Version of 7.1.15

Chapter 55

Possible worlds

In my original plans for this volume, I hoped to cover the most important consistency proofs relating to undecidable questions in measure theory. Unhappily my ignorance of forcing means that for the majority of results I have nothing useful to offer. I have therefore restricted my account to the very narrow range of ideas in which I feel I have achieved some understanding beyond what I have read in the standard texts.

For a measure theorist, by far the most important forcings are those of 'adding random reals'. I give three sections (§§552-553 and 555) to these. Without great difficulty, we can determine the behaviour of the cardinals in Cichoń's diagram (552B, 552C, 552F-552I), at least if many random reals are added. Going deeper, there are things to be said about outer measure and Sierpiński sets (552D, 552E), and extensions of Radon measures (552N). In the same section I give a version of the fundamental result that simple iteration of random real forcings gives random real forcings (552P). In §553 I collect results which are connected with other topics dealt with above (Rothberger's property, precalibers, ultrafilters, cellularity, trees, medial limits, universally measurable sets) and in which the arguments seem to me to develop properties of measure algebras which may be of independent interest. In preparation for this work, and also for §554, I start with a section (§551) devoted to a rather technical general account of forcings with quotients of σ-algebras of sets, aiming to find effective representations of names for points, sets, functions, measure algebras and filters.

Very similar ideas can also take us a long way with Cohen real forcing (§554). Here I give little more than obvious parallels to the first part of §552, with an account of Freese-Nation numbers sufficient to support Carlson's theorem that a Borel lifting for Lebesgue measure can exist when the continuum hypothesis is false (554I).

One of the most remarkable applications of random reals is in Solovay's proof that if it is consistent to suppose that there is a two-valued-measurable cardinal, then it is consistent to suppose that there is an atomlessly-measurable cardinal (555D). By taking a bit of trouble over the lemmas, we can get a good deal more, including the corresponding theorem relating supercompact cardinals to the normal measure axiom (555N); and similar techniques show the possibility of interesting power set σ-quotient algebras (555G, 555K).

I end the chapter with something quite different (§556). A familiar phenomenon in ergodic theory is that once one has proved a theorem for ergodic transformations one can expect, possibly at the cost of substantial effort, but without having to find any really new idea, a corresponding result for general measure-preserving transformations. There is more than one way to look at this, but here I present a method in which the key step, in each application, is an appeal to the main theorem of forcing. A similar approach gives a description of the completion of the asymptotic density algebra. The technical details take up a good deal of space, but are based on the same principles as those in §551, and are essentially straightforward.

Version of 2.12.13

551 Forcing with quotient algebras

In preparation for the discussion of random real forcing in the next two sections, I introduce some techniques which can be applied whenever a forcing notion is described in terms of a Loomis-Sikorski representation of its regular open algebra. The first step is just a translation of the correspondence between names for real numbers in the forcing language and members of $L^0(\mathrm{RO}(\mathbb{P}))$, as described in 5A3L, when $L^0(\mathrm{RO}(\mathbb{P}))$ can be identified with a quotient of a space $L^0(\Sigma)$ of measurable functions. More care is needed, but we can find a similar formulation of names for members of $\{0,1\}^I$ for any set I (551C). Going a step farther, it turns out that there are very useful descriptions of Baire subsets of $\{0,1\}^I$ (551D-551F), Baire measurable functions (551N), the usual measure on $\{0,1\}^I$ (551I-551J) and its measure algebra (551P). In some special cases, these methods can be used to represent iterated forcing notions (551Q). I end with a construction for a forcing extension of a filter on a countable set (551R).

551A Definition (a) A **measurable space with negligibles** is a triple $(\Omega,\Sigma,\mathcal{I})$ where Ω is a set, Σ is a σ-algebra of subsets of Ω and $\mathcal{I}$ is a σ-ideal of subsets of Ω generated by $\Sigma\cap\mathcal{I}$. In this case $\mathfrak{A}=\Sigma/\Sigma\cap\mathcal{I}$ is a Dedekind σ-complete Boolean algebra.

(b) $(\Omega,\Sigma,\mathcal{I})$ is **non-trivial** if $\Omega\notin\mathcal{I}$, so that $\mathfrak{A}\ne\{0\}$. In this case, the forcing notion $\mathbb{P}$ **associated** with $(\Omega,\Sigma,\mathcal{I})$ is $(\mathfrak{A}^+,\subseteq,\Omega^\bullet,\downarrow)$. If $\mathfrak{A}$ is Dedekind complete we can identify $\mathfrak{A}$ with the regular open algebra $\mathrm{RO}(\mathbb{P})$.

(c) $(\Omega,\Sigma,\mathcal{I})$ is ω_1**-saturated** if $\Sigma\cap\mathcal{I}$ is ω_1-saturated in Σ. In this case, $\mathfrak{A}$ is Dedekind complete.

(d) $(\Omega,\Sigma,\mathcal{I})$ is **complete** if $\mathcal{I}\subseteq\Sigma$.

551B Definition Let $(\Omega,\Sigma,\mathcal{I})$ be a non-trivial measurable space with negligibles with a Dedekind complete quotient algebra $\mathfrak{A}$, and $\mathbb{P}$ its associated forcing notion. $L^0(\mathfrak{A})$ can be regarded as a quotient of the space of Σ-measurable functions from Ω to $\mathbb{R}$. If $h:\Omega\to\mathbb{R}$ is Σ-measurable, write $\vec{h}=(h^\bullet)^{\rightarrow}$ where $h^\bullet$ is the equivalence class of h in $L^0(\mathfrak{A})$, identified with $L^0(\mathrm{RO}(\mathbb{P}))$, and $(h^\bullet)^{\rightarrow}$ is the $\mathbb{P}$-name for a real number as defined in 5A3L. Then

$$\Vdash_{\mathbb{P}}\vec{h}\text{ is a real number,}$$

and for any $\alpha\in\mathbb{Q}$

$$[\![\vec{h}>\check{\alpha}]\!]=[\![(h^\bullet)^{\rightarrow}>\check{\alpha}]\!]=[\![h^\bullet>\alpha]\!]=\{\omega:h(\omega)>\alpha\}^\bullet.$$

From 5A3Lc, we see that if h_0, h_1 are Σ-measurable real-valued functions on Ω, then

$$\Vdash_{\mathbb{P}}(h_0+h_1)^{\rightarrow}=\vec{h}_0+\vec{h}_1,\ (h_0\times h_1)^{\rightarrow}=\vec{h}_0\times\vec{h}_1,$$

and that if $\langle h_n\rangle_{n\in\mathbb{N}}$ is a sequence of measurable functions with limit h,

$$\Vdash_{\mathbb{P}}\vec{h}=\lim_{n\to\infty}\vec{h}_n\text{ in }\mathbb{R}.$$

551C Definition Let $(\Omega,\Sigma,\mathcal{I})$ be a non-trivial measurable space with negligibles with a Dedekind complete quotient algebra $\mathfrak{A}$, and $\mathbb{P}$ its associated forcing notion.

(a) If $f:\Omega\to\{0,1\}$ is Σ-measurable, let $\vec{f}$ be the $\mathbb{P}$-name

$$\{(\check{\imath},f^{-1}[\{i\}]^\bullet):i\in\{0,1\},\ f^{-1}[\{i\}]\notin\mathcal{I}\}.$$

$\Vdash_{\mathbb{P}}\vec{f}\in\{0,1\}$ and $[\![\vec{f}=\check{\imath}]\!]=f^{-1}[\{i\}]^\bullet$ for both i.

Observe that if a $\mathbb{P}$-name $\dot{x}$ and $p\in\mathfrak{A}^+$ are such that $p\Vdash_{\mathbb{P}}\dot{x}\in\{0,1\}$, then there is a measurable $f:\Omega\to\{0,1\}$ such that $p\Vdash_{\mathbb{P}}\dot{x}=\vec{f}$.

(b) Now let I be any set, and $f:\Sigma\to\{0,1\}^I$ a $(\Sigma,\mathcal{B}\mathfrak{a}_I)$-measurable function, where $\mathcal{B}\mathfrak{a}_I=\mathcal{B}\mathfrak{a}(\{0,1\}^I)$ is the Baire σ-algebra of $\{0,1\}^I$. For each $i\in I$, set $f_i(\omega)=f(\omega)(i)$ for $\omega\in\Omega$; then we have a $\mathbb{P}$-name $\vec{f_i}$. Let $\vec{f}$ be the $\mathbb{P}$-name $\{(\langle\vec{f_i}\rangle_{i\in\check{I}},\mathbb{1})\}$.

$$\Vdash_{\mathbb{P}}\vec{f}\in\{0,1\}^{\check{I}},$$

and for every $i\in I$

$$\Vdash_{\mathbb{P}}\vec{f}(\check{\imath})=\vec{f_i}.$$

(c) In the other direction, if a $\mathbb{P}$-name $\dot{x}$ and $p\in\mathfrak{A}^+$ are such that $p\Vdash_{\mathbb{P}}\dot{x}\in\{0,1\}^{\check{I}}$, then for each $i\in I$ we have a $\mathbb{P}$-name $\dot{x}(\check{\imath})$ and a measurable $f_i:\Omega\to\{0,1\}$ such that $p\Vdash_{\mathbb{P}}\dot{x}(\check{\imath})=\vec{f_i}$; setting $f(\omega)=\langle f_i(\omega)\rangle_{i\in I}$ for $\omega\in\Omega$, f is $(\Sigma,\mathcal{B}\mathfrak{a}_I)$-measurable and $p\Vdash_{\mathbb{P}}\vec{f}=\dot{x}$.

(e) Suppose that x is any point of $\{0,1\}^I$. Then we have a corresponding $\mathbb{P}$-name $\check{x}$, and $\Vdash_{\mathbb{P}} \check{x} \in \{0,1\}^{\check{I}}$. For each $i \in I$, $\Vdash_{\mathbb{P}} \check{x}(\check{\imath}) = x(i)\check{} \in \{0,1\}$. If we set $e_x(\omega) = x$ for every $\omega \in \Omega$, then $\Vdash_{\mathbb{P}} \vec{e}_x = \check{x}$.

551D Definition Let $(\Omega, \Sigma, \mathcal{I})$ be a non-trivial measurable space with negligibles with a Dedekind complete quotient algebra, and $\mathbb{P}$ its associated forcing notion. Let I be any set. If $W \subseteq \Omega \times \{0,1\}^I$, let $\vec{W}$ be the $\mathbb{P}$-name

$$\{(\vec{f}, E^\bullet) : E \in \Sigma \setminus \mathcal{I},\ f : \Omega \to \{0,1\}^I \text{ is } (\Sigma, \mathcal{B}\mathfrak{a}_I)\text{-measurable}, \\ (\omega, f(\omega)) \in W \text{ for every } \omega \in E\},$$

interpreting $\vec{f}$ as in 551C.

$$\Vdash_{\mathbb{P}} \vec{W} \subseteq \{0,1\}^{\check{I}}$$

and if $W = \Omega \times \{0,1\}^I$ then

$$\Vdash_{\mathbb{P}} \vec{W} = \{0,1\}^{\check{I}}.$$

551E Proposition Let $(\Omega, \Sigma, \mathcal{I})$ be a non-trivial measurable space with negligibles with a Dedekind complete quotient algebra, $\mathbb{P}$ its associated forcing notion, and I a set.

