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Appendix to Volume 5

Useful facts

For this volume, the most substantial ideas demanded are, naturally enough, in set theory. Fragments of general set theory are in §5A1, with cardinal arithmetic and infinitary combinatorics. §5A2 contains results from Shelah's pcf theory, restricted to those which are actually used in this book. §5A3 describes the language I will use when I discuss forcing constructions; in essence, I follow Kunen 80, but with some variations which need to be signalled.

As usual, some bits of general topology are needed; I give these in §5A4, starting with a list of cardinal functions to complement the definitions in §511. There is a tiny piece of real analysis in §5A5. In §5A6 are notes on a few undecidable propositions, mostly standard.

Version of 3.9.20

5A1 Set theory

As usual, I begin with set theory, continuing from §§2A1 and 4A1. I start with definitions and elementary remarks filling some minor gaps in the deliberately sketchy accounts in the earlier volumes (5A1A-5A1E). I give a relatively solid paragraph on cardinal arithmetic (5A1F), including an account of cofinalities of ideals $[\kappa]^{\le\lambda}$. 5A1H-5A1K are devoted to infinitary combinatorics, with the Erdős-Rado theorem and Hajnal's Free Set Theorem. 5A1L-5A1P deal with the existence of 'transversals' of various kinds in spaces of functions, that is, large sets of functions which are well separated on some combinatorial criterion. 5A1Q is a fragment of finite combinatorics, 5A1R-5A1S introduce 'stationary families' of sets and 5A1T is a remarkable property of the ordering of ω_1.

5A1A Order types (a) If X is a well-ordered set, its **order type** $\operatorname{otp} X$ is the ordinal order-isomorphic to X (2A1Dg).

If S is a set of ordinals, an ordinal-valued function f with domain S is **regressive** if $f(\xi) < \xi$ for every $\xi \in S$ (cf. 4A1Cc).

(b) The non-stationary ideal on a cardinal κ of uncountable cofinality is $(\operatorname{cf}\kappa)$-additive, because the intersection of fewer than $\operatorname{cf}\kappa$ closed cofinal sets is a closed cofinal set (4A1Bd).

(c) If κ is a cardinal, $\lambda < \operatorname{cf}\kappa$ is an infinite regular cardinal and $C \subseteq \kappa$ is a closed cofinal set, then $S = \{\xi : \xi < \kappa, \operatorname{cf}(\xi \cap C) = \lambda\}$ is stationary in κ. **P** If $C' \subseteq \kappa$ is a closed cofinal set, let $\langle\gamma_\xi\rangle_{\xi<\operatorname{otp}(C\cap C')}$ be the increasing enumeration of $C \cap C'$. Then $\operatorname{otp}(C \cap C') \ge \operatorname{cf}\kappa > \lambda$, so γ_λ is defined and belongs to $S \cap C'$. **Q**

(d) If α is an ordinal and $C \subseteq \alpha$ has closure $\overline{C}$ for the order topology of α, then $\#(\overline{C}) = \#(C)$. **P** If C is finite this is trivial. Otherwise, for any $\beta \in \overline{C} \setminus \{\sup C\}$ set $f(\beta) = \min(C \setminus \beta)$. If β, $\beta' \in \overline{C}$ and $\beta < \beta' < \sup C$ there must be a $\gamma \in C$ such that $\beta \le \gamma < \beta'$ and $f(\beta) \le \gamma < f(\beta')$ (4A2S(a-ii)); thus f is injective. So

$$\#(\overline{C}) = \#(\overline{C} \setminus \{\sup C\}) \le \#(C) \le \#(\overline{C}). \text{ } \mathbf{Q}$$

(e) If α is an ordinal, there is a closed cofinal set $C \subseteq \alpha$ such that $\operatorname{otp} C = \operatorname{cf}\alpha$. **P** If $\operatorname{cf}\alpha$ is finite then α is either 0 or a successor ordinal and the result is trivial. Otherwise, let C_0 be a cofinal subset of α with cardinality $\operatorname{cf}\alpha$. Then its closure C in the order topology of α is cofinal and $\#(C) = \operatorname{cf}\alpha$ by (d). Since any cofinal subset of C is also cofinal with α,

$$\operatorname{cf}\alpha \le \operatorname{cf}C \le \#(C) = \operatorname{cf}\alpha$$

and C has cofinality $\operatorname{cf}\alpha$. **Q**

5A1B Ordinal arithmetic (a) For ordinals ξ, η their **ordinal sum** $\xi + \eta$ is defined inductively by saying that

$\xi + 0 = \xi$,
$\xi + (\eta + 1) = (\xi + \eta) + 1$,
$\xi + \eta = \sup_{\zeta<\eta} \xi + \zeta$ for non-zero limit ordinals η.

Ordinal addition is associative, and if $\xi \le \zeta < \xi + \eta$ there is a unique $\zeta' < \eta$ such that $\zeta = \xi + \zeta'$. (Kunen 80, I.7.18; Jech 78, p. 18; Jech 03, 2.18; Just & Weese 97, §10.2.) If we identify $\mathbb{N}$ with ω, then the ordinal sum of two finite ordinals corresponds to ordinary addition on $\mathbb{N}$.

(b) For ordinals ξ, η their **ordinal product** $\xi \cdot \eta$ is defined inductively by saying

$\xi \cdot 0 = 0$,
$\xi \cdot (\eta + 1)$ is the ordinal sum $\xi \cdot \eta + \xi$,
$\xi \cdot \eta = \sup_{\zeta<\eta} \xi \cdot \zeta$ for non-zero limit ordinals η

(Kunen 80, I.7.20; Jech 78, p. 19; Jech 03, 2.19). Note that $0 \cdot \eta = 0$ and $1 \cdot \eta = \eta$ for every η, and that $\sup_{\zeta\in A} \xi \cdot \zeta = \xi \cdot (\sup A)$ for every ξ and every non-empty set A of ordinals. Ordinal multiplication is associative (Kunen 80, I.7.20; Jech 03, 2.21).

(c) For ordinals ξ, η the **ordinal power** ξ^η is defined inductively by saying that

$\xi^0 = 1$,
$\xi^{\eta+1}$ is the ordinal product $\xi^\eta \cdot \xi$,
$\xi^\eta = \sup_{\zeta<\eta} \xi^\zeta$ for non-zero limit ordinals η

(Kunen 80, I.9.5; Jech 03, 2.20). **Warning!** If ξ and η happen to be cardinals, this is quite different from the 'cardinal power' of 5A1F below.

If ξ, η are ordinals, $\eta \ne 0$ and η is greater than or equal to the ordinal product $\xi \cdot \eta$, then η is at least the ordinal power ξ^ω. **P** Note first that as multiplication is associative, we can induce on n to show that $\xi \cdot \xi^n = \xi^{n+1}$ for every n. Now we are supposing that $\eta \ge 1 = \xi^0$. If $n \in \mathbb{N}$ and $\eta \ge \xi^n$, then

$$\eta \ge \xi \cdot \eta \ge \xi \cdot \xi^n = \xi^{n+1}.$$

So $\eta \ge \xi^n$ for every n and $\eta \ge \xi^\omega$. **Q**

5A1C Concatenation It will perhaps be helpful if I describe in detail a semi-standard notation which I have already used, in special cases, at many points in Volumes 3 and 4. Suppose that σ, τ are two functions with domains α, β respectively which are ordinals (e.g., initial segments of $\mathbb{N}$, if we think of non-negative integers as finite ordinals). Then we can form their **concatenation** $\sigma^\frown\tau$, setting

$$\operatorname{dom}(\sigma^\frown\tau) = \alpha + \beta$$

(the ordinal sum),

$$(\sigma^\frown\tau)(\xi) = \sigma(\xi) \text{ if } \xi < \alpha,$$
$$(\sigma^\frown\tau)(\alpha + \eta) = \tau(\eta) \text{ if } \eta < \beta.$$

The operator $^\frown$ is associative, that is, if σ, τ, υ have ordinal domains, then $(\sigma^\frown\tau)^\frown\upsilon = \sigma^\frown(\tau^\frown\upsilon)$, so we can omit brackets and speak of $\sigma^\frown\tau^\frown\upsilon$. The empty function $\emptyset$ is an identity in the sense that

$$\emptyset^\frown\sigma = \sigma^\frown\emptyset = \sigma$$

whenever $\operatorname{dom}\sigma$ is an ordinal.

In this context, it will often be helpful to have a special notation for functions with domain the singleton set $\{0\} = 1$; I will write $<t>$ for the function with domain $\{0\}$ and value t.

We can also have infinite concatenations. If $\langle\sigma_n\rangle_{n\in\mathbb{N}}$ is a sequence of functions with ordinal domains, we can form the concatenations

$$\sigma_0^\frown\sigma_1, \quad \sigma_0^\frown\sigma_1^\frown\sigma_2, \quad \sigma_0^\frown\sigma_1^\frown\sigma_2^\frown\sigma_3, \quad \ldots$$

to get a sequence of functions each extending its predecessors. The union will be a function with domain the ordinal $\sup_{n\in\mathbb{N}}\operatorname{dom}\sigma_0+\ldots+\operatorname{dom}\sigma_n$. I will generally denote it $\sigma_0^\frown\sigma_1^\frown\sigma_2^\frown\ldots$ or in some similar form.

5A1D Well-founded sets(a) A partially ordered set P is **well-founded** if every non-empty $A\subseteq P$ has a minimal element, that is, a $p\in A$ such that $q\not<p$ for every $q\in A$.

(b) If P is a well-founded partially ordered set, we have a rank function $r:P\to\mathrm{On}$ defined by saying that

$$r(p)=\sup\{r(q)+1:q<p\}$$

for every $p\in P$ (Kunen 80, III.5.7; Jech 78, p. 21; Jech 03, 2.27). The **height** of P is the least ordinal ζ such that $r(p)<\zeta$ for every $p\in P$.

(c) A partially ordered set P is well-founded iff there is no sequence $\langle p_n\rangle_{n\in\mathbb{N}}$ in P such that $p_{n+1}<p_n$ for every $n\in\mathbb{N}$. (If $A\subseteq P$ is non-empty and has no minimal element, we can choose inductively a strictly decreasing sequence in A.)

(d) If P is a well-founded partially ordered set with height ζ, $\#(\zeta)\le\#(P)$. **P** Let $r:P\to\mathrm{On}$ be the rank function of P. Set $A_\xi=\{p:r(p)\ge\xi\}$ for ordinals ξ. If $A_\xi\ne\emptyset$, then A_ξ has a minimal element p; now $r(q)<\xi$ whenever $q<p$, so $r(p)=\xi$. The height of P is the least ordinal ζ such that $r[P]\subseteq\zeta$; now $A_\xi\ne\emptyset$ for $\xi<\zeta$, so $\zeta=r[P]$ and $\#(\zeta)\le\#(P)$. **Q**

(e) The next result really belongs with the descriptive set theory of Chapter 42, but I had no reason to call on it in that volume, so I set it out here. If X is a Polish space and $\le$ is a well-founded relation on X such that $\{(x,y):x<y\}$ is analytic, then the height of $\le$ is countable. (Kechris 95, Theorem 31.1. This is a form of the 'Kunen-Martin theorem'.)

5A1E Trees In §421 I introduced trees of sequences. For this volume a more abstract approach is useful.

(a) A **tree** is a partially ordered set T such that $\{s:s\in T,\ s\le t\}$ is well-ordered for every $t\in T$; alternatively, a well-founded partially ordered set such that $\{s:s\in T,\ s\le t\}$ is totally ordered for every $t\in T$. In particular, T has a rank function $r:T\to\mathrm{On}$ defined by saying that

$$r(t)=\operatorname{otp}\{s:s<t\}=\min\{\xi:r(s)<\xi\text{ whenever }s<t\}$$

for every $t\in T$ (5A1D). (Try to avoid using this terminology in the same sentence as that of 421Ne and 562A.)

The **levels** of T are now the sets $\{t:r(t)=\xi\}$ for $\xi\in\mathrm{On}$. A **branch** of T is a maximal totally ordered subset. A tree is **well-pruned** if it has at most one minimal element and whenever $s,t\in T$ and $r(s)<r(t)$, there is an $s'\ge s$ such that $r(s')=r(t)$. If T is a tree, a **subtree** of T is a set $T'\subseteq T$ such that $s\in T'$ whenever $s\le t\in T'$; in this case, the rank function of T' is the restriction to T' of the rank function of T.

(b)(i) Let T be a tree in which every level is finite. Then T has a branch meeting every level. **P** If T is empty, this is trivial. Otherwise, let r be the rank function of T, and $\zeta>0$ the height of T; let $\mathcal{F}$ an ultrafilter on T containing $\{t:r(t)\ge\xi\}$ for every $\xi<\zeta$. Set $C=\{t:[t,\infty[\,\in\mathcal{F}\}$. Any two elements of C are upwards-compatible, so C is totally ordered, and C meets every level of T; so C is a branch of the kind we seek. **Q**

(ii) Let $(T,\preccurlyeq')$ be a tree of height ω_1 in which every level is countable. Then there is an ordering $\preccurlyeq$ of ω_1, included in the usual ordering $\le$ of ω_1, such that $(T,\preccurlyeq')$ is isomorphic to $(\omega_1,\preccurlyeq)$. **P** Let $\langle T_\xi\rangle_{\xi<\omega_1}$ be the levels of T. Let $\le'_\xi$ be a well-ordering of T_ξ for each $\xi<\omega_1$, and define $\le'$ on T by saying that $s\le' t$ if either $r(s)<r(t)$ or $r(s)=r(t)=\xi$ and $s\le'_\xi t$; then $\le'$ is a well-ordering of T of order type ω_1. Now the order-isomorphism between $(T,\le')$ and $(\omega_1,\le)$ copies $\preccurlyeq'$ onto a tree ordering of ω_1, isomorphic to $\preccurlyeq'$, and included in the usual ordering. **Q**

(c) An **Aronszajn tree** is a tree T of height ω_1 in which every branch and every level is countable. An Aronszajn tree T is **special** if it is expressible as $\bigcup_{n\in\mathbb{N}} A_n$ where no two elements of any A_n are comparable, that is, every A_n is an up-antichain.

(d)(i) A **Souslin tree** is a tree T of height ω_1 in which every branch and every up-antichain is countable.

(ii) Every Souslin tree is a non-special Aronszajn tree.

(iii) If T is a Souslin tree, it has a subtree which is a well-pruned Souslin tree. (Kunen 80, II.5.11; Jech 78, p. 218; Jech 03, 9.13.)

(iv) **Souslin's hypothesis** is the assertion

(SH) There are no Souslin trees.

5A1F Cardinal arithmetic(a)(i) An infinite cardinal which is not regular (4A1Aa) is **singular**. A cardinal κ is a **successor cardinal** if it is of the form λ^+ (2A1Fc, 2A1Kb); otherwise it is a **limit cardinal**. κ is a **strong limit cardinal** if it is uncountable and $2^\lambda < \kappa$ for every $\lambda < \kappa$. It is **weakly inaccessible** if it is a regular uncountable limit cardinal; it is **strongly inaccessible** if moreover it is a strong limit cardinal.

(ii) If κ is a cardinal, define $\kappa^{(+\xi)}$, for ordinals ξ, by setting

$$\kappa^{(+0)} = \kappa, \quad \kappa^{(+\xi)} = \sup_{\eta<\xi}(\kappa^{(+\eta)+}) \text{ if } \xi > 0,$$

that is, $\kappa^{(+\xi)} = \omega_{\zeta+\xi}$ if $\kappa = \omega_\zeta$.

(b)(i) If $\langle\kappa_i\rangle_{i\in I}$ is a family of cardinals, its **cardinal sum** is $\#(\{(i,\xi) : i \in I,\ \xi < \kappa_i\})$, which is at most $\max(\omega, \#(I), \sup_{i\in I}\kappa_i)$.

(ii) For cardinals κ and λ, the **cardinal product** $\kappa\cdot\lambda$ is $\#(\kappa\times\lambda) \le \max(\omega,\kappa,\lambda)$.

(iii) If κ and λ are cardinals there are two natural interpretations of the formula κ^λ: (i) the set of functions from λ to κ (ii) the cardinal of this set. In this volume the latter will be the usual one, but I will try to signal this by using the phrase **cardinal power**. Of course 2^λ is always the cardinal power; the corresponding set of functions will be denoted by $\{0,1\}^\lambda$. We could also think of κ and λ as ordinals, and look at the ordinal power κ^λ as described in 5A1Bc; but I think I do this exactly three times, all at the end of §539.

(c)(i) The cardinal power κ^λ is at most $2^{\max(\omega,\kappa,\lambda)}$ for any cardinals κ and λ. (The set of functions from λ to κ is a subset of $\mathcal{P}(\lambda\times\kappa)$.)

(ii) $\mathfrak{c}^\omega = \#(((\{0,1\})^{\mathbb{N}})^{\mathbb{N}}) = \#(\{0,1\}^{\mathbb{N}\times\mathbb{N}}) = \#(\{0,1\}^{\mathbb{N}}) = \mathfrak{c}$.

(d) $\operatorname{cf} 2^\kappa > \kappa$ for every infinite cardinal κ. (Jech 03, 5.11; Jech 78, p. 46; Erdős Hajnal Máté & Rado 84, 6.9; Kunen 80, 10.41; Just & Weese 97, 11.2.24. Compare (e-v) below.)

(e)(i) If κ and λ are infinite cardinals, then, defining $[\kappa]^{\le\lambda}$ as in 3A1J,

$$\begin{aligned}\operatorname{cf}[\kappa]^{\le\lambda} &= 1 \text{ if } \lambda \ge \kappa,\\ &\ge \kappa \text{ if } \lambda < \kappa.\end{aligned}$$

(ii) Let κ, λ and θ be infinite cardinals such that $\theta \le \lambda \le \kappa$. Then $\operatorname{cf}[\kappa]^{\le\theta} \le \max(\operatorname{cf}[\kappa]^{\le\lambda}, \operatorname{cf}[\lambda]^{\le\theta})$. **P** Let $\mathcal{A} \subseteq [\kappa]^\lambda$ be a cofinal set with cardinal $\operatorname{cf}[\kappa]^\lambda = \operatorname{cf}[\kappa]^{\le\lambda}$. Then $[\kappa]^{\le\theta} = \bigcup_{A\in\mathcal{A}}[A]^{\le\theta}$, so

$$\operatorname{cf}[\kappa]^{\le\theta} \le \max(\#(\mathcal{A}), \sup_{A\in\mathcal{A}}\operatorname{cf}[A]^{\le\theta}) \le \max(\operatorname{cf}[\kappa]^{\le\lambda}, \operatorname{cf}[\lambda]^{\le\theta}). \text{ **Q**}$$

(iii) Let κ and λ be infinite cardinals. Then the cardinal power κ^λ is $\max(2^\lambda, \operatorname{cf}[\kappa]^{\le\lambda})$. **P** $\kappa^\lambda \ge 2^\lambda$ because $\kappa \ge 2$; $\kappa^\lambda \ge \#([\kappa]^{\le\lambda}) \ge \operatorname{cf}[\kappa]^{\le\lambda}$ because $f \mapsto f[\lambda]$ is a surjection from the family F of functions from λ to κ onto $[\kappa]^{\le\lambda}\setminus\{\emptyset\}$. In the other direction, if $\kappa \le \lambda$ then $F \subseteq \mathcal{P}(\lambda\times\kappa)$ so

$$\kappa^\lambda = \#(F) \le 2^\lambda = \max(2^\lambda, \operatorname{cf}[\kappa]^{\le\lambda}).$$

If $\lambda < \kappa$ let $\mathcal{A} \subseteq [\kappa]^{\le\lambda}$ be a cofinal family with cardinal $\operatorname{cf}[\kappa]^{\le\lambda}$; then $F = \bigcup_{A\in\mathcal{A}} A^\lambda$ so

$$\#(F) \le \max(\#(\mathcal{A}), \sup_{A\in\mathcal{A}} \#(A^\lambda)) = \max(\operatorname{cf}[\kappa]^{\le\lambda}, 2^\lambda). \text{ **Q**}$$

Putting this together with (i) and (c-ii) above,

$$\operatorname{cf}[\mathfrak{c}]^{\le\omega} = \max(\mathfrak{c}, \operatorname{cf}[\mathfrak{c}]^{\le\omega}) = \max(2^\omega, \operatorname{cf}[\mathfrak{c}]^{\le\omega}) = \mathfrak{c}^\omega = \mathfrak{c}.$$

(iv) If λ is an infinite cardinal and $\lambda \le \kappa < \lambda^{(+\omega)}$, then $\operatorname{cf}[\kappa]^{\le\omega} \le \max(\kappa, \operatorname{cf}[\lambda]^{\le\omega})$, with equality if $\kappa > \omega$. **P** Induce on n to see that $\operatorname{cf}[\lambda^{(+n)}]^{\le\omega} \le \max(\lambda^{(+n)}, \operatorname{cf}[\lambda]^{\le\omega})$ for every $n \in \mathbb{N}$. At the inductive step to $n > 0$,

$$\begin{aligned}\operatorname{cf}([\lambda^{(+n)}]^{\le\omega}) &= \operatorname{cf}(\bigcup_{\xi<\lambda^{(+n)}} [\xi]^{\le\omega}) \le \max(\lambda^{(+n)}, \sup_{\xi<\lambda^{(+n)}} \operatorname{cf}[\xi]^{\le\omega}) \\ &\le \max(\lambda^{(+n)}, \lambda^{(+(n-1))}, \operatorname{cf}[\lambda]^{\le\omega}) = \max(\lambda^{(+n)}, \operatorname{cf}[\lambda]^{\le\omega}). \text{ **Q**}\end{aligned}$$

If κ is uncountable, then $\operatorname{cf}[\kappa]^{\le\omega} \ge \kappa$, so $\operatorname{cf}[\kappa]^{\le\omega} = \max(\kappa, \operatorname{cf}[\lambda]^{\le\omega})$. Consequently the cardinal power κ^ω is $\max(\mathfrak{c}, \kappa, \operatorname{cf}[\lambda]^{\le\omega}) = \max(\kappa, \lambda^\omega)$.

In particular, if $\omega_1 \le \kappa < \omega_\omega$ then $\operatorname{cf}[\kappa]^{\le\omega} = \kappa$ and $\kappa^\omega = \max(\mathfrak{c}, \kappa)$. Similarly, $(\mathfrak{c}^+)^\omega = \max(\mathfrak{c}^+, \mathfrak{c}) = \mathfrak{c}^+$, $(\mathfrak{c}^{++})^\omega = \mathfrak{c}^{++}$.

(v) If κ is a singular infinite cardinal, then $\operatorname{cf}([\kappa]^{\le\operatorname{cf}\kappa}) > \kappa$. **P** Set $\lambda = \operatorname{cf}\kappa$, and let $\langle\kappa_\xi\rangle_{\xi<\lambda}$ be a strictly increasing family of cardinals with supremum κ. If $\langle A_\eta\rangle_{\eta<\kappa}$ is a family in $[\kappa]^{\le\lambda}$, then for each $\xi < \lambda$ take $\alpha_\xi \in \kappa \setminus \bigcup_{\eta<\kappa_\xi} A_\eta$; set $A = \{\alpha_\xi : \xi < \lambda\} \in [\kappa]^{\le\lambda}$; then $A \not\subseteq A_\eta$ for every $\eta < \kappa$. **Q**

(f) If λ is a regular uncountable cardinal, $\theta \ge 2$ is a cardinal and $\kappa = \sup_{\delta<\lambda} \theta^\delta$, where θ^δ is the cardinal power, then

$$\#([\kappa]^{<\lambda}) = \sup_{\delta<\lambda} \kappa^\delta = \kappa.$$

P Of course $\kappa \le \#([\kappa]^{<\lambda})$ because $\lambda \ge 2$, while

$$\#([\kappa]^{<\lambda}) = \#(\bigcup_{\delta<\lambda}[\kappa]^\delta) \le \max(\lambda, \omega, \sup_{\delta<\lambda} \#([\kappa]^\delta)) \le \sup_{\delta<\lambda} \kappa^\delta$$

because $\lambda \le \sup_{\delta<\lambda} 2^\delta \le \kappa$. If $\delta < \lambda$ then, because λ is regular,

$$\kappa^\delta = (\sup_{\zeta<\lambda} \theta^\zeta)^\delta = \sup_{\zeta<\lambda}(\theta^\zeta)^\delta \le \sup_{\zeta<\lambda} \theta^{\max(\omega,\zeta,\delta)} \le \kappa. \text{ **Q**}$$

In particular, if κ is strongly inaccessible then $\kappa^\delta \le \kappa$ for every $\delta < \kappa$. (Take $\lambda = \kappa$ and $\theta = 2$.)

(g) Let X, Y and Z be sets, with $\#(X) \le 2^{\#(Z)}$ and $0 < \#(Y) \le \#(Z)$. Then there is a function $f : X \times Z^{\mathbb{N}} \to Y$ such that whenever $\langle x_n\rangle_{n\in\mathbb{N}}$ is a sequence of distinct elements of X and $\langle y_n\rangle_{n\in\mathbb{N}}$ is a sequence in Y there is a $z \in Z^{\mathbb{N}}$ such that $f(x_n, z) = y_n$ for every $n \in \mathbb{N}$. **P** We can suppose that $X \subseteq \mathcal{P}Z$ and $Y \subseteq Z$; moreover, the case of finite X is trivial, so we can suppose that Z is infinite. For each countably infinite set $I \subseteq Z$, (c-ii) above tells us that there is a surjection $g_I : I^{\mathbb{N}} \to (\mathcal{P}I)^{\mathbb{N}} \times I^{\mathbb{N}}$. Now let $f : X \times Z^{\mathbb{N}} \to Y$ be such that

> whenever $z \in Z^{\mathbb{N}}$ is such that $I = z[\mathbb{N}]$ is infinite, $g_I(z) = (\langle a_n\rangle_{n\in\mathbb{N}}, \langle y_n\rangle_{n\in\mathbb{N}})$ and $x \in X$ is such that there is just one n for which $a_n = x \cap I$, then $f(x, z) = y_n$.

In this case, if $\langle x_n\rangle_{n\in\mathbb{N}}$ is a sequence of distinct members of X and $\langle y_n\rangle_{n\in\mathbb{N}}$ is a sequence in Y, let I be a countably infinite subset of Z containing every y_n and such that $x_m \cap I \ne x_n \cap I$ for $m < n$; let $z \in I^{\mathbb{N}}$ be such that $g_I(z) = (\langle x_n \cap I\rangle_{n\in\mathbb{N}}, \langle y_n\rangle_{n\in\mathbb{N}})$; we shall have $f(x_n, z) = y_n$ for every n. **Q**

(h) If κ is an infinite cardinal, then 2^κ is at most the cardinal power $(\sup_{\lambda<\kappa} 2^\lambda)^{\operatorname{cf}\kappa}$. **P** Let $\langle\alpha_\xi\rangle_{\xi<\operatorname{cf}\kappa}$ be a family in κ with supremum κ. Set $D = \bigcup_{\alpha<\kappa} \mathcal{P}\alpha$; then

$$\#(D) \le \max(\kappa, \sup_{\alpha<\kappa} 2^{\#(\alpha)}) = \sup_{\lambda<\kappa} 2^\lambda.$$

Let F be the set of functions from $\operatorname{cf}\kappa$ to D; we have an injection $A \mapsto \langle A \cap \alpha_\xi\rangle_{\xi<\operatorname{cf}\kappa}$ from $\mathcal{P}\kappa$ to F, so $2^\kappa \le \#(F)$. **Q**

If $\omega \le \lambda < \kappa$ and $2^\theta = 2^\lambda$ for $\lambda \le \theta < \kappa$ but $2^\kappa > 2^\lambda$ then κ is regular. **P** $2^\kappa \le (2^\lambda)^{\operatorname{cf}\kappa} = 2^{\max(\lambda,\operatorname{cf}\kappa)}$. **Q**

5A1G Three fairly simple facts (a) There is a family $\langle a_I\rangle_{I\subseteq\mathbb{N}}$ of infinite subsets of $\mathbb{N}$ such that $a_I\cap a_J$ is finite whenever I, $J\subseteq\mathbb{N}$ are distinct.

(b) Let X be a set, $f:[X]^{<\omega}\to[X]^{\le\omega}$ a function, and $Y\subseteq X$. Then there is a $Z\subseteq X$ such that $Y\subseteq Z$, $f(I)\subseteq Z$ for every $I\in[Z]^{<\omega}$, and $\#(Z)\le\max(\omega,\#(Y))$.

(c) Let $\kappa\ge\mathfrak{c}$ be a cardinal and $\mathcal{A}$ a family of countable subsets of κ such that $\#(\mathcal{A})$ is less than the cardinal power κ^ω. Then there is a countably infinite $K\subseteq\kappa$ such that $I\cap K$ is finite for every $I\in\mathcal{A}$.

proof (a) For each $n\in\mathbb{N}$, set $K_n=\{i:2^n\le i<2^{n+1}\}$, and let $f_n:\mathcal{P}n\to K_n$ be a bijection; set $a_I=\{f_n(I\cap n):n\in\mathbb{N}\}$. (Or apply 5A1Nc below with $\kappa=\omega$.)

(b) Define $\langle Z_n\rangle_{n\in\mathbb{N}}$ inductively by setting $Z_0=Y$ and $Z_{n+1}=Z_n\cup\bigcup\{f(I):I\in[Z_n]^{<\omega}\}$ for each n. Then $\#(Z_n)\le\max(\omega,\#(Y))$ for each n, so setting $Z=\bigcup_{n\in\mathbb{N}}Z_n$ we still have $\#(Z)\le\max(\omega,\#(Y))$, while $f(I)\subseteq Z$ for every $I\in[Z]^{<\omega}$.

(c) If $\#(\mathcal{A})<\kappa$ this is trivial, as we can take $K\subseteq\kappa\setminus\bigcup\mathcal{A}$. Otherwise, let $\lambda\le\kappa$ be the least cardinal such that $\#(\mathcal{A})<\lambda^\omega$. Then $\operatorname{cf}\lambda=\omega$. **P?** Otherwise,

$$\lambda^\omega=\max(\lambda,\sup_{\theta<\lambda}\theta^\omega)\le\#(\mathcal{A}).\ \mathbf{XQ}$$

Let $\langle\lambda_n\rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence of cardinals with supremum λ, starting from $\lambda_0=0$ and $\lambda_1=\omega$ (of course $\lambda>\omega$ because $\#(\mathcal{A})\ge\mathfrak{c}$). For $n\in\mathbb{N}$ let $\phi_n:[n\times\lambda_n]^{<\omega}\to\lambda_{n+1}\setminus\lambda_n$ be an injective function. For $f:\mathbb{N}\to\lambda$ define $C_f\subseteq\lambda$ by setting

$$C_f=\{\phi_n(f\cap(n\times\lambda_n)):n\in\mathbb{N}\}.$$

If f, $g\in\lambda^{\mathbb{N}}$ are distinct, then there are an $i\in\mathbb{N}$ such that $f(i)\ne g(i)$ and an $m>i$ such that both $f(i)$ and $g(i)$ are less than λ_m, so that $f\cap(n\times\lambda_n)\ne g\cap(n\times\lambda_n)$ for every $n\ge m$ and $C_f\cap C_g$ is finite. It follows that for any $I\in\mathcal{A}$ the set $B_I=\{f:f\in\lambda^{\mathbb{N}},\,C_f\cap I\text{ is infinite}\}$ has cardinal at most $\mathfrak{c}$. Since $\mathfrak{c}\le\#(\mathcal{A})<\lambda^\omega$, there must be an $f\in\lambda^{\mathbb{N}}$ such that $C_f\cap I$ is finite for every $I\in\mathcal{A}$, and we can set $K=C_f$.

5A1H Partition calculus (a) The Erdős-Rado theorem Let κ be an infinite cardinal. Set $\kappa_1=\kappa$, $\kappa_{n+1}=2^{\kappa_n}$ for $n\ge1$. If $n\ge1$, $\#(B)\le\kappa$, $\#(A)>\kappa_n$ and $f:[A]^n\to B$ is a function, there is a $C\in[A]^{\kappa^+}$ such that f is constant on $[C]^n$. (Erdős Hajnal Máté & Rado 84, 16.5; Kanamori 03, 7.3; Just & Weese 97, 15.13.)

(b) Let κ be a cardinal of uncountable cofinality, and $Q\subseteq[\kappa]^2$. Then *either* there is a stationary $A\subseteq\kappa$ such that $[A]^2\subseteq Q$ *or* there is an infinite closed $B\subseteq\kappa$ such that $[B]^2\cap Q=\emptyset$. **P** (Cf. Erdős Hajnal Máté & Rado 84, 11.3.) Let $C\subseteq\kappa$ be a closed cofinal set with $\operatorname{otp}(C)=\operatorname{cf}\kappa$ (5A1Ae). Let S_0 be $\{\alpha:\alpha\in C,\,\operatorname{cf}\alpha=\omega\}$, so that S_0 is stationary (5A1Ac). For each $\alpha\in S_0$ let $\langle f_\alpha(n)\rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence in α with supremum α. Set

$$\mathcal{I}_\alpha=\{I:I\subseteq\alpha\cap C,\,[I\cup\{\alpha\}]^2\cap Q=\emptyset,\,\#(I\cap f_\alpha(n))\le n\text{ for every }n\in\mathbb{N}\}.$$

Let I_α be a maximal member of $\mathcal{I}_\alpha$. If there is any α such that I_α is infinite, we have the second alternative, witnessed by $B=I_\alpha\cup\{\alpha\}$, and we can stop. Otherwise, there is an $n\in\mathbb{N}$ such that $S_1=\{\alpha:\alpha\in S_0,\,I_\alpha\subseteq f_\alpha(n)\}$ is stationary. As $f_\alpha(n)<\alpha$ for every $\alpha\in S_1$, the Pressing-Down Lemma (4A1Cc) tells us that there is a $\gamma<\kappa$ such that $S_2=\{\alpha:\alpha\in S_1,\,f_\alpha(n)=\gamma\}$ is stationary. Because

$$\#([\gamma\cap C]^{<\omega})\le\max(\omega,\#(\gamma\cap C))<\operatorname{cf}\kappa,$$

there is an $I\subseteq\gamma\cap C$ such that $A=\{\alpha:\alpha\in S_2,\,I_\alpha=I\}$ is stationary.

? Suppose, if possible, that $[A]^2\not\subseteq Q$. Take α, $\beta\in A$ such that $\alpha<\beta$ and $\{\alpha,\beta\}\notin Q$. We know that $[I\cup\{\alpha\}]^2$ and $[I\cup\{\beta\}]^2$ are both disjoint from Q. So $[J\cup\{\beta\}]^2$ is disjoint from Q, where $J=I\cup\{\alpha\}$. If $m\le n$,

$$f_\beta(m)\le f_\beta(n)=\gamma=f_\alpha(n)<\alpha,$$

so $\#(J\cap f_\beta(m))=\#(I_\beta\cap f_\beta(m))\le m$; while if $m>n$ then

$$\#(J\cap f_\beta(m))\le\#(I)+1=\#(I_\alpha)+1=\#(I_\alpha\cap f_\alpha(n))+1\le n+1\le m.$$

So $J\in\mathcal{I}_\beta$; but J properly includes $I=I_\beta$, so this is impossible. **X**

Thus $[A]^2\subseteq Q$ and we have the first alternative. **Q**

5A1I Δ-systems and free sets: Proposition Let κ and λ be infinite cardinals and $\langle I_\xi\rangle_{\xi<\kappa}$ a family of sets with cardinal less than λ.

(a) If $\operatorname{cf}\kappa>\lambda$, there are a $\Gamma\in[\kappa]^\kappa$ and a set J of cardinal less than κ such that $I_\xi\cap I_\eta\subseteq J$ for all distinct ξ, $\eta\in\Gamma$.

(b) If $\kappa>\lambda$ is regular and the cardinal power θ^δ is less than κ whenever $\theta<\kappa$ and $\delta<\lambda$, then there is a $\Gamma'\in[\kappa]^\kappa$ such that $\langle I_\xi\rangle_{\xi\in\Gamma'}$ is a Δ-system (definition: 4A2A).

(c) If $\kappa>\lambda$ there is a $\Gamma''\in[\kappa]^\kappa$ such that $\eta\notin I_\xi$ for any distinct ξ, $\eta\in\Gamma''$.

proof (a) ? Otherwise, choose $\langle\Gamma_\alpha\rangle_{\alpha<\lambda}$ and $\langle J_\alpha\rangle_{\alpha<\lambda}$ as follows. $J_\alpha=\bigcup_{\beta<\alpha}\bigcup_{\xi\in\Gamma_\beta}I_\xi$. Given J_α, let $\Gamma_\alpha\subseteq\kappa$ be maximal subject to the requirement that $I_\xi\cap I_\eta\subseteq J_\alpha$ for all distinct ξ, $\eta\in\Gamma_\alpha$. Then we see by induction that $\#(J_\alpha)<\kappa$ so $\#(\Gamma_\alpha)<\kappa$ for every $\alpha<\lambda$; because $\operatorname{cf}\kappa>\lambda$, $\bigcup_{\alpha<\lambda}\Gamma_\alpha$ cannot be the whole of κ.

Take any $\xi\in\kappa\setminus\bigcup_{\alpha<\lambda}\Gamma_\alpha$. As $\#(I_\xi)<\lambda$, there must be an $\alpha<\lambda$ such that $I_\xi\cap J_\alpha=I_\xi\cap J_{\alpha+1}$. As $\xi\notin\Gamma_\alpha$, there is an $\eta\in\Gamma_\alpha$ such that $I_\xi\cap I_\eta\not\subseteq J_\alpha$; but now $I_\xi\cap I_\eta\setminus J_\alpha\subseteq I_\xi\cap J_{\alpha+1}\setminus J_\alpha$. **X**

(b) Let J and Γ be as in (a). Because $\operatorname{cf}\kappa>\lambda$, there must be some cardinal $\delta<\lambda$ such that $\Gamma_1=\{\xi:\xi\in\Gamma,\,\#(I_\xi\cap J)\le\delta\}$ has cardinal κ. Now $\#([J]^{\le\delta})\le\max(2,\#(J)^\delta)<\operatorname{cf}\kappa$, so there must be a $K\subseteq J$ such that $\Gamma'=\{\xi:\xi\in\Gamma_1,\,I_\xi\cap J=K\}$ has cardinal κ; and $\langle I_\xi\rangle_{\xi\in\Gamma'}$ is a Δ-system with root K.

(c) It is enough to consider the case in which $\xi\in I_\xi$ for every $\xi<\kappa$.

