Version of 23.10.14/17.9.16

Chapter 54

Real-valued-measurable cardinals

Of the many questions in measure theory which involve non-trivial set theory, perhaps the first to have been recognised as fundamental is what I call the 'Banach-Ulam problem': is there a non-trivial measure space in which every set is measurable? In various forms, this question has arisen repeatedly in earlier volumes of this treatise (232Hc, 363S, 438A). The time has now come for an account of the developments of the last fifty years.

The measure theory of this chapter will begin in §543; the first two sections deal with generalizations to wider contexts. If ν is a probability measure with domain $\mathcal{P}X$, its null ideal is ω_1-additive and ω_1-saturated in $\mathcal{P}X$. In §541 I look at ideals $\mathcal{I} \lhd \mathcal{P}X$ such that $\mathcal{I}$ is simultaneously κ-additive and κ-saturated for some κ; this is already enough to lead us to a version of the Keisler-Tarski theorem on normal ideals (541J), a great strengthening of Ulam's theorem on inaccessibility of real-valued-measurable cardinals (541Lc), a form of Ulam's dichotomy (541P), and some very striking infinitary combinatorics (541Q-541S). In §542 I specialize to the case $\kappa = \omega_1$, still without calling on the special properties of null ideals, with more combinatorics (542E, 542I).

I have said many times in the course of this treatise that almost the first thing to ask about any measure is, what does its measure algebra look like? For an atomless probability measure with domain $\mathcal{P}X$, the Gitik-Shelah theorem (543E-543F) gives a great deal of information, associated with a tantalizing problem (543Z). §544 is devoted to the measure-theoretic consequences of assuming that there is some atomlessly-measurable cardinal, with results on repeated integration (544C, 544I, 544J), the null ideal of a normal witnessing probability (544E-544F) and regressive functions (544M).

I do not discuss consistency questions in this chapter (I will touch on some of them in Chapter 55). The ideas of §§541-544 would be in danger of becoming irrelevant if it turned out that there can be no two-valued-measurable cardinal. I have no real qualms about this. One of my reasons for confidence is the fact that very much stronger assumptions have been investigated without any hint of catastrophe. Two of these, the 'product measure extension axiom' and the 'normal measure axiom' are mentioned in §545.

One way of looking at the Gitik-Shelah theorem is to say that if X is a set and $\mathcal{I}$ is a proper σ-ideal of subsets of X, then $\mathcal{P}X/\mathcal{I}$ cannot be an atomless measurable algebra of small Maharam type. We can ask whether there are further theorems of this kind provable in ZFC. Two such results are in §547: the 'Gitik-Shelah theorem for category' (547F-547G), showing that $\mathcal{P}X/\mathcal{I}$ cannot be isomorphic to $\mathrm{RO}(\mathbb{R})$, and 547R, showing that 'σ-measurable' algebras of moderate complexity also cannot appear as power set σ-quotient algebras. This leads us to a striking fact about free sets for relations with countable equivalence classes (548C) and thence to disjoint refinements of sequences of sets (548E).

Version of 10.12.12

541 Saturated ideals

If ν is a totally finite measure with domain $\mathcal{P}X$ and null ideal $\mathcal{N}(\nu)$, then its measure algebra $\mathcal{P}X/\mathcal{N}(\nu)$ is ccc, that is to say, $\mathrm{sat}(\mathcal{P}X/\mathcal{N}(\nu)) \le \omega_1$; while the additivity of $\mathcal{N}(\nu)$ is at least ω_1. It turns out that an ideal $\mathcal{I}$ of $\mathcal{P}X$ such that $\mathrm{sat}(\mathcal{P}X/\mathcal{I}) \le \mathrm{add}\,\mathcal{I}$ is either trivial or extraordinary. In this section I present a little of the theory of such ideals. To begin with, the quotient algebra has to be Dedekind complete (541B). Further elementary ideas are in 541C (based on a method already used in §525) and 541D-541E. In a less expected direction, we have a useful fact concerning transversal numbers $\mathrm{Tr}_{\mathcal{I}}(X;Y)$ (541F).

The most remarkable properties of saturated ideals arise because of their connexions with 'normal' ideals (541G). These ideals share the properties of non-stationary ideals (541H-541I). If $\mathcal{I}$ is an $(\mathrm{add}\,\mathcal{I})$-saturated ideal of $\mathcal{P}X$, we have corresponding normal ideals on $\operatorname{add}\mathcal{I}$ (541J). Now there can be a κ-saturated normal ideal on κ only if there is a great complexity of cardinals below κ (541L).

The original expression of these ideas (Keisler & Tarski 64) concerned 'two-valued-measurable' cardinals, on which we have normal ultrafilters (541M). The dichotomy of Ulam 1930 reappears in the context of κ-saturated normal ideals (541P). For κ-saturated ideals, 'normality' implies some far-reaching extensions (541Q). Finally, I include a technical lemma concerning the covering numbers $\mathrm{cov}_{\mathrm{Sh}}(\alpha,\beta,\gamma,\delta)$ (541S).

541A Definition If $\mathfrak{A}$ is a Boolean algebra, I is an ideal of $\mathfrak{A}$ and κ is a cardinal, I will say that I is **κ-saturated** in $\mathfrak{A}$ if $\kappa \ge \mathrm{sat}(\mathfrak{A}/I)$

541B Proposition Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and I an ideal of $\mathfrak{A}$ which is $(\operatorname{add} I)^+$-saturated in $\mathfrak{A}$. Then the quotient algebra $\mathfrak{A}/I$ is Dedekind complete.

541C Proposition Let X be a set, κ a regular infinite cardinal, Σ an algebra of subsets of X such that $\bigcup\mathcal{E} \in \Sigma$ whenever $\mathcal{E} \subseteq \Sigma$ and $\#(\mathcal{E}) < \kappa$, and $\mathcal{I}$ a κ-saturated κ-additive ideal of Σ.

(a) If $\mathcal{E} \subseteq \Sigma$ there is an $\mathcal{E}' \in [\mathcal{E}]^{<\kappa}$ such that $E \setminus \bigcup\mathcal{E}' \in \mathcal{I}$ for every $E \in \mathcal{E}$.

(b) If $\langle E_\xi\rangle_{\xi<\kappa}$ is any family in $\Sigma \setminus \mathcal{I}$, and $\theta < \kappa$ is a cardinal, then $\{x : x \in X,\ \#(\{\xi : x \in E_\xi\}) \ge \theta\}$ includes a member of $\Sigma \setminus \mathcal{I}$.

(c) Suppose that no element of $\Sigma \setminus \mathcal{I}$ can be covered by κ members of $\mathcal{I}$. Then κ is a precaliber of $\Sigma/\mathcal{I}$.

541D Lemma Let X be a set, $\mathcal{I}$ an ideal of $\mathcal{P}X$, Y a set of cardinal less than $\operatorname{add}\mathcal{I}$ and κ a cardinal such that $\mathcal{I}$ is $(\operatorname{cf}\kappa)$-saturated in $\mathcal{P}X$. Then for any function $f : X \to [Y]^{<\kappa}$ there is an $M \in [Y]^{<\kappa}$ such that $\{x : f(x) \not\subseteq M\} \in \mathcal{I}$.

541E Corollary Let X be a set, $\mathcal{I}$ an ideal of $\mathcal{P}X$, Y a set of cardinal less than $\operatorname{add}\mathcal{I}$ and κ a cardinal such that $\mathcal{I}$ is $(\operatorname{cf}\kappa)$-saturated in $\mathcal{P}X$. Then for any function $g : X \to Y$ there is an $M \in [Y]^{<\kappa}$ such that $g^{-1}[Y \setminus M] \in \mathcal{I}$.

541F Lemma Let X and Y be sets, κ a regular uncountable cardinal, and $\mathcal{I}$ a proper κ-saturated κ-additive ideal of subsets of X. Then $\mathrm{Tr}_{\mathcal{I}}(X;Y)$ is attained, in the sense that there is a set $G \subseteq Y^X$ such that $\#(G) = \mathrm{Tr}_{\mathcal{I}}(X;Y)$ and $\{x : x \in X,\ g(x) = g'(x)\} \in \mathcal{I}$ for all distinct g, $g' \in G$.

541G Definition Let κ be a regular uncountable cardinal. A **normal ideal** on κ is a proper ideal $\mathcal{I}$ of $\mathcal{P}\kappa$, including $[\kappa]^{<\kappa}$, such that

$$\{\xi : \xi < \kappa,\ \xi \in \textstyle\bigcup_{\eta<\xi} I_\eta\}$$

belongs to $\mathcal{I}$ for every family $\langle I_\xi\rangle_{\xi<\kappa}$ in $\mathcal{I}$.

541H Proposition Let κ be a regular uncountable cardinal and $\mathcal{I}$ a proper ideal of $\mathcal{P}\kappa$ including $[\kappa]^{<\kappa}$. Then the following are equiveridical:

(i) $\mathcal{I}$ is normal;

(ii) $\mathcal{I}$ is κ-additive and whenever $S \in \mathcal{P}\kappa \setminus \mathcal{I}$ and $f : S \to \kappa$ is regressive, then there is an $\alpha < \kappa$ such that $\{\xi : \xi \in S,\ f(\xi) \le \alpha\}$ is not in $\mathcal{I}$;

(iii) whenever $S \in \mathcal{P}\kappa \setminus \mathcal{I}$ and $f : S \to \kappa$ is regressive, then there is a $\beta < \kappa$ such that $\{\xi : \xi \in S,\ f(\xi) = \beta\}$ is not in $\mathcal{I}$.

