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Chapter 54

Real-valued-measurable cardinals

Of the many questions in measure theory which involve non-trivial set theory, perhaps the first to have been recognised as fundamental is what I call the 'Banach-Ulam problem': is there a non-trivial measure space in which every set is measurable? In various forms, this question has arisen repeatedly in earlier volumes of this treatise (232Hc, 363S, 438A). The time has now come for an account of the developments of the last fifty years.

The measure theory of this chapter will begin in §543; the first two sections deal with generalizations to wider contexts. If ν is a probability measure with domain $\mathcal{P}X$, its null ideal is ω_1-additive and ω_1-saturated in $\mathcal{P}X$. In §541 I look at ideals $\mathcal{I} \triangleleft \mathcal{P}X$ such that $\mathcal{I}$ is simultaneously κ-additive and κ-saturated for some κ; this is already enough to lead us to a version of the Keisler-Tarski theorem on normal ideals (541J), a great strengthening of Ulam's theorem on inaccessibility of real-valued-measurable cardinals (541Lc), a form of Ulam's dichotomy (541P), and some very striking infinitary combinatorics (541Q-541S). In §542 I specialize to the case $\kappa = \omega_1$, still without calling on the special properties of null ideals, with more combinatorics (542E, 542I).

I have said many times in the course of this treatise that almost the first thing to ask about any measure is, what does its measure algebra look like? For an atomless probability measure with domain $\mathcal{P}X$, the Gitik-Shelah theorem (543E-543F) gives a great deal of information, associated with a tantalizing problem (543Z). §544 is devoted to the measure-theoretic consequences of assuming that there is some atomlessly-measurable cardinal, with results on repeated integration (544C, 544I, 544J), the null ideal of a normal witnessing probability (544E-544F) and regressive functions (544M).

I do not discuss consistency questions in this chapter (I will touch on some of them in Chapter 55). The ideas of §§541-544 would be in danger of becoming irrelevant if it turned out that there can be no two-valued-measurable cardinal. I have no real qualms about this. One of my reasons for confidence is the fact that very much stronger assumptions have been investigated without any hint of catastrophe. Two of these, the 'product measure extension axiom' and the 'normal measure axiom' are mentioned in §545.

One way of looking at the Gitik-Shelah theorem is to say that if X is a set and $\mathcal{I}$ is a proper σ-ideal of subsets of X, then $\mathcal{P}X/\mathcal{I}$ cannot be an atomless measurable algebra of small Maharam type. We can ask whether there are further theorems of this kind provable in ZFC. Two such results are in §547: the 'Gitik-Shelah theorem for category' (547F-547G), showing that $\mathcal{P}X/\mathcal{I}$ cannot be isomorphic to $\mathrm{RO}(\mathbb{R})$, and 547R, showing that 'σ-measurable' algebras of moderate complexity also cannot appear as power set σ-quotient algebras. This leads us to a striking fact about free sets for relations with countable equivalence classes (548C) and thence to disjoint refinements of sequences of sets (548E).

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541 Saturated ideals

If ν is a totally finite measure with domain $\mathcal{P}X$ and null ideal $\mathcal{N}(\nu)$, then its measure algebra $\mathcal{P}X/\mathcal{N}(\nu)$ is ccc, that is to say, $\mathrm{sat}(\mathcal{P}X/\mathcal{N}(\nu)) \le \omega_1$; while the additivity of $\mathcal{N}(\nu)$ is at least ω_1. It turns out that an ideal $\mathcal{I}$ of $\mathcal{P}X$ such that $\mathrm{sat}(\mathcal{P}X/\mathcal{I}) \le \mathrm{add}\,\mathcal{I}$ is either trivial or extraordinary. In this section I present a little of the theory of such ideals. To begin with, the quotient algebra has to be Dedekind complete (541B). Further elementary ideas are in 541C (based on a method already used in §525) and 541D-541E. In a less expected direction, we have a useful fact concerning transversal numbers $\mathrm{Tr}_{\mathcal{I}}(X;Y)$ (541F).

The most remarkable properties of saturated ideals arise because of their connexions with 'normal' ideals (541G). These ideals share the properties of non-stationary ideals (541H-541I). If $\mathcal{I}$ is an $(\mathrm{add}\,\mathcal{I})$-saturated

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ideal of $\mathcal{P}X$, we have corresponding normal ideals on $\operatorname{add}\mathcal{I}$ (541J). Now there can be a κ-saturated normal ideal on κ only if there is a great complexity of cardinals below κ (541L).

The original expression of these ideas (Keisler & Tarski 64) concerned 'two-valued-measurable' cardinals, on which we have normal ultrafilters (541M). The dichotomy of Ulam 1930 (438Ce-438Cf) reappears in the context of κ-saturated normal ideals (541P). For κ-saturated ideals, 'normality' implies some far-reaching extensions (541Q). Finally, I include a technical lemma concerning the covering numbers $\operatorname{cov}_{\mathrm{Sh}}(\alpha,\beta,\gamma,\delta)$ (541S).

541A Definition If $\mathfrak{A}$ is a Boolean algebra, I is an ideal of $\mathfrak{A}$ and κ is a cardinal, I will say that I is **κ-saturated** in $\mathfrak{A}$ if $\kappa\ge\operatorname{sat}(\mathfrak{A}/I)$; that is, if for every family $\langle a_\xi\rangle_{\xi<\kappa}$ in $\mathfrak{A}\setminus I$ there are distinct ξ, $\eta<\kappa$ such that $a_\xi\cap a_\eta\notin I$.

541B Proposition Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and I an ideal of $\mathfrak{A}$ which is $(\operatorname{add}I)^+$-saturated in $\mathfrak{A}$. Then the quotient algebra $\mathfrak{A}/I$ is Dedekind complete.

proof Take any $B\subseteq\mathfrak{A}/I$. Let C be the set of those $c\in\mathfrak{A}/I$ such that *either* $c\cap b=0$ for every $b\in B$ *or* there is a $b\in B$ such that $c\subseteq b$. Then C is order-dense in $\mathfrak{A}/I$, so there is a partition of unity $D\subseteq C$ (313K). Enumerate D as $\langle d_\xi\rangle_{\xi<\kappa}$, where

$$\kappa=\#(D)<\operatorname{sat}(\mathfrak{A}/I)\le(\operatorname{add}I)^+.$$

For each $\xi<\kappa$, choose $a_\xi\in\mathfrak{A}$ such that $a_\xi^\bullet=d_\xi$. Now $\#(\xi)<\kappa\le\operatorname{add}I$, so $\sup_{\eta<\xi}a_\xi\cap a_\eta\in I$. Set $\tilde a_\xi=a_\xi\setminus\sup_{\eta<\xi}a_\eta$; then $\tilde a_\xi^\bullet=a_\xi^\bullet=d_\xi$. Set $L=\{\xi:\xi<\kappa,\ d_\xi\subseteq b$ for some $b\in B\}$ and $a=\sup_{\xi\in L}\tilde a_\xi$ in $\mathfrak{A}$.

? If $b\in B$ and $b\not\subseteq a^\bullet$, there must be a $\xi<\kappa$ such that $d_\xi\cap(b\setminus a^\bullet)\ne 0$. But $d_\xi\in C$, so there must be a $b'\in B$ such that $d_\xi\subseteq b'$; accordingly $\xi\in L$, $\tilde a_\xi\subseteq a$ and $d_\xi\subseteq a^\bullet$. **X** Thus $a^\bullet$ is an upper bound of B in $\mathfrak{A}/I$.

? If there is an upper bound b^* of B such that $a^\bullet\not\subseteq b^*$, there must be a $\xi<\kappa$ such that $d_\xi\cap a^\bullet\setminus b^*\ne 0$. As $d_\xi\not\subseteq b^*$, $d_\xi\not\subseteq b$ for every $b\in B$, and $\xi\notin L$. But this means that $\tilde a_\xi\cap\tilde a_\eta=0$ for every $\eta\in L$, so $\tilde a_\xi\cap a=0$ (313Ba) and $d_\xi\cap a^\bullet=0$. **X**

Thus $a^\bullet=\sup B$ in $\mathfrak{A}/I$; as B is arbitrary, $\mathfrak{A}/I$ is Dedekind complete.

541C Proposition Let X be a set, κ a regular infinite cardinal, Σ an algebra of subsets of X such that $\bigcup\mathcal{E}\in\Sigma$ whenever $\mathcal{E}\subseteq\Sigma$ and $\#(\mathcal{E})<\kappa$, and $\mathcal{I}$ a κ-saturated κ-additive ideal of Σ.

(a) If $\mathcal{E}\subseteq\Sigma$ there is an $\mathcal{E}'\in[\mathcal{E}]^{<\kappa}$ such that $E\setminus\bigcup\mathcal{E}'\in\mathcal{I}$ for every $E\in\mathcal{E}$.

(b) If $\langle E_\xi\rangle_{\xi<\kappa}$ is any family in $\Sigma\setminus\mathcal{I}$, and $\theta<\kappa$ is a cardinal, then $\{x:x\in X,\ \#(\{\xi:x\in E_\xi\})\ge\theta\}$ includes a member of $\Sigma\setminus\mathcal{I}$ (and, in particular, is not empty).

(c) Suppose that no element of $\Sigma\setminus\mathcal{I}$ can be covered by κ members of $\mathcal{I}$. Then κ is a precaliber of $\Sigma/\mathcal{I}$.

proof Write $\mathfrak{A}$ for $\Sigma/\mathcal{I}$.

(a) Consider $A=\{E^\bullet:E\in\mathcal{E}\}\subseteq\mathfrak{A}$. By 514Db, there is an $\mathcal{E}'\in[\mathcal{E}]^{<\operatorname{sat}(\mathfrak{A})}$ such that $\{E^\bullet:E\in\mathcal{E}'\}$ has the same upper bounds as A. Now $\#(\mathcal{E}')<\operatorname{sat}(\mathfrak{A})\le\kappa$, so $F=\bigcup\mathcal{E}'$ belongs to Σ, and $F^\bullet$ must be an upper bound for A, that is, $E\setminus F\in\mathcal{I}$ for every $E\in\mathcal{E}$.

(b) For $\alpha\le\beta<\kappa$ set $F_{\alpha\beta}=\bigcup_{\alpha\le\xi<\beta}E_\xi\in\Sigma$. Then for every $\alpha<\kappa$ we have a $g(\alpha)<\kappa$ such that $g(\alpha)\ge\alpha$ and $E_\xi\setminus F_{\alpha,g(\alpha)}\in\mathcal{I}$ whenever $g(\alpha)\le\xi<\kappa$, by (a) and the regularity of κ.