(i) If $\operatorname{cf}\kappa>\lambda$, take Γ and J from (a). Then we can choose $\langle\xi_\delta\rangle_{\delta<\kappa}$ inductively so that

$$\xi_\delta\in\Gamma\setminus(J\cup\bigcup_{\beta<\delta}I_{\xi_\beta})$$

for every $\delta<\kappa$; and $\{\xi_\delta:\delta<\kappa\}$ will serve for Γ''.

(ii) If $\operatorname{cf}\kappa=\theta\le\lambda$, let $\langle\kappa_\alpha\rangle_{\alpha<\theta}$ be a strictly increasing family of regular cardinals with supremum κ, starting from $\kappa_0\ge\lambda^{++}$. For each $\alpha<\theta$, (i) tells us that there is an $A_\alpha\in[\kappa_\alpha]^{\kappa_\alpha}$ such that $\eta\notin I_\xi$ for any distinct ξ, $\eta\in A_\alpha$. Set

$$B_\alpha=A_\alpha\setminus\bigcup_{\beta<\alpha}(B_\beta\cup\bigcup_{\xi\in A_\beta}I_\xi);$$

then $\#(B_\alpha)=\kappa_\alpha$ for each $\alpha<\theta$. Choose $\langle C_{\alpha\gamma}\rangle_{\alpha<\theta,\gamma<\lambda^+}$ and $\langle\zeta_\alpha\rangle_{\alpha<\theta}$ inductively, as follows. Given that $\langle C_{\beta\gamma}\rangle_{\beta<\alpha,\gamma<\lambda^+}$ is disjoint, then for each $\xi\in B_\alpha$ there is a $\zeta<\lambda^+$ such that $I_\xi\cap\bigcup_{\beta<\alpha}C_{\beta\gamma}$ is empty for every $\gamma\ge\zeta$; because $\lambda^+<\operatorname{cf}\kappa_\alpha$, there is a $\zeta_\alpha<\lambda^+$ such that

$$B'_\alpha=\{\xi:\xi\in B_\alpha,\,I_\xi\cap C_{\beta\gamma}=\emptyset\text{ whenever }\beta<\alpha\text{ and }\zeta_\alpha\le\gamma<\lambda^+\}$$

has cardinal κ_α. Let $\langle C_{\alpha\gamma}\rangle_{\gamma<\lambda^+}$ be a partition of B'_α into sets with cardinal κ_α, and continue.

At the end of the induction, $\gamma=\sup_{\alpha<\theta}\zeta_\alpha$ is less than λ^+. Set $\Gamma''=\bigcup_{\alpha<\theta}C_{\alpha\gamma}$. Then $\#(\Gamma'')=\kappa$. If ξ, η are distinct members of Γ'', let α, $\beta<\theta$ be such that $\xi\in C_{\alpha\gamma}$ and $\eta\in C_{\beta\gamma}$. If $\alpha<\beta$ then $\xi\in A_\alpha$ and $\eta\in B_\beta$ so $\eta\notin I_\xi$. If $\alpha=\beta$ then both ξ and η belong to A_α so $\eta\notin I_\xi$. If $\beta<\alpha$ then $\eta\in C_{\beta\gamma}$ while $\gamma\ge\zeta_\alpha$ and $\xi\in B'_\alpha$, so $\eta\notin I_\xi$. So Γ'' will serve.

Remark (c) above is Hajnal's Free Set Theorem.

5A1J Remarks (a) I spell out the applications of these results which are used in this volume. Let κ be an infinite cardinal and $\langle I_\xi\rangle_{\xi<\kappa}$ a family of countable sets.

(i) If $\operatorname{cf}\kappa\ge\omega_2$, there are a $\Gamma\in[\kappa]^\kappa$ and a set J with cardinal less than κ such that $I_\xi\cap I_\eta\subseteq J$ for all distinct ξ, $\eta\in\Gamma$.

(ii) If κ is regular and the cardinal power λ^ω is less than κ for every $\lambda<\kappa$, there is a $\Gamma'\in[\kappa]^\kappa$ such that $\langle I_\xi\rangle_{\xi\in\Gamma'}$ is a Δ-system. (Of course κ cannot be ω_1, so we can apply 5A1Ib with $\lambda=\omega_1$.)

(iii) If $\kappa\ge\omega_2$ there is a $\Gamma''\in[\kappa]^\kappa$ such that $\eta\notin I_\xi$ for any distinct ξ, $\eta\in\Gamma''$.

(b) If, in 5A1Ic, we are willing to settle for a weaker result, there is an easier proof which generalizes to more complex systems. Let λ be an infinite cardinal. Then there is a κ_0 such that for every cardinal $\kappa\ge\kappa_0$, every $n\in\mathbb{N}$ and every function $f:[\kappa]^n\to[\kappa]^{<\lambda}$ there is an $A\in[\kappa]^{\lambda^+}$ such that $\xi\notin f(I)$ whenever $I\in[A]^n$ and $\xi\in A\setminus I$. **P** By the Erdős-Rado theorem (5A1Ha), there is a κ_0 such that for every $\kappa\ge\kappa_0$, $n\ge1$ and function $g:[\kappa]^n\to\mathbb{N}$ there is an $A\in[\kappa]^{\lambda^+}$ such that g is constant on $[A]^n$. Now, given $n\in\mathbb{N}$, $\kappa\ge\kappa_0$ and $f:[\kappa]^n\to[\kappa]^{<\lambda}$, define $g:[\kappa]^{n+1}\to\mathbb{N}$ by saying that if $J=\{\xi_0,\dots,\xi_n\}$ with $\xi_0<\xi_1<\dots<\xi_n$, then

$g(J) = \min(\{n+1\} \cup \{j : j \le n,\ \xi_j \in f(J \setminus \{\xi_j\})\}$. Let $A \in [\kappa]^{\lambda^+}$ be such that g is constant on $[A]^{n+1}$. We can suppose that A has order type λ^+. **?** Suppose that the constant value of g in $[A]^{n+1}$ is $j \le n$. Let B be the set of the first λ members of A, I_0 the set of the first j members of A and I_1 the set of the first $n-j$ members of $A \setminus B$. Then we have $g(I_0 \cup \{\xi\} \cup I_1) = j$ for every $\xi \in B \setminus I_0$, so that $B \setminus I_0 \subseteq f(I_0 \cup I_1)$; but $\#(f(I_0 \cup I_1)) < \lambda$. **X** So the constant value of g on $[A]^{n+1}$ is $n+1$, and A satisfies the required condition. **Q**

(c) In the same complex of ideas, we have an elementary fact about the case $\lambda < \kappa = \omega$. If $n \in \mathbb{N}$ and $\langle K_i \rangle_{i \in \mathbb{N}}$ is a sequence in $[\mathbb{N}]^{\le n}$, there is an infinite $\Gamma \subseteq \mathbb{N}$ such that $\langle K_i \rangle_{i \in \Gamma}$ is a Δ-system. **P** Let $K \subseteq \mathbb{N}$ be a maximal set such that $I = \{i : K \subseteq K_i\}$ is infinite; then $\{i : i \in I,\ K_i \cap L \ne \emptyset\}$ is finite for every finite $L \subseteq \mathbb{N} \setminus K$, so we can choose Γ inductively by saying that $\Gamma = \{i : i \in I,\ K_i \cap K_j = K$ whenever $j \in \Gamma \cap i\}$. **Q**

(d) In 5A1Ic, and in (a-iii) here, we have a system $\langle I_\xi \rangle_{\xi < \kappa}$ of sets and are looking for large sets Γ'' which are 'free' in the sense that $\eta \notin I_\xi$ for distinct ξ, $\eta \in \Gamma''$. If we identify $\langle I_\xi \rangle_{\xi < \kappa}$ with the set $R = \{(\xi, \eta) : \xi < \kappa$, $\eta \in I_\xi\}$ then we are asking that (ξ, η) should not belong to R for distinct ξ, $\eta \in \Gamma''$. It will be useful to apply the same idea to other kinds of relation. In particular, if $R \subseteq X \times X$ is an equivalence relation on a set X I will say that a set $A \subseteq X$ is ***R*-free** if $(x, y) \notin R$ for all distinct x, $y \in A$, that is, if A meets each equivalence class for R in at most one point.

(e) Concerning free sets for equivalence relations it will help to have some elementary facts in quotable form. Let X be a set and R an equivalence relation on X.

(i) For any cardinal κ, there is a partition $\langle X_\xi \rangle_{\xi < \kappa}$ of X into R-free sets iff every R-equivalence class has cardinal at most κ. **P** Write $\mathcal{K}$ for the set of equivalence classes under R. If $\langle X_\xi \rangle_{\xi < \kappa}$ is a partition of X into R-free sets and $K \in \mathcal{K}$, then $\#(K \cap X_\xi) \le 1$ for every $\xi < \kappa$ and $\#(K) \le \kappa$. If $\#(K) \le \kappa$ for every $K \in \mathcal{K}$, then for each $K \in \mathcal{K}$ choose an injective function $h_K : K \to \kappa$; set $h(x) = h_K(x)$ whenever $x \in K \in \mathcal{K}$; then $\langle h^{-1}[\{\xi\}] \rangle_{\xi < \kappa}$ is a partition of X into R-free sets. **Q**

(ii) If $A \subseteq X$ is R-free then $R[B] \cap R[C] = \emptyset$ whenever B, $C \subseteq A$ are disjoint. **P?** If $x \in R[B] \cap R[C]$ there are $b \in B$, $c \in C$ such that $(b, x) \in R$ and $(c, x) \in R$, so $(b, c) \in R$; but b and c are distinct members of A, which is supposed to be R-free. **XQ**

5A1K Lemma Suppose that θ, λ and κ are cardinals, with $\theta < \lambda < \operatorname{cf}\kappa$, and that S is a stationary subset of κ. Let $\langle I_\xi \rangle_{\xi \in S}$ be a family in $[\lambda]^{\le \theta}$. Then there is a set $M \subseteq \lambda$ such that $\operatorname{cf}(\#(M)) \le \theta$ and $\{\xi : \xi \in S,\ I_\xi \subseteq M\}$ is stationary in κ.

proof For $M \subseteq \lambda$, set $S_M = \{\xi : \xi \in S,\ I_\xi \subseteq M\}$. Let $M \subseteq \lambda$ be a set of minimal cardinality such that S_M is stationary in κ. Set $\delta = \#(M)$. **?** If $\operatorname{cf}\delta > \theta$, enumerate M as $\langle \alpha_\eta \rangle_{\eta < \delta}$. For each $\xi \in S_M$, set $\beta_\xi = \sup\{\eta : \alpha_\eta \in I_\xi\}$; because $\#(I_\xi) \le \theta < \operatorname{cf}\delta$, $\beta_\xi < \delta$. Because $\delta \le \lambda < \operatorname{cf}\kappa$, there is a $\beta < \delta$ such that $S' = \{\xi : \xi \in S_M,\ \beta_\xi = \beta\}$ is stationary in κ (5A1Ab). Consider $M' = \{\alpha_\eta : \eta \le \beta\}$; then $\#(M') < \#(M)$ but $S_{M'} \supseteq S'$ so is stationary in κ, contrary to the choice of M. **X**

Thus M will serve.

5A1L Lemma Let $\langle X_i \rangle_{i \in I}$ be a non-empty family of infinite sets, with product X. Then there is a set $Y \subseteq X$, with $\#(Y) = \#(X)$, such that for every finite $L \subseteq Y$ there is an $i \in I$ such that $x(i) \ne y(i)$ for any distinct x, $y \in L$.

proof Set $\kappa = \#(X)$.

(a) We can well-order I in such a way that $\#(X_i) \le \#(X_j)$ whenever $i \le j$ in I. It will therefore be enough to deal with the case in which $I = \delta$ is an ordinal and $\#(X_\alpha) \le \#(X_\beta)$ whenever $\alpha \le \beta < \delta$. I proceed by induction on δ.

(b) If δ is finite then $\kappa = \max_{\alpha < \delta} \#(X_\alpha)$ and the result is trivial, since we can take the x_ξ to be all different at a single coordinate.

(c) Suppose there is a $\gamma < \delta$ such that $\#(\delta \setminus \gamma) < \#(\delta)$. Then, in particular, the order type of $\delta \setminus \gamma$ is less than the order type of δ. Set $I_0 = \gamma$, $I_1 = \delta \setminus \gamma$ and $Y_j = \prod_{\alpha \in I_j} X_\alpha$ for both j. Then $X \cong Y_0 \times Y_1$, so $\kappa = \max(\#(Y_0), \#(Y_1))$; say $\kappa = \#(Y_j)$. By the inductive hypothesis, there is a family $\langle y_\xi\rangle_{\xi<\kappa}$ in Y_j such that for any $L \in [\kappa]^{<\omega}$ there is an $\alpha \in I_j$ such that $\xi \mapsto y_\xi(\alpha) : L \to X_\alpha$ is injective. Taking x_ξ to be any member of X extending y_ξ, for each $\xi < \kappa$, we have a suitable family $\langle x_\xi\rangle_{\xi<\kappa}$ in X, and the induction proceeds.

(d) Suppose that δ is infinite and that $\#(\delta \setminus \gamma) = \#(\delta) = \lambda$ for every $\gamma < \delta$. Enumerate $[\delta]^{<\omega}$ as $\langle J_\zeta\rangle_{\zeta<\lambda}$, and choose $\langle\alpha_\zeta\rangle_{\zeta<\lambda}$ such that

$$J_\zeta \subseteq \alpha_\zeta \in \delta \setminus \{\alpha_\eta : \eta < \zeta\}$$

for each $\alpha < \lambda$. We have

$$\#(X_{\alpha_\zeta}) \ge \max(\omega, \sup_{\beta\in J_\zeta} \#(X_\beta)) \ge \#(\prod_{\beta\in J_\zeta} X_\beta),$$

so there is an injective function $f_\zeta : \prod_{\beta\in J_\zeta} X_\beta \to X_{\alpha_\zeta}$ for each $\zeta < \lambda$. Let $\langle z_\xi\rangle_{\xi<\kappa}$ be any enumeration of X. Because all the α_ζ are distinct, we can find $x_\xi \in X$, for each $\xi < \kappa$, such that $x_\xi(\alpha_\zeta) = f_\zeta(z_\xi{\restriction}J_\zeta)$ for every ζ. Now if $L \in [\kappa]^{<\omega}$ there must be a $\zeta < \lambda$ such that $z_\xi{\restriction}J_\zeta \ne z_\eta{\restriction}J_\zeta$ for any distinct ξ, $\eta \in L$; so that $\xi \mapsto x_\xi(\alpha_\zeta)$ is injective on L. Thus $\langle x_\xi\rangle_{\xi<\kappa}$ is a suitable family in X and the induction proceeds in this case also.

5A1M Definitions(a) Let X and Y be sets and $\mathcal{I}$ an ideal of subsets of X. Write $\mathrm{Tr}_{\mathcal{I}}(X;Y)$ for the **transversal number**

$$\sup\{\#(F) : F \subseteq Y^X,\ \{x : f(x) = g(x)\} \in \mathcal{I} \text{ for all distinct } f,\ g \in F\}.$$

(b) Let κ be a cardinal. Write $\mathrm{Tr}(\kappa)$ for

$$\mathrm{Tr}_{[\kappa]^{<\kappa}}(\kappa;\kappa) = \sup\{\#(F) : F \subseteq \kappa^\kappa,\ \#(f \cap g) < \kappa \text{ for all distinct } f,\ g \in F\}.$$

5A1N Lemma (a) For any infinite cardinal κ,

$$\kappa^+ \le \mathrm{Tr}(\kappa) \le 2^\kappa.$$

(b) For any infinite cardinal κ,

$$\max(\mathrm{Tr}(\kappa), \sup_{\delta<\kappa} 2^\delta) \ge \min(2^\kappa, \kappa^{(+\omega)}).$$

(c) If κ is such that $2^\delta \le \kappa$ for every $\delta < \kappa$, then $\mathrm{Tr}(\kappa) = 2^\kappa$, and in fact there is an $F \subseteq \kappa^\kappa$ such that $\#(F) = 2^\kappa$ and $\#(f \cap g) < \kappa$ for all distinct f, $g \in F$.

(d) If X and Y are sets and $\mathcal{I}$ is a maximal proper ideal of $\mathcal{P}X$, then there is an $F \subseteq Y^X$ such that $\#(F) = \mathrm{Tr}_{\mathcal{I}}(X;Y)$ and $\{x : f(x) = g(x)\} \in \mathcal{I}$ for all distinct f, $g \in F$.

proof (a) We can build inductively a family $\langle f_\alpha\rangle_{\alpha<\kappa^+}$ in κ^κ, as follows. Given $\langle f_\alpha\rangle_{\alpha<\beta}$, where $\beta < \kappa^+$, let $\theta : \beta \to \kappa$ be any injection. Now choose $f_\beta : \kappa \to \kappa$ so that

$$f_\beta(\xi) \ne f_\alpha(\xi) \text{ whenever } \alpha < \beta \text{ and } \theta(\alpha) \le \xi.$$

This will mean that if $\alpha < \beta$, then

$$\{\xi : f_\alpha(\xi) = f_\beta(\xi)\} \subseteq \theta(\alpha)$$

has cardinal less than κ. So at the end of the induction, $F = \{f_\alpha : \alpha < \kappa^+\}$ will witness that $\mathrm{Tr}(\kappa) \ge \kappa^+$. On the other hand, $\mathrm{Tr}(\kappa) \le \#(\kappa^\kappa) = 2^\kappa$.

(b)? If not, then take $\lambda = \max(\mathrm{Tr}(\kappa), \sup_{\delta<\kappa} 2^\delta) < \min(2^\kappa, \kappa^{(+\omega)})$. For each $\xi < \kappa$ take an injective function $\phi_\xi : \mathcal{P}\xi \to \lambda$. Because $\lambda < 2^\kappa$, we have an injective function $h : \lambda^+ \to \mathcal{P}\kappa$. For $\alpha < \lambda^+$ set $g_\alpha(\xi) = \phi_\xi(h(\alpha) \cap \xi)$ for every $\xi < \kappa$; then $\langle g_\alpha\rangle_{\alpha<\lambda^+}$ is a family in λ^κ such that $\#(g_\alpha \cap g_\beta) < \kappa$ whenever $\alpha \ne \beta$.

Apply 5A1K with $S = \lambda^+$, $I_\alpha = g_\alpha[\kappa]$ to see that there is a set $M \subseteq \lambda$ with $\mathrm{cf}(\#(M)) \le \kappa$ and $S_1 = \{\alpha : \alpha < \lambda^+,\ g_\alpha[\kappa] \subseteq M\}$ stationary in λ^+. Because $\lambda < \kappa^{(+\omega)}$, we must have $\#(M) \le \kappa$. If $f : M \to \kappa$ is any injection, $\langle fg_\alpha\rangle_{\alpha\in S_1}$ will witness that $\mathrm{Tr}(\kappa) \ge \#(S_1) = \lambda^+$; which is impossible. **X**

(c) For each $\xi<\kappa$, let $\phi_\xi:\mathcal{P}\xi\to\kappa$ be injective. For $A\subseteq\kappa$, define $f_A\in\kappa^\kappa$ by writing

$$f_A(\xi)=\phi_\xi(A\cap\xi)\text{ for every }\xi<\kappa.$$

Then $F=\{f_A:A\subseteq\kappa\}$ has the required property, and $\mathrm{Tr}(\kappa)\ge 2^\kappa$; by (a), we have equality.

(d) Take any maximal set $F\subseteq Y^X$ such that $\{x:f(x)=f'(x)\}\in\mathcal{I}$ for all distinct f, $f'\in F$. Then $\#(F)=\mathrm{Tr}_{\mathcal{I}}(X;Y)$. **P** Of course $\#(F)\le\mathrm{Tr}_{\mathcal{I}}(X;Y)$. Suppose that $G\subseteq Y^X$ is such that $\{x:g(x)=g'(x)\}\in\mathcal{I}$ for all distinct g, $g'\in G$. For each $g\in G$, there must be an $f_g\in F$ such that $\{x:g(x)=f_g(x)\}\notin\mathcal{I}$; because $\mathcal{I}$ is maximal, $\{x:g(x)\ne f_g(x)\}\in\mathcal{I}$. If g, $h\in G$ are distinct, then

$$\begin{aligned}\{x:f_g(x)=f_h(x)\}&\subseteq\{x:f_g(x)\ne g(x)\}\cup\{x:g(x)=h(x)\}\cup\{x:h(x)\ne f_h(x)\}\\&\in\mathcal{I}\end{aligned}$$

so $f_g\ne f_h$. Thus we have an injective function from G to F and $\#(G)\le\#(F)$. As G is arbitrary, $\mathrm{Tr}_{\mathcal{I}}(X;Y)\le\#(F)$ and we have equality. **Q**

5A1O Almost-square-sequences: Lemma Let λ, κ be regular infinite cardinals, with $\kappa>\max(\omega_1,\lambda)$. Then we can find a stationary set $S\subseteq\kappa^+$ and a family $\langle C_\alpha\rangle_{\alpha\in S}$ of sets such that

(i) for each $\alpha\in S$, C_α is a closed cofinal set in α of order type λ;

(ii) if α, $\beta\in S$ and γ is a limit point of both C_α and C_β then $C_\alpha\cap\gamma=C_\beta\cap\gamma$.

Remark Compare the axiom $\square_\kappa$ of 5A6D below.

proof (a) For each $\gamma<\kappa^+$ fix an injection $f_\gamma:\gamma\to\kappa$. Let S_0 be the set of ordinals $\alpha<\kappa^+$ of cofinality λ; then S_0 is stationary in κ^+ (5A1Ac). For each $\alpha\in S_0$ choose a non-decreasing family $\langle N_{\alpha\delta}\rangle_{\delta<\kappa}$ of subsets of κ^+ such that

(α) $N_{\alpha0}$ is a cofinal subset of α with cardinal λ;

(β) if $\delta<\kappa$ then

$$N_{\alpha,\delta+1}=\bigcup\{f_\gamma[N_{\alpha\delta}]\cup f_\gamma^{-1}[\delta]:\gamma\in N_{\alpha\delta}\}\cup\overline{N}_{\alpha\delta}\cup\delta$$

(taking the closure $\overline{N}_{\alpha\delta}$ in the order topology of κ^+);

(γ) if $\delta<\kappa$ is a non-zero limit ordinal then $N_{\alpha\delta}=\bigcup_{\delta'<\delta}N_{\alpha\delta'}$.

Then $\#(N_{\alpha\delta})\le\max(\lambda,\#(\delta))<\kappa$ for each $\delta<\kappa$ (using 5A1Ad). Because κ is regular, $\sup(N_{\alpha\delta}\cap\kappa)<\kappa$ for every δ. It follows that $\{\delta:\delta<\kappa,\,N_{\alpha\delta}\cap\kappa=\delta\}$ is a closed cofinal set in κ, and in particular contains an ordinal of cofinality ω_1, for every $\alpha\in S_0$. Let $\delta<\kappa$ be such that $\mathrm{cf}\,\delta=\omega_1$ and

$$S_1=\{\alpha:\alpha\in S_0,\,N_{\alpha\delta}\cap\kappa=\delta\}$$

is stationary in κ^+. For $\alpha\in S_1$, set $C^*_\alpha=\alpha\cap\overline{N}_{\alpha\delta}$; then C^*_α is a closed cofinal set in α and $\#(C^*_\alpha)<\kappa$ so $\mathrm{otp}(C^*_\alpha)<\kappa$. Let $\zeta<\kappa$ be such that

$$S=\{\alpha:\alpha\in S_1,\,\mathrm{otp}(C^*_\alpha)=\zeta\}$$

is stationary in κ^+. Observe that as $\mathrm{cf}\,C^*_\alpha=\mathrm{cf}\,\alpha=\lambda$ for each $\alpha\in S$, $\mathrm{cf}\,\zeta=\lambda$.

(b) Take any closed cofinal set $C\subseteq\zeta$ of order type λ and for each $\alpha\in S$ let C_α be the image of C in C^*_α under the order-isomorphism between ζ and C^*_α. Then C_α will be a closed cofinal subset of α of order type λ.

I claim that if α, $\beta\in S$ and γ is a common limit point of C_α, C_β then $C_\alpha\cap\gamma=C_\beta\cap\gamma$.

P case 1 Suppose $\lambda=\omega$. In this case the only limit point of C_α will be α itself, and similarly for β, so that in this case we have $\alpha=\beta$ and there is nothing more to do.

case 2 Suppose $\mathrm{cf}\,\gamma=\omega<\lambda$. Then γ is a limit point of $C_\alpha\subseteq C^*_\alpha\subseteq\overline{N}_{\alpha\delta}$, so there is an increasing sequence in $N_{\alpha\delta}$ with supremum γ; as $N_{\alpha\delta}=\bigcup_{\delta'<\delta}N_{\alpha\delta'}$ and $\mathrm{cf}\,\delta=\omega_1$, this sequence lies entirely within $N_{\alpha\delta'}$ for some $\delta'<\delta$, and $\gamma\in\overline{N}_{\alpha\delta'}\subseteq N_{\alpha,\delta'+1}$. Now, for $\delta'+1\le\xi<\delta$, $N_{\alpha,\xi+1}\supseteq f_\gamma^{-1}[\xi]\cup f_\gamma[N_{\alpha\xi}]$; consequently

$$N_{\alpha\delta}\cap\gamma=f_\gamma^{-1}[N_{\alpha\delta}\cap\kappa]=f_\gamma^{-1}[\delta].$$

Similarly, $N_{\beta\delta}\cap\gamma=f_\gamma^{-1}[\delta]$. Now

$$C^*_\alpha\cap\gamma=\overline{N}_{\alpha\delta}\cap\gamma=\overline{f_\gamma^{-1}[\delta]}\cap\gamma=C^*_\beta\cap\gamma.$$

Accordingly the increasing enumerations of C^*_α and C^*_β must agree on $C^*_\alpha\cap\gamma=C^*_\beta\cap\gamma$, and $C_\alpha\cap\gamma=C_\beta\cap\gamma$.

case 3 Suppose that $\operatorname{cf}\gamma>\omega$ and $\lambda>\omega$. Because $\gamma=\sup(C_\alpha\cap\gamma)=\sup(C_\beta\cap\gamma)$,

$$D=\{\gamma':\gamma'<\gamma\text{ is a limit point of both }C_\alpha\text{ and }C_\beta,\ \operatorname{cf}\gamma'=\omega\}$$

is cofinal with γ, and

$$C_\alpha\cap\gamma=\bigcup_{\gamma'\in D}C_\alpha\cap\gamma'=C_\beta\cap\gamma,$$

using case 2. **Q**

Thus S and $\langle C_\alpha\rangle_{\alpha\in S}$ have the required properties.

5A1P Corollary Let κ, λ be regular infinite cardinals with $\lambda>\max(\omega_1,\kappa)$. Then we can find a stationary subset S of λ^+ and a family $\langle g_\alpha\rangle_{\alpha\in S}$ of functions from κ to λ^+ such that, for all distinct α, $\beta\in S$,

(i) $g_\alpha[\kappa]\subseteq\alpha$,

(ii) $\#(g_\alpha\cap g_\beta)<\kappa$,

(iii) if $\theta<\kappa$ is a limit ordinal and $g_\alpha(\theta)=g_\beta(\theta)$ then $g_\alpha\restriction\theta=g_\beta\restriction\theta$.

proof Take $\langle C_\alpha\rangle_{\alpha\in S}$ from 5A1O and let g_α be the increasing enumeration of C_α.

5A1Q A fragment of finite combinatorics turns out to be a basis for some interesting measure theory (546I).

Lemma Let I and J be non-empty finite sets, and $R\subseteq I\times J$ a relation such that $R[I]=J$. Set

$$k=\max_{x\in I}\#(R[\{x\}]),\quad l=\min_{y\in J}\#(R^{-1}[\{y\}]).$$

Then there is a $K\subseteq I$ such that $R[K]=J$ and $\#(K)\le\frac{1+\ln k}{l}\#(I)$.

proof (Bartoszyński & Judah 95, 3.3.10) **(a)** Choose $\langle I_j\rangle_{j<k}$, $\langle J_j\rangle_{j\le k}$ inductively, as follows. For each $j\le k$, set $J_j=J\setminus R[\bigcup_{i<j}I_i]$, and if $j<k$ take a maximal set $I_j\subseteq I$ such that

$$\#(R[\{x\}]\cap J_j)=k-j$$

for every $x\in I_j$,

$$R[\{x\}]\cap R[\{x'\}]\cap J_j=\emptyset$$

whenever x, $x'\in I_j$ are distinct. At the end of the induction set $K=\bigcup_{i<k}I_i$.

Now $\#(R[\{x\}]\cap J_j)\le k-j$ for every $j\le k$ and $x\in I$. **P** Induce on j. Start with $J_0=J$ and $\#(R[\{x\}])\le k$ for every x, by the definition of k. For the inductive step to $j+1\le k$, if $x\in I$ then

$$\#(R[\{x\}]\cap J_{j+1})\le\#(R[\{x\}]\cap J_j)\le k-j$$

by the inductive hypothesis, and if $\#(R[\{x\}]\cap J_j)=k-j$ then there is an $x'\in I_j$ such that $R[\{x\}]\cap R[\{x'\}]\cap J_j$ is non-empty, so that $R[\{x\}]\cap J_{j+1}$ is strictly smaller than $R[\{x\}]\cap J_j$ and cannot have more than $k-j-1$ members. **Q**

In particular, $R[\{x\}]\cap J_k=\emptyset$ for every $x\in I$, so

$$R[I]=R[I]\setminus J_k=J\setminus J_k=R[K].$$

(b) For each $j<k$, $\langle R[\{x\}]\cap J_j\rangle_{x\in I_j}$ is a partition of $J_j\setminus J_{j+1}$ into sets with cardinal $k-j$, so that $\#(J_j)-\#(J_{j+1})=(k-j)\#(I_j)$. Next, if we set $R_j=R\cap(I\times J_j)$ we see that

$$\#(R_j[\{x\}])=\#(R_j[\{x\}]\cap J_j)\le k-j,\quad\#(R_j^{-1}[\{y\}])=\#(R^{-1}[\{y\}])\ge l$$

whenever $x\in I$ and $y\in J_j$, so

$$l\#(J_j)\le\#(R_j)\le(k-j)\#(I)$$

and $\#(J_j)\le\frac{k-j}{l}\#(I)$. Accordingly

$$
\begin{aligned}
\#(K) &\le \sum_{j=0}^{k-1} \#(I_j) = \sum_{j=0}^{k-1} \tfrac{1}{k-j}(\#(J_j) - \#(J_{j+1})) \\
&= \sum_{j=1}^{k-1} \#(J_j)(\tfrac{1}{k-j} - \tfrac{1}{k-(j-1)}) + \tfrac{1}{k}\#(J_0) - \#(J_k) \\
&\le \tfrac{1}{k}\#(J_0) + \sum_{j=1}^{k-1} \tfrac{\#(J_j)}{(k-j)(k-j+1)} \le \tfrac{\#(I)}{l}(1 + \sum_{j=1}^{k-1} \tfrac{k-j}{(k-j)(k-j+1)}) \\
&= \tfrac{\#(I)}{l}(1 + \sum_{j=1}^{k-1} \tfrac{1}{k-j+1}) = \tfrac{\#(I)}{l}(1 + \sum_{j=2}^{k} \tfrac{1}{j}) \le \tfrac{1+\ln k}{l}\#(I)
\end{aligned}
$$

as required.

5A1R Stationary families of sets For the sake of a calculation which will be needed in §547, I introduce a kind of generalization of the idea of 'stationary' subset of an ordinal.

Definition If I is a set and $\mathcal{A}$ is a family of sets, I will say that $\mathcal{A}$ is **stationary over** I if for every function $f : [I]^{<\omega} \to [I]^{\le\omega}$ there is an $A \in \mathcal{A}$ such that $f(J) \subseteq A$ for every $J \in [A \cap I]^{<\omega}$.

5A1S Elementary remarks (a) If $\mathcal{A}$ is stationary over I, then $\{A \cap I : A \in \mathcal{A}\}$ is stationary over I.

(b) If $\mathcal{A}$ is stationary over I, and for every $A \in \mathcal{A}$ we are given a family $\mathcal{B}_A$ which is stationary over A, then $\bigcup_{A\in\mathcal{A}} \mathcal{B}_A$ is stationary over I.

(c) If ζ is an ordinal of uncountable cofinality, and $S \subseteq \zeta$ is stationary in the ordinary sense of 4A1C, then S is stationary over ζ in the sense of 5A1R. **P** Let $f : [\zeta]^{<\omega} \to [\zeta]^{\le\omega}$ be a function. Consider $C_f = \{\xi : \xi < \zeta,$ $f(J) \subseteq \xi$ for every $J \in [\xi]^{<\omega}\}$. Because $\operatorname{cf}\zeta > \omega$, C_f is closed and cofinal in ζ (cf. 4A1B(c-iii)), so meets S. As f is arbitrary, S is stationary over ζ. **Q**

5A1T In §539 we shall need an important property of ω_1.

Theorem (a) There is a family $\langle e_\xi\rangle_{\xi<\omega_1}$ such that $e_\xi : \xi \to \mathbb{N}$ is an injective function for each $\xi < \omega_1$ and $e_\eta \triangle (e_\xi{\restriction}\eta)$ is finite whenever $\eta < \xi < \omega_1$.

(b) There is a sequence $\langle \le_n\rangle_{n\in\mathbb{N}}$ of partial orders on ω_1 such that

$(\omega_1, \le_n)$ is a tree of height at most $n+1$ for each $n \in \mathbb{N}$,

$\eta \le_0 \xi$ iff $\eta = \xi$,

$\le_n \subseteq \le_{n+1}$ for every $n \in \mathbb{N}$,

$\bigcup_{n\in\mathbb{N}} \le_n$ is the usual well-ordering of ω_1.

proof[1] **(a)** Choose $\langle e_\xi\rangle_{\xi<\omega_1}$ inductively; as well as the obvious inductive hypothesis that $e_\eta \triangle (e_\xi{\restriction}\eta)$ is finite whenever $\eta < \xi$, we need to ensure that $\mathbb{N} \setminus e_\xi[\xi]$ is always infinite.

Start the induction with e_0 the empty function. For the inductive step to $\xi+1$, where $\xi < \omega_1$, take any $k \in \mathbb{N} \setminus e_\xi[\xi]$ and set $e_{\xi+1} = e_\xi \cup \{(\xi, k)\}$. For the inductive step to a non-zero limit ordinal $\xi < \omega_1$, choose a strictly increasing sequence $\langle \eta_n\rangle_{n\in\mathbb{N}}$ of ordinals with supremum ξ and define $\langle e'_n\rangle_{n\in\mathbb{N}}$, $\langle k_n\rangle_{n\in\mathbb{N}}$ inductively as follows. Start with $e'_0 = e_{\eta_0}$ and $k_0 \in \mathbb{N} \setminus e_{\eta_0}[\eta_0]$. Given e'_n and $k_0, \dots, k_n \in \mathbb{N}$ such that $e'_n : \eta_n \to \mathbb{N} \setminus \{k_0, \dots, k_n\}$ is injective and $e'_n \triangle e_{\eta_n}$ is finite, we see that

$$e'_n \triangle (e_{\eta_{n+1}}{\restriction}\eta_n) \subseteq (e'_n \triangle e_{\eta_n}) \cup (e_{\eta_n} \triangle (e_{\eta_{n+1}}{\restriction}\eta_n))$$

is finite while $e_{\eta_{n+1}}$ is injective and $\mathbb{N} \setminus e_{\eta_{n+1}}[\eta_{n+1}]$ is infinite. We therefore have room to adjust $e_{\eta_{n+1}}$ at finitely many points to obtain an injective function $e'_{n+1} : \eta_{n+1} \to \mathbb{N} \setminus \{k_0, \dots, k_n\}$, extending e'_n, such that $e'_{n+1} \triangle e_{\eta_{n+1}}$ is finite. Because $e'_{n+1} \triangle e_{\eta_{n+1}}$ is finite, $\mathbb{N} \setminus e'_{n+1}[\eta_{n+1}]$ is infinite and we can find $k_{n+1} \in \mathbb{N} \setminus (\{k_0, \dots, k_n\} \cup e'_{n+1}[\eta_{n+1}])$ before continuing this internal induction. Finally set $e_\xi = \bigcup_{n\in\mathbb{N}} e'_n$;

[1] I regret that I cannot recall where I saw this proof.

then $e_\xi : \xi \to \mathbb{N} \setminus \{k_i : i \in \mathbb{N}\}$ is injective and $e_{\eta_n} \triangle (e_\xi \restriction \eta_n) = e_{\eta_n} \triangle e'_n$ is finite for every $n \in \mathbb{N}$. Now if η is any ordinal less than ξ, there is an $n \in \mathbb{N}$ such that $\eta < \eta_n$, so that

$$e_\eta \triangle (e_\xi \restriction \eta) \subseteq (e_\eta \triangle (e_{\eta_n} \restriction \eta)) \cup (e_{\eta_n} \triangle (e_\xi \restriction \eta_n))$$

is finite. Thus the outer induction continues and we get a suitable family $\langle e_\xi \rangle_{\xi < \omega_1}$.

(b) Take a family $\langle e_\xi \rangle_{\xi < \omega_1}$ as in (a). For $n \in \mathbb{N}$ and $\xi, \eta < \omega_1$, say that $\eta \le_n \xi$ iff either $\eta = \xi$ or

$$\eta < \xi, \quad e_\xi(\eta) < n, \quad e_\eta \triangle (e_\xi \restriction \eta) \subseteq \omega_1 \times n.$$

We need to check the following.

(i) If $\zeta \le_n \eta \le_n \xi$ then $\zeta \le_n \xi$. **P** If either $\zeta = \eta$ or $\eta = \xi$ this is trivial. Otherwise, $\zeta < \eta < \xi$ and

$$e_\zeta \triangle (e_\xi \restriction \zeta) \subseteq (e_\zeta \triangle (e_\eta \restriction \zeta)) \cup (e_\eta \triangle (e_\xi \restriction \eta)) \subseteq \omega_1 \times n.$$

Of course $\zeta < \xi$. If $e_\xi(\zeta) = e_\eta(\zeta)$, then $e_\xi(\zeta) < n$ because $e_\eta(\zeta) < n$; otherwise, $e_\xi(\zeta) < n$ because $e_\eta \triangle (e_\xi \restriction \eta) \subseteq \omega_1 \times n$. So $\zeta \le_n \xi$. **Q**

Thus $\le_n$ is transitive; the definition ensured that it would be reflexive; and it is antisymmetric because $\le$ is.