541I Lemma Let κ be a regular uncountable cardinal.

(a) The family of non-stationary subsets of κ is a normal ideal on κ, and is included in every normal ideal on κ.

(b) If $\mathcal{I}$ is a normal ideal on κ, and $\langle I_K\rangle_{K\in[\kappa]^{<\omega}}$ is any family in $\mathcal{I}$, then $\{\xi : \xi < \kappa,\ \xi \in \bigcup_{K\in[\xi]^{<\omega}} I_K\}$ belongs to $\mathcal{I}$.

541J Theorem Let X be a set and $\mathcal{J}$ an ideal of subsets of X. Suppose that $\operatorname{add}\mathcal{J} = \kappa > \omega$ and that $\mathcal{J}$ is λ-saturated in $\mathcal{P}X$, where $\lambda \le \kappa$. Then there are $Y \subseteq X$ and $g : Y \to \kappa$ such that $\{B : B \subseteq \kappa, g^{-1}[B] \in \mathcal{J}\}$ is a λ-saturated normal ideal on κ.

541K Lemma Let κ be a regular uncountable cardinal and $\mathcal{I}$ a normal ideal on κ which is κ'-saturated in $\mathcal{P}\kappa$, where $\kappa' \le \kappa$.

(a) If $S \in \mathcal{P}\kappa \setminus \mathcal{I}$ and $f : S \to \kappa$ is regressive, then there is a set $A \in [\kappa]^{<\kappa'}$ such that $S \setminus f^{-1}[A] \in \mathcal{I}$; there is an $\alpha < \kappa$ such that $\{\xi : \xi \in S, f(\xi) \ge \alpha\} \in \mathcal{I}$.

(b) If $\lambda < \kappa$, then $\{\xi : \xi < \kappa, \operatorname{cf}\xi \le \lambda\} \in \mathcal{I}$.

(c) If for each $\xi < \kappa$ we are given a relatively closed set $C_\xi \subseteq \xi$ which is cofinal with ξ, then

$$C = \{\alpha : \alpha < \kappa, \{\xi : \alpha \notin C_\xi\} \in \mathcal{I}\}$$

is a cofinal closed set in κ.

541L Theorem Let κ be an uncountable cardinal such that there is a proper κ-saturated κ-additive ideal of $\mathcal{P}\kappa$ containing singletons.

(a) There is a κ-saturated normal ideal on κ.

(b) κ is weakly inaccessible.

(c) The set of weakly inaccessible cardinals less than κ is stationary in κ.

541M Definition (a) A regular uncountable cardinal κ is **two-valued-measurable** if there is a proper κ-additive 2-saturated ideal of $\mathcal{P}\kappa$ containing singletons.

(b) An uncountable cardinal κ is **weakly compact** if for every $S \subseteq [\kappa]^2$ there is a $D \in [\kappa]^\kappa$ such that $[D]^2$ is either included in S or disjoint from S.

541N Theorem (a) A two-valued-measurable cardinal is weakly compact.

(b) A weakly compact cardinal is strongly inaccessible.

541O Lemma Let X be a set and $\mathcal{I}$ a proper ideal of subsets of X such that $\mathcal{P}X/\mathcal{I}$ is atomless. If $\mathcal{I}$ is λ-saturated and κ-additive, with $\lambda \le \kappa$, then $\kappa \le \operatorname{cov}\mathcal{I} \le \sup_{\theta<\lambda} 2^\theta$.

541P Theorem Suppose that κ is a regular uncountable cardinal with a proper λ-saturated κ-additive ideal $\mathcal{I}$ of $\mathcal{P}\kappa$ containing singletons, where $\lambda \le \kappa$. Set $\mathfrak{A} = \mathcal{P}\kappa/\mathcal{I}$. Then

either $\kappa \le \sup_{\theta<\lambda} 2^\theta$ and $\mathfrak{A}$ is atomless

or κ is two-valued-measurable and $\mathfrak{A}$ is purely atomic.

541Q Theorem Let κ be a regular uncountable cardinal and $\mathcal{I}$ a normal ideal on κ. Let $\theta < \kappa$ be a cardinal of uncountable cofinality such that $\mathcal{I}$ is $(\operatorname{cf}\theta)$-saturated in $\mathcal{P}\kappa$, and $f : [\kappa]^{<\omega} \to [\kappa]^{<\theta}$ any function. Then there are $C \in \mathcal{I}$ and $f^* : [\kappa \setminus C]^{<\omega} \to [\kappa]^{<\theta}$ such that $f(I) \cap \eta \subseteq f^*(I \cap \eta)$ whenever $I \in [\kappa \setminus C]^{<\omega}$ and $\eta < \kappa$.

541R Corollary Let κ be a regular uncountable cardinal, $\mathcal{I}$ a normal ideal on κ, and $\theta < \kappa$ a cardinal of uncountable cofinality such that $\mathcal{I}$ is $(\operatorname{cf}\theta)$-saturated in $\mathcal{P}\kappa$.

(a) If Y is a set of cardinal less than κ and $f : [\kappa]^{<\omega} \to [Y]^{<\theta}$ a function, then there are $C \in \mathcal{I}$ and $M \in [Y]^{<\theta}$ such that $f(I) \subseteq M$ for every $I \in [\kappa \setminus C]^{<\omega}$.

(b) If Y is any set and $g : \kappa \to [Y]^{<\theta}$ a function, then there are $C \in \mathcal{I}$ and $M \in [Y]^{<\theta}$ such that $g(\xi) \cap g(\eta) \subseteq M$ for all distinct $\xi, \eta \in \kappa \setminus C$.

541S Lemma Let κ be a regular uncountable cardinal and $\mathcal{I}$ a normal ideal on κ. Suppose that γ and δ are cardinals such that $\omega \le \gamma < \delta < \kappa \le 2^\delta$, $\mathcal{I}$ is δ-saturated in $\mathcal{P}\kappa$, $2^\beta = 2^\gamma$ for $\gamma \le \beta < \delta$, but $2^\delta > 2^\gamma$. Then δ is regular and

$$2^\delta = \operatorname{cov}_{\mathrm{Sh}}(2^\gamma, \kappa, \delta^+, \delta) = \operatorname{cov}_{\mathrm{Sh}}(2^\gamma, \kappa, \delta^+, \omega_1) = \operatorname{cov}_{\mathrm{Sh}}(2^\gamma, \kappa, \delta^+, 2).$$

Version of 8.7.13

542 Quasi-measurable cardinals

As is to be expected, the results of §541 take especially dramatic forms when we look at ω_1-saturated σ-ideals. 542B-542C spell out the application of the most important ideas from §541 to this special case. In addition, we can us Shelah's pcf theory to give us some remarkable combinatorial results concerning cardinal arithmetic (542E) and cofinalities of partially ordered sets (542I-542J).

542A Definition A cardinal κ is **quasi-measurable** if κ is regular and uncountable and there is an ω_1-saturated normal ideal on κ.

542B Proposition If X is a set and $\mathcal{I}$ is a proper ω_1-saturated σ-ideal of $\mathcal{P}X$ containing singletons, then $\operatorname{add}\mathcal{I}$ is quasi-measurable.

542C Proposition If κ is a quasi-measurable cardinal, then κ is weakly inaccessible, the set of weakly inaccessible cardinals less than κ is stationary in κ, and either $\kappa \le \mathfrak{c}$ or κ is two-valued-measurable.

542D Proposition Let κ be a quasi-measurable cardinal.

(a) Let $\langle\theta_\zeta\rangle_{\zeta<\lambda}$ be a family such that $\lambda < \kappa$ is a cardinal, every θ_ζ is a regular infinite cardinal and $\lambda < \theta_\zeta < \kappa$ for every $\zeta < \lambda$. Then $\operatorname{cf}(\prod_{\zeta<\lambda}\theta_\zeta) < \kappa$.

(b) If α and γ are cardinals less than κ, then $\Theta(\alpha,\gamma)$ is less than κ.

(c) If α, β, γ and δ are cardinals, with $\alpha < \kappa$, $\gamma \le \beta$ and $\delta \ge \omega_1$, then $\operatorname{cov}_{\mathrm{Sh}}(\alpha,\beta,\gamma,\delta)$ is less than κ.

(d) $\Theta(\kappa,\kappa) = \kappa$.

542E Theorem If $\kappa \le \mathfrak{c}$ is a quasi-measurable cardinal, then

$$\{2^\gamma : \omega \le \gamma < \kappa\}$$

is finite.

542F Corollary Let $\kappa \le \mathfrak{c}$ be a quasi-measurable cardinal.

(a) There is a regular infinite cardinal $\gamma < \kappa$ such that $2^\gamma = 2^\delta$ for every cardinal δ such that $\gamma \le \delta < \kappa$; that is, $\#([\kappa]^{<\kappa}) = 2^\gamma$.

(b) Let $\mathcal{I}$ be any proper ω_1-saturated κ-additive ideal of $\mathcal{P}\kappa$ containing singletons, and $\mathfrak{A} = \mathcal{P}\kappa/\mathcal{I}$. If γ is as in (a), then the cardinal power $\tau(\mathfrak{A})^\gamma$ is equal to 2^κ.

542G Corollary Suppose that κ is a quasi-measurable cardinal.

(a) If $\kappa \le \mathfrak{c} < \kappa^{(+\omega_1)}$, then $2^\lambda \le \mathfrak{c}$ for every cardinal $\lambda < \kappa$.

