boldsymbol{x} = \langle x_\tau\rangle_{\tau\in\mathcal{T}_f}$.

(a) If $\{X_s(\omega) : s \ge 0\}$ is bounded for almost every $\omega\in\Omega$, then $\boldsymbol{x}$ is order-bounded.

(b) If $s \mapsto X_s(\omega) : [0,\infty[\, \to \mathbb{R}$ is of bounded variation for almost every $\omega\in\Omega$, then $\boldsymbol{x}$ is of bounded variation.

proof (a)(i) We need the following fact which was left as an exercise in §364. Suppose that $V:\Omega\to[0,\infty[$ is any function. Let F be the set of Σ-measurable functions $f:\Omega\to\mathbb{R}$ such that $f \le V$. Then $A = \{f^\bullet : f \in F\}$ is bounded above in L^0. **P** For $n\in\mathbb{N}$ set $c_n = \sup_{u\in A}[\![u > n]\!]$. $\mathfrak{A}$ is ccc (322G) so there is a countable set $I_n \subseteq A$ such that $c_n = \sup_{u\in I_n}[\![u > n]\!]$ (316E). Let $\langle f_n\rangle_{n\in\mathbb{N}}$ be a sequence in F such that $\bigcup_{n\in\mathbb{N}} I_n = \{f_n^\bullet : n\in\mathbb{N}\}$. Set $f(x) = \sup_{n\in\mathbb{N}} f_n(x)$ for $x \in X$; then $f\in F$ and $f_n^\bullet \le f^\bullet$ for every $n\in\mathbb{N}$. Setting $v = f^\bullet \in L^0$, $u \le v$ for every $u \in \bigcup_{n\in\mathbb{N}} I_n$, so $c_n \subseteq [\![v > n]\!]$ for every n, and $\inf_{n\in\mathbb{N}} c_n = 0$. By 364L(a-ii) again, A is bounded above in L^0. **Q**

(ii) Set $\Omega' = \{\omega : \{X_s(\omega) : s\ge0\}$ is bounded$\}$, and define

$$V(\omega) = \sup_{s\ge0}|X_s(\omega)| \text{ if } \omega\in\Omega',$$
$$= 0 \text{ if } \omega\in\Omega\setminus\Omega'.$$

Let A be the corresponding order-bounded subset of L^0 as defined in (i). If $h:\Omega\to[0,\infty[$ is a stopping time, then X_h is measurable and $|X_h(\omega)|=|X_{h(\omega)}(\omega)|\le V(\omega)$ for every $\omega\in\Omega'$, so $|X_h^\bullet|\in A$. But this means that $|x_\tau|\in A$ for every $\tau\in\mathcal{T}_f$ and $\boldsymbol{x}$ is order-bounded.

(b) Set

$$\Omega''=\{\omega:s\mapsto X_s(\omega):[0,\infty[\,\to\mathbb{R}\text{ is of bounded variation}\},$$

so that Ω'' is conegligible. For $\omega\in\Omega$ let $V(\omega)\in[0,\infty[$ be the total variation of $s\mapsto X_s(\omega):[0,\infty[\,\to\mathbb{R}$ (224A) if $\omega\in\Omega''$, and zero otherwise. As in (a-i), let F be the set of Σ-measurable functions $f:\Omega\to\mathbb{R}$ such that $f\le V$, and $A=\{f^\bullet:f\in F\}$; then A is bounded above in L^0.

Suppose that $\tau_0\le\ldots\le\tau_n$ in $\mathcal{T}_f$. Then there are stopping times $h_0,\ldots,h_n:\Omega\to[0,\infty[$ such that $h_i^\bullet=\tau_i$ for $i\le n$ (612H(a-ii)), and $h_i\le_{\text{a.e.}}h_{i+1}$ for $i<n$; replacing each h_i by $\sup_{j\le i}h_j\times\chi\Omega''$ if necessary, we can arrange that $h_0\le\ldots\le h_n$ and $h_i(\omega)=0$ whenever $i\le n$ and $\omega\in\Omega\setminus\Omega''$. In this case, $\sum_{i=0}^{n-1}|X_{h_{i+1}(\omega)}(\omega)-X_{h_i(\omega)}(\omega)|\le V(\omega)$ for each $\omega\in\Omega$. But this means that $\sum_{i=0}^{n-1}|x_{\tau_{i+1}}-x_{\tau_i}|\in A$. Thus

$$\{\textstyle\sum_{i=0}^{n-1}|x_{\tau_{i+1}}-x_{\tau_i}|:\tau_0\le\ldots\le\tau_n\text{ in }\mathcal{T}_f\}\subseteq A$$

is bounded above in $L^0(\mathfrak{A})$ and $\boldsymbol{x}$ is of bounded variation.

614X Basic exercises (a) Suppose that $\mathfrak{A}=\{0,1\}$. Let S be a non-empty subset of T, $f:S\to\mathbb{R}$ a function and $\boldsymbol{u}=\{(\check{t},f(t)\chi1):t\in S\}$ the corresponding process with domain $\mathcal{S}=\{\check{t}:t\in S\}$, as in 613W.

(i) Show that $\boldsymbol{u}$ is a simple process with breakpoint string $(\check{t}_0,\ldots,\check{t}_n)$ iff f is constant on each of the intervals

$$S\cap\,]-\infty,t_0[,\quad S\cap[t_i,t_{i+1}[\text{ for }i<n,\quad S\cap[t_n,\infty[.$$

(ii) Now suppose that $g:S\to\mathbb{R}$ is another function such that $g_\downarrow=\lim_{s\downarrow S}g(s)$ and $g_\uparrow=\lim_{s\uparrow S}g(s)$ are defined. Let $\boldsymbol{v}$ be the process corresponding to g. Show that

$$\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}=\bigl(f_\downarrow(g(t_0)-g_\downarrow)+\textstyle\sum_{i=0}^{n-1}f(t_i)(g(t_{i+1})-g(t_i))+f(t_n)(g_\uparrow-g(t_n))\bigr)\chi1$$

(b) Suppose that $\mathcal{S}$ is a non-empty sublattice of $\mathcal{T}$, $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a simple fully adapted process with a breakpoint string $(\tau_0,\ldots,\tau_n)$, and $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a fully adapted process such that $v_\downarrow=\lim_{\sigma\downarrow\mathcal{S}}v_\sigma$ is defined. Show that $\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\,d\boldsymbol{v}$ is defined and equal to

$$u_\downarrow\times(v_{\tau_0\wedge\tau}-v_\downarrow)+\textstyle\sum_{i=0}^{n-1}u_{\tau_i}\times(v_{\tau_{i+1}\wedge\tau}-v_{\tau_i\wedge\tau})+u_{\tau_n}\times(v_\tau-v_{\tau_n\wedge\tau})$$

for every $\tau\in\mathcal{S}$.

(c) In 614Fb, show that if $\boldsymbol{u}$ is simple then $\boldsymbol{v}$ is simple.

(d) Suppose that $\mathfrak{A}=\{0,1\}$, as in 613W and 614Xa, and that T is not empty. Let $f:T\to\mathbb{R}$ be a function. Show that f represents an order-bounded process iff it is bounded, and a process of bounded variation iff it is of bounded variation.

(e) Suppose that $T=[0,\infty[$. Show that the identity process (612F) and the standard Poisson process (612U) are locally of bounded variation, and that Brownian motion (612T) is locally order-bounded, but that none of these is order-bounded.

>(f) Suppose that $\mathcal{S}$ is a sublattice of $\mathcal{T}$, $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a fully adapted process, and $z\in L^0(\mathfrak{A}\cap\bigcap_{\sigma\in\mathcal{S}}\mathfrak{A}_\sigma)$. (i) Show that if $\boldsymbol{u}$ is order-bounded then $z\boldsymbol{u}$ is order-bounded and $\sup|z\boldsymbol{u}|=|z|\times\sup|\boldsymbol{u}|$. (ii) Show that if $\boldsymbol{u}$ is of bounded variation then $z\boldsymbol{u}$ is of bounded variation and $\int_{\mathcal{S}}|d(z\boldsymbol{u})|=|z|\times\int_{\mathcal{S}}|d\boldsymbol{u}|$.

(g) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u}$ a simple process with domain $\mathcal{S}$. (i) Show that $\boldsymbol{u}$ is of bounded variation. (ii) Show that the cumulative variation of $\boldsymbol{u}$ is simple.

(h) In 614Ub, show that if we set $f_\omega(s)=X_s(\omega)$ for $s\ge0$ and $\omega\in\Omega$, then $(t,\omega)\mapsto\operatorname{Var}_{[0,t]}f_\omega$ represents the cumulative variation of $\boldsymbol{x}$.

614Y Further exercises (a) Show that previsibly simple processes (612Ye) are of bounded variation.

(b) Show that Brownian motion is not locally of bounded variation. (*Hint*: for $n \ge 1$, $i < n$ set $t_{in} = i/n$; for $i < n$ set $z_{ni} = |w_{\check{t}_{i+1,n}} - w_{\check{t}_{in}}|$, $z_n = \sum_{i=0}^{n-1} z_{ni}$. Show that z_{ni} has expectation $\frac{2}{\sqrt{2n\pi}}$ and variance $\frac{1}{n}(1-\frac{2}{\pi})$, and hence, using Lindeberg's theorem (274F), that $\lim_{n\to\infty}\bar{\mu}[\![z_n \ge \frac{2n}{\sqrt{2n\pi}}]\!] = \frac{1}{2}$.)

(c) In the construction described in 613P, show that $\boldsymbol{u}$ and $\boldsymbol{v}$ are order-bounded but not of bounded variation.

614 Notes and comments This volume is supposed to be about stochastic integration. However a very large part of the work we need to do will concern the structure of various types of stochastic process. Necessarily I started with fully adapted processes (§612). These were sufficient for the definition of the Riemann-sum integral in §613. Now we have three further classes. Integration of, and with respect to, simple processes really is simple (614B-614D). From 614E to 614R, 'integration' hardly appears except in the formula $\int_{\mathcal{S}} |d\boldsymbol{v}|$, which is really just an abbreviation for

$$\sup\{\sum_{i=0}^{n-1} |v_{\tau_{i+1}} - v_{\tau_i}| : \tau_0 \le \ldots \le \tau_n \text{ in } \mathcal{S}\}$$

(614J). Again, in §615, to follow, the bulk of the section will be devoted to properties of the space of 'moderately oscillatory' processes. Only in §616 will integration again move to front centre stage. But as we go through the foundations of the theory, I will drip-feed results about integrals involving the special classes of stochastic process being examined, as in 614S-614T.

The point of 614Ba is that in the formula of 612J defining 'simple process' there is a canonical choice of the starting value u_* definable from the process $\boldsymbol{u}$. I delayed the result to this point because it speaks of 'limits', and it seems convenient to treat this as a limit for the topology of convergence in measure; but if you look at the proof you will see that the limit required is really an order-limit, and could have been described in the context of an arbitrary Dedekind complete Boolean algebra $\mathfrak{A}$, as in §612.

I hope that you will recognise 614J(i)⇔(ii) as an elaborate form of the classical result 224D concerning real functions of bounded variation. The difference now is that I have set up a definition of an integral $\int |d\boldsymbol{v}|$. We shall find that there are many reasons why Brownian motion is not locally of bounded variation. But the fact is of such importance that in 614Yb I suggest a method of proof based on ideas already covered in Volume 2.

It will transpire that often an indefinite integral inherits properties of the integrator. So far we have two cases: an indefinite integral with respect to a simple process is simple (614D) and an indefinite integral of an order-bounded process with respect to a process of bounded variation is again of bounded variation (614T). In both cases the hypothesis has to include a clause '$\int \boldsymbol{u}\,d\boldsymbol{v}$ is defined' because we are far away from any clear idea of which processes $\boldsymbol{u}$ can be expected to appear in applications.

In 614R you will see that a key step is an equality

$$\sum_{i=0}^{n-1} u_{\tau_i} \times (v_{\tau_{i+1}} - v_{\tau_i}) = -\sum_{i=0}^{n-1} v_{\tau_{i+1}} \times (u_{\tau_{i+1}} - u_{\tau_i}) - v_{\tau_0} \times u_{\tau_0} + v_{\tau_n} \times u_{\tau_n},$$

which is a manipulation which I expect you have seen before. But this is *not* a relationship between $S_I(\boldsymbol{u}, d\boldsymbol{v})$ and $S_I(\boldsymbol{v}, d\boldsymbol{u})$, because the latter would have to look at $\sum_{i=0}^{n-1} v_{\tau_i} \times (u_{\tau_{i+1}} - u_{\tau_i})$; the Riemann-sum integral of §613 insists on taking tags at the left-hand ends of intervals rather than the right-hand ends. The difference is of the form $\sum_{i=0}^{n-1}(v_{\tau_{i+1}} - v_{\tau_i}) \times (u_{\tau_{i+1}} - u_{\tau_i})$, which will appear in §617 associated with the 'covariation' of $\boldsymbol{u}$ and $\boldsymbol{v}$, and need not vanish in the limit.

This volume is devoted to the thesis that stochastic processes, at least in respect to stochastic integration, are best regarded as abstract fully adapted processes in the sense of §612. But most of the ideas of the theory were of course developed in the context of classical processes $\langle X_t\rangle_{t\ge 0}$ of real-valued functions on probability spaces. I have therefore given a couple of results (612H, 614U and 614Xa(i)) showing that the types of process I have introduced here correspond, in some sense, to natural conditions on sample paths $t \mapsto X_t(\omega)$, as in 614Xd. You will see that all the non-trivial results are one-way; given a property of typical sample paths, we can deduce a property of the corresponding abstract process. There are theorems in the other direction (e.g., 633R below), but they need different techniques, starting with the Stone representation of measure algebras (321J).

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615 Moderately oscillatory processes

I come now to the class of integrands in the basic theorem 616K, the 'moderately oscillatory' processes. I have chosen a path which starts with a natural linear space topology on the space of order-bounded processes, the ucp topology (615B). This gives a straightforward definition of the space of moderately oscillatory processes (615E) with their elementary properties (615F-615H). When the domain is finitely full, we have an alternative definition in terms of convergence along monotonic sequences of stopping times (615I-615N). Classical stochastic processes with càdlàg sample paths give rise to locally moderately oscillatory processes (615P).

615A Notation $(\mathfrak{A},\bar{\mu},T,\langle\mathfrak{A}_t\rangle_{t\in T},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ will be a stochastic integration structure, as defined in the notes to §613. For $A\subseteq\mathcal{T}$ and $\tau\in\mathcal{T}$, $A\wedge\tau=\{\sigma\wedge\tau:\sigma\in A\}$ and $A\vee\tau=\{\sigma\vee\tau:\sigma\in A\}$. For $w\in L^0=L^0(\mathfrak{A})$, $\theta(w)=\mathbb{E}(|w|\wedge\chi 1)$ as in 613Ba; limits in L^0 will be taken with respect to the topology of convergence in measure. If $\mathcal{S}$ is a sublattice of $\mathcal{T}$, $M_{\mathrm{fa}}(\mathcal{S})$ is the space of fully adapted processes with domain $\mathcal{S}$ (612I), $M_{\text{o-b}}(\mathcal{S})\subseteq M_{\mathrm{fa}}(\mathcal{S})$ is the space of order-bounded processes (614F) and $M_{\mathrm{bv}}(\mathcal{S})\subseteq M_{\text{o-b}}(\mathcal{S})$ is the space of processes of bounded variation (614Q). For an order-bounded process $=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$, $\sup|\boldsymbol{u}|=\sup_{\sigma\in\mathcal{S}}|u_\sigma|$. $\mathbf{1}$ will be the process with constant value $\chi 1$.

615B The ucp topology Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.

(a) For $\boldsymbol{u}\in M_{\text{o-b}}(\mathcal{S})$, set

$$\widehat{\theta}(\boldsymbol{u})=\theta(\sup|\boldsymbol{u}|).$$

(b) $\widehat{\theta}$ is an F-norm on $M_{\text{o-b}}(\mathcal{S})$. **P** The point is that

$$\sup|\boldsymbol{u}+\boldsymbol{v}|\le\sup|\boldsymbol{u}|+\sup|\boldsymbol{v}|,\quad \sup|\alpha\boldsymbol{u}|=|\alpha|\sup|\boldsymbol{u}|$$

for all $\boldsymbol{u}$, $\boldsymbol{v}\in M_{\text{o-b}}=M_{\text{o-b}}(\mathcal{S})$. Since θ is an F-norm on L^0, and $\theta(u)\le\theta(v)$ whenever $|u|\le|v|$, it follows at once that

$$\widehat{\theta}(\boldsymbol{u}+\boldsymbol{v})\le\widehat{\theta}(\boldsymbol{u})+\widehat{\theta}(\boldsymbol{v}),\quad \widehat{\theta}(\alpha\boldsymbol{u})\le\widehat{\theta}(\boldsymbol{u})$$

whenever $\boldsymbol{u}$, $\boldsymbol{v}\in M_{\text{o-b}}$ and $|\alpha|\le 1$, and that

$$\lim_{\alpha\to 0}\widehat{\theta}(\alpha\boldsymbol{u})=0$$

for every $\boldsymbol{u}\in M_{\text{o-b}}$. Finally, if $\widehat{\theta}(\boldsymbol{u})=0$, then $\theta(u_\sigma)=0$ and $u_\sigma=0$ for every $\sigma\in\mathcal{S}$, so $\boldsymbol{u}=0$. **Q**

(c) Accordingly $\widehat{\theta}$ defines a metrizable linear space topology (2A5B). I will call this the **ucp topology** on $M_{\text{o-b}}(\mathcal{S})$ and the associated uniformity the **ucp uniformity** on $M_{\text{o-b}}(\mathcal{S})$.

Warning! The phrase 'ucp topology' is commonly used to mean something closer to the local ucp topology described in 615Xb.

615C Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and give $M_{\text{o-b}}=M_{\text{o-b}}(\mathcal{S})$ its ucp topology.

(a) If $h:\mathbb{R}\to\mathbb{R}$ is continuous, then $\bar{h}\boldsymbol{u}\in M_{\text{o-b}}$ for every $\boldsymbol{u}\in M_{\text{o-b}}$, and $\boldsymbol{u}\mapsto\bar{h}\boldsymbol{u}:M_{\text{o-b}}\to M_{\text{o-b}}$ is continuous.

(b)(i) $(\boldsymbol{u},\boldsymbol{v})\mapsto\boldsymbol{u}\times\boldsymbol{v}:M_{\text{o-b}}\times M_{\text{o-b}}\to M_{\text{o-b}}$ is continuous.

(ii) $\boldsymbol{u}\mapsto\sup|\boldsymbol{u}|:M_{\text{o-b}}\to L^0$ is uniformly continuous.

(c) $M_{\text{o-b}}$ is complete as linear topological space.

proof (a) If $\mathcal{S}=\emptyset$ then $M_{\text{o-b}}=\{\emptyset\}$ and the result is trivial. So let us suppose that $\mathcal{S}$ is non-empty.

(i) For $x\in\mathbb{R}$, set $g(x)=\sup\{|h(y)|:|y|\le|x|\}$; then g is continuous, and $|\bar{h}(v)|\le g(w)$ whenever v, $w\in L^0$ and $|v|\le w$.

Suppose that $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ belongs to $M_{\text{o-b}}$. Set $\bar u = \sup_{\sigma\in\mathcal{S}}|u_\sigma|$. Then $|\bar h(u_\sigma)| \le g(\bar u)$ for every $\sigma\in\mathcal{S}$, so $\{\bar h(u_\sigma) : \sigma\in\mathcal{S}\}$ is order-bounded in L^0, and $\bar h\boldsymbol{u}\in M_{\text{o-b}}$.

(ii) Now take $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}} \in M_{\text{o-b}}$ and $\epsilon > 0$. Set $\bar v = \sup_{\sigma\in\mathcal{S}}|v_\sigma|$, and let $M \ge 0$ be such that $\bar\mu[\![\bar v > M]\!] \le \epsilon$. Let $\delta\in\,]0,1]$ be such that $|h(x) - h(y)| \le \epsilon$ whenever $y\in[-M-1, M+1]$ and $|x-y|\le\delta$. Then for any w, $w'\in L^0$,

$$[\![|\bar h(w) - \bar h(w')| > \epsilon]\!] \subseteq [\![|w| > M]\!] \cup [\![|w - w'| > \delta]\!].$$

Take any $\boldsymbol{u}\in M_{\text{o-b}}$ such that $\widehat\theta(\boldsymbol{v}-\boldsymbol{u}) \le \delta\epsilon$. Set $\bar u = \sup_{\sigma\in\mathcal{S}}|v_\sigma - u_\sigma|$ and $\bar w = \sup_{\sigma\in\mathcal{S}}|\bar h(v_\sigma) - \bar h(u_\sigma)|$. Then $\bar\mu[\![\bar u > \delta]\!] \le \epsilon$, so

$$\begin{aligned}[\![\bar w > \epsilon]\!] &= \sup_{\sigma\in\mathcal{S}}[\![|\bar h(v_\sigma) - \bar h(u_\sigma)| > \epsilon]\!]\\ &\subseteq \sup_{\sigma\in\mathcal{S}}[\![|v_\sigma| > M]\!] \cup \sup_{\sigma\in\mathcal{S}}[\![|v_\sigma - u_\sigma| > \delta]\!] = [\![\bar v > M]\!] \cup [\![\bar u > \delta]\!]\end{aligned}$$

has measure at most 2ϵ, and

$$\widehat\theta(\bar h\boldsymbol{u} - \bar h\boldsymbol{v}) = \theta(\bar w) \le 3\epsilon.$$

As $\boldsymbol{v}$ and ϵ are arbitrary, $\bar h : M_{\text{o-b}} \to M_{\text{o-b}}$ is continuous.

(b)(i) As the ucp topology is a linear space topology, addition and subtraction are certainly continuous. By (a), the operation $\boldsymbol{u}\mapsto\boldsymbol{u}^2$ is continuous. But this means that

$$(\boldsymbol{u},\boldsymbol{v}) \mapsto \frac{1}{2}((\boldsymbol{u}+\boldsymbol{v})^2 - \boldsymbol{u}^2 - \boldsymbol{v}^2) = \boldsymbol{u}\times\boldsymbol{v}$$

is continuous.

(ii) If $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$, $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ belong to $M_{\text{o-b}}$, then

$$|u_\sigma| \le |v_\sigma| + |u_\sigma - v_\sigma| \le \sup|\boldsymbol{v}| + \sup|\boldsymbol{u}-\boldsymbol{v}| = \sup|\boldsymbol{v}| + \sup|\boldsymbol{v}-\boldsymbol{u}|$$

for every $\sigma\in\mathcal{S}$, so $\sup|\boldsymbol{u}| \le \sup|\boldsymbol{v}| + \sup|\boldsymbol{u}-\boldsymbol{v}|$; similarly, $\sup|\boldsymbol{v}| \le \sup|\boldsymbol{u}| + \sup|\boldsymbol{u}-\boldsymbol{v}|$ and $|\sup|\boldsymbol{v}| - \sup|\boldsymbol{u}|| \le \sup|\boldsymbol{u}-\boldsymbol{v}|$. So

$$\theta(\sup|\boldsymbol{u}| - \sup|\boldsymbol{v}|) \le \theta(\sup|\boldsymbol{u}-\boldsymbol{v}|) = \widehat\theta(\boldsymbol{u}-\boldsymbol{v}).$$

As θ, $\widehat\theta$ are F-norms defining the linear space topologies of L^0 and $M_{\text{o-b}}$, $\boldsymbol{u}\mapsto\sup|\boldsymbol{u}|$ is uniformly continuous.

(c) Again, if $\mathcal{S} = \emptyset$ this is trivial, so suppose otherwise. Let $\mathcal{F}$ be a filter on $M_{\text{o-b}}$ which is Cauchy for the ucp topology.

(i) For each $\sigma\in\mathcal{S}$, define $T_\sigma : M_{\text{o-b}} \to L^0$ by saying that $T_\sigma\boldsymbol{u} = u_\sigma$ whenever $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}} \in M_{\text{o-b}}$; then T_σ is linear, and it is also continuous, because $\theta(T_\sigma\boldsymbol{u}) \le \widehat\theta(\boldsymbol{u})$ for every $\boldsymbol{u}\in M_{\text{o-b}}$. T_σ is therefore uniformly continuous for the uniformities associated with the linear space topologies here (3A4Cf[4]). Accordingly the image filter $T_\sigma[[\mathcal{F}]]$ (2A1Ib) is Cauchy (4A2Ji), and has a limit w_σ say in L^0, because L^0 is complete as linear topological space (613Bh).

(ii) $\boldsymbol{w} = \langle w_\sigma\rangle_{\sigma\in\mathcal{S}}$ is fully adapted. **P** Suppose that σ, $\tau\in\mathcal{S}$ and $a = [\![\sigma=\tau]\!]$. Then $T_\sigma\boldsymbol{u}\times\chi a = T_\tau\boldsymbol{u}\times\chi a$ for every $\boldsymbol{u}\in M_{\text{o-b}}$. As $u\mapsto u\times\chi a : L^0\to L^0$ is continuous,

$$\begin{aligned}w_\sigma\times\chi a &= (\lim_{\boldsymbol{u}\to\mathcal{F}} T_\sigma\boldsymbol{u})\times\chi a = \lim_{\boldsymbol{u}\to\mathcal{F}}(T_\sigma\boldsymbol{u}\times\chi a)\\ &= \lim_{\boldsymbol{u}\to\mathcal{F}} T_\tau\boldsymbol{u}\times\chi a = w_\tau\times\chi a,\end{aligned}$$

and $a\subseteq[\![w_\sigma = w_\tau]\!]$. As σ and τ are arbitrary, $\boldsymbol{w}$ is fully adapted. **Q**

(iii) $\boldsymbol{w}$ is order-bounded. **P** For $n\in\mathbb{N}$ set $c_n = \sup_{\sigma\in\mathcal{S}}[\![|w_\sigma| > n]\!]$. Given $\epsilon > 0$, there is a set $A\subseteq\mathcal{F}$ such that $\widehat\theta(\boldsymbol{u}-\boldsymbol{v}) \le \epsilon$ for all $\boldsymbol{u}$, $\boldsymbol{v}\in A$. Fix $\boldsymbol{v}\in A$, and set $\bar v = \sup_{\sigma\in\mathcal{S}}|T_\sigma\boldsymbol{v}|$; let $n\in\mathbb{N}$ be such that $\bar\mu[\![\bar v \ge n]\!] \le \epsilon$. For $\boldsymbol{u}\in M_{\text{o-b}}$, set $\bar v_{\boldsymbol{u}} = \sup_{\sigma\in\mathcal{S}}|T_\sigma\boldsymbol{u} - T_\sigma\boldsymbol{v}|$, so that

[4] Later editions only.

$$\theta(\bar v_{\boldsymbol{u}}) = \widehat{\theta}(\boldsymbol{u} - \boldsymbol{v}) \le \epsilon$$

for $\boldsymbol{u} \in A$.

Let I be any finite subset of $\mathcal{S}$. Then

$$\sup_{\sigma\in I} |w_\sigma| = \lim_{\boldsymbol{u}\to\mathcal{F}} \sup_{\sigma\in I} |T_\sigma(\boldsymbol{u})|$$

because the lattice operations on L^0 are continuous (613Ba), so $\langle y_\sigma\rangle_{\sigma\in I} \mapsto \sup_{\sigma\in I} |y_\sigma|$ is a continuous function from $(L^0)^I$ to L^0. There is therefore a $\boldsymbol{u} \in A$ such that $\theta(\sup_{\sigma\in I} |w_\sigma| - \sup_{\sigma\in I} |T_\sigma \boldsymbol{u}|) \le \epsilon$ and

$$\begin{aligned}
\bar\mu[\![\sup_{\sigma\in I} |w_\sigma| \ge n+2]\!] &\le \bar\mu[\![\sup_{\sigma\in I} |w_\sigma| - \sup_{\sigma\in I} |T_\sigma \boldsymbol{u}| \ge 1]\!] \\
&\qquad + \bar\mu[\![\sup_{\sigma\in I} |T_\sigma \boldsymbol{u} - T_\sigma \boldsymbol{v}| \ge 1]\!] + \bar\mu[\![\sup_{\sigma\in I} |T_\sigma \boldsymbol{v}| \ge n]\!] \\
&\le \theta(\sup_{\sigma\in I} |w_\sigma| - \sup_{\sigma\in I} |T_\sigma \boldsymbol{u}|) + \bar\mu[\![\bar v_{\boldsymbol{u}} \ge 1]\!] + \bar\mu[\![\bar v \ge n]\!] \\
&\le \epsilon + \theta(\bar v_{\boldsymbol{u}}) + \epsilon = 2\epsilon + \widehat{\theta}(\boldsymbol{u} - \boldsymbol{v}) \le 3\epsilon.
\end{aligned}$$

As I is arbitrary, $\bar\mu c_{n+2} \le 3\epsilon$. As ϵ is arbitrary, $\inf_{n\in\mathbb{N}} c_n = 0$ and $\{w_\sigma : \sigma \in \mathcal{S}\}$ is order-bounded. **Q**

(iv) Thus $\boldsymbol{w} \in M_{\text{o-b}}$. Now, given $\epsilon > 0$, again take $A \in \mathcal{F}$ such that $\widehat{\theta}(\boldsymbol{u} - \boldsymbol{v}) \le \epsilon$ for all $\boldsymbol{u}$, $\boldsymbol{v} \in A$. Then $\widehat{\theta}(\boldsymbol{w} - \boldsymbol{v}) \le \epsilon$ for all $\boldsymbol{v} \in A$. **P** For any finite $I \subseteq \mathcal{S}$,

$$\sup_{\sigma\in I} |w_\sigma - T_\sigma \boldsymbol{v}| = \lim_{\boldsymbol{u}\to\mathcal{F}} \sup_{\sigma\in I} |T_\sigma \boldsymbol{u} - T_\sigma \boldsymbol{v}|$$

and

$$\begin{aligned}
\theta(\sup_{\sigma\in I} |w_\sigma - T_\sigma \boldsymbol{v}|) &= \lim_{\boldsymbol{u}\to\mathcal{F}} \theta(\sup_{\sigma\in I} |T_\sigma \boldsymbol{u} - T_\sigma \boldsymbol{v}|) \\
&\le \sup_{\boldsymbol{u}\in A} \widehat{\theta}(\boldsymbol{u} - \boldsymbol{v}) \le \epsilon.
\end{aligned}$$

So

$$\widehat{\theta}(\boldsymbol{w} - \boldsymbol{v}) = \theta(\sup_{\sigma\in\mathcal{S}} |w_\sigma - T_\sigma \boldsymbol{v}|) = \sup_{I\in[\mathcal{S}]^{<\omega}} \theta(\sup_{\sigma\in I} |w_\sigma - T_\sigma \boldsymbol{v}|)$$

(613Ba)

$$\le \epsilon. \quad \mathbf{Q}$$

As ϵ is arbitrary, $\boldsymbol{w} = \lim \mathcal{F}$ for the ucp topology. As $\mathcal{F}$ is arbitrary, $M_{\text{o-b}}$ is complete.

615D When we have a finitely full lattice, there is an alternative approach to the ucp topology.

Lemma Let $\mathcal{S}$ be a non-empty finitely full sublattice of $\mathcal{T}$, and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a fully adapted process.

(a) If $\gamma > 0$ then

$$\bar\mu(\sup_{\tau\in\mathcal{S}} [\![|u_\tau| > \gamma]\!]) = \sup_{\tau\in\mathcal{S}} \bar\mu[\![|u_\tau| > \gamma]\!].$$

(b) If $\boldsymbol{u}$ is order-bounded, $\theta(\sup |\boldsymbol{u}|) \le 2\sqrt{\sup_{\sigma\in\mathcal{S}} \theta(u_\sigma)}$.

proof (a) If $\gamma > 0$ then

$$\bar\mu[\![\sup |\boldsymbol{u}| > \gamma]\!] = \sup_{\tau\in\mathcal{S}} \bar\mu[\![|u_\tau| > \gamma]\!].$$

P We have

$$\bar\mu[\![\sup |\boldsymbol{u}| > \gamma]\!] = \bar\mu(\sup_{\sigma\in\mathcal{S}} [\![|u_\sigma| > \gamma]\!])$$

(364L(a-ii))

$$= \sup_{I\in[\mathcal{S}]^{<\omega}} \bar\mu(\sup_{\sigma\in I} [\![|u_\sigma| > \gamma]\!]) = \sup_{I\in\mathcal{I}(\mathcal{S})} \bar\mu(\sup_{\sigma\in I} [\![|u_\sigma| > \gamma]\!]).$$

Take any $\alpha < \bar{\mu}[\![\sup|\boldsymbol{u}| > \gamma]\!]$. Let $I \in \mathcal{I}(\mathcal{S})$ be such that $\bar{\mu}(\sup_{\sigma\in I}[\![|u_\sigma| > \gamma]\!]) > \alpha$; as $\mathcal{S}$ is non-empty, we can arrange that I should be non-empty. Let $(\tau_0,\dots,\tau_n)$ be a maximal totally ordered subset of I. Then

$$[\![|u_\sigma| > \gamma]\!] \subseteq \sup_{i\le n}[\![\sigma = \tau_i]\!] \cap [\![|u_{\tau_i}| > \gamma]\!] \subseteq \sup_{i\le n}[\![|u_{\tau_i}| > \gamma]\!]$$

for every $\sigma \in I$ (611Ke), so $\alpha < \bar{\mu}(\sup_{i\le n}[\![|u_{\tau_i}| > \gamma]\!])$.

For $i \le n$, set $a_i = [\![|u_{\tau_i}| > \gamma]\!]$ and $b_i = a_i \setminus \sup_{j<i} a_j$; set $b' = 1 \setminus \sup_{i\le n} a_i$. Then a_i and b_i belong to $\mathfrak{A}_{\tau_i}$ for every $i \le n$, and $b' \in \mathfrak{A}_{\tau_n}$. So there is a $\tau \in \mathcal{T}$ such that

$$b_i \subseteq [\![\tau = \tau_i]\!] \text{ for } i \le n, \quad b' \subseteq [\![\tau = \tau_n]\!]$$

(611I), and $\tau \in \mathcal{S}$ because $\mathcal{S}$ is finitely full. Now

$$\begin{aligned}[\![|u_\tau| > \gamma]\!] &\supseteq \sup_{i\le n} b_i \cap [\![|u_{\tau_i}| > \gamma]\!] = \sup_{i\le n} b_i = \sup_{i\le n} a_i \\ &= \sup_{i\le n}[\![|u_{\tau_i}| > \gamma]\!] = \sup_{\sigma\in I}[\![|u_\sigma| > \gamma]\!]\end{aligned}$$

and $\alpha < \bar{\mu}[\![|u_\tau| > \gamma]\!]$. As α is arbitrary, this gives the result. **Q**

(b) Write α for $\sup_{\tau\in\mathcal{S}}\theta(|u_\tau|)$. If $\gamma < \frac{1}{2}\theta(\sup|\boldsymbol{u}|)$ we must have $\bar{\mu}[\![\sup|\boldsymbol{u}| > \gamma]\!] > \gamma$, so there is a $\tau \in \mathcal{S}$ such that $\bar{\mu}[\![|u_\tau| > \gamma]\!] > \gamma$, $\gamma^2 < \theta(|u_\tau|) \le \alpha$ and $\gamma \le \sqrt{\alpha}$. As γ is arbitrary, $\theta(\sup|\boldsymbol{u}|) \le 2\sqrt{\alpha}$, as claimed.

Remark What this means is that the F-norm

$$\boldsymbol{u} \mapsto \sup_{\sigma\in\mathcal{S}}\theta(u_\sigma) : M_{\text{o-b}}(\mathcal{S}) \to [0,1]$$

also defines the ucp topology on $M_{\text{o-b}}(\mathcal{S})$, since of course $\sup_{\sigma\in\mathcal{S}}\theta(u_\sigma) \le \widehat{\theta}(\boldsymbol{u})$. For a general sublattice $\mathcal{S}$, we could use the isomorphism $\boldsymbol{u} \mapsto \hat{\boldsymbol{u}} : M_{\text{o-b}}(\mathcal{S}) \to M_{\text{o-b}}(\hat{\mathcal{S}})$ and the F-norm

$$\boldsymbol{u} \mapsto \sup_{\sigma\in\hat{\mathcal{S}}}\theta(\hat{u}_\sigma) : M_{\text{o-b}}(\mathcal{S}) \to [0,1],$$

since

$$\widehat{\theta}(\hat{\boldsymbol{u}}) = \theta(\sup|\hat{\boldsymbol{u}}|) = \theta(\sup|\boldsymbol{u}|) = \widehat{\theta}(\boldsymbol{u})$$

for every $\boldsymbol{u} \in M_{\text{o-b}}(\mathcal{S})$.

615E Definition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.

(a) I will call a process with domain $\mathcal{S}$ **moderately oscillatory** if it is in the closure of $M_{\text{bv}}(\mathcal{S})$ in $M_{\text{o-b}}(\mathcal{S})$ for the ucp topology.

(b) A process $\boldsymbol{u}$ with domain $\mathcal{S}$ is **locally moderately oscillatory** if $\boldsymbol{u}{\restriction}\mathcal{S}\wedge\tau$ is moderately oscillatory for every $\tau \in \mathcal{S}$.

Remark The definitions imply directly that (locally) moderately oscillatory processes are (locally) order-bounded. Of course processes of bounded variation (e.g., simple processes, 614Q(a-iii), and in particular constant processes) are moderately oscillatory, and processes which are locally of bounded variation are locally moderately oscillatory.

615F Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\hat{\mathcal{S}}$ its covered envelope.

(a) Write $M_{\text{mo}} = M_{\text{mo}}(\mathcal{S})$ for the set of moderately oscillatory processes with domain $\mathcal{S}$.

(i) If $\mathcal{S}'$ is a sublattice of $\mathcal{S}$ then $\boldsymbol{u}{\restriction}\mathcal{S}'$ is moderately oscillatory for every $\boldsymbol{u} \in M_{\text{mo}}$.

(ii) If $h : \mathbb{R} \to \mathbb{R}$ is continuous, then $\bar{h}\boldsymbol{u} \in M_{\text{mo}}$ for every $\boldsymbol{u} \in M_{\text{mo}}$.

(iii) M_{mo} is an f-subalgebra of $M_{\text{o-b}} = M_{\text{o-b}}(\mathcal{S})$.

(iv) M_{mo} is closed in $M_{\text{o-b}}(\mathcal{S})$ for the ucp topology, so is complete for the ucp uniformity.

(v) If $\tau \in \mathcal{S}$, then a fully adapted process $\boldsymbol{u}$ with domain $\mathcal{S}$ is moderately oscillatory iff $\boldsymbol{u}{\restriction}\mathcal{S}\wedge\tau$ and $\boldsymbol{u}{\restriction}\mathcal{S}\vee\tau$ are both moderately oscillatory.

(vi) If $\boldsymbol{u} \in M_{\text{mo}}$, then its fully adapted extension to $\hat{\mathcal{S}}$ is moderately oscillatory.

(b) Write $M_{\text{lmo}} = M_{\text{lmo}}(\mathcal{S})$ for the set of locally moderately oscillatory processes with domain $\mathcal{S}$.

(i) $M_{\text{mo}} \subseteq M_{\text{lmo}}$.

(ii) If $h : \mathbb{R} \to \mathbb{R}$ is continuous, then $\bar{h}\boldsymbol{u} \in M_{\text{lmo}}$ for every $\boldsymbol{u} \in M_{\text{lmo}}$.

(iii) M_{lmo} is an f-subalgebra of the space $M_{\text{lob}} = M_{\text{lob}}(\mathcal{S})$ of locally order-bounded processes with domain $\mathcal{S}$.

(iv) If $\tau \in \mathcal{S}$, then a fully adapted process $\boldsymbol{u}$ with domain $\mathcal{S}$ is locally moderately oscillatory iff $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ and $\boldsymbol{u}\restriction\mathcal{S}\vee\tau$ are both locally moderately oscillatory.

(v) If $\boldsymbol{u} \in M_{\text{lmo}}$, then its fully adapted extension $\hat{\boldsymbol{u}}$ to $\hat{\mathcal{S}}$ is locally moderately oscillatory.

proof (a)(i) For any $\epsilon > 0$ there is a process $\boldsymbol{v} \in M_{\text{bv}}(\mathcal{S})$ such that $\theta(\sup|\boldsymbol{u}-\boldsymbol{v}|) \le \epsilon$. Now $\boldsymbol{u}\restriction\mathcal{S}'$ is order-bounded (614F(a-i)), $\boldsymbol{v}\restriction\mathcal{S}\wedge\tau$ is of bounded variation (614Lb) and $\sup|\boldsymbol{u}\restriction\mathcal{S}'-\boldsymbol{v}\restriction\mathcal{S}'| \le \sup|(\boldsymbol{u}-\boldsymbol{v})\restriction\mathcal{S}'|$, so

$$\theta(\sup|\boldsymbol{u}\restriction\mathcal{S}'-\boldsymbol{v}\restriction\mathcal{S}'|) \le \theta(\sup|(\boldsymbol{u}-\boldsymbol{v})\restriction\mathcal{S}'|) \le \epsilon.$$

As ϵ is arbitrary, $\boldsymbol{u}\restriction\mathcal{S}' \in M_{\text{mo}}(\mathcal{S}')$.

(ii)($\boldsymbol{\alpha}$) The point is that if $M \ge 0$ and $\epsilon > 0$, there is a Lipschitz function $g : \mathbb{R} \to \mathbb{R}$ such that $|h(x)-g(x)| \le \epsilon$ whenever $|x| \le M$. **P** $h\restriction[-M,M]$ is continuous, therefore uniformly continuous, and there is an $m \ge 1$ such that $|h(x)-h(y)| \le \frac{1}{2}\epsilon$ whenever $-M \le x \le y \le M$ and $y \le x+\frac{M}{m}$. Set $x_k = Mk/m$ for $-m \le k \le m$, and define g by saying that

$$\begin{aligned} g(x) &= h(-M) \text{ if } x \le -M,\\ &= h(M) \text{ if } x \ge M,\\ &= \frac{m}{M}((x-x_k)h(x_{k+1})+(x_{k+1}-x)h(x_k))\\ &\qquad\qquad\qquad \text{if } -m \le k \le m-1 \text{ and } x_k \le x \le x_{k+1}. \end{aligned}$$

Then g is Lipschitz and

$$\begin{aligned} |h(x)-g(x)| &\le |h(x)-h(x_k)|+|g(x)-h(x_k)|\\ &\le |h(x)-h(x_k)|+|h(x_{k+1})-h(x_k)| \le \epsilon \end{aligned}$$

whenever $-m \le k \le m-1$ and $x_k \le x \le x_{k+1}$. So $|h(x)-g(x)| \le \epsilon$ whenever $|x| \le M$. **Q**

($\boldsymbol{\beta}$) Now h is Borel measurable and bounded on bounded subsets of $\mathbb{R}$, so $\bar{h}\boldsymbol{u}$ is order-bounded (614F(c-i)). Take $\epsilon > 0$. Set $\eta = \min(1, \frac{\epsilon}{4})$. Then there is an $M \ge 0$ such that $\bar{\mu}[\![\sup|\boldsymbol{u}| > M]\!] \le \eta$. By ($\alpha$), there is a Lipschitz function $g : \mathbb{R} \to \mathbb{R}$ such that $|h(x)-g(x)| \le \eta$ whenever $|x| \le M$. Expressing $\boldsymbol{u}$ as $\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ and setting $a = [\![\sup|\boldsymbol{u}| \le M]\!]$, $a \subseteq [\![|\bar{h}(u_\sigma)-\bar{g}(u_\sigma)| \le \eta\chi 1]\!]$ for every $\sigma \in \mathcal{S}$, so $a \subseteq [\![\sup|\bar{h}\boldsymbol{u}-\bar{g}\boldsymbol{u}| \le \eta\chi 1]\!]$. But this means that

$$\theta(\sup|\bar{h}\boldsymbol{u}-\bar{g}\boldsymbol{u}|) \le \eta+\bar{\mu}(1\setminus a) \le 2\eta,$$

Let $\gamma \ge 1$ be such that g is γ-Lipschitz. Because $\boldsymbol{u}$ is moderately oscillatory, there is a process $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ such that $\theta(\sup|\boldsymbol{u}-\boldsymbol{v}|) \le \eta^2/\gamma$, so that $\bar{\mu}[\![\sup|\boldsymbol{u}-\boldsymbol{v}| \ge \frac{\eta}{\gamma}]\!] \le \eta$. Setting $b = [\![\sup|\boldsymbol{u}-\boldsymbol{v}| \le \frac{\eta}{\gamma}]\!]$, we have $|\bar{g}(u_\sigma)-g(v_\sigma)| \le \gamma|u_\sigma-v_\sigma|$ for every $\sigma \in \mathcal{S}$, so $\sup|\bar{g}\boldsymbol{u}-\bar{g}\boldsymbol{v}| \le \gamma\sup|\boldsymbol{u}-\boldsymbol{v}|$, $b \subseteq [\![\sup|\bar{g}\boldsymbol{u}-\bar{g}\boldsymbol{v}| \le \eta]\!]$ and

$$\theta(\sup|\bar{g}\boldsymbol{u}-\bar{g}\boldsymbol{v}|) \le \eta+\bar{\mu}(1\setminus b) \le 2\eta, \quad \theta(\sup|\bar{h}\boldsymbol{u}-\bar{g}\boldsymbol{v}|) \le 4\eta \le \epsilon.$$

And we know from 614Q(a-i) that $\bar{g}\boldsymbol{v}$ is of bounded variation. As ϵ is arbitrary, $\bar{h}\boldsymbol{u}$ is moderately oscillatory,

(iii) As $M_{\text{bv}} = M_{\text{bv}}(\mathcal{S})$ is an f-subalgebra of $M_{\text{o-b}}$ (614Q(a-ii)) and the operations $(\boldsymbol{u},\boldsymbol{v}) \to \boldsymbol{u}+\boldsymbol{v}$, $(\alpha,\boldsymbol{u}) \mapsto \alpha\boldsymbol{u}$, $(\boldsymbol{u},\boldsymbol{v}) \mapsto \boldsymbol{u}\times\boldsymbol{v}$ and $\boldsymbol{u} \mapsto |\boldsymbol{u}|$ are continuous for the ucp topology on $M_{\text{o-b}}$ (615Ca, 615Cb), $M_{\text{mo}} = \overline{M_{\text{bv}}}$ must be closed under these operations, that is, is an f-subalgebra of $M_{\text{o-b}}$.

(iv) M_{mo} is defined as the closure of $M_{\text{bv}}(\mathcal{S})$, so is surely closed. Being a closed subspace of the complete linear topological space $M_{\text{o-b}}$ (615Cc), M_{mo} is complete (3A4Fd[5]).

(v) Write $\mathcal{S}'$ for $\mathcal{S}\wedge\tau$ and $\mathcal{S}''$ for $\mathcal{S}\vee\tau$. If $\boldsymbol{u}$ is moderately oscillatory, so are $\boldsymbol{u}' = \boldsymbol{u}\restriction\mathcal{S}'$ and $\boldsymbol{u}'' = \boldsymbol{u}\restriction\mathcal{S}''$, by (i) above. If $\boldsymbol{u}'$ and $\boldsymbol{u}''$ are moderately oscillatory, take any $\epsilon > 0$. Then there are processes $\boldsymbol{v}' = \langle v'_\sigma\rangle_{\sigma\in\mathcal{S}'}$ and $\boldsymbol{v}'' = \langle v''_\sigma\rangle_{\sigma\in\mathcal{S}''}$, both of bounded variation, such that $\theta(\sup|\boldsymbol{v}'-\boldsymbol{u}'|)$ and $\theta(\sup|\boldsymbol{v}''-\boldsymbol{u}''|)$ are both at most $\frac{\epsilon}{3}$. Define $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ by saying that

[5] Later editions only.

$$v_\sigma = v'_{\sigma\wedge\tau} \times \chi[\![\sigma \le \tau]\!] + (v''_{\sigma\vee\tau} - v''_\tau + v'_\tau) \times \chi[\![\tau < \sigma]\!]$$

for $\sigma \in \mathcal{S}$. Using 611E, 611Hc and 612C, it is easy to check that $\boldsymbol{v}$ is fully adapted, while $\boldsymbol{v}\restriction\mathcal{S}' = \boldsymbol{v}'$ and $\boldsymbol{v}\restriction\mathcal{S}'' = \boldsymbol{v}'' + (v_\tau - v''_\tau)\mathbf{1}$ are both of bounded variation, so $\boldsymbol{v} \in M_{\mathrm{bv}}$ (614Lc). Now

$$\begin{aligned}|u_\sigma - v_\sigma| &= |u_\sigma - v'_\sigma| \le \sup|\boldsymbol{v}' - \boldsymbol{u}'| \text{ if } \sigma \in \mathcal{S}\wedge\tau,\\ &\le |u_\sigma - v''_\sigma| + |v''_\tau - u_\tau| + |u_\tau - v'_\tau| \le \sup|\boldsymbol{v}' - \boldsymbol{u}'| + 2\sup|\boldsymbol{v}'' - \boldsymbol{u}''| \text{ if } \sigma \in \mathcal{S}'',\end{aligned}$$

and

$$\sup_{\sigma\in\mathcal{S}'\cup\mathcal{S}''}|u_\sigma - v_\sigma| \le \sup|\boldsymbol{v}' - \boldsymbol{u}'| + 2\sup|\boldsymbol{v}'' - \boldsymbol{u}''|.$$

As $\mathcal{S}'\cup\mathcal{S}''$ covers $\mathcal{S}$ (611M(b-iv)),

$$\theta(\sup|\boldsymbol{u} - \boldsymbol{v}|) = \theta(\sup_{\sigma\in\mathcal{S}'\cup\mathcal{S}''}|u_\sigma - v_\sigma|)$$

(614Ga)

$$\le \theta(\sup|\boldsymbol{v}' - \boldsymbol{u}'|) + 2\theta(\sup|\boldsymbol{v}'' - \boldsymbol{u}''|) \le \epsilon.$$

As ϵ is arbitrary, $\boldsymbol{u}$ is moderately oscillatory.

(vi) For $\boldsymbol{u} \in M_{\mathrm{fa}}(\mathcal{S})$, let $\hat{\boldsymbol{u}}$ be its fully adapted extension to $\hat{\mathcal{S}}$. Then $\boldsymbol{u} \mapsto \hat{\boldsymbol{u}} : M_{\mathrm{fa}}(\mathcal{S}) \to M_{\mathrm{fa}}(\hat{\mathcal{S}})$ is linear, while $\hat{\boldsymbol{u}} \in M_{\text{o-b}}(\hat{\mathcal{S}})$ and $\sup|\hat{\boldsymbol{u}}| = \sup|\boldsymbol{u}|$ for every $\boldsymbol{u} \in M_{\text{o-b}}$ (614Ga); so $\boldsymbol{u} \mapsto \hat{\boldsymbol{u}} : M_{\text{o-b}} \to M_{\text{o-b}}(\hat{\mathcal{S}})$ is continuous for the ucp topologies. Also $\hat{\boldsymbol{u}} \in M_{\mathrm{bv}}(\hat{\mathcal{S}})$ for every $\boldsymbol{u} \in M_{\mathrm{bv}}$ (614Q(a-iv)), so if $\boldsymbol{u} \in M_{\mathrm{mo}} = \overline{M_{\mathrm{bv}}}$ then $\hat{\boldsymbol{u}} \in \overline{M_{\mathrm{bv}}(\hat{\mathcal{S}})} = M_{\mathrm{mo}}(\hat{\mathcal{S}})$.

(b)(i) Immediate from (a-i).

(ii) follows from (a-ii) because $(\bar{h}\boldsymbol{u})\restriction\mathcal{S}\wedge\tau = \bar{h}(\boldsymbol{u}\restriction\mathcal{S}\wedge\tau)$.

(iii) Similarly, restriction respects the algebraic and lattice operations on $M_{\mathrm{mo}}(\mathcal{S})$, so we czn use (a-iii).

(iv)($\boldsymbol{\alpha}$) If $\boldsymbol{u}$ is locally moderately oscillatory, then $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ is moderately oscillatory, therefore locally moderately oscillatory. Also, if $\tau' \in \mathcal{S}\vee\tau$,

$$(\boldsymbol{u}\restriction\mathcal{S}\vee\tau)\restriction(\mathcal{S}\vee\tau)\wedge\tau' = (\boldsymbol{u}\restriction\mathcal{S}\wedge\tau')\restriction(\mathcal{S}\wedge\tau')\vee\tau$$

is moderately oscillatory, so $\boldsymbol{u}\restriction(\mathcal{S}\vee\tau)$ is locally moderately oscillatory.

($\boldsymbol{\beta}$) Suppose that $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ and $\boldsymbol{u}\restriction\mathcal{S}\vee\tau$ are locally moderately oscillatory. Take any $\tau' \in \mathcal{S}\vee\tau$. Then $(\boldsymbol{u}\restriction\mathcal{S}\wedge\tau')\restriction(\mathcal{S}\wedge\tau')\wedge\tau = (\boldsymbol{u}\restriction\mathcal{S}\wedge\tau)\restriction(\mathcal{S}\wedge\tau)\wedge\tau'$ and $(\boldsymbol{u}\restriction\mathcal{S}\vee\tau')\restriction(\mathcal{S}\wedge\tau')\vee\tau = (\boldsymbol{u}\restriction\mathcal{S}\vee\tau)\restriction(\mathcal{S}\vee\tau)\wedge\tau'$ are moderately oscillatory, so $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau'$ is moderately oscillatory.

In general, if τ' is an arbitrary member of $\mathcal{S}$,

$$\boldsymbol{u}\restriction\mathcal{S}\wedge\tau' = (\boldsymbol{u}\restriction\mathcal{S}\wedge(\tau'\vee\tau))\restriction(\mathcal{S}\wedge(\tau'\vee\tau))\wedge\tau'$$

is moderately oscillatory, so $\boldsymbol{u}$ is locally moderately oscillatory.

(v) Take any $\tau^* \in \hat{\mathcal{S}}$ and $\epsilon \in \,]0,1]$. Then $\hat{\boldsymbol{u}}\restriction\hat{\mathcal{S}}\wedge\tau^*$ is order-bounded (614Gb) and there is a finite $I \subseteq \mathcal{S}$ such that $\bar{\mu}(\sup_{\sigma\in I}[\![\tau^* = \sigma]\!]) \ge 1 - \frac{1}{2}\epsilon$; setting $\sigma^* = \sup I$, $\bar{\mu}[\![\sigma^* < \tau^*]\!] \le \frac{1}{2}\epsilon$. As $\hat{\mathcal{S}}\wedge\sigma^*$ is the covered envelope of $\mathcal{S}\wedge\sigma^*$ (611M(e-i)), (a-vi) tells us that $\hat{\boldsymbol{u}}\restriction\hat{\mathcal{S}}\wedge\sigma^*$ is moderately oscillatory, and there is a process $\boldsymbol{v} = \langle v_\tau\rangle_{\tau\in\hat{\mathcal{S}}\wedge\sigma^*}$ of bounded variation such that $\theta(\bar{u}) \le \frac{1}{2}\epsilon$, where $\bar{u} = \sup|\boldsymbol{v} - \hat{\boldsymbol{u}}\restriction\hat{\mathcal{S}}\wedge\sigma^*|$. Setting $\hat{\boldsymbol{v}} = \langle v_{\tau\wedge\sigma^*}\rangle_{\tau\in\hat{\mathcal{S}}}$, $\hat{\boldsymbol{v}}$ is fully adapted (612Ib), $\hat{\boldsymbol{v}}\restriction\hat{\mathcal{S}}\wedge\sigma^* = \boldsymbol{v}$ is of bounded variation, and $\hat{\boldsymbol{v}}\restriction\hat{\mathcal{S}}\vee\sigma^*$ is constant; so $\hat{\boldsymbol{v}}$ is of bounded variation (614Lc). Now if $\tau \in \hat{\mathcal{S}}\wedge\tau^*$

$$\begin{aligned}[\![\tau^* \le \sigma^*]\!] &\subseteq [\![\tau \le \sigma^*]\!]\\ &\subseteq [\![\hat{v}_\tau = v_\tau]\!] \cap [\![|\hat{u}_\tau - \hat{v}_\tau| \le \bar{u}]\!]\end{aligned}$$

so

$$[\![\tau^* \le \sigma^*]\!] \subseteq [\![\sup|\hat{\boldsymbol{v}} - \hat{\boldsymbol{u}}\restriction\hat{\mathcal{S}}\wedge\tau^*| \le \bar{u}]\!]$$

and

$$\theta(\sup|\hat{\boldsymbol{v}}-\hat{\boldsymbol{u}}\restriction\hat{\mathcal{S}}\wedge\tau^*|)\le\theta(\bar u)+\bar\mu(1\setminus[\![\tau*\le\sigma^*]\!]\le\epsilon.$$

As ϵ is arbitrary, $\hat{\boldsymbol{u}}\restriction\hat{\mathcal{S}}\wedge\tau^*$ is moderately oscillatory; as τ^* is arbitrary, $\hat{\boldsymbol{u}}$ is locally moderately oscillatory.

615G Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a fully adapted process.

(a) Suppose that $\boldsymbol{u}$ is moderately oscillatory and $A\subseteq\mathcal{S}$ is non-empty and upwards-directed. Then $w=\lim_{\sigma\uparrow A}u_\sigma$ is defined. Setting $A_*=\{\rho:\rho\in\mathcal{S},\ \sup_{\sigma\in A}[\![\rho\le\sigma]\!]=1\}$,

$$\lim_{\sigma\uparrow A}\sup_{\rho\in A_*\vee\sigma}|u_\rho-w|=0.$$

(b) Suppose that $\boldsymbol{u}$ is locally moderately oscillatory and $A\subseteq\mathcal{S}$ is non-empty and downwards-directed. Then $w=\lim_{\sigma\downarrow A}u_\sigma$ is defined. Setting $A^*=\{\rho:\rho\in\mathcal{S},\ \sup_{\sigma\in A}[\![\sigma\le\rho]\!]=1\}$,

$$\lim_{\sigma\downarrow A}\sup_{\rho\in A^*\wedge\sigma}|u_\rho-w|=0.$$

proof (a)(i) To begin with, consider the case in which $\boldsymbol{u}$ is non-negative, non-decreasing and order-bounded. In this case, $B=\{u_\sigma:\sigma\in A\}\subseteq(L^0)^+$ is upwards-directed and bounded above, so has a supremum w in L^0, and

$$w=\lim_{x\uparrow B}x=\lim_{\sigma\uparrow A}u_\sigma$$

(613Ba). If $\sigma\in A$ and $\rho\in A_*\vee\sigma$, $u_\sigma\le u_\rho$ so $w-u_\rho\le w-u_\sigma$ and

$$1=\sup_{\sigma'\in A}[\![\rho\le\sigma']\!]=\sup_{\sigma\le\sigma'\in A}[\![\rho\le\sigma']\!]$$

(because A is upwards-directed)

$$\begin{aligned}&=\sup_{\sigma\le\sigma'\in A}[\![\rho=\sigma'\wedge\rho]\!]\subseteq\sup_{\sigma\le\sigma'\in A}[\![u_\rho=u_{\sigma'\wedge\rho}]\!]\\&\subseteq\sup_{\sigma\le\sigma'\in A}[\![u_\rho\le u_{\sigma'}]\!]\subseteq[\![u_\rho\le w]\!]\subseteq[\![|w-u_\rho|\le w-u_\sigma]\!]\end{aligned}$$

and $\sup_{\rho\in A_*\vee\sigma}|u_\rho-w|\le w-u_\sigma$; as $\sigma\in A_*\vee\sigma$, we have equality. So $\lim_{\sigma\uparrow A}\sup_{\rho\in A_*\vee\sigma}=\lim_{\sigma\in A}w-u_\sigma=0$.

(ii) Now suppose that $\boldsymbol{u}$ is of bounded variation. Then it can be expressed as $\boldsymbol{u}'-\boldsymbol{u}''$ where $\boldsymbol{u}'=\langle u'_\sigma\rangle_{\sigma\in\mathcal{S}}$ and $\boldsymbol{u}''=\langle u''_\sigma\rangle_{\sigma\in\mathcal{S}}$ are non-negative, non-decreasing and order-bounded. Using (i), we see that $w'=\lim_{\sigma\uparrow A}u'_\sigma$ and $w''=\lim_{\sigma\uparrow A}u''_\sigma$ are defined, so that $w=w'-w''$ is $\lim_{\sigma\uparrow A}u_\sigma$ and

$$\sup_{\rho\in A_*\vee\sigma}|u_\rho-w|\le\sup_{\rho\in A_*\vee\sigma}(|u'_\rho-w'|+|u''_\rho-w''|),$$

$$\theta(\sup_{\rho\in A_*\vee\sigma}|u_\rho-w|)\le\theta(\sup_{\rho\in A_*\vee\sigma}|u'_\rho-w'|)+\theta(\sup_{\rho\in A_*\vee\sigma}|u''_\rho-w''|)\to0$$

as $\sigma\uparrow A$.

(iii) Now suppose just that $\boldsymbol{u}$ is moderately oscillatory.

($\boldsymbol{\alpha}$) $w=\lim_{\sigma\uparrow A}u_\sigma$ is defined, **P** Given $\epsilon>0$, there is a process $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ of bounded variation such that $\theta(\bar u)\le\frac{1}{3}\epsilon$ where $\bar u=\sup|\boldsymbol{u}-\boldsymbol{v}|$. There is a $\sigma\in A$ such that $\theta(v_\tau-v_\sigma)\le\frac{1}{3}\epsilon$ whenever $\sigma\le\tau\in A$; now $\theta(u_\tau-u_\sigma)\le\epsilon$ whenever $\sigma\le\tau\in A$, As L^0 is complete, $\lim_{\sigma\uparrow A}u_\sigma$ is defined. **Q**

($\boldsymbol{\beta}$) Now take any $\epsilon>0$ and a process $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ of bounded variation such that $\theta(\bar u)\le\frac{1}{2}\epsilon$ where $\bar u=\sup|\boldsymbol{u}-\boldsymbol{v}|$. We know from (ii) that $w'=\lim_{\sigma\uparrow A}v_\sigma$ is defined and that $\lim_{\sigma\in A}\sup_{\rho\in A_*\vee\sigma}|v_\rho-w'|=0$. Now

$$|w-w'|=\lim_{\sigma\uparrow A}|u_\sigma-v_\sigma|\le\sup|\boldsymbol{u}-\boldsymbol{v}|,$$

and if $\sigma\in A$ then

$$\begin{aligned}\sup_{\rho\in A_*\vee\sigma}|u_\rho-w|&\le\sup_{\rho\in A_*\vee\sigma}|u_\rho-v_\rho|+|v_\rho-w'|+|w'-w|\\&\le2\sup|\boldsymbol{u}-\boldsymbol{v}|+\sup_{\rho\in A_*\vee\sigma}|v_\rho-w'|\end{aligned}$$

so

$$\limsup_{\sigma\uparrow A}\theta(\sup_{\rho\in A_*\vee\sigma}|u_\rho-w|)\le 2\theta(\sup|\boldsymbol{u}-\boldsymbol{v}|)+\limsup_{\sigma\uparrow A}\theta(\sup_{\rho\in A_*\vee\sigma}|v_\rho-w'|)$$
$$\le\epsilon.$$

As ϵ is arbitrary, $\lim_{\sigma\uparrow A}\theta(\sup_{\rho\in A_*\vee\sigma}|u_\rho-w|)=0$.

(b)(i) If $\boldsymbol{u}$ is actually moderately oscillatory, we can follow the same argument, inverted at the right points; for instance, the key sentence in (a-i) becomes

If $\sigma\in A$ and $\rho\in A^*\wedge\sigma$, $u_\rho\le u_\sigma$ so $u_\rho-w\le u_\sigma-w$ and

$$\begin{aligned}1&=\sup_{\sigma'\in A}[\![\sigma'\le\rho]\!]=\sup_{\sigma'\in A,\sigma'\le\sigma}[\![\sigma'\le\rho]\!]\\&=\sup_{\sigma'\in A,\sigma'\le\sigma}[\![\rho=\sigma'\vee\rho]\!]\subseteq\sup_{\sigma'\in A,\sigma'\le\sigma}[\![u_\rho=u_{\sigma'\vee\rho}]\!]\\&\subseteq\sup_{\sigma'\in A,\sigma'\le\sigma}[\![u_{\sigma'}\le u_\rho]\!]\subseteq[\![w\le u_\rho]\!]\subseteq[\![|u_\rho-w|\le u_\sigma-w]\!]\end{aligned}$$

and $\sup_{\rho\in A^*\wedge\sigma}|u_\rho-w|\le u_\sigma-w$; as $\sigma\in A^*\wedge\sigma$, we have equality.

(ii) For the general case, in which $\boldsymbol{u}$ is just locally moderately oscillatory, we can take any $\tau\in A$ and apply (i) here to $\boldsymbol{u}\restriction\mathcal{A}\wedge\tau$ and $\{\sigma:\sigma\in A,\ \sigma\le\tau\}$.

615H Corollary Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$, $\hat{\mathcal{S}}$ its covered envelope, $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a locally moderately oscillatory process, and $\hat{\boldsymbol{u}}=\langle\hat u_\sigma\rangle_{\sigma\in\hat{\mathcal{S}}}$ its fully adapted extension to $\hat{\mathcal{S}}$. Then $\boldsymbol{u}$ and $\hat{\boldsymbol{u}}$ have starting values, which are the same.

proof (a) By 615F(b-v), $\hat{\boldsymbol{u}}$ is locally moderately oscillatory. So 615Gb tells us that

$$w=\lim_{\sigma\downarrow\hat{\mathcal{S}}\wedge\tau}\hat u_\sigma=\lim_{\sigma\downarrow\hat{\mathcal{S}}}\hat u_\sigma$$

is defined and is the starting value of $\hat{\boldsymbol{u}}$.

(b) To see that w is also the starting value of $\boldsymbol{u}$, take any $\epsilon>0$. Then there is a $\tau\in\hat{\mathcal{S}}$ such that $\theta(\hat u_\sigma-w)\le\epsilon$ whenever $\sigma\in\hat{\mathcal{S}}\wedge\tau$. Next, $\bar\mu(\sup_{\sigma\in\mathcal{S}}[\![\tau=\sigma]\!])=1$, so there is a non-empty finite set $I\subseteq\mathcal{S}$ such that $\bar\mu(\sup_{\sigma\in I}[\![\tau=\sigma]\!])\ge1-\epsilon$ and $\bar\mu[\![\tau<\min I]\!]\le\epsilon$. Now if $\sigma\in\mathcal{S}$ and $\sigma\le\min I$, $\theta(\hat u_{\sigma\wedge\tau}-w)\le\epsilon$; but

$$[\![\hat u_{\sigma\wedge\tau}\ne\hat u_\sigma]\!]\subseteq[\![\tau<\sigma]\!]\subseteq[\![\tau<\min I]\!]$$

has measure at most ϵ, so $\theta(\hat u_\sigma-\hat u_{\sigma\wedge\tau})\le\epsilon$ and

$$\theta(u_\sigma-w)=\theta(\hat u_\sigma-w)\le2\epsilon.$$

As ϵ is arbitrary, w is also the starting value of $\boldsymbol{u}$.

615I Definition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a fully adapted process. I will say that $\boldsymbol{u}$ is **⥮-convergent** if

(⥮) $\lim_{n\to\infty}u_{\sigma_n}$ is defined whenever $\langle\sigma_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{S}$ which is either non-increasing or non-decreasing.

615J Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a moderately oscillatory process. Then $\boldsymbol{u}$ is ⥮-convergent.

proof 615G.

615K Lemma Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ and $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ an ⥮-convergent process. Then $\boldsymbol{u}$ is order-bounded.

proof I will work towards the contrapositive, that is, assuming that $\boldsymbol{u}$ is not order-bounded I will show that it is not $⥮$-convergent.

(a) To begin with (down to the end of (d) below) I will take it that $u_\sigma \ge 0$ for every $\sigma \in \mathcal{S}$. A couple of preliminary remarks will be useful.

(i) For any $\alpha \in \mathbb{R}$, $\{[\![u_\sigma > \alpha]\!] : \sigma \in \mathcal{S}\}$ is upwards-directed. **P** (This is where we need to know that $\mathcal{S}$ is finitely full.) Given σ, $\tau \in \mathcal{S}$, set $b = [\![u_{\sigma\wedge\tau} > \alpha]\!]$, take $\rho \in \mathcal{S}$ such that $b \subseteq [\![\rho = \sigma \wedge \tau]\!]$ and $1 \setminus b \subseteq [\![\rho = \sigma \vee \tau]\!]$ (611I again); then

$$\begin{aligned}[\![u_\rho > \alpha]\!] &= b \cup ([\![u_{\sigma\vee\tau} > \alpha]\!] \setminus b) = [\![u_{\sigma\wedge\tau} > \alpha]\!] \cup [\![u_{\sigma\vee\tau} > \alpha]\!] \\ &\supseteq ([\![\sigma \le \tau]\!] \cap [\![u_\sigma > \alpha]\!]) \cup ([\![\tau \le \sigma]\!] \cap [\![u_\sigma > \alpha]\!]) = [\![u_\sigma > \alpha]\!],\end{aligned}$$

and similarly $[\![u_\rho > \alpha]\!] \supseteq [\![u_\tau > \alpha]\!]$. **Q**

(ii) For any $\alpha \in \mathbb{R}$ and $b \in \mathfrak{A}$, $\{\sigma : \sigma \in \mathcal{S}, b \subseteq [\![u_\sigma \le \alpha]\!]\}$ is a sublattice of $\mathcal{S}$. **P** If $\sigma, \tau \in \mathcal{S}$, $b \subseteq [\![u_\sigma \le \alpha]\!]$ and $b \subseteq [\![u_\tau \le \alpha]\!]$, then

$$[\![u_{\sigma\vee\tau} \le \alpha]\!] = ([\![\sigma \le \tau]\!] \cap [\![u_\tau \le \alpha]\!]) \cup ([\![\tau \le \sigma]\!] \cap [\![u_\sigma \le \alpha]\!]) \supseteq b$$

and

$$[\![u_{\sigma\wedge\tau} \le \alpha]\!] = ([\![\sigma \le \tau]\!] \cap [\![u_\sigma \le \alpha]\!]) \cup ([\![\tau \le \sigma]\!] \cap [\![u_\tau \le \alpha]\!]) \supseteq b. \text{ **Q**}$$

(iii) Similarly, $\{\sigma : \sigma \in \mathcal{S}, b \subseteq [\![u_\sigma \ge \alpha]\!]\}$ is a sublattice of $\mathcal{S}$ for any $\alpha \in \mathbb{R}$ and $b \in \mathfrak{A}$.

(b) We are supposing that $\boldsymbol{u}$ is not order-bounded, so $\mathcal{S}$ is surely not empty, and $\{u_\sigma : \sigma \in \mathcal{S}\}$ has no upper bound in $L^0(\mathfrak{A})$. By 364L(a-i), $a = \inf_{n\in\mathbb{N}} \sup_{\sigma\in\mathcal{S}} [\![u_\sigma > n]\!]$ is non-zero. If $n \in \mathbb{N}$ and $\epsilon > 0$, there must be a finite set $K \subseteq \mathcal{S}$ such that $\bar\mu(\sup_{\sigma\in K} [\![u_\sigma > n]\!]) \ge \bar\mu a - \epsilon$; by (a-i) above, there is a single $\sigma \in \mathcal{S}$ such that $\bar\mu[\![u_\sigma > n]\!] \ge \bar\mu a - \epsilon$. We can therefore choose $\langle m_n\rangle_{n\in\mathbb{N}}$, $\langle \sigma_n\rangle_{n\in\mathbb{N}}$ inductively such that

$$m_0 = 0,$$

$$\sigma_n \in \mathcal{S}, \quad \bar\mu[\![u_{\sigma_n} > m_n + 1]\!] \ge (1 - 2^{-n-2})\bar\mu a,$$

$$m_{n+1} > m_n, \quad \textstyle\sum_{i=0}^n \bar\mu([\![u_{\sigma_i} > m_{n+1}]\!]) \le 2^{-n-2}\bar\mu a.$$

Set

$$b = \inf_{n\in\mathbb{N}} [\![u_{\sigma_n} > m_n + 1]\!] \setminus \sup_{i<n\in\mathbb{N}} [\![u_{\sigma_i} > m_n]\!]$$

so that $\bar\mu b \ge \frac{1}{2}\bar\mu a > 0$ and $b \subseteq [\![u_{\sigma_i} \le m_n]\!] \cap [\![u_{\sigma_j} \ge m_n + 1]\!]$ whenever $i < n \le j$. By (a-ii) and (a-iii),

$$b \subseteq [\![u_\sigma \le m_n]\!] \cap [\![u_\tau \ge m_n + 1]\!]$$

whenever σ is in the lattice generated by $\{\sigma_i : i \le n\}$ and τ is in the lattice generated by $\{\sigma_j : j > n\}$.

(c) case 1 Suppose that there are $k \in \mathbb{N}$ and $\epsilon > 0$ such that for every $l > k$

$$\bar\mu(b \cap \sup_{j>l} [\![\sigma_j > \sup_{k<i\le l} \sigma_i]\!]) > \epsilon.$$

Then we can choose inductively a strictly increasing sequence $\langle k_r\rangle_{r\in\mathbb{N}}$ in $\mathbb{N}$ such that $k_0 = k + 1$ and

$$\bar\mu(b \cap \sup_{k_r<j\le k_{r+1}} [\![\sigma_j > \sup_{k<i\le k_r} \sigma_i]\!]) > \epsilon$$

for every r. For each r, set $\tau_r' = \sup_{k_r<i\le k_{r+1}} \sigma_i$ and $\tau_r = \sup_{k<i\le k_r} \sigma_j$ for each i; then we always have $\tau_{r+1} = \tau_r \vee \tau' r$, so

$$[\![\tau_{r+1} = \tau_r']\!] \supseteq [\![\tau_r' > \tau_r]\!] \supseteq \sup_{k_r<j\le k_{r+1}} [\![\sigma_j > \tau_r]\!]$$

and $\bar\mu(b \cap [\![\tau_{r+1} = \tau_r']\!]) \ge \epsilon$.