Define $\langle h(\alpha)\rangle_{\alpha<\kappa}$ by setting $h(0)=0$, $h(\alpha+1)=g(h(\alpha))$ for each $\alpha<\kappa$, and $h(\alpha)=\sup_{\beta<\alpha}h(\beta)$ for non-zero limit ordinals $\alpha<\kappa$. Set $G_\alpha=F_{h(\alpha),h(\alpha+1)}$ for each α. If $\beta<\alpha$, then

$$G_\alpha\setminus G_\beta=\bigcup_{h(\alpha)\le\xi<h(\alpha+1)}E_\xi\setminus F_{h(\beta),g(h(\beta))}\in\mathcal{I}$$

because $\mathcal{I}$ is κ-additive. Consequently $G_\theta\setminus\bigcap_{\beta<\theta}G_\beta$ belongs to $\mathcal{I}$; and $G_\theta\supseteq E_{h(\theta)}\notin\mathcal{I}$, so $G=\bigcap_{\beta<\theta}G_\beta\in\Sigma\setminus\mathcal{I}$. But if $x\in G$ then $\{\xi:x\in E_\xi\}$ meets $[h(\beta),h(\beta+1)[$ for every $\beta<\theta$ and has cardinal at least θ.

(c) Let $\langle a_\xi\rangle_{\xi<\kappa}$ be a family of non-zero elements in $\mathfrak{A}$. For each $\xi<\kappa$, choose $\tilde E_\xi\in\Sigma$ such that $\tilde E_\xi^\bullet=a_\xi$. Let $\mathcal{K}$ be the family of all those finite subsets K of κ such that $H_K=\bigcap_{\xi\in K}\tilde E_\xi$ belongs to $\mathcal{I}$. Now set $E_\xi=\tilde E_\xi\setminus\bigcup\{H_K:K\in\mathcal{K},\ K\subseteq\xi\}$; then $E_\xi\in\Sigma\setminus\mathcal{I}$ and $E_\xi^\bullet=a_\xi$ for each $\xi<\kappa$.

Repeat the argument of (b). Once again we get a family $\langle G_\alpha\rangle_{\alpha<\kappa}$ in $\Sigma\setminus\mathcal{I}$ such that $G_\alpha\setminus G_\beta\in\mathcal{I}$ whenever $\beta\le\alpha<\kappa$. Now, applying (a) to $\langle X\setminus G_\alpha\rangle_{\alpha<\kappa}$, we have a $\gamma<\kappa$ such that $\bigcap_{\alpha<\gamma}G_\alpha\setminus G_\beta\in\mathcal{I}$ for every

$\beta < \kappa$. On the other hand, $G_\gamma \setminus G_\alpha \in \mathcal{I}$ for every $\alpha < \gamma$, so in fact $G_\gamma \setminus G_\beta \in \mathcal{I}$ for every $\beta < \kappa$, while $G_\gamma \notin \mathcal{I}$. At this point, recall that we are now assuming that G_γ cannot be covered by $\bigcup_{\beta<\kappa} G_\gamma \setminus G_\beta$, and there is an $x \in \bigcap_{\beta<\kappa} G_\beta$.

As in (b), it follows that $\Gamma = \{\xi : \xi < \kappa, \, x \in E_\xi\}$ has cardinal κ. If $K \subseteq \Gamma$ is finite and not empty, take $\zeta \in \Gamma$ such that $K \subseteq \zeta$. Then

$$x \in E_\zeta \cap \bigcap_{\xi\in K} \tilde{E}_\xi \subseteq E_\zeta \cap H_K;$$

it follows that $K \notin \mathcal{K}$ and $H_K \notin \mathcal{I}$, that is, that $\inf_{\xi\in K} a_\xi \neq 0$ in $\mathfrak{A}$. So $\langle a_\xi \rangle_{\xi\in\Gamma}$ is centered; as $\langle a_\xi \rangle_{\xi<\kappa}$ is arbitrary, κ is a precaliber of $\mathfrak{A}$.

541D Lemma Let X be a set, $\mathcal{I}$ an ideal of $\mathcal{P}X$, Y a set of cardinal less than $\operatorname{add}\mathcal{I}$ and κ a cardinal such that $\mathcal{I}$ is $(\operatorname{cf}\kappa)$-saturated in $\mathcal{P}X$. Then for any function $f : X \to [Y]^{<\kappa}$ there is an $M \in [Y]^{<\kappa}$ such that $\{x : f(x) \not\subseteq M\} \in \mathcal{I}$.

proof If $\kappa > \#(Y)$ this is trivial; suppose that $\kappa \le \#(Y) < \operatorname{add}\mathcal{I}$. For cardinals $\lambda < \kappa$ set $X_\lambda = \{x : \#(f(x)) = \lambda\}$. If $A = \{\lambda : X_\lambda \notin \mathcal{I}\}$ then $\mathcal{I}$ is not $\#(A)$-saturated in $\mathcal{P}X$, so $\#(A) < \operatorname{cf}\kappa$ and $\theta = \sup A$ is less than κ. Set $X' = \{x : \#(f(x)) \le \theta\}$; then $X \setminus X'$ is the union of at most $\lambda < \operatorname{add}\mathcal{I}$ members of $\mathcal{I}$, so belongs to $\mathcal{I}$.

For each $x \in X'$ let $\langle h_\xi(x) \rangle_{\xi<\theta}$ run over a set including $f(x)$. For each $\xi < \theta$,

$$Y_\xi = \{y : h_\xi^{-1}[\{y\}] \notin \mathcal{I}\}$$

has cardinal less than $\operatorname{cf}\kappa$, and because $\#(Y) < \operatorname{add}\mathcal{I}$, $h_\xi^{-1}[Y \setminus Y_\xi] \in \mathcal{I}$. Set $M = \bigcup_{\xi<\theta} Y_\xi \in [Y]^{<\kappa}$. (If κ is regular, M is the union of fewer than κ sets with cardinal less than κ, so $\#(M) < \kappa$; if κ is not regular, then M is the union of fewer than κ sets with cardinal at most $\operatorname{cf}\kappa$, so again $\#(M) < \kappa$.) Because $\theta < \operatorname{add}\mathcal{I}$,

$$\{x : f(x) \not\subseteq M\} \subseteq (X \setminus X') \cup \bigcup_{\xi<\theta} h_\xi^{-1}[Y \setminus Y_\xi] \in \mathcal{I},$$

as required.

541E Corollary Let X be a set, $\mathcal{I}$ an ideal of $\mathcal{P}X$, Y a set of cardinal less than $\operatorname{add}\mathcal{I}$ and κ a cardinal such that $\mathcal{I}$ is $(\operatorname{cf}\kappa)$-saturated in $\mathcal{P}X$. Then for any function $g : X \to Y$ there is an $M \in [Y]^{<\kappa}$ such that $g^{-1}[Y \setminus M] \in \mathcal{I}$.

proof Apply 541D to the function $x \mapsto \{g(x)\}$. (In the trivial case $\kappa = 1$, $\mathcal{I} = \mathcal{P}X$.)

541F Lemma Let X and Y be sets, κ a regular uncountable cardinal, and $\mathcal{I}$ a proper κ-saturated κ-additive ideal of subsets of X. Then $\operatorname{Tr}_{\mathcal{I}}(X;Y)$ (definition: 5A1Ma) is attained, in the sense that there is a set $G \subseteq Y^X$ such that $\#(G) = \operatorname{Tr}_{\mathcal{I}}(X;Y)$ and $\{x : x \in X, \, g(x) = g'(x)\} \in \mathcal{I}$ for all distinct g, $g' \in G$.

proof It is enough to consider the case in which $Y = \lambda$ is a cardinal. Set $\theta = \operatorname{Tr}_{\mathcal{I}}(X;\lambda)$.

(a) If $\lambda^+ < \kappa$ then $\theta \le \lambda$. **P?** Suppose, if possible, that we have a family $\langle f_\xi \rangle_{\xi<\lambda^+}$ in λ^X such that $\{x : f_\xi(x) = f_\eta(x)\} \in \mathcal{I}$ whenever $\eta < \xi < \lambda^+$. Then λ is surely infinite, so λ^+ is uncountable and regular. For each $x \in X$ there is an $\alpha_x < \lambda^+$ such that $\{f_\xi(x) : \xi < \alpha_x\} = \{f_\xi(x) : \xi < \lambda^+\}$. Setting

$$F_\alpha = \{x : x \in X, \, \alpha_x = \alpha\} \subseteq \bigcup_{\eta<\alpha} \{x : f_\eta(x) = f_\alpha(x)\},$$

we see that $F_\alpha \in \mathcal{I}$ for each $\alpha < \lambda^+$. But $X = \bigcup_{\alpha<\lambda^+} F_\alpha$ and $\lambda^+ < \kappa$, so this is impossible. **XQ**

Since we can surely find a family $\langle f_\xi \rangle_{\xi<\lambda}$ in λ^X such that $f_\xi(x) \neq f_\eta(x)$ whenever $x \in X$ and $\eta < \xi < \lambda$, we have the result when $\lambda^+ < \kappa$.

(b) We may therefore suppose from now on that $\lambda^+ \ge \kappa$.

If $H \subseteq \lambda^X$ is such that

$$F = \{f : f \in \lambda^X, \, \{x : f(x) \le h(x)\} \in \mathcal{I} \text{ for every } h \in H\} \neq \emptyset,$$

then there is an $f_0 \in F$ such that

$$\{x : f(x) < f_0(x)\} \in \mathcal{I} \text{ for every } f \in F.$$

P? If not, choose a family $\langle f_\xi\rangle_{\xi<\kappa}$ in F inductively, as follows. f_0 is to be any member of F. Given f_ξ, there is an $f\in F$ such that $\{x:f(x)<f_\xi(x)\}\notin\mathcal{I}$; set $f_{\xi+1}(x)=\min(f(x),f_\xi(x))$ for every x; then $f_{\xi+1}\in F$. Given that $f_\eta\in F$ for every $\eta<\xi$, where $\xi<\kappa$ is a non-zero limit ordinal, set $f_\xi(x)=\min_{\eta<\xi}f_\eta(x)$ for each x; then for any $h\in H$ we shall have

$$\{x:f_\xi(x)\le h(x)\}=\bigcup_{\eta<\xi}\{x:f_\eta(x)\le h(x)\}\in\mathcal{I},$$

so $f_\xi\in F$ and the induction continues.