(a) If $W \in \Sigma \widehat{\otimes} \mathcal{B}\mathfrak{a}_I$ and $f : \Omega \to \{0,1\}^I$ is $(\Sigma, \mathcal{B}\mathfrak{a}_I)$-measurable, then $\{\omega : (\omega, f(\omega)) \in W\}$ belongs to Σ, and $[\![\vec{f} \in \vec{W}]\!] = \{\omega : (\omega, f(\omega)) \in W\}^\bullet$.

(b) If V, $W \in \Sigma \widehat{\otimes} \mathcal{B}\mathfrak{a}_I$ then

$\Vdash_{\mathbb{P}} \vec{V} \cap \vec{W} = (V \cap W)^{\rightarrow}$, $\vec{V} \cup \vec{W} = (V \cup W)^{\rightarrow}$, $\vec{V} \setminus \vec{W} = (V \setminus W)^{\rightarrow}$ and $\vec{V} \triangle \vec{W} = (V \triangle W)^{\rightarrow}$.

(c) If V, $W \subseteq \Omega \times \{0,1\}^I$ and $V \subseteq W$ then

$$\Vdash_{\mathbb{P}} \vec{V} \subseteq \vec{W}.$$

(d) If $\langle W_n \rangle_{n \in \mathbb{N}}$ is a sequence in $\Sigma \widehat{\otimes} \mathcal{B}\mathfrak{a}_I$ with union W and intersection V, then

$$\Vdash_{\mathbb{P}} \bigcup_{n \in \mathbb{N}} \vec{W}_n = \vec{W} \text{ and } \bigcap_{n \in \mathbb{N}} \vec{W}_n = \vec{V}.$$

(e) Suppose that $J \subseteq I$ is countable, $z \in \{0,1\}^J$, $E \in \Sigma$ and

$$W = \{(\omega, x) : \omega \in E,\ x \in \{0,1\}^I,\ x \restriction J = z\}.$$

Then

$$E^\bullet = [\![\vec{W} = \{x : x \in \{0,1\}^{\check{I}},\ \check{z} \subseteq x\}]\!],$$

$$1 \setminus E^\bullet = [\![\vec{W} = \emptyset]\!].$$

551F Proposition Let $(\Omega, \Sigma, \mathcal{I})$ be a non-trivial measurable space with negligibles with a Dedekind complete quotient algebra, $\mathbb{P}$ its associated forcing notion, and I a set.

(a) If $W \in \Sigma \widehat{\otimes} \mathcal{B}\mathfrak{a}_I$ then

$$\Vdash_{\mathbb{P}} \vec{W} \in \mathcal{B}\mathfrak{a}_{\check{I}}.$$

(b) Suppose that $(\Omega, \Sigma, \mathcal{I})$ is ω_1-saturated, $p \in \mathfrak{A}^+$, and that $\dot{W}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{W} \in \mathcal{B}\mathfrak{a}_{\check{I}}.$$

Then there is a $W \in \Sigma \widehat{\otimes} \mathcal{B}\mathfrak{a}_I$ such that

$$p \Vdash_{\mathbb{P}} \dot{W} = \vec{W}.$$

551G Proposition Let $(\Omega, \Sigma, \mathcal{I})$ be a non-trivial measurable space with negligibles with a Dedekind complete quotient algebra $\mathfrak{A}$, $\mathbb{P}$ the associated forcing notion and I a set. Suppose that Σ is closed under Souslin's operation.

(a) If $W \in \Sigma \widehat{\otimes} \mathcal{B}\mathfrak{a}_I$ then $F = \{\omega : W[\{\omega\}] \neq \emptyset\}$ belongs to Σ and $[\![\vec{W} \neq \emptyset]\!] = F^\bullet$ in $\mathfrak{A} \cong \mathrm{RO}(\mathbb{P})$.

(b) If W, $V \in \Sigma \widehat{\otimes} \mathcal{B}\mathfrak{a}_I$ then $[\![\vec{W} = \vec{V}]\!] = \{\omega : W[\{\omega\}] = V[\{\omega\}]\}^\bullet$.

551H Examples (a) If (X, Σ, μ) is a complete locally determined measure space, then Σ is closed under Souslin's operation.

(b) If $(\Omega, \Sigma, \mathcal{I})$ is a complete ω_1-saturated measurable space with negligibles, then Σ is closed under Souslin's operation.

(c) If X is any topological space, then its Baire-property algebra $\widehat{\mathcal{B}}(X)$ is closed under Souslin's operation.

551I Theorem Let $(\Omega, \Sigma, \mathcal{I})$ be a non-trivial measurable space with negligibles with a Dedekind complete quotient algebra, $\mathbb{P}$ its associated forcing notion, and I a set. Let W be any member of $\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$. Then

(i) $h(\omega) = \nu_I W[\{\omega\}]$ is defined for every $\omega \in \Omega$, where ν_I is the usual measure of $\{0,1\}^I$;

(ii) $h : \Omega \to [0,1]$ is Σ-measurable;

(iii) $\Vdash_{\mathbb{P}} \nu_{\check{I}} \vec{W} = \vec{h}$,

where $\vec{h}$ is the $\mathbb{P}$-name for a real number defined from h as in 551B, and $\nu_{\check{I}}$ is an abbreviation for 'the usual measure on $\{0,1\}^{\check{I}}$'.

551J Corollary Let $(\Omega, \Sigma, \mathcal{I})$ be a non-trivial ω_1-saturated measurable space with negligibles, $\mathbb{P}$ its associated forcing notion, P the partially ordered set underlying $\mathbb{P}$, and I a set. If $p \in P$ and $\dot{W}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{W} \subseteq \{0,1\}^{\check{I}} \text{ is } \nu_{\check{I}}\text{-negligible},$$

then there is a $W \in \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ such that $\nu_I W[\{\omega\}] = 0$ for every $\omega \in \Omega$ and

$$p \Vdash_{\mathbb{P}} \dot{W} \subseteq \vec{W}.$$

551K Proposition Let $(\Omega, \Sigma, \mathcal{I})$ be a non-trivial measurable space with negligibles with a Dedekind complete quotient algebra, $\mathbb{P}$ the associated forcing notion, and I a set. For $H \subseteq \{0,1\}^I$ set $\tilde{H} = (\Omega \times H)^{\rightarrow}$.

(a) If $H = \{x : z \subseteq x \in \{0,1\}^I\}$, where $z \in \mathrm{Fn}_{<\omega}(I; \{0,1\})$, then

$$\Vdash_{\mathbb{P}} \tilde{H} = \{x : \check{z} \subseteq x \in \{0,1\}^{\check{I}}\}.$$

(b)(i) If G, $H \in \mathcal{B}\mathfrak{a}_I$ then

$$\Vdash_{\mathbb{P}} \tilde{G} \cup \tilde{H} = (G \cup H)^{\sim},\ \tilde{G} \cap \tilde{H} = (G \cap H)^{\sim},$$
$$\tilde{G} \setminus \tilde{H} = (G \setminus H)^{\sim},\ \tilde{G} \triangle \tilde{H} = (G \triangle H)^{\sim}.$$

(ii) If $\langle H_n \rangle_{n\in\mathbb{N}}$ is any sequence in $\mathcal{B}\mathfrak{a}_I$ then

$$\Vdash_{\mathbb{P}} \bigcup_{n\in\mathbb{N}} \tilde{H}_n = (\bigcup_{n\in\mathbb{N}} H_n)^{\sim},\ \bigcap_{n\in\mathbb{N}} \tilde{H}_n = (\bigcap_{n\in\mathbb{N}} H_n)^{\sim}.$$

(c) If $\alpha < \omega_1$ and $H \in \mathcal{B}\mathfrak{a}_\alpha(\{0,1\}^I)$, then

$$\Vdash_{\mathbb{P}} \tilde{H} \in \mathcal{B}\mathfrak{a}_{\check{\alpha}}(\{0,1\}^{\check{I}}).$$

(d) If H is measured by the usual measure ν_I of $\{0,1\}^I$, then

$$\Vdash_{\mathbb{P}} \nu_{\check{I}} \tilde{H} = (\nu_I H)^{\check{}}.$$

551M Definition Let $(\Omega, \Sigma, \mathcal{I})$ be a non-trivial measurable space with negligibles, and $\mathbb{P}$ its associated forcing notion. Let I be any set. If $\psi : \Omega \times \{0,1\}^I \to \mathbb{R}$ is $(\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I)$-measurable, let $\vec{\psi}$ be the $\mathbb{P}$-name

$$\{((\vec{f}, \vec{h}), \mathbb{1}) : f \text{ is a } (\Sigma, \mathcal{B}\mathfrak{a}_I)\text{-measurable function from } \Omega \text{ to } \{0,1\}^I,$$
$$h : \Omega \to \mathbb{R} \text{ is } \Sigma\text{-measurable},\ h(\omega) = \psi(\omega, f(\omega)) \text{ for every } \omega \in \Omega\},$$

where in this formula $\vec{f}$ is to be interpreted as a $\mathbb{P}$-name for a member of $\{0,1\}^{\check{I}}$, as in 551C, and $\vec{h}$ as a $\mathbb{P}$-name for a real number, as in 551B.

551N Proposition Let $(\Omega,\Sigma,\mathcal{I})$ be a non-trivial measurable space with negligibles with a Dedekind complete quotient algebra $\mathfrak{A}$, $\mathbb{P}$ its associated forcing notion, and I a set. Suppose that $\psi:\Omega\times\{0,1\}^I\to\mathbb{R}$ is $(\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I)$-measurable, and define $\vec{\psi}$ as in 551M.

(a) $\Vdash_{\mathbb{P}} \vec{\psi}$ is a real-valued function on $\{0,1\}^{\check{I}}$.

(b) If $\phi:\Omega\times\{0,1\}^I\to\mathbb{R}$ is another $(\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I)$-measurable function, and $\alpha\in\mathbb{R}$, then

$$\Vdash_{\mathbb{P}} (\phi+\psi)^{\vec{}}=\vec{\phi}+\vec{\psi},\ (\alpha\phi)^{\vec{}}=\check{\alpha}\vec{\phi}.$$

(c) If $\langle\psi_n\rangle_{n\in\mathbb{N}}$ is a sequence of $(\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I)$-measurable real-valued functions on $\Omega\times\{0,1\}^I$ and $\psi(\omega,x)=\lim_{n\to\infty}\psi_n(\omega,x)$ for every $\omega\in\Omega$ and $x\in\{0,1\}^I$, then

$$\Vdash_{\mathbb{P}} \vec{\psi}(x)=\lim_{n\to\infty}\vec{\psi}_n(x) \text{ for every } x\in\{0,1\}^{\check{I}}.$$

(d) If $W\in\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$, then

$$\Vdash_{\mathbb{P}} (\chi W)^{\vec{}}=\chi\vec{W}.$$

(e) $\Vdash_{\mathbb{P}} \vec{\psi}$ is $\mathcal{B}\mathfrak{a}_{\check{I}}$-measurable.

(f) If $h(\omega)=\int\psi(\omega,x)\nu_I(dx)$ is defined for every $\omega\in\Omega$, then

$$\Vdash_{\mathbb{P}} \int\vec{\psi}\,d\nu_{\check{I}} \text{ is defined and equal to } \vec{h}.$$

551O Measure algebras Let I be a set, ν_I the usual measure on $\{0,1\}^I$ and $(\mathfrak{B}_I,\bar{\nu}_I)$ its measure algebra. It will be important to appreciate that these are abbreviations for formulae in set theory with a single parameter I; so that if we have a forcing notion $\mathbb{P}$ and a $\mathbb{P}$-name τ, we shall have $\mathbb{P}$-names $\mathfrak{B}_\tau$ and $\bar{\nu}_\tau$, uniquely defined as soon as we have settled on the exact formulations we wish to apply when interpreting the basic constructions $\{\dots\}$, $\mathcal{P}$ in the forcing language. Similarly, if we write $\mathbb{P}_I=(\mathfrak{B}_I^+,\subseteq,1,\downarrow)$ for the forcing notion based on the Boolean algebra $\mathfrak{B}_I$, this also is a formula which can be interpreted in forcing languages.