On the other hand,

$$b \subseteq [\![u_{\tau_r} \le m_{k_r}]\!] \cap [\![u_{\tau_r'} \ge m_{k_r} + 1]\!] \subseteq [\![u_{\tau_r'} - u_{\tau_r} \ge 1]\!].$$

So

$$\theta(u_{\tau_{r+1}} - u_{\tau_r}) \ge \bar\mu[\![u_{\tau_{r+1}} - u_{\tau_r} \ge 1]\!] \ge \bar\mu([\![u_{\tau'_r} - u_{\tau_r} \ge 1]\!] \cap [\![\tau_{r+1} = \tau'_r]\!]$$
$$\ge \bar\mu(b \cap [\![\tau_{r+1} = \tau'_r]\!]) \ge \epsilon$$

for every r, and $\langle u_{\tau_r}\rangle_{r\in\mathbb{N}}$ is not convergent, while $\langle \tau_r\rangle_{r\in\mathbb{N}}$ is a non-decreasing sequence in $\mathcal{S}$. So in this case $\boldsymbol{u}$ is not moderately oscillatory.

(d) case 2 Suppose that for every $k \in \mathbb{N}$ and $\epsilon > 0$ there is an $l > k$ such that

$$\bar\mu(b \cap \sup_{j>l} [\![\sigma_j > \sup_{k<i\le l} \sigma_i]\!]) \le \epsilon.$$

Then we can choose a strictly increasing sequence $\langle k_r\rangle_{r\in\mathbb{N}}$ such that

$$\bar\mu(b \cap \sup_{j>k_{r+1}} [\![\sigma_j > \sup_{k_r<i\le k_{r+1}} \sigma_i]\!]) \le 2^{-r-1}\bar\mu b$$

for each r. Set $\tau'_r = \sup_{k_r<i\le k_{r+1}} \sigma_i$ for each r; then $\bar\mu(b \cap [\![\tau'_{r+1} > \tau'_r]\!]) \le 2^{-r-1}\bar\mu b$ for each r, and $c = b \setminus \sup_{r\in\mathbb{N}} [\![\tau'_{r+1} > \tau'_r]\!]$ is non-zero. Now $c \subseteq [\![\tau'_{r+1} \le \tau'_r]\!]$ for every r, so if we set $\tau_r = \inf_{i\le r} \tau'_i$, we shall have $c \subseteq [\![\tau_r = \tau'_r]\!]$ for every r. **P** Induce on r; for the inductive step to $r+1$ we have

$$c \cap [\![\tau_{r+1} = \tau'_{r+1}]\!] = c \cap [\![\tau_r \wedge \tau'_{r+1} = \tau'_{r+1}]\!] = c \cap [\![\tau'_{r+1} \le \tau_r]\!] = c \cap [\![\tau'_{r+1} \le \tau'_r]\!]$$

(by the inductive hypothesis)

$$= c. \text{ } \mathbf{Q}$$

On the other hand,

$$b \subseteq [\![u_{\tau'_r} \ge m_{k_r+1}]\!] \cap [\![u_{\tau'_r} \le m_{k_{r+1}}]\!]$$

for each r, so

$$c \subseteq [\![u_{\tau'_r} \le m_{k_{r+1}}]\!] \cap [\![u_{\tau'_{r+1}} \ge m_{k_{r+1}+1}]\!] \subseteq [\![u_{\tau'_{r+1}} - u_{\tau'_r} \ge 1]\!];$$

consequently $c \subseteq [\![u_{\tau_{r+1}-u_{\tau_r}} \ge 1]\!]$ and $\theta(u_{\tau_{r+1}} - u_{\tau_r}) \ge \bar\mu c$ for every r, while $\langle \tau_r\rangle_{r\in\mathbb{N}}$ is non-increasing. So in this case also $\boldsymbol{u}$ is not moderately oscillatory.

(e) Since case 1 and case 2 are exhaustive, this deals with the case in which $u_\sigma \ge 0$ for every σ. If $\boldsymbol{u}$ is not necessarily positive. then $|\boldsymbol{u}|$ is not order-bounded, so cannot be ⥮-convergent, and there is a monotonic sequence in $\mathcal{S}$ such that $\lim_{n\to\infty} |u_{\sigma_n}|$ is undefined. But $|\;| : L^0(\mathfrak{A}) \to L^0(\mathfrak{A})$ is continuous (613Ba), so $\lim_{n\to\infty} u_{\sigma_n}$ must be undefined, and $\boldsymbol{u}$ also is not ⥮-convergent.

615L Lemma Let $\mathcal{S}$ be a non-empty finitely full sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ an ⥮-convergent process. Suppose that $A \subseteq \mathcal{S}$ is non-empty and downwards-directed. Then $w = \lim_{\sigma\downarrow A} u_\sigma$ is defined. Setting $A^* = \{\rho : \rho \in \mathcal{S}, \sup_{\sigma\in A} [\![\sigma \le \rho]\!] = 1\}$,

$$\lim_{\sigma\downarrow A} \sup_{\rho\in A^*\wedge\sigma} |u_\rho - w| = 0.$$

proof (a) Note first that A^* is finitely full. **P** Suppose that $\tau \in \mathcal{T}$ and that $\{\tau\}$ is covered by a finite subset of A^*. Take any $a \in \mathfrak{A} \setminus \{0\}$. As $\mathcal{S}$ is finitely full, $\tau \in \mathcal{S}$. There are $\rho \in A^*$ and $\sigma \in A$ such that $b = a \cap [\![\tau = \rho]\!]$ and $c = b \cap [\![\rho \le \sigma]\!]$ are non-zero. Now $0 \ne c \subseteq a \cap [\![\tau \le \sigma]\!]$. As a is arbitrary, $\tau \in A^*$; as τ is arbitrary, A^* is finitely full. **Q**

(b) In particular, A^* is closed under $\wedge$. Consequently $w = \lim_{\sigma\downarrow A^*} u_\sigma$ is defined. **P?** Otherwise, there is an $\epsilon > 0$ such that for every $\sigma \in A^*$ there is a $\tau \in A^* \wedge \sigma$ such that $\theta(u_\tau - u_\sigma) \ge \epsilon$. Now we can choose a sequence $\langle \sigma_n\rangle_{n\in\mathbb{N}}$ in A^* such that

$$\sigma_{n+1} \le \sigma_n, \quad \theta(u_{\sigma_{n+1}} - u_{\sigma_n}) \ge \epsilon$$

for every $n \in \mathbb{N}$, and $\langle \sigma_n\rangle_{n\in\mathbb{N}}$ is a monotonic sequence in $\mathcal{S} \subseteq \hat{\mathcal{S}}$ such that $\langle \hat u_{\sigma_n}\rangle_{n\in\mathbb{N}}$ is not convergent. **XQ**

(c) $\lim_{\sigma\downarrow A} \sup_{\rho\in A^*\wedge\sigma} \theta(u_\rho - w) = 0$. **P** Let $\epsilon > 0$. Then there is a $\tau' \in A^*$ such that $\theta(u_\rho - w) \le \epsilon$ whenever $\rho \in A^* \vee \tau'$. Next, $\sup_{\sigma\in A} [\![\sigma \le \tau']\!] = 1$ and A is downwards-directed, so there is a $\sigma_0 \in A$ such that $\bar{\mu}[\![\sigma_0 \le \tau]\!] \ge 1-\epsilon$. If we now take $\sigma \in A$ and $\rho \in A^*$ such that $\rho \le \sigma \le \sigma_0$, we see that $\theta(u_{\tau'\wedge\rho} - w) \le \epsilon$ and

$$\theta(u_\rho - u_{\tau'\wedge\rho}) \le \bar{\mu}[\![\tau' < \rho]\!] \le \bar{\mu}[\![\tau' < \sigma_0]\!] \le \epsilon,$$

so $\theta(u_\rho - w) \le 2\epsilon$. **Q**

Of course it follows that $\lim_{\sigma\downarrow A} \sup_{\rho\in A,\rho\le\sigma} \theta(u_\rho - w) = 0$, and $w = \lim_{\sigma\downarrow A} u_\sigma$.

(d) In fact $\lim_{\sigma\downarrow A} \theta(\sup_{\rho\in A^*\wedge\sigma} |u_\rho - w|) = 0$. **P** Again take $\epsilon > 0$. By (iii), there is a $\sigma \in A$ such that $\theta(u_\rho - w) \le \min(\frac{1}{2}\epsilon, \frac{1}{16}\epsilon^2)$ for every $\rho \in A^* \wedge \sigma$. Set $v_\rho = u_\rho - u_\sigma$ for $\rho \in A^* \wedge \sigma$, so that

$$\boldsymbol{v} = \langle v_\rho\rangle_{\rho\in A^*\wedge\sigma} = (\boldsymbol{u} - u_\sigma\mathbf{1})\restriction A^* \wedge \sigma$$

is a fully adapted process on the finitely full sublattice $A^* \vee \sigma$ and $\theta(v_\rho) \le \frac{1}{16}\epsilon^2$ for every $\rho \in A^* \vee \sigma$. Also $\boldsymbol{u}$ is order-bounded (615K), so $\boldsymbol{v}$ also is. By 615Db, $\theta(\sup|\boldsymbol{v}|) \le \frac{1}{2}\epsilon$. Now

$$\sup_{\rho\in A^*\vee\sigma} |u_\rho - w| \le \sup_{\rho\in A^*\vee\sigma} |u_\rho - u_\sigma| + |u_\sigma - w| = \sup|\boldsymbol{v}| + |u_\sigma - w|,$$

so

$$\theta(\sup_{\rho\in A^*\vee\sigma} |u_\rho - w|) \le \theta(\sup|\boldsymbol{v}|) + \theta(u_\sigma - w) \le \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon,$$

and this is true whenever $\sigma \in A$ and $\sigma \ge \sigma_0$. **Q**

615M Construction Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ with a greatest member, $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ an $\Bbb{1}$-convergent process, and $\delta > 0$. Then there are sequences $\langle D_i\rangle_{i\in\mathbb{N}}$, $\langle y_i\rangle_{i\in\mathbb{N}}$, $\langle d_i\rangle_{i\in\mathbb{N}}$, a family $\langle c_{i\sigma}\rangle_{i\in\mathbb{N},\sigma\in\mathcal{S}}$ and a process $\tilde{\boldsymbol{u}} = \langle\tilde{u}_\sigma\rangle_{\sigma\in\mathcal{S}}$ with the following properties.

(a) $D_0 = \mathcal{S}$; for every $i \in \mathbb{N}$, $\max\mathcal{S} \in D_i \subseteq \mathcal{S}$, D_i is closed under $\wedge$, $y_i = \lim_{\sigma\downarrow D_i} u_\sigma$ and

$$D_{i+1} = \{\sigma : \sigma \in \mathcal{S},\ [\![\sigma < \max\mathcal{S}]\!] \subseteq [\![|u_\sigma - y_i| \ge \delta]\!] \text{ and there is a } \sigma' \in D_i \text{ such that } \sigma' \le \sigma\}.$$

(b) $y_i \in \bigcap_{\sigma\in D_i} L^0(\mathfrak{A}_\sigma)$ for every $i \in \mathbb{N}$.

(c)(i) For every $i \in \mathbb{N}$,

$d_i = \sup_{\sigma\in D_i} [\![\sigma < \max\mathcal{S}]\!]$,
$d_i \in \bigcap_{\sigma\in D_i} \mathfrak{A}_\sigma$,
$d_{i+1} \subseteq d_i$,
$d_{i+1} \subseteq [\![|y_{i+1} - y_i| \ge \delta]\!]$,
$1 \setminus d_i \subseteq [\![y_i = u_{\max\mathcal{S}}]\!] \cap [\![y_i = y_{i+1}]\!]$.

(ii) $\inf_{i\in\mathbb{N}} d_i = 0$.

(d)(i) If $\sigma \in \mathcal{S}$ and $i \in \mathbb{N}$,

$$c_{i\sigma} = \sup_{\tau\in D_i} [\![\tau \le \sigma]\!], \quad c_{i+1,\sigma} \subseteq c_{i\sigma}, \quad [\![\sigma = \max\mathcal{S}]\!] \subseteq c_{i\sigma} \subseteq [\![\sigma = \max\mathcal{S}]\!] \cup d_i.$$

(ii) If $i \in \mathbb{N}$ and $\sigma \in D_i$ then $c_{i\sigma} = 1$; in particular, $c_{0\sigma} = 1$ for every $\sigma \in \mathcal{S}$.

(iii) If σ, σ' in $\mathcal{S}$ then $[\![\sigma \le \sigma']\!] \cap c_{i\sigma} \subseteq c_{i\sigma'}$ for every $i \in \mathbb{N}$.

(iv) $\inf_{i\in\mathbb{N}} c_{i\sigma} = [\![\sigma = \max\mathcal{S}]\!]$ for every $\sigma \in \mathcal{S}$.

(v) If $\sigma \in \mathcal{S}$ and $i \in \mathbb{N}$ then $c_{i\sigma} \setminus c_{i+1,\sigma} \subseteq [\![|u_\sigma - y_i| < \delta]\!]$.

(e) If $\sigma \in \mathcal{S}$ then

$$c_{i\sigma} \setminus c_{i+1,\sigma} \subseteq [\![\tilde{u}_\sigma = y_i]\!]$$

for every $i \in \mathbb{N}$, and $[\![\sigma = \max\mathcal{S}]\!] \subseteq [\![\tilde{u}_\sigma = u_{\max\mathcal{S}}]\!]$.

(f) $\tilde{\boldsymbol{u}}$ is fully adapted, $\sup|\tilde{\boldsymbol{u}}| \le \sup|\boldsymbol{u}|$ and $\sup|\boldsymbol{u} - \tilde{\boldsymbol{u}}| \le \delta\chi 1$.

(g) $\tilde{\boldsymbol{u}}$ is of bounded variation.

proof (a) We start the induction with $D_0 = \mathcal{S}$. The inductive hypothesis we need is just that D_i is non-empty and closed under $\wedge$. Given this, $y_i = \lim_{\sigma\downarrow D_i} u_\sigma$ is defined (615L). Looking at the formula for D_{i+1} we see at once that $\max\mathcal{S} \in D_{i+1}$. If σ, $\tau \in D_{i+1}$, then

$$\begin{aligned}[\![\sigma\wedge\tau<\max\mathcal{S}]\!] &= ([\![\sigma\le\tau]\!]\cap[\![\sigma<\max\mathcal{S}]\!])\cup([\![\tau\le\sigma]\!]\cap[\![\tau<\max\mathcal{S}]\!])\\ &\subseteq ([\![\sigma\le\tau]\!]\cap[\![|u_\sigma-y_i|\ge\delta]\!])\cup([\![\tau\le\sigma]\!]\cap[\![|u_\tau-y_i|\ge\delta]\!])\\ &\subseteq ([\![\sigma\le\tau]\!]\cap[\![|u_{\sigma\wedge\tau}-y_i|\ge\delta]\!])\cup([\![\tau\le\sigma]\!]\cap[\![|u_{\sigma\wedge\tau}-y_i|\ge\delta]\!])\\ &\subseteq [\![|u_{\sigma\wedge\tau}-y_i|\ge\delta]\!];\end{aligned}$$

at the same time, there are σ', $\tau'\in D_i$ such that $\sigma'\le\sigma$ and $\tau'\le\tau$, so that $\sigma'\wedge\tau'\in D_i$ and $\sigma'\wedge\tau'\le\sigma\wedge\tau$. Thus $\sigma\wedge\tau\in D_{i+1}$; as σ and τ are arbitrary, D_{i+1} is closed under $\wedge$ and the induction proceeds.

(b) For every $i\in\mathbb{N}$, 613Bj tells us that $y_i\in L^0(\bigcap_{\sigma\in D_i}\mathfrak{A}_\sigma)$.

(c)(i)($\boldsymbol{\alpha}$) Define d_i as the supremum $\sup_{\sigma\in D_i}[\![\sigma<\max\mathcal{S}]\!]$ for each i. Since D_i is closed under $\wedge$,

$$d_i=\sup_{\tau\in D_i\wedge\sigma}[\![\tau<\max\mathcal{S}]\!]\in\mathfrak{A}_\sigma$$

for every $\sigma\in D_i$, and $d_i\in\bigcap_{\sigma\in D_i}\mathfrak{A}_\sigma$.

($\boldsymbol{\beta}$) For any $i\in\mathbb{N}$ and $\sigma\in D_{i+1}$, there is a $\sigma'\in D_i$ such that $\sigma'\le\sigma$, in which case $[\![\sigma<\max\mathcal{S}]\!]\subseteq[\![\sigma'<\max\mathcal{S}]\!]\subseteq d_i$; taking the supremum over such σ, $d_{i+1}\subseteq d_i$.

($\boldsymbol{\gamma}$) Whenever $\tau\le\sigma$ in D_{i+1},

$$[\![\sigma<\max\mathcal{S}]\!]\subseteq[\![\tau<\max\mathcal{S}]\!]\subseteq[\![|u_\tau-y_i|\ge\delta]\!].$$

So

$$[\![\sigma<\max\mathcal{S}]\!]\subseteq\inf_{\tau\in D_{i+1},\tau\le\sigma}[\![|u_\tau-y_i|\ge\delta]\!]\subseteq[\![|y_{i+1}-y_i|\ge\delta]\!]$$

for every $\sigma\in D_{i+1}$; taking the supremum, $d_{i+1}\subseteq[\![|y_{i+1}-y_i|\ge\delta]\!]$.

($\boldsymbol{\delta}$) For any $\sigma\in D_i$,

$$1\setminus d_i\subseteq[\![\sigma=\max\mathcal{S}]\!]\subseteq[\![u_\sigma=u_{\max\mathcal{S}}]\!],$$

so $1\setminus d_i\subseteq[\![y_i=u_{\max\mathcal{S}}]\!]$. Now we also have

$$1\setminus d_i\subseteq1\setminus d_{i+1}\subseteq[\![y_{i+1}=u_{\max\mathcal{S}}]\!],$$

so $1\setminus d_i\subseteq[\![y_i=y_{i+1}]\!]$.

(ii) For $i\in\mathbb{N}$, write $D_i^*=\bigcup_{\sigma\in D_i}\mathcal{S}\vee\sigma$; now for $\sigma\in D_i$, set

$$a_{i\sigma}=\inf_{\tau\in D_i^*,\tau\le\sigma}[\![|u_\tau-y_i|\le\tfrac13\delta]\!].$$

Of course $a_{i\sigma'}\subseteq a_{i\sigma}$ whenever $\sigma\le\sigma'$ in D_i, because $a_{i\sigma'}$ is the infimum of a larger set. Also $\sup_{\sigma'\in D_i}a_{i\sigma'}=1$. **P** Given $\eta>0$, there is a $\sigma\in D_i$ such that $\theta(\sup_{\tau\in D_i^*\wedge\sigma}|u_\tau-y_i|)\le3\eta\min(\delta,1)$ (615L again). So $b=[\![\sup_{\tau\in D_i^*\wedge\sigma}|u_\tau-y_i|>\frac13\delta]\!]$ has measure at most η. But $[\![|u_\tau-y_i|\le\frac13\delta]\!]\cup b=1$ whenever $\tau\in D_i^*$ and $\tau\le\sigma$, so $a_{i\sigma}\supseteq b$. Thus $\sup_{\sigma'\in D_i}a_{i\sigma'}$ has measure at least $1-\eta$; as η is arbitrary, $\sup_{\sigma'\in D_i}a_{i\sigma'}=1$. **Q**

? If $d=\inf_{i\in\mathbb{N}}d_i$ is non-zero, choose $\sigma_i\in D_i$, for each $i\in\mathbb{N}$, such that

$$\bar\mu(d_i\setminus(a_{i\sigma_i}\cap[\![\sigma_i<\max\mathcal{S}]\!]))\le2^{-i-2}\bar\mu d.$$

(Any element far enough down D_i will serve.) Then

$$a=\inf_{i\in\mathbb{N}}[\![\sigma_i<\max\mathcal{S}]\!]\cap a_{i\sigma_i}$$

has measure at least $\frac12\bar\mu d$ and is non-zero. For each $i\in\mathbb{N}$, $\sigma_i\wedge\sigma_{i+1}\in D_i^*$ (because there is some member of D_i less than or equal to σ_{i+1}) and

$$\begin{aligned}a\cap[\![\sigma_{i+1}\le\sigma_i]\!]\subseteq[\![\sigma_{i+1}<\max\mathcal{S}]\!]&\cap[\![u_{\sigma_{i+1}}=u_{\sigma_i\wedge\sigma_{i+1}}]\!]\\ &\cap[\![|u_{\sigma_i\wedge\sigma_{i+1}}-y_i|\le\tfrac13\delta]\!]\cap[\![|y_{i+1}-y_i|\ge\delta]\!]\end{aligned}$$

(because $a_{i\sigma_i}\subseteq[\![|u_{\sigma_i\wedge\sigma_{i+1}}-y_i|\le\frac13\delta]\!]$ and $[\![\sigma_{i+1}<\max\mathcal{S}]\!]\subseteq[\![|y_{i+1}-y_i|\ge\delta]\!]$)

$$\subseteq[\![|u_{\sigma_{i+1}}-y_i|\le\tfrac13\delta]\!]\cap[\![|u_{\sigma_{i+1}}-y_{i+1}|\le\tfrac13\delta]\!]\cap[\![|y_{i+1}-y_i|\ge\delta]\!]=0.$$

Set $\rho_i = \sup_{j\le i}\sigma_j$ for each $i\in\mathbb{N}$. Then

$$a \subseteq \inf_{j<i}\llbracket\sigma_j < \sigma_{j+1}\rrbracket \subseteq \llbracket\sigma_i = \rho_i\rrbracket \subseteq \llbracket u_{\rho_i} = u_{\sigma_i}\rrbracket$$

for every $i\in\mathbb{N}$. But we also have

$$a \subseteq \llbracket|u_{\sigma_i} - y_i| \le \tfrac{1}{3}\delta\rrbracket \cap \llbracket|y_{i+1} - y_i| \ge \delta\rrbracket$$

for each i, so $a \subseteq \llbracket|u_{\rho_{i+1}} - u_{\rho_i}| \ge \frac{1}{3}\delta\rrbracket$ for each i and $\lim_{i\to\infty} u_{\rho_i}$ is not defined. **X**

Thus $\inf_{i\in\mathbb{N}} d_i = 0$.

(d)(i) As every member of D_{i+1} dominates a member of D_i, $c_{i+1,\sigma} \subseteq c_{i\sigma}$. As $\max\mathcal{S}\in D_i$, $\llbracket\sigma = \max\mathcal{S}\rrbracket \subseteq c_{i\sigma}$. If $\tau\in D_i$,

$$\llbracket\tau\le\sigma\rrbracket \subseteq (\llbracket\tau = \max\mathcal{S}\rrbracket \cap \llbracket\tau\le\sigma\rrbracket) \cup \llbracket\tau < \max\mathcal{S}\rrbracket \subseteq \llbracket\sigma = \max\mathcal{S}\rrbracket \cup d_i;$$

taking the supremum over τ, $c_{i\sigma} \subseteq \llbracket\sigma = \max\mathcal{S}\rrbracket \cup d_i$.

(ii) If $\sigma\in D_i$ then $c_{i\sigma} \supseteq \llbracket\sigma\le\sigma\rrbracket = 1$. As $D_0 = \mathcal{S}$, $c_{0\sigma} = 1$ for every $\sigma\in\mathcal{S}$.

(iii)

$$\llbracket\sigma\le\sigma'\rrbracket \cap c_{i\sigma} = \sup_{\tau\in D_i}\llbracket\sigma\le\sigma'\rrbracket\cap\llbracket\tau\le\sigma\rrbracket \subseteq \sup_{\tau\in D_i}\llbracket\tau\le\sigma'\rrbracket = c_{i\sigma'}.$$

(iv) By (i),

$$\llbracket\sigma = \max\mathcal{S}\rrbracket \subseteq \inf_{i\in\mathbb{N}} c_{i\sigma} \subseteq \llbracket\sigma = \max\mathcal{S}\rrbracket \cup \inf_{i\in\mathbb{N}} d_i = \llbracket\sigma = \max\mathcal{S}\rrbracket$$

by (c-iii).

(v) Suppose that $\tau\in D_i$. Then

$$b = \llbracket|u_\sigma - y_i| \ge \delta\rrbracket \cap \llbracket\tau\le\sigma\rrbracket \cap \llbracket\sigma < \max\mathcal{S}\rrbracket$$

belongs to $\mathfrak{A}_\sigma$, because $y_i\times\chi\llbracket\tau\le\sigma\rrbracket \in \mathfrak{A}_\sigma$, as observed in (i) above. We therefore have a $\tau'\in\mathcal{T}$ such that

$$b \subseteq \llbracket\tau' = \sigma\rrbracket, \quad 1\setminus b \subseteq \llbracket\tau' = \max\mathcal{S}\rrbracket$$

and $\tau'\in\mathcal{S}$ because $\mathcal{S}$ is finitely full. Now

$$b \subseteq \llbracket\tau' = \sigma\rrbracket \cap \llbracket\tau\le\sigma\rrbracket \subseteq \llbracket\tau\le\tau'\rrbracket, \quad b \subseteq \llbracket\tau' = \max\mathcal{S}\rrbracket \cap \llbracket\tau\le\max\mathcal{S}\rrbracket \subseteq \llbracket\tau\le\tau'\rrbracket$$

so $\tau\le\tau'$; and

$$\llbracket\tau' < \max\mathcal{S}\rrbracket \subseteq b \subseteq \llbracket\tau' = \sigma\rrbracket \cap \llbracket|u_\sigma - y_i| \ge \delta\rrbracket \subseteq \llbracket|u_{\tau'} - y_i| \ge \delta\rrbracket$$

so $\tau'\in D_{i+1}$. But this means that

$$c_{i+1,\sigma} \supseteq \llbracket\tau'\le\sigma\rrbracket \supseteq b = \llbracket|u_\sigma - y_i| \ge \delta\rrbracket \cap \llbracket\tau\le\sigma\rrbracket \cap \llbracket\sigma < \max\mathcal{S}\rrbracket.$$

As τ is arbitrary,

$$c_{i+1,\sigma} \supseteq \llbracket|u_\sigma - y_i| \ge \delta\rrbracket \cap c_{i\sigma} \cap \llbracket\sigma < \max\mathcal{S}\rrbracket$$

and

$$\begin{aligned} c_{i\sigma}\setminus c_{i+1,\sigma} &\subseteq c_{i\sigma} \cap \llbracket\sigma < \max\mathcal{S}\rrbracket \setminus c_{i+1,\sigma} \\ &\subseteq \llbracket|u_\sigma - y_i| < \delta\rrbracket \cap \llbracket\tilde u_\sigma = y_i\rrbracket \subseteq \llbracket|u_\sigma - \tilde u_\sigma| < \delta\rrbracket. \end{aligned}$$

(e) Since $\langle c_{i\sigma}\setminus c_{i+1,\sigma}\rangle_{i\in\mathbb{N}}$ is disjoint with supremum $1\setminus\llbracket\sigma = \max\mathcal{S}\rrbracket$ ((d-iv) above), the formula here defines a member $\tilde u_\sigma$ of $L^0(\mathfrak{A})$.

(f)(i) Take $\sigma\in\mathcal{S}$. Then $\llbracket\tau\le\sigma\rrbracket \in \mathfrak{A}_\sigma$ for every $\tau\in\mathcal{S}$, so $c_{i\sigma}\in\mathfrak{A}_\sigma$ for every $i\in\mathbb{N}$; also, of course, $\llbracket\sigma = \max\mathcal{S}\rrbracket \in \mathfrak{A}_\sigma$. Next, $y_i\in L^0(\mathfrak{A}_\tau)$ for every $\tau\in D_i$, so $y_i\times\chi\llbracket\tau\le\sigma\rrbracket \in L^0(\mathfrak{A}_\sigma)$ for every $\tau\in D_i$ (612C) and $y_i\times\chi c_{i\sigma} = \lim_{\tau\downarrow D_i} y_i\times\chi\llbracket\tau\le\sigma\rrbracket$ belongs to $L^0(\mathfrak{A}_\sigma)$ for every $i\in\mathbb{N}$. Finally, $u_{\max\mathcal{S}}\times\chi\llbracket\sigma = \max\mathcal{S}\rrbracket \in L^0(\mathfrak{A}_\sigma)$. So $\tilde u_\sigma\in L^0(\mathfrak{A}_\sigma)$.

(ii) Suppose that σ, $\sigma'\in\mathcal{S}$ and $b = \llbracket\sigma = \sigma'\rrbracket$. Then $b\cap\llbracket\tau\le\sigma\rrbracket = b\cap\llbracket\tau\le\sigma'\rrbracket$ for every $\tau\in\mathcal{S}$ (611E(c-iv-α)) so $b\cap c_{i\sigma} = b\cap c_{i\sigma'}$ for every $i\in\mathbb{N}$ and $b\cap\llbracket\sigma = \max\mathcal{S}\rrbracket = a\cap\llbracket\sigma' = \max\mathcal{S}\rrbracket$. Accordingly

$$b \cap (c_{i\sigma} \setminus c_{i+1,\sigma}) \subseteq [\![\tilde{u}_\sigma = y_i]\!] \cap [\![\tilde{u}_{\sigma'} = y_i]\!] \subseteq [\![\tilde{u}_\sigma = \tilde{u}_{\sigma'}]\!]$$

for every $i \in \mathbb{N}$, and

$$b \cap [\![\sigma = \max \mathcal{S}]\!] \subseteq [\![\tilde{u}_\sigma = u_{\max \mathcal{S}}]\!] \cap [\![\tilde{u}_{\sigma'} = u_{\max \mathcal{S}}]\!] \subseteq [\![\tilde{u}_\sigma = \tilde{u}_{\sigma'}]\!].$$

So $b \subseteq [\![\tilde{u}_\sigma = \tilde{u}_{\sigma'}]\!]$. Thus $\tilde{u}$ is fully adapted.

(iii) From (d-v) we see that

$$c_{i\sigma} \setminus c_{i+1,\sigma} \subseteq [\![|u_\sigma - y_i| < \delta]\!] \cap [\![\tilde{u}_\sigma = y_i]\!] \subseteq [\![|u_\sigma - \tilde{u}_\sigma| \le \delta]\!]$$

for each i. At the top end, we surely have

$$[\![\sigma = \max \mathcal{S}]\!] \subseteq [\![u_\sigma = u_{\max \mathcal{S}}]\!] \cap [\![\tilde{u}_\sigma = u_{\max \mathcal{S}}]\!] \subseteq [\![|u_\sigma - \tilde{u}_\sigma| \le \delta]\!].$$

Once again, these parts assemble into

$$[\![|u_\sigma - \tilde{u}_\sigma| \le \delta]\!] = 1$$

and $|u_\sigma - \tilde{u}_\sigma| \le \delta\chi 1$. As σ is arbitrary, $\sup |\boldsymbol{u} - \tilde{\boldsymbol{u}}| \le \delta\chi 1$.

(iv) Writing $\bar{u}$ for $\sup |\boldsymbol{u}|$, we have

$$|y_i| = |\lim_{\sigma \downarrow D_i} u_\sigma| = \lim_{\sigma \downarrow D_i} |u_\sigma| \le \bar{u}$$

for every $i \in \mathbb{N}$. So if $\sigma \in \mathcal{S}$,

$$[\![|\tilde{u}_\sigma| \le \bar{u}]\!] \supseteq [\![\tilde{u}_\sigma = u_{\max \mathcal{S}}]\!] \cup \sup_{i \in \mathbb{N}} [\![\tilde{u}_\sigma = y_i]\!] \supseteq [\![\sigma = \max \mathcal{S}]\!] \cup \sup_{i \in \mathbb{N}} (c_{i\sigma} \setminus c_{i+1,\sigma})$$

(by (e))

$$\supseteq (\inf_{i \in \mathbb{N}} c_{i\sigma}) \cup (c_{0\sigma} \setminus \inf_{i \in \mathbb{N}} c_{i\sigma})$$

(by (d-iv))

$$= 1$$

by (d-ii). As σ is arbitrary, $\sup |\tilde{\boldsymbol{u}}| \le \bar{u}$.

(g)(i) For $n \in \mathbb{N}$ set $x_n = \sum_{i=0}^{n-1} |y_{i+1} - y_i| + \sup_{i<n} |u_{\max \mathcal{S}} - y_i|$, and write x for $x_0 \times \chi(1 \setminus d_0) \vee \sup_{n \in \mathbb{N}} x_{n+1} \times \chi(d_n \setminus d_{n+1})$; this is defined because $\langle d_n \rangle_{n \in \mathbb{N}}$ is non-increasing (c-i), so $\langle d_n \setminus d_{n+1} \rangle_{n \in \mathbb{N}}$ is disjoint. Take $\sigma_0 \le \ldots \le \sigma_m$ in $\mathcal{S}$ such that $\sigma_m = \max \mathcal{S}$.

(ii) Let $n \in \mathbb{N}$. Then

$$[\![1 \setminus d_n]\!] \subseteq [\![\textstyle\sum_{j=0}^{m-1} |\tilde{u}_{\sigma_{j+1}} - \tilde{u}_{\sigma_j}| \le x_n]\!].$$

P Let $\mathfrak{B}$ be the (finite) subalgebra of $\mathfrak{A}$ generated by

$$\{c_{i\sigma_j} : i \le n,\, j \le m\} \cup \{[\![\sigma_j < \sigma_{j+1}]\!] : j < m\} \cup \{d_n\},$$

and b an atom of $\mathfrak{B}$ disjoint from d_n. For $j < m$ either

$$b \subseteq [\![\sigma_j = \max \mathcal{S}]\!] \subseteq [\![\tilde{u}_{\sigma_j} = \tilde{u}_{\sigma_{j+1}}]\!]$$

or there is just one $i < n$ such that $b \subseteq c_{i\sigma_j} \setminus c_{i+1,\sigma_j}$. So if we set

$$\begin{aligned} K &= \{j : j < m,\, b \cap [\![\tilde{u}_{\sigma_j} \ne \tilde{u}_{\sigma_{j+1}}]\!] \ne 0\} \\ &= \{j : j < m,\, b \cap [\![\sigma_j < \sigma_{j+1}]\!] \ne 0\} = \{j : j < m,\, b \subseteq [\![\sigma_j < \sigma_{j+1}]\!]\} \end{aligned}$$

there is for each $j \in K$ an $i_j < n$ such that

$$b \subseteq c_{i_j\sigma_j} \setminus c_{i_j+1,\sigma_j} \subseteq [\![\tilde{u}_{\sigma_j} = y_{i_j}]\!]$$

and we must have $i_j < i_k$ whenever $j, k \in K$ and $j < k$. For $j \in K$ let j' be the next member of $K \cup \{m\}$ above j; then $i_{j'} > i_j$ for every $j \in K \setminus \{\max K\}$, so

$$\sum_{j=0}^{m-1}|\tilde u_{\sigma_{j+1}}-\tilde u_{\sigma_j}|\times\chi b=\sum_{j\in K}|\tilde u_{\sigma_{j+1}}-\tilde u_{\sigma_j}|\times\chi b=\sum_{j\in K}|\tilde u_{\sigma_{j'}}-\tilde u_{\sigma_j}|\times\chi b$$
$$\le\sum_{j\in K\setminus\{\max K\}}|y_{i_{j'}}-y_{i_j}|+|u_{\max\mathcal{S}}-y_{i_{\max K}}|\le x_n.$$

As b is arbitrary, we have the result. **Q**

(iii) Writing w for $\sum_{j=0}^{m-1}|\tilde u_{j+1}-\tilde u_j|$, we have

$$w=w\times\chi(1\setminus d_0)\vee\sup_{n\in\mathbb{N}}w\times\chi(d_n\setminus d_{n+1})$$

(because $\langle d_n\rangle_{n\in\mathbb{N}}$ is non-increasing and $\inf_{n\in\mathbb{N}}d_n=0$, by 615Mc)

$$\le x_0\times\chi(1\setminus d_0)\vee\sup_{n\in\mathbb{N}}x_{n+1}\times\chi(d_n\setminus d_{n+1})=x.$$

As $\sigma_0,\dots,\sigma_m$ are arbitrary, $\tilde u$ is of bounded variation.

615N Theorem Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$, and $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a fully adapted process. Then the following are equiveridical:

(i) $\boldsymbol{u}$ is moderately oscillatory;

(ii) $\boldsymbol{u}$ is ⥯-convergent;

(iii) $\langle u_{\sigma_n}\rangle_{n\in\mathbb{N}}$ is Cauchy for every monotonic sequence $\langle\sigma_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{S}$;

(iv) for every $\epsilon>0$ there is an $m\ge1$ such that whenever $\sigma_0\le\dots\le\sigma_m$ in $\mathcal{S}$ there is a $j<m$ such that $\theta(u_{\sigma_j}-u_{\sigma_{j+1}})\le\epsilon$.

proof (i)⇒(iv) Suppose that $\boldsymbol{u}$ is moderately oscillatory. Take $\epsilon>0$ and set $\eta=\frac{\epsilon}{4}$. Let $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ be a process of bounded variation such that $\theta(\sup|\boldsymbol{u}-\boldsymbol{v}|)\le\eta$. Write $\bar v$ for $\int_{\mathcal{S}}|d\boldsymbol{v}|$, and let $\gamma\ge0$ be such that $\bar\mu(\bar v>\gamma)\le\eta$. Set $m=\lceil\gamma/\eta\rceil$. If $\sigma_0\le\dots\le\sigma_m$ in $\mathcal{S}$, then $\sum_{i=0}^{m-1}|v_{\sigma_{i+1}}-v_{\sigma_i}|\le\bar v$ (614J). If $b=[\![\bar v\le\gamma]\!]$, $\sum_{i=0}^{m-1}\mathbb{E}(|v_{\sigma_{i+1}}-v_{\sigma_i}|\times\chi b)\le\gamma$ so there is a $j<m$ such that

$$\frac{\gamma}{m}\ge\mathbb{E}(|v_{\sigma_{j+1}}-v_{\sigma_j}|\times\chi b)\ge\theta(|v_{\sigma_{j+1}}-v_{\sigma_j}|\times\chi b).$$

Now

$$\theta(|u_{\sigma_{j+1}}-u_{\sigma_j}|)\le\theta(|v_{\sigma_{j+1}}-v_{\sigma_j}|)+2\theta(\sup|\boldsymbol{u}-\boldsymbol{v}|)$$
$$\le\theta(|v_{\sigma_{j+1}}-v_{\sigma_j}|\times\chi b)+\bar\mu(1\setminus b)+2\eta\le\frac{\gamma}{m}+3\eta\le4\eta=\epsilon.$$

As ϵ is arbitrary, (ii) is true.

(iv)⇒(iii) Suppose that (iv) is true. If $\langle\tau_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{S}$ such that $\langle u_{\tau_n}\rangle_{n\in\mathbb{N}}$ is not Cauchy, there are an $\epsilon>0$ and a strictly increasing sequence $\langle n_k\rangle_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that $\theta(\tau_{n_{k+1}}-\tau_{n_k})>\epsilon$ for every $k\in\mathbb{N}$. By (iv), we have an $m\ge1$ such that whenever $\sigma_0\le\dots\le\sigma_m$ in $\mathcal{S}$ there is a $j<m$ such that $\theta(u_{\sigma_j}-u_{\sigma_{j+1}})\le\epsilon$. So we cannot have either $\tau_{n_0}\le\dots\le\tau_{n_m}$ or $\tau_{n_m}\le\dots\le\tau_{n_0}$, and $\langle\tau_n\rangle_{n\in\mathbb{N}}$ is not monotonic. Turning this round, $\langle u_{\sigma_n}\rangle_{n\in\mathbb{N}}$ is convergent whenever $\langle\sigma_n\rangle_{n\in\mathbb{N}}$ is a monotonic sequence in $\mathcal{S}$, and $\boldsymbol{u}$ is ⥯-convergent.

(iii)⇒(ii) because L^0 is a complete linear topological space.

(ii)⇒(i)(α) To begin with, suppose that $\mathcal{S}$ has a greatest member. Then 615M tells us that for every $\delta>0$ there is a process $\tilde{\boldsymbol{u}}$ such that $\sup|\boldsymbol{u}-\tilde{\boldsymbol{u}}|\le\delta\chi1$, while $\tilde{\boldsymbol{u}}$ is of bounded variation (615Mg). So $\boldsymbol{u}$ is moderately oscillatory.

(β) In general, the result is trivial if $\mathcal{S}$ is empty, so suppose otherwise. Given $\epsilon>0$, there is a $\tau\in\mathcal{S}$ such that $\theta(u_\sigma-u_\tau)\le\epsilon$ for every $\sigma\in\mathcal{S}\vee\tau$. **P?** Otherwise, starting from any $\tau_0\in\mathcal{S}$, we can choose a non-decreasing sequence $\langle\tau_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{S}$ such that $\theta(u_{\tau_{n+1}}-u_{\tau_n})>\epsilon$ for every $n\in\mathbb{N}$, in which case $\langle u_{\tau_n}\rangle_{n\in\mathbb{N}}$ cannot be convergent. **XQ** Now consider $\boldsymbol{u}'=\langle u_{\sigma\wedge\tau}\rangle_{\sigma\in\mathcal{S}}$. This is fully adapted (612Ib). Since $\boldsymbol{u}$ is order-bounded (615K), so are $\boldsymbol{u}'$ and $\boldsymbol{u}-\boldsymbol{u}'$, while $\boldsymbol{u}'\restriction\mathcal{S}\wedge\tau=\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ is $\Uparrow$-convergent therefore moderately oscillatory, by (α) just above, and $\boldsymbol{u}'\restriction\mathcal{S}\vee\tau$ is constant, so also moderately oscillatory. By 615F(a-v), $\boldsymbol{u}'$ is moderately oscillatory. Now the choice of τ ensures that

$$\theta(u_\sigma-u_{\sigma\wedge\tau})=\theta(u_{\sigma\vee\tau}-u_\tau)$$

(612D(f-i))

$$\le\epsilon$$

for every $\sigma\in\mathcal{S}$, As $\mathcal{S}$ is finitely full, $\theta(\sup|\boldsymbol{u}-\boldsymbol{u}'|)\le2\sqrt{\epsilon}$ (615Db). But $\boldsymbol{u}'\in M_{\mathrm{mo}}(\mathcal{S})$ and ϵ was arbtirary, so $\boldsymbol{u}$ belongs to the closure of $M_{\mathrm{mo}}(\mathcal{S})$ for the ucp topology on $M_{\text{o-b}}(\mathcal{S})$, and is itself moderately oscillatory (615F(a-iv)).

615O Proposition Suppose that $\mathcal{S}$ is a sublattice of $\mathcal{T}$, $\boldsymbol{u}\in M_{\mathrm{mo}}(\mathcal{S})$ and $\epsilon>0$. Then there is a $\boldsymbol{u}'\in M_{\mathrm{bv}}(\mathcal{S})$ such that $\theta(\sup|\boldsymbol{u}'-\boldsymbol{u}|)\le\epsilon$ and $\sup|\boldsymbol{u}'|\le\sup|\boldsymbol{u}|$.

proof (a) If $\mathcal{S}$ is full and has a greatest member, this follows immediately from 615M. We know that $\boldsymbol{u}$ is $\Uparrow$-convergent; taking $\delta=\epsilon$ in 615M, we get a process $\tilde{\boldsymbol{u}}$ of bounded variation which will serve for $\boldsymbol{u}'$, by 615Mf and 615Mg.

(b) If we just know that $\mathcal{S}$ has a greatest member, consider the fully adapted extension $\hat{\boldsymbol{u}}$ of $\boldsymbol{u}$ to the covered envelope $\hat{\mathcal{S}}$ of $\mathcal{S}$. By 615F(a-vi), $\hat{\boldsymbol{u}}$ is moderately oscillatory; by (a) here, there is a process $\hat{\boldsymbol{u}}'\in M_{\mathrm{bv}}(\hat{\mathcal{S}})$ such that $\theta(\sup|\hat{\boldsymbol{u}}'-\hat{\boldsymbol{u}}|)\le\epsilon$ and $\sup|\hat{\boldsymbol{u}}'|\le\sup|\hat{\boldsymbol{u}}|$; now $\boldsymbol{u}'=\hat{\boldsymbol{u}}'\restriction\mathcal{S}$ is of bounded variation (614L(b-i)), $\sup|\boldsymbol{u}'-\boldsymbol{u}|\le\sup|\hat{\boldsymbol{u}}'-\hat{\boldsymbol{u}}|$ so $\theta(\sup|\boldsymbol{u}'-\boldsymbol{u}|)\le\theta(\sup|\hat{\boldsymbol{u}}'-\hat{\boldsymbol{u}}|)\le\epsilon$; and

$$\sup|\boldsymbol{u}'|\le\sup|\hat{\boldsymbol{u}}'|\le\sup|\hat{\boldsymbol{u}}|=\sup|\boldsymbol{u}|$$

by 614Ga.

(c) For the general case, if $\mathcal{S}$ is empty the result is trivial. Otherwise, express $\boldsymbol{u}$ as $\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$. By 615Ga, $u_\uparrow=\lim_{\sigma\uparrow\mathcal{S}}u_\sigma$ is defined, and there is a $\tau\in\mathcal{S}$ such that $\theta(\sup_{\sigma\in\mathcal{S}\vee\tau}|u_\sigma-u_\uparrow|)\le\frac{1}{4}\epsilon$; now $\theta(\sup_{\sigma\in\mathcal{S}\vee\tau}|u_\sigma-u_\tau|)\le\frac{1}{2}\epsilon$. Of course $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ is moderately oscillatory (615F(a-i)); by (b), we have a process $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}\wedge\tau}$ of bounded variation such that $\theta(\sup_{\sigma\in\mathcal{S}\wedge\tau}|v_\sigma-u_\sigma|)\le\frac{1}{4}\epsilon$ and $\sup|\boldsymbol{v}|\le\sup|\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$. It follows that $\theta(\sup_{\sigma\in\mathcal{S}\vee\tau}|u_\sigma-v_\tau|\le\frac{3}{4}\epsilon$.

Write $\mathcal{S}'$ for $(\mathcal{S}\wedge\tau)\cup(\mathcal{S}\vee\tau)$. Since $\sigma\le\sigma'$ whenever $\sigma\in\mathcal{S}\wedge\tau$ and $\sigma'\in\mathcal{S}\vee\tau$, $\mathcal{S}'$ is a sublattice of $\mathcal{S}$. If $\sigma\in\mathcal{S}$, then

$$[\![\sigma=\sigma\wedge\tau]\!]\cup[\![\sigma=\sigma\vee\tau]\!]\supseteq[\![\sigma\le\tau]\!]\cup[\![\tau\le\sigma]\!]=1,$$

so $\mathcal{S}'$ covers $\mathcal{S}$. Because $(\mathcal{S}\wedge\tau)\cap(\mathcal{S}\vee\tau)=\{\tau\}$, we have a family $\langle u'_\sigma\rangle_{\sigma\in\mathcal{S}'}$ defined by setting $u'_\sigma=v_\sigma$ for $\sigma\in\mathcal{S}\wedge\tau$ and $u'_\sigma=v_\tau$ for $\sigma\in\mathcal{S}\vee\tau$. If $\sigma\in\mathcal{S}\wedge\tau$ and $\sigma'\in\mathcal{S}\vee\tau$, then $\sigma\le\tau\le\sigma'$ so

$$[\![\sigma=\sigma']\!]=[\![\sigma=\tau]\!]\cap[\![\tau=\sigma']\!]\subseteq[\![u'_\sigma=u'_{tau}]\!]\cap[\![u'_\tau=u'_{\sigma'}]\!];$$

now it is easy to check that $\langle u'_\sigma\rangle_{\sigma\in\mathcal{S}'}$ is fully adapted, and therefore has a unique fully adapted extension $\boldsymbol{u}'=\langle u'_\sigma\rangle_{\sigma\in\mathcal{S}}$ (612R).

Now $\boldsymbol{u}'\restriction\mathcal{S}\wedge\tau=\boldsymbol{v}$ is of bounded variation, $\boldsymbol{u}'\restriction\mathcal{S}\vee\tau$ is constant, and

$$\sup|\boldsymbol{u}'-\boldsymbol{u}|=\sup_{\sigma\in\mathcal{S}'}|u'_\sigma-u_\sigma|$$

(614Ga)

$$=\sup_{\sigma\in\mathcal{S}\wedge\tau}|v_\sigma-u_\sigma|\vee\sup_{\sigma\in\mathcal{S}\vee\tau}|u_\sigma-v_\tau|.$$

Accordingly $\boldsymbol{u}'$ is of bounded variation (614Lc), $\theta(\sup|\boldsymbol{u}'-\boldsymbol{u}|)\le\frac{3}{4}\epsilon+\frac{1}{4}\epsilon=\epsilon$, and

$$\sup|\boldsymbol{u}'|=\sup|\boldsymbol{v}|\le\sup|\boldsymbol{u}|,$$

as required.

615P Where moderately oscillatory processes come from There is an easy condition on the structure in 612H which will ensure that the process generated there is moderately oscillatory.

Proposition Let (Ω,Σ,μ) be a complete probability space, and $\langle\Sigma_t\rangle_{t\ge0}$ a filtration of σ-subalgebras of Σ such that every μ-negligible set belongs to every Σ_t. Let $(\mathfrak{A},\bar\mu)$ be the measure algebra of μ and set $\mathfrak{A}_t=\{E^\bullet:E\in\Sigma_t\}$ for each $t\ge0$; then we have a real-time stochastic integration structure $(\mathfrak{A},\bar\mu,[0,\infty[\,,\langle\mathfrak{A}_r\rangle_{t\ge0},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$. Let $\langle X_t\rangle_{t\ge0}$ be a progressively measurable process on Ω, and $\boldsymbol{x}=\langle x_\tau\rangle_{\tau\in\mathcal{T}_f}$ the corresponding fully adapted process as described in 612H. Suppose that $\lim_{n\to\infty}X_{t_n}(\omega)$ is defined in $\mathbb{R}$ for every bounded monotonic sequence $\langle t_n\rangle_{n\in\mathbb{N}}$ in $[0,\infty[$ and every $\omega\in\Omega$. Then $\boldsymbol{x}$ is locally moderately oscillatory.

proof Take any $\sigma\in\mathcal{T}_f$.

(a) We have a stopping time $h:\Omega\to[0,\infty[$ adapted to $\langle\Sigma_t\rangle_{t\ge0}$ which represents σ in the sense that $\llbracket\sigma>t\rrbracket=\{\omega:h(\omega)>t\}^\bullet$ for every $t\ge0$ (612H(a-v)). Now $\{X_t(\omega):t\le h(\omega)\}$ is bounded for every $\omega\in\Omega$. **P?** Otherwise, there is a sequence $\langle t_n\rangle_{n\in\mathbb{N}}$ in $[0,h(\omega)]$ such that $\lim_{n\to\infty}|X_{t_n}(\omega)|=\infty$. Setting $s_0=\sup\{s:s\ge0,\ \{n:t_n\le s\}\text{ is finite}\}$, there is a a monotonic subsequence $\langle t'_n\rangle_{n\in\mathbb{N}}$ of $\langle t_n\rangle_{n\in\mathbb{N}}$ converging to s, and $\langle X_{t'_n}(\omega)\rangle_{n\in\mathbb{N}}$ has no limit in $\mathbb{R}$. **XQ**

(b) Next, setting $f(\omega)=\sup_{t\le h(\omega)}|X_t(\omega)|$ for $\omega\in\Omega$, $f:\Omega\to\mathbb{R}$ is measurable. **P** For any $\alpha\in\mathbb{R}$ and $n\in\mathbb{N}$,

$$\{(t,\omega):t\le n,\ |X_t(\omega)|>\alpha\}\in\mathcal{B}([0,n])\widehat{\otimes}\Sigma_n\subseteq\mathcal{B}([0,\infty[)\widehat{\otimes}\Sigma$$

where $\mathcal{B}([0,n])$ is the Borel σ-algebra of $[0,n]$ and $\mathcal{B}([0,\infty[)$ is the Borel σ-algebra of $[0,\infty[$. Because h is Σ-measurable,

$$\{(t,\omega):t\le h(\omega)\}=\textstyle\bigcup_{q\in\mathbb{Q}}[0,q]\times\{\omega:q\le h(\omega)\}\in\mathcal{B}([0,\infty[)\widehat{\otimes}\Sigma$$

and

$$\begin{aligned}W=&\{(t,\omega):t\le h(\omega),\ |X_t(\omega)|>\alpha\}\\=&\bigcup_{n\in\mathbb{N}}\{(t,\omega):t\le h(\omega)\}\cap\{(t,\omega):t\le n,\ |X_t(\omega)|>\alpha\}\end{aligned}$$

belongs to $\mathcal{B}([0,\infty[)\widehat{\otimes}\Sigma$. Since μ is complete, Σ is closed under Souslin's operation (431A) and contains the projection of W onto Ω (423O[6]). But this is just $\{\omega:f(\omega)>\alpha\}$. As α is arbitrary, f is Σ-measurable. **Q**

It follows that $\boldsymbol{x}\restriction\mathcal{T}\wedge\sigma$ is order-bounded. **P** If $\tau\in\mathcal{T}$ and $\tau\le\sigma$, there is a stopping time $g:\Omega\to[0,\infty[$ representing τ (612H(a-v) again) and $g\le_{\text{a.e.}}h$ (612H(a-iv)), so $\min(g,h)$ still represents τ. Since $|X_{\min(g(\omega),h(\omega))}(\omega)|\le f(\omega)$ for every $\omega\in\Omega$, $x_\tau\le f^\bullet$ in $L^0(\mathfrak{A})$. Thus $\sup|\boldsymbol{x}\restriction\mathcal{T}\wedge\sigma|$ is defined and at most $f^\bullet$. **Q**

(c) Suppose that $\langle\tau_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathcal{T}\wedge\sigma$. Choose $g_n:\Omega\to[0,\infty[$ representing τ_n for each n, and set $g'_n=\min(h,\sup_{i\le n}g_i)$ for $n\in\mathbb{N}$; then g'_n represents τ_n for each n, and $\langle g'_n(\omega)\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $[0,h(\omega)]$ for each $\omega\in\Omega$, so $g(\omega)=\lim_{n\to\infty}X_{g'_n(\omega)}(\omega)$ is defined in $\mathbb{R}$ for every $\omega\in\Omega$. But now $g^\bullet=\lim_{n\to\infty}x_{\tau_n}$ in $L^0(\mathfrak{A})$ (245Ca). So $\lim_{n\to\infty}x_{\tau_n}$ is defined.

Similarly, if $\langle\tau_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{T}\wedge\sigma$, $g_n:\Omega\to[0,\infty[$ represents τ_n for each n, and $g'_n=\min(h,\inf_{i\le n}g_i)$ for $n\in\mathbb{N}$, then g'_n represents τ_n for each n, $\langle g'_n(\omega)\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $[0,h(\omega)]$ for each $\omega\in\Omega$, and $g(\omega)=\lim_{n\to\infty}X_{g'_n(\omega)}(\omega)$ is defined in $\mathbb{R}$ for every $\omega\in\Omega$. Once again, $\lim_{n\to\infty}x_{\tau_n}$ is defined and equal to $g^\bullet$.

Putting these together with (b), $\boldsymbol{x}\restriction\mathcal{T}\wedge\sigma$ is $\upharpoonleft\!\downharpoonright$-convergent, therefore moderately oscillatory, since $\mathcal{T}\wedge\sigma$ is (finitely) full; as σ is arbitrary, $\boldsymbol{x}$ is locally moderately oscillatory on $\mathcal{T}_f$.

615Q Proposition The identity process (as described in 612F), Brownian motion (612T) and the Poisson process (612U) are all locally moderately oscillatory.

proof In the case of the identity process $\boldsymbol{\iota}=\langle\iota_\tau\rangle_{\tau\in\mathcal{T}_f}$ and the Poisson process $\boldsymbol{v}=\langle v_\tau\rangle_{\tau\in\mathcal{T}_f}$ it is enough to know that they start at $\mathbf{0}$ and are non-decreasing, so that if $\tau\in\mathcal{T}_f$ and $\langle\sigma_n\rangle_{n\in\mathbb{N}}$ is a monotonic sequence

[6] Later editions only.

in $\mathcal{T}_f \wedge \tau$, $\langle \iota_{\sigma_n} \rangle_{n\in\mathbb{N}}$ and $\langle v_{\sigma_n} \rangle_{n\in\mathbb{N}}$ are monotonic sequences in $[0,\iota_\tau]$ and $[0,v_\tau]$ respectively, and must be convergent in L^0 (613Ba). (Of course $\mathcal{T}_f \wedge \tau$ is full, so we can work directly from 614O.) As for Brownian motion, I based this on $\Omega = C([0,\infty[)_0$ and $X_t(\omega) = \omega(t)$ for $t \ge 0$ and $\omega \in \Omega$. So the sample paths $t \mapsto X_t(\omega)$ are continuous, and 615P gives the result at once.

615R Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u}$ a process of bounded variation with domain $\mathcal{S}$.

(a) If $\boldsymbol{v} \in M_{\text{mo}} = M_{\text{mo}}(\mathcal{S})$ then $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$ is defined and $|\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}| \le (\int_{\mathcal{S}} |d\boldsymbol{u}| + 2\sup|\boldsymbol{u}|) \times \sup|\boldsymbol{v}|$.

(b) $ii_{\boldsymbol{v}}(\boldsymbol{u}) \in M_{\text{mo}}$ for every $\boldsymbol{v} \in M_{\text{mo}}$, and $\boldsymbol{v} \mapsto ii_{\boldsymbol{v}}(\boldsymbol{u}) : M_{\text{mo}} \to M_{\text{mo}}$ is continuous for the ucp topology on M_{mo}.

proof Write w for $\int_{\mathcal{S}} |d\boldsymbol{u}| + 2\sup|\boldsymbol{u}|$.

(a) Let $\epsilon > 0$. Then there is a $\delta > 0$ such that $\theta(x \times w) \le \epsilon$ whenever $\theta(x) \le \delta$. Next, there is a process $\boldsymbol{v}'$ of bounded variation such that $\theta(\sup|v - v'|) \le \delta$. By 614S, $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}'$ is defined. Let $I \in \mathcal{I}(\mathcal{S})$ be such that $\theta(S_I(\boldsymbol{u}, d\boldsymbol{v}') - S_J(\boldsymbol{u}, d\boldsymbol{v}')) \le \epsilon$ whenever $I \subseteq J \in \mathcal{I}(\mathcal{S})$. Now for any $J \in \mathcal{I}(\mathcal{S})$

$$|S_J(\boldsymbol{u}, d\boldsymbol{v}) - S_J(\boldsymbol{u}, d\boldsymbol{v}'))| = |S_J(\boldsymbol{u}, d(\boldsymbol{v} - \boldsymbol{v}'))|$$
$$\le \sup(|\boldsymbol{v} - \boldsymbol{v}'|{\restriction}J) \times (\int_J |d\boldsymbol{u}| + 2\sup(|\boldsymbol{u}{\restriction}J|))$$

(614R)

$$\le \sup(|\boldsymbol{v} - \boldsymbol{v}'|) \times (\int_{\mathcal{S}} |d\boldsymbol{u}| + 2\sup(|\boldsymbol{u}|))$$

(using 614Lb)

$$= \sup(|\boldsymbol{v} - \boldsymbol{v}'| \times w),$$

so

$$\theta(S_J(\boldsymbol{u}, d\boldsymbol{v}) - S_J(\boldsymbol{u}, d\boldsymbol{v}')) \le \epsilon.$$

By the choice of I, $\theta(S_I(\boldsymbol{u}, d\boldsymbol{v}) - S_J(\boldsymbol{u}, d\boldsymbol{v})) \le 3\epsilon$ whenever $I \subseteq J \in \mathcal{I}(\mathcal{S})$. As ϵ is arbitrary, $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v} = \lim_{I\uparrow\mathcal{I}(\mathcal{S})} S_I(\boldsymbol{u}, d\boldsymbol{v})$ is defined.

At the same time, we see that $|S_J(\boldsymbol{u}, d\boldsymbol{v})| \le \sup|\boldsymbol{v}| \times w$ for every $J \in \mathcal{I}(\mathcal{S})$, so in the limit we have $|\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}| \le \sup|\boldsymbol{v}| \times w$.

(b) If $\boldsymbol{v} \in M_{\text{mo}}$ then $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$ is defined, by (a), so $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is defined everywhere on $\mathcal{S}$. Applying (a) to $\boldsymbol{u}{\restriction}\mathcal{S} \wedge \tau$ and $\boldsymbol{v}{\restriction}\mathcal{S} \wedge \tau$ for $\tau \in \mathcal{S}$, we see that

$$\sup|ii_{\boldsymbol{v}}(\boldsymbol{u})| = \sup_{\tau\in\mathcal{S}} |\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\boldsymbol{v}|$$
$$\le \sup_{\tau\in\mathcal{S}} \bigl((\int_{\mathcal{S}\wedge\tau} |d\boldsymbol{u}| + 2\sup|\boldsymbol{u}{\restriction}\mathcal{S}\wedge\tau|) \times \sup|\boldsymbol{v}{\restriction}\mathcal{S}\wedge\tau|\bigr)$$
$$\le (\int_{\mathcal{S}} |d\boldsymbol{u}| + 2\sup|\boldsymbol{u}|) \times \sup|\boldsymbol{v}|,$$

so $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is order-bounded. The same formula shows that $\boldsymbol{v} \mapsto ii_{\boldsymbol{v}}(\boldsymbol{u}) : M_{\text{mo}} \to M_{\text{o-b}} = M_{\text{o-b}}(\mathcal{S})$ is continuous for the ucp topologies on M_{mo} and $M_{\text{o-b}}$. We know that $ii_{\boldsymbol{v}}(\boldsymbol{u}) \in M_{\text{bv}}$ whenever $\boldsymbol{v} \in M_{\text{bv}}$ (614T), so $ii_{\boldsymbol{v}}(\boldsymbol{u}) \in \overline{M_{\text{bv}}} = M_{\text{mo}}$ whenever $\boldsymbol{v} \in M_{\text{mo}}$, and $\boldsymbol{v} \mapsto ii_{\boldsymbol{v}}(\boldsymbol{u}) : M_{\text{mo}} \to M_{\text{mo}}$ is continuous.

615X Basic exercises (a) Let $\bar{\nu} : \mathfrak{A} \to [0,1]$ be any functional such that $(\mathfrak{A}, \bar{\nu})$ is a probability algebra. Show that the ucp topology and uniformity on $M_{\text{o-b}}(\mathcal{S})$ defined from the associated F-norm $\theta_{\bar{\nu}}$ (613Bg) are the same as those defined from $\bar{\mu}$ and $\theta = \theta_{\bar{\mu}}$.

(b) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. Write $M_{\text{lob}} = M_{\text{lob}}(\mathcal{S})$ for the space of locally order-bounded fully adapted processes with domain $\mathcal{S}$ (615Fb). For $\tau \in \mathcal{S}$ and $\boldsymbol{u} \in M_{\text{lob}}$ set $\widehat{\theta}_\tau(\boldsymbol{u}) = \theta(\sup_{\sigma\in\mathcal{S}\wedge\tau} |u_\sigma|)$. (i) Show that $\{\widehat{\theta}_\tau : \tau \in \mathcal{S}\}$ defines a complete Hausdorff linear space topology on M_{lob} for which multiplication and the operations $\vee$, $\wedge$ are continuous; I will call this the **local ucp topology** on M_{lob}. (ii) Show that $M_{\text{lmo}}(\mathcal{S})$ is closed for the local ucp topology on M_{lob}.

(c) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and z a member of $L^0(\bigcap_{\sigma\in\mathcal{S}}\mathfrak{A}_\sigma)$. Show that, in the language of 612D(e-ii), $\boldsymbol{u}\mapsto z\boldsymbol{u}:M_{\text{o-b}}(\mathcal{S})\to M_{\text{o-b}}(\mathcal{S})$ is continuous for the ucp topology.

(d) Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ and $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a fully adapted process such that $\{u_\sigma:\sigma\in\mathcal{S}\}$ is topologically bounded in the sense of 613Bf. Show that $\boldsymbol{u}$ is order-bounded.

(e) Give examples of (i) an $\Downarrow$-convergent fully adapted process which is not order-bounded (ii) an $\Downarrow$-convergent order-bounded process which is not moderately oscillatory. (*Hint*: $\mathcal{S}=\{\check{n}:n\in\mathbb{N}\}$.)

(f) Suppose that $T=[0,\infty[$ and $\mathfrak{A}=\{0,1\}$, as in 613W. Write $\tilde{C}^{\Downarrow}$ for the set of functions $f:[0,\infty[\,\to\mathbb{R}$ such that $\lim_{s\downarrow t}f(s)$ is defined for every $t\ge 0$ and $\lim_{s\uparrow t}f(s)$ is defined for every $t>0$ (cf. 438P-438Q). Let $f:[0,\infty[\,\to\mathbb{R}$ be a function and $\boldsymbol{x}\in M_{\text{fa}}(\mathcal{T}_f)$ the corresponding process. (i) Show that the ucp topology on $M_{\text{o-b}}(\mathcal{T}_f)$ corresponds to the norm topology on $\ell^\infty([0,\infty[)$. (ii) Show that $\boldsymbol{x}$ is locally moderately oscillatory iff $f\in\tilde{C}^{\Downarrow}$, and moderately oscillatory iff $f\in\tilde{C}^{\Downarrow}$ and $\lim_{t\to\infty}f(t)$ is defined in $\mathbb{R}$.

(g) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u}$ a process of bounded variation, and $\boldsymbol{v}$ a moderately oscillatory process, both with domain $\mathcal{S}$. Show that $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is moderately oscillatory.

(h) Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ and $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a fully adapted process such that $\{u_\sigma:\sigma\in\mathcal{S}\}$ is topologically bounded (613Bf). Show that $\boldsymbol{u}$ is order-bounded.

615Y Further exercises (a) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. Show that the ucp topology on the Riesz space $M_{\text{o-b}}(\mathcal{S})$ is **locally solid** in the sense that for every neighbourhood G of 0 there is a solid neighbourhood H of 0 included in G.

(b) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. For $\boldsymbol{u}$, $\boldsymbol{v}\in M_{\text{fa}}(\mathcal{S})$ set $\rho(\boldsymbol{u},\boldsymbol{v})=\min(1,\|\boldsymbol{u}-\boldsymbol{v}\|_\infty)$ (612Sa). Show that ρ is a metric on $M_{\text{fa}}(\mathcal{S})$ under which $M_{\text{fa}}(\mathcal{S})$ is complete and addition is continuous. Show that if $\mathcal{S}$ is finitely full then $M_{\text{mo}}(\mathcal{S})$ is the closure of $M_{\text{bv}}(\mathcal{S})$ for the topology defined by ρ.

(c) In the construction described in 613P, show that the processes $\boldsymbol{u}$ and $\boldsymbol{v}$ are moderately oscillatory.

615 Notes and comments Note that the concepts of 'ucp topology', 'ucp uniformity' and 'moderately oscillatory process', like that of 'Riemann-sum integral' (613I), depend on the structure $(\mathfrak{A},L^0,\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ and on the topology of convergence in measure on L^0, but otherwise are independent of the measure $\bar{\mu}$ (615Xa).

So many processes turn out to be locally moderately oscillatory that we can quite happily regard them as the norm, at least to begin with. In Theorem 455G I showed that the most important real-valued Markov processes on $[0,\infty[$ have representations in which all paths are càdlàg functions, and 615P shows that these correspond to locally moderately oscillatory processes. There are enough examples (see 612T-612U) and preservation results (e.g., 615F) to ensure that we can hope to remain in this territory for a long time.

The words 'local' and 'locally' are going to appear repeatedly in this volume. The point is that basic theorems are often most simply expressed in terms of 'global' concepts, as in 614F, 614G, 614Q, 615C, 615Fa, 615G and 616R. But the most important applications tend to present themselves in 'local' terms; thus Brownian motion is locally order-bounded (614H) and the Poisson process is locally of bounded variation (614M).

From Theorem 615N we see that, at least if the sublattice $\mathcal{S}$ is finitely full, we have two possible definitions of 'moderately oscillatory process'; one in terms of approximation by processes of bounded variation, as in 615E, and one in terms of a kind of sequential convergence, as in 615I. The former is easier to handle (compare 615G and 615L), but the latter gives a path to a fundamental fact (616Ib). In particular, 615M can be regarded as a method of constructing a process $\tilde{\boldsymbol{u}}$ from a given moderately oscillatory process $\boldsymbol{u}$ which is not only of bounded variation but approximates $\boldsymbol{u}$ in a much finer topology than the ucp topology (615Yb). 615O looks like an insignificant variation on the definition in 615E, but I do not see a quicker way to it, and it will be useful later.

As in §614, I conclude the section with a note on a special type of integral, providing another case in which an indefinite integral inherits a property from an integrator. It looks esoteric, but happens to be a useful step towards one of the principal theorems of the next section (616K).

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616 Integrating interval functions

In this section I present a fundamental theorem on the existence of Riemann-sum integrals (616M), dealing with the case of moderately oscillatory integrands and integrating interval functions (616F). The most important integrating interval functions are those defined by integrators (616Fc, 616I). The integrators on a lattice $\mathcal{S}$ form an f-subalgebra of the space of moderately oscillatory processes with domain $\mathcal{S}$ (616P).

616A Notation $(\mathfrak{A},\bar\mu,T,\langle\mathfrak{A}_t\rangle_{t\in T},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ will be a stochastic integration structure. For a sublattice $\mathcal{S}$ of $\mathcal{T}$, $\mathcal{I}(\mathcal{S})$ is the set of finite sublattices of $\mathcal{S}$, and if $\tau\in\mathcal{T}$, $\mathcal{S}\wedge\tau=\{\sigma\wedge\tau:\sigma\in\mathcal{S}\}$ and $\mathcal{S}\vee\tau=\{\sigma\vee\tau:\sigma\in\mathcal{S}\}$. If $\boldsymbol{u}$ and $\boldsymbol{v}$ are fully adapted processes defined (at least) on a finite sublattice I of $\mathcal{T}$ and ψ is an adapted interval function defined (at least) on $I^{2\uparrow}=\{(\sigma,\tau):\sigma,\,\tau\in I,\,\sigma\le\tau\}$, then $S_I(\boldsymbol{u},d\psi)$ and $S_I(\boldsymbol{u},d\boldsymbol{v})$ will be the Riemann sums defined in 613Eb and 613Fb. For $w\in L^0=L^0(\mathfrak{A})$, $\theta(w)=\mathbb{E}(|w|\wedge\chi1)$ as in 613Ba. If $\mathcal{S}$ is a sublattice of $\mathcal{T}$, $M_{\text{fa}}(\mathcal{S})$, $M_{\text{o-b}}(\mathcal{S})$, $M_{\text{lob}}(\mathcal{S})$, $M_{\text{bv}}(\mathcal{S})$ and $M_{\text{mo}}(\mathcal{S})$ are the spaces of fully adapted processes, order-bounded processes, locally order-bounded processes, processes of bounded variation and moderately oscillatory processes with domain $\mathcal{S}$. For $\boldsymbol{u}\in M_{\text{o-b}}(\mathcal{S})$, $\sup|\boldsymbol{u}|=\sup_{\sigma\in\mathcal{S}}|u_\sigma|$. $\mathbf{1}$ will be the process with constant value $\chi1$. If $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ and $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ are fully adapted processes with the same domain $\mathcal{S}$, $[\![\boldsymbol{u}\ne\boldsymbol{v}]\!]=\sup_{\sigma\in\mathcal{S}}[\![u_\sigma\ne v_\sigma]\!]$.

616B Definition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. If ψ is an adapted interval function (613C) defined (at least) on $\mathcal{S}^{2\uparrow}$, the **capped-stake variation set of ψ over $\mathcal{S}$** is the set $Q_{\mathcal{S}}(d\psi)$ of Riemann sums $S_I(\boldsymbol{u},d\psi)$ where $I\in\mathcal{I}(\mathcal{S})$, $\boldsymbol{u}$ is a fully adapted process with domain I and $\sup|\boldsymbol{u}|\le\chi1$.

If $\boldsymbol{v}$, $\boldsymbol{w}$ are fully adapted processes defined (at least) on $\mathcal{S}$ then, corresponding to the basic interval functions of 613F, I will write $Q_{\mathcal{S}}(d\boldsymbol{v})$, $Q_{\mathcal{S}}(d\boldsymbol{v}d\boldsymbol{w})$, $Q_{\mathcal{S}}(|d\boldsymbol{v}|)$ for $Q_{\mathcal{S}}(d(\Delta\boldsymbol{v}))$, $Q_{\mathcal{S}}(d(\Delta\boldsymbol{v}\times\Delta\boldsymbol{w}))$ and $Q_{\mathcal{S}}(d|\Delta\boldsymbol{v}|)$.

616C The following elementary facts will be useful.

Lemma Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$, ψ an adapted interval function defined on $\mathcal{S}^{2\uparrow}$, and z an element of $L^0(\mathfrak{A})$. Then the following are equiveridical:

(i) $z\in Q_{\mathcal{S}}(d\psi)$;

(ii) there are $\tau_0\le\ldots\le\tau_n$ in $\mathcal{S}$ and $u_0,\ldots,u_{n-1}$ such that $u_i\in L^\infty(\mathfrak{A}_{\tau_i})$ and $|u_i|\le\chi1$ for every $i<n$ and $z=\sum_{i=0}^{n-1}u_i\times\psi(\tau_i,\tau_{i+1})$;

(iii) there are $\tau_0\le\ldots\le\tau_n$ in $\mathcal{S}$ and an order-bounded process $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ such that $\sup|\boldsymbol{u}|\le\chi1$ and $z=\sum_{i=0}^{n-1}u_{\tau_i}\times\psi(\tau_i,\tau_{i+1})$.

proof (i)$\Rightarrow$(ii) If $z=0$ we can take $n=0$ and any $\tau_0\in\mathcal{S}$. Otherwise, let $I\in\mathcal{I}(\mathcal{S})$ and $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in I}\in M_{\text{fa}}(I)$ be such that $\sup|\boldsymbol{u}|\le\chi1$ and $z=S_I(\boldsymbol{u},d\psi)$. Take a sequence $(\tau_0,\ldots,\tau_n)$ linearly generating the I-cells. Then

$$z=S_I(\boldsymbol{u},d\psi)=\textstyle\sum_{i=0}^{n-1}u_{\tau_i}\times\psi(\tau_i,\tau_{i+1})$$

(613Ec), while $\tau_0\le\ldots\le\tau_n$ and $u_{\tau_i}\in L^0(\mathfrak{A}_{\tau_i})$ and $|u_{\tau_i}|\le\chi1$ for every $i<n$.