Now consider

$$E_\xi=\{x:f_{\xi+1}(x)<f_\xi(x)\}\in\mathcal{P}X\setminus\mathcal{I}$$

for $\xi<\kappa$. By 541Cb there is an $x\in X$ such that $A=\{\xi:x\in E_\xi\}$ is infinite. But if $\langle\xi(n)\rangle_{n\in\mathbb{N}}$ is any strictly increasing sequence in A, $\langle f_{\xi(n)}(x)\rangle_{n\in\mathbb{N}}$ is a strictly decreasing sequence of ordinals, which is impossible. **X** **Q**

(c) Choose a family $\langle g_\xi\rangle_{\xi<\delta}$ in λ^X as follows. Given $\langle g_\eta\rangle_{\eta<\xi}$, set

$$F_\xi=\{f:f\in\lambda^X,\ \{x:f(x)\le g_\eta(x)\}\in\mathcal{I}\text{ for every }\eta<\xi\}.$$

If $F_\xi=\emptyset$, set $\delta=\xi$ and stop. If $F_\xi\ne\emptyset$ use (b) to find $g_\xi\in F_\xi$ such that $\{x:f(x)<g_\xi(x)\}\in\mathcal{I}$ for every $f\in F_\xi$, and continue. Note that for $\xi<\min(\lambda,\kappa)$, $\{x:g_\xi(x)\ne\xi\}\in\mathcal{I}$. **P** Induce on ξ. If $\xi<\min(\lambda,\kappa)$ and $\{x:g_\eta(x)\ne\eta\}\in\mathcal{I}$ for every $\eta<\xi$, then the constant function with value ξ belongs to F_ξ, so g_ξ is defined and $\{x:g_\xi(x)>\xi\}\in\mathcal{I}$. On the other hand, $\{x:g_\xi(x)=\eta\}\in\mathcal{I}$ for $\eta<\xi$; as $\xi<\operatorname{add}\mathcal{I}$, $\{x:g_\xi(x)<\xi\}\in\mathcal{I}$. **Q** Accordingly $\delta\ge\min(\lambda,\kappa)$.

(d) Because $g_\xi\in F_\xi$, $\{x:g_\xi(x)=g_\eta(x)\}\in\mathcal{I}$ whenever $\eta<\xi<\delta$, so $\#(\delta)\le\theta$. On the other hand, suppose that $F\subseteq\lambda^X$ is such that $\{x:f(x)=f'(x)\}\in\mathcal{I}$ for all distinct f, $f'\in F$. For each $f\in F$, set

$$\zeta'_f=\min\{\xi:\xi\le\delta,\ f\notin F_\xi\};$$

this must be defined because $F_\delta=\emptyset$. Also $F_0=\lambda^X$ and $F_\xi=\bigcap_{\eta<\xi}F_\eta$ if $\xi\le\delta$ is a non-zero limit ordinal, so ζ'_f must be a successor ordinal; let ζ_f be its predecessor. We have $f\in F_{\zeta_f}$ and

$$\{x:f(x)<g_{\zeta_f}(x)\}\in\mathcal{I},\quad\{x:f(x)\le g_{\zeta_f}(x)\}\notin\mathcal{I},$$

so that

$$E_f=\{x:f(x)=g_{\zeta_f}(x)\}\notin\mathcal{I}.$$

If f, f' are distinct members of F and $\zeta_f=\zeta_{f'}$, then $E_f\cap E_{f'}\in\mathcal{I}$. So

$$\{f:f\in F,\ \zeta_f=\zeta\}$$

must have cardinal less than κ for every $\zeta<\delta$.

If $\kappa=\lambda^+$, $\#(\{f:f\in F,\ \zeta_f=\zeta\})\le\lambda$ for every $\zeta<\delta$, so $\#(F)\le\max(\delta,\lambda)=\delta$. On the other hand, if $\kappa\le\lambda$, then $\#(F)\le\max(\delta,\kappa)=\delta$. As F is arbitrary, $\theta=\delta$ and we may take $G=\{g_\xi:\xi<\delta\}$ as our witness that $\operatorname{Tr}_{\mathcal{I}}(X;\lambda)$ is attained.

541G Definition Let κ be a regular uncountable cardinal. A **normal ideal** on κ is a proper ideal $\mathcal{I}$ of $\mathcal{P}\kappa$, including $[\kappa]^{<\kappa}$, such that

$$\{\xi:\xi<\kappa,\ \xi\in\bigcup_{\eta<\xi}I_\eta\}$$

belongs to $\mathcal{I}$ for every family $\langle I_\xi\rangle_{\xi<\kappa}$ in $\mathcal{I}$. It is easy to check that a proper ideal $\mathcal{I}$ of $\mathcal{P}\kappa$ is normal iff the dual filter $\{\kappa\setminus I:I\in\mathcal{I}\}$ is normal in the sense of 4A1Ic.

541H Proposition Let κ be a regular uncountable cardinal and $\mathcal{I}$ a proper ideal of $\mathcal{P}\kappa$ including $[\kappa]^{<\kappa}$. Then the following are equiveridical:

(i) $\mathcal{I}$ is normal;

(ii) $\mathcal{I}$ is κ-additive and whenever $S\in\mathcal{P}\kappa\setminus\mathcal{I}$ and $f:S\to\kappa$ is regressive, then there is an $\alpha<\kappa$ such that $\{\xi:\xi\in S,\ f(\xi)\le\alpha\}$ is not in $\mathcal{I}$;

(iii) whenever $S\in\mathcal{P}\kappa\setminus\mathcal{I}$ and $f:S\to\kappa$ is regressive, then there is a $\beta<\kappa$ such that $\{\xi:\xi\in S,\ f(\xi)=\beta\}$ is not in $\mathcal{I}$.

proof (i)⇒(ii) Suppose that $\mathcal{I}$ is normal.

(α) (Cf. 4A1J.) Suppose that $\langle I_\eta\rangle_{\eta<\alpha}$ is a family in $\mathcal{I}$, where $0<\alpha<\kappa$, and $I=\bigcup_{\eta<\alpha}I_\eta$. Then $I\setminus\alpha\subseteq\{\xi:\xi\in\bigcup_{\eta<\xi}I_\eta\}$ belongs to $\mathcal{I}$; as $\alpha\in\mathcal{I}$, $I\in\mathcal{I}$; as $\langle I_\eta\rangle_{\eta<\alpha}$ is arbitrary, $\mathcal{I}$ is κ-additive.

(β) Take S and f as in (ii). **?** If $I_\alpha=\{\xi:\xi\in S,\ f(\xi)\le\alpha\}$ belongs to $\mathcal{I}$ for every α, then $I=\{\xi:\xi<\kappa,\ \xi\in\bigcup_{\alpha<\xi}I_\alpha\}$ belongs to $\mathcal{I}$. But if $\xi\in S$ then $f(\xi)<\xi$ and $\xi\in I_{f(\xi)}$, so $S\subseteq I$. **X** As S and f are arbitrary, (ii) is true.

(ii)⇒(iii) Suppose (ii) is true and that S, f are as in (iii). By (ii), there is an $\alpha<\kappa$ such that $\{\xi:\xi\in S,\ f(\xi)\le\alpha\}\notin\mathcal{I}$. As $\mathcal{I}$ is κ-additive, there is a $\beta\le\alpha$ such that $\{\xi:\xi\in S,\ f(\xi)=\beta\}\notin\mathcal{I}$. As S and f are arbitrary, (iii) is true.

(iii)⇒(i) Now suppose that (iii) is true, and that $\langle I_\xi\rangle_{\xi<\kappa}$ is any family in $\mathcal{I}$; set $S=\{\xi:\xi<\kappa,\ \xi\in\bigcup_{\eta<\xi}I_\eta\}$. Then we have a regressive function $f:S\to\kappa$ such that $\xi\in I_{f(\xi)}$ for every $\xi\in S$. Since $\{\xi:\xi\in S,\ f(\xi)=\beta\}\subseteq I_\beta\in\mathcal{I}$ for every $\beta<\kappa$, (iii) tells us that $S\in\mathcal{I}$. Since we are assuming that $\mathcal{I}$ is a proper ideal including $[\kappa]^{<\kappa}$, it is normal.

541I Lemma Let κ be a regular uncountable cardinal.

(a) The family of non-stationary subsets of κ is a normal ideal on κ, and is included in every normal ideal on κ.

(b) If $\mathcal{I}$ is a normal ideal on κ, and $\langle I_K\rangle_{K\in[\kappa]^{<\omega}}$ is any family in $\mathcal{I}$, then $\{\xi:\xi<\kappa,\ \xi\in\bigcup_{K\in[\xi]^{<\omega}}I_K\}$ belongs to $\mathcal{I}$.

proof (a) Let $\mathcal{I}$ be the family of non-stationary subsets of κ.

(i) Since a subset of κ is non-stationary iff it is disjoint from some closed cofinal set (4A1Ca), any subset of a non-stationary set is non-stationary. Because the intersection of two closed cofinal sets is again a closed cofinal set (4A1Bd), $\mathcal{I}$ is an ideal. Because $\kappa\setminus\xi$ is a closed cofinal set for any $\xi<\kappa$, and κ is regular, $[\kappa]^{<\kappa}\subseteq\mathcal{I}$.

Now suppose that $\langle I_\xi\rangle_{\xi<\kappa}$ is any family in $\mathcal{I}$, and that $I=\{\xi:\xi<\kappa,\ \xi\in\bigcup_{\eta<\xi}I_\eta\}$. For each $\xi<\kappa$ let F_ξ be a closed cofinal subset of κ disjoint from I_ξ, and let F be the diagonal intersection of $\langle F_\xi\rangle_{\xi<\kappa}$; then F is a closed cofinal set (4A1B(c-ii)), and it is easy to check that F is disjoint from I, so $I\in\mathcal{I}$. Thus $\mathcal{I}$ is normal.

(ii) Let $\mathcal{J}$ be any normal ideal on κ. If $F\subseteq\kappa$ is a closed cofinal set containing 0, we have a regressive function $f:\kappa\setminus F\to F$ defined by setting $f(\xi)=\sup(F\cap\xi)$ for every $\xi\in\kappa\setminus F$. If $\alpha<\kappa$, $\{\xi:f(\xi)\le\alpha\}$ is bounded above by $\min(F\setminus\alpha)$ so belongs to $[\kappa]^{<\kappa}\subseteq\mathcal{J}$; by 541H(ii), $\kappa\setminus F$ must belong to $\mathcal{J}$. This works for any closed cofinal set containing 0; but as $\{0\}$ surely belongs to J, $\kappa\setminus F\in\mathcal{J}$ for every closed cofinal set F, that is, $\mathcal{I}\subseteq\mathcal{J}$.

(b) Set $J_\xi=\bigcup_{K\in[\xi+1]^{<\omega}}I_K$; because $\mathcal{I}$ is κ-additive, $J_\xi\in\mathcal{I}$ for each ξ. Now

$$\{\xi:\xi<\kappa,\ \xi\in\bigcup_{K\in[\xi]^{<\omega}}I_K\}=\{\xi:\xi<\kappa,\ \xi\in\bigcup_{\eta<\xi}J_\eta\}\in\mathcal{I}$$

because $\mathcal{I}$ is normal.

