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Appendix to Volume 5

Useful facts

For this volume, the most substantial ideas demanded are, naturally enough, in set theory. Fragments of general set theory are in §5A1, with cardinal arithmetic and infinitary combinatorics. §5A2 contains results from Shelah's pcf theory, restricted to those which are actually used in this book. §5A3 describes the language I will use when I discuss forcing constructions; in essence, I follow Kunen 80, but with some variations which need to be signalled.

As usual, some bits of general topology are needed; I give these in §5A4, starting with a list of cardinal functions to complement the definitions in §511. There is a tiny piece of real analysis in §5A5. In §5A6 are notes on a few undecidable propositions, mostly standard.

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5A1 Set theory

As usual, I begin with set theory, continuing from §§2A1 and 4A1. I start with definitions and elementary remarks filling some minor gaps in the deliberately sketchy accounts in the earlier volumes (5A1A-5A1E). I give a relatively solid paragraph on cardinal arithmetic (5A1F), including an account of cofinalities of ideals $[\kappa]^{\le\lambda}$. 5A1H-5A1K are devoted to infinitary combinatorics, with the Erdős-Rado theorem and Hajnal's Free Set Theorem. 5A1L-5A1P deal with the existence of 'transversals' of various kinds in spaces of functions, that is, large sets of functions which are well separated on some combinatorial criterion. 5A1Q is a fragment of finite combinatorics, 5A1R-5A1S introduce 'stationary families' of sets and 5A1T is a remarkable property of the ordering of ω_1.

5A1A Order types (a) If X is a well-ordered set, its **order type** $\operatorname{otp} X$ is the ordinal order-isomorphic to X.

If S is a set of ordinals, an ordinal-valued function f with domain S is **regressive** if $f(\xi) < \xi$ for every $\xi \in S$.

(b) The non-stationary ideal on a cardinal κ of uncountable cofinality is $(\operatorname{cf}\kappa)$-additive.

(c) If κ is a cardinal, $\lambda < \operatorname{cf}\kappa$ is an infinite regular cardinal and $C \subseteq \kappa$ is a closed cofinal set, then $S = \{\xi : \xi < \kappa, \operatorname{cf}(\xi \cap C) = \lambda\}$ is stationary in κ.

(d) If α is an ordinal and $C \subseteq \alpha$ has closure $\overline{C}$ for the order topology of α, then $\#(\overline{C}) = \#(C)$.

(e) If α is an ordinal, there is a closed cofinal set $C \subseteq \alpha$ such that $\operatorname{otp} C = \operatorname{cf}\alpha$.

5A1B Ordinal arithmetic (a) For ordinals ξ, η their **ordinal sum** $\xi + \eta$ is defined inductively by saying that

$\xi + 0 = \xi$,

$\xi + (\eta + 1) = (\xi + \eta) + 1$,

$\xi + \eta = \sup_{\zeta<\eta} \xi + \zeta$ for non-zero limit ordinals η.

Ordinal addition is associative, and if $\xi \le \zeta < \xi + \eta$ there is a unique $\zeta' < \eta$ such that $\zeta = \xi + \zeta'$. If we identify $\mathbb{N}$ with ω, then the ordinal sum of two finite ordinals corresponds to ordinary addition on $\mathbb{N}$.

(b) For ordinals ξ, η their **ordinal product** $\xi \cdot \eta$ is defined inductively by saying

$\xi \cdot 0 = 0$,

$\xi \cdot (\eta + 1)$ is the ordinal sum $\xi \cdot \eta + \xi$,

$\xi \cdot \eta = \sup_{\zeta < \eta} \xi \cdot \zeta$ for non-zero limit ordinals η.

$0 \cdot \eta = 0$ and $1 \cdot \eta = \eta$ for every η, and $\sup_{\zeta \in A} \xi \cdot \zeta = \xi \cdot (\sup A)$ for every ξ and every non-empty set A of ordinals. Ordinal multiplication is associative.

(c) For ordinals ξ, η the **ordinal power** ξ^η is defined inductively by saying that

$\xi^0 = 1$,

$\xi^{\eta+1}$ is the ordinal product $\xi^\eta \cdot \xi$,

$\xi^\eta = \sup_{\zeta < \eta} \xi^\zeta$ for non-zero limit ordinals η.

If ξ, η are ordinals, $\eta \neq 0$ and η is greater than or equal to the ordinal product $\xi \cdot \eta$, then η is at least the ordinal power ξ^ω.

5A1C Concatenation Suppose that σ, τ are two functions with domains α, β respectively which are ordinals. Then we can form their **concatenation** $\sigma^\frown\tau$, setting

$$\mathrm{dom}(\sigma^\frown\tau) = \alpha + \beta$$

(the ordinal sum),

$$(\sigma^\frown\tau)(\xi) = \sigma(\xi) \text{ if } \xi < \alpha,$$
$$(\sigma^\frown\tau)(\alpha + \eta) = \tau(\eta) \text{ if } \eta < \beta.$$

The operator $^\frown$ is associative, so we can omit brackets and speak of $\sigma^\frown\tau^\frown\upsilon$. The empty function $\emptyset$ is an identity in the sense that

$$\emptyset^\frown\sigma = \sigma^\frown\emptyset = \sigma$$

whenever $\mathrm{dom}\,\sigma$ is an ordinal.

In this context, it will often be helpful to have a special notation for functions with domain the singleton set $\{0\} = 1$; I will write $<t>$ for the function with domain $\{0\}$ and value t.

If $\langle\sigma_n\rangle_{n\in\mathbb{N}}$ is a sequence of functions with ordinal domains, we can form the concatenations

$$\sigma_0^\frown\sigma_1, \quad \sigma_0^\frown\sigma_1^\frown\sigma_2, \quad \sigma_0^\frown\sigma_1^\frown\sigma_2^\frown\sigma_3, \quad \ldots$$

to get a sequence of functions each extending its predecessors. The union will be a function with domain the ordinal $\sup_{n\in\mathbb{N}} \mathrm{dom}\,\sigma_0 + \ldots + \mathrm{dom}\,\sigma_n$. I will generally denote it $\sigma_0^\frown\sigma_1^\frown\sigma_2^\frown\ldots$ or in some similar form.

5A1D Well-founded sets(a) A partially ordered set P is **well-founded** if every non-empty $A \subseteq P$ has a minimal element.

(b) If P is a well-founded partially ordered set, we have a rank function $r : P \to \mathrm{On}$ defined by saying that

$$r(p) = \sup\{r(q) + 1 : q < p\}$$

for every $p \in P$. The **height** of P is the least ordinal ζ such that $r(p) < \zeta$ for every $p \in P$.

(c) A partially ordered set P is well-founded iff there is no sequence $\langle p_n\rangle_{n\in\mathbb{N}}$ in P such that $p_{n+1} < p_n$ for every $n \in \mathbb{N}$.

(d) If P is a well-founded partially ordered set with height ζ, $\#(\zeta) \le \#(P)$.

(e) If X is a Polish space and $\le$ is a well-founded relation on X such that $\{(x, y) : x < y\}$ is analytic, then the height of $\le$ is countable.

5A1E Trees (a) A **tree** is a partially ordered set T such that $\{s : s \in T, s \le t\}$ is well-ordered for every $t \in T$. T has a rank function $r : T \to \mathrm{On}$ defined by saying that

$$r(t) = \operatorname{otp}\{s : s < t\} = \min\{\xi : r(s) < \xi \text{ whenever } s < t\}$$

for every $t \in T$.

The **levels** of T are now the sets $\{t : r(t) = \xi\}$ for $\xi \in \mathrm{On}$. A **branch** of T is a maximal totally ordered subset. A tree is **well-pruned** if it has at most one minimal element and whenever $s, t \in T$ and $r(s) < r(t)$, there is an $s' \geq s$ such that $r(s') = r(t)$. If T is a tree, a **subtree** of T is a set $T' \subseteq T$ such that $s \in T'$ whenever $s \leq t \in T'$; in this case, the rank function of T' is the restriction to T' of the rank function of T.

(b)(i) Let T be a tree in which every level is finite. Then T has a branch meeting every level.

(ii) Let $(T, \preccurlyeq')$ be a tree of height ω_1 in which every level is countable. Then there is an ordering $\preccurlyeq$ of ω_1, included in the usual ordering $\leq$ of ω_1, such that $(T, \preccurlyeq')$ is isomorphic to $(\omega_1, \preccurlyeq)$.

(c) An **Aronszajn tree** is a tree T of height ω_1 in which every branch and every level is countable. An Aronszajn tree T is **special** if it is expressible as $\bigcup_{n\in\mathbb{N}} A_n$ where no two elements of any A_n are comparable.

(d)(i) A **Souslin tree** is a tree T of height ω_1 in which every branch and every up-antichain is countable.

(ii) Every Souslin tree is a non-special Aronszajn tree.

(iii) If T is a Souslin tree, it has a subtree which is a well-pruned Souslin tree.

(iv) **Souslin's hypothesis** is the assertion

(SH) There are no Souslin trees.

5A1F Cardinal arithmetic(a)(i) An infinite cardinal which is not regular is **singular**. A cardinal κ is a **successor cardinal** if it is of the form λ^+; otherwise it is a **limit cardinal**. κ is a **strong limit cardinal** if it is uncountable and $2^\lambda < \kappa$ for every $\lambda < \kappa$. It is **weakly inaccessible** if it is a regular uncountable limit cardinal; it is **strongly inaccessible** if moreover it is a strong limit cardinal.

(ii) If κ is a cardinal, define $\kappa^{(+\xi)}$, for ordinals ξ, by setting

$$\kappa^{(+0)} = \kappa, \quad \kappa^{(+\xi)} = \sup_{\eta<\xi}(\kappa^{(+\eta)+}) \text{ if } \xi > 0,$$

that is, $\kappa^{(+\xi)} = \omega_{\zeta+\xi}$ if $\kappa = \omega_\zeta$.

(b)(i) If $\langle\kappa_i\rangle_{i\in I}$ is a family of cardinals, its **cardinal sum** is $\#(\{(i,\xi) : i \in I,\ \xi < \kappa_i\})$, which is at most $\max(\omega, \#(I), \sup_{i\in I}\kappa_i)$.

(ii) For cardinals κ and λ, the **cardinal product** $\kappa\cdot\lambda$ is $\#(\kappa\times\lambda) \leq \max(\omega,\kappa,\lambda)$.

(iii) If κ and λ are cardinals there are two natural interpretations of the formula κ^λ: (i) the set of functions from λ to κ (ii) the cardinal of this set. In this volume the latter will be the usual one, but I will try to signal this by using the phrase **cardinal power**. 2^λ is always the cardinal power; the corresponding set of functions will be denoted by $\{0,1\}^\lambda$.

(c)(i) The cardinal power κ^λ is at most $2^{\max(\omega,\kappa,\lambda)}$ for any cardinals κ and λ.

(ii) $\mathfrak{c}^\omega = \mathfrak{c}$.

(d) $\operatorname{cf} 2^\kappa > \kappa$ for every infinite cardinal κ.

