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Chapter 56

Choice and determinacy

Nearly everyone reading this book will have been taking the axiom of choice for granted nearly all the time. This is the home territory of twentieth-century abstract analysis, and the one in which the great majority of the results have been developed. But I hope that everyone is aware that there are other ways of doing things. In this chapter I want to explore what seem to me to be the most interesting alternatives. In one sense they are minor variations on the standard approach, since I keep strictly to ideas expressible within the framework of Zermelo-Fraenkel set theory; but in other ways they are dramatic enough to rearrange our prejudices. The arguments I will present in this chapter are mostly not especially difficult by the standards of this volume, but they do depend on intuitions for which familiar results which are likely to remain valid under the new rules being considered.

Let me say straight away that the real aim of the chapter is §567, on the axiom of determinacy. The significance of this axiom is that it is (so far) the most striking rival to the axiom of choice, in that it leads us quickly to a large number of propositions directly contradicting familiar theorems; for instance, every subset of the real line is now Lebesgue measurable (567G). But we need also to know which theorems are still true, and the first six sections of the chapter are devoted to a discussion of what can be done in ZF alone (§§561-565) and with countable or dependent choice (§566). Actually §§562-565 are rather off the straight line to §567, because they examine parts of real analysis in which the standard proofs depend only on countable choice or less; but a great deal more can be done than most of us would expect, and the methods are instructive.

Going into details, §561 looks at basic facts from real analysis, functional analysis and general topology which can be proved in ZF. §562 deals with 'codable' Borel sets and functions, using Borel codes to keep track of constructions for objects, so that if we know a sequence of codes we can avoid having to make a sequence of choices. A 'Borel-coded measure' (§563) is now one which behaves well with respect to codable sequences of measurable sets; for such a measure we have an integral with versions of the convergence theorems (§564), and Lebesgue measure fits naturally into the structure (§565). In §566, with ZF + AC(ω), we are back in familiar territory, and most of the results of Volumes 1 and 2 can be proved if we are willing to re-examine some definitions and hypotheses. Finally, in §567, I look at infinite games and half a dozen of the consequences of AD, with a postscript on determinacy in the context of ZF + AC.

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561 Analysis without choice

Elementary courses in analysis are often casual about uses of weak forms of choice; a typical argument runs 'for every $\epsilon > 0$ there is an $a \in A$ such that $|a - x| \le \epsilon$, so there is a sequence in A converging to x'. This is a direct call on the countable axiom of choice: setting $A_n = \{a : a \in A,\ |a - x| \le 2^{-n}\}$, we are told that every A_n is non-empty, and conclude that $\prod_{n\in\mathbb{N}} A_n$ is non-empty. In the present section I will abjure such methods and investigate what can still be done with the ideas important in measure theory. We have useful partial versions of Tychonoff's theorem (561D), Baire's theorem (561E), Stone's theorem (561F) and Kakutani's theorem on the representation of L-spaces (561H); moreover, there is a direct construction of Haar measures, regarded as linear functionals (561G).

Unless explicitly stated otherwise, throughout this section (and the next four) I am working entirely without any form of the axiom of choice.

561A Set theory without choice In §§1A1 and 2A1 I tried to lay out some of the basic ideas of set theory without appealing to the axiom of choice except when this was clearly necessary. The most obvious point is that in the absence of choice

the union of a sequence of countable sets need not be countable

(see the note in 1A1G). In fact Feferman & Levy 63 (see Jech 73, 10.6) described a model of set theory in which $\mathbb{R}$ is the union of a sequence of countable sets. But not all is lost. The elementary arguments of 1A1E still give

$$\mathbb{N} \simeq \mathbb{Z} \simeq \mathbb{N} \times \mathbb{N} \simeq \mathbb{Q};$$

there is no difficulty in extending them to show such things as

$$\mathbb{N} \simeq [\mathbb{N}]^{<\omega} \simeq \bigcup_{n \ge 1} \mathbb{N}^n \simeq \mathbb{Q}^r \times \mathbb{Q}^r$$

for every integer $r \ge 1$. The Schröder-Bernstein theorem survives (the method in 344D is easily translated back into its original form as a proof of the ordinary Schröder-Bernstein theorem). Consequently we still have enough bijections to establish

$$\mathbb{R} \simeq \mathcal{P}\mathbb{N} \simeq \{0,1\}^{\mathbb{N}} \simeq \mathcal{P}(\mathbb{N} \times \mathbb{N}) \simeq (\mathcal{P}\mathbb{N})^{\mathbb{N}} \simeq \mathbb{R}^{\mathbb{N}} \simeq \mathbb{N}^{\mathbb{N}}.$$

Cantor's theorem that $X \not\simeq \mathcal{P}X$ is unaffected, so we still know that $\mathbb{R}$ is not countable.

We can still use transfinite recursion; see 2A1B. We still have a class On of von Neumann ordinals such that every well-ordered set is isomorphic to exactly one ordinal (2A1Dg) and therefore equipollent with exactly one initial ordinal (2A1Fb). I will say that a set X is **well-orderable** if there is a well-ordering of X. The standard arguments for Zermelo's Well-Ordering Theorem (2A1K) now tell us that for any set X the following are equiveridical:

(i) X is well-orderable;

(ii) X is equipollent with some ordinal;

(iii) there is an injective function from X into a well-orderable set;

(iv) there is a choice function for $\mathcal{P}X \setminus \{\emptyset\}$

(that is, a function f such that $f(A) \in A$ for every non-empty $A \subseteq X$). What this means is that if we are given a family $\langle A_i \rangle_{i \in I}$ of non-empty sets, and $X = \bigcup_{i \in I} A_i$ is well-orderable (e.g., because it is countable), then $\prod_{i \in I} A_i$ is not empty (it contains $\langle f(A_i) \rangle_{i \in I}$ where f is a function as in (iv) above).

Note also that while we still have a first uncountable ordinal ω_1 (the set of countable ordinals), it can have countable cofinality (561Ya). The union of a sequence of finite sets need not be countable (Jech 73, §5.4); but the union of a sequence of finite subsets of a given totally ordered set *is* countable, because we can use the total ordering to simultaneously enumerate each of the finite sets in ascending order. Consequently, if $\gamma : \omega_1 \to \mathbb{R}$ is a monotonic function there is a $\xi < \omega_1$ such that $\gamma(\xi+1) = \gamma(\xi)$. **P** It is enough to consider the case in which γ is non-decreasing. Set

$$A_n = \{\xi : \gamma(\xi) + 2^{-n} \le \gamma(\xi+1) \le n\}.$$

Then A_n has at most $2^n \max(0, n - \gamma(0))$ members, so is finite; consequently $\bigcup_{n \in \mathbb{N}} A_n$ is countable, and there is a $\xi \in \omega_1 \setminus \bigcup_{n \in \mathbb{N}} A_n$. Of course we now find that $\gamma(\xi+1) = \gamma(\xi)$. **Q**

561B Real analysis without choice In fact all the standard theorems of elementary real and complex analysis are essentially unchanged. The kind of tightening required in some proofs, to avoid explicit dependence on the existence of sequences, is similar to the adaptations needed when we move to general topological spaces. For instance, we must define 'compactness' in terms of open covers; compactness and sequential compactness, even for subsets of $\mathbb{R}$, may no longer coincide (561Xc). But we do still have the Heine-Borel theorem in the form 'a subset of $\mathbb{R}^r$ is compact iff it is closed and bounded' (provided, of course, that we understand that 'closed' is not the same thing as 'sequentially closed'); see the proof in 2A2F.

561C Some new difficulties arise when we move away from 'concrete' questions like the Prime Number Theorem and start looking at general metric spaces, or even general subsets of $\mathbb{R}$. For instance, a subset of $\mathbb{R}$, regarded as a topological space, must be second-countable but need not be separable. However we can go a long way if we take care. The following is an elementary example which will be useful below.

Lemma Let $\mathcal{E}$ be the set of non-empty closed subsets of $\mathbb{N}^{\mathbb{N}}$. Then there is a family $\langle f_F\rangle_{F\in\mathcal{E}}$ such that, for each $F\in\mathcal{E}$, f_F is a continuous function from $\mathbb{N}^{\mathbb{N}}$ to F and $f_F(\alpha)=\alpha$ for every $\alpha\in F$.

proof For $F\in\mathcal{E}$, set $T_F=\{\alpha\restriction n:\alpha\in F,\,n\in\mathbb{N}\}$. If $\alpha\in\mathbb{N}^{\mathbb{N}}\setminus F$ then, because F is closed, there is some $n\in\mathbb{N}$ such that $\beta\restriction n\ne\alpha\restriction n$ for any $\beta\in F$, that is, $\alpha\restriction n\notin T_F$. For $\sigma\in T_F$ define $\beta_{F\sigma}\in\mathbb{N}^{\mathbb{N}}$ inductively by saying that

$$\begin{aligned}\beta_{F\sigma}(n)&=\sigma(n)\text{ if }n<\#(\sigma),\\&=\inf\{i:\text{ there is some }\alpha\text{ such that }\beta_{F\sigma}\restriction n\subseteq\alpha\in F\text{ and }\alpha(n)=i\}\text{ otherwise,}\end{aligned}$$

counting $\inf\emptyset$ as 0 if necessary. We see that in fact $\beta_{F\sigma}\restriction n\in T_F$ for every $n\in\mathbb{N}$, so that $\beta_{F\sigma}\in F$.

We can therefore define $f_F:\mathbb{N}^{\mathbb{N}}\to\mathbb{N}^{\mathbb{N}}$ by setting

$$\begin{aligned}f_F(\alpha)&=\alpha\text{ if }\alpha\in F,\\&=\beta_{F,\alpha\restriction n}\text{ for the largest }n\text{ such that }\alpha\restriction n\in T_F\text{ otherwise.}\end{aligned}$$

(Because F is not empty, the empty sequence $\alpha\restriction 0$ belongs to T_F for every $\alpha\in\mathbb{N}^{\mathbb{N}}$.) We see that $f_F(\alpha)\in F$ for all F and α, and $f_F(\alpha)=\alpha$ if $\alpha\in F$. To see that f_F is always continuous, note that in fact if $\alpha\in\mathbb{N}^{\mathbb{N}}\setminus F$, and n is the largest integer such that $\alpha\restriction n$ belongs to T, then $f_F(\beta)=f_F(\alpha)$ whenever $\beta\restriction n+1=\alpha\restriction n+1$, so f_F is continuous at α. While if $\alpha\in F$, $n\in\mathbb{N}$ and $\beta\restriction n=\alpha\restriction n$, then either $\beta\in F$ so $f_F(\beta)\restriction n=f_F(\alpha)\restriction n$, or $f_F(\beta)=\beta_{F\sigma}$ where $\alpha\restriction n\subseteq\sigma\subseteq\beta_{F\sigma}$, and again $f_F(\beta)\restriction n=\beta_\sigma\restriction n=\alpha\restriction n$. So we have a suitable family of functions.

561D Tychonoff's theorem It is a classic result (Kelley 50) that Tychonoff's theorem, in a general form, is actually equivalent to the axiom of choice. But nevertheless we have useful partial results which do not depend on the axiom of choice. The following will help in the proofs of 561F and 563I.

Theorem Let $\langle X_i\rangle_{i\in I}$ be a family of compact topological spaces such that I is well-orderable. For each $i\in I$ let $\mathcal{E}_i$ be the family of non-empty closed subsets of X_i, and suppose that there is a choice function for $\bigcup_{i\in I}\mathcal{E}_i$. Then $X=\prod_{i\in I}X_i$ is compact.

proof Since I is well-orderable, we may suppose that $I=\kappa$ for some initial ordinal κ. Fix a choice function ψ for $\bigcup_{\xi<\kappa}\mathcal{E}_i$. For $\xi<\kappa$ write $\pi_\xi:X\to X_\xi$ for the coordinate map. If X is empty the result is trivial. Otherwise, let $\mathcal{F}$ be any family of closed subsets of X with the finite intersection property. I seek to define a non-decreasing family $\langle\mathcal{F}_\xi\rangle_{\xi\le\kappa}$ of filters on X such that the image filter $\pi_\xi[[\mathcal{F}_{\xi+1}]]$ (2A1Ib) is convergent for each $\xi<\kappa$. Start with $\mathcal{F}_0$ the filter generated by $\mathcal{F}$. Given $\mathcal{F}_\xi$, let F_ξ be the set of cluster points of $\pi_\xi[[\mathcal{F}_\xi]]$; because X_ξ is compact, this is a non-empty closed subset of X_ξ, and $x_\xi=\psi(F_\xi)$ is defined. Let $\mathcal{F}_{\xi+1}$ be the filter on X generated by

$$\mathcal{F}_\xi\cup\{\pi_\xi^{-1}[U]:U\text{ is a neighbourhood of }x_\xi\text{ in }X_\xi\}.$$

For limit ordinals $\xi\le\kappa$, let $\mathcal{F}_\xi$ be the filter on X generated by $\bigcup_{\eta<\xi}\mathcal{F}_\eta$.

Now $\mathcal{F}_\kappa$ is a filter including $\mathcal{F}$ converging to $x=\langle x_\xi\rangle_{\xi<\kappa}$, and x must belong to $\bigcap\mathcal{F}$. As $\mathcal{F}$ is arbitrary, X is compact.

Remark The point of the condition 'there is a choice function for $\bigcup_{i\in I}\mathcal{E}_i$' is that it is satisfied if every X_i is the unit interval $[0,1]$, for instance; we could take $\psi(E)=\min E$ for non-empty closed sets $E\subseteq[0,1]$. You will have no difficulty in devising other examples, using the technique of the proof above, or otherwise. Note that 561C shows that there is a choice function for the family $\mathcal{E}$ of non-empty closed subsets of $\mathbb{N}^{\mathbb{N}}$, since we can use the function $F\mapsto f_F(\mathbf{0})$ where $\langle f_F\rangle_{F\in\mathcal{E}}$ is the family of functions defined there.

561E Baire's theorem (a) Let (X,ρ) be a complete metric space with a well-orderable dense subset. Then X is a Baire space.

(b) Let X be a compact Hausdorff space with a well-orderable π-base. Then X is a Baire space.

proof (a) Let D be a dense subset of X with a well-ordering $\preccurlyeq$. If $\langle G_n\rangle_{n\in\mathbb{N}}$ is a sequence of dense open subsets of X, and G is a non-empty open set, define $\langle H_n\rangle_{n\in\mathbb{N}}$, $\langle x_n\rangle_{n\in\mathbb{N}}$ and $\langle\epsilon_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. $H_0=G$. Given H_n, x_n is to be the $\preccurlyeq$-first point of H_n. Given x_n and H_n, ϵ_n is to be the first rational number in $]0, 2^{-n}]$ such that $B(x_n, \epsilon_n) \subseteq H_n$. (I leave it to you to decide which rational numbers come first.) Now set $H_{n+1} = \{y : y \in G_n,\ \rho(y, x_n) < \epsilon_n\}$; continue.

At the end of the induction, $\langle x_n \rangle_{n\in\mathbb{N}}$ is a Cauchy sequence so has a limit x in X. Since $x_n \in H_n \subseteq H_m$ whenever $m \le n$, $x \in \overline{H}_{n+2} \subseteq H_{n+1} \subseteq G_n$ for every n, and x witnesses that $G \cap \bigcap_{n\in\mathbb{N}} G_n$ is non-empty. As G and $\langle G_n \rangle_{n\in\mathbb{N}}$ are arbitrary, X is a Baire space.

(b) Let $\mathcal{U}$ be a π-base for the topology of X, not containing $\emptyset$, with a well-ordering $\preccurlyeq$. If $\langle G_n \rangle_{n\in\mathbb{N}}$ is a sequence of dense open subsets of X, and G is a non-empty open set, define $\langle U_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{U}$ inductively by saying that

U_0 is the $\preccurlyeq$-first member of $\mathcal{U}$ included in G,

U_{n+1} is the $\preccurlyeq$-first member of $\mathcal{U}$ such that $\overline{U}_{n+1} \subseteq U_n \cap G_n$

for each n. Then $\bigcap_{n\in\mathbb{N}} \overline{U}_n$ is non-empty and included in $G \cap \bigcap_{n\in\mathbb{N}} G_n$.

561F Stone's Theorem Let $\mathfrak{A}$ be a well-orderable Boolean algebra. Then there is a compact Hausdorff Baire space Z such that $\mathfrak{A}$ is isomorphic to the algebra of open-and-closed subsets of Z.

proof As in 311E, let Z be the set of ring homomorphisms from $\mathfrak{A}$ onto $\mathbb{Z}_2$. Writing $\mathbb{B}$ for the set of finite subalgebras of $\mathfrak{A}$, $Z = \bigcap_{\mathfrak{B}\in\mathbb{B}} Z_{\mathfrak{B}}$ where

$$Z_{\mathfrak{B}} = \{z : z \in \mathbb{Z}_2^{\mathfrak{A}},\ z{\restriction}\mathfrak{B} \text{ is a Boolean homomorphism}\}.$$

So Z is a closed subset of the compact Hausdorff space $\{0,1\}^{\mathfrak{A}}$, and is compact. Setting $\widehat{a} = \{z : z \in Z,\ z(a) = 1\}$, the map $a \mapsto \widehat{a}$ is a Boolean homomorphism from $\mathfrak{A}$ to the algebra $\mathfrak{E}$ of open-and-closed subsets of Z. If $a \in \mathfrak{A} \setminus \{0\}$ and $\mathfrak{B}$ is a finite subalgebra of $\mathfrak{A}$, then the subalgebra $\mathfrak{C}$ generated by $\{a\} \cup \mathfrak{B}$ is still finite, and there is a Boolean homomorphism $w : \mathfrak{C} \to \mathbb{Z}_2$ such that $w(a) = 1$; extending w arbitrarily to a member of $\{0,1\}^{\mathfrak{A}}$, we obtain a $z \in Z_{\mathfrak{B}}$ such that $z(a) = 1$; as $\mathfrak{B}$ is arbitrary, there is a $z \in Z$ such that $z(a) = 1$. So the map $a \mapsto \widehat{a}$ is injective. If $G \subseteq Z$ is open and $z \in G$, there must be a finite set $A \subseteq \mathfrak{A}$ such that G includes $\{z' : z' \in Z,\ z'{\restriction}A = z{\restriction}A\}$; in this case, setting $c = \inf\{a : a \in A,\ z(a) = 1\} \setminus \sup\{a : a \in A,\ z(a) = 0\}$, $z \in \widehat{c} \subseteq G$. It follows that any member of $\mathfrak{E}$ is of the form $\widehat{a}$ for some $a \in \mathfrak{A}$, so that $a \mapsto \widehat{a}$ is an isomorphism between $\mathfrak{A}$ and $\mathfrak{E}$.

Because $\mathfrak{A}$ is well-orderable, $\mathbb{Z}_2^{\mathfrak{A}}$ and Z have well-orderable bases, and Z is a Baire space, by 561E.

561G Haar measure Now I come to something which demands a rather less sketchy treatment.

Theorem Let X be a completely regular locally compact Hausdorff topological group.

(i) There is a non-zero left-translation-invariant positive linear functional on $C_k(X)$.

(ii) If ϕ, ϕ' are non-zero left-translation-invariant positive linear functionals on $C_k(X)$ then each is a scalar multiple of the other.

proof (a) Write Φ for $\{g : g \in C_k(X)^+,\ g(e) = \|g\|_\infty = 1\}$ where e is the identity of X. For $f \in C_k(X)^+$ and $g \in \Phi$, set

$$\lceil f : g \rceil = \inf\{\sum_{i=0}^{n} \alpha_i : \alpha_0, \dots, \alpha_n \ge 0$$

$$\text{and there are } a_0, \dots, a_n \in X \text{ such that } f \le \sum_{i=0}^{n} \alpha_i a_i \bullet_l g\},$$

writing $(a\bullet_l g)(x) = g(a^{-1}x)$ as in 4A5Cc. We have to confirm that this infimum is always defined in $[0, \infty[$.
P Set $K = \overline{\{x : f(x) > 0\}}$ and $U = \{x : g(x) > \frac{1}{2}\}$, so that K is compact, U is open and $U \ne \emptyset$. Then $K \subseteq \bigcup_{a\in X} aU$, so there are $a_0, \dots, a_n \in X$ such that $K \subseteq \bigcup_{i\le n} a_i U$. In this case

$$f \le \sum_{i=0}^{n} 2\|f\|_\infty a_i \bullet_l g$$

and $\lceil f : g \rceil \le 2(n+1)\|f\|_\infty$. **Q**

It is now easy to check that

$$\lceil a\bullet_l f : g \rceil = \lceil f : g \rceil, \quad \lceil f_1 + f_2 : g \rceil \le \lceil f_1 : g \rceil + \lceil f_2 : g \rceil,$$

$$\lceil \alpha f : g\rceil = \alpha\lceil f : g\rceil, \quad \|f\|_\infty \le \lceil f : g\rceil, \quad \lceil f : h\rceil \le \lceil f : g\rceil\lceil g : h\rceil$$

whenever f, f_1, $f_2 \in C_k(X)^+$, g, $h \in \Phi$, $a \in X$ and $\alpha \in [0,\infty[$. (Compare part (c) of the proof of 441C.)

(b) Fix $g_0 \in \Phi$, and for $g \in \Phi$ set

$$\psi_g(f) = \frac{\lceil f : g\rceil}{\lceil g_0 : g\rceil}$$

for $f \in C_k(X)$. Then

$$\psi_g(a\bullet_l f) = \psi_g(f), \quad \psi_g(f_1 + f_2) \le \psi_g(f_1) + \psi_g(f_2),$$

$$\psi_g(\alpha f) = \alpha\psi_g(f), \quad \psi_g(f) \le \lceil f : g_0\rceil,$$

$$\psi_g(f) \le \psi_g(h)\lceil f : h\rceil, \quad 1 \le \psi_g(h)\lceil g_0 : h\rceil$$

whenever f, f_1, $f_2 \in C_k(X)^+$, $h \in \Phi$, $a \in X$ and $\alpha \ge 0$. For a neighbourhood U of the identity e of X, write Φ_U for the set of those $g \in \Phi$ such that $g(x) = 0$ for every $x \in X \setminus U$; because X is locally compact and completely regular, $\Phi_U \ne \emptyset$.

(c)(i) If $f_0, \dots, f_m \in C_k(X)^+$ and $\epsilon > 0$, there is a neighbourhood U of e such that

$$\sum_{j=0}^m \psi_g(f_j) \le \psi_g(\sum_{j=0}^m f_j) + \epsilon$$

whenever $g \in \Phi_U$. **P** Set $f = \sum_{j=0}^m f_j$. Let K be the compact set $\overline{\{x : f(x) \ne 0\}}$, and let $\hat f \in C_k(X)$ be such that $\chi K \le \hat f$. Let $\eta > 0$ be such that

$$(1 + (m+1)\eta)(\psi_g(f) + \eta\lceil \hat f : g_0\rceil) \le \psi_g(f) + \epsilon,$$

and set $f^* = f + \eta\hat f$. Then we can express each f_j as $f^* \times h_j$ where $h_j \in C_k(X)^+$ and $\sum_{j=0}^m h_j \le \chi X$. Let U be a neighbourhood of e such that $|h_j(x) - h_j(y)| \le \eta$ whenever $x^{-1}y \in U$ and $j \le m$ (compare 4A5Pa).

Take $g \in \Phi_U$. Let $\alpha_0, \dots, \alpha_n \ge 0$ and $a_0, \dots, a_n \in X$ be such that $f^* \le \sum_{i=0}^n \alpha_i a_i\bullet_l g$ and $\sum_{i=0}^n \alpha_i \le \lceil f^* : g\rceil + \eta$. Then, for any $x \in X$ and $j \le m$,

$$f_j(x) = f^*(x)h_j(x) \le \sum_{i=0}^n \alpha_i g(a_i^{-1}x)h_j(x) \le \sum_{i=0}^n \alpha_i g(a_i^{-1}x)(h_j(a_i) + \eta)$$

because if i is such that $g(a_i^{-1}x) \ne 0$ then $a_i^{-1}x \in U$ and $h_j(x) \le h_j(a_i)+\eta$. So $\lceil f_j : g\rceil \le \sum_{i=0}^n \alpha_i(h_j(a_i)+\eta)$. Summing over j,

$$\sum_{j=0}^m \lceil f_j : g\rceil \le \sum_{i=0}^n \alpha_i(1 + (m+1)\eta)$$

because $\sum_{j=0}^m h_j(a_i) \le 1$ for every i. As $\alpha_0, \dots, \alpha_n$ and $a_0, \dots, a_n$ are arbitrary,

$$\sum_{j=0}^m \lceil f_j : g\rceil \le (1 + (m+1)\eta)\lceil f^* : g\rceil \le (1 + (m+1)\eta)(\lceil f : g\rceil + \eta\lceil \hat f : g\rceil),$$

and

$$\begin{aligned}\sum_{j=0}^m \psi_g(f_i) &\le (1 + (m+1)\eta)(\psi_g(f) + \eta\psi_g(\hat f))\\ &\le (1 + (m+1)\eta)(\psi_g(f) + \eta\lceil \hat f : g_0\rceil) \le \psi_g(f) + \epsilon\end{aligned}$$

as required. **Q**

(ii) If $f_0, \dots, f_m \in C_k(X)^+$, $M \ge 0$ and $\epsilon > 0$, there is a neighbourhood U of e such that

$$\sum_{j=0}^m \psi_g(\gamma_j f_j) \le \psi_g(\sum_{j=0}^m \gamma_j f_j) + \epsilon$$

whenever $g \in \Phi_U$ and $\gamma_0, \dots, \gamma_m \in [0, M]$. **P** Let $\eta > 0$ be such that $\eta(1 + \sum_{j=0}^m \lceil f_j : g_0\rceil) \le \epsilon$. By (i), applied finitely often, there is a neighbourhood U of e such that

$$\sum_{j=0}^m \psi_g(\gamma_j f_j) \le \psi_g(\sum_{j=0}^m \gamma_j f_j) + \eta$$

whenever $g \in \Phi_U$ and $\gamma_0, \dots, \gamma_m \in [0, M]$ are multiples of η. Now, given arbitrary $\gamma_0, \dots, \gamma_m \in [0, M]$ and $g \in \Phi_U$, let $\gamma'_0, \dots, \gamma'_m$ be multiples of η such that $\gamma'_j \le \gamma_j < \gamma'_j + \eta$ for each j. Then

$$\sum_{j=0}^{m}\psi_g(\gamma_j f_j)\le\sum_{j=0}^{m}\psi_g(\gamma'_j f_j)+\eta\psi_g(f_j)$$
$$\le\psi_g(\sum_{j=0}^{m}\gamma'_j f_j)+\eta+\eta\sum_{j=0}^{m}\psi_g(f_j)\le\psi_g(\sum_{j=0}^{m}\gamma_j f_j)+\epsilon$$

as required. **Q**

(d) (We are coming to the magic bit.) Suppose that $f\in C_k(X)^+$, $\epsilon>0$ and that U is a neighbourhood of e such that $|f(x)-f(y)|\le\epsilon$ whenever $x^{-1}y\in U$. Then if $g\in\Phi_U$ and $\gamma>\epsilon$ there are $\alpha_0,\dots,\alpha_n\ge 0$ and $a_0,\dots,a_n\in X$ such that $\|f-\sum_{i=0}^{n}\alpha_i a_i{\bullet_l}g\|_\infty\le\gamma$. **P** For all x, $y\in X$ we have

$$(f(x)-\epsilon)g(x^{-1}y)\le f(y)g(x^{-1}y)\le(f(x)+\epsilon)g(x^{-1}y).$$

Let $\eta>0$ be such that $\eta(1+\lceil f:\overleftrightarrow{g}\rceil)\le\gamma-\epsilon$, where $\overleftrightarrow{g}(x)=g(x^{-1})$ for $x\in X$. Let V be an open neighbourhood of e such that $|g(x)-g(y)|\le\eta$ whenever $xy^{-1}\in V$. Then we have $a_0,\dots,a_n$ such that $\bigcup_{i\le n}a_iV\supseteq\overline{\{x:f(x)\ne 0\}}$, and $h_0,\dots,h_n\in C_k(X)^+$ such that $\sum_{i=0}^{n}h_i(x)=1$ whenever $f(x)>0$, while $h_i(x)=0$ if $i\le n$ and $x\notin a_iV$. By (c-ii), there is an $h\in\Phi$ such that

$$\textstyle\sum_{i=0}^{n}\psi_h(\gamma_i f\times h_i)\le\psi_h(\sum_{i=0}^{n}\gamma_i f\times h_i)+\eta$$

whenever $0\le\gamma_i\le\lceil g_0:\overleftrightarrow{g}\rceil$ for each i.

Now, for $i\le n$ and x, $y\in X$,

$$h_i(y)f(y)(g(a_i^{-1}x)-\eta)\le h_i(y)f(y)g(y^{-1}x)\le h_i(y)f(y)(g(a_i^{-1}x)+\eta).$$

Accordingly

$$(f(x)-\epsilon)(x{\bullet_l}\overleftrightarrow{g})(y)=(f(x)-\epsilon)g(y^{-1}x)\le f(y)g(y^{-1}x)=\sum_{i=0}^{n}h_i(y)f(y)g(y^{-1}x)$$
$$\le\sum_{i=0}^{n}h_i(y)f(y)(g(a_i^{-1}x)+\eta)=\eta f(y)+\sum_{i=0}^{n}h_i(y)f(y)g(a_i^{-1}x);$$

similarly,

$$(f(x)+\epsilon)(x{\bullet_l}\overleftrightarrow{g})(y)\ge\textstyle\sum_{i=0}^{n}h_i(y)f(y)g(a_i^{-1}x)-\eta f(y).$$

Fixing x for the moment, and applying the functional ψ_h to the expressions here (regarded as functions of y), we get

$$(f(x)-\epsilon)\psi_h(\overleftrightarrow{g})\le\eta\psi_h(f)+\psi_h(\textstyle\sum_{i=0}^{n}g(a_i^{-1}x)f\times h_i)$$

so

$$f(x)-\gamma\le f(x)-\epsilon-\eta\lceil f:\overleftrightarrow{g}\rceil\le f(x)-\epsilon-\eta\frac{\psi_h(f)}{\psi_h(\overleftrightarrow{g})}$$
$$\le\psi_h(\sum_{i=0}^{n}\frac{g(a_i^{-1}x)}{\psi_h(\overleftrightarrow{g})}f\times h_i)\le\sum_{i=0}^{n}\frac{g(a_i^{-1}x)}{\psi_h(\overleftrightarrow{g})}\psi_h(f\times h_i)=\sum_{i=0}^{n}\alpha_i g(a_i^{-1}x)$$

where $\alpha_i=\frac{\psi_h(f\times h_i)}{\psi_h(\overleftrightarrow{g})}$. On the other side,

$$(f(x)+\epsilon)\psi_h(\overleftrightarrow{g})+\eta\psi_h(f)\ge\psi_h(\textstyle\sum_{i=0}^{n}g(a_i^{-1}x)f\times h_i),$$

so

$$f(x)+\gamma \ge f(x)+\epsilon+\eta\frac{\psi_h(f)}{\psi_h(\overleftrightarrow{g})}+\eta$$
$$\ge \psi_h(\sum_{i=0}^n \frac{g(a_i^{-1}x)}{\psi_h(\overleftrightarrow{g})} f\times h_i)+\eta \ge \sum_{i=0}^n \frac{g(a_i^{-1}x)}{\psi_h(\overleftrightarrow{g})}\psi_h(f\times h_i)$$

(because $\frac{g(a_i^{-1}x)}{\psi_h(\overleftrightarrow{g})} \le \lceil g_0 : \overleftrightarrow{g}\rceil$ for every i)

$$=\sum_{i=0}^n \alpha_i g(a_i^{-1}x).$$

All this is valid for every $x \in X$; so

$$\|f-\sum_{i=0}^n \alpha_i a_i \bullet_l g\|_\infty \le \gamma. \quad \mathbf{Q}$$

(e) For any $f \in C_k(X)^+$ and $\epsilon > 0$ there are a $\gamma \ge 0$ and a neighbourhood U of e such that $|\psi_h(f)-\gamma| \le \epsilon$ for every $h \in \Phi_U$. **P** Let V be a compact neighbourhood of 0 and $K=\overline{\{x : f(x)+g_0(x)>0\}}$; let $f^* \in C_k(X)$ be such that $\chi(KV^{-1}V) \le f^* \le \chi X$. Let δ, $\eta > 0$ be such that

$$\delta(1+2(\delta+\lceil f : g_0\rceil)) \le \epsilon, \quad \delta \le \tfrac12, \quad \eta(1+\lceil f^* : g_0\rceil) \le \delta.$$

By (d), there are $g \in \Phi_V$, $\alpha_0,\dots,\alpha_n,\beta_0,\dots,\beta_m \ge 0$ and $a_0,\dots,a_n,b_0,\dots,b_n \in X$ such that

$$\|f-\sum_{i=0}^n \alpha_i a_i \bullet_l g\|_\infty \le \eta, \quad \|g_0-\sum_{j=0}^m \beta_j b_j \bullet_l g\|_\infty \le \eta.$$

We can suppose that all the a_i, b_j belong to KV^{-1}, since $g(a^{-1}x)=0$ if $x \in K$ and $a \notin KV^{-1}$; consequently

$$|f-\sum_{i=0}^n \alpha_i a_i \bullet_l g| \le \eta f^*, \quad |g_0-\sum_{j=0}^m \beta_j b_j \bullet_l g| \le \eta f^*.$$

Set $\alpha=\sum_{i=0}^n \alpha_i$, $\beta=\sum_{j=0}^m \beta_j$ and $\gamma=\frac{\alpha}{\beta}$. ($\beta$ is non-zero because $\|g_0\|_\infty=1$ and $\eta\|f^*\|_\infty \le \frac12$.)

Let $U \subseteq V$ be a neighbourhood of e such that

$$\sum_{i=0}^n \alpha_i \psi_h(a_i \bullet_l g) \le \psi_h(\sum_{i=0}^n \alpha_i a_i \bullet_l g)+\eta,$$
$$\sum_{j=0}^m \beta_j \psi_h(b_j \bullet_l g) \le \psi_h(\sum_{j=0}^m \beta_j b_j \bullet_l g)+\eta$$

for every $h \in \Phi_U$ ((c) above). Take any $h \in \Phi_U$. Then

$$|\psi_h(f)-\alpha\psi_h(g)| = |\psi_h(f)-\sum_{i=0}^n \alpha_i\psi_h(a_i \bullet_l g)| \le |\psi_h(f)-\psi_h(\sum_{i=0}^n \alpha_i a_i \bullet_l g)|+\eta$$
$$\le \eta\psi_h(f^*)+\eta \le \eta\lceil f^* : g_0\rceil+\eta;$$

similarly,

$$|1-\beta\psi_h(g)| = |\psi_h(g_0)-\beta\psi_h(g)| \le \eta(\lceil f^* : g_0\rceil+1).$$

But this means that

$$|\psi_h(f)-\gamma| \le \eta(1+\lceil f^* : g_0\rceil)+|\alpha\psi_h(g)-\gamma|$$
$$\le \delta+\gamma|\beta\psi_h(g)-1| \le \delta(1+\gamma).$$

Consequently

$$\gamma \le \tfrac{\psi_h(f)+\delta}{1-\delta} \le 2(\delta+\lceil f : g_0\rceil),$$

$$|\psi_h(f)-\gamma| \le \delta(1+2(\delta+\lceil f : g_0\rceil)) \le \epsilon,$$

as required. **Q**

(f) We are nearly home. Let $\mathcal{F}$ be the filter on Φ generated by $\{\Phi_U : U \text{ is a neighbourhood of } e\}$. By (e), $\phi(f)=\lim_{h\to\mathcal{F}}\psi_h(f)$ is defined for every $f \in C_k(X)^+$. From the formulae in (b) we have

$$\phi(a\bullet_l f) = \phi(f), \quad \phi(f_1 + f_2) \le \phi(f_1) + \phi(f_2), \quad \phi(\alpha f) = \alpha\phi(f)$$

whenever f, f_1, $f_2 \in C_k(X)^+$, $a \in X$ and $\alpha \ge 0$. By (c-i), we have $\phi(f_1) + \phi(f_2) \le \phi(f_1 + f_2)$ for all f_1, $f_2 \in C_k(X)^+$. So ϕ is additive and extends to an invariant positive linear functional on $C_k(X)$ which is non-zero because $\phi(g_0) = 1$.

(g) As for uniqueness, we can repeat the arguments in (e). Suppose that ϕ' is another left-translation-invariant positive linear functional on $C_k(X)$ such that $\phi'(g_0) = 1$, and $f \in C_k(X)^+$. Let K be the closure of $\{x : f(x) + g_0(x) > 0\}$ and V a compact neighbourhood of e; let $f^* \in C_k(X^*)$ be such that $\chi(KV^{-1}V) \le f^*$. Take $\epsilon > 0$. Let δ, $\eta > 0$ be such that

$$\delta \le \frac{1}{2}, \quad \delta(1 + 2(\phi(f) + 1)) \le \epsilon, \quad \delta(1 + 2(\phi'(f) + 1)) \le \epsilon,$$

$$\eta\phi(f^*) \le \delta, \quad \eta\phi'(f^*) \le \delta.$$

Then there is a neighbourhood U of e, included in V, such that $|f(x) - f(y)| \le \eta$ and $|g_0(x) - g_0(y)| \le \eta$ whenever $x^{-1}y \in U$. By (d), there are $g \in \Phi_V$, $\alpha_0, \dots, \alpha_n, \beta_0, \dots, \beta_m \ge 0$ and $a_0, \dots, a_n, b_0, \dots, b_m \in X$ such that

$$|f(x) - \sum_{i=0}^n \alpha_i (a_i\bullet_l g)(x)| \le \eta, \quad |g_0(x) - \sum_{j=0}^m \beta_j (b_j\bullet_l g)(x)| \le \eta$$

for every $x \in X$; as in (e), we may suppose that every a_i, b_j belongs to KV^{-1} so that

$$|f - \sum_{i=0}^n \alpha_i a_i\bullet_l g| \le \eta f^*, \quad |g_0 - \sum_{j=0}^m \beta_j b_j\bullet_l g| \le \eta f^*.$$

Consequently, setting $\alpha = \sum_{i=0}^n \alpha_i$, $\beta = \sum_{i=0}^n \beta_i$ and $\gamma = \alpha/\beta$,

$$|\phi(f) - \alpha\phi(g)| = |\phi(f) - \sum_{i=0}^n \alpha_i\phi(a_i\bullet_l g)| \le \eta\phi(f^*) \le \delta,$$

$$|1 - \beta\phi(g)| = |\phi(g_0) - \sum_{j=0}^m \beta_j\phi(b_j\bullet_l g)| \le \eta\phi(f^*) \le \delta.$$

So

$$|\phi(f) - \gamma| \le \eta\phi(f^*) + \gamma|\beta\phi(g) - 1| \le \eta\phi(f^*)(1 + \gamma) \le \delta(1 + \gamma)$$

and

$$\gamma \le \tfrac{\phi(f)+\delta}{1-\delta} \le 2(\phi(f) + 1),$$

$$|\phi(f) - \gamma| \le \delta(1 + 2(\phi(f) + \delta)) \le \epsilon.$$

Similarly, $|\phi'(f) - \gamma| \le \epsilon$ and $|\phi(f) - \phi'(f)| \le 2\epsilon$. As ϵ and f are arbitrary, $\phi = \phi'$.

561H Kakutani's theorem (a) Let U be an Archimedean Riesz space with a weak order unit. Then there are a Dedekind complete Boolean algebra $\mathfrak{A}$ and an order-dense Riesz subspace of $L^0(\mathfrak{A})$, containing $\chi 1$, which is isomorphic to U.

(b) Let U be an L-space with a weak order unit e. Then there is a totally finite measure algebra $(\mathfrak{A}, \bar\mu)$ such that U is isomorphic, as normed Riesz space, to $L^1(\mathfrak{A}, \bar\mu)$, and we can choose the isomorphism to match e with $\chi 1$.

proof All the required ideas are in Volume 3; but we have quite a lot of checking to do.

(a)(i) The first step is to observe that, for any Dedekind σ-complete Boolean algebra $\mathfrak{A}$, the definition of $L^0 = L^0(\mathfrak{A})$ in 364A gives no difficulties, and that the formulae of 364D can be used to define a Riesz space structure on L^0. **P** I recall the formulae in question:

$$[\![u > \alpha]\!] = \sup_{\beta > \alpha} [\![u > \beta]\!] \text{ for every } \alpha \in \mathbb{R},$$

$$\inf_{\alpha\in\mathbb{R}} [\![u > \alpha]\!] = 0, \quad \sup_{\alpha\in\mathbb{R}} [\![u > \alpha]\!] = 1,$$

$$[\![u + v > \alpha]\!] = \sup_{q\in\mathbb{Q}} [\![u > q]\!] \cap [\![v > \alpha - q]\!],$$

whenever u, $v \in L^0$ and $\alpha \in \mathbb{R}$,

$$[\![\gamma u > \alpha]\!] = [\![u > \tfrac{\alpha}{\gamma}]\!]$$

whenever $u \in L^0$, $\gamma \in \,]0,\infty[$ and $\alpha \in \mathbb{R}$. The distributive laws in 313A-313B are enough to ensure that $u+v$ and γu, so defined, belong to L^0, and also that $u+v=v+u$, $u+(v+w)=(u+v)+w$, $\gamma(u+v)=\gamma u+\gamma v$ for u, v, $w \in L^0$ and $\gamma > 0$. Defining $\mathbf{0} \in L^0$ by saying that

$$[\![\mathbf{0} > \alpha]\!] = 1 \text{ if } \alpha < 0, \ 0 \text{ if } \alpha \ge 0,$$

we can check that $u+0=u$ for every u. Defining $-u \in L^0$ by saying that

$$[\![-u > \alpha]\!] = \sup_{q\in\mathbb{Q},q>\alpha} 1 \setminus [\![u > -q]\!]$$

for $u \in L^0$ and $\alpha \in \mathbb{R}$, we find (again using the distributive laws, of course) that $u+(-u)=\mathbf{0}$; we can now define γu, for $\gamma \le 0$, by saying that $0 \cdot u = \mathbf{0}$ and $\gamma u = (-\gamma)(-u)$ if $\gamma < 0$, and we shall have a linear space. Turning to the ordering, it is nearly trivial to check that the definition

$$u \le v \iff [\![u > \alpha]\!] \subseteq [\![v > \alpha]\!] \text{ for every } \alpha \in \mathbb{R}$$

gives us a partially ordered linear space. It is a Riesz space because the formula

$$[\![u \vee v > \alpha]\!] = [\![u > \alpha]\!] \cup [\![v > \alpha]\!]$$

defines a member of L^0 which must be $\sup\{u,v\}$ in L^0. **Q**

We need to know that if $\mathfrak{A}$ is Dedekind complete, so is L^0; the argument of 364M still applies. Note also that $a \mapsto \chi a : \mathfrak{A} \to L^0$ is order-continuous, as in 364Jc.

(ii) Now suppose that U is an Archimedean Riesz space with an order unit e. Let $\mathfrak{A}$ be the band algebra of U (353B). Then we can argue as in 368E, but with the simplification that the maximal disjoint set C in $U^+ \setminus \{0\}$ is just $\{e\}$, to see that we have an injective Riesz homomorphism $T : U \to L^0(\mathfrak{A})$ defined by taking $[\![Tu > \alpha]\!]$ to be the band generated by $e \wedge (u-\alpha e)^+$ (or, if you prefer, by $(u-\alpha e)^+$, since it comes to the same thing). We shall have $T[U]$ order-dense, as before, with $Te = \chi 1$.

(b)(i) Again, the bit we have to concentrate on is the check that, starting from a totally finite measure algebra $(\mathfrak{A},\bar\mu)$, we can define $L^1(\mathfrak{A},\bar\mu)$ as in 365A. We have to be a bit careful, because already in Proposition 321C there is an appeal to AC(ω) (see 561Yi(vi)); but I think we need to know very little about measure algebras to get through the arguments here. Of course another difficulty arises at once in 365A, because I write

$$\|u\|_1 = \int_0^\infty \bar\mu[\![|u| > \alpha]\!]\,d\alpha,$$

and say that the integration is with respect to 'Lebesgue measure', which won't do, at least until I redefine Lebesgue integration as in §565. But we are integrating a monotonic function, so the integral can be thought of as an improper Riemann integral; if you like,

$$\|u\|_1 = \lim_{n\to\infty} 2^{-n}\sum_{i=1}^{4^n} \bar\mu[\![|u| > 2^{-n}i]\!] = \sup_{n\in\mathbb{N}} 2^{-n}\sum_{i=1}^{4^n} \bar\mu[\![|u| > 2^{-n}i]\!].$$

Next, we can't use the Loomis-Sikorski theorem to prove 365C, and have to go back to first principles. To see that $\| \|_1$ is subadditive, and additive on $(L^0)^+$, look first at 'simple' non-negative u, expressed as $u = \sum_{i=0}^n \alpha_i \chi a_i$, and check that $\int u = \|u\|_1 = \sum_{i=0}^n \alpha_i \bar\mu a_i$; now confirm that every element of $(L^0)^+$ is expressible as the supremum of a non-decreasing sequence of such elements, and that $\| \|_1$ is sequentially order-continuous on the left on $(L^0)^+$. (We need 321Be.) This is enough to show that L^1 is a solid linear subspace of L^0 with a Riesz norm and a sequentially order-continuous integral. (I do *not* claim, yet, that L^1 is an L-space, because I do not know, in the absence of countable choice, that every Cauchy filter on L^1 converges.)

(ii) Now let U be an L-space with a weak order unit e. As in (a), let $\mathfrak{A}$ be the band algebra of U and $T : U \to L^0$ an injective Riesz homomorphism onto an order-dense Riesz subspace of L^0 with $Te = \chi 1$. Now U is Dedekind complete (354N, 354Ee). Consequently $T[U]$ must be solid in L^0 (353L[1]).

(iii) For $a \in \mathfrak{A}$, set $\bar\mu a = \|T^{-1}(\chi a)\|$. Because the map $a \mapsto T^{-1}\chi a$ is additive and order-continuous and injective, $(\mathfrak{A},\bar\mu)$ is a measure algebra; indeed, $\bar\mu$ is actually order-continuous. So we have a space

[1]Formerly 353K.

$L^1 = L^1(\mathfrak{A},\bar\mu)$. Because $\bar\mu$ is order-continuous, 364L(a-ii) tells us that $\|w\|_1 = \sup_{v\in B}\|v\|_1$ whenever $B \subseteq L^0$ is a non-empty upwards-directed set in L^0 with supremum w in L^0.

Writing $S \subseteq L^0$ for the linear span of $\{\chi a : a \in \mathfrak{A}\}$, we see that $\|w\|_1 = \|T^{-1}w\|$ for every $w \in S$. Since S is order-dense in L^0 it is order-dense in L^1, and $T^{-1}[S]$ is order-dense in U, therefore norm-dense (354Ef).

(iv) $Tu \in L^1$ for every $u \in U^+$. **P** For $n \in \mathbb{N}$ set $a_n = [\![Tu > 2^n]\!] \setminus [\![Tu > 2^{n+1}]\!]$, $u_n = T^{-1}(\chi a_n)$. Set $w_n = \sum_{i=0}^n 2^i\chi a_i$ for $n \in \mathbb{N}$. Then $w_n \le Tu$ and $\|w_n\|_1 = \|T^{-1}w_n\| \le \|u\|$ for every n. By 364L(a-i), $w = \sup_{n\in\mathbb{N}} w_n$ is defined in L^0, and $\|w\|_1 = \sup_{n\in\mathbb{N}}\|w_n\|_1$ is finite. But $Tu \le 2w + \chi 1$, so $Tu \in L^1$. **Q**

(v) If $w \in (L^1)^+$ there is a $v \in U^+$ such that $w = Tv$ and $\|v\| = \|w\|_1$. **P** Consider $A = \{u : u \in T^{-1}[S],\ Tu \le w\}$. This is upwards-directed and norm-bounded, so has a supremum v in U (354N again), and $Tv \ge w'$ whenever $w' \in S$ and $w' \le w$. But S is order-dense in L^0 so $Tv \ge w$. Because T is order-continuous, (iii) tells us that

$$\|Tv\|_1 = \sup_{u\in A}\|Tu\|_1 = \sup_{u\in A}\|u\| = \|v\|,$$

while surely $\|w\|_1 \ge \sup_{u\in A}\|Tu\|_1$. So $Tv = w$. **Q**

(vi) Putting (iv) and (v) together, we see that $T[U] = L^1$ and that T is a normed Riesz space isomorphism, as required.

561I Hilbert spaces: Proposition Let U be a Hilbert space.

(a) If $C \subseteq U$ is a non-empty closed convex set then for any $u \in U$ there is a unique $v \in C$ such that $\|u - v\| = \min\{\|u - w\| : w \in C\}$.

(b) Every closed linear subspace of U is the image of an orthogonal projection, that is, has an orthogonal complement.

(c) Every member of U^* is of the form $u \mapsto (u|v)$ for some $v \in U$.

(d) U is reflexive.

(e) If $C \subseteq U$ is a norm-closed convex set then it is weakly closed.

proof (a) Set $\gamma = \inf\{\|u - w\| : w \in C\}$ and let $\mathcal{F}$ be the filter on U generated by sets of the form $F_\epsilon = \{w : w \in C,\ \|u - w\| \le \gamma + \epsilon\}$ for $\epsilon > 0$. Then $\mathcal{F}$ is Cauchy. **P** Suppose that $\epsilon > 0$ and w_1, $w_2 \in F_\epsilon$. Then

$$\|w_1 - w_2\|^2 = 2\|u - w_1\|^2 + 2\|u - w_2\|^2 - \|2u - w_1 - w_2\|^2 \le 4(\gamma + \epsilon)^2 - 4\gamma^2$$

(because $\frac{1}{2}(w_1 + w_2) \in C$)

$$= 8\gamma\epsilon + 4\epsilon^2.$$

So

$$\inf_{F\in\mathcal{F}} \operatorname{diam} F = \inf_{\epsilon>0} \operatorname{diam} F_\epsilon = 0. \text{ **Q**}$$

We therefore have a limit v of $\mathcal{F}$, which is in C because C is closed, and $\|u - v\| = \lim_{w\to\mathcal{F}}\|u - w\| = \gamma$. If now w is any other member of C, $\|u - \frac{1}{2}(v + w)\| \ge \gamma$ so $\|u - w\| > \gamma$.

(b) Let V be a closed linear subspace of U. By (a), we have a function $P : U \to V$ such that Pu is the unique closest element of V to u, that is, $\|u - Pu\| \le \|u - Pu + \alpha v\|$ for every $v \in V$ and $\alpha \in \mathbb{R}$. It follows that $(u - Pu|v) = 0$ for every $v \in V$, that is, that $u - Pu \in V^\perp$. As u is arbitrary, $U = V + V^\perp$; as $V \cap V^\perp = \{0\}$, P must be the projection onto V with kernel $V^\perp$, and is an orthogonal projection.

(c) Take $f \in U^*$. If $f = 0$ then $f(u) = (u|0)$ for every u. Otherwise, set $C = \{w : f(w) = 0\}$. Then C is a proper closed linear subspace of U. Take any $u_0 \in U \setminus C$. Let v_0 be the point of C nearest to u_0, and consider $u_1 = u_0 - v_0$. Then 0 is the point of C nearest to u_1, so that $(u|u_1) = 0$ for every $u \in C$. Set $v = \frac{f(u_1)}{\|u_1\|^2}u_1$; then $(u|v) = 0$ for every $u \in C$, while $f(v) = (v|v)$. So $f(u) = (u|v)$ for every $u \in U$.

(d) From (c) it follows that we can identify U^* with U and therefore U^{**} also becomes identified with U.

(e) If C is empty this is trivial. Otherwise, take any $u \in U \setminus C$. Let v be the point of C nearest to U, and set $f(w) = (w|u - v)$ for $w \in U$. Then $f(w) \le f(v) < f(u)$ for every $w \in C$. So u does not belong to the weak closure of C; as u is arbitrary, C is weakly closed.

561X Basic exercises (a) Let X be any set. (i) Show that $\ell^p(X)$, for $1 \le p \le \infty$, is a Banach space. (ii) Show that $\ell^1(X)^*$ can be identified with $\ell^\infty(X)$. (iii) Show that if $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ then $\ell^p(X)^*$ can be identified with $\ell^q(X)$.

(b) Let X be any topological space. Show that $C_b(X)$, with $\| \|_\infty$, is a Banach space.

(c) Suppose that there is an infinite subset X of $\mathbb{R}$ with no infinite countable subset (Jech 73, §10.1). Show that X is sequentially closed but not closed, second-countable but not separable, sequentially compact but not compact, sequentially complete (that is, every Cauchy sequence converges) but not complete. Show that the topology of $\mathbb{R}$ is not countably tight.

>(d) (i) Let C be the set of those $R \subseteq \mathbb{N} \times \mathbb{N}$ which are total orderings of subsets of $\mathbb{N}$. Show that C is a closed subset of $\mathcal{P}(\mathbb{N} \times \mathbb{N})$ with its usual topology. (ii) For $\xi < \omega_1$, let C_ξ be the set of those $R \in C$ which are well-orderings of order type ξ of subsets of $\mathbb{N}$. Show that C_ξ is a Borel subset of $\mathcal{P}(\mathbb{N} \times \mathbb{N})$. (*Hint*: induce on ξ.) (iii) Show that there is an injective function from ω_1 to the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$.

(e)(i) Show that every non-empty closed subset of $\mathbb{N}^\mathbb{N}$ has a lexicographically-first member. (ii) Show that if a T_1 topological space X is a continuous image of $\mathbb{N}^\mathbb{N}$, then there is an injection from X to $\mathcal{P}\mathbb{N}$.

(f) Let X be a topological space. (i) Show that if X is separable, then $X^\mathbb{R}$ is separable. (ii) Show that if X has a countable network, then $X^\mathbb{R}$ has a countable network.

(g)(i) Show that a locally compact Hausdorff space is regular. (ii) Show that a compact regular space is normal.

(h) Let U be a normed space with a well-orderable subset D such that the linear span of D is dense in U. (i) Show that if V is a linear subspace of U and $f \in V^*$, there is a $g \in U^*$, extending f, with the same norm as f. (ii) Show that the unit ball B of U^* is weak*-compact and has a well-orderable base for its topology. (iii) Show that if $K \subseteq B$ is weak*-closed then K has an extreme point.

(i) Let (X, ρ) be a separable compact metric space, and G the isometry group of X with its topology of pointwise convergence (441G). Show that G is compact. (*Hint*: $X^\mathbb{N}$ is compact.)

(j) Let X be a regular topological space and A a subset of X. Show that the following are equiveridical: (i) A is relatively compact in X; (ii) $\overline{A}$ is compact; (iii) every filter on X containing A has a cluster point.

>(k) Let $(X, \mathfrak{T})$ be a regular second-countable topological space, and write $\mathfrak{S}$ for the usual topology on $\mathbb{R}^\mathbb{N}$. (i) Show that there are a continuous function $f : X \to \mathbb{R}^\mathbb{N}$ and a function $\phi : \mathfrak{T} \to \mathfrak{S}$ such that $G = f^{-1}[\phi(G)]$ for every $G \in \mathfrak{T}$. (ii) Show that if X is Hausdorff it is metrizable.

>(l) Let X be a regular second-countable topological space, $\mathcal{C}$ the family of closed subsets of X, and $\mathcal{D}$ the set of disjoint pairs $(F_0, F_1) \in \mathcal{C} \times \mathcal{C}$. (i) Show that X is normal, and that there is a function $\psi : \mathcal{D} \to \mathcal{C}$ such that $F_0 \subseteq \operatorname{int} \psi(F_0, F_1)$ and $F_1 \cap \psi(F_0, F_1) = \emptyset$ whenever $(F_0, F_1) \in \mathcal{D}$. (ii) Show that there is a function $\phi : \mathcal{D} \to C(X)$ such $\phi(F_0, F_1)(x) = 0$, $\phi(F_0, F_1)(y) = 1$ whenever $(F_0, F_1) \in \mathcal{D}$, $x \in F_0$ and $y \in F_1$.

(m) Let X be a well-orderable discrete abelian group. Show that its dual group, as defined in 445A, is a completely regular compact Hausdorff group.

(n) Let U be a Riesz space with a Riesz norm. Let $\Delta : U^+ \to [0, \infty[$ be such that (α) Δ is non-decreasing, (β) $\Delta(\alpha u) = \alpha \Delta(u)$ whenever $u \in U^+$ and $\alpha \ge 0$, (γ) $\Delta(u + v) = \Delta(u) + \Delta(v)$ whenever $u \wedge v = 0$ (δ) $|\Delta(u) - \Delta(v)| \le \|u - v\|$ for all u, $v \in U^+$. Show that Δ has an to a member of U^*.

(o) Let U be an L-space. Show that $\|u\| = \sup\{f(u) : f \in U^*, \|f\| \le 1\}$ for every $u \in U$.

(p) Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra. Show that $L^1(\mathfrak{A}, \bar\mu)$ is a Dedekind σ-complete Riesz space and a sequentially complete normed space.

(q) Let $\mathfrak{A}$ be a Boolean algebra. Show that there are a set X, an algebra $\mathcal{E}$ of subsets of X and a surjective Boolean homomorphism from $\mathcal{E}$ onto $\mathfrak{A}$. (*Hint*: 566L.)

(r) Let U be a Hilbert space. (i) Show that a bounded sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in U is weakly convergent in U iff $\lim_{n\to\infty}(u_n|u_m)$ is defined for every $m \in \mathbb{N}$. (ii) Show that the unit ball of U is sequentially compact for the weak topology. (iii) Show that if $T : U \to U$ is a self-adjoint compact linear operator, then $T[U]$ is included in the closed linear span of $\{Tv : v \text{ is an eigenvector of } T\}$. (*Hint*: reduce to the case in which U is separable, and show that there is then a sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in the unit ball B of U such that $\lim_{n\to\infty}(Tu_n|u_n) = \sup_{u\in B}(Tu|u)$.)

(s) In 561C, show that $(F, \alpha) \mapsto f_F(\alpha) : \mathcal{E} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is continuous if $\mathcal{E}$ is given its Vietoris topology (4A2T) and $\mathbb{N}^{\mathbb{N}}$ its usual topology.

561Y Further exercises (a) Suppose that there is a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of countable sets such that $\bigcup_{n\in\mathbb{N}} A_n = \mathbb{R}$. Show that $\operatorname{cf}\omega_1 = \omega$.

(b)(i) Show that there is a bijection between ω_1 and $\mathbb{N} \times \omega_1$. (ii) Show that ω_2 is not expressible as the union of a sequence of countable sets. (iii) Show that $\mathcal{P}\omega_1$ is not expressible as the union of a sequence of countable sets. (iv) Show that $\mathcal{P}(\mathcal{P}\mathbb{N})$ is not expressible as the union of a sequence of countable sets.

(c) Suppose there is a countable family of doubleton sets with no choice function (Jech 73, §5.5). Show that (i) there is a set I such that $\{0,1\}^I$ has an open-and-closed set which is not determined by coordinates in any countable subset of I (ii) there is a compact metrizable space which is not ccc, therefore not second-countable (iii) there is a complete totally bounded metric space which is neither ccc nor compact (iv) there is a probability algebra which is not ccc.

(d) Let X be a metrizable space. Show that it is second-countable iff it has a countable π-base iff it has a countable network.

(e)(i) Let (X, ρ) be a complete metric space. Show that X has a well-orderable dense subset iff it has a well-orderable base iff it has a well-orderable π-base iff it has a well-orderable network iff there is a choice function for the family of its non-empty closed subsets. (ii) Let X be a locally compact Hausdorff space. (α) Show that if it has a well-orderable π-base then it has a well-orderable dense subset. (β) Show that if it has a well-orderable base then it is completely regular and there is a choice function for the family of its non-empty closed subsets.

(f) Let X be a metrizable space. Show that every continuous real-valued function defined on a closed subset of X has an extension to a continuous real-valued function on X.

(g)(i) Show that if $\mathfrak{A}$ is a Boolean algebra, there is an essentially unique Dedekind complete Boolean algebra $\widehat{\mathfrak{A}}$ in which $\mathfrak{A}$ can be embedded as an order-dense subalgebra. (ii) Show that if $\mathfrak{A}$ and $\mathfrak{B}$ are two Boolean algebras and $\pi : \mathfrak{A} \to \mathfrak{B}$ is an order-continuous Boolean homomorphism, π has a unique extension to an order-continuous Boolean homomorphism from $\widehat{\mathfrak{A}}$ to $\widehat{\mathfrak{B}}$. (*Hint*: take $\widehat{\mathfrak{A}}$ to be the set of pairs (A, A') of subsets of $\mathfrak{A}$ such that A is the set of lower bounds of A' and A' is the set of upper bounds of A.)

(h) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras. (i) Show that there is an essentially unique structure $(\mathfrak{A}, \langle\varepsilon_i\rangle_{i\in I})$ such that (α) $\mathfrak{A}$ is a Boolean algebra (β) $\varepsilon_i : \mathfrak{A}_i \to \mathfrak{A}$ is a Boolean homomorphism for every i (γ) whenever $\mathfrak{B}$ is a Boolean algebra and $\phi_i : \mathfrak{A}_i \to \mathfrak{B}$ is a Boolean homomorphism for every i, there is a unique Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ such that $\pi\varepsilon_i = \phi_i$ for every i. (ii) Show that if $\nu_i : \mathfrak{A}_i \to \mathbb{R}$ is additive, with $\nu_i 1 = 1$, for every $i \in I$, there is a unique additive $\nu : \mathfrak{A} \to \mathbb{R}$ such that $\nu(\inf_{i\in J}\varepsilon_i(a_i)) = \prod_{i\in J}\nu_i a_i$ whenever $J \subseteq I$ is finite and $a_i \in \mathfrak{A}_i$ for $i \in J$.

(i) Suppose that there is an infinite totally ordered set I with no countably infinite subset (Jech 73, §10.1). Let $\mathcal{E}$ be the algebra of subsets of $\{0,1\}^I$ determined by coordinates in finite sets. (i) Show that the union of any countable family of finite subsets of I is finite. (ii) Show that $\mathcal{E}$ has no countably infinite subset, so that every finitely additive real-valued functional on $\mathcal{E}$ is countably additive. (iii) Show that there is no infinite disjoint family in $\mathcal{E}$. (iv) Show that $\mathcal{E}$ is Dedekind complete. (v) Show that there is a functional μ_1 such that $(\mathcal{E},\mu_1)$ is a probability algebra and μ_1 is order-continuous. (vi) Show that there is a functional μ_2 such that $(\mathcal{E},\mu_2)$ is a probability algebra and μ_2 is not order-continuous.

(j) Let (X,ρ) be a complete metric space with a well-orderable base. Show that a subset of X is compact iff it is sequentially compact iff it is closed and totally bounded.

561 Notes and comments The arguments of this section will I hope give an idea of the kind of discipline which will be imposed for the rest of the chapter. Apart from anything else, we have to fix on the correct definitions. Typically, when defining something like 'compactness' or 'completeness', the definition to use is that which is most useful in the most general context; so that even in metrizable spaces we should prefer filters to sequences (cf. 561Xc).

We can distinguish two themes in the methods I have used here. First, in the presence of a well-ordering we can hope to adapt the standard attack on a problem; see 561D-561F. Second, if (in the presence of the axiom of choice) there is a *unique* solution to a problem, then we can hope that it is still a unique solution without choice. This is what happens in 561G and also in 561Ia-561Ic. In 561I we just go through the usual arguments with a little more care. In 561G (taken from Naimark 70) we need new ideas. But in the key step, part (d) of the proof, the two variables x and y reflect an adaptation of a repeated-integration argument as in §442. Note that the scope of 561G may be limited if we have fewer locally compact groups than we expect.

A regular second-countable Hausdorff space is metrizable (561Yf). But it may not be separable (561Xc). We do not have Urysohn's Lemma in its usual form, so cannot be sure that a locally compact Hausdorff space is completely regular; a topological group has left, right and bilateral uniformities, but a uniformity need not be defined by pseudometrics and a uniform space need not be completely regular. So in such results as 561G we may need an extra 'completely regular' in the hypotheses.

I give a version of Kakutani's theorem (561H) to show that some of the familiar patterns are distorted in possibly unexpected ways, and that occasionally it is the more abstract parts of the theory which survive best. I suppose I ought to remark explicitly that I define 'measure algebra' exactly as in 321A: a Dedekind σ-complete Boolean algebra with a strictly positive countably additive $[0,\infty]$-valued functional. I do not claim that every σ-finite measure algebra is either localizable or ccc (561Yc), nor that every measure algebra can be represented in terms of a measure space. I set up a construction of a normed Riesz space $L^1(\mathfrak{A},\bar\mu)$, but do not claim that this is an L-space. However, if we start from an L-space U with a weak order unit, we can build a measure on its band algebra and proceed to an $L^1(\mathfrak{A},\bar\mu)$ which is isomorphic to U (and is therefore an L-space).

Version of 20.10.13

562 Borel codes

The concept of 'Borel set', either in the real line or in general topological spaces, has been fundamental in measure theory since before the modern subject existed. It is at this point that the character of the subject changes if we do not allow ourselves even the countable axiom of choice. I have already mentioned the Feferman-Lévy model in which $\mathbb{R}$ is a countable union of countable sets; immediately, every subset of $\mathbb{R}$ is a countable union of countable sets and is 'Borel' on the definition of 111G. In these circumstances that definition becomes unhelpful.

An alternative which leads to a non-trivial theory, coinciding with the usual theory in the presence of AC, is the algebra of 'codable Borel sets' (562B). This is not necessarily a σ-algebra, but is closed under unions and intersections of 'codable sequences' (562K). When we come to look for measurable functions, the corresponding concept is that of 'codable Borel function' (562L); again, we do not expect the limit of an arbitrary sequence of codable Borel functions to be measurable in any useful sense, but the limit of a codable sequence of codable Borel functions is again a codable Borel function (562Ne). The same ideas can be used to give a theory of 'codable Baire sets' in any topological space (562T).

562A Trees I review some ideas from §421.

(a) Set $S^* = \bigcup_{n\ge 1}\mathbb{N}^n$. For $\sigma \in S^*$ and $T \subseteq S^*$, write T_σ for $\{\tau : \tau \in S^*,\, \sigma^\frown\tau \in T\}$ (notation: 5A1C).

(b) Let $\mathcal{T}_0$ be the family of sets $T \subseteq S^*$ such that $\sigma\restriction n \in T$ whenever $\sigma \in T$ and $n \ge 1$. Recall from 421N[2] that we have a derivation $\partial : \mathcal{T}_0 \to \mathcal{T}_0$ defined by setting

$$\partial T = \{\sigma : \sigma \in S^*,\, T_\sigma \ne \emptyset\},$$

with iterates ∂^ξ, for $\xi < \omega_1$, defined by setting

$$\partial^0 T = T, \quad \partial^\xi T = \bigcap_{\eta<\xi}\partial(\partial^\eta T) \text{ for } \xi \ge 1.$$

Now for any $T \in \mathcal{T}_0$ there is a $\xi < \omega_1$ such that $\partial^\xi T = \partial^\eta T$ whenever $\xi \le \eta < \omega_1$. **P** The argument in 421Nd assumed that ω_1 has uncountable cofinality, but we can avoid this assumption, as follows. Let $\langle\epsilon_\sigma\rangle_{\sigma\in S^*}$ be a summable family of strictly positive real numbers, and set $\gamma_T(\xi) = \sum_{\sigma\in\partial^\xi T}\epsilon_\sigma$; then $\gamma_T : \omega_1 \to [0,\infty[$ is non-increasing, so 561A tells us that there is a $\xi < \omega_1$ such that $\gamma_T(\xi+1) = \gamma_T(\xi)$, that is, $\partial^{\xi+1}T = \partial^\xi T$. Of course we now have $\partial^\eta T = \partial^\xi T$ for every $\eta \ge \xi$. **Q**

(c) We therefore still have a rank function $r : \mathcal{T}_0 \to \omega_1$ defined by saying that $r(T)$ is the least ordinal such that $\partial^{r(T)}T = \partial^{r(T)+1}T$. Now $\partial^{r(T)}T$ is empty iff there is no $\alpha \in \mathbb{N}^\mathbb{N}$ such that $\alpha\restriction n \in T$ for every $n \ge 1$. **P** The argument in 421Nf used the word 'choose'; but we can avoid this by being more specific. If $\sigma \in \partial^{r(T)}T$, then we can define a sequence $\langle\sigma_n\rangle_{n\in\mathbb{N}}$ by saying that $\sigma_0 = \sigma$ and, given $\sigma_n \in \partial^{r(T)}T$, $\sigma_{n+1} = \sigma_n{}^\frown{<}i{>}$ for the least i such that $\sigma_n{}^\frown{<}i{>} \in \partial^{r(T)}T$; $\alpha = \bigcup_{n\in\mathbb{N}}\sigma_n$ will now have $\alpha\restriction n \in T$ for every $n \ge 1$. The argument in the other direction is unchanged. **Q**

Let $\mathcal{T}$ be the set of those $T \in \mathcal{T}_0$ with no infinite branch, that is, such that $\partial^{r(T)}T = \emptyset$. Note that if $T \in \mathcal{T}$ then $r(T) = 0$ iff $T = \emptyset$, while $r(T) = 1$ iff there is a non-empty set $A \subseteq \mathbb{N}$ such that $T = \{{<}i{>} : i \in A\}$.

(d) For $T \in \mathcal{T}$, set $A_T = \{i : {<}i{>} \in T\}$. We need a fact not covered in §421: for any $T \in \mathcal{T}$, $r(T) = \sup\{r(T_{<i>}) + 1 : i \in A_T\}$. **P** An easy induction on ξ shows that $\partial^\xi(T_\sigma) = (\partial^\xi T)_\sigma$ for any $\xi < \omega_1$, $T \in \mathcal{T}_0$ and $\sigma \in S^*$. So, for $T \in \mathcal{T}$ and $\xi < \omega_1$,

$$\begin{aligned} r(T) > \xi &\Longrightarrow \partial^\xi T \ne \emptyset \\ &\Longrightarrow \exists\, i,\, {<}i{>} \in \partial^\xi T = \bigcap_{\eta<\xi}\partial^{\eta+1}T \\ &\Longrightarrow \exists\, i \in A_T,\, \partial^\eta(T_{<i>}) = (\partial^\eta T)_{<i>} \ne \emptyset\ \forall\, \eta < \xi \\ &\Longrightarrow \exists\, i \in A_T,\, r(T_{<i>}) > \eta\ \forall\, \eta < \xi \\ &\Longrightarrow \exists\, i \in A_T,\, r(T_{<i>}) \ge \xi; \end{aligned}$$

thus $r(T) \le \sup\{r(T_{<i>}) + 1 : i \in A_T\}$. In the other direction, if $i \in A_T$ and $\eta < \xi = r(T_{<i>})$, then

$$(\partial^\eta T)_{<i>} = \partial^\eta(T_{<i>}) \ne \emptyset,$$

so ${<}i{>} \in \partial^{\eta+1}T$; as η is arbitrary, ${<}i{>} \in \partial^\xi T$ and $\xi < r(T)$; as i is arbitrary, $r(T) \ge \sup\{r(T_{<i>}) + 1 : i \in A_T\}$. **Q**

562B Coding sets with trees (a) Let X be a set and $\langle E_n\rangle_{n\in\mathbb{N}}$ a sequence of subsets of X. Define $\phi : \mathcal{T} \to \mathcal{P}X$ inductively by saying that

$$\begin{aligned} \phi(T) &= \bigcup_{i\in A_T} E_i \text{ if } r(T) \le 1, \\ &= \bigcup_{i\in A_T} X \setminus \phi(T_{<i>}) \text{ if } r(T) > 1. \end{aligned}$$

By 562Ad, this definition is sound. I will call ϕ the **interpretation of Borel codes** defined by X and $\langle E_n\rangle_{n\in\mathbb{N}}$.

[2]Early editions of Volume 4 used a slightly different definition of iterated derivations, so that the 'rank' of a tree was not quite the same.

(b) Of course $\phi(\emptyset)=\emptyset$. If we set

$$T^*=\{<0>,<0>^\frown<0>,<0>^\frown<0>^\frown<0>,<1>,<1>^\frown<0>\}$$

and

$$T=\{<0>\}\cup\{<0>^\frown\sigma:\sigma\in T^*\},$$

then

$$\begin{aligned}\phi(T^*)&=(X\setminus\phi(T^*_{<0>}))\cup(X\setminus\phi(T^*_{<1>}))\\&=(X\setminus\phi(\{<0>,<0>^\frown<0>\}))\cup(X\setminus\phi(\{<0>\}))\\&=(X\setminus(X\setminus\phi(\{<0>\})))\cup(X\setminus\phi(\{<0>\}))=X,\\\phi(T)&=X\setminus\phi(T^*)=\emptyset,\end{aligned}$$

while $T\ne\emptyset$, which it will be useful to know.

(c) Now suppose that X is a second-countable topological space and that $\langle U_n\rangle_{n\in\mathbb{N}}$, $\langle V_n\rangle_{n\in\mathbb{N}}$ are two sequences running over bases for the topology of X. Let $\phi:\mathcal{T}\to\mathcal{P}X$ and $\phi':\mathcal{T}\to\mathcal{P}X$ be the interpretations of Borel codes defined by $\langle U_n\rangle_{n\in\mathbb{N}}$, $\langle V_n\rangle_{n\in\mathbb{N}}$ respectively. Then there is a function $\Theta:\mathcal{T}\to\mathcal{T}\setminus\{\emptyset\}$ such that $\phi'\Theta=\phi$. **P** Define Θ inductively, as follows. If $r(T)\le 1$, then $\phi(T)=\bigcup_{i\in A_T}U_i$ is open. If $\phi(T)\ne\emptyset$, set $\Theta(T)=\{<j>:j\in\mathbb{N},\,V_j\subseteq\phi(T)\}$; then

$$\phi'(\Theta(T))=\bigcup\{V_j:j\in\mathbb{N},\,V_j\subseteq\phi(T)\}=\phi(T).$$

If $\phi(T)=\emptyset$, take $\Theta(T)$ to be any non-empty member of $\mathcal{T}$ such that $\phi'(\Theta(T))=\emptyset$; e.g., that presented in (b) just above.

For the inductive step to $r(T)>1$, set

$$\Theta(T)=\{<i>:i\in A_T\}\cup\{<i>^\frown\sigma:i\in A_T,\,\sigma\in\Theta(T_{<i>})\};$$

then $r(\Theta(T))>1$ and

$$\begin{aligned}\phi'(\Theta(T))&=\bigcup_{i\in A_{\Theta(T)}}X\setminus\phi'(\Theta(T)_{<i>})=\bigcup_{i\in A_T}X\setminus\phi'(\Theta(T_{<i>}))\\&=\bigcup_{i\in A_T}X\setminus\phi(T_{<i>})=\phi(T),\end{aligned}$$

so the induction continues. **Q**

(There will be a substantial strengthening of this idea in 562Ma.)

(d) Now say that a **codable Borel set** in X is one expressible as $\phi(T)$ for some $T\in\mathcal{T}$, starting from some sequencen running over a base for the topology of X; in view of (c), we can restrict our calculations to a fixed enumeration of a fixed base if we wish. I will write $\mathcal{B}_c(X)$ for the family of codable Borel sets of X.

The definition of 'interpretation of Borel codes' makes it plain that any σ-algebra of subsets of X containing every open set will also contain every codable Borel set; so every codable Borel set is indeed a 'Borel set' on the definition of 111G or 4A3A.

As in the argument for (c) just above, it will sometimes be useful to know that every element of $\mathcal{B}_c(X)$ can be coded by a non-empty member of $\mathcal{T}$; we have only to check the case of the empty set, which is dealt with in the formula in (b).

562C The point of these codings is that we can define explicit functions on $\mathcal{T}$ which will have appropriate reflections in the coded sets.

