Version of 7.1.15

Chapter 55

Possible worlds

In my original plans for this volume, I hoped to cover the most important consistency proofs relating to undecidable questions in measure theory. Unhappily my ignorance of forcing means that for the majority of results I have nothing useful to offer. I have therefore restricted my account to the very narrow range of ideas in which I feel I have achieved some understanding beyond what I have read in the standard texts.

For a measure theorist, by far the most important forcings are those of 'adding random reals'. I give three sections (§§552-553 and 555) to these. Without great difficulty, we can determine the behaviour of the cardinals in Cichoń's diagram (552B, 552C, 552F-552I), at least if many random reals are added. Going deeper, there are things to be said about outer measure and Sierpiński sets (552D, 552E), and extensions of Radon measures (552N). In the same section I give a version of the fundamental result that simple iteration of random real forcings gives random real forcings (552P). In §553 I collect results which are connected with other topics dealt with above (Rothberger's property, precalibers, ultrafilters, cellularity, trees, medial limits, universally measurable sets) and in which the arguments seem to me to develop properties of measure algebras which may be of independent interest. In preparation for this work, and also for §554, I start with a section (§551) devoted to a rather technical general account of forcings with quotients of σ-algebras of sets, aiming to find effective representations of names for points, sets, functions, measure algebras and filters.

Very similar ideas can also take us a long way with Cohen real forcing (§554). Here I give little more than obvious parallels to the first part of §552, with an account of Freese-Nation numbers sufficient to support Carlson's theorem that a Borel lifting for Lebesgue measure can exist when the continuum hypothesis is false (554I).

One of the most remarkable applications of random reals is in Solovay's proof that if it is consistent to suppose that there is a two-valued-measurable cardinal, then it is consistent to suppose that there is an atomlessly-measurable cardinal (555D). By taking a bit of trouble over the lemmas, we can get a good deal more, including the corresponding theorem relating supercompact cardinals to the normal measure axiom (555N); and similar techniques show the possibility of interesting power set σ-quotient algebras (555G, 555K).

I end the chapter with something quite different (§556). A familiar phenomenon in ergodic theory is that once one has proved a theorem for ergodic transformations one can expect, possibly at the cost of substantial effort, but without having to find any really new idea, a corresponding result for general measure-preserving transformations. There is more than one way to look at this, but here I present a method in which the key step, in each application, is an appeal to the main theorem of forcing. A similar approach gives a description of the completion of the asymptotic density algebra. The technical details take up a good deal of space, but are based on the same principles as those in §551, and are essentially straightforward.

Version of 2.12.13

551 Forcing with quotient algebras

In preparation for the discussion of random real forcing in the next two sections, I introduce some techniques which can be applied whenever a forcing notion is described in terms of a Loomis-Sikorski representation of its regular open algebra. The first step is just a translation of the correspondence between names for real numbers in the forcing language and members of $L^0(\mathrm{RO}(\mathbb{P}))$, as described in 5A3L, when $L^0(\mathrm{RO}(\mathbb{P}))$ can be identified with a quotient of a space $L^0(\Sigma)$ of measurable functions. More care is needed, but we can find a similar formulation of names for members of $\{0,1\}^I$ for any set I (551C). Going a step farther, it turns out that there are very useful descriptions of Baire subsets of $\{0,1\}^I$ (551D-551F), Baire measurable functions (551N), the usual measure on $\{0,1\}^I$ (551I-551J) and its measure algebra (551P). In some special cases, these methods can be used to represent iterated forcing notions (551Q). I end with a construction for a forcing extension of a filter on a countable set (551R).

551A Definition (a) A **measurable space with negligibles** is a triple $(\Omega,\Sigma,\mathcal{I})$ where Ω is a set, Σ is a σ-algebra of subsets of Ω and $\mathcal{I}$ is a σ-ideal of subsets of Ω generated by $\Sigma\cap\mathcal{I}$. In this case $\mathfrak{A}=\Sigma/\Sigma\cap\mathcal{I}$ is a Dedekind σ-complete Boolean algebra (314C).

In this context I will use the phrase '$\mathcal{I}$-almost everywhere' to mean 'except on a set belonging to $\mathcal{I}$'.

(b) I will say that $(\Omega,\Sigma,\mathcal{I})$ is **non-trivial** if $\Omega\notin\mathcal{I}$, so that $\mathfrak{A}\ne\{0\}$. In this case, the forcing notion $\mathbb{P}$ **associated** with $(\Omega,\Sigma,\mathcal{I})$ is $(\mathfrak{A}^+,\subseteq,\Omega^\bullet,\downarrow)$ (5A3M). If $\mathfrak{A}$ is Dedekind complete we can identify $\mathfrak{A}$ with the regular open algebra $\mathrm{RO}(\mathbb{P})$ (514Sb, 5A3M).

(c) I will say that $(\Omega,\Sigma,\mathcal{I})$ is ω_1**-saturated** if $\Sigma\cap\mathcal{I}$ is ω_1-saturated in Σ in the sense of 541A, that is, if there is no uncountable disjoint family in $\Sigma\setminus\mathcal{I}$, that is, if $\mathfrak{A}$ and $\mathbb{P}$ are ccc. In this case, $\mathfrak{A}$ is Dedekind complete (316Fa, 541B).

(d) I will say that $(\Omega,\Sigma,\mathcal{I})$ is **complete** if $\mathcal{I}\subseteq\Sigma$ (cf. 211A).

Remark For an account of the general theory of measurable spaces with negligibles, see Fremlin 87.

551B Definition Let $(\Omega,\Sigma,\mathcal{I})$ be a non-trivial measurable space with negligibles with a Dedekind complete quotient algebra $\mathfrak{A}$, and $\mathbb{P}$ its associated forcing notion. Recall from 364Ib that $L^0(\mathfrak{A})$ can be regarded as a quotient of the space of Σ-measurable functions from Ω to $\mathbb{R}$. If $h:\Omega\to\mathbb{R}$ is Σ-measurable, write $\vec{h}=(h^\bullet)^\rightarrow$ where $h^\bullet$ is the equivalence class of h in $L^0(\mathfrak{A})$, identified with $L^0(\mathrm{RO}(\mathbb{P}))$, and $(h^\bullet)^\rightarrow$ is the $\mathbb{P}$-name for a real number as defined in 5A3L. Then

$$\Vdash_{\mathbb{P}}\vec{h}\text{ is a real number,}$$

and for any $\alpha\in\mathbb{Q}$

$$[\![\vec{h}>\check{\alpha}]\!]=[\![(h^\bullet)^\rightarrow>\check{\alpha}]\!]=[\![h^\bullet>\alpha]\!]=\{\omega:h(\omega)>\alpha\}^\bullet.$$

From 5A3Lc, we see that if h_0, h_1 are Σ-measurable real-valued functions on Ω, then

$$\Vdash_{\mathbb{P}}(h_0+h_1)^\rightarrow=\vec{h}_0+\vec{h}_1,\ (h_0\times h_1)^\rightarrow=\vec{h}_0\times\vec{h}_1,$$

and that if $\langle h_n\rangle_{n\in\mathbb{N}}$ is a sequence of measurable functions with limit h,

$$\Vdash_{\mathbb{P}}\vec{h}=\lim_{n\to\infty}\vec{h}_n\text{ in }\mathbb{R}.$$

551C Definition Let $(\Omega,\Sigma,\mathcal{I})$ be a non-trivial measurable space with negligibles with a Dedekind complete quotient algebra $\mathfrak{A}$, and $\mathbb{P}$ its associated forcing notion.

(a) If $f:\Omega\to\{0,1\}$ is Σ-measurable, let $\vec{f}$ be the $\mathbb{P}$-name

$$\{(\check{\imath},f^{-1}[\{i\}]^\bullet):i\in\{0,1\},\ f^{-1}[\{i\}]\notin\mathcal{I}\}.$$

Then $\Vdash_{\mathbb{P}}\vec{f}\in\{0,1\}$ and $[\![\vec{f}=\check{\imath}]\!]=f^{-1}[\{i\}]^\bullet$ for both i. (I will try always to make it clear when this definition of $\vec{f}$ is intended to overrule the definition in 551B; but we see from 551Xf that any confusion is unlikely to matter.)

Observe that if a $\mathbb{P}$-name $\dot{x}$ and $p\in\mathfrak{A}^+$ are such that $p\Vdash_{\mathbb{P}}\dot{x}\in\{0,1\}$, then there is a measurable $f:\Omega\to\{0,1\}$ such that $p\Vdash_{\mathbb{P}}\dot{x}=\vec{f}$; take $f=\chi E$ where $E\in\Sigma$ is such that $E^\bullet=[\![\dot{x}=1]\!]$ in $\mathfrak{A}$.

(b) Now let I be any set, and $f:\Sigma\to\{0,1\}^I$ a $(\Sigma,\mathcal{B}\mathfrak{a}_I)$-measurable function, where $\mathcal{B}\mathfrak{a}_I=\mathcal{B}\mathfrak{a}(\{0,1\}^I)$ is the Baire σ-algebra of $\{0,1\}^I$, that is, the σ-algebra of subsets of $\{0,1\}^I$ generated by sets of the form $\{x:x\in\{0,1\}^I,\ x(i)=1\}$ for $i\in I$ (4A3Na). For each $i\in I$, set $f_i(\omega)=f(\omega)(i)$ for $\omega\in\Omega$; then $f_i:\Omega\to\{0,1\}$ is measurable, so we have a $\mathbb{P}$-name $\vec{f_i}$ as in (a). Let $\vec{f}$ be the $\mathbb{P}$-name $\{(\langle\vec{f_i}\rangle_{i\in\check{I}},\mathbb{1})\}$ (interpreting the subformula $\langle\dots\rangle_{i\in\check{I}}$ in the forcing language, of course, by the convention of 5A3Fb). Then

$$\Vdash_{\mathbb{P}} \vec{f} \in \{0,1\}^{\check{I}},$$

and for every $i \in I$

$$\Vdash_{\mathbb{P}} \vec{f}(\check{\imath}) = \vec{f_i}.$$

(c) In the other direction, if a $\mathbb{P}$-name $\dot{x}$ and $p \in \mathfrak{A}^+$ are such that $p \Vdash_{\mathbb{P}} \dot{x} \in \{0,1\}^{\check{I}}$, then for each $i \in I$ we have a $\mathbb{P}$-name $\dot{x}(\check{\imath})$ and a measurable $f_i : \Omega \to \{0,1\}$ such that $p \Vdash_{\mathbb{P}} \dot{x}(\check{\imath}) = \vec{f_i}$; setting $f(\omega) = \langle f_i(\omega)\rangle_{i\in I}$ for $\omega \in \Omega$, f is $(\Sigma, \mathcal{B}\mathfrak{a}_I)$-measurable and $p \Vdash_{\mathbb{P}} \vec{f} = \dot{x}$.

(d) I ought to remark that there is a problem with equality for the $\mathbb{P}$-names $\vec{f}$. If, in the context of (b)-(c) above, we have two $(\Sigma, \mathcal{B}\mathfrak{a}_I)$-measurable functions f and g, and if $p \in \mathfrak{A}^+$, then

$$\begin{aligned} p \Vdash_{\mathbb{P}} \vec{f} = \vec{g} &\iff \text{ for every } i \in I,\ p \Vdash_{\mathbb{P}} \vec{f_i} = \vec{g}_i \\ &\iff \text{ for every } i \in I,\ p \subseteq \{\omega : f_i(\omega) = g_i(\omega)\}^\bullet \text{ in } \mathfrak{A}. \end{aligned}$$

In particular, $\Vdash_{\mathbb{P}} \vec{f} = \vec{g}$ iff $f_i = g_i$ $\mathcal{I}$-a.e. for every $i \in I$. If I is uncountable we can easily have $\Vdash_{\mathbb{P}} \vec{f} = \vec{g}$ while $f(\omega) \ne g(\omega)$ for every $\omega \in \Omega$. But if I is countable then we shall have

$$p \Vdash_{\mathbb{P}} \vec{f} = \vec{g} \iff p \subseteq \{\omega : f(\omega) = g(\omega)\}^\bullet.$$

For a context in which these considerations are vital, see (a-ii) of the proof of 551E.

(e) Suppose that x is any point of $\{0,1\}^I$. Then we have a corresponding $\mathbb{P}$-name $\check{x}$, and $\Vdash_{\mathbb{P}} \check{x} \in \{0,1\}^{\check{I}}$. For each $i \in I$, $\Vdash_{\mathbb{P}} \check{x}(\check{\imath}) = x(i)\check{} \in \{0,1\}$. If we set $e_x(\omega) = x$ for every $\omega \in \Omega$, then $e_x(\omega)(i) = x(i)$ for every $i \in I$ and $\omega \in \Omega$, so $\Vdash_{\mathbb{P}} \vec{e}_x(\check{\imath}) = x(i)\check{}$ for every $i \in I$, and $\Vdash_{\mathbb{P}} \vec{e}_x = \check{x}$.

551D Definition Let $(\Omega, \Sigma, \mathcal{I})$ be a non-trivial measurable space with negligibles with a Dedekind complete quotient algebra, and $\mathbb{P}$ its associated forcing notion. Let I be any set. If $W \subseteq \Omega \times \{0,1\}^I$, let $\vec{W}$ be the $\mathbb{P}$-name

$$\begin{aligned} \{(\vec{f}, E^\bullet) : E \in \Sigma \setminus \mathcal{I},\ f : \Omega \to \{0,1\}^I &\text{ is } (\Sigma, \mathcal{B}\mathfrak{a}_I)\text{-measurable}, \\ &(\omega, f(\omega)) \in W \text{ for every } \omega \in E\}, \end{aligned}$$

interpreting $\vec{f}$ as in 551C. I give the definition for arbitrary sets W, but it is useful primarily when $W \in \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$, as in most of the next proposition. Perhaps I can note straight away that

$$\Vdash_{\mathbb{P}} \vec{W} \subseteq \{0,1\}^{\check{I}}$$

and that if $W = \Omega \times \{0,1\}^I$ then

$$\Vdash_{\mathbb{P}} \vec{W} = \{0,1\}^{\check{I}}$$

(using 551Cb-551Cc).

551E Proposition Let $(\Omega, \Sigma, \mathcal{I})$ be a non-trivial measurable space with negligibles with a Dedekind complete quotient algebra $\mathfrak{A}$, $\mathbb{P}$ its associated forcing notion, and I a set.

(a) If $W \in \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ and $f : \Omega \to \{0,1\}^I$ is $(\Sigma, \mathcal{B}\mathfrak{a}_I)$-measurable, then $\{\omega : (\omega, f(\omega)) \in W\}$ belongs to Σ, and $[\![\vec{f} \in \vec{W}]\!] = \{\omega : (\omega, f(\omega)) \in W\}^\bullet$.

(b) If V, $W \in \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ then

$\Vdash_{\mathbb{P}} \vec{V} \cap \vec{W} = (V \cap W)\vec{}$, $\vec{V} \cup \vec{W} = (V \cup W)\vec{}$, $\vec{V} \setminus \vec{W} = (V \setminus W)\vec{}$ and $\vec{V} \triangle \vec{W} = (V \triangle W)\vec{}$.

(c) If V, $W \subseteq \Omega \times \{0,1\}^I$ and $V \subseteq W$ then

$$\Vdash_{\mathbb{P}} \vec{V} \subseteq \vec{W}.$$

(d) If $\langle W_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ with union W and intersection V, then

$$\Vdash_{\mathbb{P}} \bigcup_{n\in\mathbb{N}} \vec{W}_n = \vec{W} \text{ and } \bigcap_{n\in\mathbb{N}} \vec{W}_n = \vec{V}.$$

(e) Suppose that $J \subseteq I$ is countable, $z \in \{0,1\}^J$, $E \in \Sigma$ and

$$W = \{(\omega, x) : \omega \in E,\, x \in \{0,1\}^I,\, x{\restriction}J = z\}.$$

Then

$$E^\bullet = [\![\vec{W} = \{x : x \in \{0,1\}^{\check{I}},\, \check{z} \subseteq x\}]\!],$$

$$1 \setminus E^\bullet = [\![\vec{W} = \emptyset]\!].$$

proof (a)(i) Let $\mathcal{W}$ be the family of subsets of $\Omega \times \{0,1\}^I$ such that $F_W = \{\omega : (\omega, f(\omega)) \in W\} \in \Sigma$. Then $\mathcal{W}$ is a Dynkin class of subsets of $\Omega \times \{0,1\}^I$, just because Σ is a σ-algebra. If $H \in \Sigma$, $J \subseteq I$ is finite, $z \in \{0,1\}^J$ and $W = \{(\omega, x) : \omega \in H,\, z \subseteq x \in \{0,1\}^I\}$ then $F_W = H \cap \{\omega : f(\omega)(i) = z(i)$ for every $i \in J\}$ belongs to Σ because f is $(\Sigma, \mathcal{B}\mathfrak{a}_I)$-measurable, so $W \in \mathcal{W}$. By the Monotone Class Theorem (136B), $\mathcal{W}$ includes the σ-algebra generated by sets of this form, which is just $\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$.

(ii) Now suppose that $W \in \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$. If $F_W \in \mathcal{I}$ then surely $F_W^\bullet = 0 \subseteq [\![\vec{f} \in \vec{W}]\!]$. If $F_W \notin \mathcal{I}$ then $(\vec{f}, F_W^\bullet) \in \vec{W}$, $F_W^\bullet \Vdash_{\mathbb{P}} \vec{f} \in \vec{W}$ and again $F_W^\bullet \subseteq [\![\vec{f} \in \vec{W}]\!]$.

? I wish to apply 5A3E. If $F_W^\bullet \ne [\![\vec{f} \in \vec{W}]\!]$, set $p = [\![\vec{f} \in \vec{W}]\!] \setminus F_W^\bullet$. Since $p \Vdash_{\mathbb{P}} \vec{f} \in \vec{W}$ there must be a $q \in \mathfrak{A}^+$ and a $\mathbb{P}$-name $\dot{x}$ and an r stronger than both p and q such that

$$r \Vdash_{\mathbb{P}} \dot{x} = \vec{f} \text{ and } (\dot{x}, q) \in \vec{W}.$$

Now there must be a $(\Sigma, \mathcal{B}\mathfrak{a}_I)$-measurable function g and an $E \in \Sigma \setminus \mathcal{I}$ such that $\dot{x} = \vec{g}$, $q = E^\bullet$ and $(\omega, g(\omega)) \in W$ for every $\omega \in E$. In this case, $r = G^\bullet$ for some $G \subseteq E \setminus F_W$, and $r \Vdash_{\mathbb{P}} \vec{f} = \vec{g}$.

Because $W \in \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$, there is a countable set $J \subseteq I$ such that W factors through $\Omega \times \{0,1\}^J$. For each $i \in J$, we have $r \Vdash_{\mathbb{P}} \vec{f}(\check{\imath}) = \vec{g}(\check{\imath})$, that is,

$$r \subseteq [\![\vec{f}(\check{\imath}) = \vec{g}(\check{\imath})]\!] = \{\omega : f(\omega)(i) = g(\omega)(i)\}^\bullet.$$

So $f(\omega)(i) = g(\omega)(i)$ for $\mathcal{I}$-almost every $\omega \in G$. This is true for every $i \in J$, so $f(\omega){\restriction}J = g(\omega){\restriction}J$ for $\mathcal{I}$-almost every $\omega \in G$. But this means that, for $\mathcal{I}$-almost every $\omega \in G$, $(\omega, f(\omega)) \in W$ iff $(\omega, g(\omega)) \in W$. However, $G \subseteq E \setminus F_W$, so $(\omega, g(\omega)) \in W$ and $(\omega, f(\omega)) \notin W$ for every $\omega \in G$. **X**

So we must have $F_W^\bullet = [\![\vec{f} \in \vec{W}]\!]$, as claimed.

(b) These are now elementary. The point is that if a $\mathbb{P}$-name $\dot{x}$ and $p \in \mathfrak{A}^+$ are such that $p \Vdash_{\mathbb{P}} \dot{x} \in \vec{V} \cap \vec{W}$, then $p \Vdash_{\mathbb{P}} \dot{x} \in \{0,1\}^{\check{I}}$, so there is a $(\Sigma, \mathcal{B}\mathfrak{a}_I)$-measurable $f : \Omega \to \{0,1\}^I$ such that $p \Vdash_{\mathbb{P}} \dot{x} = \vec{f}$, and $p \Vdash_{\mathbb{P}} \vec{f} \in \vec{V} \cap \vec{W}$. Now (a) shows that

$$\begin{aligned}
[\![\vec{f} \in (V \cap W)^{\vec{}}]\!] &= \{\omega : (\omega, f(\omega)) \in V \cap W\}^\bullet \\
&= \{\omega : (\omega, f(\omega)) \in V\}^\bullet \cap \{\omega : (\omega, f(\omega)) \in W\}^\bullet \\
&= [\![\vec{f} \in \vec{V}]\!] \cap [\![\vec{f} \in \vec{W}]\!] \supseteq p
\end{aligned}$$

and

$$p \Vdash_{\mathbb{P}} \dot{x} = \vec{f} \in (V \cap W)^{\vec{}}.$$

As p and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \vec{V} \cap \vec{W} \subseteq (V \cap W)^{\vec{}}.$$

The other seven inequalities are equally straightforward.

(c) This is immediate from the definition in 551D, since we actually have $\vec{V} \subseteq \vec{W}$.

(d) We can repeat the method of (b). If a $\mathbb{P}$-name $\dot{x}$ and $p \in \mathfrak{A}^+$ are such that $p \Vdash_{\mathbb{P}} \dot{x} \in \bigcap_{n \in \mathbb{N}} \vec{W}_n$, then there is a $(\Sigma, \mathcal{B}\mathfrak{a}_I)$-measurable $f : \Omega \to \{0,1\}^I$ such that $p \Vdash_{\mathbb{P}} \dot{x} = \vec{f}$, and $p \Vdash_{\mathbb{P}} \vec{f} \in \vec{W}_n$ for every n. Now

$$[\![\vec{f}\in\vec{V}]\!]=\{\omega:(\omega,f(\omega))\in\bigcap_{n\in\mathbb{N}}W_n\}^\bullet$$

$$=(\bigcap_{n\in\mathbb{N}}\{\omega:(\omega,f(\omega))\in W_n\})^\bullet=\inf_{n\in\mathbb{N}}\{\omega:(\omega,f(\omega))\in W_n\}^\bullet$$

(because $\Sigma\cap\mathcal{I}$ is a σ-ideal of Σ, so $E\mapsto E^\bullet$ is sequentially order-continuous, by 313Qb)

$$\supseteq p$$

and

$$p\Vdash_{\mathbb{P}}\dot{x}=\vec{f}\in\vec{V}.$$

As p and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}}\bigcap_{n\in\mathbb{N}}\vec{W}_n\subseteq\vec{V}.$$

On the other hand, (c) tells us that

$$\Vdash_{\mathbb{P}}\vec{V}\subseteq\bigcap_{n\in\mathbb{N}}\vec{W}_n, \text{ so we have equality.}$$

Putting this together with (b) (and recalling that $\Vdash_{\mathbb{P}}(\Omega\times\{0,1\}^I)^{\rightarrow}=\{0,1\}^{\check{I}}$), we get

$$\Vdash_{\mathbb{P}}\bigcup_{n\in\mathbb{N}}\vec{W}_n=\vec{W}.$$

(e)(i) Suppose that $p\in\mathfrak{A}^+$ and that $\dot{x}$ is a $\mathbb{P}$-name such that $p\Vdash_{\mathbb{P}}\dot{x}\in\vec{W}$. Let $f:\Omega\to\{0,1\}^I$ be a $(\Sigma,\mathcal{B}\mathfrak{a}_I)$-measurable function such that $p\Vdash_{\mathbb{P}}\dot{x}=\vec{f}$ (551Cc). Then

$$p\subseteq[\![\vec{f}\in\vec{W}]\!]=\{\omega:\omega\in E,\,z\subseteq f(\omega)\}^\bullet$$

by (a) above; that is, $p\subseteq E^\bullet$ and

$$p\Vdash_{\mathbb{P}}\vec{f}(\check{\imath})=z(i)^{\check{}}=\check{z}(\check{\imath})$$

for every $i\in J$, so

$$p\Vdash_{\mathbb{P}}\check{z}\subseteq\vec{f}.$$

As p and $\dot{x}$ are arbitrary,

$$[\![\vec{W}\ne\emptyset]\!]\subseteq E^\bullet$$

and

$$\Vdash_{\mathbb{P}}\vec{W}\subseteq\{x:\check{z}\subseteq x\in\{0,1\}^{\check{I}}\}.$$

(ii) If $E\in\mathcal{I}$ then $\Vdash_{\mathbb{P}}\vec{W}=\emptyset$ and we can stop. Otherwise, suppose that $p\in\mathfrak{A}^+$ is stronger than $E^\bullet$ and that $\dot{x}$ is a $\mathbb{P}$-name such that

$$p\Vdash_{\mathbb{P}}\check{z}\subseteq\dot{x}\in\{0,1\}^{\check{I}}.$$

Let f be a $(\Sigma,\mathcal{B}\mathfrak{a}_I)$-measurable function such that $p\Vdash_{\mathbb{P}}\dot{x}=\vec{f}$. Then $p\Vdash_{\mathbb{P}}\vec{f}_i=z(i)^{\check{}}$ for each $i\in J$, where $f_i(\omega)=f(\omega)(i)$ for every ω, so $p\subseteq\{\omega:z\subseteq f(\omega)\}^\bullet$. But also $p\subseteq E^\bullet$, so

$$p\subseteq\{\omega:\omega\in E,\,z\subseteq f(\omega)\}^\bullet=[\![\vec{f}\in\vec{W}]\!],$$

and $p\Vdash_{\mathbb{P}}\dot{x}\in\vec{W}$. As p and $\dot{x}$ are arbitrary,

$$[\![\{x:\check{z}\subseteq x\}\subseteq\vec{W}]\!]\supseteq E^\bullet$$

and we have

$$[\![\{x:\check{z}\subseteq x\}=\vec{W}]\!]=E^\bullet,\quad [\![\emptyset=\vec{W}]\!]=1\setminus E^\bullet.$$

551F Proposition Let $(\Omega,\Sigma,\mathcal{I})$ be a non-trivial measurable space with negligibles with a Dedekind complete quotient algebra $\mathfrak{A}$, $\mathbb{P}$ its associated forcing notion, and I a set.

(a) If $W \in \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ then

$$\Vdash_{\mathbb{P}} \vec{W} \in \mathcal{B}\mathfrak{a}_{\check{I}}.$$

(b) Suppose that $(\Omega, \Sigma, \mathcal{I})$ is ω_1-saturated, $p \in \mathfrak{A}^+$, and that $\dot{W}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{W} \in \mathcal{B}\mathfrak{a}_{\check{I}}.$$

Then there is a $W \in \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ such that

$$p \Vdash_{\mathbb{P}} \dot{W} = \vec{W}.$$

proof (a) Let $\mathcal{W}$ be the family of those $W \in \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ such that $\Vdash_{\mathbb{P}} \vec{W} \in \mathcal{B}\mathfrak{a}_{\check{I}}$. 551Eb and 551Ed tell us that $\mathcal{W}$ is a σ-subalgebra of $\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$, and 551Ee tells us that $E \times H \in \mathcal{W}$ whenever $E \in \Sigma$ and H is a basic cylinder set in $\{0,1\}^I$. So $\mathcal{W}$ must be the whole of $\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$.

(b)(i) Suppose that $p \in \mathfrak{A}^+$ and that $\dot{W}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{W} \text{ is a basic cylinder set in } \{0,1\}^{\check{I}}.$$

Then there is a $W \in \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ such that $p \Vdash_{\mathbb{P}} \dot{W} = \vec{W}$. **P** We know that

$$p \Vdash_{\mathbb{P}} \text{there is a } z \in \mathrm{Fn}_{<\omega}(\check{I}; \{0,1\}) \text{ such that } \dot{W} = \{x : z \subseteq x \in \{0,1\}^{\check{I}}\}.$$

So there is a $\mathbb{P}$-name $\dot{z}$ such that

$$p \Vdash_{\mathbb{P}} \dot{z} \in \mathrm{Fn}_{<\omega}(\check{I}; \{0,1\}) \text{ and } \dot{W} = \{x : \dot{z} \subseteq x\};$$

adjusting $\dot{z}$ if necessary, we can suppose that

$$\Vdash_{\mathbb{P}} \dot{z} \in \mathrm{Fn}_{<\omega}(\check{I}; \{0,1\}).$$

But this means that there is a maximal antichain (that is, a partition of unity) $C \subseteq \mathfrak{A}^+$ and a family $\langle z_c \rangle_{c \in C}$ in $\mathrm{Fn}_{<\omega}(I; \{0,1\})$ such that

$$c \Vdash_{\mathbb{P}} \dot{z} = \check{z}_c$$

for every $c \in C$. Because $\mathcal{I}$ is ω_1-saturated, $\mathfrak{A}$ is ccc and C is countable. We can therefore find a partition $\langle E_c \rangle_{c \in C}$ of Ω into members of Σ such that $E_c^\bullet = c$ for every $c \in C$. Consider

$$W_c = E_c \times \{x : z_c \subseteq x \in \{0,1\}^I\} \text{ for } c \in C, \quad W = \bigcup_{c \in C} W_c.$$

Of course $W \in \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$. By 551Ee,

$$c \Vdash_{\mathbb{P}} \vec{W}_c = \{x : \check{z}_c \subseteq x\} = \{x : \dot{z} \subseteq x\}, \quad c \Vdash_{\mathbb{P}} \vec{W}_d = \emptyset$$

whenever c, $d \in C$ are distinct. Because C is countable, 551Ed tells us that

$$c \Vdash_{\mathbb{P}} \vec{W} = \bigcup_{d \in \check{C}} \vec{W}_d = \{x : \dot{z} \subseteq x\}$$

for every $c \in C$; because C is a maximal antichain,

$$\Vdash_{\mathbb{P}} \vec{W} = \{x : \dot{z} \subseteq x\}$$

and

$$p \Vdash_{\mathbb{P}} \vec{W} = \dot{W}. \ \mathbf{Q}$$

(ii) Suppose that $p \in \mathfrak{A}^+$ and that $\dot{W}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{W} \text{ is a cozero set in } \{0,1\}^{\check{I}}.$$

Then there is a $W \in \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ such that $p \Vdash_{\mathbb{P}} \dot{W} = \vec{W}$. **P** Set $p' = [\![\dot{W} = \emptyset]\!]$. If $p \subseteq p'$ we can take $W = \emptyset$ and stop. Otherwise, let $E \in \Sigma$ be such that $E^\bullet = 1 \setminus p'$. We have

$$p \setminus p' \Vdash_{\mathbb{P}} \dot{W} \text{ is the union of a sequence of basic cylinder sets,}$$

so there is a sequence $\langle \dot{W}_n \rangle_{n \in \mathbb{N}}$ of $\mathbb{P}$-names such that

$$p \setminus p' \Vdash_{\mathbb{P}} \dot{W}_n \text{ is a basic cylinder set for every } n \text{ and } \dot{W} = \bigcup_{n \in \mathbb{N}} \dot{W}_n.$$

By (i), we have for each $n\in\mathbb{N}$ a $W_n\in\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ such that $p\setminus p'\Vdash_{\mathbb{P}}\dot W_n=\vec W_n$; now $V=\bigcup_{n\in\mathbb{N}}W_n$ belongs to $\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ and $\Vdash_{\mathbb{P}}\vec V=\bigcup_{n\in\mathbb{N}}\vec W_n$, so $p\setminus p'\Vdash_{\mathbb{P}}\vec V=\dot W$. Finally, setting $W=(E\times\{0,1\}^I)\cap V$, $p\Vdash_{\mathbb{P}}\vec W=\dot W$. **Q**

(iii) Suppose that $p\in\mathfrak{A}^+$, $\alpha<\omega_1$ and that $\dot W$ is a $\mathbb{P}$-name such that

$$p\Vdash_{\mathbb{P}}\dot W\in\mathcal{B}\mathfrak{a}_{\check\alpha}(\{0,1\}^{\check I}),$$

defining $\mathcal{B}\mathfrak{a}_\alpha$ as in 5A4Ga. Then there is a $W\in\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ such that $p\Vdash_{\mathbb{P}}\dot W=\vec W$. **P** Induce on α. The case $\alpha=0$ is (ii) above. For the inductive step to $\alpha>0$, we have

$$p\Vdash_{\mathbb{P}}\dot W\in\mathcal{B}\mathfrak{a}_{\check\alpha}(\{0,1\}^{\check I}),$$

so

$$p\Vdash_{\mathbb{P}}\text{ there is a sequence }\langle W_n\rangle_{n\in\mathbb{N}}\text{ in }\textstyle\bigcup_{\beta<\check\alpha}\mathcal{B}\mathfrak{a}_\beta(\{0,1\}^{\check I})\text{ such that }\dot W=\bigcup_{n\in\mathbb{N}}\{0,1\}^{\check I}\setminus W_n;$$

let $\langle\dot W_n\rangle_{n\in\mathbb{N}}$ be a sequence of $\mathbb{P}$-names such that

$$p\Vdash_{\mathbb{P}}\dot W_n\in\textstyle\bigcup_{\beta<\check\alpha}\mathcal{B}\mathfrak{a}_\beta(\{0,1\}^{\check I})\text{ for every }n\in\mathbb{N}\text{ and }\dot W=\bigcup_{n\in\mathbb{N}}\{0,1\}^{\check I}\setminus\dot W_n.$$

For $n\in\mathbb{N}$, $\beta<\alpha$ set

$$b_{n\beta}=[\![\dot W_n\in\mathcal{B}\mathfrak{a}_{\check\beta}(\{0,1\}^{\check I})\setminus\textstyle\bigcup_{\gamma<\check\beta}\mathcal{B}\mathfrak{a}_\gamma(\{0,1\}^{\check I})]\!],$$

and choose $E_{n\beta}\in\Sigma$ such that $E^\bullet_{n\beta}=b_{n\beta}$. Writing $A_n=\{\beta:\beta<\alpha,\,b_{n\beta}\ne0\}$, $p\subseteq\sup_{\beta\in A_n}b_{n\beta}$. If $\beta\in A_n$, then

$$b_{n\beta}\Vdash_{\mathbb{P}}\dot W_n\in\mathcal{B}\mathfrak{a}_{\check\beta}(\{0,1\}^{\check I}),$$

so by the inductive hypothesis there is a $W_{n\beta}\in\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ such that $b_{n\beta}\Vdash_{\mathbb{P}}\dot W_n=\vec W_{n\beta}$. For $\beta\in\alpha\setminus A_n$ set $W_{n\beta}=\emptyset$.

Set $W_n=\bigcup_{\beta<\alpha}(E_{n\beta}\times\{0,1\}^I)\cap W_{n\beta}$. Then

$$\Vdash_{\mathbb{P}}\vec W_n=\textstyle\bigcup_{\beta<\check\alpha}(E_{n\beta}\times\{0,1\}^I)^{\rightarrow}\cap\vec W_{n\beta},$$

so if $\beta\in A_n$

$$b_{n\beta}\Vdash_{\mathbb{P}}\vec W_n=\vec W_{n\beta}=\dot W_n$$

because

$$b_{n\beta}\Vdash_{\mathbb{P}}(E_{n\gamma}\times\{0,1\}^I)^{\rightarrow}=\emptyset$$

if $\gamma<\alpha$ and $\gamma\ne\beta$, and

$$b_{n\beta}\Vdash_{\mathbb{P}}(E_{n\beta}\times\{0,1\}^I)^{\rightarrow}=\{0,1\}^{\check I}.$$

As $p\subseteq\sup_{\beta\in A_n}b_{n\beta}$,

$$p\Vdash_{\mathbb{P}}\vec W_n=\dot W_n.$$

This is true for every $n\in\mathbb{N}$. So if we set $W=\bigcup_{n\in\mathbb{N}}(\Omega\times\{0,1\}^I)\setminus W_n$, we shall have $W\in\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ and

$$p\Vdash_{\mathbb{P}}\vec W=\textstyle\bigcup_{n\in\mathbb{N}}\{0,1\}^{\check I}\setminus\vec W_n=\bigcup_{n\in\mathbb{N}}\{0,1\}^{\check I}\setminus\dot W_n=\dot W.\ \textbf{Q}$$

(iv) Finally, suppose that $p\in\mathfrak{A}^+$ and that $\dot W$ is a $\mathbb{P}$-name such that

$$p\Vdash_{\mathbb{P}}\dot W\in\mathcal{B}\mathfrak{a}_{\check I}.$$

Then there is a $W\in\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ such that $p\Vdash_{\mathbb{P}}\dot W=\vec W$. **P** Because $\mathbb{P}$ is ccc,

$$\Vdash_{\mathbb{P}}\check\omega_1\text{ is the first uncountable ordinal}$$

(5A3Nb), so

$$\Vdash_{\mathbb{P}}\mathcal{B}\mathfrak{a}(\{0,1\}^{\check I})=\textstyle\bigcup_{\alpha<\check\omega_1}\mathcal{B}\mathfrak{a}_\alpha(\{0,1\}^{\check I}).$$

For $\alpha<\omega_1$ set

$$b_\alpha = [\![\dot{W} \in \mathcal{B}\mathfrak{a}_{\check{\alpha}}(\{0,1\}^{\check{I}})]\!].$$

Then $p \subseteq \sup_{\alpha<\omega_1} b_\alpha$. Again because $\mathfrak{A}$ is ccc, there is a $\gamma < \omega_1$ such that $p \subseteq \sup_{\alpha<\gamma} b_\alpha$. If $\alpha < \gamma$ and $c_\alpha = b_\alpha \setminus \sup_{\beta<\alpha} b_\beta$ is non-zero, choose $W_\alpha \in \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ such that $c_\alpha \Vdash_{\mathbb{P}} \vec{W}_\alpha = \dot{W}$; for other $\alpha < \gamma$ set $W_\alpha = \emptyset$. Choose $F_\alpha \in \Sigma$ such that $F_\alpha^\bullet = c_\alpha$ for each α. Set $W = \bigcup_{\alpha<\gamma}(F_\alpha \times \{0,1\}^I) \cap W_\alpha \in \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$. As in (iii) just above,

$$c_\alpha \Vdash_{\mathbb{P}} \vec{W} = \vec{W}_\alpha = \dot{W}$$

whenever $c_\alpha \neq 0$, so

$$p \Vdash_{\mathbb{P}} \vec{W} = \dot{W}. \ \mathbf{Q}$$

551G I noted above that there are difficulties in computing $[\![\vec{f} = \vec{g}]\!]$ for functions f, $g : \Sigma \to \{0,1\}^I$. For W, $V \in \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ the corresponding question about $[\![\vec{W} = \vec{V}]\!]$ turns out to be simpler, at least in some important cases.

Proposition Let $(\Omega, \Sigma, \mathcal{I})$ be a non-trivial measurable space with negligibles with a Dedekind complete quotient algebra $\mathfrak{A}$, $\mathbb{P}$ the associated forcing notion and I a set. Suppose that Σ is closed under Souslin's operation.

(a) If $W \in \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ then $F = \{\omega : W[\{\omega\}] \neq \emptyset\}$ belongs to Σ and $[\![\vec{W} \neq \emptyset]\!] = F^\bullet$ in $\mathfrak{A} \cong \mathrm{RO}(\mathbb{P})$.

(b) If W, $V \in \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ then $[\![\vec{W} = \vec{V}]\!] = \{\omega : W[\{\omega\}] = V[\{\omega\}]\}^\bullet$.

proof (a)(i) The point is that there is a Σ-measurable function $f : \Omega \to \{0,1\}^I$ such that $(\omega, f(\omega)) \in W$ for every $\omega \in F$.

$\mathbf{P}(\boldsymbol{\alpha})$ Suppose first that I is countable. Let $\mathcal{V}$ be the family of subsets of $\Omega \times \{0,1\}^I$ obtainable by Souslin's operation $\mathcal{S}$ from $\{E \times H : E \in \Sigma,\ H \subseteq \{0,1\}^I \text{ is closed}\}$. The family $\mathcal{W} = \{V : V \in \mathcal{V}, (\Omega \times \{0,1\}^I) \setminus V \in \mathcal{V}\}$ is a σ-algebra and contains $E \times H$ whenever $E \in \Sigma$ and $H \subseteq \{0,1\}^I$ is open-and-closed, so $\mathcal{W} \supseteq \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ and $W \in \mathcal{W} \subseteq \mathcal{V}$. By 423N, there is a selector g for W which is measurable for the σ-algebra T of subsets of Ω generated by $\mathcal{S}(\Sigma)$; but we are supposing that this is just Σ. Also $F = \operatorname{dom} g$ belongs to $\mathrm{T} = \Sigma$. If f is any extension of g to a Σ-measurable function from Ω to $\{0,1\}^I$, then f has the required property.

$(\boldsymbol{\beta})$ For the general case, note that $W \in \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ factors through $\Omega \times \{0,1\}^J$ for some countable $J \subseteq I$, that is, there is a $W_1 \in \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_J$ such that

$$W = \{(\omega, x) : \omega \in \Omega,\ x \in \{0,1\}^I,\ (\omega, x{\restriction}J) \in W_1\}.$$

Now (α) tells us that $F_1 = \{\omega : W_1[\{\omega\}] \neq \emptyset\}$ belongs to Σ and that there is a Σ-measurable $f_1 : \Omega \to \{0,1\}^J$ such that $(\omega, f_1(\omega)) \in W_1$ for every $\omega \in F_1$. Of course $F_1 = F$, and if we set

$$\begin{aligned} f(\omega)(i) &= f_1(\omega)(i) \text{ for } \omega \in \Omega,\ i \in J, \\ &= 0 \text{ for } \omega \in \Omega,\ i \in I \setminus J, \end{aligned}$$

then f is $(\Sigma, \mathcal{B}\mathfrak{a}_I)$-measurable and $(\omega, f(\omega)) \in W$ for every $\omega \in F$. $\mathbf{Q}$

(ii) Using 551Ea, it follows that

$$[\![\vec{W} \neq \emptyset]\!] \supseteq [\![\vec{f} \in \vec{W}]\!] = \{\omega : (\omega, f(\omega)) \in W\}^\bullet = F^\bullet.$$

On the other hand, if $a = [\![\vec{W} \neq \emptyset]\!]$ is non-zero, then there is a $\mathbb{P}$-name $\dot{x}$ such that $a \Vdash_{\mathbb{P}} \dot{x} \in \vec{W}$. By 551Cc, there is a $(\Sigma, \mathcal{B}\mathfrak{a}_I)$-measurable g such that $a \Vdash_{\mathbb{P}} \dot{x} = \vec{g}$, in which case

$$a \subseteq [\![\vec{g} \in \vec{W}]\!] = \{\omega : (\omega, g(\omega)) \in W\}^\bullet \subseteq F^\bullet.$$

So $[\![\vec{W} \neq \emptyset]\!] = F^\bullet$ exactly.

(b) Apply (a) to $W \triangle V$ (using 551Eb, as usual).

551H Examples Cases in which a σ-algebra is closed under Souslin's operation, so that the conditions of 551G can be satisfied, include the following.

(a) If (X,Σ,μ) is a complete locally determined measure space, then Σ is closed under Souslin's operation (431A).

(b) If $(\Omega,\Sigma,\mathcal{I})$ is a complete ω_1-saturated measurable space with negligibles, then Σ is closed under Souslin's operation (431G).

(c) If X is any topological space, then its Baire-property algebra $\widehat{\mathcal{B}}(X)$ is closed under Souslin's operation (431Fb).

551I Theorem Let $(\Omega,\Sigma,\mathcal{I})$ be a non-trivial measurable space with negligibles with a Dedekind complete quotient algebra, $\mathbb{P}$ its associated forcing notion, and I a set. Let W be any member of $\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$. Then
(i) $h(\omega)=\nu_I W[\{\omega\}]$ is defined for every $\omega\in\Omega$, where ν_I is the usual measure of $\{0,1\}^I$;
(ii) $h:\Omega\to[0,1]$ is Σ-measurable;
(iii) $\Vdash_{\mathbb{P}}\nu_{\check I}\vec W=\vec h$,
where in this formula $\vec h$ is the $\mathbb{P}$-name for a real number defined from h as in 551B, and $\nu_{\check I}$ is an abbreviation for 'the usual measure on $\{0,1\}^{\check I}$'.

proof I follow the method of 551Ea and 551Fa.

(a) Suppose that W is of the form $E\times\{x:z\subseteq x\in\{0,1\}^I\}$, where $z\in\{0,1\}^J$ for some finite $J\subseteq I$. Then (using 551Ee)

$$E^\bullet=[\![\vec W=\{x:\check z\subseteq x\in\{0,1\}^{\check I}\}]\!]\subseteq[\![\nu_{\check I}\vec W=2^{-\#(\check J)}]\!],$$

$$1\setminus E^\bullet=[\![\vec W=\emptyset]\!]\subseteq[\![\nu_{\check I}\vec W=0]\!];$$

while also $h=2^{-\#(J)}\chi E$ so

$$E^\bullet=[\![\vec h=2^{-\#(\check J)}]\!],\quad 1\setminus E^\bullet=[\![\vec h=0]\!].$$

So in this case

$$\Vdash_{\mathbb{P}}\nu_{\check I}\vec W=\vec h.$$

(b) Now 551E shows that the set of those $W\in\mathcal{B}\mathfrak{a}_I$ for which (i)-(iii) are true is a Dynkin class, so by the Monotone Class Theorem once more we have the result.

551J Corollary Let $(\Omega,\Sigma,\mathcal{I})$ be a non-trivial ω_1-saturated measurable space with negligibles, $\mathbb{P}$ its associated forcing notion, $P=(\Sigma/\Sigma\cap\mathcal{I})^+$ the partially ordered set underlying $\mathbb{P}$, and I a set. If $p\in P$ and $\dot W$ is a $\mathbb{P}$-name such that

$$p\Vdash_{\mathbb{P}}\dot W\subseteq\{0,1\}^{\check I}\text{ is }\nu_{\check I}\text{-negligible},$$

then there is a $W\in\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ such that $\nu_I W[\{\omega\}]=0$ for every $\omega\in\Omega$ and

$$p\Vdash_{\mathbb{P}}\dot W\subseteq\vec W.$$

proof Because

$$\Vdash_{\mathbb{P}}\text{the usual measure on }\{0,1\}^{\check I}\text{ is a completion regular Radon measure},$$

we know that

$$p\Vdash_{\mathbb{P}}\text{there is a }\nu_{\check I}\text{-negligible member of }\mathcal{B}\mathfrak{a}_{\check I}\text{ including }\dot W.$$

Let $\dot V$ be a $\mathbb{P}$-name such that

$$p\Vdash_{\mathbb{P}}\dot W\subseteq\dot V\in\mathcal{B}\mathfrak{a}_{\check I}\text{ and }\nu_{\check I}\dot V=0.$$

By 551Fb, there is a $V\in\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$ such that $p\Vdash_{\mathbb{P}}\dot V=\vec V$. Set $h(\omega)=\nu_I V[\{\omega\}]$ for $\omega\in\Omega$; then

$$p\Vdash_{\mathbb{P}}\vec h=\nu_{\check I}\vec V=0$$

(551I), so $p \subseteq E^\bullet$, where $E = h^{-1}[\{0\}]$. Set $W = (E \times \{0,1\}^I) \cap V$. Then $W \in \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$, $\nu_I W[\{\omega\}] = 0$ for every ω, and (using 551Gb)

$$p \Vdash_{\mathbb{P}} \vec{W} = \vec{V} = \dot{V} \supseteq \dot{W},$$

as required.

551K We have been looking here at general sets $W \in \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$. A special case of obvious importance is when W is of the form $\Omega \times H$ where $H \in \mathcal{B}\mathfrak{a}_I$. For these it is worth refining the results slightly.

Proposition Let $(\Omega, \Sigma, \mathcal{I})$ be a non-trivial measurable space with negligibles with a Dedekind complete quotient algebra, $\mathbb{P}$ the associated forcing notion, and I a set. For $H \subseteq \{0,1\}^I$ set $\tilde{H} = (\Omega \times H)^{\rightarrow}$ as defined in 551D.

(a) If $H = \{x : z \subseteq x \in \{0,1\}^I\}$, where $z \in \mathrm{Fn}_{<\omega}(I; \{0,1\})$, then

$$\Vdash_{\mathbb{P}} \tilde{H} = \{x : \check{z} \subseteq x \in \{0,1\}^{\check{I}}\}.$$

(b)(i) If G, $H \in \mathcal{B}\mathfrak{a}_I$ then

$$\Vdash_{\mathbb{P}} \tilde{G} \cup \tilde{H} = (G \cup H)^{\sim}, \ \tilde{G} \cap \tilde{H} = (G \cap H)^{\sim},$$
$$\tilde{G} \setminus \tilde{H} = (G \setminus H)^{\sim}, \ \tilde{G} \triangle \tilde{H} = (G \triangle H)^{\sim}.$$

(ii) If $\langle H_n \rangle_{n\in\mathbb{N}}$ is any sequence in $\mathcal{B}\mathfrak{a}_I$ then

$$\Vdash_{\mathbb{P}} \bigcup_{n\in\mathbb{N}} \tilde{H}_n = (\bigcup_{n\in\mathbb{N}} H_n)^{\sim}, \ \bigcap_{n\in\mathbb{N}} \tilde{H}_n = (\bigcap_{n\in\mathbb{N}} H_n)^{\sim}.$$

(c) If $\alpha < \omega_1$ and $H \in \mathcal{B}\mathfrak{a}_\alpha(\{0,1\}^I)$, once again defining $\mathcal{B}\mathfrak{a}_\alpha$ as in 5A4Ga, then

$$\Vdash_{\mathbb{P}} \tilde{H} \in \mathcal{B}\mathfrak{a}_{\check{\alpha}}(\{0,1\}^{\check{I}}).$$

(d) If H is measured by the usual measure ν_I of $\{0,1\}^I$, then

$$\Vdash_{\mathbb{P}} \nu_{\check{I}} \tilde{H} = (\nu_I H)^{\check{}}.$$

proof (a) This is covered by 551Ee.

(b) This is a special case of parts (b) and (d) of 551E.

(c) A subset of $\{0,1\}^I$ is a cozero set iff it is empty or expressible as the union of a sequence of basic cylinder sets, so if H is a cozero set then (a) and (b-ii) tell us that

$$\Vdash_{\mathbb{P}} \tilde{H} \text{ is a cozero set in } \{0,1\}^{\check{I}}.$$

Now an induction on α shows that if $H \in \mathcal{B}\mathfrak{a}_\alpha(\{0,1\}^I)$ then

$$\Vdash_{\mathbb{P}} \tilde{H} \in \mathcal{B}\mathfrak{a}_{\check{\alpha}}(\{0,1\}^{\check{I}}).$$

(d) We have H_0, $H_1 \in \mathcal{B}\mathfrak{a}_I$ such that $H_0 \subseteq H \subseteq H_1$ and $\nu_I H_0 = \nu_I H = \nu_I H_1$. Applying 551I(iii) to $\Omega \times H_0$ and $\Omega \times H_1$,

$$\Vdash_{\mathbb{P}} \nu_{\check{I}} \tilde{H}_0 = \nu_{\check{I}} \tilde{H}_1 = (\nu_I H)^{\check{}},$$

while of course $\Vdash_{\mathbb{P}} \tilde{H}_0 \subseteq \tilde{H} \subseteq \tilde{H}_1$ (551Ec), so

$$\Vdash_{\mathbb{P}} \nu_{\check{I}} \tilde{H} = (\nu_I H)^{\check{}}.$$

551L Remark If I ask you to think of your favourite Baire set in $\{0,1\}^I$, it is likely to come with a definition; for instance, the set H of those $x \in \{0,1\}^{\mathbb{N}}$ such that $\lim_{n\to\infty} \frac{1}{n+1} \sum_{i=1}^n x(i) = \frac{1}{2}$. The point of 551K is that we shall automatically get

$$\Vdash_{\mathbb{P}} \tilde{H} = \{x : x \in \{0,1\}^{\mathbb{N}}, \lim_{n\to\infty} \frac{1}{n+1} \sum_{i=1}^n x(i) = \frac{1}{2}\}.$$

P

$$H=\bigcap_{n\in\mathbb{N}}\bigcup_{m\in\mathbb{N}}\bigcap_{k\ge m}\bigcup_{z\in L_{nk}}\{x:z\subseteq x\in\{0,1\}^{\mathbb{N}}\},$$

where

$$L_{nk}=\{z:z\in\{0,1\}^{k+1},\,|\tfrac{1}{k+1}\sum_{i=0}^{k}z_i-\tfrac{1}{2}|\le\tfrac{1}{n+1}\}.$$

So 551K tells us that

$$\Vdash_{\mathbb{P}}\tilde H=\bigcap_{n\in\mathbb{N}}\bigcup_{m\in\mathbb{N}}\bigcap_{k\ge m}\bigcup_{z\in\check L_{nk}}\{x:z\subseteq x\in\{0,1\}^{\mathbb{N}}\},$$

and of course

$$\Vdash_{\mathbb{P}}\check L_{nk}=\{z:z\in\{0,1\}^{\check k+1},\,|\tfrac{1}{\check k+1}\sum_{i=0}^{\check k}z_i-\tfrac{1}{2}|\le\tfrac{1}{\check n+1}\}.\ \mathbf{Q}$$

What I am trying to say here is that the process $H\mapsto(\Omega\times H)^{\rightharpoondown}=\tilde H$ builds a $\mathbb{P}$-name for the 'right' subset of $\{0,1\}^I$, in the sense that any adequately concrete definition of H will also, when interpreted in $V^{\mathbb{P}}$, be a definition of $\tilde H$.

551M We can go still farther.

Definition Let $(\Omega,\Sigma,\mathcal{I})$ be a non-trivial measurable space with negligibles, and $\mathbb{P}$ its associated forcing notion. Let I be any set. If $\psi:\Omega\times\{0,1\}^I\to\mathbb{R}$ is $(\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I)$-measurable, let $\vec\psi$ be the $\mathbb{P}$-name

$$\{((\vec f,\vec h),\mathbb{1}):f\text{ is a }(\Sigma,\mathcal{B}\mathfrak{a}_I)\text{-measurable function from }\Omega\text{ to }\{0,1\}^I,$$
$$h:\Omega\to\mathbb{R}\text{ is }\Sigma\text{-measurable},\ h(\omega)=\psi(\omega,f(\omega))\text{ for every }\omega\in\Omega\},$$

where in this formula $\vec f$ is to be interpreted as a $\mathbb{P}$-name for a member of $\{0,1\}^{\check I}$, as in 551C, and $\vec h$ as a $\mathbb{P}$-name for a real number, as in 551B.

551N Proposition Let $(\Omega,\Sigma,\mathcal{I})$ be a non-trivial measurable space with negligibles with a Dedekind complete quotient algebra $\mathfrak{A}$, $\mathbb{P}$ its associated forcing notion, and I a set. Suppose that $\psi:\Omega\times\{0,1\}^I\to\mathbb{R}$ is $(\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I)$-measurable, and define $\vec\psi$ as in 551M.

(a) $\Vdash_{\mathbb{P}}\vec\psi$ is a real-valued function on $\{0,1\}^{\check I}$.

(b) If $\phi:\Omega\times\{0,1\}^I\to\mathbb{R}$ is another $(\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I)$-measurable function, and $\alpha\in\mathbb{R}$, then

$$\Vdash_{\mathbb{P}}(\phi+\psi)^{\rightharpoondown}=\vec\phi+\vec\psi,\ (\alpha\phi)^{\rightharpoondown}=\check\alpha\vec\phi.$$

(c) If $\langle\psi_n\rangle_{n\in\mathbb{N}}$ is a sequence of $(\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I)$-measurable real-valued functions on $\Omega\times\{0,1\}^I$ and $\psi(\omega,x)=\lim_{n\to\infty}\psi_n(\omega,x)$ for every $\omega\in\Omega$ and $x\in\{0,1\}^I$, then

$$\Vdash_{\mathbb{P}}\vec\psi(x)=\lim_{n\to\infty}\vec\psi_n(x)\text{ for every }x\in\{0,1\}^{\check I}.$$

(d) If $W\in\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$, then

$$\Vdash_{\mathbb{P}}(\chi W)^{\rightharpoondown}=\chi\vec W.$$

(e) $\Vdash_{\mathbb{P}}\vec\psi$ is $\mathcal{B}\mathfrak{a}_{\check I}$-measurable.

(f) If $h(\omega)=\int\psi(\omega,x)\nu_I(dx)$ is defined for every $\omega\in\Omega$, then

$$\Vdash_{\mathbb{P}}\int\vec\psi\,d\nu_{\check I}\text{ is defined and equal to }\vec h.$$

proof (a)(i) Suppose that we have two members $((\vec f_0,\vec h_0),\mathbb{1})$ and $((\vec f_1,\vec h_1),\mathbb{1})$ of $\vec\psi$, and that $E\in\Sigma\setminus\mathcal{I}$ is such that $E^\bullet\Vdash_{\mathbb{P}}\vec f_0=\vec f_1$. Then $E^\bullet\Vdash_{\mathbb{P}}\vec h_0=\vec h_1$. $\mathbf{P}$ Let $J\subseteq I$ be a countable set such that ψ factors through $\Omega\times\{0,1\}^J$, in the sense that $\psi(\omega,x)=\psi(\omega,y)$ whenever $\omega\in\Omega$ and x, $y\in\{0,1\}^I$ are such that $x\restriction J=y\restriction J$. For each $i\in J$,

$$E^\bullet\Vdash_{\mathbb{P}}\vec f_0(\check\imath)=\vec f_1(\check\imath),$$

so that $f_0(\omega)(i)=f_1(\omega)(i)$ for $\mathcal{I}$-almost every $\omega\in E$. Consequently $f_0(\omega)\restriction J=f_1(\omega)\restriction J$ and

$$h_0(\omega)=\psi(\omega,f_0(\omega))=\psi(\omega,f_1(\omega))=h_1(\omega)$$

for $\mathcal{I}$-almost every $\omega \in E$; that is, $E^\bullet \Vdash_{\mathbb{P}} \vec{h}_0 = \vec{h}_1$. **Q**

It follows that

$$\Vdash_{\mathbb{P}} \vec{\psi} \text{ is a function}$$

(5A3Ea).

(ii) By the constructions in 551Cb and 551B,

$$\Vdash_{\mathbb{P}} \vec{\psi} \subseteq \{0,1\}^{\check{I}} \times \mathbb{R}.$$

(iii) If $\dot{x}$ is a $\mathbb{P}$-name and $p \in \mathfrak{A}^+$ is such that $p \Vdash_{\mathbb{P}} \dot{x} \in \{0,1\}^{\check{I}}$, then there is a $(\Sigma, \mathcal{B}\mathfrak{a}_I)$-measurable $f : \Omega \to \{0,1\}^I$ such that $p \Vdash_{\mathbb{P}} \dot{x} = \vec{f}$ (551Cc again). Setting $h(\omega) = \psi(\omega, f(\omega))$ for $\omega \in \Omega$, $((\vec{f}, \vec{h}), \mathbb{1}) \in \vec{\psi}$, so

$$p \Vdash_{\mathbb{P}} \dot{x} = \vec{f} \text{ and } (\vec{f}, \vec{h}) \in \vec{\psi}, \text{ so } \dot{x} \in \operatorname{dom}(\vec{\psi}).$$

As p and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \operatorname{dom}(\vec{\psi}) = \{0,1\}^{\check{I}}.$$

(b) This is easy. If $p \in \mathfrak{A}^+$ and $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \{0,1\}^{\check{I}}$, take a $(\Sigma, \mathcal{B}\mathfrak{a}_I)$-measurable $f : \Omega \to \{0,1\}^I$ such that $p \Vdash_{\mathbb{P}} \dot{x} = \vec{f}$; set

$$h_0(\omega) = \phi(\omega, f(\omega)), \quad h_1(\omega) = \psi(\omega, f(\omega))$$

for $\omega \in \Omega$; then

$$\begin{aligned} p \Vdash_{\mathbb{P}} (\phi + \psi)^{\rightharpoondown}(\dot{x}) &= (\phi + \psi)^{\rightharpoondown}(\vec{f}) = (h_0 + h_1)^{\rightharpoondown} \\ &= \vec{h}_0 + \vec{h}_1 = \vec{\phi}(\vec{f}) + \vec{\psi}(\vec{f}) = \vec{\phi}(\dot{x}) + \vec{\psi}(\dot{x}), \\ (\alpha\phi)^{\rightharpoondown}(\dot{x}) &= (\alpha h_0)^{\rightharpoondown} = \check{\alpha}\vec{h}_0 = \check{\alpha}\vec{\phi}(\dot{x}) \end{aligned}$$

by 5A3Lc. As p and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}} (\phi + \psi)^{\rightharpoondown} = \vec{\phi} + \vec{\psi}, \ (\alpha\phi)^{\rightharpoondown} = \check{\alpha}\vec{\phi}.$$

(c) In the same way, if $p \in \mathfrak{A}^+$ and $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \{0,1\}^{\check{I}}$, take a $(\Sigma, \mathcal{B}\mathfrak{a}_I)$-measurable $f : \Omega \to \{0,1\}^I$ such that $p \Vdash_{\mathbb{P}} \dot{x} = \vec{f}$. Set

$$h_n(\omega) = \psi_n(\omega, f(\omega)), \quad h(\omega) = \psi(\omega, f(\omega))$$

for $\omega \in \Omega$ and $n \in \mathbb{N}$; then $h = \lim_{n\to\infty} h_n$, so

$$p \Vdash_{\mathbb{P}} \vec{\psi}(\dot{x}) = \vec{h} = \lim_{n\to\infty} \vec{h}_n = \lim_{n\to\infty} \vec{\psi}_n(\dot{x}).$$

(d) Take $p \in \mathfrak{A}^+$ and a $\mathbb{P}$-name $\dot{x}$ such that $p \Vdash_{\mathbb{P}} \dot{x} \in \{0,1\}^{\check{I}}$. Let $f : \Omega \to \{0,1\}^I$ be a $(\Sigma, \mathcal{B}\mathfrak{a}_I)$-measurable function such that $p \Vdash_{\mathbb{P}} \dot{x} = \vec{f}$; set $h(\omega) = \chi W(\omega, f(\omega))$ for $\omega \in \Omega$, so that

$$\Vdash_{\mathbb{P}} \vec{h} = (\chi W)^{\rightharpoondown}(\vec{f}), \quad p \Vdash_{\mathbb{P}} \vec{h} = (\chi W)^{\rightharpoondown}(\dot{x}).$$

If $p = E^\bullet$ where $E \in \Sigma \setminus \mathcal{I}$,

$$\begin{aligned} &p \Vdash_{\mathbb{P}} (\chi W)^{\rightharpoondown}(\dot{x}) = 1 \\ &\iff p \Vdash_{\mathbb{P}} \vec{h} = 1 \\ &\iff h(\omega) = 1 \text{ for } \mathcal{I}\text{-almost every } \omega \in E \\ &\iff (\omega, f(\omega)) \in W \text{ for } \mathcal{I}\text{-almost every } \omega \in E \\ &\iff p \subseteq [\![\vec{f} \in \vec{W}]\!] \end{aligned}$$

(551Ea)

$$\iff p \Vdash_{\mathbb{P}} \dot{x} \in \vec{W};$$

similarly,

$$\begin{aligned} p \Vdash_{\mathbb{P}} (\chi W)^{\rightarrow}(\dot{x}) = 0 \\ &\iff h(\omega) = 0 \text{ for } \mathcal{I}\text{-almost every } \omega \in E \\ &\iff (\omega, f(\omega)) \notin W \text{ for } \mathcal{I}\text{-almost every } \omega \in E \\ &\iff p \Vdash_{\mathbb{P}} \dot{x} \notin \vec{W}. \end{aligned}$$

As p and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}} (\chi W)^{\rightarrow} = \chi\vec{W}.$$

(e) Assembling (a)-(d), and recalling 551Fa, we see that the result is true when ψ is a linear multiple of the indicator function of a set in $\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$, whenever ψ is a sum of such functions, and whenever ψ is the limit of a sequence of such sums; that is, whenever ψ is $(\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I)$-measurable.

(f) Similarly, (d) and 551I tell us that the result is true for the indicator function of a member of $\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I$. Once again, we can move to a linear combination of such functions, using (b), and thence to a non-negative $(\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I)$-measurable function, using (c); finally, with (b) again, we get the general case.

551O Measure algebras With a little more effort we can get a representation of the standard measure algebras in the same style. Let I be a set, ν_I the usual measure on $\{0,1\}^I$ and $(\mathfrak{B}_I, \bar{\nu}_I)$ its measure algebra. It will be important to appreciate that these are abbreviations for formulae in set theory with a single parameter I; so that if we have a forcing notion $\mathbb{P}$ and a $\mathbb{P}$-name τ, we shall have $\mathbb{P}$-names $\mathfrak{B}_\tau$ and $\bar{\nu}_\tau$, uniquely defined as soon as we have settled on the exact formulations we wish to apply when interpreting the basic constructions $\{\dots\}$, $\mathcal{P}$ in the forcing language. Similarly, if we write $\mathbb{P}_I = (\mathfrak{B}_I^+, \subseteq, 1, \downarrow)$ for the forcing notion based on the Boolean algebra $\mathfrak{B}_I$, this also is a formula which can be interpreted in forcing languages.

551P Theorem Let $(\Omega, \Sigma, \mathcal{I})$ be a non-trivial ω_1-saturated measurable space with negligibles such that Σ is closed under Souslin's operation. Let $\mathbb{P}$ be the associated forcing notion, $P = (\Sigma/\Sigma \cap \mathcal{I})^+$ its underlying partially ordered set, and I a set. Set

$$\Lambda = \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_I, \quad \mathcal{J} = \{W : W \in \Lambda,\ \nu_I W[\{\omega\}] = 0 \text{ for } \mathcal{I}\text{-almost every } \omega \in \Omega\};$$

then $\mathcal{J}$ is a σ-ideal of Λ (cf. 527B); let $\mathfrak{C}$ be the quotient algebra $\Lambda/\mathcal{J}$. For $W \in \Lambda$ and $\omega \in \Omega$ set $h_W(\omega) = \nu_I W[\{\omega\}]$. For $a \in \mathfrak{C}$ let $\vec{a}$ be the $\mathbb{P}$-name

$$\{(\vec{W}, \mathbb{1}) : W \in \Lambda,\ W^\bullet = a\}$$

where the $\mathbb{P}$-names $\vec{W}$ are defined as in 551D. Consider the $\mathbb{P}$-names

$$\dot{\mathfrak{D}} = \{(\vec{a}, \mathbb{1}) : a \in \mathfrak{C}\}, \quad \dot{\pi} = \{(((W^\bullet)^{\rightarrow}, (\vec{W})^\bullet), \mathbb{1}) : W \in \Lambda\}.$$

(a) $\Vdash_{\mathbb{P}} \dot{\pi}$ is a bijection between $\dot{\mathfrak{D}}$ and $\mathfrak{B}_{\check{I}}$.

(b) If a, $b \in \mathfrak{C}$, $V \in \Lambda$ and $V^\bullet = a$, then

$$\Vdash_{\mathbb{P}} \dot{\pi}(a \mathbin{\triangle} b)^{\rightarrow} = \dot{\pi}\vec{a} \mathbin{\triangle} \dot{\pi}\vec{b}, \quad \dot{\pi}(a \cap b)^{\rightarrow} = \dot{\pi}\vec{a} \cap \dot{\pi}\vec{b}, \quad \bar{\nu}_{\check{I}}(\dot{\pi}\vec{a}) = \vec{h}_V,$$

defining h_V and $\vec{h}_V$ as in 551I.

(c) Let $\varepsilon : \Sigma/\Sigma \cap \mathcal{I} \to \mathfrak{C}$ be the canonical map defined by the formula

$$\varepsilon(E^\bullet) = (E \times \{0,1\}^I)^\bullet \text{ for } E \in \Sigma.$$

If $p \in (\Sigma/\Sigma \cap \mathcal{I})^+$ and a, $b \in \mathfrak{C}$, then

$$p \Vdash_{\mathbb{P}} \dot{\pi}\vec{a} = \dot{\pi}\vec{b}$$

iff $a \cap \varepsilon(p) = b \cap \varepsilon(p)$.

Remarks Note that in the formula

$$\{(((W^\bullet)^\rightharpoonup,(\vec{W})^\bullet),\mathbb{1}):W\in\Lambda\}$$

the first $^\bullet$ is interpreted in the ordinary universe as the canonical map from Λ to $\mathfrak{C}$, and the second is interpreted in the forcing language as the canonical map from $\mathcal{B}\mathfrak{a}_{\tilde I}$ to $\mathfrak{B}_{\tilde I}$; while among the brackets $(\ldots)$, some are just separators, some are to be interpreted as an ordered-pair construction in the ordinary universe, and some are to be interpreted as and ordered-pair construction in the forcing language. Similarly, in the formula

$$\Vdash_{\mathbb{P}}\dot\pi(a\triangle b)^\rightharpoonup=\dot\pi\vec a\triangle\dot\pi\vec b$$

the first $\triangle$ is to be interpreted in the ordinary universe as symmetric difference in the algebra $\mathfrak{C}$, while the second is to be interpreted in the forcing language as symmetric difference in $\mathfrak{B}_{\tilde I}$.

proof (a)(i) $\Vdash_{\mathbb{P}}\dot\pi$ is a function with domain $\dot{\mathfrak{D}}$ and $\dot\pi[\dot{\mathfrak{D}}]=\dot B$.

P? Suppose, if possible, that V, $W\in\Lambda$ and $E\in\Sigma\setminus\mathcal{I}$ are such that

$$E^\bullet\Vdash_{\mathbb{P}}(V^\bullet)^\rightharpoonup=(W^\bullet)^\rightharpoonup,\ \vec V^\bullet\ne\vec W^\bullet.$$

By 551I(iii) and 551Eb,

$$E^\bullet\Vdash_{\mathbb{P}}\vec h_{V\triangle W}=\nu_{\tilde I}(V\triangle W)^\rightharpoonup\ne 0.$$

On the other hand,

$$E^\bullet\Vdash_{\mathbb{P}}\vec V\in(V^\bullet)^\rightharpoonup=(W^\bullet)^\rightharpoonup,$$

so there must be a $W_1\in\Lambda$ and an $F\in\Sigma\setminus\mathcal{I}$ such that $F^\bullet$ is stronger than $E^\bullet$, $W_1\triangle W\in\mathcal{J}$ and $F^\bullet\Vdash_{\mathbb{P}}\vec V=\vec W_1$. Now, calculating in $\Sigma/\Sigma\cap\mathcal{I}$,

$$F^\bullet\subseteq\{\omega:V[\{\omega\}]=W_1[\{\omega\}]\}^\bullet$$

(551Gb)

$$\subseteq\{\omega:\nu_I(V[\{\omega\}]\triangle W_1[\{\omega\}])=0\}^\bullet=\{\omega:\nu_I(V[\{\omega\}]\triangle W[\{\omega\}])=0\}^\bullet$$

(because $W\triangle W_1\in\mathcal{J}$, so $\nu_I(W[\{\omega\}]\triangle W_1[\{\omega\}])=0$ for $\mathcal{I}$-almost every ω)

$$=[\![\vec h_{V\triangle W}=0]\!]$$

(551B); which is impossible, because $E^\bullet\subseteq[\![\vec h_{V\triangle W}\ne 0]\!]$. **X** So 5A3E tells us that

$$\Vdash_{\mathbb{P}}\dot\pi\text{ is a function with domain }\dot{\mathfrak{D}}\text{ and }\dot\pi[\dot{\mathfrak{D}}]=\dot B,$$

where $\dot B$ is the $\mathbb{P}$-name $\{(\vec W^\bullet,\mathbb{1}):W\in\Lambda\}$.

(ii) Now $\Vdash_{\mathbb{P}}\dot\pi$ is injective. **P** I aim to use the condition (ii) in 5A3Eb. I take the argument in two bites.

(α)? Suppose, if possible, that V, $W\in\Lambda$ and $p\in P$ are such that $p\Vdash_{\mathbb{P}}\vec V^\bullet=\vec W^\bullet$ but $p\not\Vdash_{\mathbb{P}}(V^\bullet)^\rightharpoonup\subseteq(W^\bullet)^\rightharpoonup$. Then there are a $q\in P$, stronger than p, and a $\mathbb{P}$-name $\dot x$ such that

$$q\Vdash_{\mathbb{P}}\dot x\in(V^\bullet)^\rightharpoonup\setminus(W^\bullet)^\rightharpoonup.$$

By the definition in 5A3Cb, there are an $r\in P$, stronger than q, and a $V_1\in\Lambda$ such that $V_1^\bullet=V^\bullet$ and $r\Vdash_{\mathbb{P}}\dot x=\vec V_1$. Let $E\in\Sigma\setminus\mathcal{I}$ be such that $E^\bullet=r$, and set

$$W_1=(V_1\cap(E\times\{0,1\}^I))\cup(W\cap((\Omega\setminus E)\times\{0,1\}^I))\in\Lambda.$$

Then $E^\bullet\subseteq[\![\vec W_1=\vec V_1]\!]$ (551Gb again). At the same time,

$$E^\bullet\subseteq[\![\vec V^\bullet=\vec W^\bullet]\!]=[\![\vec h_{V\triangle W}=0]\!]$$

as in (i) just above. Now

$$\{\omega:\omega\in\Omega,\,\nu_I(W_1[\{\omega\}]\triangle W[\{\omega\}])>0\}$$
$$=\{\omega:\omega\in E,\,\nu_I(V_1[\{\omega\}]\triangle W[\{\omega\}])>0\}$$
$$\subseteq\{\omega:\omega\in\Omega,\,\nu_I(V_1[\{\omega\}]\triangle V[\{\omega\}])>0\}\cup\{\omega:\omega\in E,\,h_{V\triangle W}(\omega)>0\}$$
$$\in\mathcal{I}.$$

But this means that $W_1^\bullet=W^\bullet$ and $(\vec{W}_1,\mathbb{1})\in(W^\bullet)^\rightarrow$, so that

$$\Vdash_{\mathbb{P}}\vec{W}_1\in(W^\bullet)^\rightarrow,\quad r\Vdash_{\mathbb{P}}\dot{x}=\vec{V}_1=\vec{W}_1\in(W^\bullet)^\rightarrow;$$

but r is stronger than q and

$$q\Vdash_{\mathbb{P}}\dot{x}\notin(W^\bullet)^\rightarrow,$$

so we have a contradiction. **X**

(β) Thus if V, $W\in\Lambda$ and $p\in P$ are such that $p\Vdash_{\mathbb{P}}\vec{V}^\bullet=\vec{W}^\bullet$, we must have $p\Vdash_{\mathbb{P}}(V^\bullet)^\rightarrow\subseteq(W^\bullet)^\rightarrow$. Similarly, $p\Vdash_{\mathbb{P}}(W^\bullet)^\rightarrow\subseteq(V^\bullet)^\rightarrow$ and $p\Vdash_{\mathbb{P}}(W^\bullet)^\rightarrow=(V^\bullet)^\rightarrow$. Accordingly the condition 5A3Eb(ii) is satisfied and $\Vdash_{\mathbb{P}}\dot{\pi}$ is injective. **Q**

(iii) We need to check that

$$\Vdash_{\mathbb{P}}\dot{B}=\mathfrak{B}_{\check{I}}.$$

P(α) Suppose that $E\in\Sigma\setminus\mathcal{I}$ and a $\mathbb{P}$-name $\dot{x}$ are such that $E^\bullet\Vdash_{\mathbb{P}}\dot{x}\in\dot{B}$. Then there must be an $F\in\Sigma\setminus\mathcal{I}$ and a $W\in\Lambda$ such that $F^\bullet$ is stronger than $E^\bullet$ and $F^\bullet\Vdash_{\mathbb{P}}\dot{x}=\vec{W}^\bullet\in\mathfrak{B}_{\check{I}}$; as $E^\bullet$ and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}}\dot{B}\subseteq\mathfrak{B}_{\check{I}}.$$

(β) Suppose that $E\in\Sigma\setminus\mathcal{I}$ and a $\mathbb{P}$-name $\dot{x}$ are such that $E^\bullet\Vdash_{\mathbb{P}}\dot{x}\in\mathfrak{B}_{\check{I}}$. Then there must be a $\mathbb{P}$-name $\dot{W}$ such that

$$E^\bullet\Vdash_{\mathbb{P}}\dot{W}\in\mathcal{B}\mathfrak{a}_{\check{I}}\text{ and }\dot{x}=\dot{W}^\bullet.$$

By 551Fb, there is a $W\in\Lambda$ such that

$$E^\bullet\Vdash_{\mathbb{P}}\vec{W}=\dot{W},\text{ so }\dot{x}=\vec{W}^\bullet\in\dot{B};$$

as E and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}}\mathfrak{B}_{\check{I}}\subseteq\dot{B}\text{ and }\dot{B}=\mathfrak{B}_{\check{I}}.\ \mathbf{Q}$$

(iv) Putting these together,

$$\Vdash_{\mathbb{P}}\dot{\pi}\text{ is a bijection between }\dot{\mathfrak{D}}\text{ and }\mathfrak{B}_{\check{I}}.$$

(b) This is now easy. If V, $W\in\Lambda$, $a=V^\bullet$ and $b=W^\bullet$, then

$$\Vdash_{\mathbb{P}}\dot{\pi}(a\triangle b)^\rightarrow=\dot{\pi}((V\triangle W)^\bullet)^\rightarrow=((V\triangle W)^\rightarrow)^\bullet=(\vec{V}\triangle\vec{W})^\bullet$$

(551Eb)

$$=\vec{V}^\bullet\triangle\vec{W}^\bullet=\dot{\pi}\vec{a}\triangle\dot{\pi}\vec{b},$$

and similarly

$$\Vdash_{\mathbb{P}}\dot{\pi}(a\cap b)^\rightarrow=\dot{\pi}\vec{a}\cap\dot{\pi}\vec{b}.$$

Finally,

$$\Vdash_{\mathbb{P}}\bar{\nu}_{\check{I}}(\dot{\pi}\vec{a})=\bar{\nu}_{\check{I}}\vec{V}^\bullet=\nu_{\check{I}}\vec{V}=\vec{h}_V$$

by 551I(iii) again.

(c) Let $E \in \Sigma \setminus \mathcal{I}$ and V, $W \in \Lambda$ be such that $E^\bullet = p$, $V^\bullet = a$ and $W^\bullet = b$. Then (b) tells us that

$$\Vdash_{\mathbb{P}} \bar{\nu}_{\check{I}}(\dot{\pi}\vec{a} \mathbin{\triangle} \dot{\pi}\vec{b}) = \bar{\nu}_{\check{I}}\dot{\pi}(a \mathbin{\triangle} b)^{\rightarrow} = \bar{\nu}_{\check{I}}\dot{\pi}((V \mathbin{\triangle} W)^\bullet)^{\rightarrow} = \vec{h}_{V \triangle W}.$$

So

$$\begin{aligned} p \Vdash_{\mathbb{P}} \dot{\pi}\vec{a} = \dot{\pi}\vec{b} &\iff p \Vdash_{\mathbb{P}} \bar{\nu}_{\check{I}}(\dot{\pi}\vec{a} \mathbin{\triangle} \dot{\pi}\vec{b}) = 0 \\ &\iff p \Vdash_{\mathbb{P}} \vec{h}_{V \triangle W} = 0 \\ &\iff h_{V \triangle W}(\omega) = 0 \text{ for } \mathcal{I}\text{-almost every } \omega \in E \end{aligned}$$

(551B)

$$\begin{aligned} &\iff \nu_I(V[\{\omega\}] \triangle W[\{\omega\}]) = 0 \text{ for } \mathcal{I}\text{-almost every } \omega \in E \\ &\iff (E \times \{0,1\}^I) \cap (V \triangle W) \in \mathcal{J} \\ &\iff \varepsilon(p) \cap (a \mathbin{\triangle} b) = 0 \iff a \cap \varepsilon(p) = b \cap \varepsilon(p). \end{aligned}$$

551Q Iterated forcing The machinery just developed can be used to establish one of the most important properties of random real forcing.

Theorem Let $(\Omega, \Sigma, \mathcal{I})$ be a non-trivial ω_1-saturated measurable space with negligibles such that Σ is closed under Souslin's operation, $\mathbb{P}$ its associated forcing notion, and I a set. As in 551P, set $\Lambda = \Sigma \widehat{\otimes} \mathcal{B}\mathfrak{a}_I$,

$$\mathcal{J} = \{W : W \in \Lambda,\ \nu_I W[\{\omega\}] = 0 \text{ for } \mathcal{I}\text{-almost every } \omega \in \Omega\}$$

and $\mathfrak{C} = \Lambda/\mathcal{J}$. Then

$$\mathfrak{C} \cong \mathrm{RO}(\mathbb{P} * \mathbb{P}_{\check{I}}),$$

where the $\mathbb{P}$-name $\mathbb{P}_{\check{I}}$ is defined as in 551O, and $\mathbb{P} * \mathbb{P}_{\check{I}}$ is the iterated forcing notion defined in 5A3O.

proof (a) Since I wish to follow Kunen 80 as closely as possible, I should perhaps start with a remark on the interpretation of names for forcing notions. There is, strictly speaking, a distinction to be made between a name for a forcing notion, which is a name for a quadruplet of the form $(P, \le, \mathbb{1}, \Updownarrow)$, and a quadruplet of names, the first for a set, the second for a pre-order on that set, and so on; and the latter is easier to work with (Kunen 80, §VIII.5, and 5A3O). In the present case, we do not need any new manoeuvres, since the construction of the name $\mathbb{P}_{\check{I}}$ is based on $\mathbb{P}$-names for $\mathfrak{B}_{\check{I}}$, $\subseteq_{\mathfrak{B}_{\check{I}}}$ and $1_{\mathfrak{B}_{\check{I}}}$.

As usual in this section, I will write $\mathfrak{A}$ for $\Sigma/\Sigma \cap \mathcal{I}$.

(b) Now $\mathbb{P} * \mathbb{P}_{\check{I}}$ is based on the set P of pairs $(p, \dot{b})$ where $p \in \mathfrak{A}^+$, $\dot{b} \in B$ and $p \Vdash_{\mathbb{P}} \dot{b} \in \mathfrak{B}_{\check{I}}^+$; here B is the domain of the $\mathbb{P}$-name $\mathfrak{B}_{\check{I}}^+$ (5A3Ba). If we say that $(p, \dot{b}) \le (p', \dot{b}')$ if $p \subseteq p'$ and $p \Vdash_{\mathbb{P}} \dot{b} \subseteq \dot{b}'$, then P is pre-ordered by $\le$ and $\mathbb{P} * \mathbb{P}_{\check{I}}$ is active downwards.

We have a unique function $\theta : P \to \mathfrak{C}^+$ such that

$$\theta(p, \dot{b}) \subseteq \varepsilon(p), \quad p \Vdash_{\mathbb{P}} \dot{\pi}\theta(p, \dot{b})^{\rightarrow} = \dot{b},$$

whenever $(p, \dot{b}) \in P$, where ε, $\dot{\pi}$ and $\vec{a}$, for $a \in \mathfrak{C}$, are defined as in 551P. **P** If $(p, \dot{b}) \in P$, so that $p \Vdash_{\mathbb{P}} \dot{b} \in \mathfrak{B}_{\check{I}}$, there is a $\mathbb{P}$-name $\dot{b}_1$ such that

$$\Vdash_{\mathbb{P}} \dot{b}_1 \in \mathfrak{B}_{\check{I}},$$

$$p \Vdash_{\mathbb{P}} \dot{b}_1 = \dot{b}.$$

Next, there is an $a_0 \in \mathfrak{C}$ such that $\Vdash_{\mathbb{P}} \dot{\pi}\vec{a}_0 = \dot{b}_1$ (551Pa). Set $a = a_0 \cap \varepsilon(p)$. Then 551Pc tells us that

$$p \Vdash_{\mathbb{P}} \dot{\pi}\vec{a} = \dot{\pi}\vec{a}_0 = \dot{b}_1 = \dot{b} \ne 0,$$

and $a \ne 0$. To see that a is unique, observe that if $c \in \mathfrak{C}$ is such that $p \Vdash_{\mathbb{P}} \dot{\pi}\vec{c} = \dot{b}$, then $c \cap \varepsilon(p) = a \cap \varepsilon(p)$, by 551Pc again; so if $c \subseteq \varepsilon(p)$, $c = a$. We therefore can, and must, take a for $\theta(p, \dot{b})$. **Q**

(c)(i) If $(p,\dot b)$, $(p',\dot b')\in P$ and $(p,\dot b)$ is stronger than $(p',\dot b')$, then $p\subseteq p'$ and $p\Vdash_{\mathbb{P}}\dot b\subseteq\dot b'$. In this case,

$$p\Vdash_{\mathbb{P}}\dot\pi\theta(p',\dot b')^{\rightarrow}=\dot b' \text{ and } \dot\pi(\theta(p,\dot b)\cap\theta(p',\dot b'))^{\rightarrow}=\dot\pi\theta(p,\dot b)^{\rightarrow}\cap\dot\pi\theta(p',\dot b')^{\rightarrow}=\dot b\cap\dot b'=\dot b,$$

while $\theta(p,\dot b)\cap\theta(p',\dot b')\subseteq\varepsilon(p)$, so $\theta(p,\dot b)\cap\theta(p',\dot b')=\theta(p,\dot b)$ and $\theta(p,\dot b)\subseteq\theta(p',\dot b')$.