(e)(i) If κ and λ are infinite cardinals, then

$$\begin{aligned}\operatorname{cf}[\kappa]^{\leq\lambda} &= 1 \text{ if } \lambda \geq \kappa,\\ &\geq \kappa \text{ if } \lambda < \kappa.\end{aligned}$$

(ii) Let κ, λ and θ be infinite cardinals such that $\theta \leq \lambda \leq \kappa$. Then $\operatorname{cf}[\kappa]^{\leq\theta} \leq \max(\operatorname{cf}[\kappa]^{\leq\lambda}, \operatorname{cf}[\lambda]^{\leq\theta})$.

(iii) Let κ and λ be infinite cardinals. Then the cardinal power κ^λ is $\max(2^\lambda, \operatorname{cf}[\kappa]^{\leq\lambda})$.

$$\operatorname{cf}[\mathfrak{c}]^{\leq\omega} = \mathfrak{c}.$$

(iv) If λ is an infinite cardinal and $\lambda \le \kappa < \lambda^{(+\omega)}$, then $\mathrm{cf}[\kappa]^{\le\omega} \le \max(\kappa, \mathrm{cf}[\lambda]^{\le\omega})$, with equality if $\kappa > \omega$. Consequently the cardinal power κ^ω is $\max(\kappa, \lambda^\omega)$.

In particular, if $\omega_1 \le \kappa < \omega_\omega$ then $\mathrm{cf}[\kappa]^{\le\omega} = \kappa$ and $\kappa^\omega = \max(\mathfrak{c}, \kappa)$. $(\mathfrak{c}^+)^\omega = \mathfrak{c}^+$, $(\mathfrak{c}^{++})^\omega = \mathfrak{c}^{++}$.

(v) If κ is a singular infinite cardinal, then $\mathrm{cf}([\kappa]^{\le \mathrm{cf}\,\kappa}) > \kappa$.

(f) If λ is a regular uncountable cardinal, $\theta \ge 2$ is a cardinal and $\kappa = \sup_{\delta<\lambda} \theta^\delta$, where θ^δ is the cardinal power, then

$$\#([\kappa]^{<\lambda}) = \kappa.$$

In particular, if κ is strongly inaccessible then $\kappa^\delta \le \kappa$ for every $\delta < \kappa$.

(g) Let X, Y and Z be sets, with $\#(X) \le 2^{\#(Z)}$ and $0 < \#(Y) \le \#(Z)$. Then there is a function $f : X \times Z^{\mathbb{N}} \to Y$ such that whenever $\langle x_n \rangle_{n\in\mathbb{N}}$ is a sequence of distinct elements of X and $\langle y_n \rangle_{n\in\mathbb{N}}$ is a sequence in Y there is a $z \in Z^{\mathbb{N}}$ such that $f(x_n, z) = y_n$ for every $n \in \mathbb{N}$.

(h) If κ is an infinite cardinal, then 2^κ is at most the cardinal power $(\sup_{\lambda<\kappa} 2^\lambda)^{\mathrm{cf}\,\kappa}$.

If $\omega \le \lambda < \kappa$ and $2^\theta = 2^\lambda$ for $\lambda \le \theta < \kappa$ but $2^\kappa > 2^\lambda$ then κ is regular.

5A1G Three fairly simple facts (a) There is a family $\langle a_I \rangle_{I\subseteq\mathbb{N}}$ of infinite subsets of $\mathbb{N}$ such that $a_I \cap a_J$ is finite whenever I, $J \subseteq \mathbb{N}$ are distinct.

(b) Let X be a set, $f : [X]^{<\omega} \to [X]^{\le\omega}$ a function, and $Y \subseteq X$. Then there is a $Z \subseteq X$ such that $Y \subseteq Z$, $f(I) \subseteq Z$ for every $I \in [Z]^{<\omega}$, and $\#(Z) \le \max(\omega, \#(Y))$.

(c) Let $\kappa \ge \mathfrak{c}$ be a cardinal and $\mathcal{A}$ a family of countable subsets of κ such that $\#(\mathcal{A})$ is less than the cardinal power κ^ω. Then there is a countably infinite $K \subseteq \kappa$ such that $I \cap K$ is finite for every $I \in \mathcal{A}$.

5A1H Partition calculus (a) The Erdős-Rado theorem Let κ be an infinite cardinal. Set $\kappa_1 = \kappa$, $\kappa_{n+1} = 2^{\kappa_n}$ for $n \ge 1$. If $n \ge 1$, $\#(B) \le \kappa$, $\#(A) > \kappa_n$ and $f : [A]^n \to B$ is a function, there is a $C \in [A]^{\kappa^+}$ such that f is constant on $[C]^n$.

(b) Let κ be a cardinal of uncountable cofinality, and $Q \subseteq [\kappa]^2$. Then *either* there is a stationary $A \subseteq \kappa$ such that $[A]^2 \subseteq Q$ *or* there is an infinite closed $B \subseteq \kappa$ such that $[B]^2 \cap Q = \emptyset$.

5A1I Δ-systems and free sets: Proposition Let κ and λ be infinite cardinals and $\langle I_\xi \rangle_{\xi<\kappa}$ a family of sets with cardinal less than λ.

(a) If $\mathrm{cf}\,\kappa > \lambda$, there are a $\Gamma \in [\kappa]^\kappa$ and a set J of cardinal less than κ such that $I_\xi \cap I_\eta \subseteq J$ for all distinct ξ, $\eta \in \Gamma$.

(b) If $\kappa > \lambda$ is regular and the cardinal power θ^δ is less than κ whenever $\theta < \kappa$ and $\delta < \lambda$, then there is a $\Gamma' \in [\kappa]^\kappa$ such that $\langle I_\xi \rangle_{\xi\in\Gamma'}$ is a Δ-system.

(c) If $\kappa > \lambda$ there is a $\Gamma'' \in [\kappa]^\kappa$ such that $\eta \notin I_\xi$ for any distinct ξ, $\eta \in \Gamma''$.

5A1J Remarks (a) Let κ be an infinite cardinal and $\langle I_\xi \rangle_{\xi<\kappa}$ a family of countable sets.

(i) If $\mathrm{cf}\,\kappa \ge \omega_2$, there are a $\Gamma \in [\kappa]^\kappa$ and a set J with cardinal less than κ such that $I_\xi \cap I_\eta \subseteq J$ for all distinct ξ, $\eta \in \Gamma$.

(ii) If κ is regular and the cardinal power λ^ω is less than κ for every $\lambda < \kappa$, there is a $\Gamma' \in [\kappa]^\kappa$ such that $\langle I_\xi \rangle_{\xi\in\Gamma'}$ is a Δ-system.

(iii) If $\kappa \ge \omega_2$ there is a $\Gamma'' \in [\kappa]^\kappa$ such that $\eta \notin I_\xi$ for any distinct ξ, $\eta \in \Gamma''$.

(b) Let λ be an infinite cardinal. Then there is a κ_0 such that for every cardinal $\kappa \ge \kappa_0$, every $n \in \mathbb{N}$ and every function $f : [\kappa]^n \to [\kappa]^{<\lambda}$ there is an $A \in [\kappa]^{\lambda^+}$ such that $\xi \notin f(I)$ whenever $I \in [A]^n$ and $\xi \in A \setminus I$.

(c) If $n \in \mathbb{N}$ and $\langle K_i \rangle_{i\in\mathbb{N}}$ is a sequence in $[\mathbb{N}]^{\le n}$, there is an infinite $\Gamma \subseteq \mathbb{N}$ such that $\langle K_i \rangle_{i\in\Gamma}$ is a Δ-system.

(d) If $R \subseteq X \times X$ is an equivalence relation on a set X I will say that a set $A \subseteq X$ is ***R*-free** if A meets each equivalence class for R in at most one point.

(e) Let X be a set and R an equivalence relation on X.

(i) For any cardinal κ, there is a partition $\langle X_\xi\rangle_{\xi<\kappa}$ of X into R-free sets iff every R-equivalence class has cardinal at most κ.

(ii) If $A \subseteq X$ is R-free then $R[B] \cap R[C] = \emptyset$ whenever B, $C \subseteq A$ are disjoint.

5A1K Lemma Suppose that θ, λ and κ are cardinals, with $\theta < \lambda < \operatorname{cf}\kappa$, and that S is a stationary subset of κ. Let $\langle I_\xi\rangle_{\xi\in S}$ be a family in $[\lambda]^{\le\theta}$. Then there is a set $M \subseteq \lambda$ such that $\operatorname{cf}(\#(M)) \le \theta$ and $\{\xi : \xi \in S,\, I_\xi \subseteq M\}$ is stationary in κ.

5A1L Lemma Let $\langle X_i\rangle_{i\in I}$ be a non-empty family of infinite sets, with product X. Then there is a set $Y \subseteq X$, with $\#(Y) = \#(X)$, such that for every finite $L \subseteq Y$ there is an $i \in I$ such that $x(i) \ne y(i)$ for any distinct x, $y \in L$.

5A1M Definitions(a) Let X and Y be sets and $\mathcal{I}$ an ideal of subsets of X. Write $\operatorname{Tr}_{\mathcal{I}}(X;Y)$ for the **transversal number**

$$\sup\{\#(F) : F \subseteq Y^X,\ \{x : f(x) = g(x)\} \in \mathcal{I} \text{ for all distinct } f,\, g \in F\}.$$

(b) Let κ be a cardinal. Write $\operatorname{Tr}(\kappa)$ for

$$\operatorname{Tr}_{[\kappa]^{<\kappa}}(\kappa;\kappa) = \sup\{\#(F) : F \subseteq \kappa^\kappa,\ \#(f \cap g) < \kappa \text{ for all distinct } f,\, g \in F\}.$$

5A1N Lemma (a) For any infinite cardinal κ,

$$\kappa^+ \le \operatorname{Tr}(\kappa) \le 2^\kappa.$$

(b) For any infinite cardinal κ,

$$\max(\operatorname{Tr}(\kappa), \sup_{\delta<\kappa} 2^\delta) \ge \min(2^\kappa, \kappa^{(+\omega)}).$$

(c) If κ is such that $2^\delta \le \kappa$ for every $\delta < \kappa$, then $\operatorname{Tr}(\kappa) = 2^\kappa$, and in fact there is an $F \subseteq \kappa^\kappa$ such that $\#(F) = 2^\kappa$ and $\#(f \cap g) < \kappa$ for all distinct f, $g \in F$.

(d) If X and Y are sets and $\mathcal{I}$ is a maximal proper ideal of $\mathcal{P}X$, then there is an $F \subseteq Y^X$ such that $\#(F) = \operatorname{Tr}_{\mathcal{I}}(X;Y)$ and $\{x : f(x) = g(x)\} \in \mathcal{I}$ for all distinct f, $g \in F$.

5A1O Almost-square-sequences: Lemma Let λ, κ be regular infinite cardinals, with $\kappa > \max(\omega_1, \lambda)$. Then we can find a stationary set $S \subseteq \kappa^+$ and a family $\langle C_\alpha\rangle_{\alpha\in S}$ of sets such that

(i) for each $\alpha \in S$, C_α is a closed cofinal set in α of order type λ;

(ii) if α, $\beta \in S$ and γ is a limit point of both C_α and C_β then $C_\alpha \cap \gamma = C_\beta \cap \gamma$.