(ii) If $\zeta \le_n \xi$ and $\eta \le_n \xi$ then either $\zeta \le_n \eta$ or $\eta \le_n \zeta$. **P** We can suppose that $\zeta \le \eta$. If $\zeta = \eta$ or $\eta = \xi$ then of course $\zeta \le_n \eta$. If $\zeta < \eta < \xi$, then

$$e_\zeta \triangle (e_\eta \restriction \zeta) \subseteq (e_\zeta \triangle (e_\xi \restriction \zeta)) \cup (e_\eta \triangle (e_\xi \restriction \eta)) \subseteq n.$$

If $e_\eta(\zeta) = e_\xi(\zeta)$ then $e_\eta(\zeta) < n$ because $e_\xi(\zeta) < n$; otherwise $e_\eta(\zeta) < n$ because $e_\eta \triangle (e_\xi \restriction \eta) \subseteq \omega_1 \times n$. So $\zeta \le_n \eta$. **Q**

(iii) Thus $\{\eta : \eta \le_n \xi\}$ is totally ordered by $\le_n$, for every ξ. Moreover, $\{\eta : \eta \le_n \xi\} \subseteq \{\xi\} \cup \{\eta : \eta < \xi, e_\xi(\eta) < n\}$ has at most $n+1$ members, because e_ξ is injective. So $\le_n$ is well-founded and is a tree ordering with height at most n.

(iv) It is also immediate from the definition that $\langle \le_n \rangle_{n \in \mathbb{N}}$ is non-decreasing. Finally, if $\eta < \xi$, there is an $n \in \mathbb{N}$ such that $e_\xi(\eta) < n$ and $e_\eta \triangle (e_\xi \restriction \eta) \subseteq \omega_1 \times n$, because $e_\eta \triangle (e_\xi \restriction \eta)$ is itself finite. So $\le = \bigcup_{n \in \mathbb{N}} \le_n$.

Version of 25.2.21

5A2 Pcf theory

In §542 I call on some results from Shelah's pcf theory. As I have still not come across an elementary textbook for this material, I copy out part of the appendix of Fremlin 93, itself drawn largely from Burke & Magidor 90.

5A2A Reduced products We need the following elementary generalization of the construction in 351M. Let $\langle P_i \rangle_{i \in I}$ be a family of partially ordered sets with product P.

(a) Let $\mathcal{F}$ be a filter on I. We have an equivalence relation $\equiv_{\mathcal{F}}$ on P, given by saying that $f \equiv_{\mathcal{F}} g$ if $\{i : f(i) = g(i)\} \in \mathcal{F}$. I write $P|\mathcal{F}$ for the set of equivalence classes under this relation, the **partial order reduced product** of $\langle P_i \rangle_{i \in I}$ modulo $\mathcal{F}$. Now $P|\mathcal{F}$ is again a partially ordered set, writing

$$f^\bullet \le g^\bullet \iff f \le_{\mathcal{F}} g \iff \{i : f(i) \le g(i)\} \in \mathcal{F}.$$

Observe that if every P_i is totally ordered and $\mathcal{F}$ is an ultrafilter, then $P|\mathcal{F}$ is totally ordered.

(b) For any filter $\mathcal{F}$ on I we have

$$\min_{i \in I} \operatorname{add} P_i = \operatorname{add} P \le \sup_{F \in \mathcal{F}} \operatorname{add}(\prod_{i \in F} P_i) = \sup_{F \in \mathcal{F}} \min_{i \in F} \operatorname{add} P_i$$
$$\le \operatorname{add}(P|\mathcal{F}),$$

$$\mathrm{cf}(P|\mathcal{F}) \le \min_{F\in\mathcal{F}} \mathrm{cf}(\prod_{i\in F} P_i) \le \mathrm{cf}\,P.$$

P By 511Hg, $\mathrm{add}(\prod_{i\in F} P_i) = \min_{i\in F} \mathrm{add}\,P_i$ for any $F \in \mathcal{F}$, and in particular when $F = I$. For $p \in P|\mathcal{F}$ choose $f_p \in P$ such that $f_p^\bullet = p$. If $F \in \mathcal{F}$ then $p \mapsto f_p\restriction F$ is a Tukey function from $P|\mathcal{F}$ to $\prod_{i\in F} P_i$, so $P|\mathcal{F} \preccurlyeq_{\mathrm{T}} \prod_{i\in F} P_i$ and 513Ee tells us that $\mathrm{add}(\prod_{i\in F} P_i) \le \mathrm{add}(P|\mathcal{F})$ and $\mathrm{cf}(P|\mathcal{F}) \le \mathrm{cf}(\prod_{i\in F} P_i)$. Also $f \mapsto f\restriction F$ is a dual Tukey function from P to $\prod_{i\in F} P_i$, so $\prod_{i\in F} P_i \preccurlyeq_{\mathrm{T}} P$ and $\mathrm{cf}(\prod_{i\in F} P_i) \le \mathrm{cf}\,P$. **Q**

(c) Note that if $\mathcal{F}$, $\mathcal{G}$ are filters on I and $\mathcal{F} \subseteq \mathcal{G}$, then $\mathrm{add}(P|\mathcal{F}) \le \mathrm{add}(P|\mathcal{G})$ and $\mathrm{cf}(P|\mathcal{F}) \ge \mathrm{cf}(P|\mathcal{G})$. **P** If $f \le_{\mathcal{F}} g$ then $f \le_{\mathcal{G}} g$. So we have a canonical surjective order-preserving map $\psi : P|\mathcal{F} \to P|\mathcal{G}$ given by saying that $\psi(\pi_{\mathcal{F}}(f)) = \pi_{\mathcal{G}}(f)$ for every $f \in P$, where $\pi_{\mathcal{F}}(f)$, $\pi_{\mathcal{G}}(f)$ are the equivalence classes of f in $P|\mathcal{F}$ and $P|\mathcal{G}$ respectively. By 513E(b-iii), ψ is a dual Tukey function, so $P|\mathcal{G} \preccurlyeq_{\mathrm{T}} P|\mathcal{F}$ and we can use 513Ee again. **Q**

5A2B Theorem Let $\lambda > 0$ be a cardinal and $\langle\theta_\zeta\rangle_{\zeta<\lambda}$ a family of regular infinite cardinals, all greater than λ. Set $P = \prod_{\zeta<\lambda} \theta_\zeta$. For any filter $\mathcal{F}$ on λ, let $P|\mathcal{F}$ be the corresponding reduced product and $\pi_{\mathcal{F}} : P \to P|\mathcal{F}$ the canonical map. For any cardinal δ set

$$\mathfrak{F}_\delta = \{\mathcal{F} : \mathcal{F} \text{ is an ultrafilter on } \lambda,\ \mathrm{cf}(P|\mathcal{F}) = \delta\},$$

$$\mathfrak{F}^*_\delta = \bigcup_{\delta'\ge\delta} \mathfrak{F}_{\delta'};$$

if $\mathfrak{F}^*_\delta \ne \emptyset$, let $\mathcal{G}_\delta$ be the filter $\bigcap \mathfrak{F}^*_\delta$. Now

(a) if $\mathfrak{F}^*_\delta \ne \emptyset$, then $\mathrm{add}(P|\mathcal{G}_\delta) \ge \delta$;

(b) for every δ there is a set $F \in [P]^{\le\delta}$ such that $\pi_{\mathcal{F}}[F]$ is cofinal with $P|\mathcal{F}$ for every $F \in \mathfrak{F}_\delta$;

(c) $\mathfrak{F}_{\mathrm{cf}\,P} \ne \emptyset$.

proof The case of finite λ is trivial throughout, as then

$$\mathrm{cf}\,P = \max_{\zeta<\lambda} \theta_\zeta,$$

$$\mathfrak{F}_\delta = \{\mathcal{F} : \text{there is a } \zeta < \lambda \text{ such that } \{\zeta\} \in \mathcal{F} \text{ and } \theta_\zeta = \delta\},$$

$$\mathfrak{F}^*_\delta = \{\mathcal{F} : \{\zeta : \theta_\zeta \ge \delta\} \in \mathcal{F}\},$$

$$\mathcal{G}_\delta = \{G : \{\zeta : \theta_\zeta \ge \delta\} \subseteq G \subseteq \lambda\}.$$

So henceforth let us take it that λ is infinite.

If $\mathcal{F}$ is an ultrafilter on λ, then $P|\mathcal{F}$ is a non-empty totally ordered set with no greatest member, so its additivity and cofinality are the same; thus

$$\mathfrak{F}_\delta = \{\mathcal{F} : \mathcal{F} \text{ is an ultrafilter on } \lambda,\ \mathrm{add}(P|\mathcal{F}) = \delta\}$$

for every δ, and

$$\min_{\zeta\in F} \theta_\zeta \le \delta \le \mathrm{cf}(\prod_{\zeta\in F} \theta_\zeta)$$

whenever $F \in \mathcal{F} \in \mathfrak{F}_\delta$, by 5A2Ab.

Write $L = \{\zeta : \zeta < \lambda,\ \theta_\zeta = \lambda^+\}$, $M = \lambda \setminus L$. If $\mathcal{F}$ is an ultrafilter on λ and $L \in \mathcal{F}$, then $\mathrm{cf}(\prod_{\zeta\in L} \theta_\zeta) = \lambda^+$, because the set of constant functions is cofinal with $\prod_{\zeta\in L} \theta_\zeta$, so $\mathrm{cf}(P|\mathcal{F})$ must be λ^+; otherwise, $M \in \mathcal{F}$ and $\mathrm{cf}(P|\mathcal{F}) > \lambda^+$.

(a) Set $\delta' = \mathrm{add}(P|\mathcal{G}_\delta)$.

(i) δ' is a regular infinite cardinal (513C(a-i)) and

$$\delta' \ge \min_{\zeta<\lambda} \theta_\zeta > \lambda$$

by 5A2Ab again. If $\delta = \lambda^+$ then of course $\delta' \ge \delta$; so suppose that $\delta > \lambda^+$. In this case $L \notin \mathcal{F}$ for any $\mathcal{F} \in \mathfrak{F}^*_\delta$, so $M \in \mathcal{G}_\delta$ and $\delta' \ge \min_{\zeta\in M} \theta_\zeta > \lambda^+$.

(ii) ? If $\delta' < \delta$ then (translating 513C(a-i) into a statement about the pre-order $\le_{\mathcal{G}_\delta}$) there is a family $\langle f_\alpha\rangle_{\alpha<\delta'}$ in P such that $f_\alpha \le_{\mathcal{G}_\delta} f_\beta$ whenever $\alpha \le \beta < \delta'$ but there is no $f \in P$ such that $f_\alpha \le_{\mathcal{G}_\delta} f$ for every $\alpha < \delta'$. Choose $h_\xi \in P$, $\alpha_\xi < \delta'$ inductively, for $\xi < \lambda^+$, as follows. $h_0 = f_0$. Given h_ξ, set

$$B_{\xi\alpha}=\{\zeta:\zeta\in M,\,h_\xi(\zeta)\ge f_\alpha(\zeta)\}$$

for each $\alpha<\delta'$; let $\alpha_\xi<\delta'$ be such that $f_{\alpha_\xi}\not\le_{\mathcal{G}_\delta}h_\xi$, so that $B_{\xi\alpha}\notin\mathcal{G}_\delta$ when $\alpha_\xi\le\alpha<\delta'$. Choose $\mathcal{F}_\xi\in\mathfrak{F}^*_\delta$ such that $B_{\xi,\alpha_\xi}\notin\mathcal{F}_\xi$. Now, because $\operatorname{cf}(P|\mathcal{F}_\xi)\ge\delta>\delta'$, there is an $h_{\xi+1}\in P$ such that $f_\alpha\le_{\mathcal{F}_\xi}h_{\xi+1}$ for every $\alpha<\delta'$; we may take $h_{\xi+1}\ge h_\xi$.

For non-zero limit ordinals $\xi<\lambda^+$ take $h_\xi(\zeta)=\sup_{\eta<\xi}h_\eta(\zeta)$ for every $\zeta<\lambda$.

Set $\alpha=\sup_{\xi<\lambda^+}\alpha_\xi<\delta'$. Then $\langle B_{\xi\alpha}\rangle_{\xi<\lambda^+}$ is a non-decreasing family in $\mathcal{P}\lambda$. So there must be a $\xi<\lambda^+$ such that $B_{\xi\alpha}=B_{\xi+1,\alpha}$.

By the choice of $h_{\xi+1}$, $B_{\xi+1,\alpha}\in\mathcal{F}_\xi$. So $B_{\xi\alpha}\in\mathcal{F}_\xi$ and $f_\alpha\le_{\mathcal{F}_\xi}h_\xi$. Because $\alpha\ge\alpha_\xi$, $f_{\alpha_\xi}\le_{\mathcal{G}_\delta}f_\alpha$, so

$$f_{\alpha_\xi}\le_{\mathcal{F}_\xi}f_\alpha\le_{\mathcal{F}_\xi}h_\xi$$

and $B_{\xi,\alpha_\xi}\in\mathcal{F}_\xi$; contrary to the choice of $\mathcal{F}_\xi$. **X**

(b)(i) If $\mathfrak{F}_\delta=\emptyset$, we can take $F=\emptyset$; so suppose that $\mathfrak{F}_\delta$ is non-empty. As in (a-i) above, we must have $\delta\ge\min_{\zeta<\lambda}\theta_\zeta>\lambda$, and the case $\delta=\lambda^+$ is again elementary. **P** Take $F\subseteq P$ to be the set of constant functions with values less than λ^+. If $\mathcal{F}\in\mathfrak{F}_\delta$ then $M\notin\mathcal{F}$ and $L\in\mathcal{F}$. So for any $h\in P$ we have $\alpha=\sup_{\zeta\in L}h(\zeta)<\lambda^+$, and there is an $f\in F$ such that $f(\zeta)=\alpha$ for every ζ, in which case $h\le_{\mathcal{F}}f$; thus $\pi_{\mathcal{F}}[F]$ is cofinal with $P|\mathcal{F}$, as required. **Q**

So suppose from now on that $\delta>\lambda^+$, so that $M\in\mathcal{F}$ for every $\mathcal{F}\in\mathfrak{F}_\delta$. Of course δ, being the cofinality of a non-empty totally ordered set with no greatest member, is regular (511He-511Hf, 513C(a-i)). For each $\mathcal{F}\in\mathfrak{F}_\delta$, choose a family $\langle g_{\mathcal{F}\beta}\rangle_{\beta<\delta}$ in P such that $\{\pi_{\mathcal{F}}(g_{\mathcal{F}\beta}):\beta<\delta\}$ is cofinal with $P|\mathcal{F}$.

(ii) ? Suppose, if possible, that there is no F of the required type. In this case, we can find families $\langle f_{\xi\alpha}\rangle_{\xi<\lambda^+,\alpha<\delta}$ in P and $\langle\mathcal{F}_\xi\rangle_{\xi<\lambda^+}$ in $\mathfrak{F}_\delta$ such that

(α) $f_{\eta\alpha}\le_{\mathcal{F}_\xi}f_{\xi0}$ whenever $\alpha<\delta$, $\eta<\xi<\lambda^+$;

(β) $\{\pi_{\mathcal{F}_\xi}(f_{\xi\alpha}):\alpha<\delta\}$ is cofinal with $P|\mathcal{F}_\xi$ for every $\xi<\lambda^+$;

(γ) $f_{\eta\alpha}\le f_{\xi\alpha}$ whenever $\alpha<\delta$, $\eta\le\xi<\lambda^+$;

(δ) if $\xi<\lambda^+$, $\alpha<\delta$ and $\operatorname{cf}\alpha=\lambda^+$ then

$$f_{\xi\alpha}(\zeta)=\min\{\sup\nolimits_{\beta\in C}f_{\xi\beta}(\zeta):C\text{ is a closed cofinal set in }\alpha\}$$

for every $\zeta\in M$;

(ϵ) $f_{\xi\beta}\le_{\mathcal{F}_\xi}f_{\xi\alpha}$ whenever $\xi<\lambda^+$, $\beta\le\alpha<\delta$.

P Construct $\langle f_{\xi\alpha}\rangle_{\xi<\lambda^+,\alpha<\delta}$ inductively, taking $\lambda^+\times\delta$ with its lexicographic well-ordering (that is, $(\xi,\alpha)\le(\eta,\beta)$ if either $\xi<\eta$ or $\xi=\eta$ and $\alpha<\beta$). Given that $\langle f_{\eta\beta}\rangle_{(\eta,\beta)<(\xi,\alpha)}$ satisfies the inductive hypothesis so far, proceed according to the nature of α, as follows.

Zero If $\alpha=0$, then, because $\#(\xi\times\delta)\le\delta$, the counter-hypothesis tells us that there is an $\mathcal{F}_\xi\in\mathfrak{F}_\delta$ such that $\{\pi_{\mathcal{F}_\xi}(f_{\eta\beta}):\eta<\xi,\,\beta<\delta\}$ is not cofinal with $P|\mathcal{F}_\xi$. Accordingly we can find $f_{\xi0}\in P$ such that

$$f_{\eta\alpha}\le_{\mathcal{F}_\xi}f_{\xi0}\text{ whenever }\eta<\xi\text{ and }\alpha<\delta,$$

and because $\operatorname{add}P\ge\lambda^+>\#(\xi)$, we can also insist that

$$f_{\eta0}\le f_{\xi0}\text{ whenever }\eta<\xi.$$

Successor If $\alpha=\beta+1$ is a successor ordinal, set

$$f_{\xi\alpha}(\zeta)=\max(f_{\xi\beta}(\zeta),g_{\mathcal{F}_\xi\beta}(\zeta),\sup\nolimits_{\eta<\xi}f_{\eta\alpha}(\zeta))\text{ for every }\zeta<\lambda;$$

this is acceptable because $\operatorname{cf}\theta_\zeta>\lambda$ for every ζ.

Cofinality λ^+ If $\operatorname{cf}\alpha=\lambda^+$, set

$$\begin{aligned}f_{\xi\alpha}(\zeta)&=\sup_{\eta<\xi}f_{\eta\alpha}(\zeta)\text{ if }\zeta\in L,\\&=\min\{\sup_{\beta\in C}f_{\xi\beta}(\zeta):C\text{ is a closed cofinal set in }\alpha\}\text{ if }\zeta\in M.\end{aligned}$$

This time, note that if $\zeta\in M$, then $f_{\xi\alpha}(\zeta)<\theta_\zeta$ because there is a closed cofinal set in α with cardinal $\lambda^+<\theta_\zeta$.

Otherwise If α is a non-zero limit ordinal and $\operatorname{cf}\alpha\ne\lambda^+$, choose $f_{\xi\alpha}$ such that

$$f_{\eta\alpha}\le f_{\xi\alpha}\text{ for every }\eta<\xi,$$

$$f_{\xi\beta} \le_{\mathcal{F}_\xi} f_{\xi\alpha} \text{ for every } \beta < \alpha;$$

this is possible because $\operatorname{add} P$ and $\operatorname{add}(P|\mathcal{F}_\xi)$ are both at least $\lambda^+ > \max(\#(\xi), \#(\alpha))$.

Now let us work through the list of conditions to be satisfied.

(α) is written into the case $\alpha = 0$ of the induction.

(β) Because $g_{\mathcal{F}_\xi\alpha} \le f_{\xi,\alpha+1}$ for every α and $\{\pi_{\mathcal{F}_\xi}(g_{\mathcal{F}_\xi\alpha}) : \alpha < \delta\}$ is cofinal with $P|\mathcal{F}_\xi$, $\{\pi_{\mathcal{F}_\xi}(f_{\xi\alpha}) : \alpha < \delta\}$ is cofinal with $P|\mathcal{F}_\xi$.

(γ) The construction ensures that we shall have $f_{\eta\alpha}(\zeta) \le f_{\xi\alpha}(\zeta)$ in all the required cases except possibly when $\operatorname{cf}\alpha = \lambda^+$ and $\zeta \in M$. But in this case, taking $\eta < \xi$ and a closed cofinal set $C \subseteq \alpha$ such that $f_{\xi\alpha}(\zeta) = \sup_{\beta\in C} f_{\xi\beta}(\zeta)$, the inductive hypothesis will assure us that

$$f_{\eta\alpha}(\zeta) \le \sup\nolimits_{\beta\in C} f_{\eta\beta}(\zeta) \le \sup\nolimits_{\beta\in C} f_{\xi\beta}(\zeta) = f_{\xi\alpha}(\zeta),$$

so there is no problem.

(δ) is written into the formula for the inductive step when $\operatorname{cf}\alpha = \lambda^+$.

(ϵ) We certainly have $f_{\xi\alpha} \le f_{\xi,\alpha+1}$, so $f_{\xi\alpha} \le_{\mathcal{F}_\xi} f_{\xi,\alpha+1}$, for every α. If $\operatorname{cf}\alpha = \lambda^+$, then because the intersection of fewer than $\operatorname{cf}\alpha$ closed cofinal subsets of α is again a closed cofinal set in α (4A1Bd), there will be a closed cofinal set $C \subseteq \alpha$ such that $f_{\xi\alpha}(\zeta) = \sup_{\beta\in C} f_{\xi\beta}(\zeta)$ for every $\zeta \in M$. So $f_{\xi\beta} \le_{\mathcal{F}_\xi} f_{\xi\alpha}$ for every $\beta \in C$; by the inductive hypothesis, $f_{\xi\beta} \le_{\mathcal{F}_\xi} f_{\xi\alpha}$ for every $\beta < \alpha$. For other limit ordinals α, we have $f_{\xi\beta} \le_{\mathcal{F}_\xi} f_{\xi\alpha}$ for every $\beta < \alpha$ directly from the choice of $f_{\xi\alpha}$.

So the procedure works. **Q**

(iii) The next step is to choose a non-decreasing family $\langle h_\eta\rangle_{\eta<\lambda^+}$ in P and a strictly increasing family $\langle\gamma(\eta)\rangle_{\eta<\lambda^+}$ in δ such that

$\gamma(\eta) = \sup_{\eta'<\eta} \gamma(\eta')$ whenever $\eta < \lambda^+$ is a limit ordinal (in particular, $\gamma(0) = 0$);

$f_{\xi,\gamma(\eta)}(\zeta) < h_\eta(\zeta)$ whenever ξ, $\eta < \lambda^+$ and $\zeta \in M$ (choosing h_η);

$h_\eta \le_{\mathcal{F}_\xi} f_{\xi,\gamma(\eta+1)}$ whenever ξ, $\eta < \lambda^+$ (choosing $\gamma(\eta+1)$).

Set $h(\zeta) = \sup_{\eta<\lambda^+} h_\eta(\zeta)$ for $\zeta \in M$, $h(\zeta) = 0$ for $\zeta \in L$, $\alpha = \sup_{\eta<\lambda^+} \gamma(\eta) < \delta$ (because $\delta = \operatorname{cf}\delta > \lambda^+$); then $\operatorname{cf}\alpha = \lambda^+$ and $\{\gamma(\eta) : \eta < \lambda^+\}$ is a closed cofinal set in α. Thus

$$f_{\xi\alpha}(\zeta) \le \sup\nolimits_{\eta<\lambda^+} f_{\xi,\gamma(\eta)}(\zeta) \le h(\zeta)$$

for every $\xi < \lambda^+$ and $\zeta \in M$, by (ii-δ). So if we set

$$A_\xi = \{\zeta : \zeta \in M,\ f_{\xi\alpha}(\zeta) = h(\zeta)\}$$

for each $\xi < \lambda^+$, we shall have $A_\eta \subseteq A_\xi$ whenever $\eta \le \xi < \lambda^+$, by (ii-γ).

(iv) As $\#(M) \le \lambda$, there must be some $\xi < \lambda^+$ such that $A_\xi = A_{\xi+1}$. Let $C \subseteq \alpha$ be a closed cofinal set such that

$$f_{\xi+1,\alpha}(\zeta) = \sup\nolimits_{\beta\in C} f_{\xi+1,\beta}(\zeta)$$

for every $\zeta \in M$. Set $C' = \gamma^{-1}[C]$. Then C' is a closed cofinal subset of λ^+. **P** It is closed because $\gamma : \lambda^+ \to \alpha$ is order-continuous, therefore continuous (4A2Ro). Next, $\gamma[\lambda^+]$ is closed and cofinal in α, while $\operatorname{cf}\alpha = \lambda^+$ is uncountable, so $C \cap \gamma[\lambda^+]$ is cofinal with α and $\gamma[\lambda^+]$ (4A1Bd again), and C' is cofinal with λ^+. **Q**

For each $\eta \in C'$ write η' for the next member of C' greater than η; then

$$h_\eta \le_{\mathcal{F}_{\xi+1}} f_{\xi+1,\gamma(\eta+1)} \le_{\mathcal{F}_{\xi+1}} f_{\xi+1,\gamma(\eta')},$$

$$f_{\xi\alpha} \le_{\mathcal{F}_{\xi+1}} f_{\xi+1,0} = f_{\xi+1,\gamma(0)}$$

so there is a $\zeta_\eta \in M$ such that

$$h_\eta(\zeta_\eta) \le f_{\xi+1,\gamma(\eta')}(\zeta_\eta), \quad f_{\xi\alpha}(\zeta_\eta) \le f_{\xi+1,\gamma(0)}(\zeta_\eta) < h_0(\zeta_\eta) \le h(\zeta_\eta).$$

Let $\zeta \in M$ be such that

$$B = \{\eta : \eta \in C',\ \zeta_\eta = \zeta\}$$

is cofinal with λ^+. Then $f_{\xi\alpha}(\zeta) < h(\zeta)$ so $\zeta \notin A_\xi$. On the other hand,

$$f_{\xi+1,\alpha}(\zeta) = \sup\nolimits_{\beta\in C} f_{\xi+1,\beta}(\zeta) \ge \sup\nolimits_{\eta\in B} f_{\xi+1,\gamma(\eta')}(\zeta) \ge \sup\nolimits_{\eta\in B} h_\eta(\zeta) = h(\zeta)$$

because $\langle h_\eta\rangle_{\eta<\lambda^+}$ is non-decreasing. So $\zeta\in A_{\xi+1}$; which is impossible. **X**

This contradiction completes the proof of (b).

(c)(i) Set $\Delta=\{\delta:\mathfrak{F}_\delta\ne\emptyset\}$, $\mathcal{G}=\bigcup_{\delta\in\Delta}\mathcal{G}_\delta$. Since $\mathcal{G}_\delta\subseteq\mathcal{G}_{\delta'}$ for $\delta\le\delta'$ in Δ, $\mathcal{G}$ is a filter on λ and there is an ultrafilter $\mathcal{H}$ on λ including $\mathcal{G}$. For any $\delta\in\Delta$, $\mathcal{H}\supseteq\mathcal{G}_\delta$, so

$$\operatorname{cf}(P|\mathcal{H})=\operatorname{add}(P|\mathcal{H})\ge\operatorname{add}(P|\mathcal{G}_\delta)\ge\delta>\lambda,$$

using 5A2Ac and (a) above. Consequently $\delta^*=\operatorname{cf}(P|\mathcal{H})$ is the greatest element of Δ.

(ii) For each $\delta\le\delta^*$ choose a set $F_\delta\in[P]^{\le\delta}$ such that $\pi_{\mathcal{F}}[F_\delta]$ is cofinal with $P|\mathcal{F}$ for every $\mathcal{F}\in\mathfrak{F}_\delta$ (using (b) above). Set $F=\bigcup_{\delta\in\Delta}F_\delta$ and

$$G=\{\sup I:I\in[F]^{<\omega}\}\subseteq P.$$

Then $\#(G)\le\delta^*$. I claim that G is cofinal with P. **P?** Suppose otherwise; take $h\in P$ such that $h\not\le g$ for every $g\in G$. Write

$$A_g=\{\zeta:h(\zeta)>g(\zeta)\}$$

for each $g\in G$. Because G is upwards-directed, $\{A_g:g\in G\}$ is a filter base, and there is an ultrafilter $\mathcal{F}$ on λ containing every A_g. Now there is a $\delta\in\Delta$ such that $\mathcal{F}\in\mathfrak{F}_\delta$, so that $\pi_{\mathcal{F}}[F_\delta]$ is cofinal with $P|\mathcal{F}$, and there is an $f\in F_\delta$ such that $h\le_{\mathcal{F}}f$. But in this case $A=\{\zeta:h(\zeta)\le f(\zeta)\}$ and $A_f=\lambda\setminus A$ both belong to $\mathcal{F}$. **XQ**

(iii) Accordingly $\operatorname{cf}P\le\#(G)\le\delta^*$. But also of course $\delta^*=\operatorname{cf}(P|\mathcal{H})\le\operatorname{cf}P$, so $\delta^*=\operatorname{cf}P$. Now we have $\mathcal{H}\in\mathfrak{F}_{\delta^*}=\mathfrak{F}_{\operatorname{cf}P}$.

5A2C Theorem Let $\lambda>0$ be a cardinal and $\langle\theta_\zeta\rangle_{\zeta<\lambda}$ a family of regular infinite cardinals, all greater than λ. Set $P=\prod_{\zeta<\lambda}\theta_\zeta$. Let $\mathcal{F}$ be an ultrafilter on λ and κ a regular infinite cardinal with $\lambda<\kappa\le\operatorname{cf}(P|\mathcal{F})$. (In the language of 5A2B, $\mathcal{F}\in\mathfrak{F}^*_\kappa$.) Then there is a family $\langle\theta'_\zeta\rangle_{\zeta<\lambda}$ of regular infinite cardinals such that $\lambda<\theta'_\zeta\le\theta_\zeta$ for every $\zeta<\lambda$ and $\operatorname{cf}(P'|\mathcal{F})=\kappa$, where $P'=\prod_{\zeta<\lambda}\theta'_\zeta$.

proof (a) If λ is finite, there is a $\zeta<\lambda$ such that $\{\zeta\}\in\mathcal{F}$, $\operatorname{cf}(P|\mathcal{F}_\zeta)=\theta_\zeta$ and we just have to take $\theta'_\zeta=\kappa$; if $\kappa=\lambda^+$ we may take $\theta'_\zeta=\lambda^+$ for every ζ; if $\kappa=\operatorname{cf}(P|\mathcal{F})$ we may take $\theta'_\zeta=\theta_\zeta$; so let us assume that $\omega_1\le\lambda^+<\kappa<\operatorname{cf}(P|\mathcal{F})$. In this case $\{\zeta:\zeta<\lambda,\ \theta_\zeta=\lambda^+\}\notin\mathcal{F}$, by 5A2Ab, so $M=\{\zeta:\zeta<\lambda,\ \theta_\zeta>\lambda^+\}$ belongs to $\mathcal{F}$.

(b) For each ordinal $\gamma<\kappa$ choose a relatively closed cofinal set $C_\gamma\subseteq\gamma$ with $\operatorname{otp}(C_\gamma)=\operatorname{cf}\gamma$. Choose families $\langle f_\alpha\rangle_{\alpha<\kappa}$, $\langle g_{\alpha\gamma}\rangle_{\alpha,\gamma<\kappa}$ in P inductively, as follows. Given $\langle f_\beta\rangle_{\beta<\alpha}$, where $\alpha<\kappa$, and $\gamma<\kappa$, define $g_{\alpha\gamma}\in P$ by setting

$$\begin{aligned}g_{\alpha\gamma}(\zeta)&=\sup\{f_\beta(\zeta):\beta\in C_\gamma\cap\alpha\}+1\text{ if this is less than }\theta_\zeta,\\&=0\text{ otherwise.}\end{aligned}$$

Now choose $f_\alpha\in P$ such that

$$f_\beta\le_{\mathcal{F}}f_\alpha\ \forall\ \beta<\alpha,\quad g_{\alpha\gamma}\le_{\mathcal{F}}f_\alpha\ \forall\ \gamma<\kappa;$$

this is possible because $\kappa<\operatorname{cf}(P|\mathcal{F})$. Observe that if $\alpha=\beta+1$ then $C_\alpha=\{\beta\}$ so that $g_{\alpha\alpha}=f_\beta+1$ and $f_\alpha\not\le_{\mathcal{F}}f_\beta$. Continue.

(c) Suppose that for each $\zeta<\lambda$ we are given a set $S_\zeta\subseteq\theta_\zeta$ with $\#(S_\zeta)\le\lambda$. Then there is an $\alpha<\kappa$ such that

for every $h\in\prod_{\zeta<\lambda}S_\zeta$, if $f_\alpha\le_{\mathcal{F}}h$ then $f_\beta\le_{\mathcal{F}}h$ for every $\beta<\kappa$.

P? If not, then (because κ is regular) we can find a family $\langle h_\xi\rangle_{\xi<\kappa}$ in $\prod_{\zeta<\lambda}S_\zeta$ and a strictly increasing family $\langle\phi(\xi)\rangle_{\xi<\kappa}$ in κ such that

$$f_{\phi(\xi)}\le_{\mathcal{F}}h_\xi\le_{\mathcal{F}}f_{\phi(\xi+1)}\text{ for all }\xi<\kappa,$$

$$\phi(\xi)=\sup_{\eta<\xi}\phi(\eta)\text{ for limit ordinals }\xi<\kappa.$$

Set

$$C = \{\xi : \xi < \kappa,\, \phi(\xi) = \xi\},$$

so that C is a closed cofinal set in κ. Let $\alpha \in C$ be such that $\alpha = \sup(C \cap \alpha)$ and $\operatorname{cf}\alpha = \lambda^+$. Then (because $\lambda^+ \ge \omega_1$) $C \cap C_\alpha$ is cofinal with α.

For $\beta \in C \cap C_\alpha$ and $\zeta < \lambda$ we have

$$\#(C_\alpha \cap \beta) \le \operatorname{otp}(C_\alpha \cap \beta) < \operatorname{otp}(C_\alpha) = \lambda^+ \le \theta_\zeta,$$

so

$$\theta_\zeta > \sup_{\xi \in C_\alpha \cap \beta} f_\xi(\zeta) + 1 = g_{\beta\alpha}(\zeta).$$

Now

$$g_{\beta\alpha} \le_{\mathcal{F}} f_\beta = f_{\phi(\beta)} \le_{\mathcal{F}} h_\beta \le_{\mathcal{F}} f_{\phi(\beta+1)} \le_{\mathcal{F}} f_{\beta'},$$

where β' is the next member of $C \cap C_\alpha$ greater than β. So there is a $\zeta_\beta < \lambda$ such that

$$g_{\beta\alpha}(\zeta_\beta) \le h_\beta(\zeta_\beta) \le f_{\beta'}(\zeta_\beta).$$

Because $\lambda < \operatorname{cf}\alpha$ there is a $\zeta < \lambda$ such that

$$B = \{\beta : \beta \in C \cap C_\alpha,\, \zeta_\beta = \zeta\}$$

is cofinal with α. But now observe that if $\beta, \gamma \in B$ and $\beta' < \gamma$ then $\beta' \in C \cap C_\alpha \cap \gamma$ so

$$h_\beta(\zeta) \le f_{\beta'}(\zeta) < g_{\gamma\alpha}(\zeta) \le h_\gamma(\zeta).$$

It follows that

$$\lambda^+ = \#(B) = \#(\{h_\beta(\zeta) : \beta \in B\}) \le \#(S_\zeta) \le \lambda,$$

which is absurd. **XQ**

(d) Consequently $E = \{f^\bullet_\alpha : \alpha < \kappa\}$ has a least upper bound in $P|\mathcal{F}$. **P?** If not, choose a family $\langle h_\xi\rangle_{\xi<\lambda^+}$ in P inductively, as follows. Because $\kappa < \operatorname{cf}(P|\mathcal{F})$, there is an $h_0 \in P$ such that $f_\alpha \le_{\mathcal{F}} h_0$ for every $\alpha < \kappa$. Given h_ξ such that $h^\bullet_\xi = \pi_{\mathcal{F}}(h_\xi)$ is an upper bound for E in $P|\mathcal{F}$, then $h^\bullet_\xi$ cannot be the least upper bound of E, so there is an $h_{\xi+1} \in P$ such that $h^\bullet_{\xi+1}$ is an upper bound of E strictly less than $h^\bullet_\xi$. For non-zero limit ordinals $\xi < \lambda^+$, set

$$S_{\xi\zeta} = \{h_\eta(\zeta) : \eta < \xi\} \subseteq \theta_\zeta$$

for each $\zeta < \lambda$. By (c) above, there is an $\alpha_\xi < \kappa$ such that

$$\text{for every } h \in \prod_{\zeta<\lambda} S_{\xi\zeta} \text{ either } f_{\alpha_\xi} \not\le_{\mathcal{F}} h \text{ or } f_\alpha \le_{\mathcal{F}} h \ \forall\ \alpha < \kappa.$$

Set

$$h_\xi(\zeta) = \min(\{\eta : \eta \in S_{\xi\zeta},\, f_{\alpha_\xi}(\zeta) \le \eta\} \cup \{h_0(\zeta)\}) \in S_{\xi\zeta}$$

for each $\zeta < \lambda$. Then $f_{\alpha_\xi} \le_{\mathcal{F}} h_\xi$ (because $f_{\alpha_\xi}(\zeta) \le h_\xi(\zeta)$ whenever $f_{\alpha_\xi}(\zeta) \le h_0(\zeta)$) and $h_\xi \in \prod_{\zeta<\lambda} S_{\xi\zeta}$, so $f_\alpha \le_{\mathcal{F}} h_\xi$ for every $\alpha < \kappa$ and $h^\bullet_\xi$ is an upper bound for E. Also, if $\eta < \xi$, then $h_\xi(\zeta) \le h_\eta(\zeta)$ whenever $f_{\alpha_\xi}(\zeta) \le h_\eta(\zeta)$, so $h_\xi \le_{\mathcal{F}} h_\eta$. Continue.