(ii) If $(p,\dot b)$ and $(p',\dot b')\in P$ are incompatible, then $\theta(p,\dot b)\cap\theta(p',\dot b')=0$. **P?** Otherwise, writing a for $\theta(p,\dot b)\cap\theta(p',\dot b')$,

$$\varepsilon(p\cap p')=\varepsilon(p)\cap\varepsilon(p')\supseteq a\neq0$$

so $p\cap p'\neq0$ and

$$p\cap p'\Vdash_{\mathbb{P}}\dot\pi\vec a\subseteq\dot\pi\theta(p,\dot b)^{\rightarrow}\cap\dot\pi\theta(p',\dot b')^{\rightarrow}=\dot b\cap\dot b'=0;$$

as $a\subseteq\varepsilon(p\cap p')$, a must be 0; which is absurd. **XQ**

(iii) If $a\in\mathfrak{C}^+$, there is a $(p,\dot b)\in P$ such that $\theta(p,\dot b)\subseteq a$. **P** By 551Pc, $\not\Vdash_{\mathbb{P}}\dot\pi\vec a=0$, that is, there is a $p_0\in\mathfrak{C}^+$ such that $p_0\Vdash_{\mathbb{P}}\dot\pi\vec a\neq0$. Now there must be a $p\in\mathfrak{C}^+$, stronger than p_0, and a $\dot b\in B$ such that $p\Vdash_{\mathbb{P}}\dot b=\dot\pi\vec a$, in which case $(p,\dot b)\in P$ and $p\Vdash_{\mathbb{P}}\dot\pi\theta(p,\dot b)^{\rightarrow}=\dot\pi\vec a$. Accordingly

$$\theta(p,\dot b)=\theta(p,\dot b)\cap\varepsilon(p)=a\cap\varepsilon(p)\subseteq a. \textbf{ Q}$$

(d) Observe now that $\mathfrak{C}$ is ccc (527L), therefore Dedekind complete, and (c) tells us that $\theta:P\to\mathfrak{C}^+$ satisfies the conditions of 514Sa. So $\mathrm{RO}(\mathbb{P}*\mathbb{P}_{\check I})=\mathrm{RO}^{\downarrow}(P)$ is isomorphic to $\mathfrak{C}$.

551R Extending filters The following device will be useful in §553.

Proposition Let $(\Omega,\Sigma,\mathcal{I})$ be a non-trivial ω_1-saturated measurable space with negligibles, $\mathfrak{A}$ its quotient algebra, $\mathbb{P}$ the associated forcing notion, I a countable set and $\mathcal{F}$ a filter on I.

(a) For $H\in\Sigma\widehat{\otimes}\mathcal{P}I$, write $\vec H$ for the $\mathbb{P}$-name $\{(\check\imath,H^{-1}[\{i\}]^\bullet):i\in I,\,H^{-1}[\{i\}]\notin\mathcal{I}\}$.

(i) $\Vdash_{\mathbb{P}}\vec H\subseteq\check I$.

(ii) If $\dot F$ is a $\mathbb{P}$-name and $p\in\mathfrak{A}^+$ is such that $p\Vdash_{\mathbb{P}}\dot F\subseteq\check I$, then there is an $H\in\Sigma\widehat{\otimes}\mathcal{P}I$ such that $p\Vdash_{\mathbb{P}}\dot F=\vec H$.

(b) Write $\vec{\mathcal{F}}$ for the $\mathbb{P}$-name

$$\{(\vec H,E^\bullet):H\in\Sigma\widehat{\otimes}\mathcal{P}I,\,E\in\Sigma\setminus\mathcal{I},\,H[\{\omega\}]\in\mathcal{F}\text{ for every }\omega\in E\}.$$

Then

$$\Vdash_{\mathbb{P}}\vec{\mathcal{F}}\text{ is a filter on }\check I.$$

proof (a)(i) is elementary, just because

$$\Vdash_{\mathbb{P}}\check\imath\in\check I$$

for every $i\in I$.

(ii) Because $(\Omega,\Sigma,\mathcal{I})$ is ω_1-saturated, $\mathfrak{A}$ is Dedekind complete and can be identified with $\mathrm{RO}(\mathbb{P})$. We therefore have, for each $i\in I$, an $E_i\in\Sigma$ such that $E_i^\bullet$ can be identified with $[\![\check\imath\in\dot F]\!]$. Set $H=\bigcup_{i\in I}E_i\times\{i\}$. Then

$$[\![\check\imath\in\dot F]\!]=H^{-1}[\{i\}]^\bullet=[\![\check\imath\in\vec H]\!]$$

for every $i\in I$, so

$$p\Vdash_{\mathbb{P}}\dot F=\dot F\cap\check I=\vec H\cap\check I=\vec H.$$

(b)(i) By (a-i), $\Vdash_{\mathbb{P}}\vec{\mathcal{F}}\subseteq\check I$.

(ii) Since $((\Omega\times I)^{\rightarrow},\mathbb{1})\in\vec{\mathcal{F}}$ and

$$\Vdash_{\mathbb{P}}(\Omega\times I)^{\rightarrow}=\check I$$

(551D), we have

$$\Vdash_{\mathbb{P}} \check{I} \in \vec{\mathcal{F}}.$$

(iii) If $(\vec{H}, p) \in \vec{\mathcal{F}}$ then $p \Vdash_{\mathbb{P}} \vec{H} \neq \emptyset$. **P** Express p as $E^\bullet$ where $E \in \Sigma$ and $H[\{\omega\}] \in \mathcal{F}$ for every $\omega \in E$. Then $E \subseteq \bigcup_{i \in I} H^{-1}[\{i\}]$. So if $q \in \mathfrak{A}^+$ is stronger than p, there must be an $i \in I$ such that $r = q \cap H^{-1}[\{i\}]^\bullet$ is non-zero; in which case r is stronger than q and

$$r \Vdash_{\mathbb{P}} \check{\imath} \in \vec{H}, \text{ so } \vec{H} \neq \emptyset.$$

As q is arbitrary, $p \Vdash_{\mathbb{P}} \vec{H} \neq \emptyset$. **Q**

It follows at once that $\Vdash_{\mathbb{P}} \emptyset \notin \vec{\mathcal{F}}$.

(iv) If $\dot{F}_0$, $\dot{F}_1$ are $\mathbb{P}$-names and $p \in \mathfrak{A}^+$ is such that

$$p \Vdash_{\mathbb{P}} \dot{F}_0,\ \dot{F}_1 \in \vec{\mathcal{F}},$$

then

$$p \Vdash_{\mathbb{P}} \dot{F}_0 \cap \dot{F}_1 \in \vec{\mathcal{F}}.$$

P If $q \in \mathfrak{A}^+$ is stronger than p there must be $(\vec{H}_0, E_0^\bullet)$, $(\vec{H}_1, E_1^\bullet) \in \vec{\mathcal{F}}$ and an r stronger than q, $E_0^\bullet$ and $E_1^\bullet$ such that

$$r \Vdash_{\mathbb{P}} \dot{F}_0 = \vec{H}_0 \text{ and } \dot{F}_1 = \vec{H}_1.$$

Now $((H_0 \cap H_1)^{\vec{}}, (E_0 \cap E_1)^\bullet) \in \vec{\mathcal{F}}$ and

$$r \Vdash_{\mathbb{P}} \dot{F}_0 \cap \dot{F}_1 = \vec{H}_0 \cap \vec{H}_1 = (H_0 \cap H_1)^{\vec{}} \in \vec{\mathcal{F}}.$$

As q is arbitrary,

$$p \Vdash_{\mathbb{P}} \dot{F}_0 \cap \dot{F}_1 \in \vec{\mathcal{F}}. \textbf{ Q}$$

Accordingly

$$\Vdash_{\mathbb{P}} \vec{\mathcal{F}} \text{ is closed under } \cap.$$

(v) Suppose that $\dot{F}_0$, $\dot{F}_1$ are $\mathbb{P}$-names and $p \in \mathfrak{A}^+$ is such that

$$p \Vdash_{\mathbb{P}} \dot{F}_0 \subseteq \dot{F}_1 \subseteq \check{I},\ \dot{F}_0 \in \vec{\mathcal{F}}.$$

By (a-ii), there is an $H_1 \in \Sigma \widehat{\otimes} \mathcal{P}I$ such that $p \Vdash_{\mathbb{P}} \dot{F}_1 = \vec{H}_1$. If q is stronger than p, there are an $(\vec{H}_0, E_0^\bullet) \in \vec{\mathcal{F}}$ and an r stronger than both q and $E_0^\bullet$ such that

$$r \Vdash_{\mathbb{P}} \vec{H}_0 = \dot{F}_0 \subseteq \dot{F}_1 = \vec{H}_1.$$

Expressing r as $E^\bullet$ where $E \in \Sigma \setminus \mathcal{I}$, we have

$$E \cap H_0^{-1}[\{i\}] \setminus H_1^{-1}[\{i\}] \in \mathcal{I}$$

for every $i \in I$. Set

$$E_1 = E \setminus \textstyle\bigcup_{i \in I}(H_0^{-1}[\{i\}] \setminus H_1^{-1}[\{i\}]);$$

then $(\vec{H}_1, E_1^\bullet) \in \vec{\mathcal{F}}$, so

$$r = E_1^\bullet \Vdash_{\mathbb{P}} \dot{F}_1 = \vec{H}_1 \in \vec{\mathcal{F}}.$$

As q is arbitrary,

$$p \Vdash_{\mathbb{P}} \dot{F}_1 \in \vec{\mathcal{F}}.$$

As p, $\dot{F}_0$ and $\dot{F}_1$ are arbitrary,

$$\Vdash_{\mathbb{P}} \vec{\mathcal{F}} \text{ is a filter on } \check{I}.$$

551X Basic exercises (a) Let $(\Omega, \Sigma, \mathcal{I})$ be any measurable space with negligibles. Set $\hat{\Sigma} = \{E \triangle F : E \in \Sigma,\ F \in \mathcal{I}\}$. (i) Show that $(\Omega, \hat{\Sigma}, \mathcal{I})$ is a complete measurable space with negligibles; we may call it the **completion** of $(\Omega,\Sigma,\mathcal{I})$. (ii) Show that the algebras $\Sigma/\Sigma\cap\mathcal{I}$ and $\hat{\Sigma}/\mathcal{I}$ are canonically isomorphic (cf. 322Da). (iii) Show that $(\Omega,\hat{\Sigma},\mathcal{I})$ is ω_1-saturated iff $(\Omega,\Sigma,\mathcal{I})$ is.

(b) Let (Ω,Σ,μ) be a measure space and $\mathcal{N}(\mu)$ the null ideal of μ. (i) Show that $(\Omega,\Sigma,\mathcal{N}(\mu))$ is a measurable space with negligibles. (ii) Show that if the completion of (Ω,Σ,μ) (212C) is $(\Omega,\hat{\Sigma},\hat{\mu})$, then $(\Omega,\hat{\Sigma},\mathcal{N}(\hat{\mu}))$ is the completion of $(\Omega,\Sigma,\mathcal{N}(\mu))$.

(c) Let X be a topological space, $\mathcal{B}(X)$ the Borel σ-algebra of X, $\widehat{\mathcal{B}}(X)$ the Baire-property algebra of X and $\mathcal{M}$ the ideal of meager subsets of X. (i) Show that $(X,\mathcal{B}(X),\mathcal{M})$ is a measurable space with negligibles. (ii) Show that its completion is $(X,\widehat{\mathcal{B}}(X),\mathcal{M})$.

(d) Let $(\Omega,\Sigma,\mathcal{I})$ be a non-trivial measurable space with negligibles, and $\mathbb{P}$ the associated forcing notion. (i) Show that the regular open algebra of $\mathbb{P}$ can be identified with the Dedekind completion of $\Sigma/\Sigma\cap\mathcal{I}$. (ii) Show that if $\mathcal{E}$ is any coinitial subset of $\Sigma\setminus\mathcal{I}$ containing Ω, then the forcing notion $\mathcal{E}$, active downwards, has regular open algebra isomorphic to $\mathrm{RO}(\mathbb{P})$.

(e) Let $(\Omega,\Sigma,\mathcal{I})$ and $(\Upsilon,\mathrm{T},\mathcal{J})$ be measurable spaces with negligibles. Show that $(\Omega\times\Upsilon,\Sigma\widehat{\otimes}\mathrm{T},\mathcal{I}\ltimes_{\Sigma\widehat{\otimes}\mathrm{T}}\mathcal{J})$, as defined in 527Bc, is a measurable space with negligibles.

(f) Let $(\Omega,\Sigma,\mathcal{I})$ be a non-trivial measurable space with negligibles, and $\mathbb{P}$ the associated forcing notion. Let $f:\Omega\to\{0,1\}$ be a measurable function, and define $\mathbb{P}$-names $\dot{x}$, $\dot{y}$ by saying that $\dot{x}$ is the $\mathbb{P}$-name for a real number as defined from f in 551B, while $\dot{y}$ is the $\mathbb{P}$-name for a member of $\{0,1\}$ as defined in 551Ca. Show that

$$\Vdash_{\mathbb{P}} \text{regarding 0 and 1 as real numbers, } \dot{x}=\dot{y}.$$

(g) Let $(\Omega,\Sigma,\mathcal{I})$ be a non-trivial measurable space with negligibles, $\mathbb{P}$ the associated forcing notion, and I a set. Suppose that $f:\Omega\to\{0,1\}^I$ is a $(\Sigma,\mathcal{B}\mathfrak{a}_I)$-measurable function, and that $\Gamma_f\subseteq\Omega\times\{0,1\}^I$ is its graph (for once, I distinguish between f and Γ_f). Let $\vec{f}$ and $\vec{\Gamma}_f$ be the $\mathbb{P}$-names for a point of $\{0,1\}^{\check{I}}$ and a subset of $\{0,1\}^{\check{I}}$ defined by the formulae in 551C and 551D respectively. Show that $\Vdash_{\mathbb{P}}\vec{\Gamma}_f=\{\vec{f}\}$.

(h) Let $(\Omega,\Sigma,\mathcal{I})$ be a non-trivial measurable space with negligibles, and $(\Omega,\hat{\Sigma},\mathcal{I})$ its completion; write $\mathbb{P}$, $\hat{\mathbb{P}}$ for the associated forcing notions, so that $\mathbb{P}$ and $\hat{\mathbb{P}}$ are canonically isomorphic. Let I be a set, W a member of $\Sigma\widehat{\otimes}\{0,1\}^I\subseteq\hat{\Sigma}\widehat{\otimes}\{0,1\}^I$ and $\vec{W}$ the $\mathbb{P}$-name, $\hat{\vec{W}}$ the $\hat{\mathbb{P}}$-name defined by the formula in 551D. Explain what it ought to mean to say that $\Vdash_{\hat{\mathbb{P}}}\hat{\vec{W}}=\vec{W}$, and why this is true.

(i) Let $(\Omega,\Sigma,\mathcal{I})$ be a non-trivial ω_1-saturated measurable space with negligibles with a Dedekind complete quotient algebra, $\mathbb{P}$ the associated forcing notion, and I a set. Suppose that $W\in\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}(\{0,1\}^I)$ is such that $\Vdash_{\mathbb{P}}\vec{W}\in\mathcal{B}\mathfrak{a}_{\check{\alpha}}(\{0,1\}^{\check{I}})$, where $\alpha<\omega_1$. Show that $W[\{\omega\}]\in\mathcal{B}\mathfrak{a}_\alpha(\{0,1\}^I)$ for $\mathcal{I}$-almost every $\omega\in\Omega$.

551Y Further exercises (a) Investigate the difficulties which arise if we try to represent names for Borel subsets of $\{0,1\}^{\check{I}}$ as members of $\Sigma\widehat{\otimes}\{0,1\}^I$, when I is uncountable. Show that some of these are resolvable if Ω is actually the Stone space of $\mathrm{RO}(\mathbb{P})$.

(b) Let $(\Omega,\Sigma,\mathcal{I})$ be a non-trivial measurable space with negligibles, $\mathbb{P}$ the associated forcing notion and P the partially ordered set underlying $\mathbb{P}$. (i) Let $p\in P$ and a $\mathbb{P}$-name $\dot{G}$ be such that

$$p\Vdash_{\mathbb{P}}\dot{G} \text{ is a dense open subset of } \{0,1\}^\omega.$$

Show that there is a $W\in\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_\omega$ such that every vertical section of W is a dense open set and $p\Vdash_{\mathbb{P}}\dot{G}=\vec{W}$. (ii) Let $p\in P$ and a $\mathbb{P}$-name $\dot{A}$ be such that

$$p\Vdash_{\mathbb{P}}\dot{A} \text{ is a meager subset of } \{0,1\}^\omega.$$

Show that there is a $W\in\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_\omega$ such that every vertical section of W is a meager set and $p\Vdash_{\mathbb{P}}\dot{A}\subseteq\vec{W}$.

551 Notes and comments There are real metamathematical difficulties in forcing, and we need to find new compromises between formal rigour and intuitive accessibility. In the formulae of this section I am taking a path with rather more explicit declarations than is customary. The definitions of $\vec{u}$ in 5A3L, $\vec{f}$ in 551Ca and $\vec{W}$ in 551D are supposed to be $\mathbb{P}$-names in the exact sense used in KUNEN 80. This leads to rather odd sentences of the form

$$(\vec{f}, p) \in \vec{W} \text{ so } p \Vdash_{\mathbb{P}} \vec{f} \in \vec{W}$$

(as in (a-ii) of the proof of 551E, for example), in which the symbol $\in$ is being used in different ways in the two halves; but it has the advantage that we can move from W to $\vec{W}$ without further explanation, as in the statements of 551E-551J. But you will observe that elsewhere I allow such terms as $\mathcal{B}\mathfrak{a}$ and $\nu_{...}$ to enter sentences in the forcing language, since these correspond to definitions which can be expanded there. Note that I am being less strict than usual in requiring formulae to be unambiguous (see 551Xf and 551Xg).

There is always room for variation in the matter of which terms should be decorated with ˇs when they appear in expressions of the forcing language, and I have tried to be reasonably consistent; but there are particular difficulties with transferring names for families (5A3Fb), which appear here in such formulae as '$\Vdash_{\mathbb{P}} \lim_{n\to\infty} \vec{h}_n = \lim_{n\to\infty} \vec{\psi}_n(\dot{x})$' (part (c) of the proof of 551N).

I hope that it is not too confusing to have the formula $[\![\ldots]\!]$ used in two different ways, not infrequently in the same sentence: sometimes as a 'Boolean value' in the forcing sense, and sometimes in the sense of Chapter 36. If you look back at the definitions in §364 you will see that the expression $f^\bullet$ also shifts in interpretation as we move between the formally distinct algebras $\mathfrak{A}$ and $\mathrm{RO}(\mathbb{P})$. There are some particularly difficult formulae to parse in 551P; following the statement of the theorem I offer a remark on the expression $(((W^\bullet)^{\rightharpoondown}, (\vec{W})^\bullet), \mathbb{1})$, and some of the same difficulties arise in the line

$$\Vdash_{\mathbb{P}} \dot{\pi}(a \mathbin{\triangle} b)^{\rightharpoondown} = \dot{\pi}((V \triangle W)^\bullet)^{\rightharpoondown} = ((V \triangle W)^{\rightharpoondown})^\bullet = (\vec{V} \triangle \vec{W})^\bullet$$

in part (b) of the proof, where as well as the ambiguities in $^\bullet$ and $(\ldots)$ we have the symbols $\mathbin{\triangle}$ and $\triangle$ being used first for an operation in the ground-model Boolean algebra $\mathfrak{C}$, then for symmetric difference in the ordinary universe, and finally for symmetric difference in the forcing language.

Version of 29.1.14

552 Random reals I

From the point of view of a measure theorist, 'random real forcing' has a particular significance. Because the forcing notions are defined directly from the central structures of measure theory (552A), they can be expected to provide worlds in which measure-theoretic questions are answered. Thus we find ourselves with many Sierpiński sets (552E), information on cardinal functions (552C, 552F-552J), and theorems on extension of measures (552N). But there is a second reason why any measure theorist or probabilist should pay attention to random real forcing. Natural questions in the forcing language, when translated into propositions about the ground model, are likely to hinge on properties of measure algebras, giving us a new source of challenging problems. Perhaps the deepest intuitions are those associated with iterated random real forcing (552P).

552A Notation (a) As usual, if μ is a measure then $\mathcal{N}(\mu)$ will be its null ideal. It will be convenient to have a special notation for certain sets of finite functions: if I is a set, $\mathrm{Fn}_{<\omega}(I;\{0,1\})$ will be $\bigcup_{K\in[I]^{<\omega}}\{0,1\}^K$.

For any set I I will write ν_I for the usual completion regular Radon probability measure on $\{0,1\}^I$, T_I for its domain and $(\mathfrak{B}_I, \bar{\nu}_I)$ for its measure algebra; $\mathcal{B}\mathfrak{a}_I = \mathcal{B}\mathfrak{a}(\{0,1\}^I)$ will be the Baire σ-algebra of $\{0,1\}^I$. (It will sometimes be convenient, when applying the results of §551, to regard $\mathfrak{B}_I$ as the quotient $\mathcal{B}\mathfrak{a}_I/\mathcal{B}\mathfrak{a}_I \cap \mathcal{N}(\nu_I)$.) In this context, I will write $\langle e_i\rangle_{i\in I}$ for the standard generating family in $\mathfrak{B}_I$ (525A). $\mathbb{P}_I$ will be the forcing notion $\mathfrak{B}_I^+ = \mathfrak{B}_I \setminus \{0\}$, active downwards. For a formula ϕ in the corresponding forcing language I will write $[\![\phi]\!]$ for the truth value of ϕ, interpreted as a member of $\mathfrak{B}_I$ (5A3M). Note that as $\mathbb{P}_I$ is ccc (cf. 511Db), it preserves cardinals (5A3Nb).

As in §551, the formulae ν_I, $\mathfrak{B}_I$ etc. are to be regarded as formulae of set theory with one free variable into which the parameter I has been substituted, so that we have corresponding names $\nu_{\check{I}}$, $\mathfrak{B}_{\check{I}}$ in any forcing language, and in particular (once the context has established a forcing notion $\mathbb{P}$) we have $\mathbb{P}$-names $\nu_{\check{I}}$, $\mathfrak{B}_{\check{I}}$ for any ground-model set I.

(b) A great deal of the work of this chapter will involve interpretations of names for standard objects (in particular, for cardinals) in forcing languages. Reflecting suggestions in 5A3H and 5A3N, I will try to signal intended interpretations by using the superscript $\check{}$. Thus $\mathfrak{c}$ will always be an abbreviation for 'the initial ordinal equipollent with the set of subsets of the natural numbers', whether I am using the ordinary language of set theory or speaking in a forcing language; and $\check{\mathfrak{c}}$, in a forcing language, will refer to the name $\{(\check{\xi}, \mathbb{1}) : \xi < \mathfrak{c}\}$, where it is to be understood that the symbol $\mathfrak{c}$ must now be interpreted in the ordinary universe. As I shall avoid arguments involving more than one forcing notion (and, in particular, iterated forcing), there will I hope be little scope for confusion, even in such sentences as

$$\Vdash_{\mathbb{P}_\kappa} \mathfrak{b} = \check{\mathfrak{b}}$$

(552C). The leading $\Vdash_{\mathbb{P}_\kappa}$ declares that the rest of the sentence is in the language of $\mathbb{P}_\kappa$-forcing; the first $\mathfrak{b}$, and the $\check{}$, are therefore to be interpreted in that language; but the second $\mathfrak{b}$, being subject to the $\check{}$, is to be interpreted in the ground model. (Many authors would write $\mathfrak{b}^V$ at this point.) Similarly, in

$$\Vdash_{\mathbb{P}_\kappa} 2^{\check{\lambda}} = (\kappa^\lambda)\check{}$$

(552B), the subformula κ^λ is to be interpreted in the ordinary universe, but $2^{\check{\lambda}} = \#(\mathcal{P}\check{\lambda})$ is to be interpreted in the forcing language. I hope that the resulting gains in directness and conciseness will not be at the expense of leaving you uncertain of the meaning.

552B Theorem Suppose that λ and κ are infinite cardinals. Then

$$\Vdash_{\mathbb{P}_\kappa} 2^{\check{\lambda}} = (\kappa^\lambda)\check{},$$

where κ^λ is the cardinal power (interpreted in the ordinary universe, of course).

proof (a) Recall that $\#(\mathfrak{B}_\kappa) = \kappa^\omega$ (524Ma), so that

$$\#(\mathfrak{B}_\kappa^\lambda) = \#(\kappa^{\omega\times\lambda}).$$

If $\dot{A}$ is a $\mathbb{P}_\kappa$-name for a subset of $\check{\lambda}$, then we have a corresponding family $\langle [\![\check{\eta} \in \dot{A}]\!] \rangle_{\eta<\lambda}$ of truth values; and if $\dot{A}$, $\dot{B}$ are two such names, and $[\![\check{\eta} \in \dot{A}]\!] = [\![\check{\eta} \in \dot{B}]\!]$ for every $\eta < \lambda$, then

$$\Vdash_{\mathbb{P}_\kappa} \check{\eta} \in \dot{A} \iff \check{\eta} \in \dot{B}$$

for every $\eta < \lambda$, so

$$\Vdash_{\mathbb{P}_\kappa} \dot{A} = \dot{B}.$$

So

$$\Vdash_{\mathbb{P}_\kappa} 2^{\check{\lambda}} = \#(\mathcal{P}\check{\lambda}) \le \#((\mathfrak{B}_\kappa^\lambda)\check{}) = (\kappa^\lambda)\check{}.$$

(b) In the other direction, consider first the case in which $\lambda \le \kappa$. Let F be the set of all functions from λ to κ, so that $\#(F) = \kappa^\lambda$. Then there is a set $G \subseteq F$ such that $\#(G) = \kappa^\lambda$ and $\{\eta : \eta < \lambda,\ f(\eta) \ne g(\eta)\}$ is infinite whenever f, $g \in G$ are distinct. **P** If $\kappa = \kappa^\lambda$ we can take G to be the set of constant functions. Otherwise, for f, $g \in F$, say that $f =^* g$ if $\{\eta : f(\eta) \ne g(\eta)\}$ is finite; this is an equivalence relation. Let $G \subseteq F$ be a set meeting each equivalence class in just one element. Then we have $\#(\{g : g =^* f\}) = \kappa < \kappa^\lambda$ for every $f \in F$, so $\#(G) = \kappa^\lambda$, as required. **Q**

Let $\langle e_{\xi\eta} \rangle_{\xi<\kappa,\eta<\lambda}$ be a stochastically independent family in $\mathfrak{B}_\kappa$ of elements of measure $\frac{1}{2}$. For $f \in G$ let $\dot{A}_f$ be a $\mathbb{P}_\kappa$-name for a subset of $\check{\lambda}$ such that

$$[\![\check{\eta} \in \dot{A}_f]\!] = e_{f(\eta),\eta}$$

for every $\eta < \lambda$. If f, $g \in G$ are distinct, set $I = \{\eta : f(\eta) \ne g(\eta)\}$; then

$$[\![\dot{A}_f \ne \dot{A}_g]\!] = \sup_{\eta<\lambda} e_{f(\eta),\eta} \triangle e_{g(\eta),\eta} = \sup_{\eta\in I} e_{f(\eta),\eta} \triangle e_{g(\eta),\eta} = 1$$

because $\langle e_{f(\eta),\eta} \triangle e_{g(\eta),\eta} \rangle_{\eta\in I}$ is an infinite stochastically independent family of elements of measure $\frac{1}{2}$.

Thus in the forcing language we have a name for an injective function from $\check{G}$ to $\mathcal{P}\check{\lambda}$, corresponding to the map $f \mapsto \dot{A}_f$ from G to names of subsets of λ. So

$$\Vdash_{\mathbb{P}_\kappa} 2^{\check{\lambda}} \geq \#(\check{G}) = (\kappa^\lambda)\check{}.$$

Putting this together with (a), we have

$$\Vdash_{\mathbb{P}_\kappa} 2^{\check{\lambda}} = (\kappa^\lambda)\check{}.$$

(c) If $\lambda > \kappa$, then (in the ordinary universe) $2^\lambda = \kappa^\lambda$. Now

$$\Vdash_{\mathbb{P}_\kappa} (\mathcal{P}\lambda)\check{} \subseteq \mathcal{P}\check{\lambda},$$

so

$$\Vdash_{\mathbb{P}_\kappa} (\kappa^\lambda)\check{} = \#((\mathcal{P}\lambda)\check{}) \leq \#(\mathcal{P}\check{\lambda}) = 2^{\check{\lambda}},$$

and again we have

$$\Vdash_{\mathbb{P}_\kappa} 2^{\check{\lambda}} = (\kappa^\lambda)\check{}.$$

552C Theorem Let κ be any cardinal. Then

$$\Vdash_{\mathbb{P}_\kappa} \mathfrak{b} = \check{\mathfrak{b}} \text{ and } \mathfrak{d} = \check{\mathfrak{d}}.$$

proof (a) The point is that if $\dot{f}$ is any $\mathbb{P}_\kappa$-name for a member of $\mathbb{N}^\mathbb{N}$, then there is an $h \in \mathbb{N}^\mathbb{N}$ such that

$$\Vdash_{\mathbb{P}_\kappa} \dot{f} \leq^* \check{h},$$

where I write $f \leq^* g$ to mean that $\{n : g(n) < f(n)\}$ is finite, as in 522C. **P** For $n, i \in \mathbb{N}$ set $a_{ni} = [\![\dot{f}(\check{n}) = \check{\imath}]\!]$. Then $D_n = \{a_{ni} : i \in \mathbb{N}\}$ is a partition of unity in $\mathfrak{B}_\kappa$ for each $n \in \mathbb{N}$. Because $\mathfrak{B}_\kappa$ is weakly (σ, ∞)-distributive (322F), there is a partition of unity D such that $\{i : a_{ni} \cap d \neq 0\}$ is finite for each n and each $d \in D$. Let $\langle d_k \rangle_{k \in \mathbb{N}}$ be a sequence running over D and take $h(n)$ such that $a_{mi} \cap d_n = 0$ whenever $m \leq n$ and $i > h(n)$. Now

$$[\![\check{h}(\check{m}) < \dot{f}(\check{m})]\!] = [\![h(m)\check{} < \dot{f}(\check{m})]\!] = \sup\{a_{mi} : i > h(m)\} \subseteq 1 \setminus d_n$$

whenever $n \leq m$. So

$$\begin{aligned}[\![\check{h}(n) < \dot{f}(n) \text{ for infinitely many } n]\!] &= \inf_{n \in \mathbb{N}} \sup_{m \geq n} [\![h(m)\check{} < \dot{f}(\check{m})]\!] \\ &\subseteq \inf_{n \in \mathbb{N}} 1 \setminus d_n = 0,\end{aligned}$$

that is,

$$\Vdash_{\mathbb{P}_\kappa} \dot{f} \leq^* \check{h}. \text{ } \mathbf{Q}$$

(b)(i) Let $\langle \dot{f}_\xi \rangle_{\xi < \lambda}$ be a family of $\mathbb{P}_\kappa$-names for members of $\mathbb{N}^\mathbb{N}$, where $\lambda < \mathfrak{b}$. Then for each $\xi < \lambda$ we can find an $h_\xi \in \mathbb{N}^\mathbb{N}$ such that $\Vdash_{\mathbb{P}_\kappa} \dot{f}_\xi \leq^* \check{h}_\xi$. As $\lambda < \mathfrak{b}$, there is an $h \in \mathbb{N}^\mathbb{N}$ such that $h_\xi \leq^* h$ for every $\xi < \lambda$. Now $\Vdash_{\mathbb{P}_\kappa} \check{h}_\xi \leq^* \check{h}$ for every ξ, so $\Vdash_{\mathbb{P}_\kappa} \dot{f}_\xi \leq^* \check{h}$ for every ξ. As λ and $\langle \dot{f}_\xi \rangle_{\xi < \lambda}$ are arbitrary,

$$\Vdash_{\mathbb{P}_\kappa} \mathfrak{b} \geq \check{\mathfrak{b}}.$$

(ii) Let $\langle h_\xi \rangle_{\xi < \mathfrak{b}}$ be a family in $\mathbb{N}^\mathbb{N}$ which has no $\leq^*$-upper bound in $\mathbb{N}^\mathbb{N}$. Then

$$\Vdash_{\mathbb{P}_\kappa} \{\check{h}_\xi : \xi < \check{\mathfrak{b}}\} \text{ has no } \leq^*\text{-upper bound.}$$

P? Otherwise, there are a $\mathbb{P}_\kappa$-name $\dot{f}$ for a member of $\mathbb{N}^\mathbb{N}$ and an $a \in \mathfrak{B}_\kappa^+$ such that

$$a \Vdash_{\mathbb{P}_\kappa} \check{h}_\xi \leq^* \dot{f} \text{ for every } \xi < \check{\mathfrak{b}}.$$

Now there is an $h \in \mathbb{N}^\mathbb{N}$ such that $\Vdash_{\mathbb{P}_\kappa} \dot{f} \leq^* \check{h}$. There must be a $\xi < \mathfrak{b}$ such that $h_\xi \not\leq^* h$. We have $a \Vdash_{\mathbb{P}_\kappa} \check{h}_\xi \leq^* \dot{f} \leq^* \check{h}$, so there are an a', stronger than a, and an $n \in \mathbb{N}$ such that

$$a' \Vdash_{\mathbb{P}_\kappa} \check{h}_\xi(i) \leq \check{h}(i) \text{ for every } i \geq \check{n}.$$

However, there is an $i \ge n$ such that $h(i) < h_\xi(i)$, in which case

$$\Vdash_{\mathbb{P}_\kappa} \check{\imath} \ge \check{n} \text{ and } \check{h}(\check{\imath}) < \check{h}_\xi(\check{\imath});$$

which is impossible. **XQ**

So

$$\Vdash_{\mathbb{P}_\kappa} \mathfrak{b} \le \check{\mathfrak{b}}, \text{ therefore } \mathfrak{b} = \check{\mathfrak{b}}.$$

(c)(i) Let $\langle \dot{f}_\xi \rangle_{\xi<\lambda}$ be a family of $\mathbb{P}_\kappa$-names for members of $\mathbb{N}^\mathbb{N}$ where $\lambda < \check{\mathfrak{d}}$. Then for each $\xi < \lambda$ we can find an $h_\xi \in \mathbb{N}^\mathbb{N}$ such that $\Vdash_{\mathbb{P}_\kappa} \dot{f}_\xi \le^* \check{h}_\xi$. As $\lambda < \mathfrak{d}$, there is an $h \in \mathbb{N}^\mathbb{N}$ such that $h \not\le^* h_\xi$ for every $\xi < \lambda$. Now

$$\Vdash_{\mathbb{P}_\kappa} \check{h} \not\le^* \check{h}_\xi \text{ for every } \xi < \check{\lambda},$$

so

$$\Vdash_{\mathbb{P}_\kappa} \check{h} \not\le^* \dot{f}_\xi \text{ for every } \xi < \check{\lambda}.$$

As λ and $\langle \dot{f}_\xi \rangle_{\xi<\lambda}$ are arbitrary,

$$\Vdash_{\mathbb{P}_\kappa} \mathfrak{d} \ge \check{\mathfrak{d}}.$$

(ii) Let $\langle h_\xi \rangle_{\xi<\mathfrak{d}}$ be a family in $\mathbb{N}^\mathbb{N}$ which is $\le^*$-cofinal with $\mathbb{N}^\mathbb{N}$. Then

$$\Vdash_{\mathbb{P}_\kappa} \{\check{h}_\xi : \xi < \check{\mathfrak{d}}\} \text{ is } \le^*\text{-cofinal with } \mathbb{N}^\mathbb{N}.$$

P Let $\dot{f}$ be a $\mathbb{P}_\kappa$-name for a member of $\mathbb{N}^\mathbb{N}$. There are an $h \in \mathbb{N}^\mathbb{N}$ such that $\Vdash_{\mathbb{P}_\kappa} \dot{f} \le^* \check{h}$, and a $\xi < \mathfrak{d}$ such that $h \le^* h_\xi$. In this case,

$$\Vdash_{\mathbb{P}_\kappa} \dot{f} \le^* \check{h} \le^* \check{h}_\xi. \textbf{ Q}$$

So

$$\Vdash_{\mathbb{P}_\kappa} \mathfrak{d} \le \check{\mathfrak{d}}.$$

552D Lemma Let λ and κ be infinite cardinals, and A any subset of $\{0,1\}^\lambda$. Then

$$\Vdash_{\mathbb{P}_\kappa} \nu^*_{\check{\lambda}}(\check{A}) = (\nu^*_\lambda A)\check{}.$$

proof (a) ? Suppose, if possible, that

$$\neg \Vdash_{\mathbb{P}_\kappa} (\nu^*_\lambda A)\check{} \le \nu^*_{\check{\lambda}}(\check{A}).$$

Then there are an $a \in \mathfrak{B}_\kappa^+$ and a $q \in \mathbb{Q}$ such that $q < \nu^*_\lambda A$ and

$$a \Vdash_{\mathbb{P}_\kappa} \nu^*_{\check{\lambda}}(\check{A}) < \check{q}.$$

Let $\dot{E}$ be a $\mathbb{P}_\kappa$-name such that

$$a \Vdash_{\mathbb{P}_\kappa} \check{A} \subseteq \dot{E},\ \dot{E} \in \mathcal{B}\mathfrak{a}_{\check{\lambda}} \text{ and } \nu_{\check{\lambda}} \dot{E} < \check{q}.$$

Of course we can arrange that $1 \setminus a \Vdash_{\mathbb{P}_\kappa} \dot{E} = \emptyset$, so that $\Vdash_{\mathbb{P}_\kappa} \dot{E} \in \mathcal{B}\mathfrak{a}_{\check{\lambda}}$ and there is a $W \in \mathrm{T}_\kappa \widehat{\otimes} \mathcal{B}\mathfrak{a}_\lambda$ such that $\Vdash_{\mathbb{P}_\kappa} \dot{E} = \vec{W}$ (551Fb). Setting $h(x) = \nu_\lambda W[\{x\}]$ for $x \in \{0,1\}^\kappa$,

$$\Vdash_{\mathbb{P}_\kappa} \vec{h} = \nu_{\check{\lambda}} \vec{W}$$

(551I(iii)), so

$$a \Vdash_{\mathbb{P}_\kappa} \vec{h} = \nu_{\check{\lambda}} \dot{E} < \check{q}$$

and $a \subseteq \{x : h(x) < q\}^\bullet$. Take $F \in \mathrm{T}_\kappa$ such that $F^\bullet = a$; then $h(x) < q$ for almost every $x \in F$ and $(\nu_\kappa \times \nu_\lambda)(W \cap (F \times \{0,1\}^\lambda)) < q\nu_\kappa F$.

For each $y \in A$, let $e_y : \{0,1\}^\kappa \to \{0,1\}^\lambda$ be the constant function with value y. Then $\Vdash_{\mathbb{P}_\kappa} \vec{e}_y = \check{y}$ (551Ce), so

$$a \Vdash_{\mathbb{P}_\kappa} \vec{e}_y \in \check{A} \subseteq \vec{W}$$

and $(x,y)\in W$ for almost every $x\in F$. But if we set

$$H=\{y:(x,y)\in W\text{ for }\nu_\kappa\text{-almost every }x\in F\},$$

$H\in\mathrm{T}_\lambda$, $A\subseteq H$ and

$$\nu_\kappa F\cdot\nu_\lambda H\le(\nu_\kappa\times\nu_\lambda)(W\cap(F\times\{0,1\}^\lambda))<q\nu_\kappa F.$$

It follows that

$$\nu_\lambda^*A\le\nu_\lambda H<q,$$

contrary to hypothesis. **X** So

$$\Vdash_{\mathbb{P}_\kappa}(\nu_\lambda^*A)\check{}\le\nu_{\check\lambda}^*(\check A).$$

(b) In the other direction, let $E\in\mathcal{B}\mathfrak{a}_\lambda$ be such that $A\subseteq E$ and $\nu_\lambda E=\nu_\lambda^*A$, and consider $W=\{0,1\}^\kappa\times E$. Then

$$\Vdash_{\mathbb{P}_\kappa}\check A\subseteq\vec W\text{ and }\nu_{\check\lambda}\vec W=(\nu_\lambda E)\check{},$$

so

$$\Vdash_{\mathbb{P}_\kappa}\dot\nu_\lambda^*\check A\le(\nu_\lambda^*A)\check{}.$$

552E Theorem Let κ and λ be infinite cardinals, with $\kappa\ge\max(\omega_1,\lambda)$. Then

$$\Vdash_{\mathbb{P}_\kappa}\text{ there is a strongly Sierpiński set for }\nu_{\check\lambda}\text{ with cardinal }\check\kappa.$$

proof (a) As $\kappa\ge\lambda$, $\mathbb{P}_\kappa$ is isomorphic to $\mathbb{P}=\mathbb{P}_{\kappa\times\lambda}$. For each $\xi<\kappa$, let $f_\xi:\{0,1\}^{\kappa\times\lambda}\to\{0,1\}^\lambda$ be given by setting $f_\xi(x)(\eta)=x(\xi,\eta)$ for every $x\in\{0,1\}^{\kappa\times\lambda}$ and $\eta<\lambda$; then, taking $\vec f_\xi$ to be the $\mathbb{P}$-name defined by the process of 551Cb,

$$\Vdash_{\mathbb{P}}\vec f_\xi\in\{0,1\}^{\check\lambda}.$$

If ξ, $\xi'<\kappa$ are distinct, then for any finite set $I\subseteq\lambda$

$$\begin{aligned}[\![\vec f_\xi(\eta)=\vec f_{\xi'}(\eta)\text{ for every }\eta\in\check I]\!]&=\{x:f_\xi(x)(\eta)=f_{\xi'}(x)(\eta)\text{ for every }\eta\in I\}^\bullet\\&=\{x:x(\xi,\eta)=x(\xi',\eta)\text{ for every }\eta\in I\}^\bullet\end{aligned}$$

has measure $2^{-\#(I)}$, so, because λ is infinite, $\bar\nu[\![\vec f_\xi=\vec f_{\xi'}]\!]=0$ and

$$\Vdash_{\mathbb{P}}\vec f_\xi\ne\vec f_{\xi'}.$$

So, taking $\dot A$ to be the $\mathbb{P}$-name $\{(\vec f_\xi,\mathbb{1}):\xi<\kappa\}$, we have

$$\Vdash_{\mathbb{P}}\dot A\subseteq\{0,1\}^{\check\lambda}\text{ has cardinal }\check\kappa\ge\omega_1,\text{ so is uncountable.}$$

(As remarked in 5A3Nb, we do not need to distinguish between ω_1 and $\check\omega_1$ in the last formula.)

(b) Let $r\ge1$ be an integer and $\dot W$ a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}}\dot W\text{ is a subset of }(\{0,1\}^{\check\lambda})^{\check r}\text{ which is negligible for the usual measure.}$$

Then there is a Baire subset W of $\{0,1\}^{\kappa\times\lambda}\times(\{0,1\}^\lambda)^r$, negligible for the usual measure on this space, such that

$$\Vdash_{\mathbb{P}}\dot W\subseteq\vec W$$

(551J, applied to $\{0,1\}^{\lambda\times r}\cong(\{0,1\}^\lambda)^r$). Let $J\subseteq\kappa$ be a countable set such that W factors through $\{0,1\}^{J\times\lambda}\times(\{0,1\}^\lambda)^r$, that is, there is a negligible Baire set $W_1\subseteq\{0,1\}^{J\times\lambda}\times(\{0,1\}^\lambda)^r$ such that $W=\{(x,y):(x\restriction J\times\lambda,y)\in W_1\}$. If $\xi_0,\dots,\xi_{r-1}$ are distinct elements of $\kappa\setminus J$, then

$$\Vdash_{\mathbb{P}}(\vec f_{\xi_0},\dots,\vec f_{\xi_{r-1}})\notin\vec W.$$

P Applying 551Ea to the function $x\mapsto(f_{\xi_0}(x),\dots,f_{\xi_{r-1}}(x))$, we have

$$[\![(\vec{f}_{\xi_0},\dots,\vec{f}_{\xi_{r-1}})\in\vec{W}]\!]=\{x:(x,(f_{\xi_0}(x),\dots,f_{\xi_{r-1}}(x)))\in W\}^\bullet.$$

Set $K=J\cup\{\xi_0,\dots,\xi_{r-1}\}$ and for $w\in\{0,1\}^{K\times\lambda}$, $i<r$, $\eta<\lambda$ set $g_i(w)(\eta)=w(\xi_i,\eta)$. Then $w\mapsto(w{\restriction}J\times\lambda,(g_0(w),\dots,g_{r-1}(w))$ is a measure space isomorphism between $\{0,1\}^{K\times\lambda}$ and $\{0,1\}^{J\times\lambda}\times(\{0,1\}^\lambda)^r$, so

$$W_2=\{w:w\in\{0,1\}^{K\times\lambda},\ (w{\restriction}J\times\lambda,(g_0(w),\dots,g_{r-1}(w))\in W_1\}$$

is negligible. Consequently

$$\begin{aligned}\{x:(x,(f_{\xi_0}(x),\dots,f_{\xi_{r-1}}(x)))\in W\}&=\{x:(x{\restriction}J\times\lambda,(f_{\xi_0}(x),\dots,f_{\xi_{r-1}}(x)))\in W_1\}\\&=\{x:x{\restriction}K\times\lambda\in W_2\}\end{aligned}$$

is negligible and $[\![(\vec{f}_{\xi_0},\dots,\vec{f}_{\xi_{r-1}})\in\vec{W}]\!]=0$, that is,

$$\Vdash_{\mathbb{P}}(\vec{f}_{\xi_0},\dots,\vec{f}_{\xi_{r-1}})\notin\vec{W}.\ \mathbf{Q}$$

Now if we set $\dot{B}=\{(\vec{f}_\xi,\mathbb{1}):\xi\in J\}$, we have

$\Vdash_{\mathbb{P}}\dot{B}$ is a countable subset of $\dot{A}$ and $(x_0,\dots,x_{r-1})\notin\dot{W}$ whenever $x_0,\dots,x_{r-1}$ are distinct members of $\dot{A}\setminus\dot{B}$.

As $\dot{W}$ is arbitrary,

$$\Vdash_{\mathbb{P}}\dot{A}\text{ is a strongly Sierpiński set with cardinal }\check{\kappa}.$$

As $\mathbb{P}$ is isomorphic to $\mathbb{P}_\kappa$,

$$\Vdash_{\mathbb{P}_\kappa}\text{ there is a strongly Sierpiński set for }\nu_{\check{\lambda}}\text{ with cardinal }\check{\kappa}.$$

552F Theorem Let κ and λ be infinite cardinals.

(a) If either κ or λ is uncountable,

$$\Vdash_{\mathbb{P}_\kappa}\operatorname{add}\mathcal{N}(\nu_{\check{\lambda}})=\omega_1.$$

(b) $\Vdash_{\mathbb{P}_\omega}\operatorname{add}\mathcal{N}(\nu_\omega)=(\operatorname{add}\mathcal{N}(\nu_\omega))\check{}$.

proof (a)(i) If λ is uncountable, then

$$\Vdash_{\mathbb{P}_\kappa}\check{\lambda}\text{ is uncountable, so }\operatorname{add}\mathcal{N}(\nu_{\check{\lambda}})=\omega_1$$

(5A3Nb, 521Jb/523E).

(ii) If κ is uncountable, then

$$\Vdash_{\mathbb{P}_\kappa}\text{ there is a Sierpiński set for }\nu_\omega\text{, so }\omega_1\le\operatorname{add}\mathcal{N}(\nu_{\check{\lambda}})\le\operatorname{add}\mathcal{N}(\nu_\omega)=\omega_1$$

(552E, 523B, 537B(a-i)).

(b)(i) Let $\langle H_\xi\rangle_{\xi<\operatorname{add}\mathcal{N}(\nu_\omega)}$ be a family of negligible Borel sets in $\{0,1\}^\omega$ such that $A=\bigcup_{\xi<\operatorname{add}\mathcal{N}(\nu_\omega)}H_\xi$ is not negligible. Then 552D tells us that

$\Vdash_{\mathbb{P}_\omega}\check{H}_\xi$ is negligible for every $\xi<(\operatorname{add}\mathcal{N}(\nu_\omega))\check{}$, but $\check{A}=\bigcup_{\xi<(\operatorname{add}\mathcal{N}(\nu_\omega))\check{}}\check{H}_\xi$ is not, so $\operatorname{add}\mathcal{N}(\nu_\omega)\le(\operatorname{add}\mathcal{N}(\nu_\omega))\check{}$.

(ii) ? If

$$\neg\Vdash_{\mathbb{P}_\omega}\operatorname{add}\mathcal{N}(\nu_\omega)\ge(\operatorname{add}\mathcal{N}(\nu_\omega))\check{},$$

then there are an $a\in\mathfrak{B}_\kappa^+$ and a $\theta<\operatorname{add}\mathcal{N}(\nu_\omega)$ such that

$$a\Vdash_{\mathbb{P}_\omega}\operatorname{add}\mathcal{N}(\nu_\omega)=\check{\theta}.$$

Now there is a family $\langle\dot{W}_\xi\rangle_{\xi<\theta}$ of $\mathbb{P}_\kappa$-names such that

$$a\Vdash_{\mathbb{P}_\omega}\dot{W}_\xi\in\mathcal{N}(\nu_\omega)\ \forall\,\xi<\check{\theta},\ \bigcup_{\xi<\check{\theta}}\dot{W}_\xi\notin\mathcal{N}(\nu_\omega).$$

By 551J, there is for each $\xi<\theta$ a $W_\xi\in\mathrm{T}_\omega\widehat{\otimes}\mathcal{B}a_\omega$ such that

$$a \Vdash_{\mathbb{P}_\omega} \dot{W}_\xi \subseteq \vec{W}_\xi$$

and all the vertical sections of every W_ξ are negligible. But this means that W_ξ is negligible for the product measure $\nu_\omega \times \nu_\omega$. Because

$$\theta < \operatorname{add}\mathcal{N}(\nu_\omega) = \operatorname{add}\mathcal{N}(\nu_\omega \times \nu_\omega),$$

$\bigcup_{\xi<\theta} W_\xi$ also is negligible, and there is a negligible $W \in \mathrm{T}_\omega \widehat{\otimes} \mathcal{B}\mathfrak{a}_\omega$ including every W_ξ. In this case, 551I(iii) tells us that

$$\Vdash_{\mathbb{P}_\omega} \nu_\omega \vec{W} = 0,$$

so

$$a \Vdash_{\mathbb{P}_\omega} \textstyle\bigcup_{\xi<\check{\theta}} \dot{W}_\xi \subseteq \bigcup_{\xi<\check{\theta}} \vec{W}_\xi \subseteq \vec{W} \text{ is negligible. } \mathbf{X}$$

Putting this together with (i),

$$\Vdash_{\mathbb{P}_\omega} \operatorname{add}\mathcal{N}(\nu_\omega) = (\operatorname{add}\mathcal{N}(\nu_\omega))\check{}.$$

552G Theorem Let κ and λ be infinite cardinals.

(a) $\Vdash_{\mathbb{P}_\kappa} \operatorname{cov}\mathcal{N}(\nu_{\check{\lambda}}) \ge \max(\kappa, \operatorname{cov}\mathcal{N}(\nu_\lambda))\check{}$.

(b) (Pawlikowski 86) $\Vdash_{\mathbb{P}_\kappa} \operatorname{cov}\mathcal{N}(\nu_\omega) \ge \mathfrak{b}$.

(c) (Miller 82) If $\kappa \ge \mathfrak{c}$ then $\Vdash_{\mathbb{P}_\kappa} \operatorname{cov}\mathcal{N}(\nu_\omega) = \mathfrak{c}$.[1]

(d) (Miller 82) Suppose that κ and λ are uncountable. Then

$$\Vdash_{\mathbb{P}_\kappa} \operatorname{cov}\mathcal{N}(\nu_{\check{\lambda}}) \le (\sup_{\delta<\kappa} \delta^\omega)\check{},$$

where each δ^ω is the cardinal power.

proof (a)(i) If $\kappa = \omega$ then of course $\Vdash_{\mathbb{P}_\kappa} \operatorname{cov}\mathcal{N}(\nu_{\check{\lambda}}) \ge \check{\kappa}$. If κ is uncountable, then

$$\Vdash_{\mathbb{P}_\kappa} \operatorname{cov}\mathcal{N}(\nu_{\check{\kappa}}) \ge \check{\kappa}$$

by 552E and 537B(a-i), so by 523F we have

$$\Vdash_{\mathbb{P}_\kappa} \operatorname{cov}\mathcal{N}(\nu_{\check{\lambda}}) \ge \check{\kappa}.$$

(ii) ? If

$$\neg \Vdash_{\mathbb{P}_\kappa} \operatorname{cov}\mathcal{N}(\nu_{\check{\lambda}}) \ge (\operatorname{cov}\mathcal{N}(\nu_\lambda))\check{}$$

then we have an $a \in \mathfrak{B}_\kappa^+$, a cardinal $\theta < \operatorname{cov}\mathcal{N}(\nu_\lambda)$ and a family $\langle \dot{W}_\xi \rangle_{\xi<\theta}$ of $\mathbb{P}_\kappa$-names such that

$$a \Vdash_{\mathbb{P}_\kappa} \{\dot{W}_\xi : \xi < \check{\theta}\} \text{ is a cover of } \{0,1\}^{\check{\lambda}} \text{ by negligible sets.}$$

By 551J again, we have for each $\xi < \theta$ a $(\nu_\kappa \times \nu_\lambda)$-negligible set W_ξ such that $a \Vdash_{\mathbb{P}_\kappa} \dot{W}_\xi \subseteq \vec{W}_\xi$. Set

$$V_\xi = \{y : y \in \{0,1\}^\lambda,\ W_\xi^{-1}[\{y\}] \text{ is not } \nu_\kappa\text{-negligible}\};$$

then $\nu_\lambda V_\xi = 0$ for every $\xi < \theta$, so there is a $y \in \{0,1\}^\lambda \setminus \bigcup_{\xi<\theta} V_\xi$. In this case, let $e_y : \{0,1\}^\kappa \to \{0,1\}^\lambda$ be the constant function with value y. Then we have a $\mathbb{P}_\kappa$-name $\vec{e}_y$ for a member of $\{0,1\}^\lambda$, and for each $\xi < \theta$

$$[\![\vec{e}_y \in \vec{W}_\xi]\!] = \{x : (x, e_y(x)) \in W_\xi\}^\bullet = W_\xi^{-1}[\{y\}]^\bullet = 0$$

(551Ea). So

$$\Vdash_{\mathbb{P}_\kappa} \vec{e}_y \notin \vec{W}_\xi \text{ for every } \xi < \check{\theta},$$

and

$$a \Vdash_{\mathbb{P}_\kappa} \vec{e}_y \in \{0,1\}^{\check{\lambda}} \setminus \textstyle\bigcup_{\xi<\check{\theta}} \dot{W}_\xi. \ \mathbf{X}$$

We conclude that

$$\Vdash_{\mathbb{P}_\kappa} \operatorname{cov}\mathcal{N}(\nu_{\check{\lambda}}) \ge (\operatorname{cov}\mathcal{N}(\nu_\lambda))\check{}.$$

[1] Remember that the final $\mathfrak{c}$ here is to be interpreted in the forcing language.

(b)(i) Set $S_2=\bigcup_{n\in\mathbb{N}}\{0,1\}^n$ and let $\langle(\sigma_n,\tau_n,k_n)\rangle_{n\in\mathbb{N}}$ enumerate $S_2\times S_2\times\mathbb{N}$ with cofinal repetitions. Let D be the set of those $\alpha\in\mathbb{N}^{\mathbb{N}}$ such that

$\langle k_{\alpha(n)}\rangle_{n\in\mathbb{N}}$ is strictly increasing,

$\#(\sigma_{\alpha(m)})\le\alpha(n)$ whenever $n\in\mathbb{N}$ and $m<k_{\alpha(n+1)}$,

$\sum_{i=k_{\alpha(n)}}^{k_{\alpha(n+1)}-1}2^{-\#(\sigma_{\alpha(i)})-\#(\tau_{\alpha(i)})}\le 4^{-n}$ for every $n\in\mathbb{N}$.

For $\alpha\in D$ set

$$G_\alpha=\bigcap_{n\in\mathbb{N}}\bigcup_{m\ge n}\{(u,y):u,\,y\in\{0,1\}^\omega,\,u\supseteq\sigma_{\alpha(m)},\,y\supseteq\tau_{\alpha(m)}\}.$$

(ii) For every $\alpha\in D$, G_α is negligible for the product measure on $\{0,1\}^\omega\times\{0,1\}^\omega$. **P** For any $n\in\mathbb{N}$, the measure of G_α is at most

$$\begin{aligned}\sum_{m=k_{\alpha(n)}}^{\infty}2^{-\#(\sigma_{\alpha(m)})-\#(\tau_{\alpha(m)})}&\le\sum_{j=n}^{\infty}\sum_{m=k_{\alpha(j)}}^{k_{\alpha(j+1)}-1}2^{-\#(\sigma_{\alpha(m)})-\#(\tau_{\alpha(m)})}\\&\le\sum_{j=n}^{\infty}4^{-j}=\frac{4}{3}\cdot4^{-n}.\ \mathbf{Q}\end{aligned}$$

(iii) If $G\subseteq\{0,1\}^\omega\times\{0,1\}^\omega$ is negligible, there is an $\alpha\in D$ such that $G\subseteq G_\alpha$. **P** For each $i\in\mathbb{N}$, let $H_i\supseteq G$ be an open set such that $(\nu_\omega\times\nu_\omega)(H_i)\le 2^{-i}$; we can suppose that H_i is not open-and-closed. H_i can be expressed as the union of a sequence of open-and-closed sets; it can therefore be expressed as the union of a disjoint sequence of open-and-closed sets; each of these is expressible as the union of a disjoint family of sets of the form $\{u:\sigma\subseteq x\}\times\{y:\tau\subseteq y\}$ where σ, $\tau\in S_2$; so H_i is expressible as $\bigcup_{j\in\mathbb{N}}\{u:\sigma'_{ij}\subseteq u\}\times\{y:\tau'_{ij}\subseteq y\}$, with

$$\sum_{j\in\mathbb{N}}2^{-\#(\sigma'_{ij})-\#(\tau'_{ij})}\le 2^{-i}.$$

Re-indexing $\langle(\sigma'_{ij},\tau'_{ij})\rangle_{i,j\in\mathbb{N}}$ as $\langle(\sigma''_m,\tau''_m)\rangle_{m\in\mathbb{N}}$, we have

$$G\subseteq\bigcap_{n\in\mathbb{N}}\bigcup_{m\ge n}\{(u,y):\sigma''_m\subseteq u,\,\tau''_m\subseteq y\},$$

and

$$\sum_{m\in\mathbb{N}}2^{-\#(\sigma''_m)-\#(\tau''_m)}\le\sum_{i=0}^{\infty}2^{-i}<\infty.$$

Let $\gamma:\mathbb{N}\to\mathbb{N}$ be a strictly increasing function such that

$$\sum_{m=\gamma(n)}^{\gamma(n+1)-1}2^{-\#(\sigma''_m)-\#(\tau''_m)}\le 4^{-n}$$

for every $n\in\mathbb{N}$. Now choose $\langle\alpha(n)\rangle_{n\in\mathbb{N}}$ so that

$$k_{\alpha(n)}=\gamma(n),\quad\sigma_{\alpha(n)}=\sigma''_n,\quad\tau_{\alpha(n)}=\tau''_n,\quad\alpha(n)\ge\#(\sigma''_m)\text{ whenever }m<\gamma(n+1)$$

for each $n\in\mathbb{N}$. Then $\alpha\in D$ and $G\subseteq G_\alpha$. **Q**

(iv) Define $h:\mathbb{N}^{\mathbb{N}}\to\mathbb{N}^{\mathbb{N}}$ by setting $h(\beta)(n)=n+\sum_{i=0}^{n}\beta(i)$ for $\beta\in\mathbb{N}^{\mathbb{N}}$ and $n\in\mathbb{N}$. For $\beta\in\mathbb{N}^{\mathbb{N}}$ define $f_\beta:\{0,1\}^\omega\to\{0,1\}^\omega$ by setting $f_\beta(u)(n)=u(h(\beta)(n))$ for $\beta\in\mathbb{N}^{\mathbb{N}}$ and $n\in\mathbb{N}$; note that f_β is continuous. For α, $\beta\in\mathbb{N}^{\mathbb{N}}$ say that $\alpha\le^*\beta$ if $\{n:\beta(n)<\alpha(n)\}$ is finite.

(v) If $\alpha\in D$, $\beta\in\mathbb{N}^{\mathbb{N}}$ and $\alpha\le^*\beta$, then $C=\{u:u\in\{0,1\}^\omega,\,(u,f_\beta(u))\in G_\alpha\}$ is ν_ω-negligible. **P** Let n_0 be such that $\alpha(n)\le\beta(n)$ for $n\ge n_0$. C is just $\bigcap_{n\in\mathbb{N}}\bigcup_{m\ge n}C_m$ where

$$C_m=\{u:\sigma_{\alpha(m)}\subseteq u,\,\tau_{\alpha(m)}\subseteq f_\beta(u)\}$$

for each m. We know that $\langle k_{\alpha(j)}\rangle_{j\in\mathbb{N}}$ is strictly increasing; if $m\ge k_{\alpha(n_0)}$, let $j\ge n_0$ be such that $k_{\alpha(j)}\le m<k_{\alpha(j+1)}$, and set

$$\begin{aligned}C'_m&=\{u:\sigma_{\alpha(m)}\subseteq u,\,\tau_{\alpha(m)}(i)=f_\beta(u)(i)\text{ for }j\le i<\#(\tau_{\alpha(m)})\}\\&=\{u:u(i)=\sigma_{\alpha(m)}(i)\text{ for }i<\#(\sigma_{\alpha(m)}),\\&\qquad\qquad u(h(\beta)(i))=\tau_{\alpha(m)}(i)\text{ for }j\le i<\#(\tau_{\alpha(m)})\}\\&\supseteq C_m.\end{aligned}$$

We know that

$$\#(\sigma_{\alpha(m)}) \le \alpha(j) \le \beta(j) \le h(\beta)(i)$$

whenever $i \ge j$ and that $h(\beta)$ is a strictly increasing function, so

$$\nu_\omega C'_m \le 2^{-\#(\sigma_{\alpha(m)})-\#(\tau_{\alpha(m)})+j}.$$

But this means that

$$\begin{aligned}\sum_{m=k_{\alpha(n_0)}}^{\infty} \nu_\omega C_m &= \sum_{j=n_0}^{\infty} \sum_{m=k_{\alpha(j)}}^{k_{\alpha(j+1)}-1} \nu_\omega C_m \\ &\le \sum_{j=n_0}^{\infty} 2^j \sum_{m=k_{\alpha(j)}}^{k_{\alpha(j+1)}-1} 2^{-\#(\sigma_{\alpha(m)})-\#(\tau_{\alpha(m)})} \le \sum_{j=n_0}^{\infty} 2^j \cdot 4^{-j}\end{aligned}$$

is finite, and C is negligible. **Q**

(vi) Let Φ be the set of all continuous functions from $\{0,1\}^\kappa$ to $\{0,1\}^\omega$, and $\mathcal{E}$ the set of $(\nu_\kappa \times \nu_\omega)$-negligible sets in $\mathrm{T}_\kappa \widehat{\otimes} \mathcal{B}\mathfrak{a}_\omega$; let R be the relation

$$\{(W,g) : W \in \mathcal{E},\ g \in \Phi,\ \{x : (x,g(x)) \in W\} \in \mathcal{N}(\nu_\kappa)\}.$$

Then $(\mathcal{E}, R, \Phi) \preccurlyeq_{\mathrm{GT}} (\mathbb{N}^\mathbb{N}, \le^*, \mathbb{N}^\mathbb{N})$. **P** For $W \in \mathcal{E}$ set

$$\begin{aligned}V_W = \{(u,y) : {} & u,\ y \in \{0,1\}^\omega, \\ & \{v : v \in \{0,1\}^{\kappa\setminus\omega},\ (u \cup v, y) \in W\} \text{ is not } \nu_{\kappa\setminus\omega}\text{-negligible}\}.\end{aligned}$$

Then V_W is $(\nu_\omega \times \nu_\omega)$-negligible; by (iii), we can find $\phi(W) \in D$ such that $V_W \subseteq G_{\phi(W)}$. In the other direction, given $\beta \in \mathbb{N}^\mathbb{N}$, define $\psi(\beta) \in \Phi$ by saying that $\psi(\beta)(x) = f_\beta(x{\restriction}\omega)$ for $x \in \{0,1\}^\kappa$.

If $W \in \mathcal{E}$ and $\beta \in \mathbb{N}^\mathbb{N}$ are such that $\phi(W) \le^* \beta$, we have $\nu_\omega C = 0$ where $C = \{u : (u, f_\beta(u)) \in G_{\phi(W)}\}$, by (v). But if $C' = \{x : (x, \psi(\beta)(x)) \in W\}$, and $u \in \{0,1\}^\omega \setminus C$, then

$$\{v : v \in \{0,1\}^{\kappa\setminus\omega},\ u \cup v \in C'\} = \{v : v \in \{0,1\}^{\kappa\setminus\omega},\ (u \cup v, f_\beta(u)) \in W\}$$

must be $\nu_{\kappa\setminus\omega}$-negligible, since $(u, f_\beta(u)) \notin V_W$. So C' is negligible and $(W, \psi(\beta)) \in R$. As W and β are arbitrary, (ϕ, ψ) is a Galois-Tukey connection and $(\mathcal{E}, R, \Phi) \preccurlyeq_{\mathrm{GT}} (\mathbb{N}^\mathbb{N}, \le^*, \mathbb{N}^\mathbb{N})$. **Q**

Consequently $\mathrm{add}(\mathcal{E}, R, \Phi) \ge \mathfrak{b}$ (522C(i), 512Ea, 512Db).

(vii) By 552C, we do not need to distinguish between the interpretations of $\mathfrak{b}$ in the ordinary universe and in the forcing language. Suppose that $a \in \mathfrak{B}_\kappa^+$ and that $\dot{A}$ is a $\mathbb{P}_\kappa$-name such that

$$a \Vdash_{\mathbb{P}_\kappa} \dot{A} \subseteq \mathcal{N}(\nu_\omega) \text{ and } \#(\dot{A}) < \mathfrak{b}.$$

Then there are a $b \in \mathfrak{B}_\kappa^+$, stronger than a, a cardinal $\theta < \mathfrak{b}$ and a family $\langle \dot{W}_\xi \rangle_{\xi<\theta}$ of $\mathbb{P}_\kappa$-names such that

$$b \Vdash_{\mathbb{P}_\kappa} \dot{A} = \{\dot{W}_\xi : \xi < \check{\theta}\}.$$

For each $\xi < \theta$, we have a $(\nu_\kappa \times \nu_\omega)$-negligible $W_\xi \in \mathrm{T}_\kappa \widehat{\otimes} \mathcal{B}\mathfrak{a}_\omega$ such that $b \Vdash_{\mathbb{P}_\kappa} \dot{W}_\xi \subseteq \vec{W}_\xi$ (551J, as usual). Each W_ξ belongs to $\mathcal{E}$. Since $\theta < \mathfrak{b} \le \mathrm{add}(\mathcal{E}, R, \Phi)$, there is a $g \in \Phi$ such that $(W_\xi, g) \in R$ for every $\xi < \theta$, that is, $\{x : (x, g(x)) \in W_\xi\}$ is negligible for every $\xi < \theta$. But this means that

$$\Vdash_{\mathbb{P}_\kappa} \vec{g} \in \{0,1\}^\omega \setminus \vec{W}_\xi$$

for every $\xi < \theta$. So

$$b \Vdash_{\mathbb{P}_\kappa} \vec{g} \notin \bigcup \dot{A} \text{ and } \dot{A} \text{ does not cover } \{0,1\}^\omega.$$

As a and $\dot{A}$ are arbitrary,

$$\Vdash_{\mathbb{P}_\kappa} \mathrm{cov}\,\mathcal{N}(\nu_\omega) \ge \mathfrak{b}.$$

(c) Write θ for the cardinal power κ^ω, so that $\Vdash_{\mathbb{P}_\kappa} \mathfrak{c} = \check{\theta}$ (552B). **?** If

$$\neg \Vdash_{\mathbb{P}_\kappa} \mathrm{cov}\,\mathcal{N}(\nu_\omega) = \mathfrak{c},$$

then there must be an $a \in \mathfrak{B}_\kappa^+$, a cardinal $\delta < \theta$ and a family $\langle \dot{W}_\xi \rangle_{\xi<\delta}$ of $\mathbb{P}_\kappa$-names such that

$$a \Vdash_{\mathbb{P}_\kappa} \{\dot{W}_\xi : \xi < \check{\delta}\} \text{ is a cover of } \{0,1\}^\omega \text{ by negligible sets.}$$

For each $\xi < \delta$, let $W_\xi \in \mathrm{T}_\kappa \widehat{\otimes} \mathcal{B}\mathfrak{a}_\omega$ be a $(\nu_\kappa \times \nu_\omega)$-negligible set such that $a \Vdash_{\mathbb{P}_\kappa} \dot{W}_\xi \subseteq \vec{W}_\xi$; expanding it if necessary, we can suppose that W_ξ is a Baire set. Let $I_\xi \subseteq \kappa$ be a countable set such that $(u, y) \in W_\xi$ whenever $(x, y) \in W_\xi$, $u \in \{0,1\}^\kappa$ and $u \restriction I_\xi = x \restriction I_\xi$. Set

$$W'_\xi = \{(v, y) : (u, y) \in W_\xi,\ v \in \{0,1\}^\kappa,\ \{\eta : \eta < \kappa,\ u(\eta) \neq v(\eta)\} \in [I_\xi]^{<\omega}\}.$$

Then W'_ξ is still $(\nu_\kappa \times \nu_\omega)$-negligible.

Because $\kappa \geq \mathfrak{c}$ and $\delta < \kappa^\omega$, there is a countably infinite $K \subseteq \kappa$ such that $K \cap I_\xi$ is finite for every $\xi < \delta$ (5A1Gc). Enumerate K as $\langle \eta_n \rangle_{n\in\mathbb{N}}$ and define $f : \{0,1\}^\kappa \to \{0,1\}^\omega$ by setting $f(u) = \langle u(\eta_n) \rangle_{n\in\mathbb{N}}$ for $u \in \{0,1\}^\kappa$.

For each $\xi < \kappa$, $\{u : (u, f(u)) \in W_\xi\}$ is ν_κ-negligible. **P** Set $J = \kappa \setminus K$, so that $\{0,1\}^\kappa$ can be identified with $\{0,1\}^J \times \{0,1\}^K$. Because $I_\xi \setminus J$ is finite, W'_ξ is equal to

$$\{(v, y) : (u, y) \in W_\xi,\ v \in \{0,1\}^\kappa,\ \{\eta : \eta < \kappa,\ u(\eta) \neq v(\eta)\} \in [J \cap I_\xi]^{<\omega}\}$$

and can be expressed as $\{(u, y) : (u \restriction J, y) \in V\}$ where $V \subseteq \{0,1\}^J \times \{0,1\}^\omega$ must be negligible. Now the map $u \mapsto (u \restriction J, f(u)) : \{0,1\}^\kappa \to \{0,1\}^J \times \{0,1\}^\omega$ is just a copy of the map $u \mapsto (u \restriction J, u \restriction K)$, so is a measure space isomorphism between $\{0,1\}^\kappa$ and $\{0,1\}^J \times \{0,1\}^\omega$, and $V' = \{u : u \in \{0,1\}^\kappa,\ (u \restriction J, f(u)) \in V\}$ is negligible. But observe now that

$$\{u : (u, f(u)) \in W_\xi\} \subseteq \{u : (u, f(u)) \in W'_\xi\} = \{u : (u \restriction J, f(u)) \in V\} = V'$$

is negligible. **Q**

Turn now to 551E. In the language there, we have $[\![\vec{f} \in \vec{W}_\xi]\!] = 0$, that is, $\Vdash_{\mathbb{P}_\kappa} \vec{f} \notin \vec{W}_\xi$ and $a \Vdash_{\mathbb{P}_\kappa} \vec{f} \notin \dot{W}_\xi$. So

$$a \Vdash_{\mathbb{P}_\kappa} \bigcup_{\xi<\check{\delta}} \dot{W}_\xi \neq \{0,1\}^{\check{\lambda}},$$

which is impossible. **X**

So we have the result claimed.

(d)(i) If $\mathrm{cf}\,\kappa > \omega$ then $\sup_{\delta<\kappa} \delta^\omega = \kappa^\omega$; but this means that

$$\Vdash_{\mathbb{P}_\kappa} \mathrm{cov}\,\mathcal{N}_{\check{\lambda}} \leq \mathfrak{c} = (\kappa^\omega)\check{} = (\sup_{\delta<\kappa} \delta^\omega)\check{}.$$

So henceforth suppose that $\mathrm{cf}\,\kappa = \omega$. By 523B, we may also assume that $\lambda = \omega_1$.

(ii) Let D be the set of all pairs $(\boldsymbol{\xi}, y)$ where $\boldsymbol{\xi} \in \omega_1^{\mathbb{N}}$ is one-to-one and y is a Baire measurable function from $\{0,1\}^\delta$ to $\{0,1\}^\omega$ for some cardinal $\delta < \kappa$. Then $\#(D) = \sup_{\delta<\kappa} \delta^\omega$ (use 5A4G(b-ii)). For $(\boldsymbol{\xi}, y) \in D$, let $W_{\boldsymbol{\xi}y} \subseteq \{0,1\}^\kappa \times \{0,1\}^{\omega_1}$ be the set

$$\{(u, v) : \lim_{n\to\infty} \frac{1}{n} \#(\{i : i < n,\ v(\xi_i) = y(u \restriction \alpha)(i)\}) = \frac{1}{2}\},$$

where $\boldsymbol{\xi} = \langle \xi_i \rangle_{i\in\mathbb{N}}$ and $\mathrm{dom}\,y = \{0,1\}^\alpha$. Then $W_{\boldsymbol{\xi}y}$ is a Baire set; also the vertical section $W_{\boldsymbol{\xi}y}[\{u\}]$ is ν_{ω_1}-conegligible for almost every $u \in \{0,1\}^\kappa$. **P** The set

$$V = \{x : x \in \{0,1\}^{\mathbb{N}},\ \lim_{n\to\infty} \frac{1}{n} \#(\{i : i < n,\ x(i) = y(u \restriction \alpha)(i)\}) = \frac{1}{2}\}$$

is conegligible in $\{0,1\}^{\mathbb{N}}$, by the strong law of large numbers (273F). But $W_{\boldsymbol{\xi}y}[\{u\}]$ is the inverse image of V under the inverse-measure-preserving map $v \mapsto \langle v(\xi_i) \rangle_{i\in\mathbb{N}}$, so is ν_{ω_1}-conegligible. **Q**

(iii) Consequently

$$\Vdash_{\mathbb{P}_\kappa} \vec{W}_{\boldsymbol{\xi}y} \text{ is conegligible in } \{0,1\}^{\omega_1}$$

whenever $(\boldsymbol{\xi}, y) \in D$ (551I(iii)). Now

$$\Vdash_{\mathbb{P}_\kappa} \bigcap_{(\boldsymbol{\xi},y)\in\check{D}} \vec{W}_{\boldsymbol{\xi}y} \text{ is empty.}$$

P? Otherwise, there are an $a \in \mathfrak{B}_\kappa^+$ and a $\mathbb{P}_\kappa$-name $\dot{x}$ such that

$$a \Vdash_{\mathbb{P}_\kappa} \dot{x} \in \bigcap_{(\boldsymbol{\xi},y)\in\check{D}} \vec{W}_{\boldsymbol{\xi}y}.$$

Let $f:\{0,1\}^\kappa \to \{0,1\}^{\omega_1}$ be a $(\mathrm{T}_\kappa, \mathcal{B}\mathfrak{a}_{\omega_1})$-measurable function such that $a \Vdash_{\mathbb{P}_\kappa} \vec{f} = \dot{x}$ (551Cc). Set $\epsilon = \frac{1}{4}\bar{\nu}_\kappa a$ and for each $\xi < \omega_1$ let E_ξ be an open-and-closed subset of $\{0,1\}^\kappa$ such that $\bar{\nu}_\kappa(E_\xi \triangle \{u : f(u)(\xi) = 1\}) \le \epsilon$. Let $\alpha_\xi < \kappa$ be such that E_ξ is determined by coordinates less than α_ξ. Because $\operatorname{cf}\kappa = \omega$, there is a cardinal $\delta < \kappa$ such that $A = \{\xi : \alpha_\xi \le \delta\}$ is infinite; let $\boldsymbol{\xi} = \langle \xi_i \rangle_{i\in\mathbb{N}}$ enumerate a subset of A. For each $i \in \mathbb{N}$ let $F_i \subseteq \{0,1\}^\delta$ be an open-and-closed set such that $E_{\xi_i} = \{u : u{\restriction}\delta \in F_i\}$. Define $y : \{0,1\}^\delta \to \{0,1\}^{\mathbb{N}}$ by saying that $y(v)(i) = \chi F_i(v)$ for $v \in \{0,1\}^\delta$ and $i \in \mathbb{N}$; then y is Baire measurable, so $W_{\boldsymbol{\xi}y}$ is defined and $a \Vdash_{\mathbb{P}_\kappa} \vec{f} \in \vec{W}_{\boldsymbol{\xi}y}$.

Set $H = \{u : (u, f(u)) \in W_{\boldsymbol{\xi}y}\}$. Then

$$a \subseteq [\![\vec{f} \in \vec{W}_{\boldsymbol{\xi}y}]\!] = H^\bullet$$

so $\nu_\kappa H \ge \bar{\nu}_\kappa a = 4\epsilon$.

But consider the sets

$$\begin{aligned} H_i &= \{u : f(u)(\xi_i) = y(u{\restriction}\delta)(i)\} \\ &= \{0,1\}^\kappa \setminus (\{u : f(u)(\xi_i) = 1\} \triangle \{u : y(u{\restriction}\delta)(i) = 1\}) \\ &= \{0,1\}^\kappa \setminus (E_{\xi_i} \triangle \{u : f(u)(\xi_i) = 1\}) \end{aligned}$$

for $i \in \mathbb{N}$. These all have measure at least $1-\epsilon$. For $n \ge 1$ set

$$\gamma_n = \nu_\kappa\{u : \#(\{i : i < n,\, u \in H_i\}) \le \frac{2n}{3}\};$$

then

$$n(1-\epsilon) \le \sum_{i<n} \nu_\kappa H_i = \int \#(\{i : i < n,\, u \in H_i\})\nu_\kappa(du) \le \frac{2n}{3}\gamma_n + n(1-\gamma_n)$$

and $\gamma_n \le 3\epsilon$. So

$$\begin{aligned} H &= \{u : \lim_{n\to\infty} \frac{1}{n}\#(\{i : i < n,\, u \in H_i\}) = \frac{1}{2}\} \\ &\subseteq \bigcup_{m\in\mathbb{N}} \bigcap_{n\ge m} \{u : \#(\{i : i < n,\, u \in H_i\}) \le \frac{2}{3}\} \end{aligned}$$

has measure at most 3ϵ; which is impossible. **XQ**

(iv) Thus

$\Vdash_{\mathbb{P}_\kappa} \bigcap_{(\boldsymbol{\xi},y)\in\check{D}} \vec{W}_{\boldsymbol{\xi}y}$ is empty and $\{0,1\}^{\omega_1}$ can be covered by $(\sup_{\delta<\kappa}\delta^\omega)\check{}$ negligible sets,

which is what we needed to know.

552H Theorem Let κ and λ be infinite cardinals.

(a) $\Vdash_{\mathbb{P}_\kappa} \operatorname{non}\mathcal{N}(\nu_{\check{\lambda}}) \le (\operatorname{non}\mathcal{N}(\nu_\lambda))\check{}$.

(b) If $\kappa \ge \max(\lambda, \omega_1)$ then

$$\Vdash_{\mathbb{P}_\kappa} \operatorname{non}\mathcal{N}(\nu_{\check{\lambda}}) = \omega_1.$$

(c) (Pawlikowski 86)

$$\Vdash_{\mathbb{P}_\kappa} \operatorname{non}\mathcal{N}(\nu_\omega) \le \mathfrak{d}.$$

proof (a) Let $A \subseteq \{0,1\}^\lambda$ be a non-negligible set of size $\operatorname{non}\mathcal{N}(\nu_\lambda)$. Then 552D tells us that

$\Vdash_{\mathbb{P}_\kappa} \check{A}$ is a non-negligible set with cardinal $(\operatorname{non}\mathcal{N}(\nu_\lambda))\check{}$, so $\operatorname{non}\mathcal{N}(\nu_{\check{\lambda}}) \le (\operatorname{non}\mathcal{N}(\nu_\lambda))\check{}$.

(b) Put 552E and 537Ba together again, as in part (a) of the proof of 552F.

(c) Continue the argument from the end of (b-vi) of the proof of 552G above. We have $(\mathcal{E}, R, \Phi) \preccurlyeq_{\mathrm{GT}} (\mathbb{N}^{\mathbb{N}}, \le^*, \mathbb{N}^{\mathbb{N}})$, so $\operatorname{cov}(\mathcal{E}, R, \Phi) \le \mathfrak{d}$ (522C(i), 512Ea, 512Da). So there is a family $\langle g_\xi \rangle_{\xi<\mathfrak{d}}$ in Φ such that for every $W \in \mathcal{E}$ there is a $\xi < \mathfrak{d}$ such that $(W, g_\xi) \in R$, that is, $\{x : (x, g_\xi(x)) \in W\}$ is negligible, that is, $\Vdash_{\mathbb{P}_\kappa} \vec{g}_\xi \notin \vec{W}$. Now

$$\Vdash_{\mathbb{P}_\kappa} \{\vec{g}_\xi : \xi < \mathfrak{d}\} \text{ is not negligible.}$$

P? Otherwise, there are an $a \in \mathfrak{B}_\kappa^+$ and a $\mathbb{P}_\kappa$-name $\dot{W}$ such that

$$a \Vdash_{\mathbb{P}_\kappa} \dot{W} \text{ is a negligible set containing } \vec{g}_\xi \text{ for every } \xi < \mathfrak{d}.$$

By 551J once again, there is a $W \in \mathcal{E}$ such that $a \Vdash_{\mathbb{P}_\kappa} \dot{W} \subseteq \vec{W}$. But now we have a $\xi < \mathfrak{d}$ such that $\Vdash_{\mathbb{P}_\kappa} \vec{g}_\xi \notin \vec{W}$, which is impossible. **XQ** So $\Vdash_{\mathbb{P}_\kappa} \operatorname{non}\mathcal{N}(\nu_\omega) \le \mathfrak{d}$.