(a) For instance, there are functions $\Theta_0:\mathcal{T}\to\mathcal{T}$, $\Theta_1:\mathcal{T}\times\mathcal{T}\to\mathcal{T}$, $\Theta_2:\mathcal{T}\times\mathcal{T}\to\mathcal{T}$, $\Theta_3:\mathcal{T}\times\mathcal{T}\to\mathcal{T}$ such that, for any interpretation ϕ of Borel codes,

$$\phi(\Theta_0(T))=X\setminus\phi(T),\quad \phi(\Theta_1(T,T'))=\phi(T)\cup\phi(T'),$$

$$\phi(\Theta_2(T,T'))=\phi(T)\cap\phi(T'),\quad \phi(\Theta_3(T,T'))=\phi(T)\setminus\phi(T')$$

for all T, $T'\in\mathcal{T}$. **P** Let T^* be the tree described in 562Bb, so that $\phi(T^*)=X$. Set

$$\begin{aligned}\Theta_0(T)&=T^* \text{ if } T=\emptyset,\\ &=\{<0>\}\cup\{<0>^\frown\sigma:\sigma\in T\} \text{ otherwise;}\end{aligned}$$

then

$$\begin{aligned}\phi(\Theta_0(T))=\phi(T^*)=X&=X\setminus\phi(T) \text{ if } T=\emptyset,\\ &=X\setminus\phi(\Theta_0(T)_{<0>})=X\setminus\phi(T) \text{ otherwise.}\end{aligned}$$

Now set

$$\Theta_1(T,T')=\{<0>,<1>\}\cup\{<0>^\frown\sigma:\sigma\in\Theta_0(T)\}\cup\{<1>^\frown\sigma:\sigma\in\Theta_0(T')\},$$

so that

$$\begin{aligned}\phi(\Theta_1(T,T'))&=(X\setminus\phi(\Theta_1(T,T'))_{<0>})\cup(X\setminus\phi(\Theta_1(T,T'))_{<1>})\\ &=(X\setminus\phi(\Theta_0(T)))\cup(X\setminus\phi(\Theta_0(T')))\\ &=(X\setminus(X\setminus\phi(T)))\cup(X\setminus(X\setminus\phi(T')))=\phi(T)\cup\phi(T').\end{aligned}$$

So we can take

$$\Theta_2(T,T')=\Theta_0(\Theta_1(\Theta_0(T),\Theta_0(T'))),\quad \Theta_3(T,T')=\Theta_2(T,\Theta_0(T'))$$

and get

$$\phi(\Theta_2(T,T'))=X\setminus((X\setminus\phi(T))\cup(X\setminus\phi(T')))=\phi(T)\cap\phi(T'),$$

$$\phi(\Theta_3(T,T'))=\phi(T)\cap(X\setminus\phi(T'))=\phi(T)\setminus\phi(T'). \textbf{ Q}$$

(b) We can find codes for unions and intersections of sequences, provided the sequences are presented in the right way; I give a general formulation of the process. For any countable set K we have functions $\tilde\Theta_1$, $\tilde\Theta_2:\bigcup_{J\subseteq K}\mathcal{T}^J\to\mathcal{T}$ such that whenever X is a set, $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence of subsets of X and ϕ is the corresponding interpretation of Borel codes, then $\phi(\tilde\Theta_1(\tau))=\bigcup_{j\in J}\phi(\tau(j))$ and $\phi(\tilde\Theta_2(\tau))=X\cap\bigcap_{j\in J}\phi(\tau(j))$ whenever $J\subseteq K$ and $\tau\in\mathcal{T}^J$. **P** Let $\langle k_n\rangle_{n\in\mathbb{N}}$ be a sequence running over $K\cup\{\emptyset\}$. For $J\subseteq K$ and $\tau\in\mathcal{T}^J$, set $A=\{n:k_n\in J,\,\tau(k_n)\ne\emptyset\}$ and

$$\tilde\Theta_1(\tau)=\{<n>:n\in A\}\cup\{<n>^\frown\sigma:n\in A,\,\sigma\in\Theta_0(\tau(k_n))\},$$

$$\tilde\Theta_2(\tau)=\Theta_0(\tilde\Theta_1(\langle\Theta_0(\tau(j))\rangle_{j\in J})).$$

Then

$$\begin{aligned}\phi(\tilde\Theta_1(\tau))=\emptyset&=\bigcup_{j\in J}\phi(\tau(j)) \text{ if } A=\emptyset,\\ &=\bigcup_{n\in A}X\setminus\phi(\Theta_0(\tau(k_n)))=\bigcup_{n\in A}\phi(\tau(k_n))=\bigcup_{j\in J}\phi(\tau_j) \text{ otherwise,}\\ \phi(\tilde\Theta_2(\tau))&=X\setminus\bigcup_{j\in J}(X\setminus\phi(\tau_j))=X\cap\bigcap_{j\in J}\phi(\tau_j).\textbf{Q}\end{aligned}$$

(c) A more sophisticated version of two of the codings in (a) will be useful in §564. Let X be a regular second-countable space, $\langle U_n\rangle_{n\in\mathbb{N}}$ a sequence running over a base for the topology of X containing $\emptyset$, and $\phi:\mathcal{T}\to\mathcal{P}X$ the associated interpretation of Borel codes. Then there are functions Θ_1', $\Theta_2':\mathcal{T}\times\mathcal{T}\to\mathcal{T}$ such that

$$\phi(\Theta_1'(T,T'))=\phi(T)\cup\phi(T'),\quad \phi(\Theta_2'(T,T'))=\phi(T)\cap\phi(T'),$$

$$r(\Theta_1'(T,T'))=r(\Theta_2'(T,T'))=\max(r(T),r(T'))$$

for all T, $T' \in \mathcal{T}$. **P** The point is just that open sets in a regular second-countable space are F_σ. Because of the slightly awkward form taken by the definition of ϕ, we need to start with an auxiliary function. Define $T \mapsto \tilde{T} : \mathcal{T} \to \mathcal{T}$ by saying that

$$\begin{aligned}\tilde{T} &= \{<\!n\!> : \overline{U}_n \subseteq \phi(T)\} \cup \{<\!n\!>^\frown<\!i\!> : \overline{U}_n \subseteq \phi(T),\, U_i \cap U_n = \emptyset\} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{if } r(T) \le 1, \\ &= T \text{ otherwise .}\end{aligned}$$

Then $\phi(\tilde{T}) = \phi(T)$ and $r(\tilde{T}) = \max(2, r(T))$ for every T (because if $r(T) \le 1$ there is some n such that $U_n = \emptyset$ and $<\!n\!>^\frown<\!n\!> \in \tilde{T}$). Note that $\tilde{\tilde{T}} = \tilde{T}$. We also need to fix a bijection $n \mapsto (i_n, j_n)$ between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$.

Now define Θ'_1 by saying that

— if $\max(r(T), r(T')) \le 1$, $\Theta'_1(T, T') = \Theta'_1(T) \cup \Theta'_1(T')$;

— if $\max(r(T), r(T')) > 1$, then

$$\begin{aligned}\Theta'_1(T, T') &= \{<\!2n\!> : n \in A_{\tilde{T}}\} \cup \{<\!2n\!>^\frown\sigma : \sigma \in \tilde{T}_{<n>}\} \\ &\qquad \cup \{<\!2n+1\!> : n \in A_{\tilde{T}'}\} \cup \{<\!2n+1\!>^\frown\sigma : \sigma \in \tilde{T}'_{<n>}\}.\end{aligned}$$

For Θ'_2 induce on $\max(r(T), r(T'))$:

— if $T = T' = \emptyset$, $\Theta'_2(T, T') = \emptyset$;

— if $\max(r(T), r(T')) = 1$,

$$\Theta'_2(T, T') = \{<\!n\!> : U_n \subseteq \phi(T) \cap \phi(T')\};$$

— if $\max(r(T), r(T')) > 1$, set $A = \{n : i_n \in A_{\tilde{T}},\, j_n \in A_{\tilde{T}'}\}$ and

$$\Theta'_2(T, T') = \{<\!n\!> : n \in A\} \cup \{<\!n\!>^\frown\sigma : n \in A,\, \sigma \in \Theta'_1(\tilde{T}_{<i_n>}, \tilde{T}'_{<j_n>})\}$$

(interpreting $\tilde{T}'_{<j_n>}$ as $((T')^\sim)_{<j_n>}$).

These formulae work. I run through the calculations for $\Theta'_2(T, T')$ when $\max(r(T), r(T')) > 1$. We have $r(\tilde{T}) \ge 2$ and $r(\tilde{T}') \ge 2$, so $A_{\tilde{T}}$, $A_{\tilde{T}'}$ and A are non-empty,

$$\begin{aligned} r(\Theta'_2(T, T')) &= \sup_{n \in A} r(\Theta'_1(\tilde{T}_{<i_n>}, \tilde{T}'_{<j_n>})) + 1 = \sup_{i \in A_{\tilde{T}}, j \in A_{\tilde{T}'}} r(\Theta'_1(\tilde{T}_{<i>}, \tilde{T}'_{<j>})) + 1 \\ &= \sup_{i \in A_{\tilde{T}}, j \in A_{\tilde{T}'}} \max(r(\tilde{T}_{<i>}), r(\tilde{T}'_{<j>})) + 1 \\ &= \max(\sup_{i \in A_{\tilde{T}}} r(\tilde{T}_{<i>}) + 1, \sup_{j \in A_{\tilde{T}'}} r(\tilde{T}'_{<j>}) + 1) \\ &= \max(r(\tilde{T}), r(\tilde{T}')) = \max(2, r(T), r(T')) = \max(r(T), r(T'))\end{aligned}$$

and

$$\begin{aligned}\phi(\Theta'_2(T, T')) &= \bigcup_{n \in A} X \setminus \phi(\Theta'_1(\tilde{T}_{<i_n>}, \tilde{T}'_{<j_n>})) \\ &= \bigcup_{n \in A} X \setminus \phi(\Theta'_1(\tilde{\tilde{T}}_{<i_n>}, \tilde{\tilde{T}}'_{<j_n>})) = \bigcup_{n \in A} X \setminus (\phi(\tilde{T}_{<i_n>}) \cup \phi(\tilde{T}'_{<j_n>})) \\ &= \bigcup_{n \in A} (X \setminus \phi(\tilde{T}_{<i_n>})) \cap (X \setminus \phi(\tilde{T}'_{<j_n>})) \\ &= \bigcup_{i \in A_{\tilde{T}}, j \in A_{\tilde{T}'}} (X \setminus \phi(\tilde{T}_{<i_n>})) \cap (X \setminus \phi(\tilde{T}'_{<j_n>})) \\ &= (\bigcup_{i \in A_{\tilde{T}}} X \setminus \phi(\tilde{T}_{<i_n>})) \cap (\bigcup_{j \in A_{\tilde{T}'}} X \setminus \phi(\tilde{T}'_{<j_n>})) \\ &= \phi(\tilde{T}) \cap \phi(\tilde{T}') = \phi(T) \cap \phi(T'). \ \mathbf{Q}\end{aligned}$$

562D Proposition (a) If X is a second-countable space, then the family of codable Borel subsets of X is an algebra of subsets of X containing every G_δ set and every F_σ set.

(b) [AC(ω)] Every Borel set is a codable Borel set.

proof (a) Let $\langle U_n\rangle_{n\in\mathbb{N}}$ be a sequence running over a base for the topology of X, and $\phi:\mathcal{T}\to\mathcal{B}_c(X)$ the corresponding surjection. From 562Ca we see that $X\setminus E$ and $E\cup E'$ belong to $\mathcal{B}_c(X)$ for all E, $E'\in\mathcal{B}_c(X)$; since $\emptyset=\phi(\emptyset)$ belongs to $\mathcal{B}_c(X)$, $\mathcal{B}_c(X)$ is an algebra of subsets of X.

If $E\subseteq X$ is an F_σ set, there is a sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ of closed sets with union E. Set

$$T=\{{<}n{>}:n\in\mathbb{N}\}\cup\{{<}n{>}^\frown{<}i{>}:n,\,i\in\mathbb{N},\,U_i\subseteq X\setminus F_n\}.$$

Then $r(T)=2$, $\phi(T_{<n>})=X\setminus F_n$ for every n and $\phi(T)=E$.

Thus every F_σ set belongs to $\mathcal{B}_c(X)$; it follows at once that every G_δ set is also a codable Borel set.

(b) We can repeat the argument in (a), but this time in a more general form. If $\langle E_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\mathcal{B}_c(X)$, then for each $n\in\mathbb{N}$ **choose** $T^{(n)}\in\mathcal{T}\setminus\{\emptyset\}$ such that $\phi(T^{(n)})=E_n$; set

$$T=\{{<}n{>}:n\in\mathbb{N}\}\cup\{{<}n{>}^\frown\sigma:n\in\mathbb{N},\,\sigma\in T^{(n)}\};$$

then $\bigcup_{n\in\mathbb{N}}X\setminus E_n=\phi(T)$ is a codable Borel set. Because $\mathcal{B}_c(X)$ is an algebra, this is enough to show that it is a σ-algebra and therefore equal to the σ-algebra $\mathcal{B}(X)$.

562E Proposition Let X be a second-countable space and $Y\subseteq X$ a subspace of X. Then $\mathcal{B}_c(Y)=\{Y\cap E:E\in\mathcal{B}_c(X)\}$.

proof Let $\langle U_n\rangle_{n\in\mathbb{N}}$ be a sequence running over a base for the topology of X, and set $V_n=Y\cap U_n$ for each n; let $\phi_X:\mathcal{T}\to\mathcal{B}_c(X)$ and $\phi_Y:\mathcal{T}\to\mathcal{B}_c(Y)$ be the interpretations of Borel codes corresponding to $\langle U_n\rangle_{n\in\mathbb{N}}$, $\langle V_n\rangle_{n\in\mathbb{N}}$ respectively. Then an easy induction on the rank of T shows that $\phi_Y(T)=Y\cap\phi_X(T)$ for every $T\in\mathcal{T}$. So

$$\mathcal{B}_c(Y)=\phi_Y[\mathcal{T}]=\{Y\cap\phi_X(T):T\in\mathcal{T}\}=\{Y\cap E:E\in\mathcal{B}_c(X)\}.$$

***562F** I do not expect to rely on the next result, but it is interesting that two of the basic facts of descriptive set theory have versions in the new context.

Theorem (a) If X is a Hausdorff second-countable space and A, B are disjoint analytic subsets of X, there is a codable Borel set $E\subseteq X$ such that $A\subseteq E$ and $B\cap E=\emptyset$.

(b) Let X be a Polish space. Then a subset E of X is a codable Borel set iff E and $X\setminus E$ are analytic.

proof (a)(i) If either A or B is empty, this is trivial, just because $\emptyset$ and X are codable Borel sets; so suppose otherwise. Let $f:\mathbb{N}^\mathbb{N}\to X$ and $g:\mathbb{N}^\mathbb{N}\to X$ be continuous functions such that $f[\mathbb{N}^\mathbb{N}]=A$ and $g[\mathbb{N}^\mathbb{N}]=B$. Fix an enumeration $\langle(j_n,k_n)\rangle_{n\in\mathbb{N}}$ of $\mathbb{N}\times\mathbb{N}$, and a sequence $\langle U_n\rangle_{n\in\mathbb{N}}$ running over a base for the topology of X; let ϕ be the interpretation of Borel codes defined by $\langle U_n\rangle_{n\in\mathbb{N}}$. For $\sigma\in S=\bigcup_{n\in\mathbb{N}}\mathbb{N}^n$ set

$$A_\sigma=\{f(\alpha):\alpha\in\mathbb{N}^\mathbb{N},\,\alpha_i=j_{\sigma(i)}\text{ for }i<\#(\sigma)\},$$

$$B_\sigma=\{g(\alpha):\alpha\in\mathbb{N}^\mathbb{N},\,\alpha_i=k_{\sigma(i)}\text{ for }i<\#(\sigma)\}.$$

Then $A_\emptyset=A$ and $A_\sigma=\bigcup_{i\in\mathbb{N}}A_{\sigma^\frown<i>}$ for every σ, and similarly for B.

(ii) Still in the setting-up stage, we need general union and intersection operators on $\mathcal{T}$. As in 562Cb, let $\tilde\Theta_1$, $\tilde\Theta_2:\bigcup_{J\subseteq\mathbb{N}}\mathcal{T}^J\to\mathcal{T}$ be such that $\phi(\tilde\Theta_1(\tau))=\bigcup_{j\in J}\phi(\tau(j))$ and $\phi(\tilde\Theta_2(\tau))=X\cap\bigcap_{j\in J}\phi(\tau(j))$ whenever $J\subseteq\mathbb{N}$ and $\tau\in\mathcal{T}^J$.

(iii) Set

$$T=\{\sigma:\sigma\in S^*,\text{ there are no }i,\,n\in\mathbb{N}\text{ such that }n<\#(\sigma)\text{ and }A_{\sigma\restriction n}\subseteq U_i\subseteq X\setminus B_{\sigma\restriction n}\}.$$

If $\sigma\in S$ and $n\in\mathbb{N}$ then $A_{\sigma\restriction n}\supseteq A_\sigma$ and $B_{\sigma\restriction n}\supseteq B_\sigma$, so $\sigma\restriction m\in T$ whenever $\sigma\in T$ and $m\ge 1$; thus T belongs to $\mathcal{T}_0$ as defined in 562Ab. In fact $T\in\mathcal{T}$. **P?** Otherwise, by 562Ac, there is a $\gamma\in\mathbb{N}^\mathbb{N}$ such that

$\gamma\restriction n \in T$ for every $n \ge 1$. Set $\alpha = \langle\gamma_{j_n}\rangle_{n\in\mathbb{N}}$, $\beta = \langle\gamma_{k_n}\rangle_{n\in\mathbb{N}}$; then $f(\alpha) \in A$ and $g(\beta) \in B$, so $f(\alpha) \ne g(\beta)$. Because X is Hausdorff, there are i, $j \in \mathbb{N}$ such that $f(\alpha) \in U_i$, $g(\beta) \in U_j$ and $U_i \cap U_j = \emptyset$. Because f and g are continuous, there is an $n \ge 1$ such that $f(\alpha') \in U_i$ and $g(\beta') \in U_j$ whenever α', $\beta' \in \mathbb{N}^{\mathbb{N}}$, $\alpha'\restriction n = \alpha\restriction n$ and $\beta'\restriction n = \beta\restriction n$; that is, such that $A_{\gamma\restriction n} \subseteq U_i$ and $B_{\gamma\restriction n} \subseteq U_j$. But this means that $\gamma\restriction n+1 \notin T$. **XQ**

(iv) We know that $\langle\partial^\xi T\rangle_{\xi\le r(T)}$ is a non-increasing family in $\mathcal{T}$ with last member $\emptyset$, and moreover that $\partial^\xi T = \bigcap_{\eta<\xi}\partial^\eta T$ for non-zero limit ordinals $\xi \le r(T)$. So for $\sigma \in T$ there is a unique $h(\sigma) < r(T)$ such that $\sigma \in \partial^{h(\sigma)}T \setminus \partial(\partial^{h(\sigma)}T)$. I seek to define $T^{(\sigma)} \in \mathcal{T}$, for $\sigma \in T$, such that $A_\sigma \subseteq \phi(T^{(\sigma)}) \subseteq X \setminus B_\sigma$ for every σ. I do this inductively.

(v) If $h(\sigma) = 0$, that is, $\sigma \in T\setminus\partial T$, then $\sigma^\frown 0 \notin T$. So there is a first $i \in \mathbb{N}$ such that $A_\sigma \subseteq U_i \subseteq X\setminus B_\sigma$. Set $T^{(\sigma)} = \{<i>\}$, so that $\phi(T^{(\sigma)}) = U_i$ includes A_σ and is disjoint from B_σ.

(vi) Now suppose that we have $\xi \le r(t)$ such that $\xi \ge 1$ and $T^{(\sigma)}$ has been defined for every $\sigma \in T$ with $h(\sigma) < \xi$. Take $\sigma \in T$ such that $h(\sigma) = \xi$; then $\sigma^\frown{<n>} \in T$ for some, therefore every, $n \in \mathbb{N}$, while $h(\sigma^\frown{<n>}) < \xi$ and $T^{(\sigma^\frown<n>)}$ is defined for every $n \in \mathbb{N}$. Now, for each n, we have $A_{\sigma^\frown<n>} \subseteq \phi(T^{(\sigma^\frown<n>)})$. But of course $A_{\sigma^\frown<n>} = A_{\sigma^\frown<m>}$ whenever $j_m = j_n$. So we have $A_{\sigma^\frown<n>} \subseteq \bigcap_{j_m=j_n}\phi(T^{(\sigma^\frown<m>)})$ for each n, and

$$A_\sigma \subseteq \bigcup_{n\in\mathbb{N}}\bigcap_{j_m=j_n}\phi(T^{(\sigma^\frown<m>)}) = \bigcup_{j\in\mathbb{N}}\bigcap_{j_m=j}\phi(T^{(\sigma^\frown<m>)}).$$

Similarly, $B_{\sigma^\frown<n>} = B_{\sigma^\frown<m>}$ is disjoint from $\phi(T^{(\sigma^\frown<m>)})$ whenever $k_m = k_n$, so

$$B_\sigma = \bigcup_{k\in\mathbb{N}}\bigcap_{k_m=k}B_{\sigma^\frown<m>}$$

is disjoint from $\bigcap_{k\in\mathbb{N}}\bigcup_{k_m=k}\phi(T^{(\sigma^\frown<m>)})$.

On the other hand, for any j, $k \in \mathbb{N}$, there is a $p \in \mathbb{N}$ such that $j_p = j$ and $k_p = k$, so that

$$\bigcap_{j_m=j}\phi(T^{(\sigma^\frown<m>)}) \subseteq \phi(T^{(\sigma^\frown<p>)}) \subseteq \bigcup_{k_m=k}\phi(T^{(\sigma^\frown<m>)}).$$

But this means that $\bigcup_{j\in\mathbb{N}}\bigcap_{j_m=j}\phi(T^{(\sigma^\frown<m>)}) \subseteq \bigcap_{k\in\mathbb{N}}\bigcup_{k_m=k}\phi(T^{(\sigma^\frown<m>)})$ is disjoint from B_σ.

If therefore we set

$$T^{(\sigma)} = \tilde\Theta_1(\langle\tilde\Theta_2(\langle T^{(\sigma^\frown<m>)}\rangle_{j_m=j})\rangle_{j\in\mathbb{N}})$$

we shall have $A_\sigma \subseteq \phi(T^{(\sigma)}) \subseteq X \setminus B_\sigma$, and we have a formula defining a suitable tree $T^{(\sigma)}$ whenever $\sigma \in T$ and $h(\sigma) = \xi$, so we can continue the induction.

(vii) This gives us a family $\langle T^{(\sigma)}\rangle_{\sigma\in T}$ in $\mathcal{T}$. Of course what we are really looking for is a tree $T^{(\emptyset)}$. But if T is empty, this is because there is an $i \in \mathbb{N}$ such that $A \subseteq U_i \subseteq X \setminus B$; in which case U_i is a codable Borel set separating A from B. While if T is not empty, $<n> \in T$ for every $n \in \mathbb{N}$, and just as in (vi) we can set

$$T^{(\emptyset)} = \tilde\Theta_1(\langle\tilde\Theta_2(\langle T^{(<m>)}\rangle_{j_m=j})\rangle_{j\in\mathbb{N}})$$

to obtain a codable Borel set $E = \phi(T^{\emptyset})$ such that $A \subseteq E$ and $B \cap E = \emptyset$.

(b) If X is empty, this is trivial; suppose henceforth that X is not empty.

(i) If E and $X \setminus E$ are analytic, then (a) tells us that there is a codable Borel set F including E and disjoint from $X \setminus E$, so that $E = F$ is a codable Borel set. So the rest of this part of the proof will be devoted to the converse.

Let ρ be a complete metric on X inducing its topology, $\langle x_n\rangle_{n\in\mathbb{N}}$ a sequence running over a dense subset of X, and $\langle U_n\rangle_{n\in\mathbb{N}}$ a sequence running over a base for the topology of X; let ϕ be the interpretation of Borel codes defined from $\langle U_n\rangle_{n\in\mathbb{N}}$.

(ii) We need to fix on a continuous surjection from a closed subset of $\mathbb{N}^{\mathbb{N}}$ onto X; a convenient one is the following. Set

$$F = \{\alpha : \alpha \in \mathbb{N}^{\mathbb{N}},\ \rho(x_{\alpha(n+1)}, x_{\alpha(n)}) \le 2^{-n} \text{ for every } n \in \mathbb{N}\};$$

then $F \subseteq \mathbb{N}^{\mathbb{N}}$ is closed. Define $f : F \to X$ by saying that $f(\alpha) = \lim_{n\to\infty} x_{\alpha(n)}$ for every $\alpha \in \mathbb{N}^{\mathbb{N}}$. If α, $\beta \in \mathbb{N}^{\mathbb{N}}$ and $\alpha\restriction n = \beta\restriction n$ where $n \ge 1$, then $\rho(f(\alpha), f(\beta)) \le 2^{-n+2}$, so f is continuous. If $x \in X$, we can define $\alpha \in \mathbb{N}^{\mathbb{N}}$ by saying that $\alpha(n)$ is to be the least i such that $\rho(x, x_i) \le 2^{-n-1}$; then $\rho(x_{\alpha(n)}, x_{\alpha(n+1)}) \le 2^{-n-1} + 2^{-n-2} \le 2^{-n}$ for every n, so $\alpha \in F$, and of course $f(\alpha) = x$. So f is surjective.

The next thing we need is a choice function for the set $\mathcal{F}$ of non-empty closed subsets of $\mathbb{N}^{\mathbb{N}}$; I described one in 561D. Let $g : \mathcal{F} \to \mathbb{N}^{\mathbb{N}}$ be such that $g(F) \in F$ for every $F \in \mathcal{F}$.

(iii) There is a family $\langle(F_T, f_T, F'_T, f'_T)\rangle_{T\in\mathcal{T}}$ such that

$$F_T,\ F'_T \text{ are closed subsets of } \mathbb{N}^{\mathbb{N}},$$

$$f_T : F_T \to \phi(T),\ f'_T : F'_T \to X \setminus \phi(T) \text{ are continuous surjections}$$

for each $T \in \mathcal{T}$. **P** Start by fixing a homeomorphism $\alpha \mapsto \langle h_i(\alpha)\rangle_{i\in\mathbb{N}} : \mathbb{N}^{\mathbb{N}} \to (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$. Define the quadruples (F_T, f_T, F'_T, f'_T) inductively on the rank of T.

If $r(T) \le 1$ then $\phi(T)$ is open. Set

$$F_T = \{\alpha : \alpha \in F,\ x_{\alpha(0)} \in \phi(T),\ \rho(x_{\alpha(n)}, x_{\alpha(0)}) \le \tfrac{1}{2}\rho(x_{\alpha(0)}, X \setminus \phi(T)) \text{ for every } n \ge 1\}$$

(interpreting $\rho(x, \emptyset)$ as ∞ if necessary), and $f_T = f\restriction F_T$. Then F_T is a closed subset of $\mathbb{N}^{\mathbb{N}}$ and $\rho(f(\alpha), x_{\alpha(0)}) \le \frac{1}{2}\rho(x_{\alpha(0)}, X \setminus \phi(T))$, so $f(\alpha) \in \phi(T)$, for every $\alpha \in F_T$. If $x \in \phi(T)$ then we can define $\alpha \in \mathbb{N}^{\mathbb{N}}$ by taking

$$\alpha(n) = \min\{i : \rho(x_i, x) \le \min(2^{-n-1}, \tfrac{1}{5}\rho(x, X \setminus \phi(T)))\}$$

for every n, and now we find that $\alpha \in F_T$ and $f_T(\alpha) = x$. As for F'_T and f'_T, just set $F'_T = f^{-1}[X \setminus \phi(T)]$ and $f'_T = f\restriction F'_T$.

For the inductive step to $r(T) > 1$, set $A_T = \{i : <i> \in T\}$, as in 562Ad. We have $\phi(T) = \bigcup_{i\in A_T} X \setminus \phi(T_{<i>})$ and $X \setminus \phi(T) = \bigcap_{i\in A_T} \phi(T_{<i>})$, while $r(T_{<i>}) < r(T)$ for every $i \in A_T$. Set

$$F_T = \bigcup_{i\in A_T}\{<i>^\frown\alpha : \alpha \in F'_{T_{<i>}}\},$$

$$F'_T = \{\alpha : h_i(\alpha) \in F_{T_{<i>}} \text{ for every } i \in A_T,$$
$$f_{T_{<i>}}(h_i(\alpha)) = f_{T_{<j>}}(h_j(\alpha)) \text{ for all } i,\ j \in A_T\},$$

$$f_T(<i>^\frown\alpha) = f'_{T_{<i>}}(\alpha) \text{ whenever } i \in A_T \text{ and } \alpha \in F'_{T_{<i>}},$$

$$f'_T(\alpha) = f_{T_{<i>}}(h_i(\alpha)) \text{ whenever } i \in A_T \text{ and } \alpha \in F'_T.$$

It is straightforward to confirm that F_T and F'_T are closed, $f_T : F_T \to \phi(T)$ and $f'_T : F'_T \to X \setminus \phi(T)$ are continuous and $f_T[F_T] = \phi(T)$. To see that $f'_T[F'_T] = X \setminus \phi(T)$, take any $x \in X \setminus \phi(T) = \bigcap_{i\in A_T} \phi(T_{<i>})$. Then $f^{-1}_{T_{<i>}}[\{x\}]$ is a non-empty closed subset of $F_{T_{<i>}}$ for each $i \in A_T$. Set $\alpha_i = g(f^{-1}_{T_{<i>}}[\{x\}])$, so that $f_{T_{<i>}}(\alpha_i) = x$ for each $i \in A_T$. For $i \in \mathbb{N} \setminus A_T$, take $\alpha_i = \mathbf{0}$. Now $\alpha = h^{-1}(\langle\alpha_i\rangle_{i\in\mathbb{N}})$ belongs to F'_T and $f'_T(\alpha) = x$. Thus $f'_T[F'_T] = X \setminus \phi(T)$ and the induction continues. **Q**

(iv) In particular, $\phi(T) = f_T[F_T]$ is a continuous image of a closed subset of $\mathbb{N}^{\mathbb{N}}$ for every $T \in \mathcal{T}$.

(v) The definition of 'analytic set' in 423A refers to continuous images of $\mathbb{N}^{\mathbb{N}}$, so there is a final step to make. If $E \subseteq X$ is a non-empty codable Borel set, it is a continuous image of a closed subset F_T of $\mathbb{N}^{\mathbb{N}}$; but 561C tells us that F_T is a continuous image of $\mathbb{N}^{\mathbb{N}}$, so E also is, and E is analytic.

562G Resolvable sets The essence of the concept of 'codable Borel set' is that it is not enough to know, in the abstract, that a set is 'Borel'; we need to know its pedigree. For a significant number of elementary sets, however, starting with open sets and closed sets, we can determine codes from the sets themselves.

Definition (see Kuratowski 66, §12) I will say that a subset E of a topological space X is **resolvable** if there is no non-empty set $A \subseteq X$ such that $A \subseteq \overline{A \cap E} \cap \overline{A \setminus E}$.

562H Proposition Let X be a topological space, and $\mathcal{E}$ the set of resolvable subsets of X. Then $\mathcal{E}$ is an algebra of sets containing every open subset of X.

proof (a) If $G \subseteq X$ is open and $A \subseteq X$ is non-empty, then either A meets G and $A \not\subseteq \overline{A \setminus G}$, or $A \cap G = \emptyset$ then $A \not\subseteq \overline{A \cap G}$. So every open set is resolvable.

(b) If E is resolvable and $A \subseteq X$ is not empty, there is an open set G such that $A \cap G$ is not empty but one of $A \cap G \cap E$, $A \cap G \setminus E$ is empty. **P** If $A \not\subseteq \overline{A \cap E}$ take $G = X \setminus \overline{A \cap E}$; otherwise take $G = X \setminus \overline{A \setminus E}$. **Q**

(c) If E, $E' \subseteq X$ are resolvable, so is $E \cup E'$. **P** Suppose that $A \subseteq X$ is non-empty. Then there is an open set G such that $A \cap G$ is non-empty and disjoint from one of E, $X \setminus E$. Now there is an open set H such that $A \cap G \cap H$ is non-empty and disjoint from one of E', $X \setminus E'$. Consequently one of $A \cap G \cap H \cap (E \cup E')$, $A \cap G \cap H \setminus (E \cup E')$ is empty, and the open set $G \cap H$ is disjoint from $\overline{A \cap (E \cup E')} \cap \overline{A \setminus (E \cup E')}$; in which case A cannot be included in $\overline{A \cap (E \cup E')} \cap \overline{A \setminus (E \cup E')}$. As A is arbitrary, $E \cup E'$ is resolvable. **Q**

(d) Immediately from the definition in 562G we see that the complement of a resolvable set is resolvable, so $\mathcal{E}$ is an algebra of subsets of X.

562I Theorem Let X be a second-countable space, $\langle U_n \rangle_{n \in \mathbb{N}}$ a sequence running over a base for the topology of X, and $\phi : \mathcal{T} \to \mathcal{B}_c(X)$ the associated interpretation of Borel codes. Let $\mathcal{E}$ be the algebra of resolvable subsets of X. Then there is a function $\psi : \mathcal{E} \to \mathcal{T}$ such that $\phi(\psi(E)) = E$ for every $E \in \mathcal{E}$.

proof We need to start by settling on functions

$$\Theta'_1 : \mathcal{T} \times \mathbb{N} \to \mathcal{T}, \quad \Theta'_2 : \mathcal{T} \times \mathcal{T} \times \mathbb{N} \to \mathcal{T}, \quad \tilde{\Theta}'_1 : \mathcal{T}^{\mathbb{N}} \to \mathcal{T}, \quad \tilde{\Theta}'_2 : \mathcal{T}^{\mathbb{N}} \to \mathcal{T}$$

such that

$$\phi(\Theta'_1(T, n)) = \phi(T) \setminus U_n, \quad \phi(\Theta'_2(T, T', n)) = \phi(T) \cup (\phi(T') \cap U_n),$$

$$\phi(\tilde{\Theta}'_1(\tau)) = \bigcup_{i \in \mathbb{N}} \phi(\tau(i)), \quad \phi(\tilde{\Theta}'_2(\tau)) = \bigcap_{i \in \mathbb{N}} \phi(\tau(i))$$

for $T \in \mathcal{T}$, $n \in \mathbb{N}$ and $\tau \in \mathcal{T}^{\mathbb{N}}$. (We can take $\tilde{\Theta}_1$ and $\tilde{\Theta}_2$ directly from 562Cb, and

$$\Theta'_1(T, n) = \Theta_3(T, \{<n>\}), \Theta'_2(T, T', n) = \Theta_1(T, \Theta_2(T, \{<n>\}))$$

where Θ_1, Θ_2 and Θ_3 are the functions of 562Ca.)

Now, given $E \in \mathcal{E}$, define $\langle (F_\xi, \tilde{T}^{(\xi)}, T^{(\xi)}, n_\xi) \rangle_{\xi < \omega_1}$ inductively, as follows. The inductive hypothesis will be that $F_\xi \subseteq X$ is closed, $F_\xi \subseteq F_\eta$ for every $\eta \le \xi$, $\tilde{T}^{(\xi)}$, $T^{(\xi)} \in \mathcal{T}$, $\phi(\tilde{T}^{(\xi)}) = F_\xi$ and $\phi(T^{(\xi)}) = E \setminus F_\xi$. Start with $F_0 = X$, $T^{(0)} = \emptyset$, $\tilde{T}^{(0)} = \{<n> : n \in \mathbb{N}\}$. For the inductive step to $\xi + 1$,

— if $F_\xi = \emptyset$, set $n_\xi = 0$ and $(F_{\xi+1}, \tilde{T}^{(\xi+1)}, T^{(\xi+1)}) = (F_\xi, \tilde{T}^{(\xi)}, T^{(\xi)})$;

— if there is an n such that $\emptyset \ne F_\xi \cap U_n \subseteq E$, let n_ξ be the least such, and set

$$F_{\xi+1} = F_\xi \setminus U_{n_\xi}, \quad \tilde{T}^{(\xi+1)} = \Theta'_1(\tilde{T}^{(\xi)}, n_\xi), \quad T^{(\xi+1)} = \Theta'_2(T^{(\xi)}, \tilde{T}^{(\xi)}, n_\xi);$$

— otherwise, n_ξ is to be the least n such that $\emptyset \ne F_\xi \cap U_n \subseteq X \setminus E$, and

$$F_{\xi+1} = F_\xi \setminus U_{n_\xi}, \quad \tilde{T}^{(\xi+1)} = \Theta'_1(\tilde{T}^{(\xi)}, n_\xi) \quad T^{(\xi+1)} = T^{(\xi)}.$$

(Because E is resolvable, these three cases exhaust the possibilities.) It is easy to check that the inductive hypothesis remains valid at level $\xi + 1$.

For the inductive step to a non-zero limit ordinal ξ, then if there is an $\eta < \xi$ such that $F_\eta = \emptyset$, take the first such η and set $n_\xi = 0$ and $(F_\xi, T^{(\xi)}, \tilde{T}^{(\xi)}) = (F_\eta, T^{(\eta)}, \tilde{T}^{(\eta)})$. Otherwise, we must have $F_\zeta \subseteq F_\eta \setminus U_{n_\eta} \subset F_\eta$ whenever $\eta < \zeta < \xi$, so that $\eta \mapsto n_\eta : \xi \to \mathbb{N}$ is injective. Set

$$\begin{aligned} \tilde{\tau}(i) &= \tilde{T}^{(\eta)} \text{ if } \eta < \xi \text{ and } i = n_\eta, \\ &= \tilde{T}^{(0)} \text{ if there is no such } \eta, \\ \tau(i) &= T^{(\eta)} \text{ if } \eta < \xi \text{ and } i = n_\eta, \\ &= \emptyset \text{ if there is no such } \eta; \end{aligned}$$

now set

$$F_\xi = \bigcap_{\eta < \xi} F_\eta, \quad \tilde{T}^{(\xi)} = \tilde{\Theta}'_2(\tilde{\tau}), \quad T^{(\xi)} = \tilde{\Theta}'_1(\tau).$$

Again, it is easy to check that the induction proceeds.

Now, with the family $\langle(F_\xi,\tilde T^{(\xi)},T^{(\xi)},n_\xi)\rangle_{\xi<\omega_1}$ complete, observe that $\langle n_\xi\rangle_{\xi<\omega_1}$ cannot be injective. There is therefore a first $\xi=\xi_E$ for which F_{ξ_E} is empty. Set $\psi(E)=T^{(\xi_E)}$; then $\phi(\psi(E))=E\setminus F_{\xi_E}=E$, as required.

562J Codable families of sets Let X be a second-countable space and $\mathcal{B}_c(X)$ the algebra of codable Borel subsets of X. Let $\langle U_n\rangle_{n\in\mathbb{N}}$, $\langle V_n\rangle_{n\in\mathbb{N}}$ be sequences running over bases for the topology of X, and $\phi:\mathcal{T}\to\mathcal{B}_c(X)$, $\phi':\mathcal{T}\to\mathcal{B}_c(X)$ the corresponding interpretations of Borel codes. Let us say that a family $\langle E_i\rangle_{i\in I}$ is **ϕ-codable** if there is a family $\langle T^{(i)}\rangle_{i\in I}$ in $\mathcal{T}$ such that $\phi(T^{(i)})=E_i$ for every $i\in I$. Then 562Bc tells us that $\langle E_i\rangle_{i\in I}$ is ϕ-codable iff it is ϕ'-codable.

We may therefore say that a family $\langle E_i\rangle_{i\in I}$ in $\mathcal{B}_c(X)$ is **codable** if it is ϕ-codable for some, therefore any, interpretation of Borel codes defined by the procedure of 562B from a sequence running over a base for the topology of X.

Note that any finite family in $\mathcal{B}_c(X)$ is codable, and that any family of resolvable sets is codable, because we can use 562I to provide codes; also any subfamily of a codable family is codable. Slightly more generally, if $\langle E_i\rangle_{i\in I}$ is a codable family in $\mathcal{B}_c(X)$, J is a set, and $f:J\to I$ is a function, then $\langle E_{f(j)}\rangle_{j\in J}$ is codable. If $\langle E_i\rangle_{i\in I}$ and $\langle F_i\rangle_{i\in I}$ are codable families in $\mathcal{B}_c(X)$, then so are $\langle X\setminus E_i\rangle_{i\in I}$, $\langle E_i\cup F_i\rangle_{i\in I}$, $\langle E_i\cap F_i\rangle_{i\in I}$ and $\langle E_i\setminus F_i\rangle_{i\in I}$, since we have formulae to transform codes for E, F into codes for $X\setminus E$, $E\cup F$, $E\cap F$ and $E\setminus F$.

562K Proposition Let X be a second-countable space and $\langle E_n\rangle_{n\in\mathbb{N}}$ a codable sequence in $\mathcal{B}_c(X)$. Then
(a) $\bigcup_{n\in\mathbb{N}}E_n$, $\bigcap_{n\in\mathbb{N}}E_n$ belong to $\mathcal{B}_c(X)$;
(b) $\langle\bigcup_{i<n}E_i\rangle_{n\in\mathbb{N}}$ is a codable family in $\mathcal{B}_c(X)$;
(c) $\langle E_n\setminus\bigcup_{i<n}E_i\rangle_{n\in\mathbb{N}}$ is a codable family in $\mathcal{B}_c(X)$.

proof Let $\phi:\mathcal{T}\to\mathcal{B}_c(X)$ be an interpretation of Borel codes defined from a sequence running over a base for the topology of X; then we have a sequence $\langle T^{(n)}\rangle_{n\in\mathbb{N}}$ in $\mathcal{T}$ such that $\phi(T^{(n)})=E_n$ for every n, and using 562Bb we can arrange that $T^{(n)}\ne\emptyset$ for every n.

(a) Setting

$$T=\{<\!n\!>:n\in\mathbb{N}\}\cup\{<\!n\!>^\frown<\!0\!>:n\in\mathbb{N}\}\cup\{<\!n\!>^\frown<\!0\!>^\frown\sigma:n\in\mathbb{N},\,\sigma\in T^{(n)}\},$$

$$T'=\{<\!0\!>\}\cup\{<\!0\!>^\frown<\!n\!>:n\in\mathbb{N}\}\cup\{<\!0\!>^\frown<\!n\!>^\frown\sigma:n\in\mathbb{N},\,\sigma\in T^{(n)}\},$$

we have $\phi(T)=\bigcup_{n\in\mathbb{N}}E_n$ and $\phi(T')=\bigcap_{n\in\mathbb{N}}E_n$.

(b) Setting

$$\hat T^{(n)}=\{<\!i\!>:i<n\}\cup\{<\!i\!>^\frown<\!0\!>:i<n\}\cup\{<\!i\!>^\frown<\!0\!>^\frown\sigma:i<n,\,\sigma\in T^{(i)}\},$$

$\phi(\hat T^{(n)})=\bigcup_{i<n}E_i$ for every n.

(c) Setting

$$\begin{aligned}T''=\{<\!n\!>:n\in\mathbb{N}\}&\cup\{<\!n\!>^\frown<\!0\!>:n\in\mathbb{N}\}\cup\{<\!n\!>^\frown<\!0\!>^\frown\sigma:\sigma\in T^{(n)}\}\\&\cup\{<\!n\!>^\frown<\!1\!>:n\in\mathbb{N}\}\cup\{<\!n\!>^\frown<\!1\!>^\frown<\!0\!>:n\in\mathbb{N}\}\\&\cup\{<\!n\!>^\frown<\!1\!>^\frown<\!0\!>^\frown\sigma:\sigma\in\hat T^{(n)}\},\end{aligned}$$

$\phi(T'')=\bigcup_{n\in\mathbb{N}}(E_n\setminus\bigcup_{i<n}E_i)$.

562L Codable Borel functions Let X and Y be second-countable spaces. A function $f:X\to Y$ is a **codable Borel function** if $\langle f^{-1}[H]\rangle_{H\subseteq Y\text{ is open}}$ is a codable family in $\mathcal{B}_c(X)$.

562M Theorem Let X be a second-countable space, $\langle U_n\rangle_{n\in\mathbb{N}}$ a sequence running over a base for the topology of X, and $\phi:\mathcal{T}\to\mathcal{B}_c(X)$ the corresponding interpretation of Borel codes.

(a) If Y is another second-countable space, $\langle V_n\rangle_{n\in\mathbb{N}}$ a sequence running over a base for the topology of Y containing $\emptyset$, $\phi_Y:\mathcal{T}\to\mathcal{B}_c(Y)$ the corresponding interpretation of Borel codes, and $f:X\to Y$ is a function, then the following are equiveridical:

(i) f is a codable Borel function;

(ii) $\langle f^{-1}[V_n]\rangle_{n\in\mathbb{N}}$ is a codable sequence in $\mathcal{B}_c(X)$;

(iii) there is a function $\Theta:\mathcal{T}\to\mathcal{T}$ such that $\phi(\Theta(T))=f^{-1}[\phi_Y(T)]$ for every $T\in\mathcal{T}$.

(b) If Y and Z are second-countable spaces and $f:X\to Y$, $g:Y\to Z$ are codable Borel functions then $gf:X\to Z$ is a codable Borel function.

(c) If Y and Z are second-countable spaces and $f:X\to Y$, $g:X\to Z$ are codable Borel functions then $x\mapsto(f(x),g(x))$ is a codable Borel function from X to $Y\times Z$.

(d) If Y is a second-countable space then any continuous function from X to Y is a codable Borel function.

proof (a)(i)$\Rightarrow$(ii) is trivial.

(ii)$\Rightarrow$(iii) This is really a full-strength version of 562Bc. Because $\langle f^{-1}[V_n]\rangle_{n\in\mathbb{N}}$ is codable, we have a sequence $\langle T^{(n)}\rangle_{n\in\mathbb{N}}$ in $\mathcal{T}$ such that $\phi(T^{(n)})=f^{-1}[V_n]$ for every n. As in 562C, let $\Theta_0:\mathcal{T}\to\mathcal{T}$ and $\tilde{\theta}_1:\bigcup_{I\subseteq\mathbb{N}}\mathcal{T}^I\to\mathcal{T}$ be such that $\phi(\Theta_0(T))=X\setminus\phi(T)$ for every $T\in\mathcal{T}$ and $\phi(\tilde{\Theta}_1(\tau))=\bigcup_{i\in I}\phi(\tau(i))$ whenever $I\subseteq\mathbb{N}$ and $\tau\in\mathcal{T}^I$. Define $\Theta:\mathcal{T}\to\mathcal{T}$ inductively, as follows. Given $T\in\mathcal{T}$, set $A_T=\{n:{<}n{>}\in T\}$. If $r(T)=0$ set $\Theta(T)=T=\emptyset$. If $r(T)=1$ set $\Theta(T)=\tilde{\Theta}_1(\langle T^{(n)}\rangle_{n\in A_T})$, so that

$$\phi(\Theta(T))=\textstyle\bigcup_{n\in A_T}\phi(T^{(n)})=\bigcup_{n\in A_T}f^{-1}[V_n]=f^{-1}[\phi_Y(T)].$$

If $r(T)>1$ set

$$\Theta(T)=\tilde{\Theta}_1(\langle\Theta_0(\Theta(T_{<n>}))\rangle_{n\in A_T})$$

so that

$$\begin{aligned}\phi(\Theta(T))&=\bigcup_{n\in A_T}X\setminus\phi(\Theta(T_{<n>}))=\bigcup_{n\in A_T}X\setminus f^{-1}[\phi_Y(T_{<n>})]\\&=f^{-1}[\bigcup_{n\in A_T}Y\setminus\phi_Y(T_{<n>})]=f^{-1}[\phi_Y(T)]\end{aligned}$$

and the induction continues.

(iii)$\Rightarrow$(i) For open $H\subseteq Y$ set $\psi_Y(H)=\{{<}n{>}:V_n\subseteq H\}$. Taking Θ as above,

$$\phi(\Theta(\psi_Y(H))=f^{-1}[\phi_Y(\psi_Y(H))]=f^{-1}[H]$$

for every H, so $\langle\phi(\Theta(\psi_Y(H)))\rangle_{H\subseteq Y\text{ is open}}$ is a family of codes for $\langle f^{-1}[H]\rangle_{H\subseteq Y\text{is open}}$.

(b) Take $\langle V_n\rangle_{n\in\mathbb{N}}$, ϕ_Y and $\Theta:\mathcal{T}\to\mathcal{T}$ as in (a). Write $\mathfrak{U}$ for the topology of Z; then we have a function $\theta:\mathfrak{U}\to\mathcal{T}$ such that $\phi_Y(\theta(H))=g^{-1}[H]$ for every $H\in\mathfrak{U}$. Now $\langle\Theta(\theta(H))\rangle_{H\in\mathfrak{U}}$ is a coding for $\langle(gf)^{-1}[H]\rangle_{H\in\mathfrak{U}}$, so gf is codable.

(c) Let $\langle V_n\rangle_{n\in\mathbb{N}}$, $\langle W_n\rangle_{n\in\mathbb{N}}$ be sequences running over bases for the topologies of Y and Z, and $\langle(i_n,j_n)\rangle_{n\in\mathbb{N}}$ an enumeration of $\mathbb{N}\times\mathbb{N}$. Set $H_n=V_{i_n}\times W_{j_n}$; then $\langle H_n\rangle_{n\in\mathbb{N}}$ is a base for the topology of $Y\times Z$. Let $\Theta_2:\mathcal{T}\times\mathcal{T}\to\mathcal{T}$ be such that $\phi(\Theta_2(T,T'))=\phi(T)\cap\phi(T')$ for all $T,T'\in\mathcal{T}$ (562Ca). Let $\langle T^{(n)}\rangle_{n\in\mathbb{N}}$, $\langle\hat{T}^{(n)}\rangle_{n\in\mathbb{N}}$ be codings for $\langle f^{-1}[V_n]\rangle_{n\in\mathbb{N}}$, $\langle g^{-1}[W_n]\rangle_{n\in\mathbb{N}}$. Then $\langle\Theta_2(T^{(i_n)},\hat{T}^{(j_n)})\rangle_{n\in\mathbb{N}}$ is a coding for $\langle h^{-1}[H_n]\rangle_{n\in\mathbb{N}}$, where $h(x)=(f(x),g(x))$ for $x\in X$. So h is a codable Borel function.

(d) If $f:X\to Y$ is continuous, then $\langle f^{-1}[H]\rangle_{H\subseteq Y\text{ is open}}$ is a family of resolvable sets, therefore codable, as noted in 562J.

Remark Note in part (a)(ii)$\Rightarrow$(iii) of the proof the function Θ is constructed by a definite process from $\langle T^{(n)}\rangle_{n\in\mathbb{N}}$; so we shall be able to uniformize the process to define families $\langle\Theta_i\rangle_{i\in I}$ from families $\langle f_i\rangle_{i\in I}$, at least if we can reach a family $\langle T^{(i,n)}\rangle_{i\in I,n\in\mathbb{N}}$ such that $T^{(n,i)}$ codes $f_i^{-1}[V_n]$ for all $i\in I$ and $n\in\mathbb{N}$.

562N Proposition Let X be a second-countable space, and $\phi:\mathcal{T}\to\mathcal{B}_c(X)$ the interpretation of Borel codes associated with some sequence running over a base for the topology of X.

(a) If $f:X\to\mathbb{R}$ is a function, the following are equiveridical:

(i) f is a codable Borel function;

(ii) the family $\langle\{x : f(x) > \alpha\}\rangle_{\alpha\in\mathbb{R}}$ is codable;

(iii) $\langle\{x : f(x) > q\}\rangle_{q\in\mathbb{Q}}$ is codable.

(b) Write $\tilde{\mathcal{T}}$ for the set of functions $\tau : \mathbb{R} \to \mathcal{T}$ such that

$$\phi(\tau(\alpha)) = \bigcup_{\beta>\alpha} \phi(\tau(\beta)) \text{ for every } \alpha \in \mathbb{R},$$

$$\bigcap_{n\in\mathbb{N}} \phi(\tau(n)) = \emptyset, \quad \bigcup_{n\in\mathbb{N}} \phi(\tau(-n)) = X.$$

Then

(i) for every $\tau \in \tilde{\mathcal{T}}$ there is a unique codable Borel function $\tilde{\phi}(\tau) : X \to \mathbb{R}$ such that $\phi(\tau(\alpha)) = \{x : \tilde{\phi}(\tau)(x) > \alpha\}$ for every $\alpha \in \mathbb{R}$;

(ii) every codable Borel function from X to $\mathbb{R}$ is expressible as $\tilde{\phi}(\tau)$ for some $\tau \in \tilde{\mathcal{T}}$.

(c) If $\langle\tau_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\tilde{\mathcal{T}}$ such that $f(x) = \sup_{n\in\mathbb{N}} \tilde{\phi}(\tau_n)(x)$ is finite for every $x \in X$, then f is a codable Borel function.

(d) If f, $g : X \to \mathbb{R}$ are codable Borel functions and $\gamma \in \mathbb{R}$, then $f + g$, γf, $|f|$ and $f \times g$ are codable Borel functions.

(e) If $\langle\tau_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\tilde{\mathcal{T}}$, then there is a codable Borel function f such that $\liminf_{n\to\infty} \tilde{\phi}(\tau_n)(x) = f(x)$ whenever the lim inf is finite.

(f) A subset E of X belongs to $\mathcal{B}_c(X)$ iff $\chi E : X \to \mathbb{R}$ is a codable Borel function.

proof (a)(i)⇒(ii) If $f : X \to \mathbb{R}$ is codable then of course $\langle\{x : f(x) > \alpha\}\rangle_{\alpha\in\mathbb{R}}$ is codable, because it is a subfamily of $\langle f^{-1}[H]\rangle_{H\subseteq\mathbb{R} \text{ is open}}$.

(ii)⇒(iii) Similarly, if $\langle\{x : f(x) > \alpha\}\rangle_{\alpha\in\mathbb{R}}$ is codable, its subfamily $\langle\{x : f(x) > q\}\rangle_{q\in\mathbb{Q}}$ is codable.

(iii)⇒(i) If $\langle\{x : f(x) > q\}\rangle_{q\in\mathbb{Q}}$ is codable, we have a family $\langle T^{(q)}\rangle_{q\in\mathbb{Q}}$ in $\mathcal{T}$ coding it. Let $\langle(q_n, q'_n)\rangle_{n\in\mathbb{N}}$ be an enumeration of $\{(q, q') : q, q' \in \mathbb{Q}, q < q'\}$. As in 562C, we have functions

$$\Theta_3 : \mathcal{T} \times \mathcal{T} \to \mathcal{T}, \quad \tilde{\Theta}_1 : \bigcup_{I\subseteq\mathbb{Q}} \mathcal{T}^I \to \mathcal{T}, \quad \tilde{\Theta}_2 : \bigcup_{J\subseteq\mathbb{N}} \mathcal{T}^J \to \mathcal{T}$$

such that

$$\phi(\Theta_3(T, T')) = \phi(T) \setminus \phi(T'), \quad \phi(\tilde{\Theta}_1(\tau)) = \bigcup_{q\in I} \phi(\tau(q))),$$

$$\phi(\tilde{\Theta}_2(\tau)) = X \cap \bigcap_{q\in I} \phi(\tau(j))$$

for T, $T' \in \mathcal{T}$, $I \subseteq \mathbb{Q}$ and $\tau \in \mathcal{T}^I$. Now for $n \in \mathbb{N}$ consider

$$\hat{T}^{(n)} = \tilde{\Theta}_1(\langle\Theta_3(T^{(q_n)}, T^{(r)})\rangle_{r\in\mathbb{Q},r<q'_n}),$$

so that $\phi(\hat{T}^{(n)}) = f^{-1}[]q_n, q'_n[]$ for every n, and $\langle f^{-1}[]q_n, q'_n[]\rangle_{n\in\mathbb{N}}$ is codable; by 562Ma, f is a codable Borel function.

(b) This is elementary; given $\tau \in \tilde{\mathcal{T}}$ we can, and must, set $\tilde{\phi}(\tau)(x) = \sup\{\alpha : x \in \phi(\tau(\alpha))\}$ for every $x \in X$; and given f we have a coding τ for $\langle\{x : f(x) > \alpha\}\rangle_{\alpha\in\mathbb{R}}$ which must belong to $\tilde{\mathcal{T}}$ and be such that $\tilde{\phi}(\tau) = f$.

(c) Given $\langle\tau_n\rangle_{n\in\mathbb{N}}$ as described, and taking $\tilde{\Theta}_1$ as in (a)(iii)⇒(i) above,

$$\alpha \mapsto \tilde{\Theta}_1(\langle\tau_n(\alpha)\rangle_{n\in\mathbb{N}})$$

will be a Borel code for f.

(d) Use 562M(b)-(d).

(e) Let

$$\Theta_0 : \mathcal{T} \to \mathcal{T}, \quad \Theta_1 : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$$

be such that

$$\phi(\Theta_0(T)) = X \setminus \phi(T), \quad \phi(\Theta_1(T, T')) = \phi(T) \cup \phi(T')$$

for every T, $T' \in \mathcal{T}$. Now, given $\langle\tau_n\rangle_{n\in\mathbb{N}}$ as described, set

$$\tau(\alpha) = \tilde{\Theta}_1(\langle \tilde{\Theta}_1(\langle \tilde{\Theta}_2(\langle \tau_m(q) \rangle_{m \ge n}) \rangle_{q \in \mathbb{Q}, q > \alpha}) \rangle_{n \in \mathbb{N}})$$

for $\alpha \in \mathbb{R}$. Then

$$\phi(\tau(\alpha)) = \bigcup_{q > \alpha, n \in \mathbb{N}} \bigcap_{m \ge n} \{x : f_m(x) > q\} = \{x : \liminf_{n \to \infty} f_n(x) > \alpha\}$$

for each α. We don't yet have a code for a real-value function defined everywhere in X. But if we set

$$T = \tilde{\Theta}_1(\langle \Theta_3(\tau(-n), \tau(n)) \rangle_{n \in \mathbb{N}}),$$

then

$$\phi(T) = \bigcup_{n \in \mathbb{N}} \phi(\tau(-n)) \setminus \phi(\tau(n)) = \{x : \liminf_{n \to \infty} f_n(x) \text{ is finite}\}.$$

So take

$$\begin{aligned} \tau'(\alpha) &= \Theta_3(\tau(\alpha), \Theta_0(T)) \text{ if } \alpha \ge 0, \\ &= \Theta_1(\tau(\alpha), \Theta_0(T)) \text{ if } \alpha < 0; \end{aligned}$$

this will get $\tau' \in \tilde{\mathcal{T}}$ such that

$$\begin{aligned} \tilde{\phi}(\tau')(x) &= \liminf_{n \to \infty} f_n(x) \text{ if this is finite,} \\ &= 0 \text{ otherwise.} \end{aligned}$$

(f) Elementary.

562O Remarks (a) For some purposes there are advantages in coding real-valued functions by functions from $\mathbb{Q}$ to $\mathcal{T}$ rather than by functions from $\mathbb{R}$ to $\mathcal{T}$; see 364Af and 556A.

(b) As in 562C, it will be useful to observe that the constructions here are largely determinate. For instance, the function Θ of 562M(a-iii) can be built by a definite rule from the sequence $\langle T^{(n)} \rangle_{n \in \mathbb{N}}$ provided by the hypothesis (a-ii) there. What this means is that if we have a family $\langle (Y_i, \langle V_{in} \rangle_{n \in \mathbb{N}}, f_i) \rangle_{i \in I}$ such that Y_i is a second-countable space, $\langle V_{in} \rangle_{n \in \mathbb{N}}$ is a sequence running over a base for the topology of Y_i, and $f_i : X \to Y_i$ is a continuous function for each $i \in I$, then there will be a function $\tilde{\Theta} : \mathcal{T} \times I \to \mathcal{T}$ such that $\phi(\tilde{\Theta}(T, i)) = f_i^{-1}[\phi_i(T)]$ for every $i \in I$ and $T \in \mathcal{T}$, where $\phi_i : \mathcal{T} \to \mathcal{B}_c(Y_i)$ is the interpretation of Borel codes corresponding to the sequence $\langle V_{in} \rangle_{n \in \mathbb{N}}$. (Start from

$$T^{(i,n)} = \{<\!j\!> : U_j \subseteq f_i^{-1}[V_{in}]\}$$

for $i \in I$ and $n \in \mathbb{N}$, and build $\tilde{\Theta}(T, i)$ as 562M.)

(c) Similarly, when we look at 562N(d)-(e), we have something better than just existence proofs for codes for $f + g$ and $\liminf_{n \to \infty} f_n$. For instance, we have a function $\tilde{\Theta}_1 : \tilde{\mathcal{T}} \times \tilde{\mathcal{T}} \to \tilde{\mathcal{T}}$ such that $\tilde{\phi}(\tilde{\Theta}_1(\tau, \tau'))$ will always be $\tilde{\phi}(\tau) - \tilde{\phi}(\tau')$ for τ, $\tau' \in \tilde{\mathcal{T}}$. **P** We need to have

$$\phi(\tilde{\Theta}(\tau, \tau')(\alpha)) = \bigcup_{q \in \mathbb{Q}} \phi(\tau(q)) \setminus \phi(\tau'(q - \alpha))$$

for every α, and this is easy to build from a set-difference operator, as in 562Ca, and a general countable-union operator as built in 562Cb. **Q** Equally, we have a function $\tilde{\Theta}_1^* : \tilde{\mathcal{T}}^{\mathbb{N}} \to \tilde{\mathcal{T}}^{\mathbb{N}}$ such that

$$\tilde{\phi}(\tilde{\Theta}_1^*(\langle \tau_n \rangle_{n \in \mathbb{N}})(m)) = \inf_{n \ge m} \tilde{\phi}(\tau_n)$$

for every m whenever $\langle \tau_n \rangle_{n \in \mathbb{N}}$ is a sequence in $\tilde{\mathcal{T}}$ such that $\inf_{n \in \mathbb{N}} \tilde{\phi}(\tau_n)$ is defined as a real-valued function on X. **P** This time we need

$$\phi(\tilde{\Theta}_1(\langle \tau_n \rangle_{n \in \mathbb{N}})(m)(\alpha)) = \bigcup_{q \in \mathbb{Q}, q > \alpha} \bigl(X \setminus \bigcup_{n \ge m} (X \setminus \phi(\tau_n(q)))\bigr)$$

for all m and α, and once again a complementation operator and a general countable-union operator will do the trick. **Q**

562P Codable Borel equivalence (a) If X is a set, we can say that two second-countable topologies $\mathfrak{S}$, $\mathfrak{T}$ on X are **codably Borel equivalent** if the identity functions $(X, \mathfrak{S}) \to (X, \mathfrak{T})$ and $(X, \mathfrak{T}) \to (X, \mathfrak{S})$ are codable Borel functions. In this case, $\mathfrak{S}$ and $\mathfrak{T}$ give the same families of codable Borel functions and the same algebra $\mathcal{B}_c(X)$ (562Mb, 562Nf).

(b) If $(X,\mathfrak{T})$ is a second-countable space and $\langle E_n\rangle_{n\in\mathbb{N}}$ is any codable sequence in $\mathcal{B}_c(X)$, there is a topology $\mathfrak{S}$ on X, generated by a countable algebra of subsets of X, such that $\mathfrak{S}$ and $\mathfrak{T}$ are codably Borel equivalent and every E_n belongs to $\mathfrak{S}$. **P** Since there is certainly a codable sequence running over a base for the topology of X, we can suppose that such a sequence has been amalgamated with $\langle E_n\rangle_{n\in\mathbb{N}}$, so that $\{E_n : n\in\mathbb{N}\}$ includes a base for $\mathfrak{T}$. Let $\mathcal{E}$ be the algebra of subsets of X generated by $\{E_n : n\in\mathbb{N}\}$ and $\mathfrak{S}$ the topology generated by $\mathcal{E}$. As $\mathcal{E}$ is an algebra, $\mathfrak{S}$ is zero-dimensional; as $\mathcal{E}$ is countable, $\mathfrak{S}$ is second-countable.

The identity map $(X,\mathfrak{S})\to(X,\mathfrak{T})$ is continuous, therefore a codable Borel function (562Md). In the reverse direction, we have a sequence $\langle T^{(n)}\rangle_{n\in\mathbb{N}}$ of codes for $\langle E_n\rangle_{n\in\mathbb{N}}$. From these we can build, using our standard operations, codes T_I, for $I\in[\mathbb{N}]^{<\omega}$, T'_{IJ}, for I, $J\in[\mathbb{N}]^{<\omega}$, and $T''_{\mathcal{K}}$, for $\mathcal{K}\in[[\mathbb{N}]^{<\omega}\times[\mathbb{N}]^{<\omega}]^{<\omega}$, such that

T_I codes $\bigcup_{i\in I}E_i$,
T'_{IJ} codes $\bigcup_{i\in I}E_i\setminus\bigcup_{i\in J}E_i$,
$T''_{\mathcal{K}}$ codes $\bigcup_{(I,J)\in\mathcal{K}}(\bigcup_{i\in I}E_i\setminus\bigcup_{i\in J}E_i)$.

But of course $[[\mathbb{N}]^{<\omega}\times[\mathbb{N}]^{<\omega}]^{<\omega}$ is countable and the $T''_{\mathcal{K}}$ can be enumerated as a sequence $\langle T^*_n\rangle_{n\in\mathbb{N}}$ coding a sequence $\langle V_n\rangle_{n\in\mathbb{N}}$ running over $\mathcal{E}$. By 562Ma, the identity map $(X,\mathfrak{T})\to(X,\mathfrak{S})$ is a codable Borel function. **Q**

Note that $\mathfrak{S}$ here is necessarily regular; this will be useful at more than one point in the next couple of sections.

562Q Resolvable functions Let X be a topological space. I will say that a function $f:X\to[-\infty,\infty]$ is **resolvable** if whenever $\alpha<\beta$ in $\mathbb{R}$ and $A\subseteq X$ is a non-empty set, then at least one of $\{x:x\in A,\ f(x)\le\alpha\}$, $\{x:x\in A,\ f(x)\ge\beta\}$ is not dense in A.

Examples (a) Any semi-continuous function from X to $[-\infty,\infty]$ is resolvable. **P** If $f:X\to[-\infty,\infty]$ is lower semi-continuous, $A\subseteq X$ is non-empty, and $\alpha<\beta$ in $\mathbb{R}$, then $U=\{x:f(x)>\alpha\}$ is open; if $A\cap U\ne\emptyset$ then $\{x:x\in A,\ f(x)\le\alpha\}$ is not dense in A; otherwise $\{x:x\in A,\ f(x)\ge\beta\}$ is not dense in A. **Q**

(b) If $f:X\to\mathbb{R}$ is such that $\{x:f(x)>\alpha\}$ is resolvable for every α, then f is resolvable. **P** Suppose that $A\subseteq X$ is non-empty and $\alpha<\beta$ in $\mathbb{R}$. Set $E=\{x:f(x)>\alpha\}$. If $A\not\subseteq\overline{A\cap E}$, then $\{x:x\in A,\ f(x)\ge\beta\}$ is not dense in A. Otherwise $\{x:x\in A,\ f(x)\le\alpha\}=A\setminus E$ is not dense in A. **Q**

In particular, the indicator function of a resolvable set is resolvable.

(c) A function $f:\mathbb{R}\to\mathbb{R}$ which has bounded variation on every bounded set is resolvable. **P** If $A\subseteq\mathbb{R}$ is non-empty and $\alpha<\beta$ in $\mathbb{R}$, take $y\in A$. If y is isolated in A, then we have an open set U such that $U\cap A=\{y\}$, so that one of $\{x:x\in A,\ f(x)\le\alpha\}$, $\{x:x\in A,\ f(x)\ge\beta\}$ does not contain y and is not dense in A. Otherwise, y is in the closure of one of $A\cap\,]y,\infty[$, $A\cap\,]-\infty,y[$; suppose the former. For each $n\in\mathbb{N}$ set $I_n=[y+2^{-n-1},y+2^{-n}]$, $\delta_n=\mathrm{Var}_{I_n}(f)$. We have

$$\infty>\mathrm{Var}_{]y,y+1]}(f)=\sum_{n=0}^{\infty}\delta_n,$$

so there is an $n\in\mathbb{N}$ such that $\delta_m\le\frac{1}{4}(\beta-\alpha)$ for $m\ge n$. Take $m>n$ such that $I_m\cap A\ne\emptyset$, and consider $U=\mathrm{int}(I_{m-1}\cup I_m\cup I_{m+1})$. Then $\mathrm{Var}_U(f)\le\frac{3}{4}(\beta-\alpha)$ so U cannot meet both $\{x:x\in A,\ f(x)\ge\beta\}$ and $\{x:x\in A,\ f(x)\ge\alpha\}$, and one of these is not dense in A. **Q**

562R Theorem Let X be a second-countable space, $\langle U_n\rangle_{n\in\mathbb{N}}$ a sequence running over a base for the topology of X, and $\phi:\mathcal{T}\to\mathcal{B}_c(X)$ the associated interpretation of Borel codes. Let $\mathcal{R}$ be the family of resolvable real-valued functions on X. Then there is a function $\tilde\psi:\mathcal{R}\to\mathcal{T}^{\mathbb{R}}$ such that

$$\phi(\tilde\psi(f)(\alpha))=\{x:f(x)>\alpha\}$$

for every $f\in\mathcal{R}$ and $\alpha\in\mathbb{R}$.

proof (a) Start by fixing a bijection

$$k\mapsto(n_k,q_k,q'_k):\mathbb{N}\to\mathbb{N}\times\{(q,q'):q,\ q'\in\mathbb{Q},\ q<q'\}.$$

Next, fix a function $\Theta^*_1:\mathcal{T}^3\times\mathbb{N}\to\mathcal{T}$ such that

$$\phi(\Theta_1^*(T,T',T'',n)) = \phi(T) \cup (U_n \setminus (\phi(T') \cup \phi(T'')))$$

for T, T', $T'' \in \mathcal{T}$ and $n \in \mathbb{N}$, and a function $\tilde{\Theta}_1^* : \bigcup_{J\subseteq\mathbb{N}} \mathcal{T}^J \to \mathcal{T}$ such that $\phi(\tilde{\Theta}_1^*(\tau)) = \bigcup_{i\in J} \phi(\tau(i))$ whenever $J \subseteq \mathbb{N}$ and $\tau \in \mathcal{T}^J$. (See 562Ca.)

(b) Given $f \in \mathcal{R}$, define $\zeta < \omega_1$ and a family $\langle(\tau_\xi, \tau'_\xi, k_\xi)\rangle_{\xi\le\zeta}$ in $\mathcal{T}^{\mathbb{R}} \times \mathcal{T}^{\mathbb{R}} \times \mathbb{N}$ inductively, as follows. The inductive hypothesis will be that $k_\eta \ne k_\xi$ whenever $\eta < \xi < \zeta$. Start with $\tau_0(\alpha) = \tau'_0(\alpha) = \emptyset$ for every $\alpha \in \mathbb{R}$.

Inductive step to a successor ordinal $\xi + 1$ Given τ_ξ and τ'_ξ in $\mathcal{T}^{\mathbb{R}}$, then for $q < q'$ in $\mathbb{Q}$ set $F_\xi(q,q') = X \setminus (\phi(\tau_\xi(q)) \cup \phi(\tau'_\xi(q')))$. Now

— if there is a $k \in \mathbb{N} \setminus \{k_\eta : \eta < \xi\}$ such that $U_{n_k} \cap F_\xi(q_k, q'_k) \ne \emptyset$ and $f(x) \ge q_k$ for every $x \in U_{n_k} \cap F_\xi(q_k, q'_k)$, take the first such k, and set

$$\begin{aligned}
\tau_{\xi+1}(\alpha) &= \tau_\xi(\alpha) \text{ for every } \alpha \in \mathbb{R},\\
\tau'_{\xi+1}(\alpha) &= \Theta_1^*(\tau'_\xi(\alpha), \tau_\xi(q_k), \tau'_\xi(q'_k), n_k) \text{ if } \alpha \le q_k,\\
&= \tau'_\xi(\alpha) \text{ if } \alpha > q_k,\\
k_\xi &= k;
\end{aligned}$$

— if this is not so, but there is a $k \in \mathbb{N} \setminus \{k_\eta : \eta < \xi\}$ such that $U_{n_k} \cap F_\xi(q_k, q'_k) \ne \emptyset$ and $f(x) \le q'_k$ for every $x \in U_{n_k} \cap F_\xi(q_k, q'_k)$, take the first such k, and set

$$\begin{aligned}
\tau_{\xi+1}(\alpha) &= \Theta_1^*(\tau_\xi(\alpha), \tau_\xi(q_k), \tau'_\xi(q'_k), n_k) \text{ if } \alpha \ge q'_k,\\
&= \tau_\xi(\alpha) \text{ if } \alpha < q'_k,\\
\tau'_{\xi+1}(\alpha) &= \tau'_\xi(\alpha) \text{ for every } \alpha \in \mathbb{R},\\
k_\xi &= k;
\end{aligned}$$

— and if that doesn't happen either, set $\zeta = \xi$ and stop.

Inductive step to a countable limit ordinal ξ Given $\langle(\tau_\eta, \tau'_\eta, k_\eta)\rangle_{\eta<\xi}$, set $I = \{k_\eta : \eta < \xi\}$ and define $g : I \to \xi$ by setting $g(i) = \eta$ whenever $\eta < \xi$ and $k_\eta = i$. Now set

$$\tau_\xi(\alpha) = \tilde{\Theta}_1^*(\langle\tau_{g(i)}(\alpha)\rangle_{i\in I}), \quad \tau'_\xi(\alpha) = \tilde{\Theta}_1^*(\langle\tau'_{g(i)}(\alpha)\rangle_{i\in I})$$

for every $\alpha \in \mathbb{R}$.

(c) Now an induction on ξ shows that

$$\phi(\tau_\eta(\alpha)) \subseteq \phi(\tau_\xi(\alpha)), \quad \phi(\tau'_\eta(\alpha)) \subseteq \phi(\tau'_\xi(\alpha)),$$

$$\phi(\tau_\xi(\alpha)) \subseteq \{x : f(x) \le \alpha\}, \quad \phi(\tau'_\xi(\alpha)) \subseteq \{x : f(x) \ge \alpha\}$$

whenever $\eta \le \xi$, $\alpha \in \mathbb{R}$ and the codes here are defined. Next, if $k_\eta = k$ is defined, we must have $U_{n_k} \cap F_\eta(q_k, q'_k) \ne \emptyset$ and

— either $f(x) \ge q_k$ for every $x \in U_{n_k} \cap F_\eta(q_k, q'_k)$ and $\phi(\tau'_{\eta+1}(q_k)) = \phi(\tau'_\eta(q_k)) \cup (U_{n_k} \cap F_\eta(q_k, q'_k))$

— or $f(x) \le q'_k$ for every $x \in U_{n_k} \cap F_\eta(q_k, q'_k)$ and $\phi(\tau_{\eta+1}(q_k)) = \phi(\tau_\eta(q_k)) \cup (U_{n_k} \cap F_\eta(q_k, q'_k))$.

In either case, $U_{n_k} \cap F_\eta(q_k, q'_k)$ must be disjoint from $F_\xi(q_k, q'_k)$ for every $\xi > \eta$ for which F_ξ is defined; consequently we cannot have $k_\xi = k$ for any $\xi > \eta$. The induction must therefore stop.

$F_\zeta(q,q') = \emptyset$ whenever q, $q' \in \mathbb{Q}$ and $q < q'$. **P?** Otherwise, because f is resolvable, there is an $n \in \mathbb{N}$ such that $V = U_n \cap F_\zeta(q,q')$ is non-empty and either $f(x) \ge q$ for every $x \in V$ or $f(x) \le q'$ for every $x \in V$. Let $k \in \mathbb{N}$ be such that $n_k = n$, $q_k = q$ and $q'_k = q'$; then U_{n_k} meets $F_\zeta(q_k, q'_k)$ so $k \ne k_\eta$ for any $\eta < \zeta$. But this means that we ought to have proceeded according to one of the first two alternatives in the single-step inductive stage, and ought not to have stopped at ζ. **XQ**

(d) Now set

$$\tau(\alpha) = \tilde{\Theta}_1^*(\langle\tau'_\zeta(q_n)\rangle_{n\in\mathbb{N},q_n>\alpha})$$

for $\alpha \in \mathbb{R}$. Then

$$\begin{aligned}\phi(\tau(\alpha)) &= \bigcup_{n\in\mathbb{N},q_n>\alpha} \phi(\tau'_\zeta(q_n))\\ &\subseteq \bigcup_{n\in\mathbb{N},q_n>\alpha} \{x : f(x) \ge q_n\} \subseteq \{x : f(x) > \alpha\}\end{aligned}$$

for every α. **?** If α is such that $\phi(\tau(\alpha)) \subset \{x : f(x) > \alpha\}$, let $x \in X$ and $n \in \mathbb{N}$ be such that $f(x) > q'_n > q_n > \alpha$ and $x \notin \phi(\tau(\alpha))$. Then $x \notin \phi(\tau'_\zeta(q_n))$; but also $f(y) \le q'_n$ for every $y \in \phi(\tau_\zeta(q'_n))$, so $x \notin \phi(\tau_\zeta(q'_n))$ and $x \in F_\zeta(q_n, q'_n)$, which is supposed to be impossible. **X**

So we can set $\tilde\psi(f) = \tau$.