5A1P Corollary Let κ, λ be regular infinite cardinals with $\lambda > \max(\omega_1, \kappa)$. Then we can find a stationary subset S of λ^+ and a family $\langle g_\alpha\rangle_{\alpha\in S}$ of functions from κ to λ^+ such that, for all distinct α, $\beta \in S$,

(i) $g_\alpha[\kappa] \subseteq \alpha$,

(ii) $\#(g_\alpha \cap g_\beta) < \kappa$,

(iii) if $\theta < \kappa$ is a limit ordinal and $g_\alpha(\theta) = g_\beta(\theta)$ then $g_\alpha\restriction\theta = g_\beta\restriction\theta$.

5A1Q Lemma Let I and J be non-empty finite sets, and $R \subseteq I \times J$ a relation such that $R[I] = J$. Set

$$k = \max_{x\in I} \#(R[\{x\}]), \quad l = \min_{y\in J} \#(R^{-1}[\{y\}]).$$

Then there is a $K \subseteq I$ such that $R[K] = J$ and $\#(K) \le \frac{1+\ln k}{l}\#(I)$.

5A1R Stationary families of sets: Definition If I is a set and $\mathcal{A}$ is a family of sets, I will say that $\mathcal{A}$ is **stationary over** I if for every function $f : [I]^{<\omega} \to [I]^{\le\omega}$ there is an $A \in \mathcal{A}$ such that $f(J) \subseteq A$ for every $J \in [A \cap I]^{<\omega}$.

5A1S Elementary remarks (a) If $\mathcal{A}$ is stationary over I, then $\{A \cap I : A \in \mathcal{A}\}$ is stationary over I.

(b) If $\mathcal{A}$ is stationary over I, and for every $A \in \mathcal{A}$ we are given a family $\mathcal{B}_A$ which is stationary over A, then $\bigcup_{A\in\mathcal{A}} \mathcal{B}_A$ is stationary over I.

(c) If ζ is an ordinal of uncountable cofinality, and $S \subseteq \zeta$ is stationary in the ordinary sense of 4A1C, then S is stationary over ζ in the sense of 5A1R.

5A1T Theorem (a) There is a family $\langle e_\xi\rangle_{\xi<\omega_1}$ such that $e_\xi : \xi \to \mathbb{N}$ is an injective function for each $\xi < \omega_1$ and $e_\eta \triangle (e_\xi \restriction \eta)$ is finite whenever $\eta < \xi < \omega_1$.

(b) There is a sequence $\langle \le_n\rangle_{n\in\mathbb{N}}$ of partial orders on ω_1 such that

$(\omega_1, \le_n)$ is a tree of height at most $n + 1$ for each $n \in \mathbb{N}$,

$\eta \le_0 \xi$ iff $\eta = \xi$,

$\le_n \subseteq \le_{n+1}$ for every $n \in \mathbb{N}$,

$\bigcup_{n\in\mathbb{N}} \le_n$ is the usual well-ordering of ω_1.

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5A2 Pcf theory

In §542 I call on some results from Shelah's pcf theory. As I have still not come across an elementary textbook for this material, I copy out part of the appendix of Fremlin 93, itself drawn largely from Burke & Magidor 90.

5A2A Reduced products Let $\langle P_i\rangle_{i\in I}$ be a family of partially ordered sets with product P.

(a) Let $\mathcal{F}$ be a filter on I. We have an equivalence relation $\equiv_\mathcal{F}$ on P, given by saying that $f \equiv_\mathcal{F} g$ if $\{i : f(i) = g(i)\} \in \mathcal{F}$. I write $P|\mathcal{F}$ for the set of equivalence classes under this relation, the **partial order reduced product** of $\langle P_i\rangle_{i\in I}$ modulo $\mathcal{F}$. Now $P|\mathcal{F}$ is again a partially ordered set, writing

$$f^\bullet \le g^\bullet \iff f \le_\mathcal{F} g \iff \{i : f(i) \le g(i)\} \in \mathcal{F}.$$

Observe that if every P_i is totally ordered and $\mathcal{F}$ is an ultrafilter, then $P|\mathcal{F}$ is totally ordered.

(b) For any filter $\mathcal{F}$ on I we have

$$\begin{aligned}\min_{i\in I} \operatorname{add} P_i = \operatorname{add} P \le \sup_{F\in\mathcal{F}} \operatorname{add}(\prod_{i\in F} P_i) &= \sup_{F\in\mathcal{F}} \min_{i\in F} \operatorname{add} P_i \\ &\le \operatorname{add}(P|\mathcal{F}),\end{aligned}$$

$$\operatorname{cf}(P|\mathcal{F}) \le \min_{F\in\mathcal{F}} \operatorname{cf}(\textstyle\prod_{i\in F} P_i) \le \operatorname{cf} P.$$

(c) Note that if $\mathcal{F}$, $\mathcal{G}$ are filters on I and $\mathcal{F} \subseteq \mathcal{G}$, then $\operatorname{add}(P|\mathcal{F}) \le \operatorname{add}(P|\mathcal{G})$ and $\operatorname{cf}(P|\mathcal{F}) \ge \operatorname{cf}(P|\mathcal{G})$.

5A2B Theorem Let $\lambda > 0$ be a cardinal and $\langle\theta_\zeta\rangle_{\zeta<\lambda}$ a family of regular infinite cardinals, all greater than λ. Set $P = \prod_{\zeta<\lambda} \theta_\zeta$. For any filter $\mathcal{F}$ on λ, let $P|\mathcal{F}$ be the corresponding reduced product and $\pi_\mathcal{F} : P \to P|\mathcal{F}$ the canonical map. For any cardinal δ set

$$\mathfrak{F}_\delta = \{\mathcal{F} : \mathcal{F} \text{ is an ultrafilter on } \lambda,\ \operatorname{cf}(P|\mathcal{F}) = \delta\},$$

$$\mathfrak{F}^*_\delta = \textstyle\bigcup_{\delta'\ge\delta} \mathfrak{F}_{\delta'};$$

if $\mathfrak{F}^*_\delta \ne \emptyset$, let $\mathcal{G}_\delta$ be the filter $\bigcap \mathfrak{F}^*_\delta$. Now

(a) if $\mathfrak{F}^*_\delta \ne \emptyset$, then $\operatorname{add}(P|\mathcal{G}_\delta) \ge \delta$;

(b) for every δ there is a set $F \in [P]^{\le\delta}$ such that $\pi_\mathcal{F}[F]$ is cofinal with $P|\mathcal{F}$ for every $F \in \mathfrak{F}_\delta$;

(c) $\mathfrak{F}_{\operatorname{cf} P} \ne \emptyset$.

5A2C Theorem Let $\lambda > 0$ be a cardinal and $\langle\theta_\zeta\rangle_{\zeta<\lambda}$ a family of regular infinite cardinals, all greater than λ. Set $P = \prod_{\zeta<\lambda}\theta_\zeta$. Let $\mathcal{F}$ be an ultrafilter on λ and κ a regular infinite cardinal with $\lambda < \kappa \le \mathrm{cf}(P|\mathcal{F})$. Then there is a family $\langle\theta'_\zeta\rangle_{\zeta<\lambda}$ of regular infinite cardinals such that $\lambda < \theta'_\zeta \le \theta_\zeta$ for every $\zeta < \lambda$ and $\mathrm{cf}(P'|\mathcal{F}) = \kappa$, where $P' = \prod_{\zeta<\lambda}\theta'_\zeta$.

5A2D Definitions (a) Let α, β, γ and δ be cardinals. I write

$$\mathrm{cov}_{\mathrm{Sh}}(\alpha,\beta,\gamma,\delta)$$

for the least cardinal of any family $\mathcal{E} \subseteq [\alpha]^{<\beta}$ such that for every $A \in [\alpha]^{<\gamma}$ there is a $\mathcal{D} \in [\mathcal{E}]^{<\delta}$ with $A \subseteq \bigcup\mathcal{D}$. In the trivial cases in which there is no such family $\mathcal{E}$ I write $\mathrm{cov}_{\mathrm{Sh}}(\alpha,\beta,\gamma,\delta) = \infty$.

(b) For cardinals α, γ write $\Theta(\alpha,\gamma)$ for the supremum of all cofinalities

$$\mathrm{cf}(\textstyle\prod_{\zeta<\lambda}\theta_\zeta)$$

for families $\langle\theta_\zeta\rangle_{\zeta<\lambda}$ such that $\lambda < \gamma$ is a cardinal, every θ_ζ is a regular infinite cardinal and $\lambda < \theta_\zeta < \alpha$ for every $\zeta < \lambda$.

Remarks

$$\mathrm{cov}_{\mathrm{Sh}}(\alpha,\beta',\gamma,\delta') \le \mathrm{cov}_{\mathrm{Sh}}(\alpha',\beta,\gamma',\delta), \quad \Theta(\alpha,\gamma) \le \Theta(\alpha',\gamma')$$

whenever $\alpha \le \alpha'$, $\beta \le \beta'$, $\gamma \le \gamma'$ and $\delta \le \delta'$.

5A2E Lemma Let α, β, γ, γ' and δ be cardinals.
(a) If $\gamma \le \gamma' \le \beta$ and $\delta \ge 2$ then

$$\mathrm{cov}_{\mathrm{Sh}}(\alpha,\beta,\gamma,\delta) \le \mathrm{cf}([\alpha]^{<\gamma'}) \le \#([\alpha]^{<\gamma'}).$$

(b) If either $\omega \le \gamma \le \mathrm{cf}\,\alpha$ or $\omega \le \mathrm{cf}\,\alpha < \mathrm{cf}\,\delta$ then

$$\mathrm{cov}_{\mathrm{Sh}}(\alpha,\beta,\gamma,\delta) \le \max(\alpha, \sup_{\theta<\alpha}\mathrm{cov}_{\mathrm{Sh}}(\theta,\beta,\gamma,\delta)).$$

5A2F Lemma Let α, γ be cardinals. If $\alpha \le 2^\gamma$, then $\Theta(\alpha,\gamma) \le 2^\gamma$.

5A2G Theorem For any cardinals α and γ,

$$\mathrm{cov}_{\mathrm{Sh}}(\alpha,\gamma,\gamma,\omega_1) \le \max(\omega,\alpha,\Theta(\alpha,\gamma)).$$

5A2H Lemma Let γ be an infinite regular cardinal and $\alpha \ge \Theta(\gamma,\gamma)$ a cardinal. Then $\Theta(\Theta(\alpha,\gamma),\gamma) \le \Theta(\alpha,\gamma)$.

5A2I Lemma Let α and γ be cardinals. Set $\delta = \sup_{\alpha'<\alpha}\Theta(\alpha',\gamma)$.
(a) If $\mathrm{cf}\,\alpha \ge \gamma$ then $\Theta(\alpha,\gamma) \le \max(\alpha,\delta)$.
(b) If $\mathrm{cf}\,\alpha < \gamma$ then $\Theta(\alpha,\gamma) \le \max(\alpha,\delta^{\mathrm{cf}\,\alpha})$, where $\delta^{\mathrm{cf}\,\alpha}$ is the cardinal power.