Having got the family $\langle h_\xi\rangle_{\xi<\lambda^+}$, set

$$S_\zeta = \bigcup_{\xi<\lambda^+} S_{\xi\zeta} = \{h_\xi(\zeta) : \xi < \lambda^+\} \subseteq \theta_\zeta$$

for each $\zeta < \lambda$. For each $\alpha < \kappa$, $\zeta < \lambda$ set

$$g_\alpha(\zeta) = \min(\{\eta : f_\alpha(\zeta) \le \eta \in S_\zeta\} \cup \{h_0(\zeta)\}) \in S_\zeta.$$

Then, by the same arguments as above,

$$f_\alpha \le_{\mathcal{F}} g_\alpha \le_{\mathcal{F}} h_\xi \text{ for every } \alpha < \kappa,\, \xi < \lambda^+.$$

For each $\alpha < \kappa$ there is a non-zero limit ordinal $\xi < \lambda^+$ such that $g_\alpha(\zeta) \in S_{\xi\zeta}$ for every $\zeta < \lambda$, because $\langle S_{\xi\zeta}\rangle_{\xi<\lambda^+}$ is non-decreasing for each ζ. Because $\lambda^+ < \kappa$ there is a non-zero limit ordinal $\xi < \lambda^+$ such that

$$A = \{\alpha : g_\alpha(\zeta) \in S_{\xi\zeta} \ \forall\ \zeta < \lambda\}$$

is cofinal with κ. In particular, there is an $\alpha\in A$ such that $\alpha\ge\alpha_\xi$. In this case

$$f_{\alpha_\xi}\le_{\mathcal F}f_\alpha\le_{\mathcal F}g_\alpha\le_{\mathcal F}h_{\xi+1}\le_{\mathcal F}h_\xi\not\le_{\mathcal F}h_{\xi+1},$$

so there is a $\zeta<\lambda$ such that

$$f_{\alpha_\xi}(\zeta)\le f_\alpha(\zeta)\le g_\alpha(\zeta)\le h_{\xi+1}(\zeta)<h_\xi(\zeta).$$

But now observe that

$$f_{\alpha_\xi}(\zeta)\le g_\alpha(\zeta)\in S_{\xi\zeta}$$

so $h_\xi(\zeta)\le g_\alpha(\zeta)<h_\xi(\zeta)$, which is absurd. **XQ**

(e) Let $g\in P$ be such that $g^\bullet=\sup E$ in $P|\mathcal F$ and $g(\zeta)>0$ for every $\zeta<\lambda$. For each $\zeta<\lambda$ set $\hat\theta_\zeta=\operatorname{cf}(g(\zeta))<\theta_\zeta$ and choose a cofinal set $D_\zeta\subseteq g(\zeta)$ of order type $\hat\theta_\zeta$. For $\alpha<\kappa$ and $\zeta<\lambda$ set

$$\hat g_\alpha(\zeta)=\min\{\eta:f_\alpha(\zeta)\le\eta\in D_\zeta\}$$

if $f_\alpha(\zeta)<g(\zeta)$, $\min D_\zeta$ otherwise. Then $\hat g_\alpha\le_{\mathcal F}\hat g_\beta$ whenever $\alpha\le\beta<\kappa$. Also if $h\in Q=\prod_{\zeta<\lambda}D_\zeta$ then $h^\bullet<g^\bullet$ so there is an $\alpha<\kappa$ such that $h^\bullet<f^\bullet_\alpha\le\hat g^\bullet_\alpha$. Thus $\{\hat g^\bullet_\alpha:\alpha<\kappa\}$ is a cofinal subset of $\{h^\bullet:h\in Q\}$.

(f) Because each D_ζ is order-isomorphic to $\hat\theta_\zeta$, we can identify Q with $\hat P=\prod_{\zeta<\lambda}\hat\theta_\zeta$. Now if $\alpha<\kappa$, then there is a $\beta<\kappa$ such that $\hat g_\alpha\le_{\mathcal F}f_\beta$ while $f_{\beta+1}\not\le_{\mathcal F}f_\beta$ (see (b) above), so $\hat g_{\beta+1}\not\le_{\mathcal F}g_\alpha$. Because κ is regular,

$$\operatorname{cf}(\hat P|\mathcal F)=\operatorname{cf}(\{h^\bullet:h\in Q\})=\operatorname{cf}(\{\hat g^\bullet_\alpha:\alpha<\kappa\})=\kappa.$$

(g) It may be that some of the $\hat\theta_\zeta$ are less than or equal to λ. But taking $I=\{\zeta:\hat\theta_\zeta\le\lambda\}$, we have $I\notin\mathcal F$. **P?** If $I\in\mathcal F$, then for each $\zeta\in I$ set $\hat S_\zeta=D_\zeta$ and for $\zeta\in\lambda\setminus I$ set $\hat S_\zeta=\{0\}$. Then $\#(\hat S_\zeta)\le\lambda$ for every $\zeta<\lambda$. By (c), there is an $\alpha<\kappa$ such that

$$\text{for every } h\in\prod_{\zeta<\lambda}\hat S_\zeta,\text{ if } f_\alpha\le_{\mathcal F}h\text{ then }f_\beta\le_{\mathcal F}h\ \forall\ \beta<\kappa.$$

But as $f_{\alpha+1}\le_{\mathcal F}g$, and $I\in\mathcal F$, there must be an $h\in\prod_{\zeta<\lambda}\hat S_\zeta$ such that $f_\alpha\le_{\mathcal F}h$, and now $g\le_{\mathcal F}h$ because $g^\bullet$ is the least upper bound of E; but $h(\zeta)<g(\zeta)$ for every $\zeta\in I$, so this is impossible. **XQ**

So $\{\zeta:\hat\theta_\zeta>\lambda\}\in\mathcal F$. But this means that if we set

$$\theta'_\zeta=\hat\theta_\zeta\text{ when }\hat\theta_\zeta>\lambda,$$
$$=\theta_\zeta\text{ when }\hat\theta_\zeta\le\lambda$$

and $P'=\prod_{\zeta<\lambda}\theta'_\zeta$, then $P'|\mathcal F\cong\hat P|\mathcal F$ so $\operatorname{cf}(P'|\mathcal F)=\kappa$, as required.

5A2D Definitions (a) Let α, β, γ and δ be cardinals. Following Shelah 92 and Shelah 94, I write

$$\operatorname{cov}_{\mathrm{Sh}}(\alpha,\beta,\gamma,\delta)$$

for the least cardinal of any family $\mathcal E\subseteq[\alpha]^{<\beta}$ such that for every $A\in[\alpha]^{<\gamma}$ there is a $\mathcal D\in[\mathcal E]^{<\delta}$ with $A\subseteq\bigcup\mathcal D$. In the trivial cases in which there is no such family $\mathcal E$ I write $\operatorname{cov}_{\mathrm{Sh}}(\alpha,\beta,\gamma,\delta)=\infty$.

(b) For cardinals α, γ write $\Theta(\alpha,\gamma)$ for the supremum of all cofinalities

$$\operatorname{cf}(\prod_{\zeta<\lambda}\theta_\zeta)$$

for families $\langle\theta_\zeta\rangle_{\zeta<\lambda}$ such that $\lambda<\gamma$ is a cardinal, every θ_ζ is a regular infinite cardinal and $\lambda<\theta_\zeta<\alpha$ for every $\zeta<\lambda$. (This carries some of the same information as the cardinal $\operatorname{pp}_\kappa(\alpha)$ of Shelah 94, p. 41.)

Remarks (i) Immediately from the definitions, we see that

$$\operatorname{cov}_{\mathrm{Sh}}(\alpha,\beta',\gamma,\delta')\le\operatorname{cov}_{\mathrm{Sh}}(\alpha',\beta,\gamma',\delta),\quad\Theta(\alpha,\gamma)\le\Theta(\alpha',\gamma')$$

whenever $\alpha\le\alpha'$, $\beta\le\beta'$, $\gamma\le\gamma'$ and $\delta\le\delta'$.

(ii) The definition of Θ demands a moment's thought in trivial cases. If $\gamma=0$ there is no $\lambda<\gamma$, so we are taking the supremum of an empty set of cofinalities, and $\Theta(\alpha,0)=0$ for every α. If $\gamma>0$ then we are allowed $\lambda=0$ and $\prod_{\zeta<\lambda}\theta_\zeta=\{\emptyset\}$, so $\Theta(\alpha,\gamma)\ge1$ for every α. If $\gamma>1$ we are allowed $\lambda=1$, so $\Theta(\alpha^+,\gamma)\ge\alpha$ for every infinite α.

5A2E Lemma Let α, β, γ, γ' and δ be cardinals.

(a) If $\gamma \le \gamma' \le \beta$ and $\delta \ge 2$ then

$$\operatorname{cov}_{\mathrm{Sh}}(\alpha,\beta,\gamma,\delta) \le \operatorname{cf}([\alpha]^{<\gamma'}) \le \#([\alpha]^{<\gamma'}).$$

(b) If either $\omega \le \gamma \le \operatorname{cf}\alpha$ or $\omega \le \operatorname{cf}\alpha < \operatorname{cf}\delta$ then

$$\operatorname{cov}_{\mathrm{Sh}}(\alpha,\beta,\gamma,\delta) \le \max(\alpha, \sup_{\theta<\alpha} \operatorname{cov}_{\mathrm{Sh}}(\theta,\beta,\gamma,\delta)).$$

proof (a) If $\mathcal{E}$ is a cofinal subset of $[\alpha]^{<\gamma'}$ of cardinal $\operatorname{cf}([\alpha]^{<\gamma'})$, then $\mathcal{E}$ witnesses that $\operatorname{cov}_{\mathrm{Sh}}(\alpha,\gamma',\gamma',\delta) \le \operatorname{cf}([\alpha]^{<\gamma'})$. Now

$$\operatorname{cov}_{\mathrm{Sh}}(\alpha,\beta,\gamma,\delta) \le \operatorname{cov}_{\mathrm{Sh}}(\alpha,\gamma',\gamma',\delta) \le \operatorname{cf}([\alpha]^{<\gamma'}).$$

(b) Set $\kappa = \max(\alpha, \sup_{\theta<\alpha} \operatorname{cov}_{\mathrm{Sh}}(\theta,\beta,\gamma,\delta))$ and $\lambda = \operatorname{cf}\alpha$. Let $\langle\zeta_\xi\rangle_{\xi<\lambda}$ enumerate a cofinal subset of α. For each $\xi<\lambda$, let $\mathcal{E}_\xi \subseteq [\zeta_\xi]^{<\beta}$ be a set with cardinal at most κ such that for every $A \in [\zeta_\xi]^{<\gamma}$ there is a $\mathcal{D} \in [\mathcal{E}_\xi]^{<\delta}$ such that $A \subseteq \bigcup\mathcal{D}$. Set $\mathcal{E} = \bigcup_{\xi<\lambda}\mathcal{E}_\xi$, so that $\mathcal{E} \subseteq [\alpha]^{<\beta}$ has cardinal at most κ. Take $A \in [\alpha]^{<\gamma}$.

If $\omega \le \gamma \le \lambda$ then $\sup A < \alpha$ and there is a $\xi < \lambda$ such that $A \subseteq \zeta_\xi$. Now there is a $\mathcal{D} \in [\mathcal{E}_\xi]^{<\delta} \subseteq [\mathcal{E}]^{<\delta}$ such that $A \subseteq \bigcup\mathcal{D}$.

If $\omega \le \lambda < \operatorname{cf}\delta$, then for each $\xi<\lambda$ there is a $\mathcal{D}_\xi \in [\mathcal{E}_\xi]^{<\delta}$ such that $A \cap \zeta_\xi \subseteq \bigcup\mathcal{D}_\xi$. Set $\mathcal{D} = \bigcup_{\xi<\lambda}\mathcal{D}_\xi$; because $\lambda < \operatorname{cf}\delta$, $\mathcal{D} \in [\mathcal{E}]^{<\delta}$, while

$$A = \bigcup_{\xi<\lambda} A \cap \zeta_\xi \subseteq \bigcup\mathcal{D}.$$

Thus in either case $\mathcal{E}$ witnesses that $\operatorname{cov}_{\mathrm{Sh}}(\alpha,\beta,\gamma,\delta) \le \kappa$.

5A2F Lemma Let α, γ be cardinals. If $\alpha \le 2^\gamma$, then $\Theta(\alpha,\gamma) \le 2^\gamma$.

proof If $\gamma \le \omega$ then $\Theta(\alpha,\gamma) \le \max(1,\alpha) \le 2^\gamma$. If $\gamma > \omega$, $\lambda < \gamma$ and $\theta_\zeta < 2^\gamma$ for every $\zeta < \lambda$, then

$$\operatorname{cf}(\prod_{\zeta<\lambda}\theta_\zeta) \le \#(\prod_{\zeta<\lambda}\theta_\zeta) \le (2^\gamma)^\gamma = 2^\gamma.$$

5A2G Theorem For any cardinals α and γ,

$$\operatorname{cov}_{\mathrm{Sh}}(\alpha,\gamma,\gamma,\omega_1) \le \max(\omega,\alpha,\Theta(\alpha,\gamma)).$$

proof (a) To begin with (down to the end of (f) below) let us suppose that we have $\alpha \ge \gamma = \gamma_0^+ > \operatorname{cf}\alpha > \omega$, and set $\kappa = \max(\alpha,\Theta(\alpha,\gamma))$.

Take a family $\mathcal{E} \subseteq [\alpha]^{\le\gamma_0}$ such that

(i) $\mathcal{E}$ contains all singleton subsets of α;

(ii) $\mathcal{E}$ contains a cofinal subset of α;

(iii) If $E \in \mathcal{E}$ then $\{\xi : \xi+1 \in E\} \in \mathcal{E}$;

(iv) if $E \in \mathcal{E}$ then there is an $F \in \mathcal{E}$ such that $\sup(F\cap\xi) = \xi$ whenever $\xi \in E$ and $\omega \le \operatorname{cf}\xi \le \gamma_0$;

(v) if $E \in \mathcal{E}$ then $\{\xi : \xi \in E, \operatorname{cf}\xi > \gamma_0\} \in \mathcal{E}$;

(vi) if $E \in \mathcal{E}$ and $\operatorname{cf}(\prod_{\eta\in E}\eta) \le \kappa$, then $\{g : g \in \prod_{\eta\in E}\eta,\ g[E] \in \mathcal{E}\}$ is cofinal with $\prod_{\eta\in E}\eta$;

(vii) $\#(\mathcal{E}) \le \kappa$.

To see that this can be done, observe that whenever $E \in [\alpha]^{\le\gamma_0}$ there is an $F \in [\alpha]^{\le\gamma_0}$ such that $\sup(F\cap\xi) = \xi$ whenever $\xi \in E$ and $\omega \le \operatorname{cf}\xi \le \gamma_0$; thus condition (iv) can be achieved, like conditions (iii) and (v), by ensuring that $\mathcal{E}$ is closed under suitable functions from $[\alpha]^{\le\gamma_0}$ to itself; while condition (vi) requires that for each $E \in \mathcal{E}$ we have an appropriate family with cardinal at most κ included in $\mathcal{E}$.

Write $\mathcal{J}$ for the σ-ideal of $\mathcal{P}\alpha$ generated by $\mathcal{E}$. Note that if $A \in \mathcal{J}$ then $\{\xi : \xi+1 \in A\}$ belongs to $\mathcal{J}$, by (iii).

(b) ? If $\operatorname{cov}_{\mathrm{Sh}}(\alpha,\gamma,\gamma,\omega_1) > \kappa$, there must be a set in $[\alpha]^{\le\gamma_0}$ not covered by any sequence from $\mathcal{E}$, that is, not belonging to $\mathcal{J}$; that is, there is a function $f : \gamma_0 \to \alpha$ such that $f[\gamma_0] \notin \mathcal{J}$. Accordingly $\mathcal{I} = \{f^{-1}[E] : E \in \mathcal{J}\}$ is a proper σ-ideal of $\mathcal{P}\gamma_0$. By condition (a-i), $\mathcal{I}$ contains all singletons in $\mathcal{P}\gamma_0$.

Let H be the set of all functions $h : \gamma_0 \to \alpha$ such that $f(\xi) \le h(\xi)$ for every $\xi < \gamma_0$ and $h[\gamma_0] \in \mathcal{J}$. Because $\mathcal{E}$ contains a cofinal set $C \subseteq \alpha$ (condition (a-ii)), we can find an $h \in H$; just take $h : \gamma_0 \to C$ such that $f(\xi) \le h(\xi)$ for every ξ.

(c) Because $\mathcal{I}$ is a proper σ-ideal, there cannot be any sequence $\langle h_n\rangle_{n\in\mathbb{N}}$ in H such that $\{\xi : h_{n+1}(\xi) \ge h_n(\xi)\} \in \mathcal{I}$ for every $n \in \mathbb{N}$. Consequently there is an $h^* \in H$ such that

$$\{\xi : h(\xi) \ge h^*(\xi)\} \notin \mathcal{I} \text{ for every } h \in H.$$

We know that $h^*[\gamma_0] \in \mathcal{J}$; let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{E}$ covering $h^*[\gamma_0]$. For $\xi < \gamma_0$ write $\theta_\xi = \mathrm{cf}(h^*(\xi))$, so that each θ_ξ is 0 or 1 or a regular infinite cardinal less than α. Set

$$I = \{\xi : \xi < \gamma_0,\, f(\xi) = h^*(\xi)\},$$

$$I' = \{\xi : \xi < \gamma_0,\, f(\xi) < h^*(\xi),\, \theta_\xi = 1\},$$

$$I_n = \{\xi : \xi < \gamma_0,\, f(\xi) < h^*(\xi),\, \omega \le \theta_\xi \le \gamma_0,\, h^*(\xi) \in E_n\} \text{ for } n \in \mathbb{N},$$

$$J_n = \{\xi : \xi < \gamma_0,\, f(\xi) < h^*(\xi),\, \gamma_0 < \theta_\xi,\, h^*(\xi) \in E_n\} \text{ for } n \in \mathbb{N}.$$

Note that if $\theta_\xi = 0$ then $h^*(\xi) = 0 = f(\xi)$, so I, I', $\langle I_n\rangle_{n\in\mathbb{N}}$ and $\langle J_n\rangle_{n\in\mathbb{N}}$ constitute a cover of γ_0.

(d) For each $n \in \mathbb{N}$ set $G_n = \{\eta : \eta \in E_n,\, \mathrm{cf}\,\eta > \gamma_0\} \in \mathcal{E}$; note that $h^*(\xi) \in G_n$ for $\xi \in J_n$. Then $\mathrm{cf}(\prod_{\eta\in G_n} \eta) \le \Theta(\alpha,\gamma)$. **P** For $\eta \in G_n$ set $\theta'_\eta = \mathrm{cf}\,\eta$; then θ'_η is a regular cardinal and $\#(G_n) \le \gamma_0 < \theta'_\eta < \alpha$ for each $\eta \in G_n$. If for each $\eta \in G_n$ we choose a cofinal set $C_\eta \subseteq \eta$ of order type θ'_η, then

$$\mathrm{cf}(\textstyle\prod_{\eta\in G_n} \eta) = \mathrm{cf}(\prod_{\eta\in G_n} C_\eta) = \mathrm{cf}(\prod_{\eta\in G_n} \theta'_\eta) \le \Theta(\alpha,\gamma)$$

by the definition of $\Theta(\alpha,\gamma)$. **Q**

Consequently, by (a-vi),

$$\{g : g \in \textstyle\prod_{\eta\in G_n} \eta,\, g[G_n] \in \mathcal{E}\}$$

is cofinal with $\prod_{\eta\in G_n} \eta$.

(e) Define $h : \gamma_0 \to \alpha$ as follows.

(i) If $\xi \in I$ set $h(\xi) = h^*(\xi)$.

(ii) If $\xi \in I'$ let $h(\xi)$ be the predecessor of $h^*(\xi)$.

(iii) For each $n \in \mathbb{N}$ take $F_n \in \mathcal{E}$ such that $\eta = \sup(F_n \cap \eta)$ whenever $\eta \in E_n$ and $\omega \le \mathrm{cf}\,\eta \le \gamma_0$. If $\xi \in I_n \setminus \bigcup_{m<n} I_m$, take $h(\xi) \in F_n$ such that $f(\xi) \le h(\xi) < h^*(\xi)$.

(iv) For each $n \in \mathbb{N}$ and $\eta \in G_n$ set

$$g^*(\eta) = \sup\{f(\xi) : \xi < \gamma_0,\, f(\xi) < h^*(\xi) = \eta\}.$$

Then $g^*(\eta) < \eta$, because $\gamma_0 < \mathrm{cf}\,\eta$. By (d), there is a $g_n \in \prod_{\eta\in G_n} \eta$ such that $g_n[G_n] \in \mathcal{E}$ and $g^*(\eta) \le g_n(\eta)$ for every $\eta \in G_n$. So for $\xi \in J_n \setminus \bigcup_{m<n} J_m$ we may set $h(\xi) = g_n(h^*(\xi))$ and see that $h^*(\xi) \in G_n$ and

$$f(\xi) \le g^*h^*(\xi) \le g_n h^*(\xi) = h(\xi) < h^*(\xi),$$

while $h(\xi) \in g_n[G_n]$.

(f) Now we see that

$$h[\gamma_0] \subseteq h^*[\gamma_0] \cup \{\eta : \eta + 1 \in h^*[\gamma_0]\} \cup \textstyle\bigcup_{n\in\mathbb{N}} F_n \cup \bigcup_{n\in\mathbb{N}} g_n[G_n] \in \mathcal{J},$$

while $f(\xi) \le h(\xi)$ for every $\xi < \gamma_0$, so $h \in H$. Consequently

$$I = \{\xi : h(\xi) \ge h^*(\xi)\} \notin \mathcal{I}.$$

But also

$$f[I] \subseteq h^*[\gamma_0] \in \mathcal{J},$$

so $I \in \mathcal{I}$, which is absurd. **X**

(g) Thus the special case described in (a) is dealt with, and we may return to the general case. I proceed by induction on α for fixed γ.

(i) To start the induction, observe that if either $\alpha \le \omega$ or $\gamma \le \omega$ or $\alpha < \gamma$, then

$$\mathrm{cov}_{\mathrm{Sh}}(\alpha,\gamma,\gamma,\omega_1) \le \mathrm{cf}([\alpha]^{<\gamma}) \le \max(\alpha,\omega).$$

(ii) For the inductive step to α when *either* $\operatorname{cf}\alpha \ge \gamma \ge \omega$ *or* $\operatorname{cf}\alpha = \omega$, 5A2Eb tells us that

$$\begin{aligned}\operatorname{cov}_{\mathrm{Sh}}(\alpha,\gamma,\gamma,\omega_1) &\le \max(\alpha, \sup_{\alpha'<\alpha} \operatorname{cov}_{\mathrm{Sh}}(\alpha',\gamma,\gamma,\omega_1)) \\ &\le \max(\omega,\alpha, \sup_{\alpha'<\alpha} \Theta(\alpha',\gamma)) \le \max(\omega,\alpha,\Theta(\alpha,\gamma))\end{aligned}$$

by the inductive hypothesis.

(iii) For the inductive step to α when $\omega < \operatorname{cf}\alpha < \gamma \le \alpha$, observe that

$$[\alpha]^{<\gamma} = \bigcup_{\delta<\gamma} [\alpha]^{\le\delta}.$$

For each cardinal $\delta < \gamma$ we have a set $\mathcal{E}_\delta \subseteq [\alpha]^{\le\delta}$ such that $\#(\mathcal{E}_\delta) \le \operatorname{cov}_{\mathrm{Sh}}(\alpha,\delta^+,\delta^+,\omega_1)$ and every member of $[\alpha]^{\le\delta}$ can be covered by a sequence from $\mathcal{E}_\delta$. Set $\mathcal{E} = \bigcup_{\operatorname{cf}\alpha\le\delta<\gamma} \mathcal{E}_\delta$; then $\mathcal{E} \subseteq [\alpha]^{<\gamma}$ and every member of $[\alpha]^{<\gamma}$ can be covered by a sequence from $\mathcal{E}$. So

$$\begin{aligned}\operatorname{cov}_{\mathrm{Sh}}(\alpha,\gamma,\gamma,\omega_1) \le \#(\mathcal{E}) &\le \max(\gamma, \sup_{\operatorname{cf}\alpha\le\delta<\gamma} \operatorname{cov}_{\mathrm{Sh}}(\alpha,\delta^+,\delta^+,\omega_1)) \\ &\le \max(\gamma,\alpha, \sup_{\operatorname{cf}\alpha\le\delta<\gamma} \Theta(\alpha,\delta^+))\end{aligned}$$

(by (a)-(f) above)

$$\le \max(\alpha,\Theta(\alpha,\gamma)).$$

This completes the proof.

Remark This is taken from Shelah 94, Theorem II.5.4, where a stronger result is proved, giving an exact description of many of the numbers $\operatorname{cov}_{\mathrm{Sh}}(\alpha,\beta,\gamma,\delta)$ in terms of cofinalities of reduced products $\prod_{\zeta<\lambda}\theta_\zeta|\mathcal{F}$.

5A2H Lemma Let γ be an infinite regular cardinal and $\alpha \ge \Theta(\gamma,\gamma)$ a cardinal. Then $\Theta(\Theta(\alpha,\gamma),\gamma) \le \Theta(\alpha,\gamma)$.

proof (a) The case $\gamma = \omega$ is elementary, since $\Theta(\alpha,\omega) \le \alpha$ for every cardinal α. So we may suppose that γ is uncountable.

(b) ? Suppose, if possible, that $\Theta(\alpha,\gamma) < \Theta(\Theta(\alpha,\gamma),\gamma)$.

(i) $\Theta(\gamma,\gamma) \le \alpha < \Theta(\alpha,\gamma)$ so $\gamma < \alpha$ and $\Theta(\alpha,\gamma)$ is infinite.

(ii) There is a family $\langle\theta_\zeta\rangle_{\zeta<\lambda}$ of regular infinite cardinals such that $\lambda < \gamma$, $\lambda < \theta_\zeta < \Theta(\alpha,\gamma)$ for every $\zeta < \lambda$ and $\operatorname{cf}(\prod_{\zeta<\lambda}\theta_\zeta) > \Theta(\alpha,\gamma)$. As $\Theta(\alpha,\gamma) \ne 0$, $\lambda \ne 0$. By 5A2Bc, there is an ultrafilter $\mathcal{F}$ on λ such that $\operatorname{cf}(\prod_{\zeta<\lambda}\theta_\zeta|\mathcal{F}) > \Theta(\alpha,\gamma)$.

(iii) Set $L = \{\zeta : \zeta < \lambda,\ \theta_\zeta < \alpha\}$; as

$$\#(L) \le \lambda < \min(\gamma,\theta_\zeta), \quad \theta_\zeta < \alpha$$

for every $\zeta \in L$,

$$\operatorname{cf}(\textstyle\prod_{\zeta\in L}\theta_\zeta) \le \Theta(\alpha,\gamma) < \operatorname{cf}(\prod_{\zeta<\lambda}\theta_\zeta|\mathcal{F})$$

and $L \notin \mathcal{F}$ (5A2Ab). Set $M = \lambda \setminus L \in \mathcal{F}$. Let $\mathcal{F}' = \mathcal{F} \cap \mathcal{P}M$ be the induced ultrafilter on M, so that $\prod_{\zeta<\lambda}\theta_\zeta|\mathcal{F} \cong \prod_{\zeta\in M}\theta_\zeta|\mathcal{F}'$, and $\operatorname{cf}(\prod_{\zeta\in M}\theta_\zeta|\mathcal{F}') > \Theta(\alpha,\gamma)$.

(iv) For each $\zeta \in M$, we have $\theta_\zeta < \Theta(\alpha,\gamma)$, so there must be a family $\langle\theta_{\zeta\eta}\rangle_{\eta<\lambda_\zeta}$ of regular infinite cardinals with $\lambda_\zeta < \gamma$, $\lambda_\zeta < \theta_{\zeta\eta} < \alpha$ for every $\eta < \lambda_\zeta$ and $\theta_\zeta \le \operatorname{cf}(\prod_{\eta<\lambda_\zeta}\theta_{\zeta\eta})$. Again by 5A2Bc, there is an ultrafilter $\mathcal{F}_\zeta$ on λ_ζ such that $\theta_\zeta \le \operatorname{cf}(\prod_{\eta<\lambda_\zeta}\theta_{\zeta\eta}|\mathcal{F}_\zeta)$. Because

$$\lambda_\zeta < \gamma \le \alpha \le \theta_\zeta,$$

5A2C tells us that there is a family $\langle\theta'_{\zeta\eta}\rangle_{\eta<\lambda_\zeta}$ of regular infinite cardinals such that $\lambda_\zeta < \theta'_{\zeta\eta} \le \theta_{\zeta\eta}$ for every η and $\theta_\zeta = \operatorname{cf}(\prod_{\eta<\lambda_\zeta}\theta'_{\zeta\eta}|\mathcal{F}_\zeta)$.

(c)(i) Set

$$I = \{(\zeta, \eta) : \zeta \in M,\, \eta < \lambda_\zeta\},$$

$$\mathcal{H} = \{H : H \subseteq I,\, \{\zeta : \{\eta : (\zeta, \eta) \in H\} \in \mathcal{F}_\zeta\} \in \mathcal{F}'\},$$

$$P = \prod_{(\zeta,\eta)\in I} \theta'_{\zeta\eta}.$$

Then $\mathcal{H}$ is an ultrafilter on I, and $\mathrm{cf}(P|\mathcal{H}) \ge \mathrm{cf}(\prod_{\zeta\in M} \theta_\zeta|\mathcal{F}')$. **P** Let $F \subseteq P$ be a set with cardinal $\mathrm{cf}(P|\mathcal{H})$ such that $\{f^\bullet : f \in F\}$ is cofinal with $P|\mathcal{H}$. For $f \in P$ and $\zeta \in M$, define $f_\zeta \in \prod_{\eta<\lambda_\zeta} \theta'_{\zeta\eta}$ by setting $f_\zeta(\eta) = f(\zeta, \eta)$ for each $\eta < \lambda_\zeta$, and let $f^\bullet_\zeta$ be the image of f_ζ in $\prod_{\eta<\lambda_\zeta} \theta'_{\zeta\eta}|\mathcal{F}_\zeta$. For each $\zeta \in M$ let $\langle u_{\zeta\xi}\rangle_{\xi<\theta_\zeta}$ be a strictly increasing cofinal family in the totally ordered set $\prod_{\eta<\lambda_\zeta} \theta'_{\zeta\eta}|\mathcal{F}_\zeta$. Now, for $f \in F$, take a function $g_f \in \prod_{\zeta\in M} \theta_\zeta$ such that $f^\bullet_\zeta \le u_{\zeta,g_f(\zeta)}$ for every $\zeta \in M$.

If $g \in \prod_{\zeta\in M} \theta_\zeta$, then we can find an $h \in P$ such that $h^\bullet_\zeta = u_{\zeta,g(\zeta)}$ for each $\zeta \in M$. Let $f \in F$ be such that $h \le_{\mathcal{H}} f$. Then

$$\{\zeta : g(\zeta) \le g_f(\zeta)\} \supseteq \{\zeta : h^\bullet_\zeta \le f^\bullet_\zeta\} \in \mathcal{F}',$$

so $g \le_{\mathcal{F}'} g_f$. Accordingly $\{g_f : f \in F\}$ is cofinal with $\prod_{\zeta\in M} \theta_\zeta|\mathcal{F}'$ and $\mathrm{cf}(\prod_{\zeta\in M} \theta_\zeta|\mathcal{F}') \le \#(F) = \mathrm{cf}(P|\mathcal{H})$, as claimed. **Q**

(ii) Note that as $\#(M) \le \lambda < \gamma$, $\lambda_\zeta < \gamma$ for every $\zeta \in M$ and γ is uncountable and regular,

$$\#(I) \le \max(\omega, \#(M), \sup_{\zeta\in M} \lambda_\zeta) < \gamma.$$

(d)(i) Putting (c-i) and (b-iii) together, $\mathrm{cf}(P|\mathcal{H}) > \Theta(\alpha, \gamma)$. Set $J = \{(\zeta, \eta) : (\zeta, \eta) \in I,\, \theta'_{\zeta\eta} \ge \gamma\}$. Since

$$\#(J) \le \#(I) < \gamma$$

((c-ii) just above)

$$\le \theta'_{\zeta\eta} \le \theta_{\zeta\eta} < \alpha$$

whenever $(\zeta, \eta) \in J$,

$$\mathrm{cf}(\prod_{(\zeta,\eta)\in J} \theta'_{\zeta\eta}) \le \Theta(\alpha, \gamma) < \mathrm{cf}(P|\mathcal{H})$$

and $J \notin \mathcal{H}$ by 5A2Ab once more. It follows that $K = I \setminus J \in \mathcal{H}$. Set $M' = \{\zeta : \zeta \in M,\, \{\eta : (\zeta, \eta) \in K\} \in \mathcal{F}_\zeta\} \in \mathcal{F}'$.

(ii) If $\zeta \in M'$, then $F = \{\eta : \eta < \lambda_\zeta,\, \theta'_{\zeta\eta} < \gamma\}$ belongs to $\mathcal{F}_\zeta$. Now

$$\theta_\zeta = \mathrm{cf}(\prod_{\eta<\lambda_\zeta} \theta'_{\zeta\eta}|\mathcal{F}') \le \mathrm{cf}(\prod_{\eta\in F} \theta'_{\zeta\eta}) \le \Theta(\gamma, \gamma)$$

(because $\#(F) \le \lambda_\zeta < \theta'_{\zeta\eta} < \gamma$ for every $\eta \in F$)

$$\le \alpha \le \theta_\zeta$$

because $\zeta \in M$. . So in fact $\theta_\zeta = \alpha$ for $\zeta \in M'$ and we have

$$\Theta(\alpha, \gamma) < \mathrm{cf}(\prod_{\zeta\in M} \theta_\zeta|\mathcal{F}') \le \mathrm{cf}(\prod_{\zeta\in M'} \theta_\zeta) = \mathrm{cf}(\prod_{\zeta<\delta} \alpha),$$

where $\delta = \#(M')$, while at the same time α is infinite and regular.

(iii) But if α is infinite and regular and $1 \le \delta < \alpha$, $\mathrm{cf}(\prod_{\zeta<\delta} \alpha) = \alpha$. Accordingly $\Theta(\alpha, \gamma) < \alpha$; which contradicts (b-i) above. **X**

This contradiction completes the proof.

5A2I Lemma Let α and γ be cardinals. Set $\delta = \sup_{\alpha'<\alpha} \Theta(\alpha', \gamma)$.
(a) If $\mathrm{cf}\,\alpha \ge \gamma$ then $\Theta(\alpha, \gamma) \le \max(\alpha, \delta)$.
(b) If $\mathrm{cf}\,\alpha < \gamma$ then $\Theta(\alpha, \gamma) \le \max(\alpha, \delta^{\mathrm{cf}\,\alpha})$, where $\delta^{\mathrm{cf}\,\alpha}$ is the cardinal power.

proof Let $\langle\theta_\zeta\rangle_{\zeta<\lambda}$ be a family of regular infinite cardinals with $\lambda < \theta_\zeta < \alpha$ for each ζ and $\lambda < \gamma$.

case 1 If $\alpha' = \sup_{\zeta<\lambda}\theta_\zeta$ is less than α, set

$$I = \{\zeta : \zeta < \lambda,\, \theta_\zeta < \alpha'\},$$

$$J = \{\zeta : \zeta < \lambda,\, \theta_\zeta = \alpha'\}.$$

Then

$$\mathrm{cf}(\prod_{\zeta\in I}\theta_\zeta) \le \Theta(\alpha',\gamma) \le \delta, \quad \mathrm{cf}(\prod_{\zeta\in J}\theta_\zeta) \le \max(1,\alpha') \le \alpha.$$

If $\lambda = 0$ then $\mathrm{cf}(\prod_{\zeta<\lambda}\theta_\zeta) = 1 \le \alpha$; if $\lambda > 0$ then α is infinite and

$$\mathrm{cf}(\prod_{\zeta<\lambda}\theta_\zeta) \le \max(\omega, \mathrm{cf}(\prod_{\zeta\in I}\theta_\zeta), \mathrm{cf}(\prod_{\zeta\in J}\theta_\zeta)) \le \max(\alpha,\delta).$$

This is enough to deal with (a).

case 2 If $\alpha' = \alpha = 0$ then $\lambda = 0 = \delta$, so

$$\mathrm{cf}(\prod_{\zeta<\lambda}\theta_\zeta) = 1 = \delta^{\mathrm{cf}\,\alpha}.$$

So (b) is true if $\alpha = 0$.

case 3 If $\alpha' = \alpha > 0$ and $\mathrm{cf}\,\alpha < \gamma$, then $\lambda > 0$ and α is a supremum of strictly smaller infinite cardinals, so must be uncountable. Let $\langle\alpha_\xi\rangle_{\xi<\mathrm{cf}\,\alpha}$ be a strictly increasing family of cardinals with supremum α, starting from $\alpha_0 = 0$ and $\alpha_1 = \omega$ and with $\alpha_\xi = \sup_{\eta<\xi}\alpha_\eta$ for non-zero limit ordinals $\xi < \mathrm{cf}\,\alpha$. Set

$$P_\xi = \prod_{\zeta<\lambda,\alpha_\xi\le\theta_\zeta<\alpha_{\xi+1}}\theta_\zeta$$

for each $\xi < \mathrm{cf}\,\alpha$. Then

$$\mathrm{cf}\,P_\xi \le \Theta(\alpha_{\xi+1},\gamma) \le \delta$$

for each $\xi < \mathrm{cf}\,\alpha$, so

$$\mathrm{cf}(\prod_{\zeta<\lambda}\theta_\zeta) = \mathrm{cf}(\prod_{\xi<\mathrm{cf}\,\alpha}P_\xi) \le \delta^{\mathrm{cf}\,\alpha}.$$

Putting this together with case 1, we have a proof of (b) when $\alpha > 0$.

Version of 20.5.23

5A3 Forcing

My discussion of forcing is based on Kunen 80; in particular, I start from pre-ordered sets rather than Boolean algebras, and the class $V^{\mathbb{P}}$ of terms in a forcing language will consist of subsets of $V^{\mathbb{P}} \times P$. I find however that I wish to diverge almost immediately from standard formulations in a technical respect, which I describe in 5A3A, introducing what I call 'forcing notions'. I do not refer to generic filters or models of ZFC, preferring to express all results in terms of the forcing relation (5A3C). I give some space to the interpretation of names (5A3E, 5A3F) and, in particular, to names for real numbers derived from elements of $L^0(\mathrm{RO}(\mathbb{P}))$ (5A3L).

5A3A Forcing notions (a) A **forcing notion** is a quadruple $\mathbb{P} = (P, \le, \mathbb{1}, \uparrow)$ or $\mathbb{P} = (P, \le, \mathbb{1}, \downarrow)$ where $(P, \le)$ is a pre-ordered set (that is, $\le$ is a transitive reflexive relation on P), $\mathbb{1} \in P$, and

if $\mathbb{P} = (P, \le, \mathbb{1}, \uparrow)$ then $\mathbb{1} \le p$ for every $p \in P$,

if $\mathbb{P} = (P, \le, \mathbb{1}, \downarrow)$ then $p \le \mathbb{1}$ for every $p \in P$.

In this context members of P are commonly called **conditions**.

(b) I had better try to explain what I am doing here. The problem is the following. Consider two of the standard examples of pre-ordered set in this context. For a set I, $\mathrm{Fn}_{<\omega}(I;\{0,1\})$ is the set of functions from finite subsets of I to $\{0,1\}$; for a non-trivial Boolean algebra $\mathfrak{A}$, $\mathfrak{A}^+$ is the set of non-zero elements of $\mathfrak{A}$. In each case, we have a relevant direction. In $\mathrm{Fn}_{<\omega}(I;\{0,1\})$, a condition p is stronger than a condition q if p extends q, that is, if $p \supseteq q$; in $\mathfrak{A}^+$, p is stronger than q if $p \subseteq q$. So the forcing notions, in the terminology I have chosen, are

$$(\mathrm{Fn}_{<\omega}(I;\{0,1\}),\subseteq,\emptyset,\uparrow),\qquad(\mathfrak{A}^+,\subseteq,1_{\mathfrak{A}},\downarrow).$$

Generally, I will say that a forcing notion $(P,\le,\mathbb{1},\uparrow)$ is **active upwards**, while $(P,\le,\mathbb{1},\downarrow)$ is **active downwards**.