562S Codable families of codable functions (a) If X and Y are second-countable spaces, a family $\langle f_i\rangle_{i\in I}$ of functions from X to Y is a **codable family of codable Borel functions** if $\langle f_i^{-1}[H]\rangle_{i\in I,H\subseteq Y\text{ is open}}$ is a codable family in $\mathcal{B}_c(X)$.

(b) Uniformizing the arguments of 562N, it is easy to check that a family $\langle f_i\rangle_{i\in I}$ of real-valued functions on X is a codable family of codable Borel functions iff there is a family $\langle \tau_i\rangle_{i\in I}$ in $\tilde{\mathcal{T}}$ such that, in the language there, $f_i = \tilde\phi(\tau_i)$ for every i.

(c) In this language, 562Ne can be rephrased as

> if $\langle f_n\rangle_{n\in\mathbb{N}}$ is a codable sequence of real-valued codable Borel functions on X, there is a codable Borel function f such that $f(x) = \liminf_{n\to\infty} f_n(x)$ whenever the lim inf is finite,

and 562R implies that

> the family of resolvable real-valued functions on X is a codable family of codable Borel functions.

(d) If X, Y and Z are second-countable spaces, $\langle f_i\rangle_{i\in I}$ is a codable family of codable Borel functions from X to Y, and $\langle g_i\rangle_{i\in I}$ is a codable family of codable Borel functions from Y to Z, then $\langle g_i f_i\rangle_{i\in I}$ is a codable family of codable functions from X to Z; this is because the proof of 562Mb gives a recipe for calculating a code for the composition of codable functions, which can be performed simultaneously on the compositions $g_i f_i$ if we are given codes for the functions g_i and f_i.

(e) Extending the remarks in 562O(b)-(c), we see that (for instance) we can define a sequence $\langle \Phi_n\rangle_{n\in\mathbb{N}}$ such that Φ_n is a function from $\tilde{\mathcal{T}}^{n+1}$ to $\tilde{\mathcal{T}}$ for every n, and $\tilde\phi(\Phi_n(\langle\tau_i\rangle_{i\le n})) = \sum_{i=0}^n \tilde\phi(\tau_i)$ whenever $\tau_0, \dots, \tau_n \in \tilde{\mathcal{T}}$; so that if $\langle f_n\rangle_{n\in\mathbb{N}}$ is a codable sequence of codable Borel functions, then $\langle \sum_{i=0}^n f_i\rangle_{n\in\mathbb{N}}$ is codable.

562T Codable Baire sets The ideas here can be adapted to give a theory of Baire algebras in general topological spaces. Start by settling on a sequence running over a base for the topology of $\mathbb{R}^{\mathbb{N}}$, with the associated interpretation $\phi : \mathcal{T} \to \mathcal{B}_c(\mathbb{R}^{\mathbb{N}})$ of Borel codes. Let X be a topological space.

(a) A subset E of X is a **codable Baire set** if it is of the form $f^{-1}[F]$ for some continuous $f : X \to \mathbb{R}^{\mathbb{N}}$ and $F \in \mathcal{B}_c(\mathbb{R}^{\mathbb{N}})$; write $\mathcal{B}\mathfrak{a}_c(X)$ for the family of such sets. If $E \in \mathcal{B}\mathfrak{a}_c(X)$, then a **code** for E will be a pair (f, T) where $f : X \to \mathbb{R}^{\mathbb{N}}$ is continuous, $T \in \mathcal{T}$ and $E = f^{-1}[\phi(T)]$. A family $\langle E_i\rangle_{i\in I}$ in $\mathcal{B}\mathfrak{a}_c(X)$ is now a **codable** family if there is a family $\langle (f_i, T^{(i)})\rangle_{i\in I}$ such that $(f_i, T^{(i)})$ codes E_i for every i.

(b)(i) Suppose that $\langle f_i\rangle_{i\in I}$ is a countable family of continuous functions from X to $\mathbb{R}^{\mathbb{N}}$, and $\langle T^{(i)}\rangle_{i\in I}$ a family in $\mathcal{T}$. Then there are a continuous function $f : X \to \mathbb{R}^{\mathbb{N}}$ and a sequence $\langle \hat T^{(i)}\rangle_{i\in\mathbb{N}}$ in $\mathcal{T}$ such that $(f, \hat T^{(i)})$ codes the same Baire set as $(f_i, T^{(i)})$ for every $i \in I$. **P** If I is empty, this is trivial. Otherwise, $I\times\mathbb{N}$ is countably infinite, so $(\mathbb{R}^{\mathbb{N}})^I \cong \mathbb{R}^{I\times\mathbb{N}}$ is homeomorphic to $\mathbb{R}^{\mathbb{N}}$; let $h : \mathbb{R}^{\mathbb{N}} \to (\mathbb{R}^{\mathbb{N}})^I$ be a homeomorphism, and set $f(x) = h^{-1}(\langle f_i(x)\rangle_{i\in I})$ for each $x \in \mathbb{N}$. Then $f_i = \pi_i h f$ for each i, where $\pi_i(z) = z(i)$ for $z \in (\mathbb{R}^{\mathbb{N}})^I$. Now $\langle (\pi_i h)^{-1}[V_n]\rangle_{i\in I,n\in\mathbb{N}}$ is a family of open sets in $\mathbb{R}^{\mathbb{N}}$, so is codable (562I, or otherwise); let $\langle T^{(i,n)}\rangle_{i\in I,n\in\mathbb{N}}$ be a family in $\mathcal{T}$ such that $\phi(T^{(i,n)}) = (\pi_i h)^{-1}[V_n]$ whenever $n \in \mathbb{N}$ and $i \in I$. The construction of part (a)(ii)⇒(iii) in the proof of 562M gives us a family $\langle\Theta_i\rangle_{i\in I}$ of functions from $\mathcal{T}$ to $\mathcal{T}$ such that $(\pi_i h)^{-1}[\phi(T)] = \phi(\Theta_i(T))$ whenever $i \in I$ and $T \in \mathcal{T}$. So we can take $\hat{T}^{(i)} = \Theta_i(T^{(i)})$, and we shall have

$$f_i^{-1}[\phi(T^{(i)})] = f^{-1}[(\pi_i h)^{-1}[\phi(T^{(i)})]] = f^{-1}[\phi(\Theta_i(T^{(i)}))] = f^{-1}[\phi(\hat{T}^{(i)})]$$

for every i, as required. **Q**

(ii) It follows that if $\langle E_i\rangle_{i\in\mathbb{N}}$ is a codable sequence in $\mathcal{B}\mathfrak{a}_c(X)$ then $\bigcup_{i\in\mathbb{N}} E_i$ and $\bigcap_{i\in\mathbb{N}} E_i$ belong to $\mathcal{B}\mathfrak{a}_c(X)$. **P** By (i), we have a continuous $f : X \to \mathbb{R}^{\mathbb{N}}$ and a sequence $\langle\hat{T}^{(i)}\rangle_{i\in\mathbb{N}}$ in $\mathcal{T}$ such that $E_i = f^{-1}[\phi(\hat{T}^{(i)})]$ for every $i \in \mathbb{N}$. Now 562K tells us that $F = \bigcup_{i\in\mathbb{N}} \phi(\hat{T}^{(i)})$ and $F' = \bigcap_{i\in\mathbb{N}} \phi(\hat{T}^{(i)})$ are codable, so $f^{-1}[F] = \bigcup_{i\in\mathbb{N}} E_i$ and $f^{-1}[F'] = \bigcap_{i\in\mathbb{N}} E_i$ belong to $\mathcal{B}\mathfrak{a}_c(X)$. **Q**

(iii) In particular, $\mathcal{B}\mathfrak{a}_c(X)$ is closed under finite intersections; as it is certainly closed under complementation, it is an algebra of subsets of X. Every zero set belongs to $\mathcal{B}\mathfrak{a}_c(X)$. **P** If $g : X \to \mathbb{R}$ is continuous, set $f(x)(i) = g(x)$ for $x \in X$, $i \in \mathbb{N}$; then $H = \{z : z \in \mathbb{R}^{\mathbb{N}}, z(0) = 0\}$ is closed, therefore a codable Borel set, and $g^{-1}[\{0\}] = f^{-1}[H]$ is a codable Baire set. **Q**

(iv) If Y is another topological space and $g : X \to Y$ is continuous, then $\langle g^{-1}[F_i]\rangle_{i\in I}$ is a codable family in $\mathcal{B}\mathfrak{a}_c(X)$ for every codable family $\langle F_i\rangle_{i\in I}$ in $\mathcal{B}\mathfrak{a}_c(Y)$. **P** If $\langle(f_i, T^{(i)})\rangle_{i\in I}$ codes $\langle F_i\rangle_{i\in I}$, then $\langle(f_i g, T^{(i)})\rangle_{i\in I}$ codes $\langle g^{-1}[F_i]\rangle_{i\in I}$. **Q**

(c)(i) A function $f : X \to \mathbb{R}$ is a **codable Baire function** if there are a continuous $g : X \to \mathbb{R}^{\mathbb{N}}$ and a codable Borel function $h : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ such that $f = hg$. A family $\langle f_i\rangle_{i\in I}$ of codable Baire functions is a **codable** family if there is a family $\langle(g_i, h_i)\rangle_{i\in I}$ such that $g_i : X \to \mathbb{R}^{\mathbb{N}}$ is a continuous function for every $i \in I$ and $\langle h_i\rangle_{i\in I}$ is a codable family of codable Borel functions from $\mathbb{R}^{\mathbb{N}}$ to $\mathbb{R}$.

(ii) Suppose that $\langle f_n\rangle_{n\in\mathbb{N}}$ is a codable sequence of codable Baire functions from X to $\mathbb{R}$. Then there are a continuous function $g : X \to \mathbb{R}^{\mathbb{N}}$ and a codable sequence $\langle h_n\rangle_{n\in\mathbb{N}}$ of codable Borel functions from $\mathbb{R}^{\mathbb{N}}$ to $\mathbb{R}$ such that $f_n = h_n g$ for every $n \in \mathbb{N}$. **P** Let $\langle(g_n, h'_n)\rangle_{n\in\mathbb{N}}$ be such that $g_n : X \to \mathbb{R}^{\mathbb{N}}$ is continuous for every n, $\langle h'_n\rangle_{n\in\mathbb{N}}$ is a codable sequence of codable Borel functions from $\mathbb{R}^{\mathbb{N}} \to \mathbb{R}$, and $f_n = h'_n g_n$ for each n. Now $\langle\{x : h'_n(x) > q\}\rangle_{n\in\mathbb{N},q\in\mathbb{Q}}$ is a codable family in $\mathcal{B}_c(\mathbb{R}^{\mathbb{N}})$; let $\langle T'_{nq}\rangle_{n\in\mathbb{N},q\in\mathbb{Q}}$ be a family in $\mathcal{T}$ coding it. By (b-i) above, there are a continuous function $g : X \to \mathbb{R}^{\mathbb{N}}$ and a family $\langle T^{(n,q)}\rangle_{n\in\mathbb{N},q\in\mathbb{N}}$ in $\mathcal{T}$ such that

$$g^{-1}[\phi(T^{(n,q)})] = g_n^{-1}[\phi(T'_{nq})] = \{x : f_n(x) > q\}$$

for every $n \in \mathbb{N}$ and $q \in \mathbb{Q}$.

To convert $\langle T^{(n,q)}\rangle_{n\in\mathbb{N},q\in\mathbb{Q}}$ into a code for a sequence of real-valued functions on $\mathbb{R}^{\mathbb{N}}$, I copy ideas from the proof of 562N. Let

$$\Theta_0 : \mathcal{T} \to \mathcal{T}, \quad \Theta_1 : \mathcal{T} \times \mathcal{T} \to \mathcal{T},$$

$$\Theta_3 : \mathcal{T} \times \mathcal{T} \to \mathcal{T}, \quad \tilde{\Theta}_1 : \textstyle\bigcup_{I\subseteq\mathbb{Q}} \mathcal{T}^I \to \mathcal{T}$$

be such that

$$\phi(\Theta_0(T)) = X \setminus \phi(T), \quad \phi(\Theta_1(T, T')) = \phi(T) \cup \phi(T'),$$

$$\phi(\Theta_3(T, T')) = \phi(T) \setminus \phi(T'), \quad \phi(\tilde{\Theta}_1(\tau)) = \textstyle\bigcup_{q\in I} \phi(\tau(q)))$$

for T, $T' \in \mathcal{T}$, $I \subseteq \mathbb{Q}$ and $\tau \in \mathcal{T}^I$. Now, for $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$, set

$$\tau'_n(\alpha) = \tilde{\Theta}_1(\langle T^{(n,q)}\rangle_{q\in\mathbb{Q},q\ge\alpha}),$$

$$T^{(n)} = \tilde{\Theta}_1(\langle\Theta_3(\tau'_n(-k), \tau'_n(k))\rangle_{k\in\mathbb{N}}),$$

$$\begin{aligned}\tau_n(\alpha) &= \Theta_3(\tau'_n(\alpha), \Theta_0(T^{(n)})) \text{ if } \alpha \ge 0,\\ &= \Theta_1(\tau'_n(\alpha), \Theta_0(T^{(n)})) \text{ if } \alpha < 0.\end{aligned}$$

We now have a sequence $\langle\tau_n\rangle_{n\in\mathbb{N}}$ in $\tilde{\mathcal{T}}$ (as defined in 562N) coding a sequence $\langle h_n\rangle_{n\in\mathbb{N}}$ of Borel functions from $\mathbb{R}^{\mathbb{N}}$ to $\mathbb{R}$ such that $f_n = h_n g$ for every n (see 562Sb). **Q**

(iii) If $\langle f_n\rangle_{n\in\mathbb{N}}$ is a codable sequence of codable Baire functions, there is a codable Baire function f such that $f(x) = \liminf_{n\to\infty} f_n(x)$ whenever the lim inf is finite. **P** Take g and $\langle h_n\rangle_{n\in\mathbb{N}}$ as in (i); by 562Ne, there is a codable Borel function h such that $h(z) = \liminf_{n\to\infty} h_n(z)$ whenever $z \in \mathbb{R}^{\mathbb{N}}$ is such that the $\liminf$ is finite, and $f = hg : X \to \mathbb{R}$ will serve. **Q**

(iv) The family of codable Baire functions is a Riesz subspace of $\mathbb{R}^X$ containing all continuous functions and closed under multiplication. (This time, use (i) and 562Nd.)

(v) The family of continuous real-valued functions on X is a codable family of codable Baire functions. (For $f \in C(X)$, define $g_f \in C(X;\mathbb{R}^{\mathbb{N}})$ by setting $g_f(x)(n) = f(x)$ for every $x \in X$ and $n \in \mathbb{N}$; setting $\pi_0(z) = z(0)$ for $z \in \mathbb{R}^{\mathbb{N}}$, $\langle (g_f, \pi_0)\rangle_{f\in C(X)}$ is a family of codes for $C(X)$.)

(vi) If $E \subseteq X$, then $E \in \mathcal{B}\mathfrak{a}_c(X)$ iff $\chi E : X \to \mathbb{R}$ is a codable Baire function. **P**

$$\begin{aligned}
E \in \mathcal{B}\mathfrak{a}_c(X) \iff & \text{ there are a continuous } g : X \to \mathbb{R}^{\mathbb{N}} \\
& \qquad \text{and an } F \in \mathcal{B}_c(\mathbb{R}^{\mathbb{N}}) \text{ such that } E = g^{-1}[F] \\
\iff & \text{ there are a continuous } g : X \to \mathbb{R}^{\mathbb{N}} \\
& \qquad \text{and an } F \in \mathcal{B}_c(\mathbb{R}^{\mathbb{N}}) \text{ such that } \chi E = (\chi F)g \\
\iff & \text{ there are a continuous } g : X \to \mathbb{R}^{\mathbb{N}} \\
& \qquad \text{and a codable Borel function } h : \mathbb{R}^{\mathbb{N}} \text{ such that } \chi E = hg
\end{aligned}$$

(562Nf, because if $\chi E = hg$ then $\chi E = (\chi F)g$ where $F = \{y : h(y) > 0\}$)

$$\iff \chi E \text{ is a codable Baire function. } \mathbf{Q}$$

(d) If $\langle f_n\rangle_{n\in\mathbb{N}}$ is a codable sequence of codable Baire functions from X to $\mathbb{R}$, then $\langle f_n^{-1}[H]\rangle_{n\in\mathbb{N},H\subseteq\mathbb{R}\text{ is open}}$ is codable. **P** By (c-i), we have a continuous $g : X \to \mathbb{R}^{\mathbb{N}}$ and a codable sequence $\langle h_n\rangle_{n\in\mathbb{N}}$ of codable Borel functions from $\mathbb{R}^{\mathbb{N}}$ to $\mathbb{R}$ such that $f_n = h_n g$ for every n. Let $\psi : \mathcal{T} \to \mathcal{B}_c(\mathbb{R})$ be an interpretation of Borel codes corresponding to some enumeration of a base for the topology of $\mathbb{R}$. By 562Md, g is codable; by 562M(a-iii), there is a function $\Theta : \mathcal{T} \to \mathcal{T}$ such that $g^{-1}[\psi(T)] = \phi(\Theta(T))$ for every $T \in \mathcal{T}$. Now $\langle h_n^{-1}[H]\rangle_{n\in\mathbb{N},H\subseteq\mathbb{R}\text{ is open}}$ is codable, by the definition in 562S; that is, there is a family $\langle T_{nH}\rangle_{n\in\mathbb{N},H\subseteq\mathbb{R}\text{ is open}}$ in $\mathcal{T}$ such that $\phi(T_{nH}) = h_n^{-1}[H]$ for all n and H. Now

$$f_n^{-1}[H] = g^{-1}[h_n^{-1}[H]] = g^{-1}[\phi(T_{nH})] = \psi(\Theta(T_{nH}))$$

for all n and H, so we have a coding of $\langle f_n^{-1}[H]\rangle_{n\in\mathbb{N},H\subseteq\mathbb{R}\text{ is open}}$. **Q**

562U Proposition Let $(X, \mathfrak{T})$ be a second-countable space. Then there is a second-countable topology $\mathfrak{S}$ on X, codably Borel equivalent to $\mathfrak{T}$, such that $\mathcal{B}_c(X) = \mathcal{B}\mathfrak{a}_c(X, \mathfrak{S})$ and the codable families in $\mathcal{B}_c(X)$ are exactly the codable families in $\mathcal{B}\mathfrak{a}_c(X, \mathfrak{S})$.

proof (a) By 562Pb there is a topology $\mathfrak{S}$ on X, finer than $\mathfrak{T}$, generated by a countable algebra $\mathcal{E}$ of subsets of X, which is codably Borel equivalent to $\mathfrak{T}$. Let $\langle U_n\rangle_{n\in\mathbb{N}}$ be a sequence running over $\mathcal{E}$. Define $g_0 : X \to \mathbb{R}^{\mathbb{N}}$ by setting $g_0(x) = \langle \chi U_n(x)\rangle_{n\in\mathbb{N}}$ for each $x \in X$. Then g_0 is continuous. Set $W_n = \{z : z \in \mathbb{R}^{\mathbb{N}}, z(n) > 0\}$ for each n, so that $W_n \subseteq \mathbb{R}^{\mathbb{N}}$ is open and $U_n = g_0^{-1}[W_n]$; let $\langle V_n\rangle_{n\in\mathbb{N}}$ be a sequence running over a base for the topology of $\mathbb{R}^{\mathbb{N}}$ and such that $V_{2n} = W_n$ for every n. Let $\phi : \mathcal{T} \to \mathcal{B}_c(X)$, $\phi' : \mathcal{T} \to \mathcal{B}_c(\mathbb{R}^{\mathbb{N}})$ be the interpretations of Borel codes corresponding to $\langle U_n\rangle_{n\in\mathbb{N}}$, $\langle V_n\rangle_{n\in\mathbb{N}}$ respectively.

(b) We have a function $\Theta : \mathcal{T} \to \mathcal{T} \setminus \{\emptyset\}$ such that $\phi(T) = g^{-1}[\phi'(\Theta(T))]$ for every $T \in \mathcal{T}$. **P** Induce on $r(T)$. As usual, set $A_T = \{n : {<}n{>} \in T\}$. If $r(T) = 0$, take $\Theta(T) \in \mathcal{T} \setminus \{\emptyset\}$ such that $\phi'(\Theta(T)) = \emptyset$. If $r(T) = 1$, set $\Theta(T) = \{{<}2n{>} : n \in A_T\}$; then

$$\phi'(\Theta(T)) = \bigcup\{V_{2n} : n \in A_T\}, \quad g_0^{-1}[\phi'(\Theta(T))] = \bigcup\{U_n : n \in A_T\} = \phi(T).$$

If $r(T) > 1$, set

$$\Theta(T) = \{{<}i{>} : i \in A_T\} \cup \{{<}i{>}^\frown\sigma : i \in A_T,\ \sigma \in \Theta(T_{<i>})\}.\ \mathbf{Q}$$

This means that if we have any codable family in $\mathcal{B}_c(X)$, coded by a family $\langle T^{(i)}\rangle_{i\in I}$ in $\mathcal{T}$, $\langle(g_0,\Theta(T^{(i)}))\rangle_{i\in I}$ will code the same family in $\mathcal{B}\mathfrak{a}_c(X,\mathfrak{S})$.

(c) Next, there is a function $\Phi : C((X,\mathfrak{S});\mathbb{R}^{\mathbb{N}})\times\mathcal{T}\to\mathcal{T}\setminus\{\emptyset\}$ such that $g^{-1}[\phi'(T))]=\phi(\Phi(g,T))$ for every $\mathfrak{S}$-continuous $g:X\to\mathbb{R}^{\mathbb{N}}$ and $T\in\mathcal{T}$. **P** If $r(T)\le 1$ and $g^{-1}[\phi'(T)]$ is empty take $\Phi(g,T)\in\mathcal{T}\setminus\{\emptyset\}$ such that $\phi(\Phi(g,T))=\emptyset$. If $r(T)=1$ and $g^{-1}[\phi'(T)]$ is not empty set

$$\Phi(g,T)=\{<\!n\!>:U_n\subseteq g^{-1}[\phi'(T)]\}.$$

If $r(T)>1$ set

$$\Phi(g,T)=\{<\!i\!>:i\in A_T\}\cup\{<\!i\!>^\frown\sigma:i\in A_T,\ \sigma\in\Phi(g,T_{<i>})\}.\ \mathbf{Q}$$

So given any codable family in $\mathcal{B}\mathfrak{a}_c(X,\mathfrak{S})$, coded by a family $\langle(g_i,T^{(i)})\rangle_{i\in I}$ in $C((X,\mathfrak{S});\mathbb{R}^{\mathbb{N}})\times\mathcal{T}$, $\langle\Phi(g_i,T^{(i)})\rangle_{i\in I}$ will code it in $\mathcal{B}_c(X)$.

562V A different use of Borel codes will appear when we come to re-examine a result in Volume 3. I will defer the application to 566O, but the first part of the argument fits naturally into the ideas of this section.

Theorem (a) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and $\langle a_n\rangle_{n\in\mathbb{N}}$ a sequence in $\mathfrak{A}$. Then we have an interpretation $\phi:\mathcal{T}\to\mathfrak{A}$ of Borel codes such that

$$\begin{aligned}\phi(T)&=\sup_{i\in A_T}a_i\text{ if }r(T)\le 1,\\&=\sup_{i\in A_T}1\setminus\phi(T_{<i>})\text{ if }r(T)>1,\end{aligned}$$

where $A_T=\{i:<\!i\!>\in T\}$ as usual.

(b) For $n\in\mathbb{N}$, set $E_n=\{x:x\in\{0,1\}^{\mathbb{N}},\ x(n)=1\}$. Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and $\langle a_n\rangle_{n\in\mathbb{N}}$ a sequence in $\mathfrak{A}$. Let $\phi:\mathcal{T}\to\mathfrak{A}$ and $\psi:\mathcal{T}\to\mathcal{P}(\{0,1\}^{\mathbb{N}})$ be the interpretations of Borel codes corresponding to $\langle a_n\rangle_{n\in\mathbb{N}}$ and $\langle E_n\rangle_{n\in\mathbb{N}}$. If T, $T'\in\mathcal{T}$ are such that $\phi(T)\not\subseteq\phi(T')$, then $\psi(T)\not\subseteq\psi(T')$.

proof (a) Define $\phi(T)$ inductively on the rank of T, as in 562Ba.

(b) Let $\langle T^{(n)}\rangle_{n\in\mathbb{N}}$ be a sequence running over $\{T,T'\}\cup\{T_\sigma:\sigma\in S^*\}\cup\{T'_\sigma:\sigma\in S^*\}$. Define $\langle c_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. $c_0=\phi(T)\setminus\phi(T')$. Given that $c_n\in\mathfrak{A}\setminus\{0\}$, then

— if $r(T^{(n)})\le 1$ and there is an $i\in A_{T^{(n)}}$ such that $c_n\cap a_i\ne 0$, take the first such i and set $c_{n+1}=c_n\cap a_i$;

— if $r(T^{(n)})>1$ and there is an $i\in A_{T^{(n)}}$ such that $c_n\setminus\phi(T^{(n)}_{<i>})\ne 0$, take the first such i and set $c_{n+1}=c_n\setminus\phi(T^{(n)}_{<i>})$;

— otherwise, set $c_{n+1}=c_n$.

Continue.

At the end of the induction, define $x\in\{0,1\}^{\mathbb{N}}$ by saying that $x(i)=1$ iff there is an $n\in\mathbb{N}$ such that $c_n\subseteq a_i$. Now we find that, for every $m\in\mathbb{N}$,

— if $x\in\psi(T^{(m)})$ there is an $n\in\mathbb{N}$ such that $c_n\subseteq\phi(T^{(m)})$,

— if $x\notin\psi(T^{(m)})$ there is an $n\in\mathbb{N}$ such that $c_n\cap\phi(T^{(m)})=0$.

P Induce on $r(T^{(m)})$. If $r(T^{(m)})\le 1$ then

$$\begin{aligned}x\in\psi(T^{(m)})&\implies\text{there is an }i\in A_{T^{(m)}}\text{ such that }x\in E_i\\&\implies\text{there are }i\in A_{T^{(m)}},\ n\in\mathbb{N}\text{ such that }c_n\subseteq a_i\\&\implies\text{there is an }n\in\mathbb{N}\text{ such that }c_n\subseteq\phi(T^{(m)}),\end{aligned}$$

$$\begin{aligned}x\notin\psi(T^{(m)})&\implies x\notin E_i\text{ for every }i\in A_{T^{(m)}}\\&\implies c_{m+1}\not\subseteq a_i\text{ for every }i\in A_{T^{(m)}}\\&\implies c_m\cap a_i=0\text{ for every }i\in A_{T^{(m)}}\implies c_m\cap\phi(T^{(m)})=0.\end{aligned}$$

If $r(T^{(m)})>1$ then

$$\begin{aligned} x \in \psi(T^{(m)}) &\implies \text{ there is an } i \in A_{T^{(m)}} \text{ such that } x \notin \psi(T^{(m)}_{<i>}) \\ &\implies \text{ there are } i \in A_{T^{(m)}},\ n \in \mathbb{N} \text{ such that } c_n \cap \phi(T^{(m)}_{<i>}) = 0 \end{aligned}$$

(by the inductive hypothesis, because $T^{(m)}_{<i>}$ is always equal to $T^{(k)}$ for some k, and $r(T^{(m)}_{<i>}) < r(T^{(m)})$)

$$\implies \text{ there is an } n \in \mathbb{N} \text{ such that } c_n \subseteq \phi(T^{(m)}),$$

$$\begin{aligned} x \notin \psi(T^{(m)}) &\implies x \in \psi(T^{(m)}_{<i>}) \text{ for every } i \in A_{T^{(m)}} \\ &\implies \text{ for every } i \in A_{T^{(m)}} \text{ there is an } n \in \mathbb{N} \text{ such that } c_n \subseteq \phi(T^{(m)}_{<i>}) \\ &\implies c_{m+1} \not\subseteq 1 \setminus \phi(T^{(m)}_{<i>}) \text{ for every } i \in A_{T^{(m)}} \\ &\implies c_m \setminus \phi(T^{(m)}_{<i>}) = 0 \text{ for every } i \in A_{T^{(m)}} \\ &\implies c_m \cap \phi(T^{(m)}) = 0. \ \mathbf{Q} \end{aligned}$$

In particular, since both T and T' appear in the list $\langle T^{(m)} \rangle_{m\in\mathbb{N}}$, $c_n \cap \phi(T) \neq 0$ and $c_n \cap \phi(T') = 0$ for every n, $x \in \psi(T) \setminus \psi(T')$ and $\psi(T) \not\subseteq \psi(T')$.

562X Basic exercises (a) Let X be a regular second-countable space. Show that a resolvable subset of X is F_σ. (*Hint*: in the proof of 562I, show that $\phi(T^{(\xi)})$ is always F_σ.)

(b) Let X be a second-countable space and $\langle E_{ni} \rangle_{n,i\in\mathbb{N}}$ a family of resolvable subsets of X. Show that $\bigcap_{n\in\mathbb{N}} \bigcup_{i\in\mathbb{N}} E_{ni}$ is a codable Borel set.

(c) Let X be a second-countable space and $\langle E_i \rangle_{i\in I}$ a codable family in $\mathcal{B}_c(X)$. (i) Show that $\langle E_i \rangle_{i\in J}$ is codable for every $J \subseteq I$. (ii) Show that if I is countable and not empty then $\bigcup_{i\in I} E_i$ and $\bigcap_{i\in I} E_i$ are codable Borel sets. (iii) Show that if $h : J \times \mathbb{N} \to I$ is a function, where J is any other set, then $\langle \bigcup_{n\in\mathbb{N}} E_{h(j,n)} \rangle_{j\in J}$ is a codable family. (iv) Show that if $\langle F_i \rangle_{i\in I}$ is another codable family in $\mathcal{B}_c(X)$ then $\langle E_i \cap F_i \rangle_{i\in I}$ and $\langle E_i \triangle F_i \rangle_{i\in I}$ are codable families.

(d) Let X and Y be second-countable spaces and $f : X \to Y$ a function. Suppose that $\{F : F \subseteq Y,\ f^{-1}[F] \text{ is resolvable}\}$ includes a countable network for the topology of Y. Show that f is a codable Borel function.

(e) Let X be a second-countable space and $\langle E_i \rangle_{i\in I}$ a family in $\mathcal{B}_c(X)$. (i) Show that $\{J : J \subseteq I,\ \langle E_i \rangle_{i\in J}$ is codable$\}$ is an ideal of subsets of I. (ii) Show that if every E_i is resolvable then $\langle E_i \rangle_{i\in I}$ is codable.

(f) Let X be a second-countable space and $f : X \to \mathbb{R}$ a function. Show that f is a codable Borel function iff $\{(x,\alpha) : x \in X,\ \alpha < f(x)\}$ is a codable Borel subset of $X \times \mathbb{R}$.

(g) Let X be a topological space and f, $g : X \to \mathbb{R}$ resolvable functions. (i) Show that $f \vee g$ and αf are resolvable for any $\alpha \in \mathbb{R}$. (ii) Show that if f is bounded then $f + g$ is resolvable. (iii) Show that if f and g are bounded, $f \times g$ is resolvable. (iv) Show that if f and g are non-negative, then $f + g$ is resolvable. (v) Show that if $h : \mathbb{R} \to \mathbb{R}$ is continuous and $h^{-1}[\{\alpha\}]$ is finite for every $\alpha \in \mathbb{R}$, then hf is resolvable.

(h) Let $f : \mathbb{R} \to \mathbb{R}$ be such that $\lim_{t\downarrow x} f(t)$ is defined in $[-\infty, \infty]$ for every $x \in \mathbb{R}$. Show that f is resolvable.

(i) Let X be a second-countable space and $\langle f_n \rangle_{n\in\mathbb{N}}$ a sequence of resolvable real-valued functions on X. Show that there is a codable Borel function g such that $g(x) = \lim_{n\to\infty} f_n(x)$ for any x such that the limit is defined in $\mathbb{R}$.

(j) Let X be a second-countable space and Y a subspace of X. Show that a family $\langle g_i \rangle_{i\in I}$ in $\mathbb{R}^Y$ is a codable family of codable Borel functions iff there is a codable family $\langle f_i \rangle_{i\in I}$ of real-valued codable Borel functions on X such that $g_i = f_i \restriction Y$ for every $i \in I$.

>(k) Let X be a second-countable space. Show that every codable family of codable Baire subsets of X is a codable family of codable Borel subsets of X.

>(l) Let X be a regular second-countable space. Show that every codable family of codable Borel subsets of X is a codable family of codable Baire subsets of X. (*Hint*: 561Xk.)

562Y Further exercises (a) Let X be a second-countable space, Y a T_0 second-countable space and $f : X \to Y$ a function with graph $\Gamma \subseteq X \times Y$. (i) Show that if f is a codable Borel function, then Γ is a codable Borel subset of $X \times Y$. (ii) Show that if X and Y are Polish and Γ is a codable Borel subset of $X \times Y$, then f is a codable Borel function.

(b) Show that there is an analytic subset of $\mathbb{N}^{\mathbb{N}}$ which is not a codable Borel set. (*Hint*: 423M.)

(c) Show that if X is a Polish space then a subset of X is resolvable iff it is both F_σ and G_δ.

(d) Let X be a Polish space. Show that a function $f : X \to \mathbb{R}$ is resolvable iff $\{x : \alpha < f(x) < \beta\}$ is F_σ for all α, $\beta \in \mathbb{R}$.

(e) Let X be a topological space. Let Φ be the set of functions $f : X \to \mathcal{P}\mathbb{N}$ such that $\{x : n \in f(x)\}$ is open for every $n \in \mathbb{N}$. Write $\mathcal{B}'_c(X)$ for $\{f^{-1}[F] : f \in \Phi,\ F \in \mathcal{B}_c(\mathcal{P}\mathbb{N})\}$; say a family $\langle E_i\rangle_{i\in I}$ in $\mathcal{B}'_c(X)$ is codable if there is a family $\langle(f_i, F_i)\rangle_{i\in I}$ in $\Phi \times \mathcal{B}_c(\mathcal{P}\mathbb{N})$ such that $\langle F_i\rangle_{i\in I}$ is codable and $E_i = f_i^{-1}[F_i]$ for every i. (i) Show that if X is second-countable then $\mathcal{B}'_c(X) = \mathcal{B}_c(X)$ and the codable families on the definition here coincide with the codable families of 562J. (ii) Develop a theory of codable Borel sets and functions corresponding to that in 562T.

562 Notes and comments The idea of 'Borel code' is of great importance in mathematical logic, for reasons quite separate from the questions addressed here; see Jech 78, Jech 03 or Kunen 80. (Of course it is not a coincidence that an approach which is effective in the absence of the axiom of choice should also be relevant to absoluteness in the presence of choice.) Every author has his favoured formula corresponding to that in 562Ba. The particular one I have chosen is intended to be economical and direct, but is slightly awkward at the initial stages, and some proofs demand an extra moment's attention to the special case of trees of rank 1. The real motivation for the calculations here will have to wait for §565; Lebesgue measure can be defined in such a way that it is countably additive with respect to *codable* sequences of Borel sets, and there are enough of these to make the theory non-trivial.

Borel codes are wildly non-unique, which is why the concept of codable family is worth defining. But it is also important that certain sets, starting with the open sets, are self-coding in the sense that from the set we can pick out an appropriate code. 'Resolvable' sets and functions (562G, 562Q) are common enough to be very useful, and for these we can work with the objects themselves, just as we always have, and leave the coding until we need it.

The Borel codes described here can be used only in second-countable spaces. It is easy enough to find variations of the concept which can be applied in more general contexts (562Ye), though it is not obvious that there are useful theorems to be got in such a way. More relevant to the work of the next few sections is the idea of 'codable Baire set' (562T). Because any codable sequence of codable Baire sets can be factored through a single continuous function to $\mathbb{R}^{\mathbb{N}}$ (562T(b-i)), we have easy paths to the elementary results given here.

Version of 3.12.13

563 Borel measures without choice

Having decided that a 'Borel set' is to be one obtainable by a series of operations described by a Borel code, it is a natural step to say that a 'Borel measure' should be one which respects these operations (563A). In regular spaces, such measures have strong inner and outer regularity properties also based on the

Borel coding (563D-563F), and we have effective methods of constructing such measures (563H). Analytic sets are universally measurable (563I). We can use similar ideas to give a theory of Baire measures on general topological spaces (563J-563K). In the basic case, of a second-countable space with a codably σ-finite measure, we have a measure algebra with many of the same properties as in the standard theory (563M-563N).

The theory would not be very significant if there were no interesting Borel-coded measures, so you may wish to glance ahead to §565 to confirm that Lebesgue measure can be brought into the framework developed here.

563A Definitions (a) (Foreman & Wehrung 91) Let X be a second-countable space and $\mathcal{B}_c(X)$ the algebra of codable Borel subsets of X. I will say that a **Borel-coded measure** on X is a functional $\mu : \mathcal{B}_c(X) \to [0, \infty]$ such that $\mu\emptyset = 0$ and $\mu(\bigcup_{n\in\mathbb{N}} E_n) = \sum_{n=0}^{\infty} \mu E_n$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a disjoint codable family in $\mathcal{B}_c(X)$.

I will try to remember to say 'Borel-coded measure' everywhere in this section, because these are dangerously different from the 'Borel measures' of §434. Their domains are not necessarily σ-algebras and while they are finitely additive they need not be countably additive even in the sense of 326I.

(b) As usual, I will say that a subset of X is **negligible** if it is included in a set of measure 0, which here must be a codable Borel set; the terms 'conegligible', 'almost everywhere', 'null ideal' will take their meanings from this. We can now define the **completion** of μ to be the natural extension of μ to the algebra $\{E\triangle A : E \in \mathcal{B}_c(X),\ A \text{ is } \mu\text{-negligible}\}$.

(c) Some of the other definitions from the ordinary theory can be transferred without difficulty (e.g., 'totally finite', 'probability'), but we may need to make some finer distinctions. For instance, I will say that a Borel-coded measure μ is **semi-finite** if $\sup\{\mu F : F \subseteq E,\ \mu F < \infty\} = \infty$ whenever $\mu E = \infty$; we no longer have the ordinary principle of exhaustion (215A), and the definition in 211F, taken literally, may be too weak. For 'locally finite', however, 411Fa can be taken just as it is, since all open sets are measurable.

(d) For 'σ-finite' we again have to make a choice. The definition in 211C calls only for 'a sequence of measurable sets of finite measure'. Here the following will be more useful: a Borel-coded measure on X is **codably σ-finite** if there is a codable sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{B}_c(X)$ such that $X = \bigcup_{n\in\mathbb{N}} E_n$ and μE_n is finite for every n.

563B Proposition Let $(X, \mathfrak{T})$ be a second-countable space and μ a Borel-coded measure on X.

(a) Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a codable sequence in $\mathcal{B}_c(X)$.

(i) $\mu(\bigcup_{n\in\mathbb{N}} E_n) \le \sum_{n=0}^{\infty} \mu E_n$.

(ii) If $\langle E_n\rangle_{n\in\mathbb{N}}$ is non-decreasing, $\mu(\bigcup_{n\in\mathbb{N}} E_n) = \lim_{n\to\infty} \mu E_n$.

(iii) If $\langle E_n\rangle_{n\in\mathbb{N}}$ is non-increasing and μE_0 is finite, then $\mu(\bigcap_{n\in\mathbb{N}} E_n) = \lim_{n\to\infty} \mu E_n$.

(b) μ is τ-additive.

(c) Suppose that $\mathfrak{T}$ is T_1. If $\mathcal{E}$ is the algebra of resolvable subsets of X (562H), then $\mu\restriction\mathcal{E}$ is countably additive in the sense that $\mu E = \sum_{n=0}^{\infty} \mu E_n$ for any disjoint family $\langle E_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$ such that $E = \sup_{n\in\mathbb{N}} E_n$ is defined in $\mathcal{E}$.

proof (a)(i) Set $F_n = E_n \setminus \bigcup_{i<n} E_i$ for $n \in \mathbb{N}$; then $\langle F_n\rangle_{n\in\mathbb{N}}$ is codable (562Kc), so

$$\mu(\textstyle\bigcup_{n\in\mathbb{N}} E_n) = \mu(\bigcup_{n\in\mathbb{N}} F_n) = \sum_{n=0}^{\infty} \mu F_n \le \sum_{n=0}^{\infty} \mu E_n.$$

(ii) If $\langle E_n\rangle_{n\in\mathbb{N}}$ is non-decreasing then, in the language of (i), $E_n = \bigcup_{i\le n} F_i$ for each n, so

$$\lim_{n\to\infty} \mu E_n = \lim_{n\to\infty} \textstyle\sum_{i=0}^{n} \mu F_i = \mu(\bigcup_{n\in\mathbb{N}} E_n).$$

(iii) Apply (ii) to $\langle E_0 \setminus E_n\rangle_{n\in\mathbb{N}}$. (As remarked in 562J, this will be a codable sequence.)

(b) Suppose that $\mathcal{G}$ is an upwards-directed family of open sets with union H. Set $\gamma = \sup_{G\in\mathcal{G}} \mu G$. Let $\langle U_n\rangle_{n\in\mathbb{N}}$ be a sequence running over a base for the topology of X, and for $n \in \mathbb{N}$ set

$$V_n = \textstyle\bigcup\{U_i : i \le n,\ U_i \subseteq G \text{ for some } G \in \mathcal{G}\}.$$

Then $\langle V_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of open sets with union H. As every V_n is resolvable, $\langle V_n\rangle_{n\in\mathbb{N}}$ is codable and

$$\mu H=\lim_{n\to\infty}\mu V_n\le\sup_{G\in\mathcal{G}}\mu G\le\mu H$$

by (a-ii).

(c) If $\langle E_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathcal{E}$ with a supremum E in $\mathcal{E}$, then $E\supseteq\bigcup_{n\in\mathbb{N}}E_n$. If $x\in E$ then $\{x\}$ is closed, because $\mathfrak{T}$ is T_1, so $\{x\}$ is resolvable (562H) and $E\setminus\{x\}\in\mathcal{E}$; as $E\setminus\{x\}$ is not an upper bound of $\{E_n:n\in\mathbb{N}\}$, $x\in\bigcup_{n\in\mathbb{N}}E_n$. So $E=\bigcup_{n\in\mathbb{N}}E_n$. Now $\langle E_n\rangle_{n\in\mathbb{N}}$ is codable, as noted in 562J, so $\mu E=\mu(\bigcup_{n\in\mathbb{N}}E_n)=\sum_{n=0}^\infty\mu E_n$.

563C Corollary Let X be a second-countable space, μ a Borel-coded measure on X and $\langle E_n\rangle_{n\in\mathbb{N}}$ a sequence of resolvable sets in X.

(a)(i) $\bigcup_{n\in\mathbb{N}}E_n$ is measurable;

(ii) $\mu(\bigcup_{n\in\mathbb{N}}E_n)\le\sum_{n=0}^\infty\mu E_n$;

(iii) if $\langle E_n\rangle_{n\in\mathbb{N}}$ is disjoint, $\mu(\bigcup_{n\in\mathbb{N}}E_n)=\sum_{n=0}^\infty\mu E_n$;

(iv) if $\langle E_n\rangle_{n\in\mathbb{N}}$ is non-decreasing, $\mu(\bigcup_{n\in\mathbb{N}}E_n)=\lim_{n\to\infty}\mu E_n$.

(b)(i) $\bigcap_{n\in\mathbb{N}}E_n$ is measurable;

(ii) if $\langle E_n\rangle_{n\in\mathbb{N}}$ is non-increasing and $\inf_{n\in\mathbb{N}}\mu E_n$ is finite, then $\mu(\bigcap_{n\in\mathbb{N}}E_n)=\lim_{n\to\infty}\mu E_n$.

proof (a) Use 562I to find a sequence of codes for $\langle E_n\rangle_{n\in\mathbb{N}}$, and apply 563B.

(b) follows, because $X\setminus E_n$ is resolvable for each n.

563D The next lemma is primarily intended as a basis for Theorem 563H, but it will be useful in 563F.

Lemma Let $(X,\mathfrak{T})$ be a regular second-countable space and $\mu:\mathfrak{T}\to[0,\infty]$ a functional such that

$\mu\emptyset=0$,

$\mu G\le\mu H$ if $G\subseteq H$,

μ is modular (definition: 413Qc),

$\mu(\bigcup_{n\in\mathbb{N}}G_n)=\lim_{n\to\infty}\mu G_n$ for every non-decreasing sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{T}$,

$\bigcup\{G:G\in\mathfrak{T},\,\mu G<\infty\}=X$.

(a) $\mu(\bigcup_{i\in I}G_i)\le\sum_{i\in I}\mu G_i$ for every countable family $\langle G_i\rangle_{i\in I}$ in $\mathfrak{T}$.

(b) There is a function $\pi^*:\mathfrak{T}\times\mathbb{N}\to\mathfrak{T}$ such that

$$X\setminus G\subseteq\pi^*(G,k),\quad \mu(G\cap\pi^*(G,k))\le 2^{-k}$$

whenever $G\in\mathfrak{T}$ and $k\in\mathbb{N}$.

(c) Let $\phi:\mathcal{T}\to\mathcal{B}_c(X)$ be an interpretation of Borel codes defined from a sequence running over $\mathfrak{T}$, where $\mathcal{T}$ is the set of subtrees of $\bigcup_{n\in\mathbb{N}}\mathbb{N}^n$ without infinite branches (562A, 562B). Then there are functions π, $\pi':\mathcal{T}\times\mathbb{N}\to\mathfrak{T}$ such that

$$\phi(T)\subseteq\pi(T,n),\quad X\setminus\phi(T)\subseteq\pi'(T,n),\quad \mu(\pi(T,n)\cap\pi'(T,n))\le 2^{-n}$$

for every $T\in\mathcal{T}$ and $n\in\mathbb{N}$.

proof (a) This is elementary. First, $\mu(G\cup H)\le\mu G+\mu H$ for all open sets G and H, because $\mu(G\cap H)\ge 0$. Next, if $\langle G_n\rangle_{n\in\mathbb{N}}$ is a sequence of open sets with union G, then

$$\mu G=\lim_{n\to\infty}\mu(\bigcup_{i\le n}G_i)\le\lim_{n\to\infty}\sum_{i=0}^n\mu G_i=\sum_{n=0}^\infty\mu G_n.$$

Now the step to general countable I is immediate.

(b) Set $I=\{n:n\in\mathbb{N},\,\mu U_n<\infty\}$; because μ is locally finite, $\{U_n:n\in I\}$ is a base for $\mathfrak{T}$. Given $G\in\mathfrak{T}$ and $k\in\mathbb{N}$, then for $n\in I$ and $m\in\mathbb{N}$ set

$$W_{nm}=\bigcup\{U_i:i\le m,\,\overline{U}_i\subseteq U_n\cap G\}.$$

Because $\mathfrak{T}$ is regular, $\bigcup_{m\in\mathbb{N}}W_{nm}=U_n\cap G$ and $\mu(U_n\cap G)=\lim_{m\to\infty}\mu W_{nm}$. Let m_n be the least integer such that $\mu W_{nm_n}\ge\mu(U_n\cap G)-2^{-k-n-1}$. Set

$$\pi^*(G,k)=\bigcup_{n\in I}U_n\setminus\overline{W}_{nm_n}\in\mathfrak{T}.$$

Because $\overline{W}_{nm} \subseteq U_n \cap G$ for all m and n, while $\bigcup_{n\in I} U_n = X$, $\pi^*(G,k) \supseteq X \setminus G$. Now

$$\mu(G \cap \pi^*(G,k)) \le \sum_{n\in I} \mu(G \cap U_n \setminus \overline{W}_{nm_n}) \le \sum_{n=0}^{\infty} 2^{-k-n-1} = 2^{-k},$$

as required.

(c) Define $\pi(T)$ and $\pi'(T)$ inductively on the rank $r(T)$ of T.

(i) If $r(T) = 0$, set $\pi(T,n) = \emptyset$ and $\pi'(T,n) = X$ for every n. If $r(T) = 1$ then $G = \phi(T)$ is open; set $\pi(T,n) = G$ and $\pi'(T,n) = \pi^*(G,n)$ for each n.

(ii) For the inductive step to $r(T) > 1$, set $A_T = \{i : <i> \in T\}$ and $T_{<i>} = \{\sigma : <i>^\frown\sigma \in T\}$ for $i \in \mathbb{N}$, as in 562A. Set

$$\pi(T,n) = \bigcup_{i\in A_T} \pi'(T_{<i>}, n+i+2),$$

$$\pi'(T,n) = \bigcup_{i\in A_T} (\pi(T_{<i>}, n+i+2) \cap \pi'(T_{<i>}, n+i+2)) \cup \pi^*(\pi(T,n), n+1).$$

Then

$$\phi(T) = \bigcup_{i\in A_T} X \setminus T_{<i>} \subseteq \bigcup_{i\in A_T} \pi'(T_{<i>}, n+i+2) = \pi(T,n),$$

$$\begin{aligned} X \setminus \phi(T) &= \bigcap_{i\in A_T} \phi(T_{<i>}) \subseteq (\pi(T,n) \cap \bigcap_{i\in A_T} \phi(T_{<i>})) \cup \pi^*(\pi(T,n), n+1) \\ &\subseteq \Bigl(\bigcup_{i\in A_T} \pi'(T_{<i>}, n+i+2) \cap \bigcap_{i\in A_T} \pi(T_{<i>}, n+i+2)\Bigr) \cup \pi^*(\pi(T,n), n+1) \\ &\subseteq \bigcup_{i\in A_T} \bigl(\pi'(T_{<i>}, n+i+2) \cap \pi(T_{<i>}, n+i+2)\bigr) \cup \pi^*(\pi(T,n), n+1) \\ &= \pi'(T,n), \end{aligned}$$

$$\begin{aligned} \mu(\pi(T,n) \cap \pi'(T,n)) &\le \sum_{i\in A_T} \mu(\pi(T_{<i>}, n+i+2) \cap \pi'(T_{<i>}, n+i+2)) \\ &\qquad + \mu(\pi(T,n) \cap \pi^*(\pi(T,n), n+1)) \\ &\le \sum_{i\in A_T} 2^{-n-i-2} + 2^{-n-1} \le 2^{-n} \end{aligned}$$

for every n, so the induction continues.

563E Lemma Let X be a second-countable space and M a non-empty upwards-directed set of Borel-coded measures on X. For each codable Borel set $E \subseteq X$, set $\nu E = \sup_{\mu\in\mathrm{M}} \mu E$. Then ν is a Borel-coded measure on X.

proof Immediate from the definition in 563Aa.

563F Proposition Let $(X, \mathfrak{T})$ be a second-countable space and μ a Borel-coded measure on X.

(a) For any $F \in \mathcal{B}_c(X)$, we have a Borel-coded measure μ_F on X defined by saying that $\mu_F E = \mu(E \cap F)$ for every $E \in \mathcal{B}_c(X)$.

(b) We have a semi-finite Borel-coded measure μ_{sf} defined by saying that

$$\mu_{\mathrm{sf}}(E) = \sup\{\mu F : F \in \mathcal{B}_c(X),\ F \subseteq E,\ \mu F < \infty\}$$

for every $E \in \mathcal{B}_c(X)$.

(c)(i) If μ is locally finite it is codably σ-finite.

(ii) If μ is codably σ-finite, it is semi-finite and there is a totally finite Borel-coded measure ν on X with the same null ideal as μ.

(iii) If μ is codably σ-finite, there is a non-decreasing codable sequence of codable Borel sets of finite measure which covers X.

(d) If X is regular then the following are equiveridical:

(i) μ is locally finite;

(ii) μ is semi-finite, outer regular with respect to the open sets and inner regular with respect to the closed sets;

(iii) μ is semi-finite and outer regular with respect to the open sets.

(e) If X is regular and μ is semi-finite, then μ is inner regular with respect to the closed sets of finite measure.

(f) If X is Polish and μ is semi-finite, then μ is inner regular with respect to the compact sets.

(g) If μ is locally finite, and ν is another Borel-coded measure on X agreeing with μ on the open sets, then $\nu = \mu$.

proof Fix a sequence $\langle U_n\rangle_{n\in\mathbb{N}}$ running over a base for the topology of X.

(a) The point is just that $\langle E_n \cap F\rangle_{n\in\mathbb{N}}$ is a codable family whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a codable family in $\mathcal{B}_c(X)$ and F is codable. (562J again.)

(b) Writing μ_F for the Borel-coded measure corresponding to a set F of finite measure, as in (a), we have an upwards-directed family of measures; by 563E, its supremum μ_{sf} is a Borel-coded measure. If $E \subseteq X$ is a codable Borel set and $\gamma < \mu_{\text{sf}}E$, then there is a set F of finite measure such that $\mu(E \cap F) \ge \gamma$; now

$$\gamma \le \mu_{\text{sf}}(E \cap F) = \mu(E \cap F) < \infty.$$

(c)(i) Set $I = \{i : i \in \mathbb{N},\, \mu U_i < \infty\}$; then $\langle U_i\rangle_{i\in I}$ is a codable family of sets of finite measure covering X.

(ii) Let $\langle H_n\rangle_{n\in\mathbb{N}}$ be a codable sequence of sets of finite measure covering X.

(α) If $E \in \mathcal{B}_c(X)$, then $\langle E \cap \bigcup_{i\le n} H_i\rangle_{n\in\mathbb{N}}$ is a non-decreasing codable sequence with union E, so

$$\mu E = \sup_{n\in\mathbb{N}} \mu(E \cap \bigcup_{i\le n} H_i) \le \sup\{\mu F : F \subseteq E,\, \mu F < \infty\} \le \mu E.$$

As E is arbitrary, μ is semi-finite.

(β) Let $\langle \epsilon_n\rangle_{n\in\mathbb{N}}$ be a sequence of strictly positive real numbers such that $\sum_{n=0}^{\infty} \epsilon_n \mu H_n$ is finite. Set $\nu E = \sum_{n=0}^{\infty} \epsilon_n \mu(E \cap H_n)$ for $E \in \mathcal{B}_c(X)$. Of course $\nu\emptyset = 0$ and $\nu X < \infty$. If $\langle E_k\rangle_{k\in\mathbb{N}}$ is a disjoint codable sequence in $\mathcal{B}_c(X)$, then $\langle E_k \cap H_n\rangle_{k\in\mathbb{N}}$ is codable for every n, so

$$\begin{aligned}\nu(\bigcup_{k\in\mathbb{N}} E_k) &= \sum_{n=0}^{\infty} \epsilon_n \mu(\bigcup_{k\in\mathbb{N}} E_k \cap H_n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \epsilon_n \mu(E_k \cap H_n) \\ &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \epsilon_n \mu(E_k \cap H_n) = \sum_{k=0}^{\infty} \nu E_k.\end{aligned}$$

So ν is a Borel-coded measure.

If $E \in \mathcal{B}_c(X)$ and $\mu E = 0$, then of course $\nu E = \sum_{n=0}^{\infty} \epsilon_n \mu(E \cap H_n) = 0$. Conversely, if $\nu E = 0$, then $\mu(E \cap H_n) = 0$ for every n; but $\langle E \cap H_n\rangle_{n\in\mathbb{N}}$, like $\langle H_n\rangle_{n\in\mathbb{N}}$, is codable, so $\mu E = \mu(\bigcup_{n\in\mathbb{N}} E \cap H_n) = 0$. Thus μ and ν have the same sets of zero measure; it follows at once that they have the same null ideals.

(iii) All we have to note is that if $\langle E_n\rangle_{n\in\mathbb{N}}$ is a codable sequence of sets of finite measure covering X, then $\langle \bigcup_{i\le n} E_i\rangle_{n\in\mathbb{N}}$ is codable (562Kb), so gives the required non-decreasing witness.

(d)(i)$\Rightarrow$(ii)(α) Observe first that $\mu{\restriction}\mathfrak{T}$ satisfies the conditions of 563D. **P** The first three are consequences of the fact that $\mu : \mathcal{B}_c(X) \to [0, \infty]$ is additive. If $\langle G_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{T}$, it is a codable sequence of codable Borel sets, by 562I as usual; so $\mu(\bigcup_{n\in\mathbb{N}} G_n) = \lim_{n\to\infty} \mu G_n$ by 563B(a-ii). Finally, we are assuming that μ is locally finite, so the last condition is satisfied. **Q**

Take an interpretation ϕ of Borel codes and functions $\pi, \pi' : \mathcal{T} \times \mathbb{N} \to \mathfrak{T}$ as in 563Dc.

(β) If $E \in \mathcal{B}_c(X)$ and $\mu E < \gamma$, take n such that $2^{-n} \le \gamma - \mu E$. There is a $T \in \mathcal{T}$ such that $\phi(T) = E$, and now $G = \pi(T, n)$ is open, $E \subseteq G$ and

$$\mu(G \setminus E) \le \mu(G \cap \pi'(T, n)) \le 2^{-n},$$

so $\mu G \le \gamma$.

(γ) If $E \in \mathcal{B}_c(X)$ and $\gamma < \mu E$, take $T \in \mathcal{T}$ such that $\phi(T) = E$ and $n \in \mathbb{N}$ such that $2^{-n} < \mu E - \gamma$; now $F = X \setminus \pi'(T, n)$ is closed, $F \subseteq E$ and $\mu(E \setminus F) \le 2^{-n}$, so $\mu F > \gamma$. Next, if we set

$$F_m = F \cap \bigcup\{\overline{U_i} : i \le m,\, \mu\overline{U}_i < \infty\},$$

$\langle F_m\rangle_{m\in\mathbb{N}}$ will be a non-decreasing sequence of closed sets of finite measure with union F. The sets F_m are all resolvable, so $\mu F = \lim_{m\to\infty} \mu F_m$ and there is an m such that $\mu F_m \ge \gamma$, while $F_m \subseteq E$ is a set of finite measure. As E and γ are arbitrary, μ is inner regular with respect to the closed sets and also semi-finite.

(ii)$\Rightarrow$(iii) is trivial.

(iii)$\Rightarrow$(i)(α) If $x \in X$ then x belongs to some set of finite measure. **P** Set

$$F = \bigcap\{U_n : n \in \mathbb{N},\, x \in U_n\} \setminus \bigcup\{U_n : n \in \mathbb{N},\, x \notin U_n\}.$$

Then F is a codable Borel set, being the difference of G_δ sets (562Da), and the subspace topology on F is indiscrete. If $\mu F = 0$ we can stop. Otherwise, there must be an $F' \subseteq F$ such that $0 < \mu F' < \infty$; but $F' \in \mathcal{B}_c(F) = \{\emptyset, F\}$ (562E), so $F' = F$ and again F has finite measure. **Q**

(β) Now as μ is outer regular with respect to the open sets, every set of finite measure is included in an open set of finite measure. So μ must be locally finite.

(e) Suppose that $E \in \mathcal{B}_c(X)$ and $\gamma < \mu E$. Then there is an $H \in \mathcal{B}_c(X)$ such that $H \subseteq E$ and $\gamma < \mu H < \infty$. Consider the Borel-coded measure μ_H defined from μ and H as in (a). This is totally finite, so (d) tells us that it is outer regular with respect to the open sets and therefore inner regular with respect to the closed sets, and there is a closed set $F \subseteq H$ such that $\mu F = \mu_H F \ge \gamma$. As E and γ are arbitrary, μ is inner regular with respect to the closed sets of finite measure.

(f) Let ρ be a complete metric on X inducing its topology. If $E \in \mathcal{B}_c(X)$ and $\gamma < \mu E$, let $F \subseteq E$ be a closed set such that $F \subseteq E$ and $\gamma < \mu F < \infty$. For each $n \in \mathbb{N}$ set $J_n = \{i : \operatorname{diam} U_i \le 2^{-n}\}$. Define $\langle k_n\rangle_{n\in\mathbb{N}}$, $\langle F_n\rangle_{n\in\mathbb{N}}$ inductively by saying that $F_0 = F$ and

$$k_n = \min\{k : \mu(F_n \cap \bigcup_{i\in J_n\cap k} U_i) > \gamma\}, \quad F_{n+1} = F_n \cap \bigcup_{i\in J_n\cap k_n} \overline{U}_i$$

for each n; set $K = \bigcap_{n\in\mathbb{N}} F_n \subseteq E$. Then $\mu K = \lim_{n\to\infty} \mu F_n \ge \gamma$. The point is that K is compact. **P** Set $L = \prod_{n\in\mathbb{N}} J_n \cap k_n \subseteq \mathbb{N}^{\mathbb{N}}$. Then L is compact (561D). Set $L' = \{\alpha : \alpha \in L,\, F \cap \bigcap_{i\le n} \overline{U}_{\alpha(i)} \ne \emptyset$ for every $n\}$; then L' is a closed subset of L so is compact. For $\alpha \in L'$, $\{F \cap \overline{U}_{\alpha(i)} : i \in \mathbb{N}\}$ generates a filter $\mathcal{F}_\alpha$ on X which is a Cauchy filter because $\operatorname{diam}\overline{U}_{\alpha(i)} = \operatorname{diam} U_{\alpha(i)} \le 2^{-i}$ for every i; because X is ρ-complete, $f(\alpha) = \lim \mathcal{F}_\alpha$ is defined, and belongs to $F \cap \bigcap_{i\in\mathbb{N}} \overline{U}_{\alpha(i)} \subseteq K$. If $\alpha, \beta \in L'$ and $\alpha(i) = \beta(i)$, then $f(\alpha)$, $f(\beta)$ both belong to $\overline{U}_{\alpha(i)}$ so $\rho(f(\alpha), f(\beta)) \le 2^{-i}$; thus f is continuous and $f[L']$ is a compact subset of K. On the other hand, given $x \in K$, we can set $\alpha(n) = \min\{i : i \in J_n \cap k_n,\, x \in \overline{U}_i\}$ for each n, and now $\alpha \in L'$ and $f(\alpha) = x$. So $K = f[L']$ is compact. (See 561Yj.) **Q**

As E and γ are arbitrary, μ is inner regular with respect to the compact sets.

(g)(i) Consider first the case in which X is regular. In this case both μ and ν must be outer regular with respect to the open sets, by (d); as they agree on the open sets they must be equal.

(ii) Next, suppose that $\mu X = \nu X$ is finite. Let $\mathcal{E}$ be the algebra of subsets of X generated by $\{U_n : n \in \mathbb{N}\}$, and $\mathfrak{S}$ the topology generated by $\mathcal{E}$. As noted in the argument of 562Pb, $\mathfrak{S}$ is codably Borel equivalent to the original topology of X, so μ and ν are still Borel-coded measures with respect to $\mathfrak{S}$, and are still locally finite, because $\mathfrak{S}$ is finer than $\mathfrak{T}$; while $\mathfrak{S}$ is regular. Now any member of $\mathcal{E}$ is expressible in the form $E = \bigcup_{i\le n} G_i \setminus H_i$ where the G_i, H_i are open and $\langle G_i \setminus H_i\rangle_{i\le n}$ is disjoint. So

$$\mu E = \sum_{i=0}^n \mu G_i - \mu(G_i \cap H_i) = \nu E.$$

More generally, if $H \in \mathfrak{S}$, there is a non-decreasing sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$ with union H; as all the sets in $\mathcal{E}$ are resolvable, $\langle E_n\rangle_{n\in\mathbb{N}}$ is codable and

$$\mu H = \sup_{n\in\mathbb{N}} \mu E_n = \nu H.$$

Thus μ and ν agree on $\mathfrak{S}$; by (i), they are equal.

(iii) Finally, for the general case, set $V_n = \bigcup\{U_i : i \le n,\, \mu U_i < \infty\}$ for each n. Because μ is locally finite, $\bigcup_{n\in\mathbb{N}} V_n = X$. For each $n \in \mathbb{N}$ let μ_{V_n}, ν_{V_n} be the Borel-coded measures defined from V_n as in (a).

Then μ_{V_n} and ν_{V_n} are totally finite and agree on the open sets, so are equal. Now $\langle V_n\rangle_{n\in\mathbb{N}}$, being a sequence of open sets, is codable; so if $E \in \mathcal{B}_c(X)$ the sequence $\langle E\cap V_n\rangle_{n\in\mathbb{N}}$ is codable, and

$$\mu E = \lim_{n\to\infty}\mu(E\cap V_n) = \lim_{n\to\infty}\mu_{V_n}E = \nu E.$$

So in this case also we have $\mu = \nu$.

563G Proposition Let X be a set and $\theta : \mathcal{P}X \to [0,\infty]$ a submeasure.

(a)

$$\Sigma = \{E : E \subseteq X,\, \theta A = \theta(A\cap E) + \theta(A\setminus E) \text{ for every } A \subseteq X\}$$

is an algebra of subsets of X, and $\theta\restriction\Sigma$ is additive in the sense that $\theta(E\cup F) = \theta E + \theta F$ in $[0,\infty]$ whenever E, $F \in \Sigma$ are disjoint.

(b) If $E \subseteq X$ and for every $\epsilon > 0$ there is an $F \in \Sigma$ such that $E \subseteq F$ and $\theta(F\setminus E) \le \epsilon$, then $E \in \Sigma$.

proof (a) Parts (a)-(c) of the proof of 113C apply unchanged.

(b) Take any $A \subseteq X$ and $\epsilon > 0$. Let $F \in \Sigma$ be such that $E \subseteq F$ and $\theta(F\setminus E) \le \epsilon$. Then

$$\theta A \le \theta(A\cap E) + \theta(A\setminus E) \le \theta(A\cap F) + \theta(A\setminus F) + \theta(F\setminus E) \le \theta A + \epsilon.$$

As ϵ is arbitrary, $\theta A = \theta(A\cap E) + \theta(A\setminus E)$; as A is arbitrary, $E \in \Sigma$.

563H Theorem Let $(X,\mathfrak{T})$ be a regular second-countable space and $\mu : \mathfrak{T} \to [0,\infty]$ a functional such that

$\mu\emptyset = 0$,

$\mu G \le \mu H$ if $G \subseteq H$,

mu is modular,

$\mu(\bigcup_{n\in\mathbb{N}} G_n) = \lim_{n\to\infty}\mu G_n$ for every non-decreasing sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{T}$,

$\bigcup\{G : G \in \mathfrak{T},\, \mu G < \infty\} = X$.

Then μ has a unique extension to a Borel-coded measure on X.

proof (a) For $A \subseteq X$ set $\theta A = \inf\{\mu G : A \subseteq G \in \mathfrak{T}\}$. Then θ is a submeasure on $\mathcal{P}X$ (because $\mu(G\cup H) \le \mu G + \mu H$ for all G, $H \in \mathfrak{T}$), extending μ (because $\mu G \le \mu H$ if $G \subseteq H$). Set

$$\Sigma = \{E : E \subseteq X,\, \theta A = \theta(A\cap E) + \theta(A\setminus E) \text{ for every } A \subseteq X\}$$

and $\nu = \theta\restriction\Sigma$, as in 563G. Let $\phi : \mathcal{T} \to \mathcal{B}_c(X)$ be an interpretation of Borel codes and π, $\pi' : \mathcal{T}\times\mathbb{N} \to \mathfrak{T}$ corresponding functions as in 563Dc. Now $\mathcal{B}_c(X) \subseteq \Sigma$. **P** Given $T \in \mathcal{T}$, $A \subseteq X$ and $n \in \mathbb{N}$, let $G \in \mathfrak{T}$ be such that $A \subseteq G$ and $\mu G \le \theta A + 2^{-n}$. Then

$$\begin{aligned}\theta A &\le \theta(A\cap\phi(T)) + \theta(A\setminus\phi(T)) \le \theta(A\cap\pi(T,n)) + \theta(A\cap\pi'(T,n))\\ &\le \mu(G\cap\pi(T,n)) + \mu(G\cap\pi'(T,n))\\ &= \mu(G\cap(\pi(T,n)\cup\pi'(T,n)) + \mu(G\cap\pi(T,n)\cap\pi'(T,n))\\ &\le \mu G + \mu(\pi(T,n)\cap\pi'(T,n)) \le \theta A + 2^{-n+1}.\end{aligned}$$

As A and n are arbitrary, $\phi(T) \in \Sigma$. **Q**

(b) Let $\langle T_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{T}$ such that $\langle E_n\rangle_{n\in\mathbb{N}}$ is disjoint, where $E_n = \phi(T_n)$ for each n; set $E = \bigcup_{n\in\mathbb{N}} E_n$. Then, for any $k \in \mathbb{N}$, $E \subseteq \bigcup_{n\in\mathbb{N}}\pi(T_n,k+n)$, so

$$\sum_{n=0}^{\infty}\nu E_n = \lim_{n\to\infty}\nu(\bigcup_{i\le n}E_i) \le \nu E \le \nu(\bigcup_{n\in\mathbb{N}}\pi(T_n,k+n))$$

(563Da)

$$= \sum_{n=0}^{\infty}\nu E_n + \nu(\pi(T_n,k+n)\setminus E_n)$$

(563Ga)

$$\le \sum_{n=0}^{\infty} \nu E_n + \mu(\pi(T_n, k+n) \cap \pi'(T_n, k+n))$$
$$\le \sum_{n=0}^{\infty} \nu E_n + 2^{-k-n} = 2^{-k+1} + \sum_{n=0}^{\infty} \nu E_n;$$

as k is arbitrary, $\sum_{n=0}^{\infty} \nu E_n = \nu E$; as $\langle T_n \rangle_{n\in\mathbb{N}}$ is arbitrary, $\nu \restriction \mathcal{B}_c(X)$ is a Borel-coded measure extending μ.

(c) At the same time we see that if λ is any other Borel-coded measure extending μ, we must have $\lambda E \le \theta E = \nu E$ for every $E \in \mathcal{B}_c(X)$. In the other direction,

$$\lambda(\phi(T)) \ge \lambda(\pi(T, n)) - \lambda(\pi(T, n) \cap \pi'(T, n))$$
$$= \mu(\pi(T, n)) - \mu(\pi(T, n) \cap \pi'(T, n)) \ge \nu(\phi(T)) - 2^{-n}$$

for every $T \in \mathcal{T}$ and $n \in \mathbb{N}$, so $\lambda E \ge \nu E$ for every $E \in \mathcal{B}_c(X)$. Thus $\nu \restriction \mathcal{B}_c(X)$ is the only Borel-coded measure extending μ.

563I Theorem Let X be a Hausdorff second-countable space, μ a codably σ-finite Borel-coded measure on X, and $A \subseteq X$ an analytic set. Then there are a codable Borel set $E \supseteq A$ and a sequence $\langle K_n \rangle_{n\in\mathbb{N}}$ of compact subsets of A such that $E \setminus \bigcup_{n\in\mathbb{N}} K_n$ is negligible. Consequently A is measured by the completion of μ.

proof (a) By 563F(c-ii), there is a totally finite Borel-coded measure on X with the same negligible sets as μ; so it will be enough to consider the case in which μ itself is totally finite.

If A is empty, the result is trivial. So we may suppose that there is a continuous surjection $f : \mathbb{N}^{\mathbb{N}} \to A$. For $\sigma \in S = \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$ set $I_\sigma = \{\alpha : \sigma \subseteq \alpha \in \mathbb{N}^{\mathbb{N}}\}$. Fix on a sequence running over a base for the topology of X and the corresponding interpretation $\phi : \mathcal{T} \to \mathcal{B}_c(X)$ of Borel codes.

(b) For $\sigma \in S$ and $\xi < \omega_1$ define $E_{\sigma\xi}$ by saying that

$$E_{\sigma 0} = \overline{f[I_\sigma]},$$

$$E_{\sigma,\xi+1} = \bigcup_{i\in\mathbb{N}} E_{\sigma^\frown<i>,\xi},$$

$$E_{\sigma\xi} = \bigcap_{\eta<\xi} E_{\sigma\eta} \text{ if } \xi > 0 \text{ is a countable limit ordinal.}$$

Then $\langle E_{\sigma\xi} \rangle_{\xi<\omega_1}$ is a non-increasing family of sets including $f[I_\sigma]$.

(c) For every $\xi < \omega_1$, $\langle E_{\sigma\eta} \rangle_{\sigma\in S, \eta\le\xi}$ is a codable family of codable Borel sets. **P** It is enough to consider the case $\xi \ge \omega$. Because ξ is countable, we have a function $\tilde{\Theta}_1 : \bigcup_{J\subseteq\xi} \mathcal{T}^J \to \mathcal{T}$ such that $\phi(\tilde{\Theta}_1(\langle T_\eta \rangle_{\eta\in J})) = \bigcup_{\eta\in J} \phi(T_\eta)$ for every $J \subseteq \xi$ (562Cb). Also, of course, we have a function $\Theta_0 : \mathcal{T} \to \mathcal{T}$ such that $\phi(\Theta_0(T)) = X \setminus \phi(T)$ for every $T \in \mathcal{T}$. Next, all the sets $E_{\sigma 0}$ are closed, therefore resolvable. So we have a family $\langle T_{\sigma 0} \rangle_{\sigma\in S}$ in $\mathcal{T}$ such that $\phi(T_{\sigma 0}) = E_{\sigma 0}$ for every σ. Now we can set

$$T_{\sigma,\eta+1} = \tilde{\Theta}_1(\langle T_{\sigma^\frown<i>,\eta} \rangle_{i\in\mathbb{N}})$$

if $\eta < \xi$,

$$T_{\sigma\eta} = \Theta_0(\tilde{\Theta}_1(\langle \Theta_0(T_{\sigma\zeta}) \rangle_{\zeta<\eta}))$$

if $\eta \le \xi$ is a non-zero limit ordinal, and $\phi(T_{\sigma\eta})$ will be equal to $E_{\sigma\eta}$ as required. **Q**

(d) Let $\langle \epsilon_\sigma \rangle_{\sigma\in S}$ be a summable family of strictly positive real numbers, and for $\xi < \omega_1$ set

$$\gamma(\xi) = \sum_{\sigma\in S} \epsilon_\sigma \mu(E_{\sigma\xi}).$$

Then $\gamma : \omega_1 \to \mathbb{R}$ is non-increasing. There is therefore a $\xi < \omega_1$ such that $\gamma(\xi + 1) = \gamma(\xi)$ (561A), that is, $\mu(E_{\sigma,\xi+1}) = \mu(E_{\sigma\xi})$ for every $\sigma \in S$.

(e)(i) Set $E = E_{\emptyset\xi}$. Of course $A = f[I_\emptyset] \subseteq E$. Now define $\alpha_n \in \mathbb{N}^{\mathbb{N}}$, for $n \in \mathbb{N}$, as follows. Given $\langle \alpha_n(i) \rangle_{i<m}$, set

$$G_{nm} = \bigcup\{E_{\sigma\xi} : \sigma \in \mathbb{N}^m, \sigma(i) \le \alpha_n(i) \text{ for every } i < m\},$$

$$G_{nmk}=\bigcup\{E_{\sigma^\frown<j>,\xi}:\sigma\in\mathbb{N}^m,\ j\le k,\ \sigma(i)\le\alpha_n(i)\text{ for every }i<m\},$$

Then $\langle G_{nm}\rangle_{n,m\in\mathbb{N}}$ is codable, and $\lim_{k\to\infty}\mu G_{nmk}=\mu G_{nm}$ for all m, $n\in\mathbb{N}$. **P** By (c), there is a family $\langle T_{\sigma\eta}\rangle_{\sigma\in S,\eta\le\xi+1}$ in $\mathcal{T}$ such that $\phi(T_{\sigma\eta})=E_{\sigma\eta}$ whenever $\sigma\in S$ and $\eta\le\xi+1$. This time, we need a function $\tilde\Theta_1:\bigcup_{J\subseteq S}\mathcal{T}^J\to\mathcal{T}$ such that $\phi(\tilde\Theta_1(\langle T_\sigma\rangle_{\sigma\in J}))=\bigcup_{\sigma\in J}\phi(T_\sigma)$ whenever $J\subseteq S$ and $\langle T_\sigma\rangle_{\sigma\in J}$ is a family in $\mathcal{T}$, and a function $\Theta_3:\mathcal{T}\times\mathcal{T}\to\mathcal{T}$ such that $\phi(\Theta_3(T,T'))=\phi(T)\setminus\phi(T')$ for all T, $T'\in\mathcal{T}$. Setting

$$T'_{nm}=\tilde\Theta_3(\langle T_{\sigma\xi}\rangle_{\sigma\in\mathbb{N}^m,\,\sigma(i)\le\alpha_n(i)\forall i<m}),$$

$$T'_{nmk}=\tilde\Theta_3(\langle T_{\sigma^\frown<j>,\xi}\rangle_{\sigma\in\mathbb{N}^m,\,j\le k,\,\sigma(i)\le\alpha_n(i)\forall i<m}),$$

we have $\phi(T'_{nm})=G_{nm}$ and $\phi(T'_{nmk})=G_{nmk}$ for all m, n, $k\in\mathbb{N}$. In particular, all the G_{nm} and G_{nmk} are codable Borel sets, and $\langle G_{nm}\rangle_{n,m\in\mathbb{N}}$ is codable. Moreover, for any particular pair m and n, $\langle G_{nmk}\rangle_{k\in\mathbb{N}}$ is a codable sequence; we therefore have $\lim_{k\to\infty}\mu G_{nmk}=\mu G$, where $G=\bigcup_{k\in\mathbb{N}}G_{nmk}$. Next,

$$G=\bigcup\{E_{\sigma,\xi+1}:\sigma\in\mathbb{N}^m,\ \sigma(i)\le\alpha_n(i)\text{ for every }i<m\},$$

so

$$G\triangle G_{nm}\subseteq\bigcup\{E_{\sigma\xi}\setminus E_{\sigma,\xi+1}:\sigma\in S\}.$$

Since $\langle E_{\sigma\xi}\setminus E_{\sigma,\xi+1}\rangle_{\sigma\in S}$ is a countable family of negligible sets coded by $\langle\Theta_3(T_{\sigma\xi},T_{\sigma,\xi+1})\rangle_{\sigma\in S}$, $G\triangle G_{nm}$ also is negligible and

$$\mu G_{nm}=\mu G=\lim_{k\to\infty}\mu G_{nmk}.\ \mathbf{Q}$$

Take the least $\alpha_n(m)\in\mathbb{N}$ such that $\mu G_{n,m,\alpha_n(m)}\ge\mu G_{nm}-2^{-n-m}$, and continue.