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5A3 Forcing

My discussion of forcing is based on Kunen 80; in particular, I start from pre-ordered sets rather than Boolean algebras, and the class $V^{\mathbb{P}}$ of terms in a forcing language will consist of subsets of $V^{\mathbb{P}} \times P$. I find however that I wish to diverge almost immediately from standard formulations in a technical respect, which I describe in 5A3A, introducing what I call 'forcing notions'. I do not refer to generic filters or models of ZFC, preferring to express all results in terms of the forcing relation (5A3C). I give some space to the interpretation of names (5A3E, 5A3F) and, in particular, to names for real numbers derived from elements of $L^0(\mathrm{RO}(\mathbb{P}))$ (5A3L).

5A3A Forcing notions (a) A **forcing notion** is a quadruple $\mathbb{P} = (P, \le, \mathbb{1}, \uparrow)$ or $\mathbb{P} = (P, \le, \mathbb{1}, \downarrow)$ where $(P, \le)$ is a pre-ordered set (that is, $\le$ is a transitive reflexive relation on P), $\mathbb{1} \in P$, and

if $\mathbb{P} = (P, \le, \mathbb{1}, \uparrow)$ then $\mathbb{1} \le p$ for every $p \in P$,

if $\mathbb{P} = (P, \le, \mathbb{1}, \downarrow)$ then $p \le \mathbb{1}$ for every $p \in P$.

In this context members of P are called **conditions**.

(b) I will say that a forcing notion $(P, \le, \mathbb{1}, \uparrow)$ is **active upwards**, while $(P, \le, \mathbb{1}, \downarrow)$ is **active downwards**.

(d) It will no longer be helpful to talk about conditions in P being 'larger' or 'less than' others. Instead, I will use the word '**stronger**': if $\mathbb{P} = (P, \le, \mathbb{1}, \uparrow)$, then $p \in P$ will be stronger than $q \in P$ if $p \ge q$; if $\mathbb{P} = (P, \le, \mathbb{1}, \downarrow)$, then $p \in P$ will be stronger than $q \in P$ if $p \le q$.

Similarly, if $\mathbb{P} = (P, \le, \mathbb{1}, \Updownarrow)$ is a forcing notion, a subset Q of P is **dense** if for every $p \in P$ there is a $q \in Q$ such that q is stronger than p. In the same way, two conditions p, q in P are 'compatible' if there is an $r \in P$ stronger than both. We have a standard topology on P generated by sets of the form $\{q : q \text{ is stronger than } p\}$, and a corresponding regular open algebra $\mathrm{RO}(\mathbb{P})$. An antichain for $\mathbb{P}$ will be a set $A \subseteq P$ such that any two distinct conditions in A are incompatible, and $\mathbb{P}$ will be ccc if every antichain for $\mathbb{P}$ is countable. The 'saturation' $\operatorname{sat}\mathbb{P}$ of $\mathbb{P}$ will be the least cardinal κ such that there is no antichain with cardinal κ.

5A3B Forcing languages Let $\mathbb{P} = (P, \le, \mathbb{1}, \Updownarrow)$ be a forcing notion.

(a) The class of $\mathbb{P}$**-names**, that is, terms of the forcing language defined by $\mathbb{P}$, is

$$V^{\mathbb{P}} = \{A : A \text{ is a set and } A \subseteq V^{\mathbb{P}} \times P\}$$

I will say that the **domain** of a name $A \in V^{\mathbb{P}}$ is the set $\operatorname{dom} A \subseteq V^{\mathbb{P}}$ of first members of elements of A.

(b) For any set X, $\check{X}$ will be the $\mathbb{P}$-name $\{(\check{x}, \mathbb{1}) : x \in X\} \in V^{\mathbb{P}}$.

5A3C The Forcing Relation Suppose that $\mathbb{P} = (P, \le, \mathbb{1}, \Updownarrow)$ is a forcing notion, $p \in P$, ϕ, ψ are formulae of set theory, and $\dot{x}_0, \dots, \dot{x}_n \in V^{\mathbb{P}}$.

(a) $p \Vdash_{\mathbb{P}} \dot{x}_0 = \dot{x}_1$ iff

whenever $(\dot{y}, q) \in \dot{x}_0$ and $r \in P$ is stronger than both p and q, there are a $(\dot{y}', q') \in \dot{x}_1$ and an r' stronger than both r and q' such that $r' \Vdash_{\mathbb{P}} \dot{y} = \dot{y}'$,

whenever $(\dot{y}, q) \in \dot{x}_1$ and $r \in P$ is stronger than both p and q, there are a $(\dot{y}', q') \in \dot{x}_0$ and an r' stronger than both r and q' such that $r' \Vdash_{\mathbb{P}} \dot{y} = \dot{y}'$.

(b) $p \Vdash_{\mathbb{P}} \dot{x}_0 \in \dot{x}_1$ iff

whenever $q \in P$ is stronger than p there are a $(\dot{y}, q') \in \dot{x}_1$ and an r stronger than both q and q' such that $r \Vdash_{\mathbb{P}} \dot{x}_0 = \dot{y}$.

(c) $p \Vdash_{\mathbb{P}} \phi(\dot{x}_0, \dots, \dot{x}_n) \,\&\, \psi(\dot{x}_0, \dots, \dot{x}_n)$ iff

$$p \Vdash_{\mathbb{P}} \phi(\dot{x}_0, \dots, \dot{x}_n) \text{ and } p \Vdash_{\mathbb{P}} \psi(\dot{x}_0, \dots, \dot{x}_n).$$

(d) $p \Vdash_{\mathbb{P}} \neg\phi(\dot{x}_0, \dots, \dot{x}_n)$ iff

$$\text{there is no } q \text{ stronger than } p \text{ such that } q \Vdash_{\mathbb{P}} \phi(\dot{x}_0, \dots, \dot{x}_n).$$

(e) $p \Vdash_{\mathbb{P}} \exists\, x,\ \phi(x, \dot{x}_0, \dots, \dot{x}_n)$ iff

for every q stronger than p there are an r stronger than q and a $\dot{y} \in V^{\mathbb{P}}$ such that $r \Vdash_{\mathbb{P}} \phi(\dot{y}, \dot{x}_0, \dots, \dot{x}_n)$.

(f) You will see that if $p \Vdash_{\mathbb{P}} \phi$ and q is stronger than p then $q \Vdash_{\mathbb{P}} \phi$.

(g) I will write $\Vdash_{\mathbb{P}}$ for $\mathbb{1}\Vdash_{\mathbb{P}}$.

5A3D The Forcing Theorem If ϕ is any theorem of ZFC, and $\mathbb{P}$ is any forcing notion, then $\Vdash_{\mathbb{P}}\phi$.

5A3E Names for functions Let $\mathbb{P}$ be a forcing notion, P its set of conditions, and $R\subseteq V^{\mathbb{P}}\times V^{\mathbb{P}}\times P$ a set. Consider the $\mathbb{P}$-names

$$\dot f=\{((\dot x,\dot y),p):(\dot x,\dot y,p)\in R\},$$

$$\dot A=\{(\dot x,p):(\dot x,\dot y,p)\in R\},\quad \dot B=\{(\dot y,p):(\dot x,\dot y,p)\in R\}.$$

(a) The following are equiveridical:

(i) $\Vdash_{\mathbb{P}}\dot f$ is a function;

(ii) whenever $(\dot x_0,\dot y_0,p_0)$, $(\dot x_1,\dot y_1,p_1)$ belong to R, $p\in\mathbb{P}$ is stronger than both p_0 and p_1 and $p\Vdash_{\mathbb{P}}\dot x_0=\dot x_1$, then $p\Vdash_{\mathbb{P}}\dot y_0=\dot y_1$.

(b) In this case,

$$p\Vdash_{\mathbb{P}}\dot f(\dot x)=\dot y$$

whenever $(\dot x,\dot y,p)\in R$,

$$\Vdash_{\mathbb{P}}\operatorname{dom}\dot f=\dot A\text{ and }\dot f[\dot A]=\dot B,$$

and the following are equiveridical:

(i) $\Vdash_{\mathbb{P}}\dot f$ is injective;

(ii) whenever $(\dot x_0,\dot y_0,p_0)$, $(\dot x_1,\dot y_1,p_1)$ belong to R, $p\in\mathbb{P}$ is stronger than both p_0 and p_1 and $p\Vdash_{\mathbb{P}}\dot y_0=\dot y_1$, then $p\Vdash_{\mathbb{P}}\dot x_0=\dot x_1$.

5A3F More notation Let $\mathbb{P}=(P,\le,\mathbb{1},\updownarrow)$ be a forcing notion.

(a) If $\dot y_0$, $\dot y_1\in V^{\mathbb{P}}$ then $\dot x=\{(\dot y_0,\mathbb{1}),(\dot y_1,\mathbb{1})\}\in V^{\mathbb{P}}$, and

$$\Vdash_{\mathbb{P}}\dot x=\{\dot y_0,\dot y_1\}.$$

Similarly, if we think of the formula (x,y) as being an abbreviation for $\{\{x\},\{x,y\}\}$, we get a $\mathbb{P}$-name

$$\dot z=\{(\{(\dot y_0,\mathbb{1})\},\mathbb{1}),(\{(\dot y_0,\mathbb{1}),(\dot y_1,\mathbb{1})\},\mathbb{1})\}$$

such that

$$\Vdash_{\mathbb{P}}\dot z=(\dot y_0,\dot y_1).$$

(b) Now let $\langle\dot x_i\rangle_{i\in I}$ be a family of $\mathbb{P}$-names, and set

$$\dot f=\{((\check\imath,\dot x_i),\mathbb{1}):i\in I\}.$$

As in 5A3E,

$$\Vdash_{\mathbb{P}}\dot f\text{ is a function with domain }\check I,$$

and

$$\Vdash_{\mathbb{P}}\dot f(\check\imath)=\dot x_i$$

for every $i\in I$. I will use the formula

$$\langle\dot x_i\rangle_{i\in\check I}$$

to signify the $\mathbb{P}$-name $\dot f$.

(c) $\dot T=\{(\dot x_i,\mathbb{1}):i\in I\}$ is a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}}\dot x_i\in\dot T$ for every $i\in I$, and whenever $p\in\mathbb{P}$ and $\dot x$ is a $\mathbb{P}$-name such that $p\Vdash_{\mathbb{P}}\dot x\in\dot T$, there are an $i\in I$ and a q stronger than p such that $q\Vdash_{\mathbb{P}}\dot x=\dot x_i$; I will write $\{\dot x_i:i\in\check I\}$ for $\dot T$.

(d) In the same spirit, if I have a family $\langle\dot x_i\rangle_{i\in I}$ of $\mathbb{P}$-names for real numbers between 0 and 1, I will allow myself to write '$\sup_{i\in\check I}\dot x_i$' to signify a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \sup_{i\in\check I}\dot x_i=\sup\{\dot x_i:i\in\check I\}.$$

I will do the same for limits of sequences; if $\langle\dot x_n\rangle_{n\in\mathbb{N}}$ is a sequence of $\mathbb{P}$-names for real numbers, and

$$\Vdash_{\mathbb{P}} \langle\dot x_n\rangle_{n\in\mathbb{N}}\text{ is convergent},$$

then I will write '$\lim_{n\to\infty}\dot x_n$' to mean a $\mathbb{P}$-name $\dot x$ such that

$$\Vdash_{\mathbb{P}} \langle\dot x_n\rangle_{n\in\mathbb{N}}\to\dot x\in\mathbb{R}.$$

5A3G Boolean truth values Let $\mathbb{P}$ be a forcing notion and P its set of conditions.