(b) Let $\mathcal{I}$ be any proper ω_1-saturated κ-additive ideal of $\mathcal{P}\kappa$ containing singletons, and $\mathfrak{A} = \mathcal{P}\kappa/\mathcal{I}$. If $2^\lambda \le \mathfrak{c}$ for every cardinal $\lambda < \kappa$, then $\#(\mathfrak{A}) = 2^\kappa$.

542H Lemma Let κ be a quasi-measurable cardinal and $\langle\alpha_i\rangle_{i\in I}$ a countable family of ordinals less than κ and of cofinality at least ω_2. Then there is a set $F \subseteq P = \prod_{i\in I}\alpha_i$ such that

(i) F is cofinal with P;

(ii) if $\langle f_\xi\rangle_{\xi<\omega_1}$ is a non-decreasing family in F then $\sup_{\xi<\omega_1} f_\xi \in F$;

(iii) $\#(F) < \kappa$.

542I Theorem Let κ be a quasi-measurable cardinal.

(a) For any cardinal θ, $\operatorname{cf}[\kappa]^{<\theta} \le \kappa$.

(b) For any cardinal $\lambda < \kappa$, and any θ, $\operatorname{cf}[\lambda]^{<\theta} < \kappa$.

542J Corollary Let κ be a quasi-measurable cardinal. Let $\langle P_\zeta\rangle_{\zeta<\lambda}$ be a family of partially ordered sets such that $\lambda < \operatorname{add} P_\zeta \le \operatorname{cf} P_\zeta < \kappa$ for every $\zeta < \lambda$. Then $\operatorname{cf}(\prod_{\zeta<\lambda} P_\zeta) < \kappa$.

542K Proposition Let κ be a quasi-measurable cardinal.

(a) For every cardinal $\theta < \kappa$ there is a family $\mathcal{D}_\theta$ of countable sets, with cardinal less than κ, which is stationary over θ.

(b) There is a family $\mathcal{A}$ of countable sets, with cardinal at most κ, which is stationary over κ.

Version of 11.11.13

543 The Gitik-Shelah theorem

I come now to the leading case at the centre of the work of the last two sections. If our ω_1-saturated σ-ideal of sets is the null ideal of a measure with domain $\mathcal{P}X$, it has some even more striking properties than those already discussed. I will go farther into these later in the chapter. But I will begin with what is known about one of the first questions I expect a reader of this book to ask: if $(X, \mathcal{P}X, \mu)$ is a probability space, what can, or must, its measure algebra be? There can, of course, be a purely atomic part; the interesting question relates to the atomless part, if any, always remembering that we need a special act of faith to believe that there can be an atomless case. Here we find that the Maharam type of an atomless probability defined on a power set must be greater than its additivity (543F), which must itself be 'large' (541L).

543A Definitions (a) A **real-valued-measurable cardinal** is an uncountable cardinal κ such that there is a κ-additive probability measure ν on κ, defined on every subset of κ, for which all singletons are negligible. In this context I will call ν a **witnessing probability**.

(b) If κ is a regular uncountable cardinal, a probability measure ν on κ with domain $\mathcal{P}\kappa$ is **normal** if its null ideal $\mathcal{N}(\nu)$ is normal. In this case, I will say that ν is a **normal witnessing probability**.

(c) An **atomlessly-measurable cardinal** is a real-valued-measurable cardinal with an atomless witnessing probability.

543B Proposition (a) Let $(X, \mathcal{P}X, \mu)$ be a totally finite measure space in which singletons are negligible and $\mu X > 0$. Then $\kappa = \operatorname{add}\mu$ is real-valued-measurable, and there are a non-negligible $Y \subseteq X$ and a function $g : Y \to \kappa$ such that the normalized image measure $B \mapsto \frac{1}{\mu Y}\mu g^{-1}[B]$ is a normal witnessing probability on κ.

(b) Every real-valued-measurable cardinal is quasi-measurable and has a normal witnessing probability; in particular, every real-valued-measurable cardinal is uncountable and regular.

(c) If $\kappa \le \mathfrak{c}$ is a real-valued-measurable cardinal, then κ is atomlessly-measurable, and every witnessing probability on κ is atomless.

(d) If $\kappa > \mathfrak{c}$ is a real-valued-measurable cardinal, then κ is two-valued-measurable, and every witnessing probability on κ is purely atomic.

(e) A cardinal λ is measure-free iff there is no real-valued-measurable cardinal $\kappa \le \lambda$; $\mathfrak{c}$ is measure-free iff there is no atomlessly-measurable cardinal.

(f) Again suppose that $(X, \mathcal{P}X, \mu)$ is a totally finite measure space.

(i) If μ is purely atomic, $\operatorname{add}\mu$ is either ∞ or a two-valued-measurable cardinal.

(ii) If μ is not purely atomic, $\operatorname{add}\mu$ is atomlessly-measurable.

543C Theorem Suppose that $(Y, \mathcal{P}Y, \nu)$ is a σ-finite measure space and that $(X, \mathfrak{T}, \Sigma, \mu)$ is a σ-finite quasi-Radon measure space with $w(X) < \operatorname{add}\nu$. Let $f : X \times Y \to [0, \infty]$ be any function. Then

$$\overline{\int}\Bigl(\int f(x,y)\nu(dy)\Bigr)\mu(dx) \le \int\Bigl(\overline{\int} f(x,y)\mu(dx)\Bigr)\nu(dy).$$

543D Corollary Let κ be a real-valued-measurable cardinal, with witnessing probability ν, and $(X, \mathfrak{T}, \Sigma, \mu)$ a totally finite quasi-Radon measure space with $w(X) < \kappa$.

(a) If $C \subseteq X \times \kappa$ then

$$\overline{\int}\nu C[\{x\}]\mu(dx) \le \int \mu^* C^{-1}[\{\xi\}]\nu(d\xi).$$

(b) If $A \subseteq X$ and $\#(A) \le \kappa$, then there is a $B \subseteq A$ such that $\#(B) < \kappa$ and $\mu^* B = \mu^* A$.

(c) If $\langle C_\xi \rangle_{\xi<\kappa}$ is a family in $\mathcal{P}X \setminus \mathcal{N}(\mu)$ such that $\#(\bigcup_{\xi<\kappa} C_\xi) < \kappa$, then there are distinct ξ, $\eta < \kappa$ such that $\mu^*(C_\xi \cap C_\eta) > 0$.

(d) If we have a family $\langle h_\xi \rangle_{\xi<\kappa}$ of functions such that $\operatorname{dom} h_\xi$ is a non-negligible subset of X for each ξ and $\#(\bigcup_{\xi<\kappa} h_\xi) < \kappa$, then there are distinct ξ, $\eta < \kappa$ such that

$$\mu^*\{x : x \in \operatorname{dom}(h_\xi) \cap \operatorname{dom}(h_\eta),\ h_\xi(x) = h_\eta(x)\} > 0.$$

543M Lemma Let $\kappa \le \mathfrak{c}$ be a quasi-measurable cardinal and $\lambda < \min(\kappa^{(+\omega)}, 2^\kappa)$ an infinite cardinal. Set $\zeta = \max(\lambda^+, \kappa^+)$.

(a) We have an infinite cardinal $\delta < \kappa$, a stationary set $S \subseteq \zeta$, and a family $\langle g_\alpha \rangle_{\alpha \in S}$ of functions from κ to 2^δ such that $g_\alpha[\kappa] \subseteq \alpha$ for every $\alpha \in S$ and $\#(g_\alpha \cap g_\beta) < \kappa$ for distinct α, $\beta \in S$. Moreover, we can arrange that

— if $\lambda < \operatorname{Tr}(\kappa)$ (definition: 5A1Mb), then $g_\alpha[\kappa] \subseteq \kappa$ for every $\alpha \in S$;

— if $\lambda \ge \operatorname{Tr}(\kappa)$, then $g_\alpha \restriction \gamma = g_\beta \restriction \gamma$ whenever $\gamma < \kappa$ is a limit ordinal and α, $\beta \in S$ are such that $g_\alpha(\gamma) = g_\beta(\gamma)$.

(b) Now suppose that $S_1 \subseteq S$ is stationary in ζ and $\langle \theta_\alpha \rangle_{\alpha \in S_1}$ is a family of limit ordinals less than κ. Then there are a $\theta < \kappa$ and a set $Y \in [2^\delta]^{<\kappa}$ such that $S_2 = \{\alpha : \alpha \in S_1,\ \theta_\alpha = \theta,\ g_\alpha[\theta] \subseteq Y\}$ is stationary in ζ.

543E The Gitik-Shelah theorem Let κ be an atomlessly-measurable cardinal, with witnessing probability ν. Then the Maharam type of ν is at least $\min(\kappa^{(+\omega)}, 2^\kappa)$.

543F Theorem Let $(X, \mathcal{P}X, \mu)$ be an atomless semi-finite measure space. Write $\kappa = \operatorname{add} \mu$. Then the Maharam type of $(X, \mathcal{P}X, \mu)$ is at least $\min(\kappa^{(+\omega)}, 2^\kappa)$, and in particular is greater than κ.

543G Corollary Let $(X, \mathcal{P}X, \nu)$ be an atomless probability space, and $\kappa = \operatorname{add} \nu$. Let (Z, Σ, μ) be a compact probability space with Maharam type $\lambda \le \min(2^\kappa, \kappa^{(+\omega)})$ (e.g., $Z = \{0,1\}^\lambda$ with its usual measure). Then there is an inverse-measure-preserving function $f : X \to Z$.

543H Corollary If κ is an atomlessly-measurable cardinal, and (Z, μ) is a compact probability space with Maharam type at most $\min(2^\kappa, \kappa^{(+\omega)})$, then there is an extension of μ to a κ-additive measure defined on $\mathcal{P}Z$.