(c) Of course this is unconventional. It is much more usual to take all forcing notions to be active in the same direction (usually downwards) and to use local definitions (e.g., saying that '$p\le q$ if p extends q') to ensure that this will be appropriate.

However the great majority of forcing notions, like the two examples in (b) above, come with structures which strongly suggest a natural interpretation of '$\le$'; and these structures are not arbitrary, but are essential to our intuitive conception of the pre- or partial order we are studying. I prefer, therefore, to maintain the notation I would use for the same objects in any other context, and to indicate separately the orientation which is relevant when using them to build a forcing language.

(d) This approach demands further changes in the language. It will no longer be helpful to talk about conditions in P being 'larger' or 'less than' others. Instead, I will use the word '**stronger**': if $\mathbb{P}=(P,\le,\mathbb{1},\uparrow)$, then $p\in P$ will be stronger than $q\in P$ if $p\ge q$; if $\mathbb{P}=(P,\le,\mathbb{1},\downarrow)$, then $p\in P$ will be stronger than $q\in P$ if $p\le q$. (So p will be stronger than $\mathbb{1}$ for every $p\in P$.)

Similarly, the words 'cofinal' and 'coinitial' are now inappropriate, and I will turn to the word 'dense', as favoured by most authors discussing forcing; if $\mathbb{P}=(P,\le,\mathbb{1},\updownarrow)$ is a forcing notion, a subset Q of P is **dense** if for every $p\in P$ there is a $q\in Q$ such that q is stronger than p. In the same way, I can say that two conditions p, q in P are 'compatible' if there is an $r\in P$ stronger than both. We shall have a standard topology on P generated by sets of the form $\{q:q$ is stronger than $p\}$, and a corresponding regular open algebra $\mathrm{RO}(\mathbb{P})$, as in 514M. An antichain for $\mathbb{P}$ will be a set $A\subseteq P$ such that any two distinct conditions in A are incompatible, and $\mathbb{P}$ will be ccc if every antichain for $\mathbb{P}$ is countable. The 'saturation' $\operatorname{sat}\mathbb{P}$ of $\mathbb{P}$ will be the least cardinal κ such that there is no antichain with cardinal κ.

5A3B Forcing languages Let $\mathbb{P}=(P,\le,\mathbb{1},\updownarrow)$ be a forcing notion.

(a) The class of $\mathbb{P}$**-names**, that is, terms of the forcing language defined by $\mathbb{P}$, is

$$V^{\mathbb{P}}=\{A:A\text{ is a set and }A\subseteq V^{\mathbb{P}}\times P\}$$

(Kunen 80, VII.2.5)[2]. In this context, I will say that the **domain** of a name $A\in V^{\mathbb{P}}$ is the set $\operatorname{dom}A\subseteq V^{\mathbb{P}}$ of first members of elements of A.

(b) For any set X, $\check X$ will be the $\mathbb{P}$-name $\{(\check x,\mathbb{1}):x\in X\}\in V^{\mathbb{P}}$ (Kunen 80, VII.2.10).

5A3C The Forcing Relation (Kunen 80, VII.3.3) Suppose that $\mathbb{P}=(P,\le,\mathbb{1},\updownarrow)$ is a forcing notion, $p\in P$, ϕ, ψ are formulae of set theory, and $\dot x_0,\dots,\dot x_n\in V^{\mathbb{P}}$.

(a) $p\Vdash_{\mathbb{P}}\dot x_0=\dot x_1$ iff

whenever $(\dot y,q)\in\dot x_0$ and $r\in P$ is stronger than both p and q, there are a $(\dot y',q')\in\dot x_1$ and an r' stronger than both r and q' such that $r'\Vdash_{\mathbb{P}}\dot y=\dot y'$,

whenever $(\dot y,q)\in\dot x_1$ and $r\in P$ is stronger than both p and q, there are a $(\dot y',q')\in\dot x_0$ and an r' stronger than both r and q' such that $r'\Vdash_{\mathbb{P}}\dot y=\dot y'$.

Note that $p\Vdash_{\mathbb{P}}\dot x=\dot x$ for every $\mathbb{P}$-name $\dot x$ and every $p\in P$ (induce on the rank of $\dot x$).

(b) $p\Vdash_{\mathbb{P}}\dot x_0\in\dot x_1$ iff

whenever $q\in P$ is stronger than p there are a $(\dot y,q')\in\dot x_1$ and an r stronger than both q and q' such that $r\Vdash_{\mathbb{P}}\dot x_0=\dot y$.

(c) $p\Vdash_{\mathbb{P}}\phi(\dot x_0,\dots,\dot x_n)\,\&\,\psi(\dot x_0,\dots,\dot x_n)$ iff

$$p\Vdash_{\mathbb{P}}\phi(\dot x_0,\dots,\dot x_n)\text{ and }p\Vdash_{\mathbb{P}}\psi(\dot x_0,\dots,\dot x_n).$$

[2]In this section, we need the Axiom of Foundation ('for any non-empty set A, there is an $a\in A$ such that $a\cap A=\emptyset$'); here, to determine whether a set A belongs to $V^{\mathbb{P}}$, we need to induce on the rank of A.

(d) $p \Vdash_{\mathbb{P}} \neg\phi(\dot{x}_0, \dots, \dot{x}_n)$ iff

there is no q stronger than p such that $q \Vdash_{\mathbb{P}} \phi(\dot{x}_0, \dots, \dot{x}_n)$.

(e) $p \Vdash_{\mathbb{P}} \exists\, x,\, \phi(x, \dot{x}_0, \dots, \dot{x}_n)$ iff
for every q stronger than p there are an r stronger than q and a $\dot{y} \in V^{\mathbb{P}}$
such that $r \Vdash_{\mathbb{P}} \phi(\dot{y}, \dot{x}_0, \dots, \dot{x}_n)$.[3]

(f) You will see that if $p \Vdash_{\mathbb{P}} \phi$ and q is stronger than p then $q \Vdash_{\mathbb{P}} \phi$.

(g) In this context I will write $\Vdash_{\mathbb{P}}$ for $\mathbb{1} \Vdash_{\mathbb{P}}$.

5A3D The Forcing Theorem If ϕ is any theorem of ZFC, and $\mathbb{P}$ is any forcing notion, then $\Vdash_{\mathbb{P}} \phi$. (Kunen 80, VII.4.2.)

5A3E Names for functions Let $\mathbb{P}$ be a forcing notion, P its set of conditions, and $R \subseteq V^{\mathbb{P}} \times V^{\mathbb{P}} \times P$ a set. Consider the $\mathbb{P}$-names

$$\dot{f} = \{((\dot{x}, \dot{y}), p) : (\dot{x}, \dot{y}, p) \in R\},$$

$$\dot{A} = \{(\dot{x}, p) : (\dot{x}, \dot{y}, p) \in R\}, \quad \dot{B} = \{(\dot{y}, p) : (\dot{x}, \dot{y}, p) \in R\}.$$

(a) The following are equiveridical:

(i) $\Vdash_{\mathbb{P}} \dot{f}$ is a function;

(ii) whenever $(\dot{x}_0, \dot{y}_0, p_0)$, $(\dot{x}_1, \dot{y}_1, p_1)$ belong to R, $p \in \mathbb{P}$ is stronger than both p_0 and p_1 and $p \Vdash_{\mathbb{P}} \dot{x}_0 = \dot{x}_1$, then $p \Vdash_{\mathbb{P}} \dot{y}_0 = \dot{y}_1$.

(b) In this case,

$$p \Vdash_{\mathbb{P}} \dot{f}(\dot{x}) = \dot{y}$$

whenever $(\dot{x}, \dot{y}, p) \in R$,

$$\Vdash_{\mathbb{P}} \operatorname{dom} \dot{f} = \dot{A} \text{ and } \dot{f}[\dot{A}] = \dot{B},$$

and the following are equiveridical:

(i) $\Vdash_{\mathbb{P}} \dot{f}$ is injective;

(ii) whenever $(\dot{x}_0, \dot{y}_0, p_0)$, $(\dot{x}_1, \dot{y}_1, p_1)$ belong to R, $p \in \mathbb{P}$ is stronger than both p_0 and p_1 and $p \Vdash_{\mathbb{P}} \dot{y}_0 = \dot{y}_1$, then $p \Vdash_{\mathbb{P}} \dot{x}_0 = \dot{x}_1$.

Remark In the formula for $\dot{f}$ here, brackets take different meanings at different points. In the expression $((\dot{x}, \dot{y}), p)$, the inner brackets must be interpreted in the forcing language, while the outer brackets, like the brackets in the expression $(\dot{x}, \dot{y}, p) \in R$, are interpreted in the ordinary universe; Kunen 80 might write

$$\dot{f} = \{(\operatorname{op}(\dot{x}, \dot{y}), p) : (\dot{x}, \dot{y}, p) \in R\}.$$

proof Elementary.

5A3F More notation In 5A3C I took it for granted that every formula of set theory would have a version in $V^{\mathbb{P}}$. I should perhaps explain some of the versions I have in mind. Let $\mathbb{P} = (P, \le, \mathbb{1}, \Updownarrow)$ be a forcing notion.

(a) If $\dot{y}_0$, $\dot{y}_1 \in V^{\mathbb{P}}$ then $\dot{x} = \{(\dot{y}_0, \mathbb{1}), (\dot{y}_1, \mathbb{1})\} \in V^{\mathbb{P}}$, and

$$\Vdash_{\mathbb{P}} \dot{x} = \{\dot{y}_0, \dot{y}_1\};$$

so we have a suitable formal expression for pair sets in $V^{\mathbb{P}}$. Similarly, if we think of the formula (x, y) as being an abbreviation for $\{\{x\}, \{x, y\}\}$, we get a $\mathbb{P}$-name

[3] This formulation is appropriate if we wish to explore forcing without using the axiom of choice. Subject to AC, we have an alternative condition: $p \Vdash_{\mathbb{P}} \exists\, x,\, \phi(x, \dot{x}_0, \dots, \dot{x}_n)$ iff there is a $\dot{y} \in V^{\mathbb{P}}$ such that $p \Vdash_{\mathbb{P}} \phi(\dot{y}, \dot{x}_0, \dots, \dot{x}_n)$.

$$\dot{z} = \{(\{(\dot{y}_0, \mathbb{1})\}, \mathbb{1}), (\{(\dot{y}_0, \mathbb{1}), (\dot{y}_1, \mathbb{1})\}, \mathbb{1})\}$$

such that

$$\Vdash_{\mathbb{P}} \dot{z} = (\dot{y}_0, \dot{y}_1).$$

(b) Now let $\langle \dot{x}_i \rangle_{i \in I}$ be a family of $\mathbb{P}$-names, and set

$$\dot{f} = \{((\check{\imath}, \dot{x}_i), \mathbb{1}) : i \in I\}.$$

As in 5A3E,

$$\Vdash_{\mathbb{P}} \dot{f} \text{ is a function with domain } \check{I},$$

and

$$\Vdash_{\mathbb{P}} \dot{f}(\check{\imath}) = \dot{x}_i$$

for every $i \in I$. For obvious reasons I do not wish to spell this procedure out every time, and I will use the rather elliptic formula

$$\langle \dot{x}_i \rangle_{i \in \check{I}}$$

to signify the $\mathbb{P}$-name $\dot{f}$.

(c) Similarly, $\dot{T} = \{(\dot{x}_i, \mathbb{1}) : i \in I\}$ is a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}} \dot{x}_i \in \dot{T}$ for every $i \in I$, and whenever $p \in \mathbb{P}$ and $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \dot{T}$, there are an $i \in I$ and a q stronger than p such that $q \Vdash_{\mathbb{P}} \dot{x} = \dot{x}_i$; I will write $\{\dot{x}_i : i \in \check{I}\}$ for $\dot{T}$.

(d) In the same spirit, if I have a family $\langle \dot{x}_i \rangle_{i \in I}$ of $\mathbb{P}$-names for real numbers between 0 and 1, I will allow myself to write '$\sup_{i \in \check{I}} \dot{x}_i$' to signify a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \sup_{i \in \check{I}} \dot{x}_i = \sup\{\dot{x}_i : i \in \check{I}\},$$

without taking the trouble to spell out any exact formula to represent the supremum. I will do the same for limits of sequences; if $\langle \dot{x}_n \rangle_{n \in \mathbb{N}}$ is a sequence of $\mathbb{P}$-names for real numbers, and

$$\Vdash_{\mathbb{P}} \langle \dot{x}_n \rangle_{n \in \mathbb{N}} \text{ is convergent},$$

then I will write '$\lim_{n \to \infty} \dot{x}_n$' to mean a $\mathbb{P}$-name $\dot{x}$ such that

$$\Vdash_{\mathbb{P}} \langle \dot{x}_n \rangle_{n \in \mathbb{N}} \to \dot{x} \in \mathbb{R}.$$

Of course this is tolerable only because it is possible to set out a general rule for devising a suitable name $\dot{x} \in V^{\mathbb{P}}$ from the given sequence $\langle \dot{x}_n \rangle_{n \in \mathbb{N}}$. See 5A3L below.

5A3G Boolean truth values Let $\mathbb{P}$ be a forcing notion and P its set of conditions.

(a) If ϕ is a formula of set theory, and $\dot{x}_0, \dots, \dot{x}_n \in V^{\mathbb{P}}$, then

$$\{p : p \in P,\ p \Vdash_{\mathbb{P}} \phi(\dot{x}_0, \dots, \dot{x}_n)\}$$

is a regular open set in P (use 514Md); I will denote it $[\![\phi(\dot{x}_0, \dots, \dot{x}_n)]\!]$.

(b) If ϕ and ψ are formulae of set theory and $\dot{x}_0, \dots, \dot{x}_n \in V^{\mathbb{P}}$, then

$$[\![\phi(\dot{x}_0, \dots, \dot{x}_n) \,\&\, \psi(\dot{x}_0, \dots, \dot{x}_n)]\!] = [\![\phi(\dot{x}_0, \dots, \dot{x}_n)]\!] \cap [\![\psi(\dot{x}_0, \dots, \dot{x}_n)]\!]$$

and

$$[\![\neg\phi(\dot{x}_0, \dots, \dot{x}_n)]\!] = P \setminus \overline{[\![\phi(\dot{x}_0, \dots, \dot{x}_n)]\!]},$$

the complement of $[\![\phi(\dot{x}_0, \dots, \dot{x}_n)]\!]$ in $\mathrm{RO}(\mathbb{P})$.

(c) If ϕ is a formula of set theory, A is a set, and $\dot{x}_0, \dots, \dot{x}_n$ are $\mathbb{P}$-names, then

$$[\![\exists\, x \in \check{A},\ \phi(x, \dot{x}_0, \dots, \dot{x}_n)]\!] = \mathrm{int}\,\overline{\bigcup_{a \in A} [\![\phi(\check{a}, \dot{x}_0, \dots, \dot{x}_n)]\!]},$$

the supremum of $\{[\![\phi(\check{a}, \dot{x}_0, \dots, \dot{x}_n]\!] : a \in A\}$ in $\mathrm{RO}(\mathbb{P})$. (Use 5A3Ce.)

5A3H Concerning ˇs (a) The reader is entitled to an attempt at consistency on the following point of notation, among others. For any set X and any forcing notion $\mathbb{P}$ there is a corresponding $\mathbb{P}$-name $\check{X}$ (5A3Bb). We start with $\check{\emptyset} = \emptyset$. If $1 = \{\emptyset\}$ is the next von Neumann ordinal, we get a name

$$\check{1} = \{(\check{\emptyset}, \mathbb{1})\} = \{(\emptyset, \mathbb{1})\};$$

and we can check directly from 5A3C that

$$\Vdash_{\mathbb{P}} \check{1} = \{\emptyset\},$$

that is, if you like,

$$\Vdash_{\mathbb{P}} \check{1} = 1,$$

where in this formula the first 1 is interpreted in the ordinary universe and the second is interpreted in the forcing language. Similarly, if we take '2' to be an abbreviation for '$\{\emptyset, \{\emptyset\}\}$', we have

$$\Vdash_{\mathbb{P}} \check{2} = 2,$$

and so on. Indeed we get

$$\Vdash_{\mathbb{P}} \check{\omega} \text{ is the first infinite ordinal,}$$

$$\Vdash_{\mathbb{P}} \check{\mathbb{Q}} \text{ is the set of rational numbers,}$$

so the same convention would lead to

$$\Vdash_{\mathbb{P}} \check{\omega} = \omega, \check{\mathbb{N}} = \mathbb{N}, \check{\mathbb{Q}} = \mathbb{Q}.$$

(This formula does not depend on which construction of the set of rational numbers we use, provided that we use the same method both in the ordinary universe and in the forcing language.) Of course it is *not* the case (except for forcing notions of particular types) that

$$\Vdash_{\mathbb{P}} \check{\omega}_1 \text{ is the first uncountable ordinal,} \qquad \Vdash_{\mathbb{P}} \check{\mathbb{R}} \text{ is the set of real numbers.}$$

(b) For 'absolute' objects, therefore, like $\omega+7$ or $\frac{22}{7}$, appearing in sentences of a forcing language, I shall have a choice between formulations

$$(\omega+7)\check{}, \quad (\tfrac{22}{7})\check{}$$

(working directly from 5A3Bb),

$$\omega+7, \quad \tfrac{22}{7}$$

(regarding the phrases '$\omega+7$' and '$\frac{22}{7}$' as abbreviations for expressions in set theory which can be evaluated either in the ordinary universe or in the forcing language), or

$$\check{\omega} \mathbin{\check{+}} \check{7}, \quad 22\check{}\mathbin{\check{\div}}\check{7}$$

(regarding ω, $+$, 7, 22 and $\div$ as sets to which the rule of 5A3Bb can be applied, and then interpreting the combination in the forcing language). The least cluttered versions, $\omega+7$ and $\frac{22}{7}$, look better, and these will ordinarily be my choice. But it means that when you see the symbol $\mathbb{Q}$ in a sentence of the forcing language, it is likely to mean two things at once, a superposition of 'the set of rational numbers' and 'the $\mathbb{P}$-name $\check{\mathbb{Q}}$', with algebraic operations and relations attached correspondingly.

(c) 'Absoluteness' is treated properly in Kunen 80, §IV.3. I shall not attempt to even sketch the concept here. But we shall need a couple of basic examples. Let $\mathbb{P}$ be a forcing notion and P its set of conditions.

(i) If A and B are sets, $p \in P$ and $p \Vdash_{\mathbb{P}} \check{A} \subseteq \check{B}$ then $A \subseteq B$. **P** Induce on the rank of B. If $a \in A$ then $(\check{a}, \mathbb{1}) \in \check{A}$ so $\Vdash_{\mathbb{P}} \check{a} \in \check{A}$ and $p \Vdash_{\mathbb{P}} \check{a} \in \check{B}$. Now there must be a $\mathbb{P}$-name $\dot{y}$, a $q \in P$ and an r stronger than both p and q such that $(\dot{y}, q) \in \check{B}$ and $r \Vdash_{\mathbb{P}} \check{a} = \dot{y}$; in which case $\dot{y}$ must be of the form $\check{b}$ where $b \in B$, and $r \Vdash_{\mathbb{P}} \check{a} = \check{b}$. By the inductive hypothesis $a = b$ and $a \in B$. **Q**

(ii) If α, $\beta \in \mathbb{Q}$ and $p \Vdash_{\mathbb{P}} \check{\alpha} \le \check{\beta}$ then $\alpha \le \beta$. **P** I leave the proof as an exercise, since the details necessarily depend on the precise construction you use for $\mathbb{Q}$. But they resolve quickly into a handful of similar statements concerning arithmetic in $\mathbb{N}$, and (i) here, together with 5A3Eb, should be enough to deal with these. **Q**

5A3I Regular open algebras If $\mathbb{P}=(P,\le,\mathbb{1},\Updownarrow)$ is a forcing notion with regular open algebra $\mathrm{RO}(\mathbb{P})$, then we have a natural map $\iota:P\to\mathrm{RO}(\mathbb{P})^+$ defined by saying that

$$\iota(p)=\operatorname{int}\overline{\{q:q\text{ is stronger than }p\}}$$

for $p\in P$ (Kunen 80, II.3.3); and (allowing for the possible reversal of the direction of $\mathbb{P}$) ι is a dense embedding of the pre-ordered set $(P,\le)$ in the partially ordered set $(\mathrm{RO}(\mathbb{P})^+,\subseteq)$, in the sense of Kunen 80, §VII.7. Consequently, taking $\widehat{\mathbb{P}}$ to be the forcing notion $(\mathrm{RO}(\mathbb{P})^+,\subseteq,P,\downarrow)$, we shall have

$$\Vdash_{\mathbb{P}}\phi\text{ if and only if }\Vdash_{\widehat{\mathbb{P}}}\phi$$

for every sentence ϕ of set theory (Kunen 80, VII.7.11). It follows that if two forcing notions have isomorphic regular open algebras, then they force exactly the same theorems of set theory.

5A3J The following technical device will be useful at one point.

Definition Let $\mathbb{P}$ be a forcing notion. I will say that a $\mathbb{P}$-name $\dot X$ is **discriminating** if whenever $(\dot x,p)$ and $(\dot y,q)$ are distinct members of $\dot X$, and r is stronger than both p and q, then $r\Vdash_{\mathbb{P}}\dot x\ne\dot y$.

5A3K Lemma Let $\mathbb{P}$ be a forcing notion, and P its set of conditions.

(a) For any $\mathbb{P}$-name $\dot X$, there is a discriminating $\mathbb{P}$-name $\dot X_1$ such that $\Vdash_{\mathbb{P}}\dot X_1=\dot X$.

(b) Let $\dot X$ be a discriminating $\mathbb{P}$-name, and $f:\dot X\to V^{\mathbb{P}}$ a function. Let $\dot g$ be the $\mathbb{P}$-name

$$\{((\dot x,f(\dot x,p)),p):(\dot x,p)\in\dot X\}.^4$$

Then

$$\Vdash_{\mathbb{P}}\dot g\text{ is a function with domain }\dot X.$$

proof (a)(i) Set

$$\dot X_0=\{(\dot x,q):\text{there is some }p\in P\text{ such that }(\dot x,p)\in\dot X\text{ and }q\in P\text{ is stronger than }p\}.$$

Then $\Vdash_{\mathbb{P}}\dot X=\dot X_0$. **P** Because $\dot X\subseteq\dot X_0$, $\Vdash_{\mathbb{P}}\dot X\subseteq\dot X_0$. In the other direction, if $\dot x$ is a $\mathbb{P}$-name and p is such that $p\Vdash_{\mathbb{P}}\dot x\in\dot X_0$, there are an $(\dot x_1,p_1)\in\dot X_0$ and a q, stronger than both p and p_1, such that $q\Vdash_{\mathbb{P}}\dot x=\dot x_1$. Now there is a p_2 such that $(\dot x_1,p_2)\in\dot X$ and p_1 is stronger than p_2. In this case, q is stronger than p_2 and $p_2\Vdash_{\mathbb{P}}\dot x_1\in\dot X$, so $q\Vdash_{\mathbb{P}}\dot x=\dot x_1\in\dot X$, while q is stronger than p. As $\dot x$ and p are arbitrary, $\Vdash_{\mathbb{P}}\dot X_0\subseteq\dot X$. **Q**

Let $\dot X_1\subseteq\dot X_0$ be a maximal discriminating name.

(ii) Because $\dot X_1\subseteq\dot X_0$, $\Vdash_{\mathbb{P}}\dot X_1\subseteq\dot X_0=\dot X$. But we also have $\Vdash_{\mathbb{P}}\dot X\subseteq\dot X_1$. **P** Suppose that $\dot x$ is a $\mathbb{P}$-name and $p\in P$ is such that $p\Vdash_{\mathbb{P}}\dot x\in\dot X$. Then there must be an $(\dot x_1,p_1)\in\dot X$ and a q stronger than both p and p_1 such that $q\Vdash_{\mathbb{P}}\dot x=\dot x_1$. In this case, $(\dot x_1,q)\in\dot X_0$. By the maximality of $\dot X_1$, there are a $(\dot y,q')\in\dot X_1$ and an r stronger than both q and q' such that $r\Vdash_{\mathbb{P}}\dot x_1=\dot y$. Now r is stronger than p and

$$r\Vdash_{\mathbb{P}}\dot x=\dot x_1=\dot y\in\dot X_1.$$

As $\dot x$ and p are arbitrary, $\Vdash_{\mathbb{P}}\dot X\subseteq\dot X_1$. **Q**

So $\Vdash_{\mathbb{P}}\dot X_1=\dot X$, as required.

(b) Consider

$$\{(\dot x,f(\dot x,p),p):(\dot x,p)\in\dot X\}\subseteq V^{\mathbb{P}}\times V^{\mathbb{P}}\times P.$$

If $(\dot x_0,p_0)$ and $(\dot x_1,p_1)$ belong to $\dot X$, p is stronger than both p_0 and p_1, and $p\Vdash_{\mathbb{P}}\dot x_0=\dot x_1$, then $(\dot x_0,p_0)=(\dot x_1,p_1)$, because $\dot X$ is a discriminating name; so $p\Vdash_{\mathbb{P}}f(\dot x_0,p_0)=f(\dot x_1,p_1)$. By 5A3E,

$$\Vdash_{\mathbb{P}}\dot g\text{ is a function and }\operatorname{dom}\dot g=\dot X.$$

5A3L Real numbers in forcing languages Let $\mathbb{P}$ be any forcing notion, and P its set of conditions.

[4] Once again I present a formula in which some ordered pairs are to be interpreted in the ordinary universe, but another is to be interpreted in the forcing language.

(a) I have tried to avoid committing myself to any declaration of what a real number actually 'is'; in fact I believe that at the deepest level this should be regarded as an undefined concept, and that the descriptions offered by Weierstrass and Dedekind are essentially artificial. But if we are to make sense of real analysis in forcing models we must fix on some formulation, so I will say that a real number is the set of strictly smaller rational numbers. (I leave it to you to decide whether a rational number is an equivalence class of pairs of integers, or a coprime pair (m,n) where $m \in \mathbb{Z}$ and $n \in \mathbb{N} \setminus \{0\}$, or something else altogether, provided only that you fix on a construction expressible by a formula of set theory.) Observe that this has the desirable effect that

$$\Vdash_{\mathbb{P}} \check{\alpha} \text{ is a real number}$$

for every real number α.

(b) Consider the Dedekind complete Boolean algebra $\mathrm{RO}(\mathbb{P})$ and the corresponding space $L^0 = L^0(\mathrm{RO}(\mathbb{P}))$ as defined in 364A.

(i) For every $u \in L^0$, set

$$\vec{u} = \{(\check{\alpha}, p) : \alpha \in \mathbb{Q},\, p \in [\![u > \alpha]\!]\}.$$

Then

$$\Vdash_{\mathbb{P}} \vec{u} \text{ is a real number.}$$

P Asw $\Vdash_{\mathbb{P}} \check{\mathbb{Q}} = \mathbb{Q}$, $\Vdash_{\mathbb{P}} \vec{u} \subseteq \mathbb{Q}$. If $p \in P$, there are an $n \in \mathbb{Z}$ and a q stronger than p such that $q \in [\![u > n]\!]$, in which case $(\check{n}, q) \in \vec{u}$ and $q \Vdash_{\mathbb{P}} \vec{u} \neq \emptyset$; accordingly $\Vdash_{\mathbb{P}} \vec{u} \neq \emptyset$. Again, if $p \in P$, there are an $n \in \mathbb{N}$ and a q stronger than p such that $q \in [\![u \le n]\!]$, in which case $\alpha \le n$ whenever $(\check{\alpha}, r) \in \vec{u}$ and r is stronger than q, so that

$$q \Vdash_{\mathbb{P}} \check{n} \text{ is an upper bound for } \vec{u};$$

accordingly $\Vdash_{\mathbb{P}} \vec{u}$ is bounded above.

If $p \in P$ and $\dot{\alpha}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{\alpha} \in \vec{u},$$

then for any q stronger than p there are an r stronger than q and an $\alpha \in \mathbb{Q}$ such that $r \Vdash_{\mathbb{P}} \dot{\alpha} = \check{\alpha}$ and $r \in [\![u > \alpha]\!]$. Now there are a $\beta \in \mathbb{Q}$ and an r' stronger than r such that $\beta > \alpha$ and $r' \in [\![u > \beta]\!]$; in which case $r' \Vdash_{\mathbb{P}} \dot{\alpha} < \check{\beta} \in \vec{u}$. As q is arbitrary,

$$p \Vdash_{\mathbb{P}} \dot{\alpha} \text{ is not the greatest member of } \vec{u};$$

as p and $\dot{\alpha}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \vec{u} \text{ has no greatest member.}$$

If $p \in P$ and $\dot{\alpha}$, $\dot{\beta}$ are $\mathbb{P}$-names such that

$$p \Vdash_{\mathbb{P}} \dot{\alpha} \in \mathbb{Q},\ \dot{\alpha} \le \dot{\beta} \in \vec{u},$$

then for any q stronger than p there are an r stronger than q and α, $\beta \in \mathbb{Q}$ such that $(\check{\beta}, r) \in \vec{u}$ and

$$r \Vdash_{\mathbb{P}} \dot{\alpha} = \check{\alpha},\ \dot{\beta} = \check{\beta},\ \check{\alpha} \le \check{\beta}.$$

In this case, $\alpha \le \beta$ (5A3H(c-ii)), $r \in [\![u > \beta]\!] \subseteq [\![u > \alpha]\!]$, $(\check{\alpha}, r) \in \vec{u}$ and $r \Vdash_{\mathbb{P}} \dot{\alpha} = \check{\alpha} \in \vec{u}$. As q is arbitrary,

$$p \Vdash_{\mathbb{P}} \dot{\alpha} \in \vec{u};$$

as p, $\dot{\alpha}$ and $\dot{\beta}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \alpha \in \vec{u} \text{ whenever } \alpha \in \mathbb{Q} \text{ and } \alpha \le \beta \in \vec{u}, \text{ so } \vec{u} \text{ is a real number.} \quad \mathbf{Q}$$

(ii) Observe next that $[\![\vec{u} > \check{\alpha}]\!] = [\![u > \alpha]\!]$ for every $\alpha \in \mathbb{Q}$. **P** For $p \in P$,

$$
\begin{aligned}
p \in [\![\vec{u} > \check{\alpha}]\!] &\iff p \Vdash_{\mathbb{P}} \check{\alpha} < \vec{u} \\
&\iff p \Vdash_{\mathbb{P}} \check{\alpha} \in \vec{u} \\
&\iff \text{ for every } q \text{ stronger than } p \text{ there are } q' \in P,\ \beta \in \mathbb{Q} \\
&\qquad \text{and an } r \text{ stronger than both } q \text{ and } q' \\
&\qquad \text{such that } (\check{\beta}, q') \in \vec{u} \text{ and } r \Vdash_{\mathbb{P}} \check{\beta} = \check{\alpha} \\
&\iff \text{ for every } q \text{ stronger than } p \text{ there is an } r \text{ stronger than } q \\
&\qquad \text{such that } (\check{\alpha}, r) \in \vec{u} \\
&\iff \text{ for every } q \text{ stronger than } p \text{ there is an } r \text{ stronger than } q \\
&\qquad \text{such that } r \in [\![u > \alpha]\!] \\
&\iff p \in [\![u > \alpha]\!]
\end{aligned}
$$

(514Md, because $[\![u > \alpha]\!]$ is a regular open subset of P). **Q**

(iii) In the other direction, if we have a $\mathbb{P}$-name $\dot{x}$ for a real number (that is, a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}} \dot{x}$ is a real number), then there is a unique $u \in L^0$ such that $\Vdash_{\mathbb{P}} \dot{x} = \vec{u}$. **P** For every $\alpha \in \mathbb{Q}$ we have a Boolean value $[\![\dot{x} > \check{\alpha}]\!]$ belonging to $\mathrm{RO}(\mathbb{P})$ (5A3G). It is easy to see that

$$[\![\dot{x} > \check{\alpha}]\!] = \sup_{\beta \in \mathbb{Q}, \beta > \alpha} [\![\dot{x} > \check{\beta}]\!]$$

for every $\alpha \in \mathbb{Q}$,

$$\inf_{n \in \mathbb{Z}} [\![\dot{x} > \check{n}]\!] = 0, \quad \sup_{n \in \mathbb{Z}} [\![\dot{x} > \check{n}]\!] = 1.$$

We therefore have a unique $u \in L^0$ such that

$$[\![u > \alpha]\!] = [\![\dot{x} > \check{\alpha}]\!]$$

for every $\alpha \in \mathbb{R}$ (364Ae). Now $[\![\vec{u} > \check{\alpha}]\!] = [\![\dot{x} > \check{\alpha}]\!]$ for every $\alpha \in \mathbb{Q}$, that is,

$$p \Vdash_{\mathbb{P}} \vec{u} > \check{\alpha} \text{ iff } p \Vdash_{\mathbb{P}} \dot{x} > \check{\alpha}$$

for every $\alpha \in \mathbb{Q}$ and $p \in P$, that is,

$$p \Vdash_{\mathbb{P}} \check{\alpha} \in \vec{u} \text{ iff } p \Vdash_{\mathbb{P}} \check{\alpha} \in \dot{x}$$

for every $\alpha \in \mathbb{Q}$ and $p \in P$, that is (since both $\vec{u}$ and $\dot{x}$ are $\mathbb{P}$-names for subsets of $\mathbb{Q}$),

$$\Vdash_{\mathbb{P}} \vec{u} = \dot{x}. \text{ } \mathbf{Q}$$

(iv) It follows that if $\dot{x}$ is a $\mathbb{P}$-name and $p \in P$ is such that $p \Vdash_{\mathbb{P}} \dot{x} \in \mathbb{R}$, then there is a $u \in L^0$ such that $p \Vdash_{\mathbb{P}} \dot{x} = \vec{u}$. (For there is a $\mathbb{P}$-name $\dot{y}$ such that $\Vdash_{\mathbb{P}} \dot{y} \in \mathbb{R}$ and $p \Vdash_{\mathbb{P}} \dot{x} = \dot{y}$.)

(v) If $\alpha \in \mathbb{R}$, then $(\alpha\chi 1)^{\rightharpoonup} = \check{\alpha}$. **P** For any $\beta \in \mathbb{Q}$ and $p \in P$,

$$
\begin{aligned}
[\![(\alpha\chi 1)^{\rightharpoonup} > \check{\beta}]\!] = [\![\alpha\chi 1 > \beta]\!] &= 1 \text{ if } \beta < \alpha, 0 \text{ otherwise} \\
&= [\![\check{\alpha} > \check{\beta}]\!] \text{ in either case. } \mathbf{Q}
\end{aligned}
$$

(c) Suppose that u, $v \in L^0$.

(i) $[\![\vec{u} < \vec{v}]\!] = [\![v - u > 0]\!]$. **P**

$$
\begin{aligned}
[\![\vec{u} < \vec{v}]\!] &= [\![\exists\, \alpha \in \mathbb{Q},\ \vec{u} \le \alpha < \vec{v}]\!] \\
&= \sup_{\alpha \in \mathbb{Q}} [\![\vec{u} \le \check{\alpha} < \vec{v}]\!]
\end{aligned}
$$

(taking the supremum in $\mathrm{RO}(\mathbb{P})$, 5A3Gc)

$$= \sup_{\alpha \in \mathbb{Q}} ([\![\vec{v} > \check{\alpha}]\!] \setminus [\![\vec{u} > \check{\alpha}]\!])$$

(taking the relative complements in $\mathrm{RO}(\mathbb{P})$)

$$= \sup_{\alpha\in\mathbb{Q}}([\![v > \alpha]\!] \setminus [\![u > \alpha]\!]) = [\![v - u > 0]\!]. \textbf{ Q}$$

(ii) In particular, if $u \le v$ in L^0, then

$$\Vdash_{\mathbb{P}} \vec{u} \le \vec{v} \text{ in } \mathbb{R}$$

since $[\![\vec{v} < \vec{u}]\!] = 0$, and $[\![\vec{u} = \vec{v}]\!] = [\![u - v = 0]\!]$ for any u, $v \in L^0$.

(iii) $\Vdash_{\mathbb{P}} (u+v)^{\rightarrow} = \vec{u} + \vec{v}$. **P** For any $\alpha \in \mathbb{Q}$,

$$\begin{aligned}
[\![\vec{u} + \vec{v} > \check{\alpha}]\!] &= [\![\exists\ \beta \in \mathbb{Q},\ \vec{u} > \beta,\ \vec{v} > \check{\alpha} - \beta]\!] \\
&= \sup_{\beta\in\mathbb{Q}} [\![\vec{u} > \check{\beta},\ \vec{v} > \check{\alpha} - \check{\beta}]\!] \\
&= \sup_{\beta\in\mathbb{Q}}([\![\vec{u} > \check{\beta}]\!] \cap [\![\vec{v} > (\alpha - \beta)\check{}\,]\!]) \\
&= \sup_{\beta\in\mathbb{Q}}([\![u > \beta]\!] \cap [\![v > \alpha - \beta]\!]) \\
&= [\![u + v > \alpha]\!] \\
&= [\![(u+v)^{\rightarrow} > \check{\alpha}]\!]. \textbf{ Q}
\end{aligned}$$

(364D)

(iv) $\Vdash_{\mathbb{P}} (u \times v)^{\rightarrow} = \vec{u}\vec{v}$. **P** If u, $v \ge 0$ in L^0 and $\alpha \ge 0$ in $\mathbb{Q}$,

$$\begin{aligned}
[\![\vec{u}\vec{v} > \check{\alpha}]\!] &= [\![\exists\ \beta \in \mathbb{Q},\ \beta > 0,\ \vec{u} > \beta,\ \vec{v} > \frac{\check{\alpha}}{\beta}]\!] \\
&= \sup_{\beta\in\mathbb{Q},\beta>0} [\![\vec{u} > \check{\beta},\ \vec{v} > \frac{\check{\alpha}}{\check{\beta}}]\!] \\
&= \sup_{\beta\in\mathbb{Q},\beta>0} [\![\vec{u} > \check{\beta}]\!] \cap [\![\vec{v} > (\frac{\alpha}{\beta})\check{}\,]\!] \\
&= \sup_{\beta\in\mathbb{Q},\beta>0} [\![u > \beta]\!] \cap [\![v > \frac{\alpha}{\beta}]\!] \\
&= [\![u \times v > \alpha]\!] \\
&= [\![(u \times v)^{\rightarrow} > \check{\alpha}]\!].
\end{aligned}$$

So in this case

$$\Vdash_{\mathbb{P}} (u \times v)^{\rightarrow} = \vec{u}\vec{v}.$$

Since we have an appropriate distributive law in L^0 (364D), it follows from (iii) that the same is true for general u, $v \in L^0$. **Q**

(v) If $\alpha \in \mathbb{R}$, then $\Vdash_{\mathbb{P}} (\alpha u)^{\rightarrow} = \check{\alpha}\vec{u}$. (Put (iv) and (b-v) together.)