(a) If ϕ is a formula of set theory, and $\dot x_0,\dots,\dot x_n\in V^{\mathbb{P}}$, then

$$\{p:p\in P,\,p\Vdash_{\mathbb{P}}\phi(\dot x_0,\dots,\dot x_n)\}$$

is a regular open set in P; I will denote it $[\![\phi(\dot x_0,\dots,\dot x_n)]\!]$.

(b) If ϕ and ψ are formulae of set theory and $\dot x_0,\dots,\dot x_n\in V^{\mathbb{P}}$, then

$$[\![\phi(\dot x_0,\dots,\dot x_n)\,\&\,\psi(\dot x_0,\dots,\dot x_n)]\!]=[\![\phi(\dot x_0,\dots,\dot x_n)]\!]\cap[\![\psi(\dot x_0,\dots,\dot x_n)]\!]$$

and

$$[\![\neg\phi(\dot x_0,\dots,\dot x_n)]\!]=P\setminus\overline{[\![\phi(\dot x_0,\dots,\dot x_n)]\!]}.$$

(c) If ϕ is a formula of set theory, A is a set, and $\dot x_0,\dots,\dot x_n$ are $\mathbb{P}$-names, then

$$[\![\exists\,x\in\check A,\ \phi(x,\dot x_0,\dots,\dot x_n)]\!]=\operatorname{int}\overline{\bigcup_{a\in A}[\![\phi(\check a,\dot x_0,\dots,\dot x_n)]\!]},$$

the supremum of $\{[\![\phi(\check a,\dot x_0,\dots,\dot x_n]\!]:a\in A\}$ in $\mathrm{RO}(\mathbb{P})$.

5A3H Concerning $\check{}$s (a) For any set X and any forcing notion $\mathbb{P}$ there is a corresponding $\mathbb{P}$-name $\check X$. We start with $\check\emptyset=\emptyset$. If $1=\{\emptyset\}$ is the next von Neumann ordinal, we get a name

$$\check 1=\{(\check\emptyset,\mathbb{1})\}=\{(\emptyset,\mathbb{1})\};$$

and

$$\Vdash_{\mathbb{P}}\check 1=\{\emptyset\},$$

that is,

$$\Vdash_{\mathbb{P}}\check 1=1,$$

where in this formula the first 1 is interpreted in the ordinary universe and the second is interpreted in the forcing language. Similarly, if we take '2' to be an abbreviation for '$\{\emptyset,\{\emptyset\}\}$',

$$\Vdash_{\mathbb{P}}\check 2=2,$$

and so on. Indeed

$$\Vdash_{\mathbb{P}}\check\omega\text{ is the first infinite ordinal},$$

$$\Vdash_{\mathbb{P}}\check{\mathbb{Q}}\text{ is the set of rational numbers},$$

so the same convention would lead to

$$\Vdash_{\mathbb{P}}\check\omega=\omega,\check{\mathbb{N}}=\mathbb{N},\check{\mathbb{Q}}=\mathbb{Q}.$$

(c) Let $\mathbb{P}$ be a forcing notion and P its set of conditions.

(i) If A and B are sets, $p\in P$ and $p\Vdash_{\mathbb{P}}\check A\subseteq\check B$ then $A\subseteq B$.

(ii) If α, $\beta\in\mathbb{Q}$ and $p\Vdash_{\mathbb{P}}\check\alpha\le\check\beta$ then $\alpha\le\beta$.

5A3I Regular open algebras If $\mathbb{P} = (P, \le, \mathbb{1}, \updownarrow)$ is a forcing notion with regular open algebra $\mathrm{RO}(\mathbb{P})$, then we have a natural map $\iota : P \to \mathrm{RO}(\mathbb{P})^+$ defined by saying that

$$\iota(p) = \operatorname{int} \overline{\{q : q \text{ is stronger than } p\}}$$

for $p \in P$; and (allowing for the possible reversal of the direction of $\mathbb{P}$) ι is a dense embedding of the pre-ordered set $(P, \le)$ in the partially ordered set $(\mathrm{RO}(\mathbb{P})^+, \subseteq)$. Consequently, taking $\widehat{\mathbb{P}}$ to be the forcing notion $(\mathrm{RO}(\mathbb{P})^+, \subseteq, P, \downarrow)$,

$$\Vdash_{\mathbb{P}} \phi \text{ if and only if } \Vdash_{\widehat{\mathbb{P}}} \phi$$

for every sentence ϕ of set theory. It follows that if two forcing notions have isomorphic regular open algebras, then they force exactly the same theorems of set theory.

5A3J Definition Let $\mathbb{P}$ be a forcing notion. I will say that a $\mathbb{P}$-name $\dot{X}$ is **discriminating** if whenever $(\dot{x}, p)$ and $(\dot{y}, q)$ are distinct members of $\dot{X}$, and r is stronger than both p and q, then $r \Vdash_{\mathbb{P}} \dot{x} \ne \dot{y}$.

5A3K Lemma Let $\mathbb{P}$ be a forcing notion, and P its set of conditions.

(a) For any $\mathbb{P}$-name $\dot{X}$, there is a discriminating $\mathbb{P}$-name $\dot{X}_1$ such that $\Vdash_{\mathbb{P}} \dot{X}_1 = \dot{X}$.

(b) Let $\dot{X}$ be a discriminating $\mathbb{P}$-name, and $f : \dot{X} \to V^{\mathbb{P}}$ a function. Let $\dot{g}$ be the $\mathbb{P}$-name

$$\{((\dot{x}, f(\dot{x}, p)), p) : (\dot{x}, p) \in \dot{X}\}.$$

Then

$$\Vdash_{\mathbb{P}} \dot{g} \text{ is a function with domain } \dot{X}.$$

5A3L Real numbers in forcing languages Let $\mathbb{P}$ be any forcing notion, and P its set of conditions.

(a) I will say that a real number is the set of strictly smaller rational numbers.

$$\Vdash_{\mathbb{P}} \check{\alpha} \text{ is a real number}$$

for every real number α.

(b) Consider the Dedekind complete Boolean algebra $\mathrm{RO}(\mathbb{P})$ and the corresponding space $L^0 = L^0(\mathrm{RO}(\mathbb{P}))$.

(i) For every $u \in L^0$, set

$$\vec{u} = \{(\check{\alpha}, p) : \alpha \in \mathbb{Q},\ p \in [\![u > \alpha]\!]\}.$$

Then

$$\Vdash_{\mathbb{P}} \vec{u} \text{ is a real number.}$$

(ii) $[\![\vec{u} > \check{\alpha}]\!] = [\![u > \alpha]\!]$ for every $\alpha \in \mathbb{Q}$.

(iii) In the other direction, if we have a $\mathbb{P}$-name $\dot{x}$ for a real number (that is, a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}} \dot{x}$ is a real number), then there is a unique $u \in L^0$ such that $\Vdash_{\mathbb{P}} \dot{x} = \vec{u}$.

(iv) It follows that if $\dot{x}$ is a $\mathbb{P}$-name and $p \in P$ is such that $p \Vdash_{\mathbb{P}} \dot{x} \in \mathbb{R}$, then there is a $u \in L^0$ such that $p \Vdash_{\mathbb{P}} \dot{x} = \vec{u}$.

(v) If $\alpha \in \mathbb{R}$, then $(\alpha\chi 1)^{\vec{}} = \check{\alpha}$.

(c) Suppose that u, $v \in L^0$.

(i) $[\![\vec{u} < \vec{v}]\!] = [\![v - u > 0]\!]$.

(ii) In particular, if $u \le v$ in L^0, then

$$\Vdash_{\mathbb{P}} \vec{u} \le \vec{v} \text{ in } \mathbb{R},$$

and $[\![\vec{u} = \vec{v}]\!] = [\![u - v = 0]\!]$ for any u, $v \in L^0$.

(iii) $\Vdash_{\mathbb{P}} (u + v)^{\vec{}} = \vec{u} + \vec{v}$.

(iv) $\Vdash_{\mathbb{P}} (u \times v)^{\vec{}} = \vec{u}\vec{v}$.

(v) If $\alpha \in \mathbb{R}$, then $\Vdash_{\mathbb{P}} (\alpha u)^{\vec{}} = \check{\alpha}\vec{u}$.

(d)(i) Suppose that $\langle u_i\rangle_{i\in I}$ is a non-empty family in L^0 with supremum $u \in L^0$. Then

$$\Vdash_{\mathbb{P}} \vec{u} = \sup_{i\in\check{I}} \vec{u}_i \text{ in } \mathbb{R}.$$

(ii) And if $\langle u_i\rangle_{i\in I}$ is a non-empty family in L^0 with infimum $u \in L^0$, then

$$\Vdash_{\mathbb{P}} \vec{u} = \inf_{i\in\check{I}} \vec{u}_i \text{ in } \mathbb{R}$$

because $\Vdash_{\mathbb{P}} (-u)^{\vec{}} = -\vec{u}$ (using (c)).

(e) Suppose that $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in L^0, order*-convergent to $u \in L^0$. Then

$$\Vdash_{\mathbb{P}} \vec{u} = \lim_{n\to\infty} \vec{u}_n.$$

5A3M Forcing with Boolean algebras Suppose that $\mathfrak{A}$ is a Dedekind complete Boolean algebra, not $\{0\}$. $\mathbb{P} = (\mathfrak{A}^+, \subseteq, 1_{\mathfrak{A}}, \downarrow)$ is a forcing notion. We have a natural isomorphism between $\mathrm{RO}(\mathbb{P})$ and $\mathfrak{A}$, matching each $G \in \mathrm{RO}(\mathbb{P})$ with $\sup G \in \mathfrak{A}$; $\sup G$, taken in $(\mathfrak{A}^+, \subseteq)$, will belong to G unless $G = \emptyset$. I will usually identify the two algebras, so that $[\![\phi]\!]$ becomes $\sup\{a : a \in \mathfrak{A}^+,\ a \Vdash_{\mathbb{P}} \phi\}$, and $[\![\phi]\!] \Vdash_{\mathbb{P}} \phi$ except when $\Vdash_{\mathbb{P}} \neg\phi$. Note that $[\![\neg\phi]\!] = 1 \setminus [\![\phi]\!]$ (5A3Gb).