(ii) Set

$$L_n=\{\alpha:\alpha\in\mathbb{N}^\mathbb{N},\ \alpha(i)\le\alpha_n(i)\text{ for every }i\in\mathbb{N}\}.$$

Then L_n is compact (561D), and $f[L_n]\subseteq A$. Also $f[L_n]\supseteq\bigcap_{m\in\mathbb{N}}G_{nm}$. **P** If $x\in\bigcap_{m\in\mathbb{N}}G_{nm}$, then for each $m\in\mathbb{N}$ let σ_m be the lexicographically first member of $\{\sigma:\sigma\in\mathbb{N}^m,\ \sigma(i)\le\alpha_n(i)\text{ for every }i<m\}$ such that $x\in E_{\sigma_m,\xi}$, and let $\beta_m\in\mathbb{N}^\mathbb{N}$ be such that $\sigma_m\subseteq\beta_m$ and $\beta_m(i)=0$ for $i\ge m$. Then $\beta_m\in L_n$ for every m, so $\langle\beta_m\rangle_{m\in\mathbb{N}}$ has a cluster point $\alpha\in L_n$. **?** If $f(\alpha)\ne x$, we have an open neighbourhood U of $f(\alpha)$ such that $x\notin\overline{U}$. Let $m\in\mathbb{N}$ be such that $I_{\alpha\restriction m}\subseteq f^{-1}[U]$; then there is a $k\ge m$ such that $\alpha\restriction m=\beta_k\restriction m=\sigma_k\restriction m$. Now

$$x\in E_{\sigma_k,\xi}\subseteq E_{\sigma_k,0}\subseteq\overline{f[I_{\sigma_k}]}\subseteq\overline{f[I_{\alpha\restriction m}]}\subseteq\overline{U}.\ \mathbf{X}$$

So $x=f(\alpha)\in f[L_n]$. **Q**

But $\langle G_{nm}\rangle_{m\in\mathbb{N}}$ is codable, and $G_{n0}=E$, so we must have

$$\begin{aligned}\mu(E\setminus f[L_n])&\le\mu(E\setminus G_{n0})+\sum_{m=0}^\infty\mu(G_{nm}\setminus G_{n,m+1})\\&=\sum_{m=0}^\infty\mu(G_{nm}\setminus G_{n,m,\alpha_n(m)})\le\sum_{m=0}^\infty 2^{-n-m}=2^{-n+1}.\end{aligned}$$

(Of course $f[L_n]$ is compact, therefore closed, therefore measurable.)

(f) Set $K_n=f[L_n]$ for each n. Then $\langle K_n\rangle_{n\in\mathbb{N}}$ is a sequence of compact subsets of A; because the K_n are resolvable, $F=\bigcup_{n\in\mathbb{N}}K_n$ is a codable Borel set. For each n,

$$\mu(E\setminus F)\le\mu(E\setminus K_n)\le 2^{-n+1};$$

so $E\setminus F$ is negligible. Thus E and $\langle K_n\rangle_{n\in\mathbb{N}}$ have the required properties.

Of course it now follows that $E\setminus A\subseteq E\setminus F$ is negligible, so that the completion of μ measures A.

563J Baire-coded measures Working from 562T, we can develop a theory of Baire measures on general topological spaces, as follows. If X is a topological space, and $\mathcal{B}\mathfrak{a}_c(X)$ its algebra of codable Baire sets, a **Baire-coded measure** on X will be a function $\mu:\mathcal{B}\mathfrak{a}_c(X)\to[0,\infty]$ such that $\mu\emptyset=0$ and $\mu(\bigcup_{n\in\mathbb{N}}E_n)=\sum_{n=0}^\infty\mu E_n$ for every disjoint codable sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{B}\mathfrak{a}_c(X)$.

563K Proposition (a) If X and Y are topological spaces, $f : X \to Y$ is a continuous function and μ is a Baire-coded measure on X, then $F \mapsto \mu f^{-1}[F] : \mathcal{B}\mathfrak{a}_c(Y) \to [0, \infty]$ is a Baire-coded measure on Y.

(b) Suppose that μ is a Baire-coded measure on a topological space X, and $\langle E_n \rangle_{n \in \mathbb{N}}$ is a codable family in $\mathcal{B}\mathfrak{a}_c(X)$. Then

(i) $\mu(\bigcup_{n \in \mathbb{N}} E_n) \le \sum_{n=0}^{\infty} \mu E_n$;

(ii) If $\langle E_n \rangle_{n \in \mathbb{N}}$ is non-decreasing, $\mu(\bigcup_{n \in \mathbb{N}} E_n) = \lim_{n \to \infty} \mu E_n$;

(iii) If $\langle E_n \rangle_{n \in \mathbb{N}}$ is non-increasing and μE_0 is finite, then $\mu(\bigcap_{n \in \mathbb{N}} E_n) = \lim_{n \to \infty} \mu E_n$.

(c) Let X be a topological space and M a non-empty upwards-directed family of Baire-coded measures on X. Set $\nu E = \sup_{\mu \in \mathrm{M}} \mu E$ for every codable Baire set $E \subseteq X$. Then ν is a Baire-coded measure on X.

proof (a) Use 562T(b-iv).

(b) Recall that, by 562T(b-i), there must be a continuous function $f : X \to \mathbb{R}^{\mathbb{N}}$ and a codable sequence $\langle F_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{B}_c(\mathbb{R}^{\mathbb{N}})$ such that $E_n = f^{-1}[F_n]$ for every n. By (a), $F \mapsto \mu f^{-1}[F] : \mathcal{B}_c(\mathbb{R}^{\mathbb{N}}) \to [0, \infty]$ is a Borel-coded measure on $\mathbb{R}^{\mathbb{N}}$. Applying 563Ba to $\langle F_n \rangle_{n \in \mathbb{N}}$, we get the result here.

(c) As 563E.

563L Proposition Suppose that X is a topological space; write $\mathcal{G}$ for the lattice of cozero subsets of X. Let $\mu : \mathcal{G} \to [0, \infty]$ be such that

$\mu\emptyset = 0$,

$\mu G \le \mu H$ if $G \subseteq H$,

μ is modular,

$\mu(\bigcup_{n \in \mathbb{N}} G_n) = \lim_{n \to \infty} \mu G_n$ whenever $\langle G_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence in $\mathcal{G}$ and there is a sequence $\langle f_n \rangle_{n \in \mathbb{N}}$ of continuous functions from X to $\mathbb{R}$ such that $G_n = \{x : f_n(x) \ne 0\}$ for every n,[3]

$\mu G = \sup\{\mu H : H \in \mathcal{G}, H \subseteq G, \mu H < \infty\}$ for every $G \in \mathcal{G}$.

Then there is a Baire-coded measure on X extending μ; if μX is finite, then the extension is unique.

proof (a) Suppose to begin with that μX is finite.

(i) For each continuous $f : X \to \mathbb{R}^{\mathbb{N}}$, consider the functional $G \mapsto \mu f^{-1}[G]$ for open $G \subseteq \mathbb{R}^{\mathbb{N}}$. This satisfies the conditions of 563H. **P** Only the fourth requires attention. Fix a metric ρ defining the topology of $\mathbb{R}^{\mathbb{N}}$. If $\langle H_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence of open sets in $\mathbb{R}^{\mathbb{N}}$ with union H, set

$$h_n(z) = \min(1, \rho(z, \mathbb{R}^{\mathbb{N}} \setminus H_n))$$

for $n \in \mathbb{N}$ and $z \in \mathbb{R}^{\mathbb{N}}$, counting $\rho(z, \emptyset)$ as ∞ if necessary. In this case, setting $G_n = f^{-1}[H_n]$, $G_n = \{x : h_n f(x) > 0\}$ for each n; so

$$\mu f^{-1}[H] = \mu(\textstyle\bigcup_{n \in \mathbb{N}} G_n) = \lim_{n \to \infty} \mu G_n = \lim_{n \to \infty} \mu f^{-1}[H_n]. \quad \mathbf{Q}$$

There is therefore a unique Borel-coded measure ν_f on $\mathbb{R}^{\mathbb{N}}$ such that $\nu_f H = \mu f^{-1}[H]$ for every open set $G \subseteq \mathbb{R}^{\mathbb{N}}$.

(ii) If $f : X \to \mathbb{R}^{\mathbb{N}}$ is continuous and $F \in \mathcal{B}_c(\mathbb{R}^{\mathbb{N}})$, then $\nu_f F = \inf\{\mu G : f^{-1}[F] \subseteq G \in \mathcal{G}\}$. **P** By 563Fd, ν_f is outer regular with respect to the open sets, so

$$\begin{aligned} \nu_f F &= \inf\{\nu_f H : H \subseteq \mathbb{R}^{\mathbb{N}} \text{ is open}, F \subseteq H\} \\ &= \inf\{\mu f^{-1}[H] : H \subseteq \mathbb{R}^{\mathbb{N}} \text{ is open}, F \subseteq H\} \ge \inf\{\mu G : f^{-1}[F] \subseteq G \in \mathcal{G}\}. \end{aligned}$$

In the other direction, if $G \in \mathcal{G}$ and $f^{-1}[F] \subseteq G$, take any $\epsilon > 0$. There is an open set $H \subseteq \mathbb{R}^{\mathbb{N}}$ such that $\mathbb{R}^{\mathbb{N}} \setminus H \subseteq F$ and $\nu_f(F \cap H) \le \epsilon$. But this means $G \cup f^{-1}[H] = X$ and

$$\mu G \ge \mu X - \mu f^{-1}[H] = \nu_f \mathbb{R}^{\mathbb{N}} - \nu_f H \ge \nu_f F - \epsilon.$$

As ϵ is arbitrary, $\nu_f F \le G$; as G is arbitrary, $\nu_f F \le \inf\{\mu G : f^{-1}[F] \subseteq G \in \mathcal{G}\}$. **Q**

(iii) This means that if we set $\nu E = \inf\{\mu G : E \subseteq G \in \mathcal{G}\}$ for $E \in \mathcal{B}\mathfrak{a}_c(X)$, we shall have $\nu f^{-1}[F] = \nu_f F$ whenever $f : X \to \mathbb{R}^{\mathbb{N}}$ is continuous and $F \in \mathcal{B}_c(X)$. It follows that ν is a Baire-coded measure on

[3] Observe that $\bigcup_{n \in \mathbb{N}} G_n$ is a cozero set, defined by $f : X \to \mathbb{R}$ where $f(x) = \sup_{n \in \mathbb{N}} \min(2^{-n}, |f_n(x)|)$ for each x.

X. **P** Of course $\nu\emptyset = 0$. If $\langle E_n\rangle_{n\in\mathbb{N}}$ is a disjoint codable sequence in $\mathcal{B}\mathfrak{a}_c(X)$, there are a continuous $f : X \to \mathbb{R}^{\mathbb{N}}$ and a codable sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ of coded Borel sets in $\mathbb{R}^{\mathbb{N}}$ such that $E_n = f^{-1}[F_n]$ for every n, by 562T(b-i). Set $F'_n = F_n \setminus \bigcup_{i<n} F_i$ for $n \in \mathbb{N}$; then $\langle F'_n\rangle_{n\in\mathbb{N}}$ is a codable sequence (562Kc), so

$$\nu(\bigcup_{n\in\mathbb{N}} E_n) = \nu_f(\bigcup_{n\in\mathbb{N}} F'_n) = \sum_{n=0}^{\infty} \nu_f F'_n = \sum_{n=0}^{\infty} \nu E_n. \textbf{ Q}$$

Of course ν extends μ.

(iv) As for uniqueness, if ν' is any other Baire-coded measure on X extending μ, and $f : X \to \mathbb{R}^{\mathbb{N}}$ is a continuous function, then $F \mapsto \nu' f^{-1}[F]$ is a Borel-coded measure on $\mathbb{R}^{\mathbb{N}}$ which agrees with ν_f on open sets and is therefore equal to ν_f (563Fg); it follows at once that $\nu' = \nu$.

(b) For the general case, let $\mathcal{G}^f$ be $\{H : H \in \mathcal{G},\ \mu H < \infty\}$, and for $H \in \mathcal{G}^f$ define $\mu_H : \mathcal{G} \to [0,\infty[$ by setting $\mu_H G = \mu(G \cap H)$ for every $G \in \mathcal{G}$. Then μ_H satisfies all the conditions of the proposition. **P** Everything is elementary; for the hypothesis on non-decreasing sequences in $\mathcal{G}$, note that there is a continuous function $f : X \to \mathbb{R}$ such that $H = \{x : f(x) \ne 0\}$, so that if $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence of real-valued continuous function defining a sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{G}$, then $\langle f_n \times f\rangle_{n\in\mathbb{N}}$ defines $\langle G_n \cap H\rangle_{n\in\mathbb{N}}$. **Q**

There is therefore a unique Baire-coded measure ν_H on X extending μ_H. Now if H, $H' \in \mathcal{G}^f$ and $H \subseteq H'$, $\nu_H E = \nu_{H'}(E \cap H)$ for every $E \in \mathcal{B}\mathfrak{a}_c(X)$. **P** The functional $E \mapsto \nu_{H'}(E \cap H)$ is a Baire-coded measure on X extending μ_H, so must be equal to ν_H. **Q** In particular, $\nu_H E \le \nu_{H'} E$ for every codable Baire set $E \subseteq X$.

Now set $\nu E = \sup\{\nu_H E : H \in \mathcal{G}^f\}$ for $E \in \mathcal{B}\mathfrak{a}_c(X)$. By 563Kc, ν is a Baire-coded measure on X; and by the final hypothesis of this proposition, ν extends μ.

563M Measure algebras If μ is either a Borel-coded measure or a Baire-coded measure, we can form the quotient Boolean algebra $\mathfrak{A} = \operatorname{dom}\mu/\{E : \mu E = 0\}$ and the functional $\bar\mu : \mathfrak{A} \to [0,\infty]$ defined by setting $\bar\mu E^\bullet = \mu E$ for every $E \in \operatorname{dom}\mu$; as in 321H, $\bar\mu$ is a strictly positive additive functional from $\mathfrak{A}$ to $[0,\infty]$. As in §323, we have a topology and uniformity on $\mathfrak{A}$ defined by the pseudometrics $(a,b) \mapsto \bar\mu(c \cap (a \triangle b))$ for $c \in \mathfrak{A}$ of finite measure; if μ is semi-finite, the topology is Hausdorff.

563N Theorem Let X be a second-countable space, and μ a codably σ-finite Borel-coded measure on X. Let $\mathfrak{A}$ and $\bar\mu$ be as in 563M. Then $\mathfrak{A}$ is complete for its measure-algebra uniformity, therefore Dedekind complete.

proof (a) There is a codable sequence of sets of finite measure covering X. By 562Pb, we can find a codably Borel equivalent second-countable topology $\mathfrak{S}$ on X, generated by a countable algebra $\mathcal{E}$ of subsets of X, for which all these sets are open, so that μ becomes locally finite, while $\mathcal{S}$ is regular and second-countable. Let $\langle H_n\rangle_{n\in\mathbb{N}}$ be a sequence running over $\mathcal{E}$; note that $\langle H_n\rangle_{n\in\mathbb{N}}$ is codable.

(b) $\{H^\bullet : H \in \mathcal{E}\}$ is dense in $\mathfrak{A}$ for the measure-algebra topology. **P** Suppose that a, $c \in \mathfrak{A}$, $\epsilon > 0$ and $\bar\mu c < \infty$. Express a as $E^\bullet$ and c as $F^\bullet$ where E, $F \in \mathcal{B}_c(X)$. By 563Fd, there is a $G \in \mathfrak{S}$ such that $E \cap F \subseteq G$ and $\mu(G \setminus (E \cap F)) \le \epsilon$. Setting $G_n = \bigcup\{H_i : i \le n,\ H_i \subseteq G\}$, $\langle G_n \cap F\rangle_{n\in\mathbb{N}}$ is a non-decreasing codable sequence with union $G \cap F$, so there is an $n \in \mathbb{N}$ such that $\mu((G \setminus G_n) \cap F) \le \epsilon$. In this case

$$\bar\mu(c \cap (a \triangle G_n^\bullet)) = \mu(F \cap (E \triangle G_n)) \le \mu(F \cap (G \setminus G_n)) + \mu(G \setminus (E \cap F)) \le 2\epsilon,$$

while $G_n \in \mathcal{E}$. As a, c and ϵ are arbitrary, we have the result. **Q**

(c) $\mathfrak{A}$ is complete for the measure-algebra uniformity. **P** Set $\tilde H_n = \bigcup\{H_i : i \le n,\ \mu H_i < \infty\}$, $c_n = \tilde H_n^\bullet$ for each n. Let $\mathcal{F}$ be a Cauchy filter on $\mathfrak{A}$ for the measure-algebra uniformity. For each $n \in \mathbb{N}$, there is an $A \in \mathcal{F}$ such that $\bar\mu(c_n \cap (a \triangle b)) \le 2^{-n}$ for all a, $b \in A$; there is a $b_0 \in A$; and there is an $m \in \mathbb{N}$ such that $\bar\mu(c_n \cap (b_0 \triangle H_m^\bullet)) \le 2^{-n}$, so that

$$\{a : \bar\mu(c_n \cap (a \triangle H_m^\bullet)) \le 2^{-n+1}\} \in \mathcal{F}. \tag{*}$$

Let m_n be the first m for which (*) is true, and set $d_n = H_{m_n}^\bullet$. Note that

$$\bar\mu(c_i \cap (d_{i+1} \triangle d_i)) \le 3 \cdot 2^{-i}$$

for each i, because there must be an $a \in \mathfrak{A}$ such that $\bar\mu(c_i \cap (a \triangle d_i)) \le 2^{-i+1}$ and $\bar\mu(c_{i+1} \cap (a \triangle d_{i+1})) \le 2^{-i}$.

Set $E = \bigcap_{n\in\mathbb{N}} \bigcup_{i\ge n} H_{m_i}$; because $\langle\bigcup_{i\ge n} H_{m_i}\rangle_{n\in\mathbb{N}}$ is codable, $E \in \mathcal{B}_c(X)$. Set $d = E^\bullet$. If $n \in \mathbb{N}$, then

$$E \triangle H_{m_n} \subseteq \bigcup_{i\ge n} H_{m_{i+1}} \triangle H_{m_i}$$

and $\langle\tilde H_n\cap(H_{m_{i+1}}\triangle H_{m_i})\rangle_{i\in\mathbb{N}}$ is codable, so

$$\bar\mu(c_n\cap(d\triangle d_n))=\mu(\tilde H_n\cap(E\triangle H_{m_n}))\le\sum_{i=n}^{\infty}\mu(\tilde H_n\cap(H_{m_{i+1}}\triangle H_{m_i}))$$
$$\le\sum_{i=n}^{\infty}3\cdot2^{-i}=6\cdot2^{-n}.$$

Take any $c\in\mathfrak{A}$ such that $\bar\mu c$ is finite, and $\epsilon>0$. Express c as $F^\bullet$, where $\mu F<\infty$. Then $\langle F\cap\tilde H_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing codable sequence with union F, so there is an $n\in\mathbb{N}$ such that $\mu(F\setminus\tilde H_n)\le\epsilon$ and $2^{-n}\le\epsilon$. Now

$$\{a:\bar\mu(c\cap(a\triangle d))\le9\epsilon\}\supseteq\{a:\bar\mu(c_n\cap(a\triangle d))\le8\epsilon\}$$
$$\supseteq\{a:\bar\mu(c_n\cap(a\triangle d_n))\le2\epsilon\}\in\mathcal{F}.$$

As c and ϵ are arbitrary, $\mathcal{F}\to d$ for the measure-algebra topology; as $\mathcal{F}$ is arbitrary, $\mathfrak{A}$ is complete. **Q**

(d) Now suppose that $A\subseteq\mathfrak{A}$ is a non-empty set, and B the family of its upper bounds, so that B is downwards-directed. As in 323D, the filter $\mathcal{F}(B\downarrow)$ generated by $\{B\cap[0,b]:b\in B\}$ is Cauchy for the measure-algebra uniformity, so has a limit, which is $\inf B=\sup A$. As A is arbitrary, $\mathfrak{A}$ is Dedekind complete.

563X Basic exercises (a) Let X, Y be second-countable spaces, μ a Borel-coded measure on X, and $f:X\to Y$ a codable Borel function. Show that $F\mapsto\mu f^{-1}[F]:\mathcal{B}_c(Y)\to[0,\infty]$ is a Borel-coded measure on Y.

(b) Let X be a regular second-countable space and μ a locally finite Borel-coded measure on X. Show that for every $E\in\mathcal{B}_c(X)$ there are an F_σ set $F\subseteq E$ and a G_δ set $H\supseteq E$ such that $\mu(H\setminus F)=0$.

(c) Let X be a regular second-countable space. Show that a function μ is a codable Borel measure on X iff it is a codable Baire measure on X. (*Hint*: 562Xk, 562Xl.)

(d) Let X be a topological space. Show that any semi-finite Baire-coded measure on X is inner regular with respect to the zero sets.

(e) Let X be a zero-dimensional compact Hausdorff space, $\mathcal{E}$ the algebra of open-and-closed subsets of X and $\mu_0:\mathcal{E}\to[0,\infty[$ an additive functional. Show that there is a unique Baire-coded measure on X extending μ_0.

563Z Problem Suppose we define 'probability space' in the conventional way, following literally the formulations in 111A, 112A and 211B. Is it relatively consistent with ZF to suppose that every probability space is purely atomic in the sense of 211K?

563 Notes and comments The arguments above are generally drawn from those used earlier in this treatise; the new discipline required is just to systematically respect the self-denying ordinance renouncing the axiom of choice, as in part (f) of the proof of 563F. This does involve us in deeper analyses at a number of points. In 563Dc, for instance, we need functions π, π' defined on $\mathcal{T}\times\mathbb{N}$, not $\mathcal{B}_c(X)\times\mathbb{N}$, because the rank function of $\mathcal{T}$ gives us a foundation for induction. (In 563Db we can use a function π^* defined on $\mathfrak{T}\times\mathbb{N}$, but this is because we have canonical codes for open sets.) In 563I we can no longer assume the existence of measurable envelopes, let alone a whole family of them as used in the standard proof in 431A, and have to find another construction, watching carefully to make sure that we get not only a countable ordinal ξ but a codable family of sets $E_{\sigma\eta}$ leading to the measurable envelopes $E_{\sigma\xi}$; back in 561A, there was a moment when we needed to resist the temptation to suppose that a sequence in ω_1 must have a supremum in ω_1.

Note that we have to distinguish between 'negligible' and 'outer measure zero'. The natural meaning of the latter is 'for every $\epsilon>0$ there is a measurable set $E\supseteq A$ with $\mu E\le\epsilon$'. Even for outer regular measures, when a set of outer measure zero must be included in open sets of small measure, we cannot be sure that there is a sequence of such sets from which we can define a set of measure zero including A (565Xb).

In 563K I have kept the proofs short by quoting results from earlier in the section. But you may find it illuminating to look for a list of properties of codable families of codable Baire sets which would support formally independent proofs.

In 563M-563N I am taking care to avoid the phrase 'measure algebra' in the formal exposition. The reason is that the definition in §321 demands a Dedekind σ-complete algebra, and in the generality of 563M there is no reason to suppose that this will be satisfied. In the special context of 563N, of course, there is no difficulty.

There is something I ought to point out here. The problem is not that the principal arguments of §§111-113 and §§121-123 depend on the axiom of choice. If you wish, you can continue to define 'σ-algebra', 'measure', 'outer measure', 'measurable function' and 'integral' with the same forms of words as used in Volume 1, and the basic theorems, up to and including the convergence theorems, will still be true. The problem is that on these definitions the formulae of §§114-115 may not give an outer measure, and we may have nothing corresponding to Lebesgue measure. It does not quite follow that every probability space is purely atomic (there is a question here: see 563Z), but clearly we are not going to get a theory which can respond to any of the basic challenges dealt with in Volume 2 (Fundamental Theorem of Calculus, geometric measure theory, probability distributions, Fourier series), and I think it more useful to develop a new structure which can carry an effective version of the Lebesgue theory (see §565).

Version of 9.2.14

564 Integration without choice

I come now to the problem of defining an integral with respect to a Borel- or Baire-coded measure. Since a Borel-coded measure can be regarded as a Baire-coded measure on a second-countable space (562U), I will give the basic results in terms of the wider class. I seek to follow the general plan of Chapter 12, starting from simple functions and taking integrable functions to be almost-everywhere limits of sequences of simple functions (564A); the concept of 'virtually measurable' function has to be re-negotiated (564Ab). The basic convergence theorems from §123 are restricted but recognisable (564F). We also have versions of two of the representation theorems from §436 (564H, 564I).

There is a significant change when we come to the completeness of L^p spaces (564K) and the Radon-Nikodým theorem (564L), where it becomes necessary to choose sequences, and we need a well-orderable dense set of functions to pick from. Subject to this, we have workable notions of conditional expectation operator (564Mc) and product measures (564N, 564O).

564A Definitions (a) Given a topological space X and a Baire-coded measure μ on X (563J), I will write $\mathcal{B}\mathfrak{a}_c(X)^f$ for the ring of codable Baire sets of finite measure; $S = S(\mathcal{B}\mathfrak{a}_c(X)^f)$ will be the linear subspace of $\mathbb{R}^X$ generated by $\{\chi E : E \in \mathcal{B}\mathfrak{a}_c(X)^f\}$ (see 122Ab, 361D[4]). Then S is a Riesz subspace of $\mathbb{R}^X$, and also an f-algebra in the sense of 352W.

(b) I will write $\mathcal{L}^0$ for the space of real-valued functions f defined almost everywhere in X such that there is a codable Baire function $g : X \to \mathbb{R}$ such that $f =_{\text{a.e.}} g$.

(c) Let $\int : S \to \mathbb{R}$ be the positive linear functional defined by saying that $\int \chi E = \mu E$ for every $E \in \mathcal{B}\mathfrak{a}_c(X)^f$. (The arguments of 361E-361G still apply, so there is such a functional.)

(d) $\mathcal{L}^1$ will be the set of those real-valued functions f defined almost everywhere in X for which there is a codable sequence $\langle h_n\rangle_{n\in\mathbb{N}}$ in S converging to f almost everywhere and such that $\sum_{n=0}^{\infty} \int |h_{n+1} - h_n| < \infty$; I will call such functions **integrable**.

[4]§§361-362 are written on a general assumption of AC. The only essential use of it to begin with, however, is in asserting that an arbitrary Boolean ring can be faithfully represented as a ring of sets; and even that can be dispensed with for a while if we work a little harder, as in 361Ya.

564B Lemma Let X be a topological space and μ a Baire-coded measure on X.

(a) $\mathcal{L}^1 \subseteq \mathcal{L}^0$.

(b) If $\langle h_n\rangle_{n\in\mathbb{N}}$ is a non-increasing codable sequence in $S = S(\mathcal{B}\mathfrak{a}_c(X)^f)$ and $\lim_{n\to\infty} h_n(x) = 0$ for almost every x, then $\lim_{n\to\infty}\int h_n = 0$.

(c) If $\langle h_n\rangle_{n\in\mathbb{N}}$ and $\langle h'_n\rangle_{n\in\mathbb{N}}$ are two codable sequences in S such that $\lim_{n\to\infty} h_n$ and $\lim_{n\to\infty} h'_n$ are defined and equal almost everywhere, and $\sum_{n=0}^{\infty}\int |h_{n+1}-h_n|$ and $\sum_{n=0}^{\infty}\int |h'_{n+1}-h'_n|$ are both finite, then $\lim_{n\to\infty}\int h_n$ and $\lim_{n\to\infty}\int h'_n$ are defined and equal.

(d) If $\langle h_n\rangle_{n\in\mathbb{N}}$ is a codable sequence in S and $\sum_{n=0}^{\infty}\int |h_{n+1}-h_n|$ is finite, then $\langle h_n\rangle_{n\in\mathbb{N}}$ converges almost everywhere. In particular, if $\langle h_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing codable sequence in S and $\sup_{n\in\mathbb{N}}\int h_n$ is finite, $\langle h_n\rangle_{n\in\mathbb{N}}$ converges a.e.

(e) If $\langle h_n\rangle_{n\in\mathbb{N}}$ is a codable sequence in S^+ and $\liminf_{n\to\infty}\int h_n = 0$, then $\liminf_{n\to\infty} h_n = 0$ a.e.

proof (a) If $f \in \mathcal{L}^1$, there is a codable sequence $\langle h_n\rangle_{n\in\mathbb{N}}$ in S converging almost everywhere to f. Now 562T(c-iii) tells us that there is a codable Baire function g equal to $\lim_{n\to\infty} h_n$ wherever this is defined as a real number, so that $f =_{\text{a.e.}} g$ and $f \in \mathcal{L}^0$.

(b) Set $E = \{x : h_0(x) > 0\}$. Take any $\epsilon > 0$. For each $n \in \mathbb{N}$ set $E_n = \{x : h_n(x) > \epsilon\}$. Then $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-increasing codable sequence in $\mathcal{B}\mathfrak{a}_c(X)^f$ (562Td), and $\bigcap_{n\in\mathbb{N}} E_n \subseteq \{x : \lim_{n\to\infty} h_n(x) \neq 0\}$ is negligible; also E_0 has finite measure. Accordingly $\lim_{n\to\infty}\mu E_n = 0$ (563K(b-iii)). But

$$h_n \le \|h_0\|_\infty \chi E_n + \epsilon\chi E, \qquad \int h_n \le \|h_0\|_\infty \mu E_n + \epsilon\mu E$$

for every n, so $\limsup_{n\to\infty}\int h_n \le \epsilon\mu E$. As ϵ is arbitrary, $\lim_{n\to\infty}\int h_n = 0$.

(c) Since $\int$ is a positive linear functional on the Riesz space S,

$$\sum_{n=0}^{\infty}|\int h_{n+1} - \int h_n| \le \sum_{n=0}^{\infty}\int |h_{n+1}-h_n|$$

is finite, and the limit $\lim_{n\to\infty}\int h_n$ is defined in $\mathbb{R}$. Similarly, $\lim_{n\to\infty}\int h'_n$ is defined. To see that the limits are equal, set $g_n = h_n - h'_n$ for each n, so that $\lim_{n\to\infty} g_n = 0$ a.e. and $\sum_{n=0}^{\infty}\int |g_{n+1}-g_n| < \infty$. Then $\int |g_n| \le \sum_{m=n}^{\infty}\int |g_{m+1}-g_m|$ for every n. **P** For $k \ge n$, set $f_k = (|g_n| - \sum_{m=n}^{k}|g_{m+1}-g_m|)^+$. Then $0 \le f_k \le |g_{k+1}|$ for each k, so $\langle f_k\rangle_{k\ge n}$ is a non-increasing codable sequence in S converging to 0 almost everywhere. (To check that $\langle f_k\rangle_{k\ge n}$ is codable, use 562T(c-ii) and the idea of 562Se.) By (b), $\lim_{n\to\infty}\int f_k = 0$; but $\int f_k \ge \int |g_n| - \sum_{m=n}^{k}\int |g_{m+1}-g_m|$ for every k. **Q**

Consequently

$$\begin{aligned}|\lim_{n\to\infty}\int h_n - \lim_{n\to\infty}\int h'_n| &= \lim_{n\to\infty}|\int h_n - \int h'_n| \le \lim_{n\to\infty}\int |g_n| \\ &\le \lim_{n\to\infty}\sum_{m=n}^{\infty}\int |g_{m+1}-g_m| = 0\end{aligned}$$

as required.

(d) For $k \in \mathbb{N}$ let $n_k \in \mathbb{N}$ be the least integer such that $\sum_{i=n_k}^{\infty}\int |h_{i+1}-h_i| \le 2^{-k}$, and for $m \ge n_k$ set

$$G_{km} = \{x : \sum_{i=n_k}^{m}|h_{i+1}(x)-h_i(x)| \ge 1\}.$$

Then $\mu G_{km} \le 2^{-k}$, because $\chi G_{km} \le \sum_{i=n_k}^{m}|h_{i+1}-h_i|$, so $\mu G_k \le 2^{-k}$, where $G_k = \bigcup_{m\ge n_k} G_{km}$, by 563K(b-i). (Of course we have to check that all the sequences of sets and functions involved here are codable.) Accordingly, setting $E = \bigcap_{k\in\mathbb{N}} G_k$, $\mu E = 0$. But observe that if $x \in X \setminus E$ there is a $k \in \mathbb{N}$ such that $x \notin G_k$ and $\sum_{i=n_k}^{\infty}|h_{i+1}(x)-h_i(x)| \le 1$; in which case $\lim_{n\to\infty} h_n(x)$ is defined.

(e) For $k \in \mathbb{N}$ let n_k be the least integer such that $n_k > n_i$ for $i < k$ and $\int h_{n_k} \le 4^{-n_k}$. Set $G_k = \{x : h_{n_k}(x) \ge 2^{-k}\}$; then $\mu G_k \le 2^{-k}$ and $\langle G_k\rangle_{k\in\mathbb{N}}$ is codable. So $\mu(\bigcup_{k\ge n} G_k) \le 2^{-n+1}$ for every n and $E = \bigcap_{n\in\mathbb{N}}\bigcup_{k\ge n} G_k$ is negligible. But $E \supseteq \{x : \liminf_{n\to\infty} h_n(x) > 0\}$.

564C Definition Let X be a topological space and μ a Baire-coded measure on X. For $f \in \mathcal{L}^1$, define its integral $\int f$ by saying that $\int f = \lim_{n\to\infty}\int h_n$ whenever $\langle h_n\rangle_{n\in\mathbb{N}}$ is a codable sequence in $S = S(\mathcal{B}\mathfrak{a}_c(X)^f)$ converging to f almost everywhere and $\sum_{n=0}^{\infty}\int |h_{n+1}-h_n|$ is finite. By 564Bc, this definition is sound; and clearly it is consistent with the previous definition of the integral on S.

564D Lemma Let X be a topological space and $\langle f_n\rangle_{n\in\mathbb{N}}$ a codable sequence of codable Baire functions on X. Let $\langle q_i\rangle_{i\in\mathbb{N}}$ be an enumeration of $\mathbb{Q}\cap[0,\infty[$, starting with $q_0=0$. Set

$$f'_n(x)=\max\{q_i: i\le n,\, q_i\le\max(0,f_n(x))\}$$

for $n\in\mathbb{N}$ and $x\in X$. Then $\langle f'_n\rangle_{n\in\mathbb{N}}$ is a codable sequence of codable Baire functions.

proof Take a sequence running over a base for the topology of $\mathbb{R}^{\mathbb{N}}$ and the corresponding interpretation $\phi:\mathcal{T}\to\mathcal{B}_c(\mathbb{R}^{\mathbb{N}})$ of Borel codes, as in 562B, and let $\tilde\phi:\tilde T\to\mathbb{R}^X$ be the corresponding interpretation of codes for real-valued codable Borel functions, as in 562N. By 562T(c-ii), there are a continuous function $g:X\to\mathbb{R}^{\mathbb{N}}$ and a sequence $\langle\tau_n\rangle_{n\in\mathbb{N}}$ of codes such that $f_n=\tilde\phi(\tau_n)\circ g$ for every n. We need a sequence $\langle\tau'_n\rangle_{n\in\mathbb{N}}$ of codes such that

$$\begin{aligned}\phi(\tau'_n(\alpha))&=\bigcup_{\substack{i\le n\\ q_i>\alpha}}\ \bigcap_{\substack{j\in\mathbb{N}\\ q_j<q_i}}\phi(\tau_n(q_j))\text{ if }\alpha\ge 0,\\ &=X\text{ if }\alpha<0;\end{aligned}$$

and this is easy to build using complementation and general union operators as in 562C. Now take $f'_n=\tilde\phi(\tau'_n)\circ g$ for each n.

564E Theorem Let X be a topological space and μ a Baire-coded measure on X.

(a)(i) If f, $g\in\mathcal{L}^0$ and $\alpha\in\mathbb{R}$, then $f+g$, αf, $|f|$ and $f\times g$ belong to $\mathcal{L}^0$.

(ii) If $h:\mathbb{R}\to\mathbb{R}$ is a codable Borel function, $hf\in\mathcal{L}^0$ for every $f\in\mathcal{L}^0$.

(b) If f, $g\in\mathcal{L}^1$ and $\alpha\in\mathbb{R}$, then

(i) $f+g$, αf and $|f|$ belong to $\mathcal{L}^1$;

(ii) $\int f+g=\int f+\int g$, $\int\alpha f=\alpha\int f$;

(iii) if $f\le_{\text{a.e.}} g$ then $\int f\le\int g$.

(c)(i) If $f\in\mathcal{L}^0$, $g\in\mathcal{L}^1$ and $|f|\le_{\text{a.e.}} g$, then $f\in\mathcal{L}^1$.

(ii) If $E\in\mathcal{B}\mathfrak{a}_c(X)$ and $\chi E\in\mathcal{L}^1$ then μE is finite.

proof (a)(i) Use 562T(c-iv).

(ii) We know that f is equal almost everywhere to a product $f'g$ where $g:X\to\mathbb{R}^{\mathbb{N}}$ is continuous and $f':\mathbb{R}^{\mathbb{N}}\to\mathbb{R}$ is a codable Borel function. Now hf' is a codable Borel function, by 562Mb, so $hf'g$ is a codable Baire function and $hf=_{\text{a.e.}}hf'g$ belongs to $\mathcal{L}^0$.

(b)(i)-(ii) These proceed by the same arguments as in (a-i). To deal with $|f|$, we need to note that if $\langle h_n\rangle_{n\in\mathbb{N}}$ is any codable sequence in $S=S(\mathcal{B}\mathfrak{a}_c(X)^f)$ then $\sum_{n=0}^\infty\int||h_{n+1}|-|h_n||\le\sum_{n=0}^\infty\int|h_{n+1}-h_n|$.

(iii) If $f\le_{\text{a.e.}} g$, let $\langle f_n\rangle_{n\in\mathbb{N}}$, $\langle g_n\rangle_{n\in\mathbb{N}}$ be codable sequences in S converging almost everywhere to f, g respectively, and such that $\sum_{n=0}^\infty\int|f_{n+1}-f_n|$ and $\sum_{n=0}^\infty\int|g_{n+1}-g_n|$ are finite. Set $h_n=f_n\wedge g_n$ for each n. Then $\langle h_n\rangle_{n\in\mathbb{N}}$ is codable, $f=_{\text{a.e.}}\lim_{n\to\infty}h_n$, $\sum_{n=0}^\infty\int|h_{n+1}-h_n|<\infty$ and

$$\int f=\lim_{n\to\infty}\int h_n\le\lim_{n\to\infty}\int g_n=\int g.$$

(c)(i) Let $\langle h_n\rangle_{n\in\mathbb{N}}$ be a codable sequence in S such that $g=_{\text{a.e.}}\lim_{n\to\infty}h_n$ and $\sum_{n=0}^\infty\int|h_{n+1}-h_n|$ is finite. Set $h'_n=\sup_{i\le n}h_i^+$ for each n; then $\langle h'_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing codable sequence in S and $\sup_{n\in\mathbb{N}}\int h'_n$ is finite, while $|f|\le_{\text{a.e.}} g\le_{\text{a.e.}}\sup_{n\in\mathbb{N}}h'_n$. There is a codable Baire function $\tilde f$ such that $f=_{\text{a.e.}}\tilde f$. Now $\tilde f^+$ is a codable Baire function, so $\langle\tilde f^+\wedge h'_n\rangle_{n\in\mathbb{N}}$ is a codable sequence of non-negative codable Baire functions.

For each $n\in\mathbb{N}$ consider h''_n where

$$h''_n(x)=\max\{q_i: i\le n,\, q_i\le\max(0,(\tilde f^+\wedge h'_n)(x)\}$$

for $x\in X$. By 564D, $\langle h''_n\rangle_{n\in\mathbb{N}}$ is a codable sequence; it is non-decreasing and converges a.e. to $\tilde f^+=_{\text{a.e.}}f^+$. Because $0\le h''_n\le h'_n$, $h''_n\in S$ for each n, and $\sup_{n\in\mathbb{N}}\int h''_n\le\sup_{n\in\mathbb{N}}\int h'_n$ is finite; so 564Bd tells us that f^+ is integrable.

Similarly, f^- is integrable, so f is integrable.

(ii) Let $\langle h_n\rangle_{n\in\mathbb{N}}$ be a codable sequence in S such that $\chi E =_{\text{a.e.}} \lim_{n\to\infty} h_n$ and $\sum_{n=0}^{\infty}\int|h_{n+1}-h_n|$ is finite. Set $h'_n=\sup_{i\le n}h_i$ for each n; then $\langle h'_n\rangle_{n\in\mathbb{N}}$ is a codable sequence in S and $\sup_{n\in\mathbb{N}}\int h'_n$ is finite. Set $E_n=\{x:h'_n(x)>\frac{1}{2}\}$; then $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing codable sequence in $\mathcal{B}\mathfrak{a}_c(X)^f$. Also $E\setminus\bigcup_{n\in\mathbb{N}}E_n$ is negligible, so

$$\mu E\le\mu(\bigcup_{n\in\mathbb{N}}E_n)=\sup_{n\in\mathbb{N}}\mu E_n=\sup_{n\in\mathbb{N}}\int\chi E_n\le 2\sup_{n\in\mathbb{N}}\int h'_n$$

is finite.

564F I come now to versions of the fundamental convergence theorems.

Theorem Let X be a topological space and μ a Baire-coded measure on X. Suppose that $\langle f_n\rangle_{n\in\mathbb{N}}$ is a codable sequence of integrable codable Baire functions on X.

(a) If $\langle f_n\rangle_{n\in\mathbb{N}}$ is non-decreasing and $\gamma=\sup_{n\in\mathbb{N}}\int f_n$ is finite, then $f=\lim_{n\to\infty}f_n$ is defined a.e. and is integrable, and $\int f=\gamma$.

(b) If every f_n is non-negative and $\liminf_{n\to\infty}\int f_n$ is finite, then $f=\liminf_{n\to\infty}f_n$ is defined a.e. and is integrable, and $\int f\le\liminf_{n\to\infty}\int f_n$.

(c) Suppose that there is a $g\in\mathcal{L}^1$ such that $|f_n|\le_{\text{a.e.}} g$ for every n, and $f=\lim_{n\to\infty}f_n$ is defined a.e. Then $\int f$ and $\lim_{n\to\infty}\int f_n$ are defined and equal.

(d) If $\sum_{n=0}^{\infty}\int|f_{n+1}-f_n|$ is finite, then $f=\lim_{n\to\infty}f_n$ is defined a.e., and $\int f$ and $\lim_{n\to\infty}\int f_n$ are defined and equal.

(e) If $\sum_{n=0}^{\infty}\int|f_n|$ is finite, then $f=\sum_{n=0}^{\infty}f_n$ is defined a.e., and $\int f$ and $\sum_{n=0}^{\infty}\int f_n$ are defined and equal.

proof (a) Replacing f_n by f_n-f_0 for each n, we may suppose that $f_n\ge 0$ for each n. Let $\langle q_i\rangle_{i\in\mathbb{N}}$ be an enumeration of $\mathbb{Q}\cap[0,\infty[$ and set

$$h_n(x)=\max\{q_i:i\le n,\,q_i\le\max(0,f_n(x))\}$$

for each $x\in X$. Then $\langle h_n\rangle_{n\in\mathbb{N}}$ is a codable sequence of codable Baire functions (use 564D again). Moreover, h_n takes only finitely many values, all non-negative, and for $\alpha>0$ the set $E_{n\alpha}=\{x:h_n(x)>\alpha\}$ is a codable Baire set such that $\chi E_{n\alpha}\le_{\text{a.e.}}\frac{1}{\alpha}f_n$; by 564Ec, $E_{n\alpha}$ has finite measure; as α is arbitrary, $h_n\in S$.

Now $\langle h_n\rangle_{n\in\mathbb{N}}$ is non-decreasing, and $\int h_n\le\int f_n\le\gamma$ for every n; so by 564Bd $\langle h_n\rangle_{n\in\mathbb{N}}$ converges almost everywhere to an integrable function f_1, with $\int f_1\le\gamma$. Of course $f_1=_{\text{a.e.}}\lim_{n\to\infty}f_n=f$; as $f\ge_{\text{a.e.}}f_n$ for every n, $\int f=\int f_1=\gamma$ exactly.

(b) By 562T(c-ii) and 562Oc, $\langle f'_n\rangle_{n\in\mathbb{N}}$ is codable, where $f'_n=\inf_{m\ge n}f_m$ for every n, and of course

$$\int f'_n\le\inf_{m\ge n}\int f_m\le\liminf_{m\to\infty}\int f_m$$

for every n. Now $\langle f'_n\rangle_{n\in\mathbb{N}}$ is non-decreasing, so (a) tells us that $\int\liminf_{n\to\infty}f_n=\int\lim_{n\to\infty}f'_n$ is defined and equal to $\lim_{n\to\infty}\int f'_n\le\liminf_{n\to\infty}\int f_n$.

(c) Let g' be a codable Baire function such that $g'=_{\text{a.e.}}g$, and set $f'_n=\operatorname{med}(-g',f_n,g')$ for each n; once again, 562T(c-ii) and the ideas of 562Oc show that $\langle g'+f'_n\rangle_{n\in\mathbb{N}}$ is codable. So we can use (b) to see that $\int\liminf_{n\to\infty}g'+f'_n$ is defined and is at most $\liminf_{n\to\infty}\int g'+f'_n=\int g'+\liminf_{n\to\infty}\int f_n$. Subtracting g', we get $\int\liminf_{n\to\infty}f'_n\le\liminf_{n\to\infty}\int f_n$. Similarly, $\int\limsup_{n\to\infty}f'_n\ge\limsup_{n\to\infty}\int f_n$.

Once again, the sequences $\langle f_n\rangle_{n\in\mathbb{N}}$, $\langle f'_n\rangle_{n\in\mathbb{N}}$, $\langle|f'_n-f_n|\rangle_{n\in\mathbb{N}}$ and $\langle\{x:f'_n(x)\ne f_n(x)\}\rangle_{n\in\mathbb{N}}$ are all codable. Since all the sets $\{x:f'_n(x)\ne f_n(x)\}$ are negligible, so is their union; but this means that $\lim_{n\to\infty}f'_n=_{\text{a.e.}}\lim_{n\to\infty}f_n$ is defined almost everywhere. So (just as in 123C) the integrals are sandwiched, and $\int\lim_{n\to\infty}f_n=\lim_{n\to\infty}\int f_n$.

(d) Of course $\sum_{n=0}^{\infty}|\int f_{n+1}-\int f_n|$ is finite, so $\gamma=\lim_{n\to\infty}\int f_n$ is defined. Next, (a) tells us that $g=|f_0|+\sum_{n=0}^{\infty}|f_{n+1}-f_n|$ is defined a.e. and is integrable (of course this depends on our having a procedure – induction is allowed – for building a sequence of Baire codes representing $\langle|f_0|+\sum_{i=0}^{n}|f_{i+1}-f_i|\rangle_{n\in\mathbb{N}}$ out of a sequence of codes representing $\langle f_n\rangle_{n\in\mathbb{N}}$). Since $\lim_{n\to\infty}f_n(x)$ is defined whenever $g(x)$ is defined and finite, which is almost everywhere, and $|f_n|\le_{\text{a.e.}}g$ for every n, (c) gives the result we're looking for.

(e) Similarly, $\langle\sum_{i=0}^{n}f_i\rangle_{n\in\mathbb{N}}$ is codable and we can apply (d).

564G Integration over subsets: Proposition Let X be a topological space and μ a Baire-coded measure on X.

(a) If $f \in \mathcal{L}^1$, the functional $E \mapsto \int f \times \chi E : \mathcal{B}\mathfrak{a}_c(X) \to \mathbb{R}$ is additive and truly continuous with respect to μ.[5]

(c) If f, $g \in \mathcal{L}^1$, then $f \leq_{\text{a.e.}} g$ iff $\int f \times \chi E \leq \int g \times \chi E$ for every $E \in \mathcal{B}\mathfrak{a}_c(X)$. So $f =_{\text{a.e.}} g$ iff $\int f \times \chi E = \int g \times \chi E$ for every $E \in \mathcal{B}\mathfrak{a}_c(X)$.

proof (a) If $E \in \mathcal{B}\mathfrak{a}_c(X)$ then χE is a codable Baire function (use 562Nf), so that $f \times \chi E$ is integrable (564E(a-i), 564E(c-i)). Because $\chi : \mathcal{B}\mathfrak{a}_c(X) \to \mathcal{L}^0$ is additive, $E \mapsto \int f \times \chi E$ is additive. To see that it is truly continuous, take $\epsilon > 0$. There is a codable sequence $\langle h_n \rangle_{n\in\mathbb{N}}$ in $S = S(\mathcal{B}\mathfrak{a}_c(X)^f)$ such that $\sum_{n=0}^{\infty} \int |h_{n+1} - h_n| < \infty$ and $f =_{\text{a.e.}} \lim_{n\to\infty} h_n$. For each n,

$$\int |f - h_n| = \lim_{m\to\infty} \int |h_m - h_n| \leq \sum_{m=n}^{\infty} \int |h_{m+1} - h_m|,$$

so there is an n such that $\int |f - h_n| \leq \frac{1}{2}\epsilon$. Set $E = \{x : h_n(x) \neq 0\}$ and $\delta = \epsilon/(1 + 2\|h_n\|_\infty)$. Then E has finite measure. If $F \in \mathcal{B}_c(X)$ and $\mu(E \cap F) \leq \delta$, then

$$|\int f \times \chi F| \leq \int |f - h_n| + \int |h_n| \times \chi F \leq \frac{1}{2}\epsilon + \|h_n\|_\infty \mu(E \cap F) \leq \epsilon.$$

As ϵ is arbitrary, the functional is truly continuous.

(b)(i) If $f \leq_{\text{a.e.}} g$ and $E \in \mathcal{B}_c(X)$, then $f \times \chi E \leq_{\text{a.e.}} g \times \chi E$ so $\int f \times \chi E \leq \int g \times \chi E$.

(ii) If $\int f \times \chi E \leq \int g \times \chi E$ for every $E \in \mathcal{B}_c(X)$, let $\langle h_n \rangle_{n\in\mathbb{N}}$ be a codable sequence in S such that $f - g =_{\text{a.e.}} \lim_{n\to\infty} h_n$ and $\sum_{n=0}^{\infty} \int |h_{n+1} - h_n| < \infty$. For $k \in \mathbb{N}$ let n_k be the least integer such that $\sum_{m=n_k}^{\infty} \int |h_{m+1} - h_m| \leq 2^{-k}$. For m, $k \in \mathbb{N}$ set $E_{mk} = \{x : h_{n_k}(x) \geq 2^{-m}\}$. Then

$$\begin{aligned}\mu E_{mk} &\leq 2^m \int h_{n_k} \times \chi E_{mk} \leq 2^m \int (h_{n_k} - f + g) \times \chi E_{mk} \\ &= 2^m \lim_{i\to\infty} \int (h_{n_k} - h_i) \times \chi E_{mk} \leq 2^m \lim_{i\to\infty} \int |h_{n_k} - h_i| \leq 2^{m-k}.\end{aligned}$$

Also $\langle E_{mk} \rangle_{m,k\in\mathbb{N}}$ is a codable family in $\mathcal{B}\mathfrak{a}_c(X)$, so $\mu(\bigcup_{k\geq 2m} E_{mk}) \leq 2^{-m+1}$ for every m and $\mu E = 0$, where

$$E = \bigcup_{l\in\mathbb{N}} \bigcap_{m\geq l} \bigcup_{k\geq 2m} E_{mk}.$$

But for $x \in X \setminus E$, $\limsup_{k\to\infty} h_{n_k}(x) \leq 0$. Since $f - g =_{\text{a.e.}} \lim_{k\to\infty} h_{n_k}$, $f \leq_{\text{a.e.}} g$.

564H Theorem Let X be a topological space, and $f : C_b(X) \to \mathbb{R}$ a sequentially smooth positive linear functional, where $C_b(X)$ is the space of bounded continuous real-valued functions on X. Then there is a totally finite Baire-coded measure μ on X such that $f(u) = \int u\, d\mu$ for every $u \in C_b(X)$.

proof (a) For cozero sets $G \subseteq X$ set $\mu_0 G = \sup\{f(u) : u \in C_b(X),\ 0 \leq u \leq \chi G\}$. Then $\mu_0 G = \lim_{n\to\infty} f(u_n)$ whenever $G \subseteq X$ is a cozero set and $\langle u_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $C_b(X)^+$ with supremum χG in $\mathbb{R}^X$. **P** Setting $\gamma = \sup_{n\in\mathbb{N}} f(u_n)$, then of course

$$\mu_0 G \geq \gamma = \lim_{n\to\infty} f(u_n).$$

On the other hand, if $v \in C_b(X)$ and $0 \leq v \leq \chi G$, $\langle (v - u_n)^+ \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence converging to $\mathbf{0}$ pointwise, so

$$f(v) \leq f(u_n) + f(v - u_n)^+ \leq \gamma + f(v - u_n)^+ \to \gamma$$

as $n \to \infty$. As v is arbitrary, $\mu_0 G \leq \gamma$. **Q**

(b) It follows that μ_0 satisfies the conditions of 563L. **P** Of course $\mu_0 \emptyset = 0$ and μ_0 is monotonic. If G, $H \subseteq X$ are cozero sets, express them as $\{x : u(x) > 0\}$ and $\{x : v(x) > 0\}$ where u, $v \in C_b(X)^+$. Set $u_n = nu \wedge \chi X$, $v_n = nv \wedge \chi X$ for each n; then $\langle u_n \rangle_{n\in\mathbb{N}}$, $\langle v_n \rangle_{n\in\mathbb{N}}$ are non-decreasing sequences in $C_b(X)^+$

[5] The definition of 'truly continuous' in 232Ab assumed that μ was defined on a σ-algebra. I hope it is obvious that the same formulation makes sense when the domain of μ is any Boolean algebra.

converging pointwise to χG, χH respectively. Now $\langle u_n \wedge v_n\rangle_{n\in\mathbb{N}}$ and $\langle u_n \vee v_n\rangle_{n\in\mathbb{N}}$ are also non-decreasing sequences in $C_b(X)^+$ converging to $\chi(G\cap H)$, $\chi(G\cup H)$; so (a) tells us that

$$\begin{aligned}\mu_0(G\cup H)+\mu_0(G\cap H) &= \lim_{n\to\infty} f(u_n\wedge v_n)+\lim_{n\to\infty} f(u_n\vee v_n)\\ &= \lim_{n\to\infty} f(u_n\wedge v_n+u_n\vee v_n)\\ &= \lim_{n\to\infty} f(u_n+v_n)=\mu_0 G+\mu_0 H.\end{aligned}$$

As for the penultimate condition in 563L, let $\langle G_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of cozero sets such that there is a sequence $\langle v_n\rangle_{n\in\mathbb{N}}$ in $C(X)$ such that $G_n=\{x : v_n(x)\ne 0\}$ for each n. Set $u_n=\chi X\wedge n\sup_{i\le n}|v_i|$ for each n, and $G=\bigcup_{n\in\mathbb{N}}G_n$; then $\langle u_n\rangle_{n\in\mathbb{N}}\uparrow\chi G$, so

$$\mu_0 G=\lim_{n\to\infty} f(u_n)\le\lim_{n\to\infty}\mu_0 G_n\le\mu_0 G,$$

as required. **Q**

(c) We therefore have a Baire-coded measure μ on X extending μ_0. Now take any $u\in C_b(X)$ such that $0\le u\le\chi X$, and $n\ge 1$. For each $i<n$ set $G_i=\{x : u(x)>\frac{i}{n}\}$; then

$$\frac{1}{n}\sum_{i=0}^{n-1}\chi G_i\le u+\frac{1}{n}\chi X,$$

so

$$\frac{1}{n}\sum_{i=0}^{n-1}\mu G_i\le\int u\,d\mu+\frac{1}{n}\mu X.$$

Next, setting

$$v_i=u\wedge\frac{i+1}{n}\chi X-u\wedge\frac{i}{n}\chi X$$

for $i<n$, $u=\sum_{i=0}^{n-1}v_i$ and $nv_i\le\chi G_i$ for each i, so

$$f(u)=\sum_{i=0}^{n-1}f(v_i)\le\frac{1}{n}\sum_{i=0}^{n-1}\mu G_i\le\int u\,d\mu+\frac{1}{n}\mu X.$$

As n is arbitrary, $f(u)\le\int u\,d\mu$. On the other hand, $f(\chi X)=\mu X=\int\chi X\,d\mu$ and $f(\chi X-u)\le\int(\chi X-u)d\mu$; so in fact $f(u)=\int u\,d\mu$.

(d) It follows at once that $f(u)=\int u\,d\mu$ for every $u\in C_b(X)^+$ and therefore for every $u\in C_b(X)$, as required.

564I Riesz Representation Theorem Let X be a completely regular locally compact space, and $f:C_k(X)\to\mathbb{R}$ a positive linear functional, where $C_k(X)$ is the space of continuous real-valued functions with compact support. Then there is a Baire-coded measure μ on X such that $\int u\,d\mu$ is defined and equal to $f(u)$ for every $u\in C_k(X)$.

proof We can follow the plan of 564H, with minor modifications.

(a) For open sets $G\subseteq X$ write $D_G=\{u : u\in C_k(X),\ 0\le u\le\chi X,\ \operatorname{supp}u\subseteq G\}$, where $\operatorname{supp}u=\overline{\{x : u(x)\ne 0\}}$. We need to know that if G, $H\subseteq X$ are open and $K\subseteq G\cup H$, $K'\subseteq G\cap H$ are compact, there are $u\in D_G$, $v\in D_H$ such that $\chi K\le u\vee v$ and $\chi K'\le u\wedge v$. **P** Because X is completely regular, the family $\{\operatorname{int}\{x : u(x)=1\} : u\in D_G\cup D_H\}$ is an open cover of $G\cup H$ and has a finite subfamily covering K; because D_G and D_H are upwards-directed, we can reduce this finite subfamily to two terms, one corresponding to $u_1\in D_G$ and the other to $v_1\in D_H$, so that $\chi K\le u_1\vee v_1$. Next, $\{\operatorname{int}\{x : u(x)=1\} : u\in D_G\}$ is an open cover of $G\supseteq K'$, so we can find a $u_2\in D_G$ such that $\chi K'\le u_2$; similarly, there is a $v_2\in D_H$ such that $\chi K'\le v_2$; set $u=u_1\vee u_2$ and $v=v_1\wedge v_2$. **Q**

(b) For cozero $G\subseteq X$, set $\mu_0 G=\sup\{f(u) : u\in D_G\}$. If G, $H\subseteq X$ are cozero sets, $u\in D_G$ and $v\in D_H$, then $u\vee v\in D_{G\cup H}$ and $u\wedge v\in D_{G\cap H}$; this is enough to show that $\mu_0 G+\mu_0 H\le\mu_0(G\cup H)+\mu_0(G\cap H)$. If $w\in D_{G\cup H}$ and $w'\in D_{G\cap H}$, (a) tells us that there are $u\in D_G$ and $v\in D_H$ such that $u\vee v\ge\chi(\operatorname{supp}w)\ge w$ and $u\wedge v\ge\chi(\operatorname{supp}w')\ge w'$; this is what we need to show that so that $\mu_0 G+\mu_0 H\ge\mu_0(G\cup H)+\mu_0(G\cap H)$.

If $\langle G_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of cozero sets, defined from a sequence of continuous functions so that $G=\bigcup_{n\in\mathbb{N}} G_n$ is a cozero set, then $D_G=\bigcup_{n\in\mathbb{N}} D_{G_n}$ so that $\mu_0 G=\sup_{n\in\mathbb{N}}\mu_0 G_n$.

If G is a relatively compact cozero set then $\mu_0 G<\infty$. **P** There is a $w\in C_k(X)$ such that $\chi\overline{G}\le w$, so that $\mu G\le f(w)$. **Q** If G is a cozero set and $\gamma<\mu_0 G$, there is a $u\in D_G$ such that $f(u)\ge\gamma$. Now there is a $v\in D_G$ such that $\chi(\operatorname{supp}(u))\le v$, so that $\mu_0 H\ge\gamma$, where $H=\{x:v(x)>0\}$; as H is relatively compact, $\mu_0 H$ is finite. Thus $\mu_0 G=\sup\{\mu_0 H: H\subseteq G$ is a cozero set, $\mu_0 H<\infty\}$.

The other hypotheses of 563L are elementary, so we have a Baire-coded measure on X extending μ_0.

(c) If $u\in C_k(X)$ and $0\le u\le\chi X$ and $\epsilon>0$, let G be a relatively compact cozero set including $\operatorname{supp} u$, and $v\in D_G$ such that $\chi(\operatorname{supp} u)\le v$ and $f(v)\ge\mu G-\epsilon$. The argument of part (c) of the proof of 564H, with v in place of χX, shows that $f(u)\le\int u\,d\mu+\frac{1}{n}\int v\,d\mu$ for every n, so that $f(u)\le\int u\,d\mu$. On the other hand,

$$\begin{aligned} f(u)=f(v)-f(v-u)&\ge\mu G-\epsilon-\int v-u\,d\mu\\ &=\mu G-\int v\,d\mu+\int u\,d\mu-\epsilon\ge\int u\,d\mu-\epsilon. \end{aligned}$$

As ϵ is arbitrary, $f(u)=\int u\,d\mu$. Of course it follows at once that f agrees with $\int\,d\mu$ on the whole of $C_k(X)$.

564J The space L^1 Let X be a topological space and μ a Baire-coded measure on X.

(a) If f, $g\in\mathcal{L}^1$ then $f=_{\text{a.e.}} g$ iff $\int|f-g|=0$. **P** If $f=_{\text{a.e.}} g$ then $|f-g|=0$ a.e. and $\int|f-g|=0$ by the definition in 564Ad. If $\int|f-g|=0$, let f_1, g_1 be codable Baire functions such that $f=_{\text{a.e.}} f_1$ and $g=_{\text{a.e.}} g_1$ (564Ba); then $|f_1-g_1|$ is codable. For each $n\in\mathbb{N}$, set $E_n=\{x:|f_1(x)-g_1(x)|\ge 2^{-n}\}$. Then $E_n\in\mathcal{B}\mathfrak{a}_c(X)$ and $|f_1-g_1|\ge 2^{-n}\chi E_n$ so $\mu E_n=\int\chi E_n=0$. But $\langle E_n\rangle_{n\in\mathbb{N}}$ is a codable sequence so $\bigcup_{n\in\mathbb{N}} E_n=\{x:f_1(x)\ne g_1(x)\}$ is negligible and

$$f=_{\text{a.e.}} f_1=_{\text{a.e.}} g_1=_{\text{a.e.}} g.\ \mathbf{Q}$$

(b) As in §242, we have an equivalence relation $\sim$ on $\mathcal{L}^1$ defined by saying that $f\sim g$ if $f=_{\text{a.e.}} g$. The set L^1 of equivalence classes has a Riesz space structure and a Riesz norm inherited from the addition, scalar multiplication, ordering and integral on $\mathcal{L}^1$.

(c) As in §242, I will define $\int: L^1\to\mathbb{R}$ by saying that $\int f^\bullet=\int f$ for every $f\in\mathcal{L}^1$. Similarly, we can define $\int_E u$, for $u\in L^1$ and $E\in\mathcal{B}\mathfrak{a}_c(X)$, by saying that $\int_E f^\bullet=\int f\times\chi E$ for $f\in\mathcal{L}^1$.

564K In order to prove that an L^1-space is norm-complete, it seems that we need extra conditions.

Theorem Let X be a second-countable space and μ a codably σ-finite Borel-coded measure on X. Then $L^1(\mu)$ is a separable L-space.

proof (Compare 563N.)

(a) There is a codable sequence of sets of finite measure covering X. By 562Pb, we can find a codably Borel equivalent zero-dimensional second-countable topology on X for which all the these sets are open, so that μ becomes locally finite. Since this procedure does not change $\mathcal{L}^1$ and L^1, we may suppose from the beginning that X is regular and μ is locally finite. Let $\langle U_n\rangle_{n\in\mathbb{N}}$ run over a countable base for the topology of X containing $\emptyset$ and closed under finite unions.

(b) If $E\in\mathcal{B}_c(X)^f$ and $\epsilon>0$, there is an open $G\subseteq X$ such that $E\subseteq G$ and $\mu(G\setminus E)\le\epsilon$, by 563Fd. Next, $G=\bigcup\{U_n:n\in\mathbb{N}, U_n\subseteq G\}$, so there is a finite set $I\subseteq\mathbb{N}$ such that $G'=\bigcup_{n\in I}U_n\subseteq G$ and $\mu(G\setminus G')\le\epsilon$; now $\mu(G'\triangle E)\le 2\epsilon$ and $G'=U_m$ for some m.

(c) If $f\in\mathcal{L}^1$ and $\epsilon>0$, there is an $h\in S(\mathcal{B}_c(X)^f)$ such that $\int|f-h|\le\epsilon$; now there must be an $n\in\mathbb{N}$ and a family $\langle q_i\rangle_{i\le n}$ in $\mathbb{Q}$ such that $\int|h-\sum_{i=0}^n q_i\chi U_i|\le\epsilon$. The set D of such rational linear combinations of the χU_i is countable; enumerate it as $\langle h_n\rangle_{n\in\mathbb{N}}$. All the h_n are differences of semi-continuous functions, therefore resolvable, so $\langle h_n\rangle_{n\in\mathbb{N}}$ is a codable sequence; and for any $u\in L^1$ and $\epsilon>0$ there is an n such that $\|u-h_n^\bullet\|_1\le 2\epsilon$.

(d) This shows that L^1 is separable. To see that it is complete, take a Cauchy filter $\mathcal{F}$ on L^1. For each $k\in\mathbb{N}$ we can take the first $n_k\in\mathbb{N}$ such that $\{u:\|u-h_{n_k}^\bullet\|_1\le 2^{-k}\}$ belongs to $\mathcal{F}$. Now $\int|h_{n_k}-h_{n_{k+1}}|\le 2^{-k}+2^{-k-1}$ for every k, so the codable sequence $\langle h_{n_k}\rangle_{k\in\mathbb{N}}$ converges a.e. to some $f\in\mathcal{L}^1$ (564Fd), and $\int|f-h_{n_k}|\le 3\cdot 2^{-k}$ for every k. So

$$f^\bullet=\lim_{k\to\infty}h_{n_k}^\bullet=\lim\mathcal{F}.$$

(e) Thus L^1 is norm-complete. We know it is a Riesz space with a Riesz norm, so it is a Banach lattice. As for the additivity of the norm on the positive cone, we have only to observe that if f, $g\in\mathcal{L}^1$ and $f^\bullet$, $g^\bullet$ are non-negative, then

$$\begin{aligned}\|f^\bullet+g^\bullet\|_1&=\||f|^\bullet+|g|^\bullet\|_1=\|(|f|+|g|)^\bullet\|_1\\&=\int|f|+|g|=\int|f|+\int|g|=\|f^\bullet\|_1+\|g^\bullet\|_1.\end{aligned}$$

564L Radon-Nikodým theorem Let X be a second-countable space with a codably σ-finite Borel-coded measure μ. Let $\nu:\mathcal{B}_c(X)\to\mathbb{R}$ be a truly continuous additive functional. Then there is an $f\in\mathcal{L}^1(\mu)$ such that $\nu E=\int f\times\chi E$ for every $E\in\mathcal{B}_c(X)$.

proof (a) Let M be the space of bounded additive functionals on $\mathcal{B}_c(X)$; as in 362B, M is an L-space. I will write $\mathcal{L}^1$ for the Riesz space of integrable real-valued codable Borel functions on X. For $f\in\mathcal{L}^1$ and $E\in\mathcal{B}_c(X)$, set $\nu_fE=\int f\times\chi E$; this is defined by 564Ea and 564E(c-i). The map $f\mapsto\nu_f:\mathcal{L}^1\to M$ is a Riesz homomorphism, and norm-preserving in the sense that $\|\nu_f\|=\int|f|$ for every $f\in\mathcal{L}^1$. Accordingly $M_1=\{\nu_f:f\in\mathcal{L}^1\}$ is a Riesz subspace of M isomorphic, as normed Riesz space, to L^1; in particular, it is norm-complete, by 564K, therefore norm-closed.

(b) If $\nu\in M^+$ is truly continuous and $\epsilon>0$, there are an $E\in\mathcal{B}_c(X)^f$ and a $\gamma>0$ such that $\|(\nu-\gamma\nu_{\chi E})^+\|\le\epsilon$. **P** There are $E\in\mathcal{B}_c(X)$ and $\delta>0$ such that $\mu E<\infty$ and $\nu F\le\epsilon$ whenever $\mu(E\cap F)\le\delta$. Set $\gamma=\frac{\|\nu\|}{\delta}$. Then

$$\begin{aligned}(\nu-\gamma\nu_{\chi E})(F)&=\nu F-\frac{\|\nu\|}{\delta}\mu(F\cap E)\\&\le 0\text{ if }\mu(F\cap E)\ge\delta,\\&\le\epsilon\text{ otherwise.}\end{aligned}$$

So $\|(\nu-\gamma\nu_{\chi E})^+\|\le\epsilon$. **Q**

(c) Suppose that $\nu\in M$, $E\in\mathcal{B}_c(X)^f$ and $\gamma>0$ are such that $0\le\nu\le\gamma\nu_{\chi E}$. Let $\epsilon>0$. Then there are an $f\in\mathcal{L}^1$ and a $\nu'\in M^+$ such that $\|\nu-\nu_f-\nu'\|\le\epsilon$ and $\nu'\le\frac{1}{2}\gamma\nu_{\chi E}$. **P** Set $\alpha=\sup_{F\in\mathcal{B}_c(X)}\nu F-\frac{1}{2}\gamma\mu F$; let $H\in\mathcal{B}_c(X)$ be such that $\nu H-\frac{1}{2}\gamma\mu H\ge\alpha-\frac{1}{3}\epsilon$; set $f=\frac{1}{2}\gamma\chi(H\cap E)$ and $\nu'=(\nu-\nu_f)^+\wedge\frac{1}{2}\gamma\nu_{\chi E}$.

If $F\in\mathcal{B}_c(X)$ then

$$\begin{aligned}(\nu_f-\nu)(F)&=\frac{1}{2}\gamma\mu(F\cap H\cap E)-\nu F\le\frac{1}{2}\gamma\mu(F\cap H)-\nu(F\cap H)\\&=\frac{1}{2}\gamma\mu H-\nu H-\frac{1}{2}\gamma\mu(H\setminus F)+\nu(H\setminus F)\end{aligned}$$

(of course μH must be finite, as $\nu H-\frac{1}{2}\gamma\mu H$ is finite)

$$\leq -\alpha + \frac{1}{3}\epsilon + \alpha = \frac{1}{3}\epsilon,$$

$$(\nu - \nu_f - \frac{1}{2}\gamma\nu_{\chi E})(F) = \nu F - \frac{1}{2}\gamma\mu(F \cap E \cap H) - \frac{1}{2}\gamma\mu(F \cap E)$$

$$= \nu(F \cap E \setminus H) - \frac{1}{2}\gamma\mu(F \cap E \setminus H) + \nu(F \setminus E) + \nu(F \cap E \cap H) - \gamma\mu(F \cap E \cap H)$$

$$\leq \nu(F \cap E \setminus H) - \frac{1}{2}\gamma\mu(F \cap E \setminus H)$$

(because $\nu \leq \gamma\nu_{\chi E}$)

$$= \nu((F \cap E) \cup H) - \frac{1}{2}\gamma\mu((F \cap E) \cup H) - \nu H + \frac{1}{2}\gamma\mu H$$

$$\leq \alpha - (\alpha - \frac{1}{3}\epsilon) = \frac{1}{3}\epsilon.$$

So $\|(\nu_f - \nu)^+\| \leq \frac{1}{3}\epsilon$ and $\|(\nu - \nu_f - \frac{1}{2}\gamma\nu_{\chi E})^+\| \leq \frac{1}{3}\epsilon$. But this means that

$$\|\nu - \nu_f - \nu'\| = \|\nu - \nu_f - (\nu - \nu_f)^+ + ((\nu - \nu_f)^+ - \frac{1}{2}\gamma\nu_{\chi E})^+\|$$

$$\leq 2\|\nu - \nu_f - (\nu - \nu_f)^+\| + \|(\nu - \nu_f - \frac{1}{2}\gamma\nu_{\chi E})^+\|$$

$$\leq 2\|(\nu_f - \nu)^+\| + \frac{1}{3}\epsilon \leq \epsilon,$$

as required. **Q**

(d) Again suppose that $\nu \in M$, $E \in \mathcal{B}_c(X)^f$, $\gamma > 0$ and $\epsilon > 0$ are such that $0 \leq \nu \leq \gamma\nu_{\chi E}$. Then for any $n \in \mathbb{N}$ there are an $f \in \mathcal{L}^1$ and a $\nu' \in M^+$ such that $\|\nu - \nu_f - \nu'\| \leq \epsilon$ and $\nu' \leq 2^{-n}\gamma\nu_{\chi E}$. **P** Induce on n. **Q**

(e) If $\nu \in M^+$ is truly continuous and $\epsilon > 0$, there is an $f \in \mathcal{L}^1$ such that $\|\nu - \nu_f\| \leq \epsilon$. **P** By (b), there are an $E \in \mathcal{B}_c(X)^f$ and a $\gamma > 0$ such that $\|(\nu - \gamma\nu_{\chi E})^+\| \leq \frac{1}{3}\epsilon$. Let $n \in \mathbb{N}$ be such that $2^{-n}\gamma\mu E \leq \frac{1}{3}\epsilon$. By (d), we have an $f \in \mathcal{L}^1$ and a $\nu' \in M$ such that $\|(\nu \wedge \gamma\nu_{\chi E}) - \nu_f - \nu'\| \leq \frac{1}{3}\epsilon$ and $0 \leq \nu' \leq 2^{-n}\gamma\nu_{\chi E}$. But this means that

$$\|\nu - \nu_f\| \leq \|(\nu - \gamma\nu_{\chi E})^+\| + \|(\nu \wedge \gamma\nu_{\chi E}) - \nu_f\|$$

$$\leq \frac{1}{3}\epsilon + \frac{1}{3}\epsilon + \|\nu'\| \leq \frac{2}{3}\epsilon + 2^{-n}\gamma\mu E \leq \epsilon. \text{ **Q**}$$

(f) Since any truly continuous $\nu \in M$ has truly continuous positive and negative parts, the space M_{tc} of truly continuous functionals is included in the closure of $M_1 = \{\nu_f : f \in \mathcal{L}^1\}$. But I noted in (a) that M_1 is norm-isomorphic to L^1, so is complete, therefore closed, and must include M_{tc}.

564M Inverse-measure-preserving functions (a) Let X and Y be second-countable spaces, with Borel-coded measures μ and ν. Suppose that $\varphi : X \to Y$ is a codable Borel function such that $\mu\varphi^{-1}[F] = \nu F$ for every $F \in \mathcal{B}_c(Y)$. Then $h\varphi \in S_X$ and $\int h\varphi\,d\mu = \int h\,d\nu$ for every $h \in S_Y$, writing $S_X = S(\mathcal{B}_c(X)^f)$, S_Y for the spaces of simple functions. By 562Mb, $f\varphi \in \mathcal{L}^0(\mu)$ for every $f \in \mathcal{L}^0(\nu)$. By 562Sd, $\langle h_n\varphi\rangle_{n\in\mathbb{N}}$ is a codable sequence in S_X whenever $\langle h_n\rangle_{n\in\mathbb{N}}$ is a codable sequence in S_Y; consequently $f\varphi \in \mathcal{L}^1(\mu)$ whenever $f \in \mathcal{L}^1(\nu)$, and we have a norm-preserving Riesz homomorphism $T : L^1(\nu) \to L^1(\mu)$ defined by setting $Tf^\bullet = (f\varphi)^\bullet$ for $f \in \mathcal{L}^1(\mu)$.