(ii)$\Rightarrow$(iii) By 612Ka, there is a fully adapted process $\boldsymbol{u}'=\langle u'_\sigma\rangle_{\sigma\in\mathcal{S}}$ such that, for $\sigma\in\mathcal{S}$,

$$[\![u'_\sigma=u_i]\!]\supseteq[\![\tau_i\le\sigma]\!]\cap[\![\sigma<\tau_{i+1}]\!],$$

for every $i<n$, while

$$[\![u'_\sigma=0]\!]\supseteq[\![\sigma<\tau_0]\!],\quad [\![u'_\sigma=u_{\tau_n}]\!]\supseteq[\![\tau_n\le\sigma]\!].$$

Observe that $|u'_\sigma|\le\chi1$ for every σ (because

$$[\![\sigma<\tau_0]\!]\cup[\![\tau_n\le\sigma]\!]\cup\sup_{i<n}[\![\tau_i\le\sigma]\!]\cap[\![\sigma<\tau_{i+1}]\!]=1),$$

so $\boldsymbol{u}$ is order-bounded and $\sup|\boldsymbol{u}|\le\chi1$. Now if we set $I=\{\tau_0,\ldots,\tau_n\}$,

$$
\begin{aligned}
z &= \sum_{i=0}^{n-1} u_i \times \psi(\tau_i, \tau_{i+1}) = \sum_{i=0}^{n-1} u_i \times \chi([\![\tau_i < \tau_{i+1}]\!]) \times \psi(\tau_i, \tau_{i+1}) \\
&= \sum_{i=0}^{n-1} u'_{\tau_i} \times \chi([\![\tau_i < \tau_{i+1}]\!]) \times \psi(\tau_i, \tau_{i+1}) = \sum_{i=0}^{n-1} u'_{\tau_i} \times \psi(\tau_i, \tau_{i+1}).
\end{aligned}
$$

(iii)⇒(i) Setting $I = \{\tau_0, \dots, \tau_n\}$, $z = S_I(\boldsymbol{u}\restriction I, d\psi)$.

616D Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and ψ, ψ' adapted interval functions defined on $\mathcal{S}^{2\uparrow}$.

(a) $Q_{\mathcal{S}}(d\psi) = \bigcup_{I\in\mathcal{I}(\mathcal{S})} Q_I(d\psi)$.

(b) $Q_{\mathcal{S}}(d(\alpha\psi)) = \alpha Q_{\mathcal{S}}(d\psi)$ for every $\alpha \in \mathbb{R}$.

(c) $Q_{\mathcal{S}}(d(\psi + \psi')) \subseteq Q_{\mathcal{S}}(d\psi) + Q_{\mathcal{S}}(d\psi')$.

(d) If $\mathcal{S}'$ is a sublattice of $\mathcal{S}$ then $Q_{\mathcal{S}'}(d\psi) \subseteq Q_{\mathcal{S}}(d\psi)$.

(e) If $w \in Q_{\mathcal{S}}(d\psi)$, $z \in L^0(\mathfrak{A} \cap \bigcap_{\sigma\in\mathcal{S}} \mathfrak{A}_\sigma)$ and $|z| \le \chi 1$, then $z \times w \in Q_{\mathcal{S}}(\psi)$.

(f) If $\tau \in \mathcal{S}$ then $Q_{\mathcal{S}\wedge\tau}(d\psi) + Q_{\mathcal{S}\vee\tau}(d\psi) \subseteq Q_{\mathcal{S}}(d\psi)$.

proof (a) Immediate from the definition.

(b) $S_I(\boldsymbol{u}, \alpha\psi) = \alpha S_I(\boldsymbol{u}, d\psi)$ for all I and $\boldsymbol{u}$.

(c) $S_I(\boldsymbol{u}, d(\psi + \psi')) = S_I(\boldsymbol{u}, d\psi) + S_I(\boldsymbol{u}, d\psi')$ for all I and $\boldsymbol{u}$.

(d) $\mathcal{I}(\mathcal{S}') \subseteq \mathcal{I}(\mathcal{S})$.

(e) We can express w as $S_I(\boldsymbol{u}, d\psi)$ where $I \in \mathcal{I}(\mathcal{S})$, $\boldsymbol{u} \in M_{\text{fa}}(I)$ and $\sup|\boldsymbol{u}| \le \chi 1$. Now $z\boldsymbol{u} \in M_{\text{fa}}(I)$ (612D(e-ii)) and $\sup|z\boldsymbol{u}| \le |z| \times \sup|\boldsymbol{u}| \le \chi 1$, so $Q_{\mathcal{S}}(d\psi)$ contains $S_I(z\boldsymbol{u}, d\psi) = z \times w$ (613L(b-iii)).

(f) If $w \in Q_{\mathcal{S}\wedge\tau}(d\psi) + Q_{\mathcal{S}\vee\tau}(d\psi)$, then by 616C(ii) there are $\tau'_0 \le \dots \le \tau'_{n'}$ in $\mathcal{S} \wedge \tau$, $\tau''_0 \le \dots \le \tau''_{n''}$ in $\mathcal{S} \vee \tau$, $u'_i \in L^0(\mathfrak{A}_{\tau'_i})$ such that $|u'_i| \le \chi 1$ for $i < n'$, $u''_j \in L^0(\mathfrak{A}_{\tau'_j})$ such that $|u''_j| \le \chi 1$ for $j < n''$. such that

$$w = \textstyle\sum_{i=0}^{n'-1} u'_i \times \psi(\tau'_i, \tau'_{i+1}) + \sum_{j=0}^{n''-1} u''_i \times \psi(\tau''_i, \tau''_{i+1}).$$

Set $n = n' + n'' + 1$,

$$
\begin{aligned}
\tau_k &= \tau'_k \text{ for } k \le n', \\
&= \tau''_{k-n'-1} \text{ for } n' < k \le n' + n'' + 1, \\
u_k &= u'_k \text{ for } k < n', \\
&= 0 \text{ for } k = n', \\
&= u''_{k-n'-1} \text{ for } n' < k \le n' + n''.
\end{aligned}
$$

Then $\tau_0 \le \dots \le \tau_n$ in $\mathcal{S}$ while $u_{\tau_k} \in L^0(\mathfrak{A}_{\tau_k})$ and $|u_{\tau_k}| \le \chi 1$ for $k < n$ and $w = \sum_{k=0}^{n-1} u_k \times \psi(\tau_k, \tau_{k+1})$, so $w \in Q_{\mathcal{S}}(d\psi)$.

616E Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and ψ an adapted interval function on $\mathcal{S}$. Then the following are equiveridical:

(i) $Q_{\mathcal{S}}(d\psi)$ is topologically bounded;

(ii) for every $\epsilon > 0$ there is a $\delta > 0$ such that $\theta(S_I(\boldsymbol{u}, d\psi)) \le \epsilon$ whenever $I \in \mathcal{I}(\mathcal{S})$, $\boldsymbol{u} \in M_{\text{fa}}(I)$ and $\theta(\sup|\boldsymbol{u}|) \le \delta$;

(iii) for every $\epsilon > 0$ there is a $\delta > 0$ such that $\theta(S_I(\boldsymbol{u}, d\psi)) \le \epsilon$ whenever $I \in \mathcal{I}(\mathcal{S})$, $\boldsymbol{u} \in M_{\text{o-b}}(\mathcal{S})$ and $\theta(\sup|\boldsymbol{u}|) \le \delta$.

proof (i)⇒(ii) Suppose that $Q_{\mathcal{S}}(d\psi)$ is topologically bounded and $\epsilon > 0$. Let $\eta > 0$ be such that $\theta(\eta S_I(\boldsymbol{u}, d\psi)) \le \frac{1}{2}\epsilon$ whenever $I \in \mathcal{I}(\mathcal{S})$, $\boldsymbol{u} \in M_{\text{fa}}(I)$ and $\sup|\boldsymbol{u}| \le \chi 1$. Set $\delta = \frac{1}{2}\epsilon\eta$. If $I \in \mathcal{I}(\mathcal{S})$, $\boldsymbol{u} \in M_{\text{fa}}(I)$ and $\theta(\sup|\boldsymbol{u}|) \le \delta$, set $\boldsymbol{u}' = \text{med}(-\mathbf{1}, \frac{1}{\eta}\boldsymbol{u}, \mathbf{1})$. Then

$$[\![S_I(\boldsymbol{u},d\psi)\ne\eta S_I(\boldsymbol{u}',d\psi)]\!]=[\![S_I(\boldsymbol{u}-\eta\boldsymbol{u}',d\psi)\ne 0]\!]\subseteq[\![\boldsymbol{u}\ne\eta\boldsymbol{u}']\!]$$

(613Gd)

$$=[\![\sup|\boldsymbol{u}|>\eta]\!]$$

has measure at most $\frac{\delta}{\eta}=\frac{1}{2}\epsilon$. So

$$\theta(S_I(\boldsymbol{u},d\psi))\le\frac{1}{2}\epsilon+\theta(\eta S_I(\boldsymbol{u}',d\psi))\le\epsilon.$$

Thus δ witnesses that (ii) is true.

(ii)⇒(iii) If (ii) is true and $\epsilon>0$, take δ as in (ii). If $I\in\mathcal{I}(\mathcal{S})$, $\boldsymbol{u}\in M_{\text{o-b}}(\mathcal{S})$ and $\theta(\sup|\boldsymbol{u}|)\le\delta$, then $\boldsymbol{u}{\restriction}I\in M_{\text{fa}}(I)$ and $\sup|\boldsymbol{u}{\restriction}I|\le\sup|\boldsymbol{u}|$, so $\theta(\sup|\boldsymbol{u}{\restriction}I|)\le\delta$ and $\theta(S_I(\boldsymbol{u},d\psi))=\theta(S_I(\boldsymbol{u}{\restriction}I,d\psi))\le\epsilon$.

(iii)⇒(i) Suppose that (iii) is true. If $\mathcal{S}$ is empty then $Q_{\mathcal{S}}(d\psi)=\{0\}$ is certainly topologically bounded. Otherwise, given $\epsilon>0$, take δ as in (iii). Then for any $z\in Q_{\mathcal{S}}(d\psi)$ there are $\tau_0\le\ldots\le\tau_n$ in $\mathcal{S}$ and $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}\in M_{\text{o-b}}(\mathcal{S})$ such that $\sup|\boldsymbol{u}|\le\chi 1$ and $z=\sum_{i=0}^{n-1}u_{\tau_i}\times\psi(\tau_i,\tau_{i+1})$ (616C(ii)). But now $z=S_I(\boldsymbol{u},d\psi)$ where $I=\{\tau_0,\ldots,\tau_n\}$, while $\theta(\sup|\delta\boldsymbol{u}|)\le\theta(\delta\chi 1)\le\delta$ so $\theta(\delta z)=\theta(S_I(\delta\boldsymbol{u},d\psi))\le\epsilon$. As ϵ and z are arbitrary, $Q_{\mathcal{S}}(d\psi)$ is topologically bounded.

616F Definition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\psi:\mathcal{S}^{2\uparrow}\to L^0$ a function.

(a) I will say that ψ is an **integrating interval function** on $\mathcal{S}$ if

(α) ψ is a strictly adapted interval function;

(β) writing $\hat{\mathcal{S}}$ for the covered envelope of $\mathcal{S}$ and $\hat\psi:\hat{\mathcal{S}}^{2\uparrow}\to L^0$ for the strictly adapted extension of ψ (613U), $\int_{\hat{\mathcal{S}}}d\hat\psi=\int_{\hat{\mathcal{S}}}\mathbf{1}\,d\hat\psi$ is defined in the sense of 613H;

(γ) $Q_{\hat{\mathcal{S}}}(d\hat\psi)$ is topologically bounded in L^0.

(b) ψ is a **locally integrating interval function** if $\psi{\restriction}(\mathcal{S}\wedge\tau)^{2\uparrow}$ is an integrating interval function for every $\tau\in\mathcal{S}$.

(c) A fully adapted process $\boldsymbol{v}$ defined on $\mathcal{S}$ is an **integrator** if $Q_{\mathcal{S}}(d\boldsymbol{v})$ is topologically bounded in L^0, and a **local integrator** if $\boldsymbol{v}{\restriction}\mathcal{S}\wedge\tau$ is an integrator for every $\tau\in\mathcal{S}$.

I will write $M_{\text{igtr}}(\mathcal{S})$ for the set of integrators with domain $\mathcal{S}$, and $M_{\text{ligtr}}(\mathcal{S})$ for the set of local integrators with domain $\mathcal{S}$.

Remarks Evidently a strictly adapted interval function ψ on a sublattice $\mathcal{S}$ is an integrating interval function iff its adapted extension on the covered envelope of $\mathcal{S}$ is an integrating interval function.

I have given a definition of 'integrator' which does not obviously correspond directly to the definition of 'integrating interval function'. We shall see in 616I that in fact it matches exactly, but it is convenient to work with the simpler formulation here for the moment. Of course we can see already that if $\boldsymbol{v}$ is fully adapted and $\Delta\boldsymbol{v}$ is a (locally) integrating interval function, then $\boldsymbol{v}$ is a (local) integrator.

616G Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and ψ, ψ' integrating interval functions on $\mathcal{S}$.

(a) $\psi+\psi'$ and $\alpha\psi$ are integrating interval functions on $\mathcal{S}$ for every $\alpha\in\mathbb{R}$.

(b) ψ is a locally integrating interval function.

proof (a) $\psi+\psi'$ and $\alpha\psi$ are strictly adapted interval functions (613D) and

$$\int_{\mathcal{S}}d(\psi+\psi')=\int_{\mathcal{S}}d\psi+\int_{\mathcal{S}}d\psi',\quad\int_{\mathcal{S}}d(\alpha\psi)=\alpha\int_{\mathcal{S}}d\psi$$

are defined (613Jb). Next, writing $\hat\psi$ and $\hat\psi'$ for the strictly adapted extensions of ψ and ψ', as in 616Fa, $\hat\psi+\hat\psi'$ is a strictly adapted interval function extending $\psi+\psi'$, so is equal to $(\psi+\psi')\hat{}$, and

$$\int_{\hat{\mathcal{S}}}d((\psi+\psi')\hat{})=\int_{\hat{\mathcal{S}}}d(\hat\psi+\hat\psi')=\int_{\hat{\mathcal{S}}}d\hat\psi+\int_{\hat{\mathcal{S}}}d\hat\psi'$$

is defined, while

$$Q_{\hat{\mathcal{S}}}(d(\psi+\psi')\hat{\ }) = Q_{\hat{\mathcal{S}}}(d(\hat{\psi}+\hat{\psi}')) \subseteq Q_{\hat{\mathcal{S}}}(d\hat{\psi}) + Q_{\hat{\mathcal{S}}}(d\hat{\psi}')$$

(616Dc) is topologically bounded (613B(f-iii)). So $\psi+\psi'$ is an integrating interval function.

Similarly,

$$\int_{\hat{\mathcal{S}}} d((\alpha\psi)\hat{\ }) = \alpha\int_{\hat{\mathcal{S}}} d\hat{\psi}, \quad Q_{\hat{\mathcal{S}}}(d(\alpha\psi)\hat{\ }) = Q_{\hat{\mathcal{S}}}(d(\alpha\hat{\psi})) = \alpha Q_{\hat{\mathcal{S}}}(d\hat{\psi})$$

is topologically bounded and $\alpha\psi$ is an integrating interval function.

(b) If $\tau \in \mathcal{S}$, then the covered envelope of $\mathcal{S}\wedge\tau$ is $\hat{\mathcal{S}}\wedge\tau$ (611M(e-i)). $\psi' = \psi\restriction(\mathcal{S}\wedge\tau)^{2\uparrow}$ is a strictly adapted interval function (613C(b-iii)) and its strictly adapted extension $\hat{\psi}'$ to $(\hat{\mathcal{S}}\wedge\tau)^{2\uparrow}$ must be $\hat{\psi}\restriction(\hat{\mathcal{S}}\wedge\tau)^{2\uparrow}$. Now $\int_{\hat{\mathcal{S}}\wedge\tau} d\hat{\psi}' = \int_{\hat{\mathcal{S}}\wedge\tau} d\hat{\psi}$ is defined, by 613J(c-i), and

$$Q_{(\mathcal{S}\wedge\tau)\hat{\ }}(d\hat{\psi}') = Q_{\hat{\mathcal{S}}\wedge\tau}(d\hat{\psi}') = Q_{\hat{\mathcal{S}}\wedge\tau}(d\hat{\psi}) \subseteq Q_{\hat{\mathcal{S}}}(d\hat{\psi})$$

(616Dd) is topologically bounded (613B(f-iii) again).

616H The next theorem depends on some machinery.

Lemma Suppose that

$$\epsilon > 0, \quad \gamma \ge 0, \quad m \ge 1, \quad m\epsilon \ge 2\gamma,$$

$$r \ge m, \quad 1 - \frac{r!}{r^m(r-m)!} \le \tfrac{1}{2}\epsilon^m, \quad k \ge 1, \quad 2k\epsilon^m \ge \epsilon, \quad n = rk.$$

Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and ψ an adapted interval function with domain $\mathcal{S}^{2\uparrow}$.

(a) Let $\langle a_i\rangle_{i<r}$ be a family in $\mathfrak{A}$ such that $\bar{\mu}a_i \ge \epsilon$ for every $i < r$. Then there is a $J \in [r]^m$ such that $\bar{\mu}(\inf_{i\in J} a_i) \ge \frac{1}{2}\epsilon^m$.

(b) Let $\tau_0 \le \ldots \le \tau_r$ in $\mathcal{S}$ be such that $\sup\{\theta(w) : w \in Q_{\mathcal{S}\cap[\tau_i,\tau_{i+1}]}(d\psi)\} > 4\epsilon$ for every $i < r$, while $z \in L^0(\mathfrak{A}_{\tau_0})$ is such that $\bar{\mu}[\![|z| \ge \gamma]\!] \le \epsilon$. Then there is a $w \in Q_{\mathcal{S}}(d\psi)$ such that $\bar{\mu}[\![|z+w| \ge \gamma]\!] \ge \bar{\mu}[\![|z| \ge \gamma]\!] + \frac{1}{2}\epsilon^m$.

(c) Let $\tau_0 \le \ldots \le \tau_n$ in $\mathcal{S}$ be such that $\sup\{\theta(w) : w \in Q_{\mathcal{S}\cap[\tau_i,\tau_{i+1}]}(d\psi)\} > 4\epsilon$ for every $i < n$. Then there is a $w \in Q_{\mathcal{S}}(d\psi)$ such that $\bar{\mu}[\![|w| \ge \gamma]\!] \ge \epsilon$.

(d) Let $\tau_0 \le \ldots \le \tau_n$ in $\mathcal{S}$ be such that $\theta(\psi(\tau_i,\tau_{i+1})) > 4\epsilon$ for every $i < n$. Then there is a $w \in Q_{\mathcal{S}}(d\psi)$ such that $\bar{\mu}[\![|w| \ge \gamma]\!] \ge \epsilon$.

proof (a) The case $m = 1$ is trivial. Otherwise, set $u = \frac{1}{r}\sum_{i=0}^{r-1}\chi a_i$ and $q = \frac{m}{m-1}$. Then

$$(\int u^m)^{1/m} = \|u\|_m\|\chi 1\|_q \ge \int u \ge \epsilon$$

by Hölder's inequality (244Eb), so if F is the set of injective functions from m to r,

$$\begin{aligned} r^m\epsilon^m \le r^m\int u^m &= \sum_{f\in r^m}\int\prod_{i<m}\chi a_{f(i)} = \sum_{f\in r^m}\bar{\mu}(\inf_{i<m} a_{f(i)}) \\ &\le r^m - \#(F) + \sum_{f\in F}\bar{\mu}(\inf_{i<m} a_{f(i)}) \le r^m - \#(F) + \#(F)\sup_{f\in F}\bar{\mu}(\inf_{i<m} a_{f(i)}) \end{aligned}$$

and there is a $J \in [r]^m$ such that

$$\begin{aligned} \bar{\mu}(\inf_{i\in J} a_i) &\ge \frac{1}{\#(F)}(r^m\epsilon^m - r^m + \#(F)) \\ &= \frac{r^m}{\#(F)}(\epsilon^m - (1 - \frac{\#(F)}{r^m})) \ge \epsilon^m - (1 - \frac{r!}{r^m(r-m)!}) \ge \frac{1}{2}\epsilon^m, \end{aligned}$$

as required.

(b) Set $c = [\![|z| \ge \gamma]\!] \in \mathfrak{A}_{\tau_0}$; we are supposing that $\bar{\mu}c \le \epsilon$. For each $i < r$, let $w_i \in Q_{\mathcal{S}\cap[\tau_i,\tau_{i+1}]}(d\psi)$ be such that $\theta(w_i) \ge 4\epsilon$; then

$$\begin{aligned} \bar{\mu}([\![w_i \ge \epsilon]\!] \setminus c) + \bar{\mu}([\![w_i \le -\epsilon]\!] \setminus c) &\ge \bar{\mu}[\![|w_i| \ge \epsilon]\!] - \bar{\mu}c \\ &\ge \theta(w_i) - \epsilon - \epsilon \ge 2\epsilon; \end{aligned}$$

set

$$K = \{i : i < r,\, \bar\mu([\![w_i \ge \epsilon]\!] \setminus c) \ge \epsilon\}, \quad K' = r \setminus K,$$

$$\begin{aligned} a_i &= [\![w_i \ge \epsilon]\!] \setminus c \text{ for } i \in K,\\ &= [\![-w_i \ge \epsilon]\!] \setminus c \text{ for } i \in K', \end{aligned}$$

so that $\bar\mu a_i \ge \epsilon$ for every $i < r$.

By (a), there is a set $J \in [r]^m$ such that $\bar\mu(\inf_{i\in J} a_i) \ge \frac{1}{2}\epsilon^m$. Set $d = \inf_{i\in J} a_i$,

$$\begin{aligned} w'_i &= w_i \times \chi(1 \setminus c) \text{ for } i \in J \cap K,\\ &= -w_i \times \chi(1 \setminus c) \text{ for } i \in J \cap K',\\ &= 0 \text{ for } i \in r \setminus J. \end{aligned}$$

Then $w'_i \in Q_{\mathcal{S}\cap[\tau_i,\tau_{i+1}]}(d\psi)$ for every i (616De) so $w = \sum_{i=0}^{r-1} w'_i$ belongs to $Q_{\mathcal{S}}(d\psi)$ (use 616Df repeatedly). Next,

$$\begin{aligned} w'_i \times \chi d &\ge \epsilon\chi d \text{ for } i \in J,\\ &= 0 \text{ for } i \in r \setminus J, \end{aligned}$$

so $w \times \chi d \ge m\epsilon\chi d \ge 2\gamma\chi d$ while $w \times \chi c = 0$. But this means that

$$[\![|w + z| \ge \gamma]\!] \supseteq ([\![w = 0]\!] \cap [\![|z| \ge \gamma]\!]) \cup ([\![|w| \ge 2\gamma]\!] \setminus [\![|z| \ge \gamma]\!]) \supseteq c \cup d,$$

and

$$\bar\mu[\![|w + z| \ge \gamma]\!] \ge \bar\mu c + \bar\mu d \ge \bar\mu[\![|z| \ge \gamma]\!] + \frac{1}{2}\epsilon^m,$$

as required.

(c) For $j < k$ set $\mathcal{S}_j = \mathcal{S} \cap [\tau_{jr}, \tau_{(j+1)r}]$. Choose $\langle z_j\rangle_{j\le k}$, $\langle w_j\rangle_{j<k}$ as follows. $z_0 = 0$. Given that $j < k$, $z_j \in L^0(\mathfrak{A}_{\tau_{jr}})$ and that $\bar\mu[\![|z_j| \ge \gamma]\!] \ge \min(\epsilon, \frac{1}{2}j\epsilon^m)$, (b) tells us that if $\bar\mu[\![|z_j| \ge \gamma]\!] \le \epsilon$ there is a $w_j \in Q_{\mathcal{S}_j}(d\psi)$ such that $\bar\mu[\![|z_j + w_j| \ge \gamma]\!] \ge \frac{1}{2}(j+1)\epsilon^m$. If $\bar\mu[\![|z_j| \ge \gamma]\!] > \epsilon$ take $w_j = 0$. Of course $w_j \in L^0(\mathfrak{A}_{\tau_{j+1}})$ and $[\![|z_j + w_j| \ge \gamma]\!] \ge \min(\epsilon, \frac{1}{2}(j+1)\epsilon^m)$ in either case, so we can set $z_{j+1} = z_j + w_j$, and continue. At the end of the induction, set $w = z_k$.

Inducing on j, using 616Df again for the inductive step, we see that $z_j \in Q_{\mathcal{S}\wedge\tau_{jr}}(d\psi)$ for every $j \le k$. So $w \in Q_{\mathcal{S}}(d\psi)$, while $\bar\mu[\![|w| \ge \gamma]\!] \ge \min(\epsilon, \frac{1}{2}k\epsilon^m) = \epsilon$, as required.

(d) Since $\psi(\tau_i, \tau_{i+1}) \in Q_{\mathcal{S}\cap[\tau_i,\tau_{i+1}]}(d\psi)$ for every $i < n$, this is a special case of (c).

616I Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a (local) integrator.
(a) The fully adapted extension of $\boldsymbol{v}$ to the covered envelope of $\mathcal{S}$ is a (local) integrator.
(b) $\boldsymbol{v}$ is (locally) moderately oscillatory, therefore (locally) order-bounded.
(c) $\Delta\boldsymbol{v}$ is a (locally) integrating interval function.

proof (a)(i) The point is that $Q_{\hat{\mathcal{S}}}(d\hat{\boldsymbol{v}})$ is included in the topological closure of $Q_{\mathcal{S}}(d\boldsymbol{v})$. **P** Take $z \in Q_{\hat{\mathcal{S}}}(d\hat{\boldsymbol{v}})$. Express z as $S_I(\boldsymbol{y}, d\hat{\boldsymbol{v}})$ where $I \in \mathcal{I}(\hat{\mathcal{S}})$, $\boldsymbol{y} = \langle y_\tau\rangle_{\tau\in I} \in M_{\text{fa}}(I)$ and $\sup|\boldsymbol{y}| \le \chi 1$. If $I = \emptyset$ then $z = 0 \in Q_{\mathcal{S}}(d\boldsymbol{v})$. Otherwise, let $(\tau_0, \dots, \tau_n)$ enumerate a maximal totally ordered subset of I. Let $\boldsymbol{w} = \langle w_\tau\rangle_{\tau\in\hat{\mathcal{S}}}$ be the simple process defined by saying that if $\tau \in \hat{\mathcal{S}}$ then

$$[\![\tau < \tau_0]\!] \cup [\![\tau_n \le \tau]\!] \subseteq [\![w_\tau = 0]\!],$$

$$[\![\tau_i \le \tau]\!] \cap [\![\tau < \tau_{i+1}]\!] \subseteq [\![w_\tau = y_{\tau_i}]\!] \text{ for } i < n$$

(612Ka once more). Then $\int_{\hat{\mathcal{S}}\wedge\tau_0} \boldsymbol{w}\, d\hat{\boldsymbol{v}} = 0$ because

$$w_\tau \times (\hat v_{\tau'} - \hat v_\tau) = w_\tau \times (\hat v_{\tau'} - \hat v_\tau) \times \chi[\![\tau < \tau']\!] = 0$$

whenever $\tau \le \tau' \le \tau_0$, and $\int_{\hat{\mathcal{S}}\vee\tau_n} \boldsymbol{w}\, d\hat{\boldsymbol{v}} = 0$ because $w_\tau = 0$ whenever $\tau_n \le \tau$. Accordingly

$$\int_{\hat{\mathcal S}} \boldsymbol{w}\, d\hat{\boldsymbol{v}} = \int_{\hat{\mathcal S}\wedge\tau_0} \boldsymbol{w}\, d\hat{\boldsymbol{v}} + \int_{\hat{\mathcal S}\cap[\tau_0,\tau_n]} \boldsymbol{w}\, d\hat{\boldsymbol{v}} + \int_{\hat{\mathcal S}\vee\tau_n} \boldsymbol{w}\, d\hat{\boldsymbol{v}}$$

(613J(c-ii))

$$= \int_{\hat{\mathcal S}\cap[\tau_0,\tau_n]} \boldsymbol{w}\, d\hat{\boldsymbol{v}} = \sum_{i=0}^{n-1} y_{\tau_i} \times (\hat v_{\tau_{i+1}} - \hat v_{\tau_i})$$

(614Bb)

$$= S_I(\boldsymbol{y}, d\hat{\boldsymbol{v}})$$

(613Ec)

$$= z.$$

Consequently $\int_{\mathcal S}(\boldsymbol{w}\restriction\mathcal S)d\boldsymbol{v}$ is defined and equal to z (613T). So

$$z = \lim_{J\uparrow\mathcal I(\mathcal S)} S_J(\boldsymbol{w}, d\boldsymbol{v}).$$

But $\sup|\boldsymbol{w}| \le \sup_{i\le n}|y_{\tau_i}| \le \chi 1$, so $S_J(\boldsymbol{w}, d\boldsymbol{v}) \in Q_{\mathcal S}(d\boldsymbol{v})$ for every $J \in \mathcal I(\mathcal S)$ and $z \in \overline{Q_{\mathcal S}(d\boldsymbol{v})}$. As z is arbitrary, we have the result. **Q**

(ii) If $\boldsymbol{v}$ is an integrator, then $Q_{\mathcal S}(d\boldsymbol{v})$ is topologically bounded, so its closure is also topologically bounded (613B(f-iii)) and $\hat{\boldsymbol{v}}$ is an integrator.

(iii) If $\boldsymbol{v}$ is a local integrator and $\tau \in \hat{\mathcal S}$, take any $\epsilon > 0$. Then there is a $\sigma \in \mathcal S$ such that $\bar\mu[\![\sigma < \tau]\!] \le \frac12\epsilon$. We are supposing that $\boldsymbol{v}\restriction\mathcal S\wedge\sigma$ is an integrator, so $\hat{\boldsymbol{v}}\restriction\hat{\mathcal S}\wedge\sigma$ is an integrator, and there is a $\delta > 0$ such that $\theta(\delta z) \le \frac12\epsilon$ whenever $z \in Q_{\hat{\mathcal S}\wedge\sigma}(d\hat{\boldsymbol{v}})$. Now take $z \in Q_{\hat{\mathcal S}\wedge\tau}(d\hat{\boldsymbol{v}})$. By 616C(iii), there is a fully adapted process $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\hat{\mathcal S}\wedge\tau}$ with breakpoint string $(\tau_0, \dots, \tau_n)$ such that $\sup|\boldsymbol{u}| \le \chi 1$ and $z = \sum_{i=0}^{n-1} u_{\tau_i} \times (\hat v_{\tau_{i+1}} - \hat v_{\tau_i})$. Consider

$$z' = \sum_{i=0}^{n-1} u_{\tau_i\wedge\sigma} \times (\hat v_{\tau_{i+1}\wedge\sigma} - \hat v_{\tau_i\wedge\sigma}).$$

By 616C(ii), $z' \in Q_{\hat{\mathcal S}\wedge\sigma}(d\hat{\boldsymbol{v}})$, while

$$\begin{aligned}[\![z \ne z']\!] &\subseteq \sup_{i<n}[\![u_{\tau_i} \ne u_{\tau_i\wedge\sigma}]\!] \cup \sup_{i\le n}[\![\hat v_{\tau_i} \ne \hat v_{\tau_i\wedge\sigma}]\!]\\ &\subseteq \sup_{i\le n}[\![\tau_i\wedge\sigma < \tau_i]\!] \subseteq [\![\sigma < \tau]\!]\end{aligned}$$

has measure at most $\frac12\epsilon$. So

$$\theta(\delta z) \le \theta(\delta z') + \theta(\delta(z - z')) \le \frac12\epsilon + \frac12\epsilon = \epsilon.$$

As ϵ is arbitrary, $Q_{\hat{\mathcal S}\wedge\tau}(d\hat{\boldsymbol{v}})$ is topologically bounded; as τ is arbitrary, $\hat{\boldsymbol{v}}$ is a local integrator.

(b)(i) Suppose to begin with that $\boldsymbol{v}$ is an integrator. By (a), $\hat{\boldsymbol{v}}$ is an integrator. The idea is to use 616Hd with $\psi = \Delta\hat{\boldsymbol{v}}$, knowing that $Q_{\hat{\mathcal S}}(d\psi) = Q_{\hat{\mathcal S}}(d\hat{\boldsymbol{v}})$ is topologically bounded. Express $\hat{\boldsymbol{v}}$ as $\langle\hat v_\sigma\rangle_{\sigma\in\hat{\mathcal S}}$. Let $\epsilon > 0$. Then there is a $\gamma > 0$ such that $\bar\mu[\![|w| \ge \gamma]\!] < \epsilon$ for every $w \in Q_{\mathcal S}(d\hat{\boldsymbol{v}})$ (613B(f-ii)). Take m, r, $k \ge 1$ such that $m\epsilon \ge 2\gamma$, $r \ge m$, $1 - \frac{r!}{r^m(r-m)!} \le \frac12\epsilon^m$ and $2k\epsilon^m \ge \epsilon$, and set $n = rk$. Then whenever $\tau_0 \le \dots \le \tau_n$ in $\hat{\mathcal S}$, there is an $i < n$ such that $\theta(\hat v_{\tau_{i+1}} - \hat v_{\tau_i}) \le 4\epsilon$, by 616Hd. Thus $\hat{\boldsymbol{v}}$ satisfies condition (iv) of 615N and is moderately oscillatory. It follows at once that $\boldsymbol{v}$ is moderately oscillatory (615F(a-i)).

(ii) If $\boldsymbol{v}$ is a local integrator, then we can apply (i) to $\boldsymbol{v}\restriction\mathcal S\wedge\tau$, for $\tau \in \mathcal S$, to see that $\boldsymbol{v}$ is locally moderately oscillatory.

(c)(i) Again, suppose to begin with that $\boldsymbol{v}$ is an integrator. If $\mathcal S$ is full, $\Delta\boldsymbol{v}$ is certainly an integrating interval function, because we know that $Q_{\mathcal S}(d\boldsymbol{v}) = Q_{\mathcal S}(d(\Delta\boldsymbol{v}))$ is topologically bounded, and from (b) we know that $v_\uparrow = \lim_{\sigma\uparrow\mathcal S} v_\sigma$ and $v_\downarrow = \lim_{\sigma\downarrow\mathcal S} v_\sigma$ are defined (615G), so

$$\int_{\mathcal S} d(\Delta\boldsymbol{v}) = \int_{\mathcal S} d\boldsymbol{v} = v_\uparrow - v_\downarrow$$

(613N) is defined. In general, (a) tells us that $\hat{\boldsymbol{v}}$ is an integrator, so $\Delta\hat{\boldsymbol{v}}$ is an integrating interval function; but $\Delta\hat{\boldsymbol{v}}$ extends $\Delta\boldsymbol{v}$, so is the strictly adapted extension of $\Delta\boldsymbol{v}$ to $\hat{\mathcal{S}}^{2\uparrow}$, and $\Delta\boldsymbol{v}$ is an integrating interval function, as remarked in 616F.

(ii) As in (b), it follows at once that if $\boldsymbol{v}$ is a local integrator then $\Delta\boldsymbol{v}$ is a locally integrating interval function.

616J Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and ψ an integrating interval function with domain $\mathcal{S}^{2\uparrow}$. Set

$$M_\psi = \{\boldsymbol{u} : \boldsymbol{u} \in M_{\text{o-b}}(\mathcal{S}),\ \int_{\mathcal{S}} \boldsymbol{u}\,d\psi \text{ is defined}\}.$$

Then M_ψ is a closed linear subspace of $M_{\text{o-b}}(\mathcal{S})$ and we have an indefinite integral operator $ii_\psi : M_\psi \to M_{\text{igtr}}(\mathcal{S})$ which is linear and continuous for the ucp topology on $M_{\text{o-b}}(\mathcal{S})$.

proof (a) Take any $\boldsymbol{u} \in M_\psi$.

(i) Setting $y_\sigma = \int_{\mathcal{S}\wedge\sigma} \boldsymbol{u}\,d\psi$ for $\sigma \in \mathcal{S}$, and $\boldsymbol{y} = \langle y_\sigma\rangle_{\sigma\in\mathcal{S}}$, $\boldsymbol{y}$ is defined everywhere on $\mathcal{S}$ and is fully adapted (613O(b-i)). Let $\epsilon > 0$. Then there is a $\delta > 0$ such that $\theta(S_I(\boldsymbol{w}, d\psi)) \le \epsilon$ whenever $I \in \mathcal{I}(\mathcal{S})$, $\boldsymbol{w} \in M_{\text{fa}}(I)$ and $\theta(\sup|\boldsymbol{w}|) \le \delta$ (616E). We are supposing that $\boldsymbol{u}$ is order-bounded; write $\bar{u}$ for $\sup|\boldsymbol{u}|$. Let $\eta > 0$ be such that $\theta(\bar{u} \times w) \le \delta$ whenever $w \in L^0(\mathfrak{A})$ and $\theta(w) \le \eta$ (613Ba).

(ii) Suppose that $I \in \mathcal{I}(\mathcal{S})$, $\boldsymbol{w} \in M_{\text{fa}}(I)$ and $\theta(\bar{w}) \le \eta$, where $\bar{w} = \sup|\boldsymbol{w}|$. Then $\theta(S_I(\boldsymbol{w}, d\boldsymbol{y})) \le \epsilon$. **P** If $I = \emptyset$ this is trivial.

($\boldsymbol{\alpha}$) If I is non-empty, let $(\tau_0, \dots, \tau_n)$ be the increasing enumeration of a maximal totally ordered subset of I, so that $S_I(\boldsymbol{w}, d\boldsymbol{y}) = \sum_{i=0}^{n-1} w_{\tau_i} \times (y_{\tau_{i+1}} - y_{\tau_i})$. We know that $y_{\tau'} = y_\tau + \int_{\mathcal{S}\cap[\tau,\tau']} \boldsymbol{u}\,d\psi$ whenever $\tau \le \tau'$ in $\mathcal{S}$ (613Jc), so

$$S_I(\boldsymbol{w}, d\boldsymbol{y}) = \sum_{i=0}^{n-1} w_{\tau_i} \times \int_{\mathcal{S}\cap[\tau_i,\tau_{i+1}]} \boldsymbol{u}\,d\psi.$$

Let $\tilde{\boldsymbol{w}} = \langle \tilde{w}_\sigma\rangle_{\sigma\in\mathcal{S}}$ be the simple process with domain $\mathcal{S}$ defined by saying that

$$[\![\sigma < \tau_0]\!] \subseteq [\![\tilde{w}_\sigma = 0]\!], \quad [\![\tau_n \le \sigma]\!] \subseteq [\![\tilde{w}_\sigma = w_{\tau_n}]\!],$$

$$[\![\tau_i \le \sigma]\!] \cap [\![\sigma < \tau_{i+1}]\!] \subseteq [\![\tilde{w}_\sigma = w_{\tau_i}]\!]$$

for $\sigma \in \mathcal{S}$ and $i < n$ (612Ka again); then $\tilde{\boldsymbol{w}}$ extends $\boldsymbol{w}$ and $\sup|\tilde{\boldsymbol{w}}| = \sup_{i\le n}|w_{\tau_i}| = \bar{w}$. Consequently $\sup|\boldsymbol{u} \times \tilde{\boldsymbol{w}}| \le \bar{u} \times \bar{w}$, $\theta(\sup|\boldsymbol{u} \times \tilde{\boldsymbol{w}}|) \le \delta$ and $\theta(S_J(\boldsymbol{u} \times \tilde{\boldsymbol{w}}), d\psi) \le \epsilon$ for every $J \in \mathcal{I}(\mathcal{S})$.

($\boldsymbol{\beta}$) For each $i < n$ let J_i be a finite sublattice of $\mathcal{S} \cap [\tau_i, \tau_{i+1}]$ containing τ_i and τ_{i+1}. Let $(\sigma_{i0}, \dots, \sigma_{im_i})$ be the increasing enumeration of a maximal totally ordered subset of J_i. Then

$$u_{\sigma_{ij}} \times \tilde{w}_{\sigma_{ij}} \times \psi(\sigma_{ij}, \sigma_{i,j+1}) = u_{\sigma_{ij}} \times w_{\tau_i} \times \psi(\sigma_{ij}, \sigma_{i,j+1})$$

for every $j < m_i$, because

$$[\![\psi(\sigma_{ij}, \sigma_{i,j+1}) \ne 0]\!] \subseteq [\![\sigma_{ij} < \sigma_{i,j+1}]\!] \subseteq [\![\sigma_{ij} < \tau_{i+1}]\!] \subseteq [\![\tilde{w}_{\sigma_{ij}} = w_{\tau_i}]\!].$$

Now

$$\begin{aligned} S_{J_i}(\boldsymbol{u} \times \tilde{\boldsymbol{w}}, d\psi) &= \sum_{j=0}^{m_i-1} u_{\sigma_{ij}} \times \tilde{w}_{\sigma_{ij}} \times \psi(\sigma_{ij}, \sigma_{i,j+1}) \\ &= \sum_{j=0}^{m_i-1} u_{\sigma_{ij}} \times w_{\tau_i} \times \psi(\sigma_{ij}, \sigma_{i,j+1}) = w_{\tau_i} \times S_{J_i}(\boldsymbol{u}, d\psi). \end{aligned}$$

Set $J = \bigcup_{i<n} J_i$, so that $J \in \mathcal{I}(\mathcal{S})$ and $J \cap [\tau_i, \tau_{i+1}] = J_i$ for each i. Then

$$\sum_{i=0}^{n-1} w_{\tau_i} \times S_{J_i}(\boldsymbol{u}, d\psi) = \sum_{i=1}^{n-1} S_{J_i}(\boldsymbol{u} \times \tilde{\boldsymbol{w}}, d\psi) = S_J(\boldsymbol{u} \times \tilde{\boldsymbol{w}}, d\psi)$$

(613Ga) and $\theta(\sum_{i=0}^{n-1} w_{\tau_i} \times S_{J_i}(\boldsymbol{u}, d\psi)) \le \epsilon$. Taking the limit as $J_i \uparrow \mathcal{I}(\mathcal{S} \cap [\tau_i, \tau_{i+1}])$ for each i,

$$\theta(S_I(\boldsymbol{w},d\boldsymbol{y}))=\theta(\sum_{i=0}^{n-1}w_{\tau_i}\times\int_{\mathcal{S}\cap[\tau_i,\tau_{i+1}]}\boldsymbol{u}\,d\psi)\le\epsilon.\ \mathbf{Q}$$

(iii) As ϵ is arbitrary, $Q_{\mathcal{S}}(d\boldsymbol{y})$ is topologically bounded, by 616E(iii)⇒(i), and $\boldsymbol{y}$ is an integrator.

Thus ii_ψ is a function from M_ψ to $M_{\text{igtr}}(\mathcal{S})$. By 613Jb, M_ψ is a linear subspace of $M_{\text{o-b}}(\mathcal{S})$ and ii_ψ is a linear operator. Recall from 616Ib that $M_{\text{igtr}}(\mathcal{S})\subseteq M_{\text{o-b}}(\mathcal{S})$.

(b) Suppose that $\boldsymbol{u}\in M_{\text{o-b}}(\mathcal{S})$ belongs to the closure of M_ψ for the ucp topology. Let $\epsilon>0$. By 616E, there is a $\delta>0$ such that $\theta(S_I(\boldsymbol{u}',d\psi))\le\epsilon$ whenever $I\in\mathcal{I}(\mathcal{S})$, $\boldsymbol{u}'\in M_{\text{o-b}}(\mathcal{S})$ and $\theta(\sup|\boldsymbol{u}'|)\le\delta$. Now there are a $\boldsymbol{u}'\in M_\psi$ such that $\theta(\sup|\boldsymbol{u}-\boldsymbol{u}'|)\le\delta$, and a $J\in\mathcal{I}(\mathcal{S})$ such that $\theta(S_I(\boldsymbol{u}',d\psi)-\int_{\mathcal{S}}\boldsymbol{u}'d\psi)\le\epsilon$ whenever $J\subseteq I\in\mathcal{I}(\mathcal{S})$. Suppose now that $I\in\mathcal{I}(\mathcal{S})$ includes J; then

$$\begin{aligned}\theta(S_I(\boldsymbol{u},d\psi)-\int_{\mathcal{S}}\boldsymbol{u}'d\psi)&\le\theta(S_I(\boldsymbol{u},d\psi)-S_I(\boldsymbol{u}',d\psi))+\theta(S_I(\boldsymbol{u}',d\psi)-\int_{\mathcal{S}}\boldsymbol{u}'d\psi)\\&\le\epsilon+\epsilon=2\epsilon.\end{aligned}$$

As ϵ is arbitrary, $\int_{\mathcal{S}}\boldsymbol{u}\,d\psi$ is defined (613Ja), and $\boldsymbol{u}\in M_\psi$. As $\boldsymbol{u}$ is arbitrary, M_ψ is closed for the ucp topology.

(c)(i) Let $\epsilon>0$. Because ψ is an integrating interval function there is a $\delta>0$ such that $\theta(S_I(\boldsymbol{u},d\psi))\le\epsilon^2$ whenever $I\in\mathcal{I}(\mathcal{S})$, $\boldsymbol{u}\in M_{\text{fa}}(I)$ and $\theta(\sup|\boldsymbol{u}|)\le\delta$ (616E). It follows that $\bar\mu[\![|S_I(\boldsymbol{u},d\psi)|>\epsilon]\!]\le\epsilon$ whenever $I\in\mathcal{I}(\mathcal{S})$, $\boldsymbol{u}\in M_{\text{fa}}(I)$ and $\theta(\sup|\boldsymbol{u}|)\le\delta$.

(ii) (The key.) Suppose that $I\in\mathcal{I}(\mathcal{S})$ is non-empty. Let $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in I}\in M_{\text{fa}}(\mathcal{I})$ be such that $\theta(\sup|\boldsymbol{u}|)\le\delta$. Then $\bar\mu[\![\sup_{\sigma\in I}|S_{I\wedge\sigma}(\boldsymbol{u},d\psi)|>\epsilon]\!]\le\epsilon$.

$\mathbf{P}$ $(\boldsymbol{\alpha})$ Take $(\tau_0,\dots,\tau_n)$ linearly generating the I-cells. For $i\le n$ write z_i for $S_{I\wedge\tau_i}(\boldsymbol{u},d\psi)$. Since $\tau_0,\dots,\tau_i)$ linearly generates the $I\wedge\tau_i$-cella (611Kg), $z_i=\sum_{j=0}^{i-1}u_{\tau_j}\times\psi(\tau_j,\tau_{j+1})$ (613Ec). Next, set $a_i=[\![|z_i|>\epsilon]\!]$, $y_i=u_{\tau_i}\times\chi(1\setminus\sup_{j\le i}a_j)$ and $b_i=a_i\setminus\sup_{j<i}a_j$ for $i\le n$. Then $z_i\in L^0(\mathfrak{A}_{\tau_i})$ (613Gb), $a_i\in\mathfrak{A}_{\tau_i}$, $y_i\in L^0(\mathfrak{A}_{\tau_i})$ and $b_i\in\mathfrak{A}_{\tau_i}$ for all $i\le n$, while $\sup_{i\le n}|y_i|\le\sup_{\sigma\in I}|u_\sigma|$ so $\theta(\sup_{i\le n}|y_i|)\le\delta$. By 612Ka again, there is a fully adapted process $\boldsymbol{w}=\langle w_\sigma\rangle_{\sigma\in I}$ such that

$$[\![\tau_i\le\sigma]\!]\cap[\![\sigma<\tau_{i+1}]\!]\subseteq[\![w_\sigma=y_i]\!]\text{ for }i<n,\qquad[\![\tau_n=\sigma]\!]\subseteq[\![w_\sigma=\tau_n]\!]$$

for every $\sigma\in I$. (The point here is that $\tau_0\le\sigma\le\tau_n$ for every $\sigma\in I$.) Set

$$z=S_I(\boldsymbol{w},d\psi)=\sum_{j=0}^{n-1}w_{\tau_j}\times\psi(\tau_j,\tau_{j+1})$$

(613Ec again)

$$=\sum_{j=0}^{n-1}y_j\times\psi(\tau_j,\tau_{j+1})$$

because $[\![\psi(\tau_j,\tau_{j+1})\ne0]\!]\subseteq[\![\tau_j<\tau_{j+1}]\!]$ for each j, by 613C(b-i).

Now if $\sigma\in I$ then $|w_\sigma|\le\sup_{i\le n}|y_i|$ so $\theta(\sup|\boldsymbol{w}|)\le\delta$ and $\bar\mu[\![|z|>\epsilon]\!]\le\epsilon$.

$(\boldsymbol{\beta})$ If $i\le n$,

$$\begin{aligned}b_i&\subseteq\inf_{i\le j<n}[\![y_j=0]\!]\cap\inf_{j<i}[\![y_j=u_{\tau_j}]\!]\\&\subseteq[\![\sum_{j=0}^{n-1}y_j\times\psi(\tau_j,\tau_{j+1})=\sum_{j=0}^{i-1}u_{\tau_j}\times\psi(\tau_j,\tau_{j+1})]\!]=[\![z=z_i]\!];\end{aligned}$$

as $b_i\subseteq[\![|z_i|>\epsilon]\!]$, $b_i\subseteq[\![|z|>\epsilon]\!]$. Next, $\langle S_{I\wedge\sigma}(\boldsymbol{u},d\psi)\rangle_{\sigma\in I}$ is fully adapted (613Gb again). If $\sigma\in I$ then $\sup_{i\le n}[\![\sigma=\tau_i]\!]=1$ (611Ke), so

$$\begin{aligned}[\![|S_{I\wedge\sigma}(\boldsymbol{u},d\psi)|>\epsilon]\!]&=\sup_{i\le n}[\![\sigma=\tau_i]\!]\cap[\![|S_{I\wedge\sigma}(\boldsymbol{u},d\psi)|>\epsilon]\!]\\&\subseteq\sup_{i\le n}[\![|S_{I\wedge\tau_i}(\boldsymbol{u},d\psi)|>\epsilon]\!]=\sup_{i\le n}a_i=\sup_{i\le n}b_i\subseteq[\![|z|>\epsilon]\!].\end{aligned}$$

Accordingly

$$[\![\sup_{\sigma\in I}|S_{I\wedge\sigma}(\boldsymbol{u},d\psi)|>\epsilon]\!]=\sup_{\sigma\in I}[\![|S_{I\wedge\sigma}(\boldsymbol{u},d\psi)|>\epsilon]\!]$$

(364L(a-ii))

$$\subseteq[\![|z|>\epsilon]\!]$$

has measure at most ϵ. **Q**

(iii) Take $\boldsymbol{u}\in M_\psi$ such that $\theta(\sup|\boldsymbol{u}|)\le\delta$. Then $\theta(\sup_{\sigma\in I}|S_{I\wedge\sigma}(\boldsymbol{u},d\psi)|)\le 2\epsilon$ for every $I\in\mathcal{I}(\mathcal{S})$, if we interpret the supremum as zero if I is empty. Write w_σ for $\int_{\mathcal{S}\wedge\sigma}\boldsymbol{u}\,d\psi$.

($\boldsymbol{\alpha}$) If $I\in\mathcal{I}(\mathcal{S})$ then $\theta(\sup_{\sigma\in I}|w_\sigma|)\le 2\epsilon$. **P** Let $\eta>0$. For each $\sigma\in I$ let $J_\sigma\in\mathcal{I}(\mathcal{S}\wedge\sigma)$ be such that $\theta(S_K(\boldsymbol{u},d\psi)-w_\sigma)\le\eta$ whenever $J_\sigma\subseteq K\in\mathcal{I}(\mathcal{S}\wedge\sigma)$. Let K be the sublattice of $\mathcal{S}$ generated by $I\cup\bigcup_{\sigma\in I}J_\sigma$. Then $J_\sigma\subseteq K\wedge\sigma\in\mathcal{I}(\mathcal{S}\wedge\sigma)$ for every $\sigma\in I$. So

$$\begin{aligned}\theta(\sup_{\sigma\in I}|w_\sigma|)&\le\theta(\sup_{\sigma\in I}|S_{K\wedge\sigma}(\boldsymbol{u},d\psi)|+\sup_{\sigma\in I}|w_\sigma-S_{K\wedge\sigma}(\boldsymbol{u},d\psi)|)\\&\le\theta(\sup_{\sigma\in K}|S_{K\wedge\sigma}(\boldsymbol{u},d\psi)|)+\sum_{\sigma\in I}\theta(w_\sigma-S_{K\wedge\sigma}(\boldsymbol{u},d\psi))\le 2\epsilon+\eta\#(I).\end{aligned}$$

As η is arbitrary, $\theta(\sup_{\sigma\in I}|w_\sigma|)\le 2\epsilon$. **Q**

($\boldsymbol{\beta}$) We know that $ii_\psi(\boldsymbol{u})$ is order-bounded, so $\bar w=\sup_{\sigma\in\mathcal{S}}|w_\sigma|$ is defined in $L^0(\mathfrak{A})$. Now

$$\begin{aligned}\theta(\sup|ii_\psi(\boldsymbol{u})|)=\theta(\bar w)&=\sup_{I\subseteq\mathcal{S}\text{ is finite}}\theta(\sup_{\sigma\in I}|w_\sigma|)\\&=\sup_{I\in\mathcal{I}(\mathcal{S})}\theta(\sup_{\sigma\in I}|w_\sigma|)\le 2\epsilon.\end{aligned}$$

As ϵ was arbitrary, $ii_\psi:M_\psi\to M_{\text{o-b}}(\mathcal{S})$ is continuous at 0; being a linear operator, it is continuous everywhere on M_ψ, as claimed.

616K Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u}$ a moderately oscillatory process with domain $\mathcal{S}$, and $\boldsymbol{v}$ an integrator with domain $\mathcal{S}$. Then $\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}$ is defined, and $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is an integrator.

proof By 616I, $\boldsymbol{v}$ is moderately oscillatory and $\Delta\boldsymbol{v}$ is an integrating interval function. By 615Ra, $\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}$ is defined if $\boldsymbol{u}$ is of bounded variation. In the language of 616J, $M_{\text{bv}}(\mathcal{S})\subseteq M_{\Delta\boldsymbol{v}}$, while $M_{\Delta\boldsymbol{v}}$ is closed in $M_{\text{o-b}}(\mathcal{S})$ for the ucp topology. But $M_{\text{mo}}(\mathcal{S})$ is just the closure of $M_{\text{bv}}(\mathcal{S})$ in $M_{\text{o-b}}(\mathcal{S})$. so is included in $M_{\Delta\boldsymbol{v}}$, that is, $\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}$ is defined. As for the indefinite integral $ii_{\boldsymbol{v}}(\boldsymbol{u})$, this is an integrator by 616J.

616L Corollary Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. If $\boldsymbol{u}$ is a locally moderately oscillatory process and $\boldsymbol{v}$ a fully adapted process which is locally of bounded variation, both with domain $\mathcal{S}$, then $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is locally of bounded variation.

proof Apply 614T to $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ and $\boldsymbol{v}\restriction\mathcal{S}\wedge\tau$ for each $\tau\in\mathcal{S}$.

616M Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and ψ an integrating interval function on $\mathcal{S}$. Write $\boldsymbol{v}$ for $ii_\psi(\mathbf{1})=\langle\int_{\mathcal{S}\wedge\tau}d\psi\rangle_{\tau\in\mathcal{S}}$. Then $\int_{\mathcal{S}}\boldsymbol{u}\,d\psi$ is defined and equal to $\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}$ whenever $\boldsymbol{u}$ is a moderately oscillatory process with domain $\mathcal{S}$.

proof (a) I start by noting that $\boldsymbol{v}$ is well-defined and is an integrator, by 616J. If $\mathcal{S}$ is empty the result is trivial, so suppose otherwise. For the time being, I will suppose that $\boldsymbol{v}=\mathbf{0}$ and $\mathcal{S}$ is full. Note that the definition 616Fa requires $\int_{\mathcal{S}}d\psi$ to be defined, and we now have $\int_{\mathcal{S}}d\psi=\lim_{\tau\uparrow\mathcal{S}}\int_{\mathcal{S}\wedge\tau}d\psi=0$ by 613J(f-ii). Write M'_ψ for

$$\{\boldsymbol{u}:\boldsymbol{u}\in M_{\text{o-b}}(\mathcal{S}),\ \int_{\mathcal{S}}\boldsymbol{u}\,d\psi\text{ is defined and equal to }0\}.$$

Then M'_ψ is a linear subspace of $M_{\text{o-b}} = M_{\text{o-b}}(\mathcal{S})$, and by 616J it is closed for the ucp topology. Note that the constant process $\mathbf{1}$ with domain $\mathcal{S}$ belongs to M'_ψ.

(b) To begin with, suppose that $\mathcal{S}$ is full, that $\boldsymbol{v} = \mathbf{0}$, that $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ is non-increasing and that $\boldsymbol{u}$ is $\{0,1\}$-valued, that is, $[\![u_\sigma \in \{0,1\}]\!] = 1$ for every $\sigma \in \mathcal{S}$.

(i) If $I \in \mathcal{I}(\mathcal{S})$ is non-empty, there is a $\tau \in \mathcal{S}$ such that $S_I(\boldsymbol{u}, d\psi) = S_{I\wedge\tau}(\mathbf{1}, d\psi)$. **P**

($\boldsymbol{\alpha}$) Take $(\tau_0, \dots, \tau_n)$ linearly generating the I-cells; note that $\tau_n = \max I$. For $i < n$, set $a_i = [\![u_{\tau_i} = 0]\!]$ and $b_i = a_i \setminus \sup_{j<i} a_j$; set $b_n = 1 \setminus a_{n-1}$. Because $\boldsymbol{u}$ is non-increasing, $a_0 \subseteq a_1 \subseteq \dots \subseteq a_{n-1}$ and $\langle b_i \rangle_{i \le n}$ is a partition of unity in $\mathfrak{A}$. Also a_i and b_i belong to $\mathfrak{A}_{\tau_i}$ for $i < n$ and $b_n \in \mathfrak{A}_{\tau_n}$. By 611I there is a $\tau \in \mathcal{T}$ such that $b_i \subseteq [\![\tau = \tau_i]\!]$ for $i \le n$; because $\mathcal{S}$ is full, $\tau \in \mathcal{S}$.

($\boldsymbol{\beta}$) If $j < i < n$ then

$$\begin{aligned} b_i \subseteq [\![\tau = \tau_i]\!] &\subseteq [\![\tau \wedge \tau_j = \tau_j]\!] \cap [\![\tau \wedge \tau_{j+1} = \tau_{j+1}]\!] \\ &\subseteq [\![\psi(\tau \wedge \tau_j, \tau \wedge \tau_{j+1}) = \psi(\tau_j, \tau_{j+1})]\!] \end{aligned}$$

because ψ is strictly adapted, and if $i \le j < n$ then

$$\begin{aligned} b_i \subseteq [\![\tau = \tau_i]\!] &\subseteq [\![\tau \wedge \tau_j = \tau]\!] \cap [\![\tau \wedge \tau_{j+1} = \tau]\!] \\ &\subseteq [\![\psi(\tau \wedge \tau_j, \tau \wedge \tau_{j+1}) = \psi(\tau, \tau)]\!] \subseteq [\![\psi(\tau \wedge \tau_j, \tau \wedge \tau_{j+1}) = 0]\!]. \end{aligned}$$

($\boldsymbol{\gamma}$) If $i < n$ then

$$\chi b_i \times S_I(\boldsymbol{u}, d\psi) = \sum_{j=0}^{n-1} \chi b_i \times u_{\tau_j} \times \psi(\tau_j, \tau_{j+1}) = \sum_{j=0}^{i-1} \chi b_i \times u_{\tau_j} \times \psi(\tau_j, \tau_{j+1})$$

(because if $i \le j$ then $b_i \subseteq a_i \subseteq a_j$ and $b_i \subseteq [\![u_{\tau_j} = 0]\!]$)

$$= \sum_{j=0}^{i-1} \chi b_i \times \psi(\tau \wedge \tau_j, \tau \wedge \tau_{j+1})$$

(as in (β) just above)

$$= \sum_{j=0}^{n-1} \chi b_i \times \psi(\tau \wedge \tau_j, \tau \wedge \tau_{j+1})$$

(because if $i \le j < n$ then $b_i \subseteq [\![\psi(\tau \wedge \tau_j, \tau \wedge \tau_{j+1}) = 0]\!]$, by the other half of ($\beta$))

$$= \chi b_i \times S_{I\wedge\tau}(\mathbf{1}, d\psi)$$

because $(\tau \wedge \tau_0, \dots, \tau \wedge \tau_n)$ linearly generates the $(I \wedge \tau)$-cells (611Kg). Also

$$\begin{aligned} b_n &\subseteq [\![\tau = \tau_n]\!] \cap \inf_{i<n} [\![u_{\tau_i} = 1]\!] \\ &\subseteq [\![S_{I\wedge\tau}(\mathbf{1}, d\psi) = S_I(\mathbf{1}, d\psi)]\!] \cap [\![S_I(\boldsymbol{u}, d\psi) = S_I(\mathbf{1}, d\psi)]\!] \\ &\subseteq [\![S_I(\boldsymbol{u}, d\psi) = S_{I\wedge\tau}(\mathbf{1}, d\psi)]\!], \end{aligned}$$

so in fact $S_I(\boldsymbol{u}, d\psi) = S_{I\wedge\tau}(\mathbf{1}, d\psi)$. **Q**

(ii) Consequently $\boldsymbol{u} \in M'_\psi$. **P** Let $\epsilon > 0$. Then there is an $I_0 \in \mathcal{I}(\mathcal{S})$, containing $\max \mathcal{S}$, such that $\theta(S_J(\mathbf{1}, d\psi) - \theta(S_K(\mathbf{1}, d\psi)) \le \epsilon$ whenever J, $K \in \mathcal{I}(\mathcal{S})$ include I_0. If $I \in \mathcal{I}(\mathcal{S})$ includes I_0, there is a $\tau \in \mathcal{S}$ such that $S_I(\boldsymbol{u}, d\psi) = S_{I\wedge\tau}(\mathbf{1}, d\psi)$, by (i). At the same time, of course, $\theta(S_J(\mathbf{1}, d\psi) - \theta(S_K(\mathbf{1}, d\psi)) \le \epsilon$ whenever J, $K \in \mathcal{I}(\mathcal{S})$ include I. By 613V(ii-β),

$$2\epsilon \ge \theta(S_{I\wedge\tau}(\mathbf{1}, d\psi) - \int_{\mathcal{S}\wedge\tau} d\psi) = \theta(S_{I\wedge\tau}(\mathbf{1}, d\psi))$$

(because $\boldsymbol{v} = \mathbf{0}$)

$$= \theta(S_I(\boldsymbol{u}, d\psi)),$$

and this is true whenever I includes I_0. As ϵ is arbitrary, $\int_{\mathcal{S}} \boldsymbol{u}\, d\psi = 0$ and $\boldsymbol{u} \in M'_\psi$. **Q**

(c) Still supposing that $\mathcal{S}$ is full and $\boldsymbol{v} = \mathbf{0}$, I explore M'_ψ.

(i) If $\boldsymbol{u} \in M_{\text{fa}} = M_{\text{fa}}(\mathcal{S})$ is $\{0,1\}$-valued and non-decreasing, then $\boldsymbol{u} \in M'_\psi$. **P** $\mathbf{1} - \boldsymbol{u}$ is $\{0,1\}$-valued and non-increasing, so (b) tells us that it belongs to M'_ψ; as noted in (a), $\mathbf{1} \in M'_\psi$; as M'_ψ is a linear subspace, $\boldsymbol{u} \in M'_\psi$. **Q**

(ii) If $n \ge 1$ and $\boldsymbol{u} \in M_{\text{fa}}$ is non-decreasing and takes values in $\{0, 1, \dots, n\}$, then $\boldsymbol{u} \in M'_\psi$. **P** Induce on n. If $n = 1$ this is just (i) above. For the inductive step, if $\boldsymbol{u}$ takes values in $\{0, \dots, n+1\}$ then $\boldsymbol{u} \wedge n\mathbf{1}$ is non-decreasing and takes values in $\{0, \dots, n\}$, so belongs to M'_ψ by the inductive hypothesis, while $\boldsymbol{u} - \boldsymbol{u} \wedge n\mathbf{1} = (\boldsymbol{u} - n\mathbf{1})^+$ is non-decreasing and takes values in $\{0,1\}$; so their sum $\boldsymbol{u}$ belongs to M'_ψ and the induction continues. **Q**

It follows at once that if $n \ge 1$, $\delta > 0$ and $\boldsymbol{u} \in M_{\text{fa}}$ is non-decreasing and takes values in $\{0, \delta, \dots, n\delta\}$ then $\boldsymbol{u} \in M'_\psi$.

(iii) If $\boldsymbol{u} \in M_{\text{o-b}}$ is non-negative and non-decreasing it belongs to M'_ψ. **P** Write $\bar{u}$ for $\sup|\boldsymbol{u}|$, Let $\epsilon > 0$. Then there is an $n \ge 1$ such that $\bar{\mu}[\![\bar{u} > n\epsilon]\!] \le \epsilon$. Define $h : \mathbb{R} \to \mathbb{R}$ by saying that

$$\begin{aligned} h(\alpha) &= n\epsilon \text{ if } \alpha \ge n\epsilon, \\ &= i\epsilon \text{ if } i \le n \text{ and } i\epsilon \le \alpha < (i+1)\epsilon, \\ &= \alpha \text{ if } \alpha \le 0. \end{aligned}$$

Then h is non-decreasing and Borel measurable, $h(\alpha) \le \alpha$ for every α and $\alpha \le h(\alpha) + \epsilon$ if $\alpha \le n\epsilon$. So $\boldsymbol{u}' = \bar{h}\boldsymbol{u}$ is fully adapted and non-decreasing; $\boldsymbol{u}' \le \boldsymbol{u}$; because $\boldsymbol{u}$ is non-negative, $\boldsymbol{u}'$ takes values in $\{0, \epsilon, \dots, n\epsilon\}$; and $[\![\boldsymbol{u} \le n\epsilon]\!] \subseteq [\![\boldsymbol{u} \le \boldsymbol{u}' + \epsilon\mathbf{1}]\!]$. By (ii) just above, $\boldsymbol{u}' \in M'_\psi$. And

$$\mathbf{0} \le \boldsymbol{u} - \boldsymbol{u}' \le \epsilon\mathbf{1} + \chi[\![\bar{u} > n\epsilon]\!] \times \bar{u}, \qquad \theta(\sup|\boldsymbol{u} - \boldsymbol{u}'|) \le \epsilon + \bar{\mu}[\![\bar{u} > n\epsilon]\!] \le 2\epsilon.$$

As ϵ is arbitrary and M'_ψ is closed, $\boldsymbol{u} \in M'_\psi$. **Q**

(iv) Because M'_ψ is a closed linear subspace of $M_{\text{o-b}}$, it includes $M_{\text{bv}} = M_{\text{bv}}(\mathcal{S})$ and its closure $M_{\text{mo}} = M_{\text{mo}}(\mathcal{S})$. Thus we have the required result if $\mathcal{S}$ is full and $\boldsymbol{v} = \mathbf{0}$.

(d) Still supposing that $\mathcal{S}$ is full, take $\boldsymbol{u} \in M_{\text{mo}}$ and any integrating interval function ψ on $\mathcal{S}$, and set $v_\tau = \int_{\mathcal{S} \wedge \tau} d\psi$ for $\tau \in \mathcal{S}$, Then $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ is an integrator (616J) and $\Delta\boldsymbol{v}$ is an integrating interval function (616Ic), so $\psi' = \psi - \Delta\boldsymbol{v}$ also is (616Ga). For $\tau \in \mathcal{S}$,

$$\begin{aligned} \int_{\mathcal{S} \wedge \tau} d\psi' &= \int_{\mathcal{S} \wedge \tau} d\psi - \int_{\mathcal{S} \wedge \tau} d(\Delta\boldsymbol{v}) = v_\tau - \int_{\mathcal{S} \wedge \tau} d\boldsymbol{v} = v_\tau - v_\tau + \lim_{\sigma \downarrow \mathcal{S}} v_\sigma \\ &= 0 \end{aligned}$$

(613N)

(613J(f-i)). So $\mathcal{S}$ and ψ' satisfy the conditions of (a)-(c), and $\int_{\mathcal{S}} \boldsymbol{u}\, d\psi'$ is defined and equal to 0. But we know already from 616K that $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$ is defined, so $\int_{\mathcal{S}} \boldsymbol{u}\, d\psi$ is defined and equal to $\int_{\mathcal{S}} \boldsymbol{u}\, d\psi' + \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v} = \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$.

(e) Finally, if $\mathcal{S}$ is not full, write $\hat{\mathcal{S}}$ for its covered envelope, $\hat{\boldsymbol{u}}$ for the fully adapted extension of $\boldsymbol{u} \in M_{\text{mo}}$ to $\hat{\mathcal{S}}$, and $\hat{\psi}$ for the strictly adapted extension of ψ to $\hat{\mathcal{S}}^{2\uparrow}$. Then $\hat{\psi}$ is an integrating interval function (616F) and $\hat{\boldsymbol{u}}$ is moderately oscillatory (615F(a-vi)). Write $\hat{\boldsymbol{v}}$ for the indefinite integral $ii_{\hat{\psi}}(\mathbf{1})$. If $\tau \in \mathcal{S}$, $\hat{\mathcal{S}} \wedge \tau$ is the covered envelope of $\mathcal{S} \wedge \tau$ and $\int_{\hat{\mathcal{S}} \wedge \tau} \mathbf{1}\, d\hat{\psi}$ is defined, so is equal to $\int_{\mathcal{S} \wedge \tau} \mathbf{1}\, d\psi$ (613T again); thus $\boldsymbol{v} = \hat{\boldsymbol{v}} \restriction \mathcal{S}$ is the indefinite integral $ii_\psi(\mathbf{1})$. Now $\int_{\hat{\mathcal{S}}} \hat{\boldsymbol{u}}\, d\hat{\psi}$ is defined and equal to $\int_{\hat{\mathcal{S}}} \hat{\boldsymbol{u}}\, d\hat{\boldsymbol{v}}$, by (d); by 613T once more, $\int_{\mathcal{S}} \boldsymbol{u}\, d\psi$ and $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$ are defined and equal. So we have the general result.

616N So far, I have been working almost entirely with general integrating interval functions. But 616M makes it plain that we can expect that usually it will be enough to look at integration with respect to integrators, which is what most of the rest of this volume will be devoted to. As we have already seen in 616I, integrators have some special properties, which I will now set out to describe.

Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a (local) integrator. If $f : \mathbb{R} \to \mathbb{R}$ is convex, then $\bar{f}\boldsymbol{v}$ is a (local) integrator.

proof (a) To begin with (down to the end of (c) below) suppose that $\boldsymbol{v}$ is an integrator. Note straight away that f is continuous (6A1Aa), so that $\bar{f} : L^0 \to L^0$ is continuous (613Bb). As $\boldsymbol{v}$ is moderately oscillatory, therefore order-bounded (616I), we see from 615G that $\lim_{\sigma\downarrow\mathcal{S}} \bar{f}(v_\sigma) = \bar{f}(\lim_{\sigma\downarrow\mathcal{S}} v_\sigma)$, $\lim_{\sigma\uparrow\mathcal{S}} \bar{f}(v_\sigma) = \bar{f}(\lim_{\sigma\uparrow\mathcal{S}} v_\sigma)$ are defined, while $\bar{f}\boldsymbol{v}$ is order-bounded (614F(c-i)).

Write Q for $Q_{\mathcal{S}}(d\boldsymbol{v})$ and Q^* for $Q_{\mathcal{S}}(d(\bar{f}\boldsymbol{v}))$. We know that Q is topologically bounded, and we need to show that Q^* is topologically bounded.

Let g be the right derivative of f, so that g is non-decreasing and $(y - x)g(x) \le f(y) - f(x)$ for all x, $y \in \mathbb{R}$ (6A1Ab). Consequently

$$\bar{g}(v_\sigma) \times (v_{\sigma'} - v_\sigma) \le \bar{f}(v_{\sigma'}) - \bar{f}(v_\sigma)$$

for all σ, $\sigma' \in \mathcal{S}$.

By 616Ib and 614F(c-i), $\bar{f}\boldsymbol{v}$ is order-bounded; set $w = \sup|\bar{f}\boldsymbol{v}| \vee \sup|\boldsymbol{v}|$.

(b) Suppose for the moment that $|g(x)| \le M$ for every $x \in \mathbb{R}$, where $M \ge 0$. Because the solid hull of a topologically bounded set is topologically bounded, a scalar multiple of a topologically bounded set is topologically bounded and the algebraic sum of two topologically bounded sets is topologically bounded (613Bf), $A_0 = [-2w, 2w] + MQ$, the solid hull A_1 of A_0 and $A = A_1 + MQ$ are topologically bounded. Now $Q^* \subseteq A$. **P** Take $z \in Q^*$. Then there are a finite sublattice I of $\mathcal{S}$ and a fully adapted family $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in I}$ such that $|u_\sigma| \le \chi 1$ for $\sigma \in I$ and $z = S_I(\boldsymbol{u}, d(\bar{f}\boldsymbol{v}))$. If $I = \emptyset$ then $z = 0$ surely belongs to A. Otherwise, let $\tau_0 \le \ldots \le \tau_n$ linearly generate the I-cells. For $i \le n$ set

$$w_i = \bar{f}(v_{\tau_i}) - \bar{f}(v_{\tau_0}) - \textstyle\sum_{j=0}^{i-1} \bar{g}(v_{\tau_j}) \times (v_{\tau_{j+1}} - v_{\tau_j}) \in [-2w, 2w] + MQ = A_0$$

because $|\bar{f}(v_{\tau_i})| \le w$, $|\bar{f}(v_{\tau_0})| \le w$ and $|\bar{g}(v_{\tau_j}| \le M\chi 1$ for every j.