552I Theorem Let κ and λ be infinite cardinals. Set $\theta_0 = \max(\operatorname{cf}\mathcal{N}(\nu_\omega), \operatorname{cf}[\kappa]^{\le\omega}, \operatorname{cf}[\lambda]^{\le\omega})$. Then

$$\Vdash_{\mathbb{P}_\kappa} \operatorname{cf}\mathcal{N}(\nu_{\check{\lambda}}) = \check{\theta}_0.$$

proof (a) $\Vdash_{\mathbb{P}_\kappa} \operatorname{cf}\mathcal{N}(\nu_\omega) \ge \operatorname{cf}[\check{\kappa}]^{\le\omega} = (\operatorname{cf}[\kappa]^{\le\omega})\check{}$. **P** If $\kappa = \omega$ this is trivial. Otherwise it follows from 552E, 537B(a-ii) and 5A3Nd. **Q**

(b) Set

$$\theta_1 = \operatorname{cf}\mathcal{N}(\nu_\lambda) = \max(\operatorname{cf}\mathcal{N}(\nu_\omega), \operatorname{cf}[\lambda]^{\le\omega})$$

(523N). Then $\Vdash_{\mathbb{P}_\kappa} \operatorname{cf}\mathcal{N}(\nu_{\check{\lambda}}) \ge \check{\theta}_1$. **P?** Otherwise, there are $a \in \mathfrak{B}_\kappa$, $\theta < \theta_1$ and a family $\langle \dot{W}_\xi \rangle_{\xi<\theta}$ of $\mathbb{P}_\kappa$-names such that

$$a \Vdash_{\mathbb{P}_\kappa} \{\dot{W}_\xi : \xi < \check{\theta}\} \text{ is a cofinal family in } \mathcal{N}(\nu_{\check{\lambda}}).$$

For each ξ choose a $(\nu_\kappa \times \nu_\lambda)$-negligible $W_\xi \in \mathrm{T}_\kappa \widehat{\otimes} \mathcal{B}\mathfrak{a}_\lambda$ such that $a \Vdash_{\mathbb{P}_\kappa} \dot{W}_\xi \subseteq \vec{W}_\xi$. Then

$$V_\xi = \{y : y \in \{0,1\}^\lambda, \ W_\xi^{-1}[\{y\}] \text{ is not } \nu_\kappa\text{-negligible}\}$$

is ν_λ-negligible. Because $\theta < \operatorname{cf}\mathcal{N}(\nu_\lambda)$, there is a $V \in \mathcal{N}(\nu_\lambda)$ such that $V \not\subseteq V_\xi$ for every $\xi < \theta$, and (enlarging V slightly if necessary) we can arrange that $V \in \mathcal{B}\mathfrak{a}_\lambda$.

Set $W = \{0,1\}^\kappa \times V$. Then $W \in \mathrm{T}_\kappa \widehat{\otimes} \mathcal{B}\mathfrak{a}_\lambda$ and every vertical section of W is negligible, so 551I tells us that

$$\Vdash_{\mathbb{P}_\kappa} \vec{W} \text{ is negligible in } \{0,1\}^{\check{\lambda}}.$$

Accordingly

$$a \Vdash_{\mathbb{P}_\kappa} \text{there is a } \xi < \check{\theta} \text{ such that } \vec{W} \subseteq \dot{W}_\xi \subseteq \vec{W}_\xi,$$

and there must be a $b \in \mathfrak{B}_\kappa$, stronger than a, and a $\xi < \theta$ such that $b \Vdash_{\mathbb{P}_\kappa} \vec{W} \subseteq \vec{W}_\xi$. But now take any point y of $V \setminus V_\xi$ and consider the constant function e_y on $\{0,1\}^\kappa$ with value y. Then $\{x : (x, e_y(x)) \in W \setminus W_\xi\}$ is conegligible, so 551E tells us that

$$\Vdash_{\mathbb{P}_\kappa} \vec{e}_y \in \vec{W} \setminus \vec{W}_\xi, \text{ so } \vec{W} \not\subseteq \vec{W}_\xi,$$

contrary to the choice of ξ. **XQ**

(c) Now

$$\Vdash_{\mathbb{P}_\kappa} \operatorname{cf}\mathcal{N}(\nu_{\check{\lambda}}) \ge \max(\operatorname{cf}\mathcal{N}(\nu_\omega), \check{\theta}_1) \ge \max(\operatorname{cf}[\check{\kappa}]^{\le\omega}, \check{\theta}_1) = \check{\theta}_0.$$

(d) In the other direction, let $\mu = \nu_\kappa \times \nu_\lambda$ be the product measure on $\{0,1\}^\kappa \times \{0,1\}^\lambda$. Again by 523N, $\operatorname{cf}\mathcal{N}(\mu) = \theta_0$; let $\langle W_\xi \rangle_{\xi<\theta_0}$ be a cofinal family in $\mathcal{N}(\mu)$ consisting of sets in $\mathrm{T}_\kappa \widehat{\otimes} \mathcal{B}\mathfrak{a}_\lambda$. By 551J,

$$\Vdash_{\mathbb{P}_\kappa} \{\vec{W}_\xi : \xi < \check{\theta}_1\} \text{ is cofinal with } \mathcal{N}(\nu_{\check{\lambda}}), \text{ so } \operatorname{cf}\mathcal{N}(\nu_{\check{\lambda}}) \le \check{\theta}_0.$$

Putting this together with (c),

$$\Vdash_{\mathbb{P}_\kappa} \operatorname{cf}\mathcal{N}(\nu_{\check{\lambda}}) = \check{\theta}_0,$$

and the proof is complete.

552J Theorem Let κ and λ be infinite cardinals; set $\theta_0=\operatorname{shr}\mathcal{N}(\nu_\lambda)$ and let θ_1 be the cardinal power λ^ω. Then

$$\Vdash_{\mathbb{P}_\kappa} \check\theta_0\le\operatorname{shr}\mathcal{N}(\nu_{\check\lambda})\le\check\theta_1.$$

proof (a) ? Suppose, if possible, that

$$\neg\Vdash_{\mathbb{P}_\kappa} \check\theta_0\le\operatorname{shr}\mathcal{N}(\nu_{\check\lambda}).$$

Then there are an $a\in\mathfrak{B}_\kappa^+$ and a cardinal $\theta'<\theta_0$ such that

$$a\Vdash_{\mathbb{P}_\kappa}\operatorname{shr}\mathcal{N}(\nu_{\check\lambda})=\check\theta'.$$

Of course θ' is infinite. Let $A\subseteq\{0,1\}^\lambda$ be such that $\nu_\lambda^*A>0$ but $B\in\mathcal{N}(\nu_\lambda)$ for every $B\in[A]^{\le\theta'}$. By 552D,

$$\Vdash_{\mathbb{P}_\kappa}\nu_{\check\lambda}^*(\check A)>0.$$

There must therefore be a $\mathbb{P}_\kappa$-name $\dot B$ for a subset of $\check A$ with cardinal at most $\check\theta'$ such that

$$a\Vdash_{\mathbb{P}_\kappa}\nu_{\check\lambda}^*(\dot B)>0.$$

By 5A3Nc there is a $B\subseteq A$ such that $\#(B)\le\max(\omega,\theta')=\theta'$ and

$$a\Vdash_{\mathbb{P}_\kappa}\dot B\subseteq\check B.$$

By 552D, in the other direction,

$$a\Vdash_{\mathbb{P}_\kappa}0<\nu_{\check\lambda}^*(\dot B)\le\nu_{\check\lambda}^*(\check B)=(\nu_\lambda^*B)\check{}$$

and $\nu_\lambda^*B>0$, contrary to the choice of A. **X**

(b) (In this part of the proof it will be convenient to regard $\mathfrak{B}_\kappa$ as the measure algebra of $\nu_\kappa\restriction\mathcal{B}\mathfrak{a}_\kappa$.)

(i) ? Suppose, if possible, that

$$\neg\Vdash_{\mathbb{P}_\kappa}\operatorname{shr}\mathcal{N}(\nu_{\check\lambda})\le\check\theta_1.$$

Then there are an $a\in\mathfrak{B}_\kappa^+$ and a $\mathbb{P}_\kappa$-name $\dot A$ such that

$a\Vdash_{\mathbb{P}_\kappa}\dot A$ is a non-negligible subset of $\{0,1\}^{\check\lambda}$ and every subset of $\dot A$ with cardinal at most $\check\theta_1$ is negligible.

(ii) Let $\langle e_\xi\rangle_{\xi<\kappa}$ be the standard generating family in $\mathfrak{B}_\kappa$. Choose $\langle f_\xi\rangle_{\xi<\theta_1^+}$, $\langle J_\xi\rangle_{\xi<\theta_1^+}$, $\langle K_\xi\rangle_{\xi<\theta_1^+}$, $\langle W_\xi\rangle_{\xi<\theta_1^+}$ and $\langle V_\xi\rangle_{\xi<\theta_1^+}$ inductively, as follows. $K_\xi=\bigcup_{\eta<\xi}J_\eta$. Given that $\xi<\theta_1^+$ and that, for each $\eta<\xi$, $f_\eta:\{0,1\}^\kappa\to\{0,1\}^\lambda$ is a $(\mathcal{B}\mathfrak{a}_\kappa,\mathcal{B}\mathfrak{a}_\lambda)$-measurable function such that $a\Vdash_{\mathbb{P}_\kappa}\vec f_\eta\in\dot A$ (where $\vec f_\eta$ is the $\mathbb{P}_\kappa$-name for a member of $\{0,1\}^{\check\lambda}$ as defined in 551Cb), then

$$a\Vdash_{\mathbb{P}_\kappa}\{\vec f_\eta:\eta<\check\xi\}\text{ is negligible,}$$

so by 551J there is a set $W_\xi\in\mathcal{B}\mathfrak{a}_\kappa\widehat\otimes\mathcal{B}\mathfrak{a}_\lambda$, negligible for the product measure on $\{0,1\}^\kappa\times\{0,1\}^\lambda$, such that $a\Vdash_{\mathbb{P}_\kappa}\vec f_\eta\in\vec W_\xi$ for every $\eta<\xi$.

Set

$$\begin{aligned}V_\xi=\{(x,y):\ &x\in\{0,1\}^\kappa,\ y\in\{0,1\}^\lambda,\\ &\{t:t\in\{0,1\}^{\kappa\setminus K_\xi},\ ((x\restriction K_\xi)\cup t,y)\in W_\xi\}\text{ is not }\nu_{\kappa\setminus K_\xi}\text{-negligible}\}.\end{aligned}$$

Then $V_\xi\in\mathcal{B}\mathfrak{a}_\kappa\widehat\otimes\mathcal{B}\mathfrak{a}_\lambda$ is negligible, so $\Vdash_{\mathbb{P}_\kappa}\vec V_\xi\in\mathcal{N}(\nu_{\check\lambda})$ and $a\Vdash_{\mathbb{P}_\kappa}\dot A\not\subseteq\vec V_\xi$; let $f_\xi:\{0,1\}^\kappa\to\{0,1\}^\lambda$ be a $(\mathcal{B}\mathfrak{a}_\kappa,\mathcal{B}\mathfrak{a}_\lambda)$-measurable function such that $a\Vdash_{\mathbb{P}_\kappa}\vec f_\xi\in\dot A\setminus\vec V_\xi$ (551Cc). Let $J_\xi\subseteq\kappa$ be a set with cardinal at most λ such that $\{x:f_\xi(x)(\zeta)=1\}$ is determined by coordinates in J_ξ for every $\zeta<\lambda$, and continue.

(iii) If $\eta<\xi<\theta_1^+$ then $a\Vdash_{\mathbb{P}_\kappa}\vec f_\eta\in\vec V_\xi$. **P** As $J_\eta\subseteq K_\xi$, we have a function $g:\{0,1\}^{K_\xi}\to\{0,1\}^\lambda$ such that $f_\eta(x)=g(x\restriction K_\xi)$ for every $x\in\{0,1\}^\lambda$. Now, for any $s\in\{0,1\}^{K_\xi}$ and $y\in\{0,1\}^\lambda$, the set

$$\{t:t\in\{0,1\}^{\kappa\setminus K_\xi},\ (s\cup t,y)\in W_\xi\setminus V_\xi\}$$

is $\nu_{\kappa\setminus K_\xi}$-negligible, so

$$E = \{(s,t) : (s \cup t, g(s)) \in W_\xi \setminus V_\xi\}$$

is $(\nu_{K_\xi} \times \nu_{\kappa\setminus K_\xi})$-negligible. ($E \in \mathcal{B}\mathfrak{a}_{K_\xi} \widehat{\otimes} \mathcal{B}\mathfrak{a}_{\kappa\setminus K_\xi}$ because W_ξ and V_ξ belong to $\mathcal{B}\mathfrak{a}_\kappa \widehat{\otimes} \mathcal{B}\mathfrak{a}_\lambda$ and g is $(\mathcal{B}\mathfrak{a}_{K_\xi}$, $\mathcal{B}\mathfrak{a}_\lambda)$-measurable.) But if we identify $\{0,1\}^{K_\xi} \times \{0,1\}^{\kappa\setminus K_\xi}$ with $\{0,1\}^\kappa$, then E becomes $\{x : (x, f_\eta(x)) \in W_\xi \setminus V_\xi\}$. Now

$$a \subseteq [\![\vec{f}_\eta \in \vec{W}_\xi]\!] = \{x : (x, f_\eta(x)) \in W_\xi\}^\bullet$$

(551Ea)

$$\subseteq \{x : (x, f_\eta(x)) \in V_\xi\}^\bullet = [\![\vec{f}_\eta \in \vec{V}_\xi]\!],$$

and $a \Vdash_{\mathbb{P}_\kappa} \vec{f}_\eta \in \vec{V}_\xi$. **Q**

(iv) For each $\xi < \theta_1^+$, V_ξ factors through $\{0,1\}^{K_\xi} \times \{0,1\}^\lambda$ and belongs to $\mathcal{B}\mathfrak{a}_\kappa \widehat{\otimes} \mathcal{B}\mathfrak{a}_\lambda$. There is therefore a countable set $L_\xi \subseteq K_\xi$ such that V_ξ factors through $\{0,1\}^{L_\xi} \times \{0,1\}^\lambda$. Let S be the set $\{\xi : \xi < \theta_1^+$, $\operatorname{cf}\xi > \omega\}$. Because $\theta_1 \ge \omega_1$, S is stationary in θ_1^+ (5A1Ac). For each $\xi \in S$, let $g(\xi) < \xi$ be such that $L_\xi \subseteq K_{g(\xi)}$. By the Pressing-Down Lemma there is a $\gamma < \theta_1^+$ such that $S' = \{\xi : \xi \in S,\ g(\xi) = \gamma\}$ is stationary.

For $\xi \in S'$, we have a $V'_\xi \in \mathcal{B}\mathfrak{a}_{K_\gamma} \widehat{\otimes} \mathcal{B}\mathfrak{a}_\lambda$ such that

$$V_\xi = \{(x,y) : x \in \{0,1\}^\kappa,\ y \in \{0,1\}^\lambda,\ (x{\restriction}K_\gamma, y) \in V'_\xi\}.$$

But $\#(K_\gamma) \le \lambda$, so

$$\#(\mathcal{B}\mathfrak{a}_{K_\gamma} \widehat{\otimes} \mathcal{B}\mathfrak{a}_\lambda) \le \lambda^\omega = \theta_1 < \#(S'),$$

and there are ξ, $\eta \in S'$ such that $\eta < \xi$ and $V'_\eta = V'_\xi$ and $V_\eta = V_\xi$. But also

$$a \Vdash_{\mathbb{P}_\kappa} \vec{f}_\eta \in \vec{V}_\xi \setminus \vec{V}_\eta,$$

so this is impossible. **X**

552K Lemma Let I be a set. Let $q : \mathrm{Fn}_{<\omega}(I;\{0,1\}) \to [0,\infty[$ be a function such that $q(\emptyset) = 1$ and

$$q(z) = q(z \cup \{(i,0)\}) + q(z \cup \{(i,1)\})$$

whenever $z \in \mathrm{Fn}_{<\omega}(I;\{0,1\})$ and $i \in I \setminus \operatorname{dom} z$. Then there is a unique Radon measure μ on $\{0,1\}^I$ such that

$$\mu\{x : z \subseteq x \in \{0,1\}^I\} = q(z)$$

for every $z \in \mathrm{Fn}_{<\omega}(I;\{0,1\})$.

proof (a) For each $K \in [I]^{<\omega}$, let μ_K be the measure on the finite set $\{0,1\}^K$ defined by saying that $\mu_K A = \sum_{z\in A} q(z)$ for every $A \subseteq \{0,1\}^K$. For $K \subseteq L \in [I]^{<\omega}$ set $f_{KL}(z) = z{\restriction}K$ for $z \in \{0,1\}^L$; then f_{KL} is inverse-measure-preserving for μ_K and μ_L. **P** It is enough to consider the case $L = K \cup \{i\}$ where $i \in I \setminus K$. In this case, for $A \subseteq \{0,1\}^K$,

$$\mu_L f^{-1}[A] = \sum_{w\in f^{-1}[A]} q(w) = \sum_{z\in A} q(z \cup \{(i,0)\}) + q(z \cup \{(i,1)\})$$

$$= \sum_{z\in A} q(z) = \mu_K A. \text{ } \mathbf{Q}$$

(b) Let $\mathcal{E}$ be the algebra of open-and-closed sets in $\{0,1\}^I$, that is, the family $\{f_K^{-1}[A] : K \in [I]^{<\omega}$, $A \subseteq \{0,1\}^K\}$, where $f_K(x) = x{\restriction}K$ for $x \in \{0,1\}^I$. Then we can define a functional $\nu : \mathcal{E} \to [0,1]$ by setting

$$\nu f_K^{-1}[A] = \mu_K A \text{ whenever } K \in [I]^{<\omega},\ A \subseteq \{0,1\}^K;$$

by (a), this is well-defined. By 416Qa, there is a unique Radon measure μ on $\{0,1\}^I$ extending ν, so that

$$\mu\{x : z \subseteq x\} = \nu\{x : z \subseteq x\} = \mu_K\{z\} = q(z)$$

whenever $K \subseteq I$ is finite and $z \in \{0,1\}^K$.

552L Lemma Let θ be a regular infinite cardinal such that the cardinal power δ^ω is less than θ for every $\delta < \theta$, and $S \subseteq \theta$ a stationary set such that $\operatorname{cf}\xi > \omega$ for every $\xi \in S$. Let $\langle M_\xi \rangle_{\xi<\theta}$ be a family of sets with cardinal less than θ, and I a set with cardinal less than θ; suppose that for each $i \in I$ we are given a function f_i with domain S such that $f_i(\xi) \in \bigcup_{\eta<\xi} M_\eta$ for every $\xi \in S$. Then there is an ω_1-complete filter $\mathcal{F}$ on θ, containing every closed cofinal subset of θ, such that for every $i \in I$ there is a $D \in \mathcal{F}$ such that $D \subseteq S$ and f_i is constant on D.

proof (a) Set $M = \bigcup_{\xi<\theta} M_\xi$, so that $\#(M) \le \theta$; let $\langle x_\xi \rangle_{\xi<\theta}$ run over M. Set

$$F^* = \{\xi : \xi < \theta,\ \bigcup_{\eta<\xi} M_\eta = \{x_\eta : \eta < \xi\}\};$$

then F^* is a closed cofinal subset of θ, because θ is regular and uncountable. Set $S_1 = S \cap F^*$, so that S_1 is stationary. For $\xi \in S_1$ and $i \in I$ let $h_\xi(i) < \xi$ be such that $f_i(\xi) = x_{h_\xi(i)}$. For $J \in [I]^{\le\omega}$, $\xi \in S_1$ set

$$D_{\xi J} = \{\eta : \eta \in S \cap F^*,\ h_\eta \restriction J = h_\xi \restriction J\}.$$

(b) There is a $\xi \in S_1$ such that $D_{\xi J} \cap F \ne \emptyset$ for every closed cofinal set $F \subseteq \theta$ and every $J \in [I]^{\le\omega}$. **P?** Otherwise, for each $\xi \in S_1$ choose $J_\xi \in [I]^{\le\omega}$ and a closed cofinal set F_ξ not meeting $D_{\xi J_\xi}$. Let F be the diagonal intersection $\{\xi : \xi < \theta,\ \xi \in F_\eta \text{ whenever } \eta \in S_1 \cap \xi\}$, so that F is a closed cofinal set (4A1B(c-ii)) and $S_2 = S_1 \cap F$ is stationary. For $\xi \in S_2$ let $g(\xi) < \xi$ be such that $h_\xi[J_\xi] \subseteq g(\xi)$. Then there is a $\gamma < \theta$ such that $S_3 = \{\xi : \xi \in S_2,\ g(\xi) = \gamma\}$ is stationary, by the Pressing-Down Lemma (4A1Cc). Now $h_\xi \restriction J_\xi \in [I \times \gamma]^{\le\omega}$ for every $\xi \in S_1$, and $\#([I \times \gamma]^{\le\omega}) \le \max(\#(I), \gamma, \omega)^\omega < \theta$, so there are ξ, $\eta \in S_3$ such that $h_\xi \restriction J_\xi = h_\eta \restriction J_\eta$ and $\eta < \xi$. But in this case we have $\xi \in F_\eta \cap D_{\eta J_\eta}$, which is supposed to be impossible. **XQ**

(c) If now $\langle F_n \rangle_{n\in\mathbb{N}}$ is any sequence of closed cofinal sets in θ, and $\langle J_n \rangle_{n\in\mathbb{N}}$ is any sequence in $[I]^{\le\omega}$,

$$\bigcap_{n\in\mathbb{N}} D_{\xi J_n} \cap F_n = D_{\xi J} \cap F$$

is non-empty, where $J = \bigcup_{n\in\mathbb{N}} J_n$ and $F = \bigcap_{n\in\mathbb{N}} F_n$. So we have an ω_1-complete filter $\mathcal{F}$ on θ generated by

$$\{D_{\xi J} : J \in [I]^{\le\omega}\} \cup \{F : F \subseteq \theta \text{ is closed and cofinal}\}.$$

If $i \in I$ then f_i is constant on $D_{\xi,\{i\}} \in \mathcal{F}$, so we're done.

552M Proposition Let κ and λ be infinite cardinals. Then the following are equiveridical:

(i) if $\mathcal{A} \subseteq \mathcal{P}(\{0,1\}^\kappa)$ and $\#(\mathcal{A}) \le \lambda$ then there is an extension of ν_κ to a measure measuring every member of $\mathcal{A}$;

(ii) for every function $f : \{0,1\}^\kappa \to \{0,1\}^{(\kappa+\lambda)\setminus\kappa}$, there is a Baire measure μ on $\{0,1\}^{\kappa+\lambda}$ such that $\mu\{y : y \in \{0,1\}^{\kappa+\lambda},\ z \subseteq y\} = 2^{-\#(K)}$ whenever $K \in [\kappa]^{<\omega}$ and $z \in \{0,1\}^K$, and $\mu^*\{x \cup f(x) : x \in \{0,1\}^\kappa\} = 1$;

(iii) if (X, Σ, μ) is a locally compact (definition: 342Ad) semi-finite measure space with Maharam type at most κ, $\mathcal{A} \subseteq \mathcal{P}X$ and $\#(\mathcal{A}) \le \lambda$, then there is an extension of μ to a measure measuring every member of $\mathcal{A}$.

proof (i)$\Rightarrow$(ii) Assume (i). If $f : \{0,1\}^\kappa \to \{0,1\}^{(\kappa+\lambda)\setminus\kappa}$ is a function, set $\mathcal{A} = \{\{x : f(x)(\xi) = 1\} : \kappa \le \xi < \kappa + \lambda\}$, so that $\mathcal{A}$ is a family of subsets of $\{0,1\}^\kappa$ and $\#(\mathcal{A}) \le \lambda$. Let ν be a measure on $\{0,1\}^\kappa$, extending ν_κ and measuring every member of $\mathcal{A}$. Then $\{x : (x \cup f(x))(\xi) = 1\} \in \operatorname{dom}\nu$ for every $\xi \in \kappa + \lambda$, so we have a Baire measure μ on $\{0,1\}^{\kappa+\lambda}$ defined by saying that $\mu E = \nu\{x : x \cup f(x) \in E\}$ for Baire sets $E \subseteq \{0,1\}^{\kappa+\lambda}$. If $K \in [\kappa]^{<\omega}$ and $z \in \{0,1\}^\kappa$, then

$$\mu\{y : z \subseteq y\} = \nu\{x : z \subseteq x \cup f(x)\} = \nu\{x : z \subseteq x\} = \nu_\kappa\{x : z \subseteq x\} = 2^{-\#(K)};$$

while if $E \in \mathcal{B}\mathfrak{a}_{\kappa+\lambda}$ and $x \cup f(x) \in E$ for every $x \in \{0,1\}^\kappa$, then $\mu E = \nu\{0,1\}^\kappa = 1$, so $\mu^*\{x \cup f(x) : x \in \{0,1\}^\kappa\} = 1$. Thus (ii) is true.

(ii)$\Rightarrow$(i) Assume (ii). Let $\mathcal{A}$ be a family of subsets of $\{0,1\}^\kappa$ with $\#(\mathcal{A}) \le \lambda$. Let $\langle A_\eta \rangle_{\eta<\lambda}$ run over $\mathcal{A} \cup \{\emptyset\}$. Define $f : \{0,1\}^\kappa \to \{0,1\}^{(\kappa+\lambda)\setminus\kappa}$ by saying that $f(x)(\kappa + \eta) = (\chi A_\eta)(x)$ whenever $\eta < \lambda$ and $x \in \{0,1\}^\kappa$. Let μ be a Baire measure on $\{0,1\}^{\kappa+\lambda}$ satisfying the conditions of (ii). Set $g(x) = x \cup f(x)$ for $x \in \{0,1\}^\kappa$. Because $g[\{0,1\}^\kappa]$ has full outer measure for μ, we have a measure ν on $\{0,1\}^\kappa$ such that $\nu g^{-1}[E] = \mu E$ for every Baire set $E \subseteq \{0,1\}^{\kappa+\lambda}$ (234F); let $\hat\nu$ be the completion of ν. Now $A_\eta = g^{-1}[\{y : y(\kappa + \eta) = 1\}]$ is measured by ν and $\hat\nu$. Also

$$\hat{\nu}\{x : z \subseteq x\} = \nu\{x : z \subseteq x\} = \nu\{x : z \subseteq g(x)\} = \mu\{y : z \subseteq y\} = 2^{-\#(K)}$$

whenever $K \in [\kappa]^{<\omega}$ and $z \in \{0,1\}^\kappa$, so $\hat{\nu}$ extends ν_κ (254G) and is an extension of ν_κ measuring every member of $\mathcal{A}$.

(i)⇒(iii) Suppose that (i) is true.

(α) Let (X, Σ, μ) be a compact probability space of Maharam type at most κ, and $\mathcal{A}$ a family of subsets of X with cardinal at most λ. Then there is a function $h : \{0,1\}^\kappa \to X$ which is inverse-measure-preserving for ν_κ and μ. **P** By 332P, the measure algebra of μ can be embedded into $\mathfrak{B}_\kappa$; by 343B, this embedding can be realized by an inverse-measure-preserving function from $\{0,1\}^\kappa$ to X. **Q** Now $\mathcal{C} = \{h^{-1}[A] : A \in \mathcal{A}\}$ has cardinal at most λ, so there is an extension ν of ν_κ measuring every member of $\mathcal{C}$; and the image measure νh^{-1} extends μ and measures every member of $\mathcal{A}$.

(β) It follows at once that if (X, Σ, μ) is a compact totally finite measure space with Maharam type at most κ, and $\mathcal{A}$ a family of subsets of X with cardinal at most λ, then μ can be extended to every member of $\mathcal{A}$. (If $\mu X = 0$ this is trivial, and otherwise we can apply (α) to a scalar multiple of μ.)

(γ) Now suppose that (X, Σ, μ) is a locally compact semi-finite measure space with Maharam type at most κ, and $\mathcal{A}$ a family of subsets of X with cardinal at most λ. In the measure algebra $(\mathfrak{A}, \bar{\mu})$ of μ, let D be a partition of unity consisting of elements of finite measure; for $d \in D$ choose $E_d \in \Sigma$ such that $E_d^\bullet = d$. If $G \in \Sigma$ then

$$\mu G = \bar{\mu} G^\bullet = \sum_{d \in D} \bar{\mu}(d \cap G^\bullet) = \sum_{d \in D} \mu(E_d \cap G).$$

For each $d \in D$, the subspace measure μ_{E_d} on E_d is compact and totally finite and has Maharam type at most κ (put 331Hc and 322Ja together), so by (β) can be extended to a measure μ'_{E_d} measuring $A \cap E_d$ for every $A \in \mathcal{A}$. Set $\mu' F = \sum_{d \in D} \mu'_{E_d}(F \cap E_d)$ whenever $F \subseteq X$ is such that the sum is defined; then μ' is a measure on X, extending μ and measuring every set in $\mathcal{A}$, as required.

(iii)⇒(i) is trivial.

552N Theorem (Carlson 84) Let κ and λ be infinite cardinals such that κ is greater than the cardinal power λ^ω. Then

$\Vdash_{\mathbb{P}_\kappa}$ if $\mathcal{A} \subseteq \mathcal{P}(\{0,1\}^{\check{\kappa}})$ and $\#(\mathcal{A}) \le \check{\lambda}$, there is an extension of $\nu_{\check{\kappa}}$ to a measure measuring every member of $\mathcal{A}$.

proof (a) Let $\langle e_{\xi\zeta} \rangle_{\xi,\zeta<\kappa}$ be a re-indexing of the standard generating family in $\mathfrak{B}_\kappa$. For $J \subseteq \kappa \times \kappa$ let $\mathfrak{C}_J$ be the closed subalgebra of $\mathfrak{B}_\kappa$ generated by $\{e_{\xi\zeta} : (\xi, \zeta) \in J\}$. Recall that $\#(L^\infty(\mathfrak{C}_J)) \le \max(\omega, \#(J)^\omega)$ for every J (524Ma, 515Mb); we shall also need to know that every element of $\mathfrak{B}_\kappa$ belongs to $\mathfrak{C}_J$ for a smallest $J \subseteq \kappa \times \kappa$, and this J is countable (254Rc, 531Jb).

Set $I = (\kappa + \lambda) \setminus \kappa$, where $\kappa + \lambda$ is the ordinal sum, so that I is disjoint from κ and $\#(I) = \lambda$. Let $\dot{f}$ be a $\mathbb{P}_\kappa$-name for a function from $\{0,1\}^{\check{\kappa}}$ to $\{0,1\}^{\check{I}}$.

For each $\xi < \kappa$ let $\dot{x}_\xi$ be a $\mathbb{P}_\kappa$-name for a member of $\{0,1\}^{\check{\kappa}}$ such that $[\![\dot{x}_\xi(\check{\zeta}) = 1]\!] = e_{\xi\zeta}$ for every $\zeta < \kappa$. For $z \in \mathrm{Fn}_{<\omega}(\kappa + \lambda; \{0,1\})$ and $\xi < \kappa$, set

$$a_{\xi z} = [\![\check{z} \subseteq \dot{x}_\xi \cup \dot{f}(\dot{x}_\xi)]\!]$$

and let $J_{\xi z} \subseteq \kappa \times \kappa$ be the smallest set such that $a_{\xi z} \in \mathfrak{C}_{J_{\xi z}}$. Note that

$$a_{\xi z} = a_{\xi, z \restriction I} \cap \inf_{\zeta \in \kappa \cap z^{-1}[\{1\}]} e_{\xi\zeta} \setminus \sup_{\zeta \in \kappa \cap z^{-1}[\{0\}]} e_{\xi\zeta},$$

so that

$$J_{\xi z} \subseteq J_{\xi, z \restriction I} \cup (\{\xi\} \times \mathrm{dom}\, z).$$

Set $\theta = (\lambda^\omega)^+ \le \kappa$. For $\xi \le \theta$ let $L_0(\xi) \subseteq \kappa$ be the smallest set such that $\xi \subseteq L_0(\xi)$ and $J_{\eta w} \subseteq L_0(\xi) \times L_0(\xi)$ for every $\eta < \xi$ and $w \in \mathrm{Fn}_{<\omega}(I; \{0,1\})$; set $L(\xi) = L_0(\xi) \times L_0(\xi)$. Then $\#(L(\xi)) \le \max(\omega, \lambda, \#(\xi)) < \theta$ for every $\xi < \theta$, and $L(\xi) = \bigcup_{\eta<\xi} L(\eta)$ for limit $\xi \le \theta$. Set

$$D^* = \{\xi : \xi < \theta \text{ is a limit ordinal, } \xi > \sup(\theta \cap L_0(\eta)) \text{ for every } \eta < \xi\};$$

then D^* is a closed cofinal subset of θ, and $\xi \notin L_0(\xi)$ for every $\xi \in D^*$. So

$$S = \{\xi : \xi \in D^*,\ \mathrm{cf}(\xi \cap D^*) \ge \omega_1\}$$

is stationary in θ.

(b) For $J \subseteq \kappa\times\kappa$, let $P_J : L^1(\mathfrak{B}_\kappa) \to L^1(\mathfrak{C}_J) \subseteq L^1(\mathfrak{B}_\kappa)$ be the conditional expectation operator defined by saying that $P_J u \in L^1(\mathfrak{C}_J)$ and $\int_c P_J u = \int_c u$ whenever $c \in \mathfrak{C}_J$ and $u \in L^1(\mathfrak{B}_\kappa)$ (254R, 365Q[2]). We need to know that $P_{J\cap J'} = P_J P_{J'}$ for all J, $J' \subseteq \kappa$ (254Ra), and that $P_J(u\times v) = u \times P_J v$ whenever $u \in L^1(\mathfrak{C}_J)$, $v \in L^1(\mathfrak{B}_\kappa)$ and $u \times v \in L^1(\mathfrak{B}_\kappa)$ (242L). It follows that if J, J', $J'' \subseteq \kappa\times\kappa$, $u \in L^1(\mathfrak{C}_J)$ and $J\cap J' = J \cap J''$, then

$$P_{J'}(u) = P_{J'}P_J(u) = P_{J\cap J'}(u) = P_{J\cap J''}(u) = P_{J''}(u).$$

If $h : \kappa\times\kappa \to \kappa\times\kappa$ is any permutation, then we have a corresponding measure-preserving automorphism $\pi : \mathfrak{B}_\kappa \to \mathfrak{B}_\kappa$ defined by saying that $\pi e_{\xi\zeta} = e_{\xi'\zeta'}$ if $(\xi',\zeta') = h(\xi,\zeta)$, and a Banach lattice automorphism $T : L^1(\mathfrak{B}_\kappa) \to L^1(\mathfrak{B}_\kappa)$ defined by saying that $T(\chi a) = \chi\pi a$ for $a \in \mathfrak{B}_\kappa$ (see 365N[3]); note that $T\restriction L^\infty(\mathfrak{B}_\kappa)$ is multiplicative (compare the formulae in 365Nb and 364Pa).

If $J \subseteq \kappa\times\kappa$ and $u \in L^1(\mathfrak{C}_J)$, then $Tu \in L^1(\mathfrak{C}_{h[J]})$. **P** By 314H, $\mathfrak{C}_{h[J]} = \pi[\mathfrak{C}_J]$; now use the fact that $[\![Tu > \alpha]\!] = \pi[\![u > \alpha]\!]$ for every $\alpha \in \mathbb{R}$. **Q** If $h\restriction J$ is the identity, then $\pi a = a$ for every $a \in \mathfrak{C}_J$ and $Tv = v$ for every $v \in L^\infty(\mathfrak{C}_J)$. Consequently $P_J T = P_J$. **P** If $u \in L^1(\mathfrak{B}_\kappa)$ and $c \in \mathfrak{C}_J$ then

$$\begin{aligned}\int_c P_J Tu &= \int_c Tu = \int Tu \times \chi c = \int T^{-1}(Tu\times\chi c)\\ &= \int u \times T^{-1}\chi c = \int u\times\chi c = \int_c u;\end{aligned}$$

as $P_J Tu$ certainly belongs to $L^1(\mathfrak{C}_J)$, it must be equal to $P_J u$. **Q**

(c) Fix $\xi \in S$ and $w \in \mathrm{Fn}_{<\omega}(I;\{0,1\})$ for the moment. Because $\xi \cap D^*$ has uncountable cofinality and $J_{\xi w}$ is countable, there is a $g_w(\xi) \in \xi\cap D^*$ such that $J_{\xi w} \cap L(\xi) \subseteq L(g_w(\xi))$ and $\{\eta : \eta \in L_0(\xi), (\xi,\eta) \in J_{\xi w}\} \subseteq L_0(g_w(\xi))$. Let $h_{\xi w} : \kappa\times\kappa \to \kappa\times\kappa$ be the involution defined by saying that

$$\begin{aligned}h_{\xi w}(\eta,\zeta) &= (g_w(\xi),\zeta) \text{ if } \eta = \xi,\\ &= (\xi,\zeta) \text{ if } \eta = g_w(\xi),\\ &= (\eta,\zeta) \text{ otherwise;}\end{aligned}$$

note that

$$h_{\xi w}[J_{\xi w}] \cap L(\xi) \subseteq L(g_w(\xi)) \cup (\{g_w(\xi)\} \times L_0(g_w(\xi)) \subseteq L(g_w(\xi)+1),$$

while $h_{\xi w}$ is the identity on $L(g_w(\xi))$, since neither ξ nor $g_w(\xi)$ belongs to $L_0(g_w(\xi))$. Let $T_{\xi w} : L^1(\mathfrak{B}_\kappa) \to L^1(\mathfrak{B}_\kappa)$ be the corresponding Banach lattice isomorphism defined as in (b) above. Then (b) tells us that

$$P_{L(g_w(\xi))}T_{\xi w} = P_{L(g_w(\xi))}.$$

Set

$$u_{\xi w} = P_{L(g_w(\xi)+1)}T_{\xi w}(\chi a_{\xi w}) \in L^\infty(\mathfrak{C}_{L(g_w(\xi)+1)}).$$

(d) Setting $M(\eta) = \eta \times L^\infty(\mathfrak{C}_{L(\eta)})$ for $\eta < \theta$, we see that

$$\#(M(\eta)) \le \#(\eta)^\omega \le \lambda^\omega < \theta$$

whenever $\eta \ge 2$ (see (a) above), while $(g_w(\xi), u_{\xi w}) \in \bigcup_{\eta<\xi} M(\eta)$ whenever $\xi \in S$ and $w \in \mathrm{Fn}_{<\omega}(I;\{0,1\})$. Since $\#(\mathrm{Fn}_{<\omega}(I;\{0,1\})) = \lambda < \theta$, 552L tells us that there is an ω_1-complete filter $\mathcal{F}$ on θ, containing $\theta\setminus\zeta$ for every $\zeta < \theta$, such that for every $w \in \mathrm{Fn}_{<\omega}(I;\{0,1\})$ there is a $D \in \mathcal{F}$ such that $D \subseteq S$ and g_w and $\xi \mapsto u_{\xi w}$ are constant on D.

(e) For $z \in \mathrm{Fn}_{<\omega}(\kappa+\lambda;\{0,1\})$ and $\xi < \theta$, set $v_{\xi z} = P_{L(\xi)}(\chi a_{\xi z})$. Then there is a $D \in \mathcal{F}$ such that $D \subseteq S$ and $\xi \mapsto v_{\xi z}$ is constant on D. **P** Set $z' = z\restriction L_0(\theta)$, $z'' = z\restriction\kappa\setminus L_0(\theta)$ and $w = z\restriction I$, so that

[2]Formerly 365R.
[3]Formerly 365O.

$a_{\xi z} = a_{\xi z'} \cap a_{\xi z''} \cap a_{\xi w}$. Set $m = \#(z'')$, so that $a_{\xi z''} = [\![\check{z}'' \subseteq \dot{x}_\xi]\!]$ has measure 2^{-m} for every ξ. Also $a_{\xi z''} \in \mathfrak{C}_{\kappa\times(\kappa\setminus L_0(\theta))}$ is stochastically independent of $\mathfrak{C}_{L(\theta)}$, so $P_{L(\theta)}(\chi a_{\xi z''}) = 2^{-m}\chi 1$; while $a_{\xi z'}$ and $a_{\xi w}$ belong to $\mathfrak{C}_{L(\theta)}$, so, using the formulae in (b),

$$\begin{aligned} v_{\xi z} &= P_{L(\xi)}(\chi a_{\xi z}) = P_{L(\xi)}P_{L(\theta)}(\chi a_{\xi z}) \\ &= P_{L(\xi)}P_{L(\theta)}(\chi a_{\xi z'} \times \chi a_{\xi z''} \times \chi a_{\xi w}) \\ &= P_{L(\xi)}(\chi a_{\xi z'} \times \chi a_{\xi w} \times P_{L(\theta)}(\chi a_{\xi z''})) \\ &= 2^{-m}P_{L(\xi)}(\chi a_{\xi z'} \times \chi a_{\xi w}). \end{aligned}$$

Let $\xi_0 < \theta$ be such that $\operatorname{dom} z' \subseteq L_0(\xi_0)$. By (d), there are $D_0 \in \mathcal{F}$, $\zeta < \theta$ and $u \in L^\infty(\mathfrak{B}_\kappa)$ such that $D_0 \subseteq S$ and $g_w(\xi) = \zeta$ and $u_{\xi w} = u$ for every $\xi \in D_0$. For $\xi \in D_0 \setminus \xi_0$ take $h_{\xi w}$ and $T_{\xi w}$ as in (c). Then, writing $\pi_{\xi w}$ for the measure-preserving automorphism defined from $h_{\xi w}$ as in (b),

$$\begin{aligned} \pi_{\xi w}a_{\xi z'} &= \pi_{\xi w}(\inf_{z'(\eta)=1} e_{\xi\eta} \setminus \sup_{z'(\eta)=0} e_{\xi\eta}) = \inf_{z'(\eta)=1} \pi_{\xi w}e_{\xi\eta} \setminus \sup_{z'(\eta)=0} \pi_{\xi w}e_{\xi\eta} \\ &= \inf_{z'(\eta)=1} e_{\zeta\eta} \setminus \sup_{z'(\eta)=0} e_{\zeta\eta} = a_{\zeta z'}; \end{aligned}$$

consequently $T_{\xi w}(\chi a_{\xi z'}) = \chi a_{\zeta z'}$. Now

$$P_{L(\xi)}(\chi a_{\xi z'} \times \chi a_{\xi w}) = P_{L(\zeta)}(\chi a_{\xi z'} \times \chi a_{\xi w})$$

(because $\chi a_{\xi z'} \times \chi a_{\xi w} \in L^\infty(\mathfrak{C}_{(\{\xi\}\times L_0(\xi))\cup J_{\xi w}})$, and $(\{\xi\} \times L_0(\xi)) \cup J_{\xi w}) \cap L(\xi) = J_{\xi w} \cap L(\xi) \subseteq L(\zeta) \subseteq L(\xi))$

$$= P_{L(\zeta)}T_{\xi w}(\chi a_{\xi z'} \times \chi a_{\xi w})$$

(see (c) above)

$$= P_{L(\zeta)}(T_{\xi w}\chi a_{\xi z'} \times T_{\xi w}\chi a_{\xi w})$$

(because $T_{\xi w}$ is multiplicative on $L^\infty(\mathfrak{B}_\kappa)$)

$$\begin{aligned} &= P_{L(\zeta)}(\chi a_{\zeta z'} \times T_{\xi w}\chi a_{\xi w}) \\ &= P_{L(\zeta)}P_{L(\xi)}(\chi a_{\zeta z'} \times T_{\xi w}\chi a_{\xi w}) \end{aligned}$$

(because $L(\zeta) \subseteq L(\xi)$)

$$= P_{L(\zeta)}(\chi a_{\zeta z'} \times P_{L(\xi)}T_{\xi w}\chi a_{\xi w})$$

(because $a_{\zeta z'} \in \mathfrak{C}_{\{\zeta\}\times L_0(\xi)} \subseteq \mathfrak{C}_{L(\xi)}$)

$$= P_{L(\zeta)}(\chi a_{\zeta z'} \times P_{L(\zeta+1)}T_{\xi w}\chi a_{\xi w})$$

(because $T_{\xi w}\chi a_{\xi w} \in L^\infty(\mathfrak{C}_{h_{\xi w}[J_{\xi w}]})$ and $h_{\xi w}[J_{\xi w}] \cap L(\xi) \subseteq L(\zeta+1)$)

$$= P_{L(\zeta)}(\chi a_{\zeta z'} \times u_{\xi w}) = P_{L(\zeta)}(\chi a_{\zeta z'} \times u).$$

Finally, we get

$$v_{\xi z} = 2^{-m}P_{L(\xi)}(\chi a_{\xi z'} \times \chi a_{\xi w}) = 2^{-m}P_{L(\zeta)}(\chi a_{\zeta z'} \times u)$$

for every $\xi \in D = D_0 \setminus \xi_0$, so we have the required constant value. **Q**

(f) For each $z \in \mathrm{Fn}_{<\omega}(\kappa+\lambda; \{0,1\})$ set $v_z = \lim_{\xi\to\mathcal{F}} v_{\xi z}$, the limit being defined in the strong sense that $\{\xi : \xi \in S,\ v_{\xi z} = v_z\}$ belongs to $\mathcal{F}$. Observe that $0 \le v_z \le \chi 1$ and that $v_\emptyset = \chi 1$, because $a_{\xi\emptyset} = 1$ for every $\xi \in S$. If $z \in \mathrm{Fn}_{<\omega}(\kappa+\lambda; \{0,1\})$ and $\eta \in (\kappa+\lambda) \setminus \operatorname{dom} z$, then $a_{\xi,z\cup\{(\eta,0)\}}$ and $a_{\xi,z\cup\{(\eta,1)\}}$ are disjoint and have union $a_{\xi z}$, so

$$v_{\xi z} = P_{L(\xi)}(\chi a_{\xi z}) = P_{L(\xi)}(\chi a_{\xi,z\cup\{(\eta,0)\}} + \chi a_{\xi,z\cup\{(\eta,1)\}}) = v_{\xi,z\cup\{(\eta,0)\}} + v_{\xi,z\cup\{(\eta,1)\}}$$

for every $\xi \in S$, and $v_z = v_{z\cup\{(\eta,0)\}} + v_{z\cup\{(\eta,1)\}}$. Let $\vec{v}_z$ be the $\mathbb{P}_\kappa$-name for a real number corresponding to v_z as defined in 5A3L.

(g) We have a $\mathbb{P}_\kappa$-name $\dot{\mu}$ for a Baire probability measure on $\{0,1\}^{\kappa+\lambda}$ such that

$$\Vdash_{\mathbb{P}_\kappa} \dot{\mu}\{x : \check{z} \subseteq x \in \{0,1\}^{\check{\kappa}+\check{\lambda}}\} = \vec{v}_z$$

for every $z \in \mathrm{Fn}_{<\omega}(\kappa+\lambda;\{0,1\})$. **P** Start by setting

$$\dot\mu_0 = \{((\check z, \vec v_z), 1_{\mathfrak{B}_\kappa}) : z \in \mathrm{Fn}_{<\omega}(\kappa+\lambda;\{0,1\})\},$$

so that $\dot\mu_0$ is a $\mathbb{P}_\kappa$-name for a function from $\mathrm{Fn}_{<\omega}(\check\kappa+\check\lambda;\{0,1\})$ to $[0,1]$ and

$$\Vdash_{\mathbb{P}_\kappa} \dot\mu_0\check z = \vec v_z$$

for every $z \in \mathrm{Fn}_{<\omega}(\kappa+\lambda;\{0,1\})$; note that

$$\Vdash_{\mathbb{P}_\kappa} \check\kappa+\check\lambda = (\kappa+\lambda)\check{} \text{ and } \mathrm{Fn}_{<\omega}(\check\kappa+\check\lambda;\{0,1\}) = (\mathrm{Fn}_{<\omega}(\kappa+\lambda;\{0,1\}))\check{}.$$

Now (f) tells us that

$$\Vdash_{\mathbb{P}_\kappa} \dot\mu_0\emptyset = 1,$$

$$\Vdash_{\mathbb{P}_\kappa} \dot\mu_0(z) = \dot\mu_0(z\cup\{(\eta,0)\}) + \dot\mu_0(z\cup\{(\eta,1)\})$$
$$\text{whenever } z \in \mathrm{Fn}_{<\omega}(\check\kappa+\check\lambda;\{0,1\}) \text{ and } \eta \in (\check\kappa+\check\lambda)\setminus \mathrm{dom}\, z.$$

By 552K, copied into $V^{\mathbb{P}_\kappa}$,

$$\Vdash_{\mathbb{P}_\kappa} \text{ there is a Radon measure } \mu \text{ on } \{0,1\}^{\check\kappa+\check\lambda} \text{ such that}$$
$$\mu\{x : z\subseteq x\} = \dot\mu_0(z) \text{ for every } z \in \mathrm{Fn}_{<\omega}(\check\kappa+\check\lambda;\{0,1\}).$$

In fact we don't really want the Radon measure here, but its restriction to the Baire σ-algebra. Let $\dot\mu$ be a $\mathbb{P}_\kappa$-name for a Baire measure on $\{0,1\}^{\check\kappa+\check\lambda}$ such that

$$\Vdash_{\mathbb{P}_\kappa} \dot\mu\{x : \check z\subseteq x \in \{0,1\}^{\check\kappa+\check\lambda}\} = \dot\mu_0(\check z) = \vec v_z$$

for every $z \in \mathrm{Fn}_{<\omega}(\kappa+\lambda;\{0,1\})$; this is what we have been looking for. **Q**

(h) I had better check that

$$\Vdash_{\mathbb{P}_\kappa} \dot\mu\{x : \check z\subseteq x\} = (2^{-\#(z)})\check{}$$

whenever $z \in \mathrm{Fn}_{<\omega}(\kappa;\{0,1\})$. **P** If $z \in \mathrm{Fn}_{<\omega}(\kappa;\{0,1\})$ and $\xi\in S$, then $a_{\xi z}$ belongs to $\mathfrak{C}_{\{\xi\}\times\kappa}$, so is stochastically independent of $\mathfrak{C}_{L(\xi)}$, and

$$v_{\xi z} = P_{L(\xi)}(\chi a_{\xi z}) = (\bar\nu_\kappa a_{\xi z})\chi 1 = 2^{-\#(z)}\chi 1.$$

Accordingly $v_z = 2^{-\#(z)}\chi 1$ and

$$\Vdash_{\mathbb{P}_\kappa} \dot\mu\{x : \check z\subseteq x\} = \vec v_z = (2^{-\#(z)})\check{}. \text{ } \mathbf{Q}$$

(i) Finally, we come to the key fact:

$$\Vdash_{\mathbb{P}_\kappa} \{\dot x_\xi \cup \dot f(\dot x_\xi) : \xi < \check\theta\} \text{ has full outer measure for } \dot\mu.$$

P? Otherwise, there are a non-zero $b\in\mathfrak{B}_\kappa$, a rational number $q<1$ and a sequence $\langle \dot C_n\rangle_{n\in\mathbb{N}}$ of $\mathbb{P}_\kappa$-names for basic cylinder sets in $\{0,1\}^{\check\kappa+\check\lambda}$ such that

$$b \Vdash_{\mathbb{P}_\kappa} \dot x_\xi \cup \dot f(\dot x_\xi) \in \textstyle\bigcup_{n\in\mathbb{N}} \dot C_n$$

for every $\xi<\theta$, while also

$$b \Vdash_{\mathbb{P}_\kappa} \textstyle\sum_{n=0}^\infty \dot\mu\dot C_n \le \check q.$$

Because $\mathfrak{B}_\kappa$ is ccc, we can find for each $n\in\mathbb{N}$ a partition $\langle b_{ni}\rangle_{i\in\mathbb{N}}$ of unity in $\mathfrak{B}_\kappa$, and a sequence $\langle z_{ni}\rangle_{i\in\mathbb{N}}$ in $\mathrm{Fn}_{<\omega}(\kappa+\lambda;\{0,1\})$, such that $b_{ni}\Vdash_{\mathbb{P}_\kappa} \dot C_n = \{x : \check z_{ni}\subseteq x\}$ for each i. Now

$$[\![\dot x_\xi \cup \dot f(\dot x_\xi) \in \dot C_n]\!] = \sup_{i\in\mathbb{N}} b_{ni} \cap [\![\check z_{ni} \subseteq \dot x_{ni} \cup \dot f(\dot x_\xi]\!] = \sup_{i\in\mathbb{N}} b_{ni}\cap a_{\xi,z_{ni}}$$

so we must have

$$b \subseteq \sup_{i,n\in\mathbb{N}} b_{ni}\cap a_{\xi,z_{ni}}$$

for every ξ. At the same time,

$$b_{ni}\Vdash_{\mathbb{P}_\kappa}\dot\mu\dot C_n=\vec v_{z_{ni}}$$

for each n and i, so $\dot\mu\dot C_n$ is represented by $\sum_{i\in\mathbb{N}}\chi b_{ni}\times v_{z_{ni}}$ in $L^\infty(\mathfrak{B}_\kappa)$ and $\min(1,\sum_{n=0}^\infty\dot\mu\dot C_n)$ is represented by $\chi 1\wedge\sum_{n,i\in\mathbb{N}}\chi b_{ni}\times v_{z_{ni}}$. Since $[\![\sum_{n=0}^\infty\dot\mu\dot C_n\le\check q]\!]$ includes b,

$$\sum_{n,i\in\mathbb{N}}\chi b\times\chi b_{ni}\times v_{z_{ni}}\le q\chi b.$$

Let $J\subseteq\kappa\times\kappa$ be a countable set such that $b\cap b_{ni}\in\mathfrak{C}_J$ for every n, $i\in\mathbb{N}$. Then, because $\mathcal{F}$ is ω_1-complete and contains $S\setminus\zeta$ for every $\zeta<\theta$, there is a $\xi\in S$ such that

$$v_{\xi z_{ni}}=v_{z_{ni}}\text{ whenever }n,\,i\in\mathbb{N},$$

$$J\cap L(\theta)=J\cap L(\xi),\quad J\cap(\{\xi\}\times\kappa)=\emptyset.$$

Set $L=L(\theta)\cup(\{\xi\}\times\kappa)$. If n, $i\in\mathbb{N}$, then

$$\begin{aligned}\int\chi b\times\chi b_{ni}\times v_{z_{ni}}&=\int\chi b\times\chi b_{ni}\times P_{L(\xi)}\chi a_{\xi,z_{ni}}\\&=\int P_{L(\xi)}(\chi b\times\chi b_{ni}\times P_{L(\xi)}\chi a_{\xi,z_{ni}})\\&=\int P_{L(\xi)}(\chi b\times\chi b_{ni})\times P_{L(\xi)}\chi a_{\xi,z_{ni}}\\&=\int P_{L(\xi)}(\chi b\times\chi b_{ni})\times\chi a_{\xi,z_{ni}}\\&=\int P_L(\chi b\times\chi b_{ni})\times\chi a_{\xi,z_{ni}}\end{aligned}$$

(because $J\cap L=J\cap L(\xi)$)

$$=\int P_L(\chi b\times\chi b_{ni}\times\chi a_{\xi,z_{ni}})$$

(because $J_{\xi,z_{ni}}\subseteq J_{\xi,z_{ni}\restriction I}\cup(\{\xi\}\times\kappa)\subseteq L$, so $a_{\xi,z_{ni}}\in\mathfrak{C}_L$)

$$=\int\chi b\times\chi b_{ni}\times\chi a_{\xi,z_{ni}}.$$

Summing over n and i, we have

$$\begin{aligned}\bar\nu_\kappa b\le\sum_{n=0}^\infty\sum_{i=0}^\infty\bar\nu_\kappa(b\cap b_{ni}\cap a_{\xi,z_{ni}})&=\sum_{n=0}^\infty\sum_{i=0}^\infty\int\chi b\times\chi b_{ni}\times\chi a_{\xi,z_{ni}}\\&=\sum_{n=0}^\infty\sum_{i=0}^\infty\int\chi b\times\chi b_{ni}\times v_{z_{ni}}\le\int q\chi b=q\bar\nu_\kappa b,\end{aligned}$$

which is impossible. **XQ**

(j) What all this shows is that

$\Vdash_{\mathbb{P}_\kappa}$ for every $f:\{0,1\}^{\check\kappa}\to\{0,1\}^{(\check\kappa+\check\lambda)\setminus\check\kappa}$ there is a Baire measure μ on $\{0,1\}^{\check\kappa+\check\lambda}$ such that $\mu\{y:y\in\{0,1\}^{\check\kappa+\check\lambda},\,z\subseteq y\}=2^{-\#(K)}$ whenever $K\in[\check\kappa]^{<\omega}$ and $z\in\{0,1\}^K$, and $\mu^*\{x\cup f(x):x\in\{0,1\}^{\check\kappa}\}=1$.

By 552M, copied into $V^{\mathbb{P}_\kappa}$,

$\Vdash_{\mathbb{P}_\kappa}$ if $\mathcal{A}\subseteq\mathcal{P}(\{0,1\}^{\check\kappa})$ and $\#(\mathcal{A})\le\check\lambda$, there is an extension of $\nu_{\check\kappa}$ to a measure measuring every member of $\mathcal{A}$,

as required.

552O Proposition Suppose that (X,Σ,μ) is a probability space such that for every countable family $\mathcal{A}$ of subsets of X there is a measure on X extending μ and measuring every member of $\mathcal{A}$.

(a) If Y is a universally negligible (definition: 439B) second-countable T_0 space, then $\#(Y) < \operatorname{cov}\mathcal{N}(\mu)$.
(b) $\operatorname{cov}\mathcal{N}(\mu) > \operatorname{non}\mathcal{N}(\nu_\omega)$.

proof (a) ? Otherwise, let $\langle E_y\rangle_{y\in Y}$ be a cover of X by μ-negligible sets, and $f : X \to Y$ a function such that $x \in E_{f(x)}$ for every $x \in X$. Let $\mathcal{U}$ be a countable base for the topology of Y and $\mathcal{A} = \{f^{-1}[U] : U \in \mathcal{U}\}$; let $\tilde\mu$ be a measure on X extending μ and measuring every member of $\mathcal{A}$. Consider the image measure $\tilde\mu f^{-1}$ on Y. This measures every member of $\mathcal{U}$ so measures every Borel set in Y; let ν be its restriction to the Borel σ-algebra of Y. Then ν is a Borel probability measure on Y. Take any $y \in Y$. Because Y has a T_0 topology, $\mathcal{U}$ must separate the points of Y and $\{y\}$ is a Borel set; now

$$\nu\{y\} = \tilde\mu f^{-1}[\{y\}] \le \tilde\mu^* E_y \le \mu^* E_y = 0.$$

So ν is zero on singletons and witnesses that Y is not universally negligible. **X**

(b) By Grzegorek's theorem (439Fc), there is a universally negligible set $Y \subseteq [0,1]$ with cardinal $\operatorname{non}\mathcal{N}(\nu_\omega)$. (Recall that the Lebesgue null ideal is isomorphic to $\mathcal{N}(\nu_\omega)$, as noted in 522W(a-i).)

552P Theorem Let κ and λ be infinite cardinals. Then the iterated forcing notion $\mathbb{P}_\kappa * \mathbb{P}_{\check\lambda}$ has regular open algebra isomorphic to $\mathfrak{B}_{\max(\kappa,\lambda)}$.

Remark Here $\mathbb{P}_{\check\lambda}$ represents a standard $\mathbb{P}_\kappa$-name for random real forcing; see 551O.

proof In Theorem 551Q, take $\Omega = \{0,1\}^\kappa$, $\Sigma = \mathrm{T}_\kappa$, $\mathcal{I} = \mathcal{N}(\nu_\kappa)$ and $I = \lambda$. If we identify $\{0,1\}^\kappa \times \{0,1\}^\lambda$ with $\{0,1\}^{\kappa+\lambda}$, where $\kappa+\lambda$ is the ordinal sum, then $\Lambda = \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}_\lambda$ becomes a σ-algebra intermediate between $\mathcal{B}\mathfrak{a}_{\kappa+\lambda}$ and $\mathrm{T}_{\kappa+\lambda}$, while

$$\mathcal{J} = \{W : W \in \Lambda,\ \nu_\lambda W[\{x\}] = 0 \text{ for } \nu_\kappa\text{-almost every } x \in \{0,1\}^\kappa\}$$

is just $\Lambda \cap \mathcal{N}(\nu_{\kappa+\lambda})$. It follows at once that the algebra $\mathfrak{A} = \Lambda/\mathcal{J}$ is isomorphic to $\mathfrak{B}_{\kappa+\lambda}$; and 551Q tells us that $\mathrm{RO}(\mathbb{P}_\kappa * \mathbb{P}_{\check\lambda})$ is isomorphic to $\mathfrak{A}$. Since we are supposing that κ and λ are infinite, $\mathfrak{B}_{\kappa+\lambda} \cong \mathfrak{B}_{\max(\kappa,\lambda)}$ and we're done.

552X Basic exercises (a) Let κ be an infinite cardinal. Show that $\Vdash_{\mathbb{P}_\kappa} \check\kappa^\omega = (\kappa^\omega)\check{}$, where these are all cardinal powers.

(b) (Miller 82) Suppose that $\mathfrak{c} < \omega_\omega$. Show that

$$\Vdash_{\mathbb{P}_{\omega_\omega}} \operatorname{cov}\mathcal{N}(\nu_{\omega_1}) = \omega_\omega < \operatorname{cov}\mathcal{N}(\nu_\omega).$$

(c) Suppose that the continuum hypothesis is true. Show that

$$\Vdash_{\mathbb{P}_{\omega_2}} \mathfrak{c} \text{ is a precaliber of every measurable algebra.}$$

(*Hint*: 525K.)

(d) Describe Cichoń's diagram in the forcing universe $V^{\mathbb{P}_{\omega_2}}$ (i) if we start with $\mathfrak{c} = \omega_1$ (ii) if we start with $\mathfrak{m} = \mathfrak{c} = \omega_2$. Locate the shrinking number of Lebesgue measure in each case.

(e) Suppose that the continuum hypothesis is true. Show that

$\Vdash_{\mathbb{P}_{\omega_2}} \operatorname{ci}(\mathcal{P}(\{0,1\}^\omega) \setminus \mathcal{N}(\nu_\omega)) = \mathfrak{c}$, so there is a family $\langle\nu_\xi\rangle_{\xi<\mathfrak{c}}$ of additive functionals on $\mathcal{P}([0,1])$ such that $\sup_{\xi<\mathfrak{c}} \nu_\xi A$ is the Lebesgue outer measure of A for every $A \subseteq [0,1]$.

(See Lipecki 09.)

(f) Suppose that the continuum hypothesis is true. Show that there is a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of subsets of $[0,1]$ such that there is no measure extending Lebesgue measure which measures every A_n. (*Hint*: there is a sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ of functions from ω_1 to itself such that $\{f_n(\xi) : n \in \mathbb{N}\} = \{\eta : \eta \le \xi\}$ for every $\xi < \omega_1$.)

552Y Further exercises (a) Let κ and λ be infinite cardinals, and μ a Baire measure on $\{0,1\}^\lambda$. (i) Show that there is a $\mathbb{P}_\kappa$-name $\dot\mu$ for a Baire measure on $\{0,1\}^{\check\lambda}$ such that $\Vdash_{\mathbb{P}_\kappa} \dot\mu\{x : \check z \subseteq x\} = (\mu\{x : z \subseteq x\})\check{}$ for every $z \in \mathrm{Fn}_{<\omega}(\lambda;\{0,1\})$. (ii) Show that if $A \subseteq \{0,1\}^\lambda$, then $\Vdash_{\mathbb{P}_\kappa} \dot\mu^*(\vec{A}) = (\mu^* A)\check{}$.

(b)(i) Show that for any non-zero cardinals κ, λ there are cardinals $\theta^{\mathrm{cov}}_{\kappa\lambda}$ and $\theta^{\mathrm{non}}_{\kappa\lambda}$ such that

$$\Vdash_{\mathbb{P}_\kappa} \operatorname{cov}\mathcal{N}(\nu_{\check\lambda}) = \check\theta^{\mathrm{cov}}_{\kappa\lambda}, \quad \operatorname{non}\mathcal{N}(\nu_{\check\lambda}) = \check\theta^{\mathrm{non}}_{\kappa\lambda}.$$

(*Hint*: if κ is infinite, $\mathfrak{B}_\kappa$ is homogeneous.) (ii) Show that $\theta^{\mathrm{cov}}_{\kappa\lambda}$ increases with κ and decreases with λ, while $\theta^{\mathrm{non}}_{\kappa\lambda}$ decreases with κ and increases with λ. (*Hint*: 552P.)

552 Notes and comments In any forcing model, all the open questions of ZFC re-present themselves for our attention. The first and most important question concerns the continuum hypothesis, and in most cases we can say something useful. So I start with 552B: 'if you add κ random reals, then the continuum rises to κ^ω'. Any mnemonic of this kind has to come with footnotes concerning the interpretation of the terms, because we cannot rely on the formula 'κ^ω' meaning the same thing in the universe we start from and the forcing model we move to. Indeed, in general forcing models, the symbol 'κ' has to be watched, since I normally reserve it for cardinals, and cardinals sometimes collapse; but here, at least, we have a ccc forcing notion, and cardinals are preserved (5A3Nb). Actually, 'κ^ω' also is safe in the present context (552Xa); but we find this out afterwards.

One of the central properties of random real forcing concerns iteration: if you do it twice, you still have random real forcing. Of course 'iterated forcing', in a vast variety of forms, is an indispensable technique, and two-stage forcing, as in 552P, is the easiest kind. I do not expect to quote this result very often in this book, but that is because (for random reals) I am interested as much in the forcing notions themselves, and the measure algebras which are their regular open algebras, as in the propositions which are true in the forcing models. So when I see a proof which depends on repeated random real forcing my first impulse is to examine the relevant properties of measure algebras, and this generally leads to a direct proof in terms of single-stage forcing. Note the form of Theorem 552P: as in 551Q, it does not claim that the iteration $\mathbb{P}_\kappa * \mathbb{P}_{\check\lambda}$ is isomorphic to $\mathbb{P}_{\max(\kappa,\lambda)}$, but only that they have isomorphic regular open algebras, and therefore lead to the same mathematical worlds (5A3I).

A typical example is in 552J. Random real forcing does not change outer measures (552D). If we think of $\mathbb{P}_\kappa$ as an iteration $\mathbb{P}_{\kappa\setminus J} * \mathbb{P}_{\check J}$, and we have a $\mathbb{P}_\kappa$-name $\dot E$ for a 'new' negligible set, then, thinking in $V^{\mathbb{P}_J}$, the set of members of $\{0,1\}^{\check\lambda}$ contained in $\dot E$ will have to be negligible. Back in the ordinary universe, we shall have a $\mathbb{P}_J$-name for a negligible set containing every member of $\{0,1\}^{\check\lambda}$ with a $\mathbb{P}_J$-name which belongs to $\dot E$. In 552J, the idea is that if $\dot A$ is a set in $V^{\mathbb{P}_\kappa}$ and every small subset of $\dot A$ is negligible in $V^{\mathbb{P}_\kappa}$, then at every stage the set of members of $\dot A$ which have been named so far must be negligible in $V^{\mathbb{P}_\kappa}$, just because there are not very many names yet available, and therefore is also negligible in the intermediate universe of the forcing notion $\mathbb{P}_{K_\xi}$. This must be witnessed by a countable structure in the intermediate universe, and the Pressing-Down Lemma tells us that there is a stationary set of levels for which the same countable structure will serve; it follows easily that we have a name in $V^{\mathbb{P}_\kappa}$ for a negligible set including $\dot A$. I invite you to seek out the elements of the formal exposition in 552J which correspond to this sketch.

552E can also be approached as a result about iterated random real forcing. Here, $\dot A$ is just the set of 'random reals' $\dot x_\xi$ built directly from the regular open algebra $\mathfrak{B}_\kappa$. To see that this is a Sierpiński set, we need to look at a negligible set. A negligible set in $V^{\mathbb{P}_\kappa}$ is included in one which has a name $\dot C$ in $V^{\mathbb{P}_J}$ for some countable $J \subseteq \kappa$. Thinking in $V^{\mathbb{P}_J}$, all but countably many of the $\dot x_\xi$ are still random, because they are the 'random reals' of $V^{\mathbb{P}_{\kappa\setminus J}}$, and therefore do not belong to $\dot C$. The proof of 552E is no more than a formal elaboration of this idea, with the extra technical device necessary to reach 'strongly Sierpiński'.

In 552C all we need to know is that $\mathbb{P}_\kappa$ is weakly σ-distributive, and the key fact is that for every name $\dot f$ for a member of $\mathbb{N}^\mathbb{N}$ there is an h in the ordinary universe such that $\Vdash_{\mathbb{P}_\kappa} \dot f \le^* \check h$; this is why such partial orders are sometimes called 'ω^ω-bounding'. The rest of the argument is based on the same ideas as part (d) of the proof of 5A3N.

In 552F-552J I list the results known to me about the additivity, covering number, uniformity, cofinality and shrinking number of the ideals $\mathcal{N}(\nu_\lambda)$ after random real forcing. Covering number, uniformity and shrinking number are the difficult ones, and even the most basic case, when $\lambda = \omega$ and we are forcing with $\mathbb{P}_\omega$, seems not to have been completely sorted out. 552Gb and 552Hc show that there is room for surprises. My method throughout is to use the results of §551 to relate $\mathcal{N}(\nu_\lambda)$ in $V^{\mathbb{P}_\kappa}$ to $\mathcal{N}(\nu_\kappa \times \nu_\lambda)$ in the original universe. Given a $\mathbb{P}_\kappa$-name $\dot W$ for a negligible set in $\{0,1\}^{\check\lambda}$, we have a negligible $W \subseteq \{0,1\}^\kappa \times \{0,1\}^\lambda$

such that $\Vdash_{\mathbb{P}_\kappa} \dot{W} \subseteq \vec{W}$, and then a negligible $V \subseteq \{0,1\}^\lambda$, corresponding to the non-negligible horizontal sections of W, such that $(\{0,1\}^\lambda \setminus V)\check{}$ is disjoint from $\vec{W}$ and $\dot{W}$ in $V^{\mathbb{P}_\kappa}$.

In 552K-552M I give some lemmas which apparently have nothing to do with forcing. The intention is to express as much as possible of the argument of Carlson's theorem 552N as results in ZFC. In this section I am taking forcing arguments particularly laboriously; but even when you have got to the point where they seem elementary to you, I believe that it is still worth while minimising the regions in which one has to deal with more than one model of set theory at a time. In 552M the parts (ii) and (iii) contrast oddly. Part (ii) is there to serve as a combinatorial form of (i) which will be accessible for the purposes of 552N. Part (iii) is there to give a notion of the scope of 552N, and in particular to show that in random real models we have extension theorems for many measures not obviously similar to the basic measures ν_κ. I have already noted a similar result in 543G.

In §439 I described a number of examples of probability spaces (X,Σ,μ) with a countable family $\mathcal{A} \subseteq \mathcal{P}X$ such that μ has no extension to a measure measuring every member of $\mathcal{A}$. In particular, as observed in 439Xk, Grzegorek's theorem 439Fc gives us an example of a subspace of $[0,1]$ for which the subspace measure fails to be extendable to some countably-generated σ-algebra. These are ZFC examples; we really do need something like 'compactness' in 552M(iii).

Note that Carlson 84 gives a rather sharper form of Theorem 552N, carrying information about the covering numbers of the measures constructed in $V^{\mathbb{P}_\kappa}$.

Version of 3.5.14

553 Random reals II

In this section I collect some further properties of random real models which seem less directly connected with the main topics of this book than those treated in §552. The first concerns strong measure zero or 'Rothberger's property' (534C[4]) and gives a bound for the sizes of sets with this property. The second relates perfect sets in $V^{\mathbb{P}_\kappa}$ to negligible F_σ sets in the original universe; it shows that a random real model can have properties relevant to a question in §531 (553F). Following these, I discuss properties of ultrafilters and partially ordered sets which are not obviously connected with measure theory, but where the arguments needed to establish the truth of sentences in $V^{\mathbb{P}_\kappa}$ involve interesting properties of measure algebras (553G-553M). I conclude with notes on medial limits (553N) and universally measurable sets (553O).

553A Notation I repeat some formulae from 552A. For any set I, ν_I will be the usual measure on $\{0,1\}^I$, T_I its domain, $\mathcal{N}(\nu_I)$ its null ideal and $(\mathfrak{B}_I, \bar{\nu}_I)$ its measure algebra. $\mathcal{B}\mathfrak{a}_I$ will be the Baire σ-algebra of $\{0,1\}^I$. For a cardinal κ, $\mathbb{P}_\kappa$ will be the forcing notion $\mathfrak{B}_\kappa^+$, active downwards.

553B Lemma If $A \in \mathcal{R}\mathsf{bg}(\{0,1\}^{\mathbb{N}})$, then for any $f:\mathbb{N}\to\mathbb{N}$ there is a sequence $\langle z_n\rangle_{n\in\mathbb{N}}$ such that $z_n \in \{0,1\}^{f(n)}$ for each n and $A \subseteq \bigcap_{n\in\mathbb{N}}\bigcup_{m\ge n}\{x : z_m \subseteq x \in \{0,1\}^{\mathbb{N}}\}$.

proof By 534Eb[5], A has strong measure zero with respect to the metric ρ on $\{0,1\}^{\mathbb{N}}$ defined by saying that

$$\rho(x,y) = 2^{-n} \text{ if } x\restriction n = y\restriction n \text{ and } x(n) \ne y(n).$$

So for each $n \in \mathbb{N}$ we can find a sequence $\langle A_{ni}\rangle_{i\in\mathbb{N}}$ of subsets of $\{0,1\}^{\mathbb{N}}$ such that $A \subseteq \bigcup_{i\in\mathbb{N}} A_{ni}$ and $\operatorname{diam} A_{ni} \le 2^{-f(2^n(2i+1))}$ for each i. Take $x_{ni} \in A_{ni}$ if A_{ni} is non-empty, any member of $\{0,1\}^{\mathbb{N}}$ otherwise, and set $z_m = x_{ni}\restriction f(2^n(2i+1))$ if $m = 2^n(2i+1)$; take z_0 to be any member of $\{0,1\}^{f(0)}$. Then $z_m \in \{0,1\}^{f(m)}$ for every m, and if $n \in \mathbb{N}$ then

$$\begin{aligned} A \subseteq \bigcup_{i\in\mathbb{N}} A_{ni} &\subseteq \bigcup_{i\in\mathbb{N}}\{x : \rho(x,x_{ni}) \le 2^{-f(2^n(2i+1))}\} \\ &= \bigcup_{i\in\mathbb{N}}\{x : x \supseteq z_{2^n(2i+1)}\} \subseteq \bigcup_{m\ge n}\{x : x \supseteq z_m\}, \end{aligned}$$

as required.

[4]Formerly 534C, 534E.

[5]Formerly 534Fd.

553C Proposition Let κ be any cardinal. Then

$$\Vdash_{\mathbb{P}_\kappa} \#(A) \le \check{\mathfrak{c}} \text{ for every } A \in \mathcal{R}\mathrm{bg}(\{0,1\}^{\mathbb{N}}).$$

proof (See Bartoszyński & Judah 95, 8.2.11.)

(a) Let $\dot{A}$ be a $\mathbb{P}_\kappa$-name for a member of $\mathcal{R}\mathrm{bg}(\{0,1\}^{\mathbb{N}})$. Take any $f \in \mathbb{N}^{\mathbb{N}}$. Applying 553B in the forcing language, we must have

$\Vdash_{\mathbb{P}_\kappa}$ there is a sequence $\langle z_n\rangle_{n\in\mathbb{N}}$ such that $z_n \in \{0,1\}^{\check{f}(n)}$ for every $n \in \mathbb{N}$ and $\dot{A} \subseteq \bigcap_{n\in\mathbb{N}}\bigcup_{m\ge n}\{x : x \supseteq z_n\}$.

Let $\langle \dot{z}_f(n)\rangle_{n\in\mathbb{N}}$ be a sequence of $\mathbb{P}_\kappa$-names such that

$$\Vdash_{\mathbb{P}_\kappa} \dot{z}_f(n) \in \{0,1\}^{\check{f}(n)} \text{ for every } n \in \mathbb{N},$$

$$\Vdash_{\mathbb{P}_\kappa} \dot{A} \subseteq \bigcap_{n\in\mathbb{N}}\bigcup_{m\ge n}\{x : x \supseteq \dot{z}_f(n)\}.$$

(b) Let $\langle e_\xi\rangle_{\xi<\kappa}$ be the standard generating family in $\mathfrak{B}_\kappa$ (525A). Let $J \subseteq \kappa$ be a set with cardinal at most $\mathfrak{c}$ such that $[\![\dot{z}_f(n) = \check{z}]\!]$ belongs to the closed subalgebra $\mathfrak{C}_J$ generated by $\{e_\xi : \xi \in J\}$ for every $f \in \mathbb{N}^{\mathbb{N}}$ and $z \in \bigcup_{n\in\mathbb{N}}\{0,1\}^{f(n)}$. Let $P_J : L^\infty(\mathfrak{B}_\kappa) \to L^\infty(\mathfrak{C}_J)$ be the conditional expectation operator (365Q[6]).

Observe that $\mathfrak{C}_J$ and $\mathfrak{C}_J^{\mathbb{N}}$ have cardinal at most $\mathfrak{c}^\omega = \mathfrak{c}$. So we have a family $\langle \dot{y}_\eta\rangle_{\eta<\mathfrak{c}}$ of $\mathbb{P}_\kappa$-names for members of $\{0,1\}^{\mathbb{N}}$ such that whenever $\langle d_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{C}_J$ there is an $\eta < \mathfrak{c}$ such that $[\![\dot{y}_\eta(\check{n}) = 1]\!] = d_n$ for every $n \in \mathbb{N}$.

(c) Let $\dot{x}$ be a $\mathbb{P}_\kappa$-name for a member of $\{0,1\}^{\mathbb{N}}$, and suppose that $a = [\![\dot{x} \in \dot{A}]\!]$ is non-zero. For $z \in \bigcup_{n\in\mathbb{N}}\{0,1\}^n$ set $b_z = [\![\dot{x} \supseteq \check{z}]\!]$. For m, $n \in \mathbb{N}$, set

$$c_{nm} = \sup_{z\in\{0,1\}^m} [\![P_J(\chi(a \cap b_z)) > 2^{-n}]\!] \in \mathfrak{C}_J.$$

Note that if $k \le m$ and $z \in \{0,1\}^m$ then $b_z \subseteq b_{z\restriction k}$, so

$$[\![P_J(\chi(a \cap b_z)) > 2^{-n}]\!] \subseteq [\![P_J(\chi(a \cap b_{z\restriction k})) > 2^{-n}]\!],$$

and $c_{nm} \subseteq c_{nk}$; set $c_n = \inf_{m\in\mathbb{N}} c_{nm}$. **?** Suppose, if possible, that $c_n = 0$ for every n. Let $f \in \mathbb{N}^{\mathbb{N}}$ be such that $\sum_{n=0}^\infty \bar{\nu}_\kappa c_{n,f(n)} < \bar{\nu}_\kappa a$, and set

$$d = \sup_{n\in\mathbb{N}}(\sup_{z\in\{0,1\}^{f(n)}} [\![P_J(\chi(a \cap b_z)) > 2^{-n}]\!]) \in \mathfrak{C}_J.$$

Then $\bar{\nu}_\kappa d < \bar{\nu}_\kappa a$, so $a \setminus d \ne 0$; while

$$\bar{\nu}_\kappa((a \setminus d) \cap [\![\dot{x} \supseteq \dot{z}_f(n)]\!]) = \sum_{z\in\{0,1\}^{f(n)}} \bar{\nu}_\kappa(a \cap b_z \cap [\![\dot{z}_f(n) = \check{z}]\!] \setminus d)$$

(because $\Vdash_{\mathbb{P}_\kappa} \dot{z}_f(n) \in \{0,1\}^{f(n)\check{}}$)

$$= \sum_{z\in\{0,1\}^{f(n)}} \int_{[\![\dot{z}_f(n)=\check{z}]\!]\setminus d} P_J(\chi(a \cap b_z))$$

$$\le 2^{-n} \sum_{z\in\{0,1\}^{f(n)}} \bar{\nu}_\kappa[\![\dot{z}_f(n) = \check{z}]\!]$$

(because d includes $[\![P_J(\chi(a \cap b_z)) > 2^{-n}]\!]$ for every $z \in \{0,1\}^{f(n)}$)

$$= 2^{-n}$$

for every n. Consequently

$$(a \setminus d) \cap \inf_{n\in\mathbb{N}} \sup_{m\ge n} [\![\dot{x} \supseteq \dot{z}_f(m)]\!] = 0,$$

that is,

$$a \setminus d \Vdash_{\mathbb{P}_\kappa} \dot{x} \supseteq \dot{z}_f(n) \text{ for only finitely many } n.$$

[6] Formerly 365R.

But this implies that

$$a \setminus d \Vdash_{\mathbb{P}_\kappa} \dot{x} \notin \dot{A},$$

contrary to hypothesis. **X**

(d) Continuing from (c), we find that there are an $a' \in \mathfrak{B}_\kappa^+$, stronger than a, and an $\eta < \mathfrak{c}$ such that $a' \Vdash_{\mathbb{P}_\kappa} \dot{x} = \dot{y}_\eta$.

P Let $n \in \mathbb{N}$ be such that c_n, as defined in (c), is non-zero. For $m \in \mathbb{N}$ and $w \in \{0,1\}^m$, set

$$d_w = \inf_{k \ge m} \sup_{z \in \{0,1\}^k, z \supseteq w} [\![P_J(\chi(a \cap b_z)) > 2^{-n}]\!] \in \mathfrak{C}_J.$$

Then

$$\sup_{w \in \{0,1\}^m} d_w = \inf_{k \ge m} \sup_{z \in \{0,1\}^k} [\![P_J(\chi(a \cap b_z)) > 2^{-n}]\!] = c_n$$

because

$$\langle \sup_{z \in \{0,1\}^k, z \supseteq w} [\![P_J(\chi(a \cap b_z)) > 2^{-n}]\!] \rangle_{k \ge m}$$

is non-increasing for each w. In particular, $d_\emptyset = c_n$. Also $d_w = d_{w \cup \{(m,0)\}} \cup d_{w \cup \{(m,1)\}}$ for every $w \in \{0,1\}^m$. So if we set

$$d'_\emptyset = 1,$$

$$d'_{w \cup \{(m,0)\}} = d'_w \cap d_{w \cup \{(m,0)\}}, \quad d'_{w \cup \{(m,1)\}} = d'_w \setminus d_{w \cup \{(m,0)\}}$$

for every $m \in \mathbb{N}$ and $w \in \{0,1\}^m$, $d'_w \in \mathfrak{C}_J$ and $d_w = d'_w \cap c_n$ for every w, and there must be an $\eta < \mathfrak{c}$ such that

$$[\![\dot{y}_\eta(\check{n}) = 1]\!] = \sup_{w \in \{0,1\}^{n+1}, w(n)=1} d'_w$$

for every $n \in \mathbb{N}$, in which case

$$[\![\check{w} \subseteq \dot{y}_\eta]\!] = d'_w \text{ for every } w \in \bigcup_{n \in \mathbb{N}} \{0,1\}^n.$$

If $m \in \mathbb{N}$, then

$$\begin{aligned} \bar{\nu}_\kappa(a \cap [\![\dot{x} \restriction \check{m} = \dot{y}_\eta \restriction \check{m}]\!]) &= \sum_{w \in \{0,1\}^m} \bar{\nu}_\kappa(a \cap b_w \cap d'_w) \\ &= \sum_{w \in \{0,1\}^m} \int_{d'_w} P_J(\chi(a \cap b_w)) \ge \sum_{w \in \{0,1\}^m} 2^{-n} \bar{\nu}_\kappa(c_n \cap d'_w) \end{aligned}$$

(because $c_n \cap d'_w = d_w \subseteq [\![P_J(\chi(a \cap b_w)) > 2^{-n}]\!]$ for every w)

$$= 2^{-n} \bar{\nu}_\kappa c_n.$$

So if we set $a' = a \cap [\![\dot{x} = \dot{y}_\eta]\!]$, then

$$\begin{aligned} \bar{\nu}_\kappa a' &= \bar{\nu}_\kappa(\inf_{m \in \mathbb{N}} a \cap [\![\dot{x} \restriction \check{m} = \dot{y}_\eta \restriction \check{m}]\!]) \\ &= \inf_{m \in \mathbb{N}} \bar{\nu}_\kappa(a \cap [\![\dot{x} \restriction \check{m} = \dot{y}_\eta \restriction \check{m}]\!]) \ge 2^{-n} \bar{\nu}_\kappa c_n > 0, \end{aligned}$$

and $a' \ne 0$, while $a' \subseteq a$ and $a' \Vdash_{\mathbb{P}_\kappa} \dot{x} = \dot{y}_\eta$. So we have a suitable pair a', η. **Q**

(e) Putting (c) and (d) together, we see that for any name $\dot{x}$ for a member of $\{0,1\}^{\mathbb{N}}$,

$$[\![\dot{x} \in \dot{A}]\!] \subseteq \sup_{\eta < \mathfrak{c}} [\![\dot{x} = \dot{y}_\eta]\!].$$

But this means that

$$\Vdash_{\mathbb{P}_\kappa} \dot{A} \subseteq \{\dot{y}_\eta : \eta < \check{\mathfrak{c}}\}$$

and

$$\Vdash_{\mathbb{P}_\kappa} \#(\dot{A}) \le \check{\mathfrak{c}}.$$

553D Remark If $\kappa > \mathfrak{c}$ then

$\Vdash_{\mathbb{P}_\kappa}$ for any countable family of subsets of $\{0,1\}^\omega$ there is an extension of ν_ω measuring every member of the family

(552N). By 552Oa and 552Gc we see that in this case

$\Vdash_{\mathbb{P}_\kappa}$ any universally negligible subset of $\{0,1\}^\omega$ has cardinal less than $\check{\mathfrak{c}}$.