551P Theorem Let $(\Omega,\Sigma,\mathcal{I})$ be a non-trivial ω_1-saturated measurable space with negligibles such that Σ is closed under Souslin's operation. Let $\mathbb{P}$ be the associated forcing notion, P its underlying partially ordered set, and I a set. Set

$$\Lambda=\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I,\quad \mathcal{J}=\{W:W\in\Lambda,\ \nu_I W[\{\omega\}]=0 \text{ for } \mathcal{I}\text{-almost every } \omega\in\Omega\};$$

then $\mathcal{J}$ is a σ-ideal of Λ; let $\mathfrak{C}$ be the quotient algebra $\Lambda/\mathcal{J}$. For $W\in\Lambda$ and $\omega\in\Omega$ set $h_W(\omega)=\nu_I W[\{\omega\}]$. For $a\in\mathfrak{C}$ let $\vec{a}$ be the $\mathbb{P}$-name

$$\{(\vec{W},\mathbb{1}):W\in\Lambda,\ W^\bullet=a\}$$

where the $\mathbb{P}$-names $\vec{W}$ are defined as in 551D. Consider the $\mathbb{P}$-names

$$\dot{\mathfrak{D}}=\{(\vec{a},\mathbb{1}):a\in\mathfrak{C}\},\quad \dot{\pi}=\{(((W^\bullet)^{\vec{}},(\vec{W})^\bullet),\mathbb{1}):W\in\Lambda\}.$$

(a) $\Vdash_{\mathbb{P}} \dot{\pi}$ is a bijection between $\dot{\mathfrak{D}}$ and $\mathfrak{B}_{\check{I}}$.

(b) If a, $b\in\mathfrak{C}$, $V\in\Lambda$ and $V^\bullet=a$, then

$$\Vdash_{\mathbb{P}} \dot{\pi}(a\triangle b)^{\vec{}}=\dot{\pi}\vec{a}\triangle\dot{\pi}\vec{b},\quad \dot{\pi}(a\cap b)^{\vec{}}=\dot{\pi}\vec{a}\cap\dot{\pi}\vec{b},\quad \bar{\nu}_{\check{I}}(\dot{\pi}\vec{a})=\vec{h}_V,$$

defining h_V and $\vec{h}_V$ as in 551I.

(c) Let $\varepsilon:\Sigma/\Sigma\cap\mathcal{I}\to\mathfrak{C}$ be the canonical map defined by the formula

$$\varepsilon(E^\bullet)=(E\times\{0,1\}^I)^\bullet \text{ for } E\in\Sigma.$$

If $p\in(\Sigma/\Sigma\cap\mathcal{I})^+$ and a, $b\in\mathfrak{C}$, then

$$p\Vdash_{\mathbb{P}} \dot{\pi}\vec{a}=\dot{\pi}\vec{b}$$

iff $a\cap\varepsilon(p)=b\cap\varepsilon(p)$.

551Q Iterated forcing: Theorem Let $(\Omega,\Sigma,\mathcal{I})$ be a non-trivial ω_1-saturated measurable space with negligibles such that Σ is closed under Souslin's operation, $\mathbb{P}$ its associated forcing notion, and I a set. As in 551P, set $\Lambda=\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$,

$$\mathcal{J} = \{W : W \in \Lambda,\, \nu_I W[\{\omega\}] = 0 \text{ for } \mathcal{I}\text{-almost every } \omega \in \Omega\}$$

and $\mathfrak{C} = \Lambda/\mathcal{J}$. Then

$$\mathfrak{C} \cong \mathrm{RO}(\mathbb{P} * \mathbb{P}_{\check{I}}),$$

where the $\mathbb{P}$-name $\mathbb{P}_{\check{I}}$ is defined as in 551O.

551R Extending filters: Proposition Let $(\Omega, \Sigma, \mathcal{I})$ be a non-trivial ω_1-saturated measurable space with negligibles, $\mathfrak{A}$ its quotient algebra, $\mathbb{P}$ the associated forcing notion, I a countable set and $\mathcal{F}$ a filter on I.

(a) For $H \in \Sigma\widehat{\otimes}\mathcal{P}I$, write $\vec{H}$ for the $\mathbb{P}$-name $\{(\check{\imath}, H^{-1}[\{i\}]^\bullet) : i \in I,\, H^{-1}[\{i\}] \notin \mathcal{I}\}$.

(i) $\Vdash_{\mathbb{P}} \vec{H} \subseteq \check{I}$.

(ii) If $\dot{F}$ is a $\mathbb{P}$-name and $p \in \mathfrak{A}^+$ is such that $p \Vdash_{\mathbb{P}} \dot{F} \subseteq \check{I}$, then there is an $H \in \Sigma\widehat{\otimes}\mathcal{P}I$ such that $p \Vdash_{\mathbb{P}} \dot{F} = \vec{H}$.

(b) Write $\vec{\mathcal{F}}$ for the $\mathbb{P}$-name

$$\{(\vec{H}, E^\bullet) : H \in \Sigma\widehat{\otimes}\mathcal{P}I,\, E \in \Sigma \setminus \mathcal{I},\, H[\{\omega\}] \in \mathcal{F} \text{ for every } \omega \in E\}.$$

Then

$$\Vdash_{\mathbb{P}} \vec{\mathcal{F}} \text{ is a filter on } \check{I}.$$

Version of 29.1.14

552 Random reals I

From the point of view of a measure theorist, 'random real forcing' has a particular significance. Because the forcing notions are defined directly from the central structures of measure theory (552A), they can be expected to provide worlds in which measure-theoretic questions are answered. Thus we find ourselves with many Sierpiński sets (552E), information on cardinal functions (552C, 552F-552J), and theorems on extension of measures (552N). But there is a second reason why any measure theorist or probabilist should pay attention to random real forcing. Natural questions in the forcing language, when translated into propositions about the ground model, are likely to hinge on properties of measure algebras, giving us a new source of challenging problems. Perhaps the deepest intuitions are those associated with iterated random real forcing (552P).

552A Notation (a) As usual, if μ is a measure then $\mathcal{N}(\mu)$ will be its null ideal. It will be convenient to have a special notation for certain sets of finite functions: if I is a set, $\mathrm{Fn}_{<\omega}(I;\{0,1\})$ will be $\bigcup_{K\in[I]^{<\omega}}\{0,1\}^K$.

For any set I I write ν_I for the usual completion regular Radon probability measure on $\{0,1\}^I$, T_I for its domain and $(\mathfrak{B}_I, \bar{\nu}_I)$ for its measure algebra; $\mathcal{B}\mathfrak{a}_I$ will be the Baire σ-algebra of $\{0,1\}^I$. I write $\langle e_i\rangle_{i\in I}$ for the standard generating family in $\mathfrak{B}_I$. $\mathbb{P}_I$ will be the forcing notion $\mathfrak{B}_I^+ = \mathfrak{B}_I \setminus \{0\}$, active downwards. For a formula ϕ in the corresponding forcing language I write $[\![\phi]\!]$ for the truth value of ϕ, interpreted as a member of $\mathfrak{B}_I$. $\mathbb{P}_I$ preserves cardinals.

As in §551, the formulae ν_I, $\mathfrak{B}_I$ etc. are to be regarded as formulae of set theory with one free variable into which the parameter I has been substituted, so that we have corresponding names $\nu_{\check{I}}$, $\mathfrak{B}_{\check{I}}$ in any forcing language, and in particular (once the context has established a forcing notion $\mathbb{P}$) we have $\mathbb{P}$-names $\nu_{\check{I}}$, $\mathfrak{B}_{\check{I}}$ for any ground-model set I.

(b) A great deal of the work of this chapter will involve interpretations of names for standard objects in forcing languages. I try to signal intended interpretations by using the superscript ˇ.

552B Theorem Suppose that λ and κ are infinite cardinals. Then

$$\Vdash_{\mathbb{P}_\kappa} 2^{\check{\lambda}} = (\kappa^\lambda)\check{\ },$$

where κ^λ is the cardinal power.

552C Theorem Let κ be any cardinal. Then

$$\Vdash_{\mathbb{P}_\kappa} \mathfrak{b} = \check{\mathfrak{b}} \text{ and } \mathfrak{d} = \check{\mathfrak{d}}.$$

552D Lemma Let λ and κ be infinite cardinals, and A any subset of $\{0,1\}^\lambda$. Then

$$\Vdash_{\mathbb{P}_\kappa} \nu^*_{\check{\lambda}}(\check{A}) = (\nu^*_\lambda A)\check{}.$$

552E Theorem Let κ and λ be infinite cardinals, with $\kappa \geq \max(\omega_1, \lambda)$. Then

$$\Vdash_{\mathbb{P}_\kappa} \text{there is a strongly Sierpiński set for } \nu_{\check{\lambda}} \text{ with cardinal } \check{\kappa}.$$

552F Theorem Let κ and λ be infinite cardinals.
(a) If either κ or λ is uncountable,

$$\Vdash_{\mathbb{P}_\kappa} \operatorname{add}\mathcal{N}(\nu_{\check{\lambda}}) = \omega_1.$$

(b) $\Vdash_{\mathbb{P}_\omega} \operatorname{add}\mathcal{N}(\nu_\omega) = (\operatorname{add}\mathcal{N}(\nu_\omega))\check{}$.

552G Theorem Let κ and λ be infinite cardinals.
(a) $\Vdash_{\mathbb{P}_\kappa} \operatorname{cov}\mathcal{N}(\nu_{\check{\lambda}}) \geq \max(\kappa, \operatorname{cov}\mathcal{N}(\nu_\lambda))\check{}$.
(b) $\Vdash_{\mathbb{P}_\kappa} \operatorname{cov}\mathcal{N}(\nu_\omega) \geq \mathfrak{b}$.
(c) If $\kappa \geq \mathfrak{c}$ then $\Vdash_{\mathbb{P}_\kappa} \operatorname{cov}\mathcal{N}(\nu_\omega) = \mathfrak{c}$.
(d) Suppose that κ and λ are uncountable. Then

$$\Vdash_{\mathbb{P}_\kappa} \operatorname{cov}\mathcal{N}(\nu_{\check{\lambda}}) \leq (\sup_{\delta<\kappa} \delta^\omega)\check{},$$

where each δ^ω is the cardinal power.