The identification of $\mathrm{RO}(\mathbb{P})$ with $\mathfrak{A}$ itself simplifies the discussion in 5A3L. We have a $\mathbb{P}$-name $\vec{u}$ associated with each $u \in L^0(\mathfrak{A})$, and the formula

$$[\![\vec{u} = \vec{v}]\!] = [\![u - v = 0]\!] \text{ in } \mathrm{RO}(\mathbb{P}) \text{ when } u,\ v \in L^0(\mathrm{RO}(\mathbb{P}))$$

of 5A3L(c-ii) turns into

$$\text{whenever } u,\ v \in L^0(\mathfrak{A}) \text{ and } a \in \mathfrak{A}^+,\ u \times \chi a = v \times \chi a \iff a \Vdash_{\mathbb{P}} \vec{u} = \vec{v}.$$

5A3N Ordinals and cardinals Let $\mathbb{P}$ be a forcing notion, and P its set of conditions.

(a) For any ordinal α,

$$\Vdash_{\mathbb{P}} \check{\alpha} \text{ is an ordinal;}$$

if $p \in P$ and $\dot{x}$ is a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \dot{x} \text{ is an ordinal,}$$

there are a q stronger than p and an ordinal α such that

$$q \Vdash_{\mathbb{P}} \dot{x} = \check{\alpha}$$

(b) If $\mathbb{P}$ is ccc, then

$$\Vdash_{\mathbb{P}} \check{\kappa} \text{ is a cardinal}$$

for every cardinal κ. In particular,

$$\Vdash_{\mathbb{P}} \check{\omega}_1 \text{ is a cardinal, so is the first uncountable cardinal,}$$

and we can write

$$\Vdash_{\mathbb{P}} \omega_1 = \check{\omega}_1,\ \omega_2 = \check{\omega}_2$$

etc.

(c) Again suppose that $\mathbb{P}$ is ccc, and that we have a set A, a $\mathbb{P}$-name $\dot{X}$ and a cardinal κ such that

$$\Vdash_{\mathbb{P}} \dot{X} \subseteq \check{A} \text{ and } \#(\dot{X}) \le \check{\kappa}.$$

Then there is a set $B \subseteq A$ such that $\#(B) \le \max(\omega, \kappa)$ and

$$\Vdash_{\mathbb{P}} \dot{X} \subseteq \check{B}.$$

(d) If $\mathbb{P}$ is ccc, then

$$\Vdash_{\mathbb{P}} \operatorname{cf}[\check{I}]^{\le\omega} = (\operatorname{cf}[I]^{\le\omega})\check{}$$

for every set I.

5A3O Iterated forcing If $\mathbb{P}$ is a forcing notion and P its set of conditions, and we have a quadruple $\dot{\mathbb{Q}} = (\dot{Q}, \dot{\le}, \dot{1}, \dot{\epsilon})$ of $\mathbb{P}$-names such that $(\dot{1}, \mathbb{1}_{\mathbb{P}}) \in \dot{Q}$ and

$\Vdash_{\mathbb{P}} \dot{\le}$ is a pre-order on $\dot{Q}$, $\dot{\epsilon}$ is a direction of activity and every member of $\dot{Q}$ is stronger than $\dot{1}$,

then $\mathbb{P} * \dot{\mathbb{Q}}$ is the forcing notion defined by saying that its conditions are objects of the form $(p, \dot{q})$ where

$$p \in P, \quad \dot{q} \in \operatorname{dom} \dot{Q}, \quad p \Vdash_{\mathbb{P}} \dot{q} \in \dot{Q},$$

and that $(p, \dot{q})$ is stronger than $(p', \dot{q}')$ if p is stronger than p' and

$$p \Vdash_{\mathbb{P}} \dot{q} \text{ is stronger than } \dot{q}'.$$

($\mathbb{1}_{\mathbb{P}*\dot{\mathbb{Q}}} = (\mathbb{1}_{\mathbb{P}}, \dot{1})$.)

5A3P Martin's axiom Let κ be a regular uncountable cardinal such that $2^\lambda \le \kappa$ for every $\lambda < \kappa$. Then there is a ccc forcing notion $\mathbb{P}$ such that

$$\Vdash_{\mathbb{P}} \mathfrak{m} = \mathfrak{c} = \check{\kappa}.$$

5A3Q Countably closed forcings (a) Let $\mathbb{P}$ be a forcing notion, and P its set of conditions. $\mathbb{P}$ is **countably closed** if whenever $\langle p_n\rangle_{n\in\mathbb{N}}$ is a sequence in P such that p_{n+1} is stronger than p_n for every n, there is a $p \in P$ which is stronger than every p_n.

(b) If $\mathbb{P}$ is a countably closed forcing notion, then $\Vdash_{\mathbb{P}} \mathcal{P}\mathbb{N} = (\mathcal{P}\mathbb{N})\check{}$.
Consequently $\Vdash_{\mathbb{P}} \mathbb{R} = \check{\mathbb{R}}$. Similarly, $\Vdash_{\mathbb{P}} [0,1] = [0,1]\check{}$.

Version of 20.7.24

5A4 General topology

The principal new topological concepts required in this volume are some of the standard cardinal functions of topology (5A4A-5A4B). As usual, there are particularly interesting phenomena involving compact spaces (5A4C). For special purposes in §513, we need to know some non-trivial facts about metrizable spaces (5A4D). The rest of the section is made up of scraps which are either elementary or standard.

5A4A Definitions Let $(X, \mathfrak{T})$ be a topological space.

(a) The **weight** of X, $w(X)$, is the least cardinal of any base for $\mathfrak{T}$.

(b) The **π-weight** of X is $\pi(X) = \operatorname{ci}(\mathfrak{T} \setminus \{\emptyset\})$, the smallest cardinal of any π-base for $\mathfrak{T}$.

(c) The **density** $d(X)$ of X is the smallest cardinal of any dense subset of X.

(d) The **cellularity** of X is

$$c(X) = \sup\{\#(\mathcal{G}) : \mathcal{G} \subseteq \mathfrak{T} \setminus \{\emptyset\} \text{ is disjoint}\}.$$

The **saturation** of X is

$$\operatorname{sat}(X) = \sup\{\#(\mathcal{G})^+ : \mathcal{G} \subseteq \mathfrak{T} \setminus \{\emptyset\} \text{ is disjoint}\}.$$

(e) The **tightness** of X, $t(X)$, is the smallest cardinal κ such that whenever $A \subseteq X$ and $x \in \overline{A}$ there is a $B \in [A]^{\le\kappa}$ such that $x \in \overline{B}$.

(f) The **Novák number** $n(X)$ is the smallest cardinal of any family of nowhere dense subsets of X covering X; or ∞ if there is no such family.

(g)(i) The **Lindelöf number** $L(X)$ is the least cardinal κ such that every open cover of X has a subcover with cardinal at most κ.

(ii) The **hereditary Lindelöf number** $\mathrm{hL}(X)$ is $\sup_{Y\subseteq X} L(Y)$.

(h)(i) If $x \in X$, the **character of** x **in** X, $\chi(x, X)$, is the smallest cardinal of any base of neighbourhoods of x.

(ii) The **character** of X is $\chi(X) = \sup_{x\in X} \chi(x, X)$.

(i) The **network weight** of X, $\mathrm{nw}(X)$, is the smallest cardinal of any network for $\mathfrak{T}$.

5A4B Proposition Let $(X, \mathfrak{T})$ be a topological space.

(a)

$$c(X) \le d(X) \le \pi(X) \le w(X) \le \#(\mathfrak{T}) \le 2^{\mathrm{nw}(X)},$$

$$t(X) \le \chi(X) \le w(X) \le \max(\#(X), \chi(X)).$$

$\mathrm{sat}(X) = c(X)^+$ unless $\mathrm{sat}(X)$ is weakly inaccessible, in which case $\mathrm{sat}(X) = c(X)$.

(b) If Y is a subspace of X, then $w(Y) \le w(X)$, $t(Y) \le t(X)$, $\mathrm{nw}(Y) \le \mathrm{nw}(X)$ and $\chi(y, Y) \le \chi(y, X)$ for every $y \in Y$.

(c) If a topological space Y is a continuous image of X, then $d(Y) \le d(X)$, $c(Y) \le c(X)$, $t(Y) \le t(X)$, $L(Y) \le L(X)$ and $\mathrm{nw}(Y) \le \mathrm{nw}(X)$.

(d) If $\mathcal{G}$ is a family of open subsets of X, then there is a subfamily $\mathcal{H} \subseteq \mathcal{G}$ such that $\#(\mathcal{H}) < \mathrm{sat}(X)$ and $\overline{\bigcup\mathcal{H}} = \overline{\bigcup\mathcal{G}}$.

(e) Let $\langle X_i\rangle_{i\in I}$ be a family of non-empty topological spaces with product X, and λ a cardinal such that $\#(I) \le 2^\lambda$. Then

$$d(X) \le \max(\omega, \lambda, \sup_{i\in I} d(X_i)), \quad c(X) = \sup_{J\subseteq I \text{ is finite}} c(\prod_{i\in J} X_i).$$

(f) If $\mathcal{G}$ is any family of open subsets of X, there is an $\mathcal{H} \subseteq \mathcal{G}$ such that $\#(\mathcal{H}) \le \mathrm{hL}(X)$ and $\bigcup\mathcal{H} = \bigcup\mathcal{G}$.

(g) If X is Hausdorff then $\#(X) \le 2^{\max(c(X),\chi(X))}$.

(h) Suppose that X is metrizable.

(i) $d(X) = w(X)$.

(ii) $d(Y) \le d(X)$ for every $Y \subseteq X$. So any discrete subset of X has cardinal at most $d(X)$.

(iii) Let ρ be a metric on X defining its topology. Then X is separable iff there is no uncountable $A \subseteq X$ such that $\inf_{x,y\in A, x\ne y} \rho(x, y) > 0$.

5A4C Compactness Let X be a compact Hausdorff space.

(a)(i) $\mathrm{nw}(X) = w(X)$.

(ii) There is a set $Y \subseteq X$, with cardinal at most the cardinal power $d(X)^\omega$, which meets every non-empty G_δ subset of X.

(b) If X is perfectly normal it is first-countable.

(c) If $w(X) \le \kappa$, X is homeomorphic to a closed subspace of $[0, 1]^\kappa$.

(d)(i) If Y is a Hausdorff space and $f : X \to Y$ is a continuous irreducible surjection, then $d(X) = d(Y)$.

(ii) If $f : X \to \{0, 1\}^\kappa$ is a continuous irreducible surjection, where $\kappa \ge \omega$, then $\chi(x, X) \ge \kappa$ for every $x \in X$.

(iii) So if there is a continuous surjection from a closed subset of X onto $\{0, 1\}^\kappa$, there is a non-empty closed $K \subseteq X$ such that $\chi(x, K) \ge \kappa$ for every $x \in K$.

(iv) If Y and Z are Hausdorff spaces and $f : X \to Y$, $g : Y \to Z$ are continuous irreducible surjections then $gf : X \to Z$ is irreducible.

(e) If $\langle x_n\rangle_{n\in\mathbb{N}}$ is a sequence in X with at most one cluster point in X, then $\langle x_n\rangle_{n\in\mathbb{N}}$ is convergent.

(f) Let Y be a Hausdorff space and $f : X \to Y$ a continuous function. If $\mathcal{E}$ is a non-empty downwards-directed family of closed subsets of X, then $f[\bigcap \mathcal{E}] = \bigcap_{F\in\mathcal{E}} f[F]$.