The proposition here tells us that

$\Vdash_{\mathbb{P}_\kappa}$ any subset of $\{0,1\}^\omega$ with strong measure zero has cardinal at most $\check{\mathfrak{c}}$

without restriction on κ.

553E Proposition Let κ and λ be infinite cardinals, and $\dot{K}$ a $\mathbb{P}_\kappa$-name such that

$$\Vdash_{\mathbb{P}_\kappa} \dot{K} \text{ is a compact subset of } \{0,1\}^{\check{\lambda}} \text{ which is not scattered.}$$

Then there is a negligible F_σ set $G \subseteq \{0,1\}^\lambda$ such that

$$\Vdash_{\mathbb{P}_\kappa} \dot{K} \cap \tilde{G} \neq \emptyset$$

where $\tilde{G} = (\{0,1\}^\kappa \times G)^{\vec{}}$ is the $\mathbb{P}_\kappa$-name for an F_σ set in $\{0,1\}^{\check{\lambda}}$ corresponding to G as described in 551K.

proof (a) If $a \in \mathfrak{B}_\kappa^+$, $\dot{A}$ is a $\mathbb{P}_\kappa$-name such that

$$a \Vdash_{\mathbb{P}_\kappa} \dot{A} \text{ is an infinite subset of } \{0,1\}^{\check{\lambda}}$$

and $\epsilon > 0$, then there is an open-and-closed subset H of $\{0,1\}^\lambda$ such that $\nu_\lambda H \le \epsilon$ and $\bar{\nu}_\kappa(a \cap [\![\dot{A} \cap \tilde{H} = \emptyset]\!]) \le \epsilon$. **P** We may suppose that $\epsilon = 2^{-k}$ for some $k \in \mathbb{N}$. Let $\langle \dot{y}_i \rangle_{i\in\mathbb{N}}$ be a sequence of $\mathbb{P}_\kappa$-names such that

$$a \Vdash_{\mathbb{P}_\kappa} \dot{y}_i \in \dot{A} \text{ and } \dot{y}_i \neq \dot{y}_j$$

whenever i, $j \in \mathbb{N}$ are distinct. Let $N \in \mathbb{N}$ be so large that $e^{-\epsilon N} < \frac{1}{2}\epsilon$. Then

$$a \Vdash_{\mathbb{P}_\kappa} \text{there is a finite } J \subseteq \check{\lambda} \text{ such that } \dot{y}_i{\restriction}J \neq \dot{y}_j{\restriction}J \text{ whenever } i < j < \check{N},$$

that is,

$$\sup_{J\in[\lambda]^{<\omega}} [\![\dot{y}_i{\restriction}\check{J} \neq \dot{y}_j{\restriction}\check{J} \text{ whenever } i < j < \check{N}]\!] \supseteq a,$$

and there is a finite set $J \subseteq \lambda$ such that

$$\bar{\nu}_\kappa(a \setminus [\![\dot{y}_i{\restriction}\check{J} \neq \dot{y}_j{\restriction}\check{J} \text{ whenever } i < j < \check{N}]\!]) \le \frac{1}{2}\epsilon;$$

enlarging J if necessary, we can suppose that $m = \#(J)$ is such that $m \ge k$ and $(1 - \frac{N}{2^m})^{2^m\epsilon} \le \frac{1}{2}\epsilon$.

For each $i \in \mathbb{N}$, let $f_i : \{0,1\}^\kappa \to \{0,1\}^\lambda$ be a $(\mathrm{T}_\kappa, \mathcal{B}\mathfrak{a}_\lambda)$-measurable function such that (in the language of 551Cc) $a \Vdash_{\mathbb{P}_\kappa} \dot{y}_i = \vec{f_i}$. Set

$$E = \{x : f_i(x){\restriction}J \neq f_j(x){\restriction}J \text{ whenever } i < j < N\};$$

then

$$E^\bullet = [\![\dot{y}_i{\restriction}\check{J} \neq \dot{y}_j{\restriction}\check{J} \text{ whenever } i < j < \check{N}]\!],$$

and $\bar{\nu}_\kappa(a \setminus E^\bullet) \le \frac{1}{2}\epsilon$.

Let L be a subset of $\{0,1\}^J$ obtained by a stochastic process in which we pick $2^{m-k} = 2^m\epsilon$ points independently with the uniform distribution, and take L to be the set of these points. For any $x \in E$,

$$\Pr(f_i(x){\restriction}J \notin L\ \forall\, i < N) = \Pr((L \cap \{f_i(x){\restriction}J : i < N\}) = \emptyset) = (1 - \frac{N}{2^m})^{2^m\epsilon} \le \frac{1}{2}\epsilon.$$

By Fubini's theorem, there must be an $L \subseteq \{0,1\}^J$ such that $\#(L) \le 2^m\epsilon$ and

$$\nu_\kappa\{x : x \in E,\ f_i(x){\restriction}J \notin L\ \forall\, i < N\} \le \frac{1}{2}\epsilon\nu_\kappa E \le \frac{1}{2}\epsilon.$$

Set $H = \{y : y \in \{0,1\}^\lambda, y{\restriction}J \in L\}$ and $b = [\![\dot{A} \cap \tilde{H} = \emptyset]\!]$. Then H is open-and-closed, $\nu_\lambda H = 2^{-m}\#(L) \le \epsilon$ and

$$\begin{aligned} a\cap b &\subseteq a\cap [\![\dot{y}_i\notin\tilde{H}\ \forall\, i<\check{N}]\!] = a\cap\{x: f_i(x)\notin H\ \forall\, i<N\}^\bullet \\ &= a\cap\{x: f_i(x)\restriction J\notin L \text{ for every } i<N\}^\bullet \\ &\subseteq (a\setminus E^\bullet)\cup\{x: x\in E,\ f_i(x)\restriction J\notin L \text{ for every } i<N\}^\bullet \end{aligned}$$

has measure at most ϵ, as required. **Q**

(b) Because every non-scattered space has a non-empty closed subset with no isolated points, we may suppose that

$$\Vdash_{\mathbb{P}_\kappa} \dot{K} \text{ has no isolated points.}$$

For any $\epsilon\in\,]0,1[$ there is a compact negligible set $F\subseteq\{0,1\}^\lambda$ such that

$$\bar{\nu}_\kappa[\![\dot{K}\cap\tilde{F}\ne\emptyset]\!]\ge 1-\epsilon.$$

P Choose $\langle a_n\rangle_{n\in\mathbb{N}}$, $\langle H_n\rangle_{n\in\mathbb{N}}$ and $\langle\dot{K}_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. $a_0=1$ and $\dot{K}_0=\dot{K}$. Given that

$$a_n\Vdash_{\mathbb{P}_\kappa}\dot{K}_n \text{ is a non-empty compact set in } \{0,1\}^{\check{\lambda}} \text{ without isolated points}$$

and $\bar{\nu}_\kappa a_n>1-\epsilon$, let $H_n\subseteq\{0,1\}^\lambda$ be an open-and-closed set of measure at most 2^{-n} such that $a_{n+1}=a_n\cap[\![\dot{K}_n\cap\tilde{H}_n\ne\emptyset]\!]$ has measure greater than $1-\epsilon$. Now let $\dot{K}_{n+1}$ be a $\mathbb{P}_\kappa$-name such that $\Vdash_{\mathbb{P}_\kappa}\dot{K}_{n+1}=\dot{K}_n\cap\tilde{H}_n$. Because

$a_{n+1}\Vdash_{\mathbb{P}_\kappa}\dot{K}_n$ is a compact set without isolated points, $\tilde{H}_n$ is open-and-closed and $\dot{K}_n\cap\tilde{H}_n\ne\emptyset$, so $\dot{K}_{n+1}$ is a non-empty compact set without isolated points,

the induction continues.

At the end of the induction, set $F=\bigcap_{n\in\mathbb{N}}H_n$ and $a=\inf_{n\in\mathbb{N}}a_n$. Then $\bar{\nu}_\kappa a\ge 1-\epsilon$ and $\Vdash_{\mathbb{P}_\kappa}\tilde{F}=\bigcap_{n\in\mathbb{N}}\tilde{H}_n$, so

$a\Vdash_{\mathbb{P}_\kappa}\tilde{F}\cap\dot{K}$ is the intersection of the non-increasing sequence $\langle\dot{K}_n\rangle_{n\in\mathbb{N}}$ of non-empty compact sets, so is not empty.

So

$$\bar{\nu}_\kappa[\![\tilde{F}\cap\dot{K}\ne\emptyset]\!]\ge\bar{\nu}_\kappa a\ge 1-\epsilon.$$

Also, of course, $\nu_\kappa F=0$, as required. **Q**

(c) Finally, let $\langle F_n\rangle_{n\in\mathbb{N}}$ be a sequence of compact negligible sets such that

$$\bar{\nu}_\kappa[\![\dot{K}\cap\tilde{F}_n\ne\emptyset]\!]\ge 1-2^{-n}$$

for every n, and set $G=\bigcup_{n\in\mathbb{N}}F_n$; this works.

553F Corollary Suppose that $\operatorname{cf}\mathcal{N}(\nu_\omega)=\omega_1$ and that $\kappa\ge\omega_2$ is a cardinal. Then

$\Vdash_{\mathbb{P}_\kappa}\omega_1$ is a precaliber of every measurable algebra but does not have Haydon's property.

proof By 523N,

$$\operatorname{cf}\mathcal{N}(\nu_{\omega_1})=\max(\operatorname{cf}\mathcal{N}(\nu_\omega),\operatorname{cf}[\omega_1]^{\le\omega})=\omega_1;$$

let $\langle H_\xi\rangle_{\xi<\omega_1}$ be a cofinal family in $\mathcal{N}(\nu_{\omega_1})$. Now 552Ga and 525J tell us that

$\Vdash_{\mathbb{P}_\kappa}\operatorname{cov}\mathcal{N}(\nu_\lambda)>\omega_1$ for every infinite cardinal λ, so ω_1 is a precaliber of every measurable algebra.

Next, defining $\tilde{H}_\xi$ from H_ξ as in 551K and 553E,

$$\Vdash_{\mathbb{P}_\kappa}\tilde{H}_\xi\in\mathcal{N}(\nu_{\omega_1}) \text{ for every } \xi<\omega_1$$

(551Kd; remember that in this context we do not need to distinguish between ω_1 and $\check{\omega}_1$, by 5A3Nb), while

$$\Vdash_{\mathbb{P}_\kappa} \text{if } K\subseteq\{0,1\}^{\omega_1} \text{ is a non-scattered compact set then } K \text{ meets } \textstyle\bigcup_{\xi<\omega_1}\tilde{H}_\xi.$$

P Suppose that $a\in\mathfrak{B}_\kappa^+$ and $\dot{K}$ is a $\mathbb{P}_\kappa$-name such that

$$a\Vdash_{\mathbb{P}_\kappa}\dot{K}\subseteq\{0,1\}^{\omega_1} \text{ is a non-scattered compact set.}$$

If $a = 1$ set $\dot{K}' = \dot{K}$; otherwise let $\dot{K}'$ be a $\mathbb{P}_\kappa$-name such that

$$a \Vdash_{\mathbb{P}_\kappa} \dot{K}' = \dot{K}, \quad 1 \setminus a \Vdash_{\mathbb{P}_\kappa} \dot{K}' = \{0,1\}^{\omega_1}.$$

By 553E, there is a negligible set $G \subseteq \{0,1\}^{\omega_1}$ such that $\Vdash_{\mathbb{P}_\kappa} \dot{K}' \cap \tilde{G} \neq \emptyset$. Now there is a $\xi < \omega_1$ such that $G \subseteq H_\xi$, so that

$$a \Vdash_{\mathbb{P}_\kappa} \dot{K} \cap \tilde{H}_\xi \supseteq \dot{K}' \cap \tilde{G} \neq \emptyset. \quad \mathbf{Q}$$

By 531Vb,

$$\Vdash_{\mathbb{P}_\kappa} \omega_1 \text{ does not have Haydon's property.}$$

553G Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, $\mathfrak{C}$ a subalgebra of $\mathfrak{A}$, and $\langle e_n \rangle_{n \in \mathbb{N}}$ a sequence in $\mathfrak{A}$ stochastically independent of each other and of $\mathfrak{C}$. Let $I \subseteq \mathfrak{A}$ be a finite set and $\mathfrak{C}_I$ the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C} \cup I$. Then for every $\epsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that $|\bar{\mu}(b \cap e_n) - \bar{\mu}b \cdot \bar{\mu}e_n| \leq \epsilon\bar{\mu}b$ whenever $b \in \mathfrak{C}_I$ and $n \geq n_0$.

proof (a) The first step is to show that if $u \in L^1(\mathfrak{A}, \bar{\mu})$ then

for every $\epsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that $|\int_{b \cap e_n} u - \bar{\mu}e_n \cdot \int_b u| \leq \epsilon \int |u|$ whenever $b \in \mathfrak{C}$ and $n \geq n_0$.

P Consider the set U of those $u \in L^1(\mathfrak{A}, \bar{\mu})$ for which this true. This is a linear subspace of $L^1(\mathfrak{A}, \bar{\mu})$. Also it is $\| \|_1$-closed, because if $\int |v - u| \leq \frac{1}{3} \int |u|$ and $|\int_{b \cap e_n} v - \bar{\mu}e_n \cdot \int_b v| \leq \frac{1}{4}\epsilon \int |v|$ then $|\int_{b \cap e_n} u - \bar{\mu}e_n \cdot \int_b u| \leq \epsilon \int |u|$. If we take $\mathfrak{D}_m$ to be the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C} \cup \{e_n : n \leq m\}$, then $\bar{\mu}(a \cap b \cap e_n) = \bar{\mu}(a \cap b) \cdot \bar{\mu}e_n$ whenever $a \in \mathfrak{D}_m$, $b \in \mathfrak{C}$ and $n \geq m$, so $\chi a \in U$ for every $a \in \mathfrak{D}_m$. Consequently $\chi a \in U$ for every $a \in \mathfrak{D}$, where $\mathfrak{D}$ is the metric closure of $\bigcup_{m \in \mathbb{N}} \mathfrak{D}_m$ in $\mathfrak{A}$. Identifying $L^1(\mathfrak{D}, \bar{\mu} \restriction \mathfrak{D})$ with the closed linear subspace of $L^1(\mathfrak{A}, \bar{\mu})$ generated by $\{\chi a : a \in \mathfrak{D}\}$ (365Q, 365F), we see that $U \supseteq L^1(\mathfrak{D}, \bar{\mu} \restriction \mathfrak{D})$. Now suppose that u is any member of $L^1(\mathfrak{A}, \bar{\mu})$. Then we have a conditional expectation Pu of u in $L^1(\mathfrak{D}, \bar{\mu} \restriction \mathfrak{D})$ (365Q), and

$$|\int_{b \cap e_n} u - \bar{\mu}b \cdot \int_b u| = |\int_{b \cap e_n} Pu - \bar{\mu}e_n \cdot \int_b Pu|$$

for every $b \in \mathfrak{C}$ and $n \in \mathbb{N}$, while $|Pu| \leq P|u|$, so $u \in U$ because $Pu \in U$. **Q**

(b) I show now, by induction on $\#(I)$, that if $a \in \mathfrak{A}$ then

for every $\epsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that $|\bar{\mu}(a \cap b \cap e_n) - \bar{\mu}(a \cap b) \cdot \bar{\mu}e_n| \leq \epsilon\mu a$ whenever $b \in \mathfrak{C}_I$ and $n \geq n_0$.

P If I is empty, we can apply (a) with $u = \chi a$. For the inductive step to $\#(I) = k + 1$, express I as $J \cup \{c\}$ where $\#(J) = k$. Take $a \in \mathfrak{A}$. Let $n_0 \in \mathbb{N}$ be such that

$$|\bar{\mu}((a \cap c) \cap b \cap e_n) - \bar{\mu}((a \cap c) \cap b) \cdot \bar{\mu}e_n| \leq \epsilon\mu(a \cap c),$$

$$|\bar{\mu}((a \setminus c) \cap b \cap e_n) - \bar{\mu}((a \setminus c) \cap b) \cdot \bar{\mu}e_n| \leq \epsilon\mu(a \setminus c)$$

whenever $b \in \mathfrak{C}_J$ and $n \geq n_0$. Now take $b \in \mathfrak{C}_I$ and $n \geq n_0$. There are b', $b'' \in \mathfrak{C}_J$ such that $b = (b' \cap c) \cup (b'' \setminus c)$, so that

$$\begin{aligned}
|\bar{\mu}(a \cap b \cap e_n) - \bar{\mu}(a \cap b) \cdot \bar{\mu}e_n| &= |\bar{\mu}((a \cap c) \cap b' \cap e_n) - \bar{\mu}((a \cap c) \cap b') \cdot \bar{\mu}e_n \\
&\qquad + \bar{\mu}((a \setminus c) \cap b'' \cap e_n) - \bar{\mu}((a \setminus c) \cap b'') \cdot \bar{\mu}e_n| \\
&\leq |\bar{\mu}((a \cap c) \cap b' \cap e_n) - \bar{\mu}((a \cap c) \cap b') \cdot \bar{\mu}e_n| \\
&\qquad + |\bar{\mu}((a \setminus c) \cap b'' \cap e_n) - \bar{\mu}((a \setminus c) \cap b'') \cdot \bar{\mu}e_n| \\
&\leq \epsilon\bar{\mu}(a \cap c) + \epsilon\bar{\mu}(a \setminus c) = \epsilon\bar{\mu}a.
\end{aligned}$$

Thus the induction proceeds. **Q**

(c) Now the result as stated is just the case $a = 1$ in (b).

553H Theorem If $\kappa > \mathfrak{c}$, then

$$\Vdash_{\mathbb{P}_\kappa} \text{there are no rapid } p\text{-point ultrafilters, therefore no Ramsey filters on } \mathbb{N}.$$

proof (See JECH 78, §38.)

(a) Let $\langle e_{\xi n}\rangle_{\xi<\kappa,n\in\mathbb{N}}$ be a re-indexing of the standard generating family in $\mathfrak{B}_\kappa$. Let $\dot{\mathcal{F}}$ be a $\mathbb{P}_\kappa$-name for an ultrafilter, and set $\hat{a} = [\![\dot{\mathcal{F}}$ is a rapid p-point ultrafilter$]\!]$. **?** Suppose, if possible, that $\hat{a} \ne 0$. For each $f \in \mathbb{N}^\mathbb{N}$,

$$\hat{a} \Vdash_{\mathbb{P}_\kappa} \text{there is a } D \in \dot{\mathcal{F}} \text{ such that } \#(D \cap \check{f}(k)) \le k \text{ for every } k$$

(538Ad); let $\dot{D}_f$ be a $\mathbb{P}_\kappa$-name for a subset of $\mathbb{N}$ such that

$$\hat{a} \Vdash_{\mathbb{P}_\kappa} \dot{D}_f \in \dot{\mathcal{F}}$$

and

$$\hat{a} \Vdash_{\mathbb{P}_\kappa} \#(\dot{D}_f \cap f(k)\check{}\,) \le \check{k}$$

for every $k \in \mathbb{N}$. Let $J \subseteq \kappa$ be a set with cardinal at most $\mathfrak{c}$ such that $[\![\check{n} \in \dot{D}_f]\!]$ belongs to the closed subalgebra $\mathfrak{C}$ generated by $\{e_{\xi i} : \xi \in J,\ i \in \mathbb{N}\}$ for every $f \in \mathbb{N}^\mathbb{N}$ and every $n \in \mathbb{N}$, and $\hat{a}$ also belongs to $\mathfrak{C}$.

(b) Let $\zeta < \kappa$ be such that the ordinal sum $\zeta + k$ does not belong to J for any $k \in \mathbb{N}$. For each $k \in \mathbb{N}$ let $\dot{C}_k$ be a $\mathbb{P}_\kappa$-name for a subset of $\mathbb{N}$ such that $[\![\check{n} \in \dot{C}_k]\!] = e_{\zeta+k,n}$ for every $n \in \mathbb{N}$. Set $c_k = [\![\dot{C}_k \notin \dot{\mathcal{F}}]\!]$ and let $\dot{A}_k$ be a $\mathbb{P}_\kappa$-name for a subset of $\mathbb{N}$ such that

$$c_k \Vdash_{\mathbb{P}_\kappa} \dot{A}_k = \mathbb{N} \setminus \dot{C}_k \in \dot{\mathcal{F}}, \quad 1 \setminus c_k \Vdash_{\mathbb{P}_\kappa} \dot{A}_k = \dot{C}_k \in \dot{\mathcal{F}}.$$

Then $\Vdash_{\mathbb{P}_\kappa} \dot{A}_k \in \dot{\mathcal{F}}$ for every k, and $[\![\check{n} \in \dot{A}_k]\!] = c_k \mathbin{\triangle} e_{\zeta+k,n}$ for every $n \in \mathbb{N}$.

(c) For k, $n \in \mathbb{N}$ set

$$b_{kn} = [\![\check{n} \in \textstyle\bigcap_{i<\check{k}} \dot{A}_i]\!] = \inf_{i<k} [\![\check{n} \in \dot{A}_i]\!] = \inf_{i<k} c_i \mathbin{\triangle} e_{\zeta+i,n}.$$

Then we have a non-decreasing $f : \mathbb{N} \to \mathbb{N}$ such that $\bar{\nu}_\kappa(c \cap b_{kn}) \le (2^{-k+1} - 2^{-2k})\bar{\nu}_\kappa c$ whenever $c \in \mathfrak{C}$, $k \in \mathbb{N}$ and $n \ge f(k)$. **P** Define f inductively, as follows. If $k = 0$ then (interpreting $\inf\emptyset$ as 1) we have $b_{kn} = 1$ for every n so we can take $f(0) = 0$. For the inductive step to $k+1$, let $\mathfrak{C}_k$ be the closed subalgebra of $\mathfrak{B}_\kappa$ generated by $\mathfrak{C} \cup \{e_{\zeta+i,n} : i < k,\ n \in \mathbb{N}\}$ and $\mathfrak{D}_k$ the subalgebra generated by $\mathfrak{C}_k \cup \{c_i : i \le k\}$. Then $\mathfrak{C}_k$ and $\langle e_{\zeta+k,n}\rangle_{n\in\mathbb{N}}$ are stochastically independent, so Lemma 553G tells us that there is an $f(k+1) \ge f(k)$ such that

$$|\bar{\nu}_\kappa(d \cap e_{\zeta+k,n}) - \frac{1}{2}\bar{\nu}_\kappa d| \le \frac{1}{24} \cdot 2^{-k}\bar{\nu}_\kappa d \text{ whenever } d \in \mathfrak{D}_k \text{ and } n \ge f(k+1).$$

Take $n \ge f(k+1)$ and $c \in \mathfrak{C}$. Then

$$\begin{aligned}
\bar{\nu}_\kappa(c \cap b_{k+1,n}) &= \bar{\nu}_\kappa(c \cap b_{kn} \cap (c_k \mathbin{\triangle} e_{\zeta+k,n})) \\
&= \bar{\nu}_\kappa(c \cap b_{kn} \cap c_k) - 2\bar{\nu}_\kappa(c \cap b_{kn} \cap c_k \cap e_{\zeta+k,n}) + \bar{\nu}_\kappa(c \cap b_{kn} \cap e_{\zeta+k,n}) \\
&\le 2|\bar{\nu}_\kappa(c \cap b_{kn} \cap c_k \cap e_{\zeta+k,n}) - \frac{1}{2}\bar{\nu}_\kappa(c \cap b_{kn} \cap c_k)| \\
&\qquad + |\bar{\nu}_\kappa(c \cap b_{kn} \cap e_{\zeta+k,n}) - \frac{1}{2}\bar{\nu}_\kappa(c \cap b_{kn})| + \frac{1}{2}\bar{\nu}_\kappa(c \cap b_{kn}) \\
&\le \frac{1}{12} \cdot 2^{-k}\bar{\nu}_\kappa(c \cap b_{kn} \cap c_k) + \frac{1}{24} \cdot 2^{-k}\bar{\nu}_\kappa(c \cap b_{kn}) + \frac{1}{2}\bar{\nu}_\kappa(c \cap b_{kn})
\end{aligned}$$

(because all the elements $c \cap b_{kn}$ and $c \cap b_{kn} \cap c_k$ belong to $\mathfrak{D}_k$)

$$\begin{aligned}
&\le (2^{-k-3} + \frac{1}{2})\bar{\nu}_\kappa(c \cap b_{kn}) \\
&\le (2^{-k-3} + \frac{1}{2})(2^{-k+1} - 2^{-2k})\bar{\nu}_\kappa c
\end{aligned}$$

(because $n \ge f(k)$)

$$\le (2^{-k} - 2^{-2k-2})\bar{\nu}_\kappa c.$$

So the construction proceeds. **Q**

(d) Because

$$\hat{a} \Vdash_{\mathbb{P}_\kappa} \dot{\mathcal{F}} \text{ is a } p\text{-point ultrafilter and } \dot{A}_k \in \dot{\mathcal{F}} \text{ for every } k,$$

there are a $\mathbb{P}_\kappa$-name $\dot{A}$ for a subset of $\mathbb{N}$ and a $\mathbb{P}_\kappa$-name $\dot{g}$ for a function from $\mathbb{N}$ to itself such that

$$\hat{a} \Vdash_{\mathbb{P}_\kappa} \dot{A} \in \dot{\mathcal{F}} \text{ and } \dot{A} \setminus \dot{A}_i \subseteq \dot{g}(k) \text{ whenever } i < k \in \mathbb{N}.$$

Let g (in the ordinary universe) be a non-decreasing function such that $f(k) \le g(k)$ and $\bar{\nu}_\kappa([\![\dot{g}(\check{k}) > g(k)\check{}\,]\!])$ $\le 2^{-k-2}\bar{\nu}_\kappa\hat{a}$ for every k. Set $\hat{a}_1 = \hat{a} \cap [\![\dot{g} \le \check{g}]\!]$; then $\bar{\nu}_\kappa\hat{a}_1 \ge \frac{1}{2}\bar{\nu}_\kappa\hat{a}$.

(e) Take the function g from (d) and the name $\dot{D}_g$ from (a), and set $d_n = \hat{a} \cap [\![\check{n} \in \dot{D}_g]\!] \in \mathfrak{C}$ for every n. Then

$$\textstyle\sum_{n=g(k)}^{g(k+1)-1} \bar{\nu}_\kappa(d_n \cap b_{kn}) \le 2^{-k+1}(k+1)$$

for every $k \in \mathbb{N}$. **P** Set $K = g(k+1) \setminus g(k)$. We have

$$\begin{aligned}\sum_{n=g(k)}^{g(k+1)-1} \bar{\nu}_\kappa(d_n \cap b_{kn}) &= \sum_{n\in K}\sum_{I\subseteq K} \bar{\nu}_\kappa(d_n \cap b_{kn} \cap [\![\check{I} = \dot{D}_g \cap \check{K}]\!]) \\ &= \sum_{I\subseteq K}\sum_{n\in I} \bar{\nu}_\kappa(d_n \cap b_{kn} \cap [\![\check{I} = \dot{D}_g \cap \check{K}]\!])\end{aligned}$$

(because $d_n \cap [\![\check{I} = \dot{D}_g \cap \check{K}]\!] \subseteq [\![\check{n} \in \dot{D}_g]\!] \cap [\![\check{I} = \dot{D}_g \cap \check{K}]\!]$ is zero if $n \notin I$)

$$= \sum_{I\in[K]^{\le k+1}}\sum_{n\in I} \bar{\nu}_\kappa(d_n \cap b_{kn} \cap [\![\check{I} = \dot{D}_g \cap \check{K}]\!])$$

(because $d_n \cap [\![\check{I} = \dot{D}_g \cap K]\!] \subseteq \hat{a} \cap [\![\check{I} \subseteq \dot{D}_g \cap g(k+1)\check{}\,]\!]$ is zero if $\#(I) > k+1$)

$$\le 2^{-k+1} \sum_{I\in[K]^{\le k+1}}\sum_{n\in I} \bar{\nu}_\kappa(d_n \cap [\![\check{I} = \dot{D}_g \cap \check{K}]\!])$$

(because $d_n \cap [\![\check{I} = \dot{D}_g \cap \check{K}]\!] \in \mathfrak{C}$ for every n and I, and we are looking only at $n \ge g(k) \ge f(k)$)

$$\begin{aligned}&\le 2^{-k+1}(k+1) \sum_{I\in[K]^{\le k+1}} \bar{\nu}_\kappa[\![\check{I} = \dot{D}_g \cap \check{K}]\!] \\ &\le 2^{-k+1}(k+1). \textbf{ Q}\end{aligned}$$

(f) As $\hat{a} \ne 0$, $\hat{a}_1 \ne 0$. Let m be such that $\sum_{k=m}^\infty 2^{-k+1}(k+1)$ is less than $\bar{\nu}_\kappa\hat{a}_1$; then

$$\hat{a}_2 = \hat{a}_1 \setminus \sup_{k\ge m}\sup_{g(k)\le n<g(k+1)}(d_n \cap b_{kn})$$

is non-zero. Let $\dot{B}$ be a $\mathbb{P}_\kappa$-name for a subset of $\mathbb{N}$ such that $\Vdash_{\mathbb{P}_\kappa} \dot{B} = \dot{A} \cap \dot{D}_g \setminus g(m)\check{}$. Then $\hat{a} \Vdash_{\mathbb{P}_\kappa} \dot{B} \in \dot{\mathcal{F}}$. But $\hat{a}_2 \Vdash_{\mathbb{P}_\kappa} \dot{B} = \emptyset$. **P** Take any $n \in \mathbb{N}$. If $n < g(m)$ then $\Vdash_{\mathbb{P}_\kappa} n \notin \dot{B}$. If $k \ge m$ and $g(k) \le n < g(k+1)$, then

$$\hat{a}_1 \Vdash_{\mathbb{P}_\kappa} \dot{g}(\check{k}) \le g(k)\check{}, \quad \hat{a} \Vdash_{\mathbb{P}_\kappa} \dot{A} \setminus \dot{A}_i \subseteq \dot{g}(\check{k}) \text{ for every } i < \check{k},$$

so

$$\hat{a}_1 \cap [\![\check{n} \in \dot{B}]\!] \subseteq \hat{a} \cap [\![\check{n} \in \dot{A} \setminus \dot{g}(\check{k})]\!] \subseteq [\![\check{n} \in \textstyle\bigcap_{i<\check{k}} \dot{A}_i]\!] = b_{kn}.$$

Also, of course, $\Vdash_{\mathbb{P}_\kappa} \dot{B} \subseteq \dot{D}_g$, so $\hat{a}_1 \cap [\![\check{n} \in \dot{B}]\!] \subseteq d_n \cap b_{kn}$ is disjoint from $\hat{a}_2$. But this means that $\hat{a}_2 \Vdash_{\mathbb{P}_\kappa} \check{n} \notin \dot{B}$. As n is arbitrary, $\hat{a}_2 \Vdash_{\mathbb{P}_\kappa} \dot{B} = \emptyset$. **Q** Now

$$\hat{a}_2 \Vdash_{\mathbb{P}_\kappa} \emptyset \in \dot{\mathcal{F}},$$

which is impossible. **X**

(g) So $\hat{a} = 0$, that is,

$$\Vdash_{\mathbb{P}_\kappa} \dot{\mathcal{F}} \text{ is not a rapid } p\text{-point ultrafilter.}$$

As $\dot{\mathcal{F}}$ is arbitrary,

$$\Vdash_{\mathbb{P}_\kappa} \text{ there are no rapid } p\text{-point ultrafilters.}$$

(h) Finally, by 538Fa,

$$\Vdash_{\mathbb{P}_\kappa} \text{ there are no Ramsey ultrafilters.}$$

553I Lemma Suppose that $S \subseteq \omega_1^2$ is a set such that whenever $n \in \mathbb{N}$ and $\langle I_\xi \rangle_{\xi<\omega_1}$ is a family in $[\omega_1]^n$ such that $I_\xi \cap \xi = \emptyset$ for every $\xi < \omega_1$, there are $\xi < \omega_1$ and $\eta < \xi$ such that $I_\xi \times I_\eta \subseteq S$. Let P be the set

$$\{I : I \in [\omega_1]^{<\omega},\ I \cap \xi \subseteq S[\{\xi\}] \text{ for every } \xi \in I\},$$

ordered by $\subseteq$. Then P is upwards-ccc.

proof Let $\langle J_\xi \rangle_{\xi<\omega_1}$ be any family in P. Then there are distinct ξ, $\eta < \omega_1$ such that $J_\xi \cup J_\eta \in P$. **P** By the Δ-system Lemma (4A1Db), there is an uncountable set $A_0 \subseteq \omega_1$ such that $\langle J_\xi \rangle_{\xi \in A_0}$ is a Δ-system with root J say; next, there is an $n \in \mathbb{N}$ such that $A_1 = \{\xi : \xi \in A_0,\ \#(J_\xi \setminus J) = n\}$ is uncountable. If $n = 0$ then $J_\xi \cup J_\eta = J$ belongs to P for any ξ, $\eta \in A_1$ and we can stop. Otherwise, there is an uncountable $A_2 \subseteq A_1$ such that whenever ξ, $\eta \in A_2$ and $\eta < \xi$ then $\max J_\eta < \min(J_\xi \setminus J)$. Re-enumerate $\langle J_\xi \setminus J \rangle_{\xi \in A_2}$ in increasing order to get a family $\langle I_\xi \rangle_{\xi<\omega_1}$ in $[\omega_1]^n$ such that $\min I_\xi \ge \xi$ for every ξ. Our hypothesis tells us that there are $\eta < \xi$ such that $I_\xi \times I_\eta \subseteq S$. Let ξ', $\eta' < \omega_1$ be such that $I_\xi = J_{\xi'} \setminus J$ and $I_\eta = J_{\eta'} \setminus J$, and consider $I = J \cup I_\xi \cup I_\eta$. If $\alpha \in I$ and $\beta \in I \cap \alpha$,

—— *either* α, β both belong to $J_{\eta'}$ so $(\alpha, \beta) \in S$

—— *or* α, β both belong to $J_{\xi'}$ so $(\alpha, \beta) \in S$

—— *or* $\alpha \in I_\xi$ and $\beta \in I_\eta$ so $(\alpha, \beta) \in S$.

So $J_{\xi'} \cup J_{\eta'} = I$ belongs to S. **Q**

Thus P has no uncountable up-antichains and is upwards-ccc.

553J Theorem Let κ be an infinite cardinal. Then

$\Vdash_{\mathbb{P}_\kappa}$ there are two upwards-ccc partially ordered sets whose product is not upwards-ccc.

Remark If $\kappa > \omega$ this is immediate from 552E, 537F and 537G. So we have a new result only if $\kappa = \omega$.

proof (a) Let $\langle e_\xi \rangle_{\xi\in\kappa}$ be the standard generating family in $\mathfrak{B}_\kappa$. For $J \subseteq \kappa$ let $\mathfrak{C}_J$ be the closed subalgebra of $\mathfrak{B}_\kappa$ generated by $\{e_\xi : \xi \in J\}$. For $\xi < \omega_1$ let $h_\xi : \xi \to \mathbb{N}$ be an injective function.

(b) Let $\dot{S}_0$ be a $\mathbb{P}_\kappa$-name for a subset of ω_1^2 such that

$$\begin{aligned} [\![(\check{\xi}, \check{\eta}) \in \dot{S}_0]\!] &= e_{h_\xi(\eta)} \text{ if } \eta < \xi, \\ &= 0 \text{ otherwise.} \end{aligned}$$

Then

$\Vdash_{\mathbb{P}_\kappa}$ whenever $n \in \mathbb{N}$ and $\langle I_\xi \rangle_{\xi<\omega_1}$ is a family in $[\omega_1]^n$ such that $I_\xi \cap \xi = \emptyset$ for every $\xi < \omega_1$, there are $\xi < \omega_1$ and $\eta < \xi$ such that $I_\xi \times I_\eta \subseteq \dot{S}_0$.

P? Suppose, if possible, otherwise. Then we have an $n \in \mathbb{N}$, an $a \in \mathfrak{B}_\kappa^+$ and a family $\langle \dot{I}_\xi \rangle_{\xi<\omega_1}$ of $\mathbb{P}_\kappa$-names such that

$$a \Vdash_{\mathbb{P}_\kappa} \dot{I}_\eta \in [\omega_1 \setminus \eta]^{\check{n}} \text{ and } \dot{I}_\xi \times \dot{I}_\eta \not\subseteq \dot{S}_0 \text{ whenever } \eta < \xi < \omega_1.$$

For each $\xi < \omega_1$ there are $a_\xi \in \mathfrak{B}_\kappa^+$, stronger than a, and $I_\xi \in [\omega_1 \setminus \xi]^n$ such that $a_\xi \Vdash_{\mathbb{P}_\kappa} \dot{I}_\xi = \check{I}_\xi$. By 525Tc we can find an uncountable set $A_0 \subseteq \omega_1$ and an $\epsilon > 0$ such that $\bar{\nu}_\kappa(a_\xi \cap a_\eta) \ge \epsilon$ whenever ξ, $\eta \in A_0$. Next, there is an uncountable set $A_1 \subseteq A_0$ such that $I_\eta \subseteq \xi$ whenever $\xi \in A_1$ and $\eta \in \xi \cap A_1$; consequently $I_\eta \cap I_\xi = \emptyset$ whenever ξ, $\eta \in A_1$ are distinct, and $\bar{\nu}_\kappa[\![\check{I}_\xi \times \check{I}_\eta \subseteq \dot{S}_0]\!] = 2^{-n^2}$.

Let $\delta > 0$ be such that $2^{-n^2}(\epsilon - 2\delta) - 2\delta > 0$. For each $\xi \in A_1$ we can find a finite set $J_\xi \subseteq \kappa$ and an $a'_\xi \in \mathfrak{C}_{J_\xi}$ such that $\bar{\nu}_\kappa(a_\xi \mathbin{\triangle} a'_\xi) \le \delta$. Let $m \in \mathbb{N}$ be such that

$$A_2 = \{\xi : \xi \in A_1,\ J_\xi \cap \omega \subseteq m\}$$

is uncountable. Let $\xi \in A_2$ be such that $A_2 \cap \xi$ is infinite. In this case, $\langle h_\zeta[I_\eta]\rangle_{\eta\in A_2\cap\xi}$ is disjoint for each $\zeta \in I_\xi$, so we have an $\eta \in A_2 \cap \xi$ such that $h_\zeta[I_\eta] \cap m = \emptyset$ for every $\zeta \in I_\xi$. It follows that $[\![\check I_\xi \times \check I_\eta \subseteq \dot S_0]\!] \in \mathfrak{C}_{\omega\setminus m}$, while $a'_\xi \cap a'_\eta \in \mathfrak{C}_{m\cup(\kappa\setminus\omega)}$, so

$$\begin{aligned}\bar\nu_\kappa(a'_\xi \cap a'_\eta \cap [\![\check I_\xi \times \check I_\eta \subseteq \dot S_0]\!]) &= \bar\nu_\kappa(a'_\xi \cap a'_\eta)\cdot\bar\nu_\kappa[\![\check I_\xi \times \check I_\eta \subseteq \dot S_0]\!]\\ &= 2^{-n^2}\bar\nu_\kappa(a'_\xi \cap a'_\eta).\end{aligned}$$

If we set $b = a_\xi \cap a_\eta \cap [\![\check I_\xi \times \check I_\eta \subseteq \dot S_0]\!]$,

$$\begin{aligned}\bar\nu_\kappa b &\ge 2^{-n^2}\bar\nu_\kappa(a'_\xi \cap a'_\eta) - 2\delta \ge 2^{-n^2}(\bar\nu_\kappa(a_\xi \cap a_\eta) - 2\delta) - 2\delta\\ &\ge 2^{-n^2}(\epsilon - 2\delta) - 2\delta > 0.\end{aligned}$$

But now we have $b \subseteq a$ and

$$b \Vdash_{\mathbb{P}_\kappa} \dot I_\xi \times \dot I_\eta = \check I_\xi \times \check I_\eta \subseteq \dot S_0,$$

which is supposed to be impossible. **XQ**

(c) Let $\dot P_0$ be a $\mathbb{P}_\kappa$-name for a partially ordered set defined from $\dot S_0$ by the process of 553I, so that for a finite set $I \subseteq \omega_1$

$$[\![\check I \in \dot P_0]\!] = \inf_{\xi,\eta\in I,\eta<\xi} e_{h_\xi(\eta)}.$$

By 553I and (b) above,

$$\Vdash_{\mathbb{P}_\kappa} \dot P_0 \text{ is upwards-ccc.}$$

(d) Similarly, if $\dot S_1$ is a $\mathbb{P}_\kappa$-name for a subset of ω_1^2 such that

$$\begin{aligned}[\![(\check\xi,\check\eta) \in \dot S_1]\!] &= 1 \setminus e_{h_\xi(\eta)} \text{ if } \eta < \xi,\\ &= 0 \text{ otherwise,}\end{aligned}$$

and $\dot P_1$ is a $\mathbb{P}_\kappa$-name for a partially ordered set defined from $\dot S_1$ by the process of 553I, then

$$\Vdash_{\mathbb{P}_\kappa} \dot P_1 \text{ is upwards-ccc.}$$

(The point is just that $\langle 1 \setminus e_\xi\rangle_{\xi<\kappa}$ also is a stochastically independent family of elements of measure $\frac{1}{2}$.) But now observe that if $\eta < \xi < \omega_2$ then

$$[\![\{\check\xi,\check\eta\} \in \dot P_0 \cap \dot P_1]\!] = [\![(\check\xi,\check\eta) \in \dot S_0 \cap \dot S_1]\!] = e_{h_\xi(\eta)} \cap (1 \setminus e_{h_\xi(\eta)}) = 0.$$

So

$$\Vdash_{\mathbb{P}_\kappa} \{\{\{\xi\},\{\xi\}\} : \xi < \omega_1\} \text{ is an up-antichain in } \dot P_0 \times \dot P_1, \text{ and } \dot P_0 \times \dot P_1 \text{ is not upwards-ccc.}$$

Thus we have the required example.

553K I extract an elementary step from the proof of the next lemma.

Lemma Let $\mathfrak{A}$ be a Boolean algebra and $\nu : \mathfrak{A} \to [0,\infty[$ a non-negative additive functional. Then

$$\sum_{i=0}^n \nu a_i \le \nu(\sup_{i\le n} a_i) + \sum_{i<j\le n}\nu(a_i \cap a_j)$$

whenever $a_0, \dots, a_n \in \mathfrak{A}$.

proof Let d be any atom of the subalgebra of $\mathfrak{A}$ generated by $a_0, \dots, a_n$. Suppose that $\#(\{i : i \le n, d \subseteq a_i\}) = m$. Then

$$\begin{aligned}\nu(d \cap \sup_{i\le n} a_i) + \sum_{i<j\le n}\nu(d \cap a_i \cap a_j) - \sum_{i=0}^n \nu(d \cap a_i)&\\ = 0 \text{ if } m \le 1,&\\ = 1 + \frac{m(m-1)}{2} - m = \frac{1}{2}(m-1)(m-2) \ge 0 \text{ otherwise.}&\end{aligned}$$

Summing over d, we have the result.

553L Lemma Let $(\mathfrak{A},\bar\mu)$ be a probability algebra, I an uncountable set, X a non-empty set and $\mathcal{F}$ an ultrafilter on X. Let $\langle a_{ix}\rangle_{i\in I,x\in X}$ be a family in $\mathfrak{A}$ such that $\inf_{i\in I}\lim_{x\to\mathcal{F}}\bar\mu a_{ix}>0$. Then there are an uncountable set $S\subseteq I$ and a family $\langle b_i\rangle_{i\in S}$ in $\mathfrak{A}\setminus\{0\}$ such that

$$b_i\cap b_j\subseteq\sup_{x\in F}a_{ix}\cap a_{jx}$$

for all i, $j\in S$ and $F\in\mathcal{F}$.

proof (a) We can suppose that $I=\omega_1$. For each $\xi<\omega_1$ set $u_\xi=\lim_{x\to\mathcal{F}}\chi a_{\xi x}$, the limit being taken for the weak topology on $L^2(\mathfrak{A},\bar\mu)$ (§366), so that

$$\int_a u_\xi=\lim_{x\to\mathcal{F}}\bar\mu(a\cap a_{\xi x})$$

for every $a\in\mathfrak{A}$. In particular, $\int u_\xi\ge\epsilon$, where $\epsilon=\inf_{\xi<\omega_1}\lim_{x\to\mathcal{F}}\bar\mu a_{\xi x}>0$; set $b'_\xi=[\![u_\xi>\frac12\epsilon]\!]$, so that $b'_\xi\ne0$.

(b) For ξ, $\eta<\omega_1$ set

$$c_{\xi\eta}=\inf_{F\in\mathcal{F}}\sup_{x\in F}a_{\xi x}\cap a_{\eta x}.$$

For $K\subseteq\omega_1$ set

$$d_K=\inf_{\xi\in K}b'_\xi\setminus\sup_{\xi,\eta\in K\text{ are distinct}}c_{\xi\eta}.$$

If $d_K\ne0$ then $\epsilon\#(K)<3$. **P** We may suppose that K is finite and not empty; set $n=\#(K)$. We have $\int_{d_K}u_\xi>\frac12\epsilon\bar\mu d_K$ for every $\xi\in K$, so

$$F_0=\{x:x\in X,\,\bar\mu(d_K\cap a_{\xi x})\ge\tfrac12\epsilon\bar\mu d_K\text{ for every }\xi\in K\}$$

belongs to $\mathcal{F}$. Let $F\in\mathcal{F}$ be such that, setting $c'_{\xi\eta}=\sup_{x\in F}a_{\xi x}\cap a_{\eta x}$, $\bar\mu(c'_{\xi\eta}\setminus c_{\xi\eta})\le\frac{\bar\mu d_K}{n^2}$ for all ξ, $\eta\in K$. Take any $x\in F\cap F_0$. If ξ, $\eta\in K$ are distinct,

$$\bar\mu(d_K\cap a_{\xi x}\cap a_{\eta x})\le\bar\mu(c'_{\xi\eta}\setminus c_{\xi\eta})\le\frac{\bar\mu d_K}{n^2},$$

so

$$\frac{n\epsilon}{2}\bar\mu d_K\le\sum_{\xi\in K}\bar\mu(d_K\cap a_{\xi x})\le\bar\mu d_K+\sum_{\xi,\eta\in K,\xi<\eta}\bar\mu(d_K\cap a_{\xi x}\cap a_{\eta x})$$

(553K)

$$\le\bar\mu d_K+\frac{n(n-1)}{2}\cdot\frac{\bar\mu d_K}{n^2}<\frac32\bar\mu d_K$$

and $n\epsilon<3$. **Q**

(c) For each infinite $\xi<\omega_1$ there is therefore a maximal subset K_ξ of ξ such that $b_\xi=d_{K_\xi\cup\{\xi\}}$ is non-zero. Every K_ξ is finite, so there is a $K\in[\omega_1]^{<\omega}$ such that $S=\{\xi:\omega\le\xi<\omega_1,\,K_\xi=K\}$ is stationary. **P** By the Pressing-Down Lemma (4A1Cc), there is a $\zeta<\omega_1$ such that $\{\xi:\xi<\omega_1,\sup K_\xi=\zeta\}$ is stationary. As $[\zeta+1]^{<\omega}$ is countable, there will be a $K\subseteq\zeta+1$ such that $\{\xi:K_\xi=K\}$ is stationary. **Q** Now suppose that η, $\xi\in S$ and $\eta<\xi$. Then

$$b_\eta\cap b_\xi\setminus c_{\eta\xi}=d_{K_\eta\cup\{\eta\}}\cap d_{K_\xi\cup\{\xi\}}\setminus c_{\xi\eta}=d_{K\cup\{\eta,\xi\}}=0$$

because $K\cup\{\eta\}$ is a subset of ξ properly including K_ξ. So we have an appropriate family $\langle b_\xi\rangle_{\xi\in S}$.

553M Proposition (Laver 87) If $\mathfrak{m}>\omega_1$ and κ is any infinite cardinal, then

$$\Vdash_{\mathbb{P}_\kappa}\text{ every Aronszajn tree is special, so Souslin's hypothesis is true.}$$

proof (a) By 5A1E(b-ii) and 5A1E(d-ii), it is enough to show that

$$\Vdash_{\mathbb{P}_\kappa}\text{ every Aronszajn tree ordering of }\omega_1\text{ included in the usual ordering is special.}$$

Let $\dot{\preccurlyeq}$ be a $\mathbb{P}_\kappa$-name for an Aronszajn tree ordering of ω_1 included in the usual ordering of ω_1. For α, $\beta < \omega_1$ set $a_{\alpha\beta} = [\![\check{\alpha}\dot{\preccurlyeq}\check{\beta}]\!]$; note that $a_{\alpha\alpha} = 1$, $a_{\alpha\beta} = 0$ if $\beta < \alpha$ and $a_{\alpha\beta} \supseteq a_{\alpha\gamma} \cap a_{\beta\gamma}$ whenever $\alpha \le \beta \le \gamma < \omega_1$.

If $\mathcal{F}$ is an ultrafilter on ω_1 containing $\omega_1 \setminus \zeta$ for every $\zeta < \omega_1$, then $\lim_{\xi\to\mathcal{F}} \bar{\nu}_\kappa a_{\alpha\xi} = 0$ for all but countably many $\alpha < \omega_1$. **P?** Otherwise, there is an $\epsilon > 0$ such that $I = \{\alpha : \alpha < \omega_1, \lim_{\xi\to\mathcal{F}} \bar{\mu} a_{\alpha\xi} \ge \epsilon\}$ is uncountable. By 553L, there are an uncountable $S \subseteq I$ and a family $\langle b_\alpha\rangle_{\alpha\in S}$ in $\mathfrak{A} \setminus \{0\}$ such that

$$b_\alpha \cap b_\beta \subseteq \sup_{\xi\ge\beta} a_{\alpha\xi} \cap a_{\beta\xi} \subseteq a_{\alpha\beta}$$

whenever α, $\beta \in S$ and $\alpha < \beta$. Set $c = \inf_{\alpha<\omega_1} \sup_{\beta\in S\setminus\alpha} b_\beta$, so that $c \ne 0$. Let $\dot{Y}$ be a $\mathbb{P}_\kappa$-name for a subset of ω_1 such that $[\![\check{\alpha} \in \dot{Y}]\!] = b_\alpha$ for $\alpha \in S$, $[\![\check{\alpha} \in \dot{Y}]\!] = 0$ for other α. Then

$$\Vdash_{\mathbb{P}_\kappa} \alpha \mathrel{\dot{\preccurlyeq}} \beta \text{ whenever } \alpha,\, \beta \in \dot{Y} \text{ and } \alpha < \beta,$$

$$c \Vdash_{\mathbb{P}_\kappa} \dot{Y} \text{ is uncountable;}$$

so

$$c \Vdash_{\mathbb{P}_\kappa} \dot{Y} \text{ is an uncountable branch in the Aronszajn tree,}$$

which is impossible. **XQ**

(b) Let $\langle e_\xi\rangle_{\xi<\kappa}$ be the standard generating family in $\mathfrak{B}_\kappa$. Choose inductively a non-decreasing family $\langle J_\alpha\rangle_{\alpha<\omega_1}$ of countably infinite subsets of κ such that $a_{\beta\alpha}$ belongs to the closed subalgebra $\mathfrak{C}_{J_\alpha}$ of $\mathfrak{B}_\kappa$ generated by $\{e_\xi : \xi \in J_\alpha\}$ whenever $\beta \le \alpha < \omega_1$.

Let P be the partially ordered set of functions f such that

dom f is a finite subset of $\omega_1 \times \omega$,

for every $(\alpha, n) \in \operatorname{dom} f$, $f(\alpha, n) \in \mathfrak{C}_{J_\alpha}$ and $\bar{\nu}_\kappa f(\alpha, n) > \frac{1}{2}$,

$f(\alpha, n) \cap f(\beta, n) \cap a_{\beta\alpha} = 0$ whenever (α, n), $(\beta, n) \in \operatorname{dom} f$ and $\beta < \alpha$.

Say that $f \le g$ if $\operatorname{dom} f \subseteq \operatorname{dom} g$ and $g(\alpha, n) \subseteq f(\alpha, n)$ for every $(\alpha, n) \in \operatorname{dom} f$. Then $\le$ is a partial order on P.

P is upwards-ccc. **P** Let $\langle f_\xi\rangle_{\xi<\omega_1}$ be a family in P. Let $A_0 \subseteq \omega_1$ be an uncountable set such that $\langle \operatorname{dom} f_\xi\rangle_{\xi\in A_0}$ is a Δ-system with root K say; let $\epsilon > 0$, $m \in \mathbb{N}$ be such that

$$A_1 = \{\xi : \xi \in A_0,\, \#(\operatorname{dom} f_\xi) = m + \#(K),$$
$$\bar{\nu}_\kappa f_\xi(\alpha, n) \ge \frac{1}{2} + 2\epsilon \text{ whenever } (\alpha, n) \in \operatorname{dom} f_\xi\}$$

is uncountable. Let $A_2 \subseteq A_1$ be an uncountable set such that $\bar{\mu}(f_\eta(\alpha, n) \triangle f_\xi(\alpha, n)) \le \epsilon$ whenever ξ, $\eta \in A_2$ and $(\alpha, n) \in K$; such a set exists because $\mathfrak{C}_{J_\alpha}$ is metrically separable for each α. Let $A_3 \subseteq A_2$ be an uncountable set such that $\beta < \alpha$ whenever $\eta \in A_3$, $\xi \in A_3$, $\eta < \xi$, $(\beta, m) \in \operatorname{dom} f_\eta$ and $(\alpha, n) \in (\operatorname{dom} f_\xi) \setminus K$.

For $\xi \in A_3$, enumerate $(\operatorname{dom} f_\xi) \setminus K$ as $\langle(\alpha_{\xi i}, n_{\xi i})\rangle_{i<m}$. Let $\mathcal{F}$ be an ultrafilter on ω_1 containing $A_3 \setminus \zeta$ for every $\zeta < \omega_1$, and for $i < m$ let $\mathcal{F}_i$ be the ultrafilter $\{F : F \subseteq \omega_1, \{\xi : \alpha_{\xi i} \in F\} \in \mathcal{F}\}$. By (a), we have an uncountable $A_4 \subseteq A_3$ such that

$$\lim_{\xi\to\mathcal{F}_i} \bar{\nu}_\kappa a_{\alpha_{\eta j},\xi} = 0$$

for every i, $j < m$ and every $\eta \in A_4$; that is,

$$\lim_{\xi\to\mathcal{F}} \bar{\nu}_\kappa a_{\alpha_{\eta j},\alpha_{\xi i}} = 0$$

whenever i, $j < m$ and $\eta \in A_4$. But this means that we can find $\eta \in A_4$ and $\xi \in A_3$ such that $\eta < \xi$ and $\bar{\nu}_\kappa a_{\alpha_{\eta j},\alpha_{\xi i}} \le \frac{\epsilon}{m+1}$ for all i, $j < m$. Now consider the function g with domain $\operatorname{dom} f_\eta \cup \operatorname{dom} f_\xi$ such that

$$\begin{aligned} g(\alpha, n) &= f_\eta(\alpha, n) \cap f_\xi(\alpha, n) \text{ if } (\alpha, n) \in K, \\ &= f_\eta(\alpha, n) \text{ if } (\alpha, n) \in \operatorname{dom} f_\eta \setminus K, \\ &= f_\xi(\alpha_{\xi i}, n_{\xi i}) \setminus \sup_{j<m} a_{\alpha_{\eta j},\alpha_{\xi i}} \text{ if } i < m \text{ and } (\alpha, n) = (\alpha_{\xi i}, n_{\xi i}). \end{aligned}$$

Then $g(\alpha, n) \in \mathfrak{C}_{J_\alpha}$ and $\bar{\nu}_\kappa g(\alpha, n) \ge \frac{1}{2} + \epsilon$ for every $(\alpha, n) \in \operatorname{dom} g$. If (α, n) and (β, n) belong to $\operatorname{dom} g$ and $\beta < \alpha$, then

—— if both (β,n) and (α,n) belong to $\operatorname{dom} f_\eta$, then

$$g(\beta,n)\cap g(\alpha,n)\cap a_{\beta\alpha}\subseteq f_\eta(\beta,n)\cap f_\eta(\alpha,n)\cap a_{\beta\alpha}=0;$$

—— if both (β,n) and (α,n) belong to $\operatorname{dom} f_\xi$, then

$$g(\beta,n)\cap g(\alpha,n)\cap a_{\beta\alpha}\subseteq f_\xi(\beta,n)\cap f_\xi(\alpha,n)\cap a_{\beta\alpha}=0;$$

—— if $(\beta,n)=(\alpha_{\eta j},n_{\eta j})$ and $(\alpha,n)=(\alpha_{\xi i},n_{\xi i})$ then $g(\alpha,n)$ is disjoint from $a_{\alpha_{\eta j},\alpha_{\xi i}}=a_{\beta\alpha}$ so $g(\beta,n)\cap g(\alpha,n)\cap a_{\beta\alpha}=0$.

So $g\in P$ and is an upper bound for f_η and f_ξ. Thus $\langle f_\xi\rangle_{\xi<\omega_1}$ is not an up-antichain in P; as $\langle f_\xi\rangle_{\xi<\omega_1}$ is arbitrary, P is upwards-ccc. **Q**

(c) For each $\alpha<\omega_1$ let C_α be a countable metrically dense subset of $\{c:c\in\mathfrak{C}_{J_\alpha},\ \bar\nu_\kappa c\le\frac12\}$. For $\alpha<\omega_1$ and $c\in C_\alpha$, set

$$Q_{\alpha c}=\{f:f\in P\text{ and there is some }n\in\mathbb{N}\text{ such that }(\alpha,n)\in\operatorname{dom}f,$$
$$c\subseteq f(\alpha,n)\text{ and }\bar\nu_\kappa f(\alpha,n)=\frac12+\frac13\bar\nu_\kappa c\}.$$

Then $Q_{\alpha c}$ is cofinal with P. **P** Because J_α is infinite, $\mathfrak{C}_{J_\alpha}$ is atomless and there is an $a\in\mathfrak{C}_{J_\alpha}$ such that $c\subseteq a$ and $\bar\nu_\kappa a=\frac12+\frac13\bar\nu_\kappa c$. Now take n so large that $i<n$ whenever $(\alpha,i)\in\operatorname{dom}f$, and set $g=f\cup\{((\alpha,n),a)\}$; then $f\le g\in Q_{\alpha c}$. **Q**

(d) Because $\mathfrak{m}>\omega_1$, there is an upwards-directed set $R\subseteq P$ meeting $Q_{\alpha c}$ whenever $\alpha<\omega_1$ and $c\in C_\alpha$. Now, for $n\in\mathbb{N}$, let $\dot A_n$ be a $\mathbb{P}_\kappa$-name for a subset of ω_1 such that, for every $\alpha<\omega_1$,

$$[\![\check\alpha\in\dot A_n]\!]=\inf\{f(\alpha,n):f\in R,\ (\alpha,n)\in\operatorname{dom}f\}\text{ if }(\alpha,n)\in\bigcup_{f\in R}\operatorname{dom}f,$$
$$=0\text{ otherwise}$$

Then $\dot A_n$ is a name for an up-antichain for the tree order $\dot\preccurlyeq$. **P** If $\beta<\alpha<\omega_1$, then either $[\![\check\beta\in\dot A_n]\!]=0$ or $[\![\check\alpha\in\dot A_n]\!]=0$ or there are f, $g\in R$ such that $(\alpha,n)\in\operatorname{dom}f$ and $(\beta,n)\in\operatorname{dom}g$. In this case, because R is upwards-directed, there is an $h\in R$ such that both (α,n) and (β,n) belong to $\operatorname{dom}h$, so that

$$[\![\check\alpha\in\dot A_n]\!]\cap[\![\check\beta\in\dot A_n]\!]\cap[\![\check\beta\dot\preccurlyeq\check\alpha]\!]\subseteq h(\alpha,n)\cap h(\beta,n)\cap a_{\beta\alpha}=0.$$

Thus

$$\Vdash_{\mathbb{P}_\kappa}\text{ if }\alpha,\ \beta\in\dot A_n\text{ then they are }\dot\preccurlyeq\text{-incompatible upwards.}$$

As α and β are arbitrary,

$$\Vdash_{\mathbb{P}_\kappa}\dot A_n\text{ is an up-antichain. }\mathbf{Q}$$

(e) Finally,

$$\Vdash_{\mathbb{P}_\kappa}\textstyle\bigcup_{n\in\mathbb{N}}\dot A_n=\omega_1.$$

P? Otherwise, there is an $\alpha<\omega_1$ such that $a=1\setminus\sup_{n\in\mathbb{N}}[\![\check\alpha\in\dot A_n]\!]\ne0$. Observe at this point that $[\![\check\alpha\in\dot A_n]\!]\in\mathfrak{C}_{J_\alpha}$ for every n. So $a\in\mathfrak{C}_{J_\alpha}$. Let $a'\in\mathfrak{C}_{J_\alpha}$ be such that $a'\subseteq a$ and $0<\bar\nu_\kappa a'\le\frac12$, and let $c\in C_\alpha$ be such that $\bar\nu_\kappa(a'\mathbin{\triangle}c)\le\frac14\bar\nu_\kappa a'$, so that $c\ne0$ and $\bar\nu_\kappa(c\setminus a')\le\frac13\bar\nu_\kappa c$. Since R meets $Q_{\alpha c}$, there are $n\in\mathbb{N}$, $f\in R$ such that $c\subseteq f(\alpha,n)$ and $\bar\nu_\kappa f(\alpha,n)=\frac12+\frac13\bar\nu_\kappa c$.

If $g\in P$ and $f\le g$, then $g(\alpha,n)\subseteq f(\alpha,n)$ and $\bar\nu_\kappa g(\alpha,n)>\frac12$, so

$$\bar\nu_\kappa(c\setminus g(\alpha,n))\le\bar\nu_\kappa f(\alpha,n)-\bar\nu_\kappa g(\alpha,n)\le\frac13\bar\nu_\kappa c.$$

Because R is upwards-directed, $\{g(\alpha,n):g\in R,\ (\alpha,n)\in\operatorname{dom}g\}$ is downwards-directed, and

$$\bar\nu_\kappa(c\setminus[\![\check\alpha\in\dot A_n]\!])=\sup\{\bar\nu_\kappa(c\setminus g(\alpha,n)):g\in R,\ (\alpha,n)\in\operatorname{dom}g\}$$
$$=\sup\{\bar\nu_\kappa(c\setminus g(\alpha,n)):g\in R,\ f\le g\}\le\frac13\bar\nu_\kappa c.$$

Accordingly

$$\bar{\nu}_\kappa(a' \cap [\![\check{\alpha} \in \dot{A}_n]\!]) \ge \bar{\nu}_\kappa(c \cap [\![\check{\alpha} \in \dot{A}_n]\!]) - \bar{\nu}_\kappa(c \setminus a') \ge \frac{2}{3}\bar{\nu}_\kappa c - \frac{1}{3}\bar{\nu}_\kappa c > 0;$$

but $a' \subseteq a$ is supposed to be disjoint from $[\![\check{\alpha} \in \dot{A}_n]\!]$. **XQ**

So $\langle \dot{A}_n \rangle_{n\in\mathbb{N}}$ is a name for a sequence of antichains covering ω_1, and

$$\Vdash_{\mathbb{P}_\kappa} (\omega_1, \dot{\preccurlyeq}) \text{ is special,}$$

as required.

553N Proposition Suppose that there is a medial limit (definition: 538Q), and that κ is a cardinal. Then

$$\Vdash_{\mathbb{P}_\kappa} \text{there is a medial limit.}$$

proof (a) Let $\theta : \mathcal{P}\mathbb{N} \to [0,1]$ be a medial limit. Let Q be the rationally convex hull of the usual basis of ℓ^1, that is, the set of functions $v : \mathbb{N} \to \mathbb{Q} \cap [0,1]$ such that $\{n : v(n) \ne 0\}$ is finite and $\sum_{n=0}^\infty v(n) = 1$. Note that Q is absolute in the sense that

$$\Vdash_{\mathbb{P}} \check{Q} \text{ is the rationally convex hull of the usual basis of } \ell^1$$

for every forcing notion $\mathbb{P}$. Let $\mathcal{F}$ be the filter on Q which is the trace of the weak* neighbourhood filter of θ, that is, the filter generated by sets of the form

$$\{v : v \in Q,\ |\textstyle\sum_{n=0}^\infty v(n)u(n) - \fint u(n)\theta(dn)| \le \epsilon\}$$

where $u \in \ell^\infty$ and $\epsilon > 0$. (Identifying $Q \subseteq \ell^1$ with its image in $(\ell^\infty)^* \cong (\ell^1)^{**}$, the weak* closure of Q is convex, so is equal to its bipolar (4A4Eg) and is the set of positive linear functionals on ℓ^∞ taking the value 1 on the order unit $\chi\mathbb{N}$. See 363L and 538P for the notation $\fint \ldots \theta(dn)$.) Let $\vec{\mathcal{F}}$ be the $\mathbb{P}_\kappa$-name derived from $\mathcal{F}$ and $(\{0,1\}^\kappa, \mathrm{T}_\kappa, \mathcal{N}_\kappa)$ by the method of 551Rb, so that

$$\Vdash_{\mathbb{P}_\kappa} \vec{\mathcal{F}} \text{ is a filter on } \check{Q}.$$

Let $\dot{\nu}$ be a $\mathbb{P}_\kappa$-name such that

> $\Vdash_{\mathbb{P}_\kappa} \dot{\nu}$ is a bounded additive functional on $\mathcal{P}\mathbb{N}$, and identifying $\check{Q}$ with a subset of $(\ell^\infty)^*$, itself identified with the space $M(\mathcal{P}\mathbb{N})$ of bounded additive functionals on $\mathcal{P}\mathbb{N}$, $\dot{\nu}$ is a cluster point of $\vec{\mathcal{F}}$ for the weak* topology.

(b) Suppose that $a \in \mathfrak{B}_\kappa^+$ and that $\dot{\mathfrak{e}}$ is a $\mathbb{P}_\kappa$-name such that

$$a \Vdash_{\mathbb{P}_\kappa} \dot{\mathfrak{e}} \text{ is a sequence of Borel subsets of } \{0,1\}^{\mathbb{N}}.$$

By 551Fb, we have for each $n \in \mathbb{N}$ a set $W_n \in \mathrm{T}_\kappa \widehat{\otimes} \mathcal{B}\mathfrak{a}_{\mathbb{N}}$ such that

$$a \Vdash_{\mathbb{P}_\kappa} \dot{\mathfrak{e}}(\check{n}) = \vec{W}_n,$$

where $\vec{W}_n$ is defined as in 551D. Let $\lambda = \nu_\kappa \times \nu_\omega$ be the product measure on $\{0,1\}^\kappa \times \{0,1\}^{\mathbb{N}}$. Because θ is a medial limit, $\int\fint \chi W_n(x,y)\theta(dn)\lambda(d(x,y))$ is defined and equal to $\fint \lambda W_n \theta(dn)$; that is, there are a conegligible Baire set $W \subseteq \{0,1\}^\kappa \times \{0,1\}^{\mathbb{N}}$ and a Baire measurable function $\psi : \{0,1\}^\kappa \times \{0,1\}^{\mathbb{N}} \to [0,1]$ such that

$$\psi(x,y) = \fint \chi W_n(x,y)\theta(dn) = \lim_{v\to\mathcal{F}} \textstyle\sum_{n=0}^\infty v(n)\chi W_n(x,y)$$

whenever $(x,y) \in W$, and

$$\int \psi\, d\lambda = \fint \lambda W_n \theta(dn) = \lim_{v\to\mathcal{F}} \textstyle\sum_{n=0}^\infty v(n)\lambda W_n.$$

Let $\vec{W}$ and $\vec{\psi}$ be the corresponding $\mathbb{P}_\kappa$-names, as in 551D and 551M, so that

$$\Vdash_{\mathbb{P}_\kappa} \vec{W} \in \mathcal{B}\mathfrak{a}_{\mathbb{N}} \text{ and } \vec{\psi} : \{0,1\}^{\mathbb{N}} \to \mathbb{R} \text{ is Baire measurable.}$$

Moreover, since ν_κ-almost every vertical section of W must be ν_ω-conegligible,

$$\Vdash_{\mathbb{P}_\kappa} \nu_\omega \vec{W} = 1$$

(551I).

(c) Now suppose that $\dot{s}$ is a $\mathbb{P}_\kappa$-name and that $b \in \mathfrak{B}_\kappa^+$ is stronger than a and such that

$$b \Vdash_{\mathbb{P}_\kappa} \dot{s} \in \vec{W}.$$

(i) By 551Cc, there is a T_κ-measurable $f : \{0,1\}^\kappa \to \{0,1\}^{\mathbb{N}}$ such that

$$b \Vdash_{\mathbb{P}_\kappa} \dot{s} = \vec{f}.$$

Expressing b as $E^\bullet$ where $E \in \mathrm{T}_\kappa \setminus \mathcal{N}_\kappa$, $(x, f(x)) \in W$ for ν_κ-almost every $x \in E$, by 551Ea.

For each $m \in \mathbb{N}$, consider

$$C_m = \{(x, v) : x \in \{0,1\}^\kappa,\ v \in Q,\ (x, f(x)) \in W,$$
$$|\psi(x, f(x)) - \sum_{n=0}^\infty v(n)\chi W_n(x, f(x))| \le 2^{-m}\}.$$

Then $C_m \in \mathrm{T}_\kappa \widehat{\otimes} \mathcal{P}Q$; and if $x \in \{0,1\}^\kappa$ is such that $(x, f(x)) \in W$, $C_m[\{x\}] \in \mathcal{F}$. Consequently

$$b \Vdash_{\mathbb{P}_\kappa} \vec{C}_m \in \vec{\mathcal{F}},$$

where in this formula $\vec{C}_m$ is the $\mathbb{P}_\kappa$-name defined by the method of 551Ra. At the same time,

$$b \Vdash_{\mathbb{P}_\kappa} |\vec{\psi}(\dot{s}) - \textstyle\sum_{n=0}^\infty v(n)\chi(\dot{\mathfrak{e}}(n))(\dot{s})| \le 2^{-\check{m}} \text{ for every } v \in \vec{C}_m.$$

P Suppose we have a c stronger than b and a $\mathbb{P}_\kappa$-name $\dot{v}$ such that $c \Vdash_{\mathbb{P}_\kappa} \dot{v} \in \vec{C}_m$. Then there are a $G \in \mathrm{T}_\kappa \setminus \mathcal{N}_\kappa$ and a $v \in Q$ such that $G^\bullet \subseteq c$, $G^\bullet \Vdash_{\mathbb{P}_\kappa} \dot{v} = \check{v}$, and $(x, v) \in C_m$ for every $x \in G$. Setting

$$h(x) = \psi(x, f(x)), \quad h_n(x) = \chi W_n(x, f(x))$$

for $x \in \{0,1\}^\kappa$ and $n \in \mathbb{N}$, and interpreting $\vec{h}$, $\vec{h}_n$ as in 551B,

$$b \Vdash_{\mathbb{P}_\kappa} \vec{h} = \vec{\psi}(\vec{f}) = \vec{\psi}(\dot{s}),$$

and

$$b \Vdash_{\mathbb{P}_\kappa} \vec{h}_n = (\chi W_n)^{\rightarrow}(\dot{s}) = (\chi \vec{W}_n)(\dot{s})$$

(551Nd)

$$= (\chi\dot{\mathfrak{e}}(\check{n}))(\dot{s})$$

for $n \in \mathbb{N}$. For $x \in G$, moreover, $|h(x) - \sum_{n=0}^\infty v(n)h_n(x)| \le 2^{-m}$, so

$$G^\bullet \Vdash_{\mathbb{P}_\kappa} |\vec{\psi}(\dot{s}) - \textstyle\sum_{n=0}^\infty \dot{v}(n)\chi(\dot{\mathfrak{e}}(n))(\dot{s})| = |\vec{h} - \sum_{n=0}^\infty \check{v}(n)\vec{h}_n| \le 2^{-\check{m}}.$$

As c and $\dot{v}$ are arbitrary, we have the result. **Q**

(ii) As m is arbitrary,

$$b \Vdash_{\mathbb{P}_\kappa} \{v : v \in \check{Q},\ |\vec{\psi}(\dot{s}) - \textstyle\sum_{n=0}^\infty v(n)\chi(\dot{\mathfrak{e}}(n))(\dot{s})| \le \epsilon\} \in \vec{\mathcal{F}} \text{ for every } \epsilon > 0,$$

that is,

$$b \Vdash_{\mathbb{P}_\kappa} \vec{\psi}(\dot{s}) = \lim_{v \to \vec{\mathcal{F}}} \textstyle\sum_{n=0}^\infty v(n)\chi(\dot{\mathfrak{e}}(n))(\dot{s}).$$

As b and $\dot{s}$ are arbitrary,

$$a \Vdash_{\mathbb{P}_\kappa} \vec{\psi}(y) = \lim_{v \to \vec{\mathcal{F}}} \textstyle\sum_{n=0}^\infty v(n)\chi(\dot{\mathfrak{e}}(n))(y) \text{ for every } y \in \vec{W};$$

since $\Vdash_{\mathbb{P}_\kappa} \vec{W}$ is conegligible,

$$a \Vdash_{\mathbb{P}_\kappa} \vec{\psi} =_{\text{a.e.}} \lim_{v \to \vec{\mathcal{F}}} \textstyle\sum_{n=0}^\infty v(n)\chi(\dot{\mathfrak{e}}(n)).$$

Looking back at the choice of $\dot{\nu}$, we see that

$$a \Vdash_{\mathbb{P}_\kappa} \vec{\psi}(y) = \textstyle\int \chi(\dot{\mathfrak{e}}(n))(y)\dot{\nu}(dn) \text{ for } \nu_\omega\text{-almost every } y.$$

(d) As for the integral of $\vec{\psi}$, 551Nf tells us that

$$\Vdash_{\mathbb{P}_\kappa} \int \vec{\psi}\, d\nu_\omega = \vec{h},$$

where I now set $h(x) = \int \psi(x,y)\nu_\omega(dy)$ for $x \in \{0,1\}^\kappa$. Similarly, setting $h_n(x) = \nu_\omega W_n[\{x\}]$, we have

$$a \Vdash_{\mathbb{P}_\kappa} \nu_\omega \dot{\mathfrak{e}}(\check{n}) = \vec{h}_n.$$

Set

$$H = \{x : x \in \{0,1\}^\kappa,\ W[\{x\}] \text{ is conegligible in } \{0,1\}^{\mathbb{N}}\};$$

then H is conegligible in $\{0,1\}^\kappa$. Now remember that θ is a medial limit. If $x \in H$ we have $\psi(x,y) = ⨍ \chi W_n(x,y)\theta(dn)$ for every y in the conegligible set $W[\{x\}]$, so

$$\begin{aligned} h(x) &= \int \psi(x,y)\nu_\omega(dy) = \int ⨍ \chi W_n(x,y)\theta(dn)\nu_\omega(dy) \\ &= ⨍ \int \chi W_n(x,y)\nu_\omega(dy)\theta(dn) = ⨍ \nu_\omega W_n[\{x\}]\theta(dn) = \lim_{v\to\mathcal{F}} \sum_{n=0}^{\infty} v(n)h_n(x). \end{aligned}$$

So if, for $m \in \mathbb{N}$, we set

$$C'_m = \{(x,v) : x \in \{0,1\}^\kappa,\ v \in Q,\ |h(x) - \textstyle\sum_{n=0}^\infty v(n)h_n(x)| \le 2^{-m}\},$$

we shall again have $\Vdash_{\mathbb{P}_\kappa} \vec{C}'_m \in \vec{\mathcal{F}}$; and if $G \in \mathrm{T}_\kappa \setminus \mathcal{N}_\kappa$ and $v \in Q$ are such that $G^\bullet$ is stronger than p and $G^\bullet \Vdash_{\mathbb{P}_\kappa} \check{v} \in \vec{C}'_m$, then

$$G^\bullet \Vdash_{\mathbb{P}_\kappa} |\textstyle\int \vec{\psi}\, d\nu_\omega - \sum_{n=0}^\infty \check{v}(n)\nu_\omega\dot{\mathfrak{e}}(n)| \le 2^{-\check{m}}.$$

So

$$a \Vdash_{\mathbb{P}_\kappa} \{v : |\textstyle\int \vec{\psi}\, d\nu_\omega - \sum_{n=0}^\infty \check{v}(n)\nu_\omega\dot{\mathfrak{e}}(n)| \le 2^{-\check{m}}\} \in \vec{\mathcal{F}}$$

for every m, and

$$a \Vdash_{\mathbb{P}_\kappa} \textstyle\int \vec{\psi}\, d\nu_\omega = \lim_{v\to\vec{\mathcal{F}}} \sum_{n=0}^\infty \check{v}(n)\nu_\omega\dot{\mathfrak{e}}(n) = ⨍ \nu_\omega\dot{\mathfrak{e}}(n)\dot{\nu}(dn).$$

(e) As p and $\dot{\mathfrak{e}}$ are arbitrary, we see that

$$\Vdash_{\mathbb{P}_\kappa} \dot{\nu} \text{ satisfies condition (iv) of 538P, so is a medial functional.}$$

It is now easy to check that

$$\Vdash_{\mathbb{P}_\kappa} \dot{\nu} \ge 0,\ \dot{\nu}\mathbb{N} = 1 \text{ and } \dot{\nu}\{n\} = 0 \text{ for every } n \in \mathbb{N}, \text{ so } \dot{\nu} \text{ is a medial limit.}$$

This completes the proof.

553O For the most familiar classes of 'small' set – the Lebesgue null ideal, or the meager ideal of $\mathbb{R}$, for instance – it is easy to calculate the number of sets in the class; because there is a nowhere dense Lebesgue negligible set with cardinal $\mathfrak{c}$, there must be exactly $2^{\mathfrak{c}}$ meager Lebesgue negligible sets, and therefore there are just $2^{\mathfrak{c}}$ Lebesgue measurable subsets of $\mathbb{R}^r$ for any $r \ge 1$. But when we come to the ideal $\mathcal{N}_{\text{universal}} \triangleleft \mathcal{P}\mathbb{R}$ of universally negligible sets, or the algebra $\Sigma_{\text{um}} \subseteq \mathcal{P}\mathbb{R}$ of universally measurable sets, the position is much less clear. In general, since by Grzegorek's theorem (439F) we know that there is a universally negligible subset of $\mathbb{R}$ of cardinal $\operatorname{non}\mathcal{N}(\nu_\omega)$, we can say that

$$\mathfrak{c} \le 2^{\operatorname{non}\mathcal{N}(\nu_\omega)} \le \#(\mathcal{N}_{\text{universal}}) \le \#(\Sigma_{\text{um}}) \le 2^{\mathfrak{c}}.$$

It turns out that in random real models these inequalities may well collapse to the lower bound, as in (b) of the next theorem.

Theorem (Larson Neeman & Shelah 10) Let κ be an infinite cardinal.

(a) $\Vdash_{\mathbb{P}_\kappa}$ every universally measurable subset of $\{0,1\}^{\mathbb{N}}$ is expressible as the union of at most $\check{\mathfrak{c}}$ Borel sets.

(b) If the cardinal power $\kappa^{\mathfrak{c}}$ is equal to κ, then

$$\Vdash_{\mathbb{P}_\kappa} \text{there are exactly } \mathfrak{c} \text{ universally measurable subsets of } \{0,1\}^{\mathbb{N}}.$$

proof (a)(i) It will save a moment later if I note at once that we need consider only the case $\kappa > \mathfrak{c}$. **P** If $\kappa \le \mathfrak{c}$, then $\kappa^\omega = \mathfrak{c}$, so

$$\Vdash_{\mathbb{P}_\kappa} \mathfrak{c} = \check{\mathfrak{c}}$$

by 552B. But since we surely have

$\Vdash_{\mathbb{P}_\kappa}$ every universally measurable subset of $\{0,1\}^{\mathbb{N}}$ is expressible as the union of at most $\mathfrak{c}$ singleton sets,

we get the result. **Q**

So henceforth I will take it that $\kappa > \mathfrak{c}$. It will save time to have a local notation: if $M \subseteq \kappa$ and $V \subseteq \{0,1\}^\kappa \times \{0,1\}^{\mathbb{N}}$, I will say that V is M- if $(\tilde\omega, \omega') \in V$ whenever $(\omega, \omega') \in V$ and $\tilde\omega \in \{0,1\}^\kappa$ is such that $\tilde\omega\restriction M = \omega\restriction M$.

(ii) (The testing measures.) If $E \in \mathcal{B}\mathfrak{a}_\kappa \setminus \mathcal{N}(\nu_\kappa)$, $g : \{0,1\}^\kappa \to \{0,1\}^{\mathbb{N}}$ is a $\mathcal{B}\mathfrak{a}_\kappa$-measurable function, and $I \subseteq \kappa$ is a set, write Q_{IEg} for the set of pairs (V, h) where $V \in \mathcal{B}\mathfrak{a}_\kappa \widehat{\otimes} \mathcal{B}\mathfrak{a}_{\mathbb{N}}$ and $h : \{0,1\}^\kappa \to [0,1]$ is defined by saying that

$$h(\omega) = \nu_{\kappa\setminus I}\{\omega' : (\omega\restriction I) \cup \omega' \in E,\ (\omega, g((\omega\restriction I) \cup \omega')) \in V\}$$

for every $\omega \in \{0,1\}^\kappa$. Then h is $\mathcal{B}\mathfrak{a}_\kappa$-measurable for every $(V, h) \in Q_{IEg}$. **P** The function

$$(\omega, \omega') \mapsto (\omega\restriction I) \cup \omega' : \{0,1\}^\kappa \times \{0,1\}^{\kappa\setminus I} \to \{0,1\}^\kappa$$

is $(\mathcal{B}\mathfrak{a}_\kappa \widehat{\otimes} \mathcal{B}\mathfrak{a}_{\kappa\setminus I}, \mathcal{B}\mathfrak{a}_\kappa)$-measurable, because $\mathcal{B}\mathfrak{a}_\kappa = \widehat{\bigotimes}_\kappa \mathcal{P}(\{0,1\})$ (4A3Na). So

$$\{(\omega, \omega') : (\omega\restriction I) \cup \omega' \in E,\ (\omega, g((\omega\restriction I) \cup \omega')) \in V\} \in \mathcal{B}\mathfrak{a}_\kappa \widehat{\otimes} \mathcal{B}\mathfrak{a}_{\kappa\setminus I}$$

and we can apply 252P. **Q**

We can therefore set

$$\dot\mu_{IEg} = \{((\vec{V}, \vec{h}), \mathbb{1}) : (V, h) \in Q_{IEg}\},$$

and $\dot\mu_{IEg}$ will be a $\mathbb{P}_\kappa$-name, subject to the conventions I use concerning the interpretation of the brackets in the formula $((\vec{V}, \vec{h}), \mathbb{1})$.

(iii) If I, E, g and $\dot\mu_{IEg}$ are as in (ii), then

$$\Vdash_{\mathbb{P}_\kappa} \dot\mu \text{ is a } [0,1]\text{-valued function with domain } \mathcal{B}\mathfrak{a}_{\mathbb{N}}.$$

P Suppose that (V_0, h_0), $(V_1, h_1) \in Q_{IEg}$ and $p \in \mathbb{P}_\kappa$ are such that $p \Vdash_{\mathbb{P}_\kappa} \vec{V}_0 = \vec{V}_1$. Express p as $F^\bullet$ where $F \in \mathcal{B}\mathfrak{a}_\kappa$. By 551Gb, $F' = F \setminus \{\omega : V_0[\{\omega\}] = V_1[\{\omega\}]\}$ is negligible. But for $\omega \in F \setminus F'$,

$$\begin{aligned}\{\omega' : (\omega\restriction I) \cup \omega' \in E,\ (\omega, g((\omega\restriction I) \cup \omega')) \in V_0\}\\ = \{\omega' : (\omega\restriction I) \cup \omega' \in E,\ (\omega, g((\omega\restriction I) \cup \omega')) \in V_1\}\end{aligned}$$

and $h_0(\omega) = h_1(\omega)$. So $p \Vdash_{\mathbb{P}_\kappa} \vec{h}_0 = \vec{h}_1$.