552H Theorem Let κ and λ be infinite cardinals.
(a) $\Vdash_{\mathbb{P}_\kappa} \operatorname{non}\mathcal{N}(\nu_{\check{\lambda}}) \leq (\operatorname{non}\mathcal{N}(\nu_\lambda))\check{}$.
(b) If $\kappa \geq \max(\lambda, \omega_1)$ then

$$\Vdash_{\mathbb{P}_\kappa} \operatorname{non}\mathcal{N}(\nu_{\check{\lambda}}) = \omega_1.$$

(c)

$$\Vdash_{\mathbb{P}_\kappa} \operatorname{non}\mathcal{N}(\nu_\omega) \leq \mathfrak{d}.$$

552I Theorem Let κ and λ be infinite cardinals. Set $\theta_0 = \max(\operatorname{cf}\mathcal{N}(\nu_\omega), \operatorname{cf}[\kappa]^{\leq\omega}, \operatorname{cf}[\lambda]^{\leq\omega})$. Then

$$\Vdash_{\mathbb{P}_\kappa} \operatorname{cf}\mathcal{N}(\nu_{\check{\lambda}}) = \check{\theta}_0.$$

552J Theorem Let κ and λ be infinite cardinals; set $\theta_0 = \operatorname{shr}\mathcal{N}(\nu_\lambda)$ and let θ_1 be the cardinal power λ^ω. Then

$$\Vdash_{\mathbb{P}_\kappa} \check{\theta}_0 \leq \operatorname{shr}\mathcal{N}(\nu_{\check{\lambda}}) \leq \check{\theta}_1.$$

552K Lemma Let I be a set. Let $q : \mathrm{Fn}_{<\omega}(I;\{0,1\}) \to [0,\infty[$ be a function such that $q(\emptyset) = 1$ and

$$q(z) = q(z \cup \{(i,0)\}) + q(z \cup \{(i,1)\})$$

whenever $z \in \mathrm{Fn}_{<\omega}(I;\{0,1\})$ and $i \in I \setminus \operatorname{dom} z$. Then there is a unique Radon measure μ on $\{0,1\}^I$ such that

$$\mu\{x : z \subseteq x \in \{0,1\}^I\} = q(z)$$

for every $z \in \mathrm{Fn}_{<\omega}(I;\{0,1\})$.

552L Lemma Let θ be a regular infinite cardinal such that the cardinal power δ^ω is less than θ for every $\delta < \theta$, and $S \subseteq \theta$ a stationary set such that $\operatorname{cf}\xi > \omega$ for every $\xi \in S$. Let $\langle M_\xi \rangle_{\xi<\theta}$ be a family of sets with cardinal less than θ, and I a set with cardinal less than θ; suppose that for each $i \in I$ we are given a function f_i with domain S such that $f_i(\xi) \in \bigcup_{\eta<\xi} M_\eta$ for every $\xi \in S$. Then there is an ω_1-complete filter $\mathcal{F}$ on θ, containing every closed cofinal subset of θ, such that for every $i \in I$ there is a $D \in \mathcal{F}$ such that $D \subseteq S$ and f_i is constant on D.

552M Proposition Let κ and λ be infinite cardinals. Then the following are equiveridical:

(i) if $\mathcal{A} \subseteq \mathcal{P}(\{0,1\}^\kappa)$ and $\#(\mathcal{A}) \le \lambda$ then there is an extension of ν_κ to a measure measuring every member of $\mathcal{A}$;

(ii) for every function $f : \{0,1\}^\kappa \to \{0,1\}^{(\kappa+\lambda)\setminus\kappa}$, there is a Baire measure μ on $\{0,1\}^{\kappa+\lambda}$ such that $\mu\{y : y \in \{0,1\}^{\kappa+\lambda}, z \subseteq y\} = 2^{-\#(K)}$ whenever $K \in [\kappa]^{<\omega}$ and $z \in \{0,1\}^K$, and $\mu^*\{x \cup f(x) : x \in \{0,1\}^\kappa\} = 1$;

(iii) if (X, Σ, μ) is a locally compact semi-finite measure space with Maharam type at most κ, $\mathcal{A} \subseteq \mathcal{P}X$ and $\#(\mathcal{A}) \le \lambda$, then there is an extension of μ to a measure measuring every member of $\mathcal{A}$.

552N Theorem Let κ and λ be infinite cardinals such that κ is greater than the cardinal power λ^ω. Then

$\Vdash_{\mathbb{P}_\kappa}$ if $\mathcal{A} \subseteq \mathcal{P}(\{0,1\}^{\check\kappa})$ and $\#(\mathcal{A}) \le \check\lambda$, there is an extension of $\nu_{\check\kappa}$ to a measure measuring every member of $\mathcal{A}$.

552O Proposition Suppose that (X, Σ, μ) is a probability space such that for every countable family $\mathcal{A}$ of subsets of X there is a measure on X extending μ and measuring every member of $\mathcal{A}$.

(a) If Y is a universally negligible second-countable T_0 space, then $\#(Y) < \operatorname{cov}\mathcal{N}(\mu)$.

(b) $\operatorname{cov}\mathcal{N}(\mu) > \operatorname{non}\mathcal{N}(\nu_\omega)$.

552P Theorem Let κ and λ be infinite cardinals. Then the iterated forcing notion $\mathbb{P}_\kappa * \mathbb{P}_{\check\lambda}$ has regular open algebra isomorphic to $\mathfrak{B}_{\max(\kappa,\lambda)}$.

Version of 3.5.14

553 Random reals II

In this section I collect some further properties of random real models which seem less directly connected with the main topics of this book than those treated in §552. The first concerns strong measure zero or 'Rothberger's property' and gives a bound for the sizes of sets with this property. The second relates perfect sets in $V^{\mathbb{P}_\kappa}$ to negligible F_σ sets in the original universe; it shows that a random real model can have properties relevant to a question in §531 (553F). Following these, I discuss properties of ultrafilters and partially ordered sets which are not obviously connected with measure theory, but where the arguments needed to establish the truth of sentences in $V^{\mathbb{P}_\kappa}$ involve interesting properties of measure algebras (553G-553M). I conclude with notes on medial limits (553N) and universally measurable sets (553O).

553A Notation For any set I, ν_I will be the usual measure on $\{0,1\}^I$, T_I its domain, $\mathcal{N}(\nu_I)$ its null ideal and $(\mathfrak{B}_I, \bar\nu_I)$ its measure algebra. $\mathcal{B}\mathfrak{a}_I$ will be the Baire σ-algebra of $\{0,1\}^I$. For a cardinal κ, $\mathbb{P}_\kappa$ will be the forcing notion $\mathfrak{B}_\kappa^+$, active downwards.

553B Lemma If $A \in \mathcal{R}\mathrm{bg}(\{0,1\}^{\mathbb{N}})$, then for any $f : \mathbb{N} \to \mathbb{N}$ there is a sequence $\langle z_n \rangle_{n\in\mathbb{N}}$ such that $z_n \in \{0,1\}^{f(n)}$ for each n and $A \subseteq \bigcap_{n\in\mathbb{N}} \bigcup_{m\ge n} \{x : z_m \subseteq x \in \{0,1\}^{\mathbb{N}}\}$.

553C Proposition Let κ be any cardinal. Then

$$\Vdash_{\mathbb{P}_\kappa} \#(A) \le \check{\mathfrak{c}} \text{ for every } A \in \mathcal{R}\mathrm{bg}(\{0,1\}^{\mathbb{N}}).$$

553E Proposition Let κ and λ be infinite cardinals, and $\dot{K}$ a $\mathbb{P}_\kappa$-name such that

$$\Vdash_{\mathbb{P}_\kappa} \dot{K} \text{ is a compact subset of } \{0,1\}^{\check{\lambda}} \text{ which is not scattered.}$$

Then there is a negligible F_σ set $G \subseteq \{0,1\}^\lambda$ such that

$$\Vdash_{\mathbb{P}_\kappa} \dot{K} \cap \tilde{G} \neq \emptyset$$

where $\tilde{G}$ is the $\mathbb{P}_\kappa$-name for an F_σ set in $\{0,1\}^{\check{\lambda}}$ corresponding to G.

553F Corollary Suppose that $\operatorname{cf}\mathcal{N}(\nu_\omega) = \omega_1$ and that $\kappa \geq \omega_2$ is a cardinal. Then

$\Vdash_{\mathbb{P}_\kappa} \omega_1$ is a precaliber of every measurable algebra but does not have Haydon's property.

553G Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, $\mathfrak{C}$ a subalgebra of $\mathfrak{A}$, and $\langle e_n \rangle_{n \in \mathbb{N}}$ a sequence in $\mathfrak{A}$ stochastically independent of each other and of $\mathfrak{C}$. Let $I \subseteq \mathfrak{A}$ be a finite set and $\mathfrak{C}_I$ the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C} \cup I$. Then for every $\epsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that $|\bar{\mu}(b \cap e_n) - \bar{\mu}b \cdot \bar{\mu}e_n| \leq \epsilon\bar{\mu}b$ whenever $b \in \mathfrak{C}_I$ and $n \geq n_0$.

553H Theorem If $\kappa > \mathfrak{c}$, then

$$\Vdash_{\mathbb{P}_\kappa} \text{there are no rapid } p\text{-point ultrafilters, therefore no Ramsey filters on } \mathbb{N}.$$

553I Lemma Suppose that $S \subseteq \omega_1^2$ is a set such that whenever $n \in \mathbb{N}$ and $\langle I_\xi \rangle_{\xi < \omega_1}$ is a family in $[\omega_1]^n$ such that $I_\xi \cap \xi = \emptyset$ for every $\xi < \omega_1$, there are $\xi < \omega_1$ and $\eta < \xi$ such that $I_\xi \times I_\eta \subseteq S$. Let P be the set

$$\{I : I \in [\omega_1]^{<\omega},\ I \cap \xi \subseteq S[\{\xi\}] \text{ for every } \xi \in I\},$$

ordered by $\subseteq$. Then P is upwards-ccc.

553J Theorem Let κ be an infinite cardinal. Then

$\Vdash_{\mathbb{P}_\kappa}$ there are two upwards-ccc partially ordered sets whose product is not upwards-ccc.

553K Lemma Let $\mathfrak{A}$ be a Boolean algebra and $\nu : \mathfrak{A} \to [0, \infty[$ a non-negative additive functional. Then

$$\sum_{i=0}^n \nu a_i \leq \nu(\sup_{i \leq n} a_i) + \sum_{i<j\leq n} \nu(a_i \cap a_j)$$

whenever $a_0, \dots, a_n \in \mathfrak{A}$.

553L Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, I an uncountable set, X a non-empty set and $\mathcal{F}$ an ultrafilter on X. Let $\langle a_{ix} \rangle_{i \in I, x \in X}$ be a family in $\mathfrak{A}$ such that $\inf_{i \in I} \lim_{x \to \mathcal{F}} \bar{\mu}a_{ix} > 0$. Then there are an uncountable set $S \subseteq I$ and a family $\langle b_i \rangle_{i \in S}$ in $\mathfrak{A} \setminus \{0\}$ such that

$$b_i \cap b_j \subseteq \sup_{x \in F} a_{ix} \cap a_{jx}$$

for all i, $j \in S$ and $F \in \mathcal{F}$.

553M Proposition If $\mathfrak{m} > \omega_1$ and κ is any infinite cardinal, then

$$\Vdash_{\mathbb{P}_\kappa} \text{every Aronszajn tree is special, so Souslin's hypothesis is true.}$$

553N Proposition Suppose that there is a medial limit, and that κ is a cardinal. Then

$$\Vdash_{\mathbb{P}_\kappa} \text{there is a medial limit.}$$

553O Theorem Let κ be an infinite cardinal.

(a) $\Vdash_{\mathbb{P}_\kappa}$ every universally measurable subset of $\{0,1\}^{\mathbb{N}}$ is expressible as the union of at most $\check{\mathfrak{c}}$ Borel sets.