For $i < n$,

$$w_{i+1} - w_i = \bar{f}(v_{\tau_{i+1}}) - \bar{f}(v_{\tau_i}) - \bar{g}(v_{\tau_i}) \times (v_{\tau_{i+1}} - v_{\tau_i}) \ge 0.$$

Now we have

$$z = \sum_{i=0}^{n-1} u_{\tau_i} \times (\bar{f}(v_{\tau_{i+1}}) - \bar{f}(v_{\tau_i}))$$

(613Ec)

$$= \sum_{i=0}^{n-1} u_{\tau_i} \times (w_{i+1} - w_i) + \sum_{i=0}^{n-1} u_{\tau_i} \times \bar{g}(v_{\tau_i}) \times (v_{\tau_{i+1}} - v_{\tau_i}).$$

But

$$|\sum_{i=0}^{n-1} u_{\tau_i} \times (w_{i+1} - w_i)| \le \sum_{i=0}^{n-1} |u_{\tau_i}| \times |w_{i+1} - w_i| \le \sum_{i=0}^{n-1} |w_{i+1} - w_i|$$
$$= \sum_{i=0}^{n-1} w_{i+1} - w_i = w_n - w_0 = w_n \in A_0,$$

so $\sum_{i=0}^{n-1} u_{\tau_i} \times (w_{i+1} - w_i) \in A_1$; while

$$\textstyle\sum_{i=0}^{n-1} u_{\tau_i} \times \bar{g}(v_{\tau_i}) \times (v_{\tau_{i+1}} - v_{\tau_i}) \in MQ$$

because $|u_{\tau_i} \times \bar{g}(v_{\tau_i})| \le M\chi 1$ for every i. So $z \in A_1 + MQ = A$. As z is arbitrary, $Q^* \subseteq A$. **Q**

Thus in this case Q^* is topologically bounded.

(c) For general g, take $\epsilon > 0$. Let $M \ge 0$ be such that $\bar{\mu}[\![w > M]\!] \le \epsilon$. Define $f_1 : \mathbb{R} \to \mathbb{R}$ by setting

$$\begin{aligned} f_1(x) &= f(x) \text{ if } |x| \le M, \\ &= f(-M) + (-M-x)g(-M) \text{ if } x \le -M, \\ &= f(M) + (M-x)g(M) \text{ if } x \ge M. \end{aligned}$$

Then f_1 is convex and its right derivative takes values in $[g(-M), g(M)]$ so is bounded.

By (b), there is a $\delta > 0$ such that $\theta(\delta z_1) \le \epsilon$ for every $z_1 \in Q_{\mathcal{S}}(\bar{f}_1\boldsymbol{v})$. Now take $z \in Q^*$. Express z as $S_I(\boldsymbol{u}, d(\bar{f}\boldsymbol{v}))$ where I is a finite sublattice of $\mathcal{S}$ and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in I}$ is a fully adapted family such that $|u_\sigma| \le \chi 1$ for $\sigma \in I$. Set $z_1 = S_I(\boldsymbol{u}, d(\bar{f}_1\boldsymbol{v}))$. Then $z_1 \in Q_{\mathcal{S}}(\bar{f}_1\boldsymbol{v})$ and $\theta(\delta z_1) \le \epsilon$. But

$$[\![z_1 \ne z]\!] \subseteq \sup_{\sigma\in\mathcal{S}} [\![\bar{f}_1(v_\sigma) \ne \bar{f}(v_\sigma)]\!] \subseteq \sup_{\sigma\in\mathcal{S}} [\![|v_\sigma| > M]\!]$$

(because $f_1(x) = f(x)$ if $|x| \le M$)

$$\subseteq [\![w > M]\!]$$

has measure at most ϵ, and

$$\theta(\delta z) \le \theta(\delta z_1) + \bar{\mu}[\![z_1 \ne z]\!] \le 2\epsilon.$$

This is true for every $z \in Q^*$. As ϵ is arbitrary, Q^* is topologically bounded, and $\bar{f}\boldsymbol{v}$ is an integrator.

(d) If $\boldsymbol{v}$ is a local integrator, (a)-(c) show that $(\bar{f}\boldsymbol{v})\restriction\mathcal{S}\wedge\tau = \bar{f}(\boldsymbol{v}\restriction\mathcal{S}\wedge\tau)$ is an integrator for every $\tau \in \mathcal{S}$, so $\bar{f}\boldsymbol{v}$ is a local integrator.

616O Corollary If $\boldsymbol{v}$ is a (local) integrator and $f : \mathbb{R} \to \mathbb{R}$ is a function, absolutely continuous on every bounded interval in $\mathbb{R}$, such that its derivative f' has bounded variation on every bounded interval, then $\bar{f}\boldsymbol{v}$ is a (local) integrator.

proof f is expressible as a difference of two convex functions (6A1B), so $\bar{f}\boldsymbol{v}$ is the difference of two (local) integrators and is a (local) integrator.

616P Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.

(a) $M_{\text{igtr}}(\mathcal{S})$ is an f-subalgebra of the space $M_{\text{mo}}(\mathcal{S})$ of moderately oscillatory processes with domain $\mathcal{S}$.

(b)(i) Constant processes are integrators.

(ii) If $\boldsymbol{v} \in M_{\text{igtr}}(\mathcal{S})$ then $\boldsymbol{v}\restriction\mathcal{S}' \in M_{\text{igtr}}(\mathcal{S}')$ for any sublattice $\mathcal{S}'$ of $\mathcal{S}$. In particular, $\boldsymbol{v}$ is a local integrator.

(iii) Suppose that $\boldsymbol{v} \in M_{\text{fa}}(\mathcal{S})$ and for every $\epsilon > 0$ there is a $\boldsymbol{v}' \in M_{\text{igtr}}(\mathcal{S})$ such that $\bar{\mu}[\![\boldsymbol{v} \ne \boldsymbol{v}']\!] \le \epsilon$. Then $\boldsymbol{v} \in M_{\text{igtr}}(\mathcal{S})$.

(iv) If $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma\in\mathcal{S}} \in M_{\text{igtr}}(\mathcal{S})$ and $z \in L^0(\mathfrak{A} \cap \bigcap_{\sigma\in\mathcal{S}} \mathfrak{A}_\sigma)$, then $z\boldsymbol{v} = \langle z \times v_\sigma \rangle_{\sigma\in\mathcal{S}} \in M_{\text{igtr}}(\mathcal{S})$.

(v) If $\boldsymbol{v} \in M_{\text{fa}}(\mathcal{S})$ then $\boldsymbol{v} \in M_{\text{igtr}}(\mathcal{S})$ iff $\boldsymbol{v}\restriction\mathcal{S}\wedge\tau \in M_{\text{igtr}}(\mathcal{S}\wedge\tau)$ and $\boldsymbol{v}\restriction\mathcal{S}\vee\tau \in M_{\text{igtr}}(\mathcal{S}\vee\tau)$.

proof (a) We saw in 616Ib that $M_{\text{igtr}}(\mathcal{S}) \subseteq M_{\text{mo}}(\mathcal{S})$. Since a sum of topologically bounded sets is topologically bounded, 616Dc shows that $M_{\text{igtr}}(\mathcal{S})$ is closed under addition. By 616N, $\bar{h}\boldsymbol{v} \in M_{\text{igtr}}(\mathcal{S})$ whenever $h : \mathbb{R} \to \mathbb{R}$ is convex and $\boldsymbol{v} \in M_{\text{igtr}}(\mathcal{S})$, so $M_{\text{igtr}}(\mathcal{S})$ is an f-subalgebra of $M_{\text{mo}}(\mathcal{S})$ (612Bc).

(b)(i) If $\boldsymbol{v}$ is constant then $Q_{\mathcal{S}}(d\boldsymbol{v}) = \{0\}$.

(ii) As in 616Dd, $Q_{\mathcal{S}}(d(\boldsymbol{v}\restriction\mathcal{S}')) \subseteq Q_{\mathcal{S}}(d\boldsymbol{v})$.

(iii) Let $\epsilon > 0$. Then there are a $\boldsymbol{v}' \in M_{\text{igtr}}(\mathcal{S})$ such that $\bar{\mu}[\![\boldsymbol{v} \ne \boldsymbol{v}']\!] \le \frac{1}{2}\epsilon$, and a $\delta > 0$ such that $\theta(\delta z) \le \frac{1}{2}\epsilon$ for every $z \in Q_{\mathcal{S}}(d\boldsymbol{v}')$. Now suppose that $z \in Q_{\mathcal{S}}(d\boldsymbol{v})$. Then there is an $I \in \mathcal{I}(\mathcal{S})$ such that $z = S_I(\mathbf{1}, d\boldsymbol{v})$. In this case, expressing $\boldsymbol{v}$ and $\boldsymbol{v}'$ as $\langle v_\sigma \rangle_{\sigma\in\mathcal{S}}$ and $\langle v'_\sigma \rangle_{\sigma\in\mathcal{S}}$,

$$[\![S_I(\mathbf{1}, d\boldsymbol{v}') \ne z]\!] \subseteq \sup_{\sigma\in I} [\![v'_\sigma \ne v_\sigma]\!] \subseteq [\![\boldsymbol{v}' \ne \boldsymbol{v}]\!]$$

and

$$\begin{aligned} \theta(\delta z) &\le \theta(\delta z - \delta S_I(\mathbf{1}, d\boldsymbol{v}')) + \theta(\delta S_I(\mathbf{1}, d\boldsymbol{v}')) \\ &\le \bar{\mu}[\![S_I(\mathbf{1}, d\boldsymbol{v}') \ne z]\!] + \frac{1}{2}\epsilon \subseteq \bar{\mu}[\![\boldsymbol{v}' \ne \boldsymbol{v}]\!] + \frac{1}{2}\epsilon \le \epsilon. \end{aligned}$$

Accordingly $Q_{\mathcal{S}}(d\boldsymbol{v})$ is topologically bounded and $\boldsymbol{v} \in M_{\mathrm{igtr}}(\mathcal{S})$.

(iv)

$$\begin{aligned} Q_{\mathcal{S}}(d(z\boldsymbol{v})) &= \{S_I(\boldsymbol{u}, d(z\boldsymbol{v})) : I \in \mathcal{I}(\mathcal{S}),\, \boldsymbol{u} \in M_{\mathrm{fa}}(I),\, \sup|\boldsymbol{u}| \le \chi 1\} \\ &= \{z \times S_I(\boldsymbol{u}, d\boldsymbol{v}) : I \in \mathcal{I}(\mathcal{S}),\, \boldsymbol{u} \in M_{\mathrm{fa}}(I),\, \sup|\boldsymbol{u}| \le \chi 1\} \end{aligned}$$

(613L(b-iii))

$$= \{z \times w : w \in Q_{\mathcal{S}}(d\boldsymbol{v})\}.$$

As $Q_{\mathcal{S}}(d\boldsymbol{v})$ is topologically bounded, and $w \mapsto z \times w : L^0 \to L^0$ is a continuous linear operator, $Q_{\mathcal{S}}(d(z\boldsymbol{v})) = \{z \times w : w \in Q_{\mathcal{S}}(d\boldsymbol{v})\}$ is topologically bounded (3A5N(b-v)) and $z\boldsymbol{v}$ is an integrator.

(v) If $\boldsymbol{v}$ is an integrator, then $\boldsymbol{v}{\restriction}\mathcal{S} \wedge \tau$ and $\boldsymbol{v}{\restriction}\mathcal{S} \vee \tau$ are integrators, by (ii) above. On the other hand, $Q_{\mathcal{S}}(d\boldsymbol{v}) \subseteq Q_{\mathcal{S}\wedge\tau}(d\boldsymbol{v}) + Q_{\mathcal{S}\vee\tau}(d\boldsymbol{v})$. **P** Take $z \in Q_{\mathcal{S}}(d\boldsymbol{v})$. Then z is expressible as $\sum_{i=0}^{n-1} u_{\tau_i} \times (v_{\tau_{i+1}} - v_{\tau_i})$ where $\tau_0 \le \ldots \le \tau_n$ in $\mathcal{S}$ and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$ is a ful;ly adapted process such that $\sup|\boldsymbol{u}| \le \chi 1$ (616C(iii)). Now for each $i \le n$, $v_{\tau_i} = v_{\tau_i\wedge\tau} + v_{\tau_i\vee\tau} - v_\tau$ (612D(f-i)). So if $i < n$ then

$$v_{\tau_{i+1}} - v_{\tau_i} = v_{\tau_{i+1}\wedge\tau} - v_{\tau_i\wedge\tau} + v_{\tau_{i+1}\vee\tau} - v_{\tau_i\vee\tau}.$$

Define u'_i, u''_i by saying that

$$u'_i = u_{\tau_i} \times \chi[\![\tau_i \le \tau]\!], \quad u''_i = u_{\tau_i} \times \chi[\![\tau < \tau_{i+1}]\!].$$

Evidently $|u'_i| \le \chi 1$ and $|u''_i| \le \chi 1$. By 612C, $u'_i \in L^0(\mathfrak{A}_{\tau_i\wedge\tau})$; and since $[\![\tau < \tau_{i+1}]\!] \in \mathfrak{A}_\tau \subseteq \mathfrak{A}_{\tau_i\vee\tau}$, $u''_i \in L^0(\mathfrak{A}_{\tau_i\vee\tau})$. Next,

$$\begin{aligned} [\![\tau < \tau_i]\!] &\subseteq [\![u'_i = 0]\!] \cap [\![\tau_i \vee \tau = \tau_i]\!] \cap [\![\tau_{i+1} \vee \tau = \tau_{i+1}]\!] \cap [\![u''_i = u_{\tau_i}]\!] \\ &\subseteq [\![u'_i = 0]\!] \cap [\![u''_i \times (v_{\tau_{i+1}\vee\tau} - v_{\tau_i\vee\tau}) = u_{\tau_i} \times (v_{\tau_{i+1}} - v_{\tau_i})]\!] \\ &\subseteq [\![u'_i \times (v_{\tau_{i+1}\wedge\tau} - v_{\tau_i\wedge\tau}) + u''_i \times (v_{\tau_{i+1}\vee\tau} - v_{\tau_i\vee\tau}) = u_{\tau_i} \times (v_{\tau_{i+1}} - v_{\tau_i})]\!], \\ [\![\tau_i \le \tau]\!] \cap [\![\tau < \tau_{i+1}]\!] &\subseteq [\![u'_i \times (v_{\tau_{i+1}\wedge\tau} - v_{\tau_i\wedge\tau}) + u''_i \times (v_{\tau_{i+1}\vee\tau} - v_{\tau_i\vee\tau}) \\ &\qquad = u_{\tau_i} \times (v_{\tau_{i+1}\wedge\tau} - v_{\tau_i\wedge\tau} + v_{\tau_{i+1}\vee\tau} - v_{\tau_i\vee\tau})]\!] \\ &= [\![u'_i \times (v_{\tau_{i+1}\wedge\tau} - v_{\tau_i\wedge\tau}) + u''_i \times (v_{\tau_{i+1}\vee\tau} - v_{\tau_i\vee\tau}) = u_{\tau_i} \times (v_{\tau_{i+1}} - v_{\tau_i})]\!], \\ [\![\tau_{i+1} \le \tau]\!] &\subseteq [\![u'_i = u_{\tau_i}]\!] \cap [\![\tau_i \wedge \tau = \tau_i]\!] \cap [\![\tau_{i+1} \wedge \tau = \tau_{i+1}]\!] \cap [\![u''_i = 0]\!] \\ &\subseteq [\![u'_i \times (v_{\tau_{i+1}\wedge\tau} - v_{\tau_i\wedge\tau}) = u_{\tau_i} \times (v_{\tau_{i+1}} - v_{\tau_i})]\!] \cap [\![u''_i = 0]\!] \\ &\subseteq [\![u'_i \times (v_{\tau_{i+1}\wedge\tau} - v_{\tau_i\wedge\tau}) + u''_i \times (v_{\tau_{i+1}\vee\tau} - v_{\tau_i\vee\tau}) = u_{\tau_i} \times (v_{\tau_{i+1}} - v_{\tau_i})]\!], \end{aligned}$$

so

$$u'_i \times (v_{\tau_{i+1}\wedge\tau} - v_{\tau_i\wedge\tau}) + u''_i \times (v_{\tau_{i+1}\vee\tau} - v_{\tau_i\vee\tau}) = u_{\tau_i} \times (v_{\tau_{i+1}} - v_{\tau_i}).$$

Now if we write

$$z' = \sum_{i=0}^{n-1} u'_i \times (v_{\tau_{i+1}\wedge\tau} - v_{\tau_i\wedge\tau}),$$

$$z'' = \sum_{i=0}^{n-1} u''_i \times (v_{\tau_{i+1}\vee\tau} - v_{\tau_i\vee\tau}),$$

we have $z' \in Q_{\mathcal{S}\wedge\tau}(d\boldsymbol{v})$ (616C(ii)), $z'' \in Q_{\mathcal{S}\vee\tau}(d\boldsymbol{v})$ and $z = z' + z''$. As z is arbitrary, $Q_{\mathcal{S}}(d\boldsymbol{v}) \subseteq Q_{\mathcal{S}\wedge\tau}(d\boldsymbol{v}) + Q_{\mathcal{S}\vee\tau}(d\boldsymbol{v})$. **Q**

If $\boldsymbol{v}{\restriction}\mathcal{S} \wedge \tau$ and $\boldsymbol{v}{\restriction}\mathcal{S} \vee \tau$ are integrators, $Q_{\mathcal{S}\wedge\tau}(d\boldsymbol{v})$ and $Q_{\mathcal{S}\wedge\tau}(d\boldsymbol{v})$ are topologically bounded, so $Q_{\mathcal{S}}(d\boldsymbol{v})$ is topologically bounded (613B(f-iii) once more) and $\boldsymbol{v}$ is an integrator.

616Q Corollary Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.

(a) $M_{\mathrm{ligtr}}(\mathcal{S})$ is an f-subalgebra of the space $M_{\mathrm{lmo}}(\mathcal{S})$ of locally moderately oscillatory processes with domain $\mathcal{S}$.

(b)(i) If $\boldsymbol{v} \in M_{\mathrm{ligtr}}(\mathcal{S})$ then $\boldsymbol{v}{\restriction}\mathcal{S}' \in M_{\mathrm{ligtr}}(\mathcal{S}')$ for any sublattice $\mathcal{S}'$ of $\mathcal{S}$.

(ii) If $\boldsymbol{v} \in M_{\text{ligtr}}(\mathcal{S})$ and $z \in L^0(\mathfrak{A} \cap \bigcap_{\sigma\in\mathcal{S}} \mathfrak{A}_\sigma)$, then $z\boldsymbol{v} \in M_{\text{ligtr}}(\mathcal{S})$.

(c) Suppose that $\boldsymbol{u} \in M_{\text{lmo}}(\mathcal{S})$ and $\boldsymbol{v} \in M_{\text{ligtr}}(\mathcal{S})$.

(i) The indefinite integral $ii_{\boldsymbol{v}}(\boldsymbol{u})$ belongs to $M_{\text{ligtr}}(\mathcal{S})$.

(ii) Let $\hat{\mathcal{S}}$ be the covered envelope of $\mathcal{S}$, and $\hat{\boldsymbol{u}}$, $\hat{\boldsymbol{v}}$ the fully adapted extensions of $\boldsymbol{u}$, $\boldsymbol{v}$ to $\hat{\mathcal{S}}$. Then $ii_{\boldsymbol{v}}(\boldsymbol{u}) = ii_{\hat{\boldsymbol{v}}}(\hat{\boldsymbol{u}})\restriction\mathcal{S}$.

(d) Suppose that $\boldsymbol{v} \in M_{\text{fa}}(\mathcal{S})$ and $\mathcal{S}'$ is a covering ideal of $\mathcal{S}$ such that $\boldsymbol{v}\restriction\mathcal{S}' \in M_{\text{ligtr}}(\mathcal{S}')$. Then $\boldsymbol{v} \in M_{\text{ligtr}}(\mathcal{S})$.

proof (a)-(b) We just have to apply 616P to $\boldsymbol{v}\restriction\mathcal{S}\wedge\tau$ for $\tau \in \mathcal{S}$.

(c)(i) Apply 616J to $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ and $\psi = \Delta(\boldsymbol{v}\restriction\mathcal{S}\wedge\tau)$ for $\tau \in \mathcal{S}$.

(ii) By 615F(b-v) and 616Ia, $\hat{\boldsymbol{u}} \in M_{\text{lmo}}(\hat{\mathcal{S}})$ and $\hat{\boldsymbol{v}} \in M_{\text{ligtr}}(\hat{\mathcal{S}})$, so $ii_{\hat{\boldsymbol{v}}}(\hat{\boldsymbol{u}})$ is well-defined on $\hat{\mathcal{S}}$. Now if $\tau \in \mathcal{S}$,

$$\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\,d\boldsymbol{v} = \int_{\hat{\mathcal{S}}\wedge\tau} \hat{\boldsymbol{u}}\,d\hat{\boldsymbol{v}}$$

as in 613T.

(d) If $\tau \in \mathcal{S}$ and $\epsilon > 0$, there is a $\sigma \in \mathcal{S}'$ such that $\sigma \le \tau$ and $\bar{\mu}[\![\sigma < \tau]\!] \le \epsilon$. Consider the process $\boldsymbol{v}' = \langle v_{\rho\wedge\sigma}\rangle_{\rho\in\mathcal{S}\wedge\tau}$. By 612Ib, this is fully adapted, while

$$\boldsymbol{v}'\restriction\mathcal{S}\wedge\sigma = \boldsymbol{v}\restriction\mathcal{S}\wedge\sigma = \boldsymbol{v}\restriction\mathcal{S}'\wedge\sigma$$

is an integrator, and $\boldsymbol{v}'\restriction(\mathcal{S}\wedge\tau)\vee\sigma$ is constant, therefore an integrator. So $\boldsymbol{v}' \in M_{\text{igtr}}(\mathcal{S}\wedge\tau)$ (616P(b-v)), while

$$[\![\boldsymbol{v}' \ne \boldsymbol{v}\restriction\mathcal{S}\wedge\tau]\!] = \sup_{\rho\in\mathcal{S}\wedge\tau} [\![v_{\rho\wedge\sigma} \ne v_\rho]\!] \subseteq \sup_{\rho\in\mathcal{S}\wedge\tau} [\![\rho\wedge\sigma < \rho]\!] = [\![\sigma < \tau]\!]$$

has measure at most ϵ. As ϵ is arbitrary, 616P(b-iii) tells us that $\boldsymbol{v}\restriction\mathcal{S}\wedge\tau \in M_{\text{igtr}}(\mathcal{S}\wedge\tau)$. As τ is arbitrary, $\boldsymbol{v} \in M_{\text{ligtr}}(\mathcal{S})$.

616R Proposition Suppose that $\mathcal{S}$ is a sublattice of $\mathcal{T}$, and that a fully adapted process $\boldsymbol{v}$ with domain $\mathcal{S}$ is (locally) of bounded variation.

(a) $\boldsymbol{v}$ is a (local) integrator.

(b) Now suppose that $\boldsymbol{v}$ is non-decreasing and that $\boldsymbol{u}$ is a non-negative moderately oscillatory process with domain $\mathcal{S}$.

(i) If $\boldsymbol{v}$ is of bounded variation then $\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v} \ge 0$.

(ii) If $\boldsymbol{v}$ is locally of bounded variation then $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is non-decreasing.

proof (a)(i) Suppose that $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ is of bounded variation. Then $\{S_I(\mathbf{1}, |d\boldsymbol{v}|) : I \in \mathcal{I}(\mathcal{S})\}$ has an upper bound $\bar{z}$ say. If I is a finite sublattice of $\mathcal{S}$ and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in I}$ is a fully adapted process with $\sup|\boldsymbol{u}| \le \chi 1$, then

$$|u_\sigma \times (v_{\sigma'} - v_\sigma)| \le |v_{\sigma'} - v_\sigma|$$

whenever $\sigma \le \sigma'$ in I, so $|S_I(\boldsymbol{u}, d\boldsymbol{v})| \le S_I(\mathbf{1}, |d\boldsymbol{v}|) \le \bar{z}$. Thus $Q_{\mathcal{S}}(d\boldsymbol{v})$ is order-bounded in L^0 and must be bounded for the topology of convergence in measure. So $\boldsymbol{v}$ is an integrator. By 616Ib, or otherwise, $\boldsymbol{v}$ is moderately oscillatory and order-bounded.

(ii) If $\boldsymbol{v}$ is locally of bounded variation, apply (a) to $\boldsymbol{v}\restriction\mathcal{S}\wedge\tau$, for $\tau \in \mathcal{S}$, to see that $\boldsymbol{v}$ is a local integrator and therefore locally moderately oscillatory and locally order-bounded.

(b) Recall that by 616K the integral $\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}$ is defined.

(i) All the sums $S_I(\boldsymbol{u}, d\boldsymbol{v})$, for $I \in \mathcal{I}(\mathcal{S})$, are non-negative, so the limit $\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}$ also is.

(ii) If τ, $\tau' \in \mathcal{S}$ and $\tau \le \tau'$, then

$$\int_{\mathcal{S}\wedge\tau'} \boldsymbol{u}\,d\boldsymbol{v} - \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\,d\boldsymbol{v} = \int_{\mathcal{S}\cap[\tau,\tau']} \boldsymbol{u}\,d\boldsymbol{v} \ge 0.$$

616S Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v}$ a process of bounded variation with domain $\mathcal{S}$. Then $|\Delta\boldsymbol{v}|$ is an integrating interval function.

proof (a) By 613Cc and 613Db, $|\Delta\boldsymbol{v}|$ is a strictly adapted interval function on $\mathcal{S}$. We know that $\bar v = \int_{\mathcal{S}}|\Delta\boldsymbol{v}| = \int_{\mathcal{S}}|d\boldsymbol{v}|$ is defined and an upper bound of $\{\sum_{i=0}^{n-1}|v_{\tau_{i+1}}-v_{\tau_i}| : \tau_0\le\ldots\le\tau_n \text{ in } \mathcal{S}\}$ (614J). But now we see that $Q_{\mathcal{S}}(|d\boldsymbol{v}|)\subseteq[-\bar v,\bar v]$. **P** Take $z\in Q_{\mathcal{S}}(|d\boldsymbol{v}|)$. If $z=0$ then surely $z\in[-\bar v,\bar v]$. Otherwise, there are a non-empty $I\in\mathcal{I}(\mathcal{S})$ and a $\boldsymbol{u}\in M_{\mathrm{fa}}(I)$ such that $\sup|\boldsymbol{u}|\le\chi 1$ and $z=S_I(\boldsymbol{u},|d\boldsymbol{v}|)$. Take $\tau_0\le\ldots\le\tau_n$ linearly generating the I-cells. Then

$$\begin{aligned}|z| &= |\sum_{i=0}^{n-1}u_{\tau_i}\times|v_{\tau_{i+1}}-v_{\tau_i}||\\ &\le\sum_{i=0}^{n-1}|u_{\tau_i}|\times|v_{\tau_{i+1}}-v_{\tau_i}|\le\sum_{i=0}^{n-1}|v_{\tau_{i+1}}-v_{\tau_i}|\le\bar v\end{aligned}$$

and again $z\in[-\bar v,\bar v]$. **Q** So $Q_{\mathcal{S}}(|d\boldsymbol{v}|)$ is order-bounded, therefore topologically bounded.

(b) If $\mathcal{S}$ is full, this is already enough to check that $|\Delta\boldsymbol{v}|$ is an integrating interval function. If $\mathcal{S}$ is not full, then by 614Q(a-iv) the adapted extension $\hat{\boldsymbol{v}}$ of $\boldsymbol{v}$ to the covered envelope $\hat{\mathcal{S}}$ of $\mathcal{S}$ is of bounded variation, and $|\Delta\hat{\boldsymbol{v}}|$ must be the adapted extension of $|\Delta\boldsymbol{v}|$ to $\hat{\mathcal{S}}^{2\uparrow}$. Since $|\Delta\hat{\boldsymbol{v}}|$ is an integrating interval function, so is $|\Delta\boldsymbol{v}|$ (616F).

616T Corollary Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{u}$, $\boldsymbol{v}$ fully adapted processes with domain $\mathcal{S}$.

(a) If $\boldsymbol{u}$ is moderately oscillatory and $\boldsymbol{v}$ is of bounded variation, then $\int_{\mathcal{S}}\boldsymbol{u}\,|d\boldsymbol{v}|$ is defined and equal to $\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}^{\uparrow}$, where $\boldsymbol{v}^{\uparrow}$ is the cumulative variation of $\boldsymbol{v}$.

(b) If $\boldsymbol{u}$ is locally moderately oscillatory and $\boldsymbol{v}$ is locally of bounded variation, then the indefinite integrals $ii_{|\Delta\boldsymbol{v}|}(\boldsymbol{u})$ and $ii_{\boldsymbol{v}^{\uparrow}}(\boldsymbol{u})$ are equal.

proof (a) By the definition 614O, $\boldsymbol{v}^{\uparrow}=ii_{|\Delta\boldsymbol{v}|}(\mathbf{1})$, so we can apply 616M with $\psi=|\Delta\boldsymbol{v}|$.

(b) Apply (a) to $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ and $\boldsymbol{v}\restriction\mathcal{S}\wedge\tau$ for $\tau\in\mathcal{S}$.

Mnemonic $|d\boldsymbol{v}|\sim d\boldsymbol{v}^{\uparrow}$.

616X Basic exercises (a) Suppose that $T=[0,\infty[$ and that $\mathfrak{A}=\{0,1\}$, as in 613W, 614Xd and 615Xf. (i) Show that if $f:[0,\infty[\,\to\mathbb{R}$ is interpreted as a fully adapted process $\boldsymbol{v}$ with domain $\mathcal{T}_f$, then $Q_{\mathcal{T}_f}(d\boldsymbol{v})$ can be identified with either $]-\mathrm{Var}(f),\mathrm{Var}(f)[$ or $[-\mathrm{Var}(f),\mathrm{Var}(f)]$ where $\mathrm{Var}(f)$ is the total variation of f (224A). (ii) Show that $f:[0,\infty[\,\to\mathbb{R}$ corresponds to an integrator iff it corresponds to a process of bounded variation iff it is itself of bounded variation in the sense of 224A.

(b) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. Show that any simple process with domain $\mathcal{S}$ is an integrator.

(c) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v}$ a fully adapted process with domain $\mathcal{S}$. Show that $\boldsymbol{v}$ is of bounded variation iff $Q_{\mathcal{S}}(d\boldsymbol{v})$ is bounded above iff $Q_{\mathcal{S}}(|d\boldsymbol{v}|)$ is bounded above, and that in this case $\int_{\mathcal{S}}|d\boldsymbol{v}|=\sup Q_{\mathcal{S}}(d\boldsymbol{v})=\sup Q_{\mathcal{S}}(|d\boldsymbol{v}|)$. (*Hint*: 351Dc.)

616Y Further exercises >**(a)** Let $\boldsymbol{w}=\langle w_\sigma\rangle_{\sigma\in\mathcal{T}_f}$ be Brownian motion as described in 612T, based on the real-time stochastic integration structure $(\mathfrak{C},\bar\nu,\langle\mathfrak{C}_t\rangle_{t\ge0},\mathcal{T},\langle\mathfrak{C}_\tau\rangle_{\tau\in\mathcal{T}})$. (i) Show that $w_\sigma\in L^2(\bar\nu)$ for every $\sigma\in\mathcal{T}_b$ (*hint*: 477A). (ii) Show that whenever $\sigma\le\tau$ in $\mathcal{T}_b$ and $u\in L^2(\bar\nu)\cap L^0(\mathfrak{C}_\sigma)$ then $\mathbb{E}(u\times(w_\tau-w_\sigma))=0$ (*hint*: 477G). (iii) Show that if $\sigma_0\le\ldots\le\sigma_n$ in $\mathcal{T}_b$ and $u_i\in L^\infty(\mathfrak{C}_{\sigma_i})$ and $|u_i|\le\chi 1$ for $i\le n$, then (α) $\mathbb{E}(u_i\times u_j\times(w_{\sigma_{i+1}}-w_{\sigma_i})\times(w_{\tau_{j+1}}-w_{\tau_j})=0$ whenever $i<j<n$ (β)

$$\begin{aligned}\mathbb{E}((\sum_{i=0}^{n-1}u_i\times(w_{\sigma_{i+1}}-w_{\sigma_i}))^2) &= \mathbb{E}(\sum_{i=0}^{n-1}u_i^2\times(w_{\sigma_{i+1}}-w_{\sigma_i})^2)\le\sum_{i=0}^{n-1}\mathbb{E}((w_{\sigma_{i+1}}-w_{\sigma_i})^2)\\ &=\sum_{i=0}^{n-1}\mathbb{E}(w_{\sigma_{i+1}}^2-w_{\sigma_i}^2)\le\|w_{\sigma_n}\|_2^2.\end{aligned}$$

(iv) Show that if $\tau\in\mathcal{T}_b$ then $Q_{\mathcal{T}\wedge\tau}(d\boldsymbol{w})$ is $\|\,\|_2$-bounded. (v) Show that $\boldsymbol{w}\restriction\mathcal{T}_b$ is a local integrator (*hint*: 613B(f-ii)). (vi) Show that $\boldsymbol{w}$ is a local integrator (*hint*: 616Qd).

(b) Find an example in which $\mathcal{S}$ is a sublattice of $\mathcal{T}$, $\psi : \mathcal{S}^{2\uparrow} \to L^0$ is a strictly adapted interval function and $Q_{\mathcal{S}}(d\psi)$ is topologically bounded but $Q_{\hat{\mathcal{S}}}(d\hat{\psi})$ is not, where $\hat{\mathcal{S}}$ is the covered envelope of $\mathcal{S}$ and $\hat{\psi} : \hat{\mathcal{S}}^{2\uparrow} \to L^0$ is the strictly adapted extension of ψ.

(c) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u}$ a previsibly simple process with domain $\mathcal{S}$, and $\boldsymbol{v}$ an integrator with domain $\mathcal{S}$. Show that $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$ is defined.

(d) In the construction described in 613P, show that the process $\boldsymbol{v}$ is moderately oscillatory but not an integrator.

616 Notes and comments You will have noticed that I have given practically no examples in this section. It is in fact the case that all three of the central examples in §612 (the identity process, Brownian motion and Poisson processes) are local integrators. The first and third are easy (614Xe). Brownian motion, as usual, is more interesting and much more important. Its natural place in the structure of this volume is in Chapter 62 below (see 622L). But as Itô's formula (§619 below) would be of no importance without Brownian motion, or at least something very like it, I have sketched an argument in 616Ya, which I hope you will tackle straight away. Even if parts are too difficult at the moment, it should be instructive.

We shall be spending a great deal of time teasing out the nature of integrators and integrating interval functions. An elementary class of integrators, the processes of bounded variation, was treated in §614. This will take us to the statement and proof of Itô's formula, if not to its most important applications. There will be some much deeper results about integrators in §622 and §627. But from the abstract definition, we can see immediately that whether an adapted interval function is 'integrating', or a process is an 'integrator', depends on the topology of L^0, but not on the measure inducing this topology; as with integration (613I) and 'moderately oscillatory' processes (615E), the property is law-independent.

In 616K-616M, we have the central case in which we can be sure that a Riemann-sum integral $\int_{\mathcal{S}} \boldsymbol{u}\, d\psi$ or $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$ is defined; it will be sufficient to suppose that $\boldsymbol{u}$ is moderately oscillatory and ψ or $\Delta\boldsymbol{v}$ is an integrating interval function. Compared with 614C and 614S, we have a condition on $\boldsymbol{u}$ which allows very much more variety, balanced by a strong new condition on $\boldsymbol{v}$.

In 616M, we have a first example of a phenomenon which will be important on many occasions. Here we start with an integrating interval function ψ, construct an integrator $\boldsymbol{v} = ii_\psi(\mathbf{1})$, and observe that $\int_{\mathcal{S}} \boldsymbol{u}\, d\psi = \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$ for every moderately oscillatory process $\boldsymbol{u}$ with domain $\mathcal{S}$. So in this sense $d\psi$ and $d\boldsymbol{v}$ are interchangeable. This is not at all because the interval functions ψ and $\Delta\boldsymbol{v}$ are the same. For $(\sigma,\tau) \in \mathcal{S}^{2\uparrow}$, we have $\Delta\boldsymbol{v}(\sigma,\tau) = \int_{\mathcal{S}\cap[\sigma,\tau]} d\psi$, which only in special cases will be equal to $\psi(\sigma,\tau)$. But in the formulae of this theory of integration, $d\psi$ and $d\boldsymbol{v}$ are equivalent, and I will write $d\psi \sim d\boldsymbol{v}$ as an aide-memoire.

Let me try to explain the phrase 'capped-stake variation set'. One of the ways of interpreting a stochastic integral $\int_{\mathcal{S}} \boldsymbol{u}\, d\psi$ is as the net gain of a gambler who at any stopping time σ chooses to wager a stake u_σ, and whose winnings over an interval $[\sigma,\tau]$ are $u_\sigma \times \psi(\sigma,\tau)$. (So if $\psi = \Delta\boldsymbol{v}$, he gets $u_\sigma \times (v_\tau - v_\sigma)$, representing the gain on betting u_σ units in a stock whose value changes from v_σ to v_τ.) On this formulation, a simple strategy for $\boldsymbol{u}$ is one which involves only finitely many stopping times declared in advance, like a stop-loss order. A Riemann sum $S_I(\boldsymbol{u}, d\psi)$ represents his winnings if he adjusts his stake to follow his strategy $\boldsymbol{u}$ whenever a stopping time in I is reached; and we count the integral as defined if all sufficiently fine readjustment schedules give about the same result with high probability. The terms $S_I(\boldsymbol{u}, d\psi)$ calculated in the formula for $Q_{\mathcal{S}}(d\psi)$ are those corresponding to stakes (positive or negative) capped by ± 1.[7]

Version of 10.11.21

617 Integral identities and quadratic variations

We come now to proper calculus, with change-of-variable theorems. 617D-617E is a stochastic-calculus version of the result that if $\nu = f\mu$ is an indefinite-integral measure, then $\int g\, d\nu = \int g \times f\, d\mu$ (235K). Similar formulae describe the cumulative variation of an indefinite integral with respect to a process of

[7] I see that the metaphor is creaking at this point, because few investors cap their investments by the number of shares they hold, rather than the value of those shares, which would correspond to a bound on $u_\sigma \times v_\sigma$. The Dow-Jones index is exceptional in this respect.

bounded variation (617G). The next theme is 'quadratic variation' (617H). Given two integrators $\boldsymbol{v}$ and $\boldsymbol{w}$, the interval function corresponding to $d\boldsymbol{v}d\boldsymbol{w}$ gives the same integrals as a process $[\boldsymbol{v}\overset{*}{|}\boldsymbol{w}]$ (617I) which is locally of bounded variation. In particular, $(d\boldsymbol{v})^2$ mimics $d\boldsymbol{v}^*$ where the quadratic variation $\boldsymbol{v}^*$ is a non-decreasing process. Based on this, we have a second change-of-variable theorem (617P-617Q), using approximations of moderately oscillatory processes by simple processes (617B).

617A Notation As before, $(\mathfrak{A},\bar\mu,T,\langle\mathfrak{A}_t\rangle_{t\in T},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ will be a stochastic integration structure. $L^0=L^0(\mathfrak{A})$ will be given its topology of convergence in measure, with corresponding F-norm θ (613Ba). If $A\subseteq\mathcal{T}$ and $\tau\in\mathcal{T}$, $A\wedge\tau=\{\sigma\wedge\tau:\sigma\in A\}$ and $A\vee\tau=\{\sigma\vee\tau:\sigma\in A\}$. If $\mathcal{S}$ is a sublattice of $\mathcal{T}$, $\mathcal{I}(\mathcal{S})$ is the set of finite sublattices of $\mathcal{S}$, $M_{\text{fa}}(\mathcal{S})$ is the space of fully adapted processes with domain $\mathcal{S}$, $M_{\text{o-b}}(\mathcal{S})\subseteq M_{\text{fa}}(\mathcal{S})$ is the space of order-bounded processes and $M_{\text{mo}}(\mathcal{S})\subseteq M_{\text{o-b}}(\mathcal{S})$ is the space of moderately oscillatory processes. If $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}\in M_{\text{o-b}}(\mathcal{S})$, $\sup|\boldsymbol{u}|=\sup_{\sigma\in\mathcal{S}}|u_\sigma|$. $\mathbf{1}$ will denote the constant process with value $\chi1$.

617B Lemma Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ and $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a moderately oscillatory process.

(a) For each non-empty $I\in\mathcal{I}(\mathcal{S})$ there is a unique simple process $\boldsymbol{u}_I=\langle u_{I\sigma}\rangle_{\sigma\in\mathcal{S}}$ such that $\boldsymbol{u}_I$ has a breakpoint string in I, $\boldsymbol{u}_I$ and $\boldsymbol{u}$ agree on I, and $[\![\sigma<\min I]\!]\subseteq[\![u_{I\sigma}=0]\!]$ for every $\sigma\in\mathcal{S}$.

(b) Complete the definition in (a) by setting $u_{\emptyset\sigma}=0$ for every $\sigma\in\mathcal{S}$. For every integrator $\boldsymbol{v}$ with domain $\mathcal{S}$,

(i) the indefinite integral $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is the limit $\lim_{I\uparrow\mathcal{I}(\mathcal{S})}ii_{\boldsymbol{v}}(\boldsymbol{u}_I)$ for the ucp topology,

(ii) $\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}=\lim_{I\uparrow\mathcal{I}(\mathcal{S})}\int_{\mathcal{S}}\boldsymbol{u}_I d\boldsymbol{v}$ in L^0.

proof (a) Take $(\tau_0,\dots,\tau_n)$ linearly generating the I-cells, and let $\boldsymbol{u}_I=\langle u_{I\sigma}\rangle_{\sigma\in\mathcal{S}}$ be the simple process defined by the formulae of 612Ka applied to $\tau_0,\dots,\tau_n$, $u_{\tau_0},\dots,u_{\tau_n}$ and $u_*=0$. Then $u_{I\tau_n}=u_{\tau_n}$. For $i<n$,

$$\begin{aligned}[\![u_{I\tau_i}=u_{\tau_i}]\!]&\supseteq[\![\tau_i<\tau_{i+1}]\!]\cup([\![\tau_i=\tau_{i+1}]\!]\cap[\![u_{I\tau_{i+1}}=u_{\tau_i}]\!])\\&=[\![\tau_i<\tau_{i+1}]\!]\cup([\![\tau_i=\tau_{i+1}]\!]\cap[\![u_{I\tau_{i+1}}=u_{\tau_{i+1}}]\!])\end{aligned}$$

so a downwards induction shows that $[\![u_{I\tau_i}=u_{\tau_i}]\!]=1$ for every $i\le n$. Since $\{\tau_i:i\le n\}$ covers I (611Ke), $\boldsymbol{u}_I$ agrees with $\boldsymbol{u}$ on I. Also

$$[\![\sigma<\min I]\!]=[\![\sigma<\tau_0]\!]\subseteq[\![u_{I\sigma}=0]\!]$$

by the choice of u_*.

To see that $\boldsymbol{u}_I$ is unique, use 612Kb to see that $(\tau_0,\dots,\tau_n)$ must be a breakpoint string for any process satisfying the conditions.

(b)(i) If $\mathcal{S}$ is empty, the result is trivial; suppose otherwise. Let $\epsilon>0$. Let $\delta\in\,]0,1]$ be such that $4\sqrt{\delta}\le\epsilon$. By 615F(a-iii), $\boldsymbol{u}$ is order-bounded and $\sup|\boldsymbol{u}|$ is defined in L^0; let $\eta>0$ be such that $\theta(x\times\sup|\boldsymbol{u}|)\le\delta$ whenever $x\in L^0$ and $\theta(x)\le\eta$. As $\boldsymbol{v}$ is moderately oscillatory (616Ib), $\lim_{\sigma\uparrow\mathcal{S}}v_\sigma$ is defined; let $\tau^*\in\mathcal{S}$ be such that $\theta(v_\sigma-v_{\sigma'})\le\eta$ whenever σ, $\sigma'\in\mathcal{S}\vee\tau^*$. Since $\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}=\lim_{I\uparrow\mathcal{I}(\mathcal{S})}S_I(\boldsymbol{u},d\boldsymbol{v})$ is defined (616K), there is a $K\in\mathcal{I}(\mathcal{S})$ such that $\tau^*\in K$ and $\theta(S_I(\boldsymbol{u},d\boldsymbol{v})-S_J(\boldsymbol{u},d\boldsymbol{v}))\le\delta$ whenever I, $J\in\mathcal{I}(\mathcal{S})$ include K.

Take $I\in\mathcal{I}(\mathcal{S})$ such that $K\subseteq I$, and $\tau\in\mathcal{S}$. Then $\theta(S_{I\wedge\tau}(\boldsymbol{u},d\boldsymbol{v})-\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\,d\boldsymbol{v})\le2\delta$ (613V(ii-β)). Also $\theta(S_{I\wedge\tau}(\boldsymbol{u},d\boldsymbol{v})-\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}_I\,d\boldsymbol{v})\le\delta$. **P** Write $u_{I\downarrow}$, $v_\downarrow$ for $\lim_{\sigma\downarrow\mathcal{S}}u_{I\sigma}$, $\lim_{\sigma\downarrow\mathcal{S}}v_\sigma$ respectively, If $(\tau_0,\dots,\tau_n)$ linearly generates I, then $(\tau_0\wedge\tau,\dots,\tau_n\wedge\tau)$ linearly generates the $(I\wedge\tau)$-cells (611Kg) and $(\tau_0\wedge\tau,\dots,\tau_n\wedge\tau,\tau)$ is a breakpoint string for $\boldsymbol{u}_I\restriction\mathcal{S}\wedge\tau$ (612K(d-ii)), so we have

$$S_{I\wedge\tau}(\boldsymbol{u},d\boldsymbol{v})=\sum_{i=0}^{n-1}u_{\tau_i\wedge\tau}\times(v_{\tau_{i+1}\wedge\tau}-v_{\tau_i\wedge\tau}),$$

$$\begin{aligned}\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}_I d\boldsymbol{v}=u_{I\downarrow}\times(v_{\tau_0\wedge\tau}-v_\downarrow)+\sum_{i=0}^{n-1}u_{I,\tau_i\wedge\tau}\times(v_{\tau_{i+1}\wedge\tau}-v_{\tau_i\wedge\tau})\\+u_{I,\tau_n\wedge\tau}\times(v_\tau-v_{\tau_n\wedge\tau})\end{aligned}$$

(613Ec, 614C). Now

$$u_{I\downarrow} \times (v_{\tau_0\wedge\tau} - v_{\downarrow}) = \lim_{\sigma\downarrow\mathcal{S}} u_{I\sigma} \times (v_{\tau_0\wedge\tau} - v_\sigma) = \lim_{\sigma\downarrow\mathcal{S}\wedge\tau_0\wedge\tau} u_{I\sigma} \times (v_{\tau_0\wedge\tau} - v_\sigma)$$
$$= \lim_{\sigma\downarrow\mathcal{S}\wedge\tau_0\wedge\tau} u_{I\sigma} \times (v_{\tau_0\wedge\tau} - v_\sigma) \times \chi[\![\sigma < \tau_0]\!] = 0$$

because we chose u_* to be 0 in the definition of $\boldsymbol{u}_I$. Next, for each $i < n$,

$$[\![\tau_i < \tau]\!] \subseteq [\![u_{I,\tau_i\wedge\tau} = u_{I\tau_i}]\!] \cap [\![u_{\tau_i\wedge\tau} = u_{\tau_i}]\!] \subseteq [\![u_{I,\tau_i\wedge\tau} = u_{\tau_i\wedge\tau}]\!],$$
$$[\![\tau \le \tau_i]\!] \subseteq [\![\tau_{i+1} \wedge \tau = \tau_i \wedge \tau]\!] \subseteq [\![v_{\tau_{i+1}\wedge\tau} - v_{\tau_i\wedge\tau} = 0]\!],$$

so $u_{\tau_i\wedge\tau} \times (v_{\tau_{i+1}\wedge\tau} - v_{\tau_i\wedge\tau}) = u_{I,\tau_i\wedge\tau} \times (v_{\tau_{i+1}\wedge\tau} - v_{\tau_i\wedge\tau})$. Accordingly

$$\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}_I d\boldsymbol{v} - S_{I\wedge\tau}(\boldsymbol{u}, d\boldsymbol{v}) = u_{I,\tau_n\wedge\tau} \times (v_\tau - v_{\tau_n\wedge\tau}) = u_{I,\tau_n\wedge\tau} \times (v_{\tau\vee\tau_n} - v_{\tau_n})$$

(612D(f-i)), and

$$\theta(\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}_I d\boldsymbol{v} - S_{I\wedge\tau}(\boldsymbol{u}, d\boldsymbol{v})) \le \theta(\sup|\boldsymbol{u}| \times |v_{\tau\vee\tau_n} - v_{\tau_n}|) \le \delta$$

because $\tau^* \le \tau_n \le \tau \vee \tau_n$ and $\theta(v_{\tau\vee\tau_n} - v_\tau) \le \eta$. **Q**

Consequently

$$\theta(\int_{\mathcal{S}\wedge\tau} (\boldsymbol{u} - \boldsymbol{u}_I) d\boldsymbol{v}) \le \theta(S_{I\wedge\tau}(\boldsymbol{u}, d\boldsymbol{v}) - \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\boldsymbol{v}) + \theta(S_{I\wedge\tau}(\boldsymbol{u}, d\boldsymbol{v}) - \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}_I\, d\boldsymbol{v})$$
$$\le 4\delta.$$

We know that $ii_{\boldsymbol{v}}(\boldsymbol{u} - \boldsymbol{u}_I)$ is fully adapted and order-bounded (613J(e-ii), 616J, 616Ib). We are assuming that $\mathcal{S}$ is finitely full, so if we set

$$\bar{z}_I = \sup_{\tau\in\mathcal{S}} |\int_{\mathcal{S}\wedge\tau} (\boldsymbol{u} - \boldsymbol{u}_I) d\boldsymbol{v}|$$

then $\theta(\bar{z}_I) \le 2\sqrt{4\delta} \le \epsilon$ (615Db).

This is true whenever $K \subseteq I \in \mathcal{I}(\mathcal{S})$. So

$$\limsup_{I\uparrow\mathcal{S}} \theta(\sup|ii_{\boldsymbol{v}}(\boldsymbol{u}) - ii_{\boldsymbol{v}}(\boldsymbol{u}_I)|) = \limsup_{I\uparrow\mathcal{S}} \theta(\bar{z}_I) \le \epsilon.$$

Bu ϵ was arbitrary, so $ii_{\boldsymbol{v}}(\boldsymbol{u}) = \lim_{I\uparrow\mathcal{S}} ii_{\boldsymbol{v}}(\boldsymbol{u}_I)$.

(ii) Here we have only to intercept the argument of (i) to see that

$$\theta(\int_{\mathcal{S}} (\boldsymbol{u} - \boldsymbol{u}_I) d\boldsymbol{v}) = \lim_{\tau\uparrow\mathcal{S}} \theta(\int_{\mathcal{S}\wedge\tau} (\boldsymbol{u} - \boldsymbol{u}_I) d\boldsymbol{v})$$

(613J(f-ii))

$$\le 4\sqrt{\delta} \le \epsilon$$

whenever $K \subseteq I \in \mathcal{I}(\mathcal{S})$. So $\lim_{I\uparrow\mathcal{I}(\mathcal{S})} \int_{\mathcal{S}} \boldsymbol{u}_I d\boldsymbol{v} = \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$.

617D Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, ψ an integrating interval function on $\mathcal{S}$, and $\boldsymbol{u}$, $\boldsymbol{z}$ moderately oscillatory processes with domain $\mathcal{S}$.

(a) $\boldsymbol{z}\psi$, as defined in 613D, is an integrating interval function on $\mathcal{S}$.

(b) Set $\boldsymbol{w} = ii_\psi(\boldsymbol{z})$. Then $\boldsymbol{w}$ is an integrator and

$$\int_{\mathcal{S}} \boldsymbol{u}\, d(\boldsymbol{z}\psi) = \int_{\mathcal{S}} \boldsymbol{u} \times \boldsymbol{z}\, d\psi = \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{w}.$$

proof (a)(i) Suppose to begin with that $\mathcal{S}$ is full. Then $\boldsymbol{z}\psi$ is strictly adapted (613D). Now $S_I(\boldsymbol{y}, d(\boldsymbol{z}\psi)) = S_I(\boldsymbol{y} \times \boldsymbol{z}, d\psi)$ whenever $I \in \mathcal{I}(\mathcal{S})$ and $\boldsymbol{y} = \langle y_\sigma\rangle_{\sigma\in I}$ is fully adapted. **P** If $e = c(\sigma, \tau)$ is a stopping-time interval with endpoints $\sigma \le \tau$ in I, then

$$\Delta_e(\boldsymbol{y}, d(\boldsymbol{z}\psi)) = y_\sigma \times (\boldsymbol{z}\psi)(\sigma, \tau) = y_\sigma \times z_\sigma \times \psi(\sigma, \tau) = \Delta_e(\boldsymbol{y} \times \boldsymbol{z}, d\psi);$$

summing over the I-cells, we have the result. **Q**

In particular, $S_I(\mathbf{1}, d(\boldsymbol{z}\psi)) = S_I(\boldsymbol{z}, d\psi)$ for every $I \in \mathcal{I}(\mathcal{S})$, and

$$\int_{\mathcal{S}} d(\boldsymbol{z}\psi) = \lim_{I\uparrow\mathcal{I}(\mathcal{S})} S_I(\mathbf{1}, d(\boldsymbol{z}\psi)) = \lim_{I\uparrow\mathcal{I}(\mathcal{S})} S_I(\boldsymbol{z}, d\psi) = \int_{\mathcal{S}} \boldsymbol{z}\, d\psi$$

is defined. Write $\bar{z}$ for $\sup|\boldsymbol{z}|$. Given $\epsilon > 0$, there is a $\delta > 0$ such that $\theta(S_I(\boldsymbol{y}, d\psi)) \le \delta$ whenever $I \in \mathcal{I}(\mathcal{S})$, $\boldsymbol{y} \in M_{\mathrm{fa}}(I)$ and $\theta(\sup|\boldsymbol{y}|) \le \delta$. Now there is an $\eta > 0$ such that $\theta(\bar{z} \times x) \le \epsilon$ whenever $x \in L^0(\mathfrak{A})$ and $\theta(x) \le \eta$. If we now take $I \in \mathcal{I}(\mathcal{S})$ and $\boldsymbol{y} \in M_{\mathrm{fa}}(I)$ such that $\theta(\sup|\boldsymbol{y}|) \le \eta$, we have $\sup|\boldsymbol{y} \times \boldsymbol{z}| \le \bar{u} \times \sup|\boldsymbol{y}|$, so

$$\theta(\sup|\boldsymbol{y} \times \boldsymbol{z}|) \le \theta(\bar{z} \times \sup|\boldsymbol{y}|) \le \delta$$

and

$$\theta(S_I(\boldsymbol{y}, d(\boldsymbol{z}\psi))) = \theta(S_I(\boldsymbol{y} \times \boldsymbol{z}, d\psi)) \le \epsilon.$$

Thus $Q_{\mathcal{S}}(d(\boldsymbol{z}\psi))$ is topologically bounded, and $\boldsymbol{z}\psi$ is an integrating interval function.

(ii) Generally, we have the adapted extension $\hat{\psi}$ on the covered envelope $\hat{\mathcal{S}}$ and the fully adapted extension $\hat{\boldsymbol{z}}$ to $\hat{\mathcal{S}}$; as $\hat{\psi}$ is an integrating interval function and $\hat{\boldsymbol{z}}$ is moderately oscillatory, (i) tells us that $\hat{\boldsymbol{z}}\hat{\psi}$ is an integrating interval function. But $\hat{\boldsymbol{z}}\hat{\psi}$ extends $\boldsymbol{z}\psi$, so is the adapted extension of $\boldsymbol{z}\psi$ to $\hat{\mathcal{S}}^{2\uparrow}$, and $\boldsymbol{z}\psi$ also is an integrating interval function.

(b)(i) As noted in (a-i), we have $S_I(\boldsymbol{u}, d(\boldsymbol{z}\psi)) = S_I(\boldsymbol{u} \times \boldsymbol{z}, d\psi)$ for every finite sublattice I of $\mathcal{S}$; taking the limit as $I \uparrow \mathcal{I}(\mathcal{S})$, $\int_{\mathcal{S}} \boldsymbol{u}\, d(\boldsymbol{z}\psi) = \int_{\mathcal{S}} \boldsymbol{u} \times \boldsymbol{z}\, d\psi$. Applying this to $\mathcal{S} \wedge \tau$ with $\boldsymbol{u} = \mathbf{1}$, $\int_{\mathcal{S}\wedge\tau} \boldsymbol{z}\, d\psi = \int_{\mathcal{S}\wedge\tau} d(\boldsymbol{z}\psi)$ for every $\tau \in \mathcal{S}$.

(ii) By 616J again, $\boldsymbol{w}$ is an integrator, so 616K assures us that $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{w}$ is defined. Expressing $\boldsymbol{w}$ as $\langle w_\sigma \rangle_{\sigma\in\mathcal{S}}$, we see that $w_\downarrow = \lim_{\sigma\downarrow\mathcal{S}} w_\sigma$ is 0 (613J(f-ii)) so

$$\int_{\mathcal{S}\wedge\tau} d\boldsymbol{w} = w_\tau - w_\downarrow = w_\tau = \int_{\mathcal{S}\wedge\tau} \boldsymbol{z}\, d\psi = \int_{\mathcal{S}\wedge\tau} d(\boldsymbol{z}\psi)$$

for every $\tau \in \mathcal{S}$. By 616M,

$$\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{w} = \int_{\mathcal{S}} \boldsymbol{u}\, d(\boldsymbol{z}\psi).$$

Mnemonic $d(ii_\psi(\boldsymbol{z})) \sim d(\boldsymbol{z}\psi)$.

617E Corollary Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u}$, $\boldsymbol{z}$ moderately oscillatory processes with domain $\mathcal{S}$, and $\boldsymbol{v}$ an integrator with domain $\mathcal{S}$. Set $\boldsymbol{w} = ii_{\boldsymbol{v}}(\boldsymbol{z})$. Then $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{w} = \int_{\mathcal{S}} \boldsymbol{u} \times \boldsymbol{z}\, d\boldsymbol{v}$.

Mnemonic $d(ii_{\boldsymbol{v}}(\boldsymbol{z})) \sim \boldsymbol{z}\, d\boldsymbol{v}$.

617F Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{z}$ a moderately oscillatory process and $\boldsymbol{v}$ a process of bounded variation, both with domain $\mathcal{S}$. Write $\boldsymbol{w}$ for $ii_{\boldsymbol{v}}(\boldsymbol{z})$. Then $\boldsymbol{w}$ is of bounded variation and $\int_{\mathcal{S}} |d\boldsymbol{w}| = \int_{\mathcal{S}} |\boldsymbol{z}||d\boldsymbol{v}|$.

proof (a) If $\mathcal{S}$ is empty, this is trivial, so let us suppose otherwise. Express $\boldsymbol{v}$, $\boldsymbol{z}$ as $\langle v_\sigma \rangle_{\sigma\in\mathcal{S}}$ and $\langle z_\sigma \rangle_{\sigma\in\mathcal{S}}$. We know that $\boldsymbol{v}$ is an integrator (616Ra), so $\boldsymbol{w}$ is well-defined; for $\tau \in \mathcal{S}$, write $w_\tau = \int_{\mathcal{S}\wedge\tau} \boldsymbol{z}\, d\boldsymbol{v}$. By 616T, $\int_{\mathcal{S}} |\boldsymbol{z}||d\boldsymbol{v}|$ is defined. To get started, note that if $\boldsymbol{z}$ is non-negative and $\boldsymbol{v}$ is non-decreasing, $\boldsymbol{w}$ will be non-decreasing (616R(b-ii)), and we shall have

$$\int_{\mathcal{S}} |d\boldsymbol{w}| = \int_{\mathcal{S}} d\boldsymbol{w} = \int_{\mathcal{S}} \boldsymbol{z}\, d\boldsymbol{v} = \int_{\mathcal{S}} |\boldsymbol{z}||d\boldsymbol{v}|$$

by 617E.

(b) We need a couple of elementary formulae.

(i) If U is a Riesz space and u, $v \in U$, then, using identities in 352D,

$$u + v = u \wedge v + (u - v)^+ + u \wedge v + (v - u)^+ = 2(u \wedge v) + |u - v|,$$

$$|u - v| \le |u| + |v| = 2(|u| \wedge |v|) + ||u| - |v|| \le 2(|u| \wedge |v|) + |u - v|.$$

(ii) If $\boldsymbol{x}$, $\boldsymbol{y}$ are processes of bounded variation with domain $\mathcal{S}$, then applying (i) in the Riesz space $(L^0)^{\mathcal{S}^{2\uparrow}}$,

$$\Delta\boldsymbol{x}\wedge\Delta\boldsymbol{y}=\frac{1}{2}(\Delta\boldsymbol{x}+\Delta\boldsymbol{y}-|\Delta\boldsymbol{x}-\Delta\boldsymbol{y}|).$$

Now $\boldsymbol{x}-\boldsymbol{y}$ is of bounded variation (614Q(a-ii)) so $\Delta\boldsymbol{x}$, $\Delta\boldsymbol{y}$ and $\Delta\boldsymbol{x}-\Delta\boldsymbol{y}$ are integrating interval functions (616Ra, 616Ic). By 616S, $|\Delta\boldsymbol{x}-\Delta\boldsymbol{y}|$ is an integrating interval function so $\Delta\boldsymbol{x}\wedge\Delta\boldsymbol{y}=\frac{1}{2}(\Delta\boldsymbol{x}+\Delta\boldsymbol{y}-|\Delta\boldsymbol{x}-\Delta\boldsymbol{y}|)$ also is (352D, 616Ga).

Writing $d\boldsymbol{x}\wedge d\boldsymbol{y}$ for $d(\Delta\boldsymbol{x}\wedge\Delta\boldsymbol{y})$, in the spirit of 613F and 613I,

$$S_I(\mathbf{1},d(\boldsymbol{x}+\boldsymbol{y}))=S_I(\mathbf{1},|d(\boldsymbol{x}-\boldsymbol{y})|)+2S_I(\mathbf{1},d\boldsymbol{x}\wedge d\boldsymbol{y})$$

for every finite sublattice I of $\mathcal{S}$, and $\int_{\mathcal{S}}d\boldsymbol{x}\wedge d\boldsymbol{y}$ is defined and equal to $\frac{1}{2}(\int_{\mathcal{S}}d\boldsymbol{x}+\int_{\mathcal{S}}d\boldsymbol{y}-\int_{\mathcal{S}}|d(\boldsymbol{x}-\boldsymbol{y})|)$.

Similarly,

$$|\Delta(\boldsymbol{x}-\boldsymbol{y})|\le|\Delta\boldsymbol{x}|+|\Delta\boldsymbol{y}|\le|\Delta(\boldsymbol{x}-\boldsymbol{y})|+2(|\Delta\boldsymbol{x}|\wedge|\Delta\boldsymbol{y}|)$$

and

$$S_I(\mathbf{1},|d(\boldsymbol{x}-\boldsymbol{y})|)\le S_I(\mathbf{1},|d\boldsymbol{x}|)+S_I(\mathbf{1},|d\boldsymbol{y}|)\le S_I(\mathbf{1},|d(\boldsymbol{x}-\boldsymbol{y})|)+2S_I(\mathbf{1},|d\boldsymbol{x}|\wedge|d\boldsymbol{y}|)$$

for every $I\in\mathcal{I}(\mathcal{S})$, so if $\int_{\mathcal{S}}|d\boldsymbol{x}|\wedge|d\boldsymbol{y}|=0$ then $\int_{\mathcal{S}}|d(\boldsymbol{x}-\boldsymbol{y})|=\int_{\mathcal{S}}|d\boldsymbol{x}|+\int_{\mathcal{S}}|d\boldsymbol{y}|$.

(c) Next, suppose that $\mathcal{S}$ is full and $\boldsymbol{v}$ is non-negative and non-decreasing.

(i) Writing $\boldsymbol{z}^+=\langle z^+_\sigma\rangle_{\sigma\in\mathcal{S}}$ for $\boldsymbol{z}\vee\mathbf{0}$ and $\boldsymbol{z}^-=\langle z^-_\sigma\rangle_{\sigma\in\mathcal{S}}$ for $(-\boldsymbol{z})\vee\mathbf{0}$, both $\int_{\mathcal{S}}\boldsymbol{z}^+d\boldsymbol{v}$ and $\int_{\mathcal{S}}\boldsymbol{z}^-d\boldsymbol{v}$ are defined. Set $\boldsymbol{x}=ii_{\boldsymbol{v}}(\boldsymbol{z}^+)$ and $\boldsymbol{y}=ii_{\boldsymbol{v}}(\boldsymbol{z}^-)$; then $\boldsymbol{w}=\boldsymbol{x}-\boldsymbol{y}$ while $\boldsymbol{x}$ and $\boldsymbol{y}$ are of bounded variation. Now $\int_{\mathcal{S}}d\boldsymbol{x}\wedge d\boldsymbol{y}=0$. **P** Let $\epsilon>0$. Then there is an $I\in\mathcal{I}(\mathcal{S})$ such that

$$\theta(S_J(\boldsymbol{z}^+,d\boldsymbol{v})-S_K(\boldsymbol{z}^+,d\boldsymbol{v}))\le\epsilon,\quad\theta(S_J(\boldsymbol{z}^-,d\boldsymbol{v})-S_K(\boldsymbol{z}^-,d\boldsymbol{v}))\le\epsilon$$

whenever $I\subseteq J$, $K\in\mathcal{I}(\mathcal{S})$. Take any non-empty $J\in\mathcal{I}(\mathcal{S})$ including I, and a sequence $(\tau_0,\dots,\tau_n)$ linearly generating the J-cells. For each $i<n$, $a_i=[\![z_{\tau_i}\ge0]\!]$ belongs to $\mathfrak{A}_{\tau_i}\subseteq\mathfrak{A}_{\tau_{i+1}}$, so we have a $\tau'_i\in\mathcal{T}$ such that

$$a_i\subseteq[\![\tau'_i=\tau_i]\!],\quad 1\setminus a_i\subseteq[\![\tau'_i=\tau_{i+1}]\!]$$

(611I); as $\mathcal{S}$ is full, $\tau'_i\in\mathcal{S}$, while $\tau_i\le\tau'_i\le\tau_{i+1}$. Let K be the (finite) sublattice of $\mathcal{S}$ generated by $J\cup\{\tau'_i:i<n\}$; then K is included in the sublattice

$$\{\sigma:\operatorname{med}(\tau_i,\sigma,\tau_{i+1})\in\{\tau_i,\tau'_i,\tau_{i+1}\}\text{ for every }i<n\},$$

so $\{\tau_0,\tau'_0,\tau_1,\dots,\tau'_{n-1},\tau_n\}$ is a maximal totally ordered subset of K and $(\tau_0,\tau'_0,\tau_1,\dots,\tau'_{n-1},\tau_n)$ linearly generates the K-cells.

For each $i<n$,

$$[\![v_{\tau'_i}\ne v_{\tau_i}]\!]\subseteq[\![\tau_i<\tau'_i]\!]\subseteq a_i\subseteq[\![z^-_{\tau_i}=0]\!],$$

$$[\![v_{\tau_{i+1}}\ne v_{\tau'_i}]\!]\subseteq[\![\tau'_i<\tau_{i+1}]\!]\subseteq(1\setminus a_i)\cap[\![\tau'_i=\tau_i]\!]\subseteq[\![z^+_{\tau'_i}=0]\!].$$

So we have

$$0\le S_J(\mathbf{1},d\boldsymbol{x}\wedge d\boldsymbol{y})=S_K(\mathbf{1},d\boldsymbol{x}\wedge d\boldsymbol{y})$$

(because $J\subseteq K$ and J covers K, see 613S)

$$
\begin{aligned}
&= \sum_{i=0}^{n-1} \Delta_{c(\tau_i,\tau_i')}(\mathbf{1}, d\boldsymbol{x} \wedge d\boldsymbol{y}) + \Delta_{c(\tau_i',\tau_{i+1})}(\mathbf{1}, d\boldsymbol{x} \wedge d\boldsymbol{y}) \\
&\le \sum_{i=0}^{n-1} \Delta_{c(\tau_i,\tau_i')}(\mathbf{1}, d\boldsymbol{y}) + \Delta_{c(\tau_i',\tau_{i+1})}(\mathbf{1}, d\boldsymbol{x}) \\
&= \sum_{i=0}^{n-1} \int_{\mathcal{S}\cap[\tau_i,\tau_i']} \boldsymbol{z}^- d\boldsymbol{v} + \int_{\mathcal{S}\cap[\tau_i',\tau_{i+1}]} \boldsymbol{z}^+ d\boldsymbol{v} \\
&= \sum_{i=0}^{n-1} \int_{\mathcal{S}\cap[\tau_i,\tau_i']} \boldsymbol{z}^- d\boldsymbol{v} - z_{\tau_i}^- \times (v_{\tau_i'} - v_{\tau_i}) \\
&\qquad\qquad + \int_{\mathcal{S}\cap[\tau_i',\tau_{i+1}]} \boldsymbol{z}^+ d\boldsymbol{v} - z_{\tau_i'}^+ \times (v_{\tau_{i+1}} - v_{\tau_i'}) \\
&= \sum_{i=0}^{n-1} \int_{\mathcal{S}\cap[\tau_i,\tau_i']} \boldsymbol{z}^- d\boldsymbol{v} - S_{K\cap[\tau_i,\tau_i']}(\boldsymbol{z}^-, d\boldsymbol{v}) \\
&\qquad\qquad + \int_{\mathcal{S}\cap[\tau_i,\tau_i']} \boldsymbol{z}^+ d\boldsymbol{v} - S_{K\cap[\tau_i,\tau_i']}(\boldsymbol{z}^+, d\boldsymbol{v})
\end{aligned}
$$

and

$$
\begin{aligned}
\theta(S_J(\mathbf{1}, d\boldsymbol{x} \wedge d\boldsymbol{y})) &\le \theta(\sum_{i=0}^{n-1} \int_{\mathcal{S}\cap[\tau_i,\tau_i']} \boldsymbol{z}^- d\boldsymbol{v} - S_{K\cap[\tau_i,\tau_i']}(\boldsymbol{z}^-, d\boldsymbol{v})) \\
&\qquad + \theta(\sum_{i=0}^{n-1} \int_{\mathcal{S}\cap[\tau_i',\tau_{i+1}]} \boldsymbol{z}^+ d\boldsymbol{v} - S_{K\cap[\tau_i,\tau_i']}(\boldsymbol{z}^+, d\boldsymbol{v})) \\
&\le 2\epsilon
\end{aligned}
$$

(613V(i)). This is true whenever $I \subseteq J$ in $\mathcal{I}(\mathcal{S})$; as ϵ is arbitrary, $\int_{\mathcal{S}} d\boldsymbol{x} \wedge d\boldsymbol{y} = 0$. **Q**

(ii) By (b),

$$
\begin{aligned}
\int_{\mathcal{S}} |d\boldsymbol{w}| &= \int_{\mathcal{S}} d\boldsymbol{x} + \int_{\mathcal{S}} d\boldsymbol{y} - 2\int_{\mathcal{S}} d\boldsymbol{x} \wedge d\boldsymbol{y} \\
&= \int_{\mathcal{S}} \boldsymbol{z}^+ d\boldsymbol{v} + \int_{\mathcal{S}} \boldsymbol{z}^- d\boldsymbol{v} = \int_{\mathcal{S}} |\boldsymbol{z}| d\boldsymbol{v} = \int_{\mathcal{S}} |\boldsymbol{z}||d\boldsymbol{v}|,
\end{aligned}
$$

and we have the required result when $\mathcal{S}$ is full and $\boldsymbol{v}$ is non-decreasing.

(d) Next suppose only that $\mathcal{S}$ is full.

(i) Let $\boldsymbol{v}^\uparrow$ be the cumulative variation of $\boldsymbol{v}$ (614O) and set

$$\boldsymbol{v}_1 = \tfrac{1}{2}(\boldsymbol{v}^\uparrow + \boldsymbol{v}), \quad \boldsymbol{x} = ii_{\boldsymbol{v}_1}(\boldsymbol{z}), \quad \boldsymbol{v}_2 = \tfrac{1}{2}(\boldsymbol{v}^\uparrow - \boldsymbol{v}), \quad \boldsymbol{y} = ii_{\boldsymbol{v}_2}(\boldsymbol{z}).$$

Then $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are non-decreasing (614P(a-iii)), and

$$\boldsymbol{v}_1 - \boldsymbol{v}_2 = \boldsymbol{v}, \quad \boldsymbol{w} = \boldsymbol{x} - \boldsymbol{y}, \quad \boldsymbol{v}_1 + \boldsymbol{v}_2 = \boldsymbol{v}^\uparrow.$$

Express $\boldsymbol{v}_1$ and $\boldsymbol{x}$ as $\langle v_{1\sigma}\rangle_{\sigma\in\mathcal{S}}$, $\langle x_\sigma\rangle_{\sigma\in\mathcal{S}}$ respectively.