Thus $\dot\mu_{IEg}$ satisfies the condition (ii) of 5A3Ea, and

$\Vdash_{\mathbb{P}_\kappa} \dot\mu_{IEg}$ is a function with domain $\{(\vec{V}, \mathbb{1}) : V \in \mathcal{B}\mathfrak{a}_\kappa \widehat{\otimes} \mathcal{B}\mathfrak{a}_{\mathbb{N}}\} = \mathcal{B}\mathfrak{a}_{\mathbb{N}}$

where the second $\mathcal{B}\mathfrak{a}_{\mathbb{N}}$ is interpreted in the forcing language (551F). Since

$$\Vdash_{\mathbb{P}_\kappa} \vec{h} \in [0,1]$$

whenever $h : \{0,1\}^\kappa \to [0,1]$ is a $\mathcal{B}\mathfrak{a}_\kappa$-measurable function (551B),

$$\Vdash_{\mathbb{P}_\kappa} \dot\mu_{IEg} \text{ takes values in } [0,1]. \textbf{ Q}$$

Next,

$$\Vdash_{\mathbb{P}_\kappa} \dot\mu_{IEg} \text{ is countably additive, so is a Borel measure on } \{0,1\}^{\mathbb{N}}.$$

P (Compare 551M-551N.) Use the formulae of 551E. If $\dot{\boldsymbol{G}}$ is a $\mathbb{P}_\kappa$-name such that

$$\Vdash_{\mathbb{P}_\kappa} \dot{\boldsymbol{G}} \text{ is a disjoint sequence in } \mathcal{B}\mathfrak{a}_{\mathbb{N}},$$

then there is a sequence $\langle W_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{B}\mathfrak{a}_\kappa \widehat{\otimes} \mathcal{B}\mathfrak{a}_{\mathbb{N}}$ such that

$$\Vdash_{\mathbb{P}_\kappa} \dot{\boldsymbol{G}} = \langle \vec{W}_n \rangle_{n\in\mathbb{N}}.$$

If $m \ne n$, then

$$\Vdash_{\mathbb{P}_\kappa} (W_m \cap W_n)^{\vec{}} = \vec{W}_m \cap \vec{W}_n = \emptyset,$$

so $\nu_\kappa\{\omega : W_m[\{\omega\}] \cap W_n[\{\omega\}] \ne \emptyset\} = 0$ (551Ga); accordingly $\langle W_n[\{\omega\}]\rangle_{n\in\mathbb{N}}$ is disjoint for ν_κ-almost every ω. Setting $W = \bigcup_{n\in\mathbb{N}} W_n$,

$$h_n(\omega) = \nu_{\kappa\setminus I}\{\omega' : (\omega\restriction I) \cup \omega' \in E,\, (\omega, g((\omega\restriction I) \cup \omega')) \in W_n\},$$

$$h(\omega) = \nu_{\kappa\setminus I}\{\omega' : (\omega\restriction I) \cup \omega' \in E,\, (\omega, g((\omega\restriction I) \cup \omega')) \in W\}$$

for $\omega \in \{0,1\}^\kappa$ and $n \in \mathbb{N}$, we see that whenever $\langle W_n[\{\omega\}]\rangle_{n\in\mathbb{N}}$ is disjoint then

$$\langle \{\omega' : (\omega\restriction I) \cup \omega' \in E,\, (\omega, g((\omega\restriction I) \cup \omega')) \in W_n\}\rangle_{n\in\mathbb{N}}$$

is disjoint, with union

$$\{\omega' : (\omega\restriction I) \cup \omega' \in E,\, (\omega, g((\omega\restriction I) \cup \omega')) \in W\},$$

so $h(\omega) = \sum_{n=0}^\infty h_n(\omega)$. Accordingly

$$\Vdash_{\mathbb{P}_\kappa} \dot{\mu}_{IEg}(\bigcup_{n\in\mathbb{N}} \vec{W}_n) = \dot{\mu}_{IEg}\vec{W}$$

(551Ed)

$$= \vec{h} = \sum_{n=0}^\infty \vec{h}_n$$

(5A3L(c-iii), 5A3Ld)

$$= \sum_{n=0}^\infty \dot{\mu}_{IEg}\vec{W}_n.$$

As $\dot{\boldsymbol{G}}$ is arbitrary,

$$\Vdash_{\mathbb{P}_\kappa} \dot{\mu}_{IEg} \text{ is countably additive. } \mathbf{Q}$$

(iv) Still supposing that I, E, g and $\dot{\mu}_{IEg}$ are as in (ii), let J, $M \subseteq \kappa$ be such that E and g are determined by coordinates in J, and $J \cap M = I$. If $V \in \mathcal{B}\mathfrak{a}_\kappa \widehat{\otimes} \mathcal{B}\mathfrak{a}_\mathbb{N}$ is M-determined in the sense of (i) above, and

$$\Vdash_{\mathbb{P}_\kappa} \dot{\mu}_{IEg}(\vec{V}) = 0,$$

then

$$E^\bullet \Vdash_{\mathbb{P}_\kappa} \vec{g} \notin \vec{V}.$$

P We can suppose that $J \cup M = \kappa$, so that $(I, J \setminus I, M \setminus I)$ is a partition of κ. Identifying $\{0,1\}^\kappa$ with $\{0,1\}^I \times \{0,1\}^{J\setminus I} \times \{0,1\}^{M\setminus I}$, we can find $E' \in \mathcal{B}\mathfrak{a}_I \widehat{\otimes} \mathcal{B}\mathfrak{a}_{J\setminus I}$, a $\mathcal{B}\mathfrak{a}_I \widehat{\otimes} \mathcal{B}\mathfrak{a}_{J\setminus I}$-measurable function $g' : \{0,1\}^I \times \{0,1\}^{J\setminus I} \to \{0,1\}^\mathbb{N}$ and a set $V' \in \mathcal{B}\mathfrak{a}_I \widehat{\otimes} \mathcal{B}\mathfrak{a}_{M\setminus I} \widehat{\otimes} \mathcal{B}\mathfrak{a}_\mathbb{N}$ such that

$$E = \{(\omega_0, \omega_1, \omega_2) : (\omega_0, \omega_1) \in E',\, \omega_2 \in \{0,1\}^{M\setminus I}\},$$

$$g(\omega_0, \omega_1, \omega_2) = g'(\omega_0, \omega_1) \text{ for all } \omega_0 \in \{0,1\}^I,\, \omega_1 \in \{0,1\}^{J\setminus I},\, \omega_2 \in \{0,1\}^{M\setminus I},$$

$$V = \{((\omega_0, \omega_1, \omega_2), x) : ((\omega_0, \omega_2), x) \in V',\, \omega_1 \in \{0,1\}^{J\setminus I}\}.$$

The hypothesis

$$\Vdash_{\mathbb{P}_\kappa} \dot{\mu}_{IEg}(\vec{V}) = 0$$

translates into '$h = 0$ ν_κ-a.e.', where

$$h(\omega) = \nu_{\kappa\setminus I}\{\omega' : (\omega\restriction I) \cup \omega' \in E,\, (\omega, g((\omega\restriction I) \cup \omega')) \in V\},$$

that is, identifying $\{0,1\}^{\kappa\setminus I}$ with $\{0,1\}^{J\setminus I}\times\{0,1\}^{M\setminus I}$,

$$h(\omega_0,\omega_1,\omega_2)=\nu_{\kappa\setminus I}\{(\omega_1',\omega_2'):(\omega_0,\omega_1')\in E',\ ((\omega_0,\omega_2),g'(\omega_0,\omega_1'))\in V'\}.$$

So we see that

$$\nu_{\kappa\setminus I}\{(\omega_1',\omega_2'):(\omega_0,\omega_1')\in E',\ ((\omega_0,\omega_2),g'(\omega_0,\omega_1'))\in V'\}=0$$

for ν_κ-almost every $(\omega_0,\omega_1,\omega_2)$. It follows that

$$\nu_{J\setminus I}\{\omega_1':(\omega_0,\omega_1')\in E',\ ((\omega_0,\omega_2),g'(\omega_0,\omega_1'))\in V'\}=0$$

for ν_κ-almost every $(\omega_0,\omega_1,\omega_2)$, and therefore for ν_M-almost every (ω_0,ω_2), here identifying $\{0,1\}^M$ with $\{0,1\}^I\times\{0,1\}^{M\setminus I}$. Consequently

$$W=\{(\omega_0,\omega_1',\omega_2):(\omega_0,\omega_1')\in E',\ ((\omega_0,\omega_2),g'(\omega_0,\omega_1'))\in V'\}$$

is ν_κ-negligible. But W is also

$$\begin{aligned}\{(\omega_0,\omega_1,\omega_2):(\omega_0,\omega_1,\omega_2)\in E,\ ((\omega_0,\omega_1,\omega_2),g(\omega_0,\omega_1,\omega_2))\in V\}&\\ =\{\omega:\omega\in E,\ (\omega,g(\omega))\in V\}.&\end{aligned}$$

By 551Ea, $[\![\vec g\in\vec V]\!]=\{\omega:(\omega,g(\omega))\in V\}^\bullet$; we have just seen that this is disjoint from $E^\bullet$ in $\mathfrak{B}_\kappa$, so

$$E^\bullet\Vdash_{\mathbb{P}_\kappa}\vec g\notin\vec V,$$

as required. **Q**

(v) (The key.) Once again taking I, E, g and $\dot\mu_{IEg}$ as in (ii), let J, $M\subseteq\kappa$ be such that J is countable, E and g are determined by coordinates in J, $J\cap M=I$ and $M\setminus I$ is infinite. Then there are an $E'\in\mathcal{B}\mathfrak{a}_\kappa\setminus\mathcal{N}(\nu_\kappa)$ and a $\mathcal{B}\mathfrak{a}_\kappa$-measurable function $g':\{0,1\}^\kappa\to\{0,1\}^{\mathbb{N}}$, both determined by coordinates in M, such that

$$\Vdash_{\mathbb{P}_\kappa}\dot\mu_{IEg}=\dot\mu_{IE'g'}.$$

P Because J is countable, $I\subseteq M$ and $M\setminus I$ is infinite, there is a permutation α of κ such that $\alpha(\xi)=\xi$ for every $\xi\in I$ and $\alpha[J]\subseteq M$. Set

$$E'=\{\omega:\omega\in\{0,1\}^\kappa,\ \omega\alpha\in E\},\quad g'(\omega)=g(\omega\alpha)\text{ for every }\omega\in\{0,1\}^{\mathbb{N}}.$$

Because $\omega\mapsto\omega\alpha$ is an autohomeomorphism of $\{0,1\}^\kappa$, E' is a Baire set and g' is Baire measurable. If ω, $\omega'\in\{0,1\}^\kappa$ and $\omega\restriction M=\omega'\restriction M$, then $\omega\alpha\restriction J=\omega'\alpha\restriction J$; so E' and g' are both determined by coordinates in M.

Take any $V\in\mathcal{B}\mathfrak{a}_\kappa\widehat{\otimes}\mathcal{B}\mathfrak{a}_{\mathbb{N}}$ and set

$$h(\omega)=\nu_{\kappa\setminus I}\{\omega':(\omega\restriction I)\cup\omega'\in E,\ (\omega,g((\omega\restriction I)\cup\omega'))\in V\},$$

$$h'(\omega)=\nu_{\kappa\setminus I}\{\omega':(\omega\restriction I)\cup\omega'\in E',\ (\omega,g'((\omega\restriction I)\cup\omega'))\in V\}$$

for $\omega\in\{0,1\}^\kappa$. Consider the permutation $\beta=\alpha\restriction\kappa\setminus I$ of $\kappa\setminus I$, and set $\hat\beta\omega'=\omega'\beta$ for $\omega'\in\{0,1\}^{\kappa\setminus I}$. Then, for any ω,

$$\begin{aligned}\{\omega':(\omega\restriction I)\cup\omega'\in E',\ (\omega,g'((\omega\restriction I)\cup\omega'))\in V\}&\\ =\{\omega':(\omega\restriction I)\cup\hat\beta(\omega')\in E,\ (\omega,g((\omega\restriction I)\cup\hat\beta(\omega')))\in V\}&\\ =\hat\beta^{-1}[\{\omega':(\omega\restriction I)\cup\omega'\in E,\ (\omega,g((\omega\restriction I)\cup\omega'))\in V\}].&\end{aligned}$$

As $\hat\beta$ is an automorphism of $(\{0,1\}^{\kappa\setminus I},\nu_{\kappa\setminus I})$, $\{\omega':(\omega\restriction I)\cup\omega'\in E',\ (\omega,g'((\omega\restriction I)\cup\omega'))\in V\}$ and $\{\omega':(\omega\restriction I)\cup\omega'\in E,\ (\omega,g((\omega\restriction I)\cup\omega'))\in V\}$ have the same measure. Thus $h=h'$, and

$$\Vdash_{\mathbb{P}_\kappa}\dot\mu_{IEg}(\vec V)=\vec h=\vec{h'}=\dot\mu_{IE'g'}(\vec V).$$

As V is arbitrary,

$$\Vdash_{\mathbb{P}_\kappa}\dot\mu_{IEg}=\dot\mu_{IE'g'}.$$

So we have an appropriate pair E', g'. **Q**

(vi) I come at last to universally measurable sets. Let $\dot{A}$ be a $\mathbb{P}_\kappa$-name such that

$$\Vdash_{\mathbb{P}_\kappa} \dot{A} \text{ is a universally measurable subset of } \{0,1\}^{\mathbb{N}}.$$

Then there is a set $M \subseteq \kappa$ with cardinal $\mathfrak{c}$ such that whenever $I \in [M]^{\le\omega}$, $E \in \mathcal{B}\mathfrak{a}_\kappa \setminus \mathcal{N}(\nu_\kappa)$ and a Baire measurable $g : \{0,1\}^\kappa \to \{0,1\}^{\mathbb{N}}$ are determined by coordinates in M, then there are M-determined sets F, $V \in \mathcal{B}\mathfrak{a}_\kappa \widehat{\otimes} \mathcal{B}\mathfrak{a}_{\mathbb{N}}$ such that

$$\Vdash_{\mathbb{P}_\kappa} \dot{A} \triangle \vec{F} \subseteq \vec{V} \text{ and } \dot{\mu}_{IEg} \vec{V} = 0.$$

P If $I \in [\kappa]^{\le\omega}$, $E \in \mathcal{B}\mathfrak{a}_\kappa \setminus \mathcal{N}(\nu_\kappa)$ and $g : \{0,1\}^\kappa \to \{0,1\}^{\mathbb{N}}$ is Baire measurable, then

$\Vdash_{\mathbb{P}_\kappa}$ the completion of $\dot{\mu}_{IEg}$ measures $\dot{A}$, so there are Borel sets G, $H \subseteq \{0,1\}^{\mathbb{N}}$ such that $\dot{A} \triangle G \subseteq H$ and $\dot{\mu}_{IEg} H = 0$.

By 551F, as usual, there must be F, $V \in \mathcal{B}\mathfrak{a}_\kappa \widehat{\otimes} \mathcal{B}\mathfrak{a}_{\mathbb{N}}$ such that

$$\Vdash_{\mathbb{P}_\kappa} \dot{A} \triangle \vec{F} \subseteq \vec{V} \text{ and } \dot{\mu}_{IEg} \vec{V} = 0.$$

Now there will be a countable set $K(I,E,g) \subseteq \kappa$ such that F and V are both $K(I,E,g)$-determined. If we build inductively a non-decreasing family $\langle M_\xi \rangle_{\xi \le \omega_1}$ of subsets of κ with cardinal $\mathfrak{c}$ such that whenever $\xi < \omega_1$, $I \in [M_\xi]^{\le\omega}$, $E' \in \mathcal{B}\mathfrak{a}_\kappa \setminus \mathcal{N}(\nu_\kappa)$ and a Baire measurable $g' : \{0,1\}^\kappa \to \{0,1\}^{\mathbb{N}}$ are determined by coordinates in M_ξ, then $K(I,E,g) \subseteq M_{\xi+1}$ (which is possible because if $\#(M_\xi) = \mathfrak{c}$ then there are just $\mathfrak{c}$ possibilities for I, E and g), we shall be able to set $M = M_{\omega_1}$ to get a set of the type we need. **Q**

Enumerate

$$\{W : W \in \mathcal{B}\mathfrak{a}_\kappa \widehat{\otimes} \mathcal{B}\mathfrak{a}_{\mathbb{N}},\ W \text{ is } M\text{-determined}\}$$

as $\langle W_\xi \rangle_{\xi < \mathfrak{c}}$. Then

$$\Vdash_{\mathbb{P}_\kappa} \dot{A} = \bigcup \{\vec{W}_\xi : \xi < \check{\mathfrak{c}},\ \vec{W}_\xi \subseteq \dot{A}\}.$$

P Suppose that $E \in \mathcal{B}\mathfrak{a}_\kappa \setminus \mathcal{N}(\nu_\kappa)$ and a $\mathbb{P}_\kappa$-name $\dot{x}$ are such that

$$E^\bullet \Vdash_{\mathbb{P}_\kappa} \dot{x} \in \dot{A}.$$

Then there is a Baire measurable function $g : \{0,1\}^\kappa \to \{0,1\}^{\mathbb{N}}$ such that

$$E^\bullet \Vdash_{\mathbb{P}_\kappa} \dot{x} = \vec{g}$$

(551Cc). Let $J \in [\kappa]^{\le\omega}$ be such that E and g are both determined by coordinates in J, and set $I = J \cap M$. By (v) above, there are E' and g', both determined by coordinates in M, such that

$$\Vdash_{\mathbb{P}_\kappa} \dot{\mu}_{IEg} = \dot{\mu}_{IE'g'}.$$

We therefore have M-determined F, $V \in \mathcal{B}\mathfrak{a}_\kappa \widehat{\otimes} \mathcal{B}\mathfrak{a}_{\mathbb{N}}$ such that

$$\Vdash_{\mathbb{P}_\kappa} \dot{A} \triangle \vec{F} \subseteq \vec{V} \text{ and } \dot{\mu}_{IEg} \vec{V} = \dot{\mu}_{IE'g'} \vec{V} = 0.$$

Let $\xi < \mathfrak{c}$ be such that $F \setminus V = W_\xi$. By (iv),

$$E^\bullet \Vdash_{\mathbb{P}_\kappa} \vec{g} \notin \vec{V} \text{ and } \dot{x} = \vec{g} \in \vec{W}_\xi \subseteq \dot{A}.$$

As E and $\dot{x}$ are arbitrary, we have the result. **Q**

Thus

$$\Vdash_{\mathbb{P}_\kappa} \dot{A} \text{ is expressible as the union of } \check{\mathfrak{c}} \text{ Borel sets.}$$

As $\dot{A}$ is arbitrary, (a) is proved.

(b) Now all we have to do is count. We surely have

$$\Vdash_{\mathbb{P}_\kappa} \text{there are at least } \mathfrak{c} \text{ universally measurable subsets of } \{0,1\}^{\mathbb{N}}$$

just because singletons are universally measurable. In the other direction, because $\kappa^{\mathfrak{c}} = \kappa$,

$$\Vdash_{\mathbb{P}_\kappa} \mathfrak{c}^{\check{\mathfrak{c}}} = (2^\omega)^{\check{\mathfrak{c}}} = 2^{\check{\mathfrak{c}}} = \check{\kappa} = (\kappa^\omega)\check{} = \mathfrak{c}$$

(552B), while (a) tells us that

$$\Vdash_{\mathbb{P}_\kappa} \text{the number of universally measurable subsets of } \{0,1\}^{\mathbb{N}} \text{ is at most } \mathfrak{c}^{\check{\mathfrak{c}}}.$$

553X Basic exercises (a)(i) Suppose that $A\subseteq\{0,1\}^{\mathbb{N}}$ has strong measure zero, and that κ is a cardinal. Show that

$$\Vdash_{\mathbb{P}_\kappa}\check{A}\text{ has strong measure zero in }\{0,1\}^{\mathbb{N}}.$$

(ii) Repeat with $\mathbb{R}$ in place of $\{0,1\}^{\mathbb{N}}$. (iii) Suppose that $\mathfrak{m}=\mathfrak{c}>\omega_1$. Show that

$$\Vdash_{\mathbb{P}_{\omega_1}}\text{ there is a set of strong measure zero in }\mathbb{R}\text{ with cardinal greater than }\mathfrak{m}_{\text{countable}}.$$

(b) Let $W\subseteq\{0,1\}^\omega\times\{0,1\}^\omega$ be the set

$$\{(x,y):x(2n)=y(2n)\text{ for every }n\in\mathbb{N}\}.$$

Show that, for every $y\in\{0,1\}^\omega$,

$$\Vdash_{\mathbb{P}_\omega}\vec{W}\text{ is homeomorphic to }\{0,1\}^\omega\text{ and }\check{y}\notin\vec{W}.$$

(c)(i) Suppose that $\mathcal{F}$ is a p-point filter on $\mathbb{N}$, and that $\mathbb{P}$ is a ccc forcing notion. Show that

$$\Vdash_{\mathbb{P}}\text{ the filter on }\mathbb{N}\text{ generated by }\check{\mathcal{F}}\text{ is a }p\text{-point filter.}$$

(ii) Suppose that $\mathcal{F}$ is a rapid filter on $\mathbb{N}$, and that κ is a cardinal. Show that

$$\Vdash_{\mathbb{P}_\kappa}\text{ the filter on }\mathbb{N}\text{ generated by }\check{\mathcal{F}}\text{ is a rapid filter.}$$

(d) Let $\mathfrak{A}$ be a Boolean algebra and $\nu:\mathfrak{A}\to[0,\infty[$ a non-negative additive functional. Show that if $\langle a_i\rangle_{i\in I}$ is a finite family in $\mathfrak{A}$ then

$$\nu(\sup_{i\in I}a_i)\le\sum_{k=1}^m(-1)^{k+1}\sum_{J\in[I]^k}\nu(\inf_{i\in J}a_i)\text{ if }m\ge1\text{ is odd,}$$

$$\nu(\sup_{i\in I}a_i)\ge\sum_{k=1}^m(-1)^{k+1}\sum_{J\in[I]^k}\nu(\inf_{i\in J}a_i)\text{ if }m\ge1\text{ is even.}$$

(e) Let $\mathbb{P}$ be a forcing notion which satisfies Knaster's condition. (i) Show that if $(P,\le)$ is an upwards-ccc partially ordered set then

$$\Vdash_{\mathbb{P}}(\check{P},\check{\le})\text{ is upwards-ccc.}$$

(ii) Show that if $(T,\le)$ is a Souslin tree then

$$\Vdash_{\mathbb{P}}(\check{T},\check{\le})\text{ is a Souslin tree.}$$

553Y Further exercises (a) Let κ be a cardinal, $\dot{G}$ a $\mathbb{P}_\kappa$-name and $a\in\mathfrak{B}_\kappa^+$ such that

$$a\Vdash_{\mathbb{P}_\kappa}\dot{G}\text{ is a dense open subset of }\{0,1\}^\omega.$$

Show that there is a $W\in\mathrm{T}_\kappa\widehat{\otimes}\mathcal{B}\mathfrak{a}_\omega$ such that every vertical section of W is a dense open set and $a\Vdash_{\mathbb{P}_\kappa}\dot{G}=\vec{W}$.

(b) Let κ be a cardinal and $W\in\mathrm{T}_\kappa\widehat{\otimes}\mathcal{B}\mathfrak{a}_\omega$ a set such that every vertical section of W is a dense open set. Let C be the space of continuous functions from $\{0,1\}^\kappa$ to $\{0,1\}^\omega$ with the compact-open topology. Show that $\{f:f\in C,\ \{x:(x,f(x))\in W\}\text{ is conegligible}\}$ is comeager in C.

(c) Show that

$$\Vdash_{\mathbb{P}_\omega}\mathfrak{m}_{\text{countable}}\ge(\mathfrak{m}_{\text{countable}})\check{}.$$

(*Hint*: work with the ideal of meager sets in the Polish space of continuous functions from $\{0,1\}^\omega$ to itself.)

(d) Let $\mathcal{K}$ be the family of compact well-ordered subsets of $\mathbb{Q}\cap[0,\infty[$ containing 0. For s, $t\in\mathcal{K}$ say that $s\preccurlyeq t$ if $s=t\cap[0,\gamma]$ for some $\gamma\in\mathbb{R}$; for $s\in\mathcal{K}$ and $\gamma\in\mathbb{Q}$, set $A(s,\gamma)=\{t:t\in\mathcal{K},\ \max t=\gamma,\ s\preccurlyeq t\}$. (i) Show that $(\mathcal{K},\preccurlyeq)$ is a tree, and that $\operatorname{otp}(t)=r(t)+1$ for every $t\in\mathcal{K}$. (ii) Choose $\langle\mathcal{K}_\xi\rangle_{\xi<\omega_1}$, $\langle T_\xi\rangle_{\xi<\omega_1}$ inductively so that $\mathcal{K}_0=T_0=\{\{0\}\}$ and for $0<\xi<\omega_1$

$\mathcal{K}_\xi=\{t:t\in\mathcal{K},\ r(t)=\xi,\ s\in\bigcup_{\eta<\xi}T_\eta\text{ whenever }s\prec t\}$,

$T_\xi \subseteq \mathcal{K}_\xi$ is countable,

if $\eta < \xi$, $s \in T_\eta$ and $\gamma \in \mathbb{Q}$ are such that $\gamma > \max s$ and $A(s,\gamma)$ meets $\mathcal{K}_\xi$, then $A(s,\gamma)$ meets T_ξ.

Show that if $\eta < \xi < \omega_1$, $s \in T_\eta$, $\gamma \in \mathbb{Q}$ and $\gamma > \max s$, there is a $t \in T_\xi$ such that $\max t = \gamma$ and $s \preccurlyeq t$. (iii) Show that $T = \bigcup_{\xi<\omega_1} T_\xi$ is a special Aronszajn tree.

(e) Show that $\Vdash_{\mathbb{P}_\omega} \mathfrak{p} \geq (\mathfrak{m}_{\sigma\text{-linked}})\check{}$.

553Z Problem Suppose that the generalized continuum hypothesis is true. Is it the case that

$$\Vdash_{\mathbb{P}_{\omega_2}} \text{there is a Borel lifting for Lebesgue measure?}$$

(Compare 554I.)

553 Notes and comments To my mind, the chief interest of the results of this section is that they force us to explore aspects of the structures considered in new ways. We know, for instance, that if a set has Rothberger's property (in a separable metrizable space) this can be witnessed by a family of $\mathfrak{d}$ sequences. The point of 553C is that (in random real models) any family of $\mathfrak{d}$ sequences is associated with a set $\dot{Y}$ with cardinal at most the cardinal power $\mathfrak{d}^\omega = \mathfrak{c}$ (taken in the ordinary universe V), such that $\dot{Y}$ must include the given set with Rothberger's property. Remember that

$$\Vdash_{\mathbb{P}_\kappa} (\mathbb{N}^\mathbb{N})\check{} \text{ is cofinal with } \mathbb{N}^\mathbb{N}$$

(see the proof of 552C), so there is no point in looking at 'new' members of $\mathbb{N}^\mathbb{N}$ in part (a) of the proof.

In 553E, we need to distinguish between the $\mathbb{P}_\kappa$-names $\tilde{G}$ and $\check{G}$. It is quite possible to have

$$\Vdash_{\mathbb{P}_\kappa} \dot{K} \cap (\{0,1\}^\lambda)\check{} = \emptyset;$$

that is, we might have $\Vdash_{\mathbb{P}_\kappa} \dot{K} = \vec{W}$ where $W \subseteq \{0,1\}^\kappa \times \{0,1\}^\lambda$ has negligible horizontal sections (553Xb). The name $\tilde{G}$ refers not to a copy of the set G but to a re-interpretation of one (or any) of its descriptions as an F_σ set.

In 553H and 553M, we have to look quite deeply into the structure of measure algebras. Lemmas 553G and 553L are already not obvious, and the combinatorial measure theory of the proof of 553H is delicate. 553J is easier. The idea here is to 'randomize' a construction from Galvin 80, where the continuum hypothesis was used to build complementary sets S_0, S_1 with the property of 553I.

I give a bit of space to 'Aronszajn trees' because the results here express yet another contrast between random and Cohen forcing. Cohen forcing creates Souslin trees (554Yc). Random forcing preserves old Souslin trees (553Xe) but does not necessarily produce new ones (553M).

Version of 2.9.14

554 Cohen reals

Parallel to the theory of random reals as described in §§552-553, we have a corresponding theory based on category algebras rather than measure algebras. I start with the exactly matching result on cardinal arithmetic (554B), and continue with Lusin sets (balancing the Sierpiński sets of 552E) and the cardinal functions of the meager ideal of $\mathbb{R}$ (554C-554E, 554F). In the last third of the section I use the theory of Freese-Nation numbers (§518) to prove Carlson's theorem on Borel liftings (554I).

554A Notation For any set I, I will write $\widehat{\mathcal{B}}_I$ for the Baire-property algebra of $\{0,1\}^I$, $\mathcal{B}\mathfrak{a}_I$ for the Baire σ-algebra of $\{0,1\}^I$, $\mathcal{M}_I$ for the meager ideal of $\{0,1\}^I$, $\mathfrak{G}_I = \widehat{\mathcal{B}}_I/\mathcal{M}_I$ for the category algebra of $\{0,1\}^I$, and $\mathbb{Q}_I$ for the forcing notion $\mathfrak{G}_I^+ = \mathfrak{G}_I \setminus \{\emptyset\}$ active downwards. $\mathcal{C}_I$ will be the family of basic cylinder sets $\{x : z \subseteq x \in \{0,1\}^I\}$ for $z \in \mathrm{Fn}_{<\omega}(I;\{0,1\})$, and C_I the corresponding set $\{C^\bullet : C \in \mathcal{C}_I\} \subseteq \mathfrak{G}_I$; then C_I is order-dense in $\mathfrak{G}_I$ (because $\mathcal{C}_I$ is a π-base for the topology of $\{0,1\}^I$). It follows that $\tau(\mathfrak{G}_I) \leq \pi(\mathfrak{G}_I) \leq \max(\omega, \#(I))$. (These inequalities are of course equalities if I is infinite.)

554B Theorem Suppose that λ and κ are infinite cardinals. Then

$$\Vdash_{\mathbb{Q}_\kappa} 2^{\check{\lambda}} = (\kappa^\lambda)\check{}.$$

proof (Compare 552B.)

(a) Since $\mathfrak{G}_\kappa$ is ccc and has an order-dense subset C_κ with cardinal κ, $\#(\mathfrak{G}_\kappa)$ is at most the cardinal power κ^ω.

If $\dot{A}$ is a $\mathbb{Q}_\kappa$-name for a subset of $\check{\lambda}$, then we have a corresponding family $\langle [\![\check{\eta} \in \dot{A}]\!]\rangle_{\eta<\lambda}$ of truth values; and if $\dot{A}$, $\dot{B}$ are two such names, and $[\![\check{\eta} \in \dot{A}]\!] = [\![\check{\eta} \in \dot{B}]\!]$ for every $\eta < \lambda$, then

$$\Vdash_{\mathbb{Q}_\kappa} \dot{A} = \dot{B}.$$

So

$$\Vdash_{\mathbb{Q}_\kappa} 2^{\check{\lambda}} = \#(\mathcal{P}\check{\lambda}) \le \#((\mathfrak{G}_\kappa^\lambda)\check{}) = (\kappa^\lambda)\check{}.$$

(b) Consider first the case in which $\lambda \le \kappa$. Let F be the set of all functions from λ to κ, so that $\#(F) = \kappa^\lambda$. As in part (b) of the proof of 552B, there is a set $G \subseteq F$ such that $\#(G) = \kappa^\lambda$ and $\{\eta : \eta < \lambda, f(\eta) \ne g(\eta)\}$ is infinite whenever f, $g \in G$ are distinct. Let $\langle \zeta_{\xi\eta}\rangle_{\xi<\kappa,\eta<\lambda}$ be a family of distinct elements of κ and set $E_{\xi\eta} = \{x : x \in \{0,1\}^\kappa, x(\zeta_{\xi\eta}) = 1\}$ for $\xi < \kappa$ and $\eta < \lambda$. For $f \in G$ let $\dot{A}_f$ be a $\mathbb{Q}_\kappa$-name for a subset of λ such that

$$[\![\check{\eta} \in \dot{A}_f]\!] = E^\bullet_{f(\eta),\eta}$$

for every $\eta < \lambda$. If f, $g \in G$ are distinct, set $I = \{\eta : f(\eta) \ne g(\eta)\}$; then

$$[\![\dot{A}_f \ne \dot{A}_g]\!] = \sup_{\eta<\lambda} E^\bullet_{f(\eta),\eta} \,\triangle\, E^\bullet_{g(\eta),\eta} = 1$$

because $\bigcup_{\eta\in I} E_{f(\eta),\eta} \triangle E_{g(\eta),\eta}$ is a dense open set in $\{0,1\}^\kappa$.

Thus in the forcing language we have a name for an injective function from $\check{G}$ to $\mathcal{P}\lambda$, corresponding to the map $f \mapsto \dot{A}_f$ from G to names of subsets of λ. So

$$\Vdash_{\mathbb{Q}_\kappa} 2^{\check{\lambda}} \ge \#(\check{G}) = (\kappa^\lambda)\check{}.$$

Putting this together with (a), we have

$$\Vdash_{\mathbb{Q}_\kappa} 2^{\check{\lambda}} = (\kappa^\lambda)\check{}.$$

(c) If $\lambda > \kappa$, then $2^\lambda = \kappa^\lambda$. Now

$$\Vdash_{\mathbb{Q}_\kappa} (\mathcal{P}\lambda)\check{} \subseteq \mathcal{P}\check{\lambda},$$

so

$$\Vdash_{\mathbb{Q}_\kappa} (\kappa^\lambda)\check{} = \#((\mathcal{P}\lambda)\check{}) \le \#(\mathcal{P}\check{\lambda}) = 2^{\check{\lambda}},$$

and again we have

$$\Vdash_{\mathbb{Q}_\kappa} 2^{\check{\lambda}} = (\kappa^\lambda)\check{}.$$

554C Definition If X is a topological space, a subset of X is a **Lusin set** if it is uncountable but meets every meager set in a countable set; equivalently, if it is uncountable but meets every nowhere dense set in a countable set.

554D Proposition Let κ be a cardinal such that $\mathbb{R}$ has a Lusin set with cardinal κ.

(a) Writing $\mathcal{M}$ for the ideal of meager subsets of $\mathbb{R}$, $\operatorname{non}\mathcal{M} = \omega_1$ and $\mathfrak{m}_{\text{countable}} \ge \kappa$.

(b) There is a point-countable family $\mathcal{A}$ of Lebesgue-conegligible subsets of $\mathbb{R}$ with $\#(\mathcal{A}) = \kappa$.

(c) If $(\mathfrak{A}, \bar{\mu})$ is a semi-finite measure algebra which is not purely atomic, (κ, ω_1) is not a precaliber pair of $\mathfrak{A}$.

proof Let $B \subseteq \mathbb{R}$ be a Lusin set with cardinal κ.

(a) By 522Sa, $\mathfrak{m}_{\text{countable}} = \operatorname{cov}\mathcal{M}$. Any uncountable subset of B is non-meager, so $\operatorname{non}\mathcal{M} = \omega_1$. If $\mathcal{E}$ is a cover of $\mathbb{R}$ by meager sets, then each member of $\mathcal{E}$ meets B in a countable set, so

$$\kappa = \#(B) \le \max(\omega, \#(\mathcal{E}))$$

and $\#(\mathcal{E}) \ge \kappa$; thus $\operatorname{cov}\mathcal{M} \ge \kappa$.

(b) Let $E \subseteq \mathbb{R}$ be a conegligible meager set containing 0, and set $\mathcal{A} = \{x + E : x \in B\}$. Then $\mathcal{A}$ is a family of conegligible sets. If $y \in \mathbb{R}$, then $y - E$ is meager so $\{x : x \in B,\ y \in x + E\} = B \cap (y - E)$ is countable; thus $\mathcal{A}$ is point-countable. Also, each member of $\mathcal{A}$ is meager, so meets B in a countable set, and (because $B \subseteq \bigcup \mathcal{A}$)

$$\kappa = \#(B) \le \max(\omega, \#(\mathcal{A})) \le \#(\mathcal{A}) \le \#(B),$$

so $\#(\mathcal{A}) = \kappa$.

(c) Let $K \subseteq E$ be a compact set of non-zero measure. If $\Gamma \subseteq B$ is uncountable, $\bigcap_{x\in\Gamma} x + K = \emptyset$, $\{x + K : x \in \Gamma\}$ does not have the finite intersection property and $\{(x + K)^\bullet : x \in \Gamma\}$ is not centered in the measure algebra $\mathfrak{A}_L$ of Lebesgue measure. Thus $\langle (x + K)^\bullet \rangle_{x\in B}$ witnesses that (κ, ω_1) is not a precaliber pair of $\mathfrak{A}_L$.

Since $(\mathfrak{A}, \bar\mu)$ is semi-finite and not purely atomic, there is a subalgebra of a principal ideal of $\mathfrak{A}$ which is isomorphic to $\mathfrak{A}_L$, and (κ, ω_1) is not a precaliber pair of $\mathfrak{A}$, by 516Sa.

554E Theorem Let κ be an uncountable cardinal. Then

$$\Vdash_{\mathbb{Q}_\kappa} \text{there is a Lusin set } A \subseteq \mathbb{R} \text{ with cardinal } \check\kappa.$$

proof (a) (Compare 552E.) Write $\mathbb{P}$ for $\mathbb{Q}_{\kappa\times\omega}$. For each $\xi < \kappa$, let $f_\xi : \{0,1\}^{\kappa\times\omega} \to \{0,1\}^\omega$ be given by setting $f_\xi(x)(n) = x(\xi, n)$ for every $x \in \{0,1\}^{\kappa\times\lambda}$ and $n < \omega$; then, taking $\vec f_\xi$ to be the $\mathbb{P}$-name defined by the process of 551Cb,

$$\Vdash_{\mathbb{P}} \vec f_\xi \in \{0,1\}^\omega.$$

If ξ, $\xi' < \kappa$ are distinct, then, by 551Cd,

$$\begin{aligned}[\![\vec f_\xi = \vec f_{\xi'}]\!] &= \{x : f_\xi(x) = f_{\xi'}(x)\}^\bullet \\ &= \{x : x(\xi, n) = x(\xi', n) \text{ for every } n\}^\bullet = 0\end{aligned}$$

because $\{x : x(\xi, n) = x(\xi', n) \text{ for every } n\}$ is closed and nowhere dense. So, taking $\dot A$ to be the $\Vdash_{\mathbb{P}}$-name $\{(\vec f_\xi, \mathbb{1}) : \xi < \kappa\}$, we have

$$\Vdash_{\mathbb{P}} \dot A \subseteq \{0,1\}^\omega \text{ has cardinal } \check\kappa.$$

(b) Now suppose that $\dot W$ is a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \dot W \text{ is a nowhere dense zero set in } \{0,1\}^\omega.$$

By 551Fb there is a $W \in \widehat{\mathcal{B}}_{\kappa\times\omega} \widehat\otimes \mathcal{B}\mathfrak{a}_\omega$ such that, in the language of 551D, $\Vdash_{\mathbb{P}} \dot W = \vec W$. Now W is meager in $\{0,1\}^{\kappa\times\omega}$. **P** For $z \in \operatorname{Fn}_{<\omega}(\omega; \{0,1\})$ set $V_z = \{(x, y) : x \in \{0,1\}^{\kappa\times\omega},\ z \subseteq y \in \{0,1\}^\omega\}$. By 551Ee,

$$\Vdash_{\mathbb{P}} \vec V_z = \{y : \check z \subseteq y \in \{0,1\}^\omega\};$$

and as $\widehat{\mathcal{B}}_{\kappa\times\omega}$ is closed under Souslin's operation (431Fb),

$$[\![\vec W \cap \vec V_z = \emptyset]\!] = \{x : W[\{x\}] \cap \{y : y \supseteq z\} = \emptyset\}^\bullet$$

(551Ga). Now we have

$$\begin{aligned}1 &= [\![\vec{W} \text{ is nowhere dense}]\!]\\ &= \inf_{z\in\mathrm{Fn}_{<\omega}(\omega;\{0,1\})} \sup_{z'\in\mathrm{Fn}_{<\omega}(\omega;\{0,1\}),z'\supseteq z} [\![\vec{W}\cap\{y:\check{z}'\subseteq y\}=\emptyset]\!]\\ &= \inf_{z\in\mathrm{Fn}_{<\omega}(\omega;\{0,1\})} \sup_{z'\in\mathrm{Fn}_{<\omega}(\omega;\{0,1\}),z'\supseteq z} \{x:W[\{x\}]\cap\{y:y\supseteq z'\}=\emptyset\}^\bullet\\ &= \Bigl(\bigcap_{z\in\mathrm{Fn}_{<\omega}(\omega;\{0,1\})} \bigcup_{z'\in\mathrm{Fn}_{<\omega}(\omega;\{0,1\}),z'\supseteq z} \{x:W[\{x\}]\cap\{y:y\supseteq z'\}=\emptyset\}\Bigr)^\bullet\\ &= \{x:W[\{x\}] \text{ is nowhere dense}\}^\bullet.\end{aligned}$$

So $\{x:W[\{x\}]$ is meager$\}$ is comeager. Because W has the Baire property in $\{0,1\}^{\kappa\times\omega}\times\{0,1\}^\omega$ (5A4E(b-ii)), it must be meager, by the Kuratowski-Ulam theorem (527D). **Q**

(c) Continuing from (b), there is a meager Baire set $W'\supseteq W$ (5A4E(c-ii)). Let $J\subseteq\kappa$ be a countable set such that W' is determined by coordinates in $(J\times\omega)\dot\cup\,\omega$, that is, if $(x,y)\in W'$, $x'\in\{0,1\}^{\kappa\times\omega}$ and $x'\restriction J\times\omega=x\restriction J\times\omega$ then $(x',y)\in W'$. Take any $\xi\in\kappa\setminus J$. Set $L=(\kappa\setminus\{\xi\})\times\omega$ and

$$V=\{(x\restriction L,y):(x,y)\in W'\};$$

then $V\subseteq\{0,1\}^L\times\{0,1\}^\omega$ is meager (applying 527D to

$$V\times\{0,1\}^{\{\xi\}\times\omega}\subseteq\{0,1\}^L\times\{0,1\}^{\{\xi\}\times\omega}\cong\{0,1\}^{\kappa\times\omega}\times\{0,1\}^\omega).$$

Now consider the map $\phi:\{0,1\}^{\kappa\times\omega}\to\{0,1\}^L\times\{0,1\}^\omega$ defined by setting $\phi(x)=(x\restriction L,f_\xi(x))$ for $x\in\{0,1\}^{\kappa\times\omega}$. Looking back at the definition of f_ξ, we see that this is a homeomorphism. So $\phi^{-1}[V]$ must be meager, and

$$[\![\vec{f}_\xi\in\vec{W}]\!]\subseteq[\![\vec{f}_\xi\in\vec{W}']\!]=\{x:(x,f_\xi(x))\in W'\}^\bullet$$

(551Ea)

$$=\{x:(x\restriction L,f_\xi(x))\in V\}^\bullet=(\phi^{-1}[V])^\bullet=0,$$

that is, $\Vdash_{\mathbb{P}}\vec{f}_\xi\notin\vec{W}$.

This is true for every $\xi\in\kappa\setminus J$. So

$$\Vdash_{\mathbb{P}}\dot{A}\cap\dot{W}\subseteq\{\vec{f}_\xi:\xi\in\check{J}\}\text{ is countable.}$$

As $\dot{W}$ is arbitrary,

$$\Vdash_{\mathbb{P}}\dot{A}\text{ has countable intersection with every nowhere dense zero set.}$$

It follows at once that

$$\Vdash_{\mathbb{P}}\dot{A}\text{ has countable intersection with every nowhere dense set, and is a Lusin set.}$$

As $\mathbb{P}$ and $\mathbb{Q}_\kappa$ are isomorphic,

$$\Vdash_{\mathbb{Q}_\kappa}\{0,1\}^\omega\text{ has a Lusin set with cardinal }\check{\kappa}.$$

(d) The statement of the proposition referred to $\mathbb{R}$ rather than to $\{0,1\}^\omega$. But, writing $\mathcal{M}$ for the ideal of meager subsets of $\mathbb{R}$ and $\mathcal{M}_\omega$ for the ideal of meager subsets of $\{0,1\}^\omega$, $(\mathbb{R},\mathcal{M})$ and $(\{0,1\}^\omega,\mathcal{M}_\omega)$ are isomorphic (522Wb), and one will have Lusin sets iff the other does. So

$$\Vdash_{\mathbb{Q}_\kappa}\mathbb{R}\text{ has a Lusin set with cardinal }\check{\kappa}.$$

554F Corollary Let κ be a cardinal which is equal to the cardinal power κ^ω. Write $\mathcal{M}$ for the ideal of meager subsets of $\mathbb{R}$. Then

$$\Vdash_{\mathbb{Q}_\kappa}\operatorname{non}\mathcal{M}=\omega_1\text{ and }\mathfrak{m}_{\mathrm{countable}}=\mathfrak{c}.$$

proof By 554B, $\Vdash_{\mathbb{Q}_\kappa}\mathfrak{c}=\check{\kappa}$; so we have only to put 554E and 554Da together.

554G Theorem Let κ be an infinite cardinal such that $\mathrm{FN}(\mathfrak{G}_\kappa)=\omega_1$. Then

$$\Vdash_{\mathbb{Q}_\kappa} \mathrm{FN}(\mathcal{P}\mathbb{N})=\omega_1.$$

proof (a) We need to know that $\mathfrak{G}_\kappa$ is isomorphic to the simple power algebra $\mathfrak{G}_\kappa^{\mathbb{N}}$. **P** The algebra $\mathcal{E}$ of open-and-closed subsets of $\{0,1\}^\kappa$ is isomorphic to a free product of two-element algebras, so is homogeneous (316Q); $\mathfrak{G}_\kappa$ is isomorphic to the Dedekind completion of $\mathcal{E}$, so is homogeneous (316P). Now we have a partition of unity $\langle p_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{G}_\kappa$ consisting of non-zero elements, so that $\mathfrak{G}_\kappa$ is isomorphic to the simple product of the corresponding principal ideals (315F) and to $\mathfrak{G}_\kappa^{\mathbb{N}}$. **Q** There is therefore a Freese-Nation function $\theta:\mathfrak{G}_\kappa^{\mathbb{N}}\to[\mathfrak{G}_\kappa^{\mathbb{N}}]^{\le\omega}$.

For $\xi<\kappa$, set $E_\xi=\{x:x\in\{0,1\}^\kappa,\ x(\xi)=1\}$; for $J\subseteq\kappa$, let $\mathfrak{C}_J$ be the order-closed subalgebra of $\mathfrak{G}_\kappa$ generated by $\{E_\xi^\bullet:\xi\in J\}$, and let C_J be the set of elements of $\mathfrak{C}_J$ of the form $\inf_{\xi\in K}E_\xi^\bullet\setminus\sup_{\xi\in L}E_\xi^\bullet$ where K, L are disjoint finite subsets of J.

For $v\in\mathfrak{G}_\kappa^{\mathbb{N}}$ let $\vec{v}$ be the $\mathbb{Q}_\kappa$-name $\{(\check{n},v(n)):n\in\mathbb{N},\ v(n)\ne 0\}$; then $\Vdash_{\mathbb{Q}_\kappa}\vec{v}\subseteq\mathbb{N}$, and $[\![\check{n}\in\vec{v}]\!]=v(n)$ for every $n\in\mathbb{N}$.

For any $\mathbb{Q}_\kappa$-name $\dot{u}$, let $J(\dot{u})$ be a countable subset of κ such that $[\![\check{n}\in\dot{u}]\!]\in\mathfrak{C}_{J(\dot{u})}$ for every $n\in\mathbb{N}$.

(b) Let $\dot{X}$ be a discriminating $\mathbb{Q}_\kappa$-name such that $\Vdash_{\mathbb{Q}_\kappa}\dot{X}=\mathcal{P}\mathbb{N}$ (5A3Ka). For $\sigma=(\dot{u},p)\in\dot{X}$ set

$$\theta_1(\sigma)=\bigcup_{e\in C_{J(\dot{u})}}\theta(\langle[\![\check{n}\in\dot{u}]\!]\cap e\rangle_{n\in\mathbb{N}})\cup\bigcup_{e\in C_{J(\dot{u})}}\theta(\langle[\![\check{n}\in\dot{u}]\!]\cup(1\setminus e)\rangle_{n\in\mathbb{N}})\in[\mathfrak{G}_\kappa^{\mathbb{N}}]^{\le\omega},$$

$$\theta_2(\sigma)=\{(\vec{v},p):v\in\theta_1(\sigma)\},$$

so that $\theta_2(\sigma)$ is a $\mathbb{Q}_\kappa$-name and

$$\Vdash_{\mathbb{Q}_\kappa}\theta_2(\sigma)\text{ is a countable subset of }\mathcal{P}\mathbb{N}.$$

(c) Set

$$\dot{\theta}=\{((\dot{u},\theta_2(\dot{u},p)),p):(\dot{u},p)\in\dot{X}\}.$$

By 5A3Kb,

$$\Vdash_{\mathbb{Q}_\kappa}\dot{\theta}\text{ is a function with domain }\dot{X}=\mathcal{P}\mathbb{N}.$$

Next,

$$\Vdash_{\mathbb{Q}_\kappa}\dot{\theta}\text{ takes values in }[\mathcal{P}\mathbb{N}]^{\le\omega}.$$

P Suppose that $\dot{x}$ is a $\mathbb{Q}_\kappa$-name and $p\in\mathfrak{G}_\kappa^+$ is such that

$$p\Vdash_{\mathbb{Q}_\kappa}\dot{x}\text{ is a value of }\dot{\theta}.$$

Then there are a $(\dot{u},q)\in\dot{X}$ and a p' stronger than both p and q such that

$$p'\Vdash_{\mathbb{Q}_\kappa}\dot{x}=(\dot{u},\theta_2(\dot{u},q))\text{ has second member }\theta_2(\dot{u},q)\in[\mathcal{P}\mathbb{N}]^{\le\omega}.$$

As p and $\dot{x}$ are arbitrary,

$\Vdash_{\mathbb{Q}_\kappa}$ every value of $\dot{\theta}$, being the second member of an element of $\dot{\theta}$, is a countable subset of $\mathcal{P}\mathbb{N}$. **Q**

(d) In fact,

$$\Vdash_{\mathbb{Q}_\kappa}\dot{\theta}\text{ is a Freese-Nation function on }\mathcal{P}\mathbb{N}.$$

P Suppose that $\dot{A}_1$, $\dot{A}_2$ are $\mathbb{Q}_\kappa$-names and $p\in\mathfrak{G}_\kappa^+$ is such that

$$p\Vdash_{\mathbb{Q}_\kappa}\dot{A}_1\subseteq\dot{A}_2\subseteq\mathbb{N}.$$

Because $\Vdash_{\mathbb{Q}_\kappa}\dot{X}=\mathcal{P}\mathbb{N}$, there must be $(\dot{u}_1,q_1)$ and $(\dot{u}_2,q_2)\in\dot{X}$ and a p_1 stronger than p, q_1 and q_2 such that

$$p_1\Vdash_{\mathbb{Q}_\kappa}\dot{u}_1=\dot{A}_1\text{ and }\dot{u}_2=\dot{A}_2.$$

In this case, for both i, $((\dot{u}_i,\theta_2(\dot{u}_i,q_i)),q_i)\in\dot{\theta}$, so we have

$$p_1 \Vdash_{\mathbb{Q}_\kappa} \dot\theta(\dot A_i) = \dot\theta(\dot u_i) = \theta_2(\dot u_i, q_i).$$

Let $e \subseteq p_1$ be a member of C_κ, that is, a member of $\mathfrak{G}_\kappa$ which is the equivalence class of a basic cylinder set. We have

$$e \Vdash_{\mathbb{Q}_\kappa} \dot u_1 = \dot A_1 \subseteq \dot A_2 = \dot u_2,$$

so $e \cap [\![\check n \in \dot u_1]\!] \subseteq [\![\check n \in \dot u_2]\!]$ for every $n \in \mathbb{N}$. Express e as $e_1 \cap e_2 \cap e_3$ where $e_1 \in C_{J(\dot u_1)}$, $e_2 \in C_{J(\dot u_2)}$ and $e_3 \in C_{\kappa \setminus K}$, where $K = J(\dot u_1) \cup J(\dot u_2)$. For each $n \in \mathbb{N}$,

$$e_1 \cap e_2 \cap [\![\check n \in \dot u_1]\!] \setminus [\![\check n \in \dot u_2]\!]$$

belongs to $\mathfrak{C}_K$ and is disjoint from $e_3 \in \mathfrak{C}_{\kappa\setminus K} \setminus \{0\}$, so must be zero; we therefore have

$$e_1 \cap [\![\check n \in \dot u_1]\!] \subseteq [\![\check n \in \dot u_2]\!] \cup (1 \setminus e_2)$$

for every n, that is,

$$\langle [\![\check n \in \dot u_1]\!] \cap e_1 \rangle_{n\in\mathbb{N}} \subseteq \langle [\![\check n \in \dot u_2]\!] \cup (1 \setminus e_2) \rangle_{n\in\mathbb{N}}$$

in $\mathfrak{G}_\kappa^{\mathbb{N}}$. Because θ is a Freese-Nation function, there is a sequence

$$\langle a_n \rangle_{n\in\mathbb{N}} \in \theta(\langle [\![\check n \in \dot u_1]\!] \cap e_1 \rangle_{n\in\mathbb{N}}) \cap \theta(\langle [\![\check n \in \dot u_2]\!] \cup (1 \setminus e_2) \rangle_{n\in\mathbb{N}})$$

such that

$$[\![\check n \in \dot u_1]\!] \cap e_1 \subseteq a_n \subseteq [\![\check n \in \dot u_2]\!] \cup (1 \setminus e_2)$$

for every n. Now $v = \langle a_n \rangle_{n\in\mathbb{N}}$ belongs to $\theta_1(\dot u_1, p_1) \cap \theta_1(\dot u_2, p_2)$, so $(\vec v, p_i) \in \theta_2(\dot u_i, p_i)$ and

$$p_i \Vdash_{\mathbb{Q}_\kappa} \vec v \in \theta_2(\dot u_i, p_i) = \dot\theta(\dot u_i)$$

for both i. Returning to e, we have

$$e \cap [\![\check n \in \dot u_1]\!] \subseteq e \cap a_n \subseteq e \cap [\![\check n \in \dot u_2]\!]$$

for every n, because $e \subseteq e_1 \cap e_2$. So

$$e \Vdash_{\mathbb{Q}_\kappa} \dot u_1 \subseteq \vec v \subseteq \dot u_2.$$

Also e is stronger than p and

$$e \Vdash_{\mathbb{Q}_\kappa} \vec v \in \dot\theta(\dot u_1) \cap \dot\theta(\dot u_2) = \dot\theta(\dot A_1) \cap \dot\theta(\dot A_2).$$

As p, $\dot A_1$ and $\dot A_2$ are arbitrary,

$\Vdash_{\mathbb{Q}_\kappa}$ for any A, $B \subseteq \mathbb{N}$ there is a $C \in \dot\theta(A) \cap \dot\theta(B)$ such that $A \subseteq C \subseteq B$; that is, $\dot\theta$ is a Freese-Nation function. **Q**

(e) Putting (c) and (d) together, we have

$$\Vdash_{\mathbb{Q}_\kappa} \mathrm{FN}(\mathcal{P}\mathbb{N}) \le \omega_1;$$

and since the Freese-Nation number of $\mathcal{P}\mathbb{N}$ is surely uncountable (522U), this is enough.

554H Corollary Suppose that $\mathrm{FN}(\mathcal{P}\mathbb{N}) = \omega_1$ and that κ is an infinite cardinal such that

(α) $\mathrm{cf}[\lambda]^{\le\omega} \le \lambda^+$ for every cardinal $\lambda \le \kappa$,

(β) $\square_\lambda$ is true for every uncountable cardinal $\lambda \le \kappa$ of countable cofinality.

Then $\Vdash_{\mathbb{Q}_\kappa} \mathrm{FN}(\mathcal{P}\mathbb{N}) = \omega_1$.

proof Any countably generated order-closed subalgebra $\mathfrak{C}$ of $\mathfrak{G}_\kappa$ is (in the language of part (a) of the proof of 554G) included in $\mathfrak{C}_J$ for some countable $J \subseteq \kappa$, which has a countable π-base C_J; so $\mathfrak{C}_J$ and $\mathfrak{C}$ are σ-linked, and $\mathrm{FN}(\mathfrak{C}) \le \mathrm{FN}(\mathcal{P}\mathbb{N}) = \omega_1$, by 518D. By 518I, the conditions (α) and (β), together with the fact that $\tau(\mathfrak{G}_\kappa) \le \kappa$, now ensure that $\mathrm{FN}(\mathfrak{G}_\kappa) \le \omega_1$, so 554G gives the result.

554I Theorem (Carlson Frankiewicz & Zbierski 94) Suppose that the continuum hypothesis is true. Then

$$\Vdash_{\mathbb{Q}_{\omega_2}} \mathfrak{c} = \omega_2 \text{ and Lebesgue measure has a Borel lifting.}$$

proof Of course the cardinal power ω_2^ω (in the ordinary universe) is equal to $\max(\mathfrak{c}, \mathrm{cf}[\omega_2]^{\le\omega}) = \omega_2$. From 554H and 554B we see that

$$\Vdash_{\mathbb{Q}_{\omega_2}} \mathrm{FN}(\mathcal{P}\mathbb{N}) = \omega_1 \text{ and } \mathfrak{c} = \omega_2.$$

So 535E(b-ii) tells us that

$$\Vdash_{\mathbb{Q}_{\omega_2}} \text{Lebesgue measure has a Borel lifting.}$$

554X Basic exercises (a) Show that $\#(\mathfrak{G}_\kappa) = \kappa^\omega$ for every infinite cardinal κ.

(b) Show that if I is any set, every regular uncountable cardinal is a precaliber of $\mathfrak{G}_I$.

(c) Let I be any set. (i) Show that $(\mathcal{C}_I, \supseteq)$ is isomorphic to $(\mathrm{Fn}_{<\omega}(I;\{0,1\}), \subseteq)$ (definition: 552A). (ii) Show that $\mathfrak{G}_I$ can be identified with the regular open algebra $\mathrm{RO}^\uparrow(\mathrm{Fn}_{<\omega}(I;\{0,1\}))$.

(d) Let κ be an infinite cardinal such that $\mathbb{R}$ has a Lusin set with cardinal κ. Show that there is a first-countable compact Hausdorff space X such that $\kappa \in \mathrm{Mah}_{\mathrm{R}}(X)$. (*Hint*: 531N.)

(e) Devise a definition of 'strongly Lusin' set to match 537Ab, and state and prove a result corresponding to 552E. (*Hint*: 527Xf.)

(f) Describe Cichoń's diagram in the forcing universe $V^{\mathbb{Q}_{\omega_2}}$ (i) if we start with $\mathfrak{c} = \omega_1$ (ii) if we start with $\mathfrak{m} = \mathfrak{c} = \omega_2$.

554Y Further exercises (a) For how many of the results of 552F-552J can you find equivalents with respect to Cohen real forcing? (*Hint*: Bartoszyński & Judah 95.)

(b)(i) Show that there is a family $\langle e_\xi\rangle_{\xi<\omega_1}$ such that (α) for each ξ, $e_\xi \subseteq \xi\times\mathbb{N}$ is an injective function from ξ to $\mathbb{N}$ (β) if $\eta \le \xi < \omega_1$ then $e_\eta \setminus e_\xi$ is finite. (*Hint*: choose the e_ξ inductively, taking care that $\mathbb{N}\setminus e_\xi[\xi]$ is infinite for every ξ.) (ii) Set $T = \{e_\xi{\restriction}\eta : \eta, \xi < \omega_1 \text{ are successor ordinals}\}$. Show that $T\cup\{\emptyset\}$, ordered by $\subseteq$, is a special Aronszajn tree. (*Hint*: for any $n \in \mathbb{N}$, $\{t : t(\max(\mathrm{dom}\, t)) = n\}$ is an antichain.)

(c) (Todorčević 87) Let κ be an infinite cardinal. Take $\langle e_\xi\rangle_{\xi<\omega_1}$ as in 554Yb. Let $\dot{\preccurlyeq}$ be the $\mathbb{Q}_\kappa$-name

$$\{((\check\eta, \check\xi), p) : \eta \le \xi < \omega_1,\, p \in \mathbb{Q}_\kappa,\, p \subseteq \{x : xe_\eta \subseteq xe_\xi\}^\bullet\}.$$

Show that $\Vdash_{\mathbb{Q}_\kappa} (\omega_1, \dot{\preccurlyeq})$ is a Souslin tree.

554 Notes and comments The original theories of Cohen and random reals were developed in parallel; see Kunen 84 for an account of the special properties of null and meager ideals which made this possible. Thus the Sierpiński sets of random real models become Lusin sets in Cohen real models, and the horizontal gap which appears in Cichoń's diagram if we add random reals becomes a vertical gap if we add Cohen reals (552F-552I, 554F). I give a very much briefer account of Cohen reals because I am restricting attention to results which have consequences in measure theory, as in 554Dc and 554I, and (except in 554Yc/553M) I make no attempt to look for reflections of the patterns in §553, which are mostly there for the illumination they throw on the structure of measure algebras. But I do not seek out the shortest route in every case. In particular, I spell out some of the theory of Freese-Nation numbers (554G-554H) for its own sake as well as to provide a proof of Carlson's theorem 554I. Let me remind you that ω_2 has a very special place in the arguments here; see 518Rb and 535Zb.

I have written this section in terms of forcing with category algebras, partly in order to emphasize the connexion with random reals, and partly to be able to quote from §551. But of course it can equally be regarded as a fragment of the theory of forcing with partially ordered sets $\mathrm{Fn}_{<\omega}(I;\{0,1\})$ (554Xc), and there are many places (e.g. 554Yc) where this simplifies the details.

Version of 23.10.14

555 Solovay's construction of real-valued-measurable cardinals

While all the mathematical ideas of Chapter 54 were expressed as arguments in ZFC, many would be of little interest if it appeared that there could be no atomlessly-measurable cardinals. In this section I present R.M.Solovay's theorem that if there is a two-valued-measurable cardinal in the original universe, then there is a forcing notion $\mathbb{P}$ such that

$$\Vdash_{\mathbb{P}} \text{there is an atomlessly-measurable cardinal}$$

(555D). Varying $\mathbb{P}$ we find that we can force models with other kinds of quasi-measurable cardinal (555G, 555K); starting from a stronger hypothesis we can reach the normal measure axiom (555N).

555A Notation As in §§552-553, I will write $(\mathfrak{B}_\kappa, \bar{\nu}_\kappa)$ for the measure algebra of the usual measure on $\{0,1\}^\kappa$, and $\mathbb{P}_\kappa$ for the forcing notion $\mathfrak{B}_\kappa^+ = \mathfrak{B}_\kappa \setminus \{0\}$, active downwards. In this context, as in 525A, $\langle e_\eta \rangle_{\eta<\kappa}$ will be the standard generating family in $\mathfrak{B}_\kappa$.

As in §554, I will write $\mathfrak{G}_\kappa$ for the category algebra of $\{0,1\}^\kappa$, and $\mathbb{Q}_\kappa$ for the forcing notion $\mathfrak{G}_\kappa^+$, active downwards. Recall that $\mathfrak{G}_\kappa$ is isomorphic to the regular open algebra $\mathrm{RO}(\{0,1\}^\kappa)$ (514If).

555B Theorem Suppose that X is a set, and $\mathcal{I}$ a proper σ-ideal of subsets of X containing singletons. Let $\mathbb{P} = (P, \le, \mathbb{1}, \Updownarrow)$ be a ccc forcing notion, and $\dot{\mathcal{I}}$ a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \dot{\mathcal{I}} = \{J : \text{there is an } I \in \check{\mathcal{I}} \text{ such that } J \subseteq I\}.$$

Then

(a)(i) If $\dot{J}$ is a $\mathbb{P}$-name and $p \in P$ is such that $p \Vdash_{\mathbb{P}} \dot{J} \in \dot{\mathcal{I}}$, there is an $I \in \mathcal{I}$ such that $p \Vdash_{\mathbb{P}} \dot{J} \subseteq \check{I}$.

(ii)

$$\Vdash_{\mathbb{P}} \dot{\mathcal{I}} \text{ is the ideal of subsets of } \check{X} \text{ generated by } \check{\mathcal{I}}\text{; it is a proper } \sigma\text{-ideal containing singletons.}$$

(b) $\Vdash_{\mathbb{P}} \operatorname{add} \dot{\mathcal{I}} = (\operatorname{add} \mathcal{I})\check{}$.

(c) If $\mathcal{I}$ is ω_1-saturated in $\mathcal{P}X$, then

$$\Vdash_{\mathbb{P}} \dot{\mathcal{I}} \text{ is } \omega_1\text{-saturated in } \mathcal{P}\check{X}, \text{ so } \mathcal{P}\check{X}/\dot{\mathcal{I}} \text{ is ccc and Dedekind complete.}$$

(d) If $X = \lambda$ is a regular uncountable cardinal and $\mathcal{I}$ is a normal ideal on λ, then

$$\Vdash_{\mathbb{P}} \dot{\mathcal{I}} \text{ is a normal ideal on } \check{\lambda}.$$

proof (a)(i) We have

$$p \Vdash_{\mathbb{P}} \text{there is an } I \in \check{\mathcal{I}} \text{ such that } \dot{J} \subseteq I.$$

Set

$$A = \{q : \text{there is an } I \in \mathcal{I} \text{ such that } q \Vdash_{\mathbb{P}} \dot{J} \subseteq \check{I}\}.$$

If p' is stronger than p, there is a $q \in A$ stronger than p'. Let $A' \subseteq A$ be a maximal antichain. Then A' is countable and for each $q \in A'$ there is an $I_q \in \mathcal{I}$ such that $q \Vdash_{\mathbb{P}} \dot{J} \subseteq \check{I}_q$. Set $I = \bigcup_{q \in A} I_q$; because $\mathcal{I}$ is a σ-ideal, $I \in \mathcal{I}$. Now $q \Vdash_{\mathbb{P}} \dot{J} \subseteq \check{I}$ for every $q \in A'$. If p' is stronger than p there is a $q \in A'$ which is compatible with p', so $p \Vdash_{\mathbb{P}} \dot{J} \subseteq \check{I}$, as required.

(ii) Because

$$\Vdash_{\mathbb{P}} \check{\mathcal{I}} \text{ is a family of subsets of } \check{X} \text{ closed under finite unions,}$$

we have

$$\Vdash_{\mathbb{P}} \dot{\mathcal{I}} \text{ is an ideal of subsets of } \check{X}.$$

Because

$$\Vdash_{\mathbb{P}} \check{I} \in \dot{\mathcal{I}}$$

whenever $I \in \mathcal{I}$,

$$\Vdash_{\mathbb{P}} \check{\mathcal{I}} \subseteq \dot{\mathcal{I}} \text{ and } \dot{\mathcal{I}} \text{ is the ideal generated by } \check{\mathcal{I}}.$$

Since $X \notin \mathcal{I}$, (i) tells us that

$$\Vdash_{\mathbb{P}} \check{X} \notin \dot{\mathcal{I}}.$$

Since $\{x\} \in \mathcal{I}$ for every $x \in X$,

$$\Vdash_{\mathbb{P}} \{x\} \in \dot{\mathcal{I}} \text{ for every } x \in \check{X}.$$

Thus

$$\Vdash_{\mathbb{P}} \dot{\mathcal{I}} \text{ is a proper ideal of } \mathcal{P}\check{X} \text{ containing singletons.}$$

I defer the final step to (b-i) below.

(b) Set $\theta = \operatorname{add}\mathcal{I}$.

(i) Suppose that $p \in P$ and that $\dot{\mathcal{A}}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{\mathcal{A}} \subseteq \dot{\mathcal{I}} \text{ and } \#(\dot{\mathcal{A}}) < \check{\theta}.$$

Then there are a q stronger than p, a $\delta < \theta$ and a family $\langle \dot{A}_\xi \rangle_{\xi<\delta}$ of $\mathbb{P}$-names such that

$$q \Vdash_{\mathbb{P}} \dot{\mathcal{A}} = \{\dot{A}_\xi : \xi < \check{\delta}\}.$$

For each $\xi < \delta$, $q \Vdash_{\mathbb{P}} \dot{A}_\xi \in \dot{\mathcal{I}}$, so we have an $I_\xi \in \mathcal{I}$ such that $q \Vdash_{\mathbb{P}} \dot{A}_\xi \subseteq \check{I}_\xi$. Set $I = \bigcup_{\xi<\delta} I_\xi \in \mathcal{I}$. Then

$$q \Vdash_{\mathbb{P}} \dot{A}_\xi \subseteq \check{I} \text{ for every } \xi < \check{\delta}, \text{ so } \bigcup \dot{\mathcal{A}} \subseteq \check{I} \text{ and } \bigcup \dot{\mathcal{A}} \in \dot{\mathcal{I}}.$$

As p and $\dot{\mathcal{A}}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \operatorname{add}\dot{\mathcal{I}} \geq \check{\theta}.$$

In particular, since we certainly have $\theta \geq \omega_1$,

$$\Vdash_{\mathbb{P}} \dot{\mathcal{I}} \text{ is a } \sigma\text{-ideal.}$$

(ii) In the other direction, there is a family $\langle I_\xi \rangle_{\xi<\theta}$ in $\mathcal{I}$ with no upper bound in $\mathcal{I}$. Now $\Vdash_{\mathbb{P}} \check{I}_\xi \in \dot{\mathcal{I}}$ for every $\xi < \theta$. **?** If $p \in P$ is such that

$$p \Vdash_{\mathbb{P}} \bigcup_{\xi<\theta} \check{I}_\xi \in \dot{\mathcal{I}},$$

then there is an $I \in \mathcal{I}$ such that

$$p \Vdash_{\mathbb{P}} \bigcup_{\xi<\check{\theta}} \check{I}_\xi \subseteq \check{I}$$

and $\bigcup_{\xi<\theta} I_\xi \subseteq I \in \mathcal{I}$. **X** So

$$\Vdash_{\mathbb{P}} \bigcup_{\xi<\check{\theta}} \check{I}_\xi \notin \dot{\mathcal{I}} \text{ and } \operatorname{add}\dot{\mathcal{I}} \leq \check{\theta}.$$

(c) Let $p \in P$ and a family $\langle \dot{A}_\eta \rangle_{\eta<\omega_1}$ of $\mathbb{P}$-names be such that

$$p \Vdash_{\mathbb{P}} \langle \dot{A}_\eta \rangle_{\eta<\omega_1} \text{ is a disjoint family of subsets of } \check{X}.$$

For each $x \in X$, $\langle \widehat{p} \cap [\![\check{x} \in \dot{A}_\eta]\!] \rangle_{\eta<\omega_1}$ is a disjoint family in $\mathrm{RO}(\mathbb{P})$, where

$$\widehat{p} = \operatorname{int}\overline{\{q : q \text{ is stronger than } p\}}$$

is the regular open set corresponding to p. So there is an $\alpha_x < \omega_1$ such that $\widehat{p} \cap [\![\check{x} \in \dot{A}_\eta]\!] = 0$ for every $\eta \geq \alpha_x$, that is, $p \Vdash_{\mathbb{P}} \check{x} \notin \dot{A}_\eta$ for every $\eta \geq \alpha_x$. Because $\mathcal{I}$ is ω_1-saturated, therefore ω_2-additive (542B-542C), there is an $\alpha < \omega_1$ such that $I = \{x : x \in X, \alpha_x \geq \alpha\}$ belongs to $\mathcal{I}$. Now $p \Vdash_{\mathbb{P}} \check{x} \notin \dot{A}_\alpha$ for every $x \in X \setminus I$, that is,

$$p \Vdash_{\mathbb{P}} \dot{A}_\alpha \subseteq \check{I} \text{ and } \dot{A}_\alpha \in \dot{\mathcal{I}}.$$

As p and $\langle \dot{A}_\eta \rangle_{\eta<\omega_1}$ are arbitrary, $\Vdash_{\mathbb{P}} \dot{\mathcal{I}}$ is ω_1-saturated.

Thus $\Vdash_{\mathbb{P}} \mathcal{P}\check{X}/\dot{\mathcal{I}}$ is ccc. But since we know from (b) that $\Vdash_{\mathbb{P}} \dot{\mathcal{I}}$ is a σ-ideal, and of course $\Vdash_{\mathbb{P}} \mathcal{P}\check{X}$ is Dedekind complete, we have

$$\Vdash_{\mathbb{P}} \mathcal{P}\check{X}/\dot{\mathcal{I}} \text{ is Dedekind } \sigma\text{-complete, therefore Dedekind complete.}$$

(d) Suppose that $p \in P$ and that $\langle \dot{A}_\xi \rangle_{\xi<\lambda}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{A}_\xi \in \dot{\mathcal{I}} \text{ for every } \xi < \check{\lambda}.$$

For each $\xi < \lambda$ we have an $I_\xi \in \mathcal{I}$ such that $p \Vdash_{\mathbb{P}} \dot{A}_\xi \subseteq \check{I}_\xi$; let I be the diagonal union

$$\{\xi : \xi < \lambda,\, \xi \in \bigcup_{\eta<\xi} I_\eta\}.$$

Because $\mathcal{I}$ is a normal ideal on λ, $I \in \mathcal{I}$. Now suppose that q is stronger than p and that $\xi < \lambda$ is such that

$$q \Vdash_{\mathbb{P}} \check{\xi} \in \bigcup_{\eta<\check{\xi}} \dot{A}_\eta.$$

Then

$$q \Vdash_{\mathbb{P}} \check{\xi} \in \bigcup_{\eta<\check{\xi}} \check{I}_\eta,$$

so $\xi \in \bigcup_{\eta<\xi} I_\eta$ and $\xi \in I$ and $q \Vdash_{\mathbb{P}} \check{\xi} \in \check{I}$. As q and ξ are arbitrary,

$$p \Vdash_{\mathbb{P}} \text{the diagonal union of } \langle \dot{A}_\xi \rangle_{\xi<\check{\lambda}} \text{ is included in } \check{I} \text{ and belongs to } \dot{\mathcal{I}}.$$

As p and $\langle \dot{A}_\xi \rangle_{\xi<\lambda}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \dot{\mathcal{I}} \text{ is normal.}$$

555C Theorem Let $(X, \mathcal{P}X, \mu)$ be a probability space such that $\mu\{x\} = 0$ for every $x \in X$, and $\mathcal{N}$ the null ideal of μ. Let $\kappa > 0$ be a cardinal. Then we can find a $\mathbb{P}_\kappa$-name $\dot{\mu}$ such that

(i) $\Vdash_{\mathbb{P}_\kappa} \dot{\mu}$ is a probability measure with domain $\mathcal{P}\check{X}$, zero on singletons;

(ii) if $\dot{\mathcal{N}}$ is a $\mathbb{P}_\kappa$-name for the ideal of subsets of $\check{X}$ generated by $\check{\mathcal{N}}$, as in 555B, then

$$\Vdash_{\mathbb{P}_\kappa} \dot{\mathcal{N}} \text{ is the null ideal of } \dot{\mu}.$$

proof (a) For each function $\sigma : X \to \mathfrak{B}_\kappa$, write $\vec{\sigma}$ for the $\mathbb{P}_\kappa$-name

$$\{(\check{x}, \sigma(x)) : x \in X,\, \sigma(x) \neq 0\}.$$

Then

$$\Vdash_{\mathbb{P}_\kappa} \vec{\sigma} \subseteq \check{X}$$

and $[\![\check{x} \in \vec{\sigma}]\!] = \sigma(x)$ for every $x \in X$. Moreover, if $\dot{A}$ is any $\mathbb{P}_\kappa$-name such that $\Vdash_{\mathbb{P}_\kappa} \dot{A} \subseteq \check{X}$, then $\Vdash_{\mathbb{P}_\kappa} \dot{A} = \vec{\sigma}$, where $\sigma(x) = [\![\check{x} \in \dot{A}]\!]$ for $x \in X$.

(b) For $\sigma \in \mathfrak{B}_\kappa^X$, the functional

$$a \mapsto \int \bar{\nu}_\kappa(\sigma(x) \cap a)\, \mu(dx)$$

is additive and dominated by $\bar{\nu}_\kappa$, so there is a unique $u_\sigma \in L^\infty(\mathfrak{B}_\kappa)$ such that

$$\int_a u_\sigma\, d\bar{\nu}_\kappa = \int \bar{\nu}_\kappa(\sigma(x) \cap a) \mu(dx)$$

for every $a \in \mathfrak{B}_\kappa$ (365E, 365D(d-ii)), and $0 \le u_\sigma \le \chi 1$. Observe that if $\sigma, \tau \in \mathfrak{B}_\kappa^X$, $a \in \mathfrak{B}_\kappa^+$ and $a \Vdash_{\mathbb{P}_\kappa} \vec{\sigma} = \vec{\tau}$, then $a \cap \sigma(x) = a \cap \tau(x)$ for every $x \in X$, so that $u_\sigma \times \chi a = u_\tau \times \chi a$.

(c) For $u \in L^\infty(\mathfrak{B}_\kappa)$ let $\vec{u}$ be the corresponding $\mathbb{P}_\kappa$-name for a real number (5A3L-5A3M, identifying $\mathfrak{B}_\kappa$ with $\mathrm{RO}(\mathbb{P}_\kappa)$ as usual). Consider the $\mathbb{P}_\kappa$-name

$$\dot{\mu} = \{((\vec{\sigma}, \vec{u}_\sigma), \mathbb{1}) : \sigma \in \mathfrak{B}_\kappa^X\}.$$

Then

$$\Vdash_{\mathbb{P}_\kappa} \dot{\mu} \text{ is a function from } \mathcal{P}\check{X} \text{ to } [0,1].$$

P Because $0 \le u_\sigma \le \chi 1$, $\Vdash_{\mathbb{P}_\kappa} \vec{u}_\sigma \in [0,1]$ for each σ, and $\Vdash_{\mathbb{P}_\kappa} \dot\mu \subseteq \mathcal{P}\check{X} \times [0,1]$. If σ, $\tau \in \mathfrak{B}_\kappa^X$ and $a \in \mathfrak{B}_\kappa^+$ are such that $a \Vdash_{\mathbb{P}_\kappa} \vec\sigma = \vec\tau$, then $u_\sigma \times \chi a = u_\tau \times \chi a$, by (b); but this means that $a \Vdash_{\mathbb{P}_\kappa} \vec{u}_\sigma = \vec{u}_\tau$ (5A3M). So $\Vdash_{\mathbb{P}_\kappa} \dot\mu$ is a function (5A3Ea). If $a \in P$ and $\dot A$ are such that $a \Vdash_{\mathbb{P}_\kappa} \dot A \subseteq \check X$, then there is a $\sigma \in \mathfrak{B}_\kappa^X$ such that $a \Vdash_{\mathbb{P}_\kappa} \dot A = \vec\sigma$, so that $\Vdash_{\mathbb{P}_\kappa} \operatorname{dom} \dot\mu = \mathcal{P}\check X$ (5A3Eb). **Q**

(d) Now we have to check the properties of $\dot\mu$.

(i) $\Vdash_{\mathbb{P}_\kappa} \dot\mu$ is countably additive. **P** Suppose that $\langle \dot A_n\rangle_{n\in\mathbb{N}}$ is a sequence of $\mathbb{P}_\kappa$-names and $a \in \mathfrak{B}_\kappa^+$ is such that

$$a \Vdash_{\mathbb{P}_\kappa} \langle \dot A_n\rangle_{n\in\mathbb{N}} \text{ is a disjoint sequence of subsets of } \check X.$$

For each $n \in \mathbb{N}$, let $\sigma_n \in \mathfrak{B}_\kappa^X$ be such that $a \Vdash_{\mathbb{P}_\kappa} \dot A_n = \vec\sigma_n$; then $\langle a \cap \sigma_n(x)\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{B}_\kappa$ for each $x \in X$. Set $\sigma(x) = \sup_{n\in\mathbb{N}} \sigma_n(x)$ for each x. Then $[\![\check x \in \vec\sigma]\!] = \sup_{n\in\mathbb{N}} [\![\check x \in \vec\sigma_n]\!]$ for each x, so $\Vdash_{\mathbb{P}_\kappa} \vec\sigma = \bigcup_{n\in\mathbb{N}} \vec\sigma_n$. Now for any $b \subseteq a$,

$$\int_b u_\sigma d\bar\nu_\kappa = \int \bar\nu_\kappa(b \cap \sigma(x))\mu(dx) = \int \sum_{n=0}^\infty \bar\nu_\kappa(b \cap \sigma_n(x))\mu(dx)$$
$$= \sum_{n=0}^\infty \int \bar\nu_\kappa(b \cap \sigma_n(x))\mu(dx) = \sum_{n=0}^\infty \int_b u_{\sigma_n} d\bar\nu_\kappa.$$

So

$$\chi a \times u_\sigma = \sup_{n\in\mathbb{N}} \chi a \times \textstyle\sum_{i=0}^n u_{\sigma_i}$$

in $L^0(\mathfrak{B}_\kappa)$, and

$$a \Vdash_{\mathbb{P}_\kappa} \dot\mu(\textstyle\bigcup_{n\in\mathbb{N}} \dot A_n) = \dot\mu(\bigcup_{n\in\mathbb{N}} \vec\sigma_n) = \dot\mu(\vec\sigma) = \vec u_\sigma = \sum_{n=0}^\infty \vec u_{\sigma_n} = \sum_{n=0}^\infty \dot\mu(\dot A_n).$$

As a and $\langle \dot A_n\rangle_{n\in\mathbb{N}}$ are arbitrary, $\Vdash_{\mathbb{P}_\kappa} \dot\mu$ is countably additive. **Q**

(ii) Suppose that $\dot y$ is a $\mathbb{P}_\kappa$-name and $a \in \mathfrak{B}_\kappa^+$ is such that $a \Vdash_{\mathbb{P}_\kappa} \dot y \in \check X$. Take any b stronger than a and $y \in X$ such that $b \Vdash_{\mathbb{P}_\kappa} \dot y = \check y$. Set $\sigma(y) = 1$ and $\sigma(x) = 0$ for $x \in X \setminus \{y\}$. Then

$$\textstyle\int u_\sigma d\bar\nu_\kappa = \mu\{y\} = 0,$$

so

$$b \Vdash_{\mathbb{P}_\kappa} \dot\mu(\{\dot y\}) = \dot\mu(\{\check y\}) = \dot\mu(\vec\sigma) = \vec u_\sigma = 0.$$

Thus

$$\Vdash_{\mathbb{P}_\kappa} \dot\mu \text{ is zero on singletons.}$$

(iii) If $\sigma(x) = 1$ for every $x \in X$, then $u_\sigma = \chi 1$ so

$$\Vdash_{\mathbb{P}_\kappa} \dot\mu(\check X) = \vec{\chi 1} = 1.$$

(e) $\Vdash_{\mathbb{P}_\kappa} \dot{\mathcal{N}} = \{A : \dot\mu A = 0\}$. **P** Let $a \in \mathfrak{B}_\kappa^+$ and a $\mathbb{P}_\kappa$-name $\dot A$ be such that $a \Vdash_{\mathbb{P}_\kappa} \dot A \subseteq \check X$. Let $\sigma \in \mathfrak{B}_\kappa^X$ be such that $a \Vdash_{\mathbb{P}_\kappa} \dot A = \vec\sigma$; then $a \Vdash_{\mathbb{P}_\kappa} \dot\mu(\dot A) = \vec u_\sigma$.