(b) If ν is codably σ-finite, we have a conditional expectation operator in the reverse direction, as follows. For any $f \in \mathcal{L}^1(\mu)$, consider the functional λ_f defined by setting $\lambda_f F = \int f \times \chi(\varphi^{-1}[F])$ for $F \in \mathcal{B}_c(Y)$. This is additive and truly continuous. **P** Let $\epsilon > 0$. By 564Ga, there are an $E_0 \in \mathcal{B}_c(X)$ and a $\delta > 0$ such that $\mu E_0 < \infty$ and $\int |f| \times \chi E \leq \epsilon$ whenever $E \in \mathcal{B}_c(X)$ and $\mu(E \cap E_0) \leq 2\delta$. Next, there is a non-decreasing codable sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{B}_c(Y)$ such that $\nu F_n<\infty$ for every n and $Y=\bigcup_{n\in\mathbb{N}}F_n$. In this case, $\langle\varphi^{-1}[F_n]\rangle_{n\in\mathbb{N}}$ is a non-decreasing codable sequence in $\mathcal{B}_c(X)$ with union X, so there is an n such that $\mu(E_0\setminus\varphi^{-1}[F_n])\le\delta$. Now suppose that $F\in\mathcal{B}_c(Y)$ and $\nu(F\cap F_n)\le\delta$. In this case,

$$\mu(E_0\cap\varphi^{-1}[F])\le\mu(E_0\setminus\varphi^{-1}[F_n])+\mu(\varphi^{-1}[F_n\cap F])\le 2\delta,$$

so

$$|\lambda_f F|\le\int|f|\times\chi(\varphi^{-1}[F])\le\epsilon.$$

As ϵ is arbitrary, λ_f is truly continuous. **Q**

There is therefore a unique $v_f\in L^1(\nu)$ such that $\int_F v_f=\lambda_f F$ for every $F\in\mathcal{B}_c(Y)$. **P** By 564L, there is a $g\in\mathcal{L}^1(\nu)$ such that $\lambda_f F=\int g\times\chi F$ for every $F\in\mathcal{B}_c(Y)$. By 564Gb, any two such functions are equal almost everywhere, so have the same equivalence class in L^1, which we may call v_f. **Q**

We may call v_f the **conditional expectation** of f with respect to the inverse-measure-preserving function φ.

(c) Still supposing that ν is codably σ-finite, we see that $\lambda_f=\lambda_{f'}$ whenever f, $f'\in\mathcal{L}^1(\mu)$ are equal almost everywhere, so that we have an operator $P:L^1(\mu)\to L^1(\nu)$ defined by saying that $Pf^\bullet=v_f$ for every $f\in\mathcal{L}^1(\mu)$; that is, that $\int_F Pu=\int_{\varphi^{-1}[F]}u$ for every $u\in L^1(\mu)$ and $F\in\mathcal{B}_c(Y)$. Because this defines each Pu uniquely, P is linear. It is positive because if $f^\bullet\ge 0$ then $\lambda_f\ge 0$; if now $g\in\mathcal{L}^1(\nu)$ is such that $\int g\times\chi F=\lambda_f F\ge 0$ for every $F\in\mathcal{B}_c(X)$, $g\ge 0$ a.e., by 564Gb, and

$$Pf^\bullet=u_f=g^\bullet\ge 0.$$

It is elementary to check that if T is the operator of (a) above then PT is the identity operator on $L^1(\nu)$.

(d) Now consider the special case in which $Y=X$, the topology of Y is the topology generated by a codable sequence $\langle V_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{B}_c(X)^f$, $\nu=\mu\restriction\mathcal{B}_c(Y)$ and φ is the identity function. (Of course this can be done only when μ is codably σ-finite.) In this case, we can identify $L^1(\nu)$ with its image in $L^1(\mu)$ under T, and P becomes a conditional expectation operator of the kind examined in 242J.

564N Product measures: Theorem Let X and Y be second-countable spaces, and μ, ν semi-finite Borel-coded measures on X, Y respectively.

(a) There is a Borel-coded measure λ on $X\times Y$ such that $\lambda(E\times F)=\mu E\cdot\nu F$ for all $E\in\mathcal{B}_c(X)$ and $F\in\mathcal{B}_c(Y)$.

(b) If ν is codably σ-finite then we can arrange that $\iint f(x,y)\nu(dy)\mu(dx)$ is defined and equal to $\int f d\lambda$ for every λ-integrable real-valued function f.

(c) If μ and ν are both codably σ-finite then λ is uniquely defined by the formula in (a).

proof (a)(i) Start by fixing sequences $\langle U_n\rangle_{n\in\mathbb{N}}$, $\langle V_n\rangle_{n\in\mathbb{N}}$ running over bases for the topologies of X, Y respectively containing $\emptyset$, and a bijection $n\mapsto(i_n,j_n):\mathbb{N}\to\mathbb{N}$; then $\langle U_{i_n}\times V_{j_n}\rangle_{n\in\mathbb{N}}$ runs over a base for the topology of $X\times Y$ containing $\emptyset$. Let

$$\phi_X:\mathcal{T}\to\mathcal{B}_c(X),\quad\tilde{\mathcal{T}}_X\subseteq\mathcal{T}^{\mathbb{R}},\quad\tilde\phi_X:\tilde{\mathcal{T}}_X\to\mathbb{R}^X,$$

$$\phi_Y:\mathcal{T}\to\mathcal{B}_c(Y),$$

$$\phi:\mathcal{T}\to\mathcal{B}_c(X\times Y),\quad\tilde{\mathcal{T}}\subseteq\mathcal{T}^{\mathbb{R}},\quad\tilde\phi:\tilde{\mathcal{T}}\to\mathbb{R}^{X\times Y}$$

be the interpretations of codes associated with the sequences $\langle U_n\rangle_{n\in\mathbb{N}}$, $\langle V_n\rangle_{n\in\mathbb{N}}$ and $\langle U_{i_n}\times V_{j_n}\rangle_{n\in\mathbb{N}}$, as described in 562B and 562N. Let $\mathcal{R}_X$ be the space of resolvable real-valued functions on X, and $\tilde\psi_X:\mathcal{R}_X\to\tilde{\mathcal{T}}_X$ a function such that $\tilde\phi_X(\tilde\psi_X(f))=f$ for ever $f\in\mathcal{R}_X$, as in 562R. The argument will depend on the existence of a number of further functions; it may help if I lay them out explicitly. Fix a member τ_0 of $\tilde{\mathcal{T}}_X$.

(α) Let $\Theta_2':\mathcal{T}\times\mathcal{T}\to\mathcal{T}$ be such that

$$\phi(\Theta_2'(T,T'))=\phi(T)\cap\phi(T'),\quad r(\Theta_2'(T,T'))=\max(r(T),r(T'))$$

for all T, $T'\in\mathcal{T}$ (562Cc); now define $\Theta_2^*:\bigcup_{I\in[\mathbb{N}]^{<\omega}}\mathcal{T}^I\to\mathcal{T}$ by setting

$$\Theta_2^*(\langle T_i\rangle_{i\in I}) = \{\emptyset\}\cup\{<\!n\!> : n\in\mathbb{N}\} \text{ if } I=\emptyset,$$
$$= \Theta_2'(\Theta_2^*(\langle T_i\rangle_{i\in I\cap n}), T_n) \text{ if } n=\max I.$$

Then

$$\phi(\Theta_2^*(\langle T_i\rangle_{i\in I})) = (X\times Y)\cap\bigcap_{i\in I}\phi(T_i),\quad r(\Theta_2^*(\langle T_i\rangle_{i\in I})) = \max(1,\sup_{i\in I} r(T_i))$$

whenever $I\subseteq\mathbb{N}$ is finite and $T_i\in\mathcal{T}$ for $i\in I$.

(β) Let $\tilde\Theta_1 : \mathcal{T}^{\mathbb{N}}\to\mathcal{T}$ be such that

$$\phi_X(\tilde\Theta_1(\langle T_n\rangle_{n\in\mathbb{N}})) = \bigcup_{n\in\mathbb{N}}\phi_X(T_n)$$

for every sequence $\langle T_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{T}$ (526Cb).

(γ) There is a function $\Theta_2 : \tilde{\mathcal{T}}_X\times\mathcal{T}\to\tilde{\mathcal{T}}_X$ such that

$$\tilde\phi_X(\Theta_2(\tau,T')) = (\nu\phi_Y(T'))\chi X - \tilde\phi_X(\tau)$$

whenever $\tau\in\tilde{\mathcal{T}}_X$ and $T'\in\mathcal{T}$ is such that $\nu(\phi_Y(T'))$ is finite. **P** Taking $\Theta_0 : \mathcal{T}\to\mathcal{T}$ such that $\phi_X(\Theta_0(T)) = X\setminus\phi_X(T)$ for every $T\in\mathcal{T}$ (562Ca), set

$$\hat\Theta(\tau,\beta) = \tilde\Theta_1(\langle\Theta_0(\tau(\beta-2^{-n}))\rangle_{n\in\mathbb{N}})$$

for $\tau\in\tilde{\mathcal{T}}_X$ and $\beta\in\mathbb{R}$, so that $\hat\Theta$ is a function from $\tilde{\mathcal{T}}_X\times\mathbb{R}$ to $\mathcal{T}$ and

$$\phi_X(\hat\Theta(\tau,\beta)) = \bigcup_{n\in\mathbb{N}} X\setminus\phi_X(\tau(\beta-2^{-n})) = \{x : \tilde\phi_X(\tau)(x)<\beta\}$$

for $\tau\in\tilde{\mathcal{T}}_X$ and $\beta\in\mathbb{R}$. If $\nu(\phi_Y(T'))=\infty$, take $\Theta_2(\tau,T')=\tau_0$ for every $\tau\in\tilde{\mathcal{T}}_X$; otherwise set

$$\Theta_2(\tau,T')(\alpha) = \hat\Theta(\tau,\nu\phi_Y(T')-\alpha)$$

for $\tau\in\tilde{\mathcal{T}}_X$, $T'\in\mathcal{T}$ and $\alpha\in\mathbb{R}$, so that

$$\phi_X(\Theta_2(\tau,T')(\alpha)) = \{x : \tilde\phi_X(\tau)(x)<\nu\phi_Y(T')-\alpha\}$$
$$= \{x : \nu\phi_Y(T')-\tilde\phi_X(\tau)(x)>\alpha\}$$

for every α, $\Theta_2(\tau,T')\in\tilde{\mathcal{T}}_X$ and $\tilde\phi_X(\Theta_2(\tau,T'))(x) = \nu\phi_Y(T')-\tilde\phi_X(\tau)(x)$ for every $x\in X$. **Q**

(δ) Define $\Theta_1^* : \tilde{\mathcal{T}}_X^{\mathbb{N}}\to\mathcal{T}^{\mathbb{R}}$ by saying that

$$\Theta_1^*(\langle\tau_n\rangle_{n\in\mathbb{N}})(\alpha) = \tilde\Theta_1(\langle\tau_n(\alpha)\rangle_{n\in\mathbb{N}})$$

for every sequence $\langle\tau_n\rangle_{n\in\mathbb{N}}$ in $\tilde{\mathcal{T}}_X$, so that $\Theta_1^*(\langle\tau_n\rangle_{n\in\mathbb{N}})\in\tilde{\mathcal{T}}_X$ and

$$\tilde\phi_X(\tilde\Theta_1(\langle\tau_n\rangle_{n\in\mathbb{N}})) = \sup_{n\in\mathbb{N}}\tilde\phi_X(\tau_n)$$

whenever $\langle\tau_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\tilde{\mathcal{T}}_X$ such that $\sup_{n\in\mathbb{N}}\tilde\phi_X(\tau_n)$ is defined in $\mathbb{R}^X$.

(ϵ) As in 562Ob, we can find a function $\Theta^* : \mathcal{T}\times X\to\mathcal{T}$ such that

$$\phi_Y(\Theta^*(T,x)) = \{y : (x,y)\in\phi(T)\}$$

for $T\in\mathcal{T}$ and $x\in X$.

(ii) If $W\subseteq X\times Y$ is open and $F\in\mathcal{B}_c(Y)$, $x\mapsto\nu(F\cap W[\{x\}]) : X\to[0,\infty]$ is lower semi-continuous. **P** Take $\gamma\in\mathbb{R}$ and consider $G=\{x : \nu(F\cap W[\{x\}])>\gamma\}$. Given $x\in G$ let K be $\{(m,n) : x\in U_m,\ U_m\times V_n\subseteq W\}$; then $W[\{x\}]=\bigcup_{(m,n)\in K}V_n$. Now $\langle V_n\rangle_{(m,n)\in K}$ and $\langle F\cap V_n\rangle_{(m,n)\in K}$ are codable families (562J), so there is a finite set $L\subseteq K$ such that $\nu(\bigcup_{(m,n)\in L}F\cap V_n)>\gamma$ (563B(a-ii)). In this case, $H = X\cap\bigcap_{(m,n)\in L}U_m$ is an open neighbourhood of x included in G, and $\nu(F\cap W[\{x'\}])>\gamma$ for every $x'\in H$. As x is arbitrary, G is open; as γ is arbitrary, the function is lower semi-continuous. **Q**

(iii) For T, $T'\in\mathcal{T}$ and $x\in X$, set

$$h_{TT'}(x) = \nu\{y : y\in\phi_Y(T'),\ (x,y)\in\phi(T)\} = \nu(\phi_Y(T')\cap\phi_Y(\Theta^*(T,x))).$$

Then there is a function $\Theta : \mathcal{T}\times\mathcal{T}\to\tilde{\mathcal{T}}_X$ such that

$$\tilde{\phi}_X(\Theta(T,T')) = h_{TT'}$$

whenever T, $T' \in \mathcal{T}$ are such that $\nu\phi_Y(T')$ is finite. **P** If $\nu\phi_Y(T') = \infty$ set $\Theta(T,T') = \tau_0$. For other T', build Θ by induction on the rank of T, as usual. If $r(T) \le 1$, then $\phi(T)$ is open; by (ii), $h_{TT'}$ is lower semi-continuous, therefore resolvable (562Qa). So we can set $\Theta(T,T') = \tilde{\psi}_X(h_{TT'})$.

For the inductive step to $r(T) \ge 2$, set $A_T = \{n : <n> \in T\}$, so that

$$\begin{aligned}\phi(T) &= \bigcup_{n\in A_T} (X \times Y) \setminus \phi(T_{<n>}) \\ &= \bigcup_{m\in\mathbb{N}} (X \times Y) \setminus ((X \times Y) \cap \bigcap_{n\in A_T\cap m} \phi(T_{<n>})) \\ &= \bigcup_{m\in\mathbb{N}} (X \times Y) \setminus \phi(\Theta_2^*(\langle T_{<n>}\rangle_{n\in A_T\cap m}))\end{aligned}$$

and

$$h_{TT'}(x) = \lim_{m\to\infty} \nu\phi_Y(T') - h_{T^{(m)}T'}(x) = \sup_{m\in\mathbb{N}} \nu\phi_Y(T') - h_{T^{(m)}T'}(x)$$

for every x, where

$$T^{(m)} = \Theta_2^*(\langle T_{<n>}\rangle_{n\in A_T\cap m}), \quad \phi(T^{(m)}) = (X \times Y) \cap \bigcap\nolimits_{n\in A_T\cap m} \phi(T_{<n>})$$

for $m \in \mathbb{N}$. Now $r(T^{(m)}) < r(T)$ for every m, so each $\Theta(T^{(m)}, T')$ has been defined, and we can speak of $\Theta_2(\Theta(T^{(m)}, T'), T')$ for each m; we shall have

$$\begin{aligned}\tilde{\phi}_X(\Theta_2(\Theta(T^{(m)}, T'), T'))(x) &= \nu\phi_Y(T') - \tilde{\phi}_X(\Theta(T^{(m)}, T'))(x) = \nu\phi_Y(T') - h_{T^{(m)}T'}(x) \\ &= \nu\{y : y \in \phi_Y(T'),\, (x,y) \in \bigcup_{n\in A_T\cap m} (X \times Y) \setminus \phi(T_{<n>})\}\end{aligned}$$

for $m \in \mathbb{N}$ and $x \in X$. So if we set

$$\Theta(T,T') = \Theta_1^*(\langle\Theta_2(\Theta(T^{(m)}, T'), T')\rangle_{m\in\mathbb{N}}),$$

we shall have

$$\begin{aligned}\tilde{\phi}_X(\Theta(T,T')) &= \sup_{m\in\mathbb{N}} \tilde{\phi}_X(\Theta_2(\Theta(T^{(m)}, T'), T')) \\ &= \sup_{m\in\mathbb{N}} (\nu\phi_Y(T'))\chi X - \tilde{\phi}_X(\Theta(T^{(m)}, T')) \\ &= \sup_{m\in\mathbb{N}} (\nu\phi_Y(T'))\chi X - h_{T^{(m)}T'} = h_{TT'},\end{aligned}$$

as required for the induction to proceed. **Q**

(iv) Thus we see that $h_{TT'} \in \mathcal{L}^0(\mu)$ whenever T, $T' \in \mathcal{T}$ and $\nu\phi_Y(T')$ is finite.

(v) Let $\mathcal{B}_c(Y)^f$ be the ring of subsets of Y of finite measure. For $F \in \mathcal{B}_c(Y)^f$ and $W \in \mathcal{B}_c(X \times Y)$ we have T, $T' \in \mathcal{T}$ such that $\phi(T) = W$ and $\phi_Y(T') = F$, and now $\nu(F \cap W[\{x\}]) = h_{TT'}(x)$ for every $x \in X$. So we have a functional $\lambda_F : \mathcal{B}_c(X \times Y) \to [0,\infty]$ defined by saying that

$$\begin{aligned}\lambda_F W &= \int \nu(F \cap W[\{x\}])\mu(dx) \text{ if the integral is defined in } \mathbb{R}, \\ &= \infty \text{ otherwise.}\end{aligned}$$

Of course λ_F is additive. If $E \in \mathcal{B}_c(X)$ and $F' \in \mathcal{B}_c(Y)$, then

$$\begin{aligned}\lambda_F(E \times F') &= 0 = \mu E \cdot \nu(F \cap F') \text{ if } \nu(F \cap F') = 0, \\ &= \int \nu(F \cap F')\chi E\, d\mu = \mu E \cdot \nu(F \cap F') \text{ if } \mu E < \infty, \\ &= \infty = \mu E \cdot \nu(F \cap F') \text{ if } \mu E = \infty \text{ and } \nu(F \cap F') > 0.\end{aligned}$$

(To see that $E \times F' \in \mathcal{B}_c(X \times Y)$, use 562Mc.)

(vi) Now suppose that $\langle W_n \rangle_{n\in\mathbb{N}}$ is a codable disjoint sequence in $\mathcal{B}_c(X \times Y)$ with union W, and that $F \in \mathcal{B}_c(Y)^f$. We surely have $\lambda_F W \ge \sum_{n=0}^\infty \lambda_F W_n$. If $\sum_{n=0}^\infty \lambda_F W_n$ is finite, let $\langle T_n \rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{T}$ such that $\phi(T_n) = W_n$ for each n, and take $T' \in \mathcal{T}$ such that $\phi_Y(T') = F$. Then $\langle h_{T_nT'} \rangle_{n\in\mathbb{N}} = \langle \tilde{\phi}_X(\Theta(T_n, T')) \rangle_{n\in\mathbb{N}}$ is a codable sequence of integrable Borel functions, so 564Fe tells us that the sum of the integrals is the integral of the sum; but

$$\sum_{n=0}^\infty h_{T_nT'}(x) = \sum_{n=0}^\infty \nu(\phi_Y(T') \cap \phi_Y(\Theta^*(T_n, x))) = \nu(\bigcup_{n\in\mathbb{N}} \phi_Y(T') \cap \phi_Y(\Theta^*(T_n, x)))$$
$$= \nu(\bigcup_{n\in\mathbb{N}} F \cap W_n[\{x\}]) = \nu(F \cap W[\{x\}])$$

for each x, so we have

$$\lambda_F W = \int \nu(F \cap W[\{x\}])\mu(dx) = \int \sum_{n=0}^\infty h_{T_nT'} d\mu$$
$$= \sum_{n=0}^\infty \int h_{T_nT'} d\mu = \sum_{n=0}^\infty \lambda_F W_n.$$

As $\langle W_n \rangle_{n\in\mathbb{N}}$ is arbitrary, λ_F is a Borel-coded measure.

(vii) If $W \in \mathcal{B}_c(X \times Y)$ and $F \subseteq F'$ in $\mathcal{B}_c(Y)^f$, then

$$\lambda_F W = \int \nu(F \cap W[\{x\}])\mu(dx)$$

(counting $\int h\,d\mu$ as ∞ for a non-negative function $h \in \mathcal{L}^0(\mu) \setminus \mathcal{L}^1(\mu)$)

$$= \int \nu(F' \cap (W \cap (X \times F))[\{x\}])\mu(dx) = \lambda_{F'}(W \cap (X \times F)) \le \lambda_{F'} W.$$

Thus $\langle \lambda_F(W) \rangle_{F\in\mathcal{B}_c(Y)^f}$ is an upwards-directed family for each $W \in \mathcal{B}_c(X \times Y)$; let λW be its supremum. Then λ is a Borel-coded measure on $X \times Y$ (563E). Also

$$\lambda(E \times F) = \lambda_F(E \times F) = \mu E \cdot \nu_F F = \mu E \cdot \nu F$$

whenever $E \in \mathcal{B}_c(X)$ and $F \in \mathcal{B}_c(Y)^f$ have finite measure. For other measurable E and F, if either is negligible then $\lambda(E \times F) = 0$, while if one has infinite measure and the other has non-zero measure then $\lambda(E \times F) = \infty$ because μ and ν are both semi-finite.

Observe that the construction ensures that if $\lambda W < \infty$ and $W \subseteq X \times F$ for some $F \in \mathcal{B}_c(Y)^f$, then $\lambda W = \int \nu W[\{x\}]\mu(dx)$.

(b) Now suppose that ν is codably σ-finite.

(i) Let $\langle F_n \rangle_{n\in\mathbb{N}}$ be a codable sequence in $\mathcal{B}_c(Y)^f$ covering Y; since $\langle \bigcup_{i\le n} F_i \rangle_{n\in\mathbb{N}}$ also is codable, we can suppose that $\langle F_n \rangle_{n\in\mathbb{N}}$ is non-decreasing. Let $\langle T'_n \rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{T}$ such that $\phi_Y(T'_n) = F_n$ for each n. By 562Mc, as usual, $\langle X \times F_n \rangle_{n\in\mathbb{N}}$ is a codable sequence in $\mathcal{B}_c(X \times Y)$, so $\lambda W = \sup_{n\in\mathbb{N}} \lambda(W \cap (X \times F_n))$ whenever λ measures W.

(ii) Let $f : X \times Y \to [0, \infty[$ be an integrable codable Borel function. Then $\iint f(x, y)\nu(dy)\mu(dx)$ is defined and equal to $\int f d\lambda$.

$\mathbf{P}(\boldsymbol{\alpha})$ For n, $k \in \mathbb{N}$ set

$$W_{nk} = \{(x, y) : y \in F_n,\ f(x, y) \ge 2^{-n}k\};$$

then $\langle W_{nk} \rangle_{n,k\in\mathbb{N}}$ is a codable family in $\mathcal{B}_c(X \times Y)$. Let $\langle T_{nk} \rangle_{n,k\in\mathbb{N}}$ be a family in $\mathcal{T}$ such that $W_{nk} = \phi(T_{nk})$ for n, $k \in \mathbb{N}$. For $n \in \mathbb{N}$, define $v_n : X \times Y \to \mathbb{R}$ by setting

$$v_n = 2^{-n} \sum_{k=1}^{4^n} \chi W_{nk};$$

then $\langle v_n\rangle_{n\in\mathbb{N}}$ is a codable sequence of codable Borel functions on $X\times Y$. Moreover, setting $v_{nx}(y)=v_n(x,y)$, $\langle v_{nx}\rangle_{n\in\mathbb{N}}$ is a codable sequence of codable Borel functions on Y, for each $x\in X$. Now set

$$u_{nk}(x)=\nu W_{nk}[\{x\}],\quad u_n(x)=\int v_n(x,y)\nu(dy)$$

for $x\in X$ and n, $k\in\mathbb{N}$. Then, in the language of part (a) of this proof,

$$u_{nk}=\tilde{\phi}_X(\Theta(T_{nk},T'_n))$$

for all n and k, so $\langle u_{nk}\rangle_{n,k\in\mathbb{N}}$ is a codable family of codable Borel functions on X. Since

$$u_n=2^{-n}\sum_{k=1}^{4^n}u_{nk}$$

for each n, $\langle u_n\rangle_{n\in\mathbb{N}}$ is a codable sequence of codable Borel functions on X.

(β) Next, for each $n\in\mathbb{N}$,

$$\int u_n d\mu=2^{-n}\sum_{k=1}^{4^n}\int\nu W_{nk}[\{x\}]\mu(dx)=2^{-n}\sum_{k=1}^{4^n}\lambda W_{nk}$$

(by the final remark in part (a) of the proof)

$$=\int v_n d\lambda.$$

At this point, observe that $\langle v_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing codable sequence with limit f. So

$$\lim_{n\to\infty}\int u_n d\mu=\lim_{n\to\infty}\int v_n d\lambda=\int f d\lambda$$

is finite; since $\langle u_n\rangle_{n\in\mathbb{N}}$ also is non-decreasing, $u(x)=\lim_{n\to\infty}u_n(x)$ is finite for μ-almost all x, and

$$\int u\,d\mu=\lim_{n\to\infty}\int u_n d\mu=\int f d\lambda$$

(564Fa). On the other hand, for each $x\in X$, $\langle v_{nx}\rangle_{n\in\mathbb{N}}$ is a non-decreasing codable sequence with limit f_x, where $f_x(y)=f(x,y)$ for $y\in Y$; so

$$u(x)=\lim_{n\to\infty}\int v_{nx}d\nu=\int f_x d\nu$$

for almost all x, and

$$\iint f(x,y)\nu(dy)\mu(dx)=\iint f_x d\nu\mu(dx)=\int u\,d\mu=\int f d\lambda.\ \mathbf{Q}$$

(iii) It follows at once, taking the difference of positive and negative parts, that

$$\iint f(x,y)\nu(dy)\mu(dx)=\int f d\lambda$$

for every λ-integrable codable Borel function f.

(iv) In particular (or more directly), if $W\in\mathcal{B}_c(X\times Y)$ is λ-negligible, then μ-almost every vertical section of W is ν-negligible. So starting from a general λ-integrable function f, we move to a codable Borel function g such that $f=_{\text{a.e.}}g$; now $\int f(x,y)\nu(dy)$ must be defined and equal to $\int g(x,y)\nu(dy)$ for almost every x, and

$$\iint f(x,y)\nu(dy)\mu(dx)=\iint g(x,y)\nu(dy)\mu(dx)=\int g\,d\lambda=\int f d\lambda.$$

This completes the proof of (b).

(c) Let $\langle E_n\rangle_{n\in\mathbb{N}}$, $\langle F_n\rangle_{n\in\mathbb{N}}$ be codable sequences of sets of finite measure covering X, Y respectively; we may suppose that both sequences are non-decreasing. Then $\langle E_n\times F_n\rangle_{n\in\mathbb{N}}=\langle(E_n\times Y)\cap(X\times F_n)\rangle_{n\in\mathbb{N}}$ is a codable sequence (562Mc). Suppose that λ, λ' are two Borel-coded measures on $X\times Y$ agreeing on measurable rectangles. For each $n\in\mathbb{N}$ let λ_n, λ'_n be the totally finite measures defined by setting

$$\lambda_n W=\lambda(W\cap(E_n\times F_n)),\quad \lambda'_n W=\lambda'(W\cap(E_n\times F_n))$$

for $W\in\mathcal{B}_c(X\times Y)$. Now, given n, set $\mathcal{W}_n=\{W:W\in\mathcal{B}_c(X\times Y),\ \lambda_n W=\lambda'_n W\}$. Then $W\cup W'\in\mathcal{W}_n$ whenever W, $W'\in\mathcal{W}_n$ are disjoint, and $E\times F\in\mathcal{W}_n$ whenever $E\in\mathcal{B}_c(X)$ and $F\in\mathcal{B}_c(Y)$. So $\mathcal{W}_n$ includes the algebra of subsets of $X\times Y$ generated by $\{E\times F:E\in\mathcal{B}_c(X),\,F\in\mathcal{B}_c(Y)\}$. In particular, $\mathcal{W}_n$ includes any set of the form $\bigcup_{(i,j)\in K}U_i\times V_j$ where $K\subseteq\mathbb{N}\times\mathbb{N}$ is finite. But any open subset of $X\times Y$ is expressible as the union of a non-decreasing codable sequence of such sets, so also belongs to $\mathcal{W}_n$. By 563Fg, $\lambda_n=\lambda'_n$.

This is true for every $n\in\mathbb{N}$. Since

$$\lambda W=\sup_{n\in\mathbb{N}}\lambda_nW,\quad \lambda'W=\sup_{n\in\mathbb{N}}\lambda'_nW$$

for every $W\in\mathcal{B}_c(X\times Y)$, $\lambda=\lambda'$, as claimed.

564O Theorem Let $\langle(X_k,\rho_k)\rangle_{n\in\mathbb{N}}$ be a sequence of complete metric spaces, and suppose that we have a double sequence $\langle U_{ki}\rangle_{k,i\in\mathbb{N}}$ such that $\{U_{ki}:i\in\mathbb{N}\}$ is a base for the topology of X_k for each k. Let $\langle\mu_k\rangle_{n\in\mathbb{N}}$ be a sequence such that μ_k is a Borel-coded probability measure on X_k for each k. Set $X=\prod_{k\in\mathbb{N}}X_k$. Then X is a Polish space and there is a Borel-coded probability measure λ on X such that $\lambda(\prod_{k\in\mathbb{N}}E_k)=\prod_{k\in\mathbb{N}}\mu_kE_k$ whenever $\langle E_k\rangle_{k\in\mathbb{N}}\in\prod_{k\in\mathbb{N}}\mathcal{B}_c(X_k)$ and $\{k:E_k\ne X_k\}$ is finite.

proof (a)(i) Of course X is Polish; we have a complete metric ρ on X defined by saying that $\rho(x,y)=\sup_{k\in\mathbb{N}}\min(2^{-k},\rho_k(x(k),y(k)))$ for x, $y\in X$, and a countable base generated by sets of the form $\{x:x(k)\in U_{ki}\}$.

(ii) Writing $\mathcal{F}_k$ for the family of closed subsets of X_k for $k\in\mathbb{N}$, we have a choice function ζ on $\bigcup_{k\in\mathbb{N}}\mathcal{F}_k\setminus\{\emptyset\}$. **P** Given a non-empty $F\in\bigcup_{k\in\mathbb{N}}\mathcal{F}_k$, take the first k such that $F\in\mathcal{F}_k$, and define $\langle F_m\rangle_{m\in\mathbb{N}}$, $\langle i_m\rangle_{m\in\mathbb{N}}$ by saying that

$F_0=F$,

$i_m=\min\{i:i\in\mathbb{N},\,U_{ki}\cap F_m\ne\emptyset,\,\operatorname{diam}U_{ki}\le2^{-m}\}$

(taking the diameter as measured by ρ_k, of course),

$F_{m+1}=\overline{F_m\cap U_{ki_m}}$

for each m. Now $\langle F_m\rangle_{m\in\mathbb{N}}$ generates a Cauchy filter in X_k which must have a unique limit; take this limit for $\zeta(F)$. **Q**

(b)(i) Let $\mathrm{T}=\bigotimes_{k\in\mathbb{N}}\mathcal{B}_c(X_k)$ be the algebra of subsets of X generated by $\{\{x:x(k)\in E\}:k\in\mathbb{N},\,E\in\mathcal{B}_c(X_k)\}$. Note that all these sets belong to $\mathcal{B}_c(X)$, by 562Md, so $\mathrm{T}\subseteq\mathcal{B}_c(X)$. Set

$$\mathcal{C}=\{\textstyle\prod_{k\in\mathbb{N}}E_k:E_k\in\mathcal{B}_c(X_k)\text{ for every }k\in\mathbb{N},\,\{k:E_k\ne X_k\}\text{ is finite}\},$$

$$\mathcal{C}_o=\{\textstyle\prod_{k\in\mathbb{N}}G_k:G_k\subseteq X_k\text{ is open for every }k\in\mathbb{N},\,\{k:G_k\ne X_k\}\text{ is finite}\},$$

$$\mathcal{C}_c=\{\textstyle\prod_{k\in\mathbb{N}}F_k:F_k\subseteq X_k\text{ is closed for every }k\in\mathbb{N},\,\{k:F_k\ne X_k\}\text{ is finite}\}.$$

Then every member of T can be expressed as the union of a finite disjoint family in $\mathcal{C}$. $\overline{C}\in\mathcal{C}_c$ for every $C\in\mathcal{C}$, so the closure of any member of T can be expressed as the union of finitely many members of $\mathcal{C}_c$ and belongs to T. The complement of a member of $\mathcal{C}_c$ can be expressed as the union of finitely many members of $\mathcal{C}_o$, so any open set belonging to T can be expressed as the union of finitely many members of $\mathcal{C}_o$.

(ii) For $m\in\mathbb{N}$ write $\mathrm{T}_m=\bigotimes_{k\ge m}\mathcal{B}_c(X_k)$ for the algebra of subsets of $\prod_{k\ge m}X_k$ generated by sets of the form $\{x:x(k)\in E_k\}$ for $k\ge m$ and $E_k\in\mathcal{B}_c(X_k)$. Then we have an additive functional $\nu_m:\mathrm{T}_m\to[0,1]$ defined by saying that $\nu_m(\prod_{k\ge m}E_k)=\prod_{k=m}^\infty\mu_kE_k$ whenever $E_k\in\mathcal{B}_c(X_k)$ for every $k\ge m$ and $\{k:E_k\ne X_k\}$ is finite (326E). Now if $m\in\mathbb{N}$ and $W\in\mathrm{T}_m$ then

$$\nu_mW=\int\nu_{m+1}\{v:<t>^\frown v\in W\}\mu_m(dt)$$

(notation: 5A1C). **P** This is elementary for cylinder sets $W=\prod_{k\ge m}E_k$; now any other member of T_m is expressible as a finite disjoint union of such sets. **Q**

(c)(i) For open sets $W\subseteq X$ define

$$\lambda_0W=\sup\{\nu_0V:V\in\mathrm{T},\,\overline{V}\subseteq W\}.$$

Then if $\langle W_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of open sets with union X, $\lim_{n\to\infty}\lambda_0W_n=1$. **P** Starting from the double sequence $\langle U_{ki}\rangle_{k,i\in\mathbb{N}}$, it is easy to build a sequence $\langle U_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{C}_o$ which runs over a base for the topology of X. Set $W'_n=\bigcup\{U_i:i\le n,\,\overline{U}_i\subseteq W_n\}$ for each n; then $\langle W'_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of open sets belonging to $\mathrm{T}=\mathrm{T}_0$, and $\bigcup_{n\in\mathbb{N}}W'_n=X$. **?** Suppose, if possible, that $\lim_{n\to\infty}\nu_0W'_n\le1-2^{-l}$

for some $l \in \mathbb{N}$. Then we can define $\langle t_k\rangle_{k\in\mathbb{N}}$ inductively, as follows. The inductive hypothesis will be that $\nu_m V_{mn} \le 1 - 2^{-l-m}$ for every n, where

$$V_{mn} = \{v : v \in \prod_{k\ge m} X_k,\ \langle t_k\rangle_{k<m} \cup v \in W'_n\}.$$

In this case, define $f_{mn} : X_m \to [0,1]$ by setting

$$f_{mn}(t) = \nu_{m+1}\{w : w \in \prod_{k\ge m+1} X_k,\ {<}t{>}^\frown w \in V_{mn}\}.$$

By (b-ii), $\nu_m V_{mn} = \int f_{mn} d\mu_m$ for every m, while $\langle f_{mn}\rangle_{n\in\mathbb{N}}$ is non-decreasing.

Because every W'_n is a finite union of open cylinder sets, so is V_{mn}, and f_{mn} is lower semi-continuous, therefore resolvable; so

$$\int \sup_{n\in\mathbb{N}} f_{mn} d\mu_m = \sup_{n\in\mathbb{N}} \int f_{mn} d\mu_m = \lim_{n\to\infty} \nu_m V_{mn} \le 1 - 2^{-l-m}.$$

The set $F = \{t : \sup_{n\in\mathbb{N}} f_{mn}(t) \le 1 - 2^{-l-m-1}\}$ must be closed and non-empty, and we can set $t_m = \zeta(F)$, where ζ is the choice function of (a-ii). In this case,

$$V_{m+1,n} = \{w : {<}t_m{>}^\frown w \in V_{mn}\}, \quad \nu_{m+1} V_{m+1,n} = f_{mn}(t_m) \le 1 - 2^{-l-m-1}$$

for every n, and the induction continues.

At the end of the induction, however, $x = \langle t_k\rangle_{k\in\mathbb{N}}$ belongs to X, so belongs to W'_n for some n. There must be an m such that W'_n is determined by coordinates less than m, and now $V_{mn} = \prod_{k\ge m} X_k$, so $\nu_m V_{mn} = 1$; which is supposed to be impossible. **X**

We conclude that

$$1 = \lim_{n\to\infty} \nu_0 W'_n = \lim_{n\to\infty} \nu_0 \overline{W'_n} = \lim_{n\to\infty} \lambda_0 W_n$$

because $\overline{W'_n}$ is a closed member of T included in W_n for each n. **Q**

(ii) λ_0 satisfies the conditions of 563H. **P**

(α) Of course $\lambda_0\emptyset = \emptyset$ and $\lambda_0 W \le \lambda_0 W'$ whenever $W \subseteq W'$; also $\lambda_0 X = 1$ is finite.

(β) Suppose that $\langle W_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of open sets in X with union W, and $\epsilon > 0$. Then there is a closed $V \in$ T such that $V \subseteq W$ and $\nu_0 V \ge \lambda_0 W - \epsilon$. Set $W'_n = (X \setminus V) \cup W_n$ for each n; then $\langle W'_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of open sets with union X, so by (i) there are an $n \in \mathbb{N}$ such that $\lambda_0 W'_n \ge 1 - \epsilon$, and a closed $V' \in$ T such that $V' \subseteq W'_n$ and $\nu_0 V' \ge 1 - 2\epsilon$. Now $V \cap V'$ is a closed member of T included in W_n and

$$\lambda_0 W_n \ge \nu_0(V \cap V') \ge \nu_0 V - 2\epsilon \ge \lambda_0 W - 3\epsilon.$$

As ϵ is arbitrary, $\lambda_0 W \le \lim_{n\to\infty} \lambda_0 W_n$; the reverse inequality is trivial, so we have equality.

(γ) Let W, $W' \subseteq X$ be open sets. As in (i), we have non-decreasing sequences $\langle W_n\rangle_{n\in\mathbb{N}}$, $\langle W'_n\rangle_{n\in\mathbb{N}}$ of open members of T such that

$$W = \bigcup_{n\in\mathbb{N}} W_n = \bigcup_{n\in\mathbb{N}} \overline{W}_n, \quad W' = \bigcup_{n\in\mathbb{N}} W'_n = \bigcup_{n\in\mathbb{N}} \overline{W'_n}.$$

In this case

$$W \cap W' = \bigcup_{n\in\mathbb{N}} W_n \cap W'_n = \bigcup_{n\in\mathbb{N}} \overline{W_n \cap W'_n},$$

$$W \cup W' = \bigcup_{n\in\mathbb{N}} W_n \cup W'_n = \bigcup_{n\in\mathbb{N}} \overline{W_n \cup W'_n}.$$

Also

$$\lambda_0 W = \lim_{n\to\infty} \lambda_0 W_n \le \lim_{n\to\infty} \nu_0 W_n \le \lim_{n\to\infty} \nu_0 \overline{W}_n \le \lambda_0 W,$$

so these are all equal; the same applies to the sequences converging to W', $W \cap W'$ and $W \cup W'$, so

$$\begin{aligned}\lambda_0 W + \lambda_0 W' &= \lim_{n\to\infty} \nu_0 W_n + \nu_0 W'_n \\ &= \lim_{n\to\infty} \nu_0(W_n \cap W'_n) + \nu_0(W_n \cup W'_n) \\ &= \lambda_0(W \cap W') + \lambda_0(W \cup W'). \text{ } \mathbf{Q}\end{aligned}$$

(d) By 563H, we have a Borel-coded measure λ on X extending λ_0. Now λ extends ν_0. **P** If $C \in \mathcal{C}$ and $\epsilon > 0$, express C as $\prod_{k\in\mathbb{N}} E_k$ where $E_k \in \mathcal{B}_c(X_k)$ for every k and there is an m such that $E_k = X_k$ for $k > m$. For each $k \le m$, there is an open set $G_k \supseteq E_k$ such that $\mu_k G_k \le \mu_k E_k + \frac{\epsilon}{m+1}$ (563Fd again); setting $G_k = X_k$ for $k > m$ and $W = \prod_{k\in\mathbb{N}} G_k$, $C \subseteq W$ and

$$\lambda C \le \lambda W = \lambda_0 W \le \nu_0 W = \prod_{k=0}^{m} \mu_k G_k \le \nu_0 C + \epsilon.$$

As ϵ is arbitrary, $\lambda C \le \nu_0 C$. This is true for every $C \in \mathcal{C}$. As both λ and ν_0 are additive, $\lambda W \le \nu_0 W$ for every $W \in \mathrm{T}$; as $\lambda X = \nu_0 X = 1$, λ agrees with ν_0 on T. **Q**

In particular, λ agrees with ν_0 on $\mathcal{C}$, as required.

564X Basic exercises (a) Let X be a second-countable space and μ a Borel-coded measure on X. Let $E \in \mathcal{B}_c(X)$ and let μ_E be the Borel-coded measure on X defined as in 563Fa. Show that $\int f d\mu_E$ is defined and equal to $\int f \times \chi E \, d\mu$ for every $f \in \mathcal{L}^1(\mu)$.

(b) Let X be a topological space, μ a Baire-coded measure on X, and f a non-negative integrable real-valued function defined almost everywhere in X. Set $\nu E = \int f \times \chi E$ for $E \in \mathcal{B}\mathfrak{a}_c(X)$. Show that ν is a Baire-coded measure, and that $\int g \, d\nu = \int g \times f \, d\mu$ for every ν-integrable g, if we interpret $(g \times f)(x)$ as 0 when $f(x) = 0$ and $g(x)$ is undefined. (Compare 235K.)

(c) Let X be a countably compact topological space. (i) Show that $C(X) = C_b(X)$. (ii) Show that every positive linear functional $f : C(X) \to \mathbb{R}$ is sequentially smooth. (iii) Show that a norm-bounded sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ in the normed space $C(X)$ is weakly convergent to 0 iff it is pointwise convergent to 0. (iv) Prove this without using measure theory. (*Hint*: Fremlin 74, A2F. Also see 564Ya.)

(d) Let X be a topological space and μ a Baire-coded measure on X. (i) Describe constructions for normed Riesz spaces $L^p(\mu)$ for $1 < p \le \infty$. (ii) Show that if X is second-countable, μ is codably σ-finite and $1 < p < \infty$ then $L^p(\mu)$ is a Dedekind complete Banach lattice with an order-continuous norm, while $L^2(\mu)$ is a Hilbert space.

(e) In 564O, show that λ is uniquely defined. Hence show that we have commutative and associative laws for the product measure construction.

564Y Further exercises (a) Let X be a topological space and $\langle f_n \rangle_{n\in\mathbb{N}}$ a codable sequence of bounded codable Baire real-valued functions on X such that $\{\int f_n d\mu : n \in \mathbb{N}\}$ is bounded for every totally finite Baire-coded measure μ on X. (i) Show that if $\langle E_n \rangle_{n\in\mathbb{N}}$ is a disjoint codable sequence in $\mathcal{B}\mathfrak{a}_c(X)$ and μ is a Baire-coded measure on X, then $\lim_{n\to\infty} \int f_n \times \chi E_n d\mu = 0$. (ii) Now suppose in addition that $\lim_{n\to\infty} f_n(x) = 0$ for every $x \in X$. Show that if μ is a Baire-coded measure on X, then $\lim_{n\to\infty} \int f_n d\mu = 0$. (iii) Use this result to strengthen (iii) of 564Xc to 'a sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ in $C(X)$ is weakly convergent to 0 iff it is bounded for the weak topology and pointwise convergent to 0'.

(b) Let X be a locally compact completely regular topological group. Show that there is a non-zero left-translation-invariant Baire-coded measure on X.

(c) Let I be a set and $X = \{0,1\}^I$. Write Z for $\{0,1\}^{\mathbb{N}}$. For $\theta : \mathbb{N} \to I$ define $g_\theta : X \to Z$ by setting $g_\theta(x) = x\theta$ for $x \in X$. Let $\phi : \mathcal{T} \to \mathcal{B}_c(Z)$ be an interpretation of Borel codes for subsets of Z defined from a sequence running over a base for the topology of Z. Let Σ be the family of subsets of X of the form $\phi'(\theta, T) = g_\theta^{-1}[\phi(T)]$ where $\theta \in I^{\mathbb{N}}$ and $T \in \mathcal{T}$; say that a codable family in Σ is one of the form $\langle \phi'(\theta_i, T_i) \rangle_{i\in I}$. Show that there is a functional $\mu : \Sigma \to [0,1]$ such that $\mu\emptyset = 0$, $\mu(\bigcup_{n\in\mathbb{N}} E_n) = \sum_{n=0}^{\infty} \mu E_n$ whenever $\langle E_n \rangle_{n\in\mathbb{N}}$ is a disjoint codable sequence in Σ, and $\mu\{x : x{\restriction}J = w\} = 2^{-\#(J)}$ whenever $J \subseteq I$ is finite and $w \in \{0,1\}^J$.

(d) Suppose there is a disjoint sequence $\langle I_n \rangle_{n\in\mathbb{N}}$ of doubleton sets such that for every function f with domain $\mathbb{N}$ the set $\{n : f(n) \in I_n\}$ is finite (Jech 73, 4.4). Set $I = \bigcup_{n\in\mathbb{N}} I_n$ and let Σ be the algebra of subsets of $\{0,1\}^I$ determined by coordinates in finite sets. Let $\lambda : \Sigma \to [0,1]$ be the additive functional such that $\lambda\{x : z \subseteq x\} = 2^{-k}$ whenever $J \in [I]^k$ and $z \in \{0,1\}^J$. Show that there is a sequence $\langle E_n \rangle_{n\in\mathbb{N}}$ in Σ, covering $\{0,1\}^I$, such that $\sum_{n=0}^{\infty} \lambda E_n < 1$.

564 Notes and comments In the definitions of 564A, I follow the principles of earlier volumes in allowing virtually measurable functions with conegligible domains to be counted as integrable. But you will see that in 564F and elsewhere I work with real-valued Baire measurable functions defined everywhere. The point is that while, if you wish to work through the basic theorems of Fourier analysis under the new rules, you will certainly need to deal with functions which are not defined everywhere, all the main theorems will depend on establishing that you have sequences of sets and functions which are codable in appropriate senses. There is no way of coding members of $\mathcal{L}^0$ or $\mathcal{L}^1$ as I have defined them in 564A. What you will need to do is to build parallel structures, so that associated with each almost-everywhere-summable Fourier series $f(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$ you have in hand a code τ for a codable Borel function $\tilde{f}$ equal almost everywhere to f, together with a code T for a conegligible codable Borel set E included in $\{x : x \in \operatorname{dom} f,\, f(x) = \tilde{f}(x)\}$. Provided that associated with every relevant sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ you can define appropriate sequences $\langle \tau_n \rangle_{n\in\mathbb{N}}$ and $\langle T_n \rangle_{n\in\mathbb{N}}$, you can hope to deduce the reqired properties of $\langle f_n \rangle_{n\in\mathbb{N}}$ by applying 564F to the sequence coded by $\langle \tau_n \rangle_{n\in\mathbb{N}}$.

Of course there are further significant technical differences between the treatment here and the more orthodox one I have employed elsewhere. In the ordinary theory, using the axiom of choice whenever convenient, a measure μ, thought of as a function defined on a σ-algebra of sets, carries in itself all the information needed to describe the space $\mathcal{L}^0(\mu)$. In the present context, we are dealing with functions μ defined on algebras which do not directly code the topologies on which the definition relies. So it would be safer to write $\mathcal{L}^0(\mathfrak{T}, \mu)$. But of course what really matters is the collection of codable families of codable sets, and perhaps we should be thinking of a different level of abstraction. In the proof of 564N I have tried to cast the proof in a language which might be adaptable to other ways of coding sets and functions.

From 564K on, most of the results seem to depend on second-countability; it may be that something can be done with spaces which have well-orderable bases.

In the shift from 564Xc(iii) to 564Ya(iii) I find myself asking for a reason why a weakly bounded sequence in $C(X)$ should be norm-bounded. As far as I know, there is no useful general result in ZF in this direction. But in 564Ya I have suggested a method which will serve in this special context.

I offer 564Yc and 564Yd as positive and negative examples. The point is that in 564Yc there may be few sequences of functions from $\mathbb{N}$ to I, so that we get few codable sequences of sets. Of course, if I is well-orderable then $\{0,1\}^I$ is compact (561D) and we can use 564H. For well-orderable I, any continuous real-valued function on $\{0,1\}^I$ is determined by coordinates in some countable set, so that the methods of 564H and 564Yc will give the same measure.

Version of 25.4.14

565 Lebesgue measure without choice

I come now to the construction of specific non-trivial Borel-coded measures. Primary among them is of course Lebesgue measure on $\mathbb{R}^r$; we also have Hausdorff measures (565N-565O). For Lebesgue measure I begin, as in §115, with half-open intervals. The corresponding 'outer measure' may no longer be countably subadditive, so I call it 'Lebesgue submeasure'. Carathéodory's method no longer seems quite appropriate, as it smudges the distinction between 'negligible' and 'outer measure zero', so I use 563H to show that there is a Borel-coded measure agreeing with Lebesgue submeasure on open sets (565C-565D); it is the completion of this Borel-coded measure which I will call Lebesgue measure. We have a version of Vitali's theorem for well-orderable families (in particular, for countable families) of balls (565F). From this we can prove the Fundamental Theorem of Calculus in essentially its standard form (565M).

565A Definitions Throughout this section, except when otherwise stated, $r \geq 1$ will be a fixed integer. As in §115, I will say that a **half-open interval** in $\mathbb{R}^r$ is a set of the form

$$[a, b[\, = \{x : x \in \mathbb{R}^r,\ a(i) \leq x(i) < b(i) \text{ for } i < r\}$$

where a, $b \in \mathbb{R}^r$. For a half-open interval I, set $\lambda I = 0$ if $I = \emptyset$ and otherwise $\lambda I = \prod_{i=0}^{r-1} b(i) - a(i)$ where $I = [a, b[$. Now for $A \subseteq \mathbb{R}^r$ set

$$\theta A = \inf\{\textstyle\sum_{j=0}^{\infty} \lambda I_j : \langle I_j \rangle_{j\in\mathbb{N}} \text{ is a sequence of half-open intervals covering } A\}.$$

565B Proposition In the notation of 565A,

(a) the function $\theta : \mathcal{P}\mathbb{R}^r \to [0,\infty]$ is a submeasure,

(b) $\theta I = \lambda I$ for every half-open interval $I \subseteq \mathbb{R}^r$.

proof (a) As in parts (a-i) to (a-iii) of the proof of 115D, $\theta\emptyset = 0$ and $\theta A \le \theta B$ whenever $A \subseteq B$. If A, $B \subseteq \mathbb{R}^r$ and $\epsilon > 0$, we have sequences $\langle I_n\rangle_{n\in\mathbb{N}}$ and $\langle J_n\rangle_{n\in\mathbb{N}}$ of half-open intervals such that

$$A \subseteq \bigcup_{n\in\mathbb{N}} I_n, \quad B \subseteq \bigcup_{n\in\mathbb{N}} J_n,$$

$$\sum_{n=0}^\infty \lambda I_n \le \theta A + \epsilon, \quad \sum_{n=0}^\infty \lambda J_n \le \theta B + \epsilon.$$

Set $K_{2n} = I_n$, $K_{2n+1} = J_n$ for $n \in \mathbb{N}$; then $A \cup B \subseteq \bigcup_{n\in\mathbb{N}} K_n$ so

$$\theta(A \cup B) \le \sum_{n=0}^\infty \lambda K_n = \sum_{n=0}^\infty \lambda I_n + \sum_{n=0}^\infty \lambda J_n \le \theta A + \theta B + 2\epsilon.$$

As ϵ is arbitrary, $\theta(A \cup B) \le \theta A + \theta B$.

(b) The arguments of 114B/115B/115Db nowhere called on any form of the axiom of choice, so can be used unchanged.

Definition I will call the submeasure θ **Lebesgue submeasure** on $\mathbb{R}^r$.

565C Lemma Let $\mathcal{I}$ be the family of half-open intervals in $\mathbb{R}^r$; let θ be Lebesgue submeasure, and set

$$\Sigma = \{E : E \subseteq X,\, \theta A = \theta(A \cap E) + \theta(A \setminus E) \text{ for every } A \subseteq X\}, \quad \nu = \theta{\restriction}\Sigma$$

(563G).

(a) Let $\langle I_n\rangle_{n\in\mathbb{N}}$ be a disjoint sequence in $\mathcal{I}$. Then $E = \bigcup_{n\in\mathbb{N}} I_n$ belongs to Σ and $\nu E = \sum_{n=0}^\infty \nu I_n$.

(b) Every open set in $\mathbb{R}^r$ belongs to Σ.

(c) If G, $H \subseteq \mathbb{R}^r$ are open, then $\nu G + \nu H = \nu(G \cap H) + \nu(G \cup H)$.

(d) If $\langle G_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of open sets then $\nu(\bigcup_{n\in\mathbb{N}} G_n) = \lim_{n\to\infty} G_n$.

proof (a)(i) If $i < r$ and $\alpha \in \mathbb{R}$ then $\{x : x \in \mathbb{R}^r,\, x(i) < \alpha\} \in \Sigma$, as in 115F. So every half-open interval belongs to Σ. By 565Bb, $\nu I = \theta I = \lambda I$ for every $I \in \mathcal{I}$.

(ii)(α) $\theta E = \sum_{n=0}^\infty \nu I_n$. **P** Because $\langle I_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{I}$ covering E, θE is at most $\sum_{n=0}^\infty \lambda I_n \le \sum_{n=0}^\infty \nu I_n$. In the other direction,

$$\theta E \ge \sup_{n\in\mathbb{N}} \theta(\bigcup_{i\le n} I_i) = \sup_{n\in\mathbb{N}} \nu(\bigcup_{i\le n} I_i) = \sup_{n\in\mathbb{N}} \sum_{i\le n} \nu I_i = \sum_{n=0}^\infty \nu I_n. \text{ **Q**}$$

(β) $E \in \Sigma$. **P** Let $A \subseteq \mathbb{R}^r$ be such that θA is finite, and $\epsilon > 0$. We have a sequence $\langle J_m\rangle_{m\in\mathbb{N}}$ in $\mathcal{I}$ such that $A \subseteq \bigcup_{m\in\mathbb{N}} J_m$ and $\sum_{m=0}^\infty \lambda J_m \le \theta A + \epsilon$ is finite. Let m be such that $\sum_{j=m+1}^\infty \lambda J_j \le \epsilon$, and set $K = \bigcup_{j\le m} J_j$; then

$$A \cap E \subseteq (K \cap E) \cup \bigcup_{j>m} J_j,$$

so

$$\theta(A \cap E) \le \theta(K \cap E) + \theta(\bigcup_{j>m} J_j) \le \theta(K \cap E) + \sum_{j=m+1}^\infty \lambda J_j \le \theta(K \cap E) + \epsilon.$$

Similarly, $\theta(A \setminus E) \le \theta(K \setminus E) + \epsilon$. Next, by ($\alpha$) applied to $\langle J_j \cap I_i\rangle_{i\in\mathbb{N}}$ or otherwise, $\sum_{i=0}^\infty \nu(J_j \cap I_i)$ is finite for every j, so there is an $n \in \mathbb{N}$ such that $\sum_{j=0}^m \sum_{i=n+1}^\infty \nu(J_j \cap I_i) \le \epsilon$. Set $L = \bigcup_{i\le n} I_i$; then

$$K \cap E \subseteq (K \cap L) \cup \bigcup_{j\le m, i>n} J_j \cap I_i,$$

$$\begin{aligned}\theta(K \cap E) &\le \theta(K \cap L) + \theta(\bigcup_{j\le m, i>n} J_j \cap I_i)\\ &\le \theta(K \cap L) + \sum_{j=0}^m \sum_{i=n+1}^\infty \nu(J_j \cap I_i) \le \theta(K \cap L) + \epsilon.\end{aligned}$$

Assembling these,

$$\theta A \le \theta(A\cap E)+\theta(A\setminus E) \le \theta(K\cap E)+\theta(K\setminus E)+2\epsilon$$
$$\le \theta(K\cap L)+\theta(K\setminus L)+3\epsilon = \theta K+3\epsilon$$

(because we know that $L\in\Sigma$)

$$\le \theta A+3\epsilon.$$

As ϵ is arbitrary, $\theta A=\theta(A\cap E)+\theta(A\setminus E)$. This was on the assumption that θA was finite; but of course it is also true if $\theta A=\infty$. As A is arbitrary, $E\in\Sigma$. **Q**

(γ) Accordingly $\nu E=\theta E=\sum_{n=0}^{\infty}\nu I_n$.

(b) Let $\mathcal{I}_0$ be the family of dyadic half-open intervals in $\mathbb{R}^r$ of the form $\left[2^{-k}z, 2^{-k}(z+\mathbf{1})\right[$ where $k\in\mathbb{N}$, $z\in\mathbb{Z}^r$ and $\mathbf{1}=(1,\dots,1)$. Note that $\mathcal{I}_0$ is countable and that if I, $J\in\mathcal{I}_0$ then either $I\subseteq J$ or $J\subseteq I$ or $I\cap J=\emptyset$. Also any non-empty subset of $\mathcal{I}_0$ has a maximal element.

If $G\subseteq\mathbb{R}^r$ is open, set $\mathcal{J}=\{I: I\in\mathcal{I}_0,\ I\subseteq G\}$ and let $\mathcal{J}'$ be the set of maximal elements of $\mathcal{J}$. Then $\mathcal{J}'$ is disjoint and countable, so by (a-ii) $G=\bigcup\mathcal{J}=\bigcup\mathcal{J}'$ belongs to Σ.

(c) Because ν is additive on Σ,

$$\nu(G\cup H)+\nu(G\cap H)=\nu G+\nu(H\setminus G)+\nu(H\cap G)=\nu G+\nu H$$

for all open sets G, $H\subseteq\mathbb{R}^r$.

(d) This time, let $\mathcal{J}$ be $\bigcup_{n\in\mathbb{N}}\{I: I\in\mathcal{I}_0,\ I\subseteq G_n\}$; again, let $\mathcal{J}'$ be the set of maximal elements of $\mathcal{J}$. Then $G=\bigcup\mathcal{J}=\bigcup\mathcal{J}'$, so

$$\nu G=\sum_{J\in\mathcal{J}'}\nu J=\sup_{\mathcal{K}\subseteq\mathcal{J}'\text{ is finite}}\sum_{J\in\mathcal{K}}\nu J\le\sup_{n\in\mathbb{N}}\nu G_n=\lim_{n\to\infty}\nu G_n\le\nu G$$

because $\langle G_n\rangle_{n\in\mathbb{N}}$ is non-decreasing.

565D Definition Let θ and ν be as in 565C. By 563H, there is a unique Borel-coded measure μ on $\mathbb{R}^r$ such that $\mu G=\nu G=\theta G$ for every open set $G\subseteq\mathbb{R}^r$. I will say that **Lebesgue measure** on $\mathbb{R}^r$ is the completion μ_L of μ; the sets it measures will be **Lebesgue measurable**.

565E Proposition Let $\mathcal{I}$, θ, Σ, ν, μ and μ_L be as in 565A-565D.

(a) μ is the restriction of θ to the algebra $\mathcal{B}_c(\mathbb{R}^r)$ of codable Borel sets.

(b) For every $A\subseteq\mathbb{R}^r$,

$$\theta A=\inf\{\mu_L E: E\supseteq A\text{ is Lebesgue measurable}\}=\inf\{\mu G: G\supseteq A\text{ is open}\}.$$

(c) $E\in\Sigma$ and $\mu_L E=\nu E=\theta E$ whenever E is Lebesgue measurable.

(d) μ_L is inner regular with respect to the compact sets and outer regular with respect to the open sets.

proof (a) If $E\in\mathcal{B}_c(\mathbb{R}^r)$, then

$$\mu E=\inf\{\mu G: G\supseteq E\text{ is open}\}$$

(563Fd)

$$=\inf\{\theta G: G\supseteq E\text{ is open}\}\ge\theta E.$$

Next, if $I\subseteq\mathbb{R}^r$ is a half-open interval, it is a codable Borel set and

$$\lambda I=\inf\{\lambda J: J\in\mathcal{I},\ I\subseteq\operatorname{int}J\}\ge\inf\{\theta(\operatorname{int}J): J\in\mathcal{I},\ I\subseteq\operatorname{int}J\}$$
$$\ge\inf\{\theta G: G\supseteq I\text{ is open}\}=\mu I\ge\theta I=\lambda I.$$

So μ and λ agree on $\mathcal{I}$. If now E is a codable Borel set and $\epsilon>0$, there is a sequence $\langle I_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{I}$ such that $E\subseteq\bigcup_{n\in\mathbb{N}}I_n$ and $\sum_{n=0}^{\infty}\lambda I_n\le\theta E+\epsilon$. But every I_n is resolvable (because it belongs to the algebra of sets generated by the open sets), so $\langle I_n\rangle_{n\in\mathbb{N}}$ is a codable sequence (562J) and

$$\mu E \le \mu(\bigcup_{n\in\mathbb{N}} I_n) \le \sum_{n=0}^{\infty} \mu I_n = \sum_{n=0}^{\infty} \lambda I_n \le \theta E + \epsilon.$$

As E and ϵ are arbitrary, $\mu = \theta\restriction\mathcal{B}_c(\mathbb{R}^r)$.

(b) Suppose that $A \subseteq \mathbb{R}^r$. If $E \supseteq A$ is Lebesgue measurable, there are F, $H \in \mathcal{B}_c(\mathbb{R}^r)$ such that $E\triangle F \subseteq H$ and $\mu H = 0$, so that $E \subseteq F \cup H$ and

$$\theta A \le \theta(F \cup H) = \mu(F \cup H) = \mu_L E.$$

So we have

$$\theta A \le \inf\{\mu_L E : E \supseteq A \text{ is Lebesgue measurable}\} \le \inf\{\mu G : G \supseteq A \text{ is open}\}.$$

In the other direction, given $\epsilon > 0$ there is a sequence $\langle I_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{I}$, covering A, such that $\sum_{n=0}^{\infty} \lambda I_n \le \theta A+\epsilon$. As in (a) just above, $E = \bigcup_{n\in\mathbb{N}} I_n$ is a codable Borel set and $\mu E \le \sum_{n=0}^{\infty} \lambda I_n$; now there is an open $G \supseteq E$ such that $\mu G \le \mu E + \epsilon \le \theta A + 2\epsilon$. As ϵ is arbitrary,

$$\inf\{\mu G : G \supseteq A \text{ is open}\} \le \theta A$$

and we have the equalities.

(c) Suppose that E is Lebesgue measurable, $A \subseteq \mathbb{R}^r$ and $\epsilon > 0$. By (b), there is an open set $G \supseteq A$ such that $\mu G \le \theta A + \epsilon$. Now

$$\theta(A \cap E) + \theta(A \setminus E) \le \theta(G \cap E) + \theta(G \setminus E) \le \mu_L(G \cap E) + \mu_L(G \setminus E)$$

(by (b))

$$= \mu_L G = \mu G \le \theta A + \epsilon.$$

As usual, this is enough to ensure that $E \in \Sigma$. Now (b) again tells us that $\mu_L E = \theta E = \nu E$.

(d) Of course μ is locally finite, while $\mathbb{R}^r$ is a regular topological space. So 563F(d-ii) tells us that μ is inner regular with respect to the closed sets and outer regular with respect to the open sets; it follows that μ_L also is. Next, every closed set is K_σ, while compact sets are resolvable and all sequences of compact sets are codable, so $\mu F = \sup\{\mu K : K \subseteq F \text{ is compact}\}$ for every closed set $F \subseteq \mathbb{R}^r$; consequently μ_L is inner regular with respect to the compact sets.

565F Vitali's Theorem Let $\mathcal{C}$ be a well-orderable family of non-singleton closed balls in $\mathbb{R}^r$. For $\mathcal{I} \subseteq \mathcal{C}$ set

$$A_{\mathcal{I}} = \bigcap_{\delta>0} \bigcup\{C : C \in \mathcal{I},\ \operatorname{diam} C \le \delta\}.$$

Let $\mathfrak{T}$ be the family of open subsets of $\mathbb{R}^r$. Then there are functions $\Psi : \mathcal{PC} \to \mathcal{PC}$ and $\Theta : \mathcal{PC} \times \mathbb{N} \to \mathfrak{T}$ such that $\Psi(\mathcal{I}) \subseteq \mathcal{I}$, $\Psi(\mathcal{I})$ is disjoint and countable, $\mu_L(\Theta(\mathcal{I}, k)) \le 2^{-k}$ and $A_{\mathcal{I}} \subseteq \bigcup\Psi(\mathcal{I}) \cup \Theta(\mathcal{I}, k)$ whenever $\mathcal{I} \subseteq \mathcal{C}$ and $k \in \mathbb{N}$. In particular,

$$A_{\mathcal{I}} \setminus \bigcup\Psi(\mathcal{I}) \subseteq \bigcap_{k\in\mathbb{N}} \Theta(\mathcal{I}, k)$$

is negligible.

proof We use the greedy algorithm of 221A/261B, but watching more carefully. Start by fixing on a well-ordering $\preccurlyeq$ of $\mathcal{C} \cup \{\emptyset\}$. Next, for each $n \in \mathbb{N}$, set $U_n = \{x : x \in \mathbb{R}^r,\ n < \|x\| < n+1\}$, where $\| \ \|$ is the Euclidean norm on $\mathbb{R}^r$. It will be convenient to fix at this point on a family $\langle G_{kn}\rangle_{k,n\in\mathbb{N}}$ of open sets such that $\mu_L G_{kn} \le 2^{-k-n-2}$ and $\{x : \|x\| = n\} \subseteq G_{kn}$ for all k and n; for instance, G_{kn} could be an open shell with rational inner and outer radii (except for G_{k0}, which should be an open ball).[6]

[6] Of course we can still use the similarity argument from part (g) of the proof of 261B to check that thin shells have small measure.

Now define $C_{\mathcal{I}nm}$, for $\mathcal{I} \subseteq \mathcal{C}$ and m, $n \in \mathbb{N}$, by saying that

given $\langle C_{\mathcal{I}ni}\rangle_{i<m}$, $C_{\mathcal{I}nm}$ is to be the $\preccurlyeq$-first member of $\{\emptyset\} \cup (\mathcal{I} \cap \mathcal{P}U_n)$ which is disjoint from $\bigcup_{i<m} C_{\mathcal{I}ni}$ and has diameter at least

$$\frac{1}{2}\sup\{\operatorname{diam} C : C \in \{\emptyset\} \cup (\mathcal{I} \cap \mathcal{P}U_n) \text{ is disjoint from } \bigcup_{i<m} C_{\mathcal{I}ni}\}.$$

(I take the diameter of the empty set to be 0, as usual.) Set

$$\Psi(\mathcal{I}) = \{C_{\mathcal{I}nm} : n,\, m \in \mathbb{N}\} \setminus \{\emptyset\}.$$

Because the U_n are disjoint, $\Psi(\mathcal{I})$ is a disjoint subfamily of $\mathcal{I}$, and of course it is countable. Just as in 261B, we find that for each $\mathcal{I} \subseteq \mathcal{C}$ and $n \in \mathbb{N}$ we have

$$A_{\mathcal{I}} \cap U_n \subseteq \bigcup_{i<m} C_{\mathcal{I}ni} \cup \bigcup_{i\ge m} C'_{\mathcal{I}ni},$$

where for $C \in \mathcal{C}$ I write C' for the *open* ball with the same centre and *six* times the radius; $\emptyset'$ will be $\emptyset$. Just as in 261B, $\sum_{m=0}^{\infty} \mu_L C'_{\mathcal{I}nm} \le 6^r \mu_L B(\mathbf{0}, n+1)$ is finite. So, for each k and n, we can take the first m_{kn} such that $\sum_{i=m_{kn}}^{\infty} \mu_L C'_{\mathcal{I}ni} \le 2^{-n-k-2}$. Now set

$$\Theta(\mathcal{I}, k) = \bigcup_{n\in\mathbb{N}} G_{kn} \cup \bigcup_{n\in\mathbb{N}, i\ge m_{kn}} C'_{\mathcal{I}ni};$$

we shall have $A \setminus \bigcup \Psi(\mathcal{I}) \subseteq \Theta(\mathcal{I}, k)$ and

$$\mu_L\Theta(\mathcal{I}, k) \le \sum_{n=0}^{\infty} \mu_L G_{kn} + \sum_{n=0}^{\infty}\sum_{i=m_{kn}}^{\infty} \mu_L C'_{\mathcal{I}ni} \le 2^{-k},$$

as required.

565G Proposition Let $A \subseteq \mathbb{R}^r$ be any set. Then its Lebesgue submeasure is

$$\theta A = \inf\{\sum_{n=0}^{\infty} \mu_L B_n : \langle B_n\rangle_{n\in\mathbb{N}} \text{ is a sequence of closed balls covering } A\}.$$

proof Let $\epsilon > 0$. Then there is a (non-empty) open set $G \supseteq A$ with $\mu_L G \le \theta A + \epsilon$. Use Vitali's theorem, with $\mathcal{C}$ the family of closed balls with rational centres and non-zero rational radii, to see that there is a disjoint sequence $\langle C_n\rangle_{n\in\mathbb{N}}$ of balls included in G such that $\mu_L(A \setminus \bigcup_{n\in\mathbb{N}} C_n) = 0$ and $\sum_{n\in\mathbb{N}} \mu_L C_n \le \theta A + \epsilon$. Next, cover $A \setminus \bigcup_{n\in\mathbb{N}} C_n$ by a sequence of half-open intervals with measures summing to not more than ϵ, and expand these to balls with measures summing to not more than $\epsilon r^{r/2}$. Interleaving this sequence with the C_n, we get a sequence $\langle B_n\rangle_{n\in\mathbb{N}}$ of balls, covering A, with $\sum_{n=0}^{\infty} \mu_L B_n \le \theta A + (1 + r^{r/2})\epsilon$. So

$$\theta A \ge \inf\{\sum_{n=0}^{\infty} \mu_L B_n : \langle B_n\rangle_{n\in\mathbb{N}} \text{ is a sequence of closed balls covering } A\}.$$

The reverse inequality is elementary (563C(a-ii)).

565H Corollary Lebesgue measure is invariant under isometries.

proof We can see from its definition that Lebesgue submeasure is translation-invariant, so Lebesgue measure also is. Consequently two balls with the same radii have the same measure. Isometries of $\mathbb{R}^r$ take closed balls to closed balls with the same radii, so 565G gives the result.

565I Lemma (a) Writing $C_k(\mathbb{R}^r)$ for the space of continuous real-valued functions on $\mathbb{R}^r$ with compact support, $C_k(\mathbb{R}^r) \subseteq \mathcal{L}^1(\mu_L)$.

(b) There is a countable set $D \subseteq C_k(\mathbb{R}^r)$ such that $\{g^\bullet : g \in D\}$ is norm-dense in $L^1(\mu_L)$.

proof (a) This is elementary; every continuous function is resolvable, therefore a codable Borel function and belongs to $\mathcal{L}^0$; if in addition it has compact support it is dominated by an integrable function and is integrable, by 564E(c-i).

(b)(i) Let $\mathcal{U}$ be a countable base for the topology of $\mathbb{R}^r$, consisting of bounded sets and closed under finite unions. Let D_0 be the set of functions of the form $x \mapsto \max(0, 1 - 2^k\rho(x, \mathbb{R}^r \setminus U))$ for $U \in \mathcal{U}$ and $k \in \mathbb{N}$, where ρ is the Euclidean metric on $\mathbb{R}^r$, and D the set of rational linear combinations of members of D_0; then D is a countable subset of $C_k(\mathbb{R}^r)$.

(ii) If $E \subseteq \mathbb{R}^r$ is a codable Borel set of finite measure, and $\epsilon > 0$, then by 565Ed there are a compact set $K \subseteq E$ and an open set $G \supseteq E$ such that $\mu_L(G \setminus K) \le \epsilon$. Now there are a $U \in \mathcal{U}$ such that $K \subseteq U \subseteq G$ and a $g \in D_0$ such that $\chi K \le g \le \chi U$, so that $\int |g - \chi E| \le \epsilon$.

(iii) It follows that whenever f is a simple codable Borel function, in the sense of 564Aa, and $\epsilon > 0$ there is a $g \in D$ such that $\int |f - g| \le \epsilon$.

(iv) If $f \in \mathcal{L}^1$ and $\epsilon > 0$ there are a simple codable Borel function g and an $h \in D$ such that $\int |f - g| \le \frac{1}{2}\epsilon$ such that $\int |g - h| \le \frac{1}{2}\epsilon$, so that $\int |f - h| \le \epsilon$.

565J Lemma Suppose that f is an integrable function on $\mathbb{R}^r$, and that $\int_I f \ge 0$ for every half-open interval $I \subseteq \mathbb{R}^r$. Then $f(x) \ge 0$ for almost every $x \in \mathbb{R}^r$.

proof (a) Note first that any finite union E of half-open intervals is expressible as a finite disjoint union of half-open intervals. So $\int_E f \ge 0$.

(b) Suppose that g is a simple codable Borel function such that $\int_E g \le \epsilon$ whenever E is a finite union of half-open intervals. Then $\int g^+ \le \epsilon$. **P** Set $F = \{x : g(x) > 0\}$, and take any $\eta > 0$. Then there are a compact $K \subseteq F$ and an open $G \supseteq F$ such that $\mu_L(G \setminus K) \le \eta$. There is a set E, a finite union of half-open intervals, such that $K \subseteq E \subseteq G$. In this case,

$$\int g^+ - \int_E g \le \int |g \times \chi(E \triangle F)| \le \|g\|_\infty \mu_L(G \setminus K), \quad \int g^+ \le \epsilon + \eta\|g\|_\infty;$$

as η is arbitrary, we have the result. **Q**

(c) We know that there is a codable sequence $\langle g_n \rangle_{n\in\mathbb{N}}$ of simple codable Borel functions such that $f =_{\text{a.e.}} \lim_{n\to\infty} g_n$ and $\sum_{n=0}^\infty \int |g_{n+1} - g_n|$ is finite. Set $\epsilon_n = \sum_{i=n}^\infty \int |g_{i+1} - g_i|$ for each n; then $\int |f - g_n| \le \epsilon_n$, because $\int |g_m - g_n| \le \epsilon_n$ for every $m \ge n$. So if E is a finite union of half-open intervals,

$$\int_E g_n = \int g_n \times \chi E \ge \int f \times \chi E - \int |f - g_n| \ge -\epsilon_n;$$

by (a), applied to $-g_n$, $\int g_n^- \le \epsilon_n$. By 564Be,

$$f^- =_{\text{a.e.}} \lim_{n\to\infty} g_n^- =_{\text{a.e.}} \liminf_{n\to\infty} g_n^- = 0$$

almost everywhere, as required.

565K Theorem A monotonic function $f : \mathbb{R} \to \mathbb{R}$ is differentiable almost everywhere.

Remark Of course 'almost everywhere' here is with respect to Lebesgue measure on $\mathbb{R}$; in this result and the next two I am taking $r = 1$.

proof We can use the ideas in 222A if we refine them using 565F. First, $\mathcal{C}$ will be the set of closed non-trivial intervals with rational endpoints; take Ψ and Θ as in 565F. It will be enough to deal with the case of non-decreasing f. For $a < b$ in $\mathbb{R}$, set $f^*([a, b]) = [f(a), f(b)]$. I shall repeatedly use the fact that if $\mathcal{I} \subseteq \mathcal{C}$ is disjoint, then

$$\mu_L(\bigcup_{C\in\mathcal{I}} f^*(C)) = \sum_{C\in\mathcal{I}} \mu_L f^*(C),$$

because $\mathcal{I}$ is countable and $f^*(C) \cap f^*(C')$ contains at most one point for any distinct C, $C' \in \mathcal{I}$, and we can use 563C(a-iv).