5A4D Vietoris topologies: Proposition Let X be a separable metrizable space and $\mathcal{K}$ the set of compact subsets of X with the topology induced by the Vietoris topology on the set of closed subsets of X.

(a) $\mathcal{K}$ is second-countable.

(b) If Y is a topological space and $R \subseteq Y \times X$ is usco-compact, then $y \mapsto R[\{y\}] : Y \to \mathcal{K}$ is Borel measurable.

(c) There is a sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ of Borel measurable functions from $\mathcal{K}\setminus\{\emptyset\}$ to X such that $\{f_n(K) : n \in \mathbb{N}\}$ is a dense subset of K for every $K \in \mathcal{K} \setminus \{\emptyset\}$.

5A4E Category and the Baire property Let X be a topological space; write $\widehat{\mathcal{B}}(X)$ for its Baire-property algebra.

(a) Suppose that $\langle G_i\rangle_{i\in I}$ is a disjoint family of open sets and $\langle E_i\rangle_{i\in I}$ is a family of nowhere dense sets. Then $\bigcup_{i\in I} G_i \cap E_i$ is nowhere dense.

(b) Let Y be another topological space.

(i) If $A \subseteq X$ is nowhere dense in X, then $A \times Y$ is nowhere dense in $X \times Y$. So if $A \subseteq X$ is meager in X, then $A \times Y$ is meager in $X \times Y$.

(ii) $\widehat{\mathcal{B}}(X)\widehat{\otimes}\widehat{\mathcal{B}}(Y) \subseteq \widehat{\mathcal{B}}(X \times Y)$.

(iii) If Y is compact, Hausdorff and not empty, then a set $A \subseteq X$ is meager in X iff $A \times Y$ is meager in $X \times Y$.

(c) Suppose that X is completely regular and ccc.

(i) Every nowhere dense subset of X is included in a nowhere dense zero set.

(ii) Every meager subset of X is included in a meager Baire set.

(iii) Every subset of X with the Baire property is expressible as $G\triangle M$ where G is a cozero set and M is meager.

5A4F Normal and paracompact spaces (a) For a normal space X and an infinite set I, the following are equiveridical: (i) there is a continuous surjection from X onto $[0,1]^I$; (ii) there is a continuous surjection from a closed subset of X onto $\{0,1\}^I$.

(b) Suppose that X is a paracompact normal space and $\mathcal{G}$ is an open cover of X. Then there is a continuous pseudometric $\rho : X \times X \to [0,\infty[$ such that whenever $\emptyset \ne A \subseteq X$ and $\sup_{x,y\in A} \rho(x,y) \le 1$ there is a $G \in \mathcal{G}$ such that $A \subseteq G$.

5A4G Baire σ-algebras (a) Let X be a topological space. Write $\mathcal{B}\mathfrak{a}_0(X)$ for the set of cozero sets in X and for ordinals $\alpha > 0$ set

$$\mathcal{B}\mathfrak{a}_\alpha(X) = \{\textstyle\bigcup_{n\in\mathbb{N}}(X \setminus E_n) : \langle E_n\rangle_{n\in\mathbb{N}} \text{ is a sequence in } \bigcup_{\beta<\alpha} \mathcal{B}\mathfrak{a}_\beta(X)\}.$$

Then the Baire σ-algebra $\mathcal{B}\mathfrak{a}(X)$ of X is $\bigcup_{\alpha<\omega_1} \mathcal{B}\mathfrak{a}_\alpha(X)$.

(b)(i) If $\langle X_i\rangle_{i\in I}$ is a family of separable metrizable spaces with product X, then $\#(\mathcal{B}\mathfrak{a}(X)) \le \max(\mathfrak{c}, \#(I)^\omega)$.

(ii) If $\kappa \ge 2$ is a cardinal, then the set of Baire measurable functions from $\{0,1\}^\kappa$ to $\{0,1\}^\omega$ has cardinal κ^ω.

5A4H Proposition If X is a compact metrizable space and (Y,ρ) a complete separable metric space, then $C(X;Y)$, with the topology of uniform convergence, is Polish.

5A4I Compact-open topologies We shall need a couple of elementary facts concerning some spaces of continuous functions.

(a) Let X and Y be topological spaces and F a set of functions from X to Y. The **compact-open** topology on F is the topology generated by sets of the form $\{f : f \in F,\ f[K] \subseteq H\}$ where $K \subseteq X$ is compact and $H \subseteq Y$ is open.

(b) Let X be a topological space and $\langle Y_i\rangle_{i\in I}$ a family of regular spaces, with product Y. Set $\pi_i y = y(i)$ for $i \in I$ and $y \in Y$. Then $g \mapsto \langle \pi_i g\rangle_{i\in I} : C(X;Y) \to \prod_{i\in I} C(X;Y_i)$ is a homeomorphism for the compact open topologies on $C(X;Y)$ and the $C(X;Y_i)$.

(c) Let X be a compact space, and write $\mathcal{E}$ for the algebra of open-and-closed subsets of X. Then $f \mapsto f^{-1}[\{1\}]$ is a bijection between $C(X;\{0,1\})$ and $\mathcal{E}$, and the compact-open topology on $C(X;\{0,1\})$ is discrete.

5A4J Proposition Let X be a set and $\mathcal{A}$ a family of countable sets which is stationary over X. Then $\operatorname{non}\mathcal{M}(X^{\mathbb{N}}) \le \max(\#(\mathcal{A}), \operatorname{non}\mathcal{M})$.

Notation Here X is given its discrete topology and $X^{\mathbb{N}}$ the associated product topology. $\mathcal{M}(X^{\mathbb{N}})$ is its meager ideal and $\operatorname{non}\mathcal{M}(X^{\mathbb{N}})$ the corresponding uniformity, the smallest cardinal of any non-meager subset of $X^{\mathbb{N}}$. $\mathcal{M}$ is $\mathcal{M}(\mathbb{R})$.

5A4K Irreducible surjections: Lemma (a) Let Q be a topological space and K, L closed subsets of Q such that $K \subseteq \overline{Q\setminus L}$, $L \subseteq \overline{Q\setminus K}$ and $K\cup L = Q$. Set $Z = \{(x,1) : x \in K\} \cup \{(x,0) : x \in L\} \subseteq Q \times \{0,1\}$, and write $\phi : Z \to Q$ for the first-coordinate map. Then ϕ is an irreducible continuous surjection.

(b) Let θ be an ordinal, $\langle Q_\xi\rangle_{\xi<\theta}$ a family of compact Hausdorff spaces, and $\langle \phi_{\eta\xi}\rangle_{\eta\le\xi<\theta}$ a family such that $\phi_{\eta\xi} : Q_\xi \to Q_\eta$ is a continuous surjection whenever $\eta \le \xi < \theta$. Suppose that

$\phi_{\zeta\xi} = \phi_{\zeta\eta}\phi_{\eta\xi}$ whenever $\zeta \le \eta \le \xi < \theta$,

the topology of Q_ξ is generated by $\{\phi_{\eta\xi}^{-1}[U] : \eta < \xi,\ U \subseteq Q_\eta$ is open$\}$ for every non-zero limit ordinal $\xi < \theta$,

$\phi_{\xi,\xi+1}$ is irreducible whenever $\xi + 1 < \theta$. (*)

Then $\phi_{\eta\xi}$ is irreducible whenever $\eta \le \xi < \theta$.

5A4L Old friends (a) The weight of the Stone-Čech compactification $\beta\mathbb{N}$ is $\mathfrak{c}$.

(b)(i) For any infinite I, there is a continuous surjection from $\{0,1\}^I$ onto $[0,1]^I$.

(ii) There is a continuous surjection from $[0,1]$ onto $[0,1]^{\mathbb{N}}$.

(iii) For any infinite κ, $t(\{0,1\}^\kappa) = \kappa$.

(c) If X is a non-empty zero-dimensional compact metrizable space without isolated points, it is homeomorphic to $\{0,1\}^{\mathbb{N}}$.

(d) Let X be a non-empty zero-dimensional Polish space in which no non-empty open set is compact. Then X is homeomorphic to $\mathbb{N}^{\mathbb{N}}$ with its usual topology.

(e) If X is a non-empty Polish space without isolated points, then it has a dense G_δ set which is homeomorphic to $\mathbb{N}^{\mathbb{N}}$ with its usual topology.

(f) If X is any zero-dimensional Polish space there is a closed subspace of $\mathbb{N}^{\mathbb{N}}$ homeomorphic to X.

Version of 3.10.13

5A5 Real analysis

For the sake of an argument in §534 I sketch a fragment of theory.

5A5A Entire functions A real function f is **real-analytic** if its domain is an open subset G of $\mathbb{R}$ and for every $a \in G$ there are a $\delta > 0$ and a real sequence $\langle c_n \rangle_{n\in\mathbb{N}}$ such that $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ whenever $|x-a| < \delta$. It is **real-entire** if in addition its domain is the whole of $\mathbb{R}$.

We need the following facts: (i) if f and g are real-entire functions so is $f - g$; (ii) if $\langle c_n \rangle_{n\in\mathbb{N}}$ is a real sequence such that $f(x) = \sum_{n=0}^{\infty} c_n x^n$ is defined in $\mathbb{R}$ for every $x \in \mathbb{R}$, then f is real-entire; (iii) if in this expression not every c_n is zero, then every point of $F = \{x : x \in \mathbb{R},\ f(x) = 0\}$ is isolated in F, so that F is countable. If you have done a basic course in complex functions you should recognise this. If either you missed this out, or you are not sure you understand the proof of Cauchy's theorem, the following is a sketch of a real-variable argument.

(i) is elementary. For (ii), observe first that if $\langle c_n x^n \rangle_{n\in\mathbb{N}}$ is summable then $\lim_{n\to\infty} c_n x^n = 0$ so $\sum_{n=0}^{\infty} |c_n| t^n$ is finite whenever $0 \le t < |x|$. In the present case, $\sum_{n=0}^{\infty} |c_n| t^n < \infty$ for every $t \ge 0$. So if a, $x \in \mathbb{R}$,

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \Bigl|\frac{n!}{k!(n-k)!} c_n x^k a^{n-k}\Bigr| \le \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{n!}{k!(n-k)!}|c_n| R^n$$

(where $R = \max(|x|, |a|)$)

$$= \sum_{n=0}^{\infty} |c_n|(2R)^n < \infty.$$

We therefore have

$$f(x+a) = \sum_{n=0}^{\infty} c_n(x+a)^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} c_n x^k a^{n-k} = \sum_{k=0}^{\infty} c_{ak} x^k$$

where $c_{ak} = \sum_{n=k}^{\infty} \frac{n!}{k!(n-k)!} c_n a^{n-k}$ for each k. Turning this round, $f(x) = \sum_{k=0}^{\infty} c_{ak}(x-a)^k$ for every x. This shows that f is real-entire. As for (iii), if not every c_n is zero, there must be some neighbourhood of 0 in which the first non-zero term $c_n x^n$ dominates, so f is not identically zero. (The point is that $\sum_{k=0}^{\infty} |c_k| < \infty$, so there is some $\delta > 0$ such that $\sum_{k=n+1}^{\infty} |c_k \delta^{k+1}| < |c_n \delta^n|$.) In this case, if $a \in \mathbb{R}$, not every c_{ak} can be zero, and there must be some neighbourhood of a in which the first non-zero term $c_{ak}(x-a)^k$ dominates, so that there can be no zeroes of f in that neighbourhood except perhaps a itself.