(i) If $a \Vdash_{\mathbb{P}_\kappa} \dot A \in \dot{\mathcal{N}}$, then there is an $I \in \mathcal{N}$ such that $a \Vdash_{\mathbb{P}_\kappa} \vec\sigma \subseteq \check I$ (555B(a-i)), that is, $a \cap \sigma(x) = 0$ for every $x \in X \setminus I$. But this means that

$$\textstyle\int_a u_\sigma d\bar\nu_\kappa = \int \bar\nu_\kappa(\sigma(x) \cap a)\mu(dx) \le \mu I = 0,$$

and $\chi a \times u_\sigma = 0$. So

$$a \Vdash_{\mathbb{P}_\kappa} \dot\mu(\dot A) = \vec u_\sigma = 0.$$

(ii) If $a \Vdash_{\mathbb{P}_\kappa} \dot\mu(\dot A) = 0$, then $\chi a \times u_\sigma = 0$, so

$$\textstyle\int \bar\nu_\kappa(\sigma(x) \cap a)\mu(dx) = \int_a u_\sigma d\bar\nu_\kappa = 0.$$

Set $I = \{x : a \cap \sigma(x) \ne 0\}$; then $\mu I = 0$ and

$$a \Vdash_{\mathbb{P}_\kappa} \dot{A} \subseteq \check{I} \text{ so } \dot{A} \in \dot{\mathcal{N}}.$$

Putting these together we have what we need. **Q**

555D Corollary (Solovay 71) Suppose that λ is a two-valued-measurable cardinal and that $\kappa \ge \lambda$ is a cardinal. Then

$$\Vdash_{\mathbb{P}_\kappa} \check{\lambda} \text{ is atomlessly-measurable.}$$

proof Putting 555C and 555B together,

$\Vdash_{\mathbb{P}_\kappa}$ there is a probability measure μ with domain $\mathcal{P}\check{\lambda}$, zero on singletons, such that the null ideal of μ is $\check{\lambda}$-additive.

By 552B and 543Bc,

$$\Vdash_{\mathbb{P}_\kappa} \check{\lambda} \le \mathfrak{c} \text{ is a real-valued-measurable cardinal, so is atomlessly-measurable.}$$

555E Theorem Let λ be a two-valued-measurable cardinal, and $\mathcal{I}$ a λ-additive maximal proper ideal of $\mathcal{P}\lambda$ containing singletons; let μ be the $\{0,1\}$-valued probability measure on λ with null ideal $\mathcal{I}$. Let $\kappa \ge \lambda$ be a cardinal, and define $\dot{\mu}$ from μ as in Theorem 555C. Set $\theta = \mathrm{Tr}_{\mathcal{I}}(\lambda;\kappa)$ (definition: 5A1Ma). Then

$$\Vdash_{\mathbb{P}_\kappa} \dot{\mu} \text{ is Maharam-type-homogeneous with Maharam type } \check{\theta}.$$

proof (a) Let $\langle g_\alpha\rangle_{\alpha<\theta}$ be a family in κ^λ such that $\{\xi : g_\alpha(\xi) = g_\beta(\xi)\} \in \mathcal{I}$ whenever $\alpha < \beta < \theta$ (541F). Because $\lambda \le \kappa$, we can suppose that all the g_α are injective. (Just arrange that $g_\alpha(\xi)$ always belongs to some $J_\xi \in [\kappa]^\kappa$ where $\langle J_\xi\rangle_{\xi<\lambda}$ is disjoint.) For $\alpha < \theta$ and $\xi < \lambda$ set $\sigma_\alpha(\xi) = e_{g_\alpha(\xi)}$. For $\sigma \in \mathfrak{B}_\kappa^\lambda$ let $\vec{\sigma}$ be the corresponding $\mathbb{P}_\kappa$-name for a subset of λ as in the prood of 555C. Then for any non-empty finite $K \subseteq \theta$ we have

$$\Vdash_{\mathbb{P}_\kappa} \dot{\mu}(\textstyle\bigcap_{\alpha\in\check{K}} \vec{\sigma}_\alpha) = (2^{-\#(K)})\check{}.$$

P Set $\sigma(\xi) = \inf_{\alpha\in K} \sigma_\alpha(\xi)$ for each ξ, so that

$$\Vdash_{\mathbb{P}_\kappa} \vec{\sigma} = \textstyle\bigcap_{\alpha\in\check{K}} \vec{\sigma}_\alpha.$$

Set

$$I = \textstyle\bigcup_{\alpha,\beta\in K \text{ are different}} \{\xi : \xi < \lambda,\, g_\alpha(\xi) = g_\beta(\xi)\};$$

then $I \in \mathcal{I}$. If $a \in \mathfrak{B}_\kappa$, let $J \in [\kappa]^{\le\omega}$ be such that a belongs to the closed subalgebra of $\mathfrak{B}_\kappa$ generated by $\{e_\eta : \eta \in J\}$. Then

$$\{\xi : \xi < \lambda,\, \bar{\nu}_\kappa(\sigma(\xi) \cap a) \ne 2^{-\#(K)}\bar{\nu}_\kappa(a)\} \subseteq I \cup \textstyle\bigcup_{\alpha\in K} g_\alpha^{-1}[J] \in \mathcal{I},$$

so

$$\int \bar{\nu}_\kappa(\sigma(\xi) \cap a)\mu(d\xi) = 2^{-\#(K)}\bar{\nu}_\kappa(a).$$

This means that u_σ, as defined in the proof of 555C, is just $2^{-\#(K)}\chi 1$ and $\Vdash_{\mathbb{P}_\kappa} \vec{u}_\sigma = (2^{-\#(K)})\check{}$, that is,

$$\Vdash_{\mathbb{P}_\kappa} \dot{\mu}(\textstyle\bigcap_{\alpha\in\check{K}} \vec{\sigma}_\alpha) = (2^{-\#(K)})\check{}. \quad \mathbf{Q}$$

Thus

$\Vdash_{\mathbb{P}_\kappa} \langle\vec{\sigma}_\alpha\rangle_{\alpha<\check{\theta}}$ is a stochastically independent family in $\mathcal{P}\lambda$ of elements of measure $\frac{1}{2}$, and every principal ideal of the measure algebra of $\dot{\mu}$ has Maharam type at least $\check{\theta}$

(331Ja).

(b) In the other direction, suppose that $a \in \mathfrak{B}_\kappa^+$, δ is a cardinal, $t > 0$ is a rational number and $\langle\dot{A}_\alpha\rangle_{\alpha<\delta}$ is a family of $\mathbb{P}_\kappa$-names such that

$$a \Vdash_{\mathbb{P}_\kappa} \dot{A}_\alpha \subseteq \check{\lambda},\ \dot{\mu}(\dot{A}_\alpha \triangle \dot{A}_\beta) \ge 3\check{t} \text{ whenever } \alpha < \beta < \check{\delta}.$$

For each $\alpha < \delta$ let $\sigma_\alpha \in \mathfrak{B}_\kappa^\lambda$ be such that $a \Vdash_{\mathbb{P}_\kappa} \vec{\sigma}_\alpha = \dot{A}_\alpha$; then

$$\int \bar{\nu}_\kappa(a \cap (\sigma_\alpha(\xi) \triangle \sigma_\beta(\xi)))\mu(d\xi) \ge 3t\bar{\nu}_\kappa a$$

whenever $\alpha < \beta < \delta$. Let $D \subseteq \mathfrak{B}_\kappa$ be a set with cardinal κ which is dense for the measure-algebra topology (521E(a-ii)), and for $\alpha < \delta$, $\xi < \kappa$ take $d_\alpha(\xi) \in D$ such that $\bar\nu_\kappa(d_\alpha(\xi) \triangle \sigma_\alpha(\xi)) \le t\bar\nu_\kappa(a)$. Then

$$\int \bar\nu_\kappa(d_\alpha(\xi) \triangle d_\beta(\xi))\mu(d\xi) > 0$$

and $\{\xi : d_\alpha(\xi) \ne d_\beta(\xi)\} \notin \mathcal{I}$ whenever $\alpha < \beta < \delta$; as $\mathcal{I}$ is a maximal ideal, $\{\xi : d_\alpha(\xi) = d_\beta(\xi)\} \in \mathcal{I}$ whenever $\alpha < \beta < \delta$, and $\langle d_\alpha\rangle_{\alpha<\delta}$ witnesses that

$$\delta \le \mathrm{Tr}_{\mathcal{I}}(\lambda; D) = \mathrm{Tr}_{\mathcal{I}}(\lambda; \kappa) = \theta.$$

As a, t and $\langle \dot A_\alpha\rangle_{\alpha<\delta}$ are arbitrary,

$$\Vdash_{\mathbb{P}_\kappa} \text{ the Maharam type of } \dot\mu \text{ is at most } \check\theta;$$

with (a), this means that

$$\Vdash_{\mathbb{P}_\kappa} \dot\mu \text{ is Maharam-type-homogeneous with Maharam type } \check\theta.$$

555F Proposition Let λ be a two-valued-measurable cardinal and $\kappa > 0$. Let μ be a normal witnessing probability on λ and $\dot\mu$ the corresponding $\mathbb{P}_\kappa$-name for a measure on $\check\lambda$, as in 555C. Then

$\Vdash_{\mathbb{P}_\kappa}$ the covering number of the null ideal of the product measure $\dot\mu^{\mathbb{N}}$ on $\check\lambda^{\mathbb{N}}$ is $\check\lambda$.

proof (a) It may save a moment's thought later on if I remark now that if (Y, T, ν) is any measure space and $W \subseteq Y^{\mathbb{N}}$ is negligible for the product measure $\nu^{\mathbb{N}}$, then there is a family $\langle F_{ij}\rangle_{j\le i\in\mathbb{N}}$ in T such that

$$W \subseteq \bigcap_{n\in\mathbb{N}} \bigcup_{i\ge n} \{y : y \in Y^{\mathbb{N}},\ y(j) \in F_{ij} \text{ for every } j \le i\}, \quad \sum_{i=0}^\infty \prod_{j=0}^i \nu F_{ij} \le 1.$$

P For each $k \in \mathbb{N}$, let $\langle C_{ki}\rangle_{i\in\mathbb{N}}$ be a sequence of measurable cylinders such that $W \subseteq \bigcup_{i\in\mathbb{N}} C_{ki}$ and $\sum_{i=0}^\infty \lambda^{\mathbb{N}} C_{ki} \le 2^{-k-1}$; let $\langle C_i\rangle_{i\in\mathbb{N}}$ be a re-listing of the double family $\langle C_{ki}\rangle_{k,i\in\mathbb{N}}$ with enough copies of the empty set interleaved to ensure that C_i is determined by coordinates less than or equal to i for each i; express each C_i as $\{y : y(j) \in F_{ij} \text{ for } j \le i\}$. **Q**

(b) It will also help to be able to do some calculations with sequences of $\mathbb{P}_\kappa$-names for subsets of λ. For $J \subseteq \kappa$ let $\mathfrak{C}_J$ be the closed subalgebra of $\mathfrak{B}_\kappa$ generated by $\{e_\eta : \eta \in J\}$, and $P_J : L^\infty(\mathfrak{B}_\kappa) \to L^\infty(\mathfrak{C}_J)$ the corresponding conditional expectation operator; see 242J, 254R and 365Q[7] for the basic manipulations of these operators. Let $\mathcal{F}$ be the normal ultrafilter $\{F : \mu F = 1\}$.

(i) Let $\langle \dot A_{ij}\rangle_{j\le i\in\mathbb{N}}$ be a family of $\mathbb{P}_\kappa$-names such that

$$\Vdash_{\mathbb{P}_\kappa} \dot A_{ij} \subseteq \check\lambda$$

whenever $j \le i \in \mathbb{N}$. For $j \le i \in \mathbb{N}$ and $\xi < \lambda$ set $\sigma_{ij}(\xi) = [\![\check\xi \in \dot A_{ij}]\!]$. Then there is a countable set $I_\xi \subseteq \kappa$ such that $\sigma_{ij}(\xi) \in \mathfrak{C}_{I_\xi}$ whenever $j \le i \in \mathbb{N}$. By 541Rb, there are an $F_0 \in \mathcal{F}$ and a countable set $I \subseteq \kappa$ such that $I_\xi \cap I_{\xi'} \subseteq I$ for all distinct ξ, $\xi' \in F_0$.

(ii) Set $u_{ij\xi} = P_I(\chi\sigma_{ij}(\xi))$ for $\xi < \lambda$. Because $\#(L^\infty(\mathfrak{C}_I)) \le \mathfrak{c} < \lambda$ and $\mathcal{F}$ is a λ-complete ultrafilter, there are $u_{ij} \in L^\infty(\mathfrak{C}_I)$ such that $\{\xi : u_{ij\xi} = u_{ij}\}$ belongs to $\mathcal{F}$ whenever $j \le i \in \mathbb{N}$. Set

$$F = F_0 \cap \{\xi : u_{ij\xi} = u_{ij} \text{ whenever } j \le i \in \mathbb{N}\},$$

so that $F \in \mathcal{F}$.

(iii) We have

$$\Vdash_{\mathbb{P}_\kappa} \dot\mu\dot A_{ij} = \vec u_{ij}$$

whenever $j \le i \in \mathbb{N}$. **P** If $a \in \mathfrak{B}_\kappa$, there is a countable $J \subseteq \kappa$ such that a belongs to the closed subalgebra $\mathfrak{C}_J$ generated by $\{e_\eta : \eta \in J\}$; we may suppose that $I \subseteq J$. Now $\langle I_\xi \setminus I\rangle_{\xi\in F}$ is disjoint, so $\{\xi : \xi \in F,\ I_\xi \cap J \not\subseteq I\}$ is countable, and $F' = \{\xi : \xi \in F,\ I_\xi \cap J \subseteq I\}$ belongs to $\mathcal{F}$. For $\xi \in F'$ we have

$$P_J(\chi\sigma_{ij}(\xi)) = P_J P_{I_\xi}(\chi\sigma_{ij}(\xi)) = P_{J\cap I_\xi}(\chi\sigma_{ij}(\xi)) \in \mathfrak{C}_I$$

so that

[7] Formerly 365R.

$$P_J(\chi\sigma_{ij}(\xi)) = P_I P_J(\chi\sigma_{ij}(\xi)) = P_I(\chi\sigma_{ij}(\xi)) = u_{ij};$$

consequently

$$\int_a u_{ij} d\bar\nu_\kappa = \int_a \chi\sigma_{ij}(\xi) d\bar\nu_\kappa = \bar\nu_\kappa(a \cap \sigma_{ij}(\xi)).$$

Because F' is μ-conegligible,

$$\int_a u_{ij} d\bar\nu_\kappa = \int \bar\nu_\kappa(a \cap \sigma_{ij}(\xi))\mu(d\xi).$$

As this is true for every $a \in \mathfrak{B}_\kappa$, the construction in 555C gives $\Vdash_{\mathbb{P}_\kappa} \dot\mu \dot A_{ij} = \vec u_{ij}$. **Q**

(iv) Note also that if $i \in \mathbb{N}$ and $\xi_0, \dots, \xi_j \in F$ are distinct, then

$$\int \prod_{j=0}^{i} u_{ij} d\bar\nu_\kappa = \bar\nu_\kappa(\inf_{j \le i} \sigma_{ij}(\xi_j)).$$

P The algebras $\mathfrak{C}_{I \cup \{\eta\}}$, for $\eta \in \kappa \setminus I$, are relatively stochastically independent over $\mathfrak{C}_I$ in the sense of 458L; by 458H/458Le, the algebras $\mathfrak{C}_{I \cup I_\xi}$, for $\xi \in F$, are relatively stochastically independent over $\mathfrak{C}_I$; but, disentangling the definitions, this is exactly what we need to know. **Q**

(c) We are now ready for the central idea of the proof. Suppose that $\dot W$ is a $\mathbb{P}_\kappa$-name such that

$$\Vdash_{\mathbb{P}_\kappa} \dot W \subseteq \check\lambda^{\mathbb{N}} \text{ and } \dot\mu^{\mathbb{N}} \dot W = 0.$$

Then there is an $F \in \mathcal{F}$ such that

$$\Vdash_{\mathbb{P}_\kappa} \dot W \text{ is disjoint from } (F^{\mathbb{N}} \setminus \Delta)\check{}$$

where $\Delta = \bigcup_{j<k\in\mathbb{N}} \{x : x \in \lambda^{\mathbb{N}},\, x(j) = x(k)\}$. **P** By (a), we have a family $\langle \dot A_{ij} \rangle_{j \le i \in \mathbb{N}}$ of $\mathbb{P}_\kappa$-names such that

$$\Vdash_{\mathbb{P}_\kappa} \dot A_{ij} \subseteq \check\lambda \text{ whenever } j \le i \in \mathbb{N},$$

$$\Vdash_{\mathbb{P}_\kappa} \dot W \subseteq \bigcap_{n\in\mathbb{N}} \bigcup_{i \ge n} \{x : x \in \check\lambda^{\mathbb{N}},\, x(j) \in \dot A_{ij} \text{ for every } j \le i\},$$

$$\Vdash_{\mathbb{P}_\kappa} \sum_{i=0}^{\infty} \prod_{j=0}^{i} \dot\mu \dot A_{ij} \le 1.$$

Take $\sigma_{ij}(\xi)$, $I \in [\kappa]^{\le\omega}$, $u_{ij} \in \mathfrak{C}_I$ and $F \in \mathcal{F}$ as in (b). Suppose that $x \in F^{\mathbb{N}}$ and $x(j) \ne x(k)$ for distinct j, $k \in \mathbb{N}$. Then

$$[\![\check x \in \dot W]\!] \subseteq \inf_{n\in\mathbb{N}} \sup_{i \ge n} \inf_{j \le i} [\![x(j)\check{} \in \dot A_{ij}]\!] = \inf_{n\in\mathbb{N}} \sup_{i \ge n} \inf_{j \le i} \sigma_{ij}(x(j)).$$

So

$$\bar\nu_\kappa [\![\check x \in \dot W]\!] \le \inf_{n\in\mathbb{N}} \sum_{i=n}^{\infty} \bar\nu_\kappa(\inf_{j \le i} \sigma_{ij}(x(j))) = \inf_{n\in\mathbb{N}} \sum_{i=n}^{\infty} \int \prod_{j \le i} u_{ij} d\bar\nu_\kappa$$

by (b-iv). On the other hand, setting $v_i = \prod_{j \le i} u_{ij}$ for $i \in \mathbb{N}$ and $w = \sup_{m\in\mathbb{N}} \sum_{i=0}^{m} v_i$, we have

$$\Vdash_{\mathbb{P}_\kappa} \vec w = \sum_{i=0}^{\infty} \vec v_i = \sum_{i=0}^{\infty} \prod_{j=0}^{i} \dot\mu \dot A_{ij} \le 1.$$

So $w \le \chi 1$ and

$$\sum_{i=0}^{\infty} \int \prod_{j \le i} u_{ij} d\bar\nu_\kappa = \int w\, d\bar\nu_\kappa \le 1.$$

Putting these together, we see that $\bar\nu_\kappa [\![\check x \in \dot W]\!] = 0$ and $\Vdash_{\mathbb{P}_\kappa} \check x \notin \dot W$. As x is arbitrary,

$$\Vdash_{\mathbb{P}_\kappa} \dot W \text{ is disjoint from } (F^{\mathbb{N}} \setminus \Delta)\check{}. \quad \mathbf{Q}$$

(d) We are nearly home. Suppose that $a \in \mathfrak{B}_\kappa^+$ and that $\dot{\mathcal{W}}$ is a $\mathbb{P}_\kappa$-name such that

$$a \Vdash_{\mathbb{P}_\kappa} \dot{\mathcal{W}} \text{ is a family of negligible sets in } \check\lambda^{\mathbb{N}} \text{ and } \#(\dot{\mathcal{W}}) < \check\lambda.$$

Take any b stronger than a, $\theta < \lambda$ and family $\langle \dot W_\zeta \rangle_{\zeta < \theta}$ of $\mathbb{P}_\kappa$-names such that

$$b \Vdash_{\mathbb{P}_\kappa} \dot{\mathcal{W}} = \{\dot W_\zeta : \zeta < \check\theta\}.$$

For each $\zeta < \theta$ let $\dot W'_\zeta$ be a $\mathbb{P}_\kappa$-name such that

$$\Vdash_{\mathbb{P}_\kappa} \dot W'_\zeta \subseteq \check\lambda^{\mathbb{N}} \text{ is negligible}, \quad b \Vdash_{\mathbb{P}_\kappa} \dot W'_\zeta = \dot W_\zeta.$$

By (c), we have an $F_\zeta \in \mathcal{F}$ such that

$$\Vdash_{\mathbb{P}_\kappa} \dot{W}'_\zeta \cap (F_\zeta^{\mathbb{N}} \setminus \Delta)\check{} = \emptyset.$$

Because $\theta < \lambda$, $\bigcap_{\zeta<\theta} F_\zeta$ belongs to $\mathcal{F}$ and is infinite, and there is an $x \in \bigcap_{\zeta<\theta} F_\zeta^{\mathbb{N}} \setminus \Delta$. But now

$$\Vdash_{\mathbb{P}_\kappa} \check{x} \notin \bigcup_{\zeta<\check{\theta}} \dot{W}'_\zeta$$

and

$$b \Vdash_{\mathbb{P}_\kappa} \check{x} \notin \bigcup \dot{\mathcal{W}}, \text{ so } \dot{\mathcal{W}} \text{ does not cover } \check{\lambda}^{\mathbb{N}}.$$

As b is arbitrary,

$$a \Vdash_{\mathbb{P}_\kappa} \dot{\mathcal{W}} \text{ does not cover } \check{\lambda}^{\mathbb{N}};$$

as a and $\dot{\mathcal{W}}$ are arbitrary,

$$\Vdash_{\mathbb{P}_\kappa} \text{the covering number of the null ideal of the product measure on } \check{\lambda}^{\mathbb{N}} \text{ is at least } \check{\lambda}.$$

The reverse inequality is trivial, since

$$\Vdash_{\mathbb{P}_\kappa} \dot{\mu}\{\xi\} = 0$$

for every $\xi < \check{\lambda}$; so the proposition is proved.

555G Cohen forcing If we allow ourselves to start from a measurable cardinal, we can find forcing constructions for a variety of power set σ-quotient algebras besides the probability algebras provided by Theorem 555C. In view of §554, an obvious construction is the following.

Theorem Let λ be a two-valued-measurable cardinal and $\kappa \geq \lambda$ a cardinal. Let $\mathcal{I}$ be a λ-additive maximal proper ideal of subsets of λ, and $\dot{\mathcal{I}}$ a $\mathbb{Q}_\kappa$-name for the ideal of subsets of $\check{\lambda}$ generated by $\check{\mathcal{I}}$, as in 555B. Set $\theta = \mathrm{Tr}_{\mathcal{I}}(\lambda;\kappa)$. Then

$$\Vdash_{\mathbb{Q}_\kappa} \mathcal{P}\lambda/\dot{\mathcal{I}} \cong \mathfrak{G}_{\check{\theta}}.$$

proof (a) For $\eta < \kappa$ let $e_\eta \in \mathfrak{G}_\kappa$ be the equivalence class of $\{x : x \in \{0,1\}^\kappa,\ x(\eta) = 1\}$; for $L \subseteq \kappa$ let $\mathfrak{C}_L$ be the closed subalgebra of $\mathfrak{G}_\kappa$ generated by $\{e_\eta : \eta \in L\}$. For $\sigma \in \mathfrak{G}_\kappa^\lambda$ let $\vec{\sigma}$ be the $\mathbb{P}_\kappa$-name $\{(\check{\xi}, \sigma(\xi)) : \xi < \lambda,\ \sigma(\xi) \neq 0\}$, so that $\Vdash_{\mathbb{Q}_\kappa} \vec{\sigma} \subseteq \check{\lambda}$, and $[\![\check{\xi} \in \vec{\sigma}]\!] = \sigma(\xi)$ for any $\xi < \lambda$.

Write $\mathcal{F} = \{\lambda \setminus I : I \in \mathcal{I}\} = \mathcal{P}\lambda \setminus \mathcal{I}$, so that $\mathcal{F}$ is a λ-complete ultrafilter on λ.

(b) For $z \in \mathrm{Fn}_{<\omega}(\kappa;\{0,1\})$ set

$$v_z = \{x : z \subseteq x \in \{0,1\}^\kappa\}^\bullet \in \mathfrak{G}_\kappa.$$

Then $\{v_z : z \in \mathrm{Fn}_{<\omega}(\kappa;\{0,1\})\}$ is order-dense in $\mathfrak{G}_\kappa$. For $A \subseteq \lambda$ and $\tau \in \mathrm{Fn}_{<\omega}(\kappa;\{0,1\})^A$, set $\sigma_\tau(\xi) = v_{\tau(\xi)}$ if $\xi \in A$, 0 if $\xi \in \lambda \setminus A$. Note that

$$\Vdash_{\mathbb{Q}_\kappa} \vec{\sigma}_\tau \subseteq (\mathrm{dom}\,\tau)\check{}.$$

Now if $a \in \mathfrak{G}_\kappa^+$ and $\dot{C}$ is a $\mathbb{Q}_\kappa$-name such that $a \Vdash_{\mathbb{Q}_\kappa} \dot{C} \subseteq \check{\lambda}$, there is a countable set $T \subseteq \bigcup_{A\subseteq\lambda} \mathrm{Fn}_{<\omega}(\kappa;\{0,1\})^A$ such that $a \Vdash_{\mathbb{Q}_\kappa} \dot{C} = \bigcup_{\tau\in\check{T}} \vec{\sigma}_\tau$. **P** Set $D = \{\xi : \xi < \lambda,\ \Vdash_{\mathbb{Q}_\kappa} \check{\xi} \notin \dot{C}\}$. For each $\xi \in \lambda \setminus D$, choose a sequence $\langle \tau_n(\xi)\rangle_{n\in\mathbb{N}}$ in $\mathrm{Fn}_{<\omega}(\kappa;\{0,1\})$ such that $[\![\check{\xi} \in \dot{C}]\!] = \sup_{n\in\mathbb{N}} v_{\tau_n(\xi)}$. If $\xi \in D$, then $\xi \notin \mathrm{dom}\,\tau_n$ and $\sigma_{\tau_n}(\xi) = 0$ for every n; accordingly

$$[\![\check{\xi} \in \dot{C}]\!] = 0 = \sup_{n\in\mathbb{N}} [\![\check{\xi} \in \vec{\sigma}_{\tau_n}]\!].$$

While if $\xi \in \lambda \setminus D$,

$$[\![\check{\xi} \in \dot{C}]\!] = \sup_{n\in\mathbb{N}} v_{\tau_n(\xi)} = \sup_{n\in\mathbb{N}} \sigma_{\tau_n}(\xi) = \sup_{n\in\mathbb{N}} [\![\check{\xi} \in \vec{\sigma}_{\tau_n}]\!].$$

So

$$a \Vdash_{\mathbb{Q}_\kappa} \dot{C} = \bigcup_{n\in\mathbb{N}} \vec{\sigma}_{\tau_n}$$

and we can set $T = \{\tau_n : n \in \mathbb{N}\}$. **Q**

(c) There is a family $G_0\subseteq\kappa^\lambda$, with cardinal θ, such that $\{\xi:g(\xi)=g'(\xi)\}\in\mathcal{I}$ whenever g, $g'\in G$ are distinct (541F again), and we can suppose that every member of G_0 is injective (see part (a) of the proof of 555E). Let $G\supseteq G_0$ be a maximal family such that $\{\xi:g(\xi)=g'(\xi)\}$ belongs to $\mathcal{I}$ whenever g, $g'\in G$ are distinct. Then $\#(G)\le\theta$, by the definition of $\mathrm{Tr}_{\mathcal{I}}(\lambda;\kappa)$, so in fact $\#(G)=\theta$. Enumerate G as $\langle g_\alpha\rangle_{\alpha<\theta}$, and for $\alpha<\theta$, $\xi<\lambda$ set $\rho_\alpha(\xi)=e_{g_\alpha(\xi)}\in\mathfrak{G}_\kappa$.

(d) Suppose that $\alpha<\theta$ and $a\in\mathfrak{G}_\kappa^+$ are such that

$$a\Vdash_{\mathbb{Q}_\kappa}\vec{\rho}_\alpha^{\,\bullet}\text{ is neither 0 nor 1 in }\mathcal{P}\check{\lambda}/\dot{\mathcal{I}}.$$

Then $g_\alpha^{-1}[\{\eta\}]\notin\mathcal{F}$ for every $\eta<\kappa$. **P?** Otherwise, take b to be one of $a\cap e_\eta$, $a\setminus e_\eta$ and non-zero, and any $\xi\in g_\alpha^{-1}[\{\eta\}]$. If $b\subseteq e_\eta$ then $b\subseteq\rho_\alpha(\xi)$ and $b\Vdash_{\mathbb{Q}_\kappa}\check{\xi}\in\vec{\rho}_\alpha$. If $b\cap e_\eta=0$ then $\rho_\alpha(\xi)\cap b=0$ and $b\Vdash_{\mathbb{Q}_\kappa}\check{\xi}\notin\vec{\rho}_\alpha$. So

$$b\Vdash_{\mathbb{Q}_\kappa}g_\alpha^{-1}[\{\eta\}]\check{}\text{ is either included in or disjoint from }\vec{\rho}_\alpha\text{, and }\vec{\rho}_\alpha^{\,\bullet}\text{ is either 1 or 0 in }\mathcal{P}\check{\lambda}/\dot{\mathcal{I}}.$$

But this contradicts the assumption on a. **XQ**

(e) $\Vdash_{\mathbb{Q}_\kappa}\{\vec{\rho}_\alpha^{\,\bullet}:\alpha<\check{\theta}\}\setminus\{0,1\}$ is a Boolean-independent family in $\mathcal{P}\check{\lambda}/\dot{\mathcal{I}}$.
P? Otherwise, there must be disjoint finite sets J, $K\subseteq\theta$ and $a\in\mathfrak{G}_\kappa^+$ such that

$$a\Vdash_{\mathbb{Q}_\kappa}\vec{\rho}_\alpha^{\,\bullet}\text{ is neither 0 nor 1 in }\mathcal{P}\check{\lambda}/\dot{\mathcal{I}}$$

for every $\alpha\in J\cup K$, but

$$a\Vdash_{\mathbb{Q}_\kappa}\inf_{\alpha\in\check{J}}\vec{\rho}_\alpha^{\,\bullet}\setminus\sup_{\alpha\in\check{K}}\vec{\rho}_\alpha^{\,\bullet}=0.$$

In this case,

$$I=\{\xi:\xi<\lambda\text{, there are distinct }\alpha,\,\beta\in J\cup K\text{ such that }g_\alpha(\xi)=g_\beta(\xi)\}$$

belongs to $\mathcal{I}$. For $\xi<\lambda$ define $\sigma(\xi)\in\mathfrak{G}_\kappa$ by saying that

$$\begin{aligned}\sigma(\xi)&=1\text{ if }\xi\in I,\\&=\inf_{\alpha\in J}e_{g_\alpha(\xi)}\setminus\sup_{\alpha\in K}e_{g_\alpha(\xi)}\text{ otherwise.}\end{aligned}$$

For $\alpha\in J$ and $\xi\in\lambda\setminus I$, $\sigma(\xi)\subseteq\rho_\alpha(\xi)$; so

$$\Vdash_{\mathbb{Q}_\kappa}\vec{\sigma}\setminus\vec{\rho}_\alpha\text{ is included in }\check{I}\text{ and belongs to }\dot{\mathcal{I}}\text{, that is, }\vec{\sigma}^{\bullet}\subseteq\vec{\rho}_\alpha^{\,\bullet}.$$

On the other hand, if $\alpha\in K$ and $\xi\in\lambda\setminus I$, $\sigma(\xi)\cap\rho_\alpha(\xi)=0$; so

$$\Vdash_{\mathbb{Q}_\kappa}\vec{\sigma}\cap\vec{\rho}_\alpha\text{ is included in }\check{I}\text{ and belongs to }\dot{\mathcal{I}}\text{, that is, }\vec{\sigma}^{\bullet}\cap\vec{\rho}_\alpha^{\,\bullet}=0.$$

So we have

$$\Vdash_{\mathbb{Q}_\kappa}\vec{\sigma}^{\bullet}\subseteq\inf_{\alpha\in\check{J}}\vec{\rho}_\alpha^{\,\bullet}\setminus\sup_{\alpha\in\check{K}}\vec{\rho}_\alpha^{\,\bullet}$$

and

$$a\Vdash_{\mathbb{Q}_\kappa}\vec{\sigma}\in\dot{\mathcal{I}}.$$

Let $I'\in\mathcal{I}$ be such that $a\Vdash_{\mathbb{Q}_\kappa}\vec{\sigma}\subseteq\check{I}'$ (see part (a) of the proof of 555B), and $L\in[\kappa]^{\le\omega}$ such that $a\in\mathfrak{C}_L$. By (d), $g_\alpha^{-1}[\{\eta\}]\in\mathcal{I}$ for every $\alpha\in J\cup K$ and $\eta<\kappa$, so there must be a $\xi\in\lambda\setminus(I'\cup I)$ such that $g_\alpha(\xi)\notin L$ for every $\alpha\in J\cup K$. In this case, $\sigma(\xi)\in\mathfrak{C}_{\kappa\setminus L}$ so $a\cap\sigma(\xi)$ is non-zero. But $\sigma(\xi)=[\![\check{\xi}\in\vec{\sigma}]\!]$ and $a\Vdash_{\mathbb{Q}_\kappa}\check{\xi}\notin\vec{\sigma}$, so this is impossible. **XQ**

(f) $\Vdash_{\mathbb{Q}_\kappa}$ the subalgebra of $\mathcal{P}\check{\lambda}/\dot{\mathcal{I}}$ generated by $\{\vec{\rho}_\alpha^{\,\bullet}:\alpha<\check{\theta}\}$ is order-dense in $\mathcal{P}\check{\lambda}/\dot{\mathcal{I}}$.
P If $a\in\mathfrak{G}_\kappa^+$ and $\dot{c}$ is a $\mathbb{Q}_\kappa$-name such that

$$a\Vdash_{\mathbb{Q}_\kappa}\dot{c}\in(\mathcal{P}\check{\lambda}/\dot{\mathcal{I}})\setminus\{0\},$$

there is a $\mathbb{Q}_\kappa$-name $\dot{C}$ such that

$$a\Vdash_{\mathbb{Q}_\kappa}\dot{C}\subseteq\check{\lambda},\ \dot{C}\notin\dot{\mathcal{I}}\text{ and }\dot{c}=\dot{C}^{\bullet}.$$

Take a countable set $T\subseteq\bigcup_{A\subseteq\lambda}\mathrm{Fn}_{<\omega}(\kappa;\{0,1\})^A$ such that $a\Vdash_{\mathbb{Q}_\kappa}\dot{C}=\bigcup_{\tau\in\check{T}}\vec{\sigma}_\tau$, as described in (b). Since $\Vdash_{\mathbb{Q}_\kappa}\dot{\mathcal{I}}$ is a σ-ideal, there are a b stronger than a and a $\tau\in T$ such that $b\Vdash_{\mathbb{Q}_\kappa}\vec{\sigma}_\tau\notin\dot{\mathcal{I}}$. Set $F_0=\mathrm{dom}\,\tau$; since

$\Vdash_{\mathbb{Q}_\kappa} \vec{\sigma}_\tau \subseteq \check{F}_0$, $F_0 \notin \mathcal{I}$.. Since $\mathcal{I}$ is a σ-ideal, there is an $n \in \mathbb{N}$ such that $F_1 = \{\xi : \xi \in F_0,\ \#(\tau(\xi)) = n\} \notin \mathcal{I}$, and $F_1 \in \mathcal{F}$.

Let $\langle h_i \rangle_{i<n}$ be a finite sequence of functions from λ to κ such that $\operatorname{dom}\tau(\xi) = \{h_i(\xi) : i < n\}$ for every $\xi \in F_1$. As G was maximal, there is for each $i < n$ an $\alpha_i < \theta$ such that $\{\xi : g_{\alpha_i}(\xi) = h_i(\xi)\}$ belongs to $\mathcal{F}$; set $F_2 = \{\xi : \xi \in F_1,\ g_{\alpha_i}(\xi) = h_i(\xi)$ for every $i < n\}$. Note that if $i < j < n$ then $g_{\alpha_i}(\xi) \neq g_{\alpha_j}(\xi)$ for any $\xi \in F_1$, so $\alpha_i \neq \alpha_j$. Next, there must be an $L \subseteq n$ such that

$$\begin{aligned} F_3 = \{\xi : \xi \in F_2,\ &\tau(\xi)(g_{\alpha_i}(\xi)) = 1 \text{ for every } i \in L, \\ &\tau(\xi)(g_{\alpha_i}(\xi)) = 0 \text{ for every } i \in n \setminus L\} \end{aligned}$$

belongs to $\mathcal{F}$. Set $J = \{\alpha_i : i \in L\}$ and $K = \{\alpha_i : i \in n \setminus L\}$; of course $J \cap K = \emptyset$, because all the α_i are different. Then, for $\xi \in F_3$, $\operatorname{dom}\tau_\xi = \{g_{\alpha_i}(\xi) : i < n\}$ and

$$\begin{aligned} [\![\check{\xi} \in \vec{\sigma}_\tau]\!] = \sigma_\tau(\xi) = v_{\tau(\xi)} &= \inf_{i\in L} e_{g_{\alpha_i}(\xi)} \setminus \sup_{i\in n\setminus L} e_{g_{\alpha_i}(\xi)} \\ &= \inf_{\alpha\in J} e_{g_\alpha(\xi)} \setminus \sup_{\alpha\in K} e_{g_\alpha(\xi)} = \inf_{\alpha\in J} [\![\check{\xi} \in \vec{\rho}_\alpha]\!] \setminus \sup_{\alpha\in K} [\![\check{\xi} \in \vec{\rho}_\alpha]\!]. \end{aligned}$$

Accordingly

$$\Vdash_{\mathbb{Q}_\kappa} \vec{\sigma}_\tau \triangle (\textstyle\bigcap_{\alpha\in\check{J}} \vec{\rho}_\alpha \setminus \bigcup_{\alpha\in\check{K}} \vec{\rho}_\alpha) \text{ is disjoint from } \check{F}_3 \text{ and belongs to } \dot{\mathcal{I}},$$

so

$$b \Vdash_{\mathbb{Q}_\kappa} \inf_{\alpha\in\check{J}} \vec{\rho}_\alpha^\bullet \setminus \sup_{\alpha\in\check{K}} \vec{\rho}_\alpha^\bullet = \vec{\sigma}_\tau^\bullet \subseteq \dot{c}.$$

As a and $\dot{c}$ are arbitrary, this proves the result. **Q**

(g) If $\alpha < \theta$ is such that $g_\alpha \in G_0$, then

$$\Vdash_{\mathbb{Q}_\kappa} \vec{\rho}_\alpha^\bullet \neq 0.$$

P? Otherwise, there is a non-zero $a \in \mathfrak{G}_\kappa$ such that

$$a \Vdash_{\mathbb{Q}_\kappa} \vec{\rho}_\alpha^\bullet = 0,$$

that is,

$$a \Vdash_{\mathbb{Q}_\kappa} \{\xi : \vec{\rho}_\alpha(\xi) \neq 0\} \in \dot{\mathcal{I}},$$

and there is an $I \in \mathcal{I}$ such that

$$a \Vdash_{\mathbb{Q}_\kappa} \{\xi : \vec{\rho}_\alpha(\xi) \neq 0\} \subseteq \check{I}.$$

In this case, for $\xi \in \lambda \setminus I$,

$$a \Vdash_{\mathbb{P}_\kappa} \vec{\rho}_\alpha(\xi) = 0,$$

that is,

$$0 = a \cap \rho_\alpha(\xi) = a \cap e_{g_\alpha(\xi)}.$$

But $\lambda \setminus I$ is infinite and g_α is injective, so $\{g_\alpha(\xi) : \xi \in \lambda \setminus I\}$ is infinite and $a = 0$. **XQ**

Similarly,

$$\Vdash_{\mathbb{Q}_\kappa} \vec{\rho}_\alpha^\bullet \neq 1.$$

As this is true whenever $g_\alpha \in G_0$, and $\#(G_0) = \theta$, we see that

$$\Vdash_{\mathbb{Q}_\kappa} \#(\{\alpha : \vec{\rho}_\alpha^\bullet \notin \{0,1\}\}) = \check{\theta}.$$

(h) Putting (e)-(g) together, and using 555Bc, 515Nc and 514Ih,

$\Vdash_{\mathbb{Q}_\kappa} \mathcal{P}\check{\lambda}/\dot{\mathcal{I}}$ has a Boolean-independent family with cardinal $\check{\theta}$ generating an order-dense subalgebra; being Dedekind complete, it is isomorphic to $\mathrm{RO}(\{0,1\}^{\check{\theta}}) \cong \mathfrak{G}_{\check{\theta}}$.

555H Corollary Suppose that λ is a two-valued-measurable cardinal and $\kappa = 2^\lambda$. Then

$$\Vdash_{\mathbb{Q}_\kappa} \text{there is a non-trivial atomless } \sigma\text{-centered power set } \sigma\text{-quotient algebra.}$$

proof (a) Note first that $\{0,1\}^{\mathfrak{c}}$ is separable (4A2B(e-ii)), so $\mathfrak{G}_{\mathfrak{c}} \cong \mathrm{RO}(\{0,1\}^{\mathfrak{c}})$ is σ-centered (514H(b-iii)); also, of course, it is atomless.

(b) Taking $\mathcal{I}$ and $\dot{\mathcal{I}}$ as in 555G, we have

$$\Vdash_{\mathbb{Q}_\kappa} \mathcal{P}\check{\lambda}/\dot{\mathcal{I}} \cong \mathfrak{G}_{\check{\theta}}$$

where $\theta = \mathrm{Tr}_{\mathcal{I}}(\lambda;\kappa)$. But since θ lies between κ and the cardinal power $\kappa^\lambda = \kappa^\omega = \kappa$, we have

$$\Vdash_{\mathbb{Q}_\kappa} \check{\theta} = \check{\kappa} = \mathfrak{c}$$

(554B), and

$$\Vdash_{\mathbb{Q}_\kappa} \mathcal{P}\check{\lambda}/\dot{\mathcal{I}} \cong \mathfrak{G}_{\mathfrak{c}} \text{ is } \sigma\text{-centered and atomless}$$

by (a).

555I The next example relies on some interesting facts which I have not yet had any compelling reason to spell out. I must begin with a definition which has so far been confined to the exercises.

Definition A Boolean algebra $\mathfrak{A}$ has the **Egorov property** if whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of countable partitions of unity in $\mathfrak{A}$ then there is a countable partition B of unity such that $\{a : a \in A_n,\ a\cap b \ne 0\}$ is finite for every $b \in B$ and $n \in \mathbb{N}$.

555J Lemma (a) Suppose that X is a set and $\#(X) < \mathfrak{b}$. Then $\mathcal{P}X$ has the Egorov property.

(b) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra with the Egorov property and I a σ-ideal of $\mathfrak{A}$. Then $\mathfrak{A}/I$ has the Egorov property.

(c) A ccc Boolean algebra has the Egorov property iff it is weakly (σ,∞)-distributive.

proof (a) Let $\langle A_n\rangle_{n\in\mathbb{N}}$ be a sequence of countable partitions of X; enumerate each A_n as $\langle a_{ni}\rangle_{i<N_n}$ where $N_n \in \mathbb{N}\cup\{\omega\}$ for each n. For each $x \in X$ and $n \in \mathbb{N}$, let $f_x(n) < N_n$ be such that $x \in a_{n,f_x(n)}$. Because $\#(X) < \mathfrak{b}$, there is an $f : \mathbb{N} \to \mathbb{N}$ such that $\{n : f(n) < f_x(n)\}$ is finite for every $x \in X$ (522C); set $g(x) = \sup\{n : f(n) < f_x(n)\}$. Now set $b_n = \{x : x \in X,\ n = \max_{k\le g(x)} f_x(k)\}$ for each n, and $B = \{b_n : n \in \mathbb{N}\}$; then B is a partition of X, and for any m, $n \in \mathbb{N}$ we have $b_n \cap a_{mi} = \emptyset$ whenever $\max(n, f(m)) < i < N_m$. So $\{a : a \in A_m,\ b_n \cap a \ne \emptyset\}$ is finite for all m, $n \in \mathbb{N}$. As $\langle A_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathcal{P}X$ has the Egorov property.

(b) Let $\langle C_n\rangle_{n\in\mathbb{N}}$ be a sequence of countable partitions of unity in $\mathfrak{A}/I$. For each $n \in \mathbb{N}$, we can choose a countable disjoint family $A_n \subseteq \mathfrak{A}$ such that $C_n = \{a^\bullet : a \in A_n\}$; set $A'_n = A_n \cup \{1 \setminus \sup A_n\}$, so that A'_n is a countable partition of unity in $\mathfrak{A}$. Let B be a countable partition of unity in $\mathfrak{A}$ such that $\{a : a \in A'_n,\ a\cap b \ne 0\}$ is finite for every $n \in \mathbb{N}$. Then $D = \{b^\bullet : b \in B\}$ is a countable partition of unity in $\mathfrak{A}/I$ and $\{c : c \in C_n,\ c\cap d \ne 0\}$ is finite for every $d \in D$ and $n \in \mathbb{N}$.

(c) This is elementary, because every partition of unity in $\mathfrak{A}$ is countable, so the Egorov property exactly matches (ii) of 316H.

555K Główczyński's example (Główczyński 91, Balcar Jech & Pazák 05, Główczyński 08) Let λ be a two-valued-measurable cardinal, and $\mathbb{P}$ a ccc forcing notion such that

$$\Vdash_{\mathbb{P}} \check{\lambda} < \mathfrak{m}$$

(5A3P). Then, taking $\mathcal{I}$ to be the null ideal of a witnessing measure on λ, and $\dot{\mathcal{I}}$ to be a $\mathbb{P}$-name for the ideal of subsets of $\check{\lambda}$ generated by $\check{\mathcal{I}}$, as in 555B,

$\Vdash_{\mathbb{P}} \mathcal{P}\check{\lambda}/\dot{\mathcal{I}}$ is ccc, atomless, Dedekind complete, weakly (σ,∞)-distributive, has Maharam type ω and is not a Maharam algebra.

proof We know from 555B that

$$\Vdash_{\mathbb{P}} \dot{\mathcal{I}} \text{ is a } \sigma\text{-ideal, and the quotient } \mathcal{P}\check{\lambda}/\dot{\mathcal{I}} \text{ is ccc and Dedekind complete.}$$

By 541P, because $\mathfrak{m} \le \mathfrak{c}$,

$$\Vdash_{\mathbb{P}} \mathcal{P}\check{\lambda}/\dot{\mathcal{I}} \text{ is atomless.}$$

By 555J, because $\mathfrak{m} \le \mathfrak{b}$,

$\Vdash_{\mathbb{P}} \mathcal{P}\check{\lambda}$ has the Egorov property, so $\mathcal{P}\check{\lambda}/\dot{\mathcal{I}}$ has the Egorov property and is weakly (σ,∞)-distributive.

Moreover, because $\mathfrak{m} \le \mathfrak{p}$, 517Rc tells us that

$\Vdash_{\mathbb{P}} \mathcal{P}\check{\lambda}$ is σ-generated by a countable set, so $\mathcal{P}\check{\lambda}/\dot{\mathcal{I}}$ is σ-generated by a countable set and has countable Maharam type.

Finally, since we certainly have

$\Vdash_{\mathbb{P}} \check{\lambda} < \mathfrak{m} \le \mathfrak{m}_{\text{countable}} \le \mathfrak{c}$, so there is a separable metrizable topology on $\check{\lambda}$,

539I shows that

$$\Vdash_{\mathbb{P}} \text{there is no non-zero Maharam submeasure on } \mathcal{P}\check{\lambda}/\dot{\mathcal{I}}.$$

555L Supercompact cardinals and the normal measure axiom If we allow ourselves to go a good deal farther than 'measurable cardinal' we can use similar techniques to find a forcing language in which NMA is true.

Definition An uncountable cardinal κ is **supercompact** if for every set X there is a κ-additive maximal proper ideal $\mathcal{I}$ of subsets of $S = [X]^{<\kappa}$ such that

(α) $\{s : s \in S,\, x \notin s\} \in \mathcal{I}$ for every $x \in X$,

(β) if $A \subseteq S$, $A \notin \mathcal{I}$ and $f : A \to X$ is such that $f(s) \in s$ for every $s \in A$, then there is an $x \in X$ such that $\{s : s \in A,\, f(s) = x\} \notin \mathcal{I}$.

(Compare 545D.)

555M Proposition A supercompact cardinal is two-valued-measurable.

proof If κ is supercompact, let $\mathcal{I}$ be a κ-additive maximal ideal of subsets of $S = [\kappa]^{<\kappa}$ satisfying (α) of 555L. Define $f : S \to \kappa$ by setting $f(s) = \min(\kappa \setminus s)$ for $s \in S$. Then $\mathcal{J} = \{J : J \subseteq \kappa,\, f^{-1}[J] \in \mathcal{I}\}$ is a κ-additive maximal ideal of subsets of κ. If $\xi < \kappa$, then

$$f^{-1}[\{\xi\}] = \{s : f(s) = \xi\} \subseteq \{s : \xi \notin s\} \in \mathcal{I},$$

so $\{\xi\} \in \mathcal{J}$. Thus $\mathcal{J}$ contains all singletons, and witnesses that κ is two-valued-measurable.

555N Theorem (Prikry 75, Fleissner 91) Suppose that κ is a supercompact cardinal. Then

$$\Vdash_{\mathbb{P}_\kappa} \text{the normal measure axiom and the product measure extension axiom are true.}$$

proof (a) Life will be a little easier if I start by pointing out that we can work with a variation of NMA as stated in 545D. First, for a set X and an uncountable cardinal λ let $\ddagger(X,\lambda)$ be the statement

there is a λ-additive probability measure ν on $S = [X]^{<\lambda}$, with domain $\mathcal{P}S$, such that

(α) $\nu\{s : x \in s \in S\} = 1$ for every $x \in X$,

(β) if $g : S \setminus \{\emptyset\} \to X$ is such that $g(s) \in s$ for every $s \in S \setminus \{\emptyset\}$, then there is a countable set $K \subseteq X$ such that $\nu g^{-1}[K] = 1$.

Now the point is that if $\ddagger(\alpha,\mathfrak{c})$ is true for every ordinal α, then the normal measure axiom is true. **P** Let X be any set. Since, as always, we are working with the axiom of choice, X is equipollent with some ordinal and $\ddagger(X,\mathfrak{c})$ is true; let ν be a measure on $S = [X]^{<\mathfrak{c}}$ as above. Given $A \subseteq S$ and a function $f : A \to X$ which is regressive in the sense of (β) in 545D, then we can extend f to a function $g : S \setminus \{\emptyset\} \to X$ which is regressive in the sense of (β) here. If K is a countable set such that $g^{-1}[K]$ is conegligible, and A is not negligible, then there must be a $\xi \in K$ such that $A \cap g^{-1}[\{\xi\}] = f^{-1}[\{\xi\}]$ is not negligible, as required in 545D. **Q**

(b) For the time being (down to the end of (d) below), fix an ordinal α. Let $\mathcal{I}$ be a κ-additive maximal ideal of subsets of $S = [\alpha]^{<\kappa}$ as in 555L, and ν the corresponding measure on S, setting $\nu A = 0$ and $\nu(S \setminus A) = 1$ if $A \in \mathcal{I}$. By 555C, we have a corresponding $\mathbb{P}_\kappa$-name $\dot{\nu}$ for a measure on $\check{S}$. Now

$$\Vdash_{\mathbb{P}_\kappa} \check{S} \subseteq [\check{\alpha}]^{<\check{\kappa}},$$

so we have a $\mathbb{P}_\kappa$-name $\dot{\mu}$ such that

$\Vdash_{\mathbb{P}_\kappa} \dot{\mu}$ is a measure with domain $\mathcal{P}([\check{\alpha}]^{<\check{\kappa}})$ and $\dot{\mu}(D) = \dot{\nu}(D \cap \check{S})$ for every $D \subseteq [\check{\alpha}]^{<\check{\kappa}}$.

By 555C,

$\Vdash_{\mathbb{P}_\kappa} \dot{\nu}$ is a $\check{\kappa}$-additive probability measure, so $\dot{\mu}$ is a $\check{\kappa}$-additive probability measure.

(c) $\Vdash_{\mathbb{P}_\kappa} \dot{\mu}\{s : \xi \in s \in [\check{\alpha}]^{<\check{\kappa}}\} = 1$ for every $\xi < \check{\alpha}$.

P If $a \in \mathfrak{B}_\kappa^+$ and $\dot{\xi}$ are such that $a \Vdash_{\mathbb{P}} \dot{\xi} < \check{\alpha}$, take any b stronger than a and $\xi < \alpha$ such that $b \Vdash_{\mathbb{P}_\kappa} \dot{\xi} = \check{\xi}$. Now $I = \{s : s \in S,\ \xi \notin s\} \in \mathcal{I}$ so

$$b \Vdash_{\mathbb{P}_\kappa} 0 = \dot{\nu}\check{I} = \dot{\nu}\{s : s \in \check{S},\ \check{\xi} \notin s\} = \dot{\mu}\{s : s \in [\check{\alpha}]^{<\check{\kappa}},\ \check{\xi} \notin s\}$$
$$\text{and } 1 = \dot{\mu}\{s : s \in [\check{\alpha}]^{<\check{\kappa}},\ \check{\xi} \in s\}.$$

As b and ξ are arbitrary,

$$a \Vdash_{\mathbb{P}_\kappa} \dot{\mu}\{s : \dot{\xi} \in s \in [\check{\alpha}]^{<\check{\kappa}}\} = 1;$$

as a and $\dot{\xi}$ are arbitrary,

$$\Vdash_{\mathbb{P}_\kappa} \dot{\mu}\{s : \xi \in s \in [\check{\alpha}]^{<\check{\kappa}}\} = 1 \text{ for every } \xi < \check{\alpha}.\ \mathbf{Q}$$

(d) Suppose that $a \in \mathfrak{B}_\kappa^+$ and that $\dot{f}$ is a $\mathbb{P}_\kappa$-name such that

$$a \Vdash_{\mathbb{P}_\kappa} \dot{f} : [\check{\alpha}]^{<\check{\kappa}} \setminus \{\emptyset\} \to \check{\alpha} \text{ is a function and } \dot{f}(s) \in s \text{ whenever } \emptyset \neq s \in [\check{\alpha}]^{<\check{\kappa}}.$$

For each $s \in S \setminus \{\emptyset\}$, we have

$$a \Vdash_{\mathbb{P}_\kappa} \dot{f}(\check{s}) \in \check{s};$$

because $\mathbb{P}_\kappa$ is ccc, there is a non-empty countable set $J_s \subseteq s$ such that $a \Vdash_{\mathbb{P}_\kappa} \dot{f}(\check{s}) \in \check{J}_s$ (5A3Nc). Let $\langle h_n(s)\rangle_{n\in\mathbb{N}}$ be a sequence running over J_s. For each $n \in \mathbb{N}$, we have a $\beta_n < \alpha$ such that $\{s : s \in S \setminus \{\emptyset\},\ h_n(s) \neq \beta_n\} \in \mathcal{I}$. Set $K = \{\beta_n : n \in \mathbb{N}\}$; since $\mathcal{I}$ is a σ-ideal containing $\{\emptyset\}$, $I = \{s : s \in S \setminus \{\emptyset\},\ J_s \not\subseteq K\} \cup \{\emptyset\}$ belongs to $\mathcal{I}$. But in this case, $\Vdash_{\mathbb{P}_\kappa} \dot{\mu}\check{I} = \dot{\nu}\check{I} = 0$ and

$$a \Vdash_{\mathbb{P}_\kappa} \dot{f}(\check{s}) \in \check{J}_s \subseteq \check{K}$$

whenever $s \in S \setminus I$, so

$$a \Vdash_{\mathbb{P}_\kappa} \dot{\mu}(\dot{f}^{-1}[\check{K}]) = \dot{\nu}(\check{S} \cap \dot{f}^{-1}[\check{K}]) \geq \dot{\nu}(\check{S} \setminus \check{I}) = 1,$$

while of course $\Vdash_{\mathbb{P}_\kappa} \check{K}$ is countable.

(e) What this means is that

$$\Vdash_{\mathbb{P}_\kappa} \ddagger(\check{\alpha}, \check{\kappa})$$

for every ordinal α; since forcing adds no new ordinals (5A3Na),

$$\Vdash_{\mathbb{P}_\kappa} \ddagger(\alpha, \check{\kappa}) \text{ for every ordinal } \alpha.$$

But 552B, with 555M and 5A1Fc, tells us that

$$\Vdash_{\mathbb{P}_\kappa} \mathfrak{c} = \check{\kappa}, \text{ so } \ddagger(\alpha, \mathfrak{c}) \text{ for every ordinal } \alpha;$$

with (a) above and 545E, we get

$$\Vdash_{\mathbb{P}_\kappa} \text{NMA and PMEA}.$$

555O All forcing constructions of quasi-measurable cardinals start from two-valued-measurable cardinals, and there is a reason for this.

Theorem (Solovay 71) If κ is an uncountable cardinal and $\mathcal{I}$ is a proper κ-saturated κ-additive ideal of $\mathcal{P}\kappa$ containing singletons, then

$L(\mathcal{I}) \vDash \kappa$ is two-valued-measurable and the generalized continuum hypothesis is true.

Remarks The proof employs techniques not used elsewhere in this treatise, so I omit it entirely, to the point of not explaining what $L(\mathcal{I})$ is or what the symbol $\vDash$ means; I remark only that $L(\mathcal{I})$ is a proper class containing every ordinal and the set $\mathcal{I}$, and that the theorem says that the axioms of ZFC, together with 'κ is two-valued-measurable' and the generalized continuum hypothesis, are true when relativized appropriately to the class $L(\mathcal{I})$. For more, see JECH 78, p. 416, Theorem 82a.

555X Basic exercises (a) Suppose that λ is a real-valued-measurable cardinal with witnessing probability ν, and κ a cardinal. Let $\dot\mu$ be the $\mathbb{P}_\kappa$-name for a measure on $\check\lambda$ as defined in 555C. Show that

$$\Vdash_{\mathbb{P}_\kappa} \dot\mu\check A = (\nu A)\check{}$$

for any $A \subseteq \lambda$.

(b) Suppose that λ is a two-valued-measurable cardinal. Set $\kappa = 2^\lambda$. Show that

$\Vdash_{\mathbb{P}_\kappa} \check\lambda$ is an atomlessly-measurable cardinal with a witnessing probability with Maharam type $\mathfrak{c} = 2^{\check\lambda}$.

(c) Suppose that λ is a two-valued-measurable cardinal and that $\kappa = (2^\lambda)^{(+\omega)}$. Show that

$\Vdash_{\mathbb{P}_\kappa} \check\lambda$ is an atomlessly-measurable cardinal with a witnessing probability with Maharam type less than $\mathfrak{c}$.

(d) Suppose that λ is a two-valued-measurable cardinal and $\kappa = 2^\lambda$. Show that

$\Vdash_{\mathbb{P}_\kappa}$ there is a non-trivial atomless σ-linked power set σ-quotient algebra.

(e) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Show that $\mathfrak{A}$ has the Egorov property iff for every sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in $L^0 = L^0(\mathfrak{A})$ there is a sequence $\langle \alpha_n\rangle_{n\in\mathbb{N}}$ in $]0,\infty[$ such that $\{\alpha_n u_n : n \in \mathbb{N}\}$ is order-bounded in L^0.

555Y Further exercises (a) Suppose that X is a set, and $\mathcal{I}$ a proper ideal of subsets of X containing singletons. Let $\mathbb{P}$ be a forcing notion such that $\operatorname{sat}\mathbb{P} \le \operatorname{add}\mathcal{I}$, and $\dot{\mathcal{I}}$ a $\mathbb{P}$-name for the ideal of subsets of $\check X$ generated by $\check{\mathcal{I}}$, as in 555B. (i) Show that

$$\Vdash_{\mathbb{P}} \operatorname{add}\dot{\mathcal{I}} = (\operatorname{add}\mathcal{I})\check{}.$$

(ii) Suppose that $\operatorname{sat}(\mathcal{P}X/\mathcal{I}) < \operatorname{add}\mathcal{I}$. Set $\theta = \max(\operatorname{sat}\mathbb{P}, \operatorname{sat}(\mathcal{P}X/\mathcal{I}))$. Show that

$\Vdash_{\mathbb{P}} \check\theta$ is a cardinal, $\dot{\mathcal{I}}$ is $\check\theta$-saturated in $\mathcal{P}\check X$ and $\mathcal{P}\check X/\dot{\mathcal{I}}$ is Dedekind complete.

(iii) Show that if $X = \lambda$ is a regular uncountable cardinal and $\mathcal{I}$ is a normal ideal on λ, then

$\Vdash_{\mathbb{P}} \dot{\mathcal{I}}$ is a normal ideal on $\check\lambda$.

(b) In 555B, show that if $\mathcal{I}$ is θ-saturated in $\mathcal{P}X$, where θ is an uncountable cardinal such that $\operatorname{cf}[\theta]^{\le\omega} < \operatorname{add}\mathcal{I}$, then

$\Vdash_{\mathbb{P}} \dot{\mathcal{I}}$ is $\check\theta$-saturated in $\mathcal{P}\check X$.

(c) Suppose that λ is a two-valued-measurable cardinal, and that $\mathbb{P}$ is a forcing notion with $\#(\mathbb{P}) < \lambda$. Show that $\Vdash_{\mathbb{P}} \check\lambda$ is a two-valued-measurable cardinal.

(d) Suppose that κ is a two-valued-measurable cardinal, and that $\mathfrak{m} = \mathfrak{c}$. Show that

$\Vdash_{\mathbb{P}_\kappa} \mathfrak{c}$ is real-valued-measurable, $\mathfrak{b} = \mathfrak{d} = \check{\mathfrak{c}}$ and the shrinking number of the Lebesgue null ideal is at least $\check{\mathfrak{c}}$.

(e) Let κ be a cardinal. Suppose that for every set X there is a κ-additive maximal proper ideal $\mathcal{I}$ of subsets of $S = [X]^{<\kappa}$ such that

(α) $\{s : s \in S,\, x \notin s\} \in \mathcal{I}$ for every $x \in X$,

(β) if $A \subseteq S$, $A \notin \mathcal{I}$ and $f : A \to X$ is such that $f(s) \in s$ for every $s \in A$, then there is an $x \in X$ such that $\{s : s \in A,\, f(s) = x\} \notin \mathcal{I}$.

Show that κ is supercompact.

(f) Let κ be a supercompact cardinal. Show that $\square_\lambda$ is false for every $\lambda \ge \kappa$.

(g) Suppose that λ is a two-valued-measurable cardinal, and $\kappa > \lambda$ a cardinal. Show that

$\Vdash_{\mathbb{P}_\kappa}$ there is a probability measure on ω_1 with Maharam type greater than the least atomlessly-measurable cardinal.

(h) In 555C, suppose that $X = \kappa$ and that μ witnesses that κ is two-valued-measurable, that is, μ is a κ-additive $\{0,1\}$-valued measure with domain $\mathcal{P}\kappa$. For $J \subseteq \kappa$ let $P_J : L^\infty(\mathfrak{B}_\kappa) \to L^\infty(\mathfrak{B}_\kappa)$ be the corresponding conditional expectation as in part (b) of the proof of 555F. Show that for every $\sigma \in \mathfrak{B}_\kappa^\kappa$ there is a countable set $J \subseteq \kappa$ such that $\mu\{\xi : \xi < \kappa,\, u_\sigma = P_J(\chi\sigma(\xi))\} = 1$.

555Z Problems (a) In 555B, what can we say about the π-weight of $\mathcal{P}\check{X}/\dot{\mathcal{I}}$?

(b) Suppose that λ is an atomlessly-measurable cardinal with a normal witnessing probability. Let $\langle A_\eta\rangle_{\eta<\omega_1}$ be a family of non-negligible subsets of λ. Must there be a countable set meeting every A_η? (See 555F and 521Xi.)

555 Notes and comments The point of Solovay's theorems 555D and 555O is that they are relative consistency results. Continuing the discussion in the notes to §541, write '∃2vmc', '∃qmc', '∃amc' for the sentences 'there is a two-valued-measurable cardinal', 'there is a quasi-measurable cardinal' and 'there is an atomlessly-measurable cardinal'. I have already noted that there are fundamental metamathematical reasons why we cannot have a proof, in ZFC, that

if ZFC is consistent then ZFC + ∃qmc is consistent

unless ZFC is actually *inconsistent.* But 555D tells us that

if ZFC + ∃2vmc is consistent, then ZFC + ∃amc is consistent

and 555O that

if ZFC + ∃qmc is consistent, then ZFC + ∃2vmc is consistent.

Since ∃qmc is actually a consequence of both ∃2vmc and ∃amc, we see that

if one of ZFC + ∃2vmc, ZFC + ∃amc, ZFC + ∃qmc is consistent, so are the others;

that is, ∃2vmc, ∃amc and ∃qmc are equiconsistent in ZFC. Of course they are not in general *equiveridical* (unless all are disprovable); as noted in 555O, if we start from a universe in which ∃qmc is true, we can move to one in which ∃2vmc and CH are both true, so that ∃amc is false, and there are easier arguments to show that if we start with ∃amc, we can move to

∃amc + 'there are no strongly inaccessible cardinals',

so that we have ∃amc true but ∃2vmc false.

The reason for working through these equiconsistency results is to show that assertions like NMA, PMEA and ∃amc, which are of interest in measure theory and general topology, are no more dangerous than appropriate assertions about large cardinals which have been explored in depth (Jech 78, chap. 6; Kanamori 03, §22; Jech 03, §20), and for which we can have corresponding confidence that either they are consistent with ZFC, or that an eventual contradiction would lead to an earthquake, and rescue (if it came) would be from outside measure theory.

In §544 I examined some of the consequences of supposing that there is an atomlessly-measurable cardinal; for instance, that there are many Sierpiński sets (544G). It is not an accident that we get similar properties of random real models (552E). If we want to know if something might be implied by the existence of an atomlessly-measurable cardinal, the first step is to look at what can be determined in the forcing models of 555C. This is often straightforward; for instance, since $\mathfrak{d}$ is not changed by random real forcing, and since $\mathfrak{d}$ must be much lower than any two-valued-measurable cardinal, it must be much lower than any atomlessly-measurable cardinal created by random real forcing. But it is quite unclear that the same can be said about atomlessly-measurable cardinals in general (544Zd). I offer 555F as another example of a phenomenon which appears in Solovay's models but which is not known to be true for all atomlessly-measurable cardinals (555Zb).

In §§546-547 I gave some of the Gitik-Shelah results showing that non-trivial 'simple' algebras cannot be power set σ-quotient algebras. Of course this depends a good deal on what we mean by 'simple'. Looking at the basic cardinal functions, we see that (at least if there can be measurable cardinals) then there can be non-trivial ccc power set σ-quotient algebras which are σ-centered or have countable Maharam type (555H, 555K). But they are still very far away from any algebra which can be specified without (perhaps implicitly) using a two-valued-measurable cardinal at some stage. We cannot have a non-trivial power set σ-quotient algebra with countable π-weight (547Xa), but I do not see how to rule out 'small' π-weight in general (547Zb, 555Za).

Version of 3.1.15

556 Forcing with Boolean subalgebras

I propose now to describe a completely different way in which forcing can be used to throw light on problems in measure theory. Rather than finding forcing models of new mathematical universes, we look for models which will express structures of the ordinary universe in new ways. The problems to which this approach seems to be most relevant are those centered on invariant algebras: in ergodic theory, fixed-point algebras; in the theory of relative independence, the core σ-algebras.

Most of the section is taken up with development of basic machinery. The strategic plan is straightforward enough: given a specific Boolean algebra $\mathfrak{C}$ which seems to be central to a question in hand, force with $\mathfrak{C}\setminus\{0\}$, and translate the question into a question in the forcing language. In order to do this, we need an efficient scheme for automatic translation. This is what 556A-556L and 556O are setting up. The translation has to work both ways, since we need to be able to deduce properties of the ground model from properties of the forcing model.

There are four actual theorems for which I offer proofs by these methods. The first three are 556M (a strong law of large numbers), 556N (Dye's theorem on orbit-isomorphic measure-preserving transformations) and 556P (Kawada's theorem on invariant measures). In each of these, the aim is to prove a general form of the theorem from the classical special case in which the algebra $\mathfrak{C}$ is trivial. In the final example 556S (I.Farah's description of the Dedekind completion of the asymptotic density algebra $\mathfrak{Z}$), we have a natural subalgebra $\mathfrak{C}$ of $\mathfrak{Z}$ and a structure in the corresponding forcing universe to which we can apply Maharam's theorem.

556A Forcing with Boolean subalgebras Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$, and $\mathfrak{C}$ a subalgebra of $\mathfrak{A}$. Let $\mathbb{P}$ be the forcing notion $\mathfrak{C}^+=\mathfrak{C}\setminus\{0\}$, active downwards.

(a) If $a\in\mathfrak{A}$, the **forcing name for** a **over** $\mathfrak{C}$ will be the $\mathbb{P}$-name

$$\dot a=\{(\check b,p):p\in\mathfrak{C}^+,\,b\in\mathfrak{A},\,p\cap b\subseteq a\}.$$

(b) If $\mathfrak{B}$ is a Boolean subalgebra of $\mathfrak{A}$ including $\mathfrak{C}$, then the **forcing name for** $\mathfrak{B}$ **over** $\mathfrak{C}$ will be the $\mathbb{P}$-name $\{(\dot b,1):b\in\mathfrak{B}\}$, where here $1=1_{\mathfrak{A}}=1_{\mathfrak{B}}=1_{\mathfrak{C}}$.

(c) For each of the binary operations $\bigcirc=\cap,\,\cup,\,\triangle,\,\setminus$ on $\mathfrak{A}$, the **forcing name for** $\bigcirc$ **over** $\mathfrak{C}$ will be the $\mathbb{P}$-name

$$\dot\bigcirc=\{(((\dot a,\dot b),(a\bigcirc b)^\bullet),1):a,\,b\in\mathfrak{A}\}.$$

(d) The **forcing name for** $\subseteq$ **over** $\mathfrak{C}$ will be the $\mathbb{P}$-name

$$\dot\subseteq\;=\{((\dot a,\dot b),1):a,\,b\in\mathfrak{A},\,a\subseteq b\}.$$

(e) Let $\pi : \mathfrak{A} \to \mathfrak{A}$ be a ring homomorphism such that $\pi c \subseteq c$ for every $c \in \mathfrak{C}$. In this case, I will say that the **forcing name for π over $\mathfrak{C}$** is the $\mathbb{P}$-name $\{((\dot{a}, (\pi a)^{\bullet}), 1) : a \in \mathfrak{A}\}$.

(f) Now suppose that $\mathfrak{A}$ is Dedekind σ-complete. For $u \in L^0(\mathfrak{A})$, the **forcing name for u over $\mathfrak{C}$** will be the $\mathbb{P}$-name $\{((\check{\alpha}, [\![u > \alpha]\!]^{\bullet}), 1) : \alpha \in \mathbb{Q}\}$.

Remark We shall need to agree on what it is that the formula $L^0(\mathfrak{A})$ abbreviates. The primary definition in 364Aa speaks of functions from $\mathbb{R}$ to $\mathfrak{A}$. Because $\mathbb{R}$ is inadequately absolute this is not convenient here, and I will move to the alternative version in 364Af: a member u of $L^0(\mathfrak{A})$ is a family $\langle [\![u > \alpha]\!] \rangle_{\alpha \in \mathbb{Q}}$ in $\mathfrak{A}$ such that

$$[\![u > \alpha]\!] = \sup_{\beta \in \mathbb{Q}, \beta > \alpha} [\![u > \beta]\!] \text{ for every } \alpha \in \mathbb{Q},$$

$$\inf_{n \in \mathbb{N}} [\![u > n]\!] = 0, \quad \sup_{n \in \mathbb{N}} [\![u > -n]\!] = 1.$$

556B Theorem Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$, and $\mathfrak{C}$ a subalgebra of $\mathfrak{A}$. Let $\mathbb{P}$ be the forcing notion $\mathfrak{C}^+$, active downwards, and $\dot{\mathfrak{A}}$ the forcing name for $\mathfrak{A}$ over $\mathfrak{C}$.

(a) If $p \in \mathfrak{C}^+$, a, $b \in \mathfrak{A}$ and $\dot{a}$, $\dot{b}$ are the forcing names of a, b over $\mathfrak{C}$, then

$$p \Vdash_{\mathbb{P}} \dot{a} = \dot{b}$$

iff $\operatorname{upr}(p \cap (a \triangle b), \mathfrak{C}) = 0$, that is, for every q stronger than p there is an r stronger than q such that $r \cap a = r \cap b$. In particular,

$$p \Vdash_{\mathbb{P}} \dot{a} = \dot{b}$$

whenever $p \cap a = p \cap b$.

(b) Writing $\dot{\bigcirc}$ for the forcing name for $\bigcirc$ over $\mathfrak{C}$,

$$\Vdash_{\mathbb{P}} \dot{\bigcirc} \text{ is a binary operation on } \dot{\mathfrak{A}} \text{ and } \dot{a} \mathbin{\dot{\bigcirc}} \dot{b} = (a \bigcirc b)^{\bullet}$$

for each of the binary operations $\bigcirc = \cap$, $\cup$, $\triangle$ and $\setminus$ and all a, $b \in \mathfrak{A}$.

(c) All the standard identities translate. For instance,

$$\Vdash_{\mathbb{P}} x \mathbin{\dot{\cap}} (y \mathbin{\dot{\triangle}} z) = (x \mathbin{\dot{\cap}} y) \mathbin{\dot{\triangle}} (x \mathbin{\dot{\cap}} z) \text{ for all } x,\ y,\ z \in \dot{\mathfrak{A}}.$$

(d)

$\Vdash_{\mathbb{P}} \dot{\mathfrak{A}}$, with the operations $\dot{\triangle}$, $\dot{\cap}$, $\dot{\cup}$ and $\dot{\setminus}$, is a Boolean algebra, with zero $\dot{0}$ and identity $\dot{1}$.

(e)(i) Writing $\dot{\subseteq}$ for the forcing name for $\subseteq$ over $\mathfrak{C}$,

$$\Vdash_{\mathbb{P}} \dot{\subseteq} \text{ is the inclusion relation in the Boolean algebra } \dot{\mathfrak{A}}.$$

(ii) For a, $b \in \mathfrak{A}$ and $p \in \mathfrak{C}^+$,

$$p \Vdash_{\mathbb{P}} \dot{a} \mathbin{\dot{\subseteq}} \dot{b}$$

iff $\operatorname{upr}(p \cap a \setminus b, \mathfrak{C}) = 0$.

(f) If $\mathfrak{B}$ is a Boolean subalgebra of $\mathfrak{A}$ including $\mathfrak{C}$, then

$$\Vdash_{\mathbb{P}} \dot{\mathfrak{B}} \text{ is a Boolean subalgebra of } \dot{\mathfrak{A}}.$$

proof (a)(i) Recall that $\operatorname{upr}(a, \mathfrak{C}) = \inf\{c : a \subseteq c \in \mathfrak{C}\}$ if the infimum is defined in $\mathfrak{C}$ (313S). So $\operatorname{upr}(a, \mathfrak{C}) = 0$ iff for every non-zero $c \in \mathfrak{C}$ there is a $c' \in \mathfrak{C}$ such that $a \subseteq c'$ and $c \not\subseteq c'$; that is, for every non-zero $c \in \mathfrak{C}$ there is a non-zero $c' \in \mathfrak{C}$ such that $c' \subseteq c \setminus a$. In the present context, we see that for $p \in \mathfrak{C}^+$ and a, $b \in \mathfrak{A}$, $\operatorname{upr}(p \cap (a \triangle b), \mathfrak{C}) = 0$ iff for every q stronger than p there is an r stronger than q such that $r \cap (a \triangle b) = 0$.

(ii) Suppose that $p \cap a = p \cap b$, that $q \in \mathfrak{C}^+$ is stronger than p, and that $\dot{x}$ is a $\mathbb{P}$-name such that $q \Vdash_{\mathbb{P}} \dot{x} \in \dot{a}$. Then there are an $r \in \mathfrak{C}^+$, a $d \in \mathfrak{A}$ such that $(\check{d}, r) \in \dot{a}$, and an s stronger than both r and q such that $s \Vdash_{\mathbb{P}} \dot{x} = \check{d}$. In this case

$$s \cap d \subseteq p \cap r \cap d \subseteq p \cap a \subseteq b,$$

so $(\check{d},s)\in\dot{b}$ and

$$s\Vdash_{\mathbb{P}}\dot{x}=\check{d}\in\dot{b}.$$

As q and $\dot{x}$ are arbitrary,

$$p\Vdash_{\mathbb{P}}\dot{a}\text{ is a subset of }\dot{b};$$

similarly,

$$p\Vdash_{\mathbb{P}}\dot{b}\text{ is a subset of }\dot{a}\text{ and }\dot{b}=\dot{a}.$$

(iii) If $\operatorname{upr}(p\cap(a\triangle b),\mathfrak{C})=0$, then for every q stronger than p there is an r stronger than q such that $r\cap a=r\cap b$ and $r\Vdash_{\mathbb{P}}\dot{a}=\dot{b}$, by (ii). As q is arbitrary, $p\Vdash_{\mathbb{P}}\dot{a}=\dot{b}$.

(iv) Now suppose that $p\Vdash_{\mathbb{P}}\dot{a}=\dot{b}$ and that q is stronger than p. Then $(\check{a},q)\in\dot{a}$, so $q\Vdash_{\mathbb{P}}\check{a}\in\dot{a}=\dot{b}$. There must therefore be a $(\check{d},r)\in\dot{b}$ and an s stronger than both r and q such that $s\Vdash_{\mathbb{P}}\check{a}=\check{d}$; in this case $d=a$, $s\cap a\subseteq r\cap d\subseteq b$ and $s\cap a\setminus b=0$.

As q is arbitrary, $\operatorname{upr}(p\cap(a\setminus b),\mathfrak{C})=0$. Similarly, $\operatorname{upr}(p\cap(b\setminus a),\mathfrak{C})=0$. By 313Sb, $\operatorname{upr}(p\cap(a\triangle b),\mathfrak{C})=0$.

(b) Of course

$$\Vdash_{\mathbb{P}}\dot{\bigcirc}\subseteq(\dot{\mathfrak{A}}\times\dot{\mathfrak{A}})\times\dot{\mathfrak{A}},$$

just because

$$\Vdash_{\mathbb{P}}\dot{a}\in\dot{\mathfrak{A}}$$

for every $a\in\mathfrak{A}$. To see that $\dot{\bigcirc}$ is a name for a function with domain $\dot{A}\times\dot{A}$, use 5A3Ea. If $(((\dot{a}_1,\dot{b}_1),(a_1\bigcirc b_1)^\bullet),1)$ and $(((\dot{a}_2,\dot{b}_2),(a_2\bigcirc b_2)^\bullet),1)$ are two members of $\dot{\bigcirc}$, and $p\in\mathfrak{C}^+$ is such that

$$p\Vdash_{\mathbb{P}}(\dot{a}_1,\dot{b}_1)=(\dot{a}_2,\dot{b}_2),$$

then

$$\begin{aligned}\operatorname{upr}(p\cap((a_1\bigcirc b_1)\triangle(a_2\bigcirc b_2)))&\subseteq\operatorname{upr}(p\cap((a_1\triangle a_2)\cup(b_1\triangle b_2)))\\&=\operatorname{upr}(p\cap(a_1\triangle a_2))\cup\operatorname{upr}(p\cap(b_1\triangle b_2))=0\end{aligned}$$

by (a) above and 313Sb again. So

$$p\Vdash_{\mathbb{P}}(a_1\bigcirc b_1)^\bullet=(a_2\bigcirc b_2)^\bullet$$

by (a) in the other direction. Thus the condition of 5A3E(a-ii) is satisfied, and

$$\Vdash_{\mathbb{P}}\dot{\bigcirc}\text{ is a function,}$$

while of course

$$\Vdash_{\mathbb{P}}\dot{a}\dot{\bigcirc}\dot{b}=(a\bigcirc b)^\bullet$$

for all a, $b\in\mathfrak{A}$. Moreover, setting $\dot{A}=\{((\dot{a},\dot{b}),1):a,\,b\in\mathfrak{A}\}$, 5A3Eb tells us that

$$\Vdash_{\mathbb{P}}\operatorname{dom}\dot{\bigcirc}=\dot{A}=\dot{\mathfrak{A}}\times\dot{\mathfrak{A}},\text{ so }\dot{\bigcirc}\text{ is a binary operation on }\dot{\mathfrak{A}}.$$

(c) I work through only the given example. Suppose that $p\in\mathfrak{C}^+$ and that $\dot{x}$, $\dot{y}$ and $\dot{z}$ are $\mathbb{P}$-names such that

$$p\Vdash_{\mathbb{P}}\dot{x},\,\dot{y},\,\dot{z}\in\dot{\mathfrak{A}}.$$

If q is stronger than p, there are an r stronger than q and a, b, $c\in\mathfrak{A}$ such that

$$r\Vdash_{\mathbb{P}}\dot{x}=\dot{a},\ \dot{y}=\dot{b}\text{ and }\dot{z}=\dot{c}.$$

Then

$$r\Vdash_{\mathbb{P}}\dot{y}\mathbin{\dot{\triangle}}\dot{z}=\dot{b}\mathbin{\dot{\triangle}}\dot{c}=(b\triangle c)^\bullet,$$

$$r\Vdash_{\mathbb{P}}\dot{x}\mathbin{\dot{\cap}}(\dot{y}\mathbin{\dot{\triangle}}\dot{z})=(a\cap(b\triangle c))^\bullet=((a\cap b)\triangle(a\cap c))^\bullet=(\dot{x}\mathbin{\dot{\cap}}\dot{y})\mathbin{\dot{\triangle}}(\dot{x}\mathbin{\dot{\cap}}\dot{z}).$$

As q is arbitrary,

$$p \Vdash_{\mathbb{P}} \dot{x} \mathbin{\dot\cap} (\dot{y} \mathbin{\dot\triangle} \dot{z}) = (\dot{x} \mathbin{\dot\cap} \dot{y}) \mathbin{\dot\triangle} (\dot{x} \mathbin{\dot\cap} \dot{z});$$

as p, $\dot{x}$, $\dot{y}$ and $\dot{z}$ are arbitrary,

$$\Vdash_{\mathbb{P}} x \mathbin{\dot\cap} (y \mathbin{\dot\triangle} z) = (x \mathbin{\dot\cap} y) \mathbin{\dot\triangle} (x \mathbin{\dot\cap} z) \text{ for all } x,\ y,\ z \in \dot{\mathfrak{A}}.$$

(d) This is now elementary, amounting to repeated use of the technique in (c).

(e)(i) It will be enough to show that

$$\Vdash_{\mathbb{P}} \text{for all } x,\ y \in \dot{\mathfrak{A}},\ x \mathbin{\dot\subseteq} y \iff x \mathbin{\dot\cap} y = x.$$

P Suppose that $p \in \mathfrak{C}^+$ and that $\dot{x}$, $\dot{y}$ are $\mathbb{P}$-names such that

$$p \Vdash_{\mathbb{P}} \dot{x},\ \dot{y} \in \dot{\mathfrak{A}}.$$

(α) Suppose that $p \Vdash_{\mathbb{P}} \dot{x} \mathbin{\dot\subseteq} \dot{y}$. If q is stronger than p, there are an r stronger than q and a, $b \in \mathfrak{A}$ such that $a \subseteq b$ and

$$r \Vdash_{\mathbb{P}} \dot{x} = \dot{a} \text{ and } \dot{y} = \dot{b}.$$

Now

$$r \Vdash_{\mathbb{P}} \dot{x} \mathbin{\dot\cap} \dot{y} = (a \cap b)^{\bullet} = \dot{a} = \dot{x};$$

as q is arbitrary, $p \Vdash_{\mathbb{P}} \dot{x} \mathbin{\dot\cap} \dot{y} = \dot{x}$. ($\beta$) Conversely, suppose that $p \Vdash_{\mathbb{P}} \dot{x} \mathbin{\dot\cap} \dot{y} = \dot{x}$. If q is stronger than p there are r stronger than q and a, $b \in \mathfrak{A}$ such that

$$r \Vdash_{\mathbb{P}} \dot{x} = \dot{a},\ \dot{y} = \dot{b},\ (a \cap b)^{\bullet} = \dot{a};$$

now $(((a \cap b)^{\bullet}, \dot{b}), 1) \in \ \dot\subseteq$, so $\Vdash_{\mathbb{P}} (a \cap b)^{\bullet} \mathbin{\dot\subseteq} \dot{b}$ and

$$r \Vdash_{\mathbb{P}} \dot{x} = \dot{a} = (a \cap b)^{\bullet} \mathbin{\dot\subseteq} \dot{b} = \dot{y}.$$

As q is arbitrary. $p \Vdash_{\mathbb{P}} \dot{x} \mathbin{\dot\subseteq} \dot{y}$.