(ii) By (b) above and 616Ta,

$$
\begin{aligned}
2\int_{\mathcal{S}} d\boldsymbol{v}_1 \wedge d\boldsymbol{v}_2 &= \int_{\mathcal{S}} d\boldsymbol{v}_1 + \int_{\mathcal{S}} d\boldsymbol{v}_2 - \int_{\mathcal{S}} |d\boldsymbol{v}| \\
&= \int_{\mathcal{S}} d\boldsymbol{v}^\uparrow - \int_{\mathcal{S}} |d\boldsymbol{v}| = 0.
\end{aligned}
$$

Set $\bar{z} = \sup |\boldsymbol{z}|$. If $\sigma \le \tau$ in $\mathcal{S}$ and $e = c(\sigma, \tau)$ is the corresponding stopping-time interval, then

$$|\Delta_e(\boldsymbol{z}, d\boldsymbol{v}_1)| = |z_\sigma| \times (v_{1\tau} - v_{1\sigma}) \le \bar{z} \times \Delta_e(\mathbf{1}, \boldsymbol{v}_1).$$

So $|S_I(\boldsymbol{z}, d\boldsymbol{v}_1)| \le \bar{z} \times S_I(\mathbf{1}, d\boldsymbol{v}_1)$ for every $I \in \mathcal{I}(\mathcal{S})$. Again, if $e = c(\sigma,\tau)$ is a stopping-time interval with endpoints in $\mathcal{S}$,

$$|\Delta_e(\mathbf{1}, d\boldsymbol{x})| = |x_\tau - x_\sigma| = |\int_{\mathcal{S}\cap[\sigma,\tau]} \boldsymbol{z}\,d\boldsymbol{v}_1|$$
$$\le \bar{z} \times \int_{\mathcal{S}\cap[\sigma,\tau]} d\boldsymbol{v}_1 = \bar{z} \times (v_{1\tau} - v_{1\sigma}) = \bar{z} \times \Delta_e(\mathbf{1}, d\boldsymbol{v}_1).$$

Thus $|\Delta\boldsymbol{x}| \le \bar{z} \times \Delta\boldsymbol{v}_1$. Similarly, $|\Delta\boldsymbol{y}| \le \bar{z} \times \Delta\boldsymbol{v}_2$. But this means that $|\Delta\boldsymbol{x}| \wedge |\Delta\boldsymbol{y}| \le \bar{z} \times (\Delta\boldsymbol{v}_1 \wedge \Delta\boldsymbol{v}_2)$. Consequently

$$0 \le S_I(\mathbf{1}, |d\boldsymbol{x}| \wedge |d\boldsymbol{y}|) \le \bar{z} \times S_I(\mathbf{1}, d\boldsymbol{v}_1 \wedge d\boldsymbol{v}_2)$$

for every $I \in \mathcal{I}(\mathcal{S})$, and in the limit

$$0 \le \int_{\mathcal{S}} |d\boldsymbol{x}| \wedge |d\boldsymbol{y}| \le \bar{z} \times \int_{\mathcal{S}} d\boldsymbol{v}_1 \wedge d\boldsymbol{v}_2 = 0.$$

(iii) Returning to the formulae of (b), we see now that

$$\int_{\mathcal{S}} |d\boldsymbol{w}| = \int_{\mathcal{S}} |d\boldsymbol{x}| + \int_{\mathcal{S}} |d\boldsymbol{y}| = \int_{\mathcal{S}} |\boldsymbol{z}| d\boldsymbol{v}_1 + \int_{\mathcal{S}} |\boldsymbol{z}| d\boldsymbol{v}_2$$

(by (c))

$$= \int_{\mathcal{S}} |\boldsymbol{z}| d\boldsymbol{v}^{\uparrow} = \int_{\mathcal{S}} |\boldsymbol{z}||d\boldsymbol{v}|$$

by 616Ta again in its full strength.

(e) Finally, if $\mathcal{S}$ is not full, let $\hat{\mathcal{S}}$ be its covered envelope and $\hat{\boldsymbol{z}}$, $\hat{\boldsymbol{v}}$ the fully adapted extensions of $\boldsymbol{z}$ and $\boldsymbol{v}$ to $\hat{\mathcal{S}}$. Then $\hat{\boldsymbol{z}}$ is moderately oscillatory (615F(a-vi)) and $\hat{\boldsymbol{v}}$ is of bounded variation (614Q(a-iv-β)), while the fully adapted extension $\hat{\boldsymbol{w}}$ of $\boldsymbol{w}$ is $ii_{\hat{\boldsymbol{v}}}(\hat{\boldsymbol{z}})$ (616Q(c-ii)). Since $\Delta\hat{\boldsymbol{v}}$ is the strictly adapted extension of the interval function $\Delta\boldsymbol{v}$, $|\Delta\hat{\boldsymbol{v}}|$ is the strictly adapted extension of $|\Delta\boldsymbol{v}|$ (613U(b-ii)); similarly, $|\Delta\hat{\boldsymbol{w}}|$ is the extension of $|\Delta\boldsymbol{w}|$. Now

$$\int_{\mathcal{S}} |d\boldsymbol{w}| = \int_{\mathcal{S}} d|\Delta\boldsymbol{w}| = \int_{\hat{\mathcal{S}}} d|\Delta\hat{\boldsymbol{w}}|$$

(613T)

$$= \int_{\hat{\mathcal{S}}} |d\hat{\boldsymbol{w}}| = \int_{\hat{\mathcal{S}}} |\hat{\boldsymbol{z}}||d\hat{\boldsymbol{v}}|$$

((d) above)

$$= \int_{\mathcal{S}} |\boldsymbol{z}||d\boldsymbol{v}|,$$

and we have the result in the general case.

617G Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u}$ and $\boldsymbol{z}$ moderately oscillatory processes and $\boldsymbol{v}$ a process of bounded variation, all with domain $\mathcal{S}$. Write $\boldsymbol{w}$ for $ii_{\boldsymbol{v}}(\boldsymbol{z})$, and $\boldsymbol{v}^{\uparrow}$, $\boldsymbol{w}^{\uparrow}$ for the cumulative variations of $\boldsymbol{v}$ and $\boldsymbol{w}$. Then

$$\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{w}^{\uparrow} = \int_{\mathcal{S}} \boldsymbol{u}\,|d\boldsymbol{w}| = \int_{\mathcal{S}} \boldsymbol{u} \times |\boldsymbol{z}|\,|d\boldsymbol{v}| = \int_{\mathcal{S}} \boldsymbol{u} \times |\boldsymbol{z}|\,d\boldsymbol{v}^{\uparrow}.$$

proof By 617F, $\boldsymbol{w}$ is of bounded variation. Let $\boldsymbol{v}^{\uparrow}$, $\boldsymbol{w}^{\uparrow}$ be the cumulative variations of $\boldsymbol{v}$ and $\boldsymbol{w}$. If $\tau \in \mathcal{S}$, then

$$\int_{\mathcal{S}\wedge\tau} d\boldsymbol{w}^\uparrow = \int_{\mathcal{S}\wedge\tau} |d\boldsymbol{w}|$$

(616Ta once more)

$$= \int_{\mathcal{S}\wedge\tau} |\boldsymbol{z}||d\boldsymbol{v}|$$

(617F)

$$= \int_{\mathcal{S}\wedge\tau} |\boldsymbol{z}|d\boldsymbol{v}^\uparrow.$$

So $\boldsymbol{w}^\uparrow = ii_{\boldsymbol{v}^\uparrow}(|\boldsymbol{z}|)$. By 617E, $\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{w}^\uparrow = \int_{\mathcal{S}} \boldsymbol{u} \times |\boldsymbol{z}|\,d\boldsymbol{v}^\uparrow$, while $\int_{\mathcal{S}} \boldsymbol{u}\,|d\boldsymbol{w}| = \int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{w}^\uparrow$ and $\int_{\mathcal{S}} \boldsymbol{u} \times |\boldsymbol{z}|\,|d\boldsymbol{v}| = \int_{\mathcal{S}} \boldsymbol{u} \times |\boldsymbol{z}|\,d\boldsymbol{v}^\uparrow$ by 616Ta yet again.

Mnemonic $d(ii_{\boldsymbol{v}}(\boldsymbol{z})^\uparrow) \sim |\boldsymbol{z}|d\boldsymbol{v}^\uparrow$.

617H Quadratic variation Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v}$, $\boldsymbol{w}$ local integrators with domain $\mathcal{S}$.

(a)(i) If $\boldsymbol{v}$ and $\boldsymbol{w}$ are integrators, then the strictly adapted interval function $\Delta\boldsymbol{v} \times \Delta\boldsymbol{w}$ on $\mathcal{S}$ (613Da) is an integrating interval function. **P** As observed in part (a) of the proof of 613M,

$$\Delta\boldsymbol{v} \times \Delta\boldsymbol{w} = \Delta(\boldsymbol{v} \times \boldsymbol{w}) - \boldsymbol{v}\Delta\boldsymbol{w} - \boldsymbol{w}\Delta\boldsymbol{v}.$$

Now $\boldsymbol{v} \times \boldsymbol{w}$ is an integrator (616Pa), so $\Delta(\boldsymbol{v} \times \boldsymbol{w})$ is an integrating interval function (616Ic), while $\boldsymbol{v}\Delta\boldsymbol{w}$ and $\boldsymbol{w}\Delta\boldsymbol{v}$ are integrating interval functions by 617Da. So $\Delta\boldsymbol{v} \times \Delta\boldsymbol{w}$ also is, by 616Ga. **Q**

(ii) In any case, $\Delta\boldsymbol{v} \times \Delta\boldsymbol{w}$ is a locally integrating interval function (apply (i) to $\boldsymbol{v}\restriction\mathcal{S}\wedge\tau$, $\boldsymbol{w}\restriction\mathcal{S}\wedge\tau$ for $\tau \in \mathcal{S}$).

(b) The **covariation** of $\boldsymbol{v}$ and $\boldsymbol{w}$ is the indefinite integral

$$[\boldsymbol{v} \overset{*}{|} \boldsymbol{w}] = ii_{\Delta\boldsymbol{v}\times\Delta\boldsymbol{w}}(\mathbf{1}).$$

When $\boldsymbol{w} = \boldsymbol{v}$, we say that $\boldsymbol{v}^* = [\boldsymbol{v} \overset{*}{|} \boldsymbol{v}] = ii_{(\Delta\boldsymbol{v})^2}(\mathbf{1})$ is the **quadratic variation** of $\boldsymbol{v}$.

(c) Note that as

$$(\boldsymbol{v}, \boldsymbol{w}) \mapsto \Delta\boldsymbol{v} \times \Delta\boldsymbol{w}$$

is bilinear, so is $(\boldsymbol{v}, \boldsymbol{w}) \mapsto [\boldsymbol{v} \overset{*}{|} \boldsymbol{w}]$.

617I Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{v}$, $\boldsymbol{w}$ two integrators and $\boldsymbol{u}$ a moderately oscillatory process, all with domain $\mathcal{S}$. Then $[\boldsymbol{v} \overset{*}{|} \boldsymbol{w}]$ is an integrator and

$$\int_{\mathcal{S}} \boldsymbol{u}\,d[\boldsymbol{v} \overset{*}{|} \boldsymbol{w}], \quad \int_{\mathcal{S}} \boldsymbol{u}\,d(\boldsymbol{v} \times \boldsymbol{w}) - \int_{\mathcal{S}} \boldsymbol{u} \times \boldsymbol{v}\,d\boldsymbol{w} - \int_{\mathcal{S}} \boldsymbol{u} \times \boldsymbol{w}\,d\boldsymbol{v}, \quad \int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}d\boldsymbol{w}$$

are defined and equal.

proof Because $\Delta\boldsymbol{v} \times \Delta\boldsymbol{w}$ is an integrating interval function (617H(a-i)), its indefinite integral $[\boldsymbol{v} \overset{*}{|} \boldsymbol{w}]$ is an integrator (616J once more). Now

$$\int_{\mathcal{S}} \boldsymbol{u}\,d(\boldsymbol{v}\Delta\boldsymbol{w}) = \int_{\mathcal{S}} \boldsymbol{u} \times \boldsymbol{v}\,d(\Delta\boldsymbol{w}) = \int_{\mathcal{S}} \boldsymbol{u} \times \boldsymbol{v}\,d\boldsymbol{w}$$

by 617Db. Similarly, $\int_{\mathcal{S}} \boldsymbol{u}\,d(\boldsymbol{w}\Delta\boldsymbol{v}) = \int_{\mathcal{S}} \boldsymbol{u} \times \boldsymbol{w}\,d\boldsymbol{v}$, while $\int_{\mathcal{S}} \boldsymbol{u}\,d[\boldsymbol{v} \overset{*}{|} \boldsymbol{w}] = \int_{\mathcal{S}} \boldsymbol{u}\,d(\Delta\boldsymbol{v} \times \Delta\boldsymbol{w})$ by the other part of 617Db. Of course $\int_{\mathcal{S}} \boldsymbol{u}\,d(\boldsymbol{v} \times \boldsymbol{w}) = \int_{\mathcal{S}} \boldsymbol{u}\,d(\Delta(\boldsymbol{v} \times \boldsymbol{w}))$ and $\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}d\boldsymbol{w} = \int_{\mathcal{S}} \boldsymbol{u}\,d(\Delta\boldsymbol{v} \times \Delta\boldsymbol{w})$ by the definitions in 613L. Since

$$\Delta\boldsymbol{v} \times \Delta\boldsymbol{w} = \Delta(\boldsymbol{v} \times \boldsymbol{w}) - \boldsymbol{v}\Delta\boldsymbol{w} - \boldsymbol{w}\Delta\boldsymbol{v},$$

$$\int_{\mathcal{S}} \boldsymbol{u}\,d[\boldsymbol{v} \overset{*}{|} \boldsymbol{w}] = \int_{\mathcal{S}} \boldsymbol{u}\,d(\boldsymbol{v} \times \boldsymbol{w}) - \int_{\mathcal{S}} \boldsymbol{u} \times \boldsymbol{v}\,d\boldsymbol{w} - \int_{\mathcal{S}} \boldsymbol{u} \times \boldsymbol{w}\,d\boldsymbol{v} = \int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}d\boldsymbol{w}.$$

Mnemonic $d[\boldsymbol{v} \overset{*}{|} \boldsymbol{w}] \sim d(\boldsymbol{v} \times \boldsymbol{w}) - \boldsymbol{v}\,d\boldsymbol{w} - \boldsymbol{w}\,d\boldsymbol{v} = d\boldsymbol{v}d\boldsymbol{w}$.

617J Corollary Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$ and $\boldsymbol{v}$ an integrator with domain $\mathcal{S}$. Let $\boldsymbol{v}^*$ be the quadratic variation of $\boldsymbol{v}$.

(a) $\boldsymbol{v}^*$ is an integrator, and if $\boldsymbol{u}$ is a moderately oscillatory process with domain $\mathcal{S}$ then

$$\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{v}^*,\quad \int_{\mathcal{S}}\boldsymbol{u}\,d(\boldsymbol{v}^2)-2\int_{\mathcal{S}}\boldsymbol{u}\times\boldsymbol{v}\,d\boldsymbol{v},\quad \int_{\mathcal{S}}\boldsymbol{u}\,(d\boldsymbol{v})^2$$

are defined and equal.

(b)(i) Expressing $\boldsymbol{v}^*$ as $\langle v^*_\tau\rangle_{\tau\in\mathcal{S}}$, $\lim_{\tau\downarrow\mathcal{S}}v^*_\tau=0$.

(ii) $\boldsymbol{v}^*$ is non-negative, non-decreasing and order-bounded.

(c) If $\boldsymbol{w}$ is another integrator with domain $\mathcal{S}$, then $[\boldsymbol{v}\overset{*}{|}\boldsymbol{w}]$ is of bounded variation.

proof (a) This is immediate from 617I and the definition of $\boldsymbol{v}^*$ as $[\boldsymbol{v}\overset{*}{|}\boldsymbol{v}]$.

(b)(i) Immediate from 613J(f-i).

(ii) Express $\boldsymbol{v}$ as $\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$. If $\tau\le\tau'$ in $\mathcal{S}$, then

$$v^*_{\tau'}-v^*_\tau=\int_{\mathcal{S}\cap[\tau,\tau']}d\boldsymbol{v}^*=\int_{\mathcal{S}\cap[\tau,\tau']}(d\boldsymbol{v})^2.$$

But looking back at the definitions in §613, we see that

$$(\Delta\boldsymbol{v})^2(c(\sigma,\sigma'))=(v_{\sigma'}-v_\sigma)^2\ge0$$

for every stopping time interval $c(\sigma,\sigma')$ with endpoints in $\mathcal{S}$, so $S_I(\mathbf{1},(d\boldsymbol{v})^2)\ge0$ for every $I\in\mathcal{I}(\mathcal{S})$ and $\int_{\mathcal{S}\cap[\tau,\tau']}(d\boldsymbol{v})^2\ge0$. So $v^*_\tau\le v^*_{\tau'}$. As τ and τ' are arbitrary, $\boldsymbol{v}^*$ is non-decreasing.

As $\boldsymbol{v}^*$ is an integrator, $v^*_\uparrow=\lim_{\tau\uparrow\mathcal{S}}v^*_\tau$ is defined; as $\boldsymbol{v}^*$ is non-decreasing, $v^*_\downarrow\le v_\tau\le v^*_\uparrow$ for every $\tau\in\mathcal{S}$ and $\boldsymbol{v}^*$ is non-negative and order-bounded.

(c) By 614J(i), $\boldsymbol{v}^*$ is of bounded variation. Similarly, $\boldsymbol{w}^*$ and $(\boldsymbol{v}+\boldsymbol{w})^*$ are of bounded variation. Since covariation is bilinear (617Hc),

$$[\boldsymbol{v}\overset{*}{|}\boldsymbol{w}]=\frac{1}{2}([\boldsymbol{v}+\boldsymbol{w}\overset{*}{|}\boldsymbol{v}+\boldsymbol{w}]-[\boldsymbol{v}\overset{*}{|}\boldsymbol{v}]-[\boldsymbol{w}\overset{*}{|}\boldsymbol{w}])=\frac{1}{2}((\boldsymbol{v}+\boldsymbol{w})^*-\boldsymbol{v}^*-\boldsymbol{w}^*)$$

is a linear combination of processes of bounded variation and is of bounded variation (614Q(a-ii)).

Mnemonic $(d\boldsymbol{v})^2=d(\boldsymbol{v}^2)-2\boldsymbol{v}\,d\boldsymbol{v}\sim d\boldsymbol{v}^*$.

617K Remarks Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v}$, $\boldsymbol{w}$ local integrators with domain $\mathcal{S}$.

(a) Applying 617I to $\mathcal{S}\wedge\tau$, for $\tau\in\mathcal{S}$, we see that

$$ii_{[\boldsymbol{v}\overset{*}{|}\boldsymbol{w}]}(\boldsymbol{u})=ii_{\boldsymbol{v}\times\boldsymbol{w}}(\boldsymbol{u})-ii_{\boldsymbol{w}}(\boldsymbol{u}\times\boldsymbol{v})-ii_{\boldsymbol{v}}(\boldsymbol{u}\times\boldsymbol{w})$$

for every locally moderately oscillatory $\boldsymbol{u}$ with domain $\mathcal{S}$. Taking $\boldsymbol{u}=\mathbf{1}$,

$$[\boldsymbol{v}\overset{*}{|}\boldsymbol{w}]=ii_{[\boldsymbol{v}\overset{*}{|}\boldsymbol{w}]}(\mathbf{1})$$

(using 613N and 617J(b-i))

$$=ii_{\boldsymbol{v}\times\boldsymbol{w}}(\mathbf{1})-ii_{\boldsymbol{w}}(\boldsymbol{v})-ii_{\boldsymbol{v}}(\boldsymbol{w});$$

taking $\boldsymbol{v}=\boldsymbol{w}$,

$$\boldsymbol{v}^*=ii_{\boldsymbol{v}^2}(\mathbf{1})-2ii_{\boldsymbol{v}}(\boldsymbol{v}).$$

If $\mathcal{S}$ is not empty,

$$[\boldsymbol{v}\overset{*}{|}\boldsymbol{w}]=\boldsymbol{v}\times\boldsymbol{w}-(v_\downarrow\times w_\downarrow)\mathbf{1}-ii_{\boldsymbol{w}}(\boldsymbol{v})-ii_{\boldsymbol{v}}(\boldsymbol{w}),\quad \boldsymbol{v}^*=\boldsymbol{v}^2-v_\downarrow^2\mathbf{1}-2ii_{\boldsymbol{v}}(\boldsymbol{v}),$$

where $v_\downarrow$ and $w_\downarrow$ are the starting values of $\boldsymbol{v}$ and $\boldsymbol{w}$, by 613N again.

(b) Immediately from the definition in 617H, $[\boldsymbol{v}\restriction\mathcal{S}\wedge\tau\overset{*}{|}\boldsymbol{w}\restriction\mathcal{S}\wedge\tau]=[\boldsymbol{v}\overset{*}{|}\boldsymbol{w}]\restriction\mathcal{S}\wedge\tau$ for every $\tau\in\mathcal{S}$. We also have a formula for $[\boldsymbol{v}\restriction\mathcal{S}\vee\tau\overset{*}{|}\boldsymbol{w}\restriction\mathcal{S}\vee\tau]$; if $\tau'\in\mathcal{S}\vee\tau$, then

$$\int_{(\mathcal{S}\vee\tau)\wedge\tau'} d\boldsymbol{v}d\boldsymbol{w} = \int_{\mathcal{S}\cap[\tau,\tau']} d\boldsymbol{v}d\boldsymbol{w} = \int_{\mathcal{S}\wedge\tau'} d\boldsymbol{v}d\boldsymbol{w} - \int_{\mathcal{S}\wedge\tau} d\boldsymbol{v}d\boldsymbol{w},$$

so

$$[\boldsymbol{v}\restriction\mathcal{S}\vee\tau \overset{*}{\restriction} \boldsymbol{w}\restriction\mathcal{S}\vee\tau] = ([\boldsymbol{v}\overset{*}{\restriction}\boldsymbol{w}]\restriction\mathcal{S}\vee\tau) - z\mathbf{1}$$

where $z = \int_{\mathcal{S}\wedge\tau} d\boldsymbol{v}d\boldsymbol{w} \in L^0(\mathfrak{A}_\tau)$ is the final value of $[\boldsymbol{v}\overset{*}{\restriction}\boldsymbol{w}]\restriction\mathcal{S}\wedge\tau$. When $\boldsymbol{w} = \boldsymbol{v}$, this becomes

$$\boldsymbol{v}^*\restriction\mathcal{S}\vee\tau = v_\tau^*\mathbf{1} + (\boldsymbol{v}\restriction\mathcal{S}\vee\tau)^*$$

where $v_\tau^* = \int_{\mathcal{S}\wedge\tau}(d\boldsymbol{v})^2$.

(c) A perfectly elementary fact which it is worth having out in the open is that if $\boldsymbol{v} - \boldsymbol{u}$ is constant then the interval functions $\Delta\boldsymbol{u}$ and $\Delta\boldsymbol{v}$ are equal, so $(\Delta\boldsymbol{u})^2 = (\Delta\boldsymbol{v})^2$ and $\boldsymbol{u}^* = \boldsymbol{v}^*$.

617L Corollary Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v}$ a local integrator with domain $\mathcal{S}$. Let $\boldsymbol{v}^*$ be the quadratic variation of $\boldsymbol{v}$. Then $\boldsymbol{v}^*$ is non-negative, non-decreasing and locally of bounded variation. If $\boldsymbol{w}$ is another local integrator with domain $\mathcal{S}$, then $[\boldsymbol{v}\overset{*}{\restriction}\boldsymbol{w}]$ is locally of bounded variation.

proof Apply 617J to $\boldsymbol{v}\restriction\mathcal{S}\wedge\tau$ for $\tau \in \mathcal{S}$.

617M Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v}$, $\boldsymbol{w}$ local integrators with domain $\mathcal{S}$. Then $[\boldsymbol{v}\overset{*}{\restriction}\boldsymbol{w}]^2 \le \boldsymbol{v}^* \times \boldsymbol{w}^*$.

proof Of course this is just a form of Cauchy's inequality. I think it relies, however, on something not spelt out in Volume 3.

(a)(i) Suppose that α, β, $\gamma \in \mathbb{R}$ are such that $\xi^2\alpha + 2\xi\eta\beta + \eta^2\gamma \ge 0$ for all real ξ, η. Then $\alpha \ge 0$, $\gamma \ge 0$ and $\beta^2 \le \alpha\gamma$. **P** Taking $\xi = 1$, $\eta = 0$ we see that $\alpha \ge 0$. If $\alpha = 0$ then $\eta^2\gamma \pm 2\eta\beta \ge 0$ for every $\eta > 0$, so $|\beta| \le \frac{1}{2}\eta\gamma$ for every $\eta > 0$ and $\beta^2 = \alpha\gamma = 0$. Similarly, $\gamma \ge 0$ and if $\gamma = 0$ then $\beta^2 = \alpha\gamma$.

If α, $\gamma > 0$ then, taking $\xi = \sqrt{\gamma}$ and $\eta = \pm\sqrt{\alpha}$, we have $2\alpha\gamma \pm 2\sqrt{\alpha}\sqrt{\gamma}\beta \ge 0$, so $|\sqrt{\alpha\gamma}\beta| \le \alpha\gamma$ and $\beta^2 \le \alpha\gamma$, as required. **Q**

(ii) If (Ω, Σ, μ) is a probability space and f, g, $h : \Omega \to \mathbb{R}$ are measurable functions such that, for any ξ, $\eta \in \mathbb{R}$, $\xi^2 f + 2\xi\eta g + \eta^2 h \ge 0$ almost everywhere, then $f \ge 0$ a.e., $h \ge 0$ a.e. and $g^2 \le f \times h$ a.e. **P** The set Ω_0 of $\omega \in \Omega$ such that $\xi^2 f(\omega) + 2\xi\eta g(\omega) + \eta^2 h(\omega) \ge 0$ for all rational ξ and η is conegligible. Now, for $\omega \in \Omega_0$, $\xi^2 f(\omega) + 2\xi\eta g(\omega) + \eta^2 h(\omega) \ge 0$ for all real ξ and η, so (i) tells us that $f(\omega) \ge 0$, $h(\omega) \ge 0$ and $g(\omega)^2 \le f(\omega)h(\omega)$. As Ω_0 is conegligible, $f \ge 0$, $h \ge 0$ and $g^2 \le f \times h$ almost everywhere. **Q**

(iii) If u, v, $w \in L^0$ are such that $\xi^2 u + 2\xi\eta v + \eta^2 w \ge 0$ for all real ξ, η, then $u \ge 0$, $w \ge 0$ and $v^2 \le u \times w$. **P** Express $\mathfrak{A}$ as the measure algebra of a probability space (Ω, Σ, μ), identify $L^0(\mathfrak{A})$ with $L^0(\mu)$, take measurable functions f, g, h such that $f^\bullet = u$, $g^\bullet = v$ and $h^\bullet = w$, and apply (ii). **Q**

(b) Now the proposition follows at once from (a-iii) if we observe that, for any $\tau \in \mathcal{S}$ and ξ, $\eta \in \mathbb{R}$,

$$\xi^2\int_{\mathcal{S}\wedge\tau}(d\boldsymbol{v})^2 + 2\xi\eta\int_{\mathcal{S}\wedge\tau} d\boldsymbol{v}\,d\boldsymbol{w} + \eta^2\int_{\mathcal{S}\wedge\tau}(d\boldsymbol{w})^2 = \int_{\mathcal{S}\wedge\tau}(d\boldsymbol{z})^2 \ge 0 \text{ in } L^0$$

where $\boldsymbol{z} = \xi\boldsymbol{v} + \eta\boldsymbol{w}$ is a local integrator.

617N Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v}$, $\boldsymbol{w}$ local integrators with domain $\mathcal{S}$. Let $\hat{\boldsymbol{v}}$, $\hat{\boldsymbol{w}}$ be their fully adapted extensions to the covered envelope $\hat{\mathcal{S}}$ of $\mathcal{S}$. Then $[\hat{\boldsymbol{v}}\overset{*}{\restriction}\hat{\boldsymbol{w}}]$ is the fully adapted extension of $[\boldsymbol{v}\overset{*}{\restriction}\boldsymbol{w}]$ to $\hat{\mathcal{S}}$. In particular, the quadratic variation of $\hat{\boldsymbol{v}}$ is the fully adapted extension to $\hat{\mathcal{S}}$ of the quadratic variation of $\boldsymbol{v}$.

proof By 616Ia, $\hat{\boldsymbol{v}}$ and $\hat{\boldsymbol{w}}$ are local integrators. Of course $\Delta\hat{\boldsymbol{v}} \times \Delta\hat{\boldsymbol{w}}$ extends $\Delta\boldsymbol{v} \times \Delta\boldsymbol{w}$ just because $\hat{\boldsymbol{v}}$ extends $\boldsymbol{v}$ and $\hat{\boldsymbol{w}}$ extends $\boldsymbol{w}$. If $\tau \in \mathcal{S}$, then $\int_{\hat{\mathcal{S}}\wedge\tau} d(\Delta\hat{\boldsymbol{v}} \times \Delta\hat{\boldsymbol{w}})$ is defined, so is equal to $\int_{\mathcal{S}\wedge\tau} d(\Delta\boldsymbol{v} \times \Delta\boldsymbol{w})$ (613T, as $\hat{\mathcal{S}} \wedge \tau$ is the covered envelope of $\mathcal{S} \wedge \tau$). Thus $[\boldsymbol{v}\overset{*}{\restriction}\boldsymbol{w}] = [\hat{\boldsymbol{v}}\overset{*}{\restriction}\hat{\boldsymbol{w}}]\restriction\mathcal{S}$ and $[\hat{\boldsymbol{v}}\overset{*}{\restriction}\hat{\boldsymbol{w}}]$ is the fully adapted extension of $[\boldsymbol{v}\overset{*}{\restriction}\boldsymbol{w}]$ to $\hat{\mathcal{S}}$.

617O Examples Suppose that $T = [0, \infty[$.

(a) Let $\boldsymbol{\iota}=\langle\iota_\tau\rangle_{\tau\in\mathcal{T}_f}$ be the identity process (612F). Then its quadratic variation $\boldsymbol{\iota}^*$ is zero. **P** $\boldsymbol{\iota}$ is non-negative and non-decreasing, therefore locally order-bounded (614Ic) and locally of bounded variation (614Id) and a local integrator (616Ra). So $\boldsymbol{\iota}^*$ is well-defined; express it as $\langle\iota^*_\tau\rangle_{\tau\in\mathcal{T}_f}$. Take $t>0$ and $m\ge 1$, and set $\tau_i=\frac{\check{it}}{m}$, the constant stopping time at $\frac{it}{m}$, for $i\le m$; let I be any sublattice of $\mathcal{T}\wedge\check t$ containing τ_i for every $i\le m$. If e is an I-cell, then e is expressible as $c(\sigma,\tau)$ where $\tau_i\le\sigma\le\tau\le\tau_{i+1}$ for some i. Now

$$\Delta_e(\mathbf{1},(d\boldsymbol{\iota})^2)=(\iota_\tau-\iota_\sigma)^2\le\frac{t}{m}(\iota_\tau-\iota_\sigma)=\frac{t}{m}\Delta_e(\mathbf{1},d\boldsymbol{\iota}).$$

Summing over e,

$$S_I(\mathbf{1},(d\boldsymbol{\iota})^2)\le\frac{t}{m}S_I(\mathbf{1},d\boldsymbol{\iota})=\frac{t}{m}\iota_{\check t}.$$

Taking the limit as $I\uparrow\mathcal{I}(\mathcal{T}\wedge\check t)$,

$$\iota^*_{\check t}=\int_{\mathcal{T}\wedge\check t}(d\boldsymbol{\iota})^2\le\frac{t}{m}\iota_{\check t}.$$

As t and m are arbitrary, $\iota^*_\tau=0$ for every constant stopping time τ. Since $\boldsymbol{\iota}^*$ is non-decreasing, $\iota^*_\tau=0$ for every $\tau\in\mathcal{T}_b$. As $\mathcal{T}_b$ is a covering ideal of $\mathcal{T}_f$ (611Ne), $\boldsymbol{\iota}^*$ must be the zero process (612R). **Q**

(b) Let $\boldsymbol{v}=\langle v_\tau\rangle_{\tau\in\mathcal{T}_f}$ be the standard Poisson process, based on the structure $(\Omega,\Sigma,\mu,\mathfrak{A},\bar\mu,\langle\mathfrak{A}_t\rangle_{t\ge0})$ described in 612U. Then $\boldsymbol{v}$ is equal to its quadratic variation $\boldsymbol{v}^*$. **P** $\boldsymbol{v}$ is non-negative and non-decreasing, so $\boldsymbol{v}^*$ is well-defined. Express $\boldsymbol{v}^*$ as $\langle v^*_\tau\rangle_{\tau\in\mathcal{T}_f}$. Let $\langle\tau_n\rangle_{n\in\mathbb{N}}$ be the sequence of jump times of $\boldsymbol{v}$, as in 612Ue-612Uf, so that $\tau_0=\min\mathcal{T}$, $\sup_{n\in\mathbb{N}}\tau_n=\max\mathcal{T}$ and

$$[\![v_\sigma=n]\!]=[\![\tau_n\le\sigma]\!]\cap[\![\sigma<\tau_{n+1}]\!]$$

for every $n\in\mathbb{N}$ and $\sigma\in\mathcal{T}_f$.

Given $\tau\in\mathcal{T}_f$ and $\epsilon>0$, there is an $n\in\mathbb{N}$ such that $a=[\![\tau>\tau_n]\!]$ has measure at most ϵ. Set $J=\{\tau\wedge\tau_i:i\le n\}$ and take any sublattice I of $T\wedge\tau$ including J. If e is an I-cell, e is expressible as $c(\sigma,\sigma')$ where either $\tau_i\wedge\tau\le\sigma\le\sigma'\le\tau_{i+1}\wedge\tau$ for some $i<n$, or $\tau_n\wedge\tau\le\sigma\le\sigma'\le\tau$. In the first case,

$$\begin{aligned}[\![v_\sigma\ne v_{\sigma'}]\!]&\subseteq[\![\sigma<\sigma']\!]\subseteq[\![\tau_i\le\sigma]\!]\cap[\![\sigma<\tau_{i+1}]\!]\cap[\![\sigma'\le\tau_{i+1}]\!]\\&\subseteq[\![v_\sigma=i]\!]\cap[\![v_{\sigma'}\le i+1]\!]\subseteq[\![v_{\sigma'}-v_\sigma\in\{0,1\}]\!]=[\![(v_{\sigma'}-v_\sigma)^2=v_{\sigma'}-v_\sigma]\!].\end{aligned}$$

But this means that we actually have $(v_{\sigma'}-v_\sigma)^2=v_{\sigma'}-v_\sigma$ and $\Delta_e(\mathbf{1},(d\boldsymbol{v})^2)=\Delta_e(\mathbf{1},d\boldsymbol{v})$. In the second case,

$$[\![v_\sigma\ne v_{\sigma'}]\!]\subseteq[\![\tau_n<\tau]\!]=a$$

and

$$[\![\Delta_e(\mathbf{1},(d\boldsymbol{v})^2)\ne\Delta_e(\mathbf{1},d\boldsymbol{v})]\!]\subseteq a.$$

Thus we have

$$[\![\Delta_e(\mathbf{1},(d\boldsymbol{v})^2)\ne\Delta_e(\mathbf{1},d\boldsymbol{v})]\!]\subseteq a$$

for every I-cell e, and consequently

$$[\![S_I(\mathbf{1},(d\boldsymbol{v})^2)\ne S_I(\mathbf{1},d\boldsymbol{v})]\!]\subseteq a.$$

Taking the limit as $I\uparrow\mathcal{I}(\mathcal{T}\wedge\tau)$,

$$[\![v^*_\tau\ne v_\tau]\!]=[\![\int_{\mathcal{T}\wedge\tau}(d\boldsymbol{v})^2\ne\int_{\mathcal{T}\wedge\tau}d\boldsymbol{v}]\!]\subseteq a$$

has measure at most ϵ. As ϵ is arbitrary, $v^*_\tau=v_\tau$; as τ is arbitrary, $\boldsymbol{v}^*=\boldsymbol{v}$. **Q**

617P A more elaborate result of the same kind as 617E can be proved by the methods here.

Lemma Let $\mathcal{S}$ be a full sublattice of $\mathcal{T}$ with a greatest element, $\boldsymbol{z}$ a moderately oscillatory process and $\boldsymbol{v}$, $\boldsymbol{v}'$ integrators, all with domain $\mathcal{S}$. Set $\boldsymbol{w}=ii_{\boldsymbol{v}}(\boldsymbol{z})$. Then $\int_{\mathcal{S}}d\boldsymbol{w}\,d\boldsymbol{v}'=\int_{\mathcal{S}}\boldsymbol{z}\,d\boldsymbol{v}\,d\boldsymbol{v}'$.

proof Express $\boldsymbol{z}$, $\boldsymbol{v}$, $\boldsymbol{v}'$, $\boldsymbol{w}$ as $\langle z_\sigma\rangle_{\sigma\in\mathcal{S}}$, $\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$, $\langle v'_\sigma\rangle_{\sigma\in\mathcal{S}}$, $\langle w_\sigma\rangle_{\sigma\in\mathcal{S}}$.

(a) Consider first the case in which $\boldsymbol{z}$ is simple, with starting value $z_\downarrow$ and breakpoint string $(\tau_0,\dots,\tau_n)$ where $\tau_n=\max\mathcal{S}$. Then

$$\begin{aligned}w_\sigma&=z_\downarrow\times(v_\sigma-v_\downarrow)\text{ for }\sigma\le\tau_0,\\&=w_{\tau_i}+z_{\tau_i}\times(v_\sigma-v_{\tau_i})\text{ for }i<n\text{ and }\tau_i\le\sigma\le\tau_{i+1},\end{aligned}$$

so if $\sigma\le\tau$ in $\mathcal{S}$,

$$\begin{aligned}(\Delta\boldsymbol{w}\times\Delta\boldsymbol{v}')(\sigma,\tau)=z_\downarrow\times(\Delta\boldsymbol{v}\times\Delta\boldsymbol{v}')(\sigma,\tau)&=z_\sigma\times(\Delta\boldsymbol{v}\times\Delta\boldsymbol{v}')(\sigma,\tau)\text{ if }\tau\le\tau_0,\\=z_{\tau_i}\times(\Delta\boldsymbol{v}\times\Delta\boldsymbol{v}')(\sigma,\tau)&=z_\sigma\times(\Delta\boldsymbol{v}\times\Delta\boldsymbol{v}')(\sigma,\tau)\\&\qquad\text{if }i<n\text{ and }\tau_i\le\sigma\le\tau\le\tau_{i+1}.\end{aligned}$$

Consequently

$$\int_{\mathcal{S}\wedge\tau_0}d\boldsymbol{w}\,d\boldsymbol{v}'=\int_{\mathcal{S}\wedge\tau_0}\boldsymbol{z}\,d\boldsymbol{v}\,d\boldsymbol{v}',$$

$$\int_{\mathcal{S}\cap[\tau_i,\tau_{i+1}]}d\boldsymbol{w}\,d\boldsymbol{v}'=\int_{\mathcal{S}\cap[\tau_i,\tau_{i+1}]}\boldsymbol{z}\,d\boldsymbol{v}\,d\boldsymbol{v}'$$

for $i<n$, and

$$\int_{\mathcal{S}}d\boldsymbol{w}\,d\boldsymbol{v}'=\int_{\mathcal{S}}\boldsymbol{z}\,d\boldsymbol{v}\,d\boldsymbol{v}'.$$

(b) Generally, for $I\in\mathcal{I}(\mathcal{S})$, let $\boldsymbol{z}_I=\langle z_{I\sigma}\rangle_{\sigma\in\mathcal{S}}$ be the simple process with breakpoints in I agreeing with $\boldsymbol{z}$ on I as in 617B, and set $\boldsymbol{w}_I=ii_{\boldsymbol{v}}(\boldsymbol{z}_I)$. Then we know from 617I and (a) here that

$$\int_{\mathcal{S}}d(\boldsymbol{w}_I\times\boldsymbol{v}')-\int_{\mathcal{S}}\boldsymbol{w}_Id\boldsymbol{v}'-\int_{\mathcal{S}}\boldsymbol{v}'d\boldsymbol{w}_I=\int_{\mathcal{S}}d\boldsymbol{w}_Id\boldsymbol{v}'=\int_{\mathcal{S}}\boldsymbol{z}_Id\boldsymbol{v}\,d\boldsymbol{v}'.$$

(c)(i) Since 0 is the starting value of $\boldsymbol{w}_I$ and therefore of $\boldsymbol{w}_I\times\boldsymbol{v}'$, $\int_{\mathcal{S}}d(\boldsymbol{w}_I\times\boldsymbol{v}')=w_{I\max\mathcal{S}}\times v'_{\max\mathcal{S}}$ for each $I\in\mathcal{I}(\mathcal{S})$, and

$$\begin{aligned}\lim_{I\uparrow\mathcal{I}(\mathcal{S})}\int_{\mathcal{S}}d(\boldsymbol{w}_I\times\boldsymbol{v}')&=v'_{\max\mathcal{S}}\times\lim_{I\uparrow\mathcal{I}(\mathcal{S})}w_{I\max\mathcal{S}}=v'_{\max\mathcal{S}}\times\lim_{I\uparrow\mathcal{I}(\mathcal{S})}\int_{\mathcal{S}}\boldsymbol{z}_Id\boldsymbol{v}\\&=v'_{\max\mathcal{S}}\times\int_{\mathcal{S}}\boldsymbol{z}\,d\boldsymbol{v}=v'_{\max\mathcal{S}}\times w_{\max\mathcal{S}}=\int_{\mathcal{S}}d(\boldsymbol{w}\times\boldsymbol{v}').\end{aligned}$$

by 617B(b-ii).

(ii) Next, we know from 617B(b-i) that $\boldsymbol{w}=ii_{\boldsymbol{v}}(\boldsymbol{z})$ is the limit $\operatorname{ucplim}_{I\uparrow\mathcal{I}(\mathcal{S})}ii_{\boldsymbol{v}}(\boldsymbol{z}_I)$ for the ucp topology on $M_{\text{o-b}}(\mathcal{S})$. Consequently $\int_{\mathcal{S}}\boldsymbol{w}\,d\boldsymbol{v}'=\lim_{I\uparrow\mathcal{I}(\mathcal{S})}\int_{\mathcal{S}}\boldsymbol{w}_Id\boldsymbol{v}'$ (616J).

(iii) By 617E,

$$\int_{\mathcal{S}}\boldsymbol{v}'d\boldsymbol{w}_I=\int_{\mathcal{S}}\boldsymbol{v}'d(ii_{\boldsymbol{v}}(\boldsymbol{z}_I))=\int_{\mathcal{S}}\boldsymbol{v}'\times\boldsymbol{z}_I\,d\boldsymbol{v}=\int_{\mathcal{S}}\boldsymbol{z}_Id(ii_{\boldsymbol{v}}(\boldsymbol{v}')),$$

and similarly $\int_{\mathcal{S}}\boldsymbol{v}'d\boldsymbol{w}=\int_{\mathcal{S}}\boldsymbol{z}\,d(ii_{\boldsymbol{v}}(\boldsymbol{v}'))$. But this means that

$$\lim_{I\uparrow\mathcal{I}(\mathcal{S})}\int_{\mathcal{S}}\boldsymbol{v}'d\boldsymbol{w}_I=\int_{\mathcal{S}}\boldsymbol{v}'d\boldsymbol{w}$$

by 617B(b-ii) again.

(iv) On the other side, using the other part of 617I,

$$\lim_{I\uparrow\mathcal{I}(\mathcal{S})}\int_{\mathcal{S}}\boldsymbol{z}_Id\boldsymbol{v}\,d\boldsymbol{v}'=\lim_{I\uparrow\mathcal{I}(\mathcal{S})}\int_{\mathcal{S}}\boldsymbol{z}_Id[\boldsymbol{v}\overset{*}{|}\boldsymbol{v}']=\int_{\mathcal{S}}\boldsymbol{z}d[\boldsymbol{v}\overset{*}{|}\boldsymbol{v}']=\int_{\mathcal{S}}\boldsymbol{z}\,d\boldsymbol{v}\,d\boldsymbol{v}'.$$

(d) Assembling these, we find that

$$\begin{aligned}\int_{\mathcal{S}}d\boldsymbol{w}\,d\boldsymbol{v}'&=\int_{\mathcal{S}}d(\boldsymbol{w}\times\boldsymbol{v}')-\int_{\mathcal{S}}\boldsymbol{w}\,d\boldsymbol{v}'-\int_{\mathcal{S}}\boldsymbol{v}'d\boldsymbol{w}\\&=\lim_{I\uparrow\mathcal{I}(\mathcal{S})}\Bigl(\int_{\mathcal{S}}d(\boldsymbol{w}_I\times\boldsymbol{v}')-\int_{\mathcal{S}}\boldsymbol{w}_Id\boldsymbol{v}'-\int_{\mathcal{S}}\boldsymbol{v}'d\boldsymbol{w}_I\Bigr)\\&=\lim_{I\uparrow\mathcal{I}(\mathcal{S})}\int_{\mathcal{S}}\boldsymbol{z}_Id\boldsymbol{v}\,d\boldsymbol{v}'=\int_{\mathcal{S}}\boldsymbol{z}\,d\boldsymbol{v}\,d\boldsymbol{v}',\end{aligned}$$

as claimed.

617Q Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u}$, $\boldsymbol{z}$ and $\boldsymbol{z}'$ locally moderately oscillatory processes with domain $\mathcal{S}$, and $\boldsymbol{v}$, $\boldsymbol{v}'$ local integrators with domain $\mathcal{S}$. Set $\boldsymbol{w}=ii_{\boldsymbol{v}}(\boldsymbol{z})$, $\boldsymbol{w}'=ii_{\boldsymbol{v}'}(\boldsymbol{z}')$.

(a)(i) $[\boldsymbol{w}\overset{*}{|}\boldsymbol{v}']=ii_{[\boldsymbol{v}\overset{*}{|}\boldsymbol{v}']}(\boldsymbol{z})$, $\quad ii_{[\boldsymbol{w}\overset{*}{|}\boldsymbol{v}']}(\boldsymbol{u})=ii_{[\boldsymbol{v}\overset{*}{|}\boldsymbol{v}']}(\boldsymbol{u}\times\boldsymbol{z})$.

(ii) $[\boldsymbol{w}\overset{*}{|}\boldsymbol{w}']=ii_{[\boldsymbol{v}\overset{*}{|}\boldsymbol{v}']}(\boldsymbol{z}\times\boldsymbol{z}')$, $\quad ii_{[\boldsymbol{w}\overset{*}{|}\boldsymbol{w}']}(\boldsymbol{u})=ii_{[\boldsymbol{v}\overset{*}{|}\boldsymbol{v}']}(\boldsymbol{u}\times\boldsymbol{z}\times\boldsymbol{z}')$.

(iii) $\boldsymbol{w}^*=ii_{\boldsymbol{v}^*}(\boldsymbol{z}^2)$, $\quad ii_{\boldsymbol{w}^*}(\boldsymbol{u})=ii_{\boldsymbol{v}^*}(\boldsymbol{u}\times\boldsymbol{z}^2)$.

(b) If $\boldsymbol{u}$, $\boldsymbol{z}$ and $\boldsymbol{z}'$ are moderately oscillatory and $\boldsymbol{v}$, $\boldsymbol{v}'$ are integrators,

$$\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{w}\,d\boldsymbol{w}'=\int_{\mathcal{S}}\boldsymbol{u}\times\boldsymbol{z}\times\boldsymbol{z}'d\boldsymbol{v}\,d\boldsymbol{v}',\qquad\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{w}^*=\int_{\mathcal{S}}\boldsymbol{u}\times\boldsymbol{z}^2d\boldsymbol{v}^*.$$

proof (a)(i)($\boldsymbol{\alpha}$) Suppose to begin with that $\mathcal{S}$ is full. Take any $\tau\in\mathcal{S}$. Then 617P tells us that

$$\int_{\mathcal{S}\wedge\tau}d\boldsymbol{w}\,d\boldsymbol{v}'=\int_{\mathcal{S}\wedge\tau}\boldsymbol{z}\,d\boldsymbol{v}\,d\boldsymbol{v}'=\int_{\mathcal{S}\wedge\tau}\boldsymbol{z}\,d[\boldsymbol{v}\overset{*}{|}\boldsymbol{v}'].$$

But this means that the indefinite integrals $[\boldsymbol{w}\overset{*}{|}\boldsymbol{v}']$, $ii_{[\boldsymbol{v}\overset{*}{|}\boldsymbol{v}']}(\boldsymbol{z})$ are equal.

($\boldsymbol{\beta}$) In general, take $\hat{\mathcal{S}}$ to be the covered envelope of $\mathcal{S}$ and $\hat{\boldsymbol{z}}$, $\hat{\boldsymbol{v}}$, $\hat{\boldsymbol{v}}'$ and $\hat{\boldsymbol{w}}$ the fully adapted extensions of $\boldsymbol{z}$, $\boldsymbol{v}$, $\boldsymbol{v}'$ and $\boldsymbol{w}$ to $\hat{\mathcal{S}}$. Then $\hat{\boldsymbol{w}}=ii_{\hat{\boldsymbol{v}}}(\hat{\boldsymbol{z}})$ (616Q(c-ii)), so

$$[\hat{\boldsymbol{w}}\overset{*}{|}\hat{\boldsymbol{v}}']=ii_{[\hat{\boldsymbol{v}}\overset{*}{|}\hat{\boldsymbol{v}}']}(\hat{\boldsymbol{z}})$$

and

$$[\boldsymbol{w}\overset{*}{|}\boldsymbol{v}']=[\hat{\boldsymbol{w}}\overset{*}{|}\hat{\boldsymbol{v}}']\restriction\mathcal{S}$$

(617N)

$$=ii_{[\hat{\boldsymbol{v}}\overset{*}{|}\hat{\boldsymbol{v}}']}(\hat{\boldsymbol{z}})\restriction\mathcal{S}=ii_{[\hat{\boldsymbol{v}}\overset{*}{|}\hat{\boldsymbol{v}}']\restriction\mathcal{S}}(\boldsymbol{z})$$

(613T again)

$$=ii_{[\boldsymbol{v}\overset{*}{|}\boldsymbol{v}']}(\boldsymbol{z}).$$

($\boldsymbol{\gamma}$) By 617Db,

$$\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\,d[\boldsymbol{w}\overset{*}{|}\boldsymbol{v}']=\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\,d(ii_{[\boldsymbol{v}\overset{*}{|}\boldsymbol{v}']}(\boldsymbol{z}))=\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\times\boldsymbol{z}\,d[\boldsymbol{v}\overset{*}{|}\boldsymbol{v}'])$$

for every $\tau\in\mathcal{S}$, that is, $ii_{[\boldsymbol{w}\overset{*}{|}\boldsymbol{v}']}(\boldsymbol{u})=ii_{[\boldsymbol{v}\overset{*}{|}\boldsymbol{v}']}(\boldsymbol{u}\times\boldsymbol{z})$.

(ii) By (i),

$$[\boldsymbol{w}\overset{*}{|}\boldsymbol{w}']=[\boldsymbol{w}'\overset{*}{|}\boldsymbol{w}]=ii_{[\boldsymbol{v}'\overset{*}{|}\boldsymbol{w}]}(\boldsymbol{z}')=ii_{[\boldsymbol{w}\overset{*}{|}\boldsymbol{v}']}(\boldsymbol{z}')$$

$$=ii_{[\boldsymbol{v}\overset{*}{|}\boldsymbol{v}']}(\boldsymbol{z}'\times\boldsymbol{z})=ii_{[\boldsymbol{v}\overset{*}{|}\boldsymbol{v}']}(\boldsymbol{z}\times\boldsymbol{z}').$$

It follows, just as in (i-γ) above, that

$$ii_{[\boldsymbol{w}\overset{*}{|}\boldsymbol{w}']}(\boldsymbol{u})=ii_{[\boldsymbol{v}\overset{*}{|}\boldsymbol{v}']}(\boldsymbol{u}\times\boldsymbol{z}\times\boldsymbol{z}').$$

(iii) This is now just the special case $\boldsymbol{v}'=\boldsymbol{v}$, $\boldsymbol{z}'=\boldsymbol{z}$ of (ii).

(b) Since $[\boldsymbol{v}\overset{*}{|}\boldsymbol{v}']$, $\boldsymbol{w}$, $\boldsymbol{w}'$, $[\boldsymbol{w}\overset{*}{|}\boldsymbol{w}']$, $\boldsymbol{v}^*$ and $\boldsymbol{w}^*$ are all integrators (616J yet again, 617I), all the integrals are defined and

$$\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{w}\,d\boldsymbol{w}'=\int_{\mathcal{S}}\boldsymbol{u}\,d[\boldsymbol{w}\overset{*}{|}\boldsymbol{w}']=\lim_{\tau\uparrow\mathcal{S}}\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\,d[\boldsymbol{w}\overset{*}{|}\boldsymbol{w}']=\lim_{\tau\uparrow\mathcal{S}}\int_{\mathcal{S}\wedge\tau}\boldsymbol{u}\times\boldsymbol{z}\times\boldsymbol{z}'\,d[\boldsymbol{v}\overset{*}{|}\boldsymbol{v}']$$

$$=\int_{\mathcal{S}}\boldsymbol{u}\times\boldsymbol{z}\times\boldsymbol{z}'\,d[\boldsymbol{v}\overset{*}{|}\boldsymbol{v}']=\int_{\mathcal{S}}\boldsymbol{u}\times\boldsymbol{z}\times\boldsymbol{z}'\,d\boldsymbol{v}\,d\boldsymbol{v}'.$$

Taking $\boldsymbol{v}'=\boldsymbol{v}$, $\boldsymbol{z}'=\boldsymbol{z}$ and $\boldsymbol{w}'=\boldsymbol{w}$ the formulae simplify to $\int_{\mathcal{S}}\boldsymbol{u}\,d\boldsymbol{w}^*=\int_{\mathcal{S}}\boldsymbol{u}\times\boldsymbol{z}^2 d\boldsymbol{v}^*$.

Mnemonic $d[\boldsymbol{w}\overset{*}{|}\boldsymbol{w}']\sim d\boldsymbol{w}\,d\boldsymbol{w}'\sim(\boldsymbol{z}\,d\boldsymbol{v})(\boldsymbol{z}'d\boldsymbol{v}')\sim\boldsymbol{z}\times\boldsymbol{z}'\,d\boldsymbol{v}\,d\boldsymbol{v}'\sim\boldsymbol{z}\times\boldsymbol{z}'\,d[\boldsymbol{v}\overset{*}{|}\boldsymbol{v}']$.

617R Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v}$ a process with domain $\mathcal{S}$ which is locally of bounded variation. Then $\boldsymbol{v}$ and its cumulative variation have the same quadratic variation.

proof (a) I begin with an elementary fact about f-algebras which was left as an exercise in §352[8]. Suppose that u, $v\in L^0$. Then $u\times v=(u\wedge v)\times(u\vee v)$. **P**

$$u\times v=((u\wedge v)+(u-v)^+)\times((u\wedge v)+(v-u)^+)$$

(352D)

$$=(u\wedge v)\times(u\wedge v+(u-v)^++(v-u)^+)$$

(because $(u-v)^+\wedge(v-u)^+=0$, by 352D again, so $(u-v)^+\times(v-u)^+=0$, by 352W(b-i))

$$=(u\wedge v)\times(u\vee v)$$

(352D once more). **Q**

(b) Suppose for the moment that $\boldsymbol{v}$ is actually of bounded variation. Express $\boldsymbol{v}$ as $\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ and its cumulative variation as $\boldsymbol{v}^\uparrow=\langle v_\sigma^\uparrow\rangle_{\sigma\in\mathcal{S}}$. Set

$$\boldsymbol{v}_1=\tfrac{1}{2}(\boldsymbol{v}^\uparrow+\boldsymbol{v}),\quad \boldsymbol{v}_2=\tfrac{1}{2}(\boldsymbol{v}^\uparrow-\boldsymbol{v}),$$

so that $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are both non-decreasing processes (614P(a-iii)), $\boldsymbol{v}=\boldsymbol{v}_1-\boldsymbol{v}_2$ and $\boldsymbol{v}^\uparrow=\boldsymbol{v}_1+\boldsymbol{v}_2$. Let ψ be the interval function $\Delta\boldsymbol{v}^\uparrow-|\Delta\boldsymbol{v}|$ with domain $\mathcal{S}^{2\uparrow}$; then

$$\int_{\mathcal{S}\wedge\tau}d\psi=\int_{\mathcal{S}\wedge\tau}d\boldsymbol{v}^\uparrow-\int_{\mathcal{S}\wedge\tau}|d\boldsymbol{v}|=v_\tau^\uparrow-\int_{\mathcal{S}\wedge\tau}|d\boldsymbol{v}|=0$$

for every $\tau\in\mathcal{S}$ (614O). Note that if $\sigma\le\tau$ in $\mathcal{S}$ then

$$\psi(\sigma,\tau)=v_\tau^\uparrow-v_\sigma^\uparrow-|v_\tau-v_\sigma|\ge 0$$

(614P(a-i)) and, expressing $\boldsymbol{v}_1$ as $\langle v_{1\sigma}\rangle_{\sigma\in\mathcal{S}}$ and $\boldsymbol{v}_2$ as $\langle v_{2\sigma}\rangle_{\sigma\in\mathcal{S}}$,

$$\psi(\sigma,\tau)=(v_\tau^\uparrow-v_\sigma^\uparrow+v_\tau-v_\sigma)\wedge(v_\tau^\uparrow-v_\sigma^\uparrow-v_\tau+v_\sigma)=(2v_{1\tau}-2v_{1\sigma})\wedge(2v_{2\tau}-2v_{2\sigma});$$

consequently

$$(v_{1\tau}-v_{1\sigma})\times(v_{2\tau}-v_{2\sigma})=((v_{1\tau}-v_{1\sigma})\wedge(v_{2\tau}-v_{2\sigma}))\times((v_{1\tau}-v_{1\sigma})\vee(v_{2\tau}-v_{2\sigma}))$$

(by (a))

$$\le((v_{1\tau}-v_{1\sigma})\wedge(v_{2\tau}-v_{2\sigma}))\times 2\bar v=\bar v\times\psi(\sigma,\tau)$$

where $\bar v=\frac{1}{2}(\sup|\boldsymbol{v}_1|+\sup|\boldsymbol{v}_2|)$. But this means that

$$0\le S_I(\mathbf{1},d\boldsymbol{v}_1d\boldsymbol{v}_2)\le\bar v\times S_I(\mathbf{1},d\psi)\text{ for every }I\in\mathcal{I}(\mathcal{S}),$$

$$0\le\int_{\mathcal{S}\wedge\tau}d\boldsymbol{v}_1d\boldsymbol{v}_2\le\bar v\times\int_{\mathcal{S}\wedge\tau}d\psi=0\text{ for every }\tau\in\mathcal{S}$$

and $[\boldsymbol{v}_1\overset{*}{|}\boldsymbol{v}_2]=0$. Accordingly the quadratic variations

$$(\boldsymbol{v}^\uparrow)^*=[\boldsymbol{v}^\uparrow\overset{*}{|}\boldsymbol{v}^\uparrow]=[\boldsymbol{v}_1+\boldsymbol{v}_2\overset{*}{|}\boldsymbol{v}_1+\boldsymbol{v}_2]=[\boldsymbol{v}_1\overset{*}{|}\boldsymbol{v}_1]+2[\boldsymbol{v}_1\overset{*}{|}\boldsymbol{v}_2]+[\boldsymbol{v}_2\overset{*}{|}\boldsymbol{v}_2]$$

$$=[\boldsymbol{v}_1\overset{*}{|}\boldsymbol{v}_1]-2[\boldsymbol{v}_1\overset{*}{|}\boldsymbol{v}_2]+[\boldsymbol{v}_2\overset{*}{|}\boldsymbol{v}_2]=\boldsymbol{v}^*$$

are equal.

[8]Later editions only.

(c) Now the general result follows at once because

$$(\boldsymbol{v}^{\uparrow})^*\restriction\mathcal{S}\wedge\tau = (\boldsymbol{v}^{\uparrow}\restriction\mathcal{S}\wedge\tau)^*$$

(617Kb)

$$= ((\boldsymbol{v}\restriction\mathcal{S}\wedge\tau)^{\uparrow})^*$$

(614Pb)

$$= (\boldsymbol{v}\restriction\mathcal{S}\wedge\tau)^*$$

(by (b))

$$= \boldsymbol{v}^*\restriction\mathcal{S}\wedge\tau$$

for every $\tau\in\mathcal{S}$.

Mnemonic $|d\boldsymbol{v}|^2 = d\boldsymbol{v}^2$.

617X Basic exercises (a) Let $\boldsymbol{z}$ be a locally moderately oscillatory process and $\boldsymbol{v}$ a process locally of bounded variation with the same domain. Show that $ii_{\boldsymbol{v}}(\boldsymbol{z})^{\uparrow} = ii_{\boldsymbol{v}^{\uparrow}}(|\boldsymbol{z}|)$.

(b) Suppose that $T=[0,\infty[$ and $\mathfrak{A}=\{0,1\}$, as in 613W, 615Xf and 616Xa. Let $f:[0,\infty[\,\to\mathbb{R}$ be a càdlàg function such that $f\restriction[0,t]$ is of bounded variation for every $t\ge 0$. Let $\boldsymbol{v}$ be the corresponding process on $\mathcal{T}_f$ and $\boldsymbol{v}^*$ its quadratic variation. Show that $\boldsymbol{v}^*$ corresponds to the function $g:\mathbb{R}\to\mathbb{R}$ defined by saying that $g(t)=\sum_{0<s\le t}(f(s)-\lim_{s'\uparrow s}f(s'))^2$.

(c) Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$, $\boldsymbol{v}$ an integrator with domain $\mathcal{S}$, and z a member of $L^0(\bigcap_{\sigma\in\mathcal{S}}\mathfrak{A}_\sigma)$. Show that if $\boldsymbol{v}^*$ is the quadratic variation of $\boldsymbol{v}$, then (using the language of 612D(e-ii)) $z^2\boldsymbol{v}^*$ is the quadratic variation of $z\boldsymbol{v}$.

(d) Supposing that $T=[0,\infty[$, take $\boldsymbol{\iota}=\langle\iota_\tau\rangle_{\tau\in\mathcal{T}_f}$ to be the identity process. Show that

$$\int_{\mathcal{T}\wedge\tau}\boldsymbol{\iota}\,d\boldsymbol{\iota}=\frac{1}{2}\iota_\tau^2$$

for every $\tau\in\mathcal{T}_f$.

>(e) Let $\boldsymbol{v}=\langle v_\tau\rangle_{\tau\in\mathcal{T}_f}$ be the standard Poisson process. Show that

$$\int_{\mathcal{T}\wedge\tau}\boldsymbol{v}\,d\boldsymbol{v}=\frac{1}{2}(v_\tau^2-v_\tau)$$

for every $\tau\in\mathcal{T}_f$.

617 Notes and comments There is a lot of meat in this section. We have already seen the relatively straightforward, but obviously fundamental, fact that indefinite integrals are commonly local integrators (616J, 616Q(c-i)). It is a general principle, of which we shall see many examples, that an indefinite integral $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is likely to share any special properties of the (local) integrator $\boldsymbol{v}$; as a first step, $ii_{\boldsymbol{v}}(\boldsymbol{u})$ will be locally of bounded variation if $\boldsymbol{v}$ is (616L). And we know that an indefinite integral operator $ii_{\boldsymbol{v}}$ with respect to an integrator $\boldsymbol{v}$ is continuous for the ucp topology (616J).

In fact there is nothing here as difficult as the basic theorem on existence of Riemann-sum integrals (616M). Once we know that the integrals (with respect to integrating interval functions, as well as with respect to integrators) are defined, the analysis in §613 takes us a long way; if you like, Riemann-sum integrability is a very restrictive condition which we shall not escape until we come to the S-integral in Chapter 64.

In the 'mnemonics' offered in this section, we have to distinguish between $\sim$ and $=$. When I wrote

$$(d\boldsymbol{v})^2 = d(\boldsymbol{v}^2)-2\boldsymbol{v}\,d\boldsymbol{v}\sim d\boldsymbol{v}^*$$

(617J), the equality corresponds to a pair of interval functions being equal; writing $\boldsymbol{v}\Delta\boldsymbol{v}$ for the interval function $c(\sigma,\tau)\mapsto v_\sigma\times(v_\tau-v_\sigma)$, I am saying that

$$(v_\tau - v_\sigma)^2 = v_\tau^2 - v_\sigma^2 - 2v_\sigma \times (v_\tau - v_\sigma)$$

and therefore that

$$(\Delta\boldsymbol{v})^2 = \Delta(\boldsymbol{v}^2) - 2\boldsymbol{v}\Delta\boldsymbol{v},$$

$$\Delta_e(\boldsymbol{u}, (d\boldsymbol{v})^2) = \Delta_e(\boldsymbol{u}, d(\boldsymbol{v}^2)) - 2\Delta_e(\boldsymbol{u}\times\boldsymbol{v}, d\boldsymbol{v}),$$

$$S_I(\boldsymbol{u}, (d\boldsymbol{v})^2) = S_I(\boldsymbol{u}, d(\boldsymbol{v}^2)) - 2S_I(\boldsymbol{u}\times\boldsymbol{v}, d\boldsymbol{v}),$$

$$\int_{\mathcal{S}} \boldsymbol{u}\,(d\boldsymbol{v})^2 = \int_{\mathcal{S}} \boldsymbol{u}\,d(\boldsymbol{v}^2) - 2\int_{\mathcal{S}} \boldsymbol{u}\times\boldsymbol{v}\,d\boldsymbol{v}$$

for all e, I, $\boldsymbol{u}$ and $\mathcal{S}$ for which these are defined. We have to pause for a moment to decide when the integrals exist, but everything else is elementary algebra. But '$d\boldsymbol{v}^* \sim (d\boldsymbol{v})^2$' does not assert that $v_\tau^* - v_\sigma^*$ is related to $(v_\tau - v_\sigma)^2$ for any particular σ and τ. It is in the first place a claim that $\int_{\mathcal{S}} d\boldsymbol{v}^* = \int_{\mathcal{S}}(d\boldsymbol{v})^2$, and then a claim that $\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}^* = \int_{\mathcal{S}} \boldsymbol{u}\,(d\boldsymbol{v})^2$. The point is that the interval function $\Delta\boldsymbol{v}^*$ is additive, but $(\Delta\boldsymbol{v})^2$ is not. The exposition in Chapter 48 of Volume 4 tacitly assumes that when forming gauge integrals we rather expect our measures to be at least finitely additive. In the present volume we are dealing with integrals with respect to 'measures' which in some cases are most definitely not additive.

I introduced 'adapted interval functions' in §613 partly in order to have a direct definition for such expressions as $\int_{\mathcal{S}} \boldsymbol{u}|d\boldsymbol{v}|$ and $\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}d\boldsymbol{v}'$. In the contexts here, they give what turn out to be useful suggestions: if $\boldsymbol{w} = ii_{\boldsymbol{v}}(\boldsymbol{z})$, then we have mnemonics $d\boldsymbol{w} \sim \boldsymbol{z}\,d\boldsymbol{v}$ for 617E, $|d\boldsymbol{w}| \sim |\boldsymbol{z}||d\boldsymbol{v}|$ for 617G and $d\boldsymbol{w}d\boldsymbol{v}' \sim \boldsymbol{z}\,d\boldsymbol{v}d\boldsymbol{v}'$ for 617Q. But these simple-minded formulae give no hint of the pages of detailed calculations in 617F and 617P between the two expressions. The problem lies in the fact that $\Delta\boldsymbol{w}$ is not quite equal to $\boldsymbol{z}\Delta\boldsymbol{v}$, so we cannot immediately relate $|\Delta\boldsymbol{w}|$ to $|\boldsymbol{z}||\Delta\boldsymbol{v}|$ or $\Delta\boldsymbol{w}\times\Delta\boldsymbol{v}'$ to $\boldsymbol{z}\Delta\boldsymbol{v}\times\Delta\boldsymbol{v}'$.

Of course all the work of the second half of the section is dependent on being able to move between $d\boldsymbol{v}d\boldsymbol{v}'$ and $d[\boldsymbol{v}\overset{*}{|}\boldsymbol{v}']$, where $[\boldsymbol{v}\overset{*}{|}\boldsymbol{v}']$ is not merely a local integrator but actually locally of bounded variation; you will see that from 617P on I am repeatedly employing whichever expression is most immediately convenient.

In both parts of 617O, we find that the calculation of the quadratic variation from the formula

$$v_\tau^* = \int_{\mathcal{S}\wedge\tau}(d\boldsymbol{v})^2$$

(617Hb) is elementary in the sense that we just have to work carefully through the definitions, looking at very natural sublattices $\{\check{it/m} : i \le m\}$ in 617Oa and $\{\tau\wedge\tau_i : i \le n\}$ in 617Ob. Of course we could use the same ideas to give us direct calculations of the indefinite integrals $ii_{\boldsymbol{\iota}}(\boldsymbol{\iota})$ and $ii_{\boldsymbol{v}}(\boldsymbol{v})$ (617Xd-617Xe), but these would necessarily be a touch more complex. In 617Xe we have a first outright declaration of a difference between the formulae of elementary calculus and the corresponding formulae of stochastic calculus, a foretaste of Itô's formula (§619).

You will note that 617O omits any description of the quadratic variation of Brownian motion. As has happened before in this chapter, Brownian motion seems to be essentially more difficult and more interesting, as well as more important, than the Poisson process. Like the identity process, Brownian motion has no jumps (618J), but even so its quadratic variation is non-zero (624F); the argument I will give in 622L depends on Dynkin's formula in harmonic analysis (478K).

617R is based on an elementary manipulation, but I think it requires validation using ideas from §352.

Version of 8.9.12/26.8.22

618 Oscillations and jump-free processes

For the work so far, moderately oscillatory processes have been sufficiently regular for our needs. But for the next development (Itô's formula, 619C), we are going to need a new concept. In 618B I formulate a notion of 'jump-free' process corresponding to the idea of 'process with continuous sample paths' (618H).

618A Notation As previously, $(\mathfrak{A}, \bar\mu, T, \langle\mathfrak{A}_t\rangle_{t\in T}, \mathcal{T}, \langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ will be a stochastic integration structure. For a sublattice $\mathcal{S}$ of $\mathcal{T}$, $M_{\text{o-b}}(\mathcal{S})$ will be the space of order-bounded processes with domain $\mathcal{S}$ (614Fc) and $\mathcal{I}(\mathcal{S})$ the set of finite sublattices of $\mathcal{S}$. For an order-bounded process $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$, $\sup|\boldsymbol{u}|$ will be $\sup_{\sigma\in\mathcal{S}}|u_\sigma|$ (614Ea). L^∞ will be $L^\infty(\mathfrak{A})$ with its norm $\|\;\|_\infty$ as defined in §363. $L^0(\mathfrak{A})$ (§364) will be endowed with its topology and uniformity of convergence in measure, with the defining F-norm θ (613Ba). $\mathbf{1}$ will be the constant process with value $\chi 1$. If $h:\mathbb{R}\to\mathbb{R}$ is a continuous function, I will write $\bar h$ for any of the derived functions from $L^0(\mathfrak{A})$ to itself or from a space of fully adapted processes to itself (612A, 612Ia).

618B Definitions (a) Let I be a finite sublattice of $\mathcal{T}$, and $\boldsymbol{u}$ a fully adapted process defined (at least) on I. The **I-oscillation of $\boldsymbol{u}$** is

$$\mathrm{Osclln}_I(\boldsymbol{u})=\sup_{e\in\mathrm{Stio}_0(I)}\Delta_e(\mathbf{1},|d\boldsymbol{u}|),$$

where $\mathrm{Stio}_0(I)$ is the set of I-cells (611Je) and for a stopping-time interval $e=c(\sigma,\tau)$ I write $\Delta_e(\mathbf{1},|d\boldsymbol{u}|)=|u_\tau-u_\sigma|$ (613Fa). Take the supremum in $(L^0(\mathfrak{A}))^+$, so that if $\#(I)\le 1$ and $\mathrm{Stio}_0(I)$ is empty, then $\mathrm{Osclln}_I(\boldsymbol{u})=0$.

Note that if $(\tau_0,\dots,\tau_n)$ linearly generates the I-cells, then $\mathrm{Osclln}_I(\boldsymbol{u})=\sup_{i<n}|u_{\tau_{i+1}}-u_{\tau_i}|$, just because $\mathrm{Stio}_0(I)$ is the set of stopping-time intervals $\{c(\tau_i,\tau_{i+1}):i<n,\ \tau_i\ne\tau_{i+1}\}$ (611L).

(b) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ an order-bounded process (definition: 614E). Set $\bar u=\sup|\boldsymbol{u}|$.

(i) $\Delta_e(\mathbf{1},|d\boldsymbol{u}|)\le 2\bar u$ for every e in the set $\mathrm{Sti}(\mathcal{S})$ of stopping-time intervals with endpoints in $\mathcal{S}$, so $\mathrm{Osclln}_J(\boldsymbol{u})\le 2\bar u$ for every $J\in\mathcal{I}(\mathcal{S})$. We can therefore set

$$\mathrm{Osclln}^*_I(\boldsymbol{u})=\sup_{J\in\mathcal{I}(\mathcal{S}),J\supseteq I}\mathrm{Osclln}_J(\boldsymbol{u})\le 2\bar u$$

for every $I\in\mathcal{I}(\mathcal{S})$.

(ii) The **residual oscillation** $\mathrm{Osclln}(\boldsymbol{u})$ is $\inf_{I\in\mathcal{I}(\mathcal{S})}\mathrm{Osclln}^*_I(\boldsymbol{u})\le 2\bar u$. I will say that $\boldsymbol{u}$ is **jump-free** if $\mathrm{Osclln}(\boldsymbol{u})=0$. For definiteness, I add that if, in the rest of this volume, I say that a process is jump-free, I mean to imply that it is order-bounded.

(iii) I will say that $\boldsymbol{u}$ is **locally jump-free** if $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ is jump-free for every $\tau\in\mathcal{S}$.