(b) If the cardinal power $\kappa^{\mathfrak{c}}$ is equal to κ, then

$$\Vdash_{\mathbb{P}_\kappa} \text{there are exactly } \mathfrak{c} \text{ universally measurable subsets of } \{0,1\}^{\mathbb{N}}.$$

553Z Problem Suppose that the generalized continuum hypothesis is true. Is it the case that

$$\Vdash_{\mathbb{P}_{\omega_2}} \text{there is a Borel lifting for Lebesgue measure?}$$

Version of 2.9.14

554 Cohen reals

Parallel to the theory of random reals as described in §§552-553, we have a corresponding theory based on category algebras rather than measure algebras. I start with the exactly matching result on cardinal arithmetic (554B), and continue with Lusin sets (balancing the Sierpiński sets of 552E) and the cardinal functions of the meager ideal of $\mathbb{R}$ (554C-554E, 554F). In the last third of the section I use the theory of Freese-Nation numbers (§518) to prove Carlson's theorem on Borel liftings (554I).

554A Notation For any set I, I will write $\widehat{\mathcal{B}}_I$ for the Baire-property algebra of $\{0,1\}^I$, $\mathcal{M}_I$ for the meager ideal of $\{0,1\}^I$, $\mathfrak{G}_I = \widehat{\mathcal{B}}_I/\mathcal{M}_I$ for the category algebra of $\{0,1\}^I$, and $\mathbb{Q}_I$ for the forcing notion $\mathfrak{G}_I^+ = \mathfrak{G}_I \setminus \{\emptyset\}$ active downwards.

554B Theorem Suppose that λ and κ are infinite cardinals. Then

$$\Vdash_{\mathbb{Q}_\kappa} 2^{\check{\lambda}} = (\kappa^\lambda)\check{}.$$

554C Definition If X is a topological space, a subset of X is a **Lusin set** if it is uncountable but meets every meager set in a countable set.

554D Proposition Let κ be a cardinal such that $\mathbb{R}$ has a Lusin set with cardinal κ.

(a) Writing $\mathcal{M}$ for the ideal of meager subsets of $\mathbb{R}$, $\operatorname{non}\mathcal{M} = \omega_1$ and $\mathfrak{m}_{\text{countable}} \ge \kappa$.

(b) There is a point-countable family $\mathcal{A}$ of Lebesgue-conegligible subsets of $\mathbb{R}$ with $\#(\mathcal{A}) = \kappa$.

(c) If $(\mathfrak{A}, \bar{\mu})$ is a semi-finite measure algebra which is not purely atomic, (κ, ω_1) is not a precaliber pair of $\mathfrak{A}$.

554E Theorem Let κ be an uncountable cardinal. Then

$$\Vdash_{\mathbb{Q}_\kappa} \text{there is a Lusin set } A \subseteq \mathbb{R} \text{ with cardinal } \check{\kappa}.$$

554F Corollary Let κ be a cardinal which is equal to the cardinal power κ^ω. Write $\mathcal{M}$ for the ideal of meager subsets of $\mathbb{R}$. Then

$$\Vdash_{\mathbb{Q}_\kappa} \operatorname{non}\mathcal{M} = \omega_1 \text{ and } \mathfrak{m}_{\text{countable}} = \mathfrak{c}.$$

554G Theorem Let κ be an infinite cardinal such that $\mathrm{FN}(\mathfrak{G}_\kappa) = \omega_1$. Then

$$\Vdash_{\mathbb{Q}_\kappa} \mathrm{FN}(\mathcal{P}\mathbb{N}) = \omega_1.$$

554H Corollary Suppose that $\mathrm{FN}(\mathcal{P}\mathbb{N}) = \omega_1$ and that κ is an infinite cardinal such that

(α) $\operatorname{cf}[\lambda]^{\le\omega} \le \lambda^+$ for every cardinal $\lambda \le \kappa$,

(β) $\square_\lambda$ is true for every uncountable cardinal $\lambda \le \kappa$ of countable cofinality.

Then $\Vdash_{\mathbb{Q}_\kappa} \mathrm{FN}(\mathcal{P}\mathbb{N}) = \omega_1$.

554I Theorem Suppose that the continuum hypothesis is true. Then

$$\Vdash_{\mathbb{Q}_{\omega_2}} \mathfrak{c} = \omega_2 \text{ and Lebesgue measure has a Borel lifting.}$$

Version of 23.10.14

555 Solovay's construction of real-valued-measurable cardinals

While all the mathematical ideas of Chapter 54 were expressed as arguments in ZFC, many would be of little interest if it appeared that there could be no atomlessly-measurable cardinals. In this section I present R.M.Solovay's theorem that if there is a two-valued-measurable cardinal in the original universe, then there is a forcing notion $\mathbb{P}$ such that

$$\Vdash_{\mathbb{P}} \text{there is an atomlessly-measurable cardinal}$$

(555D). Varying $\mathbb{P}$ we find that we can force models with other kinds of quasi-measurable cardinal (555G, 555K); starting from a stronger hypothesis we can reach the normal measure axiom (555N).

555A Notation I will write $(\mathfrak{B}_\kappa, \bar{\nu}_\kappa)$ for the measure algebra of the usual measure on $\{0,1\}^\kappa$, and $\mathbb{P}_\kappa$ for the forcing notion $\mathfrak{B}_\kappa^+ = \mathfrak{B}_\kappa \setminus \{0\}$, active downwards. $\langle e_\eta \rangle_{\eta<\kappa}$ will be the standard generating family in $\mathfrak{B}_\kappa$.

I will write $\mathfrak{G}_\kappa$ for the category algebra of $\{0,1\}^\kappa$, and $\mathbb{Q}_\kappa$ for the forcing notion $\mathfrak{G}_\kappa^+$, active downwards. $\mathfrak{G}_\kappa$ is isomorphic to the regular open algebra $\mathrm{RO}(\{0,1\}^\kappa)$.

555B Theorem Suppose that X is a set, and $\mathcal{I}$ a proper σ-ideal of subsets of X containing singletons. Let $\mathbb{P} = (P, \le, \mathbb{1}, \Updownarrow)$ be a ccc forcing notion, and $\dot{\mathcal{I}}$ a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \dot{\mathcal{I}} = \{J : \text{there is an } I \in \check{\mathcal{I}} \text{ such that } J \subseteq I\}.$$

Then

(a)(i) If $\dot{J}$ is a $\mathbb{P}$-name and $p \in P$ is such that $p \Vdash_{\mathbb{P}} \dot{J} \in \dot{\mathcal{I}}$, there is an $I \in \mathcal{I}$ such that $p \Vdash_{\mathbb{P}} \dot{J} \subseteq \check{I}$.

(ii)

$$\Vdash_{\mathbb{P}} \dot{\mathcal{I}} \text{ is the ideal of subsets of } \check{X} \text{ generated by } \check{\mathcal{I}}\text{; it is a proper } \sigma\text{-ideal containing singletons.}$$

(b) $\Vdash_{\mathbb{P}} \operatorname{add} \dot{\mathcal{I}} = (\operatorname{add} \mathcal{I})\check{}$.

(c) If $\mathcal{I}$ is ω_1-saturated in $\mathcal{P}X$, then

$$\Vdash_{\mathbb{P}} \dot{\mathcal{I}} \text{ is } \omega_1\text{-saturated in } \mathcal{P}\check{X}, \text{ so } \mathcal{P}\check{X}/\dot{\mathcal{I}} \text{ is ccc and Dedekind complete.}$$

(d) If $X = \lambda$ is a regular uncountable cardinal and $\mathcal{I}$ is a normal ideal on λ, then

$$\Vdash_{\mathbb{P}} \dot{\mathcal{I}} \text{ is a normal ideal on } \check{\lambda}.$$

555C Theorem Let $(X, \mathcal{P}X, \mu)$ be a probability space such that $\mu\{x\} = 0$ for every $x \in X$, and $\mathcal{N}$ the null ideal of μ. Let $\kappa > 0$ be a cardinal. Then we can find a $\mathbb{P}_\kappa$-name $\dot{\mu}$ such that

(i) $\Vdash_{\mathbb{P}_\kappa} \dot{\mu}$ is a probability measure with domain $\mathcal{P}\check{X}$, zero on singletons;

(ii) if $\dot{\mathcal{N}}$ is a $\mathbb{P}_\kappa$-name for the ideal of subsets of $\check{X}$ generated by $\check{\mathcal{N}}$, then

$$\Vdash_{\mathbb{P}_\kappa} \dot{\mathcal{N}} \text{ is the null ideal of } \dot{\mu}.$$

555D Corollary Suppose that λ is a two-valued-measurable cardinal and that $\kappa \ge \lambda$ is a cardinal. Then

$$\Vdash_{\mathbb{P}_\kappa} \check{\lambda} \text{ is atomlessly-measurable.}$$

555E Theorem Let λ be a two-valued-measurable cardinal, and $\mathcal{I}$ a λ-additive maximal proper ideal of $\mathcal{P}\lambda$ containing singletons; let μ be the $\{0,1\}$-valued probability measure on λ with null ideal $\mathcal{I}$. Let $\kappa \ge \lambda$ be a cardinal, and define $\dot{\mu}$ from μ as in 555C. Set $\theta = \mathrm{Tr}_{\mathcal{I}}(\lambda; \kappa)$. Then

$$\Vdash_{\mathbb{P}_\kappa} \dot{\mu} \text{ is Maharam-type-homogeneous with Maharam type } \check{\theta}.$$

555F Proposition Let λ be a two-valued-measurable cardinal and $\kappa > 0$. Let μ be a normal witnessing probability on λ and $\dot{\mu}$ the corresponding $\mathbb{P}_\kappa$-name for a measure on $\check{\lambda}$, as in 555C. Then

$\Vdash_{\mathbb{P}_\kappa}$ the covering number of the null ideal of the product measure $\dot{\mu}^{\mathbb{N}}$ on $\check{\lambda}^{\mathbb{N}}$ is $\check{\lambda}$.

555G Cohen forcing: Theorem Let λ be a two-valued-measurable cardinal and $\kappa \ge \lambda$ a cardinal. Let $\mathcal{I}$ be a λ-additive maximal proper ideal of subsets of λ, and $\dot{\mathcal{I}}$ a $\mathbb{Q}_\kappa$-name for the ideal of subsets of $\check{\lambda}$ generated by $\check{\mathcal{I}}$. Set $\theta = \mathrm{Tr}_{\mathcal{I}}(\lambda;\kappa)$. Then

$$\Vdash_{\mathbb{Q}_\kappa} \mathcal{P}\lambda/\dot{\mathcal{I}} \cong \mathfrak{G}_{\check{\theta}}.$$

555H Corollary Suppose that λ is a two-valued-measurable cardinal and $\kappa = 2^\lambda$. Then

$\Vdash_{\mathbb{Q}_\kappa}$ there is a non-trivial atomless σ-centered power set σ-quotient algebra.

555I Definition A Boolean algebra $\mathfrak{A}$ has the **Egorov property** if whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of countable partitions of unity in $\mathfrak{A}$ then there is a countable partition B of unity such that $\{a : a \in A_n, a \cap b \ne 0\}$ is finite for every $b \in B$ and $n \in \mathbb{N}$.

555J Lemma (a) Suppose that X is a set and $\#(X) < \mathfrak{b}$. Then $\mathcal{P}X$ has the Egorov property.

(b) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra with the Egorov property and I a σ-ideal of $\mathfrak{A}$. Then $\mathfrak{A}/I$ has the Egorov property.

(c) A ccc Boolean algebra has the Egorov property iff it is weakly (σ,∞)-distributive.