541J Theorem (Solovay 71) Let X be a set and $\mathcal{J}$ an ideal of subsets of X. Suppose that $\operatorname{add}\mathcal{J}=\kappa>\omega$ and that $\mathcal{J}$ is λ-saturated in $\mathcal{P}X$, where $\lambda\le\kappa$. Then there are $Y\subseteq X$ and $g:Y\to\kappa$ such that $\{B:B\subseteq\kappa,\ g^{-1}[B]\in\mathcal{J}\}$ is a λ-saturated normal ideal on κ.

proof (Cf. 4A1K.) Let $\langle J_\xi\rangle_{\xi<\kappa}$ be a family in $\mathcal{J}$ such that $Y=\bigcup_{\xi<\kappa}J_\xi\notin\mathcal{J}$. Let F be the set of functions $f:Y\to\kappa$ such that $f^{-1}[\alpha]\in\mathcal{J}$ for every $\alpha<\kappa$. Set $f_0(y)=\min\{\xi:y\in J_\xi\}$ for $y\in Y$; then $f_0\in F$. **P** If $\alpha<\kappa$, then $f^{-1}[\alpha]=\bigcup_{\xi<\alpha}J_\xi$ belongs to $\mathcal{J}$ because $\mathcal{J}$ is κ-additive. **Q**

The point is that there is a $g\in F$ such that $\{y:y\in Y,\ f(y)<g(y)\}\in\mathcal{J}$ for every $f\in F$. **P?** Otherwise, choose f_ξ, for $0<\xi<\kappa$, as follows. Given $f_\xi\in F$, where $\xi<\kappa$, there is an $f\in F$ such that $A_\xi=\{y:f(y)<f_\xi(y)\}\notin\mathcal{J}$; set $f_{\xi+1}(y)=\min(f(y),f_\xi(y))$ for every y. Then

$$f_{\xi+1}^{-1}[\alpha]=f^{-1}[\alpha]\cup f_\xi^{-1}[\alpha]\in\mathcal{J}$$

for every $\alpha<\kappa$, so $f_{\xi+1}\in F$. Given that $f_\eta\in F$ for every $\eta<\xi$, where $\xi<\kappa$ is a non-zero limit ordinal, set $f_\xi(y)=\min\{f_\eta(y):\eta<\xi\}$ for each $y\in Y$; then

$$f_\xi^{-1}[\alpha]=\bigcup_{\eta<\xi}f_\eta^{-1}[\alpha]\in\mathcal{J}$$

for every $\alpha<\kappa$, because $\#(\xi)<\kappa=\operatorname{add}\mathcal{J}$.

This construction ensures that $\langle f_\xi(y)\rangle_{\xi<\kappa}$ is non-increasing for every y, and that $\{y:f_{\xi+1}(y)<f_\xi(y)\}=A_\xi\notin\mathcal{J}$ for every $\xi<\kappa$. But as $\mathcal{J}$ is κ-saturated in $\mathcal{P}X$, there must be a point y belonging to infinitely many A_ξ (541Cb), so that there is a strictly decreasing sequence in $\{f_\xi(y):\xi<\kappa\}$, which is impossible. **X** **Q**

Now consider $\mathcal{I}=\{B:B\subseteq\kappa,\ g^{-1}[B]\in\mathcal{J}\}$. Because $\mathcal{J}$ is λ-saturated in $\mathcal{P}X$, $\mathcal{I}$ is λ-saturated in $\mathcal{P}\kappa$. **P** If $\langle B_\xi\rangle_{\xi<\lambda}$ is a family in $\mathcal{P}\kappa\setminus\mathcal{I}$, then $\langle g^{-1}[B_\xi]\rangle_{\xi<\lambda}$ is a family in $\mathcal{P}X\setminus\mathcal{J}$, so there are distinct ξ, $\eta<\lambda$ such that $g^{-1}[B_\xi\cap B_\eta]=g^{-1}[B_\xi]\cap g^{-1}[B_\eta]$ does not belong to $\mathcal{J}$, and $B_\xi\cap B_\eta$ does not belong to $\mathcal{I}$. **Q** Next, $\mathcal{I}$ is normal. **P** Of course $\kappa=\operatorname{add}\mathcal{J}$ is regular (513C(a-i)), and we are supposing that it is uncountable. If $S\in\mathcal{P}\kappa\setminus\mathcal{I}$ and $h:S\to\kappa$ is regressive, set $f(y)=hg(y)$ if $y\in g^{-1}[S]$, $g(y)$ otherwise. Then $\{y:f(y)<g(y)\}=g^{-1}[S]\notin\mathcal{J}$, so $f\notin F$ and there is an $\alpha<\kappa$ such that $f^{-1}[\alpha]\notin\mathcal{J}$. But

$$f^{-1}[\alpha]\subseteq g^{-1}[\alpha]\cup\bigcup_{\beta<\alpha}g^{-1}[h^{-1}[\{\beta\}]];$$

as $\alpha<\operatorname{add}\mathcal{J}$, there is a $\beta<\alpha$ such that $g^{-1}[h^{-1}[\{\beta\}]]\notin\mathcal{J}$ and $h^{-1}[\{\beta\}]\notin\mathcal{I}$. As h is arbitrary, $\mathcal{I}$ is normal (541H). **Q**

541K Lemma Let κ be a regular uncountable cardinal and $\mathcal{I}$ a normal ideal on κ which is κ'-saturated in $\mathcal{P}\kappa$, where $\kappa'\le\kappa$.

(a) If $S\in\mathcal{P}\kappa\setminus\mathcal{I}$ and $f:S\to\kappa$ is regressive, then there is a set $A\in[\kappa]^{<\kappa'}$ such that $S\setminus f^{-1}[A]\in\mathcal{I}$; consequently there is an $\alpha<\kappa$ such that $\{\xi:\xi\in S,\ f(\xi)\ge\alpha\}\in\mathcal{I}$.

(b) If $\lambda<\kappa$, then $\{\xi:\xi<\kappa,\ \operatorname{cf}\xi\le\lambda\}\in\mathcal{I}$.

(c) If for each $\xi<\kappa$ we are given a relatively closed set $C_\xi\subseteq\xi$ which is cofinal with ξ, then

$$C=\{\alpha:\alpha<\kappa,\ \{\xi:\alpha\notin C_\xi\}\in\mathcal{I}\}$$

is a cofinal closed set in κ.

proof (a) Choose $\langle S_\eta\rangle_{\eta\le\gamma}$ and $\langle\alpha_\eta\rangle_{\eta<\gamma}$ inductively, as follows. $S_0=S$. If $S_\eta\in\mathcal{I}$, set $\gamma=\eta$ and stop. Otherwise, $f\restriction S_\eta$ is regressive, so (because $\mathcal{I}$ is normal) there is an $\alpha_\eta<\kappa$ such that $\{\xi:\xi\in S_\eta,\ f(\xi)=\alpha_\eta\}\notin\mathcal{I}$ (541H(iii)). Set $S_{\eta+1}=\{\xi:\xi\in S_\eta,\ f(\xi)\ne\alpha_\eta\}$. Given $\langle S_\zeta\rangle_{\zeta<\eta}$ for a non-zero limit ordinal η, set $S_\eta=\bigcap_{\zeta<\eta}S_\zeta$. Now $\langle S_\eta\setminus S_{\eta+1}\rangle_{\eta<\gamma}$ is a disjoint family in $\mathcal{P}\kappa\setminus\mathcal{I}$, so $\#(\gamma)<\kappa'$ and $A=\{\alpha_\eta:\eta<\gamma\}\in[\kappa]^{<\kappa'}$, while $S\setminus f^{-1}[A]=S_\gamma$ belongs to $\mathcal{I}$. Setting $\alpha=\sup A+1$, $\alpha<\kappa$ (because κ is regular) and $\{\xi:\xi\in S,\ f(\xi)\ge\alpha\}$ belongs to $\mathcal{I}$.

(b) ? Otherwise, set $S=\{\xi:0<\xi<\kappa,\ \operatorname{cf}\xi\le\lambda\}$ and for $\xi\in S$ choose a cofinal set $A_\xi\subseteq\xi$ with $\#(A_\xi)\le\lambda$. Let $\langle f_\eta\rangle_{\eta<\lambda}$ be a family of functions defined on S such that $A_\xi=\{f_\eta(\xi):\eta<\lambda\}$ for each $\xi\in S$. By (a), we have for each $\eta<\lambda$ an $\alpha_\eta<\kappa$ such that $B_\eta=\{\xi:\xi\in S,\ f_\eta(\xi)\ge\alpha_\eta\}\in\mathcal{I}$. Set $\alpha=\sup_{\eta<\lambda}\alpha_\eta<\kappa$; as $\lambda<\kappa=\operatorname{add}\mathcal{I}$ (541H), there is a $\xi\in S\setminus\bigcup_{\eta<\lambda}B_\eta$ such that $\xi>\alpha$. But now $A_\xi\subseteq\alpha$ is not cofinal with ξ. **X**

(c) For $\alpha<\kappa$, $0<\xi<\kappa$ set

$$\begin{aligned}f_\alpha(\xi)&=\min(C_\xi\setminus\alpha)\text{ if }\xi>\alpha,\\&=0\text{ otherwise.}\end{aligned}$$

Then f_α is regressive, so by (a) there is a $\zeta_\alpha<\kappa$ such that $\kappa\setminus f_\alpha^{-1}[\zeta_\alpha]\in\mathcal{I}$, that is, $\{\xi:C_\xi\cap\zeta_\alpha\setminus\alpha=\emptyset\}\in\mathcal{I}$. Set $\tilde C=\{\alpha:\alpha<\kappa,\ \zeta_\beta<\alpha\text{ for every }\beta<\alpha\}$; then $\tilde C$ is cofinal with κ. If $\alpha\in\tilde C$, then

$$\begin{aligned}\{\xi:\xi>\alpha,\ \alpha\notin C_\xi\}&\subseteq\{\xi:C_\xi\cap\alpha\text{ is not cofinal with }\alpha\}\\&\subseteq\{\xi:C_\xi\cap\zeta_\beta\setminus\beta=\emptyset\text{ for some }\beta<\alpha\}\end{aligned}$$

is the union of fewer than κ members of $\mathcal{I}$, so belongs to $\mathcal{I}$, and $\alpha\in C$. Thus C is cofinal with κ. If $\alpha<\kappa$ and $\alpha=\sup(C\cap\alpha)$, then

$$\{\xi : \xi > \alpha,\, \alpha \notin C_\xi\} \subseteq \{\xi : \beta \notin C_\xi \text{ for some } \beta \in C \cap \alpha\}$$

is again the union of fewer than κ members of $\mathcal{I}$, so $\alpha \in C$. Thus C is closed.

541L Theorem Let κ be an uncountable cardinal such that there is a proper κ-saturated κ-additive ideal of $\mathcal{P}\kappa$ containing singletons.

(a) There is a κ-saturated normal ideal on κ.

(b) κ is weakly inaccessible.

(c) The set of weakly inaccessible cardinals less than κ is stationary in κ.

proof (a) Let $\mathcal{J}$ be a proper κ-saturated κ-additive ideal of $\mathcal{P}\kappa$. The additivity of $\mathcal{J}$ must be exactly κ, so 541J tells us that there is a κ-saturated normal ideal $\mathcal{I}$ on κ.