(iv)

$$\begin{aligned}\mathrm{Osclln}^*_\emptyset(\boldsymbol{u})&=\sup_{J\in\mathcal{I}(\mathcal{S})}\mathrm{Osclln}_J(\boldsymbol{u})=\sup\{|u_{\sigma'}-u_\sigma|:\sigma,\ \sigma'\in\mathcal{S}\text{ and }\sigma\le\sigma'\}\\&=\sup\{|u_{\sigma'\vee\sigma}-u_{\sigma'\wedge\sigma}|:\sigma,\ \sigma'\in\mathcal{S}\}=\sup\{|u_{\sigma'}-u_\sigma|:\sigma,\ \sigma'\in\mathcal{S}\}\end{aligned}$$

by 612D(f-ii).

(v) Remarks ($\boldsymbol{\alpha}$) Note that $I\mapsto\mathrm{Osclln}^*_I(\boldsymbol{u}):\mathcal{I}(\mathcal{S})\to L^0(\mathfrak{A})$ is non-increasing, so $\mathrm{Osclln}(\boldsymbol{u})$ is the limit $\lim_{I\uparrow\mathcal{I}(\mathcal{S})}\mathrm{Osclln}^*_I(\boldsymbol{u})$ for the topology of convergence in measure, and $\boldsymbol{u}$ is jump-free iff

$$\inf_{I\in\mathcal{I}(\mathcal{S})}\theta(\mathrm{Osclln}^*_I(\boldsymbol{u}))=\lim_{I\uparrow\mathcal{I}(\mathcal{S})}\theta(\mathrm{Osclln}^*_I(\boldsymbol{u}))=0.$$

($\boldsymbol{\beta}$) In the formula

$$\mathrm{Osclln}_I(\boldsymbol{u})=\sup_{e\in\mathrm{Stio}_0(I)}\Delta_e(\mathbf{1},|d\boldsymbol{u}|),$$

the arguments I and $\boldsymbol{u}$ are explicit in the term $\mathrm{Osclln}_I(\boldsymbol{u})$. In the formula

$$\mathrm{Osclln}^*_I(\boldsymbol{u})=\sup_{J\in\mathcal{I}(\mathcal{S}),J\supseteq I}\mathrm{Osclln}_J(\boldsymbol{u}),$$

the lattice $\mathcal{S}$ is not declared overtly in the expression $\mathrm{Osclln}^*_I(\boldsymbol{v})$; a purist would prefer

$$\mathrm{Osclln}^*_I(\boldsymbol{u})=\sup_{J\in\mathcal{I}(\mathrm{dom}\,\boldsymbol{u}),J\supseteq I}\mathrm{Osclln}_J(\boldsymbol{u}),$$

just as

$$\mathrm{Osclln}(\boldsymbol{u})=\inf_{I\in\mathcal{I}(\mathrm{dom}\,\boldsymbol{u})}\mathrm{Osclln}^*_I(\boldsymbol{u}).$$

($\boldsymbol{\gamma}$) Observe that we have $\mathrm{Osclln}_I(\boldsymbol{u})=\mathrm{Osclln}(\boldsymbol{u}\restriction I)$ for every $I\in\mathcal{I}(\mathcal{S})$.

(c) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$, $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ order-bounded processes.

(i) For any $\alpha\in\mathbb{R}$,

$$\mathrm{Osclln}_I(\alpha\boldsymbol{u})=|\alpha|\,\mathrm{Osclln}_I(\boldsymbol{u})\text{ for every }I\in\mathcal{I}(\mathcal{S}),$$

$$\mathrm{Oscclln}^*_I(\alpha\boldsymbol{u}) = |\alpha|\,\mathrm{Oscclln}^*_I(\boldsymbol{u}) \text{ for every } I \in \mathcal{I}(\mathcal{S}),$$

$$\mathrm{Oscclln}(\alpha\boldsymbol{u}) = |\alpha|\,\mathrm{Oscclln}(\boldsymbol{u}).$$

(ii)

$$\mathrm{Oscclln}_I(\boldsymbol{u}+\boldsymbol{v}) \le \mathrm{Oscclln}_I(\boldsymbol{u}) + \mathrm{Oscclln}_I(\boldsymbol{v}) \text{ for every } I \in \mathcal{I}(\mathcal{S}),$$

$$\mathrm{Oscclln}^*_I(\boldsymbol{u}+\boldsymbol{v}) \le \mathrm{Oscclln}^*_I(\boldsymbol{u}) + \mathrm{Oscclln}^*_I(\boldsymbol{v}) \text{ for every } I \in \mathcal{I}(\mathcal{S}),$$

$$\mathrm{Oscclln}(\boldsymbol{u}+\boldsymbol{v}) \le \mathrm{Oscclln}(\boldsymbol{u}) + \mathrm{Oscclln}(\boldsymbol{v}).$$

(iii) Writing $\bar u$, $\bar v$ for $\sup|\boldsymbol{u}|$ and $\sup|\boldsymbol{v}|$,

$$\mathrm{Oscclln}_I(\boldsymbol{u}\times\boldsymbol{v}) \le \bar v \times \mathrm{Oscclln}_I(\boldsymbol{u}) + \bar u \times \mathrm{Oscclln}_I(\boldsymbol{v}) \text{ for every } I \in \mathcal{I}(\mathcal{S}),$$

$$\mathrm{Oscclln}^*_I(\boldsymbol{u}\times\boldsymbol{v}) \le \bar v \times \mathrm{Oscclln}^*_I(\boldsymbol{u}) + \bar u \times \mathrm{Oscclln}^*_I(\boldsymbol{v}) \text{ for every } I \in \mathcal{I}(\mathcal{S}),$$

$$\mathrm{Oscclln}(\boldsymbol{u}\times\boldsymbol{v}) \le \bar v \times \mathrm{Oscclln}(\boldsymbol{u}) + \bar u \times \mathrm{Oscclln}(\boldsymbol{v}).$$

618C We shall not need them immediately, but the following descriptions of $\mathrm{Oscclln}^*$ will be useful later on.

Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ an order-bounded fully adapted process. Let I be a non-empty finite sublattice of $\mathcal{S}$; suppose that $(\tau_0,\dots,\tau_n)$ linearly generates the I-cells.

(a) Set $\tau_{-1} = \inf\mathcal{S}$ and $\tau_{n+1} = \sup\mathcal{S}$ and

$$\begin{aligned} w = \sup\{|u_{\sigma'} - u_\sigma| : \sigma,\,\sigma' \in \mathcal{S} \text{ and there is an } i \\ \text{such that } -1 \le i \le n \text{ and } \tau_i \le \sigma \le \sigma' \le \tau_{i+1}\}, \\ w' = \sup\{|u_{\sigma'} - u_\sigma| : \sigma,\,\sigma' \in \mathcal{S} \text{ and there is an } i \\ \text{such that } -1 \le i \le n \text{ and } \sigma,\,\sigma' \in [\tau_i,\tau_{i+1}]\}. \end{aligned}$$

Then $w = w' = \mathrm{Oscclln}^*_I(\boldsymbol{u})$.

(b) Now suppose that $\boldsymbol{u}$ is non-decreasing. Set $u_\downarrow = \inf_{\sigma\in\mathcal{S}} u_\sigma$ and $u_\uparrow = \sup_{\sigma\in\mathcal{S}} u_\sigma$. Then

$$\mathrm{Oscclln}^*_I(\boldsymbol{u}) = (u_{\tau_0} - u_\downarrow) \vee \sup_{i<n}(u_{\tau_{i+1}} - u_{\tau_i}) \vee (u_\uparrow - u_{\tau_n}).$$

proof (a)(i) Suppose that $J \in \mathcal{I}(\mathcal{S})$, $I \subseteq J$ and $e \in \mathrm{Sti}_0(J)$. Then there is an i such that $-1 \le i \le n$ and $e \in \mathrm{Sti}_0(J \cap [\tau_i,\tau_{i+1}])$, by 611J(e-iii). In this case, the stopping-time interval e is expressible as $c(\sigma,\sigma')$ where $\tau_i \le \sigma \le \sigma' \le \tau_{i+1}$ and $\sigma,\,\sigma' \in \mathcal{S}$. Accordingly

$$\Delta_e(\mathbf{1},|d\boldsymbol{u}|) = |u_{\sigma'} - u_\sigma| \le w'.$$

As e is arbitrary, $\mathrm{Oscclln}_J(\boldsymbol{u}) \le w'$; as J is arbitrary, $\mathrm{Oscclln}^*_I(\boldsymbol{u}) \le w$.

(ii) Suppose that $-1 \le i \le n$ and $\tau_i \le \sigma \le \sigma' \le \tau_{i+1}$. Then $|u_{\sigma'} - u_\sigma| \le \mathrm{Oscclln}^*_I(\boldsymbol{u})$. **P** If $\sigma = \sigma'$ this is trivial. Otherwise, set $J = I \cup \{\sigma,\sigma'\}$. If $\tau \in I$ then $\mathrm{med}(\tau_i,\tau,\tau_{i+1})$ belongs to I (because $\tau\vee\tau_{-1} = \tau\wedge\tau_{n+1} = \tau \in I$), and must be either τ_i or τ_{i+1}; so either $\tau \le \sigma$ or $\sigma' \le \tau$. It follows that J is a sublattice of $\mathcal{S}$ including I and $c(\sigma,\sigma')$ is a J-cell. So

$$|u_{\sigma'} - u_\sigma| \le \mathrm{Oscclln}_J(\boldsymbol{u}) \le \mathrm{Oscclln}^*_I(\boldsymbol{u}). \text{ } \mathbf{Q}$$

Thus $w \le \mathrm{Oscclln}^*_I(\boldsymbol{u})$ and the two are equal.

(iii) Of course $w \le w'$. On the other hand, if σ, σ' both belong to $\mathcal{S} \cap [\tau_i,\tau_{i+1}]$ for some i, then so do $\sigma\wedge\sigma'$ and $\sigma\vee\sigma'$, and

$$w \ge |u_{\sigma\vee\sigma'} - u_{\sigma\wedge\sigma'}| = |u_\sigma - u_{\sigma'}|$$

(612D(f-ii)). So $w \ge w'$ and we have equality.

(b) This is now elementary, because

$$\begin{aligned}\sup\{|u_{\sigma'}-u_\sigma|:\sigma,\,\sigma'\in\mathcal{S},\,\tau_i\le\sigma\le\sigma'\le\tau_{i+1}\}&=u_{\tau_0}-u_\downarrow\text{ if }i=-1,\\&=u_{\tau_{i+1}}-u_{\tau_i}\text{ if }0\le i<n,\\&=u_\uparrow-u_{\tau_n}\text{ if }i=n.\end{aligned}$$

618D Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a locally order-bounded process.

(a) Set $v_\tau=\operatorname{Osclln}(\boldsymbol{u}\restriction\mathcal{S}\wedge\tau)$ for $\tau\in\mathcal{S}$. Then $\boldsymbol{v}=\langle v_\tau\rangle_{\tau\in\mathcal{S}}$ is a non-decreasing fully adapted process.

(b) If $\boldsymbol{u}$ is order-bounded, then

(i) $\operatorname{Osclln}(\boldsymbol{u})=\operatorname{Osclln}(\boldsymbol{u}\restriction\mathcal{S}\wedge\tau)\vee\operatorname{Osclln}(\boldsymbol{u}\restriction\mathcal{S}\vee\tau)$ for every $\tau\in\mathcal{S}$,

(ii) $\operatorname{Osclln}(\boldsymbol{u}\restriction\mathcal{S}\cap[\tau,\tau'])\le\operatorname{Osclln}(\boldsymbol{u})$ whenever $\tau\le\tau'$ in $\mathcal{S}$.

proof (a)(i) Looking at the definitions in 618B, we see that, for any $\tau\in\mathcal{S}$,

$$\operatorname{Osclln}_I(\boldsymbol{u}\restriction\mathcal{S}\wedge\tau)\in L^0(\mathfrak{A}_\tau)\text{ for every }I\in\mathcal{I}(\mathcal{S}\wedge\tau),$$

so that

$$\operatorname{Osclln}^*_I(\boldsymbol{u}\restriction\mathcal{S}\wedge\tau)\in L^0(\mathfrak{A}_\tau)\text{ for every }I\in\mathcal{I}(\mathcal{S}\wedge\tau)$$

(because $L^0(\mathfrak{A}_\tau)$ is an order-closed sublattice of $L^0(\mathfrak{A})$, by 612Ae) and

$$v_\tau=\inf_{I\in\mathcal{I}(\mathcal{S}\wedge\tau)}\operatorname{Osclln}^*_I(\boldsymbol{u}\restriction\mathcal{S}\wedge\tau)\in L^0(\mathfrak{A}_\tau).$$

(ii) If $\tau\le\tau'$ in $\mathcal{S}$, then

$$\operatorname{Osclln}_I(\boldsymbol{u})=\operatorname{Osclln}_{I\wedge\tau}(\boldsymbol{u})\vee\operatorname{Osclln}_{I\vee\tau}(\boldsymbol{u})\text{ whenever }\tau\in I\in\mathcal{I}(\mathcal{S})$$

(use 611J(e-ii)). Suppose that $\tau\in I\in\mathcal{I}(\mathcal{S}\wedge\tau')$ and that $I\wedge\tau\subseteq J\in\mathcal{I}(\mathcal{S}\wedge\tau)$. Let K be the sublattice generated by $I\cup J$; then $K\wedge\tau=J$, so

$$\operatorname{Osclln}_J(\boldsymbol{u})\le\operatorname{Osclln}_K(\boldsymbol{u})\le\operatorname{Osclln}^*_I(\boldsymbol{u}\restriction\mathcal{S}\wedge\tau').$$

As J is arbitrary,

$$v_\tau=\operatorname{Osclln}(\boldsymbol{u}\restriction\mathcal{S}\wedge\tau)\le\operatorname{Osclln}^*_{I\wedge\tau}(\boldsymbol{u}\restriction\mathcal{S}\wedge\tau)\le\operatorname{Osclln}^*_I(\boldsymbol{u}\restriction\mathcal{S}\wedge\tau').$$

As I is arbitrary,

$$v_\tau\le\operatorname{Osclln}(\boldsymbol{u}\restriction\mathcal{S}\wedge\tau')=v_{\tau'}.$$

Thus $\boldsymbol{v}$ is non-decreasing.

(iii) If $\tau\le\tau'$ in $\mathcal{S}$ and $c=[\![\tau=\tau']\!]$, then $\chi c\times|v_{\sigma'}-v_\sigma|=0$ whenever $\tau\le\sigma\le\sigma'\le\tau'$, so $\chi c\times\operatorname{Osclln}_{I\vee\tau}(\boldsymbol{u})=0$ and $\chi c\times\operatorname{Osclln}_{I\wedge\tau}(\boldsymbol{u})=\chi c\times\operatorname{Osclln}_I(\boldsymbol{u})$ whenever $\tau\in I\in\mathcal{I}(\mathcal{S}\wedge\tau')$. Consequently

$$\begin{aligned}\chi c\times v_\tau&=\chi c\times\inf_{\tau\in I\in\mathcal{I}(\mathcal{S})}\sup_{\substack{J\in\mathcal{I}(\mathcal{S})\\J\supseteq I}}\operatorname{Osclln}_{J\wedge\tau}(\boldsymbol{u})\\&=\inf_{\tau\in I\in\mathcal{I}(\mathcal{S})}\sup_{\substack{J\in\mathcal{I}(\mathcal{S})\\J\supseteq I}}\chi c\times\operatorname{Osclln}_{J\wedge\tau}(\boldsymbol{u})\\&=\inf_{\tau\in I\in\mathcal{I}(\mathcal{S})}\sup_{\substack{J\in\mathcal{I}(\mathcal{S})\\J\supseteq I}}\chi c\times\operatorname{Osclln}_J(\boldsymbol{u})=\chi c\times v_{\tau'}\end{aligned}$$

and $[\![\tau=\tau']\!]\subseteq[\![v_\tau=v_{\tau'}]\!]$. As noted in 612Db, this is enough to show that $\boldsymbol{v}$ is fully adapted.

(b)(i) If $\tau\in J\in\mathcal{I}(\mathcal{S})$ then (as noted in (a-ii) above)

$$\operatorname{Osclln}_J(\boldsymbol{u})=\operatorname{Osclln}_{J\wedge\tau}(\boldsymbol{u})\vee\operatorname{Osclln}_{J\vee\tau}(\boldsymbol{u}).$$

So if $\tau\in I\in\mathcal{I}(\mathcal{S})$ then

$$\begin{aligned}\mathrm{Oscln}^*_I(\boldsymbol{u}) &= \sup_{I\subseteq J\in\mathcal{I}(\mathcal{S})}\mathrm{Oscln}_J(\boldsymbol{u}) = \sup_{I\subseteq J\in\mathcal{I}(\mathcal{S})}\mathrm{Oscln}_{J\wedge\tau}(\boldsymbol{u})\vee\mathrm{Oscln}_{J\vee\tau}(\boldsymbol{u})\\ &= \sup_{\substack{I\wedge\tau\subseteq K\in\mathcal{I}(\mathcal{S}\wedge\tau)\\ I\vee\tau\subseteq L\in\mathcal{I}(\mathcal{S}\vee\tau)}}\mathrm{Oscln}_K(\boldsymbol{u})\vee\mathrm{Oscln}_L(\boldsymbol{u})\\ &= \mathrm{Oscln}^*_{I\wedge\tau}(\boldsymbol{u}\restriction\mathcal{S}\wedge\tau)\vee\mathrm{Oscln}^*_{I\vee\tau}(\boldsymbol{u}\restriction\mathcal{S}\vee\tau).\end{aligned}$$

Taking the infimum over I (using the distributive law 352Eb),

$$\mathrm{Oscln}(\boldsymbol{u}) = \mathrm{Oscln}(\boldsymbol{u}\restriction\mathcal{S}\wedge\tau)\vee\mathrm{Oscln}(\boldsymbol{u}\restriction\mathcal{S}\vee\tau)$$

as claimed.

(ii) Now

$$\mathrm{Oscln}(\boldsymbol{u}\restriction\mathcal{S}\cap[\tau,\tau']) = \mathrm{Oscln}(\boldsymbol{u}\restriction(\mathcal{S}\wedge\tau')\vee\tau)\le\mathrm{Oscln}(\boldsymbol{u}\restriction\mathcal{S}\wedge\tau')\le\mathrm{Oscln}(\boldsymbol{u}).$$

***618E** An elementary fact will turn out to be useful in Chapter 64.

Lemma Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$ with a greatest element, $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a jump-free process, $\tau\in\mathcal{S}$ and $\epsilon>0$. Then there is a $\tau'\in\mathcal{S}\vee\tau$ such that $[\![\tau<\tau']\!]=[\![\tau<\max\mathcal{S}]\!]$ and $\theta(\sup_{\sigma\in\mathcal{S}\cap[\tau,\tau']}|u_\sigma-u_\tau|)\le\epsilon$.

proof By 618D(b-i), $\boldsymbol{u}'=\boldsymbol{u}\restriction\mathcal{S}\vee\tau$ is jump-free. Let I be a finite sublattice of $\mathcal{S}\vee\tau$ containing τ and $\max\mathcal{S}$ and such that $\theta(\mathrm{Oscln}^*_I(\boldsymbol{u}'))\le\epsilon$. Take $(\tau_0,\dots,\tau_n)$ linearly generating the I-cells. For $i<n$ set $b_i=[\![\tau<\tau_{i+1}]\!]\setminus[\![\tau<\tau_i]\!]$; set $b_n=[\![\tau=\max\mathcal{S}]\!]$. Then $\langle b_i\rangle_{i\le n}$ is disjoint and has supremum 1, $b_i\in\mathfrak{A}_{\tau_{i+1}}$ for $i<n$, and $b_n\in\mathfrak{A}_{\max\mathcal{S}}$. By 611I there is a $\tau'\in\mathcal{T}$ such that $b_i\subseteq[\![\tau'=\tau_{i+1}]\!]$ for $i<n$ and $b_n\subseteq[\![\tau'=\max\mathcal{S}]\!]$. Because $\mathcal{S}$ is finitely full, $\tau'\in\mathcal{S}$, while $\tau\le\tau'$ and $[\![\tau<\tau']\!]=\sup_{i<n}b_i=[\![\tau<\max\mathcal{S}]\!]$. If $\sigma\in\mathcal{S}$ and $\tau\le\sigma\le\tau'$, then for $i<n$

$$\begin{aligned}b_i&\subseteq[\![\tau=\tau_i]\!]\cap[\![\tau_i\le\sigma]\!]\cap[\![\sigma\le\tau_{i+1}]\!]\subseteq[\![\tau=\tau_i]\!]\cap[\![\sigma=\mathrm{med}(\tau_i,\sigma,\tau_{i+1})]\!]\\ &\subseteq[\![|u_\sigma-u_\tau|=|u_{\mathrm{med}(\tau_i,\sigma,\tau_{i+1})}-u_{\tau_i}|]\!]\subseteq[\![|u_\sigma-u_\tau|\le\mathrm{Oscln}^*_I(\boldsymbol{u}')]\!]\end{aligned}$$

by 618Ca. At the same time,

$$b_n\subseteq[\![\tau=\sigma]\!]\subseteq[\![|u_\sigma-u_\tau|=0]\!].$$

So $|u_\sigma-u_\tau|\le\mathrm{Oscln}^*_I(\boldsymbol{u}')$. As σ is arbitrary,

$$\theta(\sup_{\sigma\in\mathcal{S}\cap[\tau,\tau']}|u_\sigma-u_\tau|)\le\theta(\mathrm{Oscln}^*_I(\boldsymbol{u}'))\le\epsilon.$$

618F Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.

(a) If $\boldsymbol{u}$, $\boldsymbol{v}$ are order-bounded processes with domain $\mathcal{S}$, then $|\mathrm{Oscln}(\boldsymbol{u})-\mathrm{Oscln}(\boldsymbol{v})|\le 2\sup|\boldsymbol{u}-\boldsymbol{v}|$.

(b) $\mathrm{Oscln}:M_{\text{o-b}}(\mathcal{S})\to L^0(\mathfrak{A})$ is uniformly continuous if $M_{\text{o-b}}(\mathcal{S})$ is given its ucp uniformity.

proof (a) By 618B(c-ii) and 618B(b-ii),

$$\mathrm{Oscln}(\boldsymbol{u})\le\mathrm{Oscln}(\boldsymbol{v})+\mathrm{Oscln}(\boldsymbol{u}-\boldsymbol{v})\le\mathrm{Oscln}(\boldsymbol{v})+2\sup|\boldsymbol{u}-\boldsymbol{v}|$$

and similarly $\mathrm{Oscln}(\boldsymbol{v})\le\mathrm{Oscln}(\boldsymbol{u})+2\sup|\boldsymbol{u}-\boldsymbol{v}|$.

(b) Consequently

$$\theta(\mathrm{Oscln}(\boldsymbol{v})-\mathrm{Oscln}(\boldsymbol{u}))\le 2\theta(\sup|\boldsymbol{u}-\boldsymbol{v}|)=2\widehat{\theta}(\boldsymbol{u}-\boldsymbol{v})$$

where $\widehat{\theta}$ is the F-norm defining the ucp linear space topology on $M_{\text{o-b}}(\mathcal{S})$ (615B). So $\mathrm{Oscln}:M_{\text{o-b}}(\mathcal{S})\to L^0(\mathfrak{A})$ is uniformly continuous.

618G Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. Write $M_{\text{j-f}}(\mathcal{S})$ for the set of jump-free fully adapted processes with domain $\mathcal{S}$.

(a) The set $M_{\text{j-f}}(\mathcal{S})$ of jump-free fully adapted processes with domain $\mathcal{S}$ is a topologically closed f-subalgebra of $M_{\text{o-b}}(\mathcal{S})$, and $\bar{h}\boldsymbol{v}\in M_{\text{j-f}}(\mathcal{S})$ whenever $\boldsymbol{v}\in M_{\text{j-f}}(\mathcal{S})$ and $h:\mathbb{R}\to\mathbb{R}$ is continuous.

(b) A (locally) jump-free fully adapted process on $\mathcal{S}$ is (locally) moderately oscillatory.

(c) If $\boldsymbol{v} \in M_{\text{j-f}}(\mathcal{S})$, then $\boldsymbol{v}\restriction\mathcal{S} \vee \tau$, $\boldsymbol{v}\restriction\mathcal{S} \wedge \tau'$ and $\boldsymbol{v}\restriction\mathcal{S} \cap [\tau, \tau']$ are jump-free whenever $\tau \le \tau'$ in $\mathcal{S}$. In particular, $\boldsymbol{v}$ is locally jump-free.

proof (a)(i) If $\boldsymbol{u}$, $\boldsymbol{v} \in M_{\text{j-f}} = M_{\text{j-f}}(\mathcal{S})$, then $\operatorname{Osclln}(\boldsymbol{u}+\boldsymbol{v}) \le \operatorname{Osclln}(\boldsymbol{u}) + \operatorname{Osclln}(\boldsymbol{v}) = 0$ (618B(c-ii)) so $\boldsymbol{u}+\boldsymbol{v} \in M_{\text{j-f}}$.

(ii) Since $\{0\}$ is closed in $L^0(\mathfrak{A})$ and $\operatorname{Osclln} : M_{\text{o-b}}(\mathcal{S}) \to L^0(\mathfrak{A})$ is continuous (618Fb), $M_{\text{j-f}}(\mathcal{S}) = \{\boldsymbol{u} : \boldsymbol{u} \in M_{\text{o-b}}(\mathcal{S}), \operatorname{Osclln}(\boldsymbol{u}) = 0\}$ is closed in $M_{\text{o-b}}(\mathcal{S})$.

(iii) If $\boldsymbol{u} \in M_{\text{j-f}}$ and $h : \mathbb{R} \to \mathbb{R}$ is uniformly continuous, then $\bar{h}\boldsymbol{u} \in M_{\text{j-f}}$. **P** By 614F(c-i), $\bar{h}\boldsymbol{u}$ is order-bounded. For $x \in \mathbb{R}$, set $g(x) = \sup\{|h(y) - h(y')| : |y - y'| \le |x|\}$; then g is Borel measurable, $\lim_{x\to 0} g(x) = 0$ and $|h(x) - h(y)| \le g(|x - y|)$ for all x, $y \in \mathbb{R}$, so $|\bar{h}(u) - \bar{h}(v)| \le \bar{g}(|u - v|)$ for all u, $v \in L^0(\mathfrak{A})$. Accordingly $\Delta_e(\mathbf{1}, |d\bar{h}\boldsymbol{u}|) \le \bar{g}(\Delta_e(\mathbf{1}, |d\boldsymbol{u}|))$ for every $e \in \operatorname{Sti}(\mathcal{S})$, $\operatorname{Osclln}_J(\bar{h}\boldsymbol{u}) \le \bar{g}(\operatorname{Osclln}_J(\boldsymbol{u}))$ for every $J \in \mathcal{I}(\mathcal{S})$ and $\operatorname{Osclln}^*_I(\bar{h}\boldsymbol{u}) \le \bar{g}(\operatorname{Osclln}^*_I(\boldsymbol{u}))$ for every $I \in \mathcal{I}(\mathcal{S})$. Since $\inf_{I\in\mathcal{I}(\mathcal{S})} \operatorname{Osclln}^*_I(\boldsymbol{u}) = 0$ and $\lim_{x\downarrow 0} g(x) = 0$, $\inf_{I\in\mathcal{I}(\mathcal{S})} \bar{g}(\operatorname{Osclln}^*_I(\boldsymbol{u})) = 0$ and $\inf_{I\in\mathcal{I}(\mathcal{S})} \operatorname{Osclln}^*_I(\bar{h}\boldsymbol{u}) = 0$ and $\bar{h}\boldsymbol{u}$ is jump-free. **Q**

(iv) If $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is jump-free and $\bar{h} : \mathbb{R} \to \mathbb{R}$ is continuous, then $\bar{h}\boldsymbol{u}$ is jump-free. **P** Again, $\bar{h}\boldsymbol{u}$ is order-bounded. Set $\bar{u} = \sup_{\sigma\in\mathcal{S}} |u_\sigma|$. Given $\epsilon > 0$, let $M \ge 0$ be such that $a = [\![\bar{u} \ge M]\!]$ has measure at most ϵ. Set

$$\begin{aligned} h_1(x) &= h(-M) \text{ if } x \le -M, \\ &= h(x) \text{ if } |x| \le M, \\ &= h(M) \text{ if } x \ge M. \end{aligned}$$

Then h_1 is uniformly continuous so $\bar{h}_1(\boldsymbol{u}) \in M_{\text{j-f}}$, by (ii). But

$$[\![\bar{h}\boldsymbol{u} \ne \bar{h}_1\boldsymbol{u}]\!] = \sup_{\sigma\in\mathcal{S}} [\![\bar{h}(u_\sigma) \ne \bar{h}_1(u_\sigma)]\!] \subseteq \sup_{\sigma\in\mathcal{S}} [\![|u_\sigma| > M]\!] = [\![\bar{u} > M]\!] \subseteq a,$$

so

$$\widehat{\theta}(\bar{h}\boldsymbol{u} - \bar{h}_1\boldsymbol{u}) \le \bar{\mu}a \le \epsilon.$$

As ϵ is arbitrary and $M_{\text{j-f}}$ is closed, $\bar{h}\boldsymbol{u} \in M_{\text{j-f}}$. **Q**

(v) Putting (i) and (iv) together, $M_{\text{j-f}}$ is an f-subalgebra of $M_{\text{o-b}}(\mathcal{S})$, by 612Bc as usual.

(b)(i) Suppose that $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is jump-free. If $\mathcal{S}$ is empty, $\boldsymbol{u}$ is certainly moderately oscillatory; suppose otherwise.

($\boldsymbol{\alpha}$) Set $\mathfrak{B} = \bigcap_{\sigma\in\mathcal{S}} \mathfrak{A}_\sigma$. Then the starting value $u_\downarrow = \lim_{\sigma\downarrow\mathcal{S}} u_\sigma$ is defined and belongs to $L^0(\mathfrak{B})$. **P** Let $\epsilon > 0$. Let $I \in \mathcal{I}(\mathcal{S})$ be non-empty and such that $\theta(\operatorname{Osclln}^*_I(\boldsymbol{u})) \le \epsilon$. If $\sigma \in \mathcal{S}$ and $\sigma \le \min I$, then $J = I \cup \{\sigma\} \in \mathcal{I}(\mathcal{S})$ and $|u_{\min I} - u_\sigma| \le \operatorname{Osclln}_J(\boldsymbol{u}) \le \operatorname{Osclln}^*_I(\boldsymbol{u})$, so $\theta(u_\sigma - u_{\min I}) \le \epsilon$. As $L^0(\mathfrak{A})$ is complete in the linear space topology of convergence in measure, this is enough to show that $u_\downarrow$ is defined. By 613Bj, $u_\downarrow \in L^0(\mathfrak{B})$. **Q**

($\boldsymbol{\beta}$) If $I \in \mathcal{I}(\mathcal{S})$ is non-empty, there is a simple process $\boldsymbol{u}'$ such that $\sup|\boldsymbol{u} - \boldsymbol{u}'| \le \operatorname{Osclln}^*_I(\boldsymbol{u})$. **P** Write w for $\operatorname{Osclln}^*_I(\boldsymbol{u})$. Let $\tau_0 \le \ldots \le \tau_n$ linearly generate the I-cells (611L). Let $\boldsymbol{u}' = \langle u'_\sigma\rangle_{\sigma\in\mathcal{S}}$ be the simple process defined on $\mathcal{S}$ by the formulae of 612Ka from $\tau_0, \ldots, \tau_n$ and $u_\downarrow, u_{\tau_0}, \ldots, u_{\tau_n}$.

If $\sigma' \le \sigma \in \mathcal{S} \wedge \tau_0$, then

$$|u_\sigma - u_{\sigma'}| \le \operatorname{Osclln}_{I\cup\{\sigma',\sigma\}}(\boldsymbol{u}) \le w;$$

taking the limit as $\sigma \downarrow \mathcal{S}$, $|u_\sigma - u_\downarrow| \le w$. If $i < n$ and $\sigma \in \mathcal{S} \cap [\tau_i, \tau_{i+1}]$, then

$$|u_\sigma - u_{\tau_i}| \le \operatorname{Osclln}_{I\cup\{\sigma\}}(\boldsymbol{u}) \le w;$$

similarly, if $\sigma \in \mathcal{S} \vee \tau_n$ then

$$|u_\sigma - u_{\tau_n}| \le \operatorname{Osclln}_{I\cup\{\sigma\}}(\boldsymbol{u}) \le w.$$

For any $\sigma \in \mathcal{S}$ we see now that

$$
\begin{aligned}
[\![\sigma < \tau_0]\!] &\subseteq [\![u_\sigma = u_{\sigma\wedge\tau_0}]\!] \cap [\![u'_\sigma = u_\downarrow]\!] \\
&\subseteq [\![|u_\sigma - u'_\sigma| = |u_{\sigma\wedge\tau_0} - u_\downarrow|]\!] \subseteq [\![|u_\sigma - u'_\sigma| \le w]\!],
\end{aligned}
$$

$$
\begin{aligned}
[\![\tau_n \le \sigma]\!] &\subseteq [\![u_\sigma = u_{\sigma\vee\tau_n}]\!] \cap [\![u'_\sigma = u_{\tau_n}]\!] \\
&\subseteq [\![|u_\sigma - u'_\sigma| = |u_{\sigma\vee\tau_n} - u_{\tau_n}|]\!] \subseteq [\![|u_\sigma - u'_\sigma| \le w]\!],
\end{aligned}
$$

and for any $i < n$

$$
\begin{aligned}
[\![\tau_i \le \sigma]\!] \cap [\![\sigma < \tau_{i+1}]\!] &\subseteq [\![u_\sigma = u_{\mathrm{med}(\tau_i,\sigma,\tau_{i+1}}]\!] \cap [\![u'_\sigma = u_{\tau_i}]\!] \\
&\subseteq [\![|u_\sigma - u'_\sigma| = |u_{\mathrm{med}(\tau_i,\sigma,\tau_{i+1})} - u_{\tau_i}|]\!] \subseteq [\![|u_\sigma - u'_\sigma| \le w]\!].
\end{aligned}
$$

Since

$$[\![\sigma < \tau_0]\!] \cup [\![\tau_n \le \sigma]\!] \cup \sup_{i<n}([\![\tau_i \le \sigma]\!] \cap [\![\sigma < \tau_{i+1}]\!]) = 1,$$

$|u_\sigma - u'_\sigma| \le w$. As σ is arbitrary, $\sup|\boldsymbol{u} - \boldsymbol{u}'| \le w$, as claimed. **Q**

(γ) Now take any $\epsilon > 0$. As $\boldsymbol{u}$ is jump-free, there is a non-empty $I \in \mathcal{I}(\mathcal{S})$ such that $\theta(\mathrm{Oscll}n^*_I(\boldsymbol{u})) \le \epsilon$. By ($\beta$), there is a simple process $\boldsymbol{u}'$ such that $\theta(\sup|\boldsymbol{u} - \boldsymbol{u}'|) \le \epsilon$. But we know that simple processes are moderately oscillatory (615E). As ϵ is arbitrary, $\boldsymbol{u}$ belongs to the closure of the set $M_{\mathrm{mo}}(\mathcal{S})$ of moderately oscillatory processes for the ucp topology on $M_{\text{o-b}}(\mathcal{S})$. But $M_{\mathrm{mo}}(CalS)$ is closed (615F(a-iv)), so $\boldsymbol{u}$ is moderately oscillatory.

(ii) If $\boldsymbol{u}$ is locally jump-free, then applying (i) we see that $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ is moderately oscillatory for every $\tau \in \mathcal{S}$, so $\boldsymbol{u}$ is locally moderately oscillatory.

(c)(i) Given $I \in \mathcal{I}(\mathcal{S})$, let I' be the sublattice generated by $I \cup \{\tau, \tau'\}$. Then $\mathrm{Oscll}n^*_{I'\vee\tau}(\boldsymbol{v}\restriction\mathcal{S}\vee\tau) \le \mathrm{Oscll}n^*_I(\boldsymbol{v})$. **P** If $I' \vee \tau \subseteq J \in \mathcal{I}(\mathcal{S}\vee\tau)$, and K is the sublattice generated by $I' \cup J$, we shall have $I \subseteq K \in \mathcal{I}(\mathcal{S})$, $K \vee \tau = J$ and $\mathrm{Sti}_0(J) \subseteq \mathrm{Sti}_0(K)$, so

$$\mathrm{Oscll}n_J(\boldsymbol{v}) \le \mathrm{Oscll}n_K(\boldsymbol{v}) \le \mathrm{Oscll}n^*_I(\boldsymbol{v}).$$

As J is arbitrary,

$$\mathrm{Oscll}n^*_{I'\vee\tau}(\boldsymbol{v}\restriction\mathcal{S}\vee\tau) \le \mathrm{Oscll}n^*_I(\boldsymbol{v}). \quad \mathbf{Q}$$

Consequently

$$\inf_{I\in\mathcal{I}(\mathcal{S}\vee\tau)} \mathrm{Oscll}n^*_I(\boldsymbol{v}\restriction\mathcal{S}\vee\tau) \le \inf_{I\in\mathcal{I}(\mathcal{S})} \mathrm{Oscll}n^*_I(\boldsymbol{v}) = 0$$

and $\boldsymbol{v}\restriction\mathcal{S}\vee\tau$ is jump-free.

(ii) Replacing '$\vee\tau$' by '$\wedge\tau'$' throughout the argument above, we see that $\boldsymbol{v}\restriction\mathcal{S}\wedge\tau'$ is jump-free. Putting these together, $\boldsymbol{v}\restriction\mathcal{S}\cap[\tau,\tau'] = (\boldsymbol{v}\restriction\mathcal{S}\vee\tau)\restriction\mathcal{S}\wedge\tau'$ is jump-free. And as τ' is arbitrary, $\boldsymbol{v}$ is locally jump-free.

618H Where jump-free processes come from: Proposition Let (Ω, Σ, μ) be a complete probability space and $\langle\Sigma_t\rangle_{t\ge0}$ a family of σ-subalgebras of Σ, all containing every negligible subset of Ω. Suppose that we are given a family $\langle X_t\rangle_{t\ge0}$ of real-valued functions on Ω such that X_t is Σ_t-measurable for every $t \ge 0$ and $t \mapsto X_t(\omega) : [0,\infty[\to \mathbb{R}$ is continuous for every $\omega \in \Omega$. Then $\langle X_t\rangle_{t\ge0}$ is progressively measurable, and if $(\mathfrak{A}, \bar\mu, \langle\mathfrak{A}_t\rangle_{t\ge0}, \mathcal{T}, \langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ and $\langle x_\sigma\rangle_{\sigma\in\mathcal{T}_f}$ are defined as in 612H, $\boldsymbol{x} = \langle x_\sigma\rangle_{\sigma\in\mathcal{T}_f}$ is locally jump-free.

proof The idea is to combine the approaches of 612H and 615P.

(a) I start by showing that we have a progressively measurable process. **P** Take any $t \ge 0$ and $\alpha \in \mathbb{R}$. Set $Q = \{qt : q \in \mathbb{Q} \cap [0,1]\}$. Then

$$
\begin{aligned}
\{(s,\omega) : s \le t,\ X_s(\omega) > \alpha\} &= \{(s,\omega) : s \le t,\ \limsup_{q\downarrow Q\cap[s,t]} X_q(\omega) > \alpha\} \\
&= \bigcup_{k\in\mathbb{N}} \bigcap_{q\in Q} \bigcup_{\substack{q'\in Q\\ q'\le q}} \{(s,\omega) : s \le t,\ s > q \text{ or } s \le q' \text{ and } X_{q'}(\omega) \ge \alpha + 2^{-k}\} \\
&\in \mathcal{B}([0,t])\widehat{\otimes}\Sigma_t
\end{aligned}
$$

where $\mathcal{B}([0,t])$ is the Borel σ-algebra of $[0,t]$. **Q**

We can therefore apply the method of 612H to define, for each stopping time $h:\Omega\to[0,\infty[$, the σ-algebra Σ_h and the Σ_h-measurable function X_h, and we find ourselves with a real-time stochastic integration structure $(\mathfrak{A},\bar\mu,[0,\infty[\,,\langle\mathfrak{A}_t\rangle_{t\ge0},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ and a fully adapted process $\boldsymbol{x}=\langle x_\sigma\rangle_{\sigma\in\mathcal{T}_f}$ such that $x_{h^\bullet}=X_h^\bullet$ for every h.

(b) The main part of the argument depends on the idea of 'hitting time' in a form similar to that of 445M. Let $h:\Omega\to[0,\infty[$ be a stopping time, and $\epsilon>0$. For $\omega\in\Omega$ set

$$f(\omega)=\min(\{h(\omega)\}\cup\{t:t\ge0,\,|X_t(\omega)|\ge\epsilon\}).$$

(I can write min rather than inf because $t\mapsto X_t(\omega)$ is continuous, so if $\{t:|X_t(\omega)|\ge\epsilon\}$ is non-empty it contains its infimum.) Then f is a stopping time. **P** For any $t\ge0$, Σ_t contains every μ-negligible set, so $(\Omega,\Sigma_t,\mu\restriction\Sigma_t)$ is a complete probability space and Σ_t is closed under Souslin's operation (431A). Next,

$$\{(s,\omega):s\le t,\,|X_s(\omega)|\ge\epsilon\}=\bigcap_{k\in\mathbb{N}}\{(s,\omega):s\le t,\,|X_s(\omega)|>\epsilon-2^{-k}\}$$

belongs to $\mathcal{B}([0,t])\widehat{\otimes}\Sigma_t$, by (a) applied to the process $(s,\omega)\mapsto|X_s(\omega)|$. So its projection $E=\{\omega:\exists\,s\in[0,t],\ |X_s(\omega)|\ge\epsilon\}$ belongs to Σ_t (423O[9]). Now

$$\{\omega:f(\omega)\le t\}=\{\omega:h(\omega)\le t\}\cup E\in\Sigma_t.$$

As t is arbitrary, f is a stopping time adapted to $\langle\Sigma_t\rangle_{t\ge0}$. **Q**

(c) Again suppose that $h:\Omega\to[0,\infty[$ is a stopping time and $\epsilon>0$. Define g_n and $X_t^{(n)}$, for $n\in\mathbb{N}$ and $t\ge0$, by setting

$$g_0(\omega)=0\text{ for every }\omega\in\Omega$$

and

$$\begin{aligned}X_t^{(n)}(\omega)&=0\text{ if }g_n(\omega)>t,\\&=X_t(\omega)-X_{g_n}(\omega)\text{ if }g_n(\omega)\le t,\end{aligned}$$

$$g_{n+1}(\omega)=\inf(\{h(\omega)\}\cup\{t:t\ge0,\,|X_t^{(n)}(\omega)|\ge\epsilon\}$$

for $n\in\mathbb{N}$, $t\ge0$ and $\omega\in\Omega$, We see immediately that $t\mapsto X_t^{(n)}(\omega)$ is always continuous. Also we can see by induction on n that every g_n is a stopping time and every $X_t^{(n)}$ is Σ_t-measurable. **P** For $n=0$ this is trivial, since of course X_t-X_0 is Σ_t-measurable and $t\mapsto X_t(\omega)-X_0(\omega)$ is always continuous. For the inductive step to $n\ge1$, g_n is a stopping time, by (b) applied to $\langle X_t^{(n-1)}\rangle_{t\ge0}$. Next, setting $F=\{\omega:g_n(\omega)\le t\}$, we have

$$\begin{aligned}F\cap\{\omega:g_n(\omega)\le s\}&=\{\omega:g_n(\omega)\le s\}\in\Sigma_s\subseteq\Sigma_t\text{ if }s\le t,\\&=F\in\Sigma_t\text{ if }t\le s\end{aligned}$$

so $F\in\Sigma_{g_n}$. If $E\in\Sigma_{g_n}$ then $E\cap F\in\Sigma_t$, so $X_{g_n}\times\chi F$ is Σ_t-measurable; while $F\in\Sigma_t$ so $X_t\times\chi F$ is Σ_t-measurable. Consequently $X_t^{(n)}=(X_t-X_{g_n})\times\chi F$ is Σ_t-measurable, and the induction continues. **Q**

We therefore have a non-decreasing sequence $\langle g_n\rangle_{n\in\mathbb{N}}$ of stopping times such that, for any $n\in\mathbb{N}$ and $\omega\in\Omega$,

— if $n=0$ then $g_n(\omega)=0$,
— $g_n(\omega)\le h(\omega)$,
— $|X_t(\omega)-X_{g_n}(\omega)|\le\epsilon$ whenever $g_n(\omega)\le t\le g_{n+1}(\omega)$

because $t\mapsto X_t(\omega)-X_{g_n}(\omega)$ is continuous, so if $|X_t(\omega)-X_{g_n}(\omega)|>\epsilon$ there is an $s\in[g_n(\omega),t[$ such that $|X_s(\omega)-X_{g_n}(\omega)|\ge\epsilon$ and $g_{n+1}(\omega)\le s<t$.

Moreover, for any ω, $\langle g_n(\omega)\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence bounded above by $h(\omega)$, so

$$\lim_{n\to\infty}X_{g_n}(\omega)=\lim_{n\to\infty}X_{g_n(\omega)}(\omega)$$

is defined and finite, in which case there must be some n such that $g_{n+1}(\omega)=h(\omega)$.

[9]Later editions only.

(d) Translating (c) into the language of the stochastic integration structure $(\mathfrak{A}, \bar{\mu}, [0,\infty[\,, \langle\mathfrak{A}_t\rangle_{t\ge 0}, \mathcal{T}, \langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$, we see that given any $\tau \in \mathcal{T}_f$ and $\epsilon > 0$ we have a non-decreasing sequence $\langle\tau_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{T}_f$ such that for every $n \in \mathbb{N}$,

— $\tau_n \le \tau$,

— $[\![\tau_n \le \sigma]\!] \cap [\![\sigma \le \tau_{n+1}]\!] \subseteq [\![|x_\sigma - x_{\tau_n}| \le \epsilon]\!]$ for every $\sigma \in \mathcal{T}_f$,

and

$$\tau_0 = \check{0} = \min \mathcal{T}_f, \quad \sup_{n\in\mathbb{N}} [\![\tau_n = \tau]\!] = 1.$$

But this means that if we take $n \in \mathbb{N}$ such that $c = 1 \setminus [\![\tau_n = \tau]\!]$ has measure at most ϵ, and set $I = \{\tau_0, \dots, \tau_n, \tau\} \in \mathcal{I}(\mathcal{T}_f \wedge \tau)$, then

$$\begin{aligned}\mathrm{Osclln}^*_I(\boldsymbol{x}{\restriction}\mathcal{T}_f \wedge \tau) &\le \sup_{i<n} \sup\{|x_{\sigma'} - x_\sigma| : \tau_i \le \sigma \le \sigma' \le \tau_{i+1}\} \\ &\qquad \vee \sup\{|x_{\sigma'} - x_\sigma| : \tau_n \le \sigma \le \sigma' \le \tau\} \\ &\le 2\epsilon\chi 1 \vee (2 \sup |\boldsymbol{x}{\restriction}\mathcal{T}_f \wedge \tau| \times \chi c)\end{aligned}$$

(618Ca)

and

$$\theta(\mathrm{Osclln}^*_I(\boldsymbol{x}{\restriction}\mathcal{T}_f \wedge \tau)) \le 2\epsilon + \bar{\mu}c \le 3\epsilon.$$

As ϵ is arbitrary, $\boldsymbol{x}{\restriction}\mathcal{T}_f \wedge \tau$ is jump-free; as τ is arbitrary, $\boldsymbol{x}$ is locally jump-free.

618I Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a locally jump-free fully adapted process. If $A \subseteq \mathcal{S}$ is non-empty and upwards-directed and $\sup A \in \mathcal{S}$, then $u_{\sup A} = \lim_{\sigma\uparrow A} u_\sigma$.

proof It is enough to consider the case in which $\sup A = \max \mathcal{S}$, so that $\boldsymbol{u}$ is actually jump-free.

(a) (The key.) Suppose that $I \in \mathcal{I}(\mathcal{S})$, that $\tau_0 \le \ldots \le \tau_n$ linearly generate the I-cells, that $\sigma \le \sigma' \in \mathcal{S}$, and that

$$a \cap [\![\sigma \le \tau_i]\!] = a \cap [\![\sigma' \le \tau_i]\!]$$

for every $i \le n$. Then $|u_\sigma - u_{\sigma'}| \times \chi a \le \mathrm{Osclln}^*_I(\boldsymbol{u})$. **P** Set $\bar{u} = \mathrm{Osclln}^*_I(\boldsymbol{u})$,

$$b_0 = [\![\sigma \le \tau_0]\!], \quad b_i = [\![\tau_{i-1} < \sigma]\!] \cap [\![\sigma \le \tau_i]\!] \text{ for } 1 \le i \le n, \quad b_{n+1} = [\![\tau_n < \sigma]\!].$$

Then $\langle b_i\rangle_{i\le n+1}$ is a partition of unity in $\mathfrak{A}$. Next, set

$$\sigma_0 = \sigma \wedge \tau_0, \quad \sigma'_0 = \sigma' \wedge \tau_0, \quad \sigma_{n+1} = \sigma \vee \tau_n, \quad \sigma'_{n+1} = \sigma' \vee \tau_n,$$

$$\sigma_i = \mathrm{med}(\tau_{i-1}, \sigma, \tau_i), \quad \sigma'_i = \mathrm{med}(\tau_{i-1}, \sigma', \tau_i) \text{ for } 1 \le i \le n.$$

If $i \le n+1$ and $\sigma_i \ne \sigma'_i$, the stopping-time interval $c(\sigma_i, \sigma'_i)$ is a J_i-cell, where J_i is the sublattice generated by $I \cup \{\sigma, \sigma'\}$, so

$$|u_{\sigma_i} - u_{\sigma'_i}| \le \mathrm{Osclln}_J(\boldsymbol{u}) \le \bar{u}.$$

Next, $b_i \subseteq [\![\sigma_i = \sigma]\!]$ and $a \cap b_i \subseteq [\![\sigma'_i = \sigma']\!]$ for each i, so

$$a = \sup_{i\le n+1} a \cap b_i \subseteq \sup_{i\le n+1} [\![u_{\sigma_i} = u_\sigma]\!] \cap [\![u_{\sigma'_i} = u_{\sigma'}]\!] \subseteq [\![|u_\sigma - u_{\sigma'}| \le \bar{u}]\!]$$

and $|u_\sigma - u_{\sigma'}| \times \chi a \le \bar{u}$. **Q**

(b) Take any $\epsilon > 0$, and let $I \in \mathcal{I}(\mathcal{S})$ be such that $\theta(\bar{u}) \le \epsilon$ where $\bar{u} = \mathrm{Osclln}^*_I(\boldsymbol{u})$. Let $(\tau_0, \dots, \tau_n)$ linearly generate the I-cells. For each $i \le n$, $[\![\sup A \le \tau_i]\!] = \inf_{\sigma\in A} [\![\sigma \le \tau_i]\!]$ for each i (611Eb), so there is a $\sigma_0 \in A$ such that $c = \sup_{i\le n} [\![\sigma_0 \le \tau_i]\!] \setminus [\![\sup A \le \tau_i]\!]$ has measure at most ϵ. If now $\sigma \in A$ and $\sigma_0 \le \sigma$, (a) tells us that

$$|u_{\sup A} - u_\sigma| \times \chi(1 \setminus c) \le \bar{u}, \quad \theta(u_{\sup A} - u_\sigma) \le \theta(\bar{u}) + \bar{\mu}c \le 2\epsilon.$$

As ϵ is arbitrary, $\lim_{\sigma\uparrow A} u_\sigma = u_{\sup A}$.

618J Examples Let us look again at our three standard examples. Take $T = [0, \infty[$.

(a) The identity process $\boldsymbol{\iota} = \langle \iota_\tau \rangle_{\tau \in \mathcal{T}_f}$ (612F) is locally jump-free. **P** We know that it is locally order-bounded (615Q, or otherwise). Take $\tau \in \mathcal{T}_f$ and $\epsilon > 0$. Let $m \in \mathbb{N}$ be such that $a = [\![\tau > m\epsilon]\!]$ has measure at most ϵ. Set $I = \{\tau \wedge \check{i\epsilon} : i \le m\}$, where $\check{i\epsilon}$ is the constant stopping time at $i\epsilon$. If J is a sublattice of $\mathcal{T} \wedge \tau$ including I and e is a J-cell, then

either there is an $i < m$ such that e can be expressed as $c(\sigma, \sigma')$ where $\tau \wedge \check{i\epsilon} \le \sigma \le \sigma' \le \tau \wedge (i \check{+} 1)\epsilon$, in which case

$$[\![\sigma < \sigma']\!] \subseteq [\![\check{i\epsilon} \le \sigma]\!] \cap [\![\sigma' \le (i \check{+} 1)\epsilon]\!]$$

and $\Delta_e(\mathbf{1}, |d\boldsymbol{\iota}|) = \iota'_\sigma - \iota_\sigma \le \epsilon\chi 1$,

or e can be expressed as $c(\sigma, \sigma')$ where $\tau \wedge \check{m\epsilon} \le \sigma \le \sigma' \le \tau$, in which case $[\![\sigma < \sigma']\!] \subseteq a$ and $\Delta_e(\mathbf{1}, |d\boldsymbol{\iota}|) = \iota'_\sigma - \iota_\sigma \le \iota_\tau \times \chi a$.

Thus $\mathrm{Osclln}^*_I(\boldsymbol{\iota}) \le \epsilon\chi 1 + \iota_\tau \times \chi a$ and $\theta(\mathrm{Osclln}^*_I(\boldsymbol{\iota})) \le \epsilon + \bar{\mu} a \le 2\epsilon$. As ϵ is arbitrary, $\boldsymbol{\iota}\restriction\mathcal{T} \wedge \tau$ is jump-free; as τ is arbitrary, $\boldsymbol{\iota}$ is jump-free. **Q**

(b) The standard Poisson process $\boldsymbol{v} = \langle v_\tau \rangle_{\tau \in \mathcal{T}_f}$ (612U) is not locally jump-free. **P** Let τ_1 be the first jump time of $\boldsymbol{v}$, as in 612Ue-612Uf, so that

$$[\![\sigma < \tau_1]\!] = [\![v_\sigma = 0]\!], \quad [\![\tau_1 \le \sigma]\!] = [\![v_\sigma \ge 1]\!]$$

for every $\sigma \in \mathcal{T}_f$. If I is any sublattice of $\mathcal{T} \wedge \tau_1$, set $J = I \cup \{\min \mathcal{T}, \tau_1\}$ and let $\sigma_0 \le \ldots \le \sigma_n$ linearly generate the J-cells; then

$$\begin{aligned}\mathrm{Osclln}^*_I(\boldsymbol{v}) &\ge \mathrm{Osclln}_J(\boldsymbol{v}) \ge \sup_{i<n} |v_{\sigma_{i+1}} - v_{\sigma_i}| \\ &\ge \sup_{i<n} \chi([\![\sigma_{i+1} = \tau_1]\!] \cap [\![\sigma_i < \tau_1]\!]) \\ &= \chi(\sup_{i<n} [\![\sigma_{i+1} = \tau_1]\!] \setminus [\![\sigma_i = \tau_1]\!]) = \chi[\![\tau_1 > 0]\!] = \chi 1\end{aligned}$$

and $\theta(\mathrm{Osclln}^*_I(\boldsymbol{v})) = 1$. As I is arbitrary, $\boldsymbol{v}\restriction\mathcal{T} \wedge \tau_1$ is not jump-free and $\boldsymbol{v}$ is not locally jump-free. **Q**

(c) Brownian motion $\boldsymbol{w} = \langle w_\tau \rangle_{\tau \in \mathcal{T}_f}$ (612T) is locally jump-free. **P** This is immediate from 618H and the continuity of Brownian sample paths, built into the description in 612Ta. **Q**

618K Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. If I, $J \in \mathcal{I}(\mathcal{S})$ and $a \in \mathfrak{A}$ are such that $J \subseteq I$ and $a \subseteq \sup_{\sigma \in J} [\![\tau = \sigma]\!]$ for every $\tau \in I$, then $a \subseteq [\![\mathrm{Osclln}_I(\boldsymbol{u}) = \mathrm{Osclln}_J(\boldsymbol{u})]\!]$ for every fully adapted process $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$.

proof The argument is essentially identical to that for 613S.

(a) The case $a = 0$ is trivial; so is the case $a \ne 0$ and $J = \emptyset$, as then I must also be empty and

$$\mathrm{Osclln}_I(\boldsymbol{u}) = \mathrm{Osclln}_J(\boldsymbol{u}) = 0.$$

So suppose otherwise. Let $(\sigma_0, \ldots, \sigma_n)$ linearly generate the J-cells. If $\tau \in I$ then

$$\begin{aligned}a \subseteq \sup_{\sigma \in J} [\![\tau = \sigma]\!] &= \sup_{\sigma \in J} \sup_{j \le n} [\![\tau = \sigma]\!] \cap [\![\sigma = \sigma_j]\!] \\ &\subseteq \sup_{j \le n} [\![\tau = \sigma_j]\!] \subseteq [\![\sigma_0 \le \tau]\!] \cap [\![\tau \le \sigma_n]\!].\end{aligned}$$

Set $I_{-1} = I \wedge \sigma_0$, $I_j = I \cap [\sigma_j, \sigma_{j+1}]$ for $0 \le j < n$, $I_n = I \vee \sigma_n$.

(b) If $\tau \le \tau'$ in I_{-1}, then

$$a \cap [\![u_{\tau'} \ne u_\tau]\!] \subseteq a \cap [\![\tau < \tau']\!] \subseteq a \cap [\![\tau < \sigma_0]\!] = 0,$$

so

$$a \cap [\![|\Delta_e(\mathbf{1}, d\boldsymbol{u})| \ne 0]\!] = 0$$

for every I_{-1}-cell e. Taking the supremum over e, $\operatorname{Osclln}_{I_{-1}}(\boldsymbol{u}) \times \chi a = 0$.

(c) In the same way, if $\tau \le \tau'$ in I_n, then

$$a \cap [\![u_{\tau'} \ne u_\tau]\!] \subseteq a \cap [\![\tau < \tau']\!] \subseteq a \cap [\![\sigma_n < \tau']\!] = 0,$$

so

$$\operatorname{Osclln}_{I_n}(\boldsymbol{u}) \times \chi a = 0.$$

(d) If $0 \le j < n$, $a \subseteq [\![\operatorname{Osclln}_{I_j}(\boldsymbol{u}) = |u_{\sigma_{j+1}} - u_{\sigma_j}|]\!]$. **P** For every $\tau \in I_j$,

$$\begin{aligned} a \cap [\![\sigma_j < \tau]\!] \cap [\![\tau < \sigma_{j+1}]\!] &\subseteq \sup_{k \le n} [\![\tau = \sigma_k]\!] \cap [\![\sigma_j < \tau]\!] \cap [\![\tau < \sigma_{j+1}]\!] \\ &\subseteq \sup_{k \le n} \cap [\![\sigma_j < \sigma_k]\!] \cap [\![\sigma_k < \sigma_{j+1}]\!] \\ &\subseteq (\sup_{k \le j} \cap [\![\sigma_j < \sigma_k]\!]) \cup (\sup_{j+1 \le k} [\![\sigma_k < \sigma_{j+1}]\!]) = 0. \end{aligned}$$

Take $(\tau_0, \dots, \tau_m)$ linearly generating the I_j-cells; then $\sigma_j = \tau_0 \le \dots \le \tau_m = \sigma_{j+1}$. For $i < m$, set $b_i = [\![\tau_i < \tau_{i+1}]\!]$, so that $\sup_{i<m} b_i = [\![\sigma_j < \sigma_{j+1}]\!]$. Now

$$\begin{aligned} a \cap b_i = a \cap [\![\tau_i < \tau_{i+1}]\!] &= a \cap [\![\sigma_j < \tau_{i+1}]\!] \cap [\![\tau_i < \sigma_{j+1}]\!] \cap [\![\tau_i < \tau_{i+1}]\!] \\ &\subseteq a \cap [\![\tau_{i+1} = \sigma_{j+1}]\!] \cap [\![\tau_i = \sigma_j]\!] \cap [\![\tau_i < \tau_{i+1}]\!] \\ &\subseteq [\![|u_{\tau_{i+1}} - u_{\tau_i}| = |u_{\sigma_{j+1}} - u_{\sigma_j}|]\!] \cap \inf_{k<i} [\![\tau_k = \tau_{k+1}]\!] \cap \inf_{i<k<m} [\![\tau_k = \tau_{k+1}]\!] \\ &\subseteq [\![|u_{\tau_{j+1}} - u_{\tau_j}| = |u_{\sigma_{j+1}} - u_{\sigma_j}|]\!] \cap \inf_{\substack{k<m \\ k \ne i}} [\![u_{\tau_k} = u_{\tau_{k+1}}]\!] \\ &\subseteq [\![|u_{\tau_{j+1}} - u_{\tau_j}| = |u_{\sigma_{j+1}} - u_{\sigma_j}|]\!] \cap [\![\operatorname{Osclln}_{I_j}(\boldsymbol{u}) = |u_{\tau_{j+1}} - u_{\tau_j}|]\!] \\ &\subseteq [\![\operatorname{Osclln}_{I_j}(\boldsymbol{u}) = |u_{\sigma_{j+1}} - u_{\sigma_j}|]\!]. \end{aligned}$$

Taking the supremum over i,

$$a \cap [\![\sigma_j < \sigma_{j+1}]\!] \subseteq [\![\operatorname{Osclln}_{I_j}(\boldsymbol{u}) = |u_{\sigma_{j+1}} - u_{\sigma_j}|]\!].$$

Since $[\![\sigma_j = \sigma_{j+1}]\!] \subseteq [\![\operatorname{Osclln}_{I_j}(\boldsymbol{u}) = 0]\!]$, $a \subseteq [\![\operatorname{Osclln}_{I_j}(\boldsymbol{u}) = |u_{\sigma_{j+1}} - u_{\sigma_j}|]\!]$. **Q**

(e) Assembling these,

$$\operatorname{Osclln}_I(\boldsymbol{u}) \times \chi a = \sup_{-1 \le j \le n} \operatorname{Osclln}_{I_j}(\boldsymbol{u}) \times \chi a$$

(618D(b-i))

$$= \sup_{0 \le j < n} |u_{\sigma_{j+1}} - u_{\sigma_j}| \times \chi a = \operatorname{Osclln}_J(\boldsymbol{u}) \times \chi a$$

and $a \subseteq [\![\operatorname{Osclln}_I(\boldsymbol{u}) = \operatorname{Osclln}_J(\boldsymbol{u})]\!]$.

618L Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ a fully adapted process, $\hat{\mathcal{S}}$ the covered envelope of $\mathcal{S}$, and $\hat{\boldsymbol{u}} = \langle \hat{u}_\sigma \rangle_{\sigma \in \hat{\mathcal{S}}}$ the fully adapted extension of $\boldsymbol{u}$ to $\hat{\mathcal{S}}$.

(a) If either $\boldsymbol{u}$ or $\hat{\boldsymbol{u}}$ is order-bounded, so is the other, and in this case $\operatorname{Osclln}(\hat{\boldsymbol{u}}) = \operatorname{Osclln}(\boldsymbol{u})$.

(b) In particular, $\boldsymbol{u}$ is jump-free iff $\hat{\boldsymbol{u}}$ is jump-free.

(c) If either $\sup_{\sigma \in \mathcal{S}} \operatorname{Osclln}(\boldsymbol{u} \restriction \mathcal{S} \wedge \sigma)$ or $\sup_{\tau \in \hat{\mathcal{S}}} \operatorname{Osclln}(\hat{\boldsymbol{u}} \restriction \hat{\mathcal{S}} \wedge \tau)$ is defined in $L^0(\mathfrak{A})$, so is the other, and they are equal.

proof If $\mathcal{S}$ is empty, this is trivial; suppose otherwise.

(a)(i) Suppose that $\boldsymbol{u}$ is order-bounded; it follows that $\hat{\boldsymbol{u}}$ is order-bounded (614G(b-i)). Let $\epsilon > 0$. Then there is a non-empty finite sublattice I of $\mathcal{S}$ such that $\theta(\operatorname{Osclln}^*_I(\boldsymbol{u}) - \operatorname{Osclln}(\boldsymbol{u})) \le \epsilon$. As in 618C, take a sequence $(\tau_0,\ldots,\tau_n)$ linearly generating the I-cells, and set $\tau_{-1}=\inf\mathcal{S}=\inf\hat{\mathcal{S}}$ and $\tau_{n+1}=\sup\mathcal{S}=\sup\hat{\mathcal{S}}$. **?** Suppose that there are an i such that $-1\le i\le n$ and ρ, $\rho'\in\hat{\mathcal{S}}$ such that $\tau_i\le\rho\le\rho'\le\tau_{i+1}$ and $|\hat{u}_{\rho'}-\hat{u}_\rho|\not\le\mathrm{Osclln}^*_I(\boldsymbol{u})$. Set $c=[\![|\hat{u}_{\rho'}-\hat{u}_\rho|>\mathrm{Osclln}^*_I(\boldsymbol{u})]\!]$, so that $c\ne 0$. Take $\sigma\in\mathcal{S}$ such that $c\cap[\![\rho=\sigma]\!]\ne 0$, and $\sigma'\in\mathcal{S}$ such that $c'=c\cap[\![\rho=\sigma]\!]\cap[\![\rho'=\sigma']\!]$ is non-zero. Then

$$c'\subseteq[\![\mathrm{med}(\tau_i,\sigma,\tau_{i+1})=\mathrm{med}(\tau_i,\rho,\tau_{i+1})]\!]\subseteq[\![\mathrm{med}(\tau_i,\sigma,\tau_{i+1})=\rho]\!]$$

and similarly $c'\subseteq[\![\mathrm{med}(\tau_i,\sigma',\tau_{i+1})=\rho']\!]$, so

$$c'\subseteq[\![|u_{\mathrm{med}(\tau_i,\sigma',\tau_{i+1})}-u_{\mathrm{med}(\tau_i,\sigma,\tau_{i+1})}|>\mathrm{Osclln}^*_I(\boldsymbol{u})]\!];$$

but $|u_{\mathrm{med}(\tau_i,\sigma',\tau_{i+1})}-u_{\mathrm{med}(\tau_i,\sigma,\tau_{i+1})}|\le\mathrm{Osclln}^*_I(\boldsymbol{u})$, by the second formula in 618Ca. So this is impossible. **X**

As ρ and ρ' are arbitrary, 618Ca tells us that $\mathrm{Osclln}^*_I(\hat{\boldsymbol{u}})\le\mathrm{Osclln}^*_I(\boldsymbol{u})$ and

$$\theta((\mathrm{Osclln}(\hat{\boldsymbol{u}})-\mathrm{Osclln}(\boldsymbol{u}))^+)\le\theta((\mathrm{Osclln}^*_I(\hat{\boldsymbol{u}})-\mathrm{Osclln}(\boldsymbol{u}))^+)\le\epsilon.$$

As ϵ is arbitrary, $\mathrm{Osclln}(\hat{\boldsymbol{u}})\le\mathrm{Osclln}(\boldsymbol{u})$.

It follows at once that if $\mathrm{Osclln}(\boldsymbol{u})=0$ then $\mathrm{Osclln}(\hat{\boldsymbol{u}})=0$, that is, if $\boldsymbol{u}$ is jump-free then $\hat{\boldsymbol{u}}$ is jump-free.

(ii) Suppose that $\hat{\boldsymbol{u}}$ is order-bounded; it follows that $\boldsymbol{u}$ is order-bounded (614G(b-i), in the other direction). Let $\epsilon>0$. Then there is a non-empty finite sublattice I of $\hat{\mathcal{S}}$ such that $\theta(\mathrm{Osclln}^*_I(\hat{\boldsymbol{u}})-\mathrm{Osclln}(\hat{\boldsymbol{u}}))\le\epsilon$. Let $J\in\mathcal{I}(\mathcal{S})$ be such that $a=\inf_{\tau\in I}\sup_{\sigma\in J}[\![\tau=\sigma]\!]$ has measure at least $1-\epsilon$. Suppose $K\in\mathcal{I}(\mathcal{S})$ includes J, and write $K\sqcup I$ for the sublattice generated by $K\cup I$. Then $a\subseteq\sup_{\sigma\in K}[\![\tau=\sigma]\!]$ for every $\tau\in K\cup I$ and therefore for every $\tau\in K\sqcup I$ (611M(b-i)). By 618K,

$$\begin{aligned}a&\subseteq[\![\mathrm{Osclln}_K(\hat{\boldsymbol{u}})=\mathrm{Osclln}_{K\sqcup I}(\hat{\boldsymbol{u}})]\!]\\&\subseteq[\![\mathrm{Osclln}_K(\hat{\boldsymbol{u}})\le\mathrm{Osclln}^*_I(\hat{\boldsymbol{u}})]\!]=[\![\mathrm{Osclln}_K(\boldsymbol{u})\le\mathrm{Osclln}^*_I(\hat{\boldsymbol{u}})]\!],\end{aligned}$$

and

$$\mathrm{Osclln}_K(\boldsymbol{u})\le\mathrm{Osclln}^*_I(\hat{\boldsymbol{u}})+2\sup|\boldsymbol{u}|\times\chi(1\setminus a).$$

As $K\supseteq J$ is arbitrary,

$$\mathrm{Osclln}(\boldsymbol{u})\le\mathrm{Osclln}^*_J(\boldsymbol{u})\le\mathrm{Osclln}^*_I(\hat{\boldsymbol{u}})+2\sup|\boldsymbol{u}|\times\chi(1\setminus a)$$

and

$$\begin{aligned}\theta((\mathrm{Osclln}(\boldsymbol{u})-\mathrm{Osclln}(\hat{\boldsymbol{u}}))^+)&\le\theta(\mathrm{Osclln}^*_I(\hat{\boldsymbol{u}})-\mathrm{Osclln}(\hat{\boldsymbol{u}})+2\sup|\boldsymbol{u}|\times\chi(1\setminus a))\\&\le\theta(\mathrm{Osclln}^*_I(\hat{\boldsymbol{u}})-\mathrm{Osclln}(\hat{\boldsymbol{u}}))+\theta(2\sup|\boldsymbol{u}|\times\chi(1\setminus a))\\&\le\epsilon+\bar{\mu}(1\setminus a)\le 2\epsilon.\end{aligned}$$

As ϵ is arbitrary, $\theta((\mathrm{Osclln}(\boldsymbol{u})-\mathrm{Osclln}(\hat{\boldsymbol{u}}))^+)=0$ and $\mathrm{Osclln}(\boldsymbol{u})\le\mathrm{Osclln}(\hat{\boldsymbol{u}})$.

(b) It follows immediately that if $\hat{\boldsymbol{u}}$ is jump-free, so is $\boldsymbol{u}$.

(c)(i) If $w=\sup_{\sigma\in\mathcal{S}}\mathrm{Osclln}(\boldsymbol{u}{\restriction}\mathcal{S}\wedge\sigma)$ is defined, then, in particular, $\boldsymbol{u}$ is locally order-bounded, so $\hat{\boldsymbol{u}}$ is locally order-bounded (614G(b-ii)). If $\sigma\in\mathcal{S}$ then $\hat{\boldsymbol{u}}{\restriction}\hat{\mathcal{S}}\wedge\sigma$ is the fully adapted extension of $\boldsymbol{u}{\restriction}\mathcal{S}$ (612Qc), so $\mathrm{Osclln}(\hat{\boldsymbol{u}}{\restriction}\hat{\mathcal{S}}\wedge\sigma)=\mathrm{Osclln}(\boldsymbol{u}{\restriction}\mathcal{S}\wedge\sigma)\le w$, by (a). But $\langle\mathrm{Osclln}(\hat{\boldsymbol{u}}{\restriction}\hat{\mathcal{S}}\wedge\tau)\rangle_{\tau\in\hat{\mathcal{S}}}$ is fully adapted (618Da), so

$$\sup_{\tau\in\hat{\mathcal{S}}}\mathrm{Osclln}(\hat{\boldsymbol{u}}{\restriction}\hat{\mathcal{S}}\wedge\tau)=\sup_{\sigma\in\mathcal{S}}\mathrm{Osclln}(\hat{\boldsymbol{u}}{\restriction}\hat{\mathcal{S}}\wedge\sigma)\le w$$

by 614G(b-i).

(ii) If $w=\sup_{\sigma\in\mathcal{S}}\mathrm{Osclln}(\hat{\boldsymbol{u}}{\restriction}\mathcal{S}\wedge\sigma)$ is defined, then $\hat{\boldsymbol{u}}$ is locally order-bounded, so $\boldsymbol{u}$ is locally order-bounded, by 614G(b-ii) again. And now

$$\sup_{\sigma\in\mathcal{S}}\mathrm{Osclln}(\boldsymbol{u}{\restriction}\mathcal{S}\wedge\sigma)=\sup_{\sigma\in\mathcal{S}}\mathrm{Osclln}(\hat{\boldsymbol{u}}{\restriction}\hat{\mathcal{S}}\wedge\sigma)\le w.$$

618M Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a moderately oscillatory process. Then $\mathrm{Osclln}(\boldsymbol{u})=\lim_{I\uparrow\mathcal{I}(\mathcal{S})}\mathrm{Osclln}_I(\boldsymbol{u})$.

proof We know that $\boldsymbol{u}$ is order-bounded (615F(a-iii)), so $\mathrm{Osclln}(\boldsymbol{u})$ is defined. If $\mathcal{S}=\emptyset$ the result is trivial; suppose otherwise.

(a) To begin with, suppose that $\boldsymbol{u}$ is of bounded variation.