555K Główczyński's example Let λ be a two-valued-measurable cardinal, and $\mathbb{P}$ a ccc forcing notion such that

$$\Vdash_{\mathbb{P}} \check{\lambda} < \mathfrak{m}.$$

Then, taking $\mathcal{I}$ to be the null ideal of a witnessing measure on λ, and $\dot{\mathcal{I}}$ to be a $\mathbb{P}$-name for the ideal of subsets of $\check{\lambda}$ generated by $\check{\mathcal{I}}$,

$\Vdash_{\mathbb{P}} \mathcal{P}\check{\lambda}/\dot{\mathcal{I}}$ is ccc, atomless, Dedekind complete, weakly (σ,∞)-distributive, has Maharam type ω and is not a Maharam algebra.

555L Supercompact cardinals and the normal measure axiom: Definition An uncountable cardinal κ is **supercompact** if for every set X there is a κ-additive maximal proper ideal $\mathcal{I}$ of subsets of $S = [X]^{<\kappa}$ such that

(α) $\{s : s \in S, x \notin s\} \in \mathcal{I}$ for every $x \in X$,

(β) if $A \subseteq S$, $A \notin \mathcal{I}$ and $f : A \to X$ is such that $f(s) \in s$ for every $s \in A$, then there is an $x \in X$ such that $\{s : s \in A, f(s) = x\} \notin \mathcal{I}$.

555M Proposition A supercompact cardinal is two-valued-measurable.

555N Theorem Suppose that κ is a supercompact cardinal. Then

$\Vdash_{\mathbb{P}_\kappa}$ the normal measure axiom and the product measure extension axiom are true.

555O Theorem If κ is an uncountable cardinal and $\mathcal{I}$ is a proper κ-saturated κ-additive ideal of $\mathcal{P}\kappa$ containing singletons, then

$L(\mathcal{I}) \vDash \kappa$ is two-valued-measurable and the generalized continuum hypothesis is true.

555Z Problems (a) In 555B, what can we say about the π-weight of $\mathcal{P}\check{X}/\dot{\mathcal{I}}$?

(b) Suppose that λ is an atomlessly-measurable cardinal with a normal witnessing probability. Let $\langle A_\eta\rangle_{\eta<\omega_1}$ be a family of non-negligible subsets of λ. Must there be a countable set meeting every A_η?

Version of 3.1.15

556 Forcing with Boolean subalgebras

I propose now to describe a completely different way in which forcing can be used to throw light on problems in measure theory. Rather than finding forcing models of new mathematical universes, we look for models which will express structures of the ordinary universe in new ways. The problems to which this approach seems to be most relevant are those centered on invariant algebras: in ergodic theory, fixed-point algebras; in the theory of relative independence, the core σ-algebras.

Most of the section is taken up with development of basic machinery. The strategic plan is straightforward enough: given a specific Boolean algebra $\mathfrak{C}$ which seems to be central to a question in hand, force with $\mathfrak{C}\setminus\{0\}$, and translate the question into a question in the forcing language. In order to do this, we need an efficient scheme for automatic translation. This is what 556A-556L and 556O are setting up. The translation has to work both ways, since we need to be able to deduce properties of the ground model from properties of the forcing model.

There are four actual theorems for which I offer proofs by these methods. The first three are 556M (a strong law of large numbers), 556N (Dye's theorem on orbit-isomorphic measure-preserving transformations) and 556P (Kawada's theorem on invariant measures). In each of these, the aim is to prove a general form of the theorem from the classical special case in which the algebra $\mathfrak{C}$ is trivial. In the final example 556S (I.Farah's description of the Dedekind completion of the asymptotic density algebra $\mathfrak{Z}$), we have a natural subalgebra $\mathfrak{C}$ of $\mathfrak{Z}$ and a structure in the corresponding forcing universe to which we can apply Maharam's theorem.

556A Forcing with Boolean subalgebras Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$, and $\mathfrak{C}$ a subalgebra of $\mathfrak{A}$. Let $\mathbb{P}$ be the forcing notion $\mathfrak{C}^+=\mathfrak{C}\setminus\{0\}$, active downwards.

(a) If $a\in\mathfrak{A}$, the **forcing name for** a **over** $\mathfrak{C}$ will be the $\mathbb{P}$-name

$$\dot a=\{(\check b,p):p\in\mathfrak{C}^+,\,b\in\mathfrak{A},\,p\cap b\subseteq a\}.$$

(b) If $\mathfrak{B}$ is a Boolean subalgebra of $\mathfrak{A}$ including $\mathfrak{C}$, then the **forcing name for** $\mathfrak{B}$ **over** $\mathfrak{C}$ will be the $\mathbb{P}$-name $\{(\dot b,1):b\in\mathfrak{B}\}$, where here $1=1_{\mathfrak{A}}=1_{\mathfrak{B}}=1_{\mathfrak{C}}$.

(c) For each of the binary operations $\bigcirc=\cap,\ \cup,\ \triangle,\ \setminus$ on $\mathfrak{A}$, the **forcing name for** $\bigcirc$ **over** $\mathfrak{C}$ will be the $\mathbb{P}$-name

$$\dot\bigcirc=\{(((\dot a,\dot b),(a\bigcirc b)^{\cdot}),1):a,\,b\in\mathfrak{A}\}.$$

(d) The **forcing name for** $\subseteq$ **over** $\mathfrak{C}$ will be the $\mathbb{P}$-name

$$\dot\subseteq\ =\{((\dot a,\dot b),1):a,\,b\in\mathfrak{A},\,a\subseteq b\}.$$

(e) Let $\pi:\mathfrak{A}\to\mathfrak{A}$ be a ring homomorphism such that $\pi c\subseteq c$ for every $c\in\mathfrak{C}$. In this case, I will say that the **forcing name for** π **over** $\mathfrak{C}$ is the $\mathbb{P}$-name $\{((\dot a,(\pi a)^{\cdot}),1):a\in\mathfrak{A}\}$.

(f) Now suppose that $\mathfrak{A}$ is Dedekind σ-complete. For $u\in L^0(\mathfrak{A})$, the **forcing name for** u **over** $\mathfrak{C}$ will be the $\mathbb{P}$-name $\{((\check\alpha,[\![u>\alpha]\!]^{\cdot}),1):\alpha\in\mathbb{Q}\}$.

Remark The primary definition in 364Aa speaks of functions from $\mathbb{R}$ to $\mathfrak{A}$. Because $\mathbb{R}$ is inadequately absolute this is not convenient here, and I will move to the alternative version: a member u of $L^0(\mathfrak{A})$ is a family $\langle[\![u>\alpha]\!]\rangle_{\alpha\in\mathbb{Q}}$ in $\mathfrak{A}$ such that

$$[\![u>\alpha]\!]=\sup_{\beta\in\mathbb{Q},\beta>\alpha}[\![u>\beta]\!]\text{ for every }\alpha\in\mathbb{Q},$$

$$\inf_{n\in\mathbb{N}}[\![u>n]\!]=0,\quad\sup_{n\in\mathbb{N}}[\![u>-n]\!]=1.$$

556B Theorem Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$, and $\mathfrak{C}$ a subalgebra of $\mathfrak{A}$. Let $\mathbb{P}$ be the forcing notion $\mathfrak{C}^+$, active downwards, and $\dot{\mathfrak{A}}$ the forcing name for $\mathfrak{A}$ over $\mathfrak{C}$.

(a) If $p \in \mathfrak{C}^+$, a, $b \in \mathfrak{A}$ and $\dot{a}$, $\dot{b}$ are the forcing names of a, b over $\mathfrak{C}$, then

$$p \Vdash_{\mathbb{P}} \dot{a} = \dot{b}$$

iff $\operatorname{upr}(p \cap (a \mathbin{\triangle} b), \mathfrak{C}) = 0$, that is, for every q stronger than p there is an r stronger than q such that $r \cap a = r \cap b$. In particular,

$$p \Vdash_{\mathbb{P}} \dot{a} = \dot{b}$$

whenever $p \cap a = p \cap b$.

(b) Writing $\dot{\bigcirc}$ for the forcing name for $\bigcirc$ over $\mathfrak{C}$,

$$\Vdash_{\mathbb{P}} \dot{\bigcirc} \text{ is a binary operation on } \dot{\mathfrak{A}} \text{ and } \dot{a} \mathbin{\dot{\bigcirc}} \dot{b} = (a \bigcirc b)^{\bullet}$$

for each of the binary operations $\bigcirc = \cap$, $\cup$, $\triangle$ and $\setminus$ and all a, $b \in \mathfrak{A}$.

(c) All the standard identities translate. For instance,

$$\Vdash_{\mathbb{P}} x \mathbin{\dot{\cap}} (y \mathbin{\dot{\triangle}} z) = (x \mathbin{\dot{\cap}} y) \mathbin{\dot{\triangle}} (x \mathbin{\dot{\cap}} z) \text{ for all } x,\ y,\ z \in \dot{\mathfrak{A}}.$$

(d)

$\Vdash_{\mathbb{P}} \dot{\mathfrak{A}}$, with the operations $\dot{\triangle}$, $\dot{\cap}$, $\dot{\cup}$ and $\dot{\setminus}$, is a Boolean algebra, with zero $\dot{0}$ and identity $\dot{1}$.

(e)(i) Writing $\dot{\subseteq}$ for the forcing name for $\subseteq$ over $\mathfrak{C}$,

$$\Vdash_{\mathbb{P}} \dot{\subseteq} \text{ is the inclusion relation in the Boolean algebra } \dot{\mathfrak{A}}.$$

(ii) For a, $b \in \mathfrak{A}$ and $p \in \mathfrak{C}^+$,

$$p \Vdash_{\mathbb{P}} \dot{a} \mathbin{\dot{\subseteq}} \dot{b}$$

iff $\operatorname{upr}(p \cap a \setminus b, \mathfrak{C}) = 0$.

(f) If $\mathfrak{B}$ is a Boolean subalgebra of $\mathfrak{A}$ including $\mathfrak{C}$, then

$$\Vdash_{\mathbb{P}} \dot{\mathfrak{B}} \text{ is a Boolean subalgebra of } \dot{\mathfrak{A}}.$$

556C Theorem Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$, and $\mathfrak{C}$ a subalgebra of $\mathfrak{A}$. Let $\mathbb{P}$ be the forcing notion $\mathfrak{C}^+$, active downwards, $\dot{\mathfrak{A}}$ the forcing name for $\mathfrak{A}$ over $\mathfrak{C}$, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a ring homomorphism such that $\pi c \subseteq c$ for every $c \in \mathfrak{C}$; write $\dot{\pi}$ for the forcing name for π over $\mathfrak{C}$.

(a)(i)

$$\Vdash_{\mathbb{P}} \dot{\pi} \text{ is a ring homomorphism from } \dot{\mathfrak{A}} \text{ to itself}$$

and

$$\Vdash_{\mathbb{P}} \dot{\pi}(\dot{a}) = (\pi a)^{\bullet}$$

for every $a \in \mathfrak{A}$.

(ii) If π is injective, $\Vdash_{\mathbb{P}} \dot{\pi}$ is injective.

(iii) If $\phi : \mathfrak{A} \to \mathfrak{A}$ is another ring homomorphism such that $\phi c \subseteq c$ for every $c \in \mathfrak{C}$, with corresponding forcing name $\dot{\phi}$, then

$$\Vdash_{\mathbb{P}} \dot{\pi}\dot{\phi} = (\pi\phi)^{\bullet}.$$

(b) Now suppose that π is a Boolean homomorphism.