(b) Of course $\kappa = \operatorname{add}\mathcal{J} = \operatorname{add}\mathcal{I}$ is regular. **?** Suppose, if possible, that $\kappa = \lambda^+$ is a successor cardinal. For each $\alpha < \kappa$ let $\phi_\alpha : \alpha \to \lambda$ be an injection. For $\beta < \kappa$ and $\xi < \lambda$ set $A_{\beta\xi} = \{\alpha : \beta < \alpha < \kappa,\, \phi_\alpha(\beta) = \xi\}$. Then $\bigcup_{\xi<\lambda} A_{\beta\xi} = \kappa \setminus (\beta+1) \notin \mathcal{I}$, so there is a $\xi_\beta < \lambda$ such that $A_{\beta,\xi_\beta} \notin \mathcal{I}$. Now there must be an $\eta < \lambda$ such that $B = \{\beta : \beta < \kappa,\, \xi_\beta = \eta\}$ has cardinal κ. But in this case $\langle A_{\beta\eta}\rangle_{\beta\in B}$ is a disjoint family in $\mathcal{P}\kappa \setminus \mathcal{I}$, and $\mathcal{I}$ is not κ-saturated in $\mathcal{P}\kappa$. **X**

Thus κ is a regular uncountable limit cardinal, i.e., is weakly inaccessible.

(c) Write R for the set of regular infinite cardinals less than κ and L for the set of limit cardinals less than κ.

(i) $\kappa \setminus R \in \mathcal{I}$. **P?** Otherwise, $A = (\kappa \setminus R) \setminus \{0,1\} \notin \mathcal{I}$. For $\xi \in A$, set $f(\xi) = \operatorname{cf}\xi$; then $f : A \to \kappa$ is regressive. Because $\mathcal{I}$ is normal, there must be a $\delta < \kappa$ such that $B = \{\xi : \xi < \kappa,\, \operatorname{cf}\xi = \delta\} \notin \mathcal{I}$. For each $\xi \in B$, let $\langle g_\eta(\xi)\rangle_{\eta<\delta}$ enumerate a cofinal subset of ξ. If $\eta < \delta$, then $g_\eta : B \to \kappa$ is regressive, so by 541Ka there is a $\gamma_\eta < \kappa$ such that $J_\eta = \{\xi : \xi \in B,\, g_\eta(\xi) \ge \gamma_\eta\} \in \mathcal{I}$. Set $\gamma = \sup_{\eta<\delta}\gamma_\eta$; as κ is regular, $\gamma < \kappa$; while $B \setminus (\gamma+1) \subseteq \bigcup_{\eta<\delta} J_\eta$ belongs to $\mathcal{I}$, which is impossible. **XQ**

(ii) $R \setminus L \in \mathcal{I}$. **P** We have a regressive function $f : R \setminus L \to \kappa$ defined by setting $f(\lambda^+) = \lambda$ for every infinite cardinal $\lambda < \kappa$. Now $f^{-1}[\{\xi\}]$ is empty or a singleton for every ξ, so always belongs to $\mathcal{I}$; because $\mathcal{I}$ is normal, $R \setminus L \in \mathcal{I}$. **Q**

(iii) Accordingly the set $R \cap L$ of weakly inaccessible cardinals less than κ cannot belong to $\mathcal{I}$ and must be stationary, by 541Ia.

541M Definition (a) A regular uncountable cardinal κ is **two-valued-measurable** (often just **measurable**) if there is a proper κ-additive 2-saturated ideal of $\mathcal{P}\kappa$ containing singletons.

Of course a proper ideal $\mathcal{I}$ of $\mathcal{P}\kappa$ is 2-saturated iff it is maximal, that is, the dual filter $\{\kappa \setminus I : I \in \mathcal{I}\}$ is an ultrafilter; thus κ is two-valued-measurable iff there is a non-principal κ-complete ultrafilter on κ. From 541J we see also that if κ is two-valued-measurable then there is a normal maximal ideal of $\mathcal{P}\kappa$, that is, there is a normal ultrafilter on κ, as considered in §4A1.

(b) An uncountable cardinal κ is **weakly compact** if for every $S \subseteq [\kappa]^2$ there is a $D \in [\kappa]^\kappa$ such that $[D]^2$ is either included in S or disjoint from S.

541N Theorem (a) A two-valued-measurable cardinal is weakly compact.

(b) A weakly compact cardinal is strongly inaccessible.

proof (a) If κ is a two-valued-measurable cardinal, there is a non-principal normal ultrafilter on κ, so 4A1L tells us that κ is weakly compact.

(b) Let κ be a weakly compact cardinal.

(i) Set $\lambda = \operatorname{cf}\kappa$; let $A \in [\kappa]^\lambda$ be a cofinal subset of κ, and $\langle\alpha_\zeta\rangle_{\zeta<\lambda}$ the increasing enumeration of A. For $\xi < \kappa$ set $f(\xi) = \min\{\zeta : \xi < \alpha_\zeta\}$; now set $S = \{I : I \in [\kappa]^2,\ f \text{ is constant on } I\}$. If $D \in [\kappa]^\kappa$, take any $\xi \in D$; then there is an $\eta \in D \setminus \alpha_{f(\xi)}$, so f is not constant on $\{\xi,\eta\}$ and $[D]^2 \not\subseteq S$. There must therefore be a $D \in [\kappa]^2$ such that $[D]^2 \cap S = \emptyset$. But in this case f is injective on D, so $\lambda \ge \#(f[D]) = \kappa$ and $\operatorname{cf}\kappa = \kappa$.

Thus κ is regular.

(ii) ? Suppose, if possible, that κ is not strongly inaccessible. Then there is a least cardinal $\lambda < \kappa$ such that $2^\lambda \ge \kappa$; let $\phi : \kappa \to \mathcal{P}\lambda$ be an injective function. Set

$$S = \{\{\xi, \eta\} : \xi < \eta < \kappa,\ \min(\phi(\xi)\triangle\phi(\eta)) \in \phi(\eta)\}.$$

Because κ is weakly compact, there is a $D \in [\kappa]^\kappa$ such that either $[D]^2 \subseteq S$ or $[D]^2 \cap S = \emptyset$. Set $B = \{\phi(\xi) \cap \gamma : \xi \in D,\ \gamma < \lambda\}$. Then

$$\#(B) \le \#(\bigcup_{\gamma<\lambda} \mathcal{P}\gamma) \le \max(\lambda, \sup_{\gamma<\lambda} 2^\gamma) < \kappa$$

because κ is regular, $\lambda < \kappa$ and $2^\gamma < \kappa$ for every $\gamma < \lambda$. So there must be an $\eta \in D$ such that $B = \{\phi(\xi) \cap \gamma : \xi \in D \cap \eta,\ \gamma < \lambda\}$. Take $\zeta \in D$ such that $\zeta > \eta$, set $\gamma = \min(\phi(\eta)\triangle\phi(\zeta))$ and take $\xi \in D \cap \eta$ such that $\phi(\xi) \cap (\gamma + 1) = \phi(\zeta) \cap (\gamma + 1)$. Now $\gamma = \min(\phi(\xi)\triangle\phi(\eta))$, so

$$\{\xi, \eta\} \in S \iff \gamma \in \phi(\eta) \iff \gamma \notin \phi(\zeta) \iff \{\eta, \zeta\} \notin S.$$

But this means that $[D]^2$ can be neither included in S nor disjoint from S; contrary to the choice of D. **X**

Thus κ is strongly inaccessible.

541O Lemma Let X be a set and $\mathcal{I}$ a proper ideal of subsets of X such that $\mathcal{P}X/\mathcal{I}$ is atomless. If $\mathcal{I}$ is λ-saturated and κ-additive, with $\lambda \le \kappa$, then $\kappa \le \operatorname{cov}\mathcal{I} \le \sup_{\theta<\lambda} 2^\theta$.

proof We may take it that $\lambda = \operatorname{sat}(\mathcal{P}X/\mathcal{I})$. If $\lambda > \kappa$ the result is trivial because $\mathcal{I}$ contains singletons. So suppose that $\lambda \le \kappa$. For each $A \in \mathcal{P}X \setminus \mathcal{I}$ choose $A' \subseteq A$ such that neither A' nor $A \setminus A'$ belongs to $\mathcal{I}$; this is possible because $\mathcal{P}X/\mathcal{I}$ is atomless. Define $\langle \mathcal{A}_\xi \rangle_{\xi<\lambda}$ inductively, as follows. $\mathcal{A}_0 = \{X\}$. Given that $\mathcal{A}_\xi \subseteq \mathcal{P}X \setminus \mathcal{I}$, then set $\mathcal{A}_{\xi+1} = \{A' : A \in \mathcal{A}_\xi\} \cup \{A \setminus A' : A \in \mathcal{A}_\xi\}$. For a non-zero limit ordinal $\xi < \lambda$, set $E_\xi = \bigcap_{\eta<\xi} \bigcup \mathcal{A}_\eta$; for $x \in E_\xi$ set $C_{\xi x} = \bigcap\{A : x \in A \in \bigcup_{\eta<\xi} \mathcal{A}_\eta\}$; set $\mathcal{A}_\xi = \{C_{\xi x} : x \in E_\xi\} \setminus \mathcal{I}$, and continue. Observe that this construction ensures that each $\mathcal{A}_\xi$ is disjoint, and that if $\eta \le \xi$ and $A \in \mathcal{A}_\xi$ then there is a $B \in \mathcal{A}_\eta$ such that $A \subseteq B$.

If $x \in X$, then $\alpha_x = \{\xi : \xi < \lambda,\ x \in \bigcup \mathcal{A}_\xi\}$ is an initial segment of λ, so is an ordinal less than or equal to λ. In fact $\alpha_x < \lambda$. **P** For each $\xi < \alpha_x$ take $A_\xi \in \mathcal{A}_\xi$ such that $x \in A_\xi$, and let B_ξ be either A'_ξ or $A_\xi \setminus A'_\xi$ and such that $x \notin B_\xi$. Then $\langle B_\xi \rangle_{\xi<\alpha_x}$ is a disjoint family in $\mathcal{P}\kappa \setminus \mathcal{I}$ so has cardinal less than λ. **Q**

Of course each α_x is a non-zero limit ordinal, because $\bigcup \mathcal{A}_\xi = \bigcup \mathcal{A}_{\xi+1}$ for each ξ. Now set $\mathcal{A} = \bigcup_{\xi<\lambda} \mathcal{A}_\xi$; then $\#(\mathcal{A}) \le \lambda$. Next, for any $x \in X$, $\mathcal{B}_x = \{A : A \in \mathcal{A},\ x \in A\}$ has cardinal less than λ and $C_x = \bigcap \mathcal{B}_x$ belongs to $\mathcal{I}$ and contains x. So $\mathcal{C} = \{C_x : x \in X\}$ has cardinal at most $\#([\lambda]^{<\lambda}) = \sup_{\theta<\lambda} 2^\theta$ (because $\lambda = \operatorname{sat}(\mathcal{P}X/\mathcal{I})$ is regular, by 514Da), and $\mathcal{C} \subseteq \mathcal{I}$ covers X, so

$$\kappa \le \operatorname{add}\mathcal{I} \le \operatorname{cov}\mathcal{I} \le \#(\mathcal{C}) \le \sup_{\theta<\lambda} 2^\theta.$$

541P Theorem (Tarski 1945, Solovay 71) Suppose that κ is a regular uncountable cardinal with a proper λ-saturated κ-additive ideal $\mathcal{I}$ of $\mathcal{P}\kappa$ containing singletons, where $\lambda \le \kappa$. Set $\mathfrak{A} = \mathcal{P}\kappa/\mathcal{I}$. Then

either $\kappa \le \sup_{\theta<\lambda} 2^\theta$ and $\mathfrak{A}$ is atomless

or κ is two-valued-measurable and $\mathfrak{A}$ is purely atomic.

proof (a) Let us begin by noting that $\mathcal{I}$ is λ-saturated iff $\lambda \ge \operatorname{sat}(\mathfrak{A})$; so it will be enough to prove the result when $\lambda = \operatorname{sat}(\mathfrak{A})$, in which case λ is either finite or regular and uncountable (514Da again).