(i) The first thing to note is that if $\mathcal{S}'$ is a sublattice of $\mathcal{S}$, then $\operatorname{Oscln}_J(\boldsymbol{u}\restriction\mathcal{S}') \le \int_{\mathcal{S}'}|d\boldsymbol{u}|$ for every $J \in \mathcal{I}(\mathcal{S})$. **P** Of course

$$\operatorname{Oscln}_\emptyset(\boldsymbol{u}\restriction\mathcal{S}') = 0 \le \int_{\mathcal{S}'}|d\boldsymbol{u}|.$$

If $J \in \mathcal{I}(\mathcal{S}')$ is non-empty and $(\tau_0,\dots,\tau_n)$ linearly generates the J-cells, then

$$\operatorname{Oscln}_J(\boldsymbol{u}) = \sup_{i<n}|u_{\tau_{i+1}} - u_{\tau_i}| \le \sum_{i=0}^{n-1}|u_{\tau_{i+1}} - u_{\tau_i}| \le \int_{\mathcal{S}'}|d\boldsymbol{u}|$$

as in 614J. **Q**

(ii) Now, for any non-empty $I \in \mathcal{I}(\mathcal{S})$, set $z_I = \int_{\mathcal{S}}|d\boldsymbol{u}| - S_I(\mathbf{1},|d\boldsymbol{u}|)$. Then $z_I \ge 0$ and $\operatorname{Oscln}^*_I(\boldsymbol{u}) \le \operatorname{Oscln}_I(\boldsymbol{u}) + z_I$. **P** Take $(\tau_0,\dots,\tau_n)$ linearly generating the I-cells, so that $\operatorname{Oscln}_I(\boldsymbol{u}) = \sup_{i<n}|u_{\tau_{i+1}} - u_{\tau_i}|$ and

$$S_I(\mathbf{1},|d\boldsymbol{u}|) = \sum_{i=0}^{n}|u_{\tau_{i+1}} - u_{\tau_i}| \le \int_{\mathcal{S}}|d\boldsymbol{u}|.$$

If $J \in \mathcal{I}(\mathcal{S})$ includes I, then

$$\operatorname{Oscln}_J(\boldsymbol{u}) = \operatorname{Oscln}(\boldsymbol{u}\restriction J)$$

(618B(b-v-γ))

$$= \operatorname{Oscln}(\boldsymbol{u}\restriction J\wedge\tau_0) \vee \sup_{i<n}\operatorname{Oscln}(\boldsymbol{u}\restriction J\cap[\tau_i,\tau_{i+1}]) \vee \operatorname{Oscln}(\boldsymbol{u}\restriction J\vee\tau_n)$$

(618D(b-i))

$$= \operatorname{Oscln}_{J\wedge\tau_0}(\boldsymbol{u}) \vee \sup_{i<n}\operatorname{Oscln}_{J\cap[\tau_i,\tau_{i+1}]}(\boldsymbol{u}) \vee \operatorname{Oscln}_{J\vee\tau_n}(\boldsymbol{u})$$

$$\le \int_{\mathcal{S}\wedge\tau_0}|d\boldsymbol{u}| \vee \sup_{i<n}\int_{\mathcal{S}\cap[\tau_i,\tau_{i+1}]}|d\boldsymbol{u}| \vee \int_{\mathcal{S}\vee\tau_n}|d\boldsymbol{u}|$$

(using (i) above)

$$\le \int_{\mathcal{S}\wedge\tau_0}|d\boldsymbol{u}| + \sum_{i=0}^{n-1}\Bigl(\int_{\mathcal{S}\cap[\tau_i,\tau_{i+1}]}|d\boldsymbol{u}| - |u_{\tau_{i+1}} - u_{\tau_i}|\Bigr)$$

$$+ \sup_{i<n}|u_{\tau_{i+1}} - u_{\tau_i}| + \int_{\mathcal{S}\vee\tau_n}|d\boldsymbol{u}|$$

$$= z_I + \operatorname{Oscln}_I(\boldsymbol{u}).$$

So

$$\operatorname{Oscln}^*_I(\boldsymbol{u}) = \sup_{I\subseteq J\in\mathcal{I}(\mathcal{S})}\operatorname{Oscln}_J(\boldsymbol{u}) \le z_I + \operatorname{Oscln}_I(\boldsymbol{u}). \ \mathbf{Q}$$

Since we always have $\operatorname{Oscln}_I(\boldsymbol{u}) \le \operatorname{Oscln}^*_I(\boldsymbol{u})$, we see that $|\operatorname{Oscln}_I(\boldsymbol{u}) - \operatorname{Oscln}^*_I(\boldsymbol{u})| \le z_I$.

(iii) Now we know that

$$\lim_{I\uparrow\mathcal{I}(\mathcal{S})}\operatorname{Oscln}^*_I(\boldsymbol{u}) = \operatorname{Oscln}(\boldsymbol{u}), \quad \lim_{I\uparrow\mathcal{I}(\mathcal{S})}S_I(\mathbf{1},|d\boldsymbol{u}|) = \int_{\mathcal{S}}|d\boldsymbol{u}|,$$

so $\lim_{I\uparrow\mathcal{I}(\mathcal{S})}z_I = 0$ and $\lim_{I\uparrow\mathcal{I}(\mathcal{S})}\operatorname{Oscln}_I(\boldsymbol{u}) = \operatorname{Oscln}(\boldsymbol{u})$,

(b) In general, if $\boldsymbol{u}$ is just moderately oscillatory, let $\epsilon > 0$. Then there is a process $\boldsymbol{v}$ of bounded variation, with domain $\mathcal{S}$, such that $\theta(z) \le \epsilon$, where $z = \sup|\boldsymbol{u} - \boldsymbol{v}|$. Now

$$|\operatorname{Oscln}_I(\boldsymbol{u}) - \operatorname{Oscln}_I(\boldsymbol{v})| \le \operatorname{Oscln}_I(\boldsymbol{u} - \boldsymbol{v}) \le z \text{ for every } I \in \mathcal{I}(\mathcal{S}),$$

$$|\operatorname{Oscln}(\boldsymbol{u}) - \operatorname{Oscln}(\boldsymbol{v})| \le \operatorname{Oscln}(\boldsymbol{u} - \boldsymbol{v}) \le z$$

(618B(c-ii)). So

$$|\operatorname{Oscln}_I(\boldsymbol{u}) - \operatorname{Oscln}(\boldsymbol{u})| \le |\operatorname{Oscln}_I(\boldsymbol{v}) - \operatorname{Oscln}(\boldsymbol{v})| + 2z$$

for every I, and

$$\limsup_{I\uparrow\mathcal{I}(\mathcal{S})}\theta(\mathrm{Osclln}_I(\boldsymbol{u})-\mathrm{Osclln}(\boldsymbol{u}))$$
$$\le\limsup_{I\uparrow\mathcal{I}(\mathcal{S})}\theta(\mathrm{Osclln}_I(\boldsymbol{v})-\mathrm{Osclln}(\boldsymbol{v}))+2\theta(z)\le 2\epsilon.$$

As ϵ is arbitrary, $\lim_{I\uparrow\mathcal{S}}\mathrm{Osclln}_I(\boldsymbol{u})=\mathrm{Osclln}(\boldsymbol{u})$.

618N The difference between a 'jump-free' process and a 'moderately oscillatory' process can be described in terms of the construction in 615M.

Lemma Let $\mathcal{S}$ be a full sublattice of $\mathcal{T}$ with a greatest element, $\boldsymbol{u}=\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ a moderately oscillatory process, and $\delta>0$. Let $\langle y_i\rangle_{i\in\mathbb{N}}$ be the sequence constructed from $\boldsymbol{u}$ and δ as in 615M. Then $|y_{i+1}-y_i|\le\mathrm{Osclln}(\boldsymbol{u})+\delta\chi 1$ for every $i\in\mathbb{N}$.

proof By 618Gb, $\boldsymbol{u}$ is moderately oscillatory, therefore $⥣$-convergent (615N), so we can indeed apply the method of 615M. Write w for $\mathrm{Osclln}(\boldsymbol{u})$. Take the sequence $\langle D_i\rangle_{i\in\mathbb{N}}$ of downwards-directed subsets of $\mathcal{S}$ constructed from $\boldsymbol{u}$ and δ there. **?** Suppose that $a=[\![|y_{i+1}-y_i|-w>\delta]\!]$ is non-zero. Then there is an $\eta>0$ such that

$$a_1=[\![|y_{i+1}-y_i|-w>\delta+2\eta]\!]$$

has measure at least 4η. Let $\tau\in D_{i+1}$ be such that

$$\theta(\sup_{\tau'\in D_{i+1},\tau'\le\tau}|u_{\tau'}-y_{i+1}|)\le\eta^2$$

(615Gb), so that $\bar\mu a_2\ge 3\eta$ where

$$a_2=a_1\setminus[\![\sup_{\tau'\in D_{i+1},\tau'\le\tau}|u_{\tau'}-y_{i+1}|>\eta]\!].$$

Let $\sigma\in D_i$ be such that $\sigma\le\tau$ and $\theta(|u_\sigma-u_{y_i}|)\le\delta\eta$, so that $\bar\mu a_3\ge 2\eta$ where

$$a_3=a_2\setminus[\![|u_\sigma-y_i|\ge\delta]\!].$$

We have a finite sublattice I of $\mathcal{S}$, containing σ and τ, such that $\theta(w-\mathrm{Osclln}^*_I(\boldsymbol{u}))\le\eta^2$; write w' for $\mathrm{Osclln}^*_I(\boldsymbol{u})$. Then $\bar\mu a_4\ge\eta$, where

$$a_4=a_3\setminus[\![|w-w'|\ge\eta]\!].$$

In particular, $a_4\ne 0$. Let $(\rho_0,\dots,\rho_n)$ linearly generate the $(I\cap[\sigma,\tau])$-cells and define $b_0,\dots,b_n$ by saying that

$$b_j=[\![|u_{\rho_j}-y_i|\ge\delta]\!]\setminus\sup_{j'<j}b_{j'}$$

for $j\le n$, and $b^*=1\setminus\sup_{j\le n}b_j$. Then $b_j\in\mathfrak{A}_{\rho_j}$ for each $j\le n$ and $b^*\in\mathfrak{A}_{\rho_n}$, so there is a $\tau'\in\mathcal{T}$ such that $b_j\subseteq[\![\tau'=\rho_j]\!]\subseteq b_j$ for $j\le n$ and $b^*\subseteq[\![\tau'=\rho_n]\!]$. Now $\tau'\in\mathcal{S}$, $\sigma\le\tau'\le\tau$ and

$$[\![|u_{\tau'}-y_i|<\delta]\!]\subseteq b^*\cap[\![|u_\tau-y_i|<\delta]\!]\subseteq[\![\tau'=\max\mathcal{S}]\!]$$

so $\tau'\in D_{i+1}$.

Since $a_4\subseteq a_3\subseteq 1\setminus b_0$, there is a $j\ge 1$ such that $b=a_4\cap[\![\tau'=\rho_j]\!]$ is non-zero. Now $|u_{\rho_j}-u_{\rho_{j-1}}|\le w'$, but at the same time $b\subseteq[\![|u_{\rho_{j-1}}-y_i|<\delta]\!]$ and $b\subseteq a_2\subseteq[\![|u_{\rho_j}-y_{i+1}|\le\eta]\!]$ and $b\subseteq[\![|w-w'|\le\eta]\!]$, so $b\subseteq[\![|y_{i+1}-y_i|-w\le\delta+2\eta]\!]$, which is impossible, as $b\subseteq a_1$. **X**

So $a=0$ and $|y_{i+1}-y_i|\le w+\delta\chi 1$, as claimed.

618O Definition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\psi:\mathcal{S}^{2\uparrow}\to L^0(\mathfrak{A})$ an adapted interval function which is order-bounded in the sense that $\{\psi(\sigma,\tau):(\sigma,\tau)\in\mathcal{S}^{2\uparrow}\}$ is bounded above and below in $L^0(\mathfrak{A})$. Following 618B, set

$$\mathrm{Osclln}_I(\psi)=\sup_{e\in\mathrm{Stio}_0(I)}\Delta_e(\mathbf{1},|d\psi|)$$

for $I\in\mathcal{I}(\mathcal{S})$ (counting $\sup\emptyset$ as 0),

$$\mathrm{Osclln}^*_I(\psi)=\sup_{J\in\mathcal{I}(\mathcal{S}),J\supseteq I}\mathrm{Osclln}_J(\psi)$$

for $I\in\mathcal{I}(\mathcal{S})$, and

$$\mathrm{Osclln}(\psi) = \inf_{I \in \mathcal{I}(\mathcal{S})} \mathrm{Osclln}^*_I(\psi).$$

Note that as $\mathrm{Osclln}^*_{I'}(\psi) \le \mathrm{Osclln}^*_I(\psi)$ whenever $I \subseteq I'$ in $\mathcal{I}(\mathcal{S})$, $\mathrm{Osclln}(\psi) = \lim_{I \uparrow \mathcal{I}(\mathcal{S})} \mathrm{Osclln}^*_I(\psi)$. Moreover, if $\boldsymbol{u}$ is an order-bounded fully adapted process and $\psi = \Delta\boldsymbol{u}$ the corresponding interval function, $\mathrm{Osclln}_I(\psi) = \mathrm{Osclln}_I(\boldsymbol{u})$ for every $I \in \mathcal{I}(\mathcal{S})$ and $\mathrm{Osclln}(\psi) = \mathrm{Osclln}(\boldsymbol{u})$.

618P Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\psi : \mathcal{S}^{2\uparrow} \to L^0(\mathfrak{A})$ a strictly adapted interval function. For $I \in \mathcal{I}(\mathcal{S})$, set $\boldsymbol{w}_I = \langle w_{I\tau}\rangle_{\tau \in \mathcal{S}}$ where $w_{I\tau} = S_{I\wedge\tau}(\mathbf{1}, d\psi)$ for $\tau \in \mathcal{S}$.

(a) For any $I \in \mathcal{I}(\mathcal{S})$, $\boldsymbol{w}_I$ is fully adapted.

(b) Suppose that $\mathcal{S}$ is finitely full, ψ is order-bounded and $\int_{\mathcal{S}} d\psi$ is defined. Then $ii_\psi(\mathbf{1})$ is order-bounded and $\mathrm{Osclln}(ii_\psi(\mathbf{1})) \le 2\,\mathrm{Osclln}(\psi)$.

(c) Suppose tht ψ is order-bounded and $\int_{\hat{\mathcal{S}}} d\hat{\psi}$ is defined, where $\hat{\mathcal{S}}$ is the covered envelope of $\mathcal{S}$ and $\hat{\psi}$ is the adapted extension of ψ to $\hat{\mathcal{S}}^\uparrow$ (613U). If $ii_\psi(\mathbf{1})$ is moderately oscillatory, then $\mathrm{Osclln}(ii_\psi(\mathbf{1})) \le \mathrm{Osclln}(\psi)$.

(d) If ψ is an order-bounded integrating interval function, then $\mathrm{Osclln}(ii_\psi(\mathbf{1})) \le \mathrm{Osclln}(\psi)$.

proof All four parts are trivial if $\mathcal{S}$ is empty, so suppose otherwise.

(a) If I is empty this is trivial, since $I \wedge \tau = \{\sigma \wedge \tau : \sigma \in I\}$ is empty for every τ. Otherwise, take a string $(\tau_0, \dots, \tau_n)$ linearly generating the I-cells. If $\tau \in \mathcal{S}$, $(\tau_0 \wedge \tau, \dots, \tau_n \wedge \tau)$ linearly generates the $(I \wedge \tau)$-cells (611Kg), and $w_{I\tau} = \sum_{i=0}^{n-1} \psi(\tau_i \wedge \tau, \tau_{i+1} \wedge \tau)$ (613Ec). Since $\psi(\tau_i \wedge \tau, \tau_{i+1} \wedge \tau) \in L^0(\mathfrak{A}_{\tau_{i+1}\wedge\tau}) \subseteq L^0(\mathfrak{A}_\tau)$ for each i, $w_{I\tau} \in L^0(\mathfrak{A}_\tau)$. Now if σ, $\tau \in \mathcal{S}$,

$$[\![\sigma = \tau]\!] \subseteq \inf_{i \le n} [\![\tau_i \wedge \sigma = \tau_i \wedge \tau]\!] \subseteq \inf_{i<n} [\![\psi(\tau_i \wedge \sigma, \tau_{i+1} \wedge \sigma) = \psi(\tau_i \wedge \tau, \tau_{i+1} \wedge \tau)]\!]$$

(because ψ is strictly adapted)

$$\subseteq [\![w_{I\sigma} = w_{I\tau}]\!].$$

Thus $\tau \mapsto w_{I\tau} : \mathcal{S} \to L^0(\mathfrak{A})$ is fully adapted.

(b)(i) Write $\bar{u}$ for $\sup_{(\sigma,\tau) \in \mathcal{S}^{2\uparrow}} |\psi(\sigma, \tau)|$. By 613O(b-i), $\boldsymbol{v} = ii_\psi(\mathbf{1})$ is defined everywhere on $\mathcal{S}$ and is fully adapted; set $v_\tau = \int_{\mathcal{S}\wedge\tau} d\psi$ and $w_{I\tau} = w_{I\tau}$ for $\tau \in \mathcal{S}$, and $\boldsymbol{w}_I = \langle w_{I\tau}\rangle_{\tau \in \mathcal{S}}$, If $I \in \mathcal{I}(\mathcal{S})$ and $\tau \in \mathcal{S}$, $|w_{I\tau}| \le \#(\mathrm{Sti}_0(I))\bar{u}$; so $\langle w_{I\tau}\rangle_{\tau\in\mathcal{S}}$ is order-bounded.

Let $\epsilon > 0$. Then there is an $I \in \mathcal{I}(\mathcal{S})$ such that $\theta(S_J(\mathbf{1}, d\psi) - S_K(\mathbf{1}, d\psi)) \le \epsilon$ whenever J, $K \in \mathcal{I}(\mathcal{S})$ include I. Now if $J \in \mathcal{I}(\mathcal{S})$ includes I_0, 613V(ii-β) tells us that $\theta(w_{J\tau} - v_\tau) \le 2\epsilon$ for every $\tau \in \mathcal{S}$; so if J, $K \in \mathcal{I}(\mathcal{S})$ include I, $\theta(w_{J\tau} - w_{K\tau}) \le 4\epsilon$. Now (a) tells us that $\tau \mapsto w_{J\tau} - w_{K\tau}$ is fully adapted, and we have just seen that it is order-bounded, so 615Db tells us that

$$\theta(\sup|\boldsymbol{w}_J - \boldsymbol{w}_K|) = \theta(\sup_{\tau\in\mathcal{S}} |w_{J\tau} - w_{K\tau}|) \le 2\sqrt{4\epsilon} = 4\sqrt{\epsilon}.$$

As ϵ is arbitrary, this shows that the filter on the space $M_{\text{o-b}} = M_{\text{o-b}}(\mathcal{S})$ generated by

$$\{\{\boldsymbol{w}_J : I \subseteq J \in \mathcal{I}(\mathcal{S})\} : I \in \mathcal{I}(\mathcal{S})\}$$

is a Cauchy filter; by 615Cc, it has a limit in $M_{\text{o-b}}$. But as $\lim_{I\uparrow\mathcal{I}(\mathcal{S})} w_{I\tau} = v_\tau$ for every $\tau \in \mathcal{S}$, this limit must be $\boldsymbol{v}$, and $\boldsymbol{v}$ is order-bounded.

(ii) Now 618Fb tells us that $\mathrm{Osclln}(\boldsymbol{v}) = \lim_{I\uparrow\mathcal{I}(\mathcal{S})} \mathrm{Osclln}(\boldsymbol{w}_I)$. If $I \in \mathcal{I}(\mathcal{S})$ is non-empty, take $(\tau_0, \dots, \tau_n)$ linearly generating the I-cells. If $J \in \mathcal{I}(\mathcal{S})$ includes I and e is a J-cell, then we can express e as $c(\sigma, \sigma')$ where either $\sigma \le \sigma' \le \tau_0$ or $\tau_n \le \sigma \le \sigma'$ or there is an $i < n$ such that $\tau_i \le \sigma \le \sigma' \le \tau_{i+1}$. Now

$$\begin{aligned} w_{I\sigma'} - w_{I\sigma} &= \sum_{j=0}^{n-1} \psi(\tau_j \wedge \sigma', \tau_{j+1} \wedge \sigma') - \psi(\tau_j \wedge \sigma, \tau_{j+1} \wedge \sigma) \\ &= 0 \text{ if } \sigma' \le \tau_0 \text{ or } \tau_n \le \sigma, \\ &= \psi(\tau_i \wedge \sigma', \tau_{i+1} \wedge \sigma') - \psi(\tau_i \wedge \sigma, \tau_{i+1} \wedge \sigma) \text{ if } \tau_i \le \sigma \le \sigma' \le \tau_{i+1} \end{aligned}$$

and in any case $|w_{I\sigma'} - w_{I\sigma}| \le 2\,\mathrm{Osclln}^*_I(\psi)$. So $\mathrm{Osclln}_J(\boldsymbol{w}_I) \le 2\,\mathrm{Osclln}(\psi)$. As J is arbitrary, $\mathrm{Osclln}^*_I(\boldsymbol{w}_I) \le 2\,\mathrm{Osclln}^*_I(\psi)$ and

$$\theta((\operatorname{Osclln}(\boldsymbol{w}_I)-2\operatorname{Osclln}(\psi))^+)\le\theta((\operatorname{Osclln}^*_I(\boldsymbol{w}_I)-2\operatorname{Osclln}(\psi))^+)$$
$$\le\theta(2\operatorname{Osclln}^*_I(\psi)-2\operatorname{Osclln}(\psi)).$$

Taking the limit as I increases through $\mathcal{I}(\mathcal{S})$,

$$\theta((\operatorname{Osclln}(\boldsymbol{w})-2\operatorname{Osclln}(\psi))^+)\le 2\lim_{I\uparrow\mathcal{I}(\mathcal{S})}\theta(\operatorname{Osclln}^*_I(\psi)-\operatorname{Osclln}(\psi))=0,$$

and $\operatorname{Osclln}(\boldsymbol{w})\le 2\operatorname{Osclln}(\psi)$, as claimed.

(c)(i) Suppose to begin with that $\mathcal{S}$ is full, so that we are in the situation of (b), but knowing that $\boldsymbol{v}$ is moderately oscillatory, so that $\operatorname{Osclln}(\boldsymbol{v})=\lim_{I\uparrow\mathcal{I}(\mathcal{S})}\operatorname{Osclln}_I(\boldsymbol{v})$ (618K). As observed in (b-i) above, $\lim_{I\uparrow\mathcal{I}(\mathcal{S})}\sup_{\tau\in\mathcal{S}}\theta(w_{I\tau}-v_\tau)=0$, and because $\mathcal{S}$ is finitely full, it follows that $\lim_{I\uparrow\mathcal{I}(\mathcal{S})}z_I=0$, where $z_I=\sup|\boldsymbol{w}_I-\boldsymbol{v}|$ Next, if $I\in\mathcal{I}(\mathcal{S}$ and $(\tau_0,\dots,\tau_n)$ linearly generate the I-cells,

$$\begin{aligned}w_{I\tau_{i+1}}-w_{I\tau_i}&=\sum_{j=0}^{n-1}\psi(\tau_j\wedge\tau_{i+1},\tau_{j+1}\wedge\tau_{i+1})-\sum_{j=0}^{n-1}\psi(\tau_j\wedge\tau_i,\tau_{j+1}\wedge\tau_i)\\&=\sum_{j=0}^{i}\psi(\tau_j\wedge\tau_{i+1},\tau_{j+1}\wedge\tau_{i+1})-\sum_{j=0}^{i-1}\psi(\tau_j\wedge\tau_i,\tau_{j+1}\wedge\tau_i)\\&=\psi(\tau_i,\tau_{i+1}),\end{aligned}$$

so

$$|v_{\tau_{i+1}}-v_{\tau_i}|\le|w_{I\tau_{i+1}}-w_{I\tau_i}|+2z_I\le|\psi(\tau_i,\tau_{i+1})|+2z_I$$

for every $i<n$. Acccordingly

$$\begin{aligned}\operatorname{Osclln}_I(\boldsymbol{v})&=\sup_{i<n}|v_{\tau_{i+1}}-v_{\tau_i}|\le\sup_{i<n}|w_{I\tau_{i+1}}-w_{I\tau_i}|+2z_I\\&\le\sup_{i<n}|\psi(\tau_i,\tau_{i+1})|+2z_I=\operatorname{Osclln}_I(\psi)+2z_I\le\operatorname{Osclln}^*_I(\psi)+2z_I\end{aligned}$$

and

$$\operatorname{Osclln}(\boldsymbol{v})\le\lim_{I\uparrow\mathcal{I}(\mathcal{S})}\operatorname{Osclln}^*_I(\psi)+\lim_{I\uparrow\mathcal{I}(\mathcal{S})}z_I=\operatorname{Osclln}(\psi).$$

(ii) For the general case, we have the adapted extension $\hat\psi$ of ψ to the covered envelope $\hat{\mathcal{S}}$ of $\mathcal{S}$.

(α) $\hat\psi$ is order-bounded. **P** Write $\bar w$ for $\sup_{(\sigma,\sigma')\in\mathcal{S}^{2\uparrow}}|\psi(\sigma,\sigma')|$. If $\tau\le\tau'$ in $\hat{\mathcal{S}}$,

$$\begin{aligned}[\![|\hat\psi(\tau,\tau')|\le\bar w]\!]&\supseteq\sup_{\sigma,\sigma'\in\mathcal{S}}[\![\sigma=\tau']\!]\cap[\![\sigma\vee\sigma'=\tau']\!]\cap[\![|\psi(\sigma,\sigma\vee\sigma')|\le\bar w]\!]\\&\supseteq\sup_{\sigma,\sigma'\in\mathcal{S}}[\![\sigma=\tau']\!]\cap[\![\sigma'=\tau']\!]=1\end{aligned}$$

and $|\hat\psi(\tau,\tau')|\le\bar w$. **Q**

(β) $\operatorname{Osclln}(\hat\psi)\le\operatorname{Osclln}(\psi)$. **P** Follow parts (i) and (ii) of the proof of 618Ca and part (a) of the proof of 618L, replacing every $\hat u_{\rho'}-\hat u_\rho$ with $\hat\psi(\rho,\rho')$ and every $u_{\sigma'}-u_\sigma$ with $\psi(\sigma,\sigma')$, to see that

— if $I\in\mathcal{I}(\mathcal{S})$, $(\tau_0,\dots,\tau_n)$ linearly generates the I-cells. $\tau_{-1}=\inf\mathcal{S}$ and $\tau_{n+1}=\sup\mathcal{S}$ then

$$\operatorname{Osclln}^*_I(\psi)=\sup\{|\psi(\sigma,\sigma')|:\sigma,\ \sigma'\in\mathcal{S}\text{ and there is an }i\text{ such that }-1\le i\le n\text{ and }\tau_i\le\sigma\le\sigma'\le\tau_{i+1}\},$$

— $\operatorname{Osclln}^*_I(\hat\psi)\le\operatorname{Osclln}^*_I(\psi)$ for every $I\in\mathcal{I}(\mathcal{S})$

and therefore $\operatorname{Osclln}(\hat\psi)\le\operatorname{Osclln}(\psi)$. **Q**

(γ) Now recall that the fully adapted extension $\hat{\boldsymbol{v}}$ of $\boldsymbol{v}$ is $ii_{\hat\psi}(\mathbf{1})$ (616Q(c-ii)) and is moderately oscillatory (615F(a-i)). So, using 618L itself and (b) above, we see that

$$\operatorname{Osclln}(\boldsymbol{v})=\operatorname{Osclln}(\hat{\boldsymbol{v}})\le\operatorname{Osclln}(\hat\psi)\le\operatorname{Osclln}(\psi),$$

as required.

(d) $\boldsymbol{v}$ is an integrator (616J) therefore moderately oscillatory (616Ib), and $\int_{\hat{\mathcal{S}}} d\hat{\psi}$ exists by the definition in 616Fa, so this is a consequence of (c).

618Q Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ a moderately oscillatory process, and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ an integrator. Then $\mathrm{Osclln}(ii_{\boldsymbol{v}}(\boldsymbol{u})) \le \sup |\boldsymbol{u}| \times \mathrm{Osclln}(\boldsymbol{v})$. In particular, $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is jump-free if $\boldsymbol{v}$ is.

proof Consider the integrating interval functions $\Delta\boldsymbol{v}$ and $\psi = \boldsymbol{u}\Delta\boldsymbol{v}$ (617Da). Write $\bar{u}$ for $\sup |\boldsymbol{u}|$. Comparing 618O with 618B, we see that $\Delta\boldsymbol{v}$ is order-bounded because $\boldsymbol{v}$ is (616Ib), and that $\mathrm{Osclln}(\Delta\boldsymbol{v}) = \mathrm{Osclln}(\boldsymbol{v})$. Next, if $\sigma \le \sigma'$ in $\mathcal{S}$,

$$|\psi(\sigma, \sigma')| = |u_\sigma \times (v_{\sigma'} - v_\sigma)| \le \bar{u} \times |v_{\sigma'} - v_\sigma|,$$

so if $I \in \mathcal{I}(\mathcal{S})$

$$\begin{aligned}\mathrm{Osclln}_I(\psi) &= \sup_{e \in \mathrm{Sti}_0(I)} \Delta_e(\mathbf{1}, |d\psi|) \\ &\le \bar{u} \times \sup_{e \in \mathrm{Sti}_0(I)} \Delta_e(\mathbf{1}, |d\boldsymbol{v}|) = \bar{u} \times \mathrm{Osclln}_I(\boldsymbol{v}).\end{aligned}$$

Accordingly $\mathrm{Osclln}^*_I(\psi) \le \bar{u} \times \mathrm{Osclln}^*_I(\boldsymbol{v})$ for every $I \in \mathcal{I}\mathcal{S})$ and $\mathrm{Osclln}(\psi) \le \bar{u} \times \mathrm{Osclln}(\boldsymbol{v})$. But now 617Db tells us that $ii_{\boldsymbol{v}}(\boldsymbol{u}) = ii_\psi(\mathbf{1})$ and 618Pc tells us that

$$\mathrm{Osclln}(ii_{\boldsymbol{v}}(\boldsymbol{u})) \le \mathrm{Osclln}(\psi) \le \bar{u} \times \mathrm{Osclln}(\boldsymbol{v}).$$

In particular, $\mathrm{Osclln}(ii_{\boldsymbol{v}}(\boldsymbol{u}))$ will be zero if $\mathrm{Osclln}(\boldsymbol{v})$ is.

618R Corollary Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ a locally moderately oscillatory process, and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ a locally jump-free local integrator. Then the indefinite integral $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is locally jump-free.

proof Apply 618Q to $\boldsymbol{u}\restriction\mathcal{S} \wedge \tau$ and $\boldsymbol{v}\restriction\mathcal{S} \wedge \tau$ for $\tau \in \mathcal{S}$, using 615F(a-i), 616P(b-ii), 618Gc and 613O(b-ii).

618S Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$, $\boldsymbol{w} = \langle w_\sigma \rangle_{\sigma \in \mathcal{S}}$ two integrators.

(a) $\mathrm{Osclln}([\boldsymbol{v} \overset{*}{|} \boldsymbol{w}]) \le \mathrm{Osclln}(\boldsymbol{v}) \times \mathrm{Osclln}(\boldsymbol{w})$.

(b) $\mathrm{Osclln}(\boldsymbol{v}^*) = (\mathrm{Osclln}(\boldsymbol{v}))^2$.

proof (a) Being integrators, $\boldsymbol{v}$ and $\boldsymbol{w}$ are order-bounded, while $[\boldsymbol{v} \overset{*}{|} \boldsymbol{w}]$ is of bounded variation (617Jc) therefore also order-bounded. Thus all three residual oscillations are well-defined.

Consider the integrating interval function $\psi = \Delta\boldsymbol{v} \times \Delta\boldsymbol{w}$ (617Ha). We have $|\psi| = |\Delta\boldsymbol{v}| \times |\Delta\boldsymbol{w}|$ so

$$\Delta_e(\mathbf{1}, |d\psi|) = \Delta_e(\mathbf{1}, |d\boldsymbol{v}|) \times \Delta_e(\mathbf{1}, |d\boldsymbol{w}|)$$

for every stopping-time interval e with endpoints in $\mathcal{S}$,

$$\begin{aligned}\mathrm{Osclln}_I(\psi) &= \sup_{e \in \mathrm{Sti}_0(I)} \Delta_e(\mathbf{1}, |d\psi|) \le \sup_{e \in \mathrm{Sti}_0(I)} \Delta_e(\mathbf{1}, |d\boldsymbol{v}|) \times \sup_{e \in \mathrm{Sti}_0(I)} \Delta_e(\mathbf{1}, |d\boldsymbol{w}|) \\ &= \mathrm{Osclln}_I(\boldsymbol{v}) \times \mathrm{Osclln}_I(\boldsymbol{w})\end{aligned}$$

for every $I \in \mathcal{I}(\mathcal{S})$,

$$\begin{aligned}\mathrm{Osclln}^*_I(\psi) &= \sup_{I \subseteq J \in \mathcal{I}(\mathcal{S})} \mathrm{Osclln}_J(\psi) \le \sup_{I \subseteq J \in \mathcal{I}(\mathcal{S})} \mathrm{Osclln}_J(d\boldsymbol{v}) \times \sup_{I \subseteq J \in \mathcal{I}(\mathcal{S})} \mathrm{Osclln}_J(d\boldsymbol{w}) \\ &= \mathrm{Osclln}^*_I(\boldsymbol{v}) \times \mathrm{Osclln}^*_I(\boldsymbol{w})\end{aligned}$$

for every $I \in \mathcal{I}(\mathcal{S})$,

$$\begin{aligned}\mathrm{Osclln}(\psi) &= \lim_{I \uparrow \mathcal{I}(\mathcal{S})} \mathrm{Osclln}^*_I(\psi) \le \lim_{I \uparrow \mathcal{I}(\mathcal{S})} \mathrm{Osclln}^*_I(d\boldsymbol{v}) \times \mathrm{Osclln}^*_I(d\boldsymbol{w}) \\ &= \lim_{I \uparrow \mathcal{I}(\mathcal{S})} \mathrm{Osclln}^*_I(d\boldsymbol{v}) \times \lim_{I \uparrow \mathcal{I}(\mathcal{S})} \mathrm{Osclln}^*_I(d\boldsymbol{w}) = \mathrm{Osclln}(\boldsymbol{v}) \times \mathrm{Osclln}(\boldsymbol{w})\end{aligned}$$

and

$$\text{Osclln}([\boldsymbol{v}\,{}^*_|\,\boldsymbol{w}]) = \text{Osclln}(ii_\psi(\mathbf{1})) \le \text{Osclln}(\psi) \le \text{Osclln}(\boldsymbol{v}) \times \text{Osclln}(\boldsymbol{w}).$$

(b) From (a) we see at once that $\text{Osclln}(\boldsymbol{v}^*) \le \text{Osclln}(\boldsymbol{v})^2$. But for the reverse inequality it seems that we need a further idea from §613.

(i) To begin with (down to the end of (v) below), suppose that $\mathcal{S}$ is full and not empty. Express $\boldsymbol{v}$, $\boldsymbol{v}^*$ as $\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ and $\langle v^*_\sigma\rangle_{\sigma\in\mathcal{S}}$. We know that

$$\text{Osclln}(\boldsymbol{v}) = \lim_{I\uparrow\mathcal{I}(\mathcal{S})}\text{Osclln}_I(\boldsymbol{v}),\quad \text{Osclln}(\boldsymbol{v}^*) = \lim_{I\uparrow\mathcal{I}(\mathcal{S})}\text{Osclln}_I(\boldsymbol{v}^*)$$

(618M),

$$\int_{\mathcal{S}}(d\boldsymbol{v})^2 = \lim_{I\uparrow\mathcal{I}(\mathcal{S})} S_I(\mathbf{1},(d\boldsymbol{v})^2) \text{ is defined,}\quad v^*_\tau = \int_{\mathcal{S}\wedge\tau}(d\boldsymbol{v})^2 \text{ for } \tau\in\mathcal{S}$$

(617H). Now $\text{Osclln}(\boldsymbol{v})^2 = \lim_{I\uparrow\mathcal{I}(\mathcal{S})}\text{Osclln}_I(\boldsymbol{v})^2$, and given $\epsilon > 0$ there is a non-empty $I \in \mathcal{I}(\mathcal{S})$ such that

$$\theta(\text{Osclln}(\boldsymbol{v})^2 - \text{Osclln}_I(\boldsymbol{v})^2) \le \epsilon,\quad \theta(\text{Osclln}(\boldsymbol{v}^*) - \text{Osclln}_I(\boldsymbol{v}^*)) \le \epsilon,$$

$$\theta(S_J(\mathbf{1},(d\boldsymbol{v})^2) - S_K(\mathbf{1},(d\boldsymbol{v})^2)) \le \epsilon \text{ whenever } J,\ K\in\mathcal{I}(\mathcal{S}) \text{ include } I.$$

From 613V(ii-β) we see that $\theta(v^*_\tau - S_{I\wedge\tau}(\mathbf{1},(d\boldsymbol{v})^2)) \le 2\epsilon$ for every $\tau\in\mathcal{S}$.

(ii) Let $(\tau_0,\dots,\tau_n)$ be a sequence linearly generating the I-cells. Translating the formulae above with the aid of the last remark in 618Ba, we have

$$\theta(\text{Osclln}(\boldsymbol{v})^2 - \sup_{i<n}(v_{\tau_{i+1}} - v_{\tau_i})^2) = \theta(\text{Osclln}(\boldsymbol{v})^2 - (\sup_{i<n}|v_{\tau_{i+1}} - v_{\tau_i}|)^2) \le \epsilon$$

and

$$\theta(\text{Osclln}(\boldsymbol{v}^*) - \sup_{i<n}|v^*_{\tau_{i+1}} - v^*_{\tau_i}|) \le \epsilon,$$

while

$$\theta(v^*_\tau - \textstyle\sum_{i<n}(v_{\tau_{i+1}\wedge\tau} - v_{\tau_i\wedge\tau})^2) \le 2\epsilon \text{ for every } \tau\in\mathcal{S}$$

because $(\tau_0\wedge\tau,\dots,\tau_n\wedge\tau)$ linearly generates the $(I\wedge\tau)$-cells (611Kg).

(iii) The next thing to note is that

$$\tau\mapsto v^*_\tau - \textstyle\sum_{i<n}(v_{\tau_{i+1}\wedge\tau} - v_{\tau_i\wedge\tau})^2 : \mathcal{S}\to L^0(\mathfrak{A})$$

is fully adapted and order-bounded, just because $\tau\mapsto v^*_\tau$ and $\tau\mapsto v_{\tau_i\wedge\tau}$ are fully adapted and order-bounded for every $i\le n$ (612Ib). Since $\mathcal{S}$ is supposed to be full, $\theta(w)\le 2\sqrt{2\epsilon}$ where

$$w = \sup_{\tau\in\mathcal{S}}|v^*_\tau - \textstyle\sum_{i<n}(v_{\tau_{i+1}\wedge\tau} - v_{\tau_i\wedge\tau})^2|$$

(615Db). Now if $j\le n$,

$$w \ge |v^*_{\tau_j} - \textstyle\sum_{i<n}(v_{\tau_{i+1}\wedge\tau_j} - v_{\tau_i\wedge\tau_j})^2| = |v^*_{\tau_j} - \sum_{i<j}(v_{\tau_{i+1}} - v_{\tau_i})^2|,$$

so if $j<n$

$$\begin{aligned}2w &\ge |v^*_{\tau_{j+1}} - \sum_{i\le j}(v_{\tau_{i+1}} - v_{\tau_i})^2| + |v^*_{\tau_j} - \sum_{i<j}(v_{\tau_{i+1}} - v_{\tau_i})^2|\\ &\ge |v^*_{\tau_{j+1}} - v^*_{\tau_j} - (v_{\tau_{j+1}} - v_{\tau_j})^2|.\end{aligned}$$

(iv) Assembling these, we have

$$\begin{aligned}\theta(\text{Osclln}(\boldsymbol{v}^*) - \text{Osclln}(\boldsymbol{v})^2) &\le \theta(\text{Osclln}(\boldsymbol{v}^*) - \sup_{i<n}(v^*_{\tau_{i+1}} - v^*_{\tau_i}))\\ &\quad + \theta(\sup_{i<n}(v^*_{\tau_{i+1}} - v^*_{\tau_i}) - \sup_{i<n}(v_{\tau_{i+1}} - v_{\tau_i})^2)\\ &\quad + \theta(\text{Osclln}(\boldsymbol{v})^2 - \sup_{i<n}(v_{\tau_{i+1}} - v_{\tau_i})^2)\\ &\le \epsilon + \theta(\sup_{i<n}|v^*_{\tau_{i+1}} - v^*_{\tau_i} - (v_{\tau_{i+1}} - v_{\tau_i})^2|) + \epsilon\\ &\le 2\epsilon + \theta(2w) \le 2\epsilon + 4\sqrt{2\epsilon}.\end{aligned}$$

(v) As ϵ was arbitrary, $\operatorname{Osclln}(\boldsymbol{v}^*)=\operatorname{Osclln}(\boldsymbol{v})^2$, at least if $\mathcal{S}$ is full and not empty.

(vi) In general, the case $\mathcal{S}=\emptyset$ is trivial. Otherwise, taking $\hat{\mathcal{S}}$ to be the covered envelope of $\mathcal{S}$ and $\hat{\boldsymbol{v}}$, $\hat{\boldsymbol{v}}^*$ to be the fully adapted extensions of $\boldsymbol{v}$, $\boldsymbol{v}^*$ to $\hat{\mathcal{S}}$, we see that $\hat{\boldsymbol{v}}^*$ is the quadratic variation of $\boldsymbol{v}$ (617N), while $\operatorname{Osclln}(\boldsymbol{v})=\operatorname{Osclln}(\hat{\boldsymbol{v}})$ and $\operatorname{Osclln}(\boldsymbol{v}^*)=\operatorname{Osclln}(\hat{\boldsymbol{v}}^*)$ (618La). So

$$\operatorname{Osclln}(\boldsymbol{v}^*)=\operatorname{Osclln}(\hat{\boldsymbol{v}}^*)=\operatorname{Osclln}(\hat{\boldsymbol{v}})^2=\operatorname{Osclln}(\boldsymbol{v})^2$$

in all cases.

618T Corollary Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$, $\boldsymbol{w}=\langle w_\sigma\rangle_{\sigma\in\mathcal{S}}$ two (local) integrators with domain $\mathcal{S}$ of which $\boldsymbol{v}$ is (locally) jump-free. Then the covariation $[\boldsymbol{v}\,{}^*_|\,\boldsymbol{w}]$ and the quadratic variation $\boldsymbol{v}^*$ are (locally) jump-free.

proof If $\boldsymbol{v}$ and $\boldsymbol{v}$ are integrators and $\boldsymbol{v}$ is jump-free then $\operatorname{Osclln}(\boldsymbol{v})=0$ so $\operatorname{Osclln}([\boldsymbol{v}\,{}^*_|\,\boldsymbol{w}])=0$ and the covariation is jump-free. If they are local integrators and $\boldsymbol{v}$ is locally jump-free then we apply this to their restrictions to $\mathcal{S}\wedge\tau$, for $\tau\in\mathcal{S}$, to see that $[\boldsymbol{v}\,{}^*_|\,\boldsymbol{w}]$ is locally jump-free. Taking $\boldsymbol{w}=\boldsymbol{v}$ we have the results for $\boldsymbol{v}^*$.

618U Theorem Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ a process of bounded variation. Let $\boldsymbol{v}^\uparrow=\langle v^\uparrow_\tau\rangle_{\tau\in\mathcal{S}}$ be its cumulative variation. Then $\operatorname{Osclln}(\boldsymbol{v}^\uparrow)$ is equal to $\operatorname{Osclln}(\boldsymbol{v})$; in particular, $\boldsymbol{v}$ is jump-free iff $\boldsymbol{v}^\uparrow$ is jump-free.

proof Consider the interval functions $\psi=\Delta\boldsymbol{v}$ and $\psi'=|\Delta\boldsymbol{v}|$. Then $\boldsymbol{v}=ii_\psi(\mathbf{1})+v_\downarrow\mathbf{1}$, where $v_\downarrow$ is the starting value of $\boldsymbol{v}$, and $\boldsymbol{v}^\uparrow=ii_{\psi'}(\mathbf{1})$. As $\Delta\boldsymbol{v}=\Delta ii_\psi(\mathbf{1})$,

$$\operatorname{Osclln}(\boldsymbol{v})=\operatorname{Osclln}(ii_\psi(\mathbf{1})=\operatorname{Osclln}(\psi)$$

as remarked in 618O; as $|\psi|=|\psi'|$, $\operatorname{Osclln}(\psi')=\operatorname{Osclln}(\psi)$; and by 618M, $\operatorname{Osclln}(\boldsymbol{v}^\uparrow)\le\operatorname{Osclln}(\psi')$, so $\operatorname{Osclln}(\boldsymbol{v}^\uparrow)\le\operatorname{Osclln}(\boldsymbol{v})$.

In the other direction, because $|v_\tau-v_\sigma|\le v^\uparrow_\tau-v^\uparrow_\sigma$ whenever $\sigma\le\tau$ in $\mathcal{S}$ (614P(a-i)), we see that

$$\operatorname{Osclln}_I(\boldsymbol{v})\le\operatorname{Osclln}_I(\boldsymbol{v}^\uparrow),\quad \operatorname{Osclln}^*_I(\boldsymbol{v})\le\operatorname{Osclln}^*_I(\boldsymbol{v}^\uparrow)$$

for every $I\in\mathcal{I}(\mathcal{S})$, so that $\operatorname{Osclln}(\boldsymbol{v})\le\operatorname{Osclln}(\boldsymbol{v}^\uparrow)$. Thus the residual oscillations are equal. Now

$$\boldsymbol{v}^\uparrow\text{ is jump-free}\iff\operatorname{Osclln}(\boldsymbol{v}^\uparrow)=0\iff\operatorname{Osclln}(\boldsymbol{v})=0\iff\boldsymbol{v}\text{ is jump-free.}$$

618V Corollary Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v}$ a fully adapted process with domain $\mathcal{S}$. Then $\boldsymbol{v}$ is jump-free and of bounded variation iff it is expressible as the difference of two non-negative non-decreasing order-bounded jump-free processes.

proof Given that $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ is of bounded variation and jump-free, let $\boldsymbol{v}^\uparrow$ be its cumulative variation and set $v_\downarrow=\lim_{\sigma\downarrow\mathcal{S}}v_\sigma$ (using 614Id); then $\boldsymbol{v}^\uparrow+|v_\downarrow|\mathbf{1}$ and $\boldsymbol{v}^\uparrow-\boldsymbol{v}+|v_\downarrow|\mathbf{1}$ are order-bounded non-negative non-decreasing processes with difference $\boldsymbol{v}$, and are jump-free if $\boldsymbol{v}$ is. In the other direction, given that $\boldsymbol{v}=\boldsymbol{v}_1-\boldsymbol{v}_2$ where $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are non-negative order-bounded jump-free non-decreasing processes, then $\boldsymbol{v}$ is of bounded variation by 614J and jump-free if $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are by 618Ga. Of course I am asking you to remember that jump-free processes are order-bounded (618B).

618X Basic exercises (a) Suppose that $T=[0,\infty[$ and $\mathfrak{A}=\{0,1\}$, as in 613W, 615Xf, 616Xa and 617Xb. Let $f:[0,\infty[\,\to\mathbb{R}$ be a function and $\boldsymbol{u}$ the corresponding process on $\mathcal{T}_f$. (i) Show that if f is bounded, then $\operatorname{Osclln}(\boldsymbol{u})$ can be identified with $\max(\sup_{t\ge0}\limsup_{s\downarrow t}|f(s)-f(t)|,\sup_{t>0}\limsup_{s\uparrow t}|f(s)-f(t)|,\inf_{t\ge0}\limsup_{s\to\infty}|f(s)-f(t)|)$. (ii) Show that $\boldsymbol{u}$ is jump-free iff f is continuous and $\lim_{t\to\infty}f(t)$ is defined.

(b) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u}$ an order-bounded fully adapted process with domain $\mathcal{S}$, and z a member of $L^0(\bigcap_{\sigma\in\mathcal{S}}\mathfrak{A}_\sigma)$. Show that, in the language of 612D(e-ii), $\operatorname{Osclln}(z\boldsymbol{u})=|z|\times\operatorname{Osclln}(\boldsymbol{u})$, so that $z\boldsymbol{u}$ is jump-free if $\boldsymbol{u}$ is.

(d) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u}$, $\boldsymbol{v}$ order-bounded processes with domain $\mathcal{S}$. Show that $[\![\mathrm{Oscln}(\boldsymbol{u}) \ne \mathrm{Oscln}(\boldsymbol{v})]\!] \subseteq [\![\boldsymbol{u}$

(e) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ an order-bounded process. Set $\boldsymbol{v} = \langle \mathrm{Oscln}(\boldsymbol{u} \restriction \mathcal{S} \wedge \tau) \rangle_{\tau \in \mathcal{S}}$, as in 618Da. Show that $\mathrm{Oscln}(\boldsymbol{v}) = \mathrm{Oscln}(\boldsymbol{u})$.

(f) Give an example of a jump-free fully adapted process $\boldsymbol{v}$ with domain $\mathcal{S}$ and a sublattice $\mathcal{S}'$ of $\mathcal{S}$ such that $\boldsymbol{v} \restriction \mathcal{S}'$ is not jump-free. (*Hint*: try $\#(\mathcal{S}') = 2$.)

(g) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ a simple process with starting value $u_\downarrow$, and I a sublattice of $\mathcal{S}$ including a breakpoint string for $\boldsymbol{u}$. Show that $\mathrm{Oscln}(\boldsymbol{u}) = \mathrm{Oscln}_I(\boldsymbol{u}) \vee |u_{\min I} - u_\downarrow|$.

(h) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{u}$ a moderately oscillatory process with domain $\mathcal{S}$. Show that $\mathrm{Oscln}(\boldsymbol{u}) = \sup_{\tau \in \mathcal{S}} \mathrm{Oscln}(\boldsymbol{u} \restriction \mathcal{S} \wedge \tau)$, so that if $\boldsymbol{u}$ is locally jump-free then it is jump-free.

(i) Let $\mathcal{S}$ be a full sublattice of $\mathcal{T}$ with a greatest element and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ a moderately oscillatory process. For $\delta > 0$ let $\langle y_{\delta i} \rangle_{i \in \mathbb{N}}$ be the sequence constructed from $\boldsymbol{u}$ and δ as in 615M. Show that $\mathrm{Oscln}(\boldsymbol{u}) = \lim_{\delta \downarrow 0} \sup_{i \in \mathbb{N}} |y_{\delta, i+1} - y_\delta|$.

618Y Further exercises (a) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\mathcal{S}'$ a sublattice of $\mathcal{S}$ which is order-convex in $\mathcal{S}$. Let $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ be an order-bounded process. Show that $\mathrm{Oscln}(\boldsymbol{u} \restriction \mathcal{S}') \le \mathrm{Oscln}(\boldsymbol{u})$.

(b) Suppose that $T = [0, \infty[$ and there is a $t \ge 0$ such that $\mathfrak{A}_t \ne \bigcap_{s > t} \mathfrak{A}_s$. Let $\boldsymbol{\iota} = \langle \iota_\sigma \rangle_{\sigma \in \mathcal{T}_f}$ be the identity process. Show there is a non-empty downwards-directed set $A \subseteq \mathcal{T}_f$ such that $\lim_{\sigma \downarrow A} \iota_\sigma \ne \iota_{\inf A}$.

(c) Let $\mathcal{S}$ be a finitely full sublattice of $\mathcal{T}$, and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ an order-bounded fully adapted process. Show that $\boldsymbol{u}$ is jump-free iff $\lim_{I \uparrow \mathcal{I}(\mathcal{S})} \mathrm{Oscln}_I(\boldsymbol{u}) = 0$ in $L^0(\mathfrak{A})$.

(d) Show that with a modification of the construction in 613P, we can find jump-free processes $\boldsymbol{u}$ and $\boldsymbol{v}$ with domain $\mathcal{T}_c$ such that $\int_{\mathcal{T}_c} \boldsymbol{u}\, d\boldsymbol{v}$ is defined but the integral $\int_{\hat{\mathcal{T}}_c} \hat{\boldsymbol{u}}\, d\hat{\boldsymbol{v}}$ is not.

618 Notes and comments In fact rather little of the work of this section is directly necessary for Itô's lemma as it will be stated in §619; only the definition is really essential, though of course it makes no sense without 618H. And since we shall need to know when an integrator is jump-free, results like 618G, 618L, 618R, 618T and 618U are worth establishing. I have taken a bit of extra trouble to calculate residual oscillations; I think that these give us a clearer notion of what is really going on. And the heavy labour of 618M leads to a fact which is not only remarkable in itself but goes to the heart of the concept of 'moderately oscillatory' process. If $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ is a simple process, then a straightforward calculation (618Xg) tells us that $\mathrm{Oscln}(\boldsymbol{u}) = \lim_{I \uparrow \mathcal{I}(\mathcal{S})} \mathrm{Oscln}_I(\boldsymbol{u})$. Similarly, it is easy to see that processes of bounded variation are integrators (616R). The point of 616K-616M and 618M is that moderately oscillatory processes are 'nearly' of bounded variation.

Note that 'jump-free', like 'simple' but for different reasons, is another property which is not automatically inherited by restrictions to sublattices (618Xf). But observe that once again we have a property of stochastic processes which is independent of the measure chosen on the underlying Boolean algebra.

618zO Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\psi : \mathcal{S}^{2\uparrow} \to L^0(\mathfrak{A})$ an order-bounded integrating interval function with indefinite integral $\boldsymbol{v} = ii_\psi(\mathbf{1})$. Then

$$\mathrm{Oscln}(\boldsymbol{v}) \le \mathrm{Oscln}(\psi).$$

proof The result is trivial if $\mathcal{S}$ is empty, so suppose otherwise.

(a) To begin with, consider the case in which $\mathcal{S}$ is full. Let $\epsilon > 0$.

(i) Since

$$\lim_{I \uparrow \mathcal{I}(\mathcal{S})} S_I(\mathbf{1}, d\psi) = \int_{\mathcal{S}} d\psi, \quad \lim_{I \uparrow \mathcal{I}(\mathcal{S})} \mathrm{Oscln}^*_I(\psi) = \mathrm{Oscln}(\psi),$$

there is a non-empty $I_0 \in \mathcal{S}$ such that whenever $I \in \mathcal{I}(\mathcal{S})$ includes I_0 then

(α) $\theta(S_J(\mathbf{1}, d\psi) - S_K(\mathbf{1}, d\psi)) \le \epsilon$ whenever J, $K \in \mathcal{I}(\mathcal{S})$ include I

(β) $\theta(\mathrm{Osclln}^*_I(\psi) - \mathrm{Osclln}(\psi)) \le \epsilon$.

Take any such I.

(ii) By 613V(ii-β), $\theta(w_{I\tau} - v_\tau) \le 2\epsilon$ for every $\tau \in \mathcal{S}$. Now $\tau \mapsto w_{I\tau} : \mathcal{S} \to L^0$ is fully adapted and order-bounded. **P** If I is empty, this is trivial. Otherwise, take a string $(\tau_0, \dots, \tau_n)$ linearly generating the I-cells. If $\tau \in \mathcal{S}$, $(\tau_0 \wedge \tau, \dots, \tau_n \wedge \tau)$ linearly generates the $(I \wedge \tau)$-cells (611Kg), and $w_{I\tau} = \sum_{i=0}^{n-1} \psi(\tau_i \wedge \tau, \tau_{i+1} \wedge \tau)$ (613Ec). Since $\psi(\tau_i \wedge \tau, \tau_{i+1} \wedge \tau) \in L^0(\mathfrak{A}_{\tau_{i+1}\wedge\tau}) \subseteq L^0(\mathfrak{A}_\tau)$ for each i, $w_{I\tau} \in L^0(\mathfrak{A}_\tau)$. Now if σ, $\tau \in \mathcal{S}$,

$$[\![\sigma = \tau]\!] \subseteq \inf_{i \le n} [\![\tau_i \wedge \sigma = \tau_i \wedge \tau]\!] \subseteq \inf_{i<n} [\![\psi(\tau_i \wedge \sigma, \tau_{i+1} \wedge \sigma) = \psi(\tau_i \wedge \tau, \tau_{i+1} \wedge \tau)]\!]$$

(because ψ is strictly adapted)

$$\subseteq [\![S_{I\wedge\sigma}(\mathbf{1}, d\psi) = w_{I\tau}]\!].$$

Thus $\tau \mapsto w_{I\tau} : \mathcal{S} \to L^0$ is fully adapted. As for order-boundedness, it is the sum of the n order-bounded functions $(\sigma, \tau) \mapsto \psi(\sigma \wedge \tau_i, \tau \wedge \tau_i)$ for $i < n$, so is order-bounded. **Q**

(iii) It follows that $\tau \mapsto w_{I\tau} - v_\tau$ is fully adapted and order-bounded, so that, writing w for $\sup_{\tau\in\mathcal{S}} |w_{I\tau} - v_\tau|$, $\theta(w) \le 2\sqrt{2\epsilon}$. Now take $(\tau_0, \dots, \tau_n)$ linearly generating the I-cells. Then

$$\begin{aligned}\mathrm{Osclln}_I(\boldsymbol{v}) &= \sup_{i<n} |v_{\tau_{i+1}} - v_{\tau_i}| \le w + \sup_{i<n} |S_{I\wedge\tau_{i+1}}(\mathbf{1}, d\psi) - S_{I\wedge\tau_i}(\mathbf{1}, d\psi)| \\ &\le w + \sup_{i<n} |\sum_{j=0}^{i} \psi(\tau_j, \tau_{j+1}) - \sum_{j=0}^{i-1} \psi(\tau_j, \tau_{j+1})| \\ &= w + \sup_{i<n} |\psi(\tau_i, \tau_{i+1})| = w + \mathrm{Osclln}_I(\psi) \le w + \mathrm{Osclln}^*_I(\psi)\end{aligned}$$

and

$$\theta((\mathrm{Osclln}_I(\boldsymbol{v}) - \mathrm{Osclln}(\psi))^+) \le \theta(w) + \theta(\mathrm{Osclln}^*_I(\psi) - \mathrm{Osclln}(\psi)) \le 2\sqrt{2\epsilon} + \epsilon.$$

(iv) This is true whenever $I_0 \subseteq I \in \mathcal{I}(\mathcal{S})$. As ϵ is arbitrary, $\lim_{I\uparrow\mathcal{I}(\mathcal{S})} (\mathrm{Osclln}_I(\boldsymbol{v}) - \mathrm{Osclln}(\psi))^+ = 0$. But because $\boldsymbol{v}$ is an integrator (616J), therefore moderately oscillatory (616Ib), $\lim_{I\uparrow\mathcal{I}(\mathcal{S})} \mathrm{Osclln}_I(\boldsymbol{v}) = \mathrm{Osclln}(\boldsymbol{v})$ (618K). So $(\mathrm{Osclln}(\boldsymbol{v}) - \mathrm{Osclln}(\psi))^+ = 0$ and $\mathrm{Osclln}(\boldsymbol{v}) \le \mathrm{Osclln}(\psi)$.

(b) For the general case, we have the adapted extension $\hat\psi$ of ψ to the covered envelope $\hat{\mathcal{S}}$ of $\mathcal{S}$, which is again an integrating interval function (616F).

(i) $\hat\psi$ is order-bounded. **P** Write $\bar w$ for $\sup_{(\sigma,\sigma')\in\mathcal{S}^{2\uparrow}} |\psi(\sigma, \sigma')|$. If $\tau \le \tau'$ in $\hat{\mathcal{S}}$,

$$\begin{aligned}[\![|\hat\psi(\tau, \tau')| \le \bar w]\!] &\supseteq \sup_{\sigma,\sigma'\in\mathcal{S}} [\![\sigma = \tau']\!] \cap [\![\sigma \vee \sigma' = \tau']\!] \cap [\![|\psi(\sigma, \sigma \vee \sigma')| \le \bar w]\!] \\ &\supseteq \sup_{\sigma,\sigma'\in\mathcal{S}} [\![\sigma = \tau']\!] \cap [\![\sigma' = \tau']\!] = 1\end{aligned}$$

and $|\hat\psi(\tau, \tau')| \le \bar w$. **Q**

(ii) $\mathrm{Osclln}(\hat\psi) \le \mathrm{Osclln}(\psi)$. **P** Follow parts (i) and (ii) of the proof of 618Ca and part (a) of the proof of 618L, replacing every $\hat u_{\rho'} - \hat u_\rho$ with $\hat\psi(\rho, \rho')$ and every $u_{\sigma'} - u_\sigma$ with $\psi(\sigma, \sigma')$, to see that

($\boldsymbol{\alpha}$) if $I \in \mathcal{I}(\mathcal{S})$, $(\tau_0, \dots, \tau_n)$ linearly generates the I-cells. $\tau_{-1} = \inf \mathcal{S}$ and $\tau_{n+1} = \sup \mathcal{S}$ then

$$\begin{aligned}\mathrm{Osclln}^*_I(\psi) = \sup\{&|\psi(\sigma, \sigma')| : \sigma,\ \sigma' \in \mathcal{S} \text{ and there is an } i \\ &\text{such that } -1 \le i \le n \text{ and } \tau_i \le \sigma \le \sigma' \le \tau_{i+1}\},\end{aligned}$$

($\boldsymbol{\beta}$) $\mathrm{Osclln}^*_I(\hat\psi) \le \mathrm{Osclln}^*_I(\psi)$ for every $I \in \mathcal{I}(\mathcal{S})$

and therefore $\mathrm{Osclln}(\hat\psi) \le \mathrm{Osclln}(\psi)$. **Q**

(iii) Now recall that the fully adapted extension $\hat{\boldsymbol{v}}$ of $\boldsymbol{v}$ is $ii_{\hat{\psi}}(\mathbf{1})$ (616Q(c-ii)). So, using 618L itself and (a) above, we see that

$$\operatorname{Osclln}(\boldsymbol{v})=\operatorname{Osclln}(\hat{\boldsymbol{v}})\le\operatorname{Osclln}(\hat{\psi})\le\operatorname{Osclln}(\psi),$$

as required.

Version of 13.3.17/29.7.19

619 Itô's formula

I give three versions of Itô's formula (619C, 619D and 619J). The last depends on elementary facts about the action of functions of more than one real variable on strings of processes (619E-619G).

619A Notation As usual, $(\mathfrak{A},\bar{\mu},T,\langle\mathfrak{A}_t\rangle_{t\in T},\mathcal{T},\langle\mathfrak{A}_\tau\rangle_{\tau\in\mathcal{T}})$ will be a stochastic integration structure. L^∞ will be $L^\infty(\mathfrak{A})$ with its norm $\|\;\|_\infty$. L^0 will be $L^0(\mathfrak{A})$ endowed with its topology of convergence in measure, and $\mathbb{E}$ will be the integral defined from $\bar{\mu}$; for $w\in L^0$, $\theta(w)$ will be $\mathbb{E}(|w|\wedge\chi 1)$. If $y\in L^0$, then $y\mathbf{1}$ will denote the constant process on $\{\sigma:\sigma\in\mathcal{T},\,y\in L^0(\mathfrak{A}_\sigma)\}$ with value y. For local integrators $\boldsymbol{v}$ and $\boldsymbol{w}$, $[\boldsymbol{v}\overset{*}{|}\boldsymbol{w}]$ will be their covariation (617Hb); $\boldsymbol{v}^*$ will be the quadratic variation of $\boldsymbol{v}$. For a sublattice $\mathcal{S}$ of $\mathcal{T}$, $M_{\mathrm{fa}}(\mathcal{S})$ will be the space of fully adapted processes with domain $\mathcal{S}$.

619B Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v}=\langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ an integrator. Then for every $\epsilon>0$ there is a $\delta>0$ such that $\theta(S_I(\boldsymbol{u},(d\boldsymbol{v})^2)\le\epsilon$ whenever $I\in\mathcal{I}(\mathcal{S})$, $\boldsymbol{u}\in M_{\mathrm{fa}}(\mathcal{I})$ and $\theta(\sup|\boldsymbol{u}|)\le\delta$.

proof As $\boldsymbol{v}$ is an integrator, it is order-bounded (616Ib) and there is a $\delta_0>0$ such that

$$\theta(S_I(\boldsymbol{u},d\boldsymbol{v}))\le\tfrac{1}{3}\epsilon\text{ whenever }I\in\mathcal{I}(\mathcal{S}),\ \boldsymbol{u}\in M_{\mathrm{fa}}(\mathcal{I})\text{ and }\theta(\sup|\boldsymbol{u}|)\le\delta_0$$

(616E). Also $\boldsymbol{v}^2$ is an integrator (616Pa), so there is a $\delta>0$ such that

$$\theta(S_I(\boldsymbol{u},d(\boldsymbol{v}^2)))\le\tfrac{1}{3}\epsilon\text{ whenever }I\in\mathcal{I}(\mathcal{S}),\ \boldsymbol{u}\in M_{\mathrm{fa}}(\mathcal{I})\text{ and }\theta(\sup|\boldsymbol{u}|)\le\delta$$

and $\theta(x\times\sup|\boldsymbol{v}|)\le\delta_0$ whenever $\theta(x)\le\delta$. Now suppose that $I\in\mathcal{I}(\mathcal{S})$, $\boldsymbol{u}\in M_{\mathrm{fa}}(\mathcal{I})$ and $\theta(\sup|\boldsymbol{u}|)\le\delta$. If $\sigma\le\tau$ in I,

$$u_\sigma\times(v_\tau-v_\sigma)^2=u_\sigma\times(v_\tau^2-v_\sigma^2)-2u_\sigma\times v_\sigma\times(v_\tau-v_\sigma).$$

Working through the definitions in §613, we see that

$$S_I(\boldsymbol{u},(d\boldsymbol{v})^2)=S_I(\boldsymbol{u},d(\boldsymbol{v}^2))-2S_I(\boldsymbol{u}\times\boldsymbol{v},d\boldsymbol{v})$$

while

$$\theta(\sup|\boldsymbol{u}\times\boldsymbol{v}|)\le\theta(\sup|\boldsymbol{u}|\times\sup|\boldsymbol{v}|)\le\delta_0,$$

so

$$\theta(S_I(\boldsymbol{u},(d\boldsymbol{v})^2))\le\theta(S_I(\boldsymbol{u},d(\boldsymbol{v}^2)))+2\theta(S_I(\boldsymbol{u}\times\boldsymbol{v},d\boldsymbol{v}))\le\frac{1}{3}\epsilon+\frac{2}{3}\epsilon=\epsilon,$$

as required.

619C Itô's Formula, first form Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{v}=\langle v_\tau\rangle_{\tau\in\mathcal{S}}$ a jump-free integrator, and $\boldsymbol{v}^*$ its quadratic variation. If $h:\mathbb{R}\to\mathbb{R}$ is a twice-differentiable function with continuous second derivative, then

$$\int_{\mathcal{S}}\bar{h}'\boldsymbol{v}\,d\boldsymbol{v}+\frac{1}{2}\int_{\mathcal{S}}\bar{h}''\boldsymbol{v}\,d\boldsymbol{v}^*$$

is defined and equal to $\bar{h}(v_\uparrow)-\bar{h}(v_\downarrow)$, where

$$v_\uparrow=\lim_{\sigma\uparrow\mathcal{S}}v_\sigma,\qquad v_\downarrow=\lim_{\sigma\downarrow\mathcal{S}}v_\sigma.$$

Remark In the formula above, $\bar{h}':L^0\to L^0$ and $\bar{h}'':L^0\to L^0$ should be read as $\overline{h'}$ and $\overline{h''}$.

proof (a) We can note straight away that $\boldsymbol{v}$ is moderately oscillatory (616Ib), so $\bar{h}'\boldsymbol{v}$ and $\bar{h}''\boldsymbol{v}$ are moderately oscillatory (615F(a-ii)), while $\bar{h}\boldsymbol{v}$ and $\boldsymbol{v}^*$ are integrators (616O, 617I), so that the integrals $\int_{\mathcal{S}}d(\bar{h}\boldsymbol{v})$, $\int_{\mathcal{S}}\bar{h}'\boldsymbol{v}\,d\boldsymbol{v}$ and $\int_{\mathcal{S}}\bar{h}''\boldsymbol{v}\,d\boldsymbol{v}^*$ are defined (616K); moreover,

$$\int_{\mathcal{S}} d(\bar{h}\boldsymbol{v}) = \lim_{\sigma\uparrow\mathcal{S}} \bar{h}(v_\sigma) - \lim_{\sigma\downarrow\mathcal{S}} \bar{h}(v_\sigma) = \bar{h}(v_\uparrow) - \bar{h}(v_\downarrow)$$

(613N, 613Bb), and

$$\int_{\mathcal{S}} \bar{h}''\boldsymbol{v}\, d\boldsymbol{v}^* = \int_{\mathcal{S}} \bar{h}''\boldsymbol{v}\,(d\boldsymbol{v})^2$$

(617I again).

(b) For the time being (down to the end of (c)), suppose that h'' is uniformly continuous. Let $\epsilon > 0$. Let $\eta > 0$ be such that $\theta(S_I(\boldsymbol{u},(d\boldsymbol{v})^2)) \le \epsilon$ whenever $I \in \mathcal{I}(\mathcal{S})$, $\boldsymbol{u} \in M_{\mathrm{fa}}(I)$ and $\theta(\sup|\boldsymbol{u}|) \le \eta$. Then there is a $\delta > 0$ such that $|h''(\alpha) - h''(\beta)| \le \eta$ whenever $|\alpha - \beta| \le 2\delta$. Now take any such α and β. By Taylor's theorem with remainder, there is a γ lying between α and β such that

$$h(\beta) = h(\alpha) + (\beta-\alpha)h'(\alpha) + \frac{1}{2}(\beta-\alpha)^2 h''(\gamma),$$

so that

$$|h(\beta) - h(\alpha) - (\beta-\alpha)h'(\alpha) - \frac{1}{2}(\beta-\alpha)^2 h''(\alpha)| \le \eta(\beta-\alpha)^2.$$

It follows that if w, $w' \in L^0$ then $[\![|w'-w| \le \delta]\!]$ is included in

$$[\![|\bar{h}(w') - \bar{h}(w) - \bar{h}'(w)\times(w'-w) - \frac{1}{2}\bar{h}''(w)\times(w'-w)^2| \le \eta(w'-w)^2]\!].$$

(c) Let $J \in \mathcal{I}(\mathcal{S})$ be such that $\theta(z) \le \delta\epsilon$, where $z = \mathrm{Osclln}^*_J(\boldsymbol{v})$ (618Bb). Take any $I \in \mathcal{I}(\mathcal{S})$ such that $I \supseteq J$. Then $\mathrm{Osclln}_I(\boldsymbol{v}) \le z$, and $a = [\![z \le \delta]\!]$ has measure at least $1-\epsilon$. Now if $e = c(\sigma,\sigma')$ is an I-cell, and we set

$$\begin{aligned} y_e &= \Delta_e(\mathbf{1}, d(\bar{h}\boldsymbol{v})) - \Delta_e(\bar{h}'\boldsymbol{v}, d\boldsymbol{v}) - \frac{1}{2}\Delta_e(\bar{h}''\boldsymbol{v},(d\boldsymbol{v})^2) \\ &= \bar{h}(v_\tau) - \bar{h}(v_\sigma) - \bar{h}'(v_\sigma)\times(v_\tau - v_\sigma) - \frac{1}{2}\bar{h}''(v_\sigma)\times(v_\tau-v_\sigma)^2, \end{aligned}$$

we have

$$|v_\tau - v_\sigma| = \Delta_e(\mathbf{1},|d\boldsymbol{v}|) \le \mathrm{Osclln}_I(\boldsymbol{v}) \le z$$

and

$$a \subseteq [\![|v_\tau - v_\sigma| \le \delta]\!] \subseteq [\![|y_e| \le \eta(v_\tau-v_\sigma)^2]\!] = [\![|y_e| \le \Delta_e(\eta\mathbf{1},(d\boldsymbol{v})^2]\!].$$

Summing over e,

$$\begin{aligned} a &\subseteq [\![\textstyle\sum_{e\in\mathrm{Stio}_0(I)}|y_e| \le \sum_{e\in\mathrm{Stio}_0(I)}\Delta_e(\eta\mathbf{1},(d\boldsymbol{v})^2)]\!] \\ &\subseteq [\![|\textstyle\sum_{e\in\mathrm{Stio}_0(I)} y_e| \le \sum_{e\in\mathrm{Stio}_0(I)}\Delta_e(\eta\mathbf{1},(d\boldsymbol{v})^2)]\!] \\ &= [\![|S_I(\mathbf{1},d(\bar{h}\boldsymbol{v})) - S_I(\bar{h}'\boldsymbol{v},d\boldsymbol{v}) - S_I(\bar{h}''\boldsymbol{v},(d\boldsymbol{v})^2| \le S_I(\eta\mathbf{1},(d\boldsymbol{v})^2)]\!], \end{aligned}$$

and

$$\theta\bigl(S_I(\mathbf{1},d(\bar{h}\boldsymbol{v})) - S_I(\bar{h}'\boldsymbol{v},d\boldsymbol{v}) - S_I(\bar{h}''\boldsymbol{v},(d\boldsymbol{v})^2)\bigr) \le \bar{\mu}(1\setminus a) + \bar{\theta}(\eta S_I(\mathbf{1},(d\boldsymbol{v})^2)) \le 2\epsilon$$

by the choice of J and η. And this is true whenever $I \supseteq J$. Accordingly, taking the limit as $I \uparrow \mathcal{I}(\mathcal{S})$, we have

$$\begin{aligned} \bar{h}(v_\uparrow) - \bar{h}(v_\downarrow) - \int_{\mathcal{S}} \bar{h}'\boldsymbol{v}\,d\boldsymbol{v} - \frac{1}{2}\int_{\mathcal{S}} \bar{h}''\boldsymbol{v}),(d\boldsymbol{v})^2 \\ &= \int_{\mathcal{S}} d(\bar{h}\boldsymbol{v}) - \int_{\mathcal{S}} \bar{h}'\boldsymbol{v}\,d\boldsymbol{v} - \frac{1}{2}\int_{\mathcal{S}} \bar{h}''\boldsymbol{v}\,(d\boldsymbol{v})^2 \\ &= \lim_{I\uparrow\mathcal{I}(\mathcal{S})} S_I(\mathbf{1},d(\bar{h}\boldsymbol{v})) - S_I(\bar{h}'\boldsymbol{v},d\boldsymbol{v}) - S_I(\bar{h}''\boldsymbol{v},(d\boldsymbol{v})^2) \\ &= 0, \end{aligned}$$

and

$$\int_{\mathcal{S}} \bar{h}'\boldsymbol{v}\,d\boldsymbol{v} + \frac{1}{2}\int_{\mathcal{S}} \bar{h}''\boldsymbol{v}\,d\boldsymbol{v}^* = \int_{\mathcal{S}} \bar{h}'\boldsymbol{v}\,d\boldsymbol{v} + \frac{1}{2}\int_{\mathcal{S}} \bar{h}''\boldsymbol{v}\,(d\boldsymbol{v})^2 = \bar{h}(v_\uparrow) - \bar{h}(v_\downarrow),$$

as required.