(i) $\pi c = c$ for every $c \in \mathfrak{C}$.

(ii) $\Vdash_{\mathbb{P}} \dot{\pi}$ is a Boolean homomorphism.

(iii) If π is surjective, $\Vdash_{\mathbb{P}} \dot{\pi}$ is surjective.

(iv) If $\pi \in \operatorname{Aut}\mathfrak{A}$ then

$$\Vdash_{\mathbb{P}} \dot{\pi} \text{ is a Boolean automorphism and } (\dot{\pi})^{-1} = (\pi^{-1})^{\bullet}.$$

(v) If the fixed-point subalgebra of π is $\mathfrak{C}$ exactly, then

$$\Vdash_{\mathbb{P}} \text{the fixed-point subalgebra of } \dot\pi \text{ is } \{0,1\}.$$

556D Regularly embedded subalgebras: Proposition Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$, and $\mathfrak{C}$ a regularly embedded subalgebra of $\mathfrak{A}$. Let $\mathbb{P}$ be the forcing notion $\mathfrak{C}^+$, active downwards, and for $a \in \mathfrak{A}$ let $\dot a$ be the forcing name for a over $\mathfrak{C}$.

(a) For $p \in \mathfrak{C}^+$ and a, $b \in \mathfrak{A}$,

$$p \Vdash_{\mathbb{P}} \dot a = \dot b$$

iff $p \cap a = p \cap b$.

(b) Let $\dot\subseteq$ be the forcing name for $\subseteq$ over $\mathfrak{C}$. Then for $p \in \mathfrak{C}^+$ and a, $b \in \mathfrak{A}$,

$$p \Vdash_{\mathbb{P}} \dot a \,\dot\subseteq\, \dot b$$

iff $p \cap a \subseteq b$.

556E Proposition Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$, $\mathfrak{C}$ a regularly embedded subalgebra of $\mathfrak{A}$, $\mathbb{P}$ the forcing notion $\mathfrak{C}^+$, active downwards, and $\dot{\mathfrak{A}}$ the forcing name for $\mathfrak{A}$ over $\mathfrak{C}$; for $a \in \mathfrak{A}$, write $\dot a$ for the forcing name for a over $\mathfrak{C}$.

(a) Let $\dot A$ be a $\mathbb{P}$-name, and set

$$B = \{q \cap a : q \in \mathfrak{C}^+,\ a \in \mathfrak{A},\ q \Vdash_{\mathbb{P}} \dot a \in \dot A\}.$$

Then for $d \in \mathfrak{A}$ and $p \in \mathfrak{C}^+$,

$$p \Vdash_{\mathbb{P}} \dot d \text{ is an upper bound for } \dot A \cap \dot{\mathfrak{A}}$$

iff $p \cap b \subseteq d$ for every $b \in B$, and

$$p \Vdash_{\mathbb{P}} \dot d = \sup(\dot A \cap \dot{\mathfrak{A}})$$

iff $p \cap d = \sup_{b \in B} p \cap b$.

(b)(i) If $\langle a_i \rangle_{i \in I}$ is a family in $\mathfrak{A}$ with supremum a, then

$$\Vdash_{\mathbb{P}} \dot a = \sup_{i \in \check I} \dot a_i.^1$$

(ii) If $\langle a_i \rangle_{i \in I}$ is a family in $\mathfrak{A}$ with infimum a, then

$$\Vdash_{\mathbb{P}} \dot a = \inf_{i \in \check I} \dot a_i.$$

(c) $\Vdash_{\mathbb{P}} \operatorname{sat}(\dot{\mathfrak{A}}) \le \operatorname{sat}(\mathfrak{A})\check{}$.[2]

(d) $\Vdash_{\mathbb{P}} \tau(\dot{\mathfrak{A}}) \le \tau(\mathfrak{A})\check{}$.

556F Quotient forcing: Proposition Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$, and $\mathfrak{C}$ a subalgebra of $\mathfrak{A}$. Let $\mathbb{P}$ be the forcing notion $\mathfrak{C}^+$, active downwards, and $\dot{\mathfrak{A}}$ the forcing name for $\mathfrak{A}$ over $\mathfrak{C}$.

(a) Consider the $\mathbb{P}$-names

$$\dot\psi = \{((\check a, \dot a), 1) : a \in \mathfrak{A}\}, \quad \dot{\mathcal{I}} = \{(\check a, p) : p \in \mathfrak{C}^+,\ a \in \mathfrak{A},\ p \cap a = 0\}.$$

Then

$$\Vdash_{\mathbb{P}} \dot\psi \text{ is a Boolean homomorphism from } \check{\mathfrak{A}} \text{ onto } \dot{\mathfrak{A}}, \text{ and its kernel is } \dot{\mathcal{I}}.$$

(b) Now suppose that $\mathfrak{C}$ is regularly embedded in $\mathfrak{A}$. Set $\dot{\mathbb{Q}} = (\dot{\mathfrak{A}}^+, \dot\subseteq, \dot 1, \check\downarrow)$ and let $\mathbb{P} * \dot{\mathbb{Q}}$ be the iterated forcing notion. Then $\mathrm{RO}(\mathbb{P} * \dot{\mathbb{Q}})$ is isomorphic to the Dedekind completion of $\mathfrak{A}$.

(c) Suppose that $\mathfrak{C}$ is regularly embedded in $\mathfrak{A}$ and that $\mathfrak{B}$ is a Boolean algebra such that

$$\Vdash_{\mathbb{P}} \dot{\mathfrak{A}} \cong \check{\mathfrak{B}}.$$

Then the Dedekind completion $\widehat{\mathfrak{A}}$ of $\mathfrak{A}$ is isomorphic to the Dedekind completion $\mathfrak{C} \widehat\otimes \mathfrak{B}$ of the free product $\mathfrak{C} \otimes \mathfrak{B}$ of $\mathfrak{C}$ and $\mathfrak{B}$.

[1] See 5A3F for a note on the interpretation of formulae of this kind.

[2] Of course I am not asserting here that '$\Vdash_{\mathbb{P}} \operatorname{sat}(\mathfrak{A})\check{}$ is a cardinal', only that '$\Vdash_{\mathbb{P}} \operatorname{sat}(\dot{\mathfrak{A}})$ is a cardinal and $\operatorname{sat}(\mathfrak{A})\check{}$ is an ordinal'.

556G Proposition Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, not $\{0\}$, $\mathfrak{C}$ a regularly embedded subalgebra of $\mathfrak{A}$, $\mathbb{P}$ the forcing notion $\mathfrak{C}^+$, active downwards, and $\dot{\mathfrak{A}}$ the forcing name for $\mathfrak{A}$ over $\mathfrak{C}$.

(a) Whenever $p \in \mathfrak{C}^+$ and $\dot{x}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{x} \in \dot{\mathfrak{A}},$$

there is an $a \in \mathfrak{A}$ such that

$$p \Vdash_{\mathbb{P}} \dot{x} = \dot{a},$$

where $\dot{a}$ is the forcing name for a over $\mathfrak{C}$.

(b) $\Vdash_{\mathbb{P}} \dot{\mathfrak{A}}$ is Dedekind complete.

556H $L^0(\mathfrak{A})$: Proposition Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, not $\{0\}$, $\mathfrak{C}$ a regularly embedded subalgebra of $\mathfrak{A}$, $\mathbb{P}$ the forcing notion $\mathfrak{C}^+$, active downwards, and $\dot{\mathfrak{A}}$ the forcing name for $\mathfrak{A}$ over $\mathfrak{C}$. For $a \in \mathfrak{A}$ let $\dot{a}$ be the forcing name for a over $\mathfrak{C}$.

(a)(i) For every $u \in L^0(\mathfrak{A})$,

$$\Vdash_{\mathbb{P}} \dot{u} \in L^0(\dot{\mathfrak{A}})$$

where $\dot{u}$ is the forcing name for u over $\mathfrak{C}$.

(ii) If u, $v \in L^0(\mathfrak{A})$ and $\Vdash_{\mathbb{P}} \dot{u} = \dot{v}$, then $u = v$.

(b) For u, $v \in L^0(\mathfrak{A})$ and $\alpha \in \mathbb{R}$,

$$\begin{aligned} \Vdash_{\mathbb{P}} \dot{u} + \dot{v} &= (u+v)^{\bullet}, \\ -\dot{u} &= (-u)^{\bullet}, \\ \dot{u} \vee \dot{v} &= (u \vee v)^{\bullet}, \\ \dot{u} \times \dot{v} &= (u \times v)^{\bullet}, \\ \check{\alpha}\dot{u} &= (\alpha u)^{\bullet}. \end{aligned}$$

If $u \le v$, then $\Vdash_{\mathbb{P}} \dot{u} \le \dot{v}$.

(c) If $\langle u_i \rangle_{i \in I}$ is a family in $L^0(\mathfrak{A})$ with supremum $u \in L^0(\mathfrak{A})$, then

$$\Vdash_{\mathbb{P}} \dot{u} = \sup_{i \in \check{I}} \dot{u}_i \text{ in } L^0(\dot{\mathfrak{A}}).$$

(d) If $p \in \mathfrak{C}^+$ and $\dot{w}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{w} \in L^0(\dot{\mathfrak{A}})$, then there is a $u \in L^0(\mathfrak{A})$ such that

$$p \Vdash_{\mathbb{P}} \dot{w} = \dot{u}.$$

(e) If $\langle u_n \rangle_{n \in \mathbb{N}}$ is a sequence in $L^0(\mathfrak{A})$, then the following are equiveridical:

(i) $\langle u_n \rangle_{n \in \mathbb{N}}$ is order*-convergent to 0,

(ii) $\Vdash_{\mathbb{P}} \langle \dot{u}_n \rangle_{n \in \mathbb{N}}$ is order*-convergent to 0.

556I Proposition Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$, and $\mathfrak{C}$ a regularly embedded subalgebra of $\mathfrak{A}$. Let $\mathbb{P}$ be the forcing notion $\mathfrak{C}^+$, active downwards, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a Boolean homomorphism fixing every point of $\mathfrak{C}$; let $\dot{\pi}$ be the forcing name for π over $\mathfrak{C}$.

(a) π is injective iff $\Vdash_{\mathbb{P}} \dot{\pi}$ is injective.

(b) If π is order-continuous, then

$$\Vdash_{\mathbb{P}} \dot{\pi} \text{ is order-continuous.}$$

(c) If π has a support $\operatorname{supp} \pi$, then

$$\Vdash_{\mathbb{P}} (\operatorname{supp} \pi)^{\bullet} \text{ is the support of } \dot{\pi}.$$

556J Theorem Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, not $\{0\}$, and $\mathfrak{C}$ a regularly embedded subalgebra of $\mathfrak{A}$. Let $\mathbb{P}$ be the forcing notion $\mathfrak{C}^+$, active downwards, and $\dot{\mathfrak{A}}$ the forcing name for $\mathfrak{A}$ over $\mathfrak{C}$.

(a) If $\dot{\theta}$ is a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \dot{\theta} \text{ is a ring homomorphism from } \dot{\mathfrak{A}} \text{ to itself,}$$

then there is a unique ring homomorphism $\pi : \mathfrak{A} \to \mathfrak{A}$ such that $\pi c \subseteq c$ for every $c \in \mathfrak{C}$ and

$$\Vdash_{\mathbb{P}} \dot{\theta} = \dot{\pi},$$

where $\dot{\pi}$ is the forcing name for π over $\mathfrak{C}$.