543I Corollary If κ is an atomlessly-measurable cardinal, with witnessing probability ν, and $2^\kappa \le \kappa^{(+\omega)}$, then $(\kappa, \mathcal{P}\kappa, \nu)$ is Maharam-type-homogeneous, with Maharam type 2^κ.

543J Proposition Let κ be an atomlessly-measurable cardinal, ν a witnessing probability on κ, and $\mathfrak{A}$ the measure algebra of ν. Then

(a) there is a $\gamma < \kappa$ such that $2^\gamma = 2^\delta$ for every cardinal δ such that $\gamma \le \delta < \kappa$;

(b) the cardinal power $\tau(\mathfrak{A})^\gamma$ is 2^κ;

(c) if $\mathfrak{c} < \kappa^{(+\omega_1)}$, then $\#(\mathfrak{A}) = \tau(\mathfrak{A})^\omega = 2^\kappa$.

543K Proposition Let κ be an atomlessly-measurable cardinal. If there is a witnessing probability on κ with Maharam type λ, then there is a Maharam-type-homogeneous normal witnessing probability ν on κ with Maharam type at most λ.

543L Proposition Suppose that ν is a Maharam-type-homogeneous witnessing probability on an atomlessly-measurable cardinal κ with Maharam type λ. Then there is a Maharam-type-homogeneous witnessing probability ν' on κ with Maharam type at least $\operatorname{Tr}_{\mathcal{N}(\nu)}(\kappa; \lambda)$.

543Z Problems Let κ be an atomlessly-measurable cardinal.

(a) Must every witnessing probability ν on κ be Maharam-type-homogeneous?

(b) Must every normal witnessing probability ν on κ be Maharam-type-homogeneous?

Version of 31.12.13

544 Measure theory with an atomlessly-measurable cardinal

As is to be expected, a witnessing measure on a real-valued-measurable cardinal has some striking properties, especially if it is normal. What is less obvious is that the mere existence of such a cardinal can have implications for apparently unrelated questions in analysis. In 544J, for instance, we see that if there is any atomlessly-measurable cardinal then we have a version of Fubini's theorem, $\iint f(x,y)dxdy = \iint f(x,y)dydx$, for many functions f on $\mathbb{R}^2$ which are not jointly measurable. In this section I explore results of this kind. We find that, in the presence of an atomlessly-measurable cardinal, the covering number of the Lebesgue null ideal is large (544B) while its uniformity is small (544G-544H). There is a second inequality on repeated integrals (544C) to add to the one already given in 543C, and which tells us something about measure-precalibers (544D); I add a couple of variations (544I-544J). Next, I give a pair of theorems (544E-544F) on a measure-combinatorial property of the filter of conegligible sets of a normal witnessing measure. Revisiting the theory of Borel measures on metrizable spaces, discussed in §438 on the assumption that no real-valued-measurable cardinal was present, we find that there are some non-trivial arguments applicable to spaces with non-measure-free weight (544K-544L).

In §541 I briefly mentioned 'weakly compact' cardinals. Two-valued-measurable cardinals are always weakly compact; atomlessly-measurable cardinals never are; but atomlessly-measurable cardinals may or may not have a significant combinatorial property which can be regarded as a form of weak compactness (544M, 544Yc). Finally, I summarise what is known about the location of an atomlessly-measurable cardinal on Cichoń's diagram (544N).

544A Notation I repeat some of my notational conventions. For a measure μ, $\mathcal{N}(\mu)$ will be its null ideal. For any set I, ν_I will be the usual measure on $\{0,1\}^I$, $\mathcal{N}_I = \mathcal{N}(\nu_I)$ its null ideal and $(\mathfrak{B}_I, \bar{\nu}_I)$ its measure algebra.

544B Proposition Let κ be an atomlessly-measurable cardinal. If (X, Σ, μ) is any locally compact semi-finite measure space with $\mu X > 0$, then $\operatorname{cov}\mathcal{N}(\mu) \geq \kappa$.

544C Theorem Let κ be a real-valued-measurable cardinal and ν a normal witnessing probability on κ; let (X, μ) be a compact probability space and $f : X \times \kappa \to [0, \infty[$ any function. Then

$$\underline{\int}\Bigl(\int f(x,\xi)\nu(d\xi)\Bigr)\mu(dx) \leq \int\Bigl(\overline{\int} f(x,\xi)\mu(dx)\Bigr)\nu(d\xi).$$

544D Corollary If κ is an atomlessly-measurable cardinal and $\omega \leq \lambda \leq \kappa$, then λ is a measure-precaliber of every probability algebra.

544E Theorem Let κ be a real-valued-measurable cardinal and ν a normal witnessing probability on κ. If (X, μ) is a quasi-Radon probability space of weight strictly less than κ, and $f : [\kappa]^{<\omega} \to \mathcal{N}(\mu)$ is any function, then

$$\bigcap_{V \subseteq \kappa, \nu V = 1} \bigcup_{I \in [V]^{<\omega}} f(I) \in \mathcal{N}(\mu).$$

544F Theorem Let κ be a real-valued-measurable cardinal with a normal witnessing probability ν. If (X, μ) is a locally compact semi-finite measure space with $\mu X > 0$ and $f : [\kappa]^{<\omega} \to \mathcal{N}(\mu)$ is a function, then there is a ν-conegligible $V \subseteq \kappa$ such that $\bigcup\{f(I) : I \in [V]^{<\omega}\} \neq X$.

544G Proposition Let κ be an atomlessly-measurable cardinal and $\omega_1 \leq \lambda < \kappa$. If (X, μ) is an atomless locally compact semi-finite measure space of Maharam type less than κ, and $\mu X > 0$, then there is a Sierpiński set $A \subseteq X$ with cardinal λ.

544H Corollary Let κ be an atomlessly-measurable cardinal.

(a) $\operatorname{non}\mathcal{N}_\theta = \omega_1$ for $\omega \le \theta < \kappa$.

(b) $\operatorname{non}\mathcal{N}_\theta \le \kappa$ for $\theta \le \min(2^\kappa, \kappa^{(+\omega)})$.

(c) $\operatorname{non}\mathcal{N}_\theta < \theta$ for $\kappa < \theta < \kappa^{(+\omega)}$.

544I Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a totally finite quasi-Radon measure space and $(Y, \mathcal{P}Y, \nu)$ a probability space; suppose that $w(X) < \operatorname{add}\nu$. Let $f : X \times Y \to \mathbb{R}$ be a bounded function such that all the sections $x \mapsto f(x, y) : X \to \mathbb{R}$ are Σ-measurable. Then the repeated integrals $\iint f(x, y)\nu(dy)\mu(dx)$ and $\iint f(x, y)\mu(dx)\nu(dy)$ are defined and equal.

544J Proposition Let κ be an atomlessly-measurable cardinal and $(X, \mathfrak{T}, \Sigma, \mu)$, $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ Radon probability spaces both of weight less than κ; let $\mu \times \nu$ be the c.l.d. product measure on $X \times Y$, and Λ its domain. Let $f : X \times Y \to \mathbb{R}$ be a function such that all its horizontal and vertical sections

$$x \mapsto f(x, y^*) : X \to \mathbb{R}, \quad y \mapsto f(x^*, y) : Y \to \mathbb{R}$$

are measurable. Then

(a) if f is bounded, the repeated integrals

$$\iint f(x, y)\mu(dx)\nu(dy), \quad \iint f(x, y)\nu(dy)\mu(dx)$$

exist and are equal;

(b) in any case, there is a Λ-measurable function $g : X \times Y \to \mathbb{R}$ such that all the sections $\{x : g(x, y^*) \ne f(x, y^*)\}$, $\{y : g(x^*, y) \ne f(x^*, y)\}$ are negligible.

544K Proposition If X is a metrizable space and μ is a σ-finite Borel measure on X, then $\operatorname{add}\mathcal{N}(\mu) \ge \operatorname{add}\mathcal{N}_\omega$.

544L Corollary Let X be a metrizable space.

(a) If $\mathcal{U}\mathfrak{n}$ is the σ-ideal of universally negligible subsets of X, then $\operatorname{add}\mathcal{U}\mathfrak{n} \ge \operatorname{add}\mathcal{N}_\omega$.

(b) If Σ_{um} is the σ-algebra of universally measurable subsets of X, then $\bigcup\mathcal{E} \in \Sigma_{\mathrm{um}}$ whenever $\mathcal{E} \subseteq \Sigma_{\mathrm{um}}$ and $\#(\mathcal{E}) < \operatorname{add}\mathcal{N}_\omega$.

544M Theorem Let κ be an atomlessly-measurable cardinal. Then the following are equiveridical:

(i) for every family $\langle f_\xi \rangle_{\xi<\kappa}$ of regressive functions defined on $\kappa \setminus \{0\}$ there is a family $\langle \alpha_\xi \rangle_{\xi<\kappa}$ in κ such that

$$\{\kappa \setminus \zeta : \zeta < \kappa\} \cup \{f_\xi^{-1}[\{\alpha_\xi\}] : \xi < \kappa\}$$

has the finite intersection property;

(ii) for every family $\langle f_\xi \rangle_{\xi<\kappa}$ in $\mathbb{N}^\kappa$ there is a family $\langle m_\xi \rangle_{\xi<\kappa}$ in $\mathbb{N}$ such that

$$\{\kappa \setminus \zeta : \zeta < \kappa\} \cup \{f_\xi^{-1}[\{m_\xi\}] : \xi < \kappa\}$$

has the finite intersection property;

(iii) $\operatorname{cov}\mathcal{N}_\kappa > \kappa$;

(iv) $\operatorname{cov}\mathcal{N}(\mu) > \kappa$ whenever (X, μ) is a locally compact semi-finite measure space and $\mu X > 0$.