(b) Suppose that $\mathfrak{A}$ is atomless. By 541O, $\kappa \le \sup_{\theta<\lambda} 2^\theta$. So in this case we have the first alternative of the dichotomy.

(c) Before continuing with an analysis of atoms in $\mathfrak{A}$, I draw out some further features of the structure discussed in the proof of 541O. We find that if $\mathfrak{A}$ is atomless then κ is not weakly compact. **P** Construct $\langle \mathcal{A}_\xi \rangle_{\xi<\lambda}$, $\mathcal{A}$ and $\mathcal{C}$ as in 541O. Consider $\alpha^* = \sup\{\xi : \xi < \lambda,\ \mathcal{A}_\xi \ne \emptyset\}$.

case 1 If $\alpha^* < \kappa$, then $\#(\mathcal{A}) < \kappa$, because κ is regular and $\#(\mathcal{A}_\xi) < \lambda \le \kappa$ for every ξ. In this case $\kappa \le \#(\mathcal{C}) \le 2^{\#(\mathcal{A})}$ and κ is not strongly inaccessible, therefore not weakly compact, by 541Nb.

case 2 If $\alpha^* = \kappa$, then $\#(\mathcal{A}') = \kappa$, where $\mathcal{A}' = \bigcup_{\xi<\kappa} \mathcal{A}_{\xi+1}$. Note that each $D \in \mathcal{A}'$ has a companion $D^* \in \mathcal{A}'$ defined by saying that if $D \in \mathcal{A}_{\xi+1}$ then $D^* = D_0 \setminus D$ where D_0 is the unique member of $\mathcal{A}_\xi$ including D. Consider the relation $S = \{(D, D') : D,\ D' \in \mathcal{A}',\ D \cap D' = \emptyset\}$. Take any $\mathcal{D} \in [\mathcal{A}']^\kappa$. Then

$[\mathcal{D}]^2 \not\subseteq S$, because $\mathcal{I}$ is κ-saturated. **?** If $[\mathcal{D}]^2 \cap S = \emptyset$, any two members of $\mathcal{D}$ meet. If D_1 and D_2 are distinct members of $\mathcal{D}$, then they cannot both belong to $\mathcal{A}_{\xi+1}$ for any ξ, so one must belong to $\mathcal{A}_{\eta+1}$ and the other to $\mathcal{A}_{\xi+1}$ where $\eta < \xi$; say $D_1 \in \mathcal{A}_{\eta+1}$ and $D_2 \in \mathcal{A}_{\xi+1}$. Now $D_2 \cup D_2^* \in \mathcal{A}_\xi$ meets D_1 and is therefore included in D_1; so $D_1^* \cap D_2^* = \emptyset$. Thus $\{D^* : D \in \mathcal{D}\}$ is a disjoint family in $\mathcal{A}$ with cardinal κ, contrary to the hypothesis that $\mathcal{I}$ is κ-saturated. **X**

Thus if $\mathcal{D} \in [\mathcal{A}']^\kappa$, $[\mathcal{D}]^2$ is neither included in nor disjoint from S. Since $\#(\mathcal{A}') = \kappa$, this shows that κ cannot be weakly compact. **Q**

(d) Now suppose that $\mathfrak{A}$ has an atom a. Let $A \in \mathcal{P}\kappa \setminus \mathcal{I}$ be such that $A^\bullet = a$. Set $\mathcal{I}_A = \{I : I \subseteq \kappa, I \cap A \in \mathcal{I}\}$; then $\mathcal{I}_A$ is a κ-additive maximal ideal of κ containing singletons, so κ is two-valued-measurable. It follows that κ is weakly compact (541Na).

? Suppose, if possible, that $\mathfrak{A}$ is not purely atomic. Then there is a $C \in \mathcal{P}\kappa \setminus \mathcal{I}$ such that $\mathcal{P}\kappa/\mathcal{I}_C$ is atomless, where $\mathcal{I}_C = \{I : I \subseteq \kappa, I \cap C \in \mathcal{I}\}$. Also $\mathcal{I}_C$ is κ-additive and λ-saturated. But this is impossible, by (c). **X** Thus $\mathfrak{A}$ is purely atomic, and we have the second alternative of the dichotomy.

541Q Theorem Let κ be a regular uncountable cardinal and $\mathcal{I}$ a normal ideal on κ. Let $\theta < \kappa$ be a cardinal of uncountable cofinality such that $\mathcal{I}$ is $(\operatorname{cf}\theta)$-saturated in $\mathcal{P}\kappa$, and $f : [\kappa]^{<\omega} \to [\kappa]^{<\theta}$ any function. Then there are $C \in \mathcal{I}$ and $f^* : [\kappa \setminus C]^{<\omega} \to [\kappa]^{<\theta}$ such that $f(I) \cap \eta \subseteq f^*(I \cap \eta)$ whenever $I \in [\kappa \setminus C]^{<\omega}$ and $\eta < \kappa$.

proof (a) I show by induction on $n \in \mathbb{N}$ that if $g : [\kappa]^{\le n} \to [\kappa]^{<\theta}$ is a function then there are $A \in \mathcal{I}$ and $g^* : [\kappa \setminus A]^{\le n} \to [\kappa]^{<\theta}$ such that $g(I) \cap \eta \subseteq g^*(I \cap \eta)$ for every $I \in [\kappa \setminus A]^{\le n}$ and $\eta < \kappa$.

P If $n = 0$ this is trivial; take $A = \emptyset$, $g^*(\emptyset) = g(\emptyset)$. For the inductive step to $n+1$, given $g : [\kappa]^{\le n+1} \to [\kappa]^{<\theta}$, then for each $\xi < \kappa$ define $g_\xi : [\kappa]^{\le n} \to [\kappa]^{<\theta}$ by setting $g_\xi(J) = g(J \cup \{\xi\})$ for every $J \in [\kappa]^{\le n}$. Set

$$D = \{\xi : \xi < \kappa, \operatorname{cf}(\xi) \ge \theta\};$$

then $\kappa \setminus D \in \mathcal{I}$ (541Kb). For $\xi \in D$ and $J \in [\kappa]^{\le n}$ set $\zeta_{J\xi} = \sup(\xi \cap g_\xi(J)) < \xi$. Then for each $J \in [\kappa]^{\le n}$ the function $\xi \mapsto \zeta_{J\xi} : D \to \kappa$ is regressive, so there is a $\zeta_J^* < \kappa$ such that $\{\xi : \zeta_{J\xi} \ge \zeta_J^*\} \in \mathcal{I}$ (541Ka). Now $\operatorname{add}\mathcal{I} = \kappa$, by 541H, so 541D tells us that there is an $h(J) \in [\zeta_J^*]^{<\theta}$ such that $\{\xi : \xi \cap g_\xi(J) \not\subseteq h(J)\} \in \mathcal{I}$. By the inductive hypothesis, there are $B \in \mathcal{I}$ and $h^* : [\kappa \setminus B]^{\le n} \to [\kappa]^{<\theta}$ such that $h(J) \cap \eta \subseteq h^*(J \cap \eta)$ for every $J \in [\kappa \setminus B]^{\le n}$ and $\eta < \kappa$.

Try setting

$$A_J = \{\xi : \xi \cap g_\xi(J) \not\subseteq h(J)\} \text{ for } J \in [\kappa]^{\le n},$$

$$A = B \cup \{\xi : \xi \in \bigcup_{J \in [\xi]^{\le n}} A_J\},$$

$$\begin{aligned} g^*(I) &= g(I) \text{ if } I \in [\kappa \setminus A]^{n+1}, \\ &= g(I) \cup h^*(I) \text{ if } I \in [\kappa \setminus A]^{\le n}. \end{aligned}$$

Then A_J always belongs to $\mathcal{I}$, by the choice of $h(J)$, so $A \in \mathcal{I}$, by 541Ib, while $g^*(I) \in [\kappa]^{<\theta}$ for every $I \in [\kappa \setminus A]^{\le n+1}$. Take $\eta < \kappa$ and $I \in [\kappa \setminus A]^{\le n+1}$. If $I \subseteq \eta$ then $g(I) \cap \eta \subseteq g^*(I) = g^*(I \cap \eta)$. Otherwise, set $\xi = \max I$ and $J = I \setminus \{\xi\}$. Then $\eta \le \xi \in \kappa \setminus A_J$, so

$$g(I) \cap \eta = g_\xi(J) \cap \xi \cap \eta \subseteq h(J) \cap \eta \subseteq h^*(J \cap \eta) = h^*(I \cap \eta) \subseteq g^*(I \cap \eta).$$

Thus the induction continues. **Q**

(b) Now applying (a) to $f \restriction [\kappa]^{\le n}$ we obtain sets $C_n \in \mathcal{I}$ and functions $f_n^* : [\kappa \setminus C_n]^{\le n} \to [\kappa]^{<\theta}$ such that $f(I) \cap \eta \subseteq f_n^*(I \cap \eta)$ whenever $I \in [\kappa \setminus C_n]^{\le n}$ and $\eta < \kappa$. Set $C = \bigcup_{n \in \mathbb{N}} C_n \in \mathcal{I}$ and $f^*(I) = \bigcup_{n \ge \#(I)} f_n^*(I)$ for each $I \in [\kappa \setminus C]^{<\omega}$. Because $\operatorname{cf}\theta > \omega$, $f(I) \in [\kappa]^{<\theta}$ for every I. If $I \in [\kappa \setminus C]^{<\omega}$ and $\eta < \kappa$, set $n = \#(I)$; then $I \in [\kappa \setminus C_n]^n$ so $f(I) \cap \eta \subseteq f_n^*(I \cap \eta) \subseteq f^*(I \cap \eta)$, as required.