(b)(i) If

$$\Vdash_{\mathbb{P}} \dot{\theta} \text{ is a Boolean homomorphism,}$$

then π is a Boolean homomorphism, and $\pi c = c$ for every $c \in \mathfrak{C}$.

(ii) If

$$\Vdash_{\mathbb{P}} \dot{\theta} \text{ is a Boolean automorphism,}$$

that π is a Boolean automorphism.

556K Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, and $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$; let $\mathbb{P}$ be the forcing notion $\mathfrak{C}^+$, active downwards, and $\dot{\mathfrak{A}}$ the forcing name for $\mathfrak{A}$ over $\mathfrak{C}$. We can identify $\mathfrak{C}$ with the regular open algebra $\mathrm{RO}(\mathbb{P})$. For $u \in L^0(\mathfrak{C})$ write $\vec{u}$ for the corresponding $\mathbb{P}$-name for a real number as described in 5A3L.

(a)(i) For each $a \in \mathfrak{A}$ there is a $u_a \in L^1(\mathfrak{C}, \bar{\mu}{\restriction}\mathfrak{C})$ defined by saying that $\int_c u_a = \bar{\mu}(a \cap c)$ for every $c \in \mathfrak{C}$.

(ii) If $p \in \mathfrak{C}^+$ and a, $b \in \mathfrak{A}$ are such that

$$p \Vdash_{\mathbb{P}} \dot{a} = \dot{b}$$

(where $\dot{a}$, $\dot{b}$ are the forcing names for a, b over $\mathfrak{C}$), then

$$p \Vdash_{\mathbb{P}} \vec{u}_a = \vec{u}_b.$$

(b) There is a $\mathbb{P}$-name $\dot{\bar{\mu}}$ such that

$$\Vdash_{\mathbb{P}} (\dot{\mathfrak{A}}, \dot{\bar{\mu}}) \text{ is a probability algebra,}$$

and

$$\Vdash_{\mathbb{P}} \dot{\bar{\mu}}\dot{a} = \vec{u}_a$$

whenever $a \in \mathfrak{A}$ and $\dot{a}$ is the corresponding forcing name over $\mathfrak{C}$.

(c) If $\pi : \mathfrak{A} \to \mathfrak{A}$ is a measure-preserving Boolean homomorphism such that $\pi c = c$ for every $c \in \mathfrak{C}$, and $\dot{\pi}$ the corresponding forcing name over $\mathfrak{C}$, then

$$\Vdash_{\mathbb{P}} \dot{\pi} : \dot{\mathfrak{A}} \to \dot{\mathfrak{A}} \text{ is measure-preserving.}$$

(d) If $\dot{\phi}$ is a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \dot{\phi} : \dot{\mathfrak{A}} \to \dot{\mathfrak{A}} \text{ is a measure-preserving Boolean automorphism}$$

then there is a measure-preserving Boolean automorphism $\pi : \mathfrak{A} \to \mathfrak{A}$ such that $\pi c = c$ for every $c \in \mathfrak{C}$ and

$$\Vdash_{\mathbb{P}} \dot{\phi} = \dot{\pi}.$$

(e) If $v \in L^1(\mathfrak{A}, \bar{\mu})$ and $u \in L^1(\mathfrak{C}, \bar{\mu}{\restriction}\mathfrak{C})$ is its conditional expectation on $\mathfrak{C}$, then

$$\Vdash_{\mathbb{P}} \dot{v} \in L^1(\dot{\mathfrak{A}}, \dot{\bar{\mu}}) \text{ and } \int \dot{v}\, d\dot{\bar{\mu}} = \vec{u}.$$

556L Relatively independent subalgebras Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra and $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$; let $\mathbb{P}$ be the forcing notion $\mathfrak{C}^+$, active downwards. Let $\dot{\bar{\mu}}$ be the forcing name for $\bar{\mu}$ described in 556K, so that $\Vdash_{\mathbb{P}} (\dot{\mathfrak{A}}, \dot{\bar{\mu}})$ is a probability algebra.

(a) For a subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ including $\mathfrak{C}$, let $\dot{\mathfrak{B}}$ be the forcing name for $\mathfrak{B}$ over $\mathfrak{C}$. If $\langle \mathfrak{B}_i \rangle_{i \in I}$ is a family of subalgebras of $\mathfrak{A}$ including $\mathfrak{C}$, then $\langle \mathfrak{B}_i \rangle_{i \in I}$ is relatively independent over $\mathfrak{C}$ iff

$$\Vdash_{\mathbb{P}} \langle \dot{\mathfrak{B}}_i \rangle_{i \in \check{I}} \text{ is stochastically independent in } \dot{\mathfrak{A}}.$$

(b) If $\langle v_i \rangle_{i \in I}$ is a family in $L^0(\mathfrak{A})$ which is relatively independent over $\mathfrak{C}$, then

$$\Vdash_{\mathbb{P}} \langle \dot{v}_i \rangle_{i \in \check{I}} \text{ is stochastically independent}$$

(writing $\dot{v}_i$ for the forcing name for v_i over $\mathfrak{C}$).

556M Laws of large numbers Consider the two statements

($\ddagger$) Let (X,Σ,μ) be a probability space, T a σ-subalgebra of Σ, and $\langle f_n\rangle_{n\in\mathbb{N}}$ a sequence in $\mathcal{L}^2(\mu)$ such that $\langle f_n\rangle_{n\in\mathbb{N}}$ is relatively independent over T and $\int_F f_n d\mu=0$ for every $n\in\mathbb{N}$ and every $F\in\mathrm{T}$. Suppose that $\langle\beta_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $]0,\infty[$, diverging to ∞, such that $\sum_{n=0}^{\infty}\frac{1}{\beta_n^2}\|f_n\|_2^2<\infty$. Then $\lim_{n\to\infty}\frac{1}{\beta_n}\sum_{i=0}^n f_i=0$ a.e.

and

($\dagger$) Let (X,Σ,μ) be a probability space and $\langle f_n\rangle_{n\in\mathbb{N}}$ an independent sequence in $\mathcal{L}^2(\mu)$ such that $\int f_n d\mu=0$ for every $n\in\mathbb{N}$. Suppose that $\langle\beta_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $]0,\infty[$, diverging to ∞, such that $\sum_{n=0}^{\infty}\frac{1}{\beta_n^2}\|f_n\|_2^2<\infty$. Then $\lim_{n\to\infty}\frac{1}{\beta_n}\sum_{i=0}^n f_i=0$ a.e.

Then if there is a proof of ($\dagger$), there must be a proof of ($\ddagger$).

556N Dye's theorem Let me state two versions of Dye's theorem: the 'full' version

($\ddagger$) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra of countable Maharam type, $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$, and π_1, π_2 two measure-preserving automorphisms of $\mathfrak{A}$ with fixed-point algebra $\mathfrak{C}$. Then there is a measure-preserving automorphism ϕ of $\mathfrak{A}$ such that $\phi c=c$ for every $c\in\mathfrak{C}$ and π_1 and $\phi\pi_2\phi^{-1}$ generate the same full subgroups of $\operatorname{Aut}\mathfrak{A}$.

and the 'simple' version

($\dagger$) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra of countable Maharam type, and π_1, π_2 two ergodic measure-preserving automorphisms of $\mathfrak{A}$. Then there is a measure-preserving automorphism ϕ of $\mathfrak{A}$ such that π_1 and $\phi\pi_2\phi^{-1}$ generate the same full subgroups of $\operatorname{Aut}\mathfrak{A}$.

Here the machinery of this section provides a proof of ($\ddagger$) from ($\dagger$).

556O Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, not $\{0\}$, and $\mathfrak{C}$ a regularly embedded subalgebra of $\mathfrak{A}$; let $\mathbb{P}$ be the forcing notion $\mathfrak{C}^+$, active downwards. Let $\dot{\mathfrak{A}}$ be the forcing name for $\mathfrak{A}$ over $\mathfrak{C}$, and for a ring homomorphism $\pi:\mathfrak{A}\to\mathfrak{A}$ such that $\pi c\subseteq c$ for every $c\in\mathfrak{C}$ let $\dot\pi$ be the forcing name for π over $\mathfrak{C}$. Let G be a subgroup of $\operatorname{Aut}\mathfrak{A}$ such that every point of $\mathfrak{C}$ is fixed by every member of G, and $\dot G$ the $\mathbb{P}$-name $\{(\dot\pi,1):\pi\in G\}$.

(a) $\Vdash_{\mathbb{P}}\dot G$ is a subgroup of $\operatorname{Aut}\dot{\mathfrak{A}}$.

(b) If $\phi:\mathfrak{A}\to\mathfrak{A}$ is a ring homomorphism such that $\phi c\subseteq c$ for every $c\in\mathfrak{C}$, and

$$\Vdash_{\mathbb{P}}\dot\phi\text{ belongs to the full local semigroup generated by }\dot G,$$

then ϕ belongs to the full local semigroup generated by G.

556P Kawada's theorem In the same way as in 556M and 556N, we have two versions of 395P:

($\ddagger$) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a fully non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$, with fixed-point subalgebra $\mathfrak{C}$, such that $\mathfrak{C}$ is a measurable algebra. Then there is a strictly positive G-invariant countably additive real-valued functional on $\mathfrak{A}$.

and

($\dagger$) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra such that $\operatorname{Aut}\mathfrak{A}$ has a subgroup G which is ergodic and fully non-paradoxical. Then there is a strictly positive G-invariant countably additive real-valued functional on $\mathfrak{A}$.

Once again, we can prove ($\ddagger$) from ($\dagger$).

556Q Lemma (a) Let $\mathfrak{A}$ be a Boolean algebra and $\bar\mu:\mathfrak{A}\to[0,1]$ a strictly positive additive functional such that $\bar\mu 1=1$. Suppose that whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$, there is an $a\in\mathfrak{A}$ such that $a\subseteq a_n$ for every n and $\bar\mu a=\inf_{n\in\mathbb{N}}\bar\mu a_n$. Then $(\mathfrak{A},\bar\mu)$ is a probability algebra.

(b) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra. Suppose that $\kappa\ge\tau(\mathfrak{A})$ is an infinite cardinal and that $\langle e_\xi\rangle_{\xi<\kappa}$ is a family in $\mathfrak{A}$ such that $\bar\mu(\inf_{\xi\in K}e_\xi)=2^{-\#(K)}$ for every finite $K\subseteq I$. Then $(\mathfrak{A},\bar\mu)$ is isomorphic to the measure algebra $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$ of the usual measure on $\{0,1\}^\kappa$.

556R Proposition Let $\mathbb{P}$ be a countably closed forcing notion. Then, for any set I, writing $(\mathfrak{B}_I, \bar{\nu}_I)$ for the measure algebra of the usual measure on $\{0,1\}^I$,

$$\Vdash_{\mathbb{P}} (\mathfrak{B}_{\check{I}}, \bar{\nu}_{\check{I}}) \cong (\check{\mathfrak{B}}_I, \check{\bar{\nu}}_I).$$

556S Theorem Let $\mathcal{Z}$ be the ideal of subsets of $\mathbb{N}$ with asymptotic density 0 and $\mathfrak{Z}$ the asymptotic density algebra $\mathcal{P}\mathbb{N}/\mathcal{Z}$. Then the Dedekind completion of $\mathfrak{Z}$ is isomorphic to the Dedekind completion of the free product $(\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}) \otimes \mathfrak{B}_{\mathfrak{c}}$.