(d)(i) Suppose that $\langle u_i \rangle_{i\in I}$ is a non-empty family in L^0 with supremum $u \in L^0$. Then

$$\Vdash_{\mathbb{P}} \vec{u} = \sup_{i\in\check{I}} \vec{u}_i \text{ in } \mathbb{R}.$$

P By (c-ii),

$$\Vdash_{\mathbb{P}} \vec{u}_i \le \vec{u}$$

for every $i \in I$, so

$$\Vdash_{\mathbb{P}} \sup_{i\in\check{I}} \vec{u}_i \le \vec{u}.$$

In the other direction, **?** suppose, if possible, that

$$\not\Vdash_{\mathbb{P}} \vec{u} \text{ is the least upper bound of } \{\vec{u}_i : i \in \check{I}\}.$$

Then there are a $p \in P$ and an $\alpha \in \mathbb{Q}$ such that

$$p \Vdash_{\mathbb{P}} \check{\alpha} < \vec{u} \text{ is an upper bound for } \{\vec{u}_i : i \in \check{I}\}.$$

In this case,

$$p \in [\![\vec{u} > \check{\alpha}]\!] = [\![u > \alpha]\!] = \sup_{i\in I} [\![u_i > \alpha]\!] = \operatorname{int}\overline{\bigcup_{i\in I} [\![u_i > \alpha]\!]}$$

(364L(a-ii), 314P). There are therefore a q stronger than p and an $i \in I$ such that $q \in [\![u_i > \alpha]\!]$; but in this case $q \Vdash_{\mathbb{P}} \vec{u}_i > \check{\alpha}$, which is impossible, because $p \Vdash_{\mathbb{P}} \vec{u}_i \le \check{\alpha}$. **X** So

$$\Vdash_{\mathbb{P}} \vec{u} = \sup_{i\in\check{I}} \vec{u}_i. \quad \mathbf{Q}$$

(ii) And if $\langle u_i\rangle_{i\in I}$ is a non-empty family in L^0 with infimum $u \in L^0$, then

$$\Vdash_{\mathbb{P}} \vec{u} = \inf_{i\in\check{I}} \vec{u}_i \text{ in } \mathbb{R}$$

because $\Vdash_{\mathbb{P}} (-u)^{\vec{}} = -\vec{u}$ (using (c)).

(e) Suppose that $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in L^0, order*-convergent (in the sense of §367) to $u \in L^0$. Then

$$\Vdash_{\mathbb{P}} \vec{u} = \lim_{n\to\infty} \vec{u}_n.$$

P By 367Gb,

$$u = \sup_{n\in\mathbb{N}} \inf_{m\ge n} u_m = \inf_{n\in\mathbb{N}} \sup_{m\ge n} u_m$$

so (d) tells us that

$$\Vdash_{\mathbb{P}} \vec{u} = \sup_{n\in\check{\mathbb{N}}} \inf_{m\ge n} \vec{u}_m = \inf_{n\in\check{\mathbb{N}}} \sup_{m\ge n} \vec{u}_m,$$

that is,

$$\Vdash_{\mathbb{P}} \vec{u} = \lim_{n\to\infty} \vec{u}_n. \quad \mathbf{Q}$$

5A3M Forcing with Boolean algebras Suppose that $\mathfrak{A}$ is a Dedekind complete Boolean algebra, not $\{0\}$. As noted in 5A3Ab, $\mathbb{P} = (\mathfrak{A}^+, \subseteq, 1_{\mathfrak{A}}, \downarrow)$ is a forcing notion. We have a natural isomorphism between $\mathrm{RO}(\mathbb{P})$ and $\mathfrak{A}$, matching each $G \in \mathrm{RO}(\mathbb{P})$ with $\sup G \in \mathfrak{A}$ (514Sb); by 514M(d-ii), $\sup G$, taken in $(\mathfrak{A}^+, \subseteq)$, will belong to G unless $G = \emptyset$. In this context, I will usually identify the two algebras, so that $[\![\phi]\!]$ becomes $\sup\{a : a \in \mathfrak{A}^+, a \Vdash_{\mathbb{P}} \phi\}$, and we shall have $[\![\phi]\!] \Vdash_{\mathbb{P}} \phi$ except when $\Vdash_{\mathbb{P}} \neg\phi$. Note that $[\![\neg\phi]\!] = 1 \setminus [\![\phi]\!]$ (5A3Gb).

The identification of $\mathrm{RO}(\mathbb{P})$ with $\mathfrak{A}$ itself simplifies some of the discussion in 5A3L. We have a $\mathbb{P}$-name $\vec{u}$ associated with each $u \in L^0(\mathfrak{A})$, and the formula

$$[\![\vec{u} = \vec{v}]\!] = [\![u - v = 0]\!] \text{ in } \mathrm{RO}(\mathbb{P}) \text{ when } u,\, v \in L^0(\mathrm{RO}(\mathbb{P}))$$

of 5A3L(c-ii) turns into

$$\text{whenever } u,\, v \in L^0(\mathfrak{A}) \text{ and } a \in \mathfrak{A}^+,\ u \times \chi a = v \times \chi a \iff a \Vdash_{\mathbb{P}} \vec{u} = \vec{v}.$$

P

$$\begin{aligned} a \Vdash_{\mathbb{P}} \vec{u} = \vec{v} &\iff a \in [\![\vec{u} = \vec{v}]\!] \text{ interpreted in } \mathrm{RO}(\mathbb{P}) \\ &\iff a \in [\![u - v = 0]\!] \text{ interpreted in } \mathrm{RO}(\mathbb{P}) \end{aligned}$$

(here thinking of u and v as members of $L^0(\mathrm{RO}(\mathbb{P}))$)

$$\iff a \subseteq [\![u - v = 0]\!] \text{ interpreted in } \mathfrak{A}$$

(now thinking of u and v as members of $L^0(\mathfrak{A})$)

$$\begin{aligned} &\iff (u - v) \times \chi a = 0 \text{ in } L^0(\mathfrak{A}) \\ &\iff u \times \chi a = v \times \chi a. \quad \mathbf{Q} \end{aligned}$$

5A3N Ordinals and cardinals Let $\mathbb{P}$ be a forcing notion, and P its set of conditions.

(a) For any ordinal α,

$$\Vdash_{\mathbb{P}} \check{\alpha} \text{ is an ordinal;}$$

moreover, if $p \in P$ and $\dot{x}$ is a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \dot{x} \text{ is an ordinal,}$$

there are a q stronger than p and an ordinal α such that

$$q \Vdash_{\mathbb{P}} \dot{x} = \check{\alpha}$$

(Jech 03, 14.23; see Kunen 80, IV.3.14).

(b) If $\mathbb{P}$ is ccc, then

$$\Vdash_{\mathbb{P}} \check{\kappa} \text{ is a cardinal}$$

for every cardinal (that is, initial ordinal) κ (Kunen 80, VII.5.6; Jech 03, 14.34). In particular,

$$\Vdash_{\mathbb{P}} \check{\omega}_1 \text{ is a cardinal, so is the first uncountable cardinal,}$$

and we can write

$$\Vdash_{\mathbb{P}} \omega_1 = \check{\omega}_1,\ \omega_2 = \check{\omega}_2$$

etc., if we are sure of being understood.

(c) Again suppose that $\mathbb{P}$ is ccc, and that we have a set A, a $\mathbb{P}$-name $\dot{X}$ and a cardinal κ such that

$$\Vdash_{\mathbb{P}} \dot{X} \subseteq \check{A} \text{ and } \#(\dot{X}) \le \check{\kappa}.$$

Then there is a set $B \subseteq A$ such that $\#(B) \le \max(\omega, \kappa)$ and

$$\Vdash_{\mathbb{P}} \dot{X} \subseteq \check{B}.$$

P Let $\dot{f}$ be a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \dot{f} \text{ is an injective function with domain } \dot{X} \text{ and } \dot{f}[\dot{X}] \subseteq \check{\kappa}.$$

For $\xi < \kappa$ set

$$B_\xi = \{a : a \in A \text{ and there is a } p \in P \text{ such that } p \Vdash_{\mathbb{P}} \check{a} \in \dot{X} \ \&\ \dot{f}(\check{a}) = \check{\xi}\}.$$

For each $a \in B_\xi$, choose $p_{\xi a} \in P$ such that

$$p_{\xi a} \Vdash_{\mathbb{P}} \check{a} \in \dot{X} \text{ and } \dot{f}(\check{a}) = \check{\xi};$$

then if a, $b \in B_\xi$ and q is stronger than both $p_{\xi a}$ and $p_{\xi b}$, we have

$$q \Vdash_{\mathbb{P}} \dot{f}(\check{a}) = \dot{f}(\check{b}) \text{ so } \check{a} = \check{b}$$

and $a = b$ (5A3Hc). Thus $\langle p_{\xi a}\rangle_{a \in B_\xi}$ is an antichain in $\mathbb{P}$ and B_ξ must be countable; setting $B = \bigcup_{\xi<\kappa} B_\xi$, $B \subseteq A$ and $\#(B) \le \max(\omega, \kappa)$.

Now suppose that $p \in P$ and that $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \dot{X}$. Then

$$p \Vdash_{\mathbb{P}} \dot{x} \in \check{A} \text{ and } \dot{f}(\dot{x}) \in \check{\kappa},$$

so there are a q stronger than p, an $a \in A$ and a $\xi < \kappa$ such that

$$q \Vdash_{\mathbb{P}} \dot{x} = \check{a} \text{ and } \dot{f}(\dot{x}) = \check{\xi}.$$

Now $a \in B_\xi \subseteq B$, so $q \Vdash_{\mathbb{P}} \dot{x} \in \check{B}$. As p and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \dot{X} \subseteq \check{B},$$

as required. **Q**

(d) If $\mathbb{P}$ is ccc, then

$$\Vdash_{\mathbb{P}} \mathrm{cf}[\check{I}]^{\le\omega} = (\mathrm{cf}[I]^{\le\omega})\check{}$$

for every set I. **P** Write κ for $\mathrm{cf}[I]^{\le\omega}$. (i) Let $\mathcal{K} \subseteq [I]^{\le\omega}$ be a cofinal family with $\#(\mathcal{K}) = \kappa$. Then $\Vdash_{\mathbb{P}} \check{\kappa}$ is a cardinal, so

$$\Vdash_{\mathbb{P}} \check{\mathcal{K}} \subseteq [\check{I}]^{\le\omega} \text{ and } \#(\check{\mathcal{K}}) = \check{\kappa}.$$

If $\dot{J}$ is a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \dot{J} \in [\check{I}]^{\le\omega},$$

then by (c) there is a countable set $K \subseteq I$ such that $\Vdash_{\mathbb{P}} \dot{J} \subseteq \check{K}$; now there is an $L \in \mathcal{K}$ such that $K \subseteq L$, and

$$\Vdash_{\mathbb{P}} \dot{J} \subseteq \check{K} \subseteq \check{L} \in \check{\mathcal{K}}.$$

As $\dot{J}$ is arbitrary,

$$\Vdash_{\mathbb{P}} \check{\mathcal{K}} \text{ is cofinal with } [\check{I}]^{\le\omega} \text{ and } \mathrm{cf}[\check{I}]^{\le\omega} \le \check{\kappa}.$$

(ii) **?** If

$$\neg \Vdash_{\mathbb{P}} \check{\kappa} \le \mathrm{cf}[\check{I}]^{\le\omega},$$

then there are a $p \in P$ and an ordinal δ such that

$$p \Vdash_{\mathbb{P}} \mathrm{cf}[\check{I}]^{\le\omega} = \check{\delta} < \check{\kappa}.$$

Now there must be a family $\langle \dot{J}_\xi \rangle_{\xi<\delta}$ of $\mathbb{P}$-names such that

$$p \Vdash_{\mathbb{P}} \{\dot{J}_\xi : \xi < \check{\delta}\} \text{ is cofinal with } [\check{I}]^{\le\omega}.$$

By (c) again, there must be for each $\xi < \kappa$ a countable $K_\xi \subseteq I$ such that $p \Vdash_{\mathbb{P}} \dot{J}_\xi \subseteq \check{K}_\xi$. Because $\delta < \mathrm{cf}[I]^{\le\omega}$, there is a $K \in [I]^{\le\omega}$ such that $K \not\subseteq K_\xi$ for every $\xi < \delta$. In this case,

$$p \Vdash_{\mathbb{P}} \check{K} \in [\check{I}]^{\le\omega} \text{ so there is a } \xi < \check{\delta} \text{ such that } \check{K} \subseteq \dot{J}_\xi,$$

and there must be a $\xi < \delta$ and a q stronger than p such that

$$q \Vdash_{\mathbb{P}} \check{K} \subseteq \dot{J}_\xi \subseteq \check{K}_\xi.$$

But this implies that $K \subseteq K_\xi$, which isn't so. **X**

We conclude that

$$\Vdash_{\mathbb{P}} \check{\kappa} \le \mathrm{cf}[\check{I}]^{\le\omega} \text{ and } \check{\kappa} = \mathrm{cf}[\check{I}]^{\le\omega}. \text{ } \mathbf{Q}$$

5A3O Iterated forcing (Kunen 80, VIII.5.2) If $\mathbb{P}$ is a forcing notion and P its set of conditions, and we have a quadruple $\dot{\mathbb{Q}} = (\dot{Q}, \dot{\le}, \dot{1}, \dot{\epsilon})$ of $\mathbb{P}$-names such that $(\dot{1}, \mathbb{1}_{\mathbb{P}}) \in \dot{Q}$ and

$\Vdash_{\mathbb{P}} \dot{\le}$ is a pre-order on $\dot{Q}$, $\dot{\epsilon}$ is a direction of activity and every member of $\dot{Q}$ is stronger than $\dot{1}$,

then $\mathbb{P} * \dot{\mathbb{Q}}$ is the forcing notion defined by saying that its conditions are objects of the form $(p, \dot{q})$ where

$$p \in P, \quad \dot{q} \in \mathrm{dom}\,\dot{Q}, \quad p \Vdash_{\mathbb{P}} \dot{q} \in \dot{Q},$$

and that $(p, \dot{q})$ is stronger than $(p', \dot{q}')$ if p is stronger than p' and

$$p \Vdash_{\mathbb{P}} \dot{q} \text{ is stronger than } \dot{q}'.$$

(Strictly speaking, I should add that $\mathbb{1}_{\mathbb{P}*\dot{\mathbb{Q}}} = (\mathbb{1}_{\mathbb{P}}, \dot{1})$.)[5]

5A3P Martin's axiom Let κ be a regular uncountable cardinal such that $2^\lambda \le \kappa$ for every $\lambda < \kappa$. Then there is a ccc forcing notion $\mathbb{P}$ such that

$$\Vdash_{\mathbb{P}} \mathfrak{m} = \mathfrak{c} = \check{\kappa}.$$

[5] This formulation gives us the freedom to take $\dot{\epsilon}$ to be non-trivial. I do not mean to suggest that it would be reasonable to take advantage of this.

(Kunen 80, VIII.6.3; Jech 03, 16.13).

5A3Q Countably closed forcings (a) Let $\mathbb{P}$ be a forcing notion, and P its set of conditions. $\mathbb{P}$ is **countably closed** if whenever $\langle p_n\rangle_{n\in\mathbb{N}}$ is a sequence in P such that p_{n+1} is stronger than p_n for every n, there is a $p\in P$ which is stronger than every p_n.

(b) If $\mathbb{P}$ is a countably closed forcing notion, then $\Vdash_{\mathbb{P}} \mathcal{P}\mathbb{N}=(\mathcal{P}\mathbb{N})\check{}$. **P** Let P be the set of conditions of $\mathbb{P}$. If $p\in P$ and $\dot x$ is a $\mathbb{P}$-name such that $p\Vdash_{\mathbb{P}}\dot x\subseteq\mathbb{N}$, choose $\langle p_n\rangle_{n\in\mathbb{N}}$ inductively in P such that $p_0=p$ and, for each $n\in\mathbb{N}$, p_{n+1} is stronger than p_n and either $p_{n+1}\Vdash_{\mathbb{P}}\check n\in\dot x$ or $p_{n+1}\Vdash_{\mathbb{P}}\check n\notin\dot x$. Let $q\in P$ be stronger than every p_n, and set $A=\{n:q\Vdash_{\mathbb{P}}\check n\in\dot x\}$. Then $q\Vdash_{\mathbb{P}}\check n\notin\dot x$ for every $n\in\mathbb{N}\setminus A$, so $q\Vdash_{\mathbb{P}}\dot x=\check A\in(\mathcal{P}\mathbb{N})\check{}$. As p and $\dot x$ are arbitrary, $\Vdash_{\mathbb{P}}\mathcal{P}\mathbb{N}\subseteq(\mathcal{P}\mathbb{N})\check{}$. For the reverse inequality, we have $\Vdash(\mathcal{P}X)\check{}\subseteq\mathcal{P}\check X$ for any forcing notion and set X, so here we have $\Vdash_{\mathbb{P}}(\mathcal{P}\mathbb{N})\check{}\subseteq\mathcal{P}\check{\mathbb{N}}=\mathcal{P}\mathbb{N}$. **Q**

Consequently $\Vdash_{\mathbb{P}}\mathbb{R}=\check{\mathbb{R}}$. **P** The argument just above shows that $\Vdash_{\mathbb{P}}\mathcal{P}\mathbb{Q}=(\mathcal{P}\mathbb{Q})\check{}$, and now it is easy to see that

$$\begin{aligned}\Vdash_{\mathbb{P}}\mathbb{R}=\{A:\emptyset\ne A\subseteq\mathbb{Q},\ &A\text{ is bounded above and has no greatest element,}\\ &q\in A\text{ whenever }q\le q'\in A\}\\ =\check{\mathbb{R}}.\ \textbf{Q}\end{aligned}$$

Similarly, $\Vdash_{\mathbb{P}}[0,1]=[0,1]\check{}$.

5A3 Notes and comments In terms of the discussion in Kunen 80, §VII.9, you will see that I follow an extreme version of the 'syntactical' approach to forcing. In the first place, this is due to a philosophical prejudice; I do not believe in models of ZF. But it seems to me that quite apart from this there is a fundamental difference between the sentences

$$\mathfrak{m}=\mathfrak{c}$$

and

$$\Vdash_{\mathbb{P}}\mathfrak{m}=\mathfrak{c}$$

associated with the fact that the symbols $\mathfrak{m}$, $\mathfrak{c}$ and even $=$ must be reinterpreted in the second version. I have tried in this section to develop a language which can express and accommodate the difference. It puts a substantial burden on the reader, especially in such formulae as $\sup_{i\in\check I}\dot x_i$ (5A3F) and $((\dot y,f(\dot y,p)),p)$ (5A3K), where you may have to read quite carefully to determine which parts of the formulae are supposed to be in the forcing language, and which are in the ordinary language of set theory. There is an additional complication in 5A3L, where I use the same symbol $[\![\ \ ]\!]$ for two quite different functions; but here at least the objects $[\![u>\alpha]\!]$, $[\![\vec u>\check\alpha]\!]$ belong to the same set RO($\mathbb{P}$), even if the formulae inside the brackets have to be parsed by very different rules. I hope that the clue of a superscripted letter $\dot x$ or $\check I$ or $\vec u$ will alert you to the need for thought. Once we have grasped this nettle, however, we are in a position to move between the two languages, as in 5A3K; and statements of results such as 5A3P can be shortened by taking it for granted that the preamble '$2^\lambda\le\kappa$ for every $\lambda<\kappa$' refers to the ground universe, while the conclusion '$\mathfrak{m}=\mathfrak{c}=\check\kappa$' is to be interpreted in the forcing universe.

Of course a large number of different types of forcing notion have been described and investigated. In 5A3N I mention some basic facts about ccc forcings. Another important class is that of countably closed forcings (5A3Q).

Version of 20.7.24

5A4 General topology

The principal new topological concepts required in this volume are some of the standard cardinal functions of topology (5A4A-5A4B). As usual, there are particularly interesting phenomena involving compact spaces (5A4C). For special purposes in §513, we need to know some non-trivial facts about metrizable spaces (5A4D). The rest of the section is made up of scraps which are either elementary or standard.

5A4A Definitions Let $(X,\mathfrak{T})$ be a topological space.

(a) The **weight** of X, $w(X)$, is the least cardinal of any base for $\mathfrak{T}$.

(b) The **π-weight** of X is $\pi(X)=\mathrm{ci}(\mathfrak{T}\setminus\{\emptyset\})$, the smallest cardinal of any π-base for $\mathfrak{T}$.

(c) The **density** $d(X)$ of X is the smallest cardinal of any dense subset of X.

(d) The **cellularity** of X is

$$c(X)=c^{\downarrow}(\mathfrak{T}\setminus\{\emptyset\})=\sup\{\#(\mathcal{G}):\mathcal{G}\subseteq\mathfrak{T}\setminus\{\emptyset\}\text{ is disjoint}\}.$$

The **saturation** of X is

$$\mathrm{sat}(X)=\mathrm{sat}^{\downarrow}(\mathfrak{T}\setminus\{\emptyset\})=\sup\{\#(\mathcal{G})^+:\mathcal{G}\subseteq\mathfrak{T}\setminus\{\emptyset\}\text{ is disjoint}\},$$

that is, the smallest cardinal κ such that there is no disjoint family of κ non-empty open sets.

(e) The **tightness** of X, $t(X)$, is the smallest cardinal κ such that whenever $A\subseteq X$ and $x\in\overline{A}$ there is a $B\in[A]^{\le\kappa}$ such that $x\in\overline{B}$. (Recall that $[A]^{\le\kappa}=\{B:B\subseteq A,\,\#(B)\le\kappa\}$.)

(f) The **Novák number** $n(X)$ is the smallest cardinal of any family of nowhere dense subsets of X covering X; or ∞ if there is no such family.

(g)(i) The **Lindelöf number** $L(X)$ is the least cardinal κ such that every open cover of X has a subcover with cardinal at most κ.

(ii) The **hereditary Lindelöf number** $\mathrm{hL}(X)$ is $\sup_{Y\subseteq X}L(Y)$.

(h)(i) If $x\in X$, the **character of x in** X, $\chi(x,X)$, is the smallest cardinal of any base of neighbourhoods of x.

(ii) The **character** of X is $\chi(X)=\sup_{x\in X}\chi(x,X)$.

(i) The **network weight** of X, $\mathrm{nw}(X)$, is the smallest cardinal of any network for $\mathfrak{T}$.

Remark Recall that X is called 'second-countable' iff $w(X)\le\omega$, 'separable' iff $d(X)\le\omega$, 'ccc' iff $c(X)\le\omega$ (that is, $\mathrm{sat}(X)\le\omega_1$), 'Lindelöf' if $L(X)\le\omega$, 'hereditarily Lindelöf' if $\mathrm{hL}(X)\le\omega$, 'first-countable' if $\chi(X)\le\omega$ and 'countably tight' iff $t(X)\le\omega$.

5A4B Proposition Let $(X,\mathfrak{T})$ be a topological space.

(a)

$$c(X)\le d(X)\le\pi(X)\le w(X)\le\#(\mathfrak{T})\le 2^{\mathrm{nw}(X)},$$

$$t(X)\le\chi(X)\le w(X)\le\max(\#(X),\chi(X)).$$

$\mathrm{sat}(X)=c(X)^+$ unless $\mathrm{sat}(X)$ is weakly inaccessible, in which case $\mathrm{sat}(X)=c(X)$.

(b) If Y is a subspace of X, then $w(Y)\le w(X)$, $t(Y)\le t(X)$, $\mathrm{nw}(Y)\le\mathrm{nw}(X)$ and $\chi(y,Y)\le\chi(y,X)$ for every $y\in Y$.

(c) If a topological space Y is a continuous image of X, then $d(Y)\le d(X)$, $c(Y)\le c(X)$, $t(Y)\le t(X)$, $L(Y)\le L(X)$ and $\mathrm{nw}(Y)\le\mathrm{nw}(X)$.

(d) If $\mathcal{G}$ is a family of open subsets of X, then there is a subfamily $\mathcal{H}\subseteq\mathcal{G}$ such that $\#(\mathcal{H})<\mathrm{sat}(X)$ and $\overline{\bigcup\mathcal{H}}=\overline{\bigcup\mathcal{G}}$.

(e) Let $\langle X_i\rangle_{i\in I}$ be a family of non-empty topological spaces with product X, and λ a cardinal such that $\#(I)\le 2^\lambda$. Then

$$d(X)\le\max(\omega,\lambda,\sup_{i\in I}d(X_i)),\quad c(X)=\sup_{J\subseteq I\text{ is finite}}c(\textstyle\prod_{i\in J}X_i).$$

(f) If $\mathcal{G}$ is any family of open subsets of X, there is an $\mathcal{H}\subseteq\mathcal{G}$ such that $\#(\mathcal{H})\le\mathrm{hL}(X)$ and $\bigcup\mathcal{H}=\bigcup\mathcal{G}$.

(g) If X is Hausdorff then $\#(X)\le 2^{\max(c(X),\chi(X))}$.

(h) Suppose that X is metrizable.

(i) $d(X)=w(X)$.

(ii) $d(Y) \le d(X)$ for every $Y \subseteq X$. So any discrete subset of X has cardinal at most $d(X)$.

(iii) Let ρ be a metric on X defining its topology. Then X is separable iff there is no uncountable $A \subseteq X$ such that $\inf_{x,y\in A,x\ne y}\rho(x,y) > 0$.

proof (a) Let $D \subseteq X$ be a dense set with cardinal $d(X)$. If $\mathcal{G} \subseteq \mathfrak{T}\setminus\{\emptyset\}$ is disjoint, we have a surjection from $D\cap\bigcup\mathcal{G}$ to $\mathcal{G}$, so $\#(\mathcal{G}) \le d(X)$; as $\mathcal{G}$ is arbitrary, $c(X) \le d(X)$.

Let $\mathcal{U}$ be a π-base for $\mathfrak{T}$ with cardinal $\pi(X)$. Then there is a set $D \subseteq X$, with cardinal at most $\#(\mathcal{U})$, meeting every non-empty member of $\mathcal{U}$; now D is dense, so $d(X) \le \#(D) \le \pi(X)$.

Any base for $\mathfrak{T}$ is a π-base for $\mathfrak{T}$, so $\pi(X) \le w(X)$. Of course $w(X) \le \#(\mathfrak{T})$.

Let $\mathcal{E}$ be a network for $\mathfrak{T}$ with cardinal $\mathrm{nw}(X)$; then $\mathfrak{T} \subseteq \{\bigcup\mathcal{E}' : \mathcal{E}'\subseteq\mathcal{E}\}$, so $\#(\mathfrak{T}) \le 2^{\#(\mathcal{E})} = 2^{\mathrm{nw}(X)}$.

If $A \subseteq X$ and $x \in \overline{A}$, there are a base $\mathcal{V}$ of neighbourhoods of X with $\#(\mathcal{V}) \le \chi(X)$ and a set $B \in [A]^{\le\chi(X)}$ meeting every member of $\mathcal{V}$, so that $x \in \overline{B}$. Thus $t(X) \le \chi(X)$.

If $\mathcal{U}$ is a base for $\mathfrak{T}$ with cardinal $w(X)$, and $x \in X$, then $\mathcal{U}_x = \{U : x \in U \in \mathcal{U}\}$ is a base of neighbourhoods of x, so $\chi(x,X) \le \#(\mathcal{U}_x) \le w(X)$; as x is arbitrary, $\chi(X) \le w(X)$.

If X is finite, every point x of X has a smallest neighbourhood V_x, and $\{V_x : x \in X\}$ is a base for $\mathfrak{T}$, so $w(X) \le \#(X)$. If X is infinite, then for each $x \in X$ choose a base $\mathcal{U}_x$ of neighbourhoods of x with $\#(\mathcal{U}_x) = \chi(x,X) \le \chi(X)$. Set $\mathcal{U} = \{\mathrm{int}\,U : U \in \bigcup_{x\in X}\mathcal{U}_x\}$; then $\mathcal{U}$ is a base for $\mathfrak{T}$ so

$$w(X) \le \#(\mathcal{U}) \le \max(\omega, \#(X), \chi(X)) = \max(\#(X), \chi(X)).$$

Taking P to be the partially ordered set $(\mathfrak{T}\setminus\{\emptyset\}, \subseteq)$, $c(X) = c^{\downarrow}(P)$ and $\mathrm{sat}(X) = \mathrm{sat}^{\downarrow}(P)$, so 513B, inverted, tells us that $\mathrm{sat}(X) = c(X)^+$ unless $\mathrm{sat}(X)$ is weakly inaccessible, in which case $\mathrm{sat}(X) = c(X)$.

(b) If $\mathcal{U}$ is a base for $\mathfrak{T}$, then $\{U\cap Y : U \in \mathcal{U}\}$ is a base for the topology of Y, so $w(Y) \le w(X)$. If $A \subseteq Y$ and $y \in Y$ belongs to the closure of A in Y it belongs to the closure of A in X so there is a $B \in [A]^{\le t(X)}$ such that $y \in \overline{B}$ taken either in X or in Y; thus $t(Y) \le t(X)$. If $\mathcal{E}$ is a network for $\mathfrak{T}$, then $\{E\cap Y : E \in \mathcal{E}\}$ is a network for the topology of Y, so $\mathrm{nw}(Y) \le \mathrm{nw}(X)$. If $y \in Y$ and $\mathcal{V}$ is a base of neighbourhoods of y in X, then $\{V\cap Y : V \in \mathcal{V}\}$ is a base of neighbourhoods of y in Y, so $\chi(y,Y) \le \chi(y,X)$.

(c) Let $f : X \to Y$ be a continuous surjection. If $D \subseteq X$ is dense, then $f[D]$ is dense in Y (3A3Eb), and $d(Y) \le \#(f[D]) \le \#(D)$; as D is arbitrary, $d(Y) \le d(X)$.

If $\mathcal{H}$ is a disjoint family of non-empty open sets in Y, then $\mathcal{G} = \{f^{-1}[H] : H \in \mathcal{H}\}$ is a disjoint family of non-empty open sets in X, so $\#(\mathcal{H}) = \#(\mathcal{G}) \le c(X)$; as $\mathcal{H}$ is arbitrary, $c(Y) \le c(X)$.

If $B \subseteq Y$ and $y \in \overline{B}$ there is an $x \in X$ such that $f(x) = y$. Now $x \in f^{-1}[\overline{B}] \subseteq \overline{f^{-1}[B]}$ so there is a $C \subseteq [f^{-1}[B]]^{\le t(X)}$ such that $x \in \overline{C}$ and $y \in \overline{f[C]}$ while $f[C] \in [B]^{\le t(X)}$. So $t(Y) \le t(X)$.

If $\mathcal{H}$ is an open cover of Y, then $\mathcal{G} = \{f^{-1}[H] : H \in \mathcal{H}\}$ is an open cover of X; let $\mathcal{G}_0 \in [\mathcal{G}]^{\le L(X)}$ be a subcover; choose $\mathcal{H}_0 \subseteq \mathcal{H}$ such that $\#(\mathcal{H}_0) = \#(\mathcal{G}_0)$ and $\mathcal{G}_0 = \{f^{-1}[H] : H \in \mathcal{H}_0\}$; then $\mathcal{H}_0$ covers Y. As $\mathcal{H}$ is arbitrary, $L(Y) \le L(X)$.

If $\mathcal{A}$ is a network for $\mathfrak{T}$, then $\{f[A] : A \in \mathcal{A}\}$ is a network for the topology of Y, so $\mathrm{nw}(Y) \le \mathrm{nw}(X)$.

(d) Let $\mathcal{V}$ be a maximal disjoint family of non-empty open sets included in members of $\mathcal{G}$. Then $\#(\mathcal{V}) < \mathrm{sat}(X)$. Let $\mathcal{H} \subseteq \mathcal{G}$ be such that $\#(\mathcal{H}) \le \#(\mathcal{V})$ and every member of $\mathcal{V}$ is included in a member of $\mathcal{H}$. If $G \in \mathcal{G}$ then $G\setminus\overline{\bigcup\mathcal{H}}$ meets no member of $\mathcal{V}$, so must be empty; so this $\mathcal{H}$ serves.

(e)(i) By Engelking 89, 2.3.15, $d(X) \le \max(\omega, \lambda, \sup_{i\in I} d(X_i))$.

(ii) Set $\kappa = \sup_{J\subseteq I\text{ is finite}} c(\prod_{i\in J} X_i)$. All the finite products $\prod_{i\in J} X_i$ are continuous images of X, so $c(X) \ge \kappa$, by (c). **?** Suppose, if possible, that $c(X) > \kappa$. Let $\mathcal{V}$ be the usual base for the topology of X, consisting of sets of the form $\prod_{i\in I} G_i$ where $G_i \subseteq X_i$ is open for every i and $\{i : G_i \ne X_i\}$ is finite. Let $\langle W_\xi\rangle_{\xi<\kappa^+}$ be a disjoint family of non-empty open sets in X. For each $\xi < \kappa$ let $W'_\xi \subseteq W_\xi$ be a non-empty member of $\mathcal{V}$, so that W'_ξ is determined by a coordinates in a finite subset I_ξ of I. By the Δ-system Lemma (4A1Db) there is a set $A \subseteq \kappa^+$, with cardinal κ^+, such that $\langle I_\xi\rangle_{\xi\in A}$ is a Δ-system with root J say. For $\xi \in A$ express W'_ξ as $U_\xi\cap V_\xi$ where U_ξ is determined by coordinates in J and V_ξ is determined by coordinates in $I_\xi\setminus J$. Now for distinct ξ, $\eta \in A$,

$$\emptyset = W'_\xi \cap W'_\eta = U_\xi\cap U_\eta\cap V_\xi\cap V_\eta.$$

Since V_ξ and V_η and $U_\xi\cap U_\eta$ are determined by coordinates in the disjoint sets $I_\xi\setminus J$, $I_\eta\setminus J$ and J respectively, one of them must be empty, and this can only be $U_\xi\cap U_\eta$. Thus $\langle U_\xi\rangle_{\xi\in A}$ is disjoint. But now observe that each U_ξ is of the form $\pi_J^{-1}[H_\xi]$ where $H_\xi \subseteq \prod_{i\in J} X_i$ is a non-empty open set and $\pi_J(x) = x\restriction J$ for every $x \in X$. So $\langle H_\xi\rangle_{\xi\in A}$ witnesses that $c(\prod_{i\in J} X_i) \ge \kappa^+$, which contradicts the definition of κ. **X**

Thus $c(X) = \sup_{J\subseteq I \text{ is finite}} c(\prod_{i\in J} X_i)$.

(f) This is just because $L(\bigcup\mathcal{G}) \le \mathrm{hL}(X)$.

(g) If X is finite, $c(X) = \#(X)$ and the result is trivial. Otherwise, set $\kappa = \max(c(X), \chi(X))$ and for each $x \in X$ let $\langle U_\xi(x)\rangle_{\xi<\kappa}$ run over a base of neighbourhoods of x consisting of open sets. Let $f : [X]^2 \to [\kappa]^2$ be such that whenever x, $y \in X$ are distinct then there are ξ, $\eta \in f(\{x,y\})$ such that $U_\xi(x)$ and $U_\eta(y)$ are disjoint. **?** If $\#(X) > 2^\kappa$ then by the Erdős-Rado theorem (5A1Ha) there is a $C \subseteq X$ such that $\#(C) > \kappa$ and f is constant on $[C]^2$; let $\{\xi,\eta\}$ be the constant value. For $x \in C$ set $G_x = U_\xi(x)\cap U_\eta(x)$; then $\langle G_x\rangle_{x\in C}$ is a disjoint family of non-empty open sets, so $c(X) \ge \#(C) > \kappa$. **X**

(h) Fix a metric ρ on X defining its topology.

(i) If $d(X) < \omega$ then X is finite and the result is trivial. Otherwise, let D be a dense subset of X with cardinal $d(X)$; setting $U(x,\epsilon) = \{y : \rho(y,x) < \epsilon\}$, $\{U(x, 2^{-n}) : x \in D,\ n \in \mathbb{N}\}$ is a base for $\mathfrak{T}$, so $w(X) \le \max(\#(D), \omega) = d(X)$. Since we know from (a) above that $d(X) \le w(X)$, we have equality.

(ii) Put (i) together with (b) above to see that $d(Y) \le d(X)$. If Y is discrete, then $\#(Y) = d(Y) \le d(X)$.

(iii)(α) If there is an uncountable $A \subseteq X$ such that $\inf_{x,y\in A,x\ne y} \rho(x,y) > 0$, then A is not separable in its subspace topology, so X is not separable, by (ii). (β) If there is no such A, then for each $n \in \mathbb{N}$ let A_n be a maximal subset of X such that $\rho(x,y) \ge 2^{-n}$ for all distinct x, $y \in A_n$. In this case $\bigcup_{n\in\mathbb{N}} A_n$ is dense, so $d(X) \le \max(\omega, \sup_{n\in\mathbb{N}} \#(A_n)) = \omega$ and X is separable.

5A4C Compactness Let X be a compact Hausdorff space.

(a)(i) $\mathrm{nw}(X) = w(X)$. (Engelking 89, 3.1.19.)

(ii) There is a set $Y \subseteq X$, with cardinal at most the cardinal power $d(X)^\omega$, which meets every non-empty G_δ subset of X. **P** Let $D \subseteq X$ be a dense set with cardinal $d(X)$. For each sequence $\omega \in D^{\mathbb{N}}$ choose a cluster point x_ω of $\langle\omega(n)\rangle_{n\in\mathbb{N}}$; set $Y = \{x_\omega : \omega \in D^{\mathbb{N}}\}$. Then $\#(Y) \le \#(D^{\mathbb{N}}) = d(X)^\omega$. If $\langle G_n\rangle_{n\in\mathbb{N}}$ is a sequence of open sets in X with non-empty intersection, take $x \in \bigcap_{n\in\mathbb{N}} G_n$ and choose inductively a sequence $\langle H_n\rangle_{n\in\mathbb{N}}$ of open sets such that $x \in H_n$ and $\overline{H}_{n+1} \subseteq H_n \cap G_n$ for every n. Let $\omega \in D^{\mathbb{N}}$ be such that $\omega(n) \in H_n$ for every n; then

$$x_\omega \in Y \cap \bigcap_{n\in\mathbb{N}} \overline{H}_n \subseteq \bigcap_{n\in\mathbb{N}} G_n.$$

As $\langle G_n\rangle_{n\in\mathbb{N}}$ is arbitrary, Y is a suitable set. **Q**

(b) If X is perfectly normal it is first-countable. (Every singleton set in X is a zero set (4A2Fi), so is a G_δ set; by 4A2Kf, X is first-countable.)