(d) This deals with the case in which h'' is uniformly continuous. For the general case, recall that there is a $\bar{v} \in L^0(\mathfrak{A})$ such that $|v_\sigma| \le \bar{v}$ for every $\sigma \in \mathcal{S}$ (616Ib). Take any $M \ge 0$. There is a smooth function $f : \mathbb{R} \to \mathbb{R}$ with compact support such that $f(x) = 1$ for every $x \in [-M, M]$. **P** By 473Eb there is a smooth function $\tilde{h}_0 : \mathbb{R} \to \mathbb{R}$ such that $\tilde{h}_0(x) = 0$ if $|x| \ge 1$ and the Lebesgue integral $\int_{-1}^1 \tilde{h}_0(t)dt$ is equal to 1. Take f to be the convolution $\tilde{h}_0 * \chi[-M-1, M+1]$; by 473De, f is smooth, while $f(x) = \int_{-1}^1 \tilde{h}_0(t)dt = 1$ if $|x| \le M$ and $f(x) = 0$ if $|x| \ge M+2$. **Q**

Consider $g = f \times h$. Then g is twice differentiable with continuous second derivative, while $g'(x) = h'(x)$ and $g''(x) = h''(x)$ whenever $|x| < M$. Since g'', like f and g, has compact support, it is uniformly continuous (4A2Jf). Accordingly

$$\bar{g}(v_\uparrow) - \bar{g}(v_\downarrow) = \int_{\mathcal{S}} \bar{g}'\boldsymbol{v}\,d\boldsymbol{v} + \frac{1}{2}\int_{\mathcal{S}} \bar{g}''\boldsymbol{v}\,d\boldsymbol{v}^*.$$

So

$$\begin{aligned}
[\![\bar{h}(v_\uparrow) - \bar{h}(v_\downarrow) \ne \textstyle\int_{\mathcal{S}} \bar{h}'\boldsymbol{v}\,d\boldsymbol{v} + \frac{1}{2}\int_{\mathcal{S}} \bar{h}''\boldsymbol{v}\,d\boldsymbol{v}^*]\!] &\\
\subseteq [\![\bar{h}(v_\uparrow) \ne \bar{g}(v_\uparrow)]\!] \cup [\![\bar{h}(v_\downarrow) \ne \bar{g}(v_\downarrow)]\!] \cup [\![\textstyle\int_{\mathcal{S}} \bar{h}'\boldsymbol{v}\,d\boldsymbol{v} \ne \int_{\mathcal{S}} \bar{g}'\boldsymbol{v}\,d\boldsymbol{v}]\!] &\\
\cup [\![\textstyle\int_{\mathcal{S}} \bar{h}''\boldsymbol{v}\,d\boldsymbol{v}^* \ne \int_{\mathcal{S}} \bar{g}''\boldsymbol{v}\,d\boldsymbol{v}^*]\!] &\\
\subseteq [\![\bar{h}\boldsymbol{v} \ne \bar{g}\boldsymbol{v}]\!] \cup [\![\bar{h}'\boldsymbol{v} \ne \bar{g}'\boldsymbol{v}]\!] \cup [\![\bar{h}''\boldsymbol{v} \ne \bar{g}''\boldsymbol{v}]\!] &
\end{aligned}$$

(613Jd twice)

$$\subseteq \sup_{\sigma \in \mathcal{S}} [\![|v_\sigma| \ge M]\!] \subseteq [\![\bar{v} \ge M]\!].$$

Letting $M \to \infty$,

$$[\![\bar{h}(v_\uparrow) - \bar{h}(v_\downarrow) \ne \int_{\mathcal{S}} \bar{h}'\boldsymbol{v}\,d\boldsymbol{v} + \frac{1}{2}\int_{\mathcal{S}} \bar{h}''\boldsymbol{v}\,d\boldsymbol{v}^*]\!] \subseteq \inf_{M \ge 0} [\![\bar{v} \ge M]\!] = 0$$

and

$$\bar{h}(v_\uparrow) - \bar{h}(v_\downarrow) = \int_{\mathcal{S}} \bar{h}'\boldsymbol{v}\,d\boldsymbol{v} + \frac{1}{2}\int_{\mathcal{S}} \bar{h}''\boldsymbol{v}\,d\boldsymbol{v}^*$$

in this case also.

619D Itô's Formula, second form Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v}$ a jump-free integrator with domain $\mathcal{S}$ and quadratic variation $\boldsymbol{v}^*$. If $\boldsymbol{u}$ is a moderately oscillatory process with domain $\mathcal{S}$, and $h : \mathbb{R} \to \mathbb{R}$ is a twice-differentiable function with continuous second derivative, then

$$\int_{\mathcal{S}} \boldsymbol{u}\,d(\bar{h}\boldsymbol{v}) = \int_{\mathcal{S}} \boldsymbol{u} \times \bar{h}'\boldsymbol{v}\,d\boldsymbol{v} + \frac{1}{2}\int_{\mathcal{S}} \boldsymbol{u} \times \bar{h}''\boldsymbol{v}\,d\boldsymbol{v}^*.$$

proof Applying 619C to $\mathcal{S} \wedge \tau$, we have

$$\bar{h}(v_\tau) - \bar{h}(v_\downarrow) = \int_{\mathcal{S}\wedge\tau} \bar{h}'\boldsymbol{v}\,d\boldsymbol{v} + \frac{1}{2}\int_{\mathcal{S}\wedge\tau} \bar{h}''\boldsymbol{v}\,d\boldsymbol{v}^*$$

for every $\tau \in \mathcal{S}$; that is, setting

$$\boldsymbol{z} = \bar{h}\boldsymbol{v} - ii_{\boldsymbol{v}}(\bar{h}'\boldsymbol{v}) - \frac{1}{2}ii_{\boldsymbol{v}^*}(\bar{h}''\boldsymbol{v}),$$

$\boldsymbol{z} = \bar{h}(v_\downarrow)\mathbf{1}{\restriction}\mathcal{S}$. But recalling that

$$\bar{h}\boldsymbol{v}, \quad ii_{\boldsymbol{v}}(\bar{h}'\boldsymbol{v}), \quad ii_{\boldsymbol{v}^*}(\bar{h}''\boldsymbol{v})$$

are all integrators (616O, 616J), it follows that

$$\int_{\mathcal{S}} \boldsymbol{u}\,d(\bar{h}\boldsymbol{v}) - \int_{\mathcal{S}} \boldsymbol{u}\,d(ii_{\boldsymbol{v}}(\bar{h}'\boldsymbol{v})) - \frac{1}{2}\int_{\mathcal{S}} \boldsymbol{u}\,d(ii_{\boldsymbol{v}^*}(\bar{h}''\boldsymbol{v})) = \int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{z} = 0.$$

Now applying 617E this becomes

$$\begin{aligned}\int_{\mathcal{S}} \boldsymbol{u}\,d(\bar{h}\boldsymbol{v}) &= \int_{\mathcal{S}} \boldsymbol{u}\,d(ii_{\boldsymbol{v}}(\bar{h}'\boldsymbol{v})) + \frac{1}{2}\int_{\mathcal{S}} \boldsymbol{u}\,d(ii_{\boldsymbol{v}^*}(\bar{h}''\boldsymbol{v}))\\ &= \int_{\mathcal{S}} \boldsymbol{u} \times \bar{h}'\boldsymbol{v}\,d\boldsymbol{v} + \frac{1}{2}\int_{\mathcal{S}} \boldsymbol{u} \times \bar{h}''\boldsymbol{v}\,d\boldsymbol{v}^*,\end{aligned}$$

as claimed.

619E Of course Itô's formula also has a multidimensional version. There are no really new ideas needed, and ordinarily I leave such adaptations to the exercises, but there are points below where we can use two-dimensional formulae to tell us interesting things about one-dimensional processes, so I spell things out here. The first step will have to be to establish an interpretation of $\bar{h}(v_1, \dots, v_k)$ where $h : \mathbb{R}^k \to \mathbb{R}$ is a Borel measurable function and $v_1, \dots, v_k \in L^0$. Since I did leave this to the exercises in §364, I give the details now.

Proposition Let $k \ge 1$ be an integer.

(a) Suppose that $u_1, \dots, u_k \in L^0$. Let $\mathcal{B}_k$ be the Borel σ-algebra of $\mathbb{R}^k$. Then there is a unique sequentially order-continuous Boolean homomorphism $\phi : \mathcal{B}_k \to \mathfrak{A}$ such that $\phi\{(\xi_1, \dots, \xi_k) : \xi_i > \alpha\} = [\![u_i > \alpha]\!]$ whenever $1 \le i \le k$ and $\alpha \in \mathbb{R}$.

In this context, write $[\![(u_1, \dots, u_k) \in E]\!]$ for ϕE, for every Borel set $E \subseteq \mathbb{R}^k$.

(b) Suppose that $h : \mathbb{R}^k \to \mathbb{R}$ is a Borel measurable function. Then there is a unique operator $\bar{h} : (L^0)^k \to L^0$ such that $[\![\bar{h}(u_1, \dots, u_k) \in F]\!] = [\![(u_1, \dots, u_k) \in h^{-1}[F]]\!]$ whenever $F \subseteq \mathbb{R}$ is a Borel set and $u_1, \dots, u_k \in L^0$.

(c) If $u_1, \dots, u_k, v_1, \dots, v_k \in L^0$, then

$$\inf_{1 \le i \le k} [\![u_i = v_i]\!] \subseteq [\![\bar{h}(u_1, \dots, u_k) = \bar{h}(v_1, \dots, v_k)]\!].$$

(d) If $h : \mathbb{R}^k \to \mathbb{R}$ is continuous, then $\bar{h} : (L^0)^k \to L^0$ is continuous for the topology of convergence in measure.

(e) Suppose that Ω is a set, Σ is a σ-algebra of subsets of Ω, $\mathcal{N}$ is a σ-ideal of Σ, and $\mathfrak{A}$ is isomorphic to the quotient Boolean algebra $\Sigma/\mathcal{N}$. Write $\mathcal{L}^0$ for the f-algebra of real-valued Σ-measurable functions on Ω, and $\mathcal{W}$ for the ideal

$$\{f : f \in \mathcal{L}^0,\ \{\omega : f(\omega) \ne 0\} \in \mathcal{N}\},$$

so that L^0 can be identified with the f-algebra quotient $\mathcal{L}^0/\mathcal{W}$ (364Ib). Write $E \mapsto E^\bullet : \Sigma \to \mathfrak{A}$ and $f \mapsto f^\bullet : \mathcal{L}^0 \to L^0$ for the homomorphisms corresponding to the identifications $\mathfrak{A} \cong \Sigma/\mathcal{N}$ and $L^0 \cong \mathcal{L}^0/\mathcal{W}$. Then if $h : \mathbb{R}^k \to \mathbb{R}$ is a Borel measurable function,

$$\bar{h}(f_1^\bullet, \dots, f_k^\bullet) = (h(f_1, \dots, f_k))^\bullet$$

for all $f_1, \dots, f_k \in \mathcal{L}^0$, defining the composition $h(f_1, \dots, f_k)$ by setting $(h(f_1, \dots, f_k))(\omega) = h(f_1(\omega), \dots, f_k(\omega))$ for every $\omega \in \Omega$.

(f) Suppose that $\langle h_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of Borel measurable functions from $\mathbb{R}^k$ to $\mathbb{R}$, and that $h(x) = \sup_{n\in\mathbb{N}} h_n(x)$ is finite for every $x \in \mathbb{R}^k$. Then $\langle \bar{h}_n(u_1, \dots, u_k) \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in L^0 with supremum $\bar{h}(u_1, \dots, u_k)$, for all $u_1, \dots, u_k \in L^0$.

(g) Now suppose that $(\mathfrak{C}, \nu)$ is another probability algebra and $\phi : \mathfrak{A} \to \mathfrak{C}$ is an order-continuous Boolean homomorphism. Let $T_\phi : L^0(\mathfrak{A}) \to L^0(\mathfrak{C})$ be the corresponding f-algebra homomorphism (612Bf). Take $u_1, \dots, u_k \in L^0(\mathfrak{A})$.

(i) If $E \in \mathcal{B}_k$ is a Borel set, then $[\![(T_\phi u_1, \dots, T_\phi u_k) \in E]\!] = \phi[\![(u_1, \dots, u_k) \in E]\!]$.

(ii) If $h : \mathbb{R}^k \to \mathbb{R}$ is Borel measurable, then $\bar{h}(T_\phi u_1, \dots, T_\phi u_k) = T_\phi \bar{h}(u_1, \dots, u_k)$.

proof Since there is a measurable space with negligibles $(\Omega, \Sigma, \mathcal{N})$, as in (e), such that $\mathfrak{A} \cong \Sigma/\mathcal{N}$ (314M), we can suppose from the beginning that $\mathfrak{A}$ is actually such a quotient; at some point, of course, we shall have to check that it won't matter which such representation is chosen.

(a)(i) If $u_1,\dots,u_k \in L^0$, let $f_1,\dots,f_k \in \mathcal{L}^0$ be such that $f_i^\bullet = u_i$ for $1 \le i \le k$. If $E \in \mathcal{B}_k$, then $\{\omega : (f_1(\omega),\dots,f_k(\omega)) \in E\} \in \Sigma$ (121K). So we can set $\phi E = \{\omega : (f_1(\omega),\dots,f_k(\omega)) \in E\}^\bullet \in \mathfrak{A}$.

Clearly $E \mapsto \{\omega : (f_1(\omega),\dots,f_k(\omega)) \in E\} : \mathcal{B}_k \to \Sigma$ is a sequentially order-continuous Boolean homomorphism, so ϕ also is. And if $1 \le i \le k$ and $\alpha \in \mathbb{R}$,

$$\phi\{(\xi_1,\dots,\xi_k) : \xi_i > \alpha\} = \{\omega : f_i(\omega) > \alpha\}^\bullet = [\![u_i > \alpha]\!]$$

(364Ib again).

(ii) Now suppose that $\phi' : \mathcal{B}_k \to \mathfrak{A}$ is another homomorphism of the same kind. Let $\mathcal{J}$ be the family of sets of the form $F = \prod_{i=1}^k \,]\alpha_i,\infty[$ where $\alpha_i \in \mathbb{R}$ for every i. For such a set,

$$\begin{aligned}\phi F &= \{\omega : f_i(\omega) > \alpha_i \text{ for } 1 \le i \le k\}^\bullet = \inf_{1\le i\le k}\{\omega : f_i(\omega) > \alpha_i\}^\bullet = \inf_{1\le i\le k}[\![u_i > \alpha_i]\!] \\ &= \inf_{1\le i\le k}\phi'\{(\xi_1,\dots,\xi_k) : \xi_i > \alpha_i\} = \phi'(\bigcap_{1\le i\le k}\{(\xi_1,\dots,\xi_k) : \xi_i > \alpha_i\}) = \phi' F.\end{aligned}$$

So ϕ and ϕ' agree on $\mathcal{J}$ and therefore on $\mathcal{J}' = \mathcal{J} \cup \{\mathbb{R}^k\}$. Set $\mathcal{A} = \{E : E \in \mathcal{B}_k,\ \phi E = \phi' E\}$. Then $\mathcal{J}' \subseteq \mathcal{A}$. If E, $F \in \mathcal{A}$ then

$$\phi(E \setminus F) = \phi E \setminus \phi F = \phi' E \setminus \phi' F = \phi'(E \setminus F)$$

and $E \setminus F \in \mathcal{A}$; if $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathcal{A}$ with union E,

$$\phi E = \phi(\bigcup_{n\in\mathbb{N}} E_n) = \sup_{n\in\mathbb{N}} \phi E_n = \sup_{n\in\mathbb{N}} \phi' E_n = \phi' E$$

and $E \in \mathcal{A}$. Since $\mathcal{J}'$ is closed under intersections, the Monotone Class Theorem in the form 313Gb tells us that $\mathcal{A}$ must include the σ-subalgebra of $\mathcal{B}_k$ generated by $\mathcal{J}$. But this is the whole of $\mathcal{B}_k$, by 121J. So $\phi' = \phi$.

This shows that ϕ is unique.

(b)(i) Write $\mathcal{B}$ for the Borel σ-algebra of $\mathbb{R}$. Suppose $u_1,\dots,u_k \in L^0$. The function $F \to h^{-1}[F] : \mathcal{B} \to \mathcal{B}_k$ is a sequentially order-continuous Boolean homomorphism, so $F \mapsto [\![(u_1,\dots,u_k) \in h^{-1}[F]]\!]$ also is. By 364F, there is a unique member of L^0, which we may call $\bar h(u_1,\dots,u_k)$, such that $[\![h(u_1,\dots,u_k) \in F]\!] = [\![(u_1,\dots,u_k) \in h^{-1}[F]]\!]$ for every $F \in \mathcal{B}$.

(ii) In the present context, we have an alternative expression for $\bar h$, as follows. If $f_1,\dots,f_k \in \mathcal{L}^0$, then $g = h(f_1,\dots,f_k)$ belongs to $\mathcal{L}^0$ (121Kb). If $F \in \mathcal{B}$ and $u_i = f_i^\bullet$ for each i, then

$$\begin{aligned}[\![\bar h(u_1,\dots,u_k) \in F]\!] &= [\![(u_1,\dots,u_k) \in h^{-1}[F]]\!] \\ &= \{\omega : (f_1(\omega),\dots,f_k(\omega)) \in h^{-1}[F]\}^\bullet\end{aligned}$$

(by the construction in (a))

$$= \{\omega : g(\omega) \in F\}^\bullet = [\![g^\bullet \in F]\!].$$

So $\bar h(f_1^\bullet,\dots,f_k^\bullet) = g^\bullet = (h(f_1,\dots,f_k))^\bullet$.

(c) Take f_i, $g_i \in \mathcal{L}^0$ such that $f_i^\bullet = u_i$ and $g_i^\bullet = v_i$ for $1 \le i \le k$. Set $f = h(f_1,\dots,f_k)$ and $g = h(g_1,\dots,g_k)$. Then, using the expression for $\bar h$ in (b-ii),

$$\begin{aligned}[\![\bar h(u_1,\dots,u_k) = \bar h(v_1,\dots,v_k)]\!] &= [\![f^\bullet = g^\bullet]\!] = \{\omega : f(\omega) = g(\omega)\}^\bullet \\ &\supseteq (\bigcap_{1\le i\le k}\{\omega : f_i(\omega) = g_i(\omega)\})^\bullet \\ &= \inf_{1\le i\le k}\{\omega : f_i(\omega) = g_i(\omega)\}^\bullet = \inf_{1\le i\le k}[\![u_i = v_i]\!].\end{aligned}$$

(d) Take $U = (u_1,\dots,u_k) \in (L^0)^k$ and $\epsilon > 0$. Let $M \ge 0$ be such that $[\![|u_i| \ge M]\!]$ has measure at most ϵ for each i. Because $h\restriction[-M-1,M+1]^k$ must be uniformly continuous, there is a $\delta \in \,]0,1]$ such that $|h(y) - h(x)| \le \epsilon$ whenever x, $y \in [-M-1,M+1]^k$ and $y - x \in [-\delta,\delta]^k$.

Suppose that $V=(v_1,\dots,v_k)\in(L^0)^k$ is such that $\theta(v_i-u_i)\le\delta\epsilon$ for every $i\le k$. Let $f_1,\dots,f_k,g_1,\dots,g_k\in\mathcal{L}^0$ be such that $f_i^\bullet=u_i$ and $g_i^\bullet=v_i$ for each i. Then

$$\delta\mu\{\omega:|g_i(\omega)-f_i(\omega)|\ge\delta\}\le\mathbb{E}(\chi\Omega\wedge|g_i-f_i|)=\mathbb{E}(\chi1\wedge|v_i-u_i|)$$
$$=\theta(v_i-u_i)\le\delta\epsilon,$$

so $\{\omega:|g_i(\omega)-f_i(\omega)|\ge\delta\}$ has measure at most ϵ, for each i. Set

$$H=\bigcap_{1\le i\le k}\{\omega:|f_i(\omega)|\le M,\,|g_i(\omega)-f_i(\omega)|\le\delta\}.$$

Then $\mu H\ge1-2k\epsilon$. For $\omega\in H$, both $x=(f_1(\omega),\dots,f_k(\omega))$ and $y=(g_1(\omega),\dots,g_k(\omega))$ belong to $[-M-1,M+1]^k$ (because $\delta\le1$), and $y-x\in[-\delta,\delta]^k$, so $|h(x)-h(y)|\le\epsilon$. But this means that

$$\theta(\bar h(V)-\bar h(U))=\mathbb{E}(\chi\Omega\wedge|h(f_1,\dots,f_k)-h(g_1,\dots,g_k)|)$$
$$\le\epsilon+\mu(\Omega\setminus H)\le(2k+1)\epsilon.$$

As U and ϵ are arbitrary, $\bar h$ is continuous.

(e) To see that any other isomorphism $\mathfrak{A}\cong\Sigma'/\mathcal{N}'$, where $(\Omega',\Sigma',\mathcal{N}')$ is another measurable space with negligibles, would lead to the same operators $\bar h:(L^0)^k\to L^0$, note that $\phi E=[\![(u_1,\dots,u_k)\in E]\!]$, for $E\in\mathcal{B}_k$ and $u_1,\dots,u_k\in L^0$, is determined from the values $[\![u_i>\alpha]\!]$ for $1\le i\le k$ and $\alpha\in\mathbb{R}$, as shown in (a). Since (following 364A) each u_i is neither more nor less than the family $\langle[\![u_i>\alpha]\!]\rangle_{\alpha\in\mathbb{R}}$, these are necessarily independent of representations of $\mathfrak{A}$. Similarly, (b) above also provides a representation-invariant description of $\bar h$.

(f) Taking $f_1,\dots,f_k\in\mathcal{L}^0$ such that $f_i^\bullet=u_i$ for each i, we have $\langle h_n(f_1,\dots,f_k)\rangle_{n\in\mathbb{N}}\uparrow h(f_1,\dots,f_k)$ in $\mathcal{L}^0$, so $\langle\bar h_n(f_1,\dots,f_k)\rangle_{n\in\mathbb{N}}\uparrow\bar h(f_1,\dots,f_k)$ in L^0.

(g)(i) Set $\mathcal{E}=\{E:E\in\mathcal{B}_k,\,[\![(T_\phi u_1,\dots,T_\phi u_k)\in E]\!]=\phi[\![(u_1,\dots,u_k)\in E]\!]\}$. If $E_i\subseteq\mathbb{R}$ is a Borel set for $i\le k$, and $E=\prod_{1\le i\le k}E_i$, then

$$[\![(T_\phi u_1,\dots,T_\phi u_k)\in E]\!]=\inf_{1\le i\le k}[\![T_\phi u_i\in E_i]\!]=\inf_{1\le i\le k}\phi[\![u_i\in E_i]\!]$$

(612Af)

$$=\phi(\inf_{1\le i\le k}[\![u_i\in E_i]\!])=\phi[\![(u_1,\dots,u_k)\in E]\!]$$

and $E\in\mathcal{E}$. Because $E\mapsto[\![(u_1,\dots,u_k)\in E]\!]:\mathcal{B}_k\to\mathfrak{A}$, $E\mapsto[\![(T_\phi u_1,\dots,T_\phi u_k)\in E]\!]:\mathcal{B}_k\to\mathfrak{C}$ and $E\mapsto\phi[\![(u_1,\dots,u_k)\in E]\!]\}:\mathcal{E}\to\mathfrak{C}$ are all sequentially order-continuous Boolean homomorphisms, $\mathcal{E}$ is a Dynkin class. So the Monotone Class Theorem (136B) tells us that $\mathcal{E}$ includes the σ-subalgebra of $\mathcal{B}_k$ generated by products of Borel subsets of $\mathbb{R}$, and is the whole of $\mathcal{B}_k$, as claimed.

(ii) For any Borel set $E\subseteq\mathbb{R}$,

$$[\![\bar h(T_\phi u_1,\dots,T_\phi u_k)\in E]\!]=[\![(T_\phi u_1,\dots,T_\phi u_k)\in h^{-1}[E]]\!]$$

(612Ac)

$$=\phi[\![(u_1,\dots,u_k)\in h^{-1}[E]]\!]$$

((i) just above)

$$=\phi[\![\bar h(u_1,\dots,u_k)\in E]\!]=[\![T_\phi\bar h(u_1,\dots,u_k)\in E]\!];$$

as E is arbitrary, $\bar h(T_\phi u_1,\dots,T_\phi u_k)=T_\phi\bar h(u_1,\dots,u_k)$.

619F Definition Let $\boldsymbol{u}_1,\dots,\boldsymbol{u}_k$ be fully adapted processes defined on sublattices $\mathcal{S}_1,\dots,\mathcal{S}_k$ of $\mathcal{T}$ and $h:\mathbb{R}^k\to\mathbb{R}$ a Borel measurable function. Regarding $\boldsymbol{U}=(\boldsymbol{u}_1,\dots,\boldsymbol{u}_k)$ as the function $\sigma\mapsto(u_{1\sigma},\dots,u_{k\sigma}):\mathcal{S}\to(L^0)^k$, where $\boldsymbol{u}_i=\langle u_{i\sigma}\rangle_{\sigma\in\mathcal{S}_i}$ for each i and $\mathcal{S}=\bigcap_{1\le i\le k}\mathcal{S}_i$, we have a composition

$$\bar h\boldsymbol{U}=\langle\bar h(u_{1\sigma},\dots,u_{k\sigma})\rangle_{\sigma\in\mathcal{S}}.$$

619G Proposition Let $\boldsymbol{u}_1,\dots,\boldsymbol{u}_k$ be fully adapted processes all with the same domain $\mathcal{S}$, and $h:\mathbb{R}^k\to\mathbb{R}$ a Borel measurable function. Write $\boldsymbol{U}$ for $(\boldsymbol{u}_1,\dots,\boldsymbol{u}_k)$.

(a) $\bar h\boldsymbol{U}$ is fully adapted.

(b) If every $\boldsymbol{u}_i$ is order-bounded and h is **locally bounded**, that is, bounded on bounded subsets of $\mathbb{R}^k$, then $\bar h\boldsymbol{U}$ is order-bounded.

(c) If every $\boldsymbol{u}_i$ is (locally) moderately oscillatory and h is continuous, then $\bar h\boldsymbol{U}$ is (locally) moderately oscillatory.

(d) If every $\boldsymbol{u}_i$ is (locally) jump-free and h is continuous, then $\bar h\boldsymbol{U}$ is (locally) jump-free.

*(e) If $\boldsymbol{z}$ is a fully adapted process with domain $\mathcal{S}$ and $\boldsymbol{z}^2=\boldsymbol{z}$, then

(i) $\boldsymbol{z}\times\bar h(\boldsymbol{z}\times\boldsymbol{u}_1,\dots,\boldsymbol{z}\times\boldsymbol{u}_k)=\boldsymbol{z}\times\bar h\boldsymbol{U}$,

(ii) and if $h(0,\dots,0)=0$, then $\bar h(\boldsymbol{z}\times\boldsymbol{u}_1,\dots,\boldsymbol{z}\times\boldsymbol{u}_k)=\boldsymbol{z}\times\bar h\boldsymbol{U}$.

proof If $\mathcal{S}$ is empty, all these are trivially true; suppose that $\mathcal{S}\ne\emptyset$. Express each $\boldsymbol{u}_i$ as $\langle u_{i\sigma}\rangle_{\sigma\in\mathcal{S}}$ and set $U_\sigma=(u_{1\sigma},\dots,u_{k\sigma})$ for $\sigma\in\mathcal{S}$.

(a) If σ, $\tau\in\mathcal{S}$,

$$[\![\sigma=\tau]\!]\subseteq\inf_{1\le i\le k}[\![u_{i\sigma}=u_{i\tau}]\!]\subseteq[\![\bar h(u_{1\sigma},\dots,u_{k\sigma})=\bar h(u_{1\tau},\dots,u_{k\tau})]\!]$$

(619Ec)

$$=[\![\bar h(U_\sigma)=\bar h(U_\tau)]\!].$$

So $\bar h\boldsymbol{U}=\langle\bar h(U_\sigma)\rangle_{\sigma\in\mathcal{S}}$ is fully adapted.

(b) (Cf. 614F(c-i).) For $x\in\mathbb{R}^k$, set $g(x)=\sup_{\|y\|\le\|x\|}|h(y)|$. Then g is Borel measurable. If $\bar u_i=\sup_{\sigma\in\mathcal{S}}|u_{i\sigma}|$ for $1\le i\le k$, then $\bar g(\bar u_1,\dots,\bar u_k)$ is an upper bound for $\{\bar h(U_\sigma):\sigma\in\mathcal{S}\}$.

(c) (Cf. 615F(a-ii)) Write $\hat{\mathcal{S}}$ for the covered envelope of $\mathcal{S}$, and for $1\le i\le k$ let $\hat{\boldsymbol{u}}_i=\langle\hat u_{i\tau}\rangle_{\tau\in\hat{\mathcal{S}}}$ be the fully adapted extension of $\boldsymbol{u}_i$ to $\hat{\mathcal{S}}$. If every $\boldsymbol{u}_i$ is moderately oscillatory and $\langle\tau_n\rangle_{n\in\mathbb{N}}$ is a monotonic sequence in $\hat{\mathcal{S}}$, then $\lim_{n\to\infty}\hat u_{i\tau_n}$ exists for every i; since $\bar h:(L^0)^k\to L^0$ is continuous (619Ec), $\lim_{n\to\infty}\bar h(\hat U_{\tau_n})$ exists, where $\hat U_\tau=(\hat u_{1\tau},\dots,\hat u_{k\tau})$ for $\tau\in\hat{\mathcal{S}}$. By 615N, $\bar h\hat{\boldsymbol{U}}=\langle\bar h(\hat U_{i\tau})\rangle_{\tau\in\hat{\mathcal{S}}}$ is moderately oscillatory. But $\bar h\hat{\boldsymbol{U}}$ extends $\bar h\boldsymbol{U}$, so $\bar h\boldsymbol{U}$ is moderately oscillatory, by 615F(a-i).

The result for locally moderately oscillatory processes follows at once.

(d)(i) (Cf. 618Ga.) Suppose first that every $\boldsymbol{u}_i$ is jump-free. We know from (b) that $\bar h\boldsymbol{U}$ is order-bounded. Set

$$\bar w=\sup_{\sigma\in\mathcal{S}}\sup_{1\le i\le k}|u_{i\sigma}|$$

$$\bar w'=\sup_{\sigma\in\mathcal{S}}|\bar h(U_\sigma)|.$$

Let $\epsilon>0$. Let $M\ge0$ be such that $\bar\mu[\![\bar w\ge M]\!]\le\epsilon$, and $\delta>0$ such that $|h(y)-h(x)|\le\epsilon$ whenever x, $y\in[-M,M]^k$ and $x-y\in[-\delta,\delta]^k$. For $1\le i\le k$, let $I_i\in\mathcal{I}(\mathcal{S})$ be such that $\theta(\mathrm{Osclln}^*_{I_i}(\boldsymbol{u}_i))\le\epsilon\delta$; set

$$a=[\![\bar w\ge M]\!]\cup\sup_{1\le i\le k}[\![\mathrm{Osclln}^*_{I_i}(\boldsymbol{u}_i))\ge\delta]\!],$$

so that $\bar\mu a\le(k+1)\epsilon$. Let I be the sublattice generated by $\bigcup_{1\le i\le k}I_i$. Then $\mathrm{Osclln}^*_I(\bar h\boldsymbol{U})\le\epsilon\chi1+2\bar w'\times\chi a$. **P** Suppose that $J\in\mathcal{I}(\mathcal{S})$ and $J\supseteq I$. Take $e=c(\sigma,\tau)\in\mathrm{Sti}_0(J)$. Then

$$\begin{aligned}[\![|\bar h(U_\tau)-\bar h(U_\sigma)|>\epsilon]\!]&\subseteq[\![\bar w\ge M]\!]\cup\sup_{1\le i\le k}[\![|u_{i\tau}-u_{i\sigma}|\ge\delta]\!]\\&\subseteq[\![\bar w\ge M]\!]\cup\sup_{1\le i\le k}[\![\mathrm{Osclln}_J(\boldsymbol{u}_i)\ge\delta]\!]\\&\subseteq[\![\bar w\ge M]\!]\cup\sup_{1\le i\le k}[\![\mathrm{Osclln}^*_I(\boldsymbol{u}_i)\ge\delta]\!]=a,\end{aligned}$$

and

$$|\bar h(U_\tau)-\bar h(U_\sigma)|\le\epsilon\chi1+2\bar w'\times\chi a.$$

As e is arbitrary, $\operatorname{Osclln}^*_J(\bar{h}\boldsymbol{U}) \le \epsilon\chi 1 + 2\bar{w}' \times \chi a$; as J is arbitrary, $\operatorname{Osclln}^*_I(\bar{h}\boldsymbol{U}) \le \epsilon\chi 1 + 2\bar{w}' \times \chi a$. **Q**

It follows that

$$\theta(\operatorname{Osclln}^*_I(\bar{h}\boldsymbol{U})) \le \epsilon + \bar{\mu}a \le (k+2)\epsilon.$$

As ϵ is arbitrary, $\bar{h}\boldsymbol{U}$ is jump-free.

(ii) As in (c), it follows at once that if every $\boldsymbol{u}_i$ is locally jump-free, then $\bar{h}\boldsymbol{U}$ is locally jump-free.

(e)(i) Express $\boldsymbol{z}$ as $\langle z_\sigma\rangle_{\sigma\in\mathcal{S}}$. Setting $d_\sigma = [\![z_\sigma = 1]\!]$, we see that that $1 \setminus d_\sigma = [\![z_\sigma = 0]\!]$ for each $\sigma \in \mathcal{S}$. If $\sigma \in \mathcal{S}$,

$$\begin{aligned}
1 \setminus d_\sigma &\subseteq [\![\bar{h}(u_{1\sigma} \times z_\sigma, \dots, u_{k\sigma} \times z_\sigma) \times z_\sigma = 0]\!] \cap [\![\bar{h}(u_{1\sigma}, \dots, u_{k\sigma}) \times z_\sigma = 0]\!]\\
&\subseteq [\![\bar{h}(u_{1\sigma} \times z_\sigma, \dots, u_{k\sigma} \times z_\sigma) \times z_\sigma = \bar{h}(u_{1\sigma}, \dots, u_{k\sigma}) \times z_\sigma]\!],\\
d_\sigma &\subseteq \inf_{1\le i\le k} [\![u_{i\sigma} \times z_\sigma = u_{i\sigma}]\!]\\
&\subseteq [\![\bar{h}(u_{1\sigma} \times z_\sigma, \dots, u_{k\sigma} \times z_\sigma) = \bar{h}(u_{1\sigma}, \dots, u_{k\sigma})]\!].
\end{aligned}$$

So $\bar{h}(u_{1\sigma} \times z_\sigma, \dots, u_{k\sigma} \times z_\sigma) \times z_\sigma = \bar{h}(u_{1\sigma}, \dots, u_{k\sigma}) \times z_\sigma$; as σ is arbitrary, $\boldsymbol{z} \times \bar{h}(\boldsymbol{z} \times \boldsymbol{u}_1, \dots, \boldsymbol{z} \times \boldsymbol{u}_k) = \boldsymbol{z} \times \bar{h}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_k)$.

(ii) Now suppose in addition that $h(0, \dots, 0) = 0$. Then for each $\sigma \in \mathcal{S}$ we have

$$\begin{aligned}
1 \setminus d_\sigma &= [\![z_\sigma = 0]\!] \cap \inf_{1\le i\le k} [\![u_{i\sigma} \times z_\sigma = 0]\!]\\
&= [\![z_\sigma = 0]\!] \cap [\![\bar{h}(u_{1\sigma} \times z_\sigma, \dots, u_{k\sigma} \times z_\sigma) = 0]\!]\\
&\subseteq [\![\bar{h}(u_{1\sigma} \times z_\sigma, \dots, u_{k\sigma} \times z_\sigma) = \bar{h}(u_{1\sigma}, \dots, u_{k\sigma}) \times z_\sigma]\!]
\end{aligned}$$

and

$$\begin{aligned}
d_\sigma &\subseteq [\![z_\sigma = \chi 1]\!] \cap [\![\bar{h}(u_{1\sigma} \times z_\sigma, \dots, u_{k\sigma} \times z_\sigma) = \bar{h}(u_{1\sigma}, \dots, u_{k\sigma})]\!]\\
&\subseteq [\![\bar{h}(u_{1\sigma} \times z_\sigma, \dots, u_{k\sigma} \times z_\sigma) = \bar{h}(u_{1\sigma}, \dots, u_{k\sigma}) \times z_\sigma]\!],
\end{aligned}$$

so that $\bar{h}(\boldsymbol{z} \times \boldsymbol{u}_1, \dots, \boldsymbol{z} \times \boldsymbol{u}_k) = \boldsymbol{z} \times \bar{h}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_k)$.

619H Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $k \ge 1$ an integer and $h : \mathbb{R}^k \to \mathbb{R}$ a continuous function. Then $\bar{h} : M_{\text{o-b}}(\mathcal{S})^k \to M_{\text{o-b}}(\mathcal{S})$ is continuous when $M_{\text{o-b}}(\mathcal{S})$ is given the ucp topology and $M_{\text{o-b}}(\mathcal{S})^k$ the corresponding product topology.

proof (Cf. 615Ca.) Take $\boldsymbol{V} = (\boldsymbol{v}_1, \dots, \boldsymbol{v}_k) \in M^k_{\text{o-b}}$, where $M_{\text{o-b}} = M_{\text{o-b}}(\mathcal{S})$, and $\epsilon > 0$. Set $\bar{v} = \sup_{1\le i\le k} \sup|\boldsymbol{v}_i|$, and let $M \ge 0$ be such that $\bar{\mu}[\![\bar{v} > M]\!] \le \epsilon$. Let $\delta \in]0, 1]$ be such that $|h(x) - h(y)| \le \delta$ whenever $x, y \in \mathbb{R}^k$, $\|x\|_\infty \le M + 1$, $\|y\|_\infty \le M + 1$ and $\|x - y\|_\infty \le \delta$. Take any $\boldsymbol{U} = (\boldsymbol{u}_1, \dots, \boldsymbol{u}_k) \in M^k_{\text{o-b}}$ such that $\theta(\sup|\boldsymbol{u}_i - \boldsymbol{v}_i|) \le \epsilon\delta$ for every i. Set $\bar{u} = \sup_{1\le i\le k} \sup|\boldsymbol{u}_i - \boldsymbol{v}_i|$ and $\bar{w} = \sup|\bar{h}\boldsymbol{U} - \bar{h}\boldsymbol{V}|$. Then

$$\bar{\mu}[\![\bar{u} > \delta]\!] \le \textstyle\sum_{i=1}^k \bar{\mu}[\![\sup|\boldsymbol{u}_i - \boldsymbol{v}_i| > \delta]\!] \le k\epsilon,$$

so, expressing $\boldsymbol{u}_i$ as $\langle u_{i\sigma}\rangle_{\sigma\in\mathcal{S}}$ and $\boldsymbol{v}_i$ as $\langle v_{i\sigma}\rangle_{\sigma\in\mathcal{S}}$ for each i.

$$\begin{aligned}
[\![\bar{w} > \epsilon]\!] &= \sup_{\sigma\in\mathcal{S}} [\![|\bar{h}(u_{1\sigma}, \dots, u_{k\sigma}) - \bar{h}(v_{1\sigma}, \dots, v_{k\sigma})| > \epsilon]\!]\\
&\subseteq \sup_{\substack{1\le i\le k\\ \sigma\in\mathcal{S}}} [\![|v_{i\sigma}| > M]\!] \cup [\![|u_{i\sigma} - v_{i\sigma}| > \delta]\!] = [\![\bar{v} > M]\!] \cup [\![\bar{u} > \delta]\!]
\end{aligned}$$

has measure at most $(k+1)\epsilon$. and

$$\theta(\sup|\bar{h}\boldsymbol{U} - \bar{h}\boldsymbol{V}|) = \theta(\bar{w}) \le (k+2)\epsilon.$$

As $\boldsymbol{V}$ and ϵ are arbitrary, $\bar{h} : M^k_{\text{o-b}} \to M_{\text{o-b}}$ is continuous.

619I Theorem Let $h : \mathbb{R}^k \to \mathbb{R}$ be a differentiable function; write $h_1, \dots, h_k$ for its partial derivatives. Suppose that every h_i is Lipschitz on every bounded set in $\mathbb{R}^k$. Let $\boldsymbol{v}_1, \dots, \boldsymbol{v}_k$ be integrators, all with the same domain $\mathcal{S}$. Then $\bar{h}(\boldsymbol{v}_1, \dots, \boldsymbol{v}_k)$ is an integrator.

proof (a) Suppose that M, $K \ge 0$ are such that $|h_i(x) - h_i(y)| \le K\|x - y\|$ whenever $1 \le i \le k$ and x, $y \in [-M, M]^k$. Then

$$|h(y) - h(x) - \sum_{i=1}^k (\eta_i - \xi_i)h_i(x)| \le K\sqrt{k}\|y - x\|^2$$

whenever $x = (\xi_1, \dots, \xi_k)$, $y = (\eta_1, \dots, \eta_k) \in [-M, M]^k$. **P** Set $g(t) = h(ty + (1 - t)x)$ for $t \in \mathbb{R}$. Then g is differentiable with

$$g'(t) = \sum_{i=1}^k (\eta_i - \xi_i)h_i(ty + (1 - t)x)$$

for every t. By the Mean Value Theorem there is a $t \in [0, 1]$ such that $g(1) - g(0) = g'(t)$, that is, $h(y) - h(x) = \sum_{i=1}^k (\eta_i - \xi_i)h_i(z)$ where $z = ty + (1 - t)x$. So

$$\begin{aligned}|h(y) - h(x) - \sum_{i=1}^k (\eta_i - \xi_i)h_i(x)| &= |\sum_{i=1}^k (\eta_i - \xi_i)(h_i(z) - h_i(x))| \\ &\le \sqrt{\sum_{i=1}^k (\eta_i - \xi_i)^2}\sqrt{\sum_{i=1}^k (h_i(z) - h_i(x))^2} \\ &\le \sqrt{k}\|y - x\| \max_{1 \le i \le k} |h_i(z) - h_i(x)| \\ &\le K\sqrt{k}\|y - x\|\|z - x\| \le K\sqrt{k}\|y - x\|^2. \quad \mathbf{Q}\end{aligned}$$

(b) Consequently, using 619Ee or otherwise,

$$\begin{aligned}&|\bar{h}(w'_1, \dots, w'_k) - \bar{h}(w_1, \dots, w_k) - \sum_{i=1}^k (w'_i - w_i) \times \bar{h}_i(w_1, \dots, w_k)| \\ &\qquad \le K\sqrt{k}\sum_{i=1}^k |w'_i - w_i|^2\end{aligned}$$

whenever $w_1, \dots, w_k, w'_1, \dots, w'_k$ belong to L^∞ and $\|w_i\|_\infty \le M$, $\|w'_i\|_\infty \le M$ for every i.

If now I is a finite sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in I}$, $\boldsymbol{w}_i = \langle w_{i\sigma} \rangle_{\sigma \in I}$ are fully adapted processes with $\|\boldsymbol{w}_i\|_\infty \le M$ for $1 \le i \le k$, and we set $\boldsymbol{W} = (\boldsymbol{w}_1, \dots, \boldsymbol{w}_k)$, $W_\sigma = (w_{1\sigma}, \dots, w_{k\sigma})$ for $\sigma \in I$, then

$$\begin{aligned}&|\Delta_e(\boldsymbol{u}, d(\bar{h}\boldsymbol{W})) - \sum_{i=1}^k \Delta_e(\boldsymbol{u} \times \bar{h}_i\boldsymbol{W}, d\boldsymbol{w}_i)| \\ &\qquad = |u_\sigma \times \big((\bar{h}(W_\tau) - \bar{h}(W_\sigma)) - \sum_{i=1}^k \bar{h}_i(W_\sigma) \times (w_{i\tau} - w_{i\sigma})\big)| \\ &\qquad \le K\sqrt{k}|u_\sigma| \times \sum_{i=1}^k (w_{i\tau} - w_{i\sigma})^2 = K\sqrt{k}\sum_{i=1}^k \Delta_e(|\boldsymbol{u}|, (d\boldsymbol{w}_i)^2)\end{aligned}$$

whenever $\sigma \le \tau$ in I and e is the stopping-time interval $c(\sigma, \tau)$. Summing over the I-cells,

$$|S_I(\boldsymbol{u}, d(\bar{h}\boldsymbol{W})) - \sum_{i=1}^k S_I(\boldsymbol{u} \times \bar{h}_i\boldsymbol{W}, d\boldsymbol{w}_i)| \le K\sqrt{k}\sum_{i=1}^k S_I(|\boldsymbol{u}|, (d\boldsymbol{w}_i)^2).$$

(c) Now suppose that $\epsilon > 0$. Set $\bar{v} = \sup_{\sigma \in \mathcal{S}, 1 \le i \le k} |v_{i\sigma}|$ and let $M \ge 0$ be such that $\bar{\mu}c \le \epsilon$ where $c = [\![\bar{v} \ge M]\!]$. Set $\boldsymbol{w}_i = \text{med}(-M\mathbf{1}, \boldsymbol{v}_i, M\mathbf{1})$ for each i, so that $\|\boldsymbol{w}_i\|_\infty \le M$ and $[\![\boldsymbol{w}_i \ne \boldsymbol{v}_i]\!] \subseteq c$ for $1 \le i \le k$; write $\boldsymbol{V} = (\boldsymbol{v}_1, \dots, \boldsymbol{v}_k)$ and $\boldsymbol{W} = (\boldsymbol{w}_1, \dots, \boldsymbol{w}_k)$.

Let $K \ge 0$ be such that $h_i \restriction [-M, M]^k$ is K-Lipschitz for each i, and $M' > 0$ such that $|h_i(x)| \le M'$ whenever $x \in [-M.M]^k$ and $1 \le i \le k$; then $\sup |\frac{1}{M'}\bar{h}_i\boldsymbol{W}| \le \chi 1$ for each i. Take $\delta > 0$ such that

$$M'\theta(S_I(\boldsymbol{u}, d\boldsymbol{v}_i)) \le \epsilon, \quad K\sqrt{k}\,\theta(S_I(\boldsymbol{u}, (d\boldsymbol{w}_i))^2) \le \epsilon$$

whenever $1 \le i \le k$, $I \in \mathcal{I}(\mathcal{S})$, $\boldsymbol{u} \in M_{\text{fa}}(I)$ and $\theta(\sup |\boldsymbol{u}|) \le \delta$ (using 619B again).

Take a finite sublattice I of $\mathcal{S}$ and $\boldsymbol{u} \in M_{\mathrm{fa}}(I)$ such that $\theta(\boldsymbol{u}) \le \delta$. Using 613Lc repeatedly, we have

$$[\![S_I(\boldsymbol{u}, d(\bar{h}\boldsymbol{V})) \ne S_I(\boldsymbol{u}, d(\bar{h}\boldsymbol{W})]\!] \subseteq c,$$

$$[\![S_I(\boldsymbol{u} \times \bar{h}_i(\boldsymbol{W}), d\boldsymbol{v}_i) \ne S_I(\boldsymbol{u} \times \bar{h}_i\boldsymbol{W}, d\boldsymbol{w}_i)]\!] \subseteq c,$$

$$[\![S_I(|\boldsymbol{u}|, (d\boldsymbol{v}_i)^2) \ne S_I(|\boldsymbol{u}|, (d\boldsymbol{w}_i)^2)]\!] \subseteq c$$

for every i, so

$$\begin{aligned}
&[\![|S_I(\boldsymbol{u}, d(\bar{h}\boldsymbol{V})) - \textstyle\sum_{i=1}^k S_I(\boldsymbol{u} \times \bar{h}_i\boldsymbol{W}, d\boldsymbol{v}_i)| > K\sqrt{k}\textstyle\sum_{i=1}^k S_I(|\boldsymbol{u}|, (d\boldsymbol{v}_i)^2)]\!]\\
&\quad\subseteq c \cup [\![|S_I(\boldsymbol{u}, d(\bar{h}\boldsymbol{W})) - \textstyle\sum_{i=1}^k S_I(\boldsymbol{u} \times \bar{h}_i\boldsymbol{W}, d\boldsymbol{w}_i)| > K\sqrt{k}\textstyle\sum_{i=1}^k S_I(|\boldsymbol{u}|, (d\boldsymbol{w}_i)^2)]\!]\\
&\quad= c,
\end{aligned}$$

and

$$\begin{aligned}
\theta(S_I(\boldsymbol{u}, d(\bar{h}\boldsymbol{V}))) &\le \sum_{i=1}^k M'\theta(S_I(\boldsymbol{u} \times \frac{1}{M'}\bar{h}_i\boldsymbol{W}, d\boldsymbol{v}_i))\\
&\qquad\qquad + \sum_{i=1}^k K\sqrt{k}\theta(S_I(|\boldsymbol{u}|, (d\boldsymbol{v}_i)^2)) + \bar{\mu}c\\
&\le (1+2k)\epsilon
\end{aligned}$$

by the choice of δ, because $\theta(\sup|\boldsymbol{u} \times \frac{1}{M'}\bar{h}_i\boldsymbol{W}|) \le \theta(\sup|\boldsymbol{u}|) \le \delta$ for each i. And this is true for every finite sublattice I of $\mathcal{S}$.

As ϵ is arbitrary, $\bar{h}\boldsymbol{V}$ is an integrator.

619J Itô's Formula, third form Let $k \ge 1$ be an integer, and $h : \mathbb{R}^k \to \mathbb{R}$ a twice-differentiable function with continuous second derivative. Denote its first partial derivatives by $h_1, \dots, h_k$ and its second partial derivatives by $h_{11}, \dots, h_{kk}$. Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v}_1, \dots, \boldsymbol{v}_k$ jump-free integrators with domain $\mathcal{S}$; let $\boldsymbol{u}$ be a moderately oscillatory process with domain $\mathcal{S}$. Write $\boldsymbol{V} = (\boldsymbol{v}_1, \dots, \boldsymbol{v}_k)$. Then

$$\int_{\mathcal{S}} \boldsymbol{u}\, d(\bar{h}\boldsymbol{V}) = \textstyle\sum_{i=1}^k \int_{\mathcal{S}} \boldsymbol{u} \times \bar{h}_i\boldsymbol{V}\, d\boldsymbol{v}_i + \frac{1}{2}\sum_{i=1}^k \sum_{j=1}^k \int_{\mathcal{S}} \boldsymbol{u} \times \bar{h}_{ij}\boldsymbol{V}\, d[\boldsymbol{v}_i \overset{*}{|} \boldsymbol{v}_j].$$

proof We can follow exactly the same lines as in 619C-619D.

(a) We know that all the $\boldsymbol{v}_i$ and $\bar{h}_i\boldsymbol{V}$ and $\bar{h}_{ij}\boldsymbol{V}$ are fully adapted and moderately oscillatory (618Gb, 619Gc), while $\bar{h}\boldsymbol{V}$ and every $[\boldsymbol{v}_i \overset{*}{|} \boldsymbol{v}_j]$ is an integrator (619I, 617I); so all the integrals are well-defined. Moreover,

$$\int_{\mathcal{S}} \boldsymbol{u} \times \bar{h}_{ij}\boldsymbol{V}\, d[\boldsymbol{v}_i \overset{*}{|} \boldsymbol{v}_j] = \int_{\mathcal{S}} \boldsymbol{u} \times \bar{h}_{ij}\boldsymbol{V}\, d\boldsymbol{v}_i d\boldsymbol{v}_j$$

for $1 \le i, j \le k$ (617I, as before). Express each $\boldsymbol{v}_i$ as $\langle v_{i\sigma}\rangle_{\sigma\in\mathcal{S}}$, and set $V_\sigma = (v_{1\sigma}, \dots, v_{k\sigma})$ for $\sigma \in \mathcal{S}$.

(b) For the time being (down to the end of (d)) suppose that all the h_{ij} are uniformly continuous. Let $\epsilon > 0$. Let $\delta_0 > 0$ be such that $\theta(S_I(\boldsymbol{z}, (d\boldsymbol{v}_i)^2)) \le \epsilon$ whenever $1 \le i \le k$, $I \in \mathcal{I}(\mathcal{S})$, $\boldsymbol{z} \in M_{\mathrm{fa}}(I)$ and $\theta(\sup|\boldsymbol{z}|) \le \delta_0$. Then there is a $\delta > 0$ such that

$$\begin{aligned}
|h(y) - h(x) - \sum_{i=1}^k(\eta_i - \xi_i)h_i(x) - \frac{1}{2}\sum_{i=1}^k\sum_{j=1}^k(\eta_i - \xi_i)(\eta_j - \xi_j)h_{ij}(x)|&\\
\le \delta_0\sum_{i=1}^k(\eta_i - \xi_i)^2&
\end{aligned}$$

whenever $x = (\xi_1, \dots, \xi_k)$, $y = (\eta_1, \dots, \eta_k)$ belong to $\mathbb{R}^k$ and $y - x \in [-\delta, \delta]^k$.

P Take δ such that $|h_{ij}(y)-h_{ij}(x)|\le\frac{2\delta_0}{k}$ whenever $1\le i,j\le k$ and $y-x\in[-\delta,\delta]^k$. Now suppose that $x=(\xi_1,\dots,\xi_k)$, $y=(\eta_1,\dots,\eta_k)$ belong to $\mathbb{R}^k$ and $y-x\in[-\delta,\delta]^k$. Set $g(t)=h(ty+(1-t)x)$ for $t\in\mathbb{R}$. Then g is twice differentiable with

$$g'(t)=\sum_{i=1}^k(\eta_i-\xi_i)h_i(ty+(1-t)x),$$
$$g''(t)=\sum_{i=1}^k\sum_{j=1}^k(\eta_i-\xi_i)(\eta_j-\xi_j)h_{ij}(ty+(1-t)x)$$

for every t. By Taylor's theorem, there is a $t\in\,]0,1[$ such that

$$g(1)=g(0)+g'(0)+\tfrac{1}{2}g''(t).$$

Setting $z=ty+(1-t)x$, we get

$$|h(y)-h(x)-\sum_{i=1}^k(\eta_i-\xi_i)h_i(x)-\frac{1}{2}\sum_{i=1}^k\sum_{j=1}^k(\eta_i-\xi_i)(\eta_j-\xi_j)h_{ij}(x)|$$
$$=\frac{1}{2}|\sum_{i=1}^k\sum_{j=1}^k(\eta_i-\xi_i)(\eta_j-\xi_j)(h_{ij}(z)-h_{ij}(x))|\le\frac{\delta_0}{k}\sum_{i=1}^k\sum_{j=1}^k|\eta_i-\xi_i||\eta_j-\xi_j|$$

(because $\|z-x\|_\infty=t\|y-x\|_\infty\le\delta$)

$$\le\frac{\delta_0}{2k}\sum_{i=1}^k\sum_{j=1}^k((\eta_i-\xi_i)^2+(\eta_j-\xi_j)^2)=\delta_0\sum_{i=1}^k(\eta_i-\xi_i)^2,$$

as required. **Q**

It follows that if $w_1,\dots,w_k,w'_0,\dots,w'_k\in L^0$ then, setting $W=(w_1,\dots,w_k)$, $W'=(w'_1,\dots,w'_k)$,

$$\bar z=\bar h(W')-\bar h(W)-\sum_{i=1}^k(w'_i-w_i)\times\bar h_i(W)$$
$$-\frac{1}{2}\sum_{i=1}^k\sum_{j=1}^k(w'_i-w_i)\times(w'_j-w_j)\times\bar h_{ij}(W),$$

we shall have

$$[\![|\bar z|>\delta_0\textstyle\sum_{i=1}^k(w'_i-w_i)^2]\!]\subseteq\sup_{1\le i\le k}[\![|w'_i-w_i|>\delta]\!].$$

(c) For $1\le i\le k$ let $J_i\in\mathcal{I}(\mathcal{S})$ be such that $\theta(\mathrm{Osclln}^*_{J_i}(\boldsymbol{v}_i))\le\delta\epsilon$, and take J to be the sublattice generated by $\bigcup_{1\le i\le k}J_i$. Set $a=\sup_{1\le i\le k}[\![\mathrm{Osclln}^*_{J_i}(\boldsymbol{v}_i)\ge\delta]\!]$, so that $\bar\mu a\le k\epsilon$. If $I\in\mathcal{I}(\mathcal{S})$, set

$$\bar z_I=S_I(\mathbf{1},d(\bar h\boldsymbol{V}))-\textstyle\sum_{i=1}^k S_I(\bar h_i\boldsymbol{V},d\boldsymbol{v}_i)-\frac{1}{2}\sum_{i=1}^k\sum_{j=1}^k S_I(\bar h_{ij}\boldsymbol{V},d\boldsymbol{v}_id\boldsymbol{v}_j).$$

Then

$$[\![|\bar z_I|>\delta_0\textstyle\sum_{i=1}^k S_I(\mathbf{1},(d\boldsymbol{v}_i)^2)]\!]\subseteq a$$

whenever $J\subseteq I\in\mathcal{I}(\mathcal{S})$.

P If $e=c(\sigma,\tau)$ is an I-cell and we set

$$\bar z_e=\Delta_e(\mathbf{1},d(\bar h\boldsymbol{V}))-\textstyle\sum_{i=1}^k\Delta_e(\bar h_i\boldsymbol{V},d\boldsymbol{v}_i)-\frac{1}{2}\sum_{i=1}^k\sum_{j=1}^k\Delta_e(\bar h_{ij}\boldsymbol{V},d\boldsymbol{v}_id\boldsymbol{v}_j),$$

then

$$\bar z_e = \bar h(V_\tau) - \bar h(V_\sigma) - \sum_{i=1}^k \bar h_i(V_\sigma)\times(v_{i\sigma'} - v_{i\sigma})$$
$$- \frac{1}{2}\sum_{i=1}^k\sum_{j=1}^k \bar h_{ij}(V_\sigma)\times(v_{i\tau} - v_{i\sigma})\times(v_{j\tau} - v_{j\sigma})$$

and

$$\begin{aligned}[\![|\bar z_e| > \delta_0 \textstyle\sum_{i=1}^k \Delta_e(\mathbf{1},(d\boldsymbol{v}_i)^2)]\!] &= [\![|\bar z_e| > \delta_0 \textstyle\sum_{i=1}^k (v_{i\tau} - v_{i\sigma})^2]\!]\\ &\subseteq \sup_{1\le i\le k}[\![|v_{i\tau} - v_{i\sigma}| > \delta]\!] \subseteq \sup_{1\le i\le k}[\![\mathrm{Osclln}_I(\boldsymbol{v}_i) > \delta]\!]\\ &\subseteq \sup_{1\le i\le k}[\![\mathrm{Osclln}^*_J(\boldsymbol{v}_i) > \delta]\!] \subseteq a.\end{aligned}$$

Summing,

$$\begin{aligned}[\![|\bar z_I| > \delta_0 \textstyle\sum_{i=1}^k S_I(\mathbf{1},(d\boldsymbol{v}_i)^2)]\!] &\subseteq \sup_{e\in\mathrm{Sti}_0(I)}[\![|\bar z_e| > \delta_0 \textstyle\sum_{i=1}^k \Delta_e(\mathbf{1},(d\boldsymbol{v}_i)^2)]\!]\\ &\subseteq a. \ \mathbf{Q}\end{aligned}$$

Consequently

$$\theta(\bar z_I) \le \theta(\textstyle\sum_{i=1}^k S_I(\delta_0\mathbf{1},(d\boldsymbol{v}_i)^2)) + \bar\mu a \le 2k\epsilon$$

because $\theta(\sup|\delta_0\mathbf{1}|) = \theta(\delta_0\chi 1) \le \delta_0$. And this is true whenever $I \in \mathcal{I}(\mathcal{S})$ includes J.

(d) As ϵ is arbitrary,

$$\begin{aligned}0 = \lim_{I\uparrow\mathcal{I}(\mathcal{S})}\bar z_I &= \int_{\mathcal{S}} d(\bar h\boldsymbol{V}) - \sum_{i=1}^k\int_{\mathcal{S}}\bar h_i\boldsymbol{V}\,d\boldsymbol{v}_i - \frac{1}{2}\sum_{i=1}^k\sum_{j=1}^k\int_{\mathcal{S}}\bar h_{ij}\boldsymbol{V}\,d\boldsymbol{v}_i d\boldsymbol{v}_j\\ &= \int_{\mathcal{S}} d(\bar h\boldsymbol{V}) - \sum_{i=1}^k\int_{\mathcal{S}}\bar h_i\boldsymbol{V}\,d\boldsymbol{v}_i - \frac{1}{2}\sum_{i=1}^k\sum_{j=1}^k\int_{\mathcal{S}}\bar h_{ij}\boldsymbol{V}\,d[\boldsymbol{v}_i\overset{*}{|}\boldsymbol{v}_j],\end{aligned}$$

and the formula is valid when $\boldsymbol{u} = \mathbf{1}$ and the h_{ij} are all uniformly continuous.

(e) As for the case in which not all the h_{ij} are uniformly continuous, let $\bar v$ be an upper bound for $\{|v_{i\sigma}| : 1 \le i \le k,\ \sigma \in \mathcal{S}\}$. Take any $\epsilon > 0$. Let $M \ge 0$ be such that $b = [\![\bar v \ge M]\!]$ has measure at most ϵ. There is a smooth function $f : \mathbb{R}^k \to \mathbb{R}$ with compact support such that $f(x) = 1$ whenever $\|x\|_\infty \le M$. **P** Take a smooth function $f_0 : \mathbb{R} \to \mathbb{R}$ with compact support such that $f_0(\xi) = 1$ for every $\xi \in [-M, M]$, and set $f(\xi_1,\dots,\xi_k) = \prod_{i=1}^k f_0(\xi_i)$ for $\xi_1,\dots,\xi_k \in \mathbb{R}$. **Q** Now set $g = f \times h$. Writing g_i, g_{ij} for the partial derivatives of g, these are all uniformly continuous and

$$\int_{\mathcal{S}} d(\bar g\boldsymbol{V}) = \textstyle\sum_{i=1}^k\int_{\mathcal{S}}\bar g_i\boldsymbol{V}\,d\boldsymbol{v}_i + \frac{1}{2}\sum_{i=1}^k\sum_{j=1}^k\int_{\mathcal{S}}\bar g_{ij}\boldsymbol{V}\,d\boldsymbol{v}_i d\boldsymbol{v}_j.$$

At the same time, $g(x) = h(x)$, $g_i(x) = h_i(x)$ and $g_{ij}(x) = h_{ij}(x)$ whenever $1 \le i, j \le k$ and $x \in \left]-M, M\right[^k$, so

$$[\![\bar g(V_\sigma) \ne \bar h(V_\sigma)]\!] \subseteq \sup_{1\le i\le k}[\![|v_{i\sigma}| \ge M]\!] \subseteq b$$

for every $\sigma \in \mathcal{S}$, and similarly

$$[\![\bar g_i(V_\sigma) \ne \bar h_i(V_\sigma)]\!] \subseteq b, \quad [\![\bar g_{ij}(V_\sigma) \ne \bar h_{ij}(V_\sigma)]\!] \subseteq b$$

whenever $1 \le i, j \le k$ and $\sigma \in \mathcal{S}$. Consequently

$$[\![\int_{\mathcal{S}} d(\bar h\boldsymbol{V}) \ne \textstyle\sum_{i=1}^k\int_{\mathcal{S}}\bar h_i\boldsymbol{V}\,d\boldsymbol{v}_i + \frac{1}{2}\sum_{i=1}^k\sum_{j=1}^k\int_{\mathcal{S}}\bar h_{ij}\boldsymbol{V}\,d\boldsymbol{v}_i d\boldsymbol{v}_j]\!] \subseteq b$$

has measure at most ϵ. As ϵ is arbitrary,

$$\int_{\mathcal{S}} d(\bar{h}\boldsymbol{V}) = \sum_{i=1}^{k} \int_{\mathcal{S}} \bar{h}_i \boldsymbol{V}\, d\boldsymbol{v}_i + \frac{1}{2}\sum_{i=1}^{k} \sum_{j=1}^{k} \int_{\mathcal{S}} \bar{h}_{ij} \boldsymbol{V}\, d\boldsymbol{v}_i d\boldsymbol{v}_j$$

in this case also. Translating into terms of covariations, we have

$$\int_{\mathcal{S}} d(\bar{h}\boldsymbol{V}) = \sum_{i=1}^{k} \int_{\mathcal{S}} \bar{h}_i \boldsymbol{V}\, d\boldsymbol{v}_i + \frac{1}{2}\sum_{i=1}^{k} \sum_{j=1}^{k} \int_{\mathcal{S}} \bar{h}_{ij} \boldsymbol{V}\, d[\boldsymbol{v}_i \overset{*}{|} \boldsymbol{v}_j].$$

(f) This deals with the case $\boldsymbol{u} = \mathbf{1}$. For other $\boldsymbol{u}$, we can use the same method as in 619D, because the components of the formula can be translated into terms of the integrators

$$\bar{h}\boldsymbol{V}, \quad ii_{\boldsymbol{v}_i}(\bar{h}_i\boldsymbol{V}), \quad ii_{[\boldsymbol{v}_i \overset{*}{|} \boldsymbol{v}_j]}(\bar{h}_{ij}\boldsymbol{V}).$$

So we get the general result as before.

619K Applying this to $\mathcal{S} \wedge \tau$ for $\tau \in \mathcal{S}$, we get the following version.

Corollary Let $k \geq 1$ be an integer, and $h : \mathbb{R}^k \to \mathbb{R}$ a twice-differentiable function with continuous second derivative. Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ locally jump-free local integrators with domain $\mathcal{S}$; let $\boldsymbol{u}$ be a locally moderately oscillatory process with domain $\mathcal{S}$. Write $\boldsymbol{V} = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k)$. Then

$$ii_{\bar{h}\boldsymbol{V}}(\boldsymbol{u}) = \sum_{i=1}^{k} ii_{\boldsymbol{v}_i}(\boldsymbol{u} \times \bar{h}_i\boldsymbol{V}) + \frac{1}{2}\sum_{i=1}^{k} \sum_{j=1}^{k} ii_{[\boldsymbol{v}_i \overset{*}{|} \boldsymbol{v}_j]}(\boldsymbol{u} \times \bar{h}_{ij}\boldsymbol{V}).$$

619X Basic exercises (a) Suppose that h_1, h_2 are Borel measurable functions from $\mathbb{R}^k \to \mathbb{R}$. Show that if $u_1, \ldots, u_k \in L^0$ then $[\![\bar{h}_1(u_1, \ldots, u_k) \leq \bar{h}_2(u_1, \ldots, u_k)]\!] = [\![(u_1, \ldots, u_k) \in E]\!]$, where $E = \{x : x \in \mathbb{R}^k, h_1(x) \leq h_2(x)\}$.

(b) Suppose that $l \geq 1$, that $h_1, \ldots, h_l$ are Borel measurable functions from $\mathbb{R}^k$ to $\mathbb{R}$, that $h : \mathbb{R}^l \to \mathbb{R}$ is Borel measurable, and that $g(x) = h(h_1(x), \ldots, h_l(x))$ for $x \in \mathbb{R}^k$. Show that $\bar{g}(u) = \bar{h}(\bar{h}_1(u), \ldots, \bar{h}_l(u))$ for $u \in L^0$.

619Y Further exercises (a) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{v} = \langle v_\tau \rangle_{\tau \in \mathcal{S}}$ a jump-free integrator, and $\boldsymbol{v}^*$ its quadratic variation. Suppose that $G \subseteq \mathbb{R}$ is an open set such that $[\![v_\tau \in G]\!] = 1$ for every $\tau \in \mathcal{S}$, and $h : G \to \mathbb{R}$ is a twice-differentiable function with continuous second derivative. Show that

$$\int_{\mathcal{S}} \boldsymbol{u}\, d(\bar{h}\boldsymbol{v}) = \int_{\mathcal{S}} \boldsymbol{u} \times \bar{h}'\boldsymbol{v}\, d\boldsymbol{v} + \frac{1}{2}\int_{\mathcal{S}} \boldsymbol{u} \times \bar{h}''\boldsymbol{v}\, d\boldsymbol{v}^*$$

whenever $\boldsymbol{u}$ is a moderately oscillatory process with domain $\mathcal{S}$.

(b) Let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k$ be simple processes all with the same domain $\mathcal{S}$, and $h : \mathbb{R}^k \to \mathbb{R}$ a continuous function. Write $\boldsymbol{U}$ for $(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k)$. Show that $\bar{h}\boldsymbol{U}$ is simple.

(c)(i) Suppose that I is a finite sublattice of $\mathcal{T}$ and $\boldsymbol{v} \in M_{\mathrm{fa}}(I)$. Show that $S_I(\mathbf{1}, |(d\boldsymbol{v})^3|) \leq \mathrm{Osclln}_I(\boldsymbol{v}) \times S_I(\mathbf{1}, (d\boldsymbol{v})^2)$. (ii) Suppose that $\mathcal{S}$ is a sublattice of $\mathcal{T}$ and $\boldsymbol{v}$ is a jump-free integrator with domain $\mathcal{S}$. Show that $\int_{\mathcal{S}} \boldsymbol{u}\,(d\boldsymbol{v})^3 = 0$ for every order-bounded process $\boldsymbol{u}$ with domain $\mathcal{S}$.

619 Notes and comments In the ordinary theory of the Riemann-Stieltjes integral, we are well accustomed to the formula $h(v(b)) - h(v(a)) = \int_a^b h'(v)dv$ for continuous functions v of bounded variation. In 619C we see that we have a correction term, which would correspond to a term $\frac{1}{2}\int_a^b h''(v)(dv)^2$. In the real-variable case we expect the term $(dv)^2$ to count as zero, as it must do if $\int_a^b |dv|$ is finite and v is continuous. The point of stochastic integration is that we integrating with respect to integrators which are *not* of bounded variation (e.g., Brownian motion), but for which we can use $\int_{\mathcal{S}} (d\boldsymbol{v})^2 = \int_{\mathcal{S}} d\boldsymbol{v}^*$ to give a general formula which will be the basis of almost every application of the theory to jump-free processes. In the proof of 619C we see that this comes from the second-order Taylor expansion

$$h(\beta) \doteq h(\alpha) + (\beta - \alpha)h'(\alpha) + \frac{1}{2}(\beta - \alpha)^2 h''(\alpha)$$

just as the familiar identity comes from the first-order expansion

$$h(\beta) \simeq h(\alpha) + (\beta - \alpha)h'(\alpha).$$

The analysis is very much harder, and the welcome simplicity of the formula depends on concepts ('moderately oscillatory process', 'jump-free integrator' and 'quadratic variation') which are far from elementary. But in Chapter 65 we shall find that, taking proper care, we can go a long way with formal manipulations.

If you look back at 617H, you will see that it amounts to a special case of 619J: we use the formula

$$\int_{\mathcal{S}} d(\boldsymbol{v} \times \boldsymbol{w}) = \int_{\mathcal{S}} \boldsymbol{v}\, d\boldsymbol{w} + \int_{\mathcal{S}} \boldsymbol{w}\, d\boldsymbol{v} + \frac{1}{2}\Bigl(\int_{\mathcal{S}} d[\boldsymbol{v} \overset{*}{\restriction} \boldsymbol{w}] + \int_{\mathcal{S}} d[\boldsymbol{w} \overset{*}{\restriction} \boldsymbol{v}]\Bigr)$$

to tell us what $[\boldsymbol{v} \overset{*}{\restriction} \boldsymbol{w}]$ should be, at least if covariation is to be symmetric. What is remarkable is that this approach extends so dramatically from multiplication to general twice-continuously-differentiable functions. In 619Yc I offer a sort of explanation of the fact that we don't have to examine third derivatives in this context.

You can see why most textbooks leave the proof of 619J to their readers. I have to admit that my formulation introduces extra obstacles in the proofs of 619I and 619J, where we need to turn inequalities concerning a function $h : \mathbb{R}^k \to \mathbb{R}$ into corresponding inequalities for $\bar{h} : (L^0)^k \to L^0$. If we allow ourselves to think of $\bar{h}$ as being defined by the formula $\bar{h}(f_1^\bullet, \dots , f_k^\bullet) = h(f_1, \dots , f_k)^\bullet$, as in 619Ee, this step becomes elementary. In §364 I went to a good deal of trouble to describe the f-algebra structure of L^0 in terms which did not depend on representations of this kind. In the context of the present volume this self-denial seems unnecessary, since in the great majority of applications our probability algebra is explicitly derived from some probability space. I hope however that even if you are impatient with measure algebras, and have been more or less frankly translating everything into expressions concerning conventional stochastic processes (as in 612H and elsewhere), you will agree that it is important to remember that trivial changes to the probability space can't affect the results we are looking at, and that it might be helpful to remember that 'trivial' can, for many of our purposes here, be interpreted as 'not affecting the measure algebra'. But to safely continue with this assumption, we have to check from time to time that our manipulations really are invariants of the measure algebras. And this is what I am trying to do in 619E-619G.

For those interested in the logical status of the theorems here, which in my view ought to include those concerned with their application to questions arising in the real world, there is another reason to have qualms about the proof of 619E given here: it depends on the Loomis-Sikorski theorem 314M, which requires a strong form of the axiom of choice. But there are no new problems beyond those already considered in §364 and Chapter 56.

Of course there is a more important problem facing us. I have offered no effective means of calculating $\boldsymbol{v}^*$ for any of the important processes $\boldsymbol{v}$ of the theory. In 617O I looked at the identity process, which is almost trivial, and the Poisson process, which is irrelevant because it's not locally jump-free. We don't seem to be ready for Brownian motion (624F), and this will be one of the aims of the next chapter.

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