(a) Again set

$$\overline{D}f(x) = \limsup_{h\to 0} \tfrac{1}{h}(f(x + h) - f(x)), \quad \underline{D}f(x) = \liminf_{h\to 0} \tfrac{1}{h}(f(x + h) - f(x))$$

for $x \in \mathbb{R}$. To see that $\overline{D}f < \infty$ a.e., set $E_m = \{x : |x| < m,\ \overline{D}f(x) > 2^m(1 + f(m) - f(-m))\}$ and

$$\begin{aligned}\mathcal{I}_m = \{[\alpha, \beta] : \alpha,\ \beta \in \mathbb{Q},\ -m < \alpha < \beta < m,&\\ f(\beta) - f(\alpha) > 2^m(1 + f(m) - f(-m))(\beta - \alpha)\}&\\ = \{C : C \in \mathcal{C},\ C \subseteq \,]-m, m[\,, \mu_L f^*(C) > 2^m(1 + f(m) - f(-m))\mu_L C\}&\end{aligned}$$

for each m. Then, in the language of 565F, $E_m \subseteq A_{\mathcal{I}_m}$. (If $x \in E_m$ and $\delta > 0$, then x is an endpoint of a non-trivial closed interval $[\alpha,\beta] \subseteq \,]-m,m[$, of length less than δ, such that $f(\beta)-f(\alpha) > 2^m(1+f(m)-f(-m))(\beta-\alpha)$. Now we can expand $[\alpha,\beta]$ slightly to get an interval $[\alpha',\beta'] \in \mathcal{I}_m$ of length at most δ.) So $E_m \subseteq \bigcup\Psi(\mathcal{I}_m) \cup \Theta(\mathcal{I}_m,k)$ for each k. $\Psi(\mathcal{I}_m)$ is a countable family of closed sets, and

$$2^m(1+f(m)-f(-m)) \sum_{C\in\Psi(\mathcal{I}_m)} \mu_L C \le \sum_{C\in\Psi(\mathcal{I}_m)} \mu_L f^*(C)$$
$$= \mu_L\Bigl(\bigcup_{C\in\Psi(\mathcal{I}_m)} f^*(C)\Bigr) \le f(m)-f(-m).$$

So $\sum_{C\in\Psi(\mathcal{I}_m)} \mu_L C \le 2^{-m}$. Setting $H_m = \Theta(\mathcal{I}_m,m) \cup \bigcup\{\operatorname{int} C : C \in \Psi(\mathcal{I}_m)\}$, H_m is open, $\mu H_m \le 2^{-m+1}$ and $E_m \setminus \mathbb{Q} \subseteq H_m$.

Set $E = \{x : \overline{D}f(x) = \infty\}$, and take any $n \in \mathbb{N}$. Then

$$E \setminus \mathbb{Q} \subseteq \textstyle\bigcup_{m\ge n} E_m \setminus \mathbb{Q} \subseteq \bigcup_{m\ge n} H_m.$$

Now 563C(a-ii) tells us that

$$\mu_L(\textstyle\bigcup_{m\ge n} H_m) \le \sum_{m=n}^{\infty} \mu H_m \le 2^{-n+1}$$

for each n, so that $E\setminus\mathbb{Q}$ is included in a negligible G_δ set and $\mu_L E = \mu_L(E\setminus\mathbb{Q}) = 0$. Thus $\overline{D}f$ is finite a.e.

(b) To see that $\overline{D}f \le_{\text{a.e.}} \underline{D}f$, we use similar ideas, but with an extra layer of complexity, corresponding to the double use of Vitali's theorem. Set $F = \{x : \underline{D}f(x) < \overline{D}f(x)\}$. Take any $\epsilon > 0$; because $\mathbb{Q}$ is countable, there is a family $\langle\epsilon_{mqq'}\rangle_{m\in\mathbb{N},q,q'\in\mathbb{Q}}$ of strictly positive numbers such that $\sum_{m\in\mathbb{N},q,q'\in\mathbb{Q}} \epsilon_{mqq'} \le \frac{1}{2}\epsilon$. For q, $q' \in \mathbb{Q}$ and m, $k \in \mathbb{N}$ let $\mathcal{I}_{mqk}$, $\mathcal{J}_{mqk}$ be

$$\{C : C \in \mathcal{C},\, C \subseteq \,]-m,m[,\, \mu_L C \le 2^{-k},\, \mu_L f^*(C) \ge q\mu_L C\},$$

$$\{C : C \in \mathcal{C},\, C \subseteq \,]-m,m[,\, \mu_L C \le 2^{-k},\, \mu_L f^*(C) \le q\mu_L C\}$$

respectively. For m, $k \in \mathbb{N}$ and q, $q' \in \mathbb{Q}$ set

$$G_{mqq'k} = \bigcup\{\operatorname{int} C : C \in \mathcal{I}_{mq'k}\} \cap \bigcup\{\operatorname{int} C : C \in \mathcal{J}_{mqk}\};$$

then $\langle G_{mqq'k}\rangle_{k\in\mathbb{N}}$ is a non-increasing sequence of open sets of finite measure. So, setting $F_{mqq'} = \bigcap_{k\in\mathbb{N}} G_{mqq'k}$, we can find a family $\langle k(m,q,q')\rangle_{m\in\mathbb{N},q,q'\in\mathbb{Q}}$ in $\mathbb{N}$ such that

$$\mu_L(G_{m,q,q',k(m,q,q')} \setminus F_{mqq'}) \le \min(1, \tfrac{q'-q}{q})\epsilon_{mqq'}$$

whenever $m \in \mathbb{N}$, q, $q' \in \mathbb{Q}$ and $0 < q < q'$ (563C(b-ii)). Write $H_{mqq'}$ for $G_{m,q,q',k(m,q,q')}$.

If $m \in \mathbb{N}$ and $0 < q < q'$ in $\mathbb{Q}$, set

$$\mathcal{J}'_{mqq'} = \{C : C \in \mathcal{C},\, C \subseteq H_{mqq'},\, \mu_L f^*(C) \le q\mu_L C\}.$$

Then $F_{mqq'} \subseteq H_{mqq'}$, so every point of $F_{mqq'}$ belongs to the interiors of arbitrarily small intervals belonging to $\mathcal{J}'_{mqq'}$; accordingly $F_{mqq'} \setminus \bigcup\Psi(\mathcal{J}'_{mqq'})$ is negligible.

Now let $\mathcal{I}'_{mqq'}$ be the set

$$\{C : C \in \mathcal{C},\, C \subseteq C' \text{ for some } C' \in \Psi(\mathcal{J}'_{mqq'}),\, \mu_L f^*(C) \ge q'\mu_L C\}.$$

Then every point of $F_{mqq'} \cap \bigcup\Psi(\mathcal{J}'_{mqq'}) \setminus \mathbb{Q}$ belongs to arbitrarily small members of $\mathcal{I}'_{mqq'}$, so $F_{mqq'} \setminus \bigcup\Psi(\mathcal{I}'_{mqq'})$ is negligible.

Now we come to the calculation at the heart of the proof. If $m \in \mathbb{N}$ and $0 < q < q'$ in $\mathbb{Q}$,

$$q'\mu_L F_{mqq'} \le q'\mu_L(\bigcup\Psi(\mathcal{I}'_{mqq'})) = q' \sum_{C\in\Psi(\mathcal{I}'_{mqq'})} \mu_L C$$
$$\le \sum_{C\in\Psi(\mathcal{I}'_{mqq'})} \mu_L f^*(C) = \mu_L\Bigl(\bigcup_{C\in\Psi(\mathcal{I}'_{mqq'})} f^*(C)\Bigr) \le \mu_L\Bigl(\bigcup_{C\in\Psi(\mathcal{J}'_{mqq'})} f^*(C)\Bigr)$$

(because every member of $\mathcal{I}'_{mqq'}$ is included in a member of $\Psi(\mathcal{J}'_{mqq'})$)

$$= \sum_{C\in\Psi(\mathcal{J}'_{mqq'})} \mu_L f^*(C) \le q \sum_{C\in\Psi(\mathcal{J}'_{mqq'})} \mu_L C$$
$$= q\mu_L(\bigcup \Psi(\mathcal{J}'_{mqq'}) \le q\mu_L H_{mqq'} \le q\mu_L F_{mqq'} + (q'-q)\epsilon_{mqq'},$$

and $\mu_L F_{mqq'} \le \epsilon_{mqq'}$, $\mu_L H_{mqq'} \le 2\epsilon_{mqq'}$. But this means that

$$F\setminus\mathbb{Q} \subseteq \bigcup_{m\in\mathbb{N},q,q'\in\mathbb{Q},0<q<q'} F_{mqq'} \subseteq \bigcup_{m\in\mathbb{N},q,q'\in\mathbb{Q},0<q<q'} H_{mqq'},$$

which has measure at most

$$\sum_{m\in\mathbb{N},q,q'\in\mathbb{Q},0<q<q'} \mu_L H_{mqq'} \le 2\sum_{m\in\mathbb{N},q,q'\in\mathbb{Q},0<q<q'} \epsilon_{mqq'} \le \epsilon.$$

The process described here gives a recipe, starting from $\epsilon > 0$, for finding an open set of measure at most ϵ including $F\setminus\mathbb{Q}$. So we can repeat this for each term of a sequence converging to 0 to define a negligible G_δ set including $F\setminus\mathbb{Q}$, and F must be negligible, as required.

565L Lemma Suppose that $F:\mathbb{R}\to\mathbb{R}$ is a bounded non-decreasing function. Then $\int F'$ is defined and is at most $\lim_{x\to\infty} F(x) - \lim_{x\to-\infty} F(x)$.

proof I copy the ideas of 222C. For each $n\in\mathbb{N}$, define $g_n:\mathbb{R}\to\mathbb{R}$ by setting $a_{nk} = 2^{-n+1}k(n+1) - n$ for $k \le 2^n$,

$$g_n(x) = \tfrac{2^{n-1}}{n+1}(F(a_{n,k+1}) - F(a_{nk})) \text{ if } k < 2^n \text{ and } a_{nk} \le x < a_{n,k+1},$$
$$= 0 \text{ if } x < -n \text{ or } x \ge n+2.$$

Then g_n is a simple Borel measurable function and $F'(x) = \lim_{n\to\infty} g_n(x)$ whenever $F'(x)$ is defined, which is almost everywhere, by 565K. Also $\int g_n = F(n+2) - F(-n)$. Because the g_n are resolvable, $\langle g_n\rangle_{n\in\mathbb{N}}$ is codable; by Fatou's Lemma (564Fb),

$$\int F' \le \liminf_{n\to\infty} \int g_n = \lim_{x\to\infty} F(x) - \lim_{x\to-\infty} F(x).$$

565M Theorem Let $F:\mathbb{R}\to\mathbb{R}$ be a function. Then the following are equiveridical:

(i) there is an integrable function f such that $F(x) = \int_{]-\infty,x[} f$ for every $x\in\mathbb{R}$,

(ii) F is of bounded variation, absolutely continuous on every bounded interval, and $\lim_{x\to-\infty} F(x) = 0$,

and in this case $F' =_{\text{a.e.}} f$.

proof (a) If f is integrable and $F(x) = \int_{]-\infty,x[} f$ for every $x\in\mathbb{R}$, take any $\epsilon > 0$. Then there is a $g\in C_k(\mathbb{R})$ such that $\int|f-g| \le \epsilon$ (565Ib). Let x_0 be such that $g(x) = 0$ for $x \le x_0$; then

$$|F(x)| \le \int |f-g| \le \epsilon$$

whenever $x \le x_0$. Set $\delta = \frac{\epsilon}{1+\|g\|_\infty}$. If $a_0 \le b_0 \le \ldots \le a_n \le b_n$ and $\sum_{i=0}^n b_i - a_i \le \delta$, then

$$\sum_{i=0}^n |F(b_i) - F(a_i)| = \sum_{i=0}^n |\int_{[a_i,b_i[} f| \le \sum_{i=0}^n \int_{[a_i,b_i[} |g| + \int_{[a_i,b_i[} |f-g|$$
$$\le \sum_{i=0}^n (b_i - a_i)\|g\|_\infty + \int |f-g| \le \delta\|g\|_\infty + \epsilon \le 2\epsilon.$$

As ϵ is arbitrary, $\lim_{x\to-\infty} F(x) = 0$ and F is absolutely continuous on every bounded interval. As for the variation of F, if $a_0 \le a_1 \le \ldots \le a_n$ then

$$\sum_{i=1}^n |F(a_i) - F(a_{i-1})| = \sum_{i=1}^n |\int_{[a_{i-1},a_i[} f| \le \sum_{i=1}^n \int_{[a_{i-1},a_i[} |f|$$
$$= \int_{[a_0,a_n[} |f| \le \int |f|,$$

so $\mathrm{Var}(F) \leq \int |f|$ is finite.

Thus (i)⇒(ii).

(b) Moreover, under the conditions of (a), $F' =_{\text{a.e.}} f$. **P** Because f is the difference of two non-negative integrable functions, it is enough to consider the case $f \geq 0$ a.e., so that F is non-decreasing. In this case, applying 565L to the function $x \mapsto \mathrm{med}(F(a), F(x), F(b))$, we see that $\int_{[a,b[} F' \leq \int_{[a,b[} f$ whenever $a \leq b$ in $\mathbb{R}$; also, applying 565L to F itself, F' is integrable. Applying 565J to $f - F'$, we see that $F' \leq_{\text{a.e.}} f$.

Recall that there is a countable subset D of $C_k(\mathbb{R})$ approximating all integrable functions in mean (565Ib). So there is a sequence $\langle g_n \rangle_{n\in\mathbb{N}}$ in D such that $\sum_{n=0}^{\infty} \int |g_n - f|$ is finite. Set $\tilde{g}_n = \sup_{i\leq n} g_i^+$ for $n \in \mathbb{N}$; then all the $\tilde{g}_n$ are continuous, therefore resolvable, and $\langle \tilde{g}_n \rangle_{n\in\mathbb{N}}$ is a codable sequence of integrable functions. By 564Fa, $g = \lim_{n\to\infty} \tilde{g}_n$ is defined a.e. and integrable. Let G_n, G be the indefinite integrals of $\tilde{g}_n$, g respectively. Then the arguments just used show that $G' \leq_{\text{a.e.}} g$. But note that each G_n, being the indefinite integral of a continuous function, has $G_n' = \tilde{g}_n$ exactly, while $G_n' \leq G'$ whenever G' is defined. So

$$g =_{\text{a.e.}} \lim_{n\to\infty} \tilde{g}_n = \lim_{n\to\infty} G_n' \leq_{\text{a.e.}} G',$$

and $g =_{\text{a.e.}} G'$.

At this point observe that $\int \liminf_{n\to\infty} |g_n - f| = 0$, by 564Fb, so $f \leq_{\text{a.e.}} g$, while $G - F$ is the indefinite integral of the essentially non-negative integrable function $g - f$. So $G' - F' \leq_{\text{a.e.}} g - f =_{\text{a.e.}} G' - f$ and $f \leq_{\text{a.e.}} F'$. So actually $f =_{\text{a.e.}} F'$, as hoped for. **Q**

(c) Now suppose that $F : \mathbb{R} \to \mathbb{R}$ is of bounded variation and absolutely continuous on every bounded interval, and that $\lim_{x\to-\infty} F(x) = 0$. By 224D and 565L, F' is integrable; set $G(x) = \int_{]-\infty,x[} F'$ and $H(x) = F(x) - G(x)$ for $x \in \mathbb{R}$. By (b), $H' = F' - G'$ is zero a.e., while H, like F and G, is absolutely continuous on every bounded interval. But this means that H is constant. **P** Suppose that $a < b$ in $\mathbb{R}$ and $\epsilon > 0$. Let $\delta \in]0, b-a[$ be such that $\sum_{i=0}^{n} |H(b_i) - H(a_i)| \leq \epsilon$ whenever $a \leq a_0 \leq b_0 \leq a_1 \leq b_1 \leq \ldots \leq a_n \leq b_n \leq b$ and $\sum_{i=0}^{n} b_i - a_i \leq \delta$ Set $E = \{x : x \in]a,b[, H'(x) = 0\}$. Let $\mathcal{C}$ be the family of non-trivial closed subintervals $[c,d]$ of $]a,b[$ with rational endpoints such that $|H(d) - H(c)| \leq \epsilon(d-c)$; then every point of E belongs to arbitrarily small members of $\mathcal{C}$. By Vitali's theorem (565F) there is a disjoint countable family $\mathcal{I} \subseteq \mathcal{C}$ such that $E \setminus \bigcup \mathcal{I}$ is negligible, so that

$$\sum_{I\in\mathcal{I}} \mu_L I = \mu_L(\bigcup \mathcal{I}) = b - a.$$

Let $\mathcal{J} \subseteq \mathcal{I}$ be a finite subset such that $\sum_{I\in\mathcal{J}} \mu_L I \geq b - a - \delta$; express $\mathcal{J}$ as $\langle [b_i, a_{i+1}] \rangle_{i<n}$ where $\langle b_i \rangle_{i<n}$ is strictly increasing. Setting $a_0 = a$ and $b_n = b$, we have $a = a_0 \leq b_0 \leq \ldots \leq a_n \leq b_n = b$ and $\sum_{i=0}^{n} b_i - a_i \leq \delta$. So

$$\begin{aligned} |H(b) - H(a)| &\leq \sum_{i=0}^{n} |H(b_i) - H(a_i)| + \sum_{i=0}^{n-1} |H(a_{i+1}) - H(b_i)| \\ &\leq \epsilon + \epsilon \sum_{i=0}^{n-1} (a_{i+1} - b_i) \leq \epsilon(1 + b - a). \end{aligned}$$

As a, b and ϵ are arbitrary, H is constant. **Q**

So $F - G$ is constant. As both F and G tend to 0 at $-\infty$, they are equal. Thus $F(x) = \int_{]-\infty,x[} F'$ for every x, and F is an indefinite integral.

565N Hausdorff measures Let (X, ρ) be a metric space and $s \in]0, \infty[$. As in §471, we can define **Hausdorff s-dimensional submeasure** $\theta_s : \mathcal{P}X \to [0, \infty]$ by writing

$$\begin{aligned} \theta_s A = \sup_{\delta>0} \inf\{\sum_{n=0}^{\infty} (\operatorname{diam} D_n)^s : &\ \langle D_n \rangle_{n\in\mathbb{N}} \text{ is a sequence of subsets of } X \text{ covering } A, \\ &\operatorname{diam} D_n \leq \delta \text{ for every } n \in \mathbb{N}\}, \end{aligned}$$

counting $\operatorname{diam} \emptyset$ as 0 and $\inf \emptyset$ as ∞. As with Lebesgue submeasure, θ_s is a submeasure.

565O Theorem Let (X, ρ) be a second-countable metric space, and $s > 0$. Then there is a Borel-coded measure μ on X such that $\mu K = \theta_s K$ whenever $K \subseteq X$ is compact and $\theta_s K$ is finite.

proof (a) To begin with, suppose that X is compact and θ_sX is finite.

(i) Let $\mathcal{U}$ be a countable base for the topology of X closed under finite unions; let $\preccurlyeq$ be a well-ordering of $\mathcal{U}$. Then for any compact $K\subseteq X$, $\delta>0$ and $\epsilon>0$, there are $U_0,\dots,U_n\in\mathcal{U}$ such that $K\subseteq\bigcup_{i\le n}U_i$, $\operatorname{diam}U_i\le\delta$ for every i and $\sum_{i=0}^n(\operatorname{diam}U_i)^s\le\theta_sK+\epsilon$. **P** There is a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of subsets of X such that $K\subseteq\bigcup_{n\in\mathbb{N}}A_n$, $\operatorname{diam}A_n\le\frac{1}{2}\delta$ for every n and $\sum_{n=0}^\infty(\operatorname{diam}A_n)^s\le\theta_sK+\frac{1}{2}\epsilon$. Let $\langle\eta_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\left]0,\frac{1}{4}\delta\right[$ such that $\sum_{n=0}^\infty(2\eta_n+\operatorname{diam}A_n)^s\le\theta_sK+\epsilon$.

For each $n\in\mathbb{N}$, set $G_n=\{x:\rho(x,A_n)<\eta_n\}$. Then $\overline{A}_n$ is a compact subset of G_n so there is a $\preccurlyeq$-first $U_n\in\mathcal{U}$ such that $\overline{A}_n\subseteq U_n\subseteq G_n$. Now $\operatorname{diam}U_n\le\min(\delta,\operatorname{diam}A_n+2^{-n-2}\epsilon)$ for each n so $\sum_{n=0}^\infty(\operatorname{diam}U_n)^s\le\theta_sK+\epsilon$. But as K is compact there is an n such that $K\subseteq\bigcup_{i\le n}U_i$. **Q**

(ii) As in the proof of 471Da, θ_s is a 'metric submeasure', that is, $\theta_s(A\cup B)=\theta_sA+\theta_sB$ whenever A, $B\subseteq X$ and $\rho(A,B)>0$. (It will be convenient here to say that $\rho(A,B)=\infty$ if either A or B is empty.) It follows that $\theta_s(\bigcup_{n\in\mathbb{N}}K_n)=\sum_{n=0}^\infty\theta_sK_n$ whenever $\langle K_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence of compact subsets of X. **P** Recall that $\rho(K,K')>0$ whenever K, K' are disjoint compact subsets of X; this is because $K\times K'$ is compact and $\rho:X\times X\to\mathbb{R}$ is continuous. So

$$\theta_s(\textstyle\bigcup_{n\in\mathbb{N}}K_n)\ge\theta_s(\bigcup_{i\le n}K_i)=\sum_{i=0}^n\theta_sK_i$$

for every $n\in\mathbb{N}$, and $\theta_s(\bigcup_{n\in\mathbb{N}}K_n)\ge\sum_{n=0}^\infty\theta_sK_n$. In the other direction, let $\epsilon>0$. Let $\preccurlyeq'$ be a well-ordering of $\bigcup_{n\in\mathbb{N}}\mathcal{U}^n$. Then for each $n\in\mathbb{N}$ there is a $\preccurlyeq'$-first finite sequence $U_{n0},\dots,U_{nm_n}$ in $\mathcal{U}$ such that $K_n\subseteq\bigcup_{i\le m_n}U_{ni}$, $\operatorname{diam}U_{ni}\le\epsilon$ for every i and $\sum_{i=0}^{m_n}(\operatorname{diam}U_{ni})^s\le\theta_sK_n+2^{-n}\epsilon$. Now $\langle U_{ni}\rangle_{n\in\mathbb{N},i\le m_n}$ witnesses that

$$\sum_{n=0}^\infty\theta_sK_n+2\epsilon\ge\inf\{\sum_{j=0}^\infty(\operatorname{diam}D_j)^s:\langle D_j\rangle_{j\in\mathbb{N}}\text{ is a sequence of subsets of }X$$
$$\text{covering }\bigcup_{n\in\mathbb{N}}K_n,\ \operatorname{diam}D_j\le\epsilon\text{ for every }j\in\mathbb{N}\}.$$

As ϵ is arbitrary, $\theta_s(\bigcup_{n\in\mathbb{N}}K_n)\le\sum_{n=0}^\infty\theta_s(K_n)$. **Q**

(iii) If $G\subseteq X$ is open and $\epsilon>0$, there is a compact set $K\subseteq G$ such that $\theta_s(G\setminus K)\le\epsilon$. **P** Set $K_0=\{x:\rho(x,X\setminus G)\ge1\}$ and for $n\ge1$ set

$$K_n=\{x:2^{-n}\le\rho(x,X\setminus G)\le2^{n+1}\}.$$

Then $\sum_{n=0}^\infty\theta_sK_{2n}$ and $\sum_{n=0}^\infty\theta_sK_{2n+1}$ are both bounded by $\theta_sX<\infty$, so there is an $n\in\mathbb{N}$ such that $\sum_{i=n}^\infty\theta_sK_n\le\epsilon$. But this means that $\theta_s(\bigcup_{i\ge n}K_n)\le\epsilon$ (apply (ii) to the odd and even terms separately). Set $K=\bigcup_{i\le n}K_i$; this works. **Q**

(iv) Writing $\mathfrak{T}$ for the topology of X, $\theta_s\restriction\mathfrak{T}$ satisfies the conditions of 563H. **P** Of course it is zero at $\emptyset$, monotonic and locally finite. If G, $H\in\mathfrak{T}$ and $\epsilon>0$, let $K\subseteq G$, $L\subseteq H$ be compact sets such that $\theta_s(G\setminus K)+\theta_s(H\setminus L)\le\epsilon$. Then $K\setminus H$, $K\cap L$ and $L\setminus G$ are disjoint compact sets and

$$(G\cup H)\setminus((K\setminus H)\cup(K\cap L)\cup(L\setminus G)),\quad(G\cap H)\setminus(K\cap L),$$

$$G\setminus((K\setminus H)\cup(K\cap L)),\quad H\setminus((L\setminus G)\cup(K\cap L))$$

are all included in $(G\setminus K)\cup(H\setminus L)$, so all have submeasure at most ϵ. But this means that $\theta_s(G\cup H)+\theta_s(G\cap H)$ and $\theta_sG+\theta_sH$ both differ from $\theta_s(K\setminus H)+2\theta_s(K\cap L)+\theta_s(L\setminus G)$ by at most 2ϵ (upwards) and differ from each other by at most 2ϵ also. As ϵ is arbitrary, we have the modularity condition.

As for the sequential order-continuity, this is elementary; if $\langle G_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with union G, and $\epsilon>0$, there is a compact $K\subseteq G$ such that $\theta_s(G\setminus K)\le\epsilon$; now $K\subseteq G_n$ for some n, and $\theta_sG\le\theta_sG_n+\epsilon$. **Q**

(v) So 563H tells us that there is a Borel-coded measure μ on X extending $\theta_s\restriction\mathfrak{T}$. Now $\mu K=\theta_sK$ for every compact $K\subseteq X$. **P**

$$\mu K+\mu(X\setminus K)=\mu X=\theta_sX\le\theta_sK+\theta_s(X\setminus K)=\theta_sK+\mu(X\setminus K),$$

so $\mu K \le \theta_s K$. On the other hand, given $\epsilon > 0$, there is a compact $L \subseteq X \setminus K$ such that $\theta_s L \ge \theta_s(X \setminus K) - \epsilon$, and now

$$\theta_s K = \theta_s(K \cup L) - \theta_s L \le \theta_s X - \theta_s(X \setminus K) + \epsilon = \mu K + \epsilon;$$

as ϵ is arbitrary, $\mu K = \theta_s K$. **Q**

(vi) Note that 563H tells us that μ is the only Borel-coded measure extending $\theta_s \restriction \mathfrak{T}$, and must therefore be the only Borel-coded measure agreeing with θ_s on the compact sets.

(b) For the general case, let $\mathcal{K}$ be $\{K : K \subseteq X$ is compact, $\theta_s K < \infty\}$. Then (a) tells us that for every $K \in \mathcal{K}$ there is a unique Borel-coded measure μ_K on K agreeing with θ_s on the compact subsets of K. If K, $L \in \mathcal{K}$ and $K \subseteq L$, $\mu_L \restriction \mathcal{B}_c(K)$ is a Borel-coded measure on K (563Fa) agreeing with θ_s on the compact subsets of K, so μ_L extends μ_K. We therefore have a Borel-coded measure μ on X defined by setting $\mu E = \sup_{K \in \mathcal{K}} \mu_K(E \cap K)$ for every $E \in \mathcal{B}_c(X)$ (cf. 563E), and μ agrees with θ_s on $\mathcal{K}$, as required.

565X Basic exercises (a)(i) Show that Lebesgue submeasure θ and Lebesgue measure are translation-invariant. (ii) Show that if $A \subseteq \mathbb{R}^r$ and $\alpha \ge 0$ then $\theta(\alpha A) = \alpha^r \theta A$. (iii) Show that if $E \subseteq \mathbb{R}^r$ is measurable and $\alpha \in \mathbb{R}$ then αE is measurable.

(b) Suppose that there is a sequence $\langle A_n \rangle_{n \in \mathbb{N}}$ of countable subsets of $[0,1]$ with union $[0,1]$. (i) Set $A = \bigcup_{m \le n} A_m + n$. Show that A belongs to the algebra Σ of 565C, that the Lebesgue submeasure of A is ∞, but that $A \cap [0,n]$ is Lebesgue negligible for every n. (ii) Set $B = \{2^{-n}x : n \in \mathbb{N},\ x \in A_n\}$. Show that B has Lebesgue submeasure 0, but is not Lebesgue negligible.

(c) Let $g : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function. For half-open intervals $I \subseteq \mathbb{R}$ define $\lambda_g I$ by setting

$$\lambda_g \emptyset = 0, \quad \lambda_g \,[a,b[\, = \lim_{x \uparrow b} g(x) - \lim_{x \uparrow a} g(x)$$

if $a < b$. For any set $A \subseteq \mathbb{R}$ set

$$\theta_g A = \inf\{\sum_{j=0}^{\infty} \lambda_g I_j : \langle I_j \rangle_{j \in \mathbb{N}} \text{ is a sequence of half-open intervals covering } A.$$

Show that θ_g is a submeasure on $\mathcal{P}\mathbb{R}$. Show that there is a Borel-coded measure μ_g on $\mathbb{R}$ agreeing with θ_g on open sets.

(d) Apply 564N to relate Lebesgue measure on $\mathbb{R}^2$ to Lebesgue measure on $\mathbb{R}$.

(e) Suppose that there is a sequence $\langle A_n \rangle_{n \in \mathbb{N}}$ of countable sets with union $[0,1]$. Show that there is a set $A \subseteq [0,1]^2$, with two-dimensional Lebesgue submeasure zero, such that all the vertical sections $A[\{x\}]$, for $x \in [0,1]$, have non-zero one-dimensional Lebesgue measure.

(f) Confirm that the principal results of §281 can be proved without the axiom of choice.

565Y Further exercises (a) Show that if X is a second-countable space and μ is a codably σ-finite Borel-coded measure on X, then there is a non-decreasing function $g : \mathbb{R} \to \mathbb{R}$ such that the Lebesgue-Stieltjes measure μ_g of 565Xc has measure algebra isomorphic to that of μ.

565 Notes and comments In these five sections I have tried to indicate, without succumbing to the temptation to re-write the whole treatise, a possible version of Lebesgue's theory which can be used in plain ZF. With the Fundamental Theorem of Calculus (565M), the Radon-Nikodým theorem (564L), Fubini's theorem (564N) and at least some infinite product measures (564O), it is clear that most of the ideas of Volume 2 should be expressible in forms not relying on the axiom of choice. We must expect restrictions of the type already found in the convergence theorems (564F); for versions of the Central Limit Theorem or the strong law of large numbers or Komlós's theorem, for instance, we should certainly start by changing any hypothesis 'let $\langle f_n \rangle_{n \in \mathbb{N}}$ be a sequence of random variables' into 'let $\langle f_n \rangle_{n \in \mathbb{N}}$ be a codable sequence of codable Borel functions'. I am not sure how to approach martingales, but the best chance of positive results will be with 'codable martingales' in which we have a full set of Borel codes for countable sets generating each of the σ-algebras involved, along the lines of 564Md. If you glance at the formulae of Chapter 28, you will see that while there are many appeals to the convergence theorems, they are generally applied to sequences of the form $\langle f \times g_n \rangle_{n\in\mathbb{N}}$ where f is integrable and the g_n are continuous; but this means that $\langle g_n \rangle_{n\in\mathbb{N}}$ is necessarily codable (562Qa, 562Sc) so that $\langle f \times g_n \rangle_{n\in\mathbb{N}}$ will be a codable sequence if f itself is a codable function.

In Volumes 3 and 4 we encounter much more solid obstacles, and I see no way in which Maharam's theorem, or the Lifting Theorem, can be made to work without something approaching the full axiom of choice, or a strong hypothesis declaring the existence of a well-orderable set at a crucial point. I give an example of such a hypothesis in the statement of Vitali's theorem (565F). But in the applications of Vitali's theorem later in this section, we can always work with a countable family of balls, for which well-orderability is not an issue. Separability and second-countability hypotheses can be expected to act in similar ways; so that, for instance, we have 565Ya, which is a kind of primitive case of Maharam's theorem.

Version of 22.8.14

566 Countable choice

With AC(ω) measure theory becomes recognisable. The definition of Lebesgue measure used in Volume 1 gives us a true countably additive Radon measure; the most important divergence from the standard theory is the possibility that every subset of $\mathbb{R}$ is Lebesgue measurable (see 567G below). With occasional exceptions (most notably, in the theory of infinite products) we can use the work of Volume 2. In Volume 3, we lose the two best theorems in the abstract theory of measure algebras, Maharam's theorem and the Lifting Theorem; but function spaces and ergodic theory are relatively unaffected. Even in Volume 4, a good proportion of the ideas can be applied in some form.

566A Nearly all mathematicians working on the topics of this treatise spend most of their time thinking in the framework of ZFC. When we move to weaker theories, we have a number of alternative strategies available.

(a) Some of the time, all we have to do is to check that our previous arguments remain valid. In the present context, moving from full ZF + AC to ZF + AC(ω), this is true of most of Volumes 1 and 2 and useful fragments thereafter. In particular, for most of the basic theory of the Lebesgue integral countable choice is adequate. Sometimes, of course, we have to trim our theorems back a bit, as in 566E, 566I, 566M, 566N, 566R and 566Xc.

(b) Some results have to be dropped altogether. For instance, we no longer have a construction of a non-Lebesgue-measurable subset of $\mathbb{R}$, and the Lifting Theorem disappears.

(c) Some results become so much weaker that they change their character entirely. For instance, the Hahn-Banach theorem, Baire's theorem, Stone's theorem and Maharam's theorem survive only in sharply restricted forms (561Xh, 561E, 561F, 566Nb).

(d) Sometimes we find that while proofs rely on the axiom of choice, the results can be proved without it, or with something much weaker. Of course this is often a reason to regard the original proof as inappropriate. Some of the ultrafilters in Volume 4 are there just to save a couple of lines of argument, and renouncing them actually brings ideas into clearer focus. But there are occasions when the less scrupulous approach makes it a good deal easier for us to develop appropriate intuitions. There is an example in the theory of the spaces $S(\mathfrak{A})$ and $L^\infty(\mathfrak{A})$ in Chapter 36. If we think of $S(\mathfrak{A})$ as a quotient of a free linear space (361Ya) and of $L^\infty(\mathfrak{A})$ as the $\| \|_\infty$-completion of $S(\mathfrak{A})$, we can prove all the basic results which come from their identification with spaces of functions on the Stone space of $\mathfrak{A}$; but for most of us such an approach would seriously complicate the process of understanding the nature of the objects being constructed. I used the representation theorems in the theory of free products (§315, §325) for the same reason.

On other occasions, we may need new ideas, as in 566F-566H, 566L and 566P-566Q. A deeper example is in 562V/566O, where I set out alternative routes to the results of 364F and 434T. Here we have quite a lot of extra distance to travel, but at the same time we see some new territory.

(e) More subtly, it may be useful to re-consider some definitions; e.g., the distinction between 'ccc' and 'countable sup property' for Boolean algebras (566Xd). I have made an effort in this book to use definitions which will be appropriate in the absence of the axiom of choice, but in a number of places this would lead to a division of a concept in potentially confusing ways.

The ordinary theory of cardinals depends so essentially on the existence of well-orderings that it is often unclear what we can do without them. However some theorems, which appear to involve the theory of infinite cardinals, can be rescued if we re-interpret the statements. Sometimes the cardinal $\mathfrak{c}$ can be simply replaced by $\mathbb{R}$ or $\mathcal{P}\mathbb{N}$ (343I, 491G). Sometimes a statement '$\#(X) \ge \mathfrak{c}$' can be replaced by 'there is an injection from $\mathcal{P}\mathbb{N}$ into X' or 'there is a surjection from X onto $\mathcal{P}\mathbb{N}$' (344H, 4A2G(j-ii)); similarly, '$\#(X) \le \mathfrak{c}$' might mean 'there is an injection from X into $\mathcal{P}\mathbb{N}$' or 'there is a surjection from $\mathcal{P}\mathbb{N}$ onto $X \cup \{\emptyset\}$' (4A1O, 4A3Fa). Of course '$\#(X) = \mathfrak{c}$' usually becomes 'there is a bijection between X and $\mathcal{P}\mathbb{N}$' (423L); but it might mean 'there are an injection from $\mathcal{P}\mathbb{N}$ into X and a surjection from $\mathcal{P}\mathbb{N}$ onto X' (4A3Fb), or the other way round, or just two surjections.

When dealing with a property which is invariant under equipollence, it may be right to drop the concept of 'cardinal' altogether, and re-phrase a definition in more primitive terms, as in 566Xl.

Elsewhere, as in 2A1Fd and 4A1E, we have results which refer to initial ordinals and hence to well-orderable sets. But the theory of cardinal functions is so bound up with the idea that cardinal numbers form a well-ordered class that much greater adjustments are necessary. I offer the following idea for consideration. For a metric space (X, ρ) and a dense set $D \subseteq X$, set

$$\mathcal{U}(X, \rho, D) = \{\{y : y \in X,\ \rho(x, y) < 2^{-n}\} : x \in D,\ n \in \mathbb{N}\},$$

so that $\mathcal{U}(X, \rho, D)$ is a base for the topology of X. The existence (in ZF) of this function $\mathcal{U}$ corresponds to the ZFC result that '$w(X)$ is at most the cardinal product $\omega \times d(X)$ for every metrizable space X'.

(f) Another way to preserve the ideas of a theorem in the new environment is to make some small variation in its hypotheses. For instance, Urysohn's Lemma, in its usual form, demands DC. So if we are working with AC(ω) alone, we cannot be sure that compact Hausdorff spaces are completely regular; similarly, there may be uniformities not definable from pseudometrics. For a general topologist, this is important. But a measure theorist may be happy to simply add 'completely regular' to the hypotheses of a theorem, as in 561G and 566Xk. In §§412-413 I repeatedly mention families $\mathcal{K}$ which are closed under disjoint finite unions. Results starting from this hypothesis tend to depend on DC; but if we take $\mathcal{K}$ to be closed under $\cup$, AC(ω) may well be enough (566D). A more dramatic change, but one which still leads to interesting results, is in 566I.

566B Volume 1 With countable choice, Lebesgue outer measure becomes an outer measure in the usual sense, so we can use Carathéodory's method to define a measure space in the sense of 112A. No further difficulties arise in the work of Chapters 11 and 12, and we can proceed exactly as before to the convergence theorems. Indeed all the theorems of Volume 1 are available, with a single exceptional feature: the construction of non-measurable sets in 134B and 134D, and a non-measurable function in 134Ib. (I will return to this point in §567.) In particular, the union of countably many countable sets is countable.

566C Volume 2 In Volume 2 also we find that arguments using more than countable choice are the exception rather than the rule. Naturally, they appear oftener in the more abstract topics of Chapter 21. One is in 211L; we can no longer be sure that a strictly localizable space is localizable, though a σ-finite measure space does have to be localizable, since the choice demanded in the proof of 211Ld can then be performed over a countable index set. There is a similar problem in 213J; a strictly localizable space might fail to have locally determined negligible sets, and might have a subset without a measurable envelope. Again, in 214Ia, it is not clear that a subspace of a strictly localizable space must be strictly localizable. In 211P I ask for a non-Borel subset of $\mathbb{R}$, and give an answer involving a non-measurable set; but with AC(ω) we have a non-Borel analytic set as in 423M. (See also 566Xb.)

A more important gap arises in the theory of infinite products of probability spaces. The first problem is that if we have an uncountable family $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ of probability spaces, there is no assurance that $\prod_{i\in I} X_i$ is non-empty. In concrete cases, this is not usually a serious worry. But there is another one. The proof of 254F makes an appeal to DC. I do not think that there can be a construction of a product measure on even a sequence of arbitrary probability spaces which does not use some form of dependent choice. However a partial version, adequate for many purposes (including the essential needs of Chapter 27), can be done with countable choice alone (566I). We can now continue through §254 with the proviso that every infinite family of probability spaces for which we consider a product measure should be a family of perfect probability spaces with non-empty product. There will be a difficulty in 254L, concerning the product of subspaces of full outer measure, where the modification essentially confines it to non-empty products of conegligible sets. For 254N, it will be helpful to know that (under the conditions of 566I) the product of perfect spaces is again perfect. The proof of this fact (451Jc) is scattered through Volume 4, but (given that we have a product probability measure) needs only countably many choices at each step.

When we come to products of probability spaces in Chapter 27, we shall again have to restrict the applications of the results, but at each point only sufficiently to ensure that we have the product probability measures discussed.

566D Exhaustion The versions of the principle of exhaustion in 215A all seem to require DC rather than AC(ω). For many applications, however, we can make do with a weaker result, as follows. I include some corollaries showing that in many familiar cases we can continue to use the intuitions developed in the main text.

Proposition [AC(ω)] (a) Let P be a partially ordered set such that $p \vee q = \sup\{p,q\}$ is defined for all p, $q \in P$, and $f : P \to \mathbb{R}$ an order-preserving function. Then there is a non-decreasing sequence $\langle p_n\rangle_{n\in\mathbb{N}}$ in P such that $\lim_{n\to\infty} f(p_n) = \sup_{p\in P} f(p)$.

(b) Let (X,Σ,μ) be a measure space and $\mathcal{E} \subseteq \Sigma$ a non-empty set such that $\sup_{E\in\mathcal{E}} \mu E$ is finite and $E \cup F \in \mathcal{E}$ for every E, $F \in \mathcal{E}$. Then there is a non-decreasing sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$ such that, setting $F = \bigcup_{n\in\mathbb{N}} F_n$, $\mu F = \sup_{E\in\mathcal{E}} \mu E$ and $E \setminus F$ is negligible for every $E \in \mathcal{E}$.

(c) Let (X,Σ,μ) be a measure space and $\mathcal{K}$ a family of sets such that

(α) $K \cup K' \in \mathcal{K}$ for all K, $K' \in \mathcal{K}$,

(β) whenever $E \in \Sigma$ is non-negligible there is a non-negligible $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq E$.

Then μ is inner regular with respect to $\mathcal{K}$.

(d)(i) Let (X,Σ,μ) be a semi-finite measure space. Then μ is inner regular with respect to the family of sets of finite measure.

(ii) Let (X,Σ,μ) be a perfect measure space. Then whenever $E \in \Sigma$, $f : X \to \mathbb{R}$ is measurable and $\gamma < \mu E$, there is a compact set $K \subseteq f[E]$ such that $\mu f^{-1}[K] \ge \gamma$.

proof (a) For each $n \in \mathbb{N}$, set $\gamma_n = \sup_{p\in P} \min(n, f(p) - 2^{-n})$. Then there is a sequence $\langle q_n\rangle_{n\in\mathbb{N}}$ in P such that $f(q_n) \ge \gamma_n$ for each n; set $p_n = \sup_{i\le n} q_i$ for each n.

(b) By (a) there is a non-decreasing sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$ such that $\sup_{n\in\mathbb{N}} \mu F_n = \sup_{E\in\mathcal{E}} \mu E$; set $F = \bigcup_{n\in\mathbb{N}} F_n$.

(c) Because μ is inner regular with respect to $\mathcal{K}$ iff it is inner regular with respect to $\mathcal{K} \cup \{\emptyset\}$, we may suppose that $\emptyset \in \mathcal{K}$. Take $F \in \Sigma$, and consider $\mathcal{E} = \{K : K \in \mathcal{K} \cap \Sigma,\ K \subseteq F\}$. **?** If $\sup_{E\in\mathcal{E}} \mu E < \mu F$, let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $\mathcal{E}$ such that $\mu(E \setminus \bigcup_{n\in\mathbb{N}} E_n) = 0$ for every $E \in \mathcal{E}$ ((b) above). Set $G = \bigcup_{n\in\mathbb{N}} E_n$; then $\mu G = \sup_{n\in\mathbb{N}} \mu E_n < \mu F$, so $\mu(F \setminus G) > 0$. But now there ought to be a non-negligible $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq F \setminus G$, in which case $K \in \mathcal{E}$ and $\mu(K \setminus G) > 0$. **X**

(d)(i) Apply (c) with $\mathcal{K}$ the family of sets of finite measure.

(ii) Apply (c) to the subspace measure μ_E and $\mathcal{K} = \{f^{-1}[K] : K \subseteq f[E] \text{ is compact}\}$.

566E The problem recurs in parts of 215B, where I list characterizations of σ-finiteness, and in 215C. It seems equally that a ccc semi-finite measure algebra may fail to be σ-finite, though a σ-finite measure algebra has to be ccc. We have a stripped-down version of 215B, with one of its fragments used in §235, as follows:

Proposition [AC(ω)] Let (X,Σ,μ) be a semi-finite measure space. Write $\mathcal{N}$ for the σ-ideal of μ-negligible sets.

(a) The following are equiveridical:

(i) μ is σ-finite;

(ii) either $\mu X = 0$ or there is a probability measure ν on X with the same domain and the same negligible sets as μ;

(iii) there is a measurable integrable function $f : X \to]0,1]$;

(iv) either $\mu X = 0$ or there is a measurable function $f : X \to]0,\infty[$ such that $\int f d\mu = 1$.

(b) If μ is σ-finite, then

(i) every disjoint family in $\Sigma \setminus \mathcal{N}$ is countable;

(ii) for every $\mathcal{E} \subseteq \Sigma$ there is a countable $\mathcal{E}_0 \subseteq \mathcal{E}$ such that $E \setminus \bigcup \mathcal{E}_0$ is negligible for every $E \in \mathcal{E}$.

(c) Suppose that μ is σ-finite, (Y, T, ν) is a semi-finite measure space, and $\phi : X \to Y$ is a (Σ, T)-measurable function such that $\mu\phi^{-1}[F] > 0$ whenever $\nu F > 0$. Then ν is σ-finite.

proof (a) Use the methods of 215B.

(b) By (a-ii), we may suppose that μ is totally finite.

(i) If $\mathcal{E} \subseteq \Sigma \setminus \mathcal{N}$ is disjoint, then $\mathcal{E}_n = \{E : E \in \mathcal{E},\ \mu E \ge 2^{-n}\}$ is finite for every n, so $\mathcal{E} = \bigcup_{n\in\mathbb{N}} \mathcal{E}_n$ is countable.

(ii) Let $\mathcal{H}$ be the set of finite unions of members of $\mathcal{E}$. By 566Db, there is a sequence $\langle H_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{H}$ such that $\mu(H \setminus \bigcup_{n\in\mathbb{N}} H_n) = 0$ for every $H \in \mathcal{H}$. For each $n \in \mathbb{N}$, choose a finite set $\mathcal{H}_n \subseteq \mathcal{E}$ such that $H_n = \bigcup \mathcal{H}_n$; then $\mathcal{E}_0 = \bigcup_{n\in\mathbb{N}} \mathcal{H}_n$ has the required properties.

(c) Again, we may suppose that μ is totally finite. For each $m \in \mathbb{N}$ let $\mathcal{H}_m$ be the set of those $F \in \mathrm{T}$ such that $\nu F < \infty$ and $\mu\phi^{-1}[F] \ge 2^{-m}$. Then any disjoint family in $\mathcal{H}_m$ has at most $\lfloor 2^m \mu X \rfloor$ members, so each $\mathcal{H}_m$ has a maximal disjoint subset; choose a sequence $\langle \mathcal{E}_m \rangle_{m\in\mathbb{N}}$ such that $\mathcal{E}_m$ is a maximal disjoint subset of $\mathcal{H}_m$ for each m. Then $\mathcal{E} = \bigcup_{m\in\mathbb{N}} \mathcal{E}_m$ is a countable family of sets of finite measure in Y. Now $Z = Y \setminus \bigcup \mathcal{E}$ is negligible. **P?** Otherwise, there is a non-negligible set F of finite measure disjoint from $\bigcup \mathcal{E}$; now there is an m such that $F \in \mathcal{H}_m$, so that $\mathcal{E}_m$ was not maximal. **XQ** So $\mathcal{E} \cup \{Z\}$ witnesses that ν is σ-finite.

566F Atomless algebras To make atomless measure spaces and measure algebras recognisable, we need a more penetrating argument than that previously used in 215D.

Lemma [AC(ω)] Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and μ a positive countably additive functional on $\mathfrak{A}$ such that $\mu 1 = 1$. Suppose that whenever $a \in \mathfrak{A}$ and $\mu a > 0$ there is a $b \subseteq a$ such that $0 < \mu b < \mu a$. Then there is a function $f : \mathfrak{A} \times [0,1] \to \mathfrak{A}$ such that $f(a,\alpha) \subseteq a$ and $\bar\mu f(a,\alpha) = \min(\alpha, \bar\mu a)$ for $a \in \mathfrak{A}$ and $\alpha \in [0,1]$, and $\alpha \mapsto f(a,\alpha)$ is non-decreasing for every $a \in \mathfrak{A}$.

proof (a) Just as in part (a) of the proof of 215D, we see by induction on n that for every $b \in \mathfrak{A}$ such that $\mu b > 0$ and every $n \in \mathbb{N}$, there is a $c \subseteq b$ such that $0 < \mu c \le 2^{-n}\mu b$.

(b) If $b \in \mathfrak{A}$ and $\mu b > 0$, there is a $c \subseteq b$ such that $\frac{1}{3}\mu b < \mu c \le \frac{2}{3}\mu b$. **P?** Otherwise, set $\gamma = \sup\{\mu c : c \subseteq b,\ \mu c \le \frac{2}{3}\mu b\}$ and let $\langle c_n \rangle_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{A}$ such that $c_n \subseteq b$ and $\gamma - 2^{-n} \le \mu c_n \le \gamma$ for every n. Set $d_n = \sup_{i\le n} c_i$ for each n, and $d = \sup_{n\in\mathbb{N}} d_n$. Inducing on n, we see that $\mu d_n \le \frac{2}{3}\mu b$ so $\mu d_n \le \frac{1}{3}\mu b$ for each n, and $\mu d \le \frac{1}{3}\mu b$. Now by (a) there is an $e \subseteq b \setminus d$ such that $0 < \mu e \le \frac{1}{3}\mu b$. In this case, $\mu(d \cup e) \le \frac{2}{3}\mu b$, so

$$\gamma \ge \mu(d \cup e) \ge \mu e + \sup_{n\in\mathbb{N}} \mu c_n > \gamma. \quad \textbf{XQ}$$

(c) For each $n \in \mathbb{N}$ there is a finite partition of unity into elements of measure at most $(\frac{2}{3})^n$. **P** Induce on n, using (b) for the inductive step. **Q**

(d) Choose a sequence $\langle C_k \rangle_{k\in\mathbb{N}}$ of finite partitions of unity such that $\mu c \le 2^{-k}$ for every $k \in \mathbb{N}$ and $c \in C_n$. Set $C = \bigcup_{k\in\mathbb{N}} C_k$; then C is countable. Moreover, whenever $a \in \mathfrak{A}$ and $\beta > 0$, there must be a $c \in C$ such that $a \cap c \ne 0$ and $\mu c \le \beta$. Let $\langle c_n \rangle_{n\in\mathbb{N}}$ be a sequence running over C.

(e) Define $\langle f_n \rangle_{n\in\mathbb{N}}$, $\langle g_n \rangle_{n\in\mathbb{N}}$ inductively by saying that, for $a \in \mathfrak{A}$ and $\alpha \in [0,1]$,

$$f_0(a,\alpha) = 0, \quad g_0(a,\alpha) = a$$

$$f_{n+1}(a,\alpha)=f_n(a,\alpha)\cup(c_n\cap g_n(a,\alpha))\text{ if }\mu(f_n(a,\alpha)\cup(c_n\cap g_n(a,\alpha)))\le\alpha,$$
$$=f_n(a,\alpha)\text{ otherwise,}$$
$$g_{n+1}(a,\alpha)=g_n(a,\alpha)\text{ if }\mu(f_n(a,\alpha)\cup(c_n\cap g_n(a,\alpha)))\le\alpha,$$
$$=f_n(a,\alpha)\cup(c_n\cap g_n(a,\alpha))\text{ otherwise.}$$

Then

$$f_n(a,\alpha)\subseteq f_{n+1}(a,\alpha)\subseteq g_{n+1}(a,\alpha)\subseteq g_n(a,\alpha)\subseteq a,$$

$$\mu f_n(a,\alpha)\le\alpha,\quad \mu g_n(a,\alpha)\ge\min(\mu a,\alpha)$$

for every $n\in\mathbb{N}$. Set $f(a,\alpha)=\sup_{n\in\mathbb{N}}f_n(a,\alpha)$. Then $f(a,\alpha)\subseteq a$ and $\mu f(a,\alpha)\le\alpha$ whenever $a\in\mathfrak{A}$ and $\alpha\in[0,1]$.

(f)(i) ? If $a\in\mathfrak{A}$ and $\alpha\in[0,1]$ are such that $\mu f(a,\alpha)<\min(\mu a,\alpha)$, set $b=\inf_{n\in\mathbb{N}}g_n(a,\alpha)$. Then $f(a,\alpha)\subseteq b$ and $\mu b\ge\min(\mu a,\alpha)$. By (d), there is an $n\in\mathbb{N}$ such that

$$c_n\cap b\setminus f(a,\alpha)\ne 0,\quad \mu c_n\le\min(\mu a,\alpha)-\mu f(a,\alpha).$$

In this case, $\mu(f_n(a,\alpha)\cup(c_n\cap g_n(a,\alpha)))\le\min(\mu a,\alpha)$ so

$$f(a,\alpha)\supseteq f_{n+1}(a,\alpha)\supseteq c_n\cap g_n(a,\alpha))\supseteq c_n\cap b,$$

which is impossible. **X**

So $\mu f(a,\alpha)=\min(\mu a,\alpha)$ for all $a\in\mathfrak{A}$ and $\alpha\in[0,1]$.

(ii) If $a\in\mathfrak{A}$ and $0\le\alpha\le\beta\le1$, then for every $n\in\mathbb{N}$
either $f_n(a,\alpha)=f_n(a,\beta)$ and $g_n(a,\alpha)=g_n(a,\beta)$
or $g_n(a,\alpha)\subseteq f_n(a,\beta)$.

(Induce on n.) So $f(a,\alpha)\subseteq f(a,\beta)$.

Thus we have a suitable function f.

566G Vitali's theorem The arguments I presented for Vitali's theorem in 221A/261B and 471N-471O, and for the similar result in 472B, involve the inductive construction of a sequence, which ordinarily is a signal that DC is being used. In 565F I suggested a weaker form of Vitali's theorem which is adequate for its most important applications in measure theory. With AC(ω), however, we can get most of the results as previously stated, if we refine our methods slightly.

(a) In 261B, we have a family $\mathcal{I}$ of closed balls in $\mathbb{R}^r$ and we wish to choose inductively a disjoint sequence $\langle I_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{I}$ such that

$$\operatorname{diam}I_n\ge\frac{1}{2}\sup\{\operatorname{diam}I:I\in\mathcal{I},\ I\cap\bigcup_{i<n}I_i=\emptyset\}$$

for every n. We have already reduced the problem to the case in which $\sup_{I\in\mathcal{I}}\operatorname{diam}I$ is finite and for any finite disjoint subset of $\mathcal{I}$ there is a member of $\mathcal{I}$ disjoint from all of them. Let $\langle G_m\rangle_{m\in\mathbb{N}}$ run over the family of all open balls with centres in $\mathbb{Q}^r$ and rational radii. For $m\in\mathbb{N}$ set $\mathcal{K}_m=\mathcal{I}\cap\mathcal{P}G_m$, and let $\mathcal{I}'\subseteq\mathcal{I}$ be a countable set such that $\sup_{I\in\mathcal{I}'\cap\mathcal{K}_m}\operatorname{diam}I=\sup_{I\in\mathcal{K}_m}\operatorname{diam}I$ for every $m\in\mathbb{N}$ such that $\mathcal{K}_m$ is non-empty; this can be found with countably many choices.

Now, when we come to choose I_n, we can always pick a member of $\mathcal{I}'$. **P** If $\mathcal{I}_n=\{I:I\in\mathcal{I},\ I\cap\bigcup_{i<n}I_i=\emptyset\}$, $\gamma_n=\sup_{I\in\mathcal{I}_n}\operatorname{diam}I$ and $I\in\mathcal{I}_n$ is such that $\operatorname{diam}I>\frac{1}{2}\gamma_n$, there is an $m\in\mathbb{N}$ such that $I\subseteq G_m\subseteq\mathbb{R}^r\setminus\bigcup_{i<n}I_i$, in which case there is an $I'\in\mathcal{I}'\cap\mathcal{K}_m$ such that $\operatorname{diam}I'\ge\frac{1}{2}\gamma_n$, and I' is eligible to be I_n. **Q** Because $\mathcal{I}'$ is well-orderable, we can set out a rule for making these choices, and the argument can proceed as written, without recourse to the devices of §565.

(b) A similar trick can be used in 472B. Here, given a family $\mathcal{I}$ of closed balls, we wish to choose a sequence $\langle B_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{I}$ such that the centre of B_n does not belong to $\bigcup_{i<n}B_i$ and, subject to this, the diameter of B_n is nearly as large as it could be. This time, take $\mathcal{K}_m$ to be the set of members of $\mathcal{I}$ with

centres in G_m, use countably many choices to find a countable set $\mathcal{I}' \subseteq \mathcal{I}$ with adequately large intersections with every $\mathcal{K}_m$, and choose $\langle B_n\rangle_{n\in\mathbb{N}}$ from $\mathcal{I}'$.

At the next step, in 472C, we have to do this repeatedly, but the same method works; in fact, we can work inside a fixed family $\mathcal{I}'$ chosen as above. (See 472Yd.)

(c) The version in 471N-471O is not manageable in quite the same way. If, however, we assume that the metric spaces there are locally compact and separable, we can use the same idea as in (a) above to limit our search to countable subfamilies of the given family $\mathcal{F}$.

566H Bounded additive functionals We come to another obstacle in the proof of 231E. The argument given there relies on DC to show that a countably additive functional is bounded. But we can avoid this, at the cost of an extra manoeuvre, as follows.

Lemma [AC(ω)] Let $\mathfrak{A}$ be a Boolean algebra and $\nu:\mathfrak{A}\to\mathbb{R}$ an additive functional such that $\{\nu a_n : n\in\mathbb{N}\}$ is bounded for every disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$. Then ν is bounded.

proof ? Suppose, if possible, otherwise. Then there is a sequence $\langle b_n\rangle_{n\in\mathbb{N}}$ in Σ such that $|\nu b_n|\ge 2^n n$ for every $n\in\mathbb{N}$. For each $n\in\mathbb{N}$, let $\mathfrak{B}_n$ be the subalgebra of $\mathfrak{A}$ generated by $\{b_i : i<n\}$; then $\mathfrak{B}_n$ has at most 2^n atoms, so there must be an atom a of $\mathfrak{B}_n$ such that $|\nu(a\cap b_n)|\ge n$. Choose a sequence $\langle c_n\rangle_{n\in\mathbb{N}}$ such that c_n is an atom of $\mathfrak{B}_n$ and $|\nu d_n|\ge n$ for every n, where $d_n=c_n\cap b_n$; note that d_n is an atom of $\mathfrak{B}_{n+1}$, so that if $n<m$ then either $d_m\subseteq d_n$ or $d_m\cap d_n=0$. By Ramsey's theorem (4A1G), there is an infinite $I\subseteq\mathbb{N}$ such that

either $\langle d_n\rangle_{n\in I}$ is disjoint

or $d_m\subseteq d_n$ whenever m, $n\in I$ and $n<m$.

Now the first alternative is certainly impossible, because $\{\nu d_n : n\in I\}$ is unbounded. So we have the second. But in this case we can define a strictly increasing sequence $\langle n_k\rangle_{k\in\mathbb{N}}$ in I such that $n_{k+1}\ge k+|\nu d_{n_k}|$ for each k. Set $a_k=d_{n_k}\setminus d_{n_{k+1}}$ for each k; then $\langle a_k\rangle_{k\in\mathbb{N}}$ is disjoint and $|\nu a_k|\ge k$ for each k, so again we have a contradiction. **X**

566I Infinite products: Theorem [AC(ω)] Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of perfect probability spaces such that $X=\prod_{i\in I}X_i$ is non-empty. Then there is a complete probability measure λ on X such that

(i) if $E_i\in\Sigma_i$ for every $i\in I$, and $\{i : E_i\ne X_i\}$ is countable, then $\lambda(\prod_{i\in I}E_i)$ is defined and equal to $\prod_{i\in I}\mu_i E_i$;

(ii) λ is inner regular with respect to $\widehat{\bigotimes}_{i\in I}\Sigma_i$.

proof The only point at which the construction in 254A-254F needs re-examination is in the proof that the standard outer measure on X gives it outer measure 1.

(a) I recall the definitions. For a cylinder $C=\prod_{i\in I}C_i$, set $\theta_0 C=\prod_{i\in I}\mu_i C_i$; for $A\subseteq X$, set

$$\theta A=\inf\{\sum_{n=0}^\infty\theta_0 C_n : C_n\in\mathcal{C}\text{ for every }n\in\mathbb{N},\ A\subseteq\bigcup_{n\in\mathbb{N}}C_n\};$$

λ will be the measure defined from θ by Carathéodory's method.

? Suppose, if possible, that $\theta X<1$. Then we have a sequence $\langle C_n\rangle_{n\in\mathbb{N}}$ of cylinder sets, covering X, with $\sum_{n=0}^\infty\theta C_n=1-2\epsilon$ where $\epsilon>0$. Express each C_n as $\prod_{i\in I}E_{ni}$ where $J_n=\{i : E_{ni}\ne X_i\}$ is finite; let J be the countable set $\bigcup_{n\in\mathbb{N}}J_n$; take $K=\#(J)$ (identifying $\mathbb{N}$ with ω), and a bijection $k\mapsto i_k : K\to J$.

For each $k\in K$ and $n\in\mathbb{N}$, set $L_k=\{i_j : j<k\}\subseteq J$ and $\alpha_{nk}=\prod_{i\in I\setminus L_k}\mu_i E_{ni}$. If J is finite, $L_{\#(J)}=J$ and $\alpha_{n,\#(J)}=1$ for every n. We have $\alpha_{n0}=\theta_0 C_n$ for each n, so $\sum_{n=0}^\infty\alpha_{n0}=1-2\epsilon$. For $n\in\mathbb{N}$, $k\in K$ and $t\in X_{i_k}$ set

$$\begin{aligned}f_{nk}(t)&=\alpha_{n,k+1}\text{ if }t\in E_{n,i_k},\\&=0\text{ otherwise.}\end{aligned}$$

Then

$$\int f_{nk}d\mu_{i_k}=\alpha_{n,k+1}\mu_{i_k}E_{ni_k}=\alpha_{nk}.$$

(b) For each $k \in K$, let $h_k : X_{i_k} \to \mathbb{R}$ be the Marczewski functional defined by setting

$$h_k(t) = \sum_{n=0}^{\infty} 3^{-n} \chi E_{ni_k}(t)$$

for $t \in X_k$. Because μ_k is perfect, there is for each $k \in K$ a compact set $Q \subseteq h_k[X_{i_k}]$ such that $\mu_{i_k} h_k^{-1}[Q] \ge 1 - 2^{-k}\epsilon$. Choose $\langle Q_k \rangle_{k\in K}$ such that $Q_k \subseteq h_k[X_{i_k}]$ is compact and $\mu_{i_k} h_k^{-1}[Q_k] \ge 1 - 2^{-k}\epsilon$ for every $k \in K$. Observe that if $k \in K$ and $n \in \mathbb{N}$ then $f_{nk} = \alpha_{nk}\chi E_{ni_k}$ is of the form $\alpha_{nk} g_n h_k$ where $g_n : \mathbb{R} \to [0, 1]$ is continuous.

(c) Define non-empty sets $F_k \subseteq H_k \subseteq X_{i_k}$ inductively, for $k \in K$, as follows. The inductive hypothesis will be that $\sum_{n\in M_k} \alpha_{nk} \le 1 - 2^{-k+1}\epsilon$, where $M_k = \{n : n \in \mathbb{N},\ F_j \subseteq E_{ni_j}$ whenever $j < k\}$; of course $M_0 = \mathbb{N}$, so the induction starts. Given that $k \in K$ and that

$$1 - 2^{-k+1}\epsilon \ge \sum_{n\in M_k} \alpha_{nk} = \sum_{n\in M_k} \int f_{nk} d\mu_{i_k} = \int (\sum_{n\in M_k} f_{nk}) d\mu_{i_k},$$

the set

$$H_k = \{t : t \in X_{i_k},\ \sum_{n\in M_k} f_{nk}(t) \le 1 - 2^{-k}\epsilon\}$$

must have measure greater than $2^{-k}\epsilon$ and meets $h_k^{-1}[Q_k]$. But observe that $\sum_{n\in M_k} f_{nk} = g'_k h_k$ where $g'_k = \sum_{n\in M_k} \alpha_{nk} g_n$ is lower semi-continuous, so that $H_k = h_k^{-1}[G_k]$ where $G_k = \{\alpha : g'_k(\alpha) \le 1 - 2^{-k}\epsilon\}$ is closed. Since H_k meets $h_k^{-1}[Q_k]$, $Q_k \cap G_k$ is non-empty and has a least member β_k; set $F_k = h_k^{-1}[\{\beta_k\}]$. Because $Q_k \subseteq h_k[X_{i_k}]$, F_k is non-empty.

Examine

$$M_{k+1} = \{n : n \in M_k,\ F_k \subseteq E_{ni_k}\}.$$

There certainly is some $t^* \in F_k$, and because $h_k{\restriction}F_k$ is constant, $M_{k+1} = \{n : n \in M_k,\ t^* \in E_{ni_k}\}$. In this case

$$\sum_{n\in M_{k+1}} \alpha_{n,k+1} = \sum_{n\in M_k} f_{nk}(t^*) \le 1 - 2^{-k}\epsilon$$

and the induction proceeds.

(d) At the end of the induction, either finite or infinite, choose $t_k \in F_k$ for $k \in K$. We are supposing that X has a member x^*; define $x \in X$ by setting $x(i_k) = t_k$ for $k \in K$ and $x(i) = x^*(i)$ for $i \in I \setminus J$. Then there is supposed to be an $m \in \mathbb{N}$ such that $x \in C_m$, so that $m \in M_k$ for every k. But at some stage we shall have $J_m \subseteq L_k$ (allowing $k = \#(K)$ if K is finite) and $\alpha_{mk} = 1$, which is impossible. **X**

566J In particular, 566I applies to all products $\{0,1\}^I$ and $[0,1]^I$ with their usual measures. For these we have Kakutani's theorem that the usual measures are topological measures (415E), which turns out to be valid with countable choice alone.

Theorem [AC(ω)] (a) Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i) \rangle_{i\in I}$ be a family of metrizable Radon probability spaces such that every μ_i is strictly positive and $X = \prod_{i\in I} X_i$ is non-empty. Then the product measure on X is a quasi-Radon measure.

(b) If I is well-orderable then the product measure on $\{0,1\}^I$ is a completion regular Radon measure.

proof (a)(i) We had better check immediately that every X_i is separable. The point is that because μ_i is a totally finite measure inner regular with respect to the compact sets, there is a conegligible K_σ set; because μ_i is strictly positive, this is dense; and countable choice is enough to ensure that a compact metrizable space is second-countable, therefore separable. It follows that $\prod_{i\in J} X_i$ is separable, therefore second-countable, for every countable $J \subseteq I$.

(ii) Because every μ_i is a Radon measure it is perfect, so we have a product probability measure on X. Now we can repeat the argument of 416Ua.

(b) Put (a), 561D and G together.

566K Volume 3 Turning to the concerns of Volume 3, the elementary theory of measure algebras is not radically changed. But Lemma 311D is hopelessly lost; we no longer have Stone spaces, and need to re-examine any proof which appears to rely on them. Another result which changes is 313K; order-dense sets, as defined in 313J, need no longer give rise to partitions of unity. So a localizable measure algebra does not need to be isomorphic to a simple product of totally finite measure algebras. Similarly, condition (ii) of 316H is no longer sufficient to prove weak (σ,∞)-distributivity. However some of the constructions which I described in terms of Stone spaces, in particular, the Loomis-Sikorski theorem, the Dedekind completion of a Boolean algebra, the localization of a semi-finite measure algebra, free products and measure-algebra free products, can be done by other methods which remain effective with AC(ω) at most; see 566L, 561Yg, 323Xh and 325Yc.

The theory of ccc algebras is rather different (566M, 566Xd). Maharam's theorem (331I, 332B) is surely unprovable without something like the full axiom of choice; and the Lifting Theorem (341K) is equally inaccessible under the rules of this section. We do however have useful special cases of results in Chapters 33 and 34 (566N).

A good start can be made on the elementary theory of Riesz spaces without any form of the axiom of choice (see 561H), and with AC(ω) we can go a long way, as in 566Q. What is missing is the Hahn-Banach theorem (for non-separable spaces) and many representation theorems. Similarly, the function spaces of Chapter 36 are recognisable, provided that (for general Boolean algebras $\mathfrak{A}$) we think of $S(\mathfrak{A})$ as a quotient space of the free linear space generated by $\mathfrak{A}$, and of $L^\infty(\mathfrak{A})$ as the $\|\,\|_\infty$-completion of $S(\mathfrak{A})$. Of course we have to take care at every point to avoid the use of Stone spaces. One place at which this involves us in a new argument is in 566O. Most of the arguments of Chapter 24 remain valid, so the basic theory of L^p spaces in §§365-366 survives. What is perhaps surprising is that if we take the trouble we can still reach the most important results on weak compactness (566P, 566Q).

In the ergodic theory of Chapter 38, a good proportion of the classical results survive. There are difficulties with some of the extensions of the classical theory in §§381-382. For instance, the definition of 'full subgroup' of the group of automorphisms of a Boolean algebra in 381Be assumes that order-dense sets include partitions of unity. If not, this definition may fail to be equivalent to the formulation in 381Ia. The latter would seem to be the more natural one to use. However, the definition as given seems to work for the principal needs of Chapter 38 (see 381I).

Frolík's theorem in the generality 382D-382E needs something approaching AC, and with AC(ω) alone there seems no hope of getting results for general Dedekind complete algebras along the lines of the main theorems of §382. For measurable algebras, however, we do have a version of 382Eb (566R).

Many of the later results of Chapter 38 are equally robust, at least in their leading applications to measure algebras. We have to remember that we do not know that measurable algebras have many involutions, and even among those which do there is no assurance that 382Q will be true. So in §§383-384 we find ourselves restricted rather further, to those measurable algebras in which every non-zero element is the support of an involution; but these include the standard examples (566N).

566L The Loomis-Sikorski theorem [AC(ω)] (a) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Then there are a set X, a σ-algebra Σ of subsets of X and a σ-ideal $\mathcal{I}$ of Σ such that $\mathfrak{A}\cong\Sigma/\mathcal{I}$.

(b) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. Then it is isomorphic to the measure algebra of a measure space.

proof (a)(i) Set $X=\{0,1\}^{\mathfrak{A}}$, and $\Sigma=\widehat{\bigotimes}_{\mathfrak{A}}\mathcal{P}(\{0,1\})$. For $a\in\mathfrak{A}$ set $\widehat{a}=\{x:x\in X,\,x(a)=1\}\in\Sigma$. Let $\mathcal{I}$ be the σ-ideal of Σ generated by sets of the form

$$\widehat{a\mathbin{\triangle}b}\triangle\widehat{a}\triangle\widehat{b},\qquad(\inf_{n\in\mathbb{N}}a_n)\widehat{\ }\triangle\bigcap_{n\in\mathbb{N}}\widehat{a_n}$$

for a, $b\in\mathfrak{A}$ and sequences $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$, together with the set $\{x:x(1)=0\}$.

(ii) (The key.) $\widehat{a}\notin\mathcal{I}$ for any $a\in\mathfrak{A}\setminus\{0\}$. **P** If $E\in\mathcal{I}$ then (using AC(ω)) we can find sequences $\langle a_n\rangle_{n\in\mathbb{N}}$, $\langle b_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$, together with a double sequence $\langle c_{ni}\rangle_{n,i\in\mathbb{N}}$, such that, setting $c_n=\inf_{i\in\mathbb{N}}c_{ni}$ for each n,

$$F=\{x:x(1)=0\}\cup\bigcup_{n\in\mathbb{N}}\bigl(\widehat{a_n\mathbin{\triangle}b_n}\triangle\widehat{a_n}\triangle\widehat{b_n}\bigr)\cup\bigcup_{n\in\mathbb{N}}\bigl(\widehat{c_n}\triangle\bigcap_{i\in\mathbb{N}}\widehat{c_{ni}}\bigr)$$

includes E. Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}$ generated by

$$\{a\}\cup\{a_n:n\in\mathbb{N}\}\cup\{b_n:n\in\mathbb{N}\}\cup\{c_n:n\in\mathbb{N}\}\cup\{c_{ni}:n,i\in\mathbb{N}\}.$$

Then $\mathfrak{B}$ is countable, so we can choose inductively a sequence $\langle d_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{B}\setminus\{0\}$ such that $d_0=a$ and, for each $n\in\mathbb{N}$,

$d_{n+1} \subseteq d_n$,
either $d_{n+1} \subseteq a_n$ or $d_{n+1} \cap a_n = 0$,
either $d_{n+1} \subseteq b_n$ or $d_{n+1} \cap b_n = 0$,
either $d_{n+1} \subseteq c_n$ or there is an $i \in \mathbb{N}$ such that $d_{n+1} \cap c_{ni} = 0$.

Define $x \in X$ by saying that

$$x(d) = 1 \text{ if } d \supseteq d_n \text{ for some } n \in \mathbb{N},$$
$$= 0 \text{ otherwise.}$$

Then $x \in \widehat{a} \setminus F$ and $\widehat{a} \not\subseteq E$; as E is arbitrary, $\widehat{a} \notin \mathcal{I}$. **Q**

(iii) Set

$$\Sigma_0 = \{E : E \in \Sigma,\ E \triangle \widehat{a} \in \mathcal{I} \text{ for some } a \in \mathfrak{A}\}.$$

Then Σ_0 is closed under symmetric difference and countable intersections and contains X (because $X \triangle \widehat{1} \in \mathcal{I}$). So Σ_0 is a σ-algebra of sets; as it contains $\widehat{a}$ for every $a \in \mathfrak{A}$, it is equal to Σ.

(iv) From (ii) we see that $\widehat{a \triangle b}$, and therefore $\widehat{a} \triangle \widehat{b}$, do not belong to $\mathcal{I}$ for any distinct a, $b \in \mathfrak{A}$. With (iii), this tells us that we have a function $\pi : \Sigma \to \mathfrak{A}$ defined by setting $\pi E = a$ whenever $E \triangle \widehat{a} \in \mathcal{I}$. Now $\pi X = 1$ and π preserves symmetric difference and countable infima, so is a sequentially order-continuous Boolean homomorphism; its kernel is $\mathcal{I}$, so $\mathfrak{A} \cong \Sigma/\mathcal{I}$, as required.

(b) This is now easy; we can use the familiar argument of 321J.

566M Measure algebras: Proposition [AC(ω)] (a) Let $\mathfrak{A}$ be a measurable algebra.
(i) For any $A \subseteq \mathfrak{A}$ there is a countable $B \subseteq A$ with the same upper bounds as A.
(ii) $\mathfrak{A}$ is Dedekind complete.
(iii) If $D \subseteq \mathfrak{A}$ is order-dense and $c \in D$ whenever $c \subseteq d \in D$, there is a partition of unity included in D.
(b) Let $(\mathfrak{A}, \bar{\mu})$ be a σ-finite measure algebra and $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$ such that $(\mathfrak{B}, \bar{\mu}{\restriction}\mathfrak{B})$ is a semi-finite measure algebra. Then $(\mathfrak{B}, \bar{\mu}{\restriction}\mathfrak{B})$ is a σ-finite measure algebra.

proof (a) (Cf. 322G, 316E, 322Cc.) Let $\bar{\mu}$ be such that $(\mathfrak{A}, \bar{\mu})$ is a totally finite measure algebra.