(c) If $w(X) \le \kappa$, X is homeomorphic to a closed subspace of $[0,1]^\kappa$. (Engelking 89, 3.2.5.)

(d)(i) If Y is a Hausdorff space and $f : X \to Y$ is a continuous irreducible surjection, then $d(X) = d(Y)$. **P** We know that $d(Y) \le d(X)$ (5A4Bc). In the other direction, let $D \subseteq Y$ be a dense set with cardinal $d(Y)$, and $C \subseteq X$ a set with cardinal $\#(D)$ such that $f[C] = D$. If $G \subseteq X$ is open and not empty, $f[X \setminus G]$ is a closed proper subset of Y (because f is irreducible), so $D \not\subseteq f[X \setminus G]$ and $C \not\subseteq X \setminus G$. As G is arbitrary, C is dense, and witnesses that $d(X) \le d(Y)$. **Q**

(ii) If $f : X \to \{0,1\}^\kappa$ is a continuous irreducible surjection, where $\kappa \ge \omega$, then $\chi(x,X) \ge \kappa$ for every $x \in X$. **P** Let $\mathcal{V}$ be a base of neighbourhoods of x with cardinal $\chi(x,X)$. For each $\xi < \kappa$, set $G_\xi = \{y : y \in X,\ f(y)(\xi) = f(x)(\xi)\}$. For $V \in \mathcal{V}$, set $I_V = \{\xi : \xi < \kappa,\ V \subseteq G_\xi\}$; then $f[V] \subseteq \{z : z \in \{0,1\}^\kappa,\ z\restriction I_V = f(x)\restriction I_V\}$; but $f[X \setminus V]$ is a closed proper subset of $\{0,1\}^\kappa$, so $\mathrm{int}\, f[V]$ is non-empty and I_V is finite. As $\mathcal{V}$ is a base of neighbourhoods of x, $\kappa = \bigcup_{V\in\mathcal{V}} I_V$. As κ is infinite, $\mathcal{V}$ is infinite, and $\kappa \le \#(\mathcal{V}) = \chi(x,X)$. **Q**

(iii) So if there is a continuous surjection from a closed subset of X onto $\{0,1\}^\kappa$, there is a non-empty closed $K \subseteq X$ such that $\chi(x,K) \ge \kappa$ for every $x \in K$. **P** Let $f : F \to \{0,1\}^\kappa$ be a continuous surjection, where $F \subseteq X$ is closed. By 4A2G(i-i), there is a closed $K \subseteq F$ such that $f{\restriction}K$ is an irreducible surjection onto $\{0,1\}^\kappa$, and we can use (ii). **Q**

(iv) If Y and Z are Hausdorff spaces and $f : X \to Y$, $g : Y \to Z$ are continuous irreducible surjections then $gf : X \to Z$ is irreducible. (If $F \subseteq X$ is a closed proper subset, then $f[F]$ is a closed proper subset of Y and $g[f[F]] \ne Z$.)

(e) If $\langle x_n\rangle_{n\in\mathbb{N}}$ is a sequence in X with at most one cluster point in X, then $\langle x_n\rangle_{n\in\mathbb{N}}$ is convergent. **P** Because X is compact, $\langle x_n\rangle_{n\in\mathbb{N}}$ has at least one cluster point; let x be such a point. **?** If $\langle x_n\rangle_{n\in\mathbb{N}}$ does not converge to x, let G be an open set containing x such that $J = \{n : n \in \mathbb{N},\, x_n \notin G\}$ is infinite. Then there must be a point y in $\bigcap_{n\in\mathbb{N}} \overline{\{x_i : i \in J \setminus n\}}$; and now y is a cluster point of $\langle x_n\rangle_{n\in\mathbb{N}}$ in $X \setminus G$, so cannot be equal to x. **XQ**

(f) Let Y be a Hausdorff space and $f : X \to Y$ a continuous function. If $\mathcal{E}$ is a non-empty downwards-directed family of closed subsets of X, then $f[\bigcap\mathcal{E}] = \bigcap_{F\in\mathcal{E}} f[F]$. **P** Of course $f[\bigcap\mathcal{E}] \subseteq \bigcap_{F\in\mathcal{E}} f[F]$. If $y \in \bigcap_{F\in\mathcal{E}} f[F]$, then $\{F \cap f^{-1}[\{y\}] : F \in \mathcal{E}\}$ is a downwards-directed family of closed subsets of X, so has non-empty intersection; and any point of the intersection witnesses that $y \in f[\bigcap\mathcal{E}]$. **Q**

5A4D Vietoris topologies: Proposition Let X be a separable metrizable space and $\mathcal{K}$ the set of compact subsets of X with the topology induced by the Vietoris topology on the set of closed subsets of X (4A2T).

(a) $\mathcal{K}$ is second-countable.

(b) If Y is a topological space and $R \subseteq Y \times X$ is usco-compact, then $y \mapsto R[\{y\}] : Y \to \mathcal{K}$ is Borel measurable.

(c) There is a sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ of Borel measurable functions from $\mathcal{K}\setminus\{\emptyset\}$ to X such that $\{f_n(K) : n \in \mathbb{N}\}$ is a dense subset of K for every $K \in \mathcal{K} \setminus \{\emptyset\}$.

proof (a) Let $\mathcal{U}$ be a countable base for the topology of X. Let $\mathcal{V}$ be the family of sets of the form

$$\{K : K \in \mathcal{K},\, K \cap U_i \ne \emptyset \text{ for } i < n,\, K \subseteq \textstyle\bigcup_{i<n} U_i\}$$

where $n \in \mathbb{N}$ and $U_i \in \mathcal{U}$ for $i < n$; then $\mathcal{V}$ is a countable family of open sets in $\mathcal{K}$ and is a base for the topology of $\mathcal{K}$.

(b) If $G \subseteq X$ is open, then

$$\{y : y \in Y,\, R[\{y\}] \subseteq G\} = Y \setminus R^{-1}[X \setminus G]$$

is open. Also G can be expressed as $\bigcup_{n\in\mathbb{N}} F_n$ where every F_n is closed, so

$$\{y : R[\{y\}] \cap G \ne \emptyset\} = \textstyle\bigcup_{n\in\mathbb{N}} R^{-1}[F_n]$$

is F_σ, therefore Borel. Thus

$$\mathcal{W} = \{W : W \subseteq \mathcal{K},\, \{y : R[\{y\}] \in W\} \text{ is Borel}\}$$

includes a subbase for the topology of $\mathcal{K}$. It therefore includes a base; because $\mathcal{K}$ is second-countable, therefore hereditarily Lindelöf, every open set is a countable union of members of $\mathcal{W}$ and belongs to $\mathcal{W}$, that is, $y \mapsto R[\{y\}]$ is Borel measurable.

(c)(i) Note first that if $G \subseteq X$ is open, then $K \mapsto \overline{K \cap G} : \mathcal{K} \to \mathcal{K}$ is Borel measurable. **P** If $H \subseteq X$ is open, then

$$\{K : \overline{K \cap G} \cap H \ne \emptyset\} = \{K : K \cap (G \cap H) \ne \emptyset\}$$

is open. Next, we can express H as the union $\bigcup_{n\in\mathbb{N}} H_n$ of a non-decreasing sequence of open sets such that $\overline{H}_n \subseteq H$ for every n, so

$$\{K : \overline{K \cap G} \subseteq H\} = \textstyle\bigcup_{n\in\mathbb{N}}\{K : K \cap G \subseteq \overline{H}_n\} = \bigcup_{n\in\mathbb{N}}\{K : K \cap (G \setminus \overline{H}_n) = \emptyset\}$$

is F_σ, therefore Borel. As in (b), this is enough. **Q**

(ii) Let $\langle U_n\rangle_{n\in\mathbb{N}}$ run over a base for the topology of X. For each $n \in \mathbb{N}$ define $g_n : \mathcal{K} \to \mathcal{K}$ by setting

$$g_n(K) = \overline{K \cap U_n} \text{ if } K \cap U_n \neq \emptyset,$$
$$= K \text{ otherwise.}$$

Since $\{K : K \cap U_n \neq \emptyset\}$ is open, (i) tells us that g_n is Borel measurable. Set $h_n = g_n \ldots g_1 g_0$; then h_n also is Borel measurable, for each n. Now, for each $K \in \mathcal{K} \setminus \{\emptyset\}$, $\langle h_n(K)\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of non-empty compact sets, so has non-empty intersection. Morover, for each n, $h_n(K)$ is either disjoint from U_n or included in $\overline{U}_n$; so $\bigcap_{n\in\mathbb{N}} h_n(K)$ has exactly one point; call this point $f(K)$. Of course $f(K) \in h_0(K) \subseteq K$.

Now $f : \mathcal{K} \setminus \{\emptyset\} \to X$ is Borel measurable. **P** If $F \subseteq X$ is closed, then

$$f^{-1}[F] = \bigcap_{n\in\mathbb{N}}\{K : F \cap h_n(K) \neq \emptyset\}$$

is a Borel set because every h_n is Borel measurable and $\{K : F \cap K = \emptyset\}$ is open. **Q**

(iii) Set $f_n = fg_n$ for each n. Then $f_n(K) \in K$ for every $n \in \mathbb{N}$ and $K \in \mathcal{K} \setminus \{\emptyset\}$, $f_n : \mathcal{K} \setminus \{\emptyset\} \to X$ is Borel measurable for each n, and $f_n(K) \in \overline{K \cap U_n}$ whenever $K \cap U_n \neq \emptyset$; so $\{f_n(K) : n \in \mathbb{N}\}$ is dense in K for every $K \in \mathcal{K} \setminus \{\emptyset\}$.

5A4E Category and the Baire property Let X be a topological space; write $\widehat{\mathcal{B}}(X)$ for its Baire-property algebra (4A3R[6]).

(a) Suppose that $\langle G_i\rangle_{i\in I}$ is a disjoint family of open sets and $\langle E_i\rangle_{i\in I}$ is a family of nowhere dense sets. Then $\bigcup_{i\in I} G_i \cap E_i$ is nowhere dense. (Elementary; see (a-i) of the proof of 4A3S[7].)

(b) Let Y be another topological space.

(i) If $A \subseteq X$ is nowhere dense in X, then $A \times Y$ is nowhere dense in $X \times Y$. ($\overline{A \times Y} = \overline{A} \times Y$.) So if $A \subseteq X$ is meager in X, then $A \times Y$ is meager in $X \times Y$.

(ii) $\widehat{\mathcal{B}}(X)\widehat{\otimes}\widehat{\mathcal{B}}(Y) \subseteq \widehat{\mathcal{B}}(X \times Y)$. **P** If $E \in \widehat{\mathcal{B}}(X)$, let $G \subseteq X$ be such that $E \triangle G$ is meager; then

$$E \times Y = (G \times Y)\triangle((E\triangle G) \times Y) \in \widehat{\mathcal{B}}(X \times Y).$$

Similarly, $X \times F \in \widehat{\mathcal{B}}(X \times Y)$ for every $F \in \widehat{\mathcal{B}}(Y)$. Because $\widehat{\mathcal{B}}(X \times Y)$ is a σ-algebra of sets, it includes $\widehat{\mathcal{B}}(X)\widehat{\otimes}\widehat{\mathcal{B}}(Y)$. **Q**

(iii) If Y is compact, Hausdorff and not empty, then a set $A \subseteq X$ is meager in X iff $A \times Y$ is meager in $X \times Y$. **P** We saw in (i) that if A is meager then $A \times Y$ is meager. In the other direction, if $A \times Y$ is meager in $X \times Y$, let $\langle W_n\rangle_{n\in\mathbb{N}}$ be a sequence of dense open subsets of $X \times Y$ such that $\bigcap_{n\in\mathbb{N}} W_n$ is disjoint from $A \times Y$. Choose $\langle V_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. $V_0 = X \times Y$. Given that V_n is an open subset of $X \times Y$ such that $\pi_1[V_n]$ is dense in X, where π_1 is the projection from $X \times Y$ onto X, then $\pi_1[V_n \cap W_n]$ is dense in X. Set

$$\mathcal{V}_n = \{G \times H : G \subseteq X \text{ is open}, H \subseteq Y \text{ is open}, G \times \overline{H} \subseteq V_n \cap W_n\}.$$

Observe that if $V \in \mathcal{V}_n$ and $x \in \pi_1[V]$ then $\pi_1[V] \times \overline{V[\{x\}]} \subseteq W_n$. Because Y is regular, $\bigcup \mathcal{V}_n$ is dense in $V_n \cap W_n$ and $\pi_1[\bigcup \mathcal{V}_n]$ is dense in X. Let $\mathcal{V}'_n \subseteq \mathcal{V}_n$ be a maximal family such that $\pi_1[V] \cap \pi_1[V']$ is empty whenever V, $V' \in \mathcal{V}'_n$ are disjoint; because $G' \times H \in \mathcal{V}_n$ whenever $G \times H \in \mathcal{V}_n$ and G' is an open subset of G, $\pi_1[\bigcup \mathcal{V}'_n]$ is dense in X. Set $V_{n+1} = \bigcup \mathcal{V}'_n$, and continue.

If $x \in \bigcap_{n\in\mathbb{N}} \pi_1[V_n]$, $\langle\overline{V_n[\{x\}]}\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of non-empty closed subsets of Y, so there is a $y \in \bigcap_{n\in\mathbb{N}} \overline{V_n[\{x\}]}$, because Y is compact. For each n, $x \in \pi_1[V_{n+1}]$ and there is a $V \in \mathcal{V}'_n$ such that $x \in \pi_1[V]$, so $V_{n+1}[\{x\}] = V[\{x\}]$, $y \in \overline{V[\{x\}]}$ and $(x, y) \in W_n$. Thus $x \in \pi_1[\bigcap_{n\in\mathbb{N}} W_n]$ and $x \notin A$. As x is arbitrary, A is disjoint from $\bigcap_{n\in\mathbb{N}} \pi_1[V_n]$ and is meager. **Q**

(c) Suppose that X is completely regular and ccc.

[6] Formerly 4A3Q.

[7] Later editions only.

(i) Every nowhere dense subset of X is included in a nowhere dense zero set. **P** If $E \subseteq X$ is nowhere dense, let $\mathcal{G}$ be a maximal disjoint family of cozero sets included in $X \setminus E$. Because X is ccc, $\mathcal{G}$ is countable, and $G = \bigcup \mathcal{G}$ is a cozero set. Because X is completely regular, $X \setminus (G \cup E)$ is nowhere dense and $X \setminus G$ is a nowhere dense zero set including E. **Q**

(ii) Every meager subset of X is included in a meager Baire set. (By (i), it is included in the union of a sequence of nowhere dense zero sets.)

(iii) Every subset of X with the Baire property is expressible as $G \triangle M$ where G is a cozero set and M is meager. **P** If $E \in \widehat{\mathcal{B}}(X)$, express it as $G_0 \triangle M_0$ where G_0 is open and M_0 is meager. Let $\mathcal{H}$ be a maximal disjoint family of cozero subsets of G_0; as X is ccc, $\mathcal{H}$ is countable and $G = \bigcup \mathcal{H}$ is a cozero set; as X is completely regular, $G_0 \subseteq \overline{G}$ and $G_0 \setminus G$ is nowhere dense. So $M = (G_0 \setminus G) \triangle M_0$ is meager and $E = G \triangle M$ is in the required form. **Q**

5A4F Normal and paracompact spaces (a) For a normal space X and an infinite set I, the following are equiveridical: (i) there is a continuous surjection from X onto $[0,1]^I$; (ii) there is a continuous surjection from a closed subset of X onto $\{0,1\}^I$. **P** (i)$\Rightarrow$(ii) is elementary, as $\{0,1\}^I$ is a closed subset of $[0,1]^I$. So suppose that (ii) is true. The map $x \mapsto \sum_{n=0}^{\infty} 2^{-n-1}$ is a continuous surjection from $\{0,1\}^{\mathbb{N}}$ onto $[0,1]$; there is therefore a continuous surjection from $\{0,1\}^{I \times \mathbb{N}}$ onto $[0,1]^I$; but I is infinite, so $\{0,1\}^I$ is homeomorphic to $\{0,1\}^{I \times \mathbb{N}}$. We therefore have a continuous surjection from $\{0,1\}^I$ onto $[0,1]^I$. Accordingly there is a continuous surjection f from a closed subset F of X onto $[0,1]^I$. Set $f_i(x) = f(x)(i)$ for $x \in F$ and $i \in I$; by Tietze's theorem (4A2F(d-ix)), there is a continuous $g_i : X \to [0,1]$ extending f_i; now $x \mapsto \langle g_i(x) \rangle_{i \in I} : X \to [0,1]^I$ is a continuous surjection, and (i) is true. **Q**

(b) Suppose that X is a paracompact normal space and $\mathcal{G}$ is an open cover of X. Then there is a continuous pseudometric $\rho : X \times X \to [0,\infty[$ such that whenever $\emptyset \ne A \subseteq X$ and $\sup_{x,y \in A} \rho(x,y) \le 1$ there is a $G \in \mathcal{G}$ such that $A \subseteq G$. **P** If X is empty this is trivial; suppose otherwise. There is a locally finite open cover $\mathcal{H}$ of X refining $\mathcal{G}$. Enumerate $\mathcal{H}$ as $\langle H_\xi \rangle_{\xi<\kappa}$. By 4A2F(d-vi), there is a family $\langle U_\xi \rangle_{\xi<\kappa}$ of open sets, covering X, such that $\overline{U}_\xi \subseteq H_\xi$ for every $\xi < \kappa$.

Now for each $\xi < \kappa$ there is a continuous function $f_\xi : X \to [0,1]$ such that $f_\xi(x) = 1$ for $x \in \overline{U}_\xi$ and $f_\xi(x) = 0$ for $x \in X \setminus H_\xi$ (4A2F(d-i)). Set $\rho(x,y) = 2\sum_{\xi<\kappa} |f_\xi(x) - f_\xi(y)|$ for x, $y \in X$. Then ρ is a pseudometric on X, and is continuous because $\langle f_\xi^{-1}[\,]0,1]\,] \rangle_{\xi<\kappa}$ is locally finite. If $A \subseteq X$ is a non-empty set such that $\rho(x,y) \le 1$ for all x, $y \in A$, take any $x \in A$. There is a $\xi < \kappa$ such that $x \in U_\xi$. If $y \in A$, $1 - f_\xi(y) = |f_\xi(x) - f_\xi(y)| \le \frac{1}{2}$ so $y \in H_\xi$. Let $G \in \mathcal{G}$ be such that $H_\xi \subseteq G$; then $A \subseteq G$, as required. **Q**

5A4G Baire σ-algebras (a) Let X be a topological space. Write $\mathcal{B}\mathfrak{a}_0(X)$ for the set of cozero sets in X and for ordinals $\alpha > 0$ set

$$\mathcal{B}\mathfrak{a}_\alpha(X) = \{\bigcup_{n\in\mathbb{N}}(X \setminus E_n) : \langle E_n \rangle_{n\in\mathbb{N}} \text{ is a sequence in } \bigcup_{\beta<\alpha} \mathcal{B}\mathfrak{a}_\beta(X)\}.$$

Then the Baire σ-algebra $\mathcal{B}\mathfrak{a}(X)$ of X is $\bigcup_{\alpha<\omega_1} \mathcal{B}\mathfrak{a}_\alpha(X)$. **P** Inducing on α, we see that $\mathcal{B}\mathfrak{a}_\alpha(X)$ is included in the Baire σ-algebra of X for every α; and $\bigcup_{\alpha<\omega_1} \mathcal{B}\mathfrak{a}_\alpha(X)$ is a σ-algebra of sets containing every cozero set, so includes the Baire σ-algebra. **Q**

(b)(i) If $\langle X_i \rangle_{i\in I}$ is a family of separable metrizable spaces with product X, then $\#(\mathcal{B}\mathfrak{a}(X)) \le \max(\mathfrak{c}, \#(I)^\omega)$. **P** By 4A3Na, $\mathcal{B}\mathfrak{a}(X) = \widehat{\bigotimes}_{i\in I}\mathcal{B}(X_i)$, where $\mathcal{B}(X_i)$ is the Borel σ-algebra of X_i for each i. By 4A3Fa, $\#(\mathcal{B}(X_i)) \le \mathfrak{c}$ for each i, so

$$\mathcal{E} = \{\{x : x \in X,\, x(i) \in E\} : i \in E,\, E \in \mathcal{B}(X_i)\}$$

has cardinal at most $\max(\mathfrak{c}, \#(I))$ and the σ-algebra $\mathcal{B}\mathfrak{a}(X)$ it generates has cardinal at most $\max(\mathfrak{c}, \#(I))^\omega = \max(\mathfrak{c}, \#(I)^\omega)$ (4A1O). **Q**

(ii) If $\kappa \ge 2$ is a cardinal, then the set F of Baire measurable functions from $\{0,1\}^\kappa$ to $\{0,1\}^\omega$ has cardinal κ^ω. **P** The map

$$f \mapsto \langle \{x : f(x)(i) = 1\} \rangle_{i\in\mathbb{N}} : F \to \mathcal{B}\mathfrak{a}(\{0,1\}^\kappa)^{\mathbb{N}}$$

is bijective, so

$$\#(F) = \#(\mathcal{B}\mathfrak{a}(\{0,1\}^\kappa))^\omega \le (\max(\mathfrak{c},\kappa)^\omega)^\omega = \kappa^\omega.$$

Of course $\#(\mathcal{B}\mathfrak{a}(\{0,1\})^\kappa) \ge \kappa$ so $\#(F) = \kappa^\omega$. **Q**

5A4H Proposition If X is a compact metrizable space and (Y,ρ) a complete separable metric space, then $C(X;Y)$, with the topology of uniform convergence, is Polish. (Engelking 89, 4.3.13 and 4.2.18.)

5A4I Compact-open topologies We shall need a couple of elementary facts concerning some spaces of continuous functions.

(a) Let X and Y be topological spaces and F a set of functions from X to Y. The **compact-open** topology on F is the topology generated by sets of the form $\{f : f \in F,\ f[K] \subseteq H\}$ where $K \subseteq X$ is compact and $H \subseteq Y$ is open. (Cf. Engelking 89, §3.4.)

(b) Let X be a topological space and $\langle Y_i\rangle_{i\in I}$ a family of regular spaces, with product Y. Set $\pi_i y = y(i)$ for $i \in I$ and $y \in Y$. Then $g \mapsto \langle \pi_i g\rangle_{i\in I} : C(X;Y) \to \prod_{i\in I} C(X;Y_i)$ is a homeomorphism for the compact open topologies on $C(X;Y)$ and the $C(X;Y_i)$. **P** (α) A function $g : X \to Y$ is continuous iff $\pi_i g$ is continuous for every $i \in I$ (3A3Ib), so $g \mapsto \langle \pi_i g\rangle_{i\in I} : C(X;Y) \to \prod_{i\in I} C(X;Y_i)$ is a bijection; write $\phi : \prod_{i\in I} C(X;Y_i) \to C(X;Y)$ for its inverse. (β) If $j \in I$, $K \subseteq X$ is compact and $H \subseteq Y_j$ is open, then $\{g : (\pi_j g)[K] \subseteq H\} = \{g : g[K] \subseteq \pi_j^{-1}[H]\}$ is open in $C(X;Y)$. By 4A2B(a-ii), $g \mapsto \pi_j g : C(X;Y) \to C(X;Y_j)$ is continuous. As j is arbitrary, $g \mapsto \langle \pi_i g\rangle_{i\in I}$ is continuous. (γ) Now suppose that $K \subseteq X$ is compact and $W \subseteq Y$ is open, and consider $U = \{\boldsymbol{g} : \boldsymbol{g} \in \prod_{i\in I} C(X;Y_i),\ \phi(\boldsymbol{g})[K] \subseteq W\}$. Take any $\boldsymbol{h} = \langle h_i\rangle_{i\in I}$ in U. Set

$$\begin{aligned}\mathcal{H} = \{\prod_{i\in I} H_i : H_i \subseteq Y_i \text{ is open for every } i \in I,\ \{i : H_i \ne Y_i\} \text{ is finite},\\ \text{there is a family } \langle G_i\rangle_{i\in I} \text{ such that } G_i \text{ is an open subset of } Y_i\\ \text{including } \overline{H}_i \text{ for every } i \in I \text{ and } \prod_{i\in I} G_i \subseteq W\}.\end{aligned}$$

Then $\mathcal{H}$ is a family of open subsets of Y and because every Y_i is regular we have $W = \bigcup\mathcal{H}$. Now $\phi(\boldsymbol{h})[K]$ is a compact subset of W so there are $H^{(0)},\dots,H^{(n)} \in \mathcal{H}$ covering $\phi(\boldsymbol{h})[K]$. Set $K_k = K \cap \phi(\boldsymbol{h})^{-1}[\overline{H^{(k)}}]$ for $k \le n$; then $K_0,\dots,K_n$ are compact subsets of X with union K.

For each $k \in K$ take $\langle H_{ki}\rangle_{i\in I}$, $\langle G_{ki}\rangle_{i\in I}$ such that

$$H^{(k)} = \textstyle\prod_{i\in I} H_{ki},$$

$$H_{ki},\ G_{ki} \text{ are open subsets of } Y_i \text{ and } \overline{H}_{ki} \subseteq G_{ki} \text{ for each } i \in I,$$

$$J_k = \{i : H_{ki} \ne Y_i\} \text{ is finite},\qquad \textstyle\prod_{i\in I} G_{ki} \subseteq W.$$

We have

$$\phi(\boldsymbol{h})[K_k] \subseteq \overline{H^{(k)}} \subseteq \textstyle\prod_{i\in I} G_{ki},$$

so $\pi_i[\phi(\boldsymbol{h})[K_k]] \subseteq G_{ki}$ for every $i \in I$. Consider the set

$$V = \{\langle g_i\rangle_{i\in I} : \langle g_i\rangle_{i\in I} \in \textstyle\prod_{i\in I} C(X;Y_i),\ g_i[K_k] \subseteq G_{ki} \text{ whenever } k \le n \text{ and } i \in J_k\}.$$

Then V is an open subset of $\prod_{i\in I} C(X;Y_i)$ containing $\boldsymbol{h}$. If $\boldsymbol{g} = \langle g_i\rangle_{i\in I}$ belongs to V and $k \le n$, then

$$\phi(\boldsymbol{g})[K_k] \subseteq \textstyle\prod_{i\in I} g_i[K_k] \subseteq \prod_{i\in I} G_{ki} \subseteq W$$

because $G_{ki} = Y_i$ for $i \in I \setminus J_k$. So $\phi(\boldsymbol{g})[K] = \bigcup_{k\le n} \phi(\boldsymbol{g})[K_k]$ is included in W and $\boldsymbol{g} \in U$. Thus we have $\boldsymbol{h} \in V \subseteq U$ and U is a neighbourhood of $\boldsymbol{h}$. As $\boldsymbol{h}$ is arbitrary, U is open. As K and W are arbitrary, ϕ is continuous and $g \mapsto \langle \pi_i g\rangle_{i\in I}$ is a homeomorphism. **Q**

(c) Let X be a compact space, and write $\mathcal{E}$ for the algebra of open-and-closed subsets of X. Then $f \mapsto f^{-1}[\{1\}]$ is a bijection between $C(X;\{0,1\})$ and $\mathcal{E}$, and the compact-open topology on $C(X;\{0,1\})$ is discrete. **P** (α) $f \mapsto f^{-1}[\{1\}] : \{0,1\}^X \to \mathcal{P}X$ is a bijection, and a function $f : X \to \{0,1\}$ is continuous iff $f^{-1}[\{0\}]$ and $f^{-1}[\{1\}]$ are open, that is, iff $f^{-1}[\{1\}] \in \mathcal{E}$. So we have a bijection $f \mapsto f^{-1}[\{1\}] : C(X;\{0,1\}) \to \mathcal{E}$. (β) If $g \in C(X;\{0,1\})$ then $K_0 = g^{-1}[\{0\}]$ and $K_1 = g^{-1}[\{1\}]$ are closed, therefore compact, and

$$\{g\} = \{f : f \in C(X;\{0,1\}),\ f[K_0] \subseteq \{0\},\ f[K_1] \subseteq \{1\}\}$$

is open; accordingly the compact-open topology on $C(X;\{0,1\})$ is discrete. **Q**

5A4J In §547 we shall need a bound on the uniformities of certain meager ideals.

Proposition Let X be a set and $\mathcal{A}$ a family of countable sets which is stationary over X (definition: 5A1R). Then $\operatorname{non}\mathcal{M}(X^{\mathbb{N}}) \le \max(\#(\mathcal{A}), \operatorname{non}\mathcal{M})$.

Notation Here X is given its discrete topology and $X^{\mathbb{N}}$ the associated product topology. $\mathcal{M}(X^{\mathbb{N}})$ is its meager ideal and $\operatorname{non}\mathcal{M}(X^{\mathbb{N}})$ the corresponding uniformity, the smallest cardinal of any non-meager subset of $X^{\mathbb{N}}$. $\mathcal{M}$ is $\mathcal{M}(\mathbb{R})$.

proof (a) We can suppose that $\emptyset \ne A \subseteq X$ for every $A \in \mathcal{A}$. For each $A \in \mathcal{A}$, let $F_A \subseteq A^{\mathbb{N}}$ be a non-meager set with cardinal at most $\operatorname{non}\mathcal{M}$. (If A is a singleton, take $F_A = A^{\mathbb{N}}$; otherwise, $\operatorname{non}\mathcal{M}(A^{\mathbb{N}}) = \operatorname{non}\mathcal{M}(\mathbb{R}) = \operatorname{non}\mathcal{M}$ by 522Wb.) Set $F = \bigcup_{A\in\mathcal{A}} F_A$; then $\#(F) \le \max(\#(\mathcal{A}), \operatorname{non}\mathcal{M})$.

(b) F is non-meager in $X^{\mathbb{N}}$. **P** Let $\langle G_n\rangle_{n\in\mathbb{N}}$ be a sequence of dense open subsets of $X^{\mathbb{N}}$. For each $z \in \bigcup_{k\in\mathbb{N}} X^k$ and $n \in \mathbb{N}$ choose $w_{zn} \in \bigcup_{k\in\mathbb{N}} X^k$ such that $z \subseteq w_{zn}$ and $\{x : w_{zn} \subseteq x \in X^{\mathbb{N}}\} \subseteq G_n$. For each $I \in [X]^{<\omega}$ set $f(I) = \bigcup\{w_{zn}[\mathbb{N}] : z \in \bigcup_{k\in\mathbb{N}} I^k,\ n \in \mathbb{N}\}$. Let $A \in \mathcal{A}$ be such that $f(I) \subseteq A$ for every finite $I \subseteq A$.

Take any $n \in \mathbb{N}$. If $z \in \bigcup_{k\in\mathbb{N}} A^k$, set $I = z[\operatorname{dom} z] \in [A]^{<\omega}$; then $w_{zn} \in \bigcup_{k\in\mathbb{N}} f(I)^k \subseteq \bigcup_{k\in\mathbb{N}} A^k$ and $x \in G_n$ whenever $w_{zn} \subseteq x \in A^{\mathbb{N}}$. So $G_n \cap A^{\mathbb{N}}$ is dense in $A^{\mathbb{N}}$. Also, of course, $G_n \cap A^{\mathbb{N}}$ is open in $A^{\mathbb{N}}$. As this is true for every $n \in \mathbb{N}$,

$$\emptyset \ne F_A \cap \textstyle\bigcap_{n\in\mathbb{N}} G_n \subseteq F \cap \bigcap_{n\in\mathbb{N}} G_n.$$

As $\langle G_n\rangle_{n\in\mathbb{N}}$ is arbitrary, F is non-meager. **Q**

So F witnesses that $\operatorname{non}\mathcal{M}(X^{\mathbb{N}}) \le \max(\#(\mathcal{A}), \operatorname{non}\mathcal{M})$.

5A4K Irreducible surjections: Lemma (a) Let Q be a topological space and K, L closed subsets of Q such that $K \subseteq \overline{Q \setminus L}$, $L \subseteq \overline{Q \setminus K}$ and $K \cup L = Q$. Set $Z = \{(x,1) : x \in K\} \cup \{(x,0) : x \in L\} \subseteq Q \times \{0,1\}$, and write $\phi : Z \to Q$ for the first-coordinate map. Then ϕ is an irreducible continuous surjection.

(b) Let θ be an ordinal, $\langle Q_\xi\rangle_{\xi<\theta}$ a family of compact Hausdorff spaces, and $\langle\phi_{\eta\xi}\rangle_{\eta\le\xi<\theta}$ a family such that $\phi_{\eta\xi} : Q_\xi \to Q_\eta$ is a continuous surjection whenever $\eta \le \xi < \theta$. Suppose that

$\phi_{\zeta\xi} = \phi_{\zeta\eta}\phi_{\eta\xi}$ whenever $\zeta \le \eta \le \xi < \theta$,

the topology of Q_ξ is generated by $\{\phi_{\eta\xi}^{-1}[U] : \eta < \xi,\ U \subseteq Q_\eta$ is open$\}$ for every non-zero limit ordinal $\xi < \theta$,

$\phi_{\xi,\xi+1}$ is irreducible whenever $\xi + 1 < \theta$. (*)

Then $\phi_{\eta\xi}$ is irreducible whenever $\eta \le \xi < \theta$.

proof (a) ϕ is a surjection because $K \cup L = Q$, and is continuous by the definition of the product topology on $Q \times \{0,1\}$. Suppose that $W \subseteq Z$ is open and not empty.

If $x \in K$ and $(x,1) \in W$, then there is an open set $U \subseteq Q$ such that $(x,1) \in U \times \{1\} \subseteq W$. In this case, $x \in U \cap K \subseteq \overline{Q \setminus L}$ and $U \setminus L$ is not empty. But $\phi[Z \setminus W]$ does not meet $U \setminus L$. **P?** Otherwise, take $z \in Z \setminus W$ such that $\phi(z) \in U \setminus L$. Then z must be equal to $(\phi(z), 1)$ and $z \in U \times \{1\} \subseteq W$, which is absurd. **XQ** So $\phi[Z \setminus W] \ne Q$.

Similarly, $\phi[Z \setminus W] \ne Q$ if $W \cap \{(y,0) : y \in L\} \ne \emptyset$. But as $K \cup L = Q$, this means that $\phi[Z \setminus W]$ can never be Q.

Thus $\phi[F] \ne Q$ for any closed proper subset F of Z, and $\phi : Z \to Q$ is irreducible.

(b) Fixing $\eta < \theta$, induce on ξ. As $\phi_{\eta\eta} = \phi_{\eta\eta}^2 : Q_\eta \to Q_\eta$ is surjective, $\phi_{\eta\eta}$ is the identity map on Q_η and the case $\xi = \eta$ is trivial. For the inductive step to a successor ordinal $\xi + 1$, $\phi_{\eta,\xi+1} = \phi_{\eta,\xi}\phi_{\xi,\xi+1}$ is the composition of two irreducible continuous surjections, by the inductive hypothesis and (*), so is irreducible by 5A4C(d-iv). For the inductive step to a limit ordinal $\xi \in \,]\eta,\theta[$, take a non-empty open set $G \subseteq Q_\xi$; then there must be $\eta_0,\ldots,\eta_n\in[\eta,\xi[$ and open sets $G_i\subseteq Q_{\eta_i}$, for $i\le n$, such that $\emptyset\ne\bigcap_{i\le n}\phi^{-1}_{\eta_i\xi}[G_i]\subseteq G$. We can take it that $\eta_0\le\ldots\le\eta_n$. Setting $H=\bigcap_{i\le n}\phi^{-1}_{\eta_i\eta_n}[G_i]\subseteq Q_{\eta_n}$,

$$\phi^{-1}_{\eta_n\xi}[H]=\bigcap_{i\le n}\phi^{-1}_{\eta_n\xi}[\phi^{-1}_{\eta_i\eta_n}[G_i]]=\bigcap_{i\le n}\phi^{-1}_{\eta_i\xi}[G_i]$$

is a non-empty open subset of G. So

$$\phi_{\eta\xi}[Q_\xi\setminus G]=\phi_{\eta\eta_n}[\phi_{\eta_n\xi}[Q_\xi\setminus G]]\subseteq\phi_{\eta\eta_n}[Q_{\eta_n}\setminus H]\ne Q_\eta$$

because $\phi_{\eta\eta_n}$ is irreducible. As G is arbitrary, $\phi_{\eta\xi}$ is irreducible and the induction continues.

5A4L Old friends (a) The weight of the Stone-Čech compactification $\beta\mathbb{N}$ is $\mathfrak{c}$. (Engelking 89, 3.6.11.)

(b)(i) For any infinite I, there is a continuous surjection from $\{0,1\}^I$ onto $[0,1]^I$. (Immediate from 5A4Fa, or otherwise.)

(ii) There is a continuous surjection from $[0,1]$ onto $[0,1]^{\mathbb{N}}$. (The Cantor set $C\subseteq[0,1]$ is homeomorphic to $\{0,1\}^{\mathbb{N}}$ (4A2Uc), so again 5A4Fa gives the result. Or see 416Yi.)