541R Corollary Let κ be a regular uncountable cardinal, $\mathcal{I}$ a normal ideal on κ, and $\theta < \kappa$ a cardinal of uncountable cofinality such that $\mathcal{I}$ is $(\operatorname{cf}\theta)$-saturated in $\mathcal{P}\kappa$.

(a) If Y is a set of cardinal less than κ and $f : [\kappa]^{<\omega} \to [Y]^{<\theta}$ a function, then there are $C \in \mathcal{I}$ and $M \in [Y]^{<\theta}$ such that $f(I) \subseteq M$ for every $I \in [\kappa \setminus C]^{<\omega}$.

(b) If Y is any set and $g : \kappa \to [Y]^{<\theta}$ a function, then there are $C \in \mathcal{I}$ and $M \in [Y]^{<\theta}$ such that $g(\xi) \cap g(\eta) \subseteq M$ for all distinct ξ, $\eta \in \kappa \setminus C$.

proof (a) We may suppose that $Y \subseteq \kappa$. In this case, by 541Q, we have a $C_0 \in \mathcal{I}$ and an $f^* : [\kappa \setminus C_0]^{<\omega} \to [\kappa]^{<\theta}$ such that $f(I) \cap \eta \subseteq f^*(I \cap \eta)$ whenever $I \subseteq \kappa \setminus C_0$ is finite and $\eta < \kappa$. Let $\gamma < \kappa$ be such that $Y \subseteq \gamma$ and set $M = Y \cap f^*(\emptyset)$, $C = C_0 \cup \gamma$. Then $M \in [Y]^{<\theta}$, $C \in \mathcal{I}$ and if $I \in [\kappa \setminus C]^{<\omega}$ then

$$f(I) = f(I) \cap \gamma \subseteq Y \cap f^*(I \cap \gamma) \cap \gamma = M.$$

(b) Since $\bigcup_{\xi<\kappa} g(\xi)$ has cardinal at most κ, we may again suppose that $Y \subseteq \kappa$. Apply 541Q with $f(\{\xi\}) = g(\xi)$ for $\xi < \kappa$. Taking C and f^* from 541Q, set $M = Y \cap f^*(\emptyset)$. Set $F = \{\xi : \xi < \kappa,\ g(\eta) \subseteq \xi$ for every $\eta < \xi\}$; then F is a closed cofinal subset of κ (because $\theta \le \kappa$ and κ is regular), so $C' = C \cup (\kappa \setminus F) \in \mathcal{I}$ (541Ia). If ξ, η belong to $\kappa \setminus C' = F \setminus C$ and $\eta < \xi$, then

$$g(\xi) \cap g(\eta) \subseteq \xi \cap g(\xi) = \xi \cap f(\{\xi\}) \subseteq f^*(\xi \cap \{\xi\}) = f^*(\emptyset),$$

so $g(\xi) \cap g(\eta) \subseteq M$. Thus C' serves.

541S Lemma Let κ be a regular uncountable cardinal and $\mathcal{I}$ a normal ideal on κ. Suppose that γ and δ are cardinals such that $\omega \le \gamma < \delta < \kappa \le 2^\delta$, $\mathcal{I}$ is δ-saturated in $\mathcal{P}\kappa$, $2^\beta = 2^\gamma$ for $\gamma \le \beta < \delta$, but $2^\delta > 2^\gamma$. Then δ is regular and

$$2^\delta = \operatorname{cov}_{\mathrm{Sh}}(2^\gamma, \kappa, \delta^+, \delta) = \operatorname{cov}_{\mathrm{Sh}}(2^\gamma, \kappa, \delta^+, \omega_1) = \operatorname{cov}_{\mathrm{Sh}}(2^\gamma, \kappa, \delta^+, 2).$$

proof By 5A1Fh, δ is regular. Of course

$$\operatorname{cov}_{\mathrm{Sh}}(2^\gamma, \kappa, \delta^+, \delta) \le \operatorname{cov}_{\mathrm{Sh}}(2^\gamma, \kappa, \delta^+, \omega_1) \le \operatorname{cov}_{\mathrm{Sh}}(2^\gamma, \kappa, \delta^+, 2) \le \#([2^\gamma]^{\le\delta}) \le 2^\delta$$

(5A2D, 5A2Ea). For the reverse inequality, let $\mathcal{E} \subseteq [2^\gamma]^{<\kappa}$ be a set with cardinal $\operatorname{cov}_{\mathrm{Sh}}(2^\gamma, \kappa, \delta^+, \delta)$ such that every member of $[2^\gamma]^{\le\delta}$ is covered by fewer than δ members of $\mathcal{E}$. For each ordinal $\xi < \delta$ let $\phi_\xi : \mathcal{P}\xi \to 2^\gamma$ be an injective function. For $A \subseteq \delta$ define $f_A : \delta \to 2^\gamma$ by

$$f_A(\xi) = \phi_\xi(A \cap \xi) \text{ for every } \xi < \delta.$$

Choose $E_A \in \mathcal{E}$ such that $f_A^{-1}[E_A]$ is cofinal with δ; such must exist because δ is regular and $f_A[\delta]$ can be covered by fewer than δ members of $\mathcal{E}$.

? If $2^\delta > \#(\mathcal{E})$ then there must be an $E \in \mathcal{E}$ and an $\mathcal{A} \subseteq \mathcal{P}\delta$ such that $\#(\mathcal{A}) = \kappa$ and $E_A = E$ for every $A \in \mathcal{A}$. For each pair A, B of distinct members of $\mathcal{A}$ set $\xi_{AB} = \min(A \triangle B) < \delta$. By 541Ra, there is a set $\mathcal{B} \subseteq \mathcal{A}$, with cardinal κ, such that $M = \{\xi_{AB} : A,\ B \in \mathcal{B},\ A \ne B\}$ has cardinal less than δ. Set $\zeta = \sup M < \delta$. Next, for each $A \in \mathcal{B}$, take $\eta_A > \zeta$ such that $f_A(\eta_A) \in E$. Let $\eta < \delta$ be such that $\mathcal{C} = \{A : A \in \mathcal{B},\ \eta_A = \eta\}$ has cardinal κ. Then we have a map

$$A \mapsto f_A(\eta) = \phi_\eta(A \cap \eta) : \mathcal{C} \to E$$

which is injective, because if A, B are distinct members of $\mathcal{C}$ then $\xi_{AB} \le \zeta < \eta$, so $A \cap \eta \ne B \cap \eta$. So $\#(E) \ge \kappa$; but $E \in \mathcal{E} \subseteq [2^\gamma]^{<\kappa}$. **X**

As $\mathcal{E}$ is arbitrary, $\operatorname{cov}_{\mathrm{Sh}}(2^\gamma, \kappa, \delta^+, \delta) \ge 2^\delta$.

541X Basic exercises (a) Let κ be a regular infinite cardinal. Show that $\mathcal{P}\kappa/[\kappa]^{<\kappa}$ is not Dedekind complete, so $[\kappa]^{<\kappa}$ is not κ^+-saturated in $\mathcal{P}\kappa$. (*Hint*: construct a disjoint family $\langle A_\xi\rangle_{\xi<\kappa}$ in $[\kappa]^\kappa$; show that if $\#(A_\xi \setminus A) < \kappa$ for every ξ there is a $B \in [A]^\kappa$ such that $\#(B \cap A_\xi) < \kappa$ for every ξ.)

(b) Let $\mathfrak{A}$ be a Boolean algebra and I an ideal of $\mathfrak{A}$. Suppose there is a cardinal κ such that I is κ-additive and κ^+-saturated and $\mathfrak{A}$ is Dedekind κ^+-complete in the sense that $\sup A$ is defined in $\mathfrak{A}$ whenever $A \in [\mathfrak{A}]^{\le\kappa}$. Show that $\mathfrak{A}/I$ is Dedekind complete.

(c) Suppose that X and Y are sets and $\mathcal{I}$, $\mathcal{J}$ ideals of subsets of X, Y respectively. Suppose that κ is an infinite cardinal such that both $\mathcal{I}$ and $\mathcal{J}$ are κ-saturated and κ^+-additive. Show that $\mathcal{I} \ltimes \mathcal{J}$ (definition: 527Ba) is κ-saturated and κ^+-additive.

(d) Simplify the argument of 541D to give a direct proof of 541E in the case in which κ is regular.

>(e) Show that there is a two-valued-measurable cardinal iff there are a set I and a non-principal ω_1-complete ultrafilter on I.

>(f) Let κ be a two-valued-measurable cardinal and $\mathcal{I}$ a normal maximal proper ideal of $\mathcal{P}\kappa$. (i) Show that if $S \subseteq [\kappa]^{<\omega}$, there is a $C \in \mathcal{I}$ such that, for each $n \in \mathbb{N}$, $[\kappa \setminus C]^n$ is either disjoint from S or included in it. (ii) Show that if $\#(Y) < \kappa$, and $f : [\kappa]^{<\omega} \to Y$ is any function, then there is a $C \in \mathcal{I}$ such that f is constant on $[\kappa \setminus C]^n$ for each $n \in \mathbb{N}$. (*Hint*: 4A1L.)

(g) Let κ be a regular uncountable cardinal, $\mathcal{I}$ a κ-saturated normal ideal on κ and $f : [\kappa]^2 \to \kappa$ a function. Show that there are a $C \in \mathcal{I}$ and an $f^* : \kappa \to \kappa$ such that whenever $\eta \in \kappa \setminus C$ and $\xi \in \eta \setminus C$ then either $f(\{\xi, \eta\}) \ge \eta$ or $f(\{\xi, \eta\}) < f^*(\xi)$.

541Y Further exercises (a) Show that if κ is a regular uncountable cardinal and $S \subseteq \kappa$ is stationary, then S can be partitioned into κ stationary sets. (*Hint*: reduce to the case in which there is a κ-saturated normal ideal $\mathcal{I}$ of κ containing $\kappa \setminus S$. Define $f : S \to \kappa$ inductively by saying that

$$f(\xi) = \min(\bigcup\{\kappa \setminus f[C] : C \subseteq \xi \text{ is relatively closed and cofinal}\}.$$

Set $S_\gamma = f^{-1}[\{\gamma\}]$. Apply 541Kc with $C_\xi \cap S_\gamma = \emptyset$ for $\xi \in S_\gamma$ to show that $S_\gamma \in \mathcal{I}$ for every γ. Hence show that S_γ is stationary for every γ. See Solovay 71.)

(b) Show that if κ is two-valued-measurable and $\mathcal{F}$ is a normal ultrafilter on κ, then the set of weakly compact cardinals less than κ belongs to $\mathcal{F}$.