(i) Let A^* be the set of suprema of finite subsets of A, and set $\gamma = \sup_{a \in A^*} \bar{\mu} a$. There is a sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ in A^* such that $\sup_{n \in \mathbb{N}} \bar{\nu} a_n = \gamma$; let $B \subseteq A$ be a countable set such that every a_n is the supremum of a finite subset of B. Then any upper bound c of B is an upper bound of A. **P** Take $d \in A$. Then $a \cup a_n \in A^*$, so

$$\bar{\mu}(a \setminus c) \le \bar{\mu}(a \setminus a_n) = \bar{\mu}(a \cup a_n) - \bar{\mu} a_n \le \gamma - \bar{\mu} a_n$$

for every n, and $\bar{\mu}(a \setminus c) = 0$, that is, $a \subseteq c$. **Q**

(ii) follows at once from (i), since $\mathfrak{A}$ is Dedekind σ-complete.

(iii) By (i), there is a sequence $\langle d_n \rangle_{n \in \mathbb{N}}$ in D with supremum 1; now $\langle d_n \setminus \sup_{i<n} d_i \rangle_{n \in \mathbb{N}}$ is a partition of unity included in D.

(b) (Cf. 322Nc.) Write $\mathfrak{B}^f$ for the ring $\{b : b \in \mathfrak{B},\ \bar{\mu} b < \infty\}$. Let $\langle a_n \rangle_{n \in \mathbb{N}}$ be a non-decreasing sequence in $\mathfrak{A}$, with supremum 1, such that $\bar{\mu} a_n < \infty$ for every n. For each $n \in \mathbb{N}$, set $\alpha_n = \sup\{\bar{\mu}(b \cap a_n) : b \in \mathfrak{B}^f\}$; choose a sequence $\langle b_n \rangle_{n \in \mathbb{N}}$ in $\mathfrak{B}^f$ such that $\bar{\mu}(b_n \cap a_n) \ge \alpha_n - 2^{-n}$ for every n. **?** If 1 is not the supremum of $\{b_n : n \in \mathbb{N}\}$ in $\mathfrak{B}$, let $b \in \mathfrak{B} \setminus \{0\}$ be such that $b \cap b_n = 0$ for every n. Because $\bar{\mu}{\restriction}\mathfrak{B}$ is semi-finite, there is a non-zero $b' \in \mathfrak{B}^f$ included in b. But now $0 < \bar{\mu} b' = \sup_{n \in \mathbb{N}} \bar{\mu}(b' \cap a_n)$, so there is an $n \in \mathbb{N}$ such that $\bar{\mu}(b' \cap a_n) > 2^{-n}$; in which case $b' \cup b_n \in \mathfrak{B}^f$ and $\bar{\mu}((b' \cup b_n) \cap a_n) > \alpha_n$, which is impossible. **X**

566N Characterizing the usual measure on $\{0,1\}^{\mathbb{N}}$: Theorem [AC(ω)] (a) Let (X, Σ, μ) be an atomless, perfect, complete, countably separated probability space. Then it is isomorphic to $\{0,1\}^{\mathbb{N}}$ with its usual measure.
(b) Let $(\mathfrak{A}, \bar{\mu})$ be an atomless probability algebra of countable Maharam type. Then it is isomorphic to the measure algebra of the usual measure on $\{0,1\}^{\mathbb{N}}$.

(c) An atomless measurable algebra of countable Maharam type is homogeneous.

(d) For any infinite set I, the measure algebra of the usual measure on $\{0,1\}^I$ is homogeneous.

proof (a) (Cf. 344I.) Write ν for the usual measure on $Y = \{0,1\}^{\mathbb{N}}$, and T for its domain.

(i) Let $\mathcal{H}$ be a countable subset of Σ separating the points of X, and $\langle E_n\rangle_{n\in\mathbb{N}}$ a sequence running over $\mathcal{H}$ with cofinal repetitions. Let $f : \Sigma \times [0,1] \to \Sigma$ and be a function as in 566F. Define $g : \Sigma \times \mathbb{N} \to \mathfrak{A}$ by setting

$$g(E,n) = f(E \cap E_n, \frac{1}{2}\bar\mu E) \text{ if } \bar\mu(E \cap E_n) \ge \frac{1}{2}\bar\mu E,$$

$$= f(E \setminus E_n, \frac{1}{2}\bar\mu E) \text{ otherwise }.$$

Define $\langle \mathcal{E}_n\rangle_{n\in\mathbb{N}}$ inductively by saying that $\mathcal{E}_0 = \{X\}$ and

$$\mathcal{E}_{n+1} = \{g(E,n) : E \in \mathcal{E}_n\} \cup \{E \setminus g(E,n) : E \in \mathcal{E}_n\}$$

for each n. Then each $\mathcal{E}_n$ is a partition of unity consisting of 2^n elements of measure 2^{-n}. Set $G_n = \bigcup_{E\in\mathcal{E}_n} g(E,n)$ for each n, and let Σ_0 be the σ-subalgebra of Σ generated by $\{G_n : n \in \mathbb{N}\}$.

For $H \in \Sigma$ and $n \in \mathbb{N}$, set

$$\gamma_n(H) = 2^{-n}\#(\{E : E \in \mathcal{E}_n,\ E \cap H \ne \emptyset \text{ and } E \not\subseteq H\}.$$

Then $\gamma_{n+1}(H) \le \gamma_n(H)$ for every n, and $\gamma_{n+1}(E_n) \le \frac{1}{2}\gamma_n(E_n)$. Since every member of $\mathcal{H}$ appears infinitely often as an E_n, $\lim_{n\to\infty}\gamma_n(H) = 0$ for every $H \in \mathcal{H}$. But this means that if $H \in \mathcal{H}$ and we set $H' = \bigcup\{E : E \in \bigcup_{n\in\mathbb{N}}\mathcal{E}_n,\ E \subseteq H\}$ and $H'' = X \setminus \bigcup\{E : E \in \bigcup_{n\in\mathbb{N}}\mathcal{E}_n,\ H \cap E = \emptyset\}$, then H' and H'' both belong to Σ_0, $H' \subseteq H \subseteq H''$ and $H'' \setminus H'$ is negligible.

(ii) Define $\phi_0 : X \to Y$ by setting $\phi_0(x) = \langle\chi G_n(x)\rangle_{n\in\mathbb{N}}$ for $x \in Y$. Then ϕ_0 is Σ_0-measurable. Consider the image measure $\mu\phi_0^{-1}$. This is a topological measure, and because μ is perfect and complete (and Y is homeomorphic to a subset of $\mathbb{R}$) $\mu\phi_0^{-1}$ is a Radon measure. If $n \in \mathbb{N}$ and $z \in \{0,1\}^n$ then $\phi_0^{-1}\{y : z \subseteq y \in Y\}$ belongs to $\mathcal{E}_n$ and has measure 2^{-n}, so $\mu\phi_0^{-1}$ and ν agree on such sets; both being Radon measures, they must be equal.

(iii) Now observe that $\Sigma_0 = \{\phi_0^{-1}[F] : F \subseteq Y \text{ is Borel}\}$. We have seen that if $H \in \mathcal{H}$ there are H', $H'' \in \Sigma_0$ such that $H' \subseteq H \subseteq H''$ and $H'' \setminus H'$ is negligible. Set $X_1 = X \setminus \bigcup_{H\in\mathcal{H}} H \setminus H'$, so that $X_1 \subseteq X$ is μ-conegligible. Now $\phi_0\restriction X_1$ is injective. **P** If x, x' are distinct members of X_0, there is an $H \in \mathcal{H}$ containing one and not the other; as neither belongs to $H \setminus H'$, H' contains one and not the other; as $H' = \phi_0^{-1}[F]$ for some $F \subseteq Y$, $\phi_0(x) \ne \phi_0(x')$. **Q** We also find that $\phi_0[X_1]$ is ν-conegligible. **P** Because $\nu = \mu\phi_0^{-1}$, $\phi_0[X]$ is ν-conegligible. For each $H \in \mathcal{H}$,

$$\nu\phi_0[H \setminus H'] = \mu\phi_0^{-1}[\phi_0[H \setminus H']] \le \mu\phi_0^{-1}[\phi_0[H'' \setminus H']] = \mu(H'' \setminus H')$$

(because $H'' \setminus H' \in \Sigma_0$ so is the inverse image of a subset of Y)

$$= 0.$$

So $\phi_0[\bigcup_{H\in\mathcal{H}} H \setminus H']$ is ν-negligible and $\phi_0[X_1]$ is ν-conegligible. **Q**

(iv) It follows that if we set $\phi_1 = \phi_0\restriction X_1$ then the subspace measure ν_{Y_1} is just the image measure $\mu_{X_1}\phi_1^{-1}$. **P** If $F \subseteq Y_1$ then

$$\mu_{X_1}\phi_1^{-1}[F] = \mu(X_1 \cap \phi_0^{-1}[F]) = \mu\phi_0^{-1}[F] = \nu F = \nu_{Y_1}F$$

if any of these is defined. **Q** But as ϕ_1 is a bijection, this means that it is an isomorphism between (X_1, μ_{X_1}) and (Y_1, ν_{Y_1}).

(v) There is no reason to suppose that $X \setminus X_1$ and $Y \setminus Y_1$ are equipollent, so ϕ_1 may not be directly extendable to an isomorphism between X and Y. However, there is a negligible subset D of Y_1 which is equipollent with $\mathbb{R}$. **P** Let $K \subseteq Y_1$ be a non-negligible compact set. Set $S_2 = \bigcup_{n\in\mathbb{N}}\{0,1\}^n$ and define $\langle K_z\rangle_{z\in S_2}$ inductively, as follows. $K_\emptyset = K$. Given that $z \in \{0,1\}^n$ and that K_z is a non-negligible compact set, take the first $m \geq 2n+2$ such that $J = \{w : w \in \{0,1\}^m,\ \nu\{y : w \subseteq y \in K_z\} > 0\}$ has more than one member, let w, w' be the lexicographically two first members of J, and set

$$K_{z^\frown 0} = \{y : w \subseteq y \in K_z\}, \quad K_{z^\frown 1} = \{y : w' \subseteq y \in K_z\};$$

continue. This will ensure that $0 < \nu K_z \leq 4^{-n}$ for every $z \in \{0,1\}^n$. Set $D = \bigcap_{n\in\mathbb{N}} \bigcup_{z\in\{0,1\}^n} K_z$; then D is negligible and equipollent with $\{0,1\}^{\mathbb{N}}$ and $\mathbb{R}$. **Q**

Now set $X_2 = X_1 \setminus \phi_1^{-1}[D]$ and $Y_2 = Y_1 \setminus D$. $\phi_2 = \phi_1 \restriction X_2$ is an isomorphism between the conegligible sets X_2 and Y_2 with their subspace measures. Since $\mathcal{H}$ separates the points of X, we surely have an injective function from $X \setminus X_2$ to $\mathbb{R}$, while we also have an injective function from $\mathbb{R}$ to $\phi_1^{-1}[D] \subseteq X \setminus X_2$. So $X \setminus X_2$ is equipollent with $\mathbb{R}$. Similarly, $Y \setminus Y_2$ is equipollent with $\mathbb{R}$. So $\phi_2 : X_2 \to Y_2$ can be extended to a bijection $\phi : X \to Y$, which will be the required isomorphism between (X, Σ, μ) and (Y, T, ν).

(b) (Cf. 331I.) We can use the same idea as in (a). Let $\langle a_n \rangle_{n\in\mathbb{N}}$ be a sequence running over a τ-generating set $A \subseteq \mathfrak{A}$ with cofinal repetitions. Let $f : \mathfrak{A} \times [0,1] \to \mathfrak{A}$ be a function as in 566F. Define $g : \mathfrak{A} \times \mathbb{N} \to \mathfrak{A}$ by setting

$$g(a,n) = f(a \cap a_n, \frac{1}{2}\bar{\mu}a) \text{ if } \bar{\mu}(a \cap a_n) \geq \frac{1}{2}\bar{\mu}a,$$

$$= f(a \setminus a_n, \frac{1}{2}\bar{\mu}a) \text{ otherwise }.$$

Define $\langle B_n \rangle_{n\in\mathbb{N}}$ inductively by saying that $B_0 = \{1\}$ and

$$B_{n+1} = \{g(b,n) : b \in B_n\} \cup \{b \setminus g(b,n) : b \in B_n\}$$

for each n. Then each B_n is a partition of unity consisting of 2^n elements of measure 2^{-n}. Let $\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{n\in\mathbb{N}} B_n$; then $\mathfrak{B}$ is isomorphic to the measure algebra of the usual measure on $\{0,1\}^{\mathbb{N}}$.

For $a \in \mathfrak{A}$ and $n \in \mathbb{N}$, set

$$\gamma_n(a) = 2^{-n}\#(\{b : b \in B_n,\ a \cap b \notin \{0,b\}\}).$$

Then $\gamma_{n+1}(a) \leq \gamma_n(a)$ for every n, and $\gamma_{n+1}(a_n) \leq \frac{1}{2}\gamma_n(a_n)$. Since every member of A appears infinitely often as an a_n, $\lim_{n\to\infty} \gamma_n(a) = 0$ for every $a \in A$. But this means that $A \subseteq \mathfrak{B}$ and $\mathfrak{B} = \mathfrak{A}$. So we have the required isomorphism.

(c) (Cf. 331N.) If $\mathfrak{A}$ is such an algebra, any non-zero principal ideal of $\mathfrak{A}$ is atomless and of countable Maharam type and supports a probability measure, so must be isomorphic to the measure algebra of the usual measure on $\{0,1\}^{\mathbb{N}}$ and to $\mathfrak{A}$.

(d) For $J \subseteq I$, write ν_J for the usual measure on $\{0,1\}^J$, T_J for its domain and $(\mathfrak{B}_J, \bar{\nu}_J)$ for its measure algebra. If $a \in \mathfrak{B}_I$ is non-zero, then it is of the form $E^\bullet$ for some $E \in \mathrm{T}_I$ determined by coordinates in a countable subset J of I. Identifying $\{0,1\}^I$ with $\{0,1\}^J \times \{0,1\}^{I\setminus J}$, we have an $F \in \mathrm{T}_J$ such that $E = F \times \{0,1\}^{I\setminus J}$. Let $b \in \mathfrak{B}_J$ be the equivalence class of F. Now we can think of the probability algebra free product $\mathfrak{B}_J \widehat{\otimes} \mathfrak{B}_{I\setminus J}$ as the metric completion of the algebraic free product $\mathfrak{B}_J \otimes \mathfrak{B}_{I\setminus J}$, and as such isomorphic to $\mathfrak{B}_I$ under an isomorphism which identifies the principal ideal $(\mathfrak{B}_I)_a$ with $(\mathfrak{B}_J)_b \widehat{\otimes} \mathfrak{B}_{I\setminus J}$. By (b), $((\mathfrak{B}_J)_b, \bar{\nu}_J \restriction (\mathfrak{B}_J)_b)$ is isomorphic, up to a scalar multiple of the measure, to $(\mathfrak{B}_J, \bar{\nu}_J)$; so we have

$$(\mathfrak{B}_I)_a \cong (\mathfrak{B}_J)_b \widehat{\otimes} \mathfrak{B}_{I\setminus J} \cong \mathfrak{B}_J \widehat{\otimes} \mathfrak{B}_{I\setminus J} \cong \mathfrak{B}_I.$$

As a is arbitrary, $\mathfrak{B}_I$ is homogeneous.

566O Boolean values: Proposition [AC(ω)] (a) Let $\mathfrak{B}$ be the algebra of open-and-closed subsets of $\{0,1\}^{\mathbb{N}}$, and $\mathcal{B}(\{0,1\}^{\mathbb{N}})$ the Borel σ-algebra. If $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra and $\pi : \mathfrak{B} \to \mathfrak{A}$ is a Boolean homomorphism, π has a unique extension to a sequentially order-continuous Boolean homomorphism from $\mathcal{B}(\{0,1\}^{\mathbb{N}})$ to $\mathfrak{A}$.

(b) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Then there is a bijection between $L^0 = L^0(\mathfrak{A})$ and the set Φ of sequentially order-continuous Boolean homomorphisms from the algebra $\mathcal{B}(\mathbb{R})$ of Borel subsets of $\mathbb{R}$ to $\mathfrak{A}$, defined by saying that $u \in L^0$ corresponds to $\phi \in \Phi$ iff $[\![u > \alpha]\!] = \phi(]\alpha, \infty[)$ for every $\alpha \in \mathbb{R}$.

(c) Let $(\mathfrak{A},\bar\mu)$ be a localizable measure algebra. Write Σ_{um} for the algebra of universally measurable subsets of $\mathbb{R}$. Then for any $u\in L^0=L^0(\mathfrak{A})$, we have a sequentially order-continuous Boolean homomorphism $E\mapsto[\![u\in E]\!]:\Sigma_{\mathrm{um}}\to\mathfrak{A}$ such that

$$\begin{aligned}[\![u\in E]\!]&=\sup\{[\![u\in F]\!]:F\subseteq E\text{ is Borel}\}=\sup\{[\![u\in K]\!]:K\subseteq E\text{ is compact}\}\\&=\inf\{[\![u\in F]\!]:F\supseteq E\text{ is Borel}\}=\inf\{[\![u\in G]\!]:G\supseteq E\text{ is open}\}\end{aligned}$$

for every $E\in\Sigma_{\mathrm{um}}$, while

$$[\![u\in\,]\alpha,\infty[\,]\!]=[\![u>\alpha]\!]$$

for every $\alpha\in\mathbb{R}$.

proof (a) As in §562, let $\mathcal{T}$ be the set of trees without infinite branches in $S^*=\bigcup_{n\ge1}\mathbb{N}^n$. For $n\in\mathbb{N}$ set $E_n=\{x:x\in\{0,1\}^{\mathbb{N}},\,x(n)=1\}\in\mathfrak{B}$ and $a_n=\pi E_n\in\mathfrak{A}$. Let $\phi:\mathcal{T}\to\mathfrak{A}$ and $\psi:\mathcal{T}\to\mathcal{B}(\{0,1\}^{\mathbb{N}})$ be the corresponding interpretations of Borel codes, as in 562V. Then $\phi(T)=\phi(T')$ whenever $\psi(T)=\psi(T')$ (562V), and (using AC(ω)) it is easy to check that $\psi[\mathcal{T}]=\mathcal{B}(\{0,1\}^{\mathbb{N}})$ (cf. 562Db), so we have a function $\tilde\pi:\mathcal{B}(\{0,1\}^{\mathbb{N}})\to\mathfrak{A}$ defined by saying that $\tilde\pi(\psi(T))=\phi(T)$ for every $T\in\mathcal{T}$. Now if $\langle F_n\rangle_{n\in\mathbb{N}}$ is any sequence of Borel subsets of $\{0,1\}^{\mathbb{N}}$, we have a $T\in\mathcal{T}$ such that $F_n=\psi(T_{<n>})$ for every n and no $T_{<n>}$ is empty (see 562Bb). In this case

$$\begin{aligned}\tilde\pi(\bigcup_{n\in\mathbb{N}}\{0,1\}^{\mathbb{N}}\setminus F_n)&=\tilde\pi(\psi(T))=\phi(T)\\&=\sup_{n\in\mathbb{N}}1\setminus\phi(T_{<n>})=\sup_{n\in\mathbb{N}}1\setminus\tilde\pi F_n.\end{aligned}$$

So $\tilde\pi$ is a sequentially order-continuous Boolean homomorphism. Since it agrees with π on $\{E_n:n\in\mathbb{N}\}$ it must agree with π on $\mathfrak{B}$.

Of course the extension is unique because if $\tilde\pi':\mathcal{B}(\{0,1\}^{\mathbb{N}}\to\mathfrak{A}$ is any sequentially order-continuous Boolean homomorphism extending π then $\{E:\tilde\pi'E=\tilde\pi E\}$ is a σ-algebra of sets including $\mathfrak{B}$ and therefore containing every open set.

(b) (Cf. 364F.) Let $\mathcal{E}$ be the algebra of subsets of $\mathbb{R}$ generated by sets of the form $]q,\infty[$ for $q\in\mathbb{Q}$. Then $\mathcal{E}$ is an atomless countable Boolean algebra, so is isomorphic to the algebra $\mathfrak{B}$; let $\theta:\mathfrak{B}\to\mathcal{E}$ be an isomorphism. Define $f:\{0,1\}^{\mathbb{N}}\to[-\infty,\infty]$ by setting $f(x)=\sup\{q:q\in\mathbb{Q},\,x\in\theta^{-1}]q,\infty[\}$. Then f is Borel measurable.

Take any u in L^0. It is easy to check that we have a Boolean homomorphism $\pi:\mathcal{E}\to\mathfrak{A}$ defined by saying that $\pi\,]q,\infty[\,=[\![u>q]\!]$ for every $q\in\mathbb{Q}$. By (a), there is a sequentially order-continuous Boolean homomorphism $\tilde\pi:\mathcal{B}(\{0,1\}^{\mathbb{N}})\to\mathfrak{A}$ extending $\pi\theta:\mathfrak{B}\to\mathfrak{A}$. Set $\phi E=\tilde\pi(f^{-1}[E])$ for $E\in\mathcal{B}(\mathbb{R})$.

If $\alpha\in\mathbb{R}$ then

$$\begin{aligned}\phi(]\alpha,\infty[)&=\tilde\pi\{x:f(x)>\alpha\}=\tilde\pi(\bigcup\{\theta^{-1}]q,\infty[:q\in\mathbb{Q},\,q>\alpha\})\\&=\sup\{\tilde\pi(\theta^{-1}]q,\infty[):q\in\mathbb{Q},\,q>\alpha\}\\&=\sup\{\pi\,]q,\infty[:q\in\mathbb{Q},\,q>\alpha\}\\&=\sup\{[\![u>q]\!]:q\in\mathbb{Q},\,q>\alpha\}=[\![u>\alpha]\!].\end{aligned}$$

It follows that

$$\begin{aligned}\phi\mathbb{R}&=\sup_{n\in\mathbb{N}}\tilde\pi(f^{-1}]-n,\infty[)\setminus\inf_{n\in\mathbb{N}}\tilde\pi(f^{-1}]n,\infty[)\\&=\sup_{n\in\mathbb{N}}[\![u>-n]\!]\setminus\inf_{n\in\mathbb{N}}[\![u>n]\!]=1,\end{aligned}$$

and therefore that $\phi\in\Phi$. For the rest of the argument we can follow the method of 364F.

(c) (Cf. 434T.)

(i) To begin with, consider the case in which $\bar\mu$ is totally finite. In this case, we have a non-decreasing function $g:\mathbb{R}\to[0,\infty[$ defined by saying that $g(\alpha)=\bar\mu1-\bar\mu[\![u>\alpha]\!])$ for $\alpha\in\mathbb{R}$. Let ν_g be the corresponding

Lebesgue-Stieltjes measure (114Xa), and $(\mathfrak{C}, \bar{\nu}_g)$ its measure algebra. Note that g is continuous on the right, so that $\nu_g\,]\alpha, \beta] = \bar{\mu}[\![u > \alpha]\!] - \bar{\mu}[\![u > \beta]\!]$ whenever $\alpha \le \beta$ in $\mathbb{R}$. Let $\mathfrak{D}$ be the subalgebra of $\mathfrak{C}$ generated by $\{]-\infty, \alpha]^\bullet : \alpha \in \mathbb{R}\}$. Then we have a measure-preserving Boolean homomorphism $\pi : \mathfrak{D} \to \mathfrak{A}$ defined uniquely by saying that $\pi(]\alpha, \infty[^\bullet) = [\![u > \alpha]\!]$ for $\alpha \in \mathbb{R}$. Because $\mathfrak{D}$ is dense in $\mathfrak{C}$ for the measure-algebra topology, π has a unique extension to a measure-preserving Boolean homomorphism $\tilde{\pi} : \mathfrak{C} \to \mathfrak{A}$.

Because $\Sigma_{\mathrm{um}} \subseteq \operatorname{dom} \nu_g$, we can define $[\![u \in E]\!]$ to be $\tilde{\pi} E^\bullet$ for $E \in \Sigma_{\mathrm{um}}$, and this will give us a sequentially order-continuous Boolean homomorphism from Σ_{um} to $\mathfrak{A}$ such that $[\![u \in\,]\alpha, \infty[\,]\!] = [\![u > \alpha]\!]$ for every α. As for the other formulae, they are immediate from the facts that ν_g is inner regular with respect to the compact sets and outer regular with respect to the open sets.

(ii) We need to observe that these properties uniquely define $[\![u \in E]\!]$. **P** Let $\mathcal{E}$ be the algebra of subsets of $\mathbb{R}$ generated by $\{]\alpha, \infty[: \alpha \in \mathbb{R}\}$. The requirement $[\![u \in\,]\alpha, \infty[\,]\!] = [\![u > \alpha]\!]$ determines the values of $[\![u \in E]\!]$ for $E \in \mathcal{E}$. Next, if $G \subseteq \mathbb{R}$ is open and $K \subseteq G$ is compact there is an $E \in \mathcal{E}$ such that $K \subseteq E \subseteq G$. Consequently $[\![u \in K]\!] = \inf\{[\![u \in E]\!] : E \in \mathcal{E},\, E \supseteq K\}$ is fixed for every compact $K \subseteq \mathbb{R}$. Finally, the inner regularity condition $[\![u \in E]\!] = \sup\{[\![u \in K]\!] : K \subseteq E$ is compact$\}$ determines $[\![u \in E]\!]$ for other $E \in \Sigma_{\mathrm{um}}$. **Q**

(iii) Now turn to the general case of a localizable measure algebra $(\mathfrak{A}, \bar{\mu})$ and $u \in L^0(\mathfrak{A})$. Let $\mathfrak{A}^f$ be the ideal of elements of finite measure. Then for each $a \in \mathfrak{A}^f$ we have a corresponding homomorphism $E \mapsto [\![u \in E]\!]_a$ from Σ_{um} to the principal ideal $\mathfrak{A}_a$. If $a \subseteq b \in \mathfrak{A}^f$, we can use the uniqueness described in (ii) to see that $[\![u \in E]\!]_a = a \cap [\![u \in E]\!]_b$ for every E. So if we set $[\![u \in E]\!] = \sup_{a \in \mathfrak{A}^f} [\![u \in E]\!]_a$, we shall have $[\![u \in E]\!]_a = a \cap [\![u \in E]\!]$ whenever $a \in \mathfrak{A}^f$ and $E \in \Sigma_{\mathrm{um}}$. It is now easy to check that $E \mapsto [\![u \in E]\!]$ has the required properties.

566P Weak compactness In the absence of Tychonoff's theorem, the theory of weak compactness in normed spaces becomes uncertain. However AC(ω) is enough to give a couple of the principal results involving classical Banach spaces, starting with Hilbert space.

Theorem [AC(ω)] Let U be a Hilbert space. Then bounded sets in U are relatively weakly compact.

proof If U is finite-dimensional, this is trivial; so let us suppose that U is infinite-dimensional. Let $A \subseteq U$ be a bounded set, and $\overline{A}$ its closure for the weak topology; let $\mathcal{F}_0$ be a family of weakly closed subsets of $\overline{A}$ with the finite intersection property, and $\mathcal{F}$ the filter on U generated by $\mathcal{F}_0$.

(a) For closed subspaces V of U, let $P_V : U \to V$ be the orthogonal projection from U onto V (561Ib), and set $\gamma_V = \liminf_{u \to \mathcal{F}} \|P_V u\|^2$. Because $\mathcal{F}$ contains a bounded set, $\gamma_V \le \gamma_U < \infty$ for every V. If V_0, V_1 are orthogonal subspaces of U, then $\|P_{V_0+V_1} u\|^2 = \|P_{V_0} u\|^2 + \|P_{V_1} u\|^2$ for every $u \in U$, so $\gamma_{V_0+V_1} \ge \gamma_{V_0} + \gamma_{V_1}$.

(b) Set $\gamma = \sup\{\gamma_V : V$ is a finite-dimensional linear subspace of $U\}$, and choose a sequence $\langle V_n \rangle_{n\in\mathbb{N}}$ of finite-dimensional subspaces of U such that $\gamma = \sup_{n\in\mathbb{N}} \gamma_{V_n}$; because U is infinite-dimensional, we can do this in such a way that $\dim V_n \ge n$ for each n. Let W be the closed linear span of $\bigcup_{n\in\mathbb{N}} V_n$. If V is a finite-dimensional linear subspace of $W^\perp$, then

$$\gamma \ge \gamma_{V+V_n} \ge \gamma_V + \gamma_{V_n}$$

for every n, so $\gamma_V = 0$.

(c) If $F \in \mathcal{F}$, $V \subseteq W^\perp$ is a finite-dimensional linear subspace, and $\epsilon > 0$, then $F \cap \{u : \|P_V u\| \le \epsilon\}$ is non-empty. We can therefore extend $\mathcal{F}$ to the filter $\mathcal{G}$ generated by sets of this type, and $\lim_{u\to\mathcal{G}} (u|w) = 0$ for every $w \in W^\perp$.

(d) Let $\langle e_n \rangle_{n\in\mathbb{N}}$ be an orthonormal basis for W. Define $\langle \mathcal{G}_n \rangle_{n\in\mathbb{N}}$ as follows. $\mathcal{G}_0 = \mathcal{G}$. Given that $\mathcal{G}_n$ is a filter on U containing a bounded set, set $\alpha_n = \liminf_{u\to\mathcal{G}_n} (u|e_n)$, and let $\mathcal{G}_{n+1}$ be the filter generated by $\mathcal{G}_n \cup \{\{u : (u|e_n) < \alpha\} : \alpha > \alpha_n\}$; then $\alpha_n = \lim_{u\to\mathcal{G}_{n+1}} (u|e_n)$. Set $\mathcal{H} = \bigcup_{n\in\mathbb{N}} \mathcal{G}_n$; then $\alpha_n = \lim_{u\to\mathcal{H}} (u|e_n)$ for each n.

For any $n \in \mathbb{N}$,

$$\sum_{i=0}^n \alpha_i^2 = \sum_{i=0}^n \lim_{u\to\mathcal{H}} (u|e_i)^2 = \lim_{u\to\mathcal{H}} \sum_{i=0}^n (u|e_i)^2 \le \limsup_{u\to\mathcal{H}} \|u\|^2 < \infty$$

because $\mathcal{H}$ contains a bounded set. So $\sum_{n=0}^\infty \alpha_n^2$ is finite and $v = \sum_{n=0}^\infty \alpha_n e_n$ is defined in U.

(e) Now

$$(v|e_n) = \alpha_n = \lim_{u\to\mathcal{H}}(u|e_n)$$

for every n; again because $\mathcal{H}$ contains a bounded set, $(v|w) = \lim_{u\to\mathcal{H}}(u|w)$ for every $w \in W$. On the other hand, if $w \in W^\perp$,

$$\lim_{u\to\mathcal{H}}(u|w) = \lim_{u\to\mathcal{G}}(u|w) = 0 = (v|w).$$

Since $W + W^\perp = U$, $\lim_{u\to\mathcal{H}}(u|w) = (v|w)$ for every $w \in U$. By 561Ic, v is the limit of $\mathcal{H}$ for the weak topology on U, and must belong to every member of $\mathcal{F}_0$.

As $\mathcal{F}_0$ is arbitrary, $\overline{A}$ is weakly compact and A is relatively weakly compact.

566Q Theorem [AC(ω)] Let U be an L-space. Then a subset of U is weakly relatively compact iff it is uniformly integrable.

proof (a)(i)(α) Recall that U is a Banach lattice with an order-continuous norm (354N), so is Dedekind complete (354Ee) and all its bands are projection bands (353J[7]); for a band V in U, let $P_V : U \to V$ be the band projection onto V.

(β) If $u \in U$ there is an $f \in U^*$ such that $\|f\| \le 1$ and $f(u) = \|u\|$. **P** Let V be the band generated by u^+ and $W = V^\perp$ its band complement. Set $f(v) = \int P_V v - \int P_W v$ for $v \in U$. Since $\|v\| = \|P_V v\| + \|P_W v\|$ for every $v \in U$, $\|f\| \le 1$. Also $P_V u = u^+$ and $P_W u = -u^-$ so $f(u) = \int |u| = \|u\|$. **Q**

(γ) If $A \subseteq U$ is weakly bounded it is norm-bounded. **P?** Otherwise, choose for each $n \in \mathbb{N}$ a $u_n \in A$ and $f_n \in U^*$ such that $\|u_n\| \ge n$, $\|f_n\| = 1$ and $f_n(u_n) = \|u_n\| \ge n$. For $f \in U^*$ set $\rho_A(f) = \sup_{u\in A}|f(u)|$. Define $\langle n_k\rangle_{k\in\mathbb{N}}$ by setting $n_k = \lceil 2\cdot 3^k(k + \sum_{i=0}^{k-1} 3^{k-i}\rho_A(f_{n_i}))\rceil$ for each k. Set $f = \sum_{i=0}^\infty 3^{-i}f_{n_i}$. Then for any $k \in \mathbb{N}$ we have

$$\begin{aligned}\rho_A(f) \ge f(u_{n_k}) &= \sum_{i=0}^\infty 3^{-i}f_{n_i}(u_{n_k})\\ &\ge 3^{-k}f_{n_k}(u_{n_k}) - \sum_{i=0}^{k-1} 3^{-i}\rho_A(f_{n_i}) - \sum_{i=k+1}^\infty 3^{-i}\|u_{n_k}\|\\ &= \frac{1}{2\cdot 3^k}\|u_{n_k}\| - \sum_{i=0}^{k-1} 3^{-i}\rho_A(f_{n_i}) \ge k. \text{ \textbf{XQ}}\end{aligned}$$

(ii) Now let $K \subseteq U$ be a weakly relatively countably compact set. Let $\mathfrak{A}$ be the band algebra of U. For $V \in \mathfrak{A}$ set $\nu V = \sup_{u\in K}\|P_V u\|$ (counting $\sup\emptyset$ as 0). Then ν is a submeasure on $\mathfrak{A}$. By (i-γ), K is norm-bounded and ν is finite-valued; set $\alpha = \nu U = \sup_{u\in K}\|u\|$.

ν is exhaustive. **P?** Otherwise, let $\langle V_n\rangle_{n\in\mathbb{N}}$ be a disjoint sequence in $\mathfrak{A}$ such that $\epsilon = \frac{1}{6}\inf_{n\in\mathbb{N}}\nu V_n$ is greater than 0. For each $n \in \mathbb{N}$ choose $u_n \in K$ and $f_n \in U^*$ such that $\|f_n\| \le 1$ and $f_n(P_n u_n) = \|P_n u_n\| \ge 5\epsilon$, where here I write P_n for P_{V_n}. Let v_0 be a cluster point of $\langle u_n\rangle_{n\in\mathbb{N}}$ in U for the weak topology of U. Note that $\sum_{n=0}^\infty \|P_n u\| \le \|u\|$ for any $u \in U$; let $m \in \mathbb{N}$ be such that $\sum_{n=m}^\infty \|P_n v_0\| \le \epsilon$. For $n \in \mathbb{N}$, set $g_n(u) = f_n(P_n u)$ for $u \in U$.

We can now build a strictly increasing sequence $\langle n_k\rangle_{k\in\mathbb{N}}$ such that

$$n_0 \ge m,$$

$$\sum_{i=0}^{k-1} |g_{n_i}(u_{n_k})| \le \epsilon + \sum_{i=0}^{k-1} |g_{n_i}(v_0)|,$$

$$\|P_{n_k}u_{n_i}\| \le 2^{-k}\epsilon \text{ whenever } i < k$$

for every $k \in \mathbb{N}$. Let v_1 be a weak cluster point of $\langle u_{n_k}\rangle_{k\in\mathbb{N}}$, and $l \in \mathbb{N}$ such that $\sum_{k=l}^\infty \|P_{n_k}v_1\| \le \epsilon$. Set $g = \sum_{k=l}^\infty g_{n_k}$; this is defined in U^* because

$$\sum_{k=l}^\infty |g_{n_k}(u)| \le \sum_{k=l}^\infty \|P_{n_k}u\| \le \|u\|$$

[7] Formerly 353I.

for every $u\in U$. Of course $|g(v_1)|\le\epsilon$. On the other hand, for any $k\ge l$,

$$\begin{aligned}g(u_{n_k})&=g_{n_k}(u_{n_k})+\sum_{i=l}^{k-1}g_{n_i}(u_{n_k})+\sum_{i=k+1}^{\infty}g_{n_i}(u_{n_k})\\&\ge 5\epsilon-\sum_{i=0}^{k-1}|g_{n_i}(u_{n_k})|-\sum_{i=k+1}^{\infty}\|P_{n_i}u_{n_k}\|\\&\ge 5\epsilon-\sum_{i=0}^{k-1}|g_{n_i}(v_0)|-\epsilon-\sum_{i=k+1}^{\infty}2^{-i}\epsilon\\&\ge 4\epsilon-\sum_{n=m}^{\infty}\|P_nv_0\|-2^{-k}\epsilon\ge 2\epsilon\end{aligned}$$

and v_1 cannot be a weak cluster point of $\langle u_{n_k}\rangle_{k\in\mathbb{N}}$. **XQ**

(iii) In fact ν is uniformly exhaustive. **P?** Otherwise, let $\epsilon>0$ be such that there are arbitrarily long disjoint strings in $\mathfrak{A}$ of elements of submeasure greater than 2ϵ. Set $q(n)=\lceil\frac{2^nn\alpha}{\epsilon}\rceil$ for each n, and choose a family $\langle V_{ni}\rangle_{n\in\mathbb{N},i\le q(n)}$ such that $\langle V_{ni}\rangle_{i\le q(n)}$ is a disjoint family in $\mathfrak{A}$ for each n and $\nu V_{ni}>2\epsilon$ for all n and i; adapting the temporary notation of (ii), I set $P_{ni}=P_{V_{ni}}$ for $i\le q(n)$. Now choose $u_{ni}\in K$ such that $\|P_{ni}u_{ni}\|\ge 2\epsilon$ for all i and n. Because $\sum_{i=0}^{q(n)}\|P_{ni}u\|\le\|u\|$ for every $u\in U$ and $n\in\mathbb{N}$, we can define inductively a sequence $\langle i_n\rangle_{n\in\mathbb{N}}$ such that $i_n\le q(n)$ and $\|P_{ni_n}u_{mi_m}\|\le 2^{-n}\epsilon$ whenever $m<n$.

Now set

$$W_{mn}=V_{mi_m}\cap\bigcap_{m<k\le n}V_{ki_k}^{\perp},\quad Q_{mn}=P_{W_{mn}}$$

for $m\le n$,

$$W_m=\bigcap_{n\ge m}W_{mn},\quad Q_m=P_{W_m}$$

for $m\in\mathbb{N}$. For any $u\in U$ and $m\le n$,

$$|P_{mi_m}u|=P_{mi_m}|u|\le Q_{mn}|u|+\sum_{k=m+1}^{n}P_{ki_k}|u|,$$

$$\|P_{mi_m}u\|\le\|Q_{mn}u\|+\sum_{k=m+1}^{n}\|P_{ki_k}u\|,$$

so

$$\begin{aligned}\|Q_{mn}u_{mi_m}\|&\ge\|P_{mi_m}u_{mi_m}\|-\sum_{k=m+1}^{n}\|P_{ki_k}u_{mi_m}\|\\&\ge 2\epsilon-\sum_{k=m+1}^{\infty}2^{-k}\epsilon\ge\epsilon.\end{aligned}$$

Next, if $u\ge 0$, $\langle Q_{mn}u\rangle_{n\ge m}$ is a non-increasing sequence, and its infimum belongs to $\bigcap_{n\ge m}W_{mn}$, so must be equal to Q_mu; accordingly Q_mu is the norm-limit of $\langle Q_{mn}u\rangle_{n\ge m}$. The same is therefore true for every $u\in U$, and in particular

$$\|Q_mu_{mi_m}\|=\lim_{n\to\infty}\|Q_{mn}u_{mi_m}\|\ge\epsilon.$$

Consequently $\nu W_m\ge\epsilon$. But $W_m\cap W_n\subseteq V_{ni_n}^{\perp}\cap V_{ni_n}=\{0\}$ whenever $n>m$, so this contradicts (ii). **XQ**

(iv) Now take any $\epsilon>0$. Then there is a $u^*\in U^+$ such that $\int(|u|-u^*)^+\le\epsilon$ for every $u\in K$. **P** If $\alpha\le\epsilon$ we can take $u^*=0$ and stop. Otherwise, there is a largest $n\in\mathbb{N}$ such that there are disjoint $V_0,\dots,V_n\in\mathfrak{A}$ such that $\nu V_i>\epsilon$ for every n. Take $u_0,\dots,u_n\in K$ such that $\|P_{V_i}u_i\|>\epsilon$ for each i. Let $\gamma>0$ be such that $\|P_{V_i}u_i\|-\frac{\alpha}{\gamma}>\epsilon$ for every $i\le n$, and set $u^*=\gamma\sum_{i=0}^{n}|u_i|$. **?** Suppose that $u\in K$ is such that $\int(|u|-u^*)^+>\epsilon$. Let W be the band generated by $(|u|-u^*)^+$, so that $\nu W\ge\|P_Wu\|>\epsilon$. For each $i\le n$, set $W_i=V_i\cap W^{\perp}$; then

$$|P_W u_i| \le \frac{1}{\gamma} P_W u^* \le \frac{1}{\gamma}|u|, \quad |P_{W_i} u_i| \ge |P_{V_i} u_i| - \frac{1}{\gamma}|u|,$$

$$\nu W_i \ge \|P_{W_i} u_i\| \ge \|P_{V_i} u_i\| - \frac{\alpha}{\gamma} > \epsilon.$$

But now $W_0, \dots, W_n, W$ witnesses that n was not maximal. **X** So $\sup_{u\in K} \int(|u| - u^*)^+ \le \epsilon$, as required. **Q**

As ϵ is arbitrary, K is uniformly integrable. Thus every relatively weakly compact subset of U is uniformly integrable.

(b)(i) In the reverse direction, suppose to begin with that $(\mathfrak{A}, \bar\mu)$ is a totally finite measure algebra, and that $A \subseteq L^1 = L^1(\mathfrak{A}, \bar\mu)$ is uniformly integrable; let $\mathcal{F}$ be a filter on L^1 containing A. Write $\mathcal{V}$ for the set of neighbourhoods of 0 for the weak topology $\mathfrak{T}_s(L^1, (L^1)^*)$.[8]

(α) For each $n \in \mathbb{N}$ let $M_n \ge 0$ be such that $\|(|u| - M_n\chi 1)^+\|_1 \le 2^{-n}$ for every $u \in A$, and define sets $K_n \subseteq [-M_n\chi 1, M_n\chi 1]$ and filters $\mathcal{F}_n$ as follows. $\mathcal{F}_0 = \mathcal{F}$. Given that $\mathcal{F}_n$ contains A, define $\phi_n : L^1 \to L^2 = L^2(\mathfrak{A}, \bar\mu)$ by setting $\phi_n(u) = \mathrm{med}(-M_n\chi 1, u, M_n\chi 1)$ for each $u \in L^1$, and consider the filter $\phi_n[[\mathcal{F}_n]]$. This is a filter on the Hilbert space L^2 containing the $\| \|_2$-bounded set $[-M_n\chi 1, M_n\chi 1]$, so the set K_n^* of its $\mathfrak{T}_s(L^2, L^2)$-cluster points is non-empty, by 566P; as K_n^* is $\mathfrak{T}_s(L^2, L^2)$-closed, it is $\mathfrak{T}_s(L^2, L^2)$-compact. As $[-M_n\chi 1, M_n\chi 1]$ is $\| \|_2$-closed and convex, it is $\mathfrak{T}_s(L^2, L^2)$-closed (561Ie) and includes K_n^*. Set $\gamma_n = \inf\{\|u\|_2 : u \in K_n^*\}$. As all the sets $\{u : \|u\|_2 \le \alpha\}$, for $\alpha > \gamma_n$, are $\mathfrak{T}_s(L^2, L^2)$-closed and meet K_n^*, $K_n = \{u : u \in K_n^*, \|u\|_2 \le \gamma_n\}$ is non-empty.

Suppose that $G \in \mathcal{V}$. Because the embedding $L^2 \subsetneqq L^1$ is norm-continuous, it is weakly continuous, and $G \cap L^2$ is a $\mathfrak{T}_s(L^2, L^2)$-neighbourhood of 0. It follows that $x + G$ meets every member of $\phi_n[[\mathcal{F}_n]]$ for every $x \in K_n^*$; so $K_n + G$ meets every member of $\phi_n[[\mathcal{F}_n]]$. We can therefore extend $\mathcal{F}_n$ to the filter $\mathcal{F}_{n+1}$ generated by

$$\mathcal{F}_n \cup \{\phi_n^{-1}[K_n + G] : G \in \mathcal{V}\}$$

and continue.

(β) Set $\mathcal{G} = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ and $B = \{u : u \in L^1, \|u\|_1 \le 1\}$. Then for each $n \in \mathbb{N}$ there is a finite set $J \subseteq L^1$ such that $J + G + 2^{-n+1}B \in \mathcal{G}$ for every $G \in \mathcal{V}$. **P** K_n is a $\mathfrak{T}_s(L^2, L^2)$-closed subset of K_n^*, so is $\mathfrak{T}_s(L^2, L^2)$-compact; also it is included in the sphere $S = \{u : \|u\|_2 = \gamma_n\}$. Because $\| \|_2$ is locally uniformly rotund, it is a Kadec norm (467B) and the norm and weak topologies on S coincide; consequently K_n is $\| \|_2$-compact. Since $\| \|_1$ and $\| \|_2$ give rise to the same topology on any $\| \|_\infty$-bounded set, K_n is $\| \|_1$-compact. There is therefore a finite set $J \subseteq K_n$ such that $K_n \subseteq J + 2^{-n}B$.

Take any $G \in \mathcal{V}$. Then $\|u - \phi_n(u)\| = \|(|u| - M_n\chi 1)^+\| \le 2^{-n}$ for every $u \in A$, so

$$J + G + 2^{-n+1}B \supseteq (K_n + G) + 2^{-n}B \supseteq A \cap \phi_n^{-1}[K_n + G] \in \mathcal{F}_{n+1} \subseteq \mathcal{G}. \textbf{ Q}$$

(γ) For each $n \in \mathbb{N}$ choose a minimal finite set $J_n \subseteq L^1$ such that $J_n + G + 2^{-n+1}B \in \mathcal{G}$ for every $G \in \mathcal{V}$. Note that $(x + G + 2^{-n+1}B) \cap D$ must be non-empty whenever $n \in \mathbb{N}$, $x \in J_n$, $G \in \mathcal{V}$ and $D \in \mathcal{G}$. **P?** Otherwise,

$$(J_n \setminus \{x\}) + G' + 2^{-n+1}B \supseteq (J_n + (G \cap G') + 2^{-n+1}B) \cap D$$

belongs to $\mathcal{G}$ for every $G' \in \mathcal{V}$, and J_n was not minimal. **XQ**

(δ) For any $n \in \mathbb{N}$ and $u \in J_n$ there is a $v \in J_{n+1}$ such that $\|u - v\|_1 \le 2^{-n+1} + 2^{-n}$. **P?** Otherwise, by (a-i-β) above, we can choose for each $v \in J_{n+1}$ an $f_v \in (L^1)^*$ such that $\|f_v\| = 1$ and $f_v(v - u) = \|v - u\| = 2^{-n+1} + 2^{-n} + \delta_v$ where $\delta_v > 0$; set

$$G = \{w : |f_v(w)| < \tfrac{1}{2}\delta_v \text{ for every } v \in J_{n+1}\} \in \mathcal{V}.$$

Then $u + G + 2^{-n+1}B$ does not meet $J_{n+1} + G + 2^{-n}B$, contradicting (γ) here. **XQ**

(ε) Because $\bigcup_{n\in\mathbb{N}} J_n$ is countable, therefore well-orderable, we can define inductively a sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ such that $u_n \in J_n$ and $\|u_n - u_{n+1}\| \le 2^{-n+1} + 2^{-n}$ for every n. Now $\langle u_n\rangle_{n\in\mathbb{N}}$ is Cauchy, so has a

[8] Of course $(L^1)^*$ can be identified with $L^\infty(\mathfrak{A})$, but if you don't wish to trace through the arguments for this, and confirm that they can be carried out without appealing to anything more than $\mathrm{AC}(\omega)$, you can defer the exercise for the time being.

limit u in L^1. If $G\in\mathcal{V}$ there is an $n\in\mathbb{N}$ such that $u+G\supseteq u_n+\frac{1}{2}G+2^{-n}B$, so $u+G$ meets every member of $\mathcal{G}$; thus u is a weak cluster point of $\mathcal{G}$ and of $\mathcal{F}$. As $\mathcal{F}$ is arbitrary, A is relatively weakly compact.

(ii) Now suppose that U is an arbitrary L-space and $A\subseteq U$ is a uniformly integrable set. Then we can choose a sequence $\langle e_n\rangle_{n\in\mathbb{N}}$ in U^+ such that $\|(|u|-e_n)^+\|\le 2^{-n}$ for every $n\in\mathbb{N}$ and $u\in A$. Set $e=\sum_{n=0}^\infty\frac{1}{1+2^n\|e_n\|}e_n$ in U, and let V be the band in U generated by e. Then $A\subseteq V$, and of course A is uniformly integrable in V. By 561Hb, we have a totally finite measure algebra $(\mathfrak{A},\bar\mu)$ and a normed Riesz space isomorphism $T:V\to L^1(\mathfrak{A},\bar\mu)$; now $T[A]$ is uniformly integrable in $L^1(\mathfrak{A},\bar\mu)$, therefore relatively weakly compact, by (i). But this means that A is relatively weakly compact in V; as the embedding $V\subsetneqq U$ is weakly continuous, A is relatively weakly compact in U.

This completes the proof.

566R Automorphisms of measurable algebras: Theorem [AC(ω)] Let $\mathfrak{A}$ be a measurable algebra.

(a) Every automorphism of $\mathfrak{A}$ has a separator.

(b) Every $\pi\in\operatorname{Aut}\mathfrak{A}$ is a product of at most three exchanging involutions belonging to the full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π.

proof (a) (Cf. 382Eb.) Take $\pi\in\operatorname{Aut}\mathfrak{A}$. Let $\bar\mu$ be such that $(\mathfrak{A},\bar\mu)$ is a totally finite measure algebra. For $a\in\mathbb{N}$ set $\psi(a)=\sup_{n\in\mathbb{Z}}\pi^na$, so that $\pi(\psi(a))=\psi(a)$. Note that if $a\cap\psi(b)=0$ then $\psi(a)\cap\psi(b)=0$. Set $A=\{a:a\in\mathfrak{A},\,a\cap\pi a=0\}$ and choose a sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in A such that

$$\sup_{n\in\mathbb{N}}\bar\mu(\psi(a_n))=\sup_{a\in A}\bar\mu(\psi(a)).$$

Define $\langle b_n\rangle_{n\in\mathbb{N}}$, $\langle c_n\rangle_{n\in\mathbb{N}}$ by saying that

$$c_0=0,\quad b_n=a_n\setminus\psi(c_n),\quad c_{n+1}=b_n\cup c_n$$

for each n. Inducing on n we see that b_n and c_n belong to A and that $\psi(c_{n+1})\supseteq\psi(a_n)$ for every n. Set $c=\sup_{n\in\mathbb{N}}c_n$; then $c\in A$ and $\psi(c)\supseteq\psi(a_n)$ for every n.

Now c is a separator for π. **P?** Otherwise, there is a non-zero $d\subseteq 1\setminus\psi(c)$ such that $d\cap\pi d=0$ (381Ei). In this case $d\cup c\in A$ and

$$\bar\mu(\psi(d\cup c))>\bar\mu(\psi(c))=\sup_{n\in\mathbb{N}}\bar\mu(\psi(a_n))=\sup_{a\in A}\bar\mu(\psi(a)),$$

which is impossible. **XQ**

(b) We can now work through the proofs of 382A-382M to confirm that there is no essential use of anything beyond countable choice there, so long as we suppose that we are working with measurable algebras. (There is an inductive construction in the proof of 382J. To do this with AC(ω) rather than DC, we need to check that every element of the construction can be made determinate following an initial countable set of choices; in the case there, we need to check that the existence assertions of 382D and 382I can be represented as functions, as in 566Xh and 566Xj.) Since the proof of 382K speaks of the Stone representation theorem, there seems to be a difficulty here, unless we take the alternative route suggested in 382Yb. But note that while the general Stone theorem has a strength little short of full AC, the representation of a *countable* Boolean algebra $\mathfrak{B}$ as the algebra of open-and-closed subsets of a compact Hausdorff Baire space can be done in ZF alone (561F). In part (f) of the proof of 382K, therefore, take $\mathfrak{B}$ to be a countable subalgebra of $\mathfrak{A}$ such that

e_n, u'_n, u''_n, v'_l, v''_l, d_{lj}, d'_{lj}, $\operatorname{supp}(\pi\phi)^k$, $\operatorname{supp}(\pi\phi_1)^k\in\mathfrak{B}$ whenever $n\in\mathbb{N}$ and j, k, $l\ge1$,

c_0, c_1, $\operatorname{supp}\phi_2\in\mathfrak{B}$,

$\mathfrak{B}$ is closed under the functions π, ϕ_1, ϕ_2, ϕ and $\tilde\pi_n$ for $n\in\mathbb{N}$,

and let Z be the Stone space of $\mathfrak{B}$. Now we can perform the arguments of the rest of the proof in Z to show that $c_0=\inf_{n\ge1}\operatorname{supp}(\pi\phi)^n$ is zero, as required.

566S Volume 4 In Volume 4, naturally, a rather larger proportion of the ideas become inaccessible without strong forms of the axiom of choice. Since we are missing the most useful representation theorems, many results have to be abandoned altogether. More subtly, we seem to lose the result that Radon measures are localizable (416B). Nevertheless, a good deal can still be done, if we follow the principles set out in 566Ae-566Af. Most notably, we have a workable theory of Haar measure on completely regular locally compact topological groups, because the Riesz representation theorems of §436 are still available, and we can use 561G instead of 441C. I should remark, however, that in the absence of Tychonoff's theorem we may have fewer compact groups than we expect. And the theory of dual groups in §445 depends heavily on AC.

The descriptive set theory of Chapter 42 is hardly touched, and enough of the rest of the volume survives to make it worth checking any point of particular interest. Most of Chapter 46 depends heavily on the Hahn-Banach theorem and therefore becomes limited to cases in which we have a good grasp of dual spaces, as in 561Xh. There are some difficulties in the geometric measure theory of arbitrary metric spaces in §471, but the rest of the chapter seems to stand up. The abstract theory of gauge integrals in §482 is expressed in forms which need DC at least, but I think that the basic facts about the Henstock integral (§483) are unaffected. There are some interesting challenges in Chapter 49, but there the eclectic nature of the arguments means that we cannot expect much of the theory to keep its shape.

566T I give one result which may not be obvious and helps to keep things in order.

Proposition [AC(ω)] Let I be any set, and X a separable metrizable space. Then the Baire σ-algebra $\mathcal{B}\mathfrak{a}(X^I)$ of X^I is equal to the σ-algebra $\widehat{\bigotimes}_I\mathcal{B}(X)$ generated by sets of the form $\{x : x(i) \in E\}$ for $i \in I$ and Borel sets $E \subseteq X$.

proof (a) Every open set in X is a cozero set, so $\mathcal{B}(X) = \mathcal{B}\mathfrak{a}(X)$ and $\{x : x(i) \in E\} \in \mathcal{B}\mathfrak{a}(X^I)$ whenever $i \in I$ and $E \in \mathcal{B}(X)$; accordingly $\widehat{\bigotimes}_I\mathcal{B}(X) \subseteq \mathcal{B}\mathfrak{a}(X^I)$.

(b) Fix a sequence $\langle U_n\rangle_{n\in\mathbb{N}}$ running over a base for the topology of X. For $\sigma \in S = \bigcup_{J\in[I]^{<\omega}} \mathbb{N}^J$ set

$$C_\sigma = \{x : x \in X,\, x(i) \in U_{\sigma(i)} \text{ for every } i \in \operatorname{dom}\sigma\} \in \widehat{\bigotimes}_I\mathcal{B}(X).$$

Then $\{C_\sigma : \sigma \in S\}$ is a base for the topology of X^I. If $W \subseteq X^I$ is a regular open set, there is a countable set $R \subseteq S$ such that $W = \bigcup_{\sigma\in R} C_\sigma$. **P** Let R^* be the set of those $\sigma \in S$ such that $C_\sigma \subseteq W$, and R the set of minimal members of R^* (ordering S by extension of functions). Then every member of R^* extends some member of R, so

$$\bigcup_{\sigma\in R} C_\sigma = \bigcup_{\sigma\in R^*} C_\sigma = W.$$

For $n \in \mathbb{N}$ set $R_n = \{\sigma : \sigma \in R,\, \#(\sigma) = n,\, \sigma(i) < n \text{ for every } i \in \operatorname{dom}\sigma\}$.

? Suppose, if possible, that $n \in \mathbb{N}$ and R_n is infinite. Then there is a sequence $\langle\sigma_k\rangle_{k\in\mathbb{N}}$ of distinct elements of R_n; set $J_k = \operatorname{dom}\sigma_k$ for each k. Let $M \subseteq \mathbb{N}$ be an infinite set such that $\langle J_k\rangle_{k\in M}$ is a Δ-system with root J say. Then there is a $\sigma \in n^J$ such that $M' = \{k : k \in M,\, \sigma_k{\restriction}J = \sigma\}$ is infinite.

In this case, however,

$$C_\sigma \subseteq \operatorname{int}\overline{\bigcup_{k\in M'} C_{\sigma_k}} \subseteq \operatorname{int}\overline{W} = W$$

and $\sigma \in R^*$, so that $\sigma_k \notin R$ for $k \in M'$; which is impossible. **X**

Thus every R_n is countable and $R = \bigcup_{n\in\mathbb{N}} R_n$ is countable. **Q**

(c) This shows that every regular open subset of X^I is a countable union of open cylinder sets and belongs to $\widehat{\bigotimes}_I\mathcal{B}(X)$. Consequently every cozero set belongs to $\widehat{\bigotimes}_I\mathcal{B}(X)$. **P** If $f : X^I \to \mathbb{R}$ is continuous, then for each rational $q > 0$ set $W_q = \operatorname{int}\{x : |f(x)| \ge q\}$. Then W_q is a regular open set so belongs to $\widehat{\bigotimes}_I\mathcal{B}(X)$. But now $\{x : f(x) \ne 0\} = \bigcup_{q\in\mathbb{Q},q>0} W_q$ is the union of countably many sets in $\widehat{\bigotimes}_I\mathcal{B}(X)$ and itself belongs to $\widehat{\bigotimes}_I\mathcal{B}(X)$. **Q**

So $\widehat{\bigotimes}_I\mathcal{B}(X) \supseteq \mathcal{B}\mathfrak{a}(X^I)$ and the two are equal.

566U Dependent choice If we allow ourselves to use the stronger principle DC rather than AC(ω) alone, we get some useful simplifications. The difficulties with the principle of exhaustion in §215 and 566D above disappear, and there is no longer any obstacle to the construction of product measures in 254F, provided only that we know we have a non-empty product space. So a typical theorem on product measures will now begin 'let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of probability spaces such that $X = \prod_{i\in I} X_i$ is non-empty'. Later, we now have Baire's theorem (both for complete metric spaces and for locally compact Hausdorff spaces) and

Urysohn's Lemma (so we can drop the formulation 'completely regular locally compact topological group'). The most substantial gap in Volume 4 which is now filled seems to be in the abstract theory of gauge integrals in §482. But I cannot point to a result which is essential to the structure of this treatise and can be proved in ZF + DC but not in ZF + AC(ω).

566X Basic exercises (a) [AC(ω)] Let (X,ρ) be a metric space. (i) Show that X is compact iff it is sequentially compact iff it is countably compact iff it is complete and totally bounded. (ii) Show that if X is separable then every subspace of X is separable.

(b) [AC(ω)] Show that there is a surjection from $\mathbb{R}$ onto its Borel σ-algebra, so that there must be a non-Borel subset of $\mathbb{R}$.

(c)(i) (Cf. 313K) Let $\mathfrak{A}$ be a Boolean algebra, and $D\subseteq\mathfrak{A}$ an order-dense set. Show that $a=\sup\{d:d\in D,\,d\subseteq a\}$ for every $a\in\mathfrak{A}$. (ii) (Cf. 322Eb) Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra. Show that $a=\sup\{b:b\subseteq a,\,\bar\mu b<\infty\}$ for every $a\in\mathfrak{A}$.

(d) Let us say that a Boolean algebra $\mathfrak{A}$ has the **countable sup property** if for every $A\subseteq\mathfrak{A}$ there is a countable $B\subseteq A$ with the same upper bounds as A. (i) Show that a Dedekind σ-complete Boolean algebra with the countable sup property is Dedekind complete. (ii) Show that a countably additive functional on a Boolean algebra with the countable sup property is completely additive.

(e) [AC(ω)] Show that if there is a translation-invariant lifting for Lebesgue measure then there is a subset of $\mathbb{R}$ which is not Lebesgue measurable. (*Hint*: 345F.)

(f) [AC(ω)] Show that if $1<p<\infty$ and $(\mathfrak{A},\bar\mu)$ is a measure algebra, the unit ball of $L^p(\mathfrak{A},\bar\mu)$ (§366) is weakly compact. (*Hint*: part (b) of the proof of 566Q.)

(g) [AC(ω)] (i) Let $\mathfrak{A}$ be a measurable algebra. Show that the unit ball of $L^\infty=L^\infty(\mathfrak{A})$ is compact for $\mathfrak{T}_s(L^\infty,(L^\infty)^\times)$ (definition: 3A5Ea). (ii) Let U be an L-space with a weak order unit. Show that the unit ball of U^* is weak*-compact. (*Hint*: 561Hb.)

(h) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Show that there is a function $f:\operatorname{Aut}\mathfrak{A}\times\mathfrak{A}\to\mathfrak{A}$ such that if $\pi\in\operatorname{Aut}\mathfrak{A}$ and a is a separator for π then $a\cap f(\pi,a)=0$ and $f(\pi,a)\cup\pi f(\pi,a)\cup\pi^2f(\pi,a)$ is the support of π. (*Hint*: 382D.)

(i) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a well-orderable subgroup of $\operatorname{Aut}\mathfrak{A}$. Let G^* be the full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by G. Show that there is a function $f:G^*\times G\to\mathfrak{A}$ such that $\langle f(\pi,\phi)\rangle_{\phi\in G}$ is a partition of unity for each $\pi\in G^*$ and $\pi a=\phi a$ whenever $\pi\in G^*$, $\phi\in G$ and $a\subseteq f(\pi,\phi)$. (*Hint*: 381I.)

(j) [AC(ω)] Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a countable subgroup of $\operatorname{Aut}\mathfrak{A}$ such that every member of G has a separator. Let G^* be the full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by G. Show that there is a function $g:G^*\to\mathfrak{A}$ such that $g(\pi)$ is a separator for π for every $\pi\in G^*$. (*Hint*: 566Xi, 382Id.)

(k) [AC(ω)] Let $(X,\mathfrak{T})$ be a completely regular locally compact Hausdorff space, and $f:C_k(X)\to\mathbb{R}$ a positive linear functional. Show that there is a unique Radon measure μ on X such that $f(u)=\int u\,d\mu$ for every $u\in C_k(X)$.

(l) [AC(ω)] Say that a set X is **measure-free** if whenever μ is a probability measure with domain $\mathcal{P}X$ there is an $x\in X$ such that $\mu\{x\}>0$. (i) Show that the following are equiveridical: (α) $\mathbb{R}$ is not measure-free; (β) there is a semi-finite measure space $(X,\mathcal{P}X,\mu)$ which is not purely atomic; (γ) there is a measure μ on $[0,1]$ extending Lebesgue measure and measuring every subset of $[0,1]$. (ii) Prove 438B for point-finite families $\langle E_i\rangle_{i\in I}$ such that the index set I is measure-free.

566Y Further exercises (a) [AC(ω)] Show that if U is an L-space, and $\langle u_n\rangle_{n\in\mathbb{N}}$ is a bounded sequence in U, then there are a subsequence $\langle v_n\rangle_{n\in\mathbb{N}}$ of $\langle u_n\rangle_{n\in\mathbb{N}}$ and a $w\in U$ such that $\langle\frac{1}{n+1}\sum_{i=0}^n w_i\rangle_{n\in\mathbb{N}}$ is order*-convergent to w for every subsequence $\langle w_n\rangle_{n\in\mathbb{N}}$ of $\langle v_n\rangle_{n\in\mathbb{N}}$. (*Hint*: in the proof of 276H, show that we can find a countably-generated filter to replace the ultrafilter $\mathcal{F}$.)

(b) [AC(ω)] Let X be a completely regular compact Hausdorff topological group and μ a left Haar measure on X. Show that if $w \in L^2(\mu)$ then $u \mapsto u * w : L^2(\mu) \to C(X)$ is a compact linear operator. (*Hint*: 444V.)

(c) [AC(ω)] Let $\langle(X_i, \langle U_{in}\rangle_{n\in\mathbb{N}})\rangle_{i\in I}$ be a family such that X_i is a separable metrizable space and $\langle U_{in}\rangle_{n\in\mathbb{N}}$ is a base for the topology of X_i for each $i \in I$. Show that $\mathcal{B}\mathfrak{a}(\prod_{i\in I} X_i) = \widehat{\bigotimes}_{i\in I}\mathcal{B}(X_i)$.

(d) [DC] Let U be an inner product space and $K \subseteq U$ a convex weakly compact set. Show that K has an extreme point.

566Z Problem Is it relatively consistent with ZF + AC(ω) to suppose that there is a non-zero atomless rigid measurable algebra?

566 Notes and comments In this section I have taken a lightning tour through the material of Volumes 1 to 4, pausing over a rather odd selection of results, mostly chosen to exhibit the alternative arguments which are available. In the first place, I am trying to suggest something of the quality of the world of measure theory, and of analysis in general, under this particular set of rules. Perhaps I should say that my real objective is the next section, with DC rather than AC(ω), because DC is believed to be compatible with the axiom of determinacy, and ZF + DC + AD is not a poor relation of ZFC, as ZF + AC(ω) sometimes seems to be, but a potential rival.

I have a second reason for taking all this trouble, which is a variation on one of the reasons for 'generalization' as found in twentieth-century pure mathematics. When we 'generalize' an argument, moving (for example) from metric spaces to topological spaces, or from Lebesgue measure to abstract measures, we are usually stimulated by some particular question which demands the new framework. But the process frequently has a lasting value which is quite independent of its motivation. It forces us to re-examine the nature of the proofs we are using, discarding or adapting those steps which depend on the original context, and isolating those which belong in some other class of ideas. In the same way, renouncing the use of AC forces us to look more closely at critical points, and decide which of them correspond to some deeper principle.

Something I have not attempted to do is to look for models in which my favourite theorems are actually false. An interesting class of problems is concerned with 'exact engineering', that is, finding combinatorial propositions which will be equivalent, in ZF, to given results which are not provable in ZF. For instance, Baire's theorem for complete metric spaces is actually equivalent to DC (Blair 77), while Baire's theorem for compact Hausdorff spaces may be weaker (Fossy & Morillon 98). I am not presenting any such results here. However, if we take Maharam's theorem as an example of a central result of measure theory with ZFC which is surely unprovable without a strong form of AC, we can ask just how false it can be; and I offer 566Z as a sample target.

Version of 31.10.14

567 Determinacy

So far, this chapter has been looking at set theories which are weaker than the standard theory ZFC, and checking which of the principal results of measure theory can still be proved. I now turn to an axiom which directly contradicts the axiom of choice, and leads to a very different world. This is AD, the 'axiom of determinacy', defined in terms of strategies for infinite games (567A-567C). The first step is to confirm that we automatically have a weak version of countable choice which is enough to make Lebesgue measure well-behaved (567D-567E). Next, in separable metrizable spaces all subsets are universally measurable and have the Baire property (567G). Consequently (at least when we can use AC(ω)) linear operators between Banach spaces are bounded (567H), additive functionals on σ-complete Boolean algebras are countably additive (567J), and many L-spaces are reflexive (567K). In a different direction, we find that ω_1 is two-valued-measurable (567L) and that there are many surjections from $\mathbb{R}$ onto ordinals (567M).

At the end of the section I include two celebrated results in ZFC (567N, 567O) which depend on some of the same ideas.

567A Infinite games I return to an idea introduced in §451.

(a) Let X be a non-empty set and A a subset of $X^{\mathbb{N}}$. In the corresponding infinite game $\mathrm{Game}(X,A)$, players I and II choose members of X alternately, so that I chooses $x(0), x(2), \ldots$ and II chooses $x(1), x(3), \ldots$; a **play** of the game is an element of $X^{\mathbb{N}}$; player I wins the play x if $x \in A$, otherwise II wins. A **strategy** for I is a function $\sigma : \bigcup_{n\in\mathbb{N}} X^n \to X$; a play $x \in X^{\mathbb{N}}$ is **consistent** with σ if $x(2n) = \sigma(\langle x(2i+1)\rangle_{i<n})$ for every n, that is, if I uses the function σ to decide his move from the previous moves by his opponent; σ is a **winning strategy** if every play consistent with σ belongs to A, that is, if I wins whenever he follows the strategy σ. Similarly, a strategy for II is a function $\tau : \bigcup_{n\ge1} X^n \to X$; a play x is consistent with τ if $x(2n+1) = \tau(\langle x(2i)\rangle_{i\le n})$ for every n; and τ is a winning strategy for II if $x \notin A$ whenever $x \in X^{\mathbb{N}}$ and x is consistent with τ.

(b) A set $A \subseteq X^{\mathbb{N}}$ is **determined** if either I or II has a winning strategy in $\mathrm{Game}(X,A)$. Note that we need to know the set X as well as the set A to specify the game in question.

(c) It will sometimes be convenient to describe games with 'rules', so that the players are required to choose points in subsets of X (determined by the moves so far) at each move. Such a description can be regarded as specifying A in the form $(A'\cup G)\setminus H$, where G is the set of plays in which II is the first to break a rule, H is the set of plays in which I is the first to break a rule, and A' is the set of plays in which both obey the rules and I wins.

(d) Not infrequently the 'rules' will specify different sets for the moves of the two players, so that I always chooses a point in X_1 and II always chooses a point in X_2; setting $X = X_1 \cup X_2$ we can reduce this to the formalization above.

567B Theorem Let X be a non-empty well-orderable set. Give X its discrete topology and $X^{\mathbb{N}}$ the product topology. If $F \subseteq X^{\mathbb{N}}$ is closed then $\mathrm{Game}(X,F)$ is determined.

proof (a) Fix a well-ordering $\preccurlyeq$ of X. Define $\langle W_\xi\rangle_{\xi\in\mathrm{On}}$ by setting

$$W_0 = \{w : w \in \textstyle\bigcup_{n\in\mathbb{N}} X^{2n+1},\ w \not\subseteq x \text{ for any } x \in F\},$$

$$W_\xi = \{w : w \in \bigcup_{n\in\mathbb{N}} X^{2n+1}, \text{ there is some } t \in X \text{ such that}$$

$$w^\frown{<}t{>}^\frown{<}u{>} \in \bigcup_{\eta<\xi} W_\eta \text{ for every } u \in X\}$$

if $\xi > 0$. (See 5A1C for the notation here.) If $w \in W_0$ then of course $w^\frown{<}t{>}^\frown{<}u{>} \in W_0$ for all t, $u \in X$; so $W_0 \subseteq W_1$, and of course that $W_\xi \subseteq W_{\xi'}$ whenever $1 \le \xi \le \xi'$ in On. There is therefore an ordinal ζ such that $W_{\zeta+1} = W_\zeta$; write W for W_ζ.

For $w \in W$, let $r(w) \le \zeta$ be the least ordinal such that $w \in W_{r(w)}$. If $r(w) > 0$ then there is some $t \in X$ such that such that $w^\frown{<}t{>}^\frown{<}u{>} \in \bigcup_{\eta<r(w)} W_\eta$, that is, $r(w^\frown{<}t{>}^\frown{<}u{>}) < r(w)$, for every $u \in X$.

Let V be the set of those $v \in \bigcup_{n\in\mathbb{N}} X^{2n}$ such that there is a $u \in X$ such that $v^\frown{<}u{>} \notin W$. Observe that if $w \in \bigcup_{n\in\mathbb{N}} X^{2n+1} \setminus W$ then $w \notin W_{\zeta+1}$ so $w^\frown{<}t{>} \in V$ for every $t \in X$.

(b) Suppose that $\emptyset \in V$. Define $\sigma : \bigcup_{n\in\mathbb{N}} X^n \to X$ inductively by saying that

$\sigma(\emptyset)$ is the $\preccurlyeq$-least member t of X such that the one-element sequence $\emptyset^\frown{<}t{>}$ does not belong to W,

if $v \in X^{n+1}$ and $w = (\sigma(v{\restriction}0), v(0), \sigma(v{\restriction}1), v(1), \ldots, \sigma(v{\restriction}n), v(n)) \in V$, take $\sigma(v)$ to be the $\preccurlyeq$-least member t of X such that $w^\frown{<}t{>} \notin W$,

for other $v \in X^{n+1}$ take $\sigma(v)$ to be the $\preccurlyeq$-least member of X.

Then σ is a winning strategy for I. **P** If x is a play consistent with σ, then an induction on n shows that $x{\restriction}2n \in V$ and $x{\restriction}2n+1 \notin W$ for every n. In particular, $x{\restriction}2n+1 \notin W_0$, that is, there is a member of F extending $x{\restriction}2n+1$, for every n. As F is closed, $x \in F$ and I wins the play x. **Q**

(c) Suppose that $\emptyset \notin V$, that is, $w \in W$ for every $w \in X^1$. Define $\tau : \bigcup_{n\ge1} X^n \to X$ inductively by saying if $v \in X^n$ and $w = (v(0), \tau(v\restriction 1), v(1), \tau(v\restriction 2), \dots, v(n-1))$ belongs to $W \setminus W_0$, then $\tau(v)$ is the $\preccurlyeq$-least $t \in X$ such that $r(w^\frown{<}t{>}^\frown{<}u{>}) < r(w)$ for every $u \in X$,

for other $v \in X^n$, $\tau(v)$ is the $\preccurlyeq$-least member of X.

Then τ is a winning strategy for II. **P** Let x be a play consistent with τ. Then an induction on n tells us that

$$x\restriction 2n+1 \in W, \quad \text{if } x\restriction 2n+1 \notin W_0 \text{ then } r(x\restriction 2n+3) < r(x\restriction 2n+1)$$

for every $n \in \mathbb{N}$. Since $\langle r(x\restriction 2n+1)\rangle_{n\in\mathbb{N}}$ cannot be strictly decreasing, there is some $n \in \mathbb{N}$ such that $x\restriction 2n+1 \in W_0$ and $x \notin F$. Thus II wins the play x. **Q**

(d) Putting (b) and (c) together we see that F is determined.

567C The axiom of determinacy (a) The standard 'axiom of determinacy' is the statement

(AD) Every subset of $\mathbb{N}^{\mathbb{N}}$ is determined.

Evidently it will follow that every subset of $X^{\mathbb{N}}$ is determined for any countable set X. (If $X \subseteq \mathbb{N}$, a game on X can be regarded as a game on $\mathbb{N}$ in which there is a rule that the players must always choose points in X. See also 567Xc.)

(b) At the same time, it will be useful to consider a weak form of the axiom of countable choice: for any set X, write $\mathrm{AC}(X;\omega)$ for the statement

$\prod_{n\in\mathbb{N}} A_n \neq \emptyset$ whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of non-empty subsets of X.

567D Theorem (Mycielski 64) AD implies $\mathrm{AC}(\mathbb{R};\omega)$.

proof Since we know that $\mathbb{R}$ is equipollent with $\mathbb{N}^{\mathbb{N}}$, we can look at $\mathrm{AC}(\mathbb{N}^{\mathbb{N}};\omega)$. Let $\langle A_n\rangle_{n\in\mathbb{N}}$ be a sequence of non-empty subsets of $\mathbb{N}^{\mathbb{N}}$. Set

$$A = \{x : x \in \mathbb{N}^{\mathbb{N}},\ \langle x(2n+1)\rangle_{n\in\mathbb{N}} \notin A_{x(0)}\}.$$

Then I has no winning strategy in $\mathrm{Game}(\mathbb{N}, A)$, because if σ is a strategy for I in $\mathrm{Game}(\mathbb{N}, A)$ set $k = \sigma(\emptyset)$; there is a point $y \in A_k$, and II need only play $x(2n+1) = y(n)$ for each n.

So II has a winning strategy τ say. Define $g : \mathbb{N} \to \mathbb{N}^{\mathbb{N}}$ by saying that $g(n)(i) = \tau(e_{ni})$ for n, $i \in \mathbb{N}$, where $e_{ni} \in \mathbb{N}^{i+1}$, $e_{ni}(0) = n$, $e_{ni}(j) = 0$ for $1 \le j \le i$. If now $n \in \mathbb{N}$, I plays $(n, 0, 0, \dots)$ and II follows the strategy τ, the resulting play $(n, g(n)(0), 0, g(n)(1), 0, \dots)$ must not belong to A so $g(n) \in A_n$.