As p, $\dot{x}$ and $\dot{y}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \text{for } x,\ y \in \dot{\mathfrak{A}},\ x \mathbin{\dot\subseteq} y \iff x \mathbin{\dot\cap} y = x. \quad \mathbf{Q}$$

(ii) Now, for a, $b \in \mathfrak{A}$ and $p \in \mathfrak{C}^+$,

$$\begin{aligned} &p \Vdash_{\mathbb{P}} \dot{a} \mathbin{\dot\subseteq} \dot{b} \\ &\qquad \text{iff } p \Vdash_{\mathbb{P}} \dot{a} \mathbin{\dot\cap} \dot{b} = \dot{a} \\ &\qquad \text{iff } p \Vdash_{\mathbb{P}} (a \cap b)^{\bullet} = \dot{a} \\ &\qquad \text{iff } \operatorname{upr}(p \cap (a \triangle (a \cap b)), \mathfrak{C}) = 0 \\ &\qquad \text{iff } \operatorname{upr}(p \cap a \setminus b, \mathfrak{C}) = 0. \end{aligned}$$

(f) This should now be easy. As $\dot{\mathfrak{B}} \subseteq \dot{\mathfrak{A}}$, $\Vdash_{\mathbb{P}} \dot{\mathfrak{B}} \subseteq \dot{\mathfrak{A}}$. If $p \in \mathfrak{C}^+$ and $\dot{x}$, $\dot{y}$ are $\mathbb{P}$-names such that $p \Vdash_{\mathbb{P}} \dot{x}$, $\dot{y} \in \dot{\mathfrak{B}}$, then for every q stronger than p there are r stronger than q and a, $b \in \mathfrak{B}$ such that $r \Vdash_{\mathbb{P}} \dot{x} = \dot{a}$ and $\dot{y} = \dot{b}$. In this case

$$r \Vdash_{\mathbb{P}} \dot{x} \mathbin{\dot\cap} \dot{y} = (a \cap b)^{\bullet} \in \dot{\mathfrak{B}},\ \dot{x} \mathbin{\dot\triangle} \dot{y} = (a \triangle b)^{\bullet} \in \dot{\mathfrak{B}};$$

as q is arbitrary,

$$p \Vdash_{\mathbb{P}} \dot{x} \mathbin{\dot\cap} \dot{y},\ \dot{x} \mathbin{\dot\triangle} \dot{y} \in \dot{\mathfrak{B}}.$$

As p, $\dot{x}$ and $\dot{y}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \dot{\mathfrak{B}} \text{ is a subring of } \dot{\mathfrak{A}};$$

as we also have $\Vdash_{\mathbb{P}} \dot{1} \in \dot{\mathfrak{B}}$, we get

$$\Vdash_{\mathbb{P}} \dot{\mathfrak{B}} \text{ is a subalgebra of } \dot{\mathfrak{A}}.$$

556C Theorem Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$, and $\mathfrak{C}$ a subalgebra of $\mathfrak{A}$. Let $\mathbb{P}$ be the forcing notion $\mathfrak{C}^+$, active downwards, $\dot{\mathfrak{A}}$ the forcing name for $\mathfrak{A}$ over $\mathfrak{C}$, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a ring homomorphism such that $\pi c \subseteq c$ for every $c \in \mathfrak{C}$; write $\dot{\pi}$ for the forcing name for π over $\mathfrak{C}$.

(a)(i)

$$\Vdash_{\mathbb{P}} \dot{\pi} \text{ is a ring homomorphism from } \dot{\mathfrak{A}} \text{ to itself}$$

and

$$\Vdash_{\mathbb{P}} \dot{\pi}(\dot{a}) = (\pi a)^{\bullet}$$

for every $a \in \mathfrak{A}$.

(ii) If π is injective, $\Vdash_{\mathbb{P}} \dot{\pi}$ is injective.

(iii) If $\phi : \mathfrak{A} \to \mathfrak{A}$ is another ring homomorphism such that $\phi c \subseteq c$ for every $c \in \mathfrak{C}$, with corresponding forcing name $\dot{\phi}$, then

$$\Vdash_{\mathbb{P}} \dot{\pi}\dot{\phi} = (\pi\phi)^{\bullet}.$$

(b) Now suppose that π is a Boolean homomorphism.

(i) $\pi c = c$ for every $c \in \mathfrak{C}$.

(ii) $\Vdash_{\mathbb{P}} \dot{\pi}$ is a Boolean homomorphism.

(iii) If π is surjective, $\Vdash_{\mathbb{P}} \dot{\pi}$ is surjective.

(iv) If $\pi \in \operatorname{Aut}\mathfrak{A}$ then

$$\Vdash_{\mathbb{P}} \dot{\pi} \text{ is a Boolean automorphism and } (\dot{\pi})^{-1} = (\pi^{-1})^{\bullet}.$$

(v) If the fixed-point subalgebra of π is $\mathfrak{C}$ exactly, then

$$\Vdash_{\mathbb{P}} \text{ the fixed-point subalgebra of } \dot{\pi} \text{ is } \{0, 1\}.$$

proof (a)(i)($\boldsymbol{\alpha}$) It will help to note straight away that $\pi c = c \cap \pi 1$ for every $c \in \mathfrak{C}$. **P** The hypothesis is that $\pi c \subseteq c$; because π is a ring homomorphism, $\pi c \subseteq \pi 1$, so $\pi c \subseteq c \cap \pi 1$. Since also

$$\pi c = \pi 1 \setminus \pi(1 \setminus c) \supseteq \pi 1 \setminus (1 \setminus c) = c \cap \pi 1,$$

we have equality. **Q** Consequently

$$c \cap \pi a = c \cap \pi(1 \cap a) = c \cap \pi 1 \cap \pi a = \pi c \cap \pi a = \pi(c \cap a)$$

whenever $c \in \mathfrak{C}$ and $a \in \mathfrak{A}$.

($\boldsymbol{\beta}$) $\Vdash_{\mathbb{P}} \dot{\pi}$ is a function from $\dot{\mathfrak{A}}$ to itself. **P** Of course $\Vdash_{\mathbb{P}} \dot{\pi} \subseteq \dot{\mathfrak{A}} \times \dot{\mathfrak{A}}$. Suppose that $p \in \mathfrak{C}^+$ and that $((\dot{a}, (\pi a)^{\bullet}), 1)$, $((\dot{b}, (\pi b)^{\bullet}), 1)$ are two members of $\dot{\pi}$ such that $p \Vdash_{\mathbb{P}} \dot{a} = \dot{b}$. Then for every q stronger than p there is an r stronger than q such that $r \cap a = r \cap b$ (556Ba), in which case

$$r \cap \pi a = \pi(r \cap a) = \pi(r \cap b) = r \cap \pi b.$$

This shows that $p \Vdash_{\mathbb{P}} (\pi a)^{\bullet} = (\pi b)^{\bullet}$, by 556Ba in the other direction. As a and b are arbitrary, the condition of 5A3Ea is satisfied, with $\dot{A}$ there exactly equal to $\dot{\mathfrak{A}}$ here, and

$$\Vdash_{\mathbb{P}} \dot{\pi} \text{ is a function with domain } \dot{\mathfrak{A}}. \textbf{ Q}$$

If $a \in \mathfrak{A}$ then $((\dot{a}, (\pi a)^{\bullet}), 1) \in \dot{\pi}$ so

$$\Vdash_{\mathbb{P}} (\dot{a}, (\pi a)^{\bullet}) \in \dot{\pi} \text{ and } \dot{\pi}(\dot{a}) = (\pi a)^{\bullet}.$$

($\boldsymbol{\gamma}$) $\Vdash_{\mathbb{P}} \dot{\pi}$ is a ring homomorphism. **P** Writing $\bigcirc$ for either $\cap$ or $\triangle$,

$$\begin{aligned}\Vdash_{\mathbb{P}} \dot{\pi}(\dot{a} \mathbin{\dot{\bigcirc}} \dot{b}) &= \dot{\pi}(a \bigcirc b)^{\bullet} = (\pi(a \bigcirc b))^{\bullet} \\ &= (\pi a \bigcirc \pi b)^{\bullet} = (\pi a)^{\bullet} \mathbin{\dot{\bigcirc}} (\pi b)^{\bullet} = (\dot{\pi}\dot{a}) \mathbin{\dot{\bigcirc}} (\dot{\pi}\dot{b})\end{aligned}$$

for all a, $b \in \mathfrak{A}$. If now $p \in \mathfrak{C}^+$ and $\dot{x}$, $\dot{y}$ are $\mathbb{P}$-names such that

$$p \Vdash_{\mathbb{P}} \dot{x},\ \dot{y} \in \dot{\mathfrak{A}},$$

then for any q stronger than p there are r stronger than q and a, $b \in \mathfrak{A}$ such that

$$r \Vdash_{\mathbb{P}} \dot{x} = \dot{a} \text{ and } \dot{y} = \dot{b},$$

in which case

$$r \Vdash_{\mathbb{P}} \dot{\pi}(\dot{x} \mathbin{\dot{\bigcirc}} \dot{b}) = \dot{\pi}(\dot{a} \mathbin{\dot{\bigcirc}} \dot{b}) = \dot{\pi}(\dot{a}) \mathbin{\dot{\bigcirc}} \dot{\pi}(\dot{b}) = \dot{\pi}(\dot{x}) \mathbin{\dot{\bigcirc}} \dot{\pi}(\dot{y}).$$

As q is arbitrary,

$$p \Vdash_{\mathbb{P}} \dot{\pi}(\dot{x} \mathbin{\dot{\bigcirc}} \dot{y}) = \dot{\pi}(\dot{x}) \mathbin{\dot{\bigcirc}} \dot{\pi}(\dot{y});$$

as p, $\dot{x}$ and $\dot{y}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \dot{\pi}(x \mathbin{\dot{\bigcirc}} y) = (\dot{\pi}x) \mathbin{\dot{\bigcirc}} (\dot{\pi}y) \text{ for all } x,\, y \in \dot{\mathfrak{A}}.$$

As this is true for both $\dot{\bigcirc} = \dot{\cap}$ and $\dot{\bigcirc} = \dot{\triangle}$,

$$\Vdash_{\mathbb{P}} \dot{\pi} \text{ is a ring homomorphism. } \mathbf{Q}$$

(ii) Let $p \in \mathfrak{C}^+$ and a $\mathbb{P}$-name $\dot{x}$ be such that

$$p \Vdash_{\mathbb{P}} \dot{x} \in \dot{\mathfrak{A}} \text{ and } \dot{\pi}\dot{x} = 0.$$

For any q stronger than p, there are an r stronger than q and an $a \in \mathfrak{A}$ such that

$$r \Vdash_{\mathbb{P}} \dot{a} = \dot{x}, \text{ therefore } 0 = \dot{\pi}\dot{a} = (\pi a)^{\bullet}.$$

By 556Ba, there is an s stronger than r such that $s \cap \pi a = 0$; since $\pi s \subseteq s$, $\pi(s \cap a) = 0$. As π is injective, $s \cap a = 0$ and (using 556Ba again)

$$s \Vdash_{\mathbb{P}} \dot{x} = \dot{a} = 0.$$

As q is arbitrary, $p \Vdash_{\mathbb{P}} \dot{x} = 0$; as p and $\dot{x}$ are arbitrary, $\Vdash_{\mathbb{P}} \dot{\pi}$ is injective.

(iii) Suppose that $p \in \mathfrak{C}^+$ and that $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \dot{\mathfrak{A}}$. For any q stronger than p, there are an r stronger than q and an $a \in \mathfrak{A}$ such that $r \Vdash_{\mathbb{P}} \dot{x} = \dot{a}$, so that

$$r \Vdash_{\mathbb{P}} \dot{\pi}(\dot{\phi}(\dot{x})) = \dot{\pi}(\dot{\phi}(\dot{a})) = \dot{\pi}((\phi a)^{\bullet}) = (\pi\phi a)^{\bullet} = (\pi\phi)^{\bullet}(\dot{a}) = (\pi\phi)^{\bullet}(\dot{x}).$$

As q is arbitrary,

$$p \Vdash_{\mathbb{P}} \dot{\pi}(\dot{\phi}(\dot{x})) = (\pi\phi)^{\bullet}(\dot{x});$$

as p and $\dot{x}$ are arbitrary, $\Vdash_{\mathbb{P}} \dot{\pi}\dot{\phi} = (\pi\phi)^{\bullet}$.

(b)(i) I observed in (a-i-α) above that $\pi c = c \cap \pi 1$ for every $c \in \mathfrak{C}$, so if $\pi 1 = 1$ we shall have $\pi c = c$ for every $c \in \mathfrak{C}$.

(ii) I pointed out in 556Bd that

$$\Vdash_{\mathbb{P}} \dot{1} \text{ is the identity of } \dot{\mathfrak{A}},$$

and we now have

$$\Vdash_{\mathbb{P}} \dot{\pi}(\dot{1}) = (\pi 1)^{\bullet} = \dot{1}, \text{ so } \dot{\pi} \text{ is a Boolean homomorphism.}$$

(iii) Let $p \in \mathfrak{C}^+$ and a $\mathbb{P}$-name $\dot{x}$ be such that $p \Vdash_{\mathbb{P}} \dot{x} \in \dot{\mathfrak{A}}$. For any q stronger than p, there are an r stronger than q and an $a \in \mathfrak{A}$ such that $r \Vdash_{\mathbb{P}} \dot{a} = \dot{x}$. Now there is a $b \in \mathfrak{A}$ such that $a = \pi b$, in which case

$$r \Vdash_{\mathbb{P}} \dot{x} = \dot{a} = \dot{\pi}\dot{b} \in \dot{\pi}[\dot{\mathfrak{A}}].$$

As q is arbitrary, $p \Vdash_{\mathbb{P}} \dot{x} \in \dot{\pi}[\dot{\mathfrak{A}}]$; as p and $\dot{x}$ are arbitrary, $\Vdash_{\mathbb{P}} \dot{\pi}$ is surjective.

(iv) By (a-iii),

$$\Vdash_{\mathbb{P}} \dot{\pi}(\pi^{-1})^{\bullet} = (\pi^{-1})^{\bullet}\dot{\pi} = \dot{\iota}$$

where $\iota : \mathfrak{A} \to \mathfrak{A}$ is the identity automorphism. But

$$\Vdash_{\mathbb{P}} \dot{\iota} \text{ is the identity on } \dot{\mathfrak{A}}.$$

P If $p \in \mathfrak{C}^+$ and a $\mathbb{P}$-name $\dot{x}$ are such that $p \Vdash_{\mathbb{P}} \dot{x} \in \dot{\mathfrak{A}}$, then for any q stronger than p there are an r stronger than q and an $a \in \mathfrak{A}$ such that

$$r \Vdash_{\mathbb{P}} \dot{x} = \dot{a} = (\iota a)^{\bullet} = \dot{\iota}\dot{a} = \dot{\iota}\dot{x}.$$

As q is arbitrary, $p \Vdash_{\mathbb{P}} \dot{\iota}\dot{x} = \dot{x}$; as p and $\dot{x}$ are arbitrary, $\Vdash_{\mathbb{P}} \dot{\iota}$ is the identity. **Q** Putting these together,

$$\Vdash_{\mathbb{P}} (\pi^{-1})^{\bullet} \text{ is the inverse of } \dot{\pi}.$$

(v) Suppose that $p \in \mathfrak{C}^+$ and a $\mathbb{P}$-name $\dot{x}$ are such that

$$p \Vdash_{\mathbb{P}} \dot{x} \in \dot{\mathfrak{A}} \text{ and } \dot{\pi}\dot{x} = \dot{x}.$$

For any q stronger than p there are an r stronger than q and an $a \in \mathfrak{A}$ such that

$$r \Vdash_{\mathbb{P}} \dot{a} = \dot{x} = \dot{\pi}\dot{x} = \dot{\pi}\dot{a} = (\pi a)^{\bullet},$$

and an s stronger than r such that $s \cap a = s \cap \pi a$. Now $s \cap a = \pi(s \cap a)$ and $s \cap a \in \mathfrak{C}$. If $s \cap a = 0$, set $s' = s$; then $s' \Vdash_{\mathbb{P}} \dot{x} = 0$. Otherwise, set $s' = s \cap a$; then $s' \Vdash_{\mathbb{P}} \dot{x} = 1$. Thus in either case we have an s' stronger than q such that $s' \Vdash_{\mathbb{P}} \dot{x} \in \{0, 1\}$. As q is arbitrary, $p \Vdash_{\mathbb{P}} \dot{x} \in \{0, 1\}$; as p and $\dot{x}$ are arbitrary, $\Vdash_{\mathbb{P}} \dot{\pi}$ has fixed-point subalgebra $\{0, 1\}$.

556D Regularly embedded subalgebras I am trying to set these results out in maximal generality, as usual. However it seems that we need to move almost at once to the case in which our subalgebra is regularly embedded, and we have more effective versions of 556Ba and 556B(e-ii).

Proposition Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$, and $\mathfrak{C}$ a regularly embedded subalgebra of $\mathfrak{A}$. Let $\mathbb{P}$ be the forcing notion $\mathfrak{C}^+$, active downwards, and for $a \in \mathfrak{A}$ let $\dot{a}$ be the forcing name for a over $\mathfrak{C}$.

(a) For $p \in \mathfrak{C}^+$ and a, $b \in \mathfrak{A}$,

$$p \Vdash_{\mathbb{P}} \dot{a} = \dot{b}$$

iff $p \cap a = p \cap b$.

(b) Let $\dot{\subseteq}$ be the forcing name for $\subseteq$ over $\mathfrak{C}$. Then for $p \in \mathfrak{C}^+$ and a, $b \in \mathfrak{A}$,

$$p \Vdash_{\mathbb{P}} \dot{a} \mathrel{\dot{\subseteq}} \dot{b}$$

iff $p \cap a \subseteq b$.

proof The point is just that $\operatorname{upr}(a, \mathfrak{C}) = 0$ only when $a = 0$, because infima in $\mathfrak{C}$ are also infima in $\mathfrak{A}$ (313N); so that $\operatorname{upr}(p \cap a \setminus b, \mathfrak{C}) = 0$ iff $p \cap a \subseteq b$, and $\operatorname{upr}(p \cap (a \triangle b), \mathfrak{C}) = 0$ iff $p \cap a = p \cap b$.

556E Proposition Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$, $\mathfrak{C}$ a regularly embedded subalgebra of $\mathfrak{A}$, $\mathbb{P}$ the forcing notion $\mathfrak{C}^+$, active downwards, and $\dot{\mathfrak{A}}$ the forcing name for $\mathfrak{A}$ over $\mathfrak{C}$; for $a \in \mathfrak{A}$, write $\dot{a}$ for the forcing name for a over $\mathfrak{C}$.

(a) Let $\dot{A}$ be a $\mathbb{P}$-name, and set

$$B = \{q \cap a : q \in \mathfrak{C}^+, \, a \in \mathfrak{A}, \, q \Vdash_{\mathbb{P}} \dot{a} \in \dot{A}\}.$$

Then for $d \in \mathfrak{A}$ and $p \in \mathfrak{C}^+$,

$$p \Vdash_{\mathbb{P}} \dot{d} \text{ is an upper bound for } \dot{A} \cap \dot{\mathfrak{A}}$$

iff $p \cap b \subseteq d$ for every $b \in B$, and

$$p \Vdash_{\mathbb{P}} \dot{d} = \sup(\dot{A} \cap \dot{\mathfrak{A}})$$

iff $p \cap d = \sup_{b \in B} p \cap b$.

(b)(i) If $\langle a_i \rangle_{i \in I}$ is a family in $\mathfrak{A}$ with supremum a, then

$$\Vdash_{\mathbb{P}} \dot{a} = \sup_{i \in \check{I}} \dot{a}_i.$$ [8]

(ii) If $\langle a_i \rangle_{i \in I}$ is a family in $\mathfrak{A}$ with infimum a, then

$$\Vdash_{\mathbb{P}} \dot{a} = \inf_{i \in \check{I}} \dot{a}_i.$$

[8] See 5A3F for a note on the interpretation of formulae of this kind.

(c) $\Vdash_{\mathbb{P}} \operatorname{sat}(\dot{\mathfrak{A}}) \le \operatorname{sat}(\mathfrak{A})\check{}$.[9]

(d) $\Vdash_{\mathbb{P}} \tau(\dot{\mathfrak{A}}) \le \tau(\mathfrak{A})\check{}$.

proof (a)(i) ? Suppose, if possible, that $b \in B$, $p \cap b \not\subseteq d$ and

$$p \Vdash_{\mathbb{P}} \dot{d} \text{ is an upper bound for } \dot{A} \cap \dot{\mathfrak{A}}.$$

Let $q \in \mathfrak{C}^+$, $a \in \mathfrak{A}$ be such that $b = q \cap a$ and $q \Vdash_{\mathbb{P}} \dot{a} \in \dot{A}$. Then $p \cap q \ne 0$, so $p \cap q \in \mathfrak{C}^+$ and

$$p \cap q \Vdash_{\mathbb{P}} \dot{a} \in \dot{A} \cap \dot{\mathfrak{A}}, \text{ therefore } \dot{a} \mathrel{\dot{\subseteq}} \dot{d}.$$

It follows that $p \cap q \cap a \subseteq d$ (556Db); but this contradicts the choice of p and b. **X**

Thus $p \cap b \subseteq d$ whenever $b \in B$ and $p \Vdash_{\mathbb{P}} \dot{d}$ is an upper bound for $\dot{A} \cap \dot{\mathfrak{A}}$.

(ii) Next, suppose that $p \in \mathfrak{C}^+$ and $d \in \mathfrak{A}$ are such that $p \cap b \subseteq d$ for every $b \in B$. Suppose that q is stronger than p and that $\dot{x}$ is a $\mathbb{P}$-name such that $q \Vdash_{\mathbb{P}} \dot{x} \in \dot{A} \cap \dot{\mathfrak{A}}$. If r is stronger than q, there are an s stronger than r and an $a \in \mathfrak{A}$ such that $s \Vdash_{\mathbb{P}} \dot{x} = \dot{a}$. In this case, $s \cap a \in B$ and $s \cap a = p \cap s \cap a \subseteq d$, so

$$s \Vdash_{\mathbb{P}} \dot{x} = \dot{a} = (s \cap a)^{\bullet} \mathrel{\dot{\subseteq}} \dot{d}.$$

As r is arbitrary, $q \Vdash_{\mathbb{P}} \dot{x} \mathrel{\dot{\subseteq}} \dot{d}$; as q and $\dot{x}$ are arbitrary,

$$p \Vdash_{\mathbb{P}} \dot{d} \text{ is an upper bound for } \dot{A}.$$

(iii) Putting these together, we see that $p \cap b \subseteq d$ for every $b \in B$ iff $p \Vdash_{\mathbb{P}} \dot{d}$ is an upper bound for $\dot{A} \cap \dot{\mathfrak{A}}$.

(iv) Now suppose that $p \Vdash_{\mathbb{P}} \dot{d} = \sup(\dot{A} \cap \dot{\mathfrak{A}})$. We know that d, and therefore $p \cap d$, is an upper bound of $\{p \cap b : b \in B\}$. If e is any other upper bound of $\{p \cap b : b \in B\}$, then

$$p \Vdash_{\mathbb{P}} \dot{e} \text{ is an upper bound of } \dot{A}, \text{ so } \dot{d} \mathrel{\dot{\subseteq}} \dot{e}$$

and $p \cap d \subseteq e$, by 556Db again; thus $p \cap d = \sup_{b \in B} p \cap b$.

(v) Finally, suppose that $p \cap d = \sup_{b \in B} p \cap b$. Suppose that q is stronger than p and that $\dot{x}$ is a $\mathbb{P}$-name such that

$$q \Vdash_{\mathbb{P}} \dot{x} \in \dot{\mathfrak{A}} \text{ is an upper bound of } \dot{A} \cap \dot{\mathfrak{A}}.$$

If r is stronger than q, there are a s stronger than r and a $c \in \mathfrak{A}$ such that $s \Vdash_{\mathbb{P}} \dot{x} = \dot{c}$. In this case, by (i), we must have $s \cap b \subseteq c$ for every $b \in B$; accordingly $s \cap d \subseteq c$ (313Ba), so that

$$s \Vdash_{\mathbb{P}} \dot{d} \mathrel{\dot{\subseteq}} \dot{c} = \dot{x}.$$

As r is arbitrary,

$$q \Vdash_{\mathbb{P}} \dot{d} \mathrel{\dot{\subseteq}} \dot{x};$$

as q and $\dot{x}$ are arbitrary,

$$p \Vdash_{\mathbb{P}} \dot{d} = \sup(\dot{A} \cap \dot{\mathfrak{A}}).$$

(b)(i) Of course

$$\Vdash_{\mathbb{P}} \dot{a}_i \mathrel{\dot{\subseteq}} \dot{a}$$

for every $i \in I$, so that

$$\Vdash_{\mathbb{P}} \dot{a} \text{ is an upper bound for } \{\dot{a}_i : i \in \check{I}\}.$$

(Formally speaking: if $p \in \mathfrak{C}^+$ and $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \{\dot{a}_i : i \in \check{I}\}$, then for any q stronger than p there are an r stronger than q and an $i \in I$ such that $r \Vdash_{\mathbb{P}} \dot{x} = \dot{a}_i \mathrel{\dot{\subseteq}} \dot{a}$; hence $p \Vdash_{\mathbb{P}} \dot{x} \mathrel{\dot{\subseteq}} \dot{a}$.) In the other direction, suppose that $p \in \mathfrak{C}^+$ and that $\dot{x}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{a}_i \subseteq \dot{x} \in \dot{\mathfrak{A}} \text{ for every } i \in \check{I}.$$

[9] Of course I am not asserting here that '$\Vdash_{\mathbb{P}} \operatorname{sat}(\mathfrak{A})\check{}$ is a cardinal', only that '$\Vdash_{\mathbb{P}} \operatorname{sat}(\dot{\mathfrak{A}})$ is a cardinal and $\operatorname{sat}(\mathfrak{A})\check{}$ is an ordinal'.

For any q stronger than p there are an r stronger than q and a $b\in\mathfrak{A}$ such that $r\Vdash_{\mathbb{P}}\dot{x}=\dot{b}$. Now, for any $i\in I$,

$$r\Vdash_{\mathbb{P}}\check{\imath}\in\check{I},\ \dot{a}_i\mathbin{\dot{\subseteq}}\dot{x}=\dot{b}$$

and therefore $r\cap a_i\subseteq b$. As i is arbitrary, $r\cap a\subseteq b$ and $r\Vdash_{\mathbb{P}}\dot{a}\mathbin{\dot{\subseteq}}\dot{x}$. As q is arbitrary, $p\Vdash_{\mathbb{P}}\dot{a}\mathbin{\dot{\subseteq}}\dot{x}$; as p and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}}\dot{a}\text{ is the least upper bound of }\{\dot{a}_i:i\in\check{I}\}.$$

(ii) Now

$$\begin{aligned}\Vdash_{\mathbb{P}}\inf_{i\in\check{I}}\dot{a}_i&=1\mathbin{\dot{\setminus}}\sup_{i\in\check{I}}(1\mathbin{\dot{\setminus}}\dot{a}_i)=1\mathbin{\dot{\setminus}}\sup_{i\in\check{I}}(1\setminus a_i)^{\bullet}\\&=1\mathbin{\dot{\setminus}}(\sup_{i\in I}(1\setminus a_i))^{\bullet}=(1\setminus(\sup_{i\in I}(1\setminus a_i)))^{\bullet}=(\inf_{i\in I}a_i)^{\bullet}.\end{aligned}$$

(c) ? Otherwise, there are a $p\in\mathfrak{C}^+$ and a family $\langle\dot{x}_\xi\rangle_{\xi<\kappa}$, where $\kappa=\operatorname{sat}(\mathfrak{A})$, such that

$$p\Vdash_{\mathbb{P}}\dot{x}_\xi\in\dot{\mathfrak{A}}^+\text{ for every }\xi<\check{\kappa}\text{ and }\dot{x}_\xi\mathbin{\dot{\cap}}\dot{x}_\eta=0\text{ whenever }\xi<\eta<\check{\kappa}.$$

For each $\xi<\kappa$ choose q_ξ stronger than p and $a_\xi\in\mathfrak{A}$ such that $q_\xi\Vdash_{\mathbb{P}}\dot{x}_\xi=\dot{a}_\xi$. Then $q_\xi\Vdash_{\mathbb{P}}\dot{a}_\xi\neq0$, so $b_\xi=q_\xi\cap a_\xi$ is non-zero. As $\operatorname{sat}(\mathfrak{A})=\kappa$, there must be $\xi<\eta<\kappa$ such that $b_\xi\cap b_\eta\neq0$. Set $r=q_\xi\cap q_\eta$; then $r\in\mathfrak{C}^+$ is stronger than p and

$$r\Vdash_{\mathbb{P}}\dot{x}_\xi\mathbin{\dot{\cap}}\dot{x}_\eta=\dot{a}_\xi\mathbin{\dot{\cap}}\dot{a}_\eta=(a_\xi\cap a_\eta)^{\bullet}\neq0$$

by 556Da, because $r\cap a_\xi\cap a_\eta\neq0$. **X**

(d) Let $A\subseteq\mathfrak{A}$ be a set with cardinal $\kappa=\tau(\mathfrak{A})$ which τ-generates $\mathfrak{A}$. Let $\dot{A}$ be the $\mathbb{P}$-name $\{(\dot{a},1):a\in A\}$; then

$$\Vdash_{\mathbb{P}}\dot{A}\ \tau\text{-generates }\dot{\mathfrak{A}}\text{ and }\#(\dot{A})\le\check{\kappa}.$$

P (i) Suppose that $p\in\mathfrak{C}^+$ and $\dot{x}$, $\dot{\mathfrak{B}}$ are $\mathbb{P}$-names such that

$$p\Vdash_{\mathbb{P}}\dot{\mathfrak{B}}\text{ is an order-closed subalgebra of }\dot{\mathfrak{A}}\text{ including }\dot{A},\text{ and }\dot{x}\in\dot{\mathfrak{A}}.$$

Consider $\mathfrak{D}=\{a:a\in\mathfrak{A},\ p\Vdash_{\mathbb{P}}\dot{a}\in\dot{\mathfrak{B}}\}$. Then $\mathfrak{D}$ is a subalgebra of $\mathfrak{A}$, by 556Bb, and is order-closed by (b) here; also $A\subseteq\mathfrak{D}$, so $\mathfrak{D}=\mathfrak{A}$. Next, for any q stronger than p there are an r stronger than q and an $a\in\mathfrak{A}$ such that $r\Vdash_{\mathbb{P}}\dot{x}=\dot{a}$; since $a\in\mathfrak{D}$, $p\Vdash_{\mathbb{P}}\dot{a}\in\dot{\mathfrak{B}}$ and

$$r\Vdash_{\mathbb{P}}\dot{x}\in\dot{\mathfrak{B}}.$$

As p, $\dot{x}$ and $\dot{\mathfrak{B}}$ are arbitrary,

$$\Vdash_{\mathbb{P}}\dot{A}\ \tau\text{-generates }\dot{\mathfrak{A}}.$$

(ii) If $\langle a_\xi\rangle_{\xi<\kappa}$ enumerates A, then

$$\Vdash_{\mathbb{P}}\dot{A}=\{\dot{a}_\xi:\xi<\check{\kappa}\}\text{ and }\#(\dot{A})\le\#(\check{\kappa})\le\check{\kappa}.\ \mathbf{Q}$$

Accordingly

$$\Vdash_{\mathbb{P}}\tau(\dot{\mathfrak{A}})\le\check{\kappa}.$$

556F Quotient forcing In 556A-556B I have gone to pains to describe names $\dot{\mathfrak{A}}$, $\dot{\triangle}$, $\dot{\cap}$, $\dot{0}$, $\dot{1}$ constituting a Boolean algebra. Of course we also have much simpler names $\check{\mathfrak{A}}$, $\check{\triangle}$, $\check{\cap}$, $\check{0}$, $\check{1}$ also constituting a Boolean algebra in the forcing language, and these must obviously be related in some way to the construction here. I think the details are worth bringing into the open.

Proposition Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$, and $\mathfrak{C}$ a subalgebra of $\mathfrak{A}$. Let $\mathbb{P}$ be the forcing notion $\mathfrak{C}^+$, active downwards, and $\dot{\mathfrak{A}}$ the forcing name for $\mathfrak{A}$ over $\mathfrak{C}$.

(a) Consider the $\mathbb{P}$-names

$$\dot{\psi}=\{((\check{a},\dot{a}),1):a\in\mathfrak{A}\},\quad\dot{\mathcal{I}}=\{(\check{a},p):p\in\mathfrak{C}^+,\ a\in\mathfrak{A},\ p\cap a=0\}.$$

Then

$$\Vdash_{\mathbb{P}} \dot{\psi} \text{ is a Boolean homomorphism from } \check{\mathfrak{A}} \text{ onto } \dot{\mathfrak{A}}, \text{ and its kernel is } \dot{\mathcal{I}}.$$

(b) Now suppose that $\mathfrak{C}$ is regularly embedded in $\mathfrak{A}$. Set $\dot{\mathbb{Q}} = (\dot{\mathfrak{A}}^+, \dot{\subseteq}, \dot{1}, \dot{\downarrow})$ and let $\mathbb{P} * \dot{\mathbb{Q}}$ be the iterated forcing notion (5A3O). Then $\mathrm{RO}(\mathbb{P} * \dot{\mathbb{Q}})$ is isomorphic to the Dedekind completion of $\mathfrak{A}$.

(c) Suppose that $\mathfrak{C}$ is regularly embedded in $\mathfrak{A}$ and that $\mathfrak{B}$ is a Boolean algebra such that

$$\Vdash_{\mathbb{P}} \dot{\mathfrak{A}} \cong \check{\mathfrak{B}}.$$

Then the Dedekind completion $\widehat{\mathfrak{A}}$ of $\mathfrak{A}$ is isomorphic to the Dedekind completion $\mathfrak{C}\widehat{\otimes}\mathfrak{B}$ of the free product $\mathfrak{C} \otimes \mathfrak{B}$ of $\mathfrak{C}$ and $\mathfrak{B}$.

proof (a)(i) It is elementary that

$$\Vdash_{\mathbb{P}} \dot{\psi} : \check{\mathfrak{A}} \to \dot{\mathfrak{A}} \text{ is a surjective function}$$

just because $\dot{\mathfrak{A}} = \{(\dot{a}, 1) : a \in \mathfrak{A}\}$. By 556Bb,

$$\Vdash_{\mathbb{P}} \dot{\psi} \text{ is a ring homomorphism; being surjective, it is a Boolean homomorphism.}$$

(ii)($\boldsymbol{\alpha}$) If $p \in \mathfrak{C}^+$ and $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \dot{\mathcal{I}}$, then there are a $q \in \mathfrak{C}^+$, an $a \in \mathfrak{A}$, and an r stronger than both p and q such that

$$q \cap a = 0, \quad r \Vdash_{\mathbb{P}} \dot{x} = \check{a}.$$

In this case, $r \cap a = 0$ so, by 556Ba,

$$r \Vdash_{\mathbb{P}} 0 = \dot{a} = \dot{\psi}(\check{a}) = \dot{\psi}(\dot{x}).$$

As p and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \dot{\mathcal{I}} \text{ is included in the kernel of } \dot{\psi}.$$

($\boldsymbol{\beta}$) If $p \in \mathfrak{C}^+$ and $\dot{x}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{x} \in \check{\mathfrak{A}} \text{ and } \dot{\psi}(\dot{x}) = 0,$$

then there are an $a \in \mathfrak{A}$ and a q stronger than p such that

$$q \Vdash_{\mathbb{P}} \dot{x} = \check{a} \text{ and } \dot{a} = \dot{\psi}(\check{a}) = \dot{\psi}(\dot{x}) = 0.$$

Now there is an r stronger than q such that $r \cap a = 0$, so that $(\check{a}, r) \in \dot{\mathcal{I}}$ and

$$r \Vdash_{\mathbb{P}} \dot{x} = \check{a} \in \dot{\mathcal{I}}.$$

As p and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \text{the kernel of } \dot{\psi} \text{ is included in } \dot{\mathcal{I}}, \text{ so they coincide.}$$

(b)(i) In order to use the description of iterated forcing in 5A3O, we need to set out an exact $\mathbb{P}$-name for $\dot{\mathfrak{A}}^+$. If we say that $\dot{\mathfrak{A}}^+$ abbreviates $\{x : x \in \dot{\mathfrak{A}},\ x \neq 0\}$, and use the formulation of Comprehension in Kunen 80, Theorem 4.2, we get

$$\dot{\mathfrak{A}}^+ = \{(\dot{x}, p) : \dot{x} \in \operatorname{dom} \dot{\mathfrak{A}},\ p \in \mathfrak{C}^+,\ p \Vdash_{\mathbb{P}} \dot{x} \in \dot{\mathfrak{A}} \text{ and } \dot{x} \neq 0\}.$$

Now 556Ab specifies $\operatorname{dom} \dot{\mathfrak{A}}$ to be $\{\dot{a} : a \in \mathfrak{A}\}$, so we get

$$\begin{aligned} \dot{\mathfrak{A}}^+ &= \{(\dot{a}, p) : a \in \mathfrak{A},\ p \in \mathfrak{C}^+,\ p \Vdash_{\mathbb{P}} \dot{a} \in \dot{\mathfrak{A}} \text{ and } \dot{a} \neq 0\} \\ &= \{(\dot{a}, p) : a \in \mathfrak{A},\ p \in \mathfrak{C}^+,\ p \Vdash_{\mathbb{P}} \dot{a} \neq 0\}, \end{aligned}$$

$$\operatorname{dom} \dot{\mathfrak{A}}^+ = \{\dot{a} : a \in \mathfrak{A},\ \not\Vdash_{\mathbb{P}} \dot{a} = 0\} = \{\dot{a} : a \in \mathfrak{A}^+\}$$

by 556Da.

(ii) 5A3O now tells us that the underlying set of $\mathbb{P} * \dot{\mathbb{Q}}$ is to be

$$P = \{(p, \dot{a}) : p \in \mathfrak{C}^+,\ a \in \mathfrak{A}^+,\ p \Vdash_{\mathbb{P}} \dot{a} \neq 0\}.$$

For $p \in \mathfrak{C}^+$ and $a \in \mathfrak{A}$,

$$\begin{aligned} p \Vdash_{\mathbb{P}} \dot{a} \neq 0 &\iff \text{for every } q \text{ stronger than } p,\ q \not\Vdash_{\mathbb{P}} \dot{a} = 0 \\ &\iff \text{for every non-zero } q \subseteq p,\ q \cap a \neq 0 \end{aligned}$$

(556Da). So P is just

$$\{(p, \dot{a}) : p \in \mathfrak{C}^+,\ a \in \mathfrak{A},\ q \cap a \neq 0 \text{ whenever } q \in \mathfrak{C} \text{ and } 0 \neq q \subseteq p\}.$$

Next, for $(p, \dot{a})$, $(q, \dot{b}) \in P$,

$$\begin{aligned} (p, \dot{a}) \text{ is stronger than } (q, \dot{b}) &\iff p \subseteq q \text{ and } p \Vdash_{\mathbb{P}} \dot{a} \mathrel{\dot{\subseteq}} \dot{b} \\ &\iff p \subseteq q \text{ and } p \cap a \subseteq b \end{aligned}$$

(556Db).

(iii) We can define a function $f : P \to \mathfrak{A}^+$ by setting

$$f(p, \dot{a}) = p \cap a$$

whenever $(p, \dot{a}) \in P$. **P** If you look at the definition of $\dot{a}$ in 556A, you will see that $((b, 1), 1) = (\check{b}, 1)$ belongs to $\dot{a}$ iff $b \subseteq a$, so that $\dot{a} = \dot{b}$ only when $a = b$; thus f is a function from P to $\mathfrak{A}$. And the definition of P ensures that $f(p, \dot{a}) \neq 0$ whenever $(p, \dot{a}) \in P$. **Q**

(iv)(α) If $(p, \dot{a})$ is stronger than $(q, \dot{b})$ in P, then $p \subseteq q$ and $p \cap a \subseteq b$, so $f(p, \dot{a}) \subseteq f(q, b)$.

(β) If $a \in \mathfrak{A}^+$, then (because $\mathfrak{C}$ is regularly embedded) $C = \{q : a \subseteq q \in \mathfrak{C}\}$ does not have infimum 0 in $\mathfrak{C}$; let $p \in \mathfrak{C}^+$ be a lower bound for C. Then $(p, \dot{a}) \in P$, and $f(p, \dot{a}) \subseteq a$. Thus $f[P]$ is order-dense in $\mathfrak{A}$.

(γ) If $(p, \dot{a})$, $(q, \dot{b})$ are incompatible in P, then $f(p, \dot{a}) \cap f(q, \dot{b}) = 0$. **P?** Otherwise, $c = p \cap a \cap q \cap b$ is non-zero. Let $r \in \mathfrak{C}^+$ be such that $(r, \dot{c}) \in P$. Since $r \setminus (p \cap q) \Vdash_{\mathbb{P}} \dot{c} = 0$, $r \subseteq p \cap q$; since $c \subseteq a \cap b$, $(r, \dot{c})$ is stronger than both $(p, \dot{a})$ and $(q, \dot{b})$, which is supposed to be impossible. **XQ**

(v) Thinking of $\mathfrak{A}^+$ as an order-dense subset of $\widehat{\mathfrak{A}}$, and of f as a function from P to $\widehat{\mathfrak{A}}^+$, 514Sa tells us that

$$\mathrm{RO}(\mathbb{P} * \dot{\mathbb{Q}}) = \mathrm{RO}(P) \cong \widehat{\mathfrak{A}},$$

as claimed.

(c) For free products of Boolean algebras, see §315; for Dedekind completions, see §314. This part can be regarded as a corollary of (b) (see 556Ya-556Yb), but can also be approached directly, as follows.

(i) Let $\dot{\theta}$ be a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \dot{\theta} : \dot{\mathfrak{A}} \to \check{\mathfrak{B}} \text{ is an isomorphism.}$$

Set

$$R = \{(p, b, a) : p \in \mathfrak{C}^+,\ b \in \mathfrak{B}^+,\ a \in \mathfrak{A}^+,\ a \subseteq p \text{ and } p \Vdash_{\mathbb{P}} \dot{\theta}(\dot{a}) = \check{b}\},$$

and give R the ordering induced by the product partial ordering of $\mathfrak{C}^+ \times \mathfrak{B}^+ \times \mathfrak{A}^+$.

(ii) $\mathrm{RO}^{\downarrow}(R) \cong \mathfrak{C} \widehat{\otimes} \mathfrak{B}$. **P** Define $f : R \to (\mathfrak{C} \widehat{\otimes} \mathfrak{B})^+$ by setting $f(p, b, a) = p \otimes b$.

(α) Of course $f(p, b, a) \subseteq f(p', b', a')$ whenever $(p, b, a) \leq (p', b', a')$.

(β) If (p_0, b_0, a_0) and (p_1, b_1, a_1) belong to R and $f(p_0, b_0, a_0) \cap f(p_1, b_1, a_1) \neq 0$, set $p = p_0 \cap p_1$, $b = b_0 \cap b_1$ and $a = a_0 \cap a_1$. Then $p \in \mathfrak{C}^+$, $b \in \mathfrak{B}^+$, $a \subseteq p$ and

$$p \Vdash_{\mathbb{P}} \dot{\theta}(\dot{a}) = \dot{\theta}(\dot{a}_0 \mathbin{\dot{\cap}} \dot{a}_1) = \dot{\theta}(\dot{a}_0) \mathbin{\check{\cap}} \dot{\theta}(\dot{a}_1) = \check{b}_0 \mathbin{\check{\cap}} \check{b}_1 = \check{b}.$$

As $p \Vdash_{\mathbb{P}} \dot{\theta}(\dot{a}) \neq \check{0}$, a cannot be 0, and $(p, b, a) \in R$; so (p_0, b_0, a_0) and (p_1, b_1, a_1) are downwards compatible in R.

(γ) If $d \in (\mathfrak{C}\widehat{\otimes}\mathfrak{B})^+$, there are $p_0 \in \mathfrak{C}^+$, $b \in \mathfrak{B}^+$ such that $p_0 \otimes b \subseteq d$. Now there is a $\mathbb{P}$-name $\dot{x}$ such that

$$\Vdash_{\mathbb{P}} \dot{x} \in \dot{\mathfrak{A}} \text{ and } \dot{\theta}(\dot{x}) = \check{b}.$$

Let p stronger than p_0 and $a_0 \in \mathfrak{A}$ be such that $p \Vdash_{\mathbb{P}} \dot{a}_0 = \dot{x}$, and set $a = p \cap a_0$; then

$$p \Vdash_{\mathbb{P}} \dot{a} = \dot{a}_0 \text{ so } \dot{\theta}(\dot{a}) = \dot{\theta}(\dot{a}_0) = \check{b}.$$

As in (β), it follows that $a \neq 0$, so that $(p, b, a) \in R$; now $f(p, b, a) \subseteq d$. As d is arbitrary, $f[R]$ is order-dense in $\mathfrak{C}\widehat{\otimes}\mathfrak{B}$.

(δ) Thus f satisfies the conditions of 514Sa and $\mathrm{RO}^{\downarrow}(R) \cong \mathfrak{C}\widehat{\otimes}\mathfrak{B}$. **Q**

(iii) $\mathrm{RO}^{\downarrow}(R) \cong \widehat{\mathfrak{A}}$. **P** Define $g : R \to \widehat{\mathfrak{A}}^+$ by setting $g(p, b, a) = a$ for $(p, b, a) \in R$.

(α) Of course $g(p, b, a) \subseteq g(p', b', a')$ whenever $(p, b, a) \leq (p', b', a')$ in R.

(β) Suppose that (p_0, b_0, a_0), $(p_1, b_1, a_1) \in R$ and that $a = a_0 \cap a_1 \neq 0$. Set $p = p_0 \cap p_1$ and $b = b_0 \cap b_1$. Then $p \supseteq a \neq 0$ and

$$p \Vdash_{\mathbb{P}} \dot{\theta}(\dot{a}) = \check{b}$$

as in (ii-β). Since $p \cap a \neq 0$, $p \not\Vdash_{\mathbb{P}} \dot{a} = \dot{0}$ (556Da), so $p \not\Vdash_{\mathbb{P}} \check{b} = \check{0}$ and $b \neq 0$. Thus $(p, b, a) \in R$ and (p_0, b_0, a_0), (p_1, b_1, a_1) are compatible downwards in R.

(γ) If $d \in \widehat{\mathfrak{A}}^+$, there is an $a_0 \in \mathfrak{A}^+$ such that $a_0 \subseteq d$. In this case, $\not\Vdash_{\mathbb{P}} \dot{a}_0 = \dot{0}$ so there is a $p_0 \in \mathfrak{C}^+$ such that $p_0 \Vdash_{\mathbb{P}} \dot{a}_0 \neq \dot{0}$. Now $p_0 \Vdash_{\mathbb{P}} \dot{\theta}(\dot{a}_0) \in \check{\mathfrak{B}}$ so there are a p stronger than p_0 and a $b \in \mathfrak{B}$ such that $p \Vdash_{\mathbb{P}} \dot{\theta}(\dot{a}_0) = \check{b}$. Set $a = p \cap a_0$; then

$$p \Vdash_{\mathbb{P}} \dot{a} = \dot{a}_0 \neq \dot{0} \text{ and } \check{b} = \dot{\theta}(\dot{a}) \neq \check{0}.$$

Consequently a and b are both non-zero and $(p, b, a) \in R$, while $g(p, b, a) \subseteq d$.

(δ) Thus g satisfies the conditions of 514Sa and $\mathrm{RO}^{\downarrow}(R) \cong \widehat{\mathfrak{A}}$. **Q**

(iv) Putting these together, $\mathfrak{C}\widehat{\otimes}\mathfrak{B}$ and $\widehat{\mathfrak{A}}$ are isomorphic.

556G Proposition Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, not $\{0\}$, $\mathfrak{C}$ a regularly embedded subalgebra of $\mathfrak{A}$, $\mathbb{P}$ the forcing notion $\mathfrak{C}^+$, active downwards, and $\dot{\mathfrak{A}}$ the forcing name for $\mathfrak{A}$ over $\mathfrak{C}$.

(a) Whenever $p \in \mathfrak{C}^+$ and $\dot{x}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{x} \in \dot{\mathfrak{A}},$$

there is an $a \in \mathfrak{A}$ such that

$$p \Vdash_{\mathbb{P}} \dot{x} = \dot{a},$$

where $\dot{a}$ is the forcing name for a over $\mathfrak{C}$.

(b) $\Vdash_{\mathbb{P}} \dot{\mathfrak{A}}$ is Dedekind complete.

proof (a) Set

$$B = \{q \cap b : q \in \mathfrak{C}^+ \text{ is stronger than } p,\ b \in \mathfrak{A},\ q \Vdash_{\mathbb{P}} \dot{b} = \dot{x}\}, \quad a = \sup B.$$

Then 556Ea tells us that

$$p \Vdash_{\mathbb{P}} \dot{a} = \sup\{\dot{x}\} = \dot{x}.$$

(b) Suppose that $p \in \mathfrak{C}^+$ and that $\dot{A}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{A} \subseteq \dot{\mathfrak{A}}$. Set

$$B = \{q \cap a : a \in \mathfrak{A},\ q \in \mathfrak{C}^+ \text{ and } q \Vdash_{\mathbb{P}} \dot{a} \in \dot{A}\}, \quad d = \sup B.$$

Then $p \cap d = \sup_{b \in B} p \cap b$, so $p \Vdash_{\mathbb{P}} \dot{d} = \sup \dot{A}$, by 556Ea. As p and $\dot{A}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \dot{\mathfrak{A}} \text{ is Dedekind complete.}$$

556H $L^0(\mathfrak{A})$: Proposition Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, not $\{0\}$, $\mathfrak{C}$ a regularly embedded subalgebra of $\mathfrak{A}$, $\mathbb{P}$ the forcing notion $\mathfrak{C}^+$, active downwards, and $\dot{\mathfrak{A}}$ the forcing name for $\mathfrak{A}$ over $\mathfrak{C}$. For $a \in \mathfrak{A}$ let $\dot{a}$ be the forcing name for a over $\mathfrak{C}$.

(a)(i) For every $u \in L^0(\mathfrak{A})$,

$$\Vdash_{\mathbb{P}} \dot{u} \in L^0(\dot{\mathfrak{A}})$$

where $\dot{u}$ is the forcing name for u over $\mathfrak{C}$.

(ii) If u, $v \in L^0(\mathfrak{A})$ and $\Vdash_{\mathbb{P}} \dot{u} = \dot{v}$, then $u = v$.

(b) For u, $v \in L^0(\mathfrak{A})$ and $\alpha \in \mathbb{R}$,

$$\begin{aligned}\Vdash_{\mathbb{P}} \dot{u} + \dot{v} &= (u+v)^\bullet,\\ -\dot{u} &= (-u)^\bullet,\\ \dot{u} \vee \dot{v} &= (u \vee v)^\bullet,\\ \dot{u} \times \dot{v} &= (u \times v)^\bullet,\\ \check{\alpha}\dot{u} &= (\alpha u)^\bullet.\end{aligned}$$

If $u \le v$, then $\Vdash_{\mathbb{P}} \dot{u} \le \dot{v}$.

(c) If $\langle u_i \rangle_{i \in I}$ is a family in $L^0(\mathfrak{A})$ with supremum $u \in L^0(\mathfrak{A})$, then

$$\Vdash_{\mathbb{P}} \dot{u} = \sup_{i \in \check{I}} \dot{u}_i \text{ in } L^0(\dot{\mathfrak{A}}).$$

(d) If $p \in \mathfrak{C}^+$ and $\dot{w}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{w} \in L^0(\dot{\mathfrak{A}})$, then there is a $u \in L^0(\mathfrak{A})$ such that

$$p \Vdash_{\mathbb{P}} \dot{w} = \dot{u}.$$

(e) If $\langle u_n \rangle_{n \in \mathbb{N}}$ is a sequence in $L^0(\mathfrak{A})$, then the following are equiveridical:

(i) $\langle u_n \rangle_{n \in \mathbb{N}}$ is order*-convergent to 0 (definition: 367A),

(ii) $\Vdash_{\mathbb{P}} \langle \dot{u}_n \rangle_{n \in \mathbb{N}}$ is order*-convergent to 0.

proof (a)(i) Examining the definition in 556Af, we see that we have

$$\Vdash_{\mathbb{P}} \dot{u} \text{ is a function from } \mathbb{Q} \text{ to } \dot{\mathfrak{A}} \text{ and } \dot{u}(\check{\alpha}) = [\![u > \alpha]\!]^\bullet$$

for every $\alpha \in \mathbb{Q}$. Now 556Eb tells us that, for every $\alpha \in \mathbb{Q}$,

$$\begin{aligned}\Vdash_{\mathbb{P}} \dot{u}(\check{\alpha}) = [\![u > \alpha]\!]^\bullet &= (\sup_{\beta \in \mathbb{Q}, \beta > \alpha} [\![u > \beta]\!])^\bullet = \sup_{\beta \in \mathbb{Q}, \beta > \check{\alpha}} [\![u > \beta]\!]^\bullet = \sup_{\beta \in \mathbb{Q}, \beta > \check{\alpha}} \dot{u}(\check{\beta}),\\ 0 &= (\inf_{n \in \mathbb{N}} [\![u > n]\!])^\bullet = \inf_{n \in \mathbb{N}} [\![u > n]\!]^\bullet = \inf_{n \in \mathbb{N}} \dot{u}(\check{n}),\\ 1 &= (\sup_{n \in \mathbb{N}} [\![u > -n]\!])^\bullet = \sup_{n \in \mathbb{N}} [\![u > -n]\!]^\bullet = \sup_{n \in \mathbb{N}} \dot{u}(-\check{n}),\end{aligned}$$

so

$$\Vdash_{\mathbb{P}} \dot{u} \in L^0(\dot{\mathfrak{A}}),$$

and I can write $[\![\dot{u} > \check{\alpha}]\!]$ for the $\mathbb{P}$-name $\dot{u}(\check{\alpha})$, so that

$$\Vdash_{\mathbb{P}} [\![\dot{u} > \check{\alpha}]\!] = [\![u > \alpha]\!]^\bullet$$

for every $\alpha \in \mathbb{Q}$.

(ii) For any $\alpha \in \mathbb{Q}$,

$$\Vdash_{\mathbb{P}} [\![u > \alpha]\!]^\bullet = [\![\dot{u} > \check{\alpha}]\!] = [\![\dot{v} > \check{\alpha}]\!] = [\![u > \alpha]\!]^\bullet.$$

By 556Da, $[\![u > \alpha]\!] = [\![v > \alpha]\!]$. As α is arbitrary, $u = v$.

(b)(i) Suppose u, $v \in L^0(\mathfrak{A})$. By 364D, we have

$$[\![u + v > \alpha]\!] = \sup_{\beta \in \mathbb{Q}} [\![u > \beta]\!] \cap [\![v > \alpha - \beta]\!]$$

for every $\alpha \in \mathbb{Q}$. If α, $\beta \in \mathbb{Q}$,

$$\Vdash_{\mathbb{P}} [\![\dot{u} > \check{\beta}]\!] \dot{\cap} [\![\dot{v} > \check{\alpha} - \check{\beta}]\!] = [\![u > \beta]\!]^{\bullet} \dot{\cap} [\![v > \alpha - \beta]\!]^{\bullet} = ([\![u > \beta]\!] \cap [\![v > \alpha - \beta]\!])^{\bullet}.$$

Taking the supremum over β, as in 556E(b-i),

$$\Vdash_{\mathbb{P}} [\![\dot{u} + \dot{v} > \check{\alpha}]\!] = \sup_{\beta \in \mathbb{Q}} [\![\dot{u} > \check{\beta}]\!] \dot{\cap} [\![\dot{v} > \check{\alpha} - \check{\beta}]\!] = \sup_{\beta \in \check{\mathbb{Q}}} [\![\dot{u} > \beta]\!] \dot{\cap} [\![\dot{v} > \check{\alpha} - \beta]\!]$$
$$= (\sup_{\beta \in \mathbb{Q}} [\![u > \beta]\!] \cap [\![v > \alpha - \beta]\!])^{\bullet} = [\![u + v > \alpha]\!]^{\bullet} = [\![(u + v)^{\bullet} > \check{\alpha}]\!]$$

for every $\alpha \in \mathbb{Q}$, and

$$\Vdash_{\mathbb{P}} \dot{u} + \dot{v} = (u + v)^{\bullet}.$$

(ii) Concerning $u \vee v$, we have

$$\Vdash_{\mathbb{P}} [\![(u \vee v)^{\bullet} > \check{\alpha}]\!] = [\![u \vee v > \alpha]\!]^{\bullet} = [\![u > \alpha]\!]^{\bullet} \dot{\cup} [\![v > \alpha]\!]^{\bullet}$$
$$= [\![\dot{u} > \check{\alpha}]\!] \dot{\cup} [\![\dot{v} > \check{\alpha}]\!] = [\![\dot{u} \vee \dot{v} > \check{\alpha}]\!]$$

for every u, $v \in L^0(\mathfrak{A})$ and $\alpha \in \mathbb{Q}$, so

$$\Vdash_{\mathbb{P}} \dot{u} \vee \dot{v} = (u \vee v)^{\bullet};$$

it follows that if $u \geq 0$ then $\Vdash_{\mathbb{P}} \dot{u} = \dot{u} \vee 0 \geq 0$.

(iii) If u, $v \in L^0(\mathfrak{A})^+$, $\alpha \in \mathbb{Q}$ and $\alpha \geq 0$, then, just as in (i),

$$\Vdash_{\mathbb{P}} [\![\dot{u} \times \dot{v} > \check{\alpha}]\!] = \sup_{\beta \in \mathbb{Q}, \beta > 0} [\![\dot{u} > \check{\beta}]\!] \dot{\cap} [\![\dot{v} > \frac{\check{\alpha}}{\check{\beta}}]\!] = (\sup_{\beta \in \mathbb{Q}, \beta > 0} [\![u > \beta]\!] \cap [\![v > \frac{\alpha}{\beta}]\!])^{\bullet}$$
$$= [\![u \times v > \alpha]\!]^{\bullet} = [\![(u \times v)^{\bullet} > \check{\alpha}]\!];$$

so $\Vdash_{\mathbb{P}} \dot{u} \times \dot{v} = (u \times v)^{\bullet}$. Using the distributive law we see that the same is true for all u, $v \in L^0(\mathfrak{A})$.

(iv) Take $\alpha \in \mathbb{R}$ and set $w = \alpha\chi 1 \in L^0(\mathfrak{A})$. If $\beta \in \mathbb{Q}$ and $\beta < \alpha$, then

$$\Vdash_{\mathbb{P}} [\![\check{\alpha}\chi 1 > \check{\beta}]\!] = 1 = \dot{1} = [\![w > \beta]\!]^{\bullet} = [\![\dot{w} > \check{\beta}]\!];$$

while if $\beta \geq \alpha$,

$$\Vdash_{\mathbb{P}} [\![\check{\alpha}\chi 1 > \check{\beta}]\!] = 0 = \dot{0} = [\![w > \beta]\!]^{\bullet} = [\![\dot{w} > \check{\beta}]\!].$$

So

$$\Vdash_{\mathbb{P}} [\![\check{\alpha}\chi 1 > \beta]\!] = [\![\dot{w} > \beta]\!] \text{ for every } \beta \in \mathbb{Q}, \text{ and } \check{\alpha}\chi 1 = \dot{w} = (\alpha\chi 1)^{\bullet}.$$

Putting this together with (iii), we have

$$\Vdash_{\mathbb{P}} \check{\alpha}\dot{u} = (\check{\alpha}\chi 1) \times \dot{u} = (\alpha\chi 1)^{\bullet} \times \dot{u} = (\alpha\chi 1 \times u)^{\bullet} = (\alpha u)^{\bullet}$$

for every $u \in L^0(\mathfrak{A})$. In particular, taking $\alpha = -1$, $\Vdash_{\mathbb{P}} -\dot{u} = (-u)^{\bullet}$.

(v) Finally, if $u \leq v$ then $u \vee v = v$, so

$$\Vdash_{\mathbb{P}} \dot{u} \vee \dot{v} = \dot{v} \text{ and } \dot{u} \leq \dot{v}.$$

(c)(i) It will help to note that the criterion in 364L(a-ii)

if $A \subseteq L^0(\mathfrak{A})$ is non-empty, then $v \in L^0(\mathfrak{A})$ is the supremum of A in $L^0(\mathfrak{A})$ iff $[\![v > \alpha]\!] = \sup_{u \in A} [\![u > \alpha]\!]$ in $\mathfrak{A}$ for every $\alpha \in \mathbb{R}$

can be replaced by

if $A \subseteq L^0(\mathfrak{A})$ is non-empty, then $v \in L^0(\mathfrak{A})$ is the supremum of A in $L^0(\mathfrak{A})$ iff $[\![v > \alpha]\!] = \sup_{u \in A} [\![u > \alpha]\!]$ in $\mathfrak{A}$ for every $\alpha \in \mathbb{Q}$.

P If the weaker condition is satisfied, and $\alpha \in \mathbb{R}$, then

$$[\![v > \alpha]\!] = \sup_{\beta \in \mathbb{Q}, \beta \geq \alpha} [\![v > \beta]\!] = \sup_{\beta \in \mathbb{Q}, \beta \geq \alpha} \sup_{u \in A} [\![u > \beta]\!]$$
$$= \sup_{u \in A} \sup_{\beta \in \mathbb{Q}, \beta \geq \alpha} [\![u > \beta]\!] = \sup_{u \in A} [\![u > \alpha]\!]. \textbf{ Q}$$

(ii) Now 556E(b-i) tells us that

$$\Vdash_{\mathbb{P}} [\![\dot{u} > \check{\alpha}]\!] = \sup_{i\in\check{I}} [\![\dot{u}_i > \check{\alpha}]\!]$$

for every $\alpha \in \mathbb{Q}$, so

$$\Vdash_{\mathbb{P}} \dot{u} = \sup_{i\in\check{I}} \dot{u}_i.$$

(d) For each $\alpha \in \mathbb{Q}$ we have an $a_\alpha \in \mathfrak{A}$ such that $p \Vdash_{\mathbb{P}} \dot{a}_\alpha = [\![\dot{w} > \check{\alpha}]\!]$ (556Ga); since $p \Vdash_{\mathbb{P}} \dot{a} = (a \cap p)^\bullet$ for every $a \in \mathfrak{A}$, we can suppose that $a_\alpha \subseteq p$ for every α. Now we find that if $\alpha \in \mathbb{Q}$ and $b_\alpha = \sup_{\beta\in\mathbb{Q},\beta>\alpha} a_\alpha$, then

$$p \Vdash_{\mathbb{P}} \dot{b}_\alpha = \sup_{\beta\in\mathbb{Q},\beta>\check{\alpha}} [\![\dot{w} > \beta]\!] = \dot{a}_\alpha,$$

so $a_\alpha = b_\alpha$. Similarly, if $b = \inf_{n\in\mathbb{N}} a_n$ and $c = \sup_{n\in\mathbb{N}} a_{-n}$,

$$p \Vdash_{\mathbb{P}} \dot{b} = \inf_{n\in\mathbb{N}} [\![\dot{w} > n]\!] = 0,$$
$$\dot{c} = \sup_{n\in\mathbb{N}} [\![\dot{w} > -n]\!] = 1$$

and $b = 0$, $c = p$. It is now easy to check that there is a $u \in L^0(\mathfrak{A})$ such that

$$[\![u > \alpha]\!] = a_\alpha \text{ if } \alpha \in \mathbb{Q} \text{ and } \alpha > 0,$$
$$= a_\alpha \cup (1 \setminus p) \text{ for other } \alpha \in \mathbb{Q},$$

and that $p \Vdash_{\mathbb{P}} \dot{u} = \dot{w}$.

(e) Recall that $\langle u_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 iff $\langle u_n\rangle_{n\in\mathbb{N}}$ is order-bounded and $0 = \inf_{n\in\mathbb{N}} \sup_{m\ge n} |u_n|$ (367G); and we shall have a similar formulation in the forcing language. So if $\langle u_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0, then

$$\Vdash_{\mathbb{P}} \sup_{m\ge\check{n}} |\dot{u}_m| = (\sup_{m\ge n} |u_m|)^\bullet$$

for every $n \in \mathbb{N}$, and

$$\Vdash_{\mathbb{P}} \inf_{n\in\mathbb{N}} \sup_{m\ge n} |\dot{u}_m| = (\inf_{n\in\mathbb{N}} \sup_{m\ge n} |u_m|)^\bullet = 0, \text{ so } \langle \dot{u}_n\rangle_{n\in\mathbb{N}} \to^* 0.$$

Conversely, if $\Vdash_{\mathbb{P}} \langle \dot{u}_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0, then $\Vdash_{\mathbb{P}} \langle \dot{u}_n\rangle_{n\in\mathbb{N}}$ is order-bounded, and there is a $\mathbb{P}$-name $\dot{w}$ such that

$$\Vdash_{\mathbb{P}} \dot{w} \in L^0(\dot{\mathfrak{A}}),\ |\dot{u}_n| \le \dot{w} \text{ for every } n \in \mathbb{N}.$$

By (d), there is a $v \in L^0(\mathfrak{A})$ such that $\Vdash_{\mathbb{P}} \dot{w} = \dot{v}$, so that

$$\Vdash_{\mathbb{P}} (v \vee |u_n|)^\bullet = \dot{w} \vee |\dot{u}_n| = \dot{v} \text{ for every } n \in \mathbb{N}$$

and $|u_n| \le v$ for every n (use (a-ii)). We can therefore repeat the calculation just above to see that

$$\Vdash_{\mathbb{P}} (\inf_{n\in\mathbb{N}} \sup_{m\ge n} |u_m|)^\bullet = \inf_{n\in\mathbb{N}} \sup_{m\ge n} |\dot{u}_m| = 0,$$

so that $\inf_{n\in\mathbb{N}} \sup_{m\ge n} |u_m| = 0$ and $\langle u_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0.

556I Proposition Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$, and $\mathfrak{C}$ a regularly embedded subalgebra of $\mathfrak{A}$. Let $\mathbb{P}$ be the forcing notion $\mathfrak{C}^+$, active downwards, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a Boolean homomorphism fixing every point of $\mathfrak{C}$; let $\dot{\pi}$ be the forcing name for π over $\mathfrak{C}$.

(a) π is injective iff $\Vdash_{\mathbb{P}} \dot{\pi}$ is injective.

(b) If π is order-continuous, then

$$\Vdash_{\mathbb{P}} \dot{\pi} \text{ is order-continuous.}$$

(c) If π has a support $\operatorname{supp}\pi$ (definition: 381Bb), then

$$\Vdash_{\mathbb{P}} (\operatorname{supp}\pi)^\bullet \text{ is the support of } \dot{\pi}.$$

proof (a) We saw in 556C(a-ii) that if π is injective then $\Vdash_{\mathbb{P}} \dot{\pi}$ is injective. Now suppose that π is not injective; let $a \in \mathfrak{A}^+$ be such that $\pi a = 0$. Then $\Vdash_{\mathbb{P}} \dot{\pi}\dot{a} = 0$. $1 \cap a \ne 0$, so $\not\Vdash_{\mathbb{P}} \dot{a} = 0$, by 556Da, and

$$\not\Vdash_{\mathbb{P}} \dot\pi \text{ is injective.}$$

(b) Take $p \in \mathfrak{C}^+$ and a $\mathbb{P}$-name $\dot A$ such that

$$p \Vdash_{\mathbb{P}} \dot A \subseteq \dot{\mathfrak{A}} \text{ and } \sup \dot A = 1.$$

Set $B = \{q \cap a : q \in \mathfrak{C}^+,\ a \in \mathfrak{A},\ q \Vdash_{\mathbb{P}} \dot a \in \dot A\}$. Then $\sup_{b \in B} p \cap b = p \cap 1 = p$, by 556Ea. Because π is order-continuous,

$$p \cap 1 = p = \pi p = \sup_{b \in B} \pi(p \cap b) = \sup_{b \in B} p \cap \pi b.$$

Consider

$$C = \{q \cap a : q \in \mathfrak{C}^+,\ a \in \mathfrak{A},\ q \Vdash_{\mathbb{P}} \dot a \in \dot\pi[\dot A]\}.$$

Then $\pi[B] \subseteq C$. **P** If $q \in \mathfrak{C}^+$, $a \in \mathfrak{A}$ and $q \Vdash_{\mathbb{P}} \dot a \in \dot A$, then

$$q \Vdash_{\mathbb{P}} (\pi a)^{\bullet} = \dot\pi \dot a \in \dot\pi[\dot A]$$

so

$$\pi(q \cap a) = q \cap \pi a \in C. \textbf{ Q}$$

Accordingly

$$\{p \cap c : c \in C\} \supseteq \{p \cap \pi b : b \in B\}$$

must have supremum p, and $p \Vdash_{\mathbb{P}} \sup \dot\pi[\dot A] = 1$.

As p and $\dot A$ are arbitrary,

$$\Vdash_{\mathbb{P}} \sup \dot\pi[\dot A] = 1 \text{ whenever } \dot A \subseteq \mathfrak{A} \text{ and } \sup \dot A = 1, \text{ so } \dot\pi \text{ is order-continuous}$$

(313L(b-iii)).

(c)(i) $\Vdash_{\mathbb{P}}$ if $x \in \dot{\mathfrak{A}}$ and $x \mathbin{\dot\cap} (\operatorname{supp}\pi)^{\bullet} = 0$ then $\dot\pi x = x$. **P** Take $p \in \mathfrak{C}^+$ and a $\mathbb{P}$-name $\dot x$ such that

$$p \Vdash_{\mathbb{P}} \dot x \in \dot{\mathfrak{A}} \text{ and } \dot x \mathbin{\dot\cap} (\operatorname{supp}\pi)^{\bullet} = 0.$$

For any q stronger than p there are an r stronger than q and an $a \in \mathfrak{A}$ such that

$$r \Vdash_{\mathbb{P}} \dot x = \dot a,\ (a \cap \operatorname{supp}\pi)^{\bullet} = 0;$$

now $r \cap a \cap \operatorname{supp}\pi = 0$ (556Da). In this case,

$$r \Vdash_{\mathbb{P}} \dot\pi \dot x = \dot\pi (r \cap a)^{\bullet} = (\pi(r \cap a))^{\bullet} = (r \cap a)^{\bullet} = \dot x.$$

As q is arbitrary, $p \Vdash_{\mathbb{P}} \dot\pi \dot x = \dot x$; as p and $\dot x$ are arbitrary, we have the result. **Q**

Now 381Ei, applied in the forcing language, tells us that

$$\Vdash_{\mathbb{P}} (\operatorname{supp}\pi)^{\bullet} \text{ supports } \dot\pi.$$

(ii) $\Vdash_{\mathbb{P}}$ if $x \in \dot{\mathfrak{A}}$ supports $\dot\pi$, then $x \mathbin{\dot\supseteq} (\operatorname{supp}\pi)^{\bullet}$. **P** Take $p \in \mathfrak{C}^+$ and a $\mathbb{P}$-name $\dot x$ such that

$$p \Vdash_{\mathbb{P}} \dot x \in \dot{\mathfrak{A}} \text{ supports } \dot\pi.$$

Then for any q stronger than p we have an r stronger than q and an $a \in \mathfrak{A}$ such that $r \Vdash_{\mathbb{P}} \dot x = \dot a$. Set $b = a \cup (1 \setminus r)$; then $r \Vdash_{\mathbb{P}} \dot b = \dot a$ supports $\dot\pi$. **?** If b does not support π, then there is a non-zero $d \subseteq 1 \setminus b$ such that $d \cap \pi d = 0$ (381Ei again). Since $r \cap d = d \neq 0$, there is an s stronger than r such that $s \Vdash_{\mathbb{P}} \dot d \neq 0$. Now

$$s \Vdash_{\mathbb{P}} \dot d \mathbin{\dot\cap} \dot\pi \dot d = (d \cap \pi d)^{\bullet} = 0, \text{ while } \dot d \mathbin{\dot\cap} \dot a = 0 \text{ and } \dot a \text{ supports } \dot\pi,$$

which is impossible. **X**

So $b \supseteq \operatorname{supp}\pi$ and

$$r \Vdash_{\mathbb{P}} \dot x = \dot b \mathbin{\dot\supseteq} (\operatorname{supp}\pi)^{\bullet}.$$

As q is arbitrary,

$$p \Vdash_{\mathbb{P}} \dot x \mathbin{\dot\supseteq} (\operatorname{supp}\pi)^{\bullet};$$

as p and $\dot x$ are arbitrary, we have the result. **Q**

Putting this together with (i),

$$\Vdash_{\mathbb{P}} (\operatorname{supp}\pi)^{\bullet} \text{ is the least element of } \dot{\mathfrak{A}} \text{ supporting } \dot{\pi}, \text{ and is the support of } \dot{\pi}.$$

556J Theorem Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, not $\{0\}$, and $\mathfrak{C}$ a regularly embedded subalgebra of $\mathfrak{A}$. Let $\mathbb{P}$ be the forcing notion $\mathfrak{C}^+$, active downwards, and $\dot{\mathfrak{A}}$ the forcing name for $\mathfrak{A}$ over $\mathfrak{C}$.

(a) If $\dot{\theta}$ is a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \dot{\theta} \text{ is a ring homomorphism from } \dot{\mathfrak{A}} \text{ to itself,}$$

then there is a unique ring homomorphism $\pi : \mathfrak{A} \to \mathfrak{A}$ such that $\pi c \subseteq c$ for every $c \in \mathfrak{C}$ and

$$\Vdash_{\mathbb{P}} \dot{\theta} = \dot{\pi},$$

where $\dot{\pi}$ is the forcing name for π over $\mathfrak{C}$.

(b)(i) If

$$\Vdash_{\mathbb{P}} \dot{\theta} \text{ is a Boolean homomorphism,}$$

then π is a Boolean homomorphism, and $\pi c = c$ for every $c \in \mathfrak{C}$.

(ii) If

$$\Vdash_{\mathbb{P}} \dot{\theta} \text{ is a Boolean automorphism,}$$

that π is a Boolean automorphism.

proof (a)(i) For each $a \in \mathfrak{A}$, 556Ga tells us that there is a $b \in \mathfrak{A}$ such that

$$\Vdash_{\mathbb{P}} \dot{\theta}(\dot{a}) = \dot{b};$$

by 556Da, this defines b uniquely, so we have a unique function $\pi : \mathfrak{A} \to \mathfrak{A}$ defined by the rule

$$\text{for every } a \in \mathfrak{A},\ \Vdash_{\mathbb{P}} \dot{\theta}(\dot{a}) = (\pi a)^{\bullet}.$$

(ii) Now, for $\bigcirc = \triangle$ or $\bigcirc = \cap$, and a, $b \in \mathfrak{A}$,

$$\begin{aligned}\Vdash_{\mathbb{P}} (\pi(a \bigcirc b))^{\bullet} &= \dot{\theta}((a \bigcirc b)^{\bullet}) = \dot{\theta}(\dot{a} \mathbin{\dot{\bigcirc}} \dot{b}) \\ &= \dot{\theta}\dot{a} \mathbin{\dot{\bigcirc}} \dot{\theta}\dot{b} = (\pi a)^{\bullet} \mathbin{\dot{\bigcirc}} (\pi b)^{\bullet} = (\pi a \bigcirc \pi b)^{\bullet}\end{aligned}$$

and $\pi(a \bigcirc b) = \pi a \bigcirc \pi b$. So π is a ring homomorphism.

(iii) If $c \in \mathfrak{C}$ then $\pi c \subseteq c$. **P** If $c = 1$ this is trivial. Otherwise,

$$1 \setminus c \Vdash_{\mathbb{P}} \dot{c} = 0,\ (\pi c)^{\bullet} = \dot{\theta}0 = 0,$$

so $(1 \setminus c) \cap \pi c = 0$ and $\pi c \subseteq c$. **Q**

(iv) We can therefore speak of the forcing name $\dot{\pi}$ (556Ae, 556C). If $p \in \mathfrak{C}^+$ and $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \dot{\mathfrak{A}}$, let $a \in \mathfrak{A}$ be such that $p \Vdash_{\mathbb{P}} \dot{x} = \dot{a}$ (556Ga again); then

$$p \Vdash_{\mathbb{P}} \dot{\theta}(\dot{x}) = \dot{\theta}(\dot{a}) = (\pi a)^{\bullet} = \dot{\pi}(\dot{a}) = \dot{\pi}(\dot{x}).$$

As p and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \dot{\theta} = \dot{\pi}.$$

(b)(i) If $\Vdash_{\mathbb{P}} \dot{\theta}$ is a Boolean homomorphism, then

$$\Vdash_{\mathbb{P}} (\pi 1)^{\bullet} = \dot{\theta}\dot{1} = \dot{1}$$

and $\pi 1 = 1$. Now

$$\pi c \subseteq c = 1 \setminus (1 \setminus c) \subseteq 1 \setminus \pi(1 \setminus c) = \pi 1 \setminus (\pi 1 \setminus \pi c) = \pi c$$

so $\pi c = c$ for every $c \in \mathfrak{C}$.

(ii) If $\Vdash_{\mathbb{P}} \dot\theta$ is a Boolean automorphism, then the same arguments tell us that there is a Boolean homomorphism $\phi : \mathfrak{A} \to \mathfrak{A}$ such that $\phi c = c$ for every $c \in \mathfrak{C}$ and $\Vdash_{\mathbb{P}} \dot\phi = \dot\theta^{-1}$. But in this case

$$\Vdash_{\mathbb{P}} (\pi\phi)^{\bullet} = \dot\pi\dot\phi = \dot\theta\dot\theta^{-1} = \dot\iota$$

where ι is the identity automorphism on $\mathfrak{A}$; by the uniqueness of the representing homomorphisms of $\mathfrak{A}$, $\pi\phi = \iota$. Similarly, $\phi\pi = \iota$ and $\phi = \pi^{-1}$, so that π is an automorphism.

556K Theorem Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra, and $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$; let $\mathbb{P}$ be the forcing notion $\mathfrak{C}^+$, active downwards, and $\dot{\mathfrak{A}}$ the forcing name for $\mathfrak{A}$ over $\mathfrak{C}$. We can identify $\mathfrak{C}$ with the regular open algebra RO($\mathbb{P}$) (514Sb). For $u \in L^0(\mathfrak{C})$ write $\vec u$ for the corresponding $\mathbb{P}$-name for a real number as described in 5A3L.

(a)(i) For each $a \in \mathfrak{A}$ there is a $u_a \in L^1(\mathfrak{C}, \bar\mu{\restriction}\mathfrak{C})$ defined by saying that $\int_c u_a = \bar\mu(a \cap c)$ for every $c \in \mathfrak{C}$.

(ii) If $p \in \mathfrak{C}^+$ and a, $b \in \mathfrak{A}$ are such that

$$p \Vdash_{\mathbb{P}} \dot a = \dot b$$

(where $\dot a$, $\dot b$ are the forcing names for a, b over $\mathfrak{C}$), then

$$p \Vdash_{\mathbb{P}} \vec u_a = \vec u_b.$$

(b) There is a $\mathbb{P}$-name $\dot{\bar\mu}$ such that

$$\Vdash_{\mathbb{P}} (\dot{\mathfrak{A}}, \dot{\bar\mu}) \text{ is a probability algebra,}$$

and

$$\Vdash_{\mathbb{P}} \dot{\bar\mu}\dot a = \vec u_a$$

whenever $a \in \mathfrak{A}$ and $\dot a$ is the corresponding forcing name over $\mathfrak{C}$.

(c) If $\pi : \mathfrak{A} \to \mathfrak{A}$ is a measure-preserving Boolean homomorphism such that $\pi c = c$ for every $c \in \mathfrak{C}$, and $\dot\pi$ the corresponding forcing name over $\mathfrak{C}$, then

$$\Vdash_{\mathbb{P}} \dot\pi : \dot{\mathfrak{A}} \to \dot{\mathfrak{A}} \text{ is measure-preserving.}$$

(d) If $\dot\phi$ is a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \dot\phi : \dot{\mathfrak{A}} \to \dot{\mathfrak{A}} \text{ is a measure-preserving Boolean automorphism}$$

then there is a measure-preserving Boolean automorphism $\pi : \mathfrak{A} \to \mathfrak{A}$ such that $\pi c = c$ for every $c \in \mathfrak{C}$ and

$$\Vdash_{\mathbb{P}} \dot\phi = \dot\pi.$$

(e) If $v \in L^1(\mathfrak{A}, \bar\mu)$ and $u \in L^1(\mathfrak{C}, \bar\mu{\restriction}\mathfrak{C})$ is its conditional expectation on $\mathfrak{C}$, then

$$\Vdash_{\mathbb{P}} \dot v \in L^1(\dot{\mathfrak{A}}, \dot{\bar\mu}) \text{ and } \int \dot v \, d\dot{\bar\mu} = \vec u.$$

proof (a)(i) This is just the Radon-Nikodým theorem (365E).

(ii) If $p \Vdash_{\mathbb{P}} \dot a = \dot b$, then $p \cap a = p \cap b$ (556Da). Consequently

$$\int_c u_a \times \chi p = \int_{c \cap p} u_a = \bar\mu(c \cap p \cap a) = \bar\mu(c \cap p \cap b) = \int_c u_b \times \chi p$$

whenever $c \in \mathfrak{C}$, and $u_a \times \chi p = u_b \times \chi p$; by 5A3M,

$$p \Vdash_{\mathbb{P}} \vec u_a = \vec u_b.$$

(b)(i) Note first the elementary properties of the conditional expectation $a \mapsto u_a : \mathfrak{A} \to L^1(\mathfrak{C}, \bar\mu{\restriction}\mathfrak{C})$: it is additive and positive and order-continuous, and $0 \le u_a \le \chi 1$ for every a. (To extract these facts efficiently from the presentation in §365, note that $u_a = P(\chi a)$, where $P : L^1(\mathfrak{A}, \bar\mu) \to L^1(\mathfrak{C}, \bar\mu{\restriction}\mathfrak{C})$ is the conditional expectation operator of 365Q[10].) In particular,

$$\Vdash_{\mathbb{P}} \vec u_a \in [0, 1]$$

[10] Formerly 365R.

for every $a \in \mathfrak{A}$. It is also worth observing that if $c \in \mathfrak{C}$ and $a \in \mathfrak{A}$ then $u_{a\cap c} = u_a \times \chi c$ (see 365Oc[11]).

(ii) Now consider the $\mathbb{P}$-name

$$\dot{\bar{\mu}} = \{((\dot{a}, \vec{u}_a), 1) : a \in \mathfrak{A}\}.$$

We have quite a lot to check, of course. First, $\dot{\bar{\mu}}$ is a name for a function with domain $\dot{\mathfrak{A}}$. **P** If $((\dot{a}, \vec{u}_a), 1)$ and $((\dot{b}, \vec{u}_b), 1)$ are two members of $\dot{\bar{\mu}}$, and $p \in \mathfrak{C}^+$ is such that $p \Vdash_{\mathbb{P}} \dot{a} = \dot{b}$, then $p \cap a = p \cap b$, so $p \Vdash_{\mathbb{P}} \vec{u}_a = \vec{u}_b$, by (a-ii) above. By 5A3E, $\Vdash_{\mathbb{P}} \dot{\bar{\mu}}$ is a function. Also $\Vdash_{\mathbb{P}} \operatorname{dom} \dot{\bar{\mu}} = \dot{A}$, where $\dot{A} = \{(\dot{a}, 1) : a \in \mathfrak{A}\} = \dot{\mathfrak{A}}$. **Q**

(iii) We have

$$\Vdash_{\mathbb{P}} \dot{\bar{\mu}}\dot{a} = \vec{u}_a \in [0, 1]$$

for every $a \in \mathfrak{A}$, so

$$\Vdash_{\mathbb{P}} \dot{\bar{\mu}} \text{ is a function from } \dot{\mathfrak{A}} \text{ to } [0, 1].$$

Since $u_1 = \chi 1$,

$$\Vdash_{\mathbb{P}} \dot{\bar{\mu}}1 = \vec{u}_1 = 1.$$

(iv) Next, $\Vdash_{\mathbb{P}} \dot{\bar{\mu}}$ is additive. **P** Suppose that $p \in \mathfrak{C}^+$ and $\dot{x}$, $\dot{y}$ are $\mathbb{P}$-names such that

$$p \Vdash_{\mathbb{P}} \dot{x},\ \dot{y} \in \dot{\mathfrak{A}} \text{ are disjoint.}$$

By 556Ga there are a, $b \in \mathfrak{A}$ such that

$$p \Vdash_{\mathbb{P}} \dot{x} = \dot{a},\ \dot{y} = \dot{b},\ (a \cap b)^{\bullet} = \dot{x} \mathbin{\dot\cap} \dot{y} = 0.$$

So $p \cap a \cap b = 0$ and

$$\chi p \times u_{a\cup b} = u_{p\cap(a\cup b)} = u_{p\cap a} + u_{p\cap b} = \chi p \times u_a + \chi p \times u_b = \chi p \times (u_a + u_b);$$

it follows that

$$p \Vdash_{\mathbb{P}} \dot{\bar{\mu}}(\dot{x} \mathbin{\dot\cup} \dot{y}) = \dot{\bar{\mu}}(\dot{a} \mathbin{\dot\cup} \dot{b}) = \dot{\bar{\mu}}(a \cup b)^{\bullet} = \vec{u}_{a\cup b} = (u_a + u_b)^{\rightarrow}$$

(5A3M)

$$= \vec{u}_a + \vec{u}_b = \dot{\bar{\mu}}\dot{x} + \dot{\bar{\mu}}\dot{y}.$$

As p, $\dot{x}$ and $\dot{y}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \dot{\bar{\mu}} \text{ is additive. } \mathbf{Q}$$

(v) Suppose that $p \in \mathfrak{C}^+$ and that $\dot{A}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{A} \subseteq \dot{\mathfrak{A}} \text{ is closed under } \dot\cup \text{ and has supremum } 1.$$

Then for every rational number $\alpha < 1$ there are an $r \in \mathfrak{C}^+$, stronger than p, and a $d \in \mathfrak{A}$ such that

$$r \Vdash_{\mathbb{P}} \dot{d} \in \dot{A} \text{ and } \dot{\bar{\mu}}\dot{d} \ge \check{\alpha}.$$

P Set

$$B = \{q \cap a : q \in \mathfrak{C}^+,\ a \in \mathfrak{A},\ q \Vdash_{\mathbb{P}} \dot{a} \in \dot{A}\},$$

so that $p \subseteq \sup B$ (556Ea). Because $\bar{\mu}$ is completely additive, there are $b_0, \ldots, b_{n-1} \in B$ such that $\bar{\mu}(p \cap \sup_{i<n} b_i) > \alpha\bar{\mu}p$. Express each b_i as $q_i \cap a_i$ where q_i is stronger than p and $q_i \Vdash_{\mathbb{P}} \dot{a}_i \in \dot{A}$. For $J \subseteq n$ set

$$c_J = p \cap \inf_{i\in J} q_i \setminus \sup_{i\in n\setminus J} q_i, \qquad d_J = \sup_{i\in J} a_i;$$

then $\langle c_J \rangle_{J\subseteq n}$ is disjoint and

$$p \cap \sup_{i<n} b_i = \sup_{\emptyset\ne J\subseteq n} c_J \cap d_J.$$

[11] Formerly 365Pc.