544N Cichoń's diagram and other cardinals (a) Returning to the concerns of Chapter 52, any atomlessly-measurable cardinal κ is necessarily connected with the structures there. $\kappa \le \operatorname{cov}\mathcal{N}_\lambda$ for every λ; all the cardinals on the bottom line of Cichoń's diagram, and the Martin numbers $\mathfrak{m}$, $\mathfrak{p}$ etc., must be ω_1, while all the cardinals on the top line must be at least κ. $\mathrm{FN}(\mathcal{P}\mathbb{N})$ must be at least κ.

(b) If κ is an atomlessly-measurable cardinal, then $\mathfrak{b} < \kappa$.

(c) If κ is an atomlessly-measurable cardinal, then $\operatorname{cf}\mathfrak{d} \ne \kappa$.

(d) $\operatorname{cf}\mathcal{N}_\kappa = \max(\kappa, \operatorname{cf}\mathcal{N}_\omega)$ for any quasi-measurable cardinal κ.

544Z Problems (a) In 543C, can we replace '$w(X) < \operatorname{add}\nu$' with '$\tau(\mu) < \operatorname{add}\nu$'? More concretely, suppose that (Z,λ) is the Stone space of $(\mathfrak{B}_\omega,\bar{\nu}_\omega)$, κ is an atomlessly-measurable cardinal and ν a normal witnessing probability on κ. Let $C \subseteq \kappa \times Z$ be such that $\lambda C[\{\xi\}] = 0$ for every $\xi < \kappa$. $\{z : \nu C^{-1}[\{z\}] > 0\}$ has inner measure zero. But does it have to be negligible?

(b) Suppose that κ is an atomlessly-measurable cardinal. Must there be a Sierpiński set $A \subseteq \{0,1\}^\omega$ with cardinal κ? (See 552E.)

(c) Suppose that κ is an atomlessly-measurable cardinal. Can $\operatorname{non}\mathcal{N}_\kappa$ be greater than ω_1? What if $\kappa = \mathfrak{c}$? (See 552H.)

(d) Can there be an atomlessly-measurable cardinal less than $\mathfrak{d}$? (See the notes to §555.)

(e) Can there be an atomlessly-measurable cardinal less than or equal to $\operatorname{shr}\mathcal{N}_\omega$? (See 555Yd.)

(f) Suppose that there is an atomlessly-measurable cardinal. Does it follow that $\operatorname{cov}\mathcal{N}_\omega = \mathfrak{c}$?

Version of 10.2.14

545 PMEA and NMA

One of the reasons for supposing that it is consistent to assume that there are measurable cardinals is that very much stronger axioms have been studied at length without any contradiction appearing. Here I mention two such axioms which have obvious consequences in measure theory.

545A Theorem The following are equiveridical:
(i) for every cardinal λ, there is a probability space $(X,\mathcal{P}X,\mu)$ with $\tau(\mu) \geq \lambda$ and $\operatorname{add}\mu \geq \mathfrak{c}$;
(ii) for every cardinal λ, there is an extension of the usual measure ν_λ on $\{0,1\}^\lambda$ to a $\mathfrak{c}$-additive probability measure with domain $\mathcal{P}(\{0,1\}^\lambda)$;
(iii) for every semi-finite locally compact measure space (X,Σ,μ), there is an extension of μ to a $\mathfrak{c}$-additive measure with domain $\mathcal{P}X$.

545B Definition PMEA (the '**product measure extension axiom**') is the assertion that the statements (i)-(iii) of 545A are true.

545C Proposition If PMEA is true, then $\mathfrak{c}$ is atomlessly-measurable.

545D Definition NMA (the '**normal measure axiom**') is the statement

For every set I there is a $\mathfrak{c}$-additive probability measure ν on $S = [I]^{<\mathfrak{c}}$, with domain $\mathcal{P}S$, such that
(α) $\nu\{s : i \in s \in S\} = 1$ for every $i \in I$,
(β) if $A \subseteq S$, $\nu A > 0$ and $f : A \to I$ is such that $f(s) \in s$ for every $s \in A$, then there is an $i \in I$ such that $\nu\{s : s \in A,\ f(s) = i\} > 0$.

545E Proposition NMA implies PMEA.

545F Proposition Suppose that NMA is true. Let $\mathfrak{A}$ be a Boolean algebra such that whenever $s \in [\mathfrak{A}]^{<\mathfrak{c}}$ there is a subalgebra $\mathfrak{B} \subseteq \mathfrak{A}$, including s, with a strictly positive countably additive functional. Then there is a strictly positive countably additive functional on $\mathfrak{A}$.

545G Corollary Suppose that NMA is true. Let $\mathfrak{A}$ be a Boolean algebra such that every $s \in [\mathfrak{A}]^{<\mathfrak{c}}$ is included in a subalgebra of $\mathfrak{A}$ which is, in itself, a measurable algebra. Then $\mathfrak{A}$ is a measurable algebra.

Version of 3.2.21

546 Power set σ-quotient algebras

One way of interpreting the Gitik-Shelah theorem (543E) is to say that it shows that 'simple' atomless probability algebras cannot be of the form $\mathcal{P}X/\mathcal{N}(\mu)$. Similarly, the results of §541-§542 show that any ccc Boolean algebra expressible as the quotient of a power set by a non-trivial σ-ideal involves us in dramatic complexities, though it is not clear when these must appear in the quotient algebra itself. In the next section I will present further examples of algebras which cannot appear in this way. To prepare for these I collect some general facts about quotients of power set algebras.

546A(a) Definition A **power set σ-quotient algebra** is a Boolean algebra which is isomorphic to an algebra of the form $\mathcal{P}X/\mathcal{I}$ where X is a set and $\mathcal{I}$ is a σ-ideal of subsets of X.

(b) A **normal power set σ-quotient algebra** is a Boolean algebra which is isomorphic to an algebra of the form $\mathcal{P}\kappa/\mathcal{I}$ where κ is a regular uncountable cardinal and $\mathcal{I}$ is a normal ideal of $\mathcal{P}\kappa$.

(c) As in §522, I will write $\operatorname{non}\mathcal{M}$ for the uniformity of the meager ideal of $\mathbb{R}$, $\operatorname{non}\mathcal{N}$ for the uniformity of the Lebesgue null ideal and $\operatorname{cov}\mathcal{N}$ for the covering number of the Lebesgue null ideal. If κ is a cardinal, ν_κ will be the usual measure on $\{0,1\}^\kappa$. $\mathcal{N}_\kappa$ its null ideal and $\mathfrak{B}_\kappa$ its measure algebra; $\mathfrak{G}_\kappa$ will be the category algebra of $\{0,1\}^\kappa$. Note that the covering number and uniformity of $\mathcal{N}_\omega$ are $\operatorname{cov}\mathcal{N}$ and $\operatorname{non}\mathcal{N}$ respectively, while $\operatorname{non}\mathcal{M}$ is the uniformity of the meager ideal $\mathcal{M}_\omega$ of $\{0,1\}^{\mathbb{N}} = \{0,1\}^\omega$.

546B Proposition (a) Any power set σ-quotient algebra is Dedekind σ-complete.

(b) If $\mathfrak{A}$ is a power set σ-quotient algebra, $\mathfrak{B}$ is a Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{B}$ is a surjective sequentially order-continuous Boolean homomorphism, then $\mathfrak{B}$ is a power set σ-quotient algebra. In particular, any principal ideal of a power set σ-quotient algebra is a power set σ-quotient algebra.

(c) The simple product of any family of power set σ-quotient algebras is a power set σ-quotient algebra.

546C Proposition A non-zero principal ideal of a normal power set σ-quotient algebra is a normal power set σ-quotient algebra.

546D Lemma Let X be a set, $\mathcal{I}$ a proper σ-ideal of subsets of X containing singletons, and $\mathfrak{A}$ the quotient algebra $\mathcal{P}X/\mathcal{I}$. Write κ for $\operatorname{add}\mathcal{I}$.

(a) There are an $a \in \mathfrak{A}\setminus\{0\}$, a σ-subalgebra $\mathfrak{C}$ of the principal ideal $\mathfrak{A}_a$ and a κ-additive ideal $\mathcal{J}$ of $\mathcal{P}\kappa$, containing singletons, such that $\mathfrak{C} \cong \mathcal{P}\kappa/\mathcal{J}$.

(b) If $\mathfrak{A}$ is atomless and ccc then $\kappa \le \mathfrak{c}$ is quasi-measurable, $\mathfrak{C}$ is atomless and we can arrange that $\mathcal{J}$ should be a normal ideal, so that $\mathfrak{C}$ is a normal power set σ-quotient algebra.

546E Proposition The measure algebra of Lebesgue measure on $\mathbb{R}$ is not a power set σ-quotient algebra.

546F Definition Let $\mathfrak{A}$ be a Boolean algebra. I will say that an **e-h family** in $\mathfrak{A}$ is a double sequence $\langle e_{ij}\rangle_{i,j\in\mathbb{N}}$ in $\mathfrak{A}$ such that

$$\langle e_{ij}\rangle_{j\in\mathbb{N}} \text{ is disjoint for every } i \in \mathbb{N}$$

and

$$\sup_{i\in\mathbb{N}} e_{i,f(i)} = 1$$

for every $f \in \mathbb{N}^{\mathbb{N}}$.