Version of 13.2.17

5A6 Special axioms

This section contains very brief accounts of some of the undecidable propositions and special axioms which are used in this volume, with a few of their most basic consequences: the generalized continuum hypothesis, the axiom of constructibility, Jensen's Covering Lemma, square principles, Chang's transfer principle, Todorčević's p-ideal dichotomy and the filter dichotomy.

5A6A The generalized continuum hypothesis (a) The generalized continuum hypothesis is the assertion

$$\text{(GCH)} \quad 2^\kappa = \kappa^+ \text{ for every infinite cardinal } \kappa.$$

(b) If GCH is true, then for infinite cardinals κ, λ

$$\begin{aligned}\mathrm{cf}[\kappa]^{\le\lambda} &= 1 \text{ if } \kappa \le \lambda,\\ &= \kappa \text{ if } \lambda < \mathrm{cf}\,\kappa,\\ &= \kappa^+ \text{ otherwise.}\end{aligned}$$

(c) If GCH is true, then for infinite cardinals κ and λ, the cardinal power κ^λ is 2^λ if $\kappa \le \lambda$, κ if $\lambda < \mathrm{cf}\,\kappa$, κ^+ otherwise.

5A6B L, $0^\sharp$ and Jensen's Covering Lemma (a)(i) Let L be the class of **constructible** sets. The **axiom of constructibility** is

$(V{=}L)$ Every set is constructible.

$V{=}L$ implies GCH .

(ii) Every ordinal belongs to L; if A, $B \in L$ then $A \cap B \in L$; if κ is a cardinal, then $\#(L \cap \mathcal{P}\kappa) \le \kappa^+$.

(b) $0^\sharp$, if it exists, is a set of sentences in a countable formal language. I will write '$\exists 0^\sharp$' for the assertion '$0^\sharp$ exists'.

Jensen's Covering Lemma is the assertion

(CL) for every uncountable set A of ordinals, there is a constructible set of the same cardinality including A.

Jensen's Covering Theorem is

CL iff not-$\exists 0^\sharp$.

(c) CL implies that $\Vdash_{\mathbb{P}}$ CL for every forcing notion $\mathbb{P}$ of the kind considered in §5A3.

5A6C Theorem Assume that CL is true.

(a) For infinite cardinals κ and λ,

$$\begin{aligned}\mathrm{cf}[\kappa]^{\le\lambda} &= 1 \text{ if } \kappa \le \lambda,\\ &= \kappa \text{ if } \lambda < \mathrm{cf}\,\kappa,\\ &= \kappa^+ \text{ otherwise.}\end{aligned}$$

(b) If κ and λ are infinite cardinals, then the cardinal power κ^λ is 2^λ if $\kappa \le 2^\lambda$, κ if $\lambda < \mathrm{cf}\,\kappa$ and $2^\lambda \le \kappa$, and κ^+ otherwise.

5A6D Square principles (a)(i) Let Sing be the class of non-zero limit ordinals which are not regular cardinals. Global Square is the statement

there is a family $\langle C_\xi\rangle_{\xi\in\mathrm{Sing}}$ such that
for every $\xi \in \mathrm{Sing}$, C_ξ is a closed cofinal set in ξ;
$\mathrm{otp}\,C_\xi < \xi$ for every $\xi \in \mathrm{Sing}$;
if $\xi \in \mathrm{Sing}$ and $\zeta > 0$ is such that $\zeta = \sup(\zeta \cap C_\xi)$, then $\zeta \in \mathrm{Sing}$ and $C_\zeta = \zeta \cap C_\xi$.

(ii) For an infinite cardinal κ, let $\square_\kappa$ be the statement

there is a family $\langle C_\xi\rangle_{\xi<\kappa^+}$ of sets such that
for every $\xi < \kappa^+$, $C_\xi \subseteq \xi$ is a closed cofinal set in ξ;
if $\mathrm{cf}\,\xi < \kappa$ then $\#(C_\xi) < \kappa$;
whenever $\xi < \kappa^+$ and $\zeta < \xi$ is such that $\zeta = \sup(\zeta \cap C_\xi)$, then $C_\zeta = \zeta \cap C_\xi$.

(b) $V{=}L$ implies Global Square. Global Square implies that $\square_\kappa$ is true for every infinite cardinal κ. CL implies that $\square_\kappa$ is true for every singular infinite cardinal κ.

(c) If κ is an uncountable cardinal and $\langle C_\xi\rangle_{\xi<\kappa^+}$ is a family as in (a-ii), then $\operatorname{otp} C_\xi \le \kappa$ for every $\xi < \kappa^+$.

5A6E Lemma Suppose that κ is an uncountable cardinal with countable cofinality such that $\square_\kappa$ is true. Then there is a family $\langle I_\xi\rangle_{\xi<\kappa^+}$ of countably infinite subsets of κ such that

$I_\xi \cap I_\eta$ is finite whenever $\eta < \xi < \kappa^+$,

$\{\xi : \xi < \kappa^+, I \cap I_\xi$ is infinite$\}$ is countable for every countable $I \subseteq \kappa$.

5A6F Chang's transfer principle (a) If λ_0, λ_1, κ_0 and κ_1 are cardinals, then $(\kappa_1, \lambda_1) \twoheadrightarrow (\kappa_0, \lambda_0)$ means

whenever $f : [\kappa_1]^{<\omega} \to \lambda_1$ is a function, there is an $A \in [\kappa_1]^{\kappa_0}$ such that $\#(f[[A]^{<\omega}]) \le \lambda_0$.

I write $\mathrm{CTP}(\kappa, \lambda)$ for the statement

$$(\kappa, \lambda) \twoheadrightarrow (\omega_1, \omega).$$

(b) Suppose that $\mathrm{CTP}(\kappa, \lambda)$ is true.

(i) If $f : [\kappa]^{<\omega} \to [\lambda]^{\le\omega}$ is a function, then there is an uncountable $A \subseteq \kappa$ such that $\bigcup\{f(I) : I \in [A]^{<\omega}\}$ is countable.

(ii) If $\langle A_\xi\rangle_{\xi<\kappa}$ is any family of countable subsets of λ, then there is a countable $A \subseteq \lambda$ such that $\{\xi : A_\xi \subseteq A\}$ is uncountable.

(c) CL implies that $\mathrm{CTP}(\kappa, \lambda)$ is false except when $\lambda \le \omega$.

5A6G Todorčević's p-ideal dichotomy (a) Let X be a set and $\mathcal{I}$ an ideal of subsets of X. Then $\mathcal{I}$ is a **p-ideal** if for every sequence $\langle I_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{I}$ there is an $I \in \mathcal{I}$ such that $I_n \setminus I$ is finite for every $n \in \mathbb{N}$.

(b) Now Todorčević's p-ideal dichotomy is the statement

(TPID) whenever X is a set and $\mathcal{I} \subseteq [X]^{\le\omega}$ is a p-ideal of countable subsets of X, then *either* there is a $B \in [X]^{\omega_1}$ such that $[B]^{\le\omega} \subseteq \mathcal{I}$ *or* X is expressible as $\bigcup_{n\in\mathbb{N}} X_n$ where $\mathcal{I} \cap \mathcal{P}X_n \subseteq [X_n]^{<\omega}$ for every $n \in \mathbb{N}$.

This is a consequence of the Proper Forcing Axiom, and implies that $\square_\kappa$ is false for every $\kappa \ge \omega_1$.

***5A6H Analytic p-ideals: Theorem** Suppose that the Proper Forcing Axiom is true. Take a non-empty set $D \subseteq [0, \infty[^{\mathbb{N}}$ and set

$$\mathcal{I} = \{I : I \subseteq \mathbb{N}, \lim_{n\to\infty} \sup_{z\in D} \textstyle\sum_{i\in I\setminus n} z(i) = 0\},$$

so that $\mathcal{I}$ is an ideal of subsets of $\mathbb{N}$. Let $\mathfrak{A}$ be the quotient Boolean algebra $\mathcal{P}\mathbb{N}/\mathcal{I}$. Then for every $\pi \in \operatorname{Aut}\mathfrak{A}$ there are sets I, $J \in \mathcal{I}$ and a bijection $h : \mathbb{N} \setminus I \to \mathbb{N} \setminus J$ representing π in the sense that $\pi(A^\bullet) = (h^{-1}[A])^\bullet$ for every $A \subseteq \mathbb{N}$.

5A6I $\mathfrak{u}$, $\mathfrak{g}$ and the filter dichotomy: Definitions (a) The **ultrafilter number** $\mathfrak{u}$ is the least cardinal of any filter base generating a free ultrafilter on $\mathbb{N}$.

(b)(i) A family A of infinite subsets of $\mathbb{N}$ is **groupwise dense** if

(α) whenever $a \in A$, $a' \in [\mathbb{N}]^\omega$ and $a' \setminus a$ is finite, then $a' \in A$,

(β) whenever $\phi : \mathbb{N} \to \mathbb{N}$ is finite-to-one, there is an infinite $c \subseteq \mathbb{N}$ such that $\phi^{-1}[c] \in A$.

(A function $f : X \to Y$ is 'finite-to-one' if $f^{-1}[\{y\}]$ is finite for every $y \in Y$.)

(ii) The **groupwise density number** $\mathfrak{g}$ is the least cardinal of any collection $\mathbb{A}$ of groupwise dense subsets of $[\mathbb{N}]^\omega$ such that $\bigcap\mathbb{A} = \emptyset$.

(c) For filters $\mathcal{F}$ on X and $\mathcal{G}$ on Y, say that $\mathcal{F} \le_{\mathrm{RB}} \mathcal{G}$ if there is a finite-to-one $\phi : Y \to X$ such that $\mathcal{F} = \phi[[\mathcal{G}]]$. (This is the **Rudin-Blass ordering** of filters.) $\mathcal{F} \le_{\mathrm{RB}} \mathcal{F}$ for every filter $\mathcal{F}$, and if $\mathcal{F} \le_{\mathrm{RB}} \mathcal{G}$ and $\mathcal{G} \le_{\mathrm{RB}} \mathcal{H}$ then $\mathcal{F} \le_{\mathrm{RB}} \mathcal{H}$ (and $\mathcal{F} \le_{\mathrm{RK}} \mathcal{G}$).

(d) The **filter dichotomy** is the statement

(FD) For every free filter $\mathcal{F}$ on $\mathbb{N}$ either $\mathcal{F}_{\mathrm{Fr}} \leq_{\mathrm{RB}} \mathcal{F}$, where $\mathcal{F}_{\mathrm{Fr}}$ is the Fréchet filter, or there is an ultrafilter $\mathcal{G}$ on $\mathbb{N}$ such that $\mathcal{G} \leq_{\mathrm{RB}} \mathcal{F}$.

***5A6J Proposition** If $\mathfrak{u} < \mathfrak{g}$ then the filter dichotomy is true.