Accordingly

$$\sum_{\emptyset\ne J\subseteq n}\bar\mu(c_J\cap d_J)>\alpha\bar\mu p$$

and there must be a non-empty $J\subseteq n$ such that $c_J\ne 0$ and

$$\alpha\bar\mu c_J<\bar\mu(c_J\cap d_J)=\int_{c_J}u_{d_J}.$$

So $r=c_J\cap[\![u_{d_J}\ge\alpha]\!]$ is non-zero. Set $d=d_J$; then

$$r\Vdash_{\mathbb{P}}\dot a_i\in\dot A\text{ for every }i\in\check J,\text{ therefore }\dot d=\sup_{i\in\check J}\dot a_i\in\dot A,$$

and

$$r\Vdash_{\mathbb{P}}\dot{\bar\mu}\dot d=\vec u_{d_J}\ge\check\alpha.\ \mathbf{Q}$$

(vi) It follows that

$$\Vdash_{\mathbb{P}}\dot{\bar\mu}\text{ is completely additive.}$$

P Suppose that $p\in\mathfrak{C}^+$ and that $\dot A$ is a $\mathbb{P}$-name such that

$$p\Vdash_{\mathbb{P}}\dot A\subseteq\dot{\mathfrak{A}}\text{ is closed under }\dot\cup\text{ and has supremum 1.}$$

Then for every rational $\alpha<1$ and every q stronger than p there is an r stronger than q such that

$$r\Vdash_{\mathbb{P}}\text{ there is an }x\in\dot A\text{ such that }\dot{\bar\mu}x\ge\check\alpha;$$

as q is arbitrary,

$$p\Vdash_{\mathbb{P}}\text{ there is an }x\in\dot A\text{ such that }\dot{\bar\mu}x\ge\check\alpha;$$

as α is arbitrary,

$$p\Vdash_{\mathbb{P}}\sup_{x\in\dot A}\dot{\bar\mu}x=1.$$

As p and $\dot A$ are arbitrary,

$$\Vdash_{\mathbb{P}}\sup_{x\in A}\dot{\bar\mu}x=1\text{ whenever }A\subseteq\dot{\mathfrak{A}}\text{ is closed under }\dot\cup\text{ and has supremum 1.}$$

We know that

$$\Vdash_{\mathbb{P}}\dot{\bar\mu}\text{ is additive and }\dot{\bar\mu}1=1,$$

so we can turn this over to get

$\Vdash_{\mathbb{P}}\inf_{x\in A}\dot{\bar\mu}x=0$ whenever $A\subseteq\dot{\mathfrak{A}}$ is closed under $\dot\cap$ and has infimum 0, therefore $\dot{\bar\mu}$ is completely additive. **Q**

(vii) Since we already know that

$$\Vdash_{\mathbb{P}}\dot{\mathfrak{A}}\text{ is Dedekind complete}$$

(556Gb), we have all the elements needed for

$$\Vdash_{\mathbb{P}}(\dot{\mathfrak{A}},\dot{\bar\mu})\text{ is a probability algebra.}$$

(c) The point is that if $a\in\mathfrak{A}$ then $u_{\pi a}=u_a$. **P** For any $c\in\mathfrak{C}$,

$$\int_c u_{\pi a}=\bar\mu(c\cap\pi a)=\bar\mu(\pi(c\cap a))=\bar\mu(c\cap a)=\int_c u_a.\ \mathbf{Q}$$

Now suppose that $p\in\mathfrak{C}^+$ and $\dot x$ is a $\mathbb{P}$-name such that $p\Vdash_{\mathbb{P}}\dot x\in\dot{\mathfrak{A}}$. Then there is an $a\in\mathfrak{A}$ such that $p\Vdash_{\mathbb{P}}\dot a=\dot x$ (556Ga again), and

$$p\Vdash_{\mathbb{P}}\dot{\bar\mu}(\dot\pi\dot x)=\dot{\bar\mu}(\dot\pi\dot a)=\dot{\bar\mu}(\pi a)^\bullet=\vec u_{\pi a}=\vec u_a=\dot{\bar\mu}\dot x.$$

As p and $\dot x$ are arbitrary,

$$\Vdash_{\mathbb{P}}\dot\pi\text{ is measure-preserving.}$$

(d) By 556J, there is a unique $\pi\in\operatorname{Aut}\mathfrak{A}$ such that $\pi c=c$ for every $c\in\mathfrak{C}$ and $\Vdash_{\mathbb{P}}\dot\phi=\dot\pi$. In this case, for any $a\in\mathfrak{A}$,

$$\Vdash_{\mathbb{P}} \vec{u}_a = \dot{\bar{\mu}}\dot{a} = \dot{\bar{\mu}}(\dot{\phi}\dot{a}) = \dot{\bar{\mu}}(\dot{\pi}\dot{a}) = \dot{\bar{\mu}}(\pi a)^{\bullet} = \vec{u}_{\pi a}.$$

So $u_a = u_{\pi a}$ (5A3M again) and

$$\bar{\mu}a = \int u_a = \int u_{\pi a} = \bar{\mu}(\pi a).$$

Thus π is measure-preserving.

(e) Let $P : L^1(\mathfrak{A}, \bar{\mu}) \to L^1(\mathfrak{C}, \bar{\mu}\restriction\mathfrak{C})$ be the conditional expectation operator, and let U be the set of those $v \in L^1(\mathfrak{A}, \bar{\mu})$ such that

$$\Vdash_{\mathbb{P}} \dot{v} \in L^1(\dot{\mathfrak{A}}, \dot{\bar{\mu}}) \text{ and } \int \dot{v}\, d\dot{\bar{\mu}} = \vec{Pv}.$$

By (a), $\chi a \in U$ for every $a \in \mathfrak{A}$; by 556Hb, U is closed under addition and rational multiplication; by 556Hc $\sup_{n\in\mathbb{N}} v_n \in U$ for every non-decreasing sequence $\langle v_n\rangle_{n\in\mathbb{N}}$ in U. So $U = L^1(\mathfrak{A}, \bar{\mu})$, as required.

556L Relatively independent subalgebras Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra and $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$; let $\mathbb{P}$ be the forcing notion $\mathfrak{C}^+$, active downwards. Let $\dot{\bar{\mu}}$ be the forcing name for $\bar{\mu}$ described in 556K, so that $\Vdash_{\mathbb{P}} (\dot{\mathfrak{A}}, \dot{\bar{\mu}})$ is a probability algebra.

(a) For a subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ including $\mathfrak{C}$, let $\dot{\mathfrak{B}}$ be the forcing name for $\mathfrak{B}$ over $\mathfrak{C}$. If $\langle \mathfrak{B}_i\rangle_{i\in I}$ is a family of subalgebras of $\mathfrak{A}$ including $\mathfrak{C}$, then $\langle \mathfrak{B}_i\rangle_{i\in I}$ is relatively independent over $\mathfrak{C}$ (definition: 458La) iff

$$\Vdash_{\mathbb{P}} \langle \dot{\mathfrak{B}}_i\rangle_{i\in\check{I}} \text{ is stochastically independent in } \dot{\mathfrak{A}}.$$

(b) If $\langle v_i\rangle_{i\in I}$ is a family in $L^0(\mathfrak{A})$ which is relatively independent over $\mathfrak{C}$, then

$$\Vdash_{\mathbb{P}} \langle \dot{v}_i\rangle_{i\in\check{I}} \text{ is stochastically independent}$$

(writing $\dot{v}_i$ for the forcing name for v_i over $\mathfrak{C}$).

proof (a)(i) Suppose that $\langle \mathfrak{B}_i\rangle_{i\in I}$ is relatively independent over $\mathfrak{C}$. Let $p \in \mathfrak{C}^+$ and $\dot{J}$, $\langle \dot{x}_j\rangle_{j\in\dot{J}}$ be $\mathbb{P}$-names such that

$$p \Vdash_{\mathbb{P}} \dot{J} \in [\check{I}]^{<\omega} \text{ is non-empty, } \dot{x}_j \in \dot{\mathfrak{B}}_j \text{ for every } j \in \dot{J}.$$

Then for every q stronger than p there are an r stronger than q and a family $\langle b_j\rangle_{j\in J}$ such that J is a non-empty finite subset of I, $b_j \in \mathfrak{B}_j$ for every $j \in J$, and

$$r \Vdash_{\mathbb{P}} \dot{J} = \check{J} \text{ and } \dot{x}_j = \dot{b}_j \text{ for every } j \in \check{J}.$$

Set $a = \inf_{j\in J} b_j$. For each $j \in J$ let $u_{b_j} \in L^1(\mathfrak{C}, \bar{\mu}\restriction\mathfrak{C})$ be the conditional expectation of χb_j on $\mathfrak{C}$, as in 556Ka; then $u_a = \prod_{i\in J} u_{b_j}$, because $\langle \mathfrak{B}_i\rangle_{i\in I}$ is relatively stochastically independent. But this means that

$$\begin{aligned} r \Vdash_{\mathbb{P}} \dot{\bar{\mu}}(\inf_{j\in\dot{J}} \dot{x}_j) &= \dot{\bar{\mu}}(\inf_{j\in\check{J}} \dot{b}_j) = \dot{\bar{\mu}}\dot{a} = \vec{u}_a \\ &= \prod_{j\in\check{J}} \vec{u}_{b_j} = \prod_{j\in\check{J}} \dot{\bar{\mu}}\dot{b}_j = \prod_{j\in\dot{J}} \dot{\bar{\mu}}\dot{x}_j. \end{aligned}$$

As q is arbitrary,

$$p \Vdash_{\mathbb{P}} \dot{\bar{\mu}}(\inf_{j\in\dot{J}} \dot{x}_j) = \prod_{j\in\dot{J}} \dot{\bar{\mu}}\dot{x}_j;$$

as p and $\langle \dot{x}_j\rangle_{j\in\dot{J}}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \langle \dot{\mathfrak{B}}_i\rangle_{i\in\check{I}} \text{ is independent.}$$

(ii) Now suppose that

$$\Vdash_{\mathbb{P}} \langle \dot{\mathfrak{B}}_i\rangle_{i\in\check{I}} \text{ is independent.}$$

Take a finite set $J \subseteq I$ and $\langle b_j\rangle_{j\in J} \in \prod_{j\in J} \mathfrak{B}_j$. Again set $a = \inf_{j\in J} b_j$ and let u_{b_j} be the conditional expectation of χb_j on $\mathfrak{C}$ for each j. Then

$$\Vdash_{\mathbb{P}} (\prod_{j\in J} u_{b_j})^{\rightarrow} = \prod_{j\in\check{J}} \vec{u}_{b_j} = \prod_{j\in\check{J}} \dot{\bar{\mu}}\dot{b}_j$$
$$= \dot{\bar{\mu}}(\inf_{j\in\check{J}} \dot{b}_j) = \dot{\bar{\mu}}(\inf_{j\in J} b_j)^{\bullet} = \vec{u}_a,$$

so $\prod_{j\in J} u_{b_j} = u_a$. As $\langle b_j\rangle_{j\in J}$ is arbitrary, $\langle\mathfrak{B}_i\rangle_{i\in I}$ is relatively independent over $\mathfrak{C}$.

(b) For each $i \in I$, let $\mathfrak{A}_i$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{[\![v_i > \alpha]\!] : \alpha \in \mathbb{Q}\}$, and $\mathfrak{B}_i$ the closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_i \cup \mathfrak{C}$. Then $\langle\mathfrak{B}_i\rangle_{i\in I}$ is relatively independent over $\mathfrak{C}$ (458Ld), so $\Vdash_{\mathbb{P}} \langle\dot{\mathfrak{B}}_i\rangle_{i\in\check{I}}$ is independent, by (a) here. Now we have

$$\Vdash_{\mathbb{P}} [\![\dot{v}_i > \alpha]\!] = [\![v_i > \alpha]\!]^{\bullet} \in \dot{\mathfrak{B}}_i$$

whenever $\alpha \in \mathbb{Q}$ and $i \in I$, so

$$\Vdash_{\mathbb{P}} [\![\dot{v}_i > \alpha]\!] \in \dot{\mathfrak{B}}_i \text{ for every } \alpha \in \mathbb{Q} \text{ and } i \in \check{I}, \text{ and } \langle\dot{v}_i\rangle_{i\in\check{I}} \text{ is independent.}$$

556M Laws of large numbers As an elementary example to show that we can use this machinery to extend a classical result, I give the following. Consider the two statements

(‡) Let (X,Σ,μ) be a probability space, T a σ-subalgebra of Σ, and $\langle f_n\rangle_{n\in\mathbb{N}}$ a sequence in $\mathcal{L}^2(\mu)$ such that $\langle f_n\rangle_{n\in\mathbb{N}}$ is relatively independent over T and $\int_F f_n d\mu = 0$ for every $n \in \mathbb{N}$ and every $F \in$ T. Suppose that $\langle\beta_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $]0,\infty[$, diverging to ∞, such that $\sum_{n=0}^{\infty} \frac{1}{\beta_n^2}\|f_n\|_2^2 < \infty$. Then $\lim_{n\to\infty} \frac{1}{\beta_n}\sum_{i=0}^n f_i = 0$ a.e.

and

(†) Let (X,Σ,μ) be a probability space and $\langle f_n\rangle_{n\in\mathbb{N}}$ an independent sequence in $\mathcal{L}^2(\mu)$ such that $\int f_n d\mu = 0$ for every $n \in \mathbb{N}$. Suppose that $\langle\beta_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $]0,\infty[$, diverging to ∞, such that $\sum_{n=0}^{\infty} \frac{1}{\beta_n^2}\|f_n\|_2^2 < \infty$. Then $\lim_{n\to\infty} \frac{1}{\beta_n}\sum_{i=0}^n f_i = 0$ a.e.

In 273D I presented (†) as the basic strong law of large numbers from which the other standard forms could be derived. (‡) may be found in Volume 4 as an exercise (458Ye). What I propose to do is to show how (‡) can be deduced, not exactly from (†), but from (†) in a forcing model; relying on the fundamental theorem of forcing to confirm that if (†) is true in its ordinary sense, then its interpretation in any forcing language will again be true.

proof (a) In order to avoid explanations involving names for real numbers, it seems helpful to re-word (‡). Consider the version

$(‡)_1$ Let (X,Σ,μ) be a probability space, T a σ-subalgebra of Σ, and $\langle f_n\rangle_{n\in\mathbb{N}}$ a sequence in $\mathcal{L}^2(\mu)$ such that $\langle f_n\rangle_{n\in\mathbb{N}}$ is relatively independent over T and $\int_F f_n d\mu = 0$ for every $n \in \mathbb{N}$ and every $F \in$ T. Suppose that $\langle\beta_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathbb{Q}\cap\,]0,\infty[$, diverging to ∞, such that $\sum_{n=0}^{\infty} \frac{1}{\beta_n^2}\|f_n\|_2^2 < \infty$. Then $\lim_{n\to\infty} \frac{1}{\beta_n}\sum_{i=0}^n f_i = 0$ a.e.

Then $(‡)_1$ implies (‡). **P** Given the structure of (‡), with general $\beta_n > 0$, let $\delta_n \in \mathbb{Q}\cap\,]0,\beta_n]$ be such that

$$\frac{1}{\delta_n^2}\|f_n\|_2^2 \le \frac{1}{\beta_n^2}\|f_n\|_2^2 + 2^{-n}$$

for every n. Set $\gamma_n = \sup_{m\le n}\delta_m$ for each n; then $\langle\gamma_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathbb{Q}\cap\,]0,\infty[$ and $\sum_{n=0}^{\infty}\frac{1}{\gamma_n^2}\|f_n\|_2^2$ is finite, so $(‡)_1$ tells us that

$$\lim_{n\to\infty}\frac{1}{\beta_n}\sum_{i=0}^n f_i = \lim_{n\to\infty}\frac{\gamma_n}{\beta_n}\cdot\frac{1}{\gamma_n}\sum_{i=0}^n f_i = 0 \text{ a.e. } \mathbf{Q}$$

(b) Now formulate the assertions $(‡)_1$ and (†) in terms of measure algebras; we get

$(‡)'$ Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra, $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$, and $\langle v_n\rangle_{n\in\mathbb{N}}$ a sequence in $L^2(\mathfrak{A},\bar{\mu})$ such that $\langle v_n\rangle_{n\in\mathbb{N}}$ is relatively independent over $\mathfrak{C}$ and $Pv_n = 0$ for every $n \in \mathbb{N}$, where $P : L^1(\mathfrak{A},\bar{\mu}) \to L^1(\mathfrak{C},\bar{\mu}\restriction\mathfrak{C})$ is the conditional expectation operator. Suppose that $\langle\beta_n\rangle_{n\in\mathbb{N}}$

is a non-decreasing sequence in $\mathbb{Q}\cap]0,\infty[$, diverging to ∞, such that $\sum_{n=0}^{\infty}\frac{1}{\beta_n^2}\|v_n\|_2^2<\infty$. Then $\langle\frac{1}{\beta_n}\sum_{i=0}^n v_i\rangle_{n\in\mathbb{N}}$ is order*-convergent to 0.

and

($\dagger$)$'$ Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and $\langle v_n\rangle_{n\in\mathbb{N}}$ an independent sequence in $L^2(\mathfrak{A},\bar\mu)$ such that $\int v_n d\bar\mu=0$ for every n. Suppose that $\langle\beta_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $]0,\infty[$, diverging to ∞, such that $\sum_{n=0}^{\infty}\frac{1}{\beta_n^2}\|v_n\|_2^2<\infty$. Then $\langle\frac{1}{\beta_n}\sum_{i=0}^n v_i\rangle_{n\in\mathbb{N}}$ is order*-convergent to 0.

(As usual, the conversions are just a matter of applying the Loomis-Sikorski theorem, with 367F to translate order*-convergence in L^0 into almost-everywhere convergence of functions.)

(c) Assuming ($\dagger$)$'$, take a structure $(\mathfrak{A},\bar\mu,\mathfrak{C},\langle v_n\rangle_{n\in\mathbb{N}},\langle\beta_n\rangle_{n\in\mathbb{N}})$ as in ($\ddagger$)$'$, let $\mathbb{P}$ be the forcing notion $\mathfrak{C}^+$, active downwards, and consider the corresponding forcing names $\dot{\mathfrak{A}}$, $\dot{\bar\mu}$ and $\langle\dot v_n\rangle_{n\in\mathbb{N}}$. Let $P:L^1(\mathfrak{A},\bar\mu)\to L^1(\mathfrak{C},\bar\mu\restriction\mathfrak{C})$ be the conditional expectation operator. For each $n\in\mathbb{N}$,

$$\Vdash_{\mathbb{P}}\dot v_n\times\dot v_n=(v_n\times v_n)^{\bullet}\in L^1(\dot{\mathfrak{A}},\dot{\bar\mu}),\quad \|\dot v_n\|_2^2=\int\dot v_n^2d\dot{\bar\mu}=(P(v_n^2))^{\rightharpoondown}$$

by 556Hb and 556Ke. Now

$$\sum_{n=0}^{\infty}\frac{1}{\beta_n}\int P(v_n^2)d(\bar\mu\restriction\mathfrak{C})\le\sum_{n=0}^{\infty}\frac{1}{\beta_n}\|v_n\|_2^2<\infty,$$

so

$$v=\sum_{i=n}^{\infty}\frac{1}{\beta_n}P(v_n^2)$$

is defined in $L^0(\mathfrak{C})$, and

$$\Vdash_{\mathbb{P}}\sum_{n=0}^{\infty}\frac{1}{\check\beta_n}\|\dot v_n\|_2^2\le\vec v\text{ is finite.}$$

At the same time,

$$\Vdash_{\mathbb{P}}\int\dot v_n d\dot{\bar\mu}=\overrightarrow{Pv_n}=0\text{ for every }n\in\mathbb{N},$$

and by 556Lb

$$\Vdash_{\mathbb{P}}\langle\dot v_n\rangle_{n\in\mathbb{N}}\text{ is independent.}$$

Applying ($\dagger$)$'$ in the forcing language,

$$\Vdash_{\mathbb{P}}\langle(\frac{1}{\check\beta_n}\sum_{i=0}^n v_i)^{\bullet}\rangle_{n\in\mathbb{N}}=\langle\frac{1}{\check\beta_n}\sum_{i=0}^n\dot v_i\rangle_{n\in\mathbb{N}}\text{ order*-converges to 0 in }L^0(\dot{\mathfrak{A}}),$$

so $\langle\frac{1}{\beta_n}\sum_{i=0}^n v_i\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in $L^0(\mathfrak{A})$, by 556He.

Thus ($\ddagger$)$'$ is true, and we're home.

556N Dye's theorem Now for something from Volume 3. Let me state two versions of Dye's theorem (388L): the 'full' version

($\ddagger$) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra of countable Maharam type, $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$, and π_1, π_2 two measure-preserving automorphisms of $\mathfrak{A}$ with fixed-point algebra $\mathfrak{C}$. Then there is a measure-preserving automorphism ϕ of $\mathfrak{A}$ such that $\phi c=c$ for every $c\in\mathfrak{C}$ and π_1 and $\phi\pi_2\phi^{-1}$ generate the same full subgroups of $\operatorname{Aut}\mathfrak{A}$.

and the 'simple' version

($\dagger$) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra of countable Maharam type, and π_1, π_2 two ergodic measure-preserving automorphisms of $\mathfrak{A}$. Then there is a measure-preserving automorphism ϕ of $\mathfrak{A}$ such that π_1 and $\phi\pi_2\phi^{-1}$ generate the same full subgroups of $\operatorname{Aut}\mathfrak{A}$.

Here also the machinery of this section provides a proof of ($\ddagger$) from ($\dagger$).

proof (a) Assume ($\dagger$). Take $(\mathfrak{A},\bar\mu)$, $\mathfrak{C}$, π_1 and π_2 as in ($\ddagger$). Let $\mathbb{P}$ be the forcing notion $\mathfrak{C}^+$, active downwards, and let $\dot{\mathfrak{A}}$, $\dot\pi_1$ and $\dot\pi_2$ be the forcing names for $\mathfrak{A}$, π_1 and π_2 over $\mathfrak{C}$. By 556C(b-iv), 556C(b-v), 556Gb, 556Kb and 556Kc, and using 372Pc,

$\Vdash_{\mathbb{P}}$ there is a measure on $\dot{\mathfrak{A}}$ with respect to which it is a probability measure and $\dot{\pi}_1$ and $\dot{\pi}_2$ are measure-preserving automorphisms with fixed-point subalgebra $\{0,1\}$, so are ergodic, because $\dot{\mathfrak{A}}$ is Dedekind complete.

By 556Ed,

$$\Vdash_{\mathbb{P}} \dot{\mathfrak{A}} \text{ has countable Maharam type.}$$

By (†), applied in the forcing universe,

$\Vdash_{\mathbb{P}}$ there is a measure-preserving automorphism θ of $\dot{\mathfrak{A}}$ such that $\dot{\pi}_1$ and $\theta\dot{\pi}_2\theta^{-1}$ generate the same full subgroups of $\operatorname{Aut}\dot{\mathfrak{A}}$.

Let $\dot{\theta}$ be a $\mathbb{P}$-name such that

$\Vdash_{\mathbb{P}} \dot{\theta}$ is a measure-preserving automorphism of $\dot{\mathfrak{A}}$ such that $\dot{\pi}_1$ and $\dot{\theta}\dot{\pi}_2\dot{\theta}^{-1}$ generate the same full subgroups of $\operatorname{Aut}\dot{\mathfrak{A}}$.

By 556Kd, there is a $\phi \in \operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ such that $\phi c = c$ for every $c \in \mathfrak{C}$ and $\Vdash_{\mathbb{P}} \dot{\theta} = \dot{\phi}$, so that, setting $\pi_3 = \phi\pi_2\phi^{-1}$,

$$\Vdash_{\mathbb{P}} \dot{\pi}_1 \text{ and } \dot{\pi}_3 \text{ generate the same full subgroups of } \operatorname{Aut}\dot{\mathfrak{A}}$$

(using 556C(a-iii) and 556C(b-iv)).

(b) Since

$$\Vdash_{\mathbb{P}} \dot{\pi}_3 \text{ belongs to the full subgroup of } \operatorname{Aut}\dot{\mathfrak{A}} \text{ generated by } \dot{\pi}_1,$$

we can apply 381I(c-iv) in the forcing language to get

$$\Vdash_{\mathbb{P}} \inf_{n\in\mathbb{Z}} \operatorname{supp}(\dot{\pi}_1^n\dot{\pi}_3) = 0.$$

Now by 556Ic we know that

$$\Vdash_{\mathbb{P}} \operatorname{supp}(\dot{\pi}_1^n\dot{\pi}_3) = (\operatorname{supp}(\pi_1^n\pi_3))^{\bullet}$$

for every $n \in \mathbb{Z}$ (of course we need to check that $\Vdash_{\mathbb{P}} \dot{\pi}_1^n\dot{\pi}_3 = (\pi_1^n\pi_3)^{\bullet}$; but this is easily deduced from 556C(a-iii), an induction on n for $n \ge 0$, and 556C(b-iv)). So

$$\Vdash_{\mathbb{P}} (\inf_{n\in\mathbb{Z}} \operatorname{supp}(\pi_1^n\pi_3))^{\bullet} = \inf_{n\in\mathbb{Z}}(\operatorname{supp}(\pi_1^n\pi_3)^{\bullet})$$

(556E(b-ii))

$$= \inf_{n\in\mathbb{Z}} \operatorname{supp}(\pi_1^n\pi_3)^{\bullet} = \inf_{n\in\mathbb{Z}} \operatorname{supp}(\dot{\pi}_1^n\dot{\pi}_3) = 0.$$

By 556Da, as usual, $\inf_{n\in\mathbb{Z}} \operatorname{supp}(\pi_1^n\pi_3) = 0$; by 381I(c-iv), in the other direction and in the ordinary universe, π_3 belongs to the full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π_1. Similarly, π_1 belongs to the full subgroup generated by π_3, so π_1 and π_3 generate the same full subgroups, as required by (‡).

556O For the next result, I prepare the ground with a note on 'full local semigroups' as defined in §395.

Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, not $\{0\}$, and $\mathfrak{C}$ a regularly embedded subalgebra of $\mathfrak{A}$; let $\mathbb{P}$ be the forcing notion $\mathfrak{C}^+$, active downwards. Let $\dot{\mathfrak{A}}$ be the forcing name for $\mathfrak{A}$ over $\mathfrak{C}$, and for a ring homomorphism $\pi : \mathfrak{A} \to \mathfrak{A}$ such that $\pi c \subseteq c$ for every $c \in \mathfrak{C}$ let $\dot{\pi}$ be the forcing name for π over $\mathfrak{C}$. Let G be a subgroup of $\operatorname{Aut}\mathfrak{A}$ such that every point of $\mathfrak{C}$ is fixed by every member of G, and $\dot{G}$ the $\mathbb{P}$-name $\{(\dot{\pi}, 1) : \pi \in G\}$.

(a) $\Vdash_{\mathbb{P}} \dot{G}$ is a subgroup of $\operatorname{Aut}\dot{\mathfrak{A}}$.

(b) If $\phi : \mathfrak{A} \to \mathfrak{A}$ is a ring homomorphism such that $\phi c \subseteq c$ for every $c \in \mathfrak{C}$, and

$$\Vdash_{\mathbb{P}} \dot{\phi} \text{ belongs to the full local semigroup generated by } \dot{G},$$

then ϕ belongs to the full local semigroup generated by G.

proof (a)(i) If $p \in \mathfrak{C}^+$ and $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \dot{G}$, then for every q stronger than p there must be an r stronger than q and a $\pi \in G$ such that

$$r \Vdash_{\mathbb{P}} \dot{x} = \dot{\pi} \in \operatorname{Aut} \dot{\mathfrak{A}}.$$

(556C(b-iv)). As q is arbitrary, $p \Vdash_{\mathbb{P}} \dot{x} \in \operatorname{Aut} \dot{\mathfrak{A}}$; as p and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \dot{G} \subseteq \operatorname{Aut} \dot{\mathfrak{A}}.$$

(ii) Writing ι for the identity automorphism of $\mathfrak{A}$,

$$\Vdash_{\mathbb{P}} \dot{\iota} \in \dot{G} \text{ is the identity automorphism of } \dot{\mathfrak{A}}$$

(see part (b-iv) of the proof of 556C). If $p \in \mathfrak{C}^+$ and $\dot{x}$, $\dot{y}$ are $\mathbb{P}$-names such that $p \Vdash_{\mathbb{P}} \dot{x}, \dot{y} \in \dot{G}$, then for every q stronger than p there are r stronger than q and $\pi_1, \pi_2 \in G$ such that

$$r \Vdash_{\mathbb{P}} \dot{x} = \dot{\pi}_1, \, \dot{y} = \dot{\pi}_2, \, \dot{x} \cdot \dot{y} = \dot{\pi}_1 \dot{\pi}_2 = (\pi_1 \pi_2)^{\bullet} \in \dot{G}, \, \dot{x}^{-1} = (\dot{\pi}_1)^{-1} = (\pi_1^{-1})^{\bullet} \in \dot{G}$$

(556C(a-iii), 556C(b-iv)), because $\pi_1 \pi_2$ and π_1^{-1} belong to G. As q is arbitrary,

$$p \Vdash_{\mathbb{P}} \dot{x} \cdot \dot{y} \text{ and } \dot{x}^{-1} \text{ belong to } \dot{G};$$

as p, $\dot{x}$ and $\dot{y}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \dot{G} \text{ is a subgroup of } \operatorname{Aut} \dot{\mathfrak{A}}.$$

(b) Take any non-zero $a \in \mathfrak{A}$. Then there is a $p \in \mathfrak{C}^+$ such that $p \Vdash_{\mathbb{P}} \dot{a} \neq 0$ (556Da). Since

$$p \Vdash_{\mathbb{P}} \dot{\phi} \text{ belongs to the full local semigroup generated by } \dot{G},$$

there must be $\mathbb{P}$-names $\dot{x}$, $\dot{\theta}$ such that

$$p \Vdash_{\mathbb{P}} \dot{x} \in \dot{\mathfrak{A}} \setminus \{0\}, \, \dot{x} \mathrel{\dot{\subseteq}} \dot{a}, \, \dot{\theta} \in \dot{G}, \, \dot{\theta} y = \dot{\phi} y \text{ whenever } y \mathrel{\dot{\subseteq}} \dot{x}$$

(395B(a-ii)). Now there are a q stronger than p and $b \in \mathfrak{A}$, $\pi \in G$ such that

$$q \Vdash_{\mathbb{P}} \dot{b} = \dot{x}, \, \dot{\pi} = \dot{\theta}.$$

Since $q \Vdash_{\mathbb{P}} \dot{b} \neq 0$, $q \cap b \neq 0$. Suppose that $d \subseteq q \cap b$. Then

$$q \Vdash_{\mathbb{P}} \dot{d} \mathrel{\dot{\subseteq}} \dot{x}, \text{ so } (\pi d)^{\bullet} = \dot{\pi} \dot{d} = \dot{\theta} \dot{d} = \dot{\phi} \dot{d} = (\phi d)^{\bullet}$$

and

$$\pi d = q \cap \pi d$$

(see (a-i-α) of the proof of 556C)

$$= q \cap \phi d$$

(556Da, because $\phi d \subseteq \phi q \subseteq q$)

$$= \phi d.$$

Thus π and ϕ agree on the principal ideal $\mathfrak{A}_{q \cap b}$, while $q \cap b \subseteq a$ is non-zero. As a is arbitrary, ϕ belongs to the full local semigroup generated by G, by 395B(a-ii) in the other direction.

556P Kawada's theorem In the same way as in 556M and 556N, we have two versions of 395P:

(‡) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a fully non-paradoxical subgroup of $\operatorname{Aut} \mathfrak{A}$, with fixed-point subalgebra $\mathfrak{C}$, such that $\mathfrak{C}$ is a measurable algebra. Then there is a strictly positive G-invariant countably additive real-valued functional on $\mathfrak{A}$.

and

(†) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra such that $\operatorname{Aut} \mathfrak{A}$ has a subgroup G which is ergodic and fully non-paradoxical. Then there is a strictly positive G-invariant countably additive real-valued functional on $\mathfrak{A}$.

Once again, I claim that we can prove (‡) from (†).

proof (a) Take $\mathfrak{A}$, G and $\mathfrak{C}$ as in (‡). If $\mathfrak{A} = \{0\}$, the result is trivial; so let us suppose from now on that $\mathfrak{A} \neq \{0\}$. Let $\bar{\lambda}$ be a functional such that $(\mathfrak{C}, \bar{\lambda})$ is a probability algebra. Let $\mathbb{P}$ be the forcing notion $\mathfrak{C}^+$, active downwards, and let $\dot{\mathfrak{A}}$ be the forcing name for $\mathfrak{A}$ over $\mathfrak{C}$; for $\pi \in \operatorname{Aut}\mathfrak{A}$ let $\dot{\pi}$ be the forcing name for π over $\mathfrak{C}$. Let $\dot{G}$ be the $\mathbb{P}$-name $\{(\dot{\pi}, 1) : \pi \in G\}$.

(b) $\Vdash_{\mathbb{P}} \dot{G}$ is an ergodic subgroup of $\operatorname{Aut}\dot{\mathfrak{A}}$. **P** I noted in 556Oa that

$$\Vdash_{\mathbb{P}} \dot{G} \text{ is a subgroup of } \operatorname{Aut}\dot{\mathfrak{A}}.$$

For its ergodicity, copy the argument of 556C(b-v). Suppose that $p \in \mathfrak{C}^+$ and $\dot{x}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{x} \in \dot{\mathfrak{A}} \text{ and } \theta(\dot{x}) = \dot{x} \text{ for every } \theta \in \dot{G}.$$

For any q stronger than p there are an r stronger than q and an $a \in \mathfrak{A}$ such that $r \Vdash_{\mathbb{P}} \dot{x} = \dot{a}$. Take any $\pi \in G$. Then

$$r \Vdash_{\mathbb{P}} \dot{\pi} \in \dot{G},\ (\pi a)^{\bullet} = \dot{\pi}\dot{x} = \dot{x} = \dot{a},$$

so

$$\pi(r \cap a) = r \cap \pi a = r \cap a$$

(556Da). As π is arbitrary, $r \cap a \in \mathfrak{C}$. If $r \cap a \neq 0$, then $r \cap a \Vdash_{\mathbb{P}} \dot{a} = 1$; if $r \cap a = 0$, then $r \Vdash_{\mathbb{P}} \dot{a} = 0$. In either case, we have an s stronger than r such that $s \Vdash_{\mathbb{P}} \dot{x} \in \{0, 1\}$. As q is arbitrary, $p \Vdash_{\mathbb{P}} \dot{x} \in \{0, 1\}$; as p and $\dot{x}$ are arbitrary,

$\Vdash_{\mathbb{P}} \dot{G}$ has fixed-point subalgebra $\{0, 1\}$, so is ergodic, because $\dot{\mathfrak{A}}$ is Dedekind complete

(556Gb, 395Gf). **Q**

(c) $\Vdash_{\mathbb{P}} \dot{G}$ is fully non-paradoxical.
P (i) ? Otherwise,

$$\not\Vdash_{\mathbb{P}} \dot{G} \text{ satisfies condition (i) of 395D,}$$

and there must be a $p \in \mathfrak{C}^+$ and $\mathbb{P}$-names $\dot{\theta}$, $\dot{x}$ such that

$p \Vdash_{\mathbb{P}} \dot{x} \in \dot{\mathfrak{A}} \setminus \{1\}$, $\dot{\theta}$ is a Boolean homomorphism from $\dot{\mathfrak{A}}$ to the principal ideal generated by $\dot{x}$, and $\dot{\theta}$ belongs to the full local semigroup generated by $\dot{G}$.

In order to apply 556J and 556O as stated we need a $\mathbb{P}$-name $\dot{\theta}_1$ such that $\Vdash_{\mathbb{P}} \dot{\theta}_1$ is a ring homomorphism. If $p = 1$, take $\dot{\theta}_1 = \dot{\theta}$; otherwise, take $\dot{\theta}_1$ such that

$$p \Vdash_{\mathbb{P}} \dot{\theta}_1 = \dot{\theta}, \quad 1 \setminus p \Vdash_{\mathbb{P}} \dot{\theta}_1 \text{ is the identity automorphism.}$$

Then

$\Vdash_{\mathbb{P}} \dot{\theta}_1 : \dot{\mathfrak{A}} \to \dot{\mathfrak{A}}$ is a ring homomorphism belonging to the full local semigroup generated by $\dot{G}$.

(ii) By 556J there is a unique ring homomorphism $\phi : \mathfrak{A} \to \mathfrak{A}$ such that $\phi c \subseteq c$ for every $c \in \mathfrak{C}$ and $\Vdash_{\mathbb{P}} \dot{\theta}_1 = \dot{\phi}$. By 556Ob, ϕ belongs to the full local semi-group generated by G. Since G is fully non-paradoxical, $\phi 1 = 1$ and $\Vdash_{\mathbb{P}} \dot{\theta}_1 1 = \dot{\phi} 1 = 1$. But $p \Vdash_{\mathbb{P}} \dot{\theta}_1 1 = \dot{x} \neq 1$. **XQ**

(d) Applying (†) in the forcing language, we see that

$\Vdash_{\mathbb{P}}$ there is a strictly positive $\dot{G}$-invariant countably additive functional on $\dot{\mathfrak{A}}$, therefore there is a there is a strictly positive $\dot{G}$-invariant countably additive functional on $\dot{\mathfrak{A}}$ taking values in $[0, 1]$.

Let $\dot{\nu}$ be a $\mathbb{P}$-name such that

$\Vdash_{\mathbb{P}} \dot{\nu}$ is a strictly positive $\dot{G}$-invariant $[0, 1]$-valued countably additive functional on $\dot{\mathfrak{A}}$.

For each $a \in \mathfrak{A}$, $\Vdash_{\mathbb{P}} \dot{\nu}\dot{a} \in [0, 1]$, so there is a unique $u_a \in L^0(\mathfrak{C})^+$ such that

$$\Vdash_{\mathbb{P}} \dot{\nu}\dot{a} = \vec{u}_a$$

(5A3M once more), and $0 \leq u_a \leq \chi 1$. Set $\mu a = \int u_a d\bar{\lambda}$ for $a \in \mathfrak{A}$.

(e) μ is a strictly positive G-invariant countably additive functional on $\mathfrak{A}$.

P (i) If a, $b \in \mathfrak{A}$ are disjoint,

$$\Vdash_{\mathbb{P}} \dot a \mathbin{\dot\cap} \dot b = 0, \text{ so } \vec u_a + \vec u_b = \dot\nu \dot a + \dot\nu \dot b = \dot\nu(\dot a \mathbin{\dot\cup} \dot b) = \dot\nu(a \cup b)^{\bullet} = \vec u_{a\cup b}$$

(using 556Bb); it follows that $u_a + u_b = u_{a\cup b}$ and $\mu a + \mu b = \mu(a \cup b)$. Thus μ is additive.

(ii) If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing family in $\mathfrak{A}$ with supremum a, then, by 556Be and 556Eb,

$\Vdash_{\mathbb{P}} \langle \dot a_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\dot{\mathfrak{A}}$ with supremum $\dot a$, so $\langle \vec u_{a_n}\rangle_{n\in\mathbb{N}} = \langle \dot\nu \dot a_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $[0,1]$ with supremum $\vec u_a = \dot\nu \dot a$.

Now $\langle u_{a_n}\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $L^0(\mathfrak{C})$ with supremum u_a (5A3Ld), so $\mu a = \sup_{n\in\mathbb{N}} \mu a_n$. Thus μ is countably additive.

(iii) Because $u_a \ge 0$, $\mu a \ge 0$ for every $a \in \mathfrak{A}$. If $\mu a = 0$, then $u_a = 0$ so

$$\Vdash_{\mathbb{P}} \dot\nu \dot a = \vec u_a = 0, \text{ therefore } \dot a = 0, \text{ because } \dot\nu \text{ is strictly positive,}$$

and $a = 0$ (556Da). Thus μ is strictly positive.

(iv) Suppose that $\pi \in G$ and $a \in \mathfrak{A}$. Then

$$\Vdash_{\mathbb{P}} \dot\pi \in \dot G \text{ and } \dot\nu \text{ is } \dot G\text{-invariant, so } \vec u_{\pi a} = \dot\nu(\pi a)^{\bullet} = \dot\nu(\dot\pi \dot a) = \dot\nu \dot a = \vec u_a.$$

So $u_{\pi a} = u_a$ and $\mu(\pi a) = \mu a$. Thus μ is G-invariant. **Q**

Accordingly μ is a functional as required by (‡).

556Q For the final application of the methods of this section, I turn to a result of a quite different kind. Here the structure under consideration, the asymptotic density algebra $\mathfrak{Z}$, is off the main line of this treatise, but has some important measure-theoretic properties (see §491); and it turns out that there is a remarkable identification of its Dedekind completion (556S) which can be established by applying Maharam's theorem in a suitable forcing universe of the kind considered here. I start with a couple of easy lemmas, one just a restatement of ideas from Volume 3, and the other a straightforward property of a basic class of forcing notions.

Lemma (a) Let $\mathfrak{A}$ be a Boolean algebra and $\bar\mu : \mathfrak{A} \to [0,1]$ a strictly positive additive functional such that $\bar\mu 1 = 1$. Suppose that whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$, there is an $a \in \mathfrak{A}$ such that $a \subseteq a_n$ for every n and $\bar\mu a = \inf_{n\in\mathbb{N}} \bar\mu a_n$. Then $(\mathfrak{A}, \bar\mu)$ is a probability algebra.

(b) Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra. Suppose that $\kappa \ge \tau(\mathfrak{A})$ is an infinite cardinal and that $\langle e_\xi\rangle_{\xi<\kappa}$ is a family in $\mathfrak{A}$ such that $\bar\mu(\inf_{\xi\in K} e_\xi) = 2^{-\#(K)}$ for every finite $K \subseteq I$. Then $(\mathfrak{A}, \bar\mu)$ is isomorphic to the measure algebra $(\mathfrak{B}_\kappa, \bar\nu_\kappa)$ of the usual measure on $\{0,1\}^\kappa$.

proof (a) Let $A \subseteq \mathfrak{A}$ be a non-empty countable set. Let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a sequence running over A, and set $b_n = \inf_{i\le n} a_i$ for each n. There is a $b \in \mathfrak{A}$, a lower bound for $\{b_n : n \in \mathbb{N}\}$ and therefore for A, such that $\bar\mu b = \inf_{n\in\mathbb{N}} \bar\mu b_n$. If $c \in \mathfrak{A}$ is any lower bound for A, then $b \cup c \subseteq b_n$ for every n, so

$$\bar\mu b + \bar\mu(c \setminus b) = \bar\mu(b \cup c) \le \inf_{n\in\mathbb{N}} \bar\mu b_n = \bar\mu b,$$

and $\bar\mu(c \setminus b) = 0$; as $\bar\mu$ is strictly positive, $c \subseteq b$. Thus $b = \inf A$. As A is arbitrary, $\mathfrak{A}$ is Dedekind σ-complete. But this is the only clause missing from the definition of 'probability algebra'.

(b) By 331Ja, $\tau(\mathfrak{A}_d) \ge \kappa$ for every non-zero $d \in \mathfrak{A}$. So $\mathfrak{A}$ is Maharam-type-homogeneous, with Maharam type κ, and $(\mathfrak{A}, \bar\mu) \cong (\mathfrak{B}_\kappa, \bar\mu_\kappa)$ (331I).

556R Proposition Let $\mathbb{P}$ be a countably closed forcing notion. Then, for any set I, writing $(\mathfrak{B}_I, \bar\nu_I)$ for the measure algebra of the usual measure on $\{0,1\}^I$,

$$\Vdash_{\mathbb{P}} (\mathfrak{B}_{\check I}, \bar\nu_{\check I}) \cong (\check{\mathfrak{B}}_I, \check{\bar\nu}_I).$$

proof If I is finite, this is elementary (and does not rely on $\mathbb{P}$ being countably closed), so I shall suppose that I is infinite.

(a)

$\Vdash_{\mathbb{P}}$ if $\langle x_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\check{\mathfrak{B}}_I$, there is an $x \in \check{\mathfrak{B}}_I$ such that $x \mathrel{\check\subseteq} x_n$ for every n and $\check{\bar\nu}_I(x) = \inf_{n\in\mathbb{N}} \check{\bar\nu}_I(x_n)$.

P Let p be a condition of $\mathbb{P}$ and $\langle\dot{x}_n\rangle_{n\in\mathbb{N}}$ a sequence of $\mathbb{P}$-names such that

$$p \Vdash_{\mathbb{P}} \dot{x}_n \in \check{\mathfrak{B}}_I \text{ and } \dot{x}_{n+1} \check{\subseteq} \dot{x}_n$$

for every n. If q is stronger than p, we can choose $\langle q_n\rangle_{n\in\mathbb{N}}$, $\langle b_n\rangle_{n\in\mathbb{N}}$ inductively so that $q_0 = q$ and, for each n, q_{n+1} is stronger than q_n, $b_n \in \mathfrak{B}_I$ and $q_{n+1} \Vdash_{\mathbb{P}} \dot{x}_n = \check{b}_n$. In this case,

$$q_{n+1} \Vdash_{\mathbb{P}} \check{b}_{n+1} = \dot{x}_{n+1} \check{\subseteq} \dot{x}_n = \check{b}_n,$$

so $b_{n+1} \subseteq b_n$ for each n. Setting $b = \inf_{n\in\mathbb{N}} b_n$, $\bar{\nu}_I b = \inf_{n\in\mathbb{N}} \bar{\nu}_I b_n$. Also, because $\mathbb{P}$ is countably closed, there is a condition r stronger than any q_n. So

$$r \Vdash_{\mathbb{P}} \check{b} \check{\subseteq} \check{b}_n = \dot{x}_n \text{ for every } n \in \mathbb{N},\ \check{\bar{\nu}}_I(\check{b}) = \inf_{n\in\mathbb{N}} \check{\bar{\nu}}_I(\dot{x}_n).$$

As q is arbitrary,

$p \Vdash_{\mathbb{P}}$ there is an $x \in \check{\mathfrak{B}}_I$ such that $x \check{\subseteq} \dot{x}_n$ for every n and $\check{\bar{\nu}}_I(x) = \inf_{n\in\mathbb{N}} \check{\bar{\nu}}_I(\dot{x}_n)$.

As p and $\langle\dot{x}_n\rangle_{n\in\mathbb{N}}$ are arbitrary,

$\Vdash_{\mathbb{P}}$ if $\langle x_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\check{\mathfrak{B}}_I$, there is an $x \in \check{\mathfrak{B}}_I$ such that $x \check{\subseteq} x_n$ for every n and $\check{\bar{\nu}}_I(x) = \inf_{n\in\mathbb{N}} \check{\bar{\nu}}_I(x_n)$. **Q**

Since we certainly have

$\Vdash_{\mathbb{P}} \check{\mathfrak{B}}_I$ is a Boolean algebra and $\check{\bar{\nu}} : \check{\mathfrak{B}}_I \to [0,1]$ is a strictly positive additive functional such that $\check{\bar{\nu}}_I 1 = 1$,

556Qa, applied in the forcing universe, tells us that

$$\Vdash_{\mathbb{P}} (\check{\mathfrak{B}}, \check{\bar{\nu}}_I) \text{ is a probability algebra.}$$

(b) Let $\langle e_i\rangle_{i\in I}$ be the standard generating family in $\mathfrak{B}_I$. Then $\bar{\nu}_I(\inf_{i\in K} e_i) = 2^{-\#(K)}$ for every finite set $K \subseteq I$, so

$$\Vdash_{\mathbb{P}} \check{\bar{\nu}}_I(\inf_{i\in K} \check{e}_i) = 2^{-\#(K)} \text{ for every finite set } K \subseteq \check{I}.$$

Next, if $\mathfrak{D}$ is the subalgebra of $\mathfrak{B}_I$ generated by $\{e_i : i \in I\}$, then $\mathfrak{D}$ is dense in $\mathfrak{B}_I$ for the measure metric. Now

$\Vdash_{\mathbb{P}} \check{\mathfrak{D}}$ is the subalgebra of $\check{\mathfrak{B}}_I$ generated by $\{\check{e}_i : i \in \check{I}\}$ and $\check{\mathfrak{D}}$ is metrically dense in $\check{\mathfrak{B}}_I$, so $\tau(\check{\mathfrak{B}}_I) \le \#(\check{I})$. By 556Q, $(\check{\mathfrak{B}}_I, \check{\bar{\nu}}_I) \cong (\mathfrak{B}_{\#(\check{I})}, \bar{\nu}_{\#(\check{I})}) \cong (\mathfrak{B}_{\check{I}}, \bar{\nu}_{\check{I}})$,

as required.

556S Theorem (Farah 06) Let $\mathcal{Z}$ be the ideal of subsets of $\mathbb{N}$ with asymptotic density 0 and $\mathfrak{Z}$ the asymptotic density algebra $\mathcal{P}\mathbb{N}/\mathcal{Z}$. Then the Dedekind completion of $\mathfrak{Z}$ is isomorphic to the Dedekind completion of the free product $(\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}) \otimes \mathfrak{B}_{\mathfrak{c}}$.

proof (a) For $n \in \mathbb{N}$, set $I_n = \{i : 2^n \le i < 2^{n+1}\}$, so that $\langle I_n\rangle_{n\in\mathbb{N}}$ is a partition of $\mathbb{N}\setminus\{0\}$, and $\#(I_n) = 2^n$ for every $n \in \mathbb{N}$. Recall that

$$\mathcal{Z} = \{J : J \subseteq \mathbb{N},\ \lim_{n\to\infty} 2^{-n}\#(J \cap I_n) = 0\}$$

(491Ab). The notation of this proof will be slightly less appalling if I write b_J for $J^\bullet \in \mathfrak{Z}$ when $J \subseteq \mathbb{N}$ and c_K for $(\bigcup_{n\in K} I_n)^\bullet$ when $K \subseteq \mathbb{N}$.

Set

$$\mathfrak{C} = \{c_K : K \subseteq \mathbb{N}\}.$$

Because $K \mapsto c_K : \mathcal{P}\mathbb{N} \to \mathfrak{Z}$ is a Boolean homomorphism, $\mathfrak{C}$ is a subalgebra of $\mathfrak{Z}$. Now $\mathfrak{C} \cong \mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}$. **P** If $K \subseteq \mathbb{N}$, then

$$c_K = 0 \iff \bigcup_{n\in K} I_n \in \mathcal{Z} \iff K \text{ is finite.}$$

So the Boolean homomorphism $K \mapsto c_K$ induces a Boolean isomorphism $\pi : \mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega} \to \mathfrak{C}$ defined by setting $\pi(K^\bullet) = c_K$ for every $K \subseteq \mathbb{N}$. **Q**

For $p \in \mathfrak{C}^+$, set

$$\mathcal{F}_p = \{K : K \subseteq \mathbb{N},\ p \subseteq c_K\},$$

so that $\mathcal{F}_p$ is a filter on $\mathbb{N}$ containing every cofinite set. Note that if $p \subseteq q$ then $\mathcal{F}_p$ is finer than $\mathcal{F}_q$.

(b) $\mathfrak{C}$ is regularly embedded in $\mathfrak{Z}$. **P** Suppose that $A \subseteq \mathfrak{C}$ has infimum 0 in $\mathfrak{C}$, and that $b \in \mathfrak{Z}^+$. Let $J_0 \in \mathcal{P}\mathbb{N} \setminus \mathcal{Z}$ be such that $b = b_{J_0}$. Then $\limsup_{n\to\infty} 2^{-n}\#(J_0 \cap I_n) > 0$, so there is an $\epsilon > 0$ such that $K = \{n : \#(J_0 \cap I_n) \ge 2^n\epsilon\}$ is infinite. c_K cannot be a lower bound of A in $\mathfrak{C}$, so there is an $L \subseteq \mathbb{N}$ such that $c_L \in A$ and $c_K \not\subseteq c_L$, that is, $K \setminus L$ is infinite. Set $J = \bigcup_{n\in K\setminus L} J_0 \cap I_n$; then $\#(J \cap I_n) \ge 2^n\epsilon$ for infinitely many n, so $J \notin \mathcal{Z}$ and $0 \ne b_J \subseteq b$. On the other hand, $b_J \cap c_L = 0$. So $b \not\subseteq c_L$ and b is not a lower bound of A in $\mathfrak{Z}$. As b is arbitrary, A has infimum 0 in $\mathfrak{Z}$; as A is arbitrary, the embedding $\mathfrak{C} \subsetneqq \mathfrak{Z}$ is order-continuous (313L(b-v)), and $\mathfrak{C}$ is regularly embedded in $\mathfrak{Z}$. **Q**.

(c) Let $\mathbb{P}$ be the forcing notion $\mathfrak{C}^+$, active downwards. Then $\mathbb{P}$ is countably closed. **P** Let $\langle p_m\rangle_{m\in\mathbb{N}}$ be a non-increasing sequence in $\mathfrak{C}^+$. For each $m \in \mathbb{N}$, let $K_m \subseteq \mathbb{N}$ be such that $p_m = c_{K_m}$. Then $K_{m+1} \setminus K_m$ is finite for each m. Let $\langle n_k\rangle_{k\in\mathbb{N}}$ be a strictly increasing sequence such that $n_k \in K_m$ whenever $m \le k \in \mathbb{N}$, and set $K = \{n_k : k \in \mathbb{N}\}$; then c_K belongs to $\mathfrak{C}^+$, and $c_K \subseteq p_m$ for every $m \in \mathbb{N}$. **Q**

(d)(i) Let $\dot{\mathfrak{Z}}$ be the forcing name for $\mathfrak{Z}$ over $\mathfrak{C}$, and for $b \in \mathfrak{Z}$ let $\dot{b}$ be the forcing name for b over $\mathfrak{C}$. Let $\dot\nu$ be the $\mathbb{P}$-name

$$\{((\dot b_J, \check\alpha), p) : p \in \mathfrak{C}^+,\ J \subseteq \mathbb{N},\ \lim_{n\to\mathcal{F}_p} 2^{-n}\#(J \cap I_n) \text{ is defined and equal to } \alpha\}.$$

(ii) $\Vdash_{\mathbb{P}} \dot\nu$ is a function. **P** Suppose that (J_0, α_0, p_0) and $(J_1, \alpha_1, p_1) \in \mathcal{P}\mathbb{N} \times \mathbb{R} \times \mathfrak{C}^+$ are such that

$$\lim_{n\to\mathcal{F}_{p_0}} 2^{-n}\#(J_0 \cap I_n) = \alpha_0, \quad \lim_{n\to\mathcal{F}_{p_1}} 2^{-n}\#(J_1 \cap I_n) = \alpha_1,$$

and that $p \in \mathfrak{C}^+$, $p \subseteq p_0 \cap p_1$ and $p \Vdash_{\mathbb{P}} \dot b_{J_0} = \dot b_{J_1}$. Then $p \cap b_{J_0} = p \cap b_{J_1}$ (556Da). Express p as c_K, where $K \subseteq \mathbb{N}$; then $\bigcup_{n\in K} I_n \cap (J_0 \triangle J_1) \in \mathcal{Z}$, so $\lim_{n\in K, n\to\infty} 2^{-n}\#(I_n \cap (J_0 \triangle J_1)) = 0$, that is, $\lim_{n\to\mathcal{F}_p} 2^{-n}\#(I_n \cap (J_0 \triangle J_1)) = 0$. But this means that

$$\alpha_0 = \lim_{n\to\mathcal{F}_p} 2^{-n}\#(I_n \cap J_0) = \lim_{n\to\mathcal{F}_p} 2^{-n}\#(I_n \cap J_1) = \alpha_1,$$

and surely $p \Vdash_{\mathbb{P}} \check\alpha_0 = \check\alpha_1$. Thus the condition of 5A3Ea is satisfied and

$$\Vdash_{\mathbb{P}} \dot\nu \text{ is a function. } \mathbf{Q}$$

(iii) $\Vdash_{\mathbb{P}} \operatorname{dom}\dot\nu = \dot{\mathfrak{Z}}$. **P** Setting

$$\dot A = \{(\dot b_J, p) : p \in \mathfrak{C}^+,\ J \subseteq \mathbb{N},\ \lim_{n\to\mathcal{F}_p} 2^{-n}\#(J \cap I_n) \text{ is defined}\},$$

5A3E tells us that $\Vdash_{\mathbb{P}} \operatorname{dom}\dot\nu = \dot A$. Of course $\Vdash_{\mathbb{P}} \dot A \subseteq \dot{\mathfrak{Z}}$ just because $\Vdash_{\mathbb{P}} \dot b_J \in \dot{\mathfrak{Z}}$ for every $J \subseteq \mathbb{N}$. In the other direction, if $p \in \mathfrak{C}^+$ and $\dot x$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot x \in \dot{\mathfrak{Z}}$, there are a q stronger than p and a $b \in \mathfrak{Z}$ such that $q \Vdash_{\mathbb{P}} \dot x = \dot b$. Express q as c_K and b as b_J where $K \subseteq \mathbb{N}$ is infinite and $J \subseteq \mathbb{N}$. Then there is an infinite $L \subseteq K$ such that $\lim_{n\in L, n\to\infty} 2^{-n}\#(J \cap I_n)$ is defined, that is, $(\dot b, r) \in \dot A$, where $r = c_L$. So $r \Vdash_{\mathbb{P}} \dot x = \dot b \in \dot A$. As p and $\dot x$ are arbitrary,

$$\Vdash_{\mathbb{P}} \dot{\mathfrak{Z}} \subseteq \dot A \text{ and } \operatorname{dom}\dot\nu = \dot{\mathfrak{Z}}. \ \mathbf{Q}$$

(iv) Of course $\lim_{n\to\mathcal{F}_p} 2^{-n}\#(J \cap I_n)$, if it is defined, must belong to $[0,1]$. So

$$\Vdash_{\mathbb{P}} \dot\nu \text{ is a function from } \dot{\mathfrak{Z}} \text{ to } [0,1].$$

Next, $((\dot 1, \check 1), 1) \in \dot\nu$ (if you can work out how to interpret each 1 in this formula), so $\Vdash_{\mathbb{P}} \dot\nu\dot 1 = \check 1 = 1$.

(v) $\Vdash_{\mathbb{P}} \dot\nu$ is additive. **P** Suppose that $p \in \mathfrak{C}^+$ and that $\dot x_0$, $\dot x_1$ are $\mathbb{P}$-names such that

$$p \Vdash_{\mathbb{P}} \dot x_0,\ \dot x_1 \in \dot{\mathfrak{Z}} \text{ are disjoint.}$$

If p_1 is stronger than p there are q_0, q_0', q_1, $r \in \mathbb{P}$, J_0, $J_1 \subseteq \mathbb{N}$ and α_0, $\alpha_1 \in \mathbb{R}$ such that

$$((\dot b_{J_0}, \check\alpha_0), q_0) \in \dot\nu, \quad q_0' \text{ is stronger than both } q_0 \text{ and } p_1, \quad q_0' \Vdash_{\mathbb{P}} \dot b_{J_0} = \dot x_0,$$

$$((\dot b_{J_1}, \check\alpha_1), q_1) \in \dot\nu, \quad r \text{ is stronger than both } q_1 \text{ and } q_0', \quad r \Vdash_{\mathbb{P}} \dot b_{J_1} = \dot x_1.$$

As $r \Vdash_{\mathbb{P}} (b_{J_0} \cap b_{J_1})^\bullet = 0$, $r \cap b_{J_0} \cap b_{J_1} = 0$. Express r as c_K, where $K \in [\mathbb{N}]^\omega$. Then $J_0 \cap J_1 \cap \bigcup_{n\in K} I_n \in \mathcal{Z}$, so $\lim_{n\to\mathcal{F}_r} 2^{-n}\#(J_0 \cap J_1 \cap I_n) = 0$. At the same time,

$$\lim_{n\to\mathcal{F}_r} 2^{-n}\#(J_0\cap I_n)=\lim_{n\to\mathcal{F}_{q_0}} 2^{-n}\#(J_0\cap I_n)=\alpha_0,$$

$$\lim_{n\to\mathcal{F}_r} 2^{-n}\#(J_1\cap I_n)=\lim_{n\to\mathcal{F}_{q_1}} 2^{-n}\#(J_1\cap I_n)=\alpha_1,$$

so

$$\begin{aligned}\lim_{n\to\mathcal{F}_r} &2^{-n}\#((J_0\cup J_1)\cap I_n)\\ &=\lim_{n\to\mathcal{F}_r} 2^{-n}\#(J_0\cap I_n)+\lim_{n\to\mathcal{F}_r} 2^{-n}\#(J_1\cap I_n)-\lim_{n\to\mathcal{F}_r} 2^{-n}\#(J_0\cap J_1\cap I_n)\\ &=\alpha_0+\alpha_1\end{aligned}$$

and $(((\dot b_{J_0\cup J_1},(\alpha_0+\alpha_1)\check{}\,),r)\in\dot\nu$. Accordingly

$$r\Vdash_{\mathbb{P}}\dot\nu(\dot x_0\mathbin{\dot\cup}\dot x_1)=\dot\nu(\dot b_{J_0\cup J_1})=(\alpha_0+\alpha_1)\check{}=\check\alpha_0+\check\alpha_1=\dot\nu(\dot x_0)+\dot\nu(\dot x_1),$$

while $r\subseteq p_1$. As p_1 is arbitrary,

$$p\Vdash_{\mathbb{P}}\dot\nu(\dot x_0\mathbin{\dot\cup}\dot x_1)=\dot\nu(\dot x_0)+\dot\nu(\dot x_1).$$

As p, $\dot x_0$ and $\dot x_1$ are arbitrary,

$$\Vdash_{\mathbb{P}}\dot\nu\text{ is additive. }\mathbf{Q}$$

(vi) $\Vdash_{\mathbb{P}}\dot\nu$ is strictly positive. **P** Let $p\in\mathfrak{C}^+$ and a $\mathbb{P}$-name $\dot x$ be such that $p\Vdash_{\mathbb{P}}\dot x\in\dot{\mathfrak{Z}}$ and $\dot x\ne\dot 0$. If q is stronger than p there are a q' stronger than q and a $J\subseteq\mathbb{N}$ such that $q'\Vdash_{\mathbb{P}}\dot x=\dot b_J$. Express q' as c_K where $K\in[\mathbb{N}]^\omega$. As $q'\Vdash_{\mathbb{P}}\dot b_J\ne\dot 0$, $q'\cap b_J\ne 0$ and $\bigcup_{n\in K}I_n\cap J\notin\mathcal{Z}$. Accordingly $\limsup_{n\in K,n\to\infty}2^{-n}\#(I_n\cap J)>0$ and there is an infinite $L\subseteq K$ such that $\alpha=\lim_{n\in L,n\to\infty}2^{-n}\#(I_n\cap J)$ is defined and greater than 0. Set $r=c_L$; then $((\dot b_J,\check\alpha),r)\in\dot\nu$, so

$$r\Vdash_{\mathbb{P}}\dot\nu(\dot x)=\dot\nu(\dot b_J)=\check\alpha>0,$$

while r is stronger than q. As q is arbitrary, $p\Vdash_{\mathbb{P}}\dot\nu(\dot x)>0$; as p and $\dot x$ are arbitrary, $\Vdash_{\mathbb{P}}\dot\nu$ is strictly positive. **Q**

(vii)

$$\begin{aligned}\Vdash_{\mathbb{P}}\ &\text{if }\langle x_k\rangle_{k\in\mathbb{N}}\text{ is a non-increasing sequence in }\dot{\mathfrak{Z}}\\ &\qquad\text{there is an }x\in\dot{\mathfrak{Z}}\text{ such that }x\mathbin{\dot\subseteq}x_k\text{ for every }k\text{ and }\dot\nu(x)=\inf_{k\in\mathbb{N}}\dot\nu(x_k).\end{aligned}$$

P Let $p\in\mathfrak{C}^+$ and a sequence $\langle\dot x_k\rangle_{k\in\mathbb{N}}$ of $\mathbb{P}$-names be such that

$$p\Vdash_{\mathbb{P}}\langle\dot x_n\rangle_{n\in\mathbb{N}}\text{ is a non-increasing sequence in }\dot{\mathfrak{Z}}.$$

Let q be stronger than p. Then we can choose $\langle q_k\rangle_{k\in\mathbb{N}}$, $\langle q_k'\rangle_{k\in\mathbb{N}}$, $\langle q_k''\rangle_{k\in\mathbb{N}}$, $\langle J_k\rangle_{k\in\mathbb{N}}$ and $\langle\alpha_k\rangle_{k\in\mathbb{N}}$ inductively so that $q_0'=q$ and

$$q_k''\text{ is stronger than }q_k',\ J_k\subseteq\mathbb{N}\text{ and }q_k''\Vdash_{\mathbb{P}}\dot x_k=\dot b_{J_k},$$

$$q_k\text{ is stronger than }q_k'',\ \alpha_k\in[0,1],\ \lim_{n\to\mathcal{F}_{q_k}}2^{-n}\#(J_k\cap I_n)=\alpha_k$$

(compare (iii) above),

$$q_{k+1}'=q_k$$

for every $k\in\mathbb{N}$.

Because $\mathbb{P}$ is countably closed, there is an $r\in\mathfrak{C}^+$ stronger than every q_k. In this case, $((\dot b_{J_k},\check\alpha_k),r)\in\dot\nu$ for every k, so

$$r\Vdash_{\mathbb{P}}\inf_{k\in\mathbb{N}}\dot\nu(\dot x_k)=\inf_{k\in\mathbb{N}}\check\alpha_k=\check\alpha$$

where $\alpha=\inf_{k\in\mathbb{N}}\alpha_k$. Express r as c_K where $K\subseteq\mathbb{N}$ is infinite. For each $k\in\mathbb{N}$,

$$r\Vdash_{\mathbb{P}}\dot b_{J_{k+1}}=\dot x_{k+1}\mathbin{\dot\subseteq}\dot x_k=\dot b_{J_k},$$

so $r\cap b_{J_{k+1}}\setminus b_{J_k}=0$ and

$$\lim_{n\in K,n\to\infty} 2^{-n}\#(I_n \cap J_{k+1} \setminus J_k) = 0,$$

while

$$\lim_{n\in K,n\to\infty} 2^{-n}\#(I_n \cap J_k) = \lim_{n\to\mathcal{F}_{q_k}} 2^{-n}\#(I_n \cap J_k) = \alpha_k \ge \alpha.$$

We can therefore find a strictly increasing sequence $\langle n_k\rangle_{k\in\mathbb{N}}$ in K such that

$$2^{-n_k}\#(I_{n_k} \cap J'_k) \ge \alpha - 2^{-k}$$

for every k, where $J'_k = \bigcap_{j\le k} J_j$. Set $r' = (\bigcup_{k\in\mathbb{N}} I_{n_k})^\bullet$ and $J = \bigcup_{k\in\mathbb{N}} I_{n_k} \cap J'_k$. Then $J \setminus J_k$ is finite, so $r' \Vdash_{\mathbb{P}} \dot{b}_J \subseteq \dot{x}_k$ for every k. Also $((\dot{b}_J, \check{\alpha}), r') \in \dot{\nu}$, so

$$r' \Vdash_{\mathbb{P}} \dot{\nu}(\dot{b}_J) = \check{\alpha} = \inf_{k\in\mathbb{N}} \dot{\nu}(\dot{x}_k).$$

Thus

$$r' \Vdash_{\mathbb{P}} \text{ there is a lower bound } x \text{ for } \{\dot{x}_k : k \in \mathbb{N}\} \text{ such that } \dot{\nu}(x) = \inf_{k\in\mathbb{N}} \dot{\nu}(\dot{x}_k).$$

As q is arbitrary,

$$p \Vdash_{\mathbb{P}} \text{ there is a lower bound } x \text{ for } \{\dot{x}_k : k \in \mathbb{N}\} \text{ such that } \dot{\nu}(x) = \inf_{k\in\mathbb{N}} \dot{\nu}(\dot{x}_k).$$

As p and $\langle \dot{x}_k\rangle_{k\in\mathbb{N}}$ are arbitrary,

$\Vdash_{\mathbb{P}}$ if $\langle x_k\rangle_{k\in\mathbb{N}}$ is a non-increasing sequence in $\dot{\mathfrak{Z}}$ there is an $x \in \dot{\mathfrak{Z}}$ such that $x \subseteq x_k$ for every k and $\dot{\nu}(x) = \inf_{k\in\mathbb{N}} \dot{\nu}(x_k)$. **Q**

(viii)

$\Vdash_{\mathbb{P}}$ there is a family $\langle x_L\rangle_{L\in\mathcal{P}\mathbb{N}}$ in $\dot{\mathfrak{A}}$ such that $\dot{\nu}(\inf_{L\in\mathcal{L}} x_L) = 2^{-\#(\mathcal{L})}$ for every finite set $\mathcal{L} \subseteq \mathcal{P}\mathbb{N}$.

P Let $\langle M_L\rangle_{L\in\mathcal{P}\mathbb{N}}$ be an almost disjoint family of infinite subsets of $\mathbb{N}$ (5A1Ga). For each $n \in \mathbb{N}$, let $\langle K_{ni}\rangle_{i<n}$ be a family of subsets of I_n such that $\#(\bigcap_{i\in J} K_{ni}) = 2^{n-\#(J)}$ for every non-empty set $J \subseteq n$; such a family exists because $\#(I_n) = 2^n$. For $L \subseteq \mathbb{N}$, set

$$A_L = \bigcup_{n\in\mathbb{N},n>\min M_L} K_{n,\max(n\cap M_L)}, \quad a_L = b_{A_L} \in \mathfrak{Z}.$$

If $\mathcal{L} \subseteq \mathcal{P}\mathbb{N}$ is finite and not empty, let $n_0 \in \mathbb{N}$ be such that $M_L \cap M_{L'} \subseteq n_0$ whenever L, $L' \in \mathcal{L}$ are distinct, and $n_1 \ge n_0$ such that $M_L \cap n_1 \setminus n_0 \ne \emptyset$ for every $L \in \mathcal{L}$. Then $\max(n \cap M_L) \ne \max(n \cap M_{L'})$ whenever L, $L' \in \mathcal{L}$ are distinct and $n \ge n_1$. So

$$\begin{aligned}\lim_{n\to\infty} 2^{-n}\#(I_n \cap \bigcap_{L\in\mathcal{L}} A_L) &= \lim_{n\to\infty} 2^{-n}\#(I_n \cap \bigcap_{L\in\mathcal{L}} K_{n,\max(n\cap M_L)})\\ &= \lim_{n\to\infty} 2^{-n}2^{n-\#(\mathcal{L})} = 2^{-\#(\mathcal{L})}.\end{aligned}$$

Of course the same formula is valid when $\mathcal{L} = \emptyset$.

It follows that

$$\Vdash_{\mathbb{P}} \dot{\nu}(\inf_{L\in\check{\mathcal{L}}} \dot{a}_L) = 2^{-\#(\check{\mathcal{L}})}$$

for every finite $\mathcal{L} \subseteq \mathcal{P}\mathbb{N}$. Accordingly

$$\Vdash_{\mathbb{P}} \dot{\nu}(\inf_{L\in\mathcal{L}} \dot{a}_L) = 2^{-\#(\mathcal{L})} \text{ for every finite set } \mathcal{L} \subseteq (\mathcal{P}\mathbb{N})\check{}.$$

But we know also that

$$\Vdash_{\mathbb{P}} \mathcal{P}\mathbb{N} = (\mathcal{P}\mathbb{N})\check{}$$

(5A3Qb). So the family $\langle \dot{a}_L\rangle_{L\in\mathcal{P}\mathbb{N}}$ of $\mathbb{P}$-names, when interpreted as a $\mathbb{P}$-name $\langle \dot{a}_L\rangle_{L\in(\mathcal{P}\mathbb{N})\check{}}$ as in 5A3Fb, can also be regarded as a $\mathbb{P}$-name for a function defined on the whole power set of the set of natural numbers. If we do this, we get

$$\Vdash_{\mathbb{P}} \dot{\nu}(\inf_{L\in\mathcal{L}} \dot{a}_L) = 2^{-\#(\mathcal{L})} \text{ for every finite set } \mathcal{L} \subseteq \mathcal{P}\mathbb{N},$$

witnessing the truth of the result we seek. **Q**

(ix) $\Vdash_{\mathbb{P}} \#(\dot{\mathfrak{A}}) \le \mathfrak{c}$. **P** Since

$$\dot{\mathfrak{Z}} = \{(\dot{a}, 1) : a \in \mathfrak{Z}\} = \{(\dot{b}_J, 1) : J \in \mathcal{P}\mathbb{N}\}$$

(556Ab), we get

$$\Vdash_{\mathbb{P}} \dot{\mathfrak{Z}} = \{\dot{b}_J : J \in (\mathcal{P}\mathbb{N})\check{}\}, \text{ so } \#(\dot{\mathfrak{Z}}) \le \#((\mathcal{P}\mathbb{N})\check{}) \le \#(\mathcal{P}\mathbb{N}) = \mathfrak{c}. \ \mathbf{Q}$$

(e) Assembling the facts in (d), we see that

$$\Vdash_{\mathbb{P}} (\dot{\mathfrak{Z}}, \dot{\nu}) \text{ satisfies the conditions of 556Q with } \kappa = \mathfrak{c}, \text{ so } \dot{\mathfrak{Z}} \cong \mathfrak{B}_{\mathfrak{c}}\ .$$

But we also have

$$\Vdash_{\mathbb{P}} \mathfrak{B}_{\mathfrak{c}} \text{ is isomorphic to } \mathfrak{B}_{\mathcal{P}\mathbb{N}} = \mathfrak{B}_{(\mathcal{P}\mathbb{N})\check{}} \cong (\mathfrak{B}_{\mathcal{P}\mathbb{N}})\check{}$$

by 556R. As $\mathfrak{C}$ is regularly embedded in $\mathfrak{Z}$, we can apply 556Fc to see that $\widehat{\mathfrak{Z}}$ is isomorphic to the Dedekind completed free product $\mathfrak{C}\widehat{\otimes}\mathfrak{B}_{\mathcal{P}\mathbb{N}}$ and therefore to $(\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega})\widehat{\otimes}\mathfrak{B}_{\mathfrak{c}}$, by (a).

This completes the proof.

556X Basic exercises (a) Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$, and $\mathfrak{C}$ a Boolean subalgebra of $\mathfrak{A}$ which is not regularly embedded; let $\mathbb{P}$ be the forcing notion $\mathfrak{C}^+$, active downwards, and let $\dot{\mathfrak{A}}$ be the forcing name for $\mathfrak{A}$ over $\mathfrak{C}$. Show that there is an $a \in \mathfrak{A} \setminus \{0\}$ such that $\Vdash_{\mathbb{P}} \dot{a} = 0$, where $\dot{a}$ is the forcing name for a over $\mathfrak{C}$.

(b) Let $\mathbb{P}$ be a countably closed forcing notion. (i) Show that $\Vdash_{\mathbb{P}} \omega_1 = \check{\omega}_1$. (ii) Show that $\Vdash_{\mathbb{P}} [\check{I}]^{\le\omega} = ([I]^{\le\omega})\check{}$ for every set I. (iii) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Show that $\Vdash_{\mathbb{P}} \check{\mathfrak{A}}$ is Dedekind σ-complete. (iv) Let (X, ρ) be a complete metric space. Show that $\Vdash_{\mathbb{P}} (\check{X}, \check{\rho})$ is a complete metric space.

(c) Show that the Dedekind completion $\widehat{\mathfrak{Z}}$ of the asymptotic density algebra is a homogeneous Boolean algebra. (*Hint*: 316Q, 316P.)

556Y Further exercises (a) Let $\mathbb{P}$ be a forcing notion, and $\dot{\mathbb{Q}}_1$, $\dot{\mathbb{Q}}_2$ two $\mathbb{P}$-names for forcing notions such that

$$\Vdash_{\mathbb{P}} \mathrm{RO}(\dot{\mathbb{Q}}_1) \cong \mathrm{RO}(\dot{\mathbb{Q}}_2).$$

Show that $\mathrm{RO}(\mathbb{P} * \dot{\mathbb{Q}}_1) \cong \mathrm{RO}(\mathbb{P} * \dot{\mathbb{Q}}_2)$.

(b) Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing notions. Show that $\mathrm{RO}(\mathbb{P} * \check{\mathbb{Q}}) \cong \mathrm{RO}(\mathbb{P})\widehat{\otimes}\mathrm{RO}(\mathbb{Q})$.

(c) Give an example of a Dedekind σ-complete Boolean algebra $\mathfrak{A}$ with an order-closed subalgebra $\mathfrak{C}$ such that

$$\Vdash_{\mathbb{P}} \dot{\mathfrak{A}} \text{ is not Dedekind } \sigma\text{-complete},$$

where $\mathbb{P}$ is the forcing notion $\mathfrak{C}^+$, active downwards, and $\dot{\mathfrak{A}}$ is the forcing name for $\mathfrak{A}$ over $\mathfrak{C}$.

(d) Show that the argument of 556Q is sufficient to take us from (†) there to Theorem 395N, as well as to 395P.

(e) Show that if the Proper Forcing Axiom is true then the asymptotic density algebra $\mathfrak{Z}$ is not homogeneous. (*Hint*: 5A6H.)

(f) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, $\mathfrak{C}$ a closed subalgebra and $\mathbb{P}$ the forcing notion $\mathfrak{C}^+$ active downwards. Set $q(t) = -t \ln t$ for $t > 0$, 0 for $t \le 0$ (cf. 385A). Let A be a finite partition of unity in $\mathfrak{A}$, and $\dot{A}$ the $\mathbb{P}$-name $\{(\dot{a}, 1) : a \in A\}$. (i) Confirm that the definition of q can be interpreted in the forcing universe $V^{\mathbb{P}}$. (ii) Show that if $u \in L^0(\mathfrak{C})$ then $\Vdash_{\mathbb{P}} (\bar{q}(u))^{\rightarrow} = q(\vec{u})$. (iii) Set $v = \sum_{a \in A} \bar{q}(P\chi a)$ where P is the conditional expectation associated with $\mathfrak{C}$ (cf. 385D). Show that

$$\Vdash_{\mathbb{P}} \dot{A} \text{ is a partition of unity in } (\dot{\mathfrak{A}}, \dot{\bar{\mu}}) \text{ and its entropy is } \vec{v}.$$

(iv) Re-examine Lemma 385Ga in the light of this.

556 Notes and comments I did not positively instruct you to do so in the introduction to this section, but I expect that most readers will have passed rather quickly over the nineteen $\dot{}$-infested pages up to 556L, and looked at the target theorems from 556M on. In each of the first three we have a pair (‡), (†) of propositions, (†) being the special case of (‡) in which an algebra T or $\mathfrak{C}$ is trivial. If, as I hope, you are already acquainted with at least one of the assertions (‡), you will know that it can be proved by essentially the same methods as the corresponding (†), but with some non-trivial technical changes. These technical changes, already incorporated in the proofs of 388L/556N and 395P/556P in Volume 3, and indicated in §458 for 458Ye/556M, certainly do not amount to nineteen pages of mathematics in total; moreover, they explore ideas which are of independent interest. So I cannot on this evidence claim that the approach gives quick proofs of otherwise inaccessible results.

What I do claim is that the general method gives a way of looking at a recurrent phenomenon. Throughout the theory of measure-preserving transformations, ergodic transformations have a special place; and one comes to expect that once one has answered a question for ergodic transformations, the general case will be easy to determine. Similarly, every theorem about independent random variables ought to have a form applying to relatively independent variables. Indeed there are standard techniques for developing such extensions, based on disintegrations, as in §§458-459. What I have tried to do here is to develop a completely different approach, and in the process to indicate a new aspect of the theory of forcing. I note that the method demands preliminary translations into the language of measure algebras, which suits my prejudices as already expressed at length in Volume 3.

The message is that everything works. There are no royal roads in mathematics, and to use this one you will have to master some non-trivial machinery. But perhaps just knowing that a machine exists will give you the confidence to attack similar problems in your own way. I offer an example in 556Yf. Note that this depends on the fact that the ordinary functions of elementary calculus have definitions which can be interpreted in any forcing universe.

The great bulk of the work of this section consists of routine checks that natural formulae are in fact valid. You will see that some simple ideas recur repeatedly, but the details demand a certain amount of attention. At the very beginning, in finding a forcing name $\dot{a}$ for an element of a Boolean algebra, we have to take care that we are exactly following our preferred formulation of what a name 'is'. (If my preferred formulation is not yours, you have some work to do, but it should not be difficult, and might be enlightening.) It is not surprising that regularly embedded subalgebras have a special status (556D); it is worth taking a moment to think about why it matters so much (556Xa). In 556H, I do not think it is obvious that $\mathfrak{A}$ must be Dedekind complete, rather than just Dedekind σ-complete, to make the ideas work in the straightforward way that they do (556Yc). When we come to measure algebras (556K), we need to be sure that we have a description of forcing names for real numbers which is compatible with the apparatus here. Again and again, we have sentences with clauses in both the forcing language and in ordinary language, and we must keep the pieces properly segregated in our minds.

The last fifth of the section (556Q-556S) is quite hard work for the result we get, but I think it is particularly instructive, in that it cannot be regarded as a technical extension of a simpler and more important result. It is a good example of a theorem proved by a method unavoidably dependent on the Forcing Theorem (5A3D), and for which it is not at all clear that a proof avoiding forcing can be made manageably simple. Such a proof must exist, but the obvious route to it involves teasing out the requisite parts of the proof of Maharam's theorem, and translating them into properties of the set

$$\{(J,\alpha,K) : K\in[\mathbb{N}]^{\omega},\, J\subseteq\mathbb{N},\, \lim_{n\in K,n\to\infty}2^{-n}\#(J\cap I_n)=\alpha\}$$

as in part (d) of the proof of 556S, but going very much farther. My own experience is that facing up to such challenges is often profitable, but for the moment I am happy to present an adaptation of Farah's original proof.

An easy corollary of Theorem 556S is that $\widehat{\mathfrak{Z}}$ is homogeneous (556Xc). This is striking in view of the fact that $\mathfrak{Z}$ itself may or may not be homogeneous. If the continuum hypothesis is true, then $\mathfrak{Z}$ is indeed homogeneous (Farah 03); but if the Proper Forcing Axiom is true, then $\mathfrak{Z}$ is *not* homogeneous (556Ye), even though its completion is.

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