546G Lemma Let $\mathfrak{A}$ be a Boolean algebra and $\langle e_{ij}\rangle_{i,j\in\mathbb{N}}$ an e-h family in $\mathfrak{A}$. Then $\sup_{i\ge n} e_{i,f(i)} = 1$ for every $f \in \mathbb{N}^{\mathbb{N}}$ and $n \in \mathbb{N}$.

546H Free products and completed free products For Boolean algebras $\mathfrak{A}$, $\mathfrak{B}$ I write $\mathfrak{A}\otimes\mathfrak{B}$ for their free product and $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$ for its Dedekind completion. $\mathfrak{G}_\kappa \cong \mathfrak{G}_\kappa\widehat{\otimes}\mathfrak{G}_\omega$ for any infinite κ.

546I Lemma Let X be a set, $\mathcal{I}$ a proper σ-ideal of subsets of X, and $\mathfrak{A}$ the quotient algebra $\mathcal{P}X/\mathcal{I}$. Write κ for $\operatorname{add}\mathcal{I}$.

(a)(i) If $\mathfrak{A}$ has an atomless order-closed subalgebra which is a measurable algebra, then

(α) $\kappa \le \operatorname{non}\mathcal{M}$,

(β) $\operatorname{non}\mathcal{N} \le \#(X)$,

(γ) $\operatorname{cov}\mathcal{N} \ge \operatorname{cov}\mathcal{I} \ge \kappa$.

(ii) If $\mathfrak{G}_\omega$ can be regularly embedded in $\mathfrak{A}$, then $\kappa \le \operatorname{non}\mathcal{N}$.

(b) If $\mathfrak{A}$ has a non-trivial principal ideal with an e-h family, then $\operatorname{non}\mathcal{M} \le \#(X)$.

(c) Suppose that $\kappa \le \operatorname{non}\mathcal{M}$. Then there is an $a \in \mathfrak{A} \setminus \{0\}$ such that for every family $\langle a_{\xi n}\rangle_{\xi<\kappa,n\in\mathbb{N}}$ in $\mathfrak{A}$ there is a family $\langle c_{m\sigma}\rangle_{m\in\mathbb{N},\sigma\in S_2}$ in $\mathfrak{A}$ such that

$$\inf_{\sigma\in S_2} c_{m,\tau^\frown\sigma} = c_{m\tau}, \quad \sup_{\sigma\in S_2} c_{m,\tau^\frown\sigma} = 1$$

for every $\tau \in S_2$ and $m \in \mathbb{N}$, and

$$a \cap \inf_{m\in\mathbb{N}} \sup_{\sigma\in S_2} \bigl(c_{m\sigma} \cap \inf_{\substack{i<\#(\sigma)\\ \sigma(i)=1}} a_{\xi i} \setminus \sup_{\substack{i<\#(\sigma)\\ \sigma(i)=0}} a_{\xi i}\bigr) = 0$$

for every $\xi < \kappa$.

(d) Suppose that $\kappa = \operatorname{non}\mathcal{N}$. Then $\mathfrak{A}$ has a countably generated order-closed subalgebra which is not a measurable algebra.

(e) Suppose that $\mathfrak{A}$ is isomorphic to $\mathfrak{C}\widehat{\otimes}\mathfrak{G}_\omega$ where $\mathfrak{C}$ is a ccc Boolean algebra. Then $\operatorname{cov}\mathcal{N} = \omega_1$.

546J Theorem Let $\mathfrak{A}$ be a Boolean algebra such that $\mathfrak{B}_\omega$ can be regularly embedded in $\mathfrak{A}$ and $\mathfrak{A} \cong \mathfrak{C}\widehat{\otimes}\mathfrak{G}_\omega$ for some ccc Boolean algebra $\mathfrak{C}$. Then $\mathfrak{A}$ is not a power set σ-quotient algebra.

546K Corollary If λ, κ are infinite cardinals then $\mathfrak{B}_\lambda\widehat{\otimes}\mathfrak{G}_\kappa$ is not a power set σ-quotient algebra.

Version of 24.10.20

547 Cohen algebras and σ-measurable algebras

I examine the conditions under which two classes of algebra can be power set σ-quotient algebras. In the first, shorter, part of the section (547B-547G) I look at Cohen algebras. I then turn to 'σ-measurable' algebras (547H-547S).

547A Notation If I is a set, $\mathfrak{G}_I$ will be the category algebra of $\{0,1\}^I$. If $\mathcal{I}$ is an ideal of subsets of X and $\mathcal{J}$ an ideal of subsets of Y, then

$$\mathcal{I} \ltimes \mathcal{J} = \{W : W \subseteq X \times Y,\ \{x : W[\{x\}] \notin \mathcal{J}\} \in \mathcal{I}\},$$

$$\mathcal{I} \rtimes \mathcal{J} = \{W : W \subseteq X \times Y,\ \{y : W^{-1}[\{y\}] \notin \mathcal{I}\} \in \mathcal{J}\}\ .$$

If X is a topological space, $\mathcal{M}(X)$ will be its meager ideal, $\mathcal{B}\mathfrak{a}(X)$ its Borel σ-algebra and $\widehat{\mathcal{B}}(X)$ its Baire-property algebra. If $\mathfrak{A}$ is a Boolean algebra, $\mathfrak{C}$ is a subalgebra of $\mathfrak{A}$ and $a \in \mathfrak{A}$, then $\operatorname{upr}(a,\mathfrak{C}) = \min\{c : a \subseteq c \in \mathfrak{C}\}$ if this is defined.

I will write S_2 for $\bigcup_{n\in\mathbb{N}}\{0,1\}^n$, ordered by $\subseteq$. For $\sigma \in S_2$, set $I_\sigma = \{z : \sigma \subseteq z \in \{0,1\}^{\mathbb{N}}\}$; then σ, $\tau \in S_2$ are (upwards) incompatible iff neither extends the other iff $I_\sigma \cap I_\tau = \emptyset$, while $\{I_\sigma : \sigma \in S_2\}$ is a base for the usual topology of $\{0,1\}^{\mathbb{N}}$.

547B Lemma Suppose that κ is a regular uncountable cardinal and $\mathcal{I}$ is a κ-additive ideal of subsets of κ such that $\mathcal{P}\kappa/\mathcal{I}$ is harmless. Then $\mathcal{P}(\kappa\times\kappa)/\mathcal{I}\ltimes\mathcal{I}$ is harmless.

547C Proposition Suppose that κ is a regular uncountable cardinal and $\mathcal{I}$ is a κ-additive ideal of subsets of κ such that $\mathcal{P}\kappa/\mathcal{I}$ is harmless. Let X be a ccc topological space of π-weight less than κ. Then $\mathcal{M}(X) \rtimes \mathcal{I} \subseteq \mathcal{M}(X) \ltimes \mathcal{I}$.

547D Corollary Suppose that κ is a regular uncountable cardinal and $\mathcal{I}$ is a proper κ-additive ideal of subsets of κ, containing singletons, such that $\mathcal{P}\kappa/\mathcal{I}$ is harmless. Let X be a ccc topological space of π-weight less than κ.

(a) Suppose that $\langle A_\xi\rangle_{\xi<\kappa}$ is a non-decreasing family of subsets of X with union A. Then there is a $\theta<\kappa$ such that $E\cap A_\theta$ is non-meager whenever $E\subseteq X$ is a set with the Baire property and $E\cap A$ is not meager.

(b) If $\langle A_\xi\rangle_{\xi<\kappa}$ is a family in $\mathcal{P}X\setminus\mathcal{M}(X)$ such that $\#(\bigcup_{\xi<\kappa}A_\xi)<\kappa$, then there are distinct ξ, $\eta<\kappa$ such that $A_\xi\cap A_\eta\notin\mathcal{M}(X)$.

(c) If we have a family $\langle h_\xi\rangle_{\xi<\kappa}$ of functions such that $\operatorname{dom}h_\xi$ is a non-meager subset of X for each ξ and $\#(\bigcup_{\xi<\kappa}h_\xi)<\kappa$, then there are distinct ξ, $\eta<\kappa$ such that $\{x:h_\xi(x)$ and $h_\eta(x)$ are defined and equal$\}$ is non-meager.

547E Lemma Suppose that X, I are sets, $\mathcal{I}$ is a σ-ideal of subsets of X and ϕ is a sequentially order-continuous Boolean homomorphism from $\mathfrak{G}_I$ to $\mathcal{P}X/\mathcal{I}$. Then there is a function $f:X\to\{0,1\}^I$ such that $f^{-1}[E]^\bullet=\phi E^\bullet$ in $\mathfrak{A}$ for every E in the Baire-property algebra $\widehat{\mathcal{B}}$ of $\{0,1\}^I$.

547F The Gitik-Shelah theorem for Cohen algebras: Theorem Let κ be a regular uncountable cardinal and $\mathcal{I}$ a κ-additive ideal of subsets of κ such that $\mathcal{P}\kappa/\mathcal{I}$ is isomorphic to $\mathfrak{G}_\lambda$ for some infinite cardinal λ. Then $\lambda\ge\min(\kappa^{(+\omega)},2^\kappa)$.

547G Corollary (a) $\mathfrak{G}_\omega$ is not a power set σ-quotient algebra.

(b) $\mathfrak{G}_{\omega_1}$ is not a power set σ-quotient algebra.

547H Definitions Let $\mathfrak{A}$ be a Boolean algebra.