567E Consequences of AC($\mathbb{R};\omega$) Suppose that $\mathrm{AC}(\mathbb{R};\omega)$ is true.

(a) If a set X is the image of a subset Y of $\mathbb{R}$ under a function f, then $\mathrm{AC}(X;\omega)$ is true. **P** If $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of non-empty subsets of X, then there is an $x \in \prod_{n\in\mathbb{N}} f^{-1}[A_n]$, and $\langle f(x(n))\rangle_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} A_n$. **Q**

(b) In particular, taking $S^* = \bigcup_{n\ge1} \mathbb{N}^n$ as in §562, $\mathrm{AC}(\mathcal{P}S^*;\omega)$ is true. It follows that (in any second-countable space X) every sequence of codable Borel sets is codable and the family of codable Borel sets is a σ-algebra, coinciding with the Borel σ-algebra $\mathcal{B}(X)$ on its ordinary definition. Moreover, since $\mathcal{B}(X)$ is an image of $\mathcal{P}S^*$, we have $\mathrm{AC}(\mathcal{B}(X);\omega)$, countable choice for collections of Borel sets. Similarly, the family of codable Borel functions becomes the ordinary family of Borel-measurable functions, and we have countable choice for sets of Borel real-valued functions on X.

(c) Consequently the results of §562-565 give us large parts of the elementary theory of Borel measures on second-countable spaces. At the same time, if X is second-countable, the union of a sequence of meager subsets of X is meager (because we have countable choice for sequences of nowhere dense closed sets), so the Baire-property algebra of X is a σ-algebra.

(d) We also find that the supremum of a sequence of countable ordinals is again countable. **P** Let $\langle \xi_n\rangle_{n\in\mathbb{N}}$ be a sequence in ω_1. Using $\mathrm{AC}(\mathbb{R};\omega)$), we can choose for each $n \in \mathbb{N}$ a subset $\preccurlyeq_n$ of $\mathbb{N}\times\mathbb{N}$ which is a well-ordering of $\mathbb{N}$ with order type $\max(\omega, \xi_n)$. Now we have a well-ordering $\preccurlyeq$ of $\mathbb{N}^2$ defined by saying that $(i, j) \preccurlyeq (i', j')$ if $i < i'$ or $i = i'$ and $j \preccurlyeq_i j'$. In this case, the order type ξ of $\preccurlyeq$ will be greater than or equal to every ξ_n, so that $\sup_{n\in\mathbb{N}} \xi_n \le \xi$ is countable. **Q**

567F Lemma (see Mycielski & Świerczkowski 64) [AC($\mathbb{R};\omega$)] Suppose that $A \subseteq \{0,1\}^{\mathbb{N}}$ is a continuous image of a subset B of $\{0,1\}^{\mathbb{N}}$ such that $(h^{-1}[B] \cap F) \cup H \subseteq \mathbb{N}^{\mathbb{N}}$ is determined whenever $h : \mathbb{N}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ is continuous, $F \subseteq \mathbb{N}^{\mathbb{N}}$ is closed and $H \subseteq \mathbb{N}^{\mathbb{N}}$ is open.

(a) A is universally measurable.

(b) A has the Baire property in $\{0,1\}^{\mathbb{N}}$.

proof Fix a continuous surjection $f : B \to A$. Let $\mathcal{E}$ be the countable algebra of subsets of $\{0,1\}^{\mathbb{N}}$ determined by coordinates in finite sets, that is to say, the algebra of open-and-closed subsets of $\{0,1\}^{\mathbb{N}}$ (311Xh).

(a)(i) Let μ be a Borel probability measure on $\{0,1\}^{\mathbb{N}}$ and $\hat{\mu}$ its completion. If $Z \subseteq \{0,1\}^{\mathbb{N}}$ is closed and not negligible, then at least one of $Z \cap A$, $Z \setminus A$ has non-zero inner measure.

P Let $\langle E_n\rangle_{n\in\mathbb{N}}$ enumerate $\mathcal{E}$. Set $\epsilon_n = 2^{-2n-2}\mu Z$ for $n \in \mathbb{N}$. In $(\{0,1\} \times \mathcal{E})^{\mathbb{N}}$ consider the game in which the players choose $(k_0, K_0), (k_1, K_1), \dots$ such that $K_0 = Z$ and for each $n \in \mathbb{N}$

$$k_n \in \{0,1\}, \quad K_n \in \mathcal{E}, \quad \mu K_{2n+1} \le \epsilon_n.$$

I wins if $y = \langle k_{2n}\rangle_{n\in\mathbb{N}}$ belongs to B and $f(y) \notin \bigcup_{n\in\mathbb{N}} K_{2n+1}$. Observe that when $y \in B$, $f(y) \in \bigcup_{n\in\mathbb{N}} K_{2n+1}$ iff there is an $m \in \mathbb{N}$ such that $f(w) \in \bigcup_{i<m} K_{2i+1}$ whenever $w \in B$ and $w{\restriction}m = y{\restriction}m$; so I wins iff $y \in B$ and at every stage $((k_0, K_0), \dots, (k_{2m}, K_{2m}))$ there is a $w \in B$ such that $w(i) = k_{2i}$ for $i < m$ and $f(w) \notin \bigcup_{i<m} K_{2i+1}$. So the payoff set D of plays $\langle(k_n, K_n)\rangle_{n\in\mathbb{N}}$ won by I is of the form $(h^{-1}[B] \cap F) \cup H$ where $h : (\{0,1\} \times \mathcal{E})^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ is continuous, $F \subseteq (\{0,1\} \times \mathcal{E})^{\mathbb{N}}$ is closed and $H \subseteq (\{0,1\} \times \mathcal{E})^{\mathbb{N}}$ is open. (Here H is the set of plays which are won because II is the first to break a rule.) Consequently D is determined.

case 1 Suppose that I has a winning strategy σ. For each play $\langle(k_n, K_n)\rangle_{n\in\mathbb{N}}$ consistent with σ, $f(\langle k_{2n}\rangle_{n\in\mathbb{N}})$ is defined and belongs to A. Since the set of plays consistent with σ is a closed subset of $(\{0,1\} \times \mathcal{E})^{\mathbb{N}}$, the set C of points obtainable in this way is an analytic subset of Z, therefore measured by $\hat{\mu}$ (563I). **?** If $\hat{\mu}C = 0$, then there is an open set $G \supseteq C$ such that $\mu G < \epsilon_0$ (563Fd). In this case, II can play in such a way that

$$K_{2n+1} \subseteq G, \quad \mu(G \setminus \textstyle\bigcup_{i\le n} K_{2i+1}) < \epsilon_{n+1},$$

$$\text{if } E_n \subseteq G \text{ then } E_n \subseteq \textstyle\bigcup_{i\le n} K_{2i+1}$$

for every n. But now, taking I's responses under σ, we have a play of Game($\{0,1\} \times \mathcal{E}, D$) in which $\bigcup_{n\in\mathbb{N}} K_{2n+1} = G$ includes C, so contains $f(\langle k_n\rangle_{n\in\mathbb{N}})$, and is won by II; which is supposed to be impossible. **X**

So in this case $\mu_*A \ge \mu C > 0$.

case 2 Suppose that II has a winning strategy τ. For each $n \in \mathbb{N}$ and $u \in \{0,1\}^n$, let $L(u)$ be the second component of $\tau(\langle(u(i), \emptyset)\rangle_{i<n})$; set $G = \bigcup_{n\in\mathbb{N}} \bigcup_{u\in\{0,1\}^n} L(u)$, so that $\mu G \le \sum_{n=0}^{\infty} 2^n \epsilon_n < \mu Z$. If we take any $y \in B$, then we have a play $\langle(k_n, K_n)\rangle_{n\in\mathbb{N}}$ of Game($\{0,1\} \times \mathcal{E}, D$), consistent with τ, in which $k_{2n} = y(n)$ and $K_{2n} = \emptyset$ for each n. Since II wins this play, $f(y)$ must belong to

$$\textstyle\bigcup_{n\in\mathbb{N}} K_{2n+1} = \bigcup_{n\in\mathbb{N}} L(y{\restriction}n) \subseteq G.$$

As y is arbitrary, $A \subseteq G$ and $\mu_*(Z \setminus A) \ge \mu(Z \setminus G) > 0$. **Q**

(ii) Write $\mathcal{K}$ for the family of compact sets $K \subseteq \{0,1\}^{\mathbb{N}}$ such that $A \cap K$ is Borel. If $E \subseteq \{0,1\}^{\mathbb{N}}$ and $\mu E > 0$, there is a $K \in \mathcal{K}$ such that $K \subseteq E$ and $\mu K > 0$. **P** There is a closed $Z \subseteq E$ such that $\mu Z > 0$ (563Fd again). By (i), at least one of $\mu_*(Z \cap A)$, $\mu_*(Z \setminus A)$ is non-zero, and there is a compact set K of non-zero measure which is included in one of $Z \cap A$, $Z \setminus A$. But now $K \in \mathcal{K}$. **Q**

Now (because we have countable choice for subsets of $\mathcal{K}$) there is a sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$ such that $\sup_{n\in\mathbb{N}} \mu K_n = \sup_{K\in\mathcal{K}} \mu K$; setting $E = \{0,1\}^{\mathbb{N}} \setminus \bigcup_{n\in\mathbb{N}} K_n$, E must be negligible, while $A \setminus E$ is a Borel set; so A is measured by $\hat{\mu}$. As μ is arbitrary, A is universally measurable.

(b)(i) If $V \in \mathcal{E} \setminus \{\emptyset\}$ then either $V \cap A$ is meager or there is a $V' \in \mathcal{E} \setminus \{\emptyset\}$ such that $V' \subseteq V$ and $V' \setminus A$ is meager. **P** Set $\mathcal{U} = \{E : E \in \mathcal{E} \setminus \{\emptyset\},\ E \subseteq V\}$ and let $\preccurlyeq$ be a well-ordering of $\mathcal{U}$ (in order type ω, if you like). Consider the game on $\{0,1\} \times \mathcal{U}$ in which the players choose $(k_0, U_0), (k_1, U_1), \dots$ such that, for each $n \in \mathbb{N}$,

$$k_n \in \{0,1\}, \quad U_n \in \mathcal{U}, \quad U_{n+1} \subseteq U_n.$$

I wins if $y = \langle k_{2n} \rangle_{n \in \mathbb{N}}$ belongs to B and $f(y) \in \bigcap_{n \in \mathbb{N}} U_n$. Because the U_n are all open-and-closed, this game is determined for the same reasons as the game of (a).

case 1 Suppose that I has a winning strategy σ; say that $\sigma_2(w) \in \mathcal{U}$ is the second component of $\sigma(w)$ for each $w \in \bigcup_{n \in \mathbb{N}}(\{0,1\} \times \mathcal{U})^n$. For each $n \in \mathbb{N}$ let $\mathcal{U}_n$ be the set of those $U \in \mathcal{U}$ such that $v \restriction n = v' \restriction n$ for all v, $v' \in U$. Let $(k', V') = \sigma(\emptyset)$ be I's first move when following σ. Let Q be the set of positions in the game consistent with σ and with II to move, that is, finite sequences

$$q = \langle (k_i, U_i) \rangle_{i \le 2n} \in (\{0,1\} \times \mathcal{U})^{2n+1}$$

such that $(k_{2m}, U_{2m}) = \sigma(\langle (k_{2i+1}, U_{2i+1}) \rangle_{i<m})$ for every $m \le n$ and $\langle U_i \rangle_{i \le 2n}$ is non-increasing. For such a q, set $V_q = U_{2n}$ and

$$W_q = \bigcup \{\sigma_2(\langle (k_{2i+1}, U_{2i+1}) \rangle_{i<n} {}^\frown <(k,U)>) : k \in \{0,1\}, U \in \mathcal{U}_n, U \subseteq V_q\}.$$

Then W_q is an open subset of V_q; but also it is dense in V_q, because if $W \subseteq V_q$ is open and not empty there is a $U \in \mathcal{U}_n$ included in W and $\sigma_2(\langle (k_{2i+1}, U_{2i+1}) \rangle_{i<n} {}^\frown <(k,U)>)$ is a non-empty subset of U. Q is countable, so $E = \bigcap_{q \in Q} W_q \cup (\{0,1\}^{\mathbb{N}} \setminus \overline{V}_q)$ is comeager in $\{0,1\}^{\mathbb{N}}$.

? If $V' \setminus A$ is not meager, there is an $x \in E \cap V' \setminus A$. Define $\langle (k_n, U_n) \rangle_{n \in \mathbb{N}}$ inductively, as follows. $(k_0, U_0) = (k', V')$. Given that $q = \langle (k_i, U_i) \rangle_{i \le 2n}$ belongs to Q and $x \in V_q$, then $x \in W_q$ so there are $k \in \{0,1\}$, $U \in \mathcal{U}_n$ such that $x \in \sigma_2(\langle (k_{2i+1}, U_{2i+1}) \rangle_{i<n} {}^\frown <(k,U)>)$; take the lexicographically first such pair (k, U) for (k_{2n+1}, U_{2n+1}), and set $(k_{2n+2}, U_{2n+2}) = \sigma(\langle (k_{2i+1}, U_{2i+1}) \rangle_{i \le n})$. Then $q' = \langle (k_i, U_i) \rangle_{i \le 2n+2}$ belongs to Q and $V_{q'} = U_{2n+2} = \sigma_2(\langle (k_{2i+1}, U_{2i+1}) \rangle_{i \le n})$ contains x, so the induction can continue.

At the end of this induction, $\langle (k_n, U_n) \rangle_{n \in \mathbb{N}}$ will be a play of the game consistent with σ in which the only point of $\bigcap_{n \in \mathbb{N}} U_n$ is x and does not belong to A. So either $y = \langle k_{2n} \rangle_{n \in \mathbb{N}}$ does not belong to B or $f(y) \notin \bigcap_{n \in \mathbb{N}} U_n$; in either case, II wins the play; which is supposed to be impossible. **X**

So in this case $V' \setminus A$ is meager.

case 2 Suppose that II has a winning strategy τ; say that $\tau_2(w)$ is the second component of $\tau(w)$ for each $w \in \bigcup_{n \ge 1}(\{0,1\} \times \mathcal{U})^n$. Let Q be the set of objects

$$q = (\langle (k_i, U_i) \rangle_{i<2n}, k)$$

such that $\langle (k_i, U_i) \rangle_{i<2n}$ is a finite sequence in $\{0,1\} \times \mathcal{U}$ consistent with τ (allowing the empty string when $n = 0$) and $k \in \{0,1\}$. For such a q, set $V_q = U_{2n-1}$ (if $n > 0$) or $V_q = V$ (if $n = 0$); set

$$W_q = \bigcup \{\tau_2(\langle (k_{2i}, U_{2i}) \rangle_{i<n} {}^\frown <(k,U)>) : U \in \mathcal{U}, U \subseteq V_q\},$$

so that W_q is a dense subset of V_q. Q is countable, so $E = \bigcap_{q \in Q} W_q \cup (\{0,1\}^{\mathbb{N}} \setminus \overline{V}_q)$ is comeager.

? If there is an x in $A \cap V \cap E$, let $y \in B$ be such that $f(y) = x$, and define $\langle (k_n, U_n) \rangle_{n \in \mathbb{N}}$ as follows. Given that $q = (\langle (k_i, U_i) \rangle_{i<2n}, y(n))$ belongs to Q and $x \in V_q$, then $x \in W_q$ so there is a $U \in \mathcal{U}$ such that $x \in \tau_2(\langle (k_{2i}, U_{2i}) \rangle_{i<n} {}^\frown <(y(n), U)>)$; take the $\preccurlyeq$-first such U for U_{2n}, set $k_{2n} = y(n)$ and $(k_{2n+1}, U_{2n+1}) = \tau(\langle (k_{2i}, U_{2i}) \rangle_{i \le n})$, so that $q' = (\langle (k_i, U_i) \rangle_{i \le 2n+1}, y(n+1))$ belongs to Q and $V_{q'} = U_{2n+1} = \tau_2(\langle (k_{2i}, U_{2i}) \rangle_{i \le n})$ contains x.

At the end of this induction, $\langle (k_n, U_n) \rangle_{n \in \mathbb{N}}$ will be a play of the game consistent with τ in which $f(\langle k_{2n} \rangle_{n \in \mathbb{N}}) = x \in \bigcap_{n \in \mathbb{N}} U_n$, so that I wins, which is supposed to be impossible. **X**

Thus in this case $A \cap V$ must be meager. **Q**

(ii) Now let G be the union of those $V \in \mathcal{E}$ such that $V \setminus A$ is meager; then $G \setminus A$ is meager. (This is where we need AC($\mathbb{R}$; ω).) If $V \in \mathcal{E}$ and $V \subseteq \{0,1\}^{\mathbb{N}} \setminus G$, then $V' \setminus A$ is non-meager for every whenever $V \in \mathcal{E} \setminus \{\emptyset\}$ and $V' \subseteq V$, so $V \cap A$ is meager; accordingly $G' \cap A$ is meager, where $G' = \{0,1\}^{\mathbb{N}} \setminus \overline{G}$. But this means that $G \triangle A \subseteq (G \setminus A) \cup (G' \cap A) \cup (\overline{G} \setminus G)$ is meager and A has the Baire property.

567G Theorem [AD] In any Hausdorff second-countable space, every subset is universally measurable and has the Baire property.

proof Let X be a Hausdorff second-countable space, $\langle U_n \rangle_{n \in \mathbb{N}}$ a sequence running over a base for the topology of X, and $A \subseteq X$.

(a) Define $g : X \to \{0,1\}^{\mathbb{N}}$ by setting $g(x) = \langle \chi U_n(x) \rangle_{n\in\mathbb{N}}$ for $x \in X$; then g is injective and Borel measurable. If μ is a Borel probability measure on X, we have a Borel probability measure $\nu = \mu g^{-1} \restriction \mathcal{B}(\{0,1\}^{\mathbb{N}})$ on $\{0,1\}^{\mathbb{N}}$. By 567Fa, $g[A]$ is measured by the completion $\hat{\nu}$ of ν; let F, $H \subseteq \{0,1\}^{\mathbb{N}}$ be Borel sets such that $\nu H = 0$ and $g[A] \triangle F \subseteq H$; then $A \triangle g^{-1}[F] \subseteq g^{-1}[H]$ is μ-negligible, so A is measured by $\hat{\mu}$. As μ is arbitrary, A is universally measurable.

(b) Set $G = \bigcup\{U_n : n \in \mathbb{N}, U_n \cap A$ has the Baire property$\}$, so that $G \cap A$ has the Baire property. (Remember that as we have a bijection between X and a subset of $\mathbb{R}$, we have countable choice for subsets of X, so that the ideal of meager subsets of X is a σ-ideal and the Baire-property algebra is a σ-algebra.) Set $V = X \setminus (\bigcup_{n\in\mathbb{N}} \partial U_n \cup \overline{G})$; then $G \cup V$ is comeager in X, and $A \setminus V$ has the Baire property. If V is empty, we can stop. Otherwise, let $\mathcal{V}$ be the countable algebra of subsets of V generated by $\{V \cap U_n : n \in \mathbb{N}\}$. Since $A \cap U$ does not have the Baire property (in X) for any non-empty relatively open subset U of V, V has no isolated points and $\mathcal{V}$ is atomless. So $\mathcal{V}$ is isomorphic to the algebra of open-and-closed subsets of $\{0,1\}^{\mathbb{N}}$ (316M) and there is a Boolean-independent sequence $\langle V_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{V}$ generating $\mathcal{V}$. Define $h : V \to \{0,1\}^{\mathbb{N}}$ by setting $h(x) = \langle \chi V_n(x) \rangle_{n\in\mathbb{N}}$ for $x \in V$. Then $h[V]$ is dense in $\{0,1\}^{\mathbb{N}}$ and $h^{-1}[H]$ is dense in V for every dense open set $H \subseteq \{0,1\}^{\mathbb{N}}$; consequently $h^{-1}[M]$ is meager in V and in X whenever $M \subseteq \{0,1\}^{\mathbb{N}}$ is either nowhere dense or meager. By 567Fb, $h[A]$ has the Baire property in $\{0,1\}^{\mathbb{N}}$; express it as $H \triangle M$ where H is open and M is meager; then $A \cap V = h^{-1}[h[A]] = h^{-1}[H] \triangle h^{-1}[M]$ has the Baire property in X, so A has the Baire property in X, as required.

567H Theorem (a) [AD] Let X be a Polish group and Y a topological group which is either separable or Lindelöf. Then every group homomorphism from X to Y is continuous.

(b) [AD+AC(ω)] Let X be an abelian topological group which is complete under a metric defining its topology, and Y a topological group which is either separable or Lindelöf. Then every group homomorphism from X to Y is continuous.

(c) [AD+AC(ω)] Let X be a complete metrizable linear topological space, Y a linear topological space and $T : X \to Y$ a linear operator. Then T is continuous. In particular, every linear operator between Banach spaces is a bounded operator.

proof (a)(i) Let $f : X \to Y$ be a homomorphism, and V a neighbourhood of the identity in Y. Let W be an open neighbourhood of the identity in Y such that $W^{-1}W \subseteq V$. Then there is countable family $\mathcal{H}$ of left translates of W which covers Y. **P** If Y is separable, let D be a countable dense subset of Y, and set $\mathcal{H} = \{yW : y \in D\}$. If Y is Lindelöf, we have only to note that $\{yW : y \in Y\}$ is an open cover of Y, so has a countable subcover. **Q**

(ii) Since X is a Baire space (561Ea), and the ideal of meager subsets of X is a σ-ideal (see part (b) of the proof of 567G), and $\{f^{-1}[H] : H \in \mathcal{H}\}$ is a countable cover of X, there is an $H \in \mathcal{H}$ such that $E = f^{-1}[H]$ is non-meager. Now $E^{-1}E$ is a neighbourhood of the identity in X. **P** By 567G, E has the Baire property; let G be a non-empty open set in X such that $G \setminus E$ is meager. Set $U = \{x : Gx \cap G \neq \emptyset\}$; then U is a neighbourhood of the identity in X. If $x \in U$, then

$$Gx \cap G \subseteq (Ex \cap E) \cup (Gx \setminus Ex) \cup (G \setminus E) = (Ex \cap E) \cup (G \setminus E)x \cup (G \setminus E).$$

Since $Gx \cap G$ is non-meager, while $G \setminus E$ and $(G \setminus E)x$ are meager, $Ex \cap E \neq \emptyset$ and $x \in E^{-1}E$. Thus $E^{-1}E \supseteq U$ is a neighbourhood of the identity. **Q**

(iii) Let $y \in Y$ be such that $H = yW$. If x, $z \in E$, $y^{-1}f(x)$ and $y^{-1}f(z)$ both belong to W, so

$$f(x^{-1}z) = f(x)^{-1}f(z) \in W^{-1}yy^{-1}W = W^{-1}W \subseteq V.$$

Thus $f^{-1}[V] \supseteq E^{-1}E$ is a neighbourhood of the identity in X. As V is arbitrary, f is continuous at the identity, therefore continuous.

(b) ? Otherwise, there is a neighbourhood V of the identity e_Y of Y such that $f^{-1}[V]$ is not a neighbourhood of the identity e_X of X. Let ρ be a metric on X, defining its topology, under which X is complete. Then for each $n \in \mathbb{N}$ we can choose an $x_n \in X$ such that $\rho(x_n, e_X) \leq 2^{-n}$ and $f(x_n) \notin V$. (This is where we need AC(ω).) For finite $J \subseteq \mathbb{N}$ set $u_J = \prod_{n\in J} x_n$, starting from $u_\emptyset = e_X$. We can define an infinite $I \subseteq \mathbb{N}$ inductively by saying that

$$I = \{n : \text{whenever } J \subseteq I \cap n \text{ then } \rho(u_J, u_J x_n) \leq 2^{-\#(I\cap n)}\}.$$

This will ensure that $v_K = \lim_{n\to\infty} u_{K\cap n}$ is defined for every $K \subseteq I$. Note that $v_{K\cup\{m\}} = v_K x_m$ whenever $m \in I$ and $K \subseteq I \setminus \{m\}$ (this is where we need to know that X is abelian).

Give $\mathcal{P}I$ its usual topology. Let W be a neighbourhood of e_Y such that $W^{-1}W \subseteq V$. By the argument of (a) above, applied to the map $K \mapsto f(v_K) : \mathcal{P}I \to Y$, there is a $y \in Y$ such that $E = \{K : K \subseteq I, f(v_K) \in yW\}$ is non-meager in $\mathcal{P}I$. Looking at the topological group $(\mathcal{P}I, \triangle)$, we see that there is a neighbourhood U of $\emptyset$ in $\mathcal{P}I$ included in $\{K\triangle L : K, L \in E\}$. Taking any sufficiently large $n \in I$, we have $\{n\} \in U$, so there must be a $K \in E$ such that $n \notin K$ and $K \cup \{n\} \in E$. In this case $f(v_K) \in yW$, $f(v_{K\cup\{n\}}) \in yW$ and

$$f(x_n) = f(v_K^{-1} v_{K\cup\{n\}}) = f(v_K)^{-1} f(v_{K\cup\{n\}}) \in W^{-1}W \subseteq V,$$

which is impossible. **X**

(c) ? Otherwise, there is a neighbourhood V of 0 in Y such that $T^{-1}[V]$ is not a neighbourhood of 0 in X; we can suppose that $\alpha y \in V$ whenever $y \in V$ and $|\alpha| \le 1$. Let ρ be a metric on X, defining its topology, under which X is complete. Let W be a neighbourhood of 0 in Y such that $W - W \in V$. Then for each $n \in \mathbb{N}$ we can choose an $x_n \in X \setminus nT^{-1}[V]$ such that $\rho(x_n, 0) \le 2^{-n}$. Define $I \in [\mathbb{N}]^\omega$ and $\langle v_K\rangle_{K\subseteq I}$ as in (b), but using additive notation rather than multiplicative. This time we are not supposing that Y is separable. However, there must be an $m \in \mathbb{N}$ such that $E = \{K : T(v_K) \in mW\}$ is non-meager. As before, we can find $n \in I \setminus m$ and $K \in E$ such that $n \notin L$ and $K \cup \{n\} \in E$. So the calculation gives

$$Tx_n = Tv_{K\cup\{n\}} - Tv_K \in mW - mW \subseteq mV \subseteq nV,$$

again contrary to the choice of x_n. **X**

567I Proposition [AC($\mathbb{R};\omega$)] Let $\widehat{\mathcal{B}}$ be the Baire-property algebra of $\mathcal{P}\mathbb{N}$. Then every $\widehat{\mathcal{B}}$-measurable real-valued additive functional on $\mathcal{P}\mathbb{N}$ is of the form $a \mapsto \sum_{n\in a} \gamma_n$ for some $\langle\gamma_n\rangle_{n\in\mathbb{N}} \in \ell^1$.

proof As noted in 567Ec, $\widehat{\mathcal{B}}$ is a σ-algebra of subsets of $\mathcal{P}\mathbb{N}$.

(a)(i) If $G \subseteq \mathcal{P}\mathbb{N}$ is a dense open set and $m \in \mathbb{N}$, there are an $m' > m$ and an $L \subseteq m' \setminus m$ such that $\{a : a \subseteq \mathbb{N}, a \cap m' \setminus m = L\} \subseteq G$. **P** The set $H = \{b : b \subseteq \mathbb{N} \setminus m, I \cup b \in G$ for every $I \subseteq m\}$ is a dense open subset of $\mathcal{P}(\mathbb{N} \setminus m)$, so there are an $m' > m$ and an $L \subseteq m' \setminus m$ such that $H \supseteq \{b : b \subseteq \mathbb{N} \setminus m, b \cap m' \setminus m = L\}$; this pair m', L works. **Q**

(ii) If $G \subseteq \mathcal{P}\mathbb{N}$ is comeager, there are a strictly increasing sequence $\langle m_n\rangle_{n\in\mathbb{N}}$ in $\mathbb{N}$ and sets $L_n \subseteq m_{n+1} \setminus m_n$, for $n \in \mathbb{N}$, such that

$$G \supseteq \{a : a \subseteq \mathbb{N}, a \cap m_{n+1} \setminus m_n = L_n \text{ for infinitely many } n\}.$$

P Let $\langle G_n\rangle_{n\in\mathbb{N}}$ be a non-increasing sequence of dense open sets such that $G \supseteq \bigcap_{n\in\mathbb{N}} G_n$, and choose $\langle m_n\rangle_{n\in\mathbb{N}}$, $\langle L_n\rangle_{n\in\mathbb{N}}$ inductively such that $m_n < m_{n+1}$, $L_n \subseteq m_{n+1} \setminus m_n$ and $\{a : a \subseteq \mathbb{N}, a \cap m_{n+1} \setminus m_n = L_n\} \subseteq G_n$ for every n. **Q**

(iii) If $G \subseteq \mathcal{P}\mathbb{N}$ is comeager, and $a \subseteq \mathbb{N}$, then there are b_0, b_0', b_1, $b_1' \in G$ such that

$$b_0 \subseteq b_0', \quad b_1 \subseteq b_1', \quad (b_0' \setminus b_0) \cap (b_1' \setminus b_1) = \emptyset, \quad (b_0' \setminus b_0) \cup (b_1' \setminus b_1) = a.$$

P Let $\langle m_n\rangle_{n\in\mathbb{N}}$ and $\langle L_n\rangle_{n\in\mathbb{N}}$ be as in (ii). Set

$$b_0 = \bigcup_{n\in\mathbb{N}} L_{2n}, \quad b_0' = b_0 \cup (a \cap m_0) \cup \bigcup_{n\in\mathbb{N}} a \cap m_{2n+2} \setminus m_{2n+1},$$

$$b_1 = \bigcup_{n\in\mathbb{N}} L_{2n+1}, \quad b_1' = b_1 \cup \bigcup_{n\in\mathbb{N}} a \cap m_{2n+1} \setminus m_{2n}. \textbf{ Q}$$

So if $\nu : \mathcal{P}\mathbb{N} \to \mathbb{R}$ is additive, $\sup_{a\subseteq\mathbb{N}} |\nu a| \le 4 \sup_{b\in G} |\nu b|$.

(iv) If $G \subseteq \mathcal{P}\mathbb{N}$ is comeager, there is a disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in G. **P** Take $\langle m_n\rangle_{n\in\mathbb{N}}$ and $\langle L_n\rangle_{n\in\mathbb{N}}$ as in (ii), and set $a_n = \bigcup_{i\in\mathbb{N}} L_{2^n(2i+1)}$ for each n. **Q**

(b) If $\nu : \mathcal{P}\mathbb{N} \to \mathbb{R}$ is additive and $\widehat{\mathcal{B}}$-measurable, it is bounded. **P** Let $M \in \mathbb{N}$ be such that $E = \{a : |\nu a| \le M\}$ is non-meager. Then there are an $m \in \mathbb{N}$ and $J \subseteq m$ such that $V_{mJ} \setminus E$ is meager, where $V_{mJ} = \{a : a \cap m = J\}$. For $K \subseteq m$, $a \subseteq \mathbb{N}$ set $\phi_K(a) = a\triangle K$; then ϕ_K is an autohomeomorphism of $\mathcal{P}\mathbb{N}$, so $\phi_K[V_{mJ} \setminus E]$ is meager. Let G be the comeager set $\mathcal{P}\mathbb{N} \setminus \bigcup_{K\subseteq m} \phi_K[V_{mJ} \setminus E]$. Set $\delta = \sum_{i<m} |\nu\{i\}|$;

then $|\nu\phi_K(a) - \nu a| \le \delta$ whenever $K \subseteq m$ and $a \subseteq \mathbb{N}$. If $b \in G$, set $K = (b \cap m)\triangle J$; then $\phi_K(b) \in V_{mJ} \setminus (V_{mJ} \setminus E) \subseteq E$, so $|\nu b| \le M + \delta$. So (a-iii) tells us that $|\nu a| \le 4(M+\delta)$ for every $a \subseteq \mathbb{N}$, and ν is bounded. **Q**

(c) If $\nu : \mathcal{P}\mathbb{N} \to \mathbb{R}$ is additive and $\widehat{\mathcal{B}}$-measurable and $\nu\{n\} = 0$ for every $n \in \mathbb{N}$, then $E = \{a : \nu a \ge \epsilon\}$ is meager for every $\epsilon > 0$. **P?** Otherwise, let $m \in \mathbb{N}$ and $J \subseteq m$ be such that $V_{mJ} \setminus E$ is meager. Let G be the comeager set $\mathcal{P}\mathbb{N} \setminus \bigcup_{K\subseteq m} \phi_K[V_{mJ} \setminus E]$, as in (b). This time, $\nu a = \nu\phi_K(a)$ whenever $K \subseteq m$ and $a \subseteq \mathbb{N}$, so $\nu a \ge \epsilon$ for every $a \in G$. But (a-iv) tells us that there is a disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in G, and now $\sup_{n\in\mathbb{N}} \nu(\bigcup_{i\le n} a_i) = \infty$, contradicting (b). **XQ**

(d) If $\nu : \mathcal{P}\mathbb{N} \to \mathbb{R}$ is additive and $\widehat{\mathcal{B}}$-measurable and $\nu\{n\} = 0$ for every $n \in \mathbb{N}$, then $\nu = 0$. **P** By (c), applied to ν and $-\nu$, $G = \{a : \nu a = 0\}$ is comeager. By (a-iii), ν must be identically zero. **Q**

(e) Now suppose that ν is any additive $\widehat{\mathcal{B}}$-measurable functional. Set $\gamma_n = \nu\{n\}$ for each n. By (b), $\langle\gamma_n\rangle_{n\in\mathbb{N}} \in \ell^1$. Setting $\nu' a = \nu a - \sum_{n\in a}\gamma_n$ for $a \subseteq \mathbb{N}$, ν' is still additive and $\widehat{\mathcal{B}}$-measurable, and $\nu'\{n\} = 0$ for every n, so (d) tells us that $\nu' = 0$ and $\nu a = \sum_{n\in a}\gamma_n$ for every a, as required.

567J Proposition [AD] A finitely additive functional on a Dedekind σ-complete Boolean algebra is countably additive.

proof Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, ν a finitely additive functional on $\mathfrak{A}$ and $\langle a_n\rangle_{n\in\mathbb{N}}$ a disjoint sequence in $\mathfrak{A}$ with supremum a. Set $\lambda c = \nu(\sup_{n\in c} a_n)$ for $c \subseteq \mathbb{N}$. Then λ is an additive functional on $\mathcal{P}\mathbb{N}$. By 567G, it is $\widehat{\mathcal{B}}(\mathcal{P}\mathbb{N})$-measurable; by 567I,

$$\nu a = \lambda\mathbb{N} = \sum_{n=0}^{\infty} \lambda\{n\} = \sum_{n=0}^{\infty} \nu a_n.$$

567K Theorem [AD+AC(ω)] If U is an L-space with a weak order unit, it is reflexive.

proof By 561Hb, U is isomorphic to $L^1(\mathfrak{A},\bar\mu)$ for some totally finite measure algebra $(\mathfrak{A},\bar\mu)$; now U^* can be identified with $L^\infty(\mathfrak{A})$. Next, $L^\infty(\mathfrak{A})^*$ can be identified with the space of bounded finitely additive functionals on $\mathfrak{A}$, as in 363K; by 567J, these are all countably additive. Because we have countable choice, $\mathfrak{A}$ is ccc (566M), so countably additive functionals are completely additive and correspond to members of L^1, as in 365Ea. Thus the canonical embedding of U in U^{**} is surjective.

567L Theorem (R.M.Solovay) [AD] ω_1 is two-valued-measurable.

Remark The definition in 541M speaks of 'regular uncountable cardinals'. In the present context I will use the formulation 'an initial ordinal κ is two-valued-measurable if there is a proper κ-additive 2-saturated ideal $\mathcal{I}$ of $\mathcal{P}\kappa$ containing singletons', where here 'κ-additive' means that $\bigcup_{\eta<\xi} J_\eta \in \mathcal{I}$ whenever $\xi < \kappa$ and $\langle J_\eta\rangle_{\eta<\xi}$ is a family in $\mathcal{I}$.

proof (a) Let $\mathrm{Str_I}$ be the set of strategies for player I in games of the form $\Gamma(\mathbb{N},.)$, that is, $\mathrm{Str_I}$ is the set of functions from $\bigcup_{n\in\mathbb{N}} \mathbb{N}^n$ to $\mathbb{N}$; for $\sigma \in \mathrm{Str_I}$ and $x \in \mathbb{N}^\mathbb{N}$, let $\sigma * x \in \mathbb{N}^\mathbb{N}$ be the play in which I follows the strategy σ and II plays the sequence x, that is,

$$(\sigma * x)(2n) = \sigma(x\restriction n), \quad (\sigma * x)(2n+1) = x(n)$$

for $n \in \mathbb{N}$. Similarly, let $\mathrm{Str_{II}}$ be the set of functions from $\bigcup_{n\ge1} \mathbb{N}^n$ to $\mathbb{N}$ and for $\tau \in \mathrm{Str_{II}}$, $x \in \mathbb{N}^\mathbb{N}$, $n \in \mathbb{N}$ set

$$(\tau * x)(2n) = x(n), \quad (\tau * x)(2n+1) = \tau(x\restriction(n+1)).$$

We can find bijections $g : \mathbb{N}^\mathbb{N} \to \mathrm{Str_I} \cup \mathrm{Str_{II}}$ and $h : \mathbb{N}^\mathbb{N} \to \mathrm{WO}(\mathbb{N})$, where $\mathrm{WO}(\mathbb{N}) \subseteq \mathcal{P}(\mathbb{N}^2)$ is the set of well-orderings of $\mathbb{N}$. **P** Since $S = \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$ and $S^* = \bigcup_{n\ge1} \mathbb{N}^n$ are countably infinite, $\mathrm{Str_I} = \mathbb{N}^S$ and $\mathrm{Str_{II}} = \mathbb{N}^{S^*}$ are equipollent with $\mathbb{N}^\mathbb{N}$. As $\mathcal{P}(\mathbb{N}^2) \sim \mathcal{P}\mathbb{N} \sim \mathbb{N}^\mathbb{N}$, there is an injection from $\mathrm{WO}(\mathbb{N})$ to $\mathbb{N}^\mathbb{N}$. In the reverse direction, there are an injection from $\mathbb{N}^\mathbb{N}$ to the set F of permutations of $\mathbb{N}$, and an injection from F to $\mathrm{WO}(\mathbb{N})$; so the Schroeder-Bernstein theorem tells us that $\mathrm{WO}(\mathbb{N}) \sim \mathbb{N}^\mathbb{N}$. **Q**

Define $f : \mathrm{WO}(\mathbb{N}) \to \omega_1$ by saying that $f(\preccurlyeq) = \mathrm{otp}(\mathbb{N},\preccurlyeq)$ for $\preccurlyeq \in \mathrm{WO}(\mathbb{N})$.

(b) For $x \in \mathbb{N}^\mathbb{N}$ let $L_x \subseteq \mathbb{N}^\mathbb{N}$ be the smallest set such that

$x \in L_x$,
whenever y, $z \in L_x$ then $g(y) * z \in L_x$,
whenever $y \in L_x$ then $\langle y(2n)\rangle_{n\in\mathbb{N}}$ and $\langle y(2^k(2n+1))\rangle_{n\in\mathbb{N}}$ belong to L_x for every $k \in \mathbb{N}$.

Observe that L_x is countable and that $L_y \subseteq L_x$ whenever $y \in L_x$. For $x \in \mathbb{N}^{\mathbb{N}}$, set $C_x = \{y : y \in \mathbb{N}^{\mathbb{N}}, x \in L_y\}$.

(c) For any sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in $\mathbb{N}^{\mathbb{N}}$ there is an $x \in \mathbb{N}^{\mathbb{N}}$ such that $C_x \subseteq \bigcap_{n\in\mathbb{N}} C_{x_n}$. **P** Set $x(0) = 0$ and $x(2^k(2n+1)) = x_k(n)$ for k, $n \in \mathbb{N}$. Then $x_k \in L_x$ for every k. So if $y \in C_x$ and $n \in \mathbb{N}$, we have $x_n \in L_x \subseteq L_y$ and $y \in C_{x_n}$. **Q**

Let $\mathcal{F}$ be the filter on $\mathbb{N}^{\mathbb{N}}$ generated by $\{C_x : x \in \mathbb{N}^{\mathbb{N}}\}$; then (because $\mathrm{AC}(\mathbb{R};\omega)$ is true) $\mathcal{F}$ is closed under countable intersections.

(d) Suppose that $A \subseteq \mathbb{N}^{\mathbb{N}}$ is such that whenever $x \in A$ and $L_y = L_x$ then $y \in A$.

(i) If I has a winning strategy in $\mathrm{Game}(\mathbb{N}, A)$ then $A \in \mathcal{F}$. **P** Let $\sigma \in \mathrm{Str}_{\mathrm{I}}$ be a winning strategy for I, and consider $x = g^{-1}(\sigma) \in \mathbb{N}^{\mathbb{N}}$. Suppose that $y \in \mathbb{N}^{\mathbb{N}}$ and $x \in L_y$, and consider $z = \sigma * y \in A$. As $z = g(x) * y$ belongs to L_y, $L_z \subseteq L_y$; on the other hand, $y(n) = z(2n+1)$ for every n, so $y \in L_z$ and $L_y \subseteq L_z$. So $L_y = L_z$ and $y \in A$. As y is arbitrary, $C_x \subseteq A$ and $A \in \mathcal{F}$. **Q**

(ii) If II has a winning strategy in $\mathrm{Game}(\mathbb{N}, A)$ then $\mathbb{N}^{\mathbb{N}} \setminus A \in \mathcal{F}$. **P** Let $\tau \in \mathrm{Str}_{\mathrm{II}}$ be a winning strategy for II, and consider $x = g^{-1}(\tau) \in \mathbb{N}^{\mathbb{N}}$. Suppose that $y \in \mathbb{N}^{\mathbb{N}}$ and $x \in L_y$, and consider $z = \tau * y \in \mathbb{N}^{\mathbb{N}} \setminus A$. As before, $L_z \subseteq L_y$; this time, $y(n) = z(2n)$ for every n so $y \in L_z$ and $L_y \subseteq L_z$. So $y \notin A$. As y is arbitrary, $C_x \subseteq \mathbb{N}^{\mathbb{N}} \setminus A$ and $\mathbb{N}^{\mathbb{N}} \setminus A \in \mathcal{F}$. **Q**

(e) For $x \in \mathbb{N}^{\mathbb{N}}$ set $\phi(x) = \sup_{y\in L_x} f(h(y))$; because L_x is countable, $\phi(x) < \omega_1$ (567Ed). Let $\mathcal{G}$ be the image filter $\phi[[\mathcal{F}]]$. Because $\mathcal{F}$ is closed under countable intersections, so is $\mathcal{G}$. If $B \subseteq \omega_1$ then $\phi^{-1}[B]$ satisfies the condition of (d), so that one of $\phi^{-1}[B]$, $\mathbb{N}^{\mathbb{N}} \setminus \phi^{-1}[B]$ belongs to $\mathcal{F}$ and one of B, $\omega_1 \setminus B$ belongs to $\mathcal{G}$; as B is arbitrary, $\mathcal{G}$ is an ultrafilter.

(f) Finally, $\mathcal{G}$ does not contain any singletons. **P** If $\xi < \omega_1$, there is an $x \in \mathbb{N}^{\mathbb{N}}$ such that $f(h(x)) = \xi + 1$. Now $C_x \in \mathcal{F}$ so $\phi[C_x] \in \mathcal{G}$. If $y \in C_x$ then $x \in L_y$ so $\xi + 1 \le \phi(y)$; accordingly $\xi \notin \phi[C_x]$ and $\{\xi\} \notin \mathcal{G}$. **Q** So $\mathcal{G}$ (or, if you like, the ideal $\{\omega_1 \setminus B : B \in \mathcal{G}\}$) witnesses that ω_1 is two-valued-measurable.

567M Theorem (Moschovakis 70) [AD] Let α be an ordinal such that there is a surjection from $\mathcal{P}\mathbb{N}$ onto α. Then there is a surjection from $\mathcal{P}\mathbb{N}$ onto $\mathcal{P}\alpha$.

proof The formulae will run slightly more smoothly if we work with surjections from $\mathbb{N}^{\mathbb{N}}$ rather than from $\mathcal{P}\mathbb{N}$; of course this makes no difference to the result.

(a) We may suppose that α is uncountable. Let $f : \mathbb{N}^{\mathbb{N}} \to \alpha$ be a surjection. I seek to define inductively a family $\langle g_\xi\rangle_{\xi\le\alpha}$ such that g_ξ is a surjection from $\mathbb{N}^{\mathbb{N}}$ onto $\mathcal{P}\xi$ for every $\xi \le \alpha$. As in the proof of 567L, let $\mathrm{Str}_{\mathrm{I}}$ be the set of functions from $\bigcup_{n\in\mathbb{N}} \mathbb{N}^n$ to $\mathbb{N}$, and $\mathrm{Str}_{\mathrm{II}}$ the set of functions from $\bigcup_{n\ge1} \mathbb{N}^n$ to $\mathbb{N}$; fix a surjection $h : \mathbb{N}^{\mathbb{N}} \to \mathrm{Str}_{\mathrm{I}} \cup \mathrm{Str}_{\mathrm{II}}$. For $\sigma \in \mathrm{Str}_{\mathrm{I}}$, $\tau \in \mathrm{Str}_{\mathrm{II}}$ and $x \in \mathbb{N}^{\mathbb{N}}$ let $\sigma * x$, $\tau * x$ be the plays in games on $\mathbb{N}$ as described in the proof of 567L.

(b) Start by setting $g_n(x) = n \cap x[\mathbb{N}]$ for $x \in \mathbb{N}^{\mathbb{N}}$ and $n \in \mathbb{N}$.

(c) For the inductive step to a non-zero limit ordinal $\xi \le \alpha$, given $\langle g_\eta\rangle_{\eta<\xi}$, then for $x \in \mathbb{N}^{\mathbb{N}}$ set

$$\eta_x = f(\langle x(4n)\rangle_{n\in\mathbb{N}}), \quad \zeta_x = f(\langle x(4n+1)\rangle_{n\in\mathbb{N}}),$$

$$\begin{aligned} E_x &= g_{\eta_x}(\langle x(4n+2)\rangle_{n\in\mathbb{N}}) \text{ if } \eta_x < \xi, \\ &= \emptyset \text{ otherwise,} \\ F_x &= g_{\zeta_x}(\langle x(4n+3)\rangle_{n\in\mathbb{N}}) \text{ if } \zeta_x < \xi, \\ &= \emptyset \text{ otherwise.} \end{aligned}$$

Next, for $D \subseteq \xi$, set

$$A_D = \{x : x \in \mathbb{N}^{\mathbb{N}},\, \eta_x < \xi,\, E_x = D \cap \eta_x$$
$$\text{and either } \zeta_x \le \eta_x \text{ or } \zeta_x \ge \xi \text{ or } F_x \ne D \cap \zeta_x\}.$$

(The idea is that the players are competing to see who can capture the largest initial segment of D with the pair (η_x, E_x) determined by I's moves or the pair (ζ_x, F_x) determined by II's moves; for definiteness, if neither correctly defines an initial segment, then II wins, while if they seize the same segment $(\eta_x, E_x) = (\zeta_x, F_x)$, then I wins.) Finally, define $g : \mathrm{Str_I} \cup \mathrm{Str_{II}} \to \mathcal{P}\xi$ by setting

$$g(\sigma) = \bigcup\{D : \sigma \text{ is a winning strategy for I in Game}(\mathbb{N}, A_D)\} \text{ if } \sigma \in \mathrm{Str_I},$$

$$g(\tau) = \bigcup\{D : \tau \text{ is a winning strategy for II in Game}(\mathbb{N}, A_D)\} \text{ if } \tau \in \mathrm{Str_{II}}.$$

We find that g is a surjection onto $\mathcal{P}\xi$. **P** Take any $D \subseteq \xi$.

case 1 Suppose that I has a winning strategy σ in Game$(\mathbb{N}, A_D)$. Then $D \subseteq g(\sigma)$. **?** If $D \ne g(\sigma)$, there is a D', distinct from D, such that σ is a winning strategy for I in Game$(\mathbb{N}, D')$. Let $\zeta < \xi$ be such that $D \cap \zeta \ne D' \cap \zeta$. Then there is a $z \in \mathbb{N}^{\mathbb{N}}$ such that $f(\langle z(2n)\rangle_{n\in\mathbb{N}}) = \zeta$ and $g_\zeta(\langle z(2n+1)\rangle_{n\in\mathbb{N}}) = D \cap \zeta$. In this case, taking $x = \sigma * z$, we have $x(4n+1) = z(2n)$ and $x(4n+3) = z(2n+1)$ for every n, so $\zeta_x = \zeta$ and $F_x = D \cap \zeta$. Since $x \in A_D$, we have $\eta_x < \xi$, $E_x = D \cap \eta_x$ and $\zeta \le \eta_x$. But also $x \in A_{D'}$, so $E_x = D' \cap \eta_x$ and $D \cap \zeta = D' \cap \zeta$, contrary to the choice of ζ. **X** Thus $D = g(\sigma)$.

case 2 Suppose that II has a winning strategy τ in Game$(\mathbb{N}, A_D)$. Then $D \subseteq g(\tau)$. **?** If $D \ne g(\tau)$, there is a D', distinct from D, such that τ is a winning strategy for II in Game$(\mathbb{N}, D')$. Let $\zeta < \xi$ be such that $D \cap \zeta \ne D' \cap \zeta$. Again, there is a $z \in \mathbb{N}^{\mathbb{N}}$ such that $f(\langle z(2n)\rangle_{n\in\mathbb{N}}) = \zeta$ and $g_\zeta(\langle z(2n+1)\rangle_{n\in\mathbb{N}}) = D \cap \zeta$. This time, taking $x = \tau * z$, we have $x(4n) = z(2n)$ and $x(4n+2) = z(2n+1)$ for every n, so $\eta_x = \zeta$ and $E_x = D \cap \eta_x$. Since $x \notin A_D$, we must have $\eta_x < \zeta_x < \xi$ and $F_x = D \cap \zeta_x$; since also $x \notin A_{D'}$, $F_x = D' \cap \zeta_x$; so that $D \cap \zeta = F_x \cap \zeta = D' \cap \zeta$, which is impossible. **X** Thus $D = g(\tau)$.

Thus in either case $D \in g[\mathrm{Str_I} \cup \mathrm{Str_{II}}]$. As D is arbitrary, $g[\mathrm{Str_I} \cup \mathrm{Str_{II}}] = \mathcal{P}\xi$. **Q**

Setting $g_\xi = gh$, the induction proceeds.

(d) For the inductive step to $\xi + 1$ where $\omega \le \xi < \alpha$, set

$$h_\xi(0) = \xi,\ h_\xi(n) = n - 1 \text{ for } n \in \omega \setminus \{0\},\ h_\xi(\eta) = \eta \text{ if } \omega \le \eta < \xi,$$

$$g_{\xi+1}(x) = h_\xi[g_\xi(x)] \text{ for } x \in \mathbb{N}^{\mathbb{N}}.$$

(e) At the end of the induction, g_α is the required surjection onto $\mathcal{P}\alpha$.

567N Theorem (Martin 70) [AC] Assume that there is a two-valued-measurable cardinal. Then every coanalytic subset of $\mathbb{N}^{\mathbb{N}}$ is determined.

proof Let $A \subseteq \mathbb{N}^{\mathbb{N}}$ be a coanalytic set.

(a) Set $S^* = \bigcup_{n\ge1} \mathbb{N}^n$. For v, $v' \in S^*$ say that $v \preccurlyeq v'$ if either v extends v' or there is an $n < \min(\#(v), \#(v'))$ such that $v{\restriction}n = v'{\restriction}n$ and $v(n) < v'(n)$. Then $\preccurlyeq$ is a total order, and its restriction to $\mathbb{N}^n$ is the lexicographic well-ordering for each $n \ge 1$.

For $w \in \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$, set $I_w = \{x : w \subseteq x \in \mathbb{N}^{\mathbb{N}}\}$. Fix an enumeration $\langle v_i\rangle_{i\in\mathbb{N}}$ of S^* such that $\#(v_i) \le i + 1$ for every $i \in \mathbb{N}$.

(b) $A' = \mathbb{N}^{\mathbb{N}} \setminus A$ is Souslin-F (423Eb); express it as

$$A' = \bigcup_{y\in\mathbb{N}^{\mathbb{N}}} \bigcap_{n\ge1} F_{y\restriction n}$$

where F_v is closed for every $v \in S^*$. Replacing F_v by $\bigcap_{1\le i\le\#(v)} F_{v\restriction i}$ if necessary, we may suppose that $F_v \subseteq F_{v'}$ whenever $v \supseteq v'$, as in 421Cf[9].

For $x \in \mathbb{N}^{\mathbb{N}}$, set

$$T_x = \{v : v \in S^*,\, I_{x\restriction\#(v)} \cap F_v \ne 0\},$$

[9]Later editions only.

and define a relation $\preccurlyeq_x$ on S^* by saying that

$$\begin{aligned} \upsilon \preccurlyeq_x \upsilon' \iff & \text{ either } \upsilon,\ \upsilon' \in T_x \text{ and } \upsilon \preccurlyeq \upsilon' \\ & \text{ or } \upsilon \in T_x \text{ and } \upsilon' \notin T_x \\ & \text{ or } \upsilon,\ \upsilon' \notin T_x \text{ and } i \le j \text{ where } \upsilon = \upsilon_i,\ \upsilon' = \upsilon_j. \end{aligned}$$

Then $\preccurlyeq_x$ is a total ordering, since it copies the total ordering $\preccurlyeq$ on T_x and the well-ordering induced by the enumeration $\langle \upsilon_i \rangle_{i\in\mathbb{N}}$ on $S^* \setminus T_x$, and puts one below the other.

Note that if $n \in \mathbb{N}$, x, $y \in \mathbb{N}^{\mathbb{N}}$ are such that $x{\restriction}n = y{\restriction}n$, and $i < n$, then $x{\restriction}\#(\upsilon_i) = y{\restriction}\#(\upsilon_i)$, so $\upsilon_i \in T_x$ iff $\upsilon_i \in T_y$. Consequently, for i, $j < n$, $\upsilon_i \preccurlyeq_x \upsilon_j$ iff $\upsilon_i \preccurlyeq_y \upsilon_j$. It follows that for every $w \in \mathbb{N}^n$ we have a total ordering $\preccurlyeq'_w$ of n defined by saying that $i \preccurlyeq'_w j$ iff $\upsilon_i \preccurlyeq_x \upsilon_j$ whenever $x \in I_w$.

(c) If $x \in \mathbb{N}^{\mathbb{N}}$ and $\preccurlyeq_x$ is not a well-ordering, then $x \notin A$. **P** Let $D \subseteq S^*$ be a non-empty set with no $\preccurlyeq_x$-least member. Then $D \cap T_x$ is an initial segment of D. Since $S^* \setminus T_x$ is certainly well-ordered by $\preccurlyeq_x$, $D \cap T_x \ne \emptyset$. Define $\langle D_n \rangle_{n\in\mathbb{N}}$, $\langle y(n) \rangle_{n\in\mathbb{N}}$ as follows. $D_0 = D \cap T_x$. Given that D_n is a non-empty initial segment of D and that $\upsilon \supseteq y{\restriction}n$ for every $\upsilon \in D_n$, then $y{\restriction}n$ cannot be the least member of D, so $D_n \ne \{y{\restriction}n\}$; set $y(n) = \min\{\upsilon(n) : \upsilon \in D_n \setminus \{y{\restriction}n\}\}$,

$$D_{n+1} = \{\upsilon : \upsilon \in D_n \setminus \{y{\restriction}n\},\ \upsilon(n) = y(n)\}.$$

Because $\preccurlyeq_x$ agrees with $\preccurlyeq$ on T_x, D_{n+1} is a non-empty initial segment of D, and the induction continues.

If m, $n \in \mathbb{N}$, then there is an $\upsilon \in T_x$ such that $\upsilon \supseteq y{\restriction}\max(m,n)$, and

$$I_{x{\restriction}m} \cap F_{y{\restriction}n} \supseteq I_{x{\restriction}\#(\upsilon)} \cap F_\upsilon \ne \emptyset.$$

As m is arbitrary and $F_{y{\restriction}n}$ is closed, $x \in F_{y{\restriction}n}$; as n is arbitrary, $x \in A'$ and $x \notin A$. **Q**

(d) Let κ be a two-valued-measurable cardinal, and give $\mathbb{N} \times \kappa$ its discrete topology. In $(\mathbb{N} \times \kappa)^{\mathbb{N}}$ consider the set F of sequences $\langle (x(n), \xi(n)) \rangle_{n\in\mathbb{N}}$ such that

$$\text{whenever } i,\ j \in \mathbb{N},\ \upsilon_i \subset \upsilon_j \text{ and } x \in F_{\upsilon_j}, \text{ then } \xi(2j) < \xi(2i).$$

Then F is closed for the product topology; by 567B, F is determined.

(e) Suppose I has a winning strategy σ in the game $\mathrm{Game}(\mathbb{N} \times \kappa, F)$. Then I has a winning strategy in $\mathrm{Game}(\mathbb{N}, A)$. **P** For $\langle k_i \rangle_{i<n} \in \mathbb{N}^n$ take $\sigma'(\langle k_i \rangle_{i<n})$ to be the first component of $\sigma(\langle (k_i, 0) \rangle_{i<n})$. If x is any play of $\mathrm{Game}(\mathbb{N}, A)$ consistent with σ', then for each n set $\xi(2n+1) = 0$ and let $\xi(2n)$ be the second component of $\sigma(\langle (x(2i+1), 0) \rangle_{i<n})$. Then $\langle (x(n), \xi(n)) \rangle_{n\in\mathbb{N}}$ is a play of $\mathrm{Game}(\mathbb{N} \times \kappa, F)$ consistent with σ, so is won by I. **?** If $x \notin A$, let $y \in \mathbb{N}^{\mathbb{N}}$ be such that $x \in F_{y{\restriction}n}$ for every $n \in \mathbb{N}$. Set $I = \{i : i \in \mathbb{N},\ y \supseteq \upsilon_i\}$; then I is infinite, and there is an infinite $J \subseteq I$ such that $\upsilon_i \subset \upsilon_j$ whenever i, $j \in J$ and $i < j$, while $x \in F_{\upsilon_j}$ for every $j \in J$. But now we see that $\xi(2j) < \xi(2i)$ whenever $i < j$ in J, which is impossible. **X**

Thus $x \in A$; as x was arbitrary, σ' is a winning strategy for I in $\mathrm{Game}(\mathbb{N}, A)$. **Q**

(f) Suppose II has a winning strategy τ in $\mathrm{Game}(\mathbb{N}\times\kappa, F)$. Then II has a winning strategy in $\mathrm{Game}(\mathbb{N}, A)$. **P** Fix a normal κ-additive ultrafilter $\mathcal{F}$ on κ (541Ma). For $w = (k_0, \dots, k_{2n}) \in \mathbb{N}^{2n+1}$ consider the function $f_w : [\kappa]^{n+1} \to \mathbb{N}$ defined by saying that $f_w(J)$ is to be the first component of $\tau(\langle (k_{2i}, \xi_i) \rangle_{i\le n})$ where $(\xi_0, \dots, \xi_n)$ is that enumeration of J such that, for i, $j \le n$, $\xi_i \le \xi_j$ iff $i \preccurlyeq'_w j$. Then for each $m \in \mathbb{N}$ there is a $C_{wm} \in \mathcal{F}$ such that either $f_w(J) \le m$ for every $J \in [C_{wm}]^{n+1}$ or $f_w(J) > m$ for every $J \in [C_{wm}]^{n+1}$ (4A1L). Setting $C = \bigcap_{m,n\in\mathbb{N}} \bigcap_{w\in\mathbb{N}^{2n+1}} C_{wm}$, $C \in \mathcal{F}$ and every f_w is constant on $[C]^{n+1}$. Let $\rho(w)$ be the constant value of $f_w{\restriction}[C]^{n+1}$.

Define $\tau' : \bigcup_{n\ge1} \mathbb{N}^n \to \mathbb{N}$ inductively, saying that $\tau'(k_0, \dots, k_n) = \rho(w)$ whenever $w(2i) = k_i$ for $i \le n$ and $w(2i+1) = \tau'(k_0, \dots, k_i)$ for $i < n$. Suppose that x is a play of $\mathrm{Game}(\mathbb{N}, A)$ consistent with τ'. **?** If $x \in A$, then $\preccurlyeq_x$ is a well-ordering, by (c). The order type of $(S^*, \preccurlyeq_x)$ is countable, so is surely less than $\mathrm{otp}(C) = \kappa$, and we have a function $\theta : S^* \to C$ such that $\theta(\upsilon) \le \theta(\upsilon')$ iff $\upsilon \preccurlyeq_x \upsilon'$.

Define $\langle \xi(n) \rangle_{n\in\mathbb{N}}$ by saying that

$$\begin{aligned} \xi(n) &= \theta(\upsilon_j) \text{ if } n = 2j \text{ is even },\\ &= \text{ the second component of } \tau((x(0), \xi(0)), (x(2), \xi(2)), \dots, (x(2j), \xi(2j))) \\ &\qquad \text{if } n = 2j+1 \text{ is odd.} \end{aligned}$$

For $i, j \le n \in \mathbb{N}$, setting $w = x{\restriction}2n+1$, we have $\xi(2i), \xi(2j) \in C$ and

$$\xi(2i) \le \xi(2j) \iff \theta(\upsilon_i) \le \theta(\upsilon_j) \iff \upsilon_i \preccurlyeq_x \upsilon_j \iff i \preccurlyeq'_w j.$$

So

$$x(2n+1) = \rho(w) = f_w(\{\xi(2i) : i \le n\})$$

is the first component of $\tau((x(0),\xi(0)),\dots,(x(2n),\xi(2n)))$; thus $\langle(x(n),\xi(n))\rangle_{n\in\mathbb{N}}$ is a play of Game$(\mathbb{N}\times\kappa, F)$ consistent with τ, and is won by II. There must therefore be $i, j \in \mathbb{N}$ such that $\upsilon_i \subset \upsilon_j$, $x \in F_{\upsilon_j}$ and and $\xi(2i) \le \xi(2j)$. Now $\upsilon_j \in T_x$ and $\upsilon_i \preccurlyeq_x \upsilon_j$, so $\upsilon_i \preccurlyeq \upsilon_j$; which is impossible. **X**

So $x \notin A$; as x is arbitrary, τ' is a winning strategy for II in Game$(\mathbb{N}, A)$. **Q**

(g) Putting (d), (e) and (f) together, we see that A is determined.

567O Corollary [AC] If there is a two-valued-measurable cardinal, then every PCA ($= \mathbf{\Sigma}^1_2$) subset of any Polish space is universally measurable.

proof (a) Let $A \subseteq \{0,1\}^{\mathbb{N}}$ be PCA. Then there is a coanalytic subset B of $\mathbb{N}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$ such that A is the projection of B. Of course this means that there is a coanalytic subset B' of $\{0,1\}^{\mathbb{N}}$ such that A is a continuous image of B', since $\mathbb{N}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$ is homeomorphic to a G_δ subset of $\{0,1\}^{\mathbb{N}}$, and any homeomorphism must carry B to a coanalytic subset of $\{0,1\}^{\mathbb{N}}$, by 423Tc. Now $(h^{-1}[B'] \cap F) \cup H$ is coanalytic whenever $h : \mathbb{N}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ is continuous, $F \subseteq \mathbb{N}^{\mathbb{N}}$ is closed and $H \subseteq \mathbb{N}^{\mathbb{N}}$ is open; by 567N, $(\mathbb{N}, (h^{-1}[B'] \cap F) \cup H)$ is always determined; by 567F, A is measured by the usual measure ν on $\{0,1\}^{\mathbb{N}}$.

(b) Now suppose that X is a Polish space, $A \subseteq X$ is a PCA set and μ is a Borel probability measure on X with completion $\hat\mu$. Then there is a Borel measurable function $f : \{0,1\}^{\mathbb{N}} \to X$ such that $\hat\mu$ is the image measure νf^{-1}. **P** Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be the measure algebras of μ, ν respectively. Then $\mathfrak{A}$ has Maharam type at most $w(X) = \omega$ (531Aa), so there is a measure-preserving Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ (332N). Now $\hat\mu$ is a Radon measure (433Cb), so there is a function $f_0 : \{0,1\}^{\mathbb{N}} \to X$ such that $f_0^{-1}[E]$ is measured by ν, and $\nu f_0^{-1}[E] = \hat\mu E$, whenever E is measured by $\hat\mu$ (416Wb). In this case, f_0 is almost continuous (433E) and there is a sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ of compact subsets of $\{0,1\}^{\mathbb{N}}$ such that $\lim_{n\to\infty} \nu K_n = 1$ and $f_0{\restriction}K_n$ is continuous for every n. Fix any $x_0 \in X$ and set $f(z) = f_0(z)$ for $z \in \bigcup_{n\in\mathbb{N}} K_n$, x for other $z \in \{0,1\}^{\mathbb{N}}$; then f is Borel measurable and equal ν-a.e. to f_0, so f also is inverse-measure-preserving for ν and $\hat\mu$. Finally, because f likewise is almost continuous, the image measure νf^{-1} on X is a Radon measure (418I), and must be exactly $\hat\mu$ (416Eb). **Q**

Since $f^{-1}[A]$ is PCA (423Td), ν measures $f^{-1}[A]$ and $\hat\mu$ measures A. As μ is arbitrary, A is universally measurable.

567X Basic exercises (a) Let X be a non-empty well-orderable set, with its discrete topology, and $G \subseteq X^{\mathbb{N}}$ an open set. Show that G is determined.

(b) [AC($\mathbb{R};\omega$)] Let $A \subseteq \mathbb{N}^{\mathbb{N}}$ be such that $\{x : {<}n{>}^\frown x \in A\}$ is determined for every $n \in \mathbb{N}$. Show that $\mathbb{N}^{\mathbb{N}} \setminus A$ is determined.

(c) Show that AD is true iff every subset of $\{0,1\}^{\mathbb{N}}$ is determined. (*Hint*: For $x \in \{0,1\}^{\mathbb{N}}$ set $I_x = \{n : x(2n) = 1\}$, $J_x = \{n : x(2n+1) = 1\}$; set $C_{\mathrm{I}} = \{x : \sup I_x > \sup J_x\}$, $D = \{x : I_x$ and J_x are both infinite$\}$. Define $f : D \to \mathbb{N}^{\mathbb{N}}$ by setting $f(x)(0) = \min I_x$, $f(x)(2n+1) = \min\{k : f(x)(2n) + k \in J_x\}$, $f(x)(2n+2) = \min\{k : f(x)(2n+1) + k + 1 \in I_x\}$. Show that if $A \subseteq \mathbb{N}^{\mathbb{N}}$ and $C_{\mathrm{I}} \cup f^{-1}[A]$ is determined, then A is determined.)

(d) [AC($\mathbb{R};\omega$)] (i) Show that the intersection of a sequence of closed cofinal subsets of ω_1 is cofinal. (ii) Show that we have a unique topological probability measure on ω_1 which is zero on singletons and inner regular with respect to the closed sets.

(e) [AD] Show that if $f : [0,1]^2 \to \mathbb{R}$ is a bounded function, then $\iint f(x,y)dxdy$ and $\iint f(x,y)dydx$ are defined and equal, where the integrations are with respect to Lebesgue measure on $[0,1]$.

(f) [AD] Let μ be a Radon measure on a Polish space X, and $\mathcal{E}$ a well-ordered family of subsets of X. Show that $\mu(\bigcup\mathcal{E}) = \sup_{E\in\mathcal{E}} \mu E$. (*Hint*: 567Xe.)

(g) [AD+AC(ω)] Show that there are no interesting Sierpiński sets, in the sense that every atomless probability space has an uncountable negligible subset.

(h) [AD] Show that every semi-finite measure space is perfect.

(i) [AD] Show that if X is a separable Banach space and Y is a normed space then every linear operator from X to Y is bounded.

(j) [DC] Let I be a set, and $\widehat{\mathcal{B}}$ the Baire-property algebra of $\mathcal{P}I$ with its usual topology. Show that every $\widehat{\mathcal{B}}$-measurable real-valued finitely additive functional on $\mathcal{P}I$ is completely additive. (*Hint*: remember to prove that $\mathcal{P}I$ is a Baire space.)

(k) [AD] (i) Show that there is no non-principal ultrafilter on $\mathbb{N}$. (ii) Show that $\{0,1\}^{\mathbb{R}}$ is not compact.

(l) [AD] Show that there is no linear lifting for Lebesgue measure on $\mathbb{R}$. (*Hint*: 567J.)

(m) [AD] (i) Show that $\ell^1(\mathbb{R})$ is not reflexive. (ii) Show that $\ell^1(\omega_1)$ is not reflexive.

(n) [AD] (i) Show that there is no injective function from ω_1 to $\mathbb{R}$. (ii) Show that there is no family $\langle f_\xi\rangle_{\xi<\omega_1}$ such that f_ξ is an injective function from ξ to $\mathbb{N}$ for every $\xi<\omega_1$. (iii) Show that there is no function $f:\omega_1\times\mathbb{N}\to\omega_1$ such that $\{f(\xi,n):n\in\mathbb{N}\}$ is a cofinal subset of ξ for every non-zero limit ordinal $\xi<\omega_1$. (*Hint*: 567L.)

(o) (i) Show that there is a set $A\subseteq\omega_1^{\mathbb{N}}$ such that Game(ω_1,A) is not determined. (*Hint*: Set II the task of enumerating $x(0)$; see 567D and 567Xn.) (ii) Show that there is a set $A\subseteq(\mathcal{P}\mathbb{R})^{\mathbb{N}}$ such that Game$(\mathcal{P}\mathbb{R},A)$ is not determined.

(p) [AD] Show that there is a surjective function from $\mathbb{R}$ to $\mathcal{B}(\mathbb{R})$, but no injective function from $\mathcal{B}(\mathbb{R})$ to $\mathbb{R}$. (*Hint*: 567E, 561Xd.)

(q) [AC] Show that if there is a two-valued-measurable cardinal and $A\subseteq\mathbb{N}^{\mathbb{N}}$ is analytic then A is determined.

(r) [AC] Suppose that there is a two-valued-measurable cardinal. Show that every PCA subset of $\mathbb{R}$ has the Baire property.

567Y Further exercises (a) Let X be a non-empty set and $A\subseteq X^{\mathbb{N}}$. A **quasi-strategy for I** in Game(X,A) is a function $\sigma:\bigcup_{n\in\mathbb{N}}X^n\to\mathcal{P}X\setminus\{\emptyset\}$; it is a winning quasi-strategy if $x\in A$ whenever $x\in X^{\mathbb{N}}$ and $x(2n)\in\sigma(\langle x(2i+1)\rangle_{i<n})$ for every n. Similarly, a winning quasi-strategy for II is a function $\tau:\bigcup_{n\ge1}X^n\to\mathcal{P}X\setminus\{\emptyset\}$ such that $x\notin A$ whenever $x\in X^{\mathbb{N}}$ and $x(2n+1)\in\tau(\langle x(2i)\rangle_{i<n})$ for every n. (i) Show that if X is any non-empty discrete space and $F\subseteq X^{\mathbb{N}}$ is closed then at least one player has a winning quasi-strategy in Game(X,F). (ii) Show that DC is true iff there is no game Game(X,A) such that both players have winning quasi-strategies.

(b) [AD] Show that every uncountable subset of $\mathbb{R}$ has a non-empty perfect subset. (*Hint*: Let $A\subseteq\{0,1\}^{\mathbb{N}}$. Enumerate $\bigcup_{n\in\mathbb{N}}\{0,1\}^n$ as $\langle v_j\rangle_{j\in\mathbb{N}}$. For $x\in\mathbb{N}^{\mathbb{N}}$ set

$$f(x)=v_{x(0)}{}^\frown{<}\min(1,x(1)){>}^\frown v_{x(2)}{}^\frown{<}\min(1,x(3)){>}^\frown\ldots.$$

Consider Game$(\mathbb{N},f^{-1}[A])$.)

(c) [AD] Let X be an analytic Hausdorff space, and $c:\mathcal{P}X\to[0,\infty]$ a submodular Choquet capacity. Show that $c(A)=\sup\{c(K):K\subseteq A$ is compact$\}$ for every $A\subseteq X$. (Cf. 479Yj.)

(d) [AC($\mathbb{R}$;ω)] Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and $\nu : \mathfrak{A} \to \mathbb{R}$ an additive functional which is Borel measurable for the order-sequential topology on $\mathfrak{A}$. Show that ν is countably additive.

(e) Let Θ be the least ordinal such that there is no surjection from $\mathcal{P}\mathbb{N}$ onto Θ. (i) [AC(ω)] Show that $\operatorname{cf}\Theta > \omega$. (ii) [AD] Show that $\Theta = \omega_\Theta$.

(f) [AC] Suppose that there is a two-valued-measurable cardinal. Show that every uncountable PCA subset of $\mathbb{R}$ has a non-empty perfect subset.

567 Notes and comments The consequences of the axiom of determinacy are so striking that the question of its consistency is particularly pressing. In fact W.H.Woodin has determined its consistency strength, in terms of large cardinals (Kanamori 03, 32.16, or Jech 03, 33.27), and this is less than that of the existence of a supercompact cardinal; so it seems safe enough.

In ZFC, 567B is most naturally thought of as a basic special case of Martin's theorem that every Borel subset of $X^{\mathbb{N}}$, for any discrete space X, is determined (Martin 75, or Kechris 95, 20.5). The idea of the proof is that if II has no winning strategy, then all I has to do is to avoid positions from which II can win. But for a proof in ZF we need more than this. It would not be enough to show that for every first move by I, there is a winning strategy for II from the resulting position; we should need to show that these can be pieced together as a single function $\tau : \bigcup_{n\ge1} X^n \to X$. Turning this round, AD must imply a weak form of the axiom of choice (567D; see also 567Xo). In the particular case of 567B, we have a basic set W_0 of winning positions for II with a trivial family $\langle\tau_w\rangle_{w\in W_0}$ of strategies. (Starting from a position in W_0, II can simply play the $\preccurlyeq$-least point of X to get a position from which I cannot avoid W_0.) From these we can work backwards to construct a family $\langle\tau_w\rangle_{w\in W}$ of strategies, where $W = \bigcup_{\xi\in\mathrm{On}} W_\xi$; so that if $<u> \in W$ for every $u \in X$, we can assemble these into a winning strategy for II in $\mathrm{Game}(X, F)$.

The central result of the section is I suppose 567G. From the point of view of a real analyst like myself, as opposed to a logician or set theorist, this is the door into a different world, explored in 567H-567K, 567Xe-567Xm and 567Yb. In 567I we have a result which is already interesting in ZFC. Recall that in ZFC there are non-trivial additive functionals on $\mathcal{P}\mathbb{N}$ which are measurable in the sense of §464 (464Jb); none of them can be Baire-property-measurable.

I have not talked about 'automatic continuity' in this book. If you have seen anything of this subject you will recognise the three parts of 567H as versions of standard results on homomorphisms which are measurable in some sense. I do not know whether the hypothesis 'abelian' is necessary in 567Hb. If you like, 567J can also be thought of as an automatic-continuity result.

You will see that 567H-567J depend on 567Gb rather than on 567Ga; that is, on category rather than on measure. It is not clear how much can be proved if we assume, as an axiom, that every subset of $\mathbb{R}$ is Lebesgue measurable (together with AC(ω) at least, of course), rather than that every subset of $\mathbb{R}$ has the Baire property.

In 567L far more is true, at least with AD+DC; ω_2, as well as ω_1, is two-valued-measurable, and the filter on ω_1 generated by the closed cofinal sets is an ultrafilter (Kanamori 03, §28, or Jech 03, Theorem 33.12). I am not sure what we should think of as a 'real-valued-measurable cardinal' in this context. In the language of 566Xl, AD implies that $\mathbb{R}$ is not measure-free, and Lebesgue measure is κ-additive for every initial ordinal κ (567Xf). For further combinatorial consequences of AD, see Kanamori 03. Note that AD implies CH in the form 'every uncountable subset of $\mathbb{R}$ is equipollent with $\mathbb{R}$' (567Yb). But the relationship of $\mathbb{R}$ with ω_1 is quite different. ZF is enough to build a surjection from $\mathbb{R}$ onto ω_1. AD implies that there is no injection from ω_1 into $\mathbb{R}$ (567Xn) but that there are surjections from $\mathbb{R}$ onto much larger initial ordinals (567M, 567Ye).

In 567N-567O I return to the world of ZFC; they are in this section because they depend on 567B and 567F. Once again, much more is known about determinacy compatible with AC, and may be found in Kanamori 03 or Jech 03.

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