(iii) For any infinite κ, $t(\{0,1\}^\kappa)=\kappa$. **P** $t(\{0,1\}^\kappa)\le w(\{0,1\}^\kappa)\le\kappa$ (5A4Ba, 4A2D(a-ii))). In the other directon, set $A=\{\chi I:I\in[\kappa]^{<\omega}\}$; then $\chi\kappa\in\overline{A}\setminus\overline{B}$ for every $B\in[A]^{<\kappa}$, so $\kappa\le t(\{0,1\}^\kappa)$. **Q**

(c) If X is a non-empty zero-dimensional compact metrizable space without isolated points, it is homeomorphic to $\{0,1\}^{\mathbb{N}}$. **P** Let $\mathfrak{B}$ be the algebra of open-and-closed subsets of X. By 311J, X is homeomorphic to the Stone space of $\mathfrak{B}$. Because X has no isolated points, $\mathfrak{B}$ is atomless (316Lb). We know that X is second-countable (4A2P(a-ii)); let $\mathcal{U}$ be a countable base for its topology; then every member of $\mathfrak{B}$ is open, so expressible as a union of members of $\mathcal{U}$, and compact, so expressible as the union of a finite subset of $\mathcal{U}$. Accordingly $\mathfrak{B}$ is countable; and as $X\ne\emptyset$, $\mathfrak{B}\ne\{0\}$. By 316M, $\mathfrak{B}$ is isomorphic to the algebra of open-and-closed subsets of $\{0,1\}^{\mathbb{N}}$; by 311J again, X and $\{0,1\}^{\mathbb{N}}$ are homeomorphic. **Q**

(d) Let X be a non-empty zero-dimensional Polish space in which no non-empty open set is compact. Then X is homeomorphic to $\mathbb{N}^{\mathbb{N}}$ with its usual topology. **P** Let ρ be a complete metric on X defining its topology. (i) If $U\subseteq X$ is a non-empty open set and $\epsilon>0$, there is a partition $\langle U_n\rangle_{n\in\mathbb{N}}$ of U into non-empty open-and-closed sets of diameter at most ϵ. To see this, note that as U is not compact, there is a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in U with no cluster point in U (4A2Le). Let $\mathcal{V}$ be the family of subsets of U, of diameter at most ϵ, which are open-and-closed in X and contain x_i for at most finitely many i. Because X is zero-dimensional, $\mathcal{V}$ is a base for the subspace topology of U. Because U is Lindelöf (4A2P(a-iii)), there is a sequence $\langle V_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{V}$ covering U; set $V'_n=V_n\setminus\bigcup_{i<n}V_i$ for each n, so that $\langle V'_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathcal{V}$ covering U. Because no V'_n can contain infinitely many of the x_i, $I=\{n:V'_n\ne\emptyset\}$ is infinite, and we can re-enumerate $\langle V'_n\rangle_{n\in I}$ as $\langle U_n\rangle_{n\in\mathbb{N}}$ to get an appropriate sequence. (ii) Now set $S=\bigcup_{n\in\mathbb{N}}\mathbb{N}^n$ and define $\langle U_\sigma\rangle_{\sigma\in S}$ inductively in such a way that $U_\emptyset=X$ and

> $\langle U_{\sigma^\frown<n>}\rangle_{n\in\mathbb{N}}$ is a partition of U_σ into non-empty open-and-closed sets of diameter at most 2^{-k} whenever $k\in\mathbb{N}$ and $\sigma\in\mathbb{N}^k$.

For $\alpha\in\mathbb{N}^{\mathbb{N}}$, $\langle U_{\alpha\restriction k}\rangle_{k\in\mathbb{N}}$ is a non-increasing sequence of non-empty closed sets and $\operatorname{diam}U_{\alpha\restriction k}\le 2^{-k+1}$ for every $k\ge 1$, so there is exactly one point in $\bigcap_{k\in\mathbb{N}}U_{\alpha\restriction k}$; let $f(\alpha)$ be this point. This defines a function $f:\mathbb{N}^{\mathbb{N}}\to X$. (iii) Because $\langle U_{\sigma^\frown<n>}\rangle_{n\in\mathbb{N}}$ is a partition of U_σ for every σ, f is a bijection. (iv) If $G\subseteq X$ is open, then

$$f^{-1}[G]=\{\alpha:\text{there is some }k\in\mathbb{N}\text{ such that }U_{\alpha\restriction k}\subseteq G\}$$

is open, so f is continuous. (v) If $H\subseteq\mathbb{N}^{\mathbb{N}}$ is open, then

$$f[H]=\bigcup\{U_\sigma:\sigma\in S,\ \{\alpha:\sigma\subseteq\alpha\in\mathbb{N}^{\mathbb{N}}\}\subseteq H\}$$

is open, so f is a homeomorphism. **Q**

(e) If X is a non-empty Polish space without isolated points, then it has a dense G_δ set which is homeomorphic to $\mathbb{N}^{\mathbb{N}}$ with its usual topology. **P** Let $\mathcal{U}$ be a countable base for the topology of X, and D a countable dense subset of X; set

$$Y=X\setminus(D\cup\bigcup_{U\in\mathcal{U}}\partial U)$$

where $\partial U = \overline{U} \setminus U$ is the boundary of U. Then Y is a G_δ set in X, so is Polish (4A2Qd). Because X has no isolated points, Y is comeager in X and is dense and not empty. Because $\{U \cap Y : U \in \mathcal{U}\}$ is a base for the topology of Y consisting of relatively open-and-closed sets, Y is zero-dimensional. If $V \subseteq Y$ is a non-empty relatively open set, let $G \subseteq X$ be an open set such that $G \cap Y = V$; then $D \cap G$ is non-empty, so there is a sequence in V converging (in X) to a point in $D \cap G \subseteq X \setminus V$, and V cannot be compact. By (d), Y is homeomorphic to $\mathbb{N}^{\mathbb{N}}$. **Q**

(f) If X is any zero-dimensional Polish space there is a closed subspace of $\mathbb{N}^{\mathbb{N}}$ homeomorphic to X. **P** Let ρ be a complete metric on X defining its topology. For each $n \in \mathbb{N}$, $\mathcal{V}_n = \{V : V \subseteq X$ is open-and-closed, $\operatorname{diam}(V) \le 2^{-n}\}$ is an open cover of X; let $\langle V_{ni}\rangle_{i\in\mathbb{N}}$ be a sequence in $\mathcal{V}_n$ covering X; set $U_{ni} = V_{ni} \setminus \bigcup_{j<i} V_{nj}$ for $i \in \mathbb{N}$, so that $\langle U_{ni}\rangle_{i\in\mathbb{N}}$ is a partition of X into open-and-closed sets of diameter at most 2^{-n}. For $n \in \mathbb{N}$ and $x \in U_{ni}$, set $f(x)(n) = i$; then f is a continuous function from X to $\mathbb{N}^{\mathbb{N}}$. If $x, y \in X$, $n \in \mathbb{N}$ and $f(x)(n) = f(y)(n) = i$, then $\rho(x,y) \le \operatorname{diam} U_{ni} \le 2^{-n}$, so f is injective. Because $\{f[U_{ni}] : n, i \in \mathbb{N}\} = \{f[X] \cap \{\alpha : \alpha(n) = i\} : n, i \in \mathbb{N}\}$ is a subbase for the topology of $f[X]$, while $\{U_{ni} : n, i \in \mathbb{N}\}$ is a subbase for the topology of X, f is a homeomorphism between X and $f[X]$. Finally, to see that $f[X]$ is a closed subset of $\mathbb{N}^{\mathbb{N}}$, take a sequence $\langle x_m\rangle_{m\in\mathbb{N}}$ in X such that $\langle f(x(m))\rangle_{m\in\mathbb{N}} \to \alpha$ in $\mathbb{N}^{\mathbb{N}}$. For any $n \in \mathbb{N}$ there is a $k \in \mathbb{N}$ such that $f(x(m))(n) = \alpha(n)$ for every $m \ge k$, so that $x(m) \in U_{n,\alpha(n)}$ for $m \ge k$, and $\rho(x(m), x(k)) \le 2^{-n}$ for $m \ge k$. Thus $\langle x(m)\rangle_{m\in\mathbb{N}}$ is a Cauchy sequence in X with a limit $x \in X$, and $\alpha = f(x)$ belongs to $f[X]$. **Q**

Version of 3.10.13

5A5 Real analysis

For the sake of an argument in §534 I sketch a fragment of theory.

5A5A Entire functions A real function f is **real-analytic** if its domain is an open subset G of $\mathbb{R}$ and for every $a \in G$ there are a $\delta > 0$ and a real sequence $\langle c_n\rangle_{n\in\mathbb{N}}$ such that $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ whenever $|x-a| < \delta$. It is **real-entire** if in addition its domain is the whole of $\mathbb{R}$.

We need the following facts: (i) if f and g are real-entire functions so is $f - g$; (ii) if $\langle c_n\rangle_{n\in\mathbb{N}}$ is a real sequence such that $f(x) = \sum_{n=0}^{\infty} c_n x^n$ is defined in $\mathbb{R}$ for every $x \in \mathbb{R}$, then f is real-entire; (iii) if in this expression not every c_n is zero, then every point of $F = \{x : x \in \mathbb{R}, f(x) = 0\}$ is isolated in F, so that F is countable. If you have done a basic course in complex functions you should recognise this. If either you missed this out, or you are not sure you understood the proof of Cauchy's theorem, the following is a sketch of a real-variable argument.

(i) is elementary. For (ii), observe first that if $\langle c_n x^n\rangle_{n\in\mathbb{N}}$ is summable then $\lim_{n\to\infty} c_n x^n = 0$ so $\sum_{n=0}^{\infty} |c_n| t^n$ is finite whenever $0 \le t < |x|$. In the present case, $\sum_{n=0}^{\infty} |c_n| t^n < \infty$ for every $t \ge 0$. So if $a, x \in \mathbb{R}$,

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \Bigl|\frac{n!}{k!(n-k)!} c_n x^k a^{n-k}\Bigr| \le \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} |c_n| R^n$$

(where $R = \max(|x|, |a|)$)

$$= \sum_{n=0}^{\infty} |c_n| (2R)^n < \infty.$$

We therefore have

$$f(x+a) = \sum_{n=0}^{\infty} c_n(x+a)^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} c_n x^k a^{n-k} = \sum_{k=0}^{\infty} c_{ak} x^k$$

where $c_{ak} = \sum_{n=k}^{\infty} \frac{n!}{k!(n-k)!} c_n a^{n-k}$ for each k. Turning this round, $f(x) = \sum_{k=0}^{\infty} c_{ak}(x-a)^k$ for every x. This shows that f is real-entire. As for (iii), if not every c_n is zero, there must be some neighbourhood of 0 in which the first non-zero term $c_n x^n$ dominates, so f is not identically zero. (The point is that $\sum_{k=0}^{\infty} |c_k| < \infty$, so there is some $\delta > 0$ such that $\sum_{k=n+1}^{\infty} |c_k \delta^{k+1}| < |c_n \delta^n|$.) In this case, if $a \in \mathbb{R}$, not every c_{ak} can be zero, and there must be some neighbourhood of a in which the first non-zero term $c_{ak}(x-a)^k$ dominates, so that there can be no zeroes of f in that neighbourhood except perhaps a itself.

Version of 13.2.17

5A6 Special axioms

This section contains very brief accounts of some of the undecidable propositions and special axioms which are used in this volume, with a few of their most basic consequences: the generalized continuum hypothesis, the axiom of constructibility, Jensen's Covering Lemma, square principles, Chang's transfer principle, Todorčević's p-ideal dichotomy and the filter dichotomy.

5A6A The generalized continuum hypothesis (a) The generalized continuum hypothesis is the assertion

$$\text{(GCH)}\quad 2^\kappa=\kappa^+ \text{ for every infinite cardinal } \kappa.$$

(b) If GCH is true, then for infinite cardinals κ, λ

$$\begin{aligned}\mathrm{cf}[\kappa]^{\le\lambda}&=1 \text{ if } \kappa\le\lambda,\\ &=\kappa \text{ if } \lambda<\mathrm{cf}\,\kappa,\\ &=\kappa^+ \text{ otherwise.}\end{aligned}$$

P If $\kappa\le\lambda$, use 5A1F(e-i). If $\lambda<\kappa$ then

$$\mathrm{cf}[\kappa]^{\le\lambda}\le\#([\kappa]^{\le\lambda})\le\#(\mathcal{P}\kappa)=2^\kappa=\kappa^+.$$

If $\lambda<\theta=\mathrm{cf}\,\kappa$, then $[\kappa]^{\le\lambda}=\bigcup_{\xi<\kappa}[\xi]^{\le\lambda}$ so

$$\mathrm{cf}[\kappa]^{\le\lambda}\le\max(\kappa,\sup_{\xi<\kappa}\mathrm{cf}[\xi]^{\le\lambda})\le\max(\kappa,\sup_{\xi<\kappa}\#(\xi)^+)=\kappa$$

and we have equality (using the other part of 5A1F(e-i)). If $\lambda=\theta$ then 5A1F(e-v) tells us that $\mathrm{cf}[\kappa]^\lambda$ is greater than κ, so must be κ^+. If $\theta<\lambda<\kappa$ then, by 5A1F(e-ii),

$$\kappa<\mathrm{cf}[\kappa]^{\le\theta}\le\max(\mathrm{cf}[\kappa]^{\le\lambda},\mathrm{cf}[\lambda]^{\le\theta})\le\max(\mathrm{cf}[\kappa]^{\le\lambda},\lambda^+)\le\max(\mathrm{cf}[\kappa]^{\le\lambda},\kappa),$$

so again $\mathrm{cf}[\kappa]^{\le\lambda}=\kappa^+$. **Q**

(c) If GCH is true, then for infinite cardinals κ and λ, the cardinal power κ^λ is 2^λ if $\kappa\le\lambda$, κ if $\lambda<\mathrm{cf}\,\kappa$, κ^+ otherwise. (Put (b) and 5A1F(e-iii) together.)

5A6B L, $0^\sharp$ and Jensen's Covering Lemma (a)(i) Let L be the class of **constructible** sets (Jech 03, §13; Jech 78, §12; Kanamori 03, §3; Kunen 80, chap. VI). The **axiom of constructibility** is

$$(V{=}L)\quad \text{Every set is constructible.}$$

$V{=}L$ implies GCH (Jech 03, 13.20; Jech 78, Theorem 34; Kunen 80, §VI.4).

(ii) I will call on the following three properties of L in the remarks below. To make sense of them you will of course need to look at the proper definition. Only the third has any real content. Every ordinal belongs to L; if A, $B\in L$ then $A\cap B\in L$; if κ is a cardinal, then $\#(L\cap\mathcal{P}\kappa)\le\kappa^+$.

(b) $0^\sharp$, if it exists, is a set of sentences in a countable formal language (Jech 03, §18; Kanamori 03, §9). I will not attempt to explain further; I mention $0^\sharp$ only so that you will be able to explore the literature for proofs of the assertions below. I will write '$\exists 0^\sharp$' for the assertion '$0^\sharp$ exists'.

Jensen's Covering Lemma is the assertion

(CL) for every uncountable set A of ordinals, there is a constructible set of the same cardinality including A.

Now Jensen's Covering Theorem is

$$\text{CL iff not-}\exists 0^\sharp$$

(Jech 03, Theorem 18.30.)

(c) The importance to us of $0^\sharp$ is that there are relatively direct proofs that $V{=}L$ implies not-$\exists 0^\sharp$ (Jech 03, §18), and that not-$\exists 0^\sharp$ is true in any set forcing extension of a model of not-$\exists 0^\sharp$; see Jech 03, Exercise 18.2 or Jech 78, Exercise 30.2. So CL implies that $\Vdash_{\mathbb{P}}$ CL for every forcing notion $\mathbb{P}$ of the kind considered in §5A3.

5A6C Theorem Assume that CL is true.
(a) For infinite cardinals κ and λ,

$$\begin{aligned}\operatorname{cf}[\kappa]^{\le\lambda} &= 1 \text{ if } \kappa \le \lambda,\\ &= \kappa \text{ if } \lambda < \operatorname{cf}\kappa,\\ &= \kappa^+ \text{ otherwise.}\end{aligned}$$

(b) If κ and λ are infinite cardinals, then the cardinal power κ^λ is 2^λ if $\kappa \le 2^\lambda$, κ if $\lambda < \operatorname{cf}\kappa$ and $2^\lambda \le \kappa$, and κ^+ otherwise.

proof (a)(i) If $\omega_1 \le \lambda < \kappa$ then $\operatorname{cf}[\kappa]^{\le\lambda} \le \kappa^+$. **P** By CL, every $A \in [\kappa]^\lambda$ is included in a $B \in [L]^\lambda$; now $\kappa \in L$ so $B \cap \kappa \in L$ and $A \subseteq B \cap \kappa \in [\kappa]^\lambda$. Thus $L \cap [\kappa]^\lambda$ is cofinal with $[\kappa]^\lambda$ and $[\kappa]^{\le\lambda}$. But $\#(L \cap \mathcal{P}\kappa) \le \kappa^+$ (5A6B(a-ii)), so $\operatorname{cf}[\kappa]^{\le\lambda} \le \kappa^+$. **Q**

(ii) It follows that $\operatorname{cf}[\kappa]^{\le\lambda} \le \kappa^+$ for all infinite cardinals κ and λ. **P** The case $\lambda \ge \kappa$ is trivial, so only the case $\lambda = \omega < \kappa$ remains. But

$$\operatorname{cf}[\kappa]^{\le\omega} \le \max(\operatorname{cf}[\kappa]^{\omega_1}, \operatorname{cf}[\omega_1]^{\le\omega}) \le \max(\kappa^+, \omega_1) = \kappa^+$$

by 5A1F(e-ii) and (i). **Q**

(iii) If $\lambda < \operatorname{cf}\kappa$ then $\operatorname{cf}[\kappa]^{\le\lambda} \le \kappa$. **P** $[\kappa]^{\le\lambda} = \bigcup_{\xi<\kappa}[\xi]^{\le\lambda}$, so

$$\operatorname{cf}[\kappa]^{\le\lambda} \le \max(\kappa, \sup_{\xi<\kappa}\operatorname{cf}[\xi]^{\le\lambda}) \le \max(\kappa, \sup_{\xi<\kappa}\#(\xi)^+) = \kappa. \text{ } \mathbf{Q}$$

(iv) If $\operatorname{cf}\kappa \le \lambda < \kappa$ then $\operatorname{cf}[\kappa]^{\le\lambda} > \kappa$. **P** Set $\theta = \operatorname{cf}\kappa$. Then

$$\begin{aligned}\kappa < \operatorname{cf}[\kappa]^{\le\theta} &\le \max(\operatorname{cf}[\kappa]^{\le\lambda}, \operatorname{cf}[\lambda]^{\le\theta})\\ &\le \max(\operatorname{cf}[\kappa]^{\le\lambda}, \lambda^+) \le \max(\operatorname{cf}[\kappa]^{\le\lambda}, \kappa)\end{aligned}$$

(5A1F(e-v), 5A1F(e-ii), (ii) above), so $\operatorname{cf}[\kappa]^{\le\lambda} > \kappa$. **Q**
Putting this together with 5A1F(e-i), (ii) and (iii) we have the result.

(b) As 5A6Ac.

5A6D Square principles (a)(i) Let Sing be the class of non-zero limit ordinals which are not regular cardinals. Global Square is the statement

there is a family $\langle C_\xi\rangle_{\xi\in\mathrm{Sing}}$ such that
for every $\xi \in$ Sing, C_ξ is a closed cofinal set in ξ;
$\operatorname{otp} C_\xi < \xi$ for every $\xi \in$ Sing;
if $\xi \in$ Sing and $\zeta > 0$ is such that $\zeta = \sup(\zeta \cap C_\xi)$, then $\zeta \in$ Sing and $C_\zeta = \zeta \cap C_\xi$.

(ii) For an infinite cardinal κ, let $\square_\kappa$ be the statement
there is a family $\langle C_\xi\rangle_{\xi<\kappa^+}$ of sets such that
for every $\xi < \kappa^+$, $C_\xi \subseteq \xi$ is a closed cofinal set in ξ;
if $\operatorname{cf}\xi < \kappa$ then $\#(C_\xi) < \kappa$;
whenever $\xi < \kappa^+$ and $\zeta < \xi$ is such that $\zeta = \sup(\zeta \cap C_\xi)$, then $C_\zeta = \zeta \cap C_\xi$.

(b) $V{=}L$ implies Global Square (Friedman & Koepke 97). Global Square implies that $\square_\kappa$ is true for every infinite cardinal κ (Devlin 84, VI.6.2). CL implies that $\square_\kappa$ is true for every singular infinite cardinal κ (Devlin 84, V.5.6).

(c) If κ is an uncountable cardinal and $\langle C_\xi\rangle_{\xi<\kappa^+}$ is a family as in (a-ii), then $\operatorname{otp} C_\xi \le \kappa$ for every $\xi < \kappa^+$. **P** If $\operatorname{cf}\xi < \kappa$ this is immediate from the second clause of $\square_\kappa$. Otherwise, κ is regular, and $\operatorname{cf} C_\xi = \kappa$. **?** If $\operatorname{otp} C_\xi > \kappa$ then $\operatorname{otp} C_\xi > \kappa+\omega$ so there is a $\zeta \in C_\xi$ such that $\operatorname{otp}(\zeta\cap C_\xi) = \kappa+\omega$; but now $\operatorname{cf}\zeta = \omega < \kappa$ and $C_\zeta = \zeta \cap C_\xi$ has cardinal κ, which is not allowed. **XQ**

5A6E Lemma Suppose that κ is an uncountable cardinal with countable cofinality such that $\square_\kappa$ is true. Then there is a family $\langle I_\xi\rangle_{\xi<\kappa^+}$ of countably infinite subsets of κ such that

$I_\xi \cap I_\eta$ is finite whenever $\eta < \xi < \kappa^+$,

$\{\xi : \xi < \kappa^+,\, I \cap I_\xi$ is infinite$\}$ is countable for every countable $I \subseteq \kappa$.

proof Let $\langle C_\xi\rangle_{\xi<\kappa^+}$ be a family as in 5A6D(a-ii). Let $\langle\kappa_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of infinite cardinals less than κ with supremum κ. Define $\langle f_\xi\rangle_{\xi<\kappa^+}$ in $\prod_{n\in\mathbb{N}}\kappa_n^+$ as follows. $f_0(n) = 0$ for every n. Given f_ξ, set $f_{\xi+1}(n) = f_\xi(n)+1$ for every n. Given $\langle f_\eta\rangle_{\eta<\xi}$, where $\xi < \kappa^+$ is a non-zero limit ordinal, set

$$f_\xi(n) = \sup\{f_\eta(n) : \eta \in C_\xi,\, \#(\eta\cap C_\xi) < \kappa_n\}$$

for each n; because $\{\eta : \eta \in C_\xi,\, \#(\eta\cap C_\xi) < \kappa_n\}$ has cardinal at most κ_n, $f_\xi(n) < \kappa_n^+$. Continue.

We find that if $\eta < \xi < \kappa^+$ then $\{n : f_\xi(n) \le f_\eta(n)\}$ is finite. **P** Induce on ξ. If $\xi = 0$ there is nothing to prove. If $\xi = \zeta+1$ then $\{n : f_\xi(n) \le f_\eta(n)\} = \{n : f_\zeta(n) < f_\eta(n)\}$ is finite. If ξ is a non-zero limit ordinal, let $\zeta \in C_\xi$ be such that $\eta < \zeta$. Because $\operatorname{otp} C_\xi \le \kappa$ (5A6Dc), $\#(\zeta\cap C_\xi) < \kappa_m$ for some m. Now $f_\xi(n) \ge f_\zeta(n)$ for every $n \ge m$, so $\{n : f_\xi(n) \le f_\eta(n)\} \subseteq m \cup \{n : f_\zeta(n) \le f_\eta(n)\}$ is finite. **Q**

If $I \subseteq \mathbb{N}\times\kappa$ is countable, then $B = \{\xi : \xi < \kappa^+,\, I \cap f_\xi$ is infinite$\}$ is countable, where in this formula I am identifying f_ξ with its graph, as usual. **P?** Otherwise, let $B' \subseteq B$ be a set with order type ω_1, and set $\xi = \sup B' < \kappa^+$. Set

$$I' = \{(n,\alpha) : (n,\alpha) \in I,\, \alpha \le f_\eta(n) \text{ for some } \eta \in C_\xi\}.$$

Because I and I' are countable, while $\operatorname{cf} C_\xi = \operatorname{cf}\xi = \omega_1$, there is a $\zeta \in C_\xi$ such that $\zeta = \sup(\zeta\cap C_\xi)$ and

$$I' = \{(n,\alpha) : (n,\alpha) \in I,\, \alpha \le f_\eta(n) \text{ for some } \eta \in \zeta\cap C_\xi\}.$$

Take $\eta \in B'$ such that $\eta > \zeta$, and $\zeta' \in C_\xi$ such that $\zeta' > \eta$. Then there is an $m \in \mathbb{N}$ such that $\#(\zeta\cap C_\xi) < \kappa_m$ and $f_\zeta(n) < f_\eta(n) < f_{\zeta'}(n)$ for every $n \ge m$. As $\eta \in B$, there is an $n \ge m$ such that $(n, f_\eta(n)) \in I$; as $f_\eta(n) < f_{\zeta'}(n)$, $(n, f_\eta(n)) \in I'$ and there is an $\eta' \in \zeta\cap C_\xi$ such that $f_\eta(n) \le f_{\eta'}(n)$. But now we have $\eta' \in C_\zeta$ and $\#(\eta'\cap C_\zeta) \le \#(C_\zeta) < \kappa_n$ and $f_\zeta(n) < f_{\eta'}(n)$, contrary to the choice of f_ζ. **XQ**

Thus if we set $I_\xi = f_\xi$ for $\xi < \kappa^+$ we have an appropriate family of sets in $\mathbb{N}\times\kappa^+$ which can be transferred to κ^+ by any bijection.

5A6F Chang's transfer principle (a) If λ_0, λ_1, κ_0 and κ_1 are cardinals, then $(\kappa_1,\lambda_1) \twoheadrightarrow (\kappa_0,\lambda_0)$ means

whenever $f : [\kappa_1]^{<\omega} \to \lambda_1$ is a function, there is an $A \in [\kappa_1]^{\kappa_0}$ such that $\#(f[[A]^{<\omega}]) \le \lambda_0$.

For the original model-theoretic version of this principle, and the proof that it comes to the same thing, see Kanamori 03, 8.1. For various combinatorial consequences, see 'Chang's conjecture' in Erdős Hajnal Máté & Rado 84.

In this book, I write $\mathrm{CTP}(\kappa,\lambda)$ for the statement

$$(\kappa,\lambda) \twoheadrightarrow (\omega_1,\omega).$$

What is commonly called 'Chang's conjecture' is $\mathrm{CTP}(\omega_2,\omega_1)$. For a model of GCH + $\mathrm{CTP}(\omega_{\omega+1},\omega_\omega)$, see Levinski Magidor & Shelah 90.

(b) Suppose that $\mathrm{CTP}(\kappa,\lambda)$ is true.

(i) If $f : [\kappa]^{<\omega} \to [\lambda]^{\le\omega}$ is a function, then there is an uncountable $A \subseteq \kappa$ such that $\bigcup\{f(I) : I \in [A]^{<\omega}\}$ is countable. **P** Enumerate $\mathbb{N}\times\mathbb{N}$ as $\langle(k_n,m_n)\rangle_{n\in\mathbb{N}}$ in such a way that $m_n \le n$ for every $n \in \mathbb{N}$. For $I \in [\kappa]^{<\omega}$ let $\langle f_k(I)\rangle_{k\in\mathbb{N}}$ be a sequence running over $f(I)\cup\{0\}$. (I am passing over the trivial case $\lambda = 0$.) Now, for $n \in \mathbb{N}$ and $I \in [\kappa]^n$, enumerate I in ascending order as $\langle\xi_i\rangle_{i<n}$ and set $g(I) = f_{k_n}(\{\xi_i : i < m_n\})$. There is an uncountable $A \subseteq \kappa$ such that $B = \{g(I) : I \in [A]^{<\omega}\}$ is countable; we may suppose that A has order type ω_1. If $J \in [A]^{<\omega}$ and $k \in \mathbb{N}$, let $n \in \mathbb{N}$ be such that $k_n = k$ and $m_n = \#(J)$; let $I \in [A]^n$ be such that J consists of the first m_n elements of I; then $f_k(J) = g(I)$ belongs to B. As J and k are arbitrary, $\bigcup\{f(I) : I \in [A]^{<\omega}\} \subseteq B$ is countable. **Q**

(ii) If $\langle A_\xi\rangle_{\xi<\kappa}$ is any family of countable subsets of λ, then there is a countable $A \subseteq \lambda$ such that $\{\xi : A_\xi \subseteq A\}$ is uncountable. **P** In (i), take $f(I) = \bigcup_{\xi\in I} A_\xi$ for $I \in [\kappa]^{<\omega}$. **Q**

(c) CL implies that $\mathrm{CTP}(\kappa, \lambda)$ is false except when $\lambda \le \omega$ (Kanamori 03, 8.3, 21.1 and 21.4).

5A6G Todorčević's p-ideal dichotomy (a) Let X be a set and $\mathcal{I}$ an ideal of subsets of X. Then $\mathcal{I}$ is a **p-ideal** if for every sequence $\langle I_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{I}$ there is an $I \in \mathcal{I}$ such that $I_n \setminus I$ is finite for every $n \in \mathbb{N}$. (Compare 538Ab.)

(b) Now Todorčević's p-ideal dichotomy is the statement

(TPID) whenever X is a set and $\mathcal{I} \subseteq [X]^{\le\omega}$ is a p-ideal of countable subsets of X, then *either* there is a $B \in [X]^{\omega_1}$ such that $[B]^{\le\omega} \subseteq \mathcal{I}$ *or* X is expressible as $\bigcup_{n\in\mathbb{N}} X_n$ where $\mathcal{I} \cap \mathcal{P}X_n \subseteq [X_n]^{<\omega}$ for every $n \in \mathbb{N}$.

This is a consequence of the Proper Forcing Axiom, and implies that $\square_\kappa$ is false for every $\kappa \ge \omega_1$ (Todorčević 00).

***5A6H Analytic p-ideals: Theorem** Suppose that the Proper Forcing Axiom is true. Take a non-empty set $D \subseteq [0, \infty[^{\mathbb{N}}$ and set

$$\mathcal{I} = \{I : I \subseteq \mathbb{N}, \lim_{n\to\infty} \sup_{z\in D} \textstyle\sum_{i\in I\setminus n} z(i) = 0\},$$

so that $\mathcal{I}$ is an ideal of subsets of $\mathbb{N}$. Let $\mathfrak{A}$ be the quotient Boolean algebra $\mathcal{P}\mathbb{N}/\mathcal{I}$. Then for every $\pi \in \operatorname{Aut}\mathfrak{A}$ there are sets I, $J \in \mathcal{I}$ and a bijection $h : \mathbb{N} \setminus I \to \mathbb{N} \setminus J$ representing π in the sense that $\pi(A^\bullet) = (h^{-1}[A])^\bullet$ for every $A \subseteq \mathbb{N}$. (Farah 00, 3.4.6.)

5A6I $\mathfrak{u}$, $\mathfrak{g}$ and the filter dichotomy: Definitions (a) The **ultrafilter number** $\mathfrak{u}$ is the least cardinal of any filter base generating a free ultrafilter on $\mathbb{N}$, that is, $\min\{\operatorname{ci}\mathcal{F} : \mathcal{F}$ is a free ultrafilter on $\mathbb{N}\}$.

(b)(i) A family A of infinite subsets of $\mathbb{N}$ is **groupwise dense** if

(α) whenever $a \in A$, $a' \in [\mathbb{N}]^\omega$ and $a' \setminus a$ is finite, then $a' \in A$,

(β) whenever $\phi : \mathbb{N} \to \mathbb{N}$ is finite-to-one, there is an infinite $c \subseteq \mathbb{N}$ such that $\phi^{-1}[c] \in A$.

(A function $f : X \to Y$ is 'finite-to-one' if $f^{-1}[\{y\}]$ is finite for every $y \in Y$.)

(ii) The **groupwise density number** $\mathfrak{g}$ is the least cardinal of any collection $\mathbb{A}$ of groupwise dense subsets of $[\mathbb{N}]^\omega$ such that $\bigcap\mathbb{A} = \emptyset$.

(iii) For a model in which $\omega_1 = \mathfrak{u} < \mathfrak{g}$ see Blass & Laflamme 89.

(c) For filters $\mathcal{F}$ on X and $\mathcal{G}$ on Y, say that $\mathcal{F} \le_{\mathrm{RB}} \mathcal{G}$ if there is a finite-to-one $\phi : Y \to X$ such that $\mathcal{F} = \phi[[\mathcal{G}]]$. (This is the **Rudin-Blass ordering** of filters.) Note that $\mathcal{F} \le_{\mathrm{RB}} \mathcal{F}$ for every filter $\mathcal{F}$, and if $\mathcal{F} \le_{\mathrm{RB}} \mathcal{G}$ and $\mathcal{G} \le_{\mathrm{RB}} \mathcal{H}$ then $\mathcal{F} \le_{\mathrm{RB}} \mathcal{H}$ (and $\mathcal{F} \le_{\mathrm{RK}} \mathcal{G}$, of course).

(d) The **filter dichotomy** is the statement

(FD) For every free filter $\mathcal{F}$ on $\mathbb{N}$ either $\mathcal{F}_{\mathrm{Fr}} \le_{\mathrm{RB}} \mathcal{F}$, where $\mathcal{F}_{\mathrm{Fr}}$ is the Fréchet filter, or there is an ultrafilter $\mathcal{G}$ on $\mathbb{N}$ such that $\mathcal{G} \le_{\mathrm{RB}} \mathcal{F}$.

***5A6J Proposition** (Blass & Laflamme 89) If $\mathfrak{u} < \mathfrak{g}$ then the filter dichotomy is true.

proof Let $\mathcal{F}$ be a free filter on $\mathbb{N}$ such that $\mathcal{F}_{\mathrm{Fr}} \not\le_{\mathrm{RB}} \mathcal{F}$, where $\mathcal{F}_{\mathrm{Fr}}$ is the Fréchet filter.

(a) For subsets a, b, c of $\mathbb{N}$ I will say that b **interpolates between** a **and** c, and write $(a(b)c)$, if whenever $i \in a$, $k \in c$ and $i \le k$ then there is a $j \in b$ such that $i \le j \le k$. Now if $b \subseteq \mathbb{N}$ is infinite,

$$A_b = \{a : a \in [\mathbb{N}]^\omega,\ (a(b)c) \text{ for some } c \in \mathcal{F}\}$$

is groupwise dense. **P** (α) If $a \in A_b$, $a' \in [\mathbb{N}]^\omega$ and $a' \setminus a$ is finite, let $c \in \mathcal{F}$ be such that $(a(b)c)$. Let $j_0 \in b$ be such that $a' \setminus a \subseteq j_0$, and set $c' = c \setminus j_0 \in \mathcal{F}$. If $i \in a'$, $k \in c'$ and $i \le k$, either $i \le j_0$ and $i \le j_0 \le k$, or $i \in a$ and there is a $j \in b$ such that $i \le j \le k$; thus $(a'(b)c')$ and $a' \in A_b$. (β) If $\phi : \mathbb{N} \to \mathbb{N}$ is finite-to-one, we can choose strictly increasing sequences $\langle l_r \rangle_{r\in\mathbb{N}}$, $\langle m_r \rangle_{r\in\mathbb{N}}$ in $\mathbb{N}$ such that

$l_0 \in \phi[\mathbb{N}]$,

given l_r, $m_r \in b$ and $i < m_r$ whenever $\phi(i) = l_r$,

given m_r, $l_{r+1} \in \phi[\mathbb{N}]$ and $i \ge m_r$ whenever $\phi(i) = l_{r+1}$.

Set $\psi(i) = \#(\{r : m_r \le i\})$ for $i \in \mathbb{N}$; then $\psi : \mathbb{N} \to \mathbb{N}$ is finite-to-one. As $\mathcal{F}$ is free, $\mathcal{F}_{\mathrm{Fr}} \subseteq \psi[[\mathcal{F}]]$; as $\mathcal{F}_{\mathrm{Fr}} \not\le_{\mathrm{RB}} \mathcal{F}$, $\psi[[\mathcal{F}]] \ne \mathcal{F}_{\mathrm{Fr}}$ and there is an infinite set $d \subseteq \mathbb{N}$ such that $d' = \mathbb{N} \setminus d \in \psi[[\mathcal{F}]]$. Of course we can suppose that $0 \notin d$.

If $i \in \psi^{-1}[d]$, $k \in \psi^{-1}[d']$ and $i \le k$, then $\psi(i) \in d$, $\psi(k) \in d'$ and $\psi(i) \le \psi(k)$, so $\psi(i) < \psi(k)$; setting $r = \psi(i)$, we have $i < m_r \le k$, while $m_r \in b$. This shows that $(\psi^{-1}[d](b)\psi^{-1}[d'])$, so that $\psi^{-1}[d] \in A_b$. Next, setting $c = \{l_r : r \in d\}$, c is infinite; and if $i \in \phi^{-1}[c]$, we have an $r \in d$ such that $\phi(i) = l_r$, in which case (as $r > 0$) $m_{r-1} \le i < m_r$ and $\psi(i) = r$. So $\phi^{-1}[c] \subseteq \psi^{-1}[d]$; also $\phi^{-1}[c]$ is infinite (because $l_r \in \phi[\mathbb{N}]$ for every r), so $\phi^{-1}[c] \in A_b$. As ϕ is arbitrary, A_b is groupwise dense. **Q**

(b)(i) Now let $\mathcal{G}$ be a non-principal ultrafilter on $\mathbb{N}$ which has a filter base B with cardinal $\mathfrak{u}$. Because $\mathfrak{u} < \mathfrak{g}$, there is an $a \in \bigcap_{b\in B} A_b$. This time, set $\phi(i) = \#(a \cap (i+1))$ for $i \in \mathbb{N}$; as a is infinite, ϕ is finite-to-one. Set $m = \min a$. If $b \in B$, $c \subseteq \mathbb{N}$ and $(a(b)c)$, then $\phi[c \setminus m] \subseteq \phi[b]$. **P** If $k \in c \setminus m$, set $i = \max(a \cap (k+1))$; then $i \le k$ so there is a $j \in b$ such that $i \le j \le k$, and now $a \cap (j+1) = a \cap (k+1) = a \cap (i+1)$ so $\phi(k) = \phi(j) \in \phi[b]$. **Q**

(ii) It follows that $\phi[[\mathcal{G}]] \subseteq \phi[[\mathcal{F}]]$. **P** If $G \in \phi[[\mathcal{G}]]$ there is a $b \in B$ such that $b \subseteq \phi^{-1}[G]$. Now $a \in A_b$ so there is a $c \in \mathcal{F}$ such that $(a(b)c)$; in this case, setting $m = \min a$, $c \setminus m \in \mathcal{F}$ and $\phi[c \setminus m] \in \phi[[\mathcal{F}]]$. By (i) just above, $\phi[c \setminus m] \subseteq \phi[b]$, while $\phi[b] \subseteq G$. So $G \in \phi[[\mathcal{F}]]$. As G is arbitrary, $\phi[[\mathcal{G}]] \subseteq \phi[[\mathcal{F}]]$. **Q**

(iii) We supposed that $\mathcal{G}$ was an ultrafilter, so $\phi[[\mathcal{G}]]$ is an ultrafilter (2A1N) and must be equal to $\phi[[\mathcal{F}]] \le_{\mathrm{RB}} \mathcal{F}$. Thus we have the second alternative in the statement of FD. As $\mathcal{F}$ is arbitrary, FD is true.

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