(a) I will say that $\mathfrak{A}$ is **σ-measurable**, with **witnessing sequence** $\langle\mathfrak{B}_n\rangle_{n\in\mathbb{N}}$, if $\mathfrak{A}$ is Dedekind complete, every $\mathfrak{B}_n$ is an order-closed subalgebra of $\mathfrak{A}$ which is, in itself, a measurable algebra, and $\bigcup_{n\in\mathbb{N}}\mathfrak{B}_n$ is order-dense in $\mathfrak{A}$.

(b) If $\mathfrak{A}$ is σ-measurable algebra, I will say that

$$\tau_{\sigma\text{-m}}(\mathfrak{A})=\min\{\textstyle\sum_{n=0}^\infty\tau(\mathfrak{B}_n):\langle\mathfrak{B}_n\rangle_{n\in\mathbb{N}}\text{ is a witnessing sequence for }\mathfrak{A}\},$$

where the sums here are cardinal sums.

547I Examples (a) Every measurable algebra is σ-measurable.

(b) If $\mathfrak{A}$ is Dedekind complete and has countable π-weight, it is σ-measurable. In particular, $\mathfrak{G}_\omega$ is σ-measurable.

(c) If $\mathfrak{A}$ is a measurable algebra and $\mathfrak{B}$ is a Boolean algebra with countable π-weight, then the Dedekind completion $\mathfrak{C}$ of the free product $\mathfrak{A}\otimes\mathfrak{B}$ is σ-measurable, with $\tau_{\sigma\text{-m}}(\mathfrak{C})\le\max(\omega,\tau(\mathfrak{A}))$.

547J Lemma Let $\mathfrak{A}$ be a Boolean algebra, and $\mathfrak{B}$ a regularly embedded subalgebra of $\mathfrak{A}$ which is a measurable algebra. Then $\mathfrak{B}_a=\{b\cap a:b\in\mathfrak{B}\}$ is a measurable algebra, with $\tau(\mathfrak{B}_a)\le\tau(\mathfrak{B})$, for every $a\in\mathfrak{A}$.

547K Proposition Let $\mathfrak{A}$ be a σ-measurable Boolean algebra with witnessing sequence $\langle\mathfrak{B}_n\rangle_{n\in\mathbb{N}}$.

(a) $\mathfrak{A}$ satisfies Knaster's condition, so is ccc.

(b) $a=\inf_{n\in\mathbb{N}}\operatorname{upr}(a,\mathfrak{B}_n)$ for every $a\in\mathfrak{A}$.

(c) $\tau(\mathfrak{A})\le\tau_{\sigma\text{-m}}(\mathfrak{A})$.

(d) If $a\in\mathfrak{A}$ then the principal ideal $\mathfrak{A}_a$ generated by a is σ-measurable, with $\tau_{\sigma\text{-m}}(\mathfrak{A}_a)\le\tau_{\sigma\text{-m}}(\mathfrak{A})$.

(e) If $\mathfrak{A}$ is actually measurable, then $\tau_{\sigma\text{-m}}(\mathfrak{A})=\tau(\mathfrak{A})$.

(f) $\pi(\mathfrak{A})\le\sum_{n=0}^\infty\pi(\mathfrak{B}_n)$.

547L Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\langle\mathfrak{C}_n\rangle_{n\in\mathbb{N}}$ a sequence of σ-subalgebras of $\mathfrak{A}$. For each $n\in\mathbb{N}$ let $\nu_n:\mathfrak{C}_n\to[0,\infty[$ be a countably additive functional. Suppose that for every $a\in\mathfrak{A}$ there are an $n\in\mathbb{N}$ and a $c\in\mathfrak{C}_n$ such that $c\subseteq a$ and $\nu_n c>0$. Then $\mathfrak{A}$ is a σ-measurable algebra and $\tau_{\sigma\text{-m}}(\mathfrak{A})\le\max(\omega,\sup_{n\in\mathbb{N}}\tau(\mathfrak{C}_n/\nu_n^{-1}[\{0\}]))$.

547M Proposition Let $\mathfrak{A}$ be a Boolean algebra.

(a) If $\mathfrak{A}$ is ccc and Dedekind complete and is not a measurable algebra, then it has a non-zero principal ideal which is nowhere measurable.

(b) If $\mathfrak{A}$ is nowhere measurable and Dedekind complete and $\mathfrak{B}$ is an order-closed subalgebra of $\mathfrak{A}$ which is a measurable algebra, then for any $a\in\mathfrak{A}$ there is a $d\subseteq a$ such that $\operatorname{upr}(d,\mathfrak{B})=\operatorname{upr}(a\setminus d,\mathfrak{B})=\operatorname{upr}(a,\mathfrak{B})$.

(c) Suppose that $\mathfrak{A}$ is nowhere measurable and Dedekind complete, $\nu:\mathfrak{A}\to[0,\infty[$ is a submeasure and $\mathfrak{B}_0,\dots,\mathfrak{B}_n$ are order-closed subalgebras of $\mathfrak{A}$ which are all measurable algebras. Then for any $a\in\mathfrak{A}$ there is a $d\subseteq a$ such that $\operatorname{upr}(a\setminus d,\mathfrak{B}_i)=\operatorname{upr}(a,\mathfrak{B}_i)$ for every $i\le n$ and $\nu d\ge 2^{-n-1}\nu a$.

(d) If $\mathfrak{A}$ is nowhere measurable and Dedekind complete and $\mathfrak{B}_0,\dots,\mathfrak{B}_n$ are order-closed subalgebras of $\mathfrak{A}$ which are all measurable algebras, then there are disjoint $a_0,\dots,a_n\in\mathfrak{A}$ such that $\operatorname{upr}(a_i,\mathfrak{B}_i)=1$ for every $i\le n$.

(e) If $\mathfrak{A}$ is nowhere measurable and Dedekind complete, $\mathfrak{B}_0,\dots,\mathfrak{B}_n$ are order-closed subalgebras of $\mathfrak{A}$ which are all measurable algebras and $a\in\mathfrak{A}$, then there is a $d\subseteq a$ such that $\operatorname{upr}(d,\mathfrak{B}_i)=\operatorname{upr}(a\setminus d,\mathfrak{B}_i)=\operatorname{upr}(a,\mathfrak{B}_i)$ for every $i\le n$.

547N Proposition If $\mathfrak{A}$ is a non-trivial nowhere measurable σ-measurable algebra, then $\mathfrak{G}_\omega$ can be regularly embedded in $\mathfrak{A}$.

547O Lemma Let $(\mathfrak{A},\bar\mu)$ be a probability algebra. Suppose that $\langle a_\sigma\rangle_{\sigma\in S_2}$ is a family in $\mathfrak{A}$ such that

$$a_\sigma=a_{\sigma^\frown<0>}\cup a_{\sigma^\frown<1>}=\sup_{\tau\in S_2,\tau\supseteq\sigma}a_{\tau^\frown<0>}\cap a_{\tau^\frown<1>}$$

for every $\sigma\in S_2$. For $A\subseteq S_2$ set $c_A=\sup_{\sigma\in A}a_\sigma$.

(a) For every $k\ge1$ and $\epsilon>0$ there is an $m\in\mathbb{N}$ such that

$$\bar\mu(\sup_{I\in[\{0,1\}^m]^k}\inf_{\sigma\in I}a_\sigma)\ge\bar\mu a_\emptyset-\epsilon.$$

(b) For A, $B\subseteq S_2$ I will say that $A\perp B$ if σ and τ are incompatible whenever $\sigma\in A$ and $\tau\in B$. Now for any $\epsilon>0$ there are finite A, $B\subseteq S_2$ such that $A\perp B$ and $\bar\mu c_A$, $\bar\mu c_B$ are both at least $\bar\mu a_\emptyset-\epsilon$.

(c) For any $\epsilon>0$ there is a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of finite subsets of S_2 such that $\bar\mu(\inf_{n\in\mathbb{N}}c_{A_n})\ge\bar\mu a_\emptyset-\epsilon$ and $A_m\perp A_n$ whenever $m<n$ in $\mathbb{N}$.

547P Proposition Let $\mathfrak{A}$ be a σ-measurable Boolean algebra. If $\mathfrak{C}$ is an order-closed subalgebra of $\mathfrak{A}$ of countable Maharam type, there is a $c\in\mathfrak{C}$ such that the principal ideal $\mathfrak{C}_c$ has an e-h family and its complement $\mathfrak{C}_{1\setminus c}$ is a measurable algebra.

547Q Lemma Let $\mathfrak{A}$ be a σ-measurable algebra. Set $\lambda=\max(\omega,\tau_{\sigma\text{-m}}(\mathfrak{A}))$ and $\kappa=\operatorname{non}\mathcal{M}(\lambda^{\mathbb{N}})$, where in this product λ is given its discrete topology. Then there is a family $\langle a_{\xi n}\rangle_{\xi<\kappa,n\in\mathbb{N}}$ in $\mathfrak{A}$ such that

whenever $\langle d_{m\sigma}\rangle_{m\in\mathbb{N},\sigma\in S_2}$ is a family in $\mathfrak{A}$ such that

$$d_{m\tau}\subseteq d_{m,\tau^\frown\sigma}\text{ for every }\sigma\in S_2,\quad\sup_{\sigma\in S_2}d_{m,\tau^\frown\sigma}=1$$

for every $\tau\in S_2$ and $m\in\mathbb{N}$, there is a $\xi<\kappa$ such that

$$\sup_{\sigma\in S_2}\big(d_{m\sigma}\cap\inf_{\substack{i<\#(\sigma)\\ \sigma(i)=1}}a_{\xi i}\setminus\sup_{\substack{i<\#(\sigma)\\ \sigma(i)=0}}a_{\xi i}\big)=1$$

for every $m\in\mathbb{N}$.

547R Theorem Suppose that X is a set and $\mathcal{I}$ is a proper σ-ideal of subsets of X such that the quotient algebra $\mathfrak{A}=\mathcal{P}X/\mathcal{I}$ is atomless and σ-measurable. Then $\tau_{\sigma\text{-m}}(\mathfrak{A})>\operatorname{add}\mathcal{I}$.

547S Corollary If a non-zero σ-measurable algebra $\mathfrak{A}$ is also an atomless power set σ-quotient algebra, then there is a quasi-measurable cardinal less than $\tau_{\sigma\text{-m}}(\mathfrak{A})$.

547Z Problems (a) Can $\mathfrak{G}_{\omega_2}$ be a power set σ-quotient algebra?

(b) Can there be a power set σ-quotient algebra $\mathfrak{A}$ such that $c(\mathfrak{A}) = \omega$ and $\pi(\mathfrak{A}) = \omega_1$?

(c) Let $\mathfrak{A}$ be a non-purely-atomic Maharam algebra with countable Maharam type. Can $\mathfrak{A}$ be a power set σ-quotient algebra? What about algebras constructed as in 394B-394M?

(d) Find a σ-measurable algebra with an order-closed subalgebra which is not σ-measurable.

(e) Is there a σ-measurable algebra $\mathfrak{A}$ such that $\tau(\mathfrak{A}) < \tau_{\sigma\text{-m}}(\mathfrak{A})$?

Version of 30.6.21

548 Selectors and disjoint refinements

We come now to a remarkable result (548C) which is a minor extension of the principal theorem of Kumar & Shelah 17. This leads directly to 548E, which is a corresponding elaboration of the main theorem of Gitik & Shelah 01. Both of these results apply to spaces whose Maharam types are not too large, so give interesting facts about Lebesgue measure not dependent on special axioms (548F). A similar restriction on shrinking number leads to further results of this kind (548G-548H) which are not necessarily applicable to Lebesgue measure. If we choose an easier target other methods are available (548I-548K).

548A Lemma Let X be a set, $\mathcal{I}$ a σ-ideal of subsets of X, $\langle (Y_n, \mathrm{T}_n, \nu_n)\rangle_{n\in\mathbb{N}}$ a sequence of totally finite measure spaces and $\langle f_n\rangle_{n\in\mathbb{N}}$ a sequence of functions, each f_n being an injective function from a subset of X to Y_n. Suppose that for every $A \in \mathcal{P}X \setminus \mathcal{I}$ there are an $n \in \mathbb{N}$ and an $F \in \mathrm{T}_n$ such that $\nu_n F > 0$ and $F \subseteq f_n[A]$. Then $\mathfrak{A} = \mathcal{P}X/\mathcal{I}$ is σ-measurable and $\tau_{\sigma\text{-m}}(\mathfrak{A}) \le \max(\omega, \sup_{n\in\mathbb{N}} \tau(\nu_n))$.

548B Lemma Lat (X, Σ, μ) be a probability space and $\langle X_i\rangle_{i\in\mathbb{N}}$ a partition of X. Then μ can be extended to a probability measure ν measuring every X_i and with Maharam type $\tau(\nu) \le \max(\omega, \tau(\mu))$.

548C Theorem Suppose that (X, Σ, μ) is an atomless σ-finite measure space and that there is no quasi-measurable cardinal less than the Maharam type of μ. Let $R \subseteq X \times X$ be an equivalence relation with countable equivalence classes. Then there is an R-free set which has full outer measure in X.

548D Lemma Let (X, Σ, μ) be a measure space. Then the following are equiveridical:

(i) whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of subsets of X there is a disjoint sequence $\langle A'_n\rangle_{n\in\mathbb{N}}$ of sets such that $A'_n \subseteq A_n$ and $\mu^*(A'_n) = \mu^*(A_n)$ for every $n \in \mathbb{N}$;

(ii) whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of subsets of X there is a set $D \subseteq X$ such that $\mu^*(A_n \cap D) = \mu^*(A_n \setminus D) = \mu^* A_n$ for every $n \in \mathbb{N}$.

548E Theorem Let (X, Σ, μ) be an atomless σ-finite measure space such that there is no quasi-measurable cardinal less than the Maharam type of μ. Then for any sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of subsets of X there is a disjoint sequence $\langle A'_n\rangle_{n\in\mathbb{N}}$ such that $A'_n \subseteq A_n$ and $\mu^* A'_n = \mu^* A_n$ for every $n \in \mathbb{N}$.

548F Corollary Let μ be Lebesgue measure on $\mathbb{R}$.

(a) Let X be a subset of $\mathbb{R}$ and $\sim$ an equivalence relation on X with countable equivalence classes. Then there is a subset of X, with full outer measure for the subspace measure on X, which meets each equivalence class in at most one point.

(b) Let $\langle A_n\rangle_{n\in\mathbb{N}}$ be any sequence of subsets of $\mathbb{R}$. Then there is a disjoint sequence $\langle A'_n\rangle_{n\in\mathbb{N}}$ such that A'_n is a subset of A_n, with the same outer measure as A_n, for every n.

(c) Let $\langle A_n\rangle_{n\in\mathbb{N}}$ be any sequence of subsets of $\mathbb{R}$. Then there is a $D \subseteq \mathbb{R}$ such that $\mu^*(A_n \cap D) = \mu^*(A_n \setminus D) = \mu^* A_n$ for every $n \in \mathbb{N}$.

548G Lemma Let (X,Σ,μ) be a totally finite measure space in which singleton sets are negligible and suppose that there is no quasi-measurable cardinal less than or equal to the shrinking number $\operatorname{shr}\mathcal{N}(\mu)$. Then for any $A\subseteq X$ there is a disjoint family $\langle A_\xi\rangle_{\xi<\omega_1}$ of subsets of A such that $\mu^*A_\xi=\mu^*A$ for every $\xi<\omega_1$.

548H Proposition Let (X,Σ,μ) be a totally finite measure space. Suppose that for every $A\subseteq X$ there is a partition $\langle A_\xi\rangle_{\xi<\omega_1}$ of A such that $\mu^*A_\xi=\mu^*A$ for every $\xi<\omega_1$.

(a) If R is an equivalence relation on X with countable equivalence classes, there is an R-free set with full outer measure.

(b) For any sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of subsets of X there is a disjoint sequence $\langle A'_n\rangle_{n\in\mathbb{N}}$ such that $A'_n\subseteq A_n$ and $\mu^*A'_n=\mu^*A_n$ for every $n\in\mathbb{N}$.

(c) For any sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of subsets of X, there is a $D\subseteq X$ such that $\mu^*(A_n\cap D)=\mu^*(A_n\setminus D)=\mu^*A_n$ for every $n\in\mathbb{N}$.

548I Proposition Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and κ a cardinal. Then the following are equiveridical:

(i) $\kappa<\pi(\mathfrak{A}_d)$ for every $d\in\mathfrak{A}^+=\mathfrak{A}\setminus\{0\}$, writing $\mathfrak{A}_d$ for the principal ideal generated by d;

(ii) whenever $A\subseteq\mathfrak{A}^+$ and $\#(A)\le\kappa$ there is a $b\in\mathfrak{A}$ such that $a\cap b$ and $a\setminus b$ are both non-zero for every $a\in A$.

548J Proposition Let (X,Σ,μ) be a strictly localizable measure space with null ideal $\mathcal{N}(\mu)$, and κ a cardinal such that

(*) whenever $\mathcal{E}\in[\Sigma\setminus\mathcal{N}(\mu)]^{\le\kappa}$ and $F\in\Sigma\setminus\mathcal{N}(\mu)$, there is a non-negligible measurable $G\subseteq F$ such that $E\setminus G$ is non-negligible for every $E\in\mathcal{E}$.

Then whenever $\langle A_\xi\rangle_{\xi<\kappa}$ is a family of non-negligible subsets of X, there is a $G\in\Sigma$ such that $A_\xi\cap G$ and $A_\xi\setminus G$ are non-negligible for every $\xi<\kappa$.

548K Corollary Let (X,Σ,μ) be an atomless quasi-Radon measure space and $\langle A_\xi\rangle_{\xi<\omega_1}$ a family of non-negligible subsets of X. Then there is a $D\subseteq X$ such that $A_\xi\cap D$ and $A_\xi\setminus D$ are non-negligible for every $\xi<\omega_1$.

548Z Problems (a) Suppose that $\langle A_\xi\rangle_{\xi<\omega_1}$ is a family of subsets of $[0,1]$. Must there be a set $D\subseteq[0,1]$ such that $\mu^*(A_\xi\cap D)=\mu^*(A_\xi\setminus D)=\mu^*A_\xi$ for every $\xi<\omega_1$, where μ is Lebesgue measure on $[0,1]$?

(b) Suppose that there is no quasi-measurable cardinal. Let (X,Σ,μ) be an atomless probability space and $\langle A_\xi\rangle_{\xi<\omega_1}$ a family of subsets of X. Must there be a disjoint family $\langle A'_\xi\rangle_{\xi<\omega_1}$ such that $A'_\xi\subseteq A_\xi$ and $\mu^*A'_\xi=\mu^*A_\xi$ for every $\xi<\omega_1$?

(c) (P.Komjath) Suppose that $X\subseteq\mathbb{R}^2$. Must there be a set $A\subseteq X$, of the same Lebesgue outer measure as X, such that $\|x-y\|\notin\mathbb{Q}$ whenever x, $y\in A$ are distinct? (See 548Xb.)