541 Notes and comments The ordinary principle of exhaustion (215A) can be regarded as an expression of ω_1-saturation (compare 316E and 514Db). In 541B-541E we have versions of results already given in special cases; but note that 541B, for instance, goes a step farther than the arguments offered in 314C and 316Fa can reach. 541Ca corresponds to 215B(iv); 541Cb-541Cc are associated with 516Q and 525C. In all this work you will probably find it helpful to use the words 'negligible' and 'conegligible' and 'almost everywhere', so that the conclusion of 541E becomes '$g(x) \in M$ a.e.(x)'. I don't use this language in the formal exposition because of the obvious danger of confusing a reader who is skimming through without reading introductions to sections very carefully; but in the principal applications I have in mind, $\mathcal{I}$ will indeed be a null ideal. The cardinals $\mathrm{Tr}_{\mathcal{I}}(X; Y)$ will appear only occasionally in this book, but are of great importance in infinitary combinatorics generally. Note that the key step in the proof of 541F (part (b) of the proof) develops an idea from the proof of 541J.

For the special purposes of §438 I mentioned 'normal filters' in §4A1; I have now attempted an introduction to the general theory of normal filters and ideals. The central observation of Keisler & Tarski 64 was that if κ is uncountable then any κ-complete non-principal ultrafilter gives rise to a normal ultrafilter on κ. It was noticed very quickly that something similar happens if we have a κ-complete filter of conegligible sets in a totally finite measure space; the extension of the idea to general κ-additive κ-saturated ideals is in Solovay 71. In this chapter I speak oftener of ideals than of filters but the ideas are necessarily identical. Observe that the Pressing-Down Lemma (4A1Cc) is the special case of 541H(iii) when $\mathcal{I}$ is the ideal of non-stationary sets (541Ia).

541Lb here is a re-working of Ulam's theorem (438C, Ulam 1930). The dramatic further step in 541Lc derives from Keisler & Tarski 64. The proof of 541Lc already makes it plain that much more can be said; for extensions of these ideas, see Fremlin 93, Levy 71 and Baumgartner Taylor & Wagon 77. In 541P we have an extension of Ulam's dichotomy (438C, 543B). 'Weak compactness' of a cardinal corresponds to Ramsey's theorem (4A1G); the idea was the basis of the proof of 451Q. Here I treat it as a purely combinatorial concept, but its real importance is in model theory (Kanamori 03, §4).

541Q is a fairly strong version of one of the typical properties of saturated normal ideals. The simplest not-quite-trivial case is when we have a function $f:[\kappa]^2\to\kappa$. In this case we find that if we discard an appropriate negligible set C then, for the remaining doubletons $I\in[\kappa\setminus C]^2$, $f(I)$ is either greater than or equal to $\max I$ or in a 'small' set determined by $\min I$. In this form, with the appropriate definition of 'small', it is enough for the ideal to be κ-saturated (541Xg). In the intended applications of 541Q, however, we shall be looking at functions $f:[\kappa]^{<\omega}\to[\kappa]^{\le\omega}$ and shall need to start from an ω_1-saturated ideal to obtain the full strength of the result.

Shelah's four-cardinal covering numbers $\mathrm{cov}_{\mathrm{Sh}}$ are not immediately digestible; in §5A2 I give the basic pcf theory linking them to cofinalities of products. 541S is here because it relies on a normal ideal being saturated.

Perhaps I have not yet sufficiently emphasized that there is a good reason why I have given no examples of normal ideals other than the non-stationary ideals, and no discussion of the saturation of those beyond Solovay's theorem 541Ya. We have in fact come to an area of mathematics which demands further acts of faith. I will continue, whenever possible, to express ideas as arguments in ordinary ZFC; but in most of the principal theorems the hypotheses will include assertions which can be satisfied only in rather special models of set theory. Moreover, these are special in a new sense. By and large, the assumptions used in the first three chapters of this volume (Martin's axiom, the continuum hypothesis, and so on) have been proved to be relatively consistent with ZFC (indeed, with ZF); that is, we know how to convert any proof in ZFC that '$\mathfrak{m}=\omega_1$' into a proof in ZF that '$0=1$'. The formal demonstration that this can be done is of course normally expressed in a framework reducible to Zermelo-Fraenkel set theory; but it is sufficiently compelling to be itself part of the material which must be encompassed by any formal system claiming to represent twenty-first century mathematics. In the present chapter, however, we are coming to results like 541P which have no content unless there can be non-trivial κ-saturated κ-additive ideals. And such objects are known to be strange in a different way from Souslin lines.

To describe this difference I turn to the simplest of the new propositions. Write ∃sic, ∃wic for the sentences 'there is a strongly inaccessible cardinal', 'there is a weakly inaccessible cardinal'. Of course ∃sic implies ∃wic, while 'there is a cardinal which is not measure-free' also implies ∃wic, by Ulam's theorem. We have no such implications in the other direction, but it is easy to adapt Gödel's argument for the relative consistency of GCH to show that if ZFC + ∃wic is consistent so is ZFC + GCH + ∃wic, while ZFC + GCH + ∃wic implies ∃sic. But we find also that there is a proof in ZFC + ∃sic that 'ZFC is consistent'. So if there were a proof in ZFC that 'if ZFC is consistent, then ZFC + ∃sic is consistent', there would be a proof in ZFC + ∃sic that 'ZFC + ∃sic is consistent'; by Gödel's incompleteness theorem, this would give us a proof that ZFC + ∃sic was in fact *inconsistent* (and therefore that ZFC and ZF are inconsistent).

The last paragraph is expressed crudely, in a language which blurs some essential distinctions; for a more careful account see Kunen 80, §IV.10. But what I am trying to say is that any theory involving inaccessible cardinals – and the theory of this chapter involves unthinkably many such cardinals – necessarily leads us to propositions which are not merely unprovable, but have high 'consistency strength'; we have long strings $\phi_0,\dots,\phi_n$ of statements such that (i) if ZFC + ϕ_{i+1} is consistent, so is ZFC + ϕ_i (ii) there can be no proof of the reverse unless ZF is inconsistent.

We do not (and in my view cannot) know for sure that ZF is consistent. It has now survived for a hundred years or so, which is empirical evidence of a sort. I do not suppose that the century of my own birth was also the century in which the structure of formal mathematics was determined for eternity; I hope and trust that mathematicians will come to look on ZFC as we now think of Euclidean geometry, as a glorious achievement and an enduring source of inspiration but inadequate for the expression of many of our deepest ideas. But (arguing from the weakest of historical analogies) I suggest that if and when a new paradox shakes the foundations of mathematics, it will be because some new Cantor has sought to extend apparently trustworthy methods to a totally new context. And I think that the mathematicians of that generation will stretch their ingenuity to the utmost to find a resolution of the paradox which is conservative, in that it retains as much as possible of their predecessors' ideas, subject perhaps to re-writing a good many proofs and tut-tutting over the naivety of essays such as this.

I think indeed (I am going a bit farther here) that they will be as reluctant to discard measurable cardinals as our forebears were to discard Cantor's cardinals. There is a flourishing theory of large cardinals in which very much stronger statements than 'there is a two-valued-measurable cardinal' have been explored in depth without catastrophe. (I mention a couple of these in §545; another is the Axiom of Determinacy in §567.)

Occasionally a proof that there are no measurable cardinals is announced; but the last real fright was in 1976, and most of these arguments have easily been shown not to reach the claimed conclusion. My best guess is that measurable cardinals are safe. But even if I am wrong, and they are irreconcilable with ZFC as now formulated, it does not follow that ZFC will be kept and measurable cardinals discarded. It could equally happen that one of the axioms of ZF will be modified; or, at least, that a modified form will become a recognized option. This is a partisan view from somebody who has a substantial investment to protect. But if you wish to prove me wrong, I do not see how you can do so without giving part of your own life to the topic.

Version of 8.7.13

542 Quasi-measurable cardinals

As is to be expected, the results of §541 take especially dramatic forms when we look at ω_1-saturated σ-ideals. 542B-542C spell out the application of the most important ideas from §541 to this special case. In addition, we can us Shelah's pcf theory to give us some remarkable combinatorial results concerning cardinal arithmetic (542E) and cofinalities of partially ordered sets (542I-542J).

542A Definition A cardinal κ is **quasi-measurable** if κ is regular and uncountable and there is an ω_1-saturated normal ideal on κ.

542B Proposition If X is a set and $\mathcal{I}$ is a proper ω_1-saturated σ-ideal of $\mathcal{P}X$ containing singletons, then $\operatorname{add}\mathcal{I}$ is quasi-measurable.

proof This is immediate from the special case $\lambda = \omega_1$ of 541J.

542C Proposition If κ is a quasi-measurable cardinal, then κ is weakly inaccessible, the set of weakly inaccessible cardinals less than κ is stationary in κ, and either $\kappa \le \mathfrak{c}$ or κ is two-valued-measurable.

proof By 541L, κ is weakly inaccessible and the set of weakly inaccessible cardinals less than κ is stationary in κ. Let $\mathcal{I}$ be an ω_1-saturated normal ideal on κ and $\mathfrak{A} = \mathcal{P}\kappa/\mathcal{I}$. Then 541P tells us that either $\mathfrak{A}$ is atomless and $\kappa \le \mathfrak{c}$ or $\mathfrak{A}$ is purely atomic and κ is two-valued-measurable.

542D Proposition Let κ be a quasi-measurable cardinal.

(a) Let $\langle\theta_\zeta\rangle_{\zeta<\lambda}$ be a family such that $\lambda < \kappa$ is a cardinal, every θ_ζ is a regular infinite cardinal and $\lambda < \theta_\zeta < \kappa$ for every $\zeta < \lambda$. Then $\operatorname{cf}(\prod_{\zeta<\lambda}\theta_\zeta) < \kappa$.

(b) If α and γ are cardinals less than κ, then $\Theta(\alpha,\gamma)$ (definition: 5A2Db) is less than κ.

(c) If α, β, γ and δ are cardinals, with $\alpha < \kappa$, $\gamma \le \beta$ and $\delta \ge \omega_1$, then $\operatorname{cov}_{\text{Sh}}(\alpha,\beta,\gamma,\delta)$ (definition: 5A2Da) is less than κ.

(d) $\Theta(\kappa,\kappa) = \kappa$.

proof Fix an ω_1-saturated normal ideal $\mathcal{I}$ on κ.

(a) ? Suppose, if possible, otherwise. Then λ is surely infinite. By 5A2Bc, there is an ultrafilter $\mathcal{F}$ on λ such that $\operatorname{cf}(\prod_{\zeta<\lambda}\theta_\zeta) = \operatorname{cf}(\prod_{\zeta<\lambda}\theta_\zeta|\mathcal{F})$, where the reduced product $\prod_{\zeta<\lambda}\theta_\zeta|\mathcal{F}$ is defined in 5A2A; by 5A2C, there is a family $\langle\theta'_\zeta\rangle_{\zeta<\lambda}$ of regular cardinals such that $\lambda < \theta'_\zeta \le \theta_\zeta$ for each ζ and $\operatorname{cf}(\prod_{\zeta<\lambda}\theta'_\zeta|\mathcal{F}) = \kappa$. Let $\langle p_\xi\rangle_{\xi<\kappa}$ be a cofinal family in $P = \prod_{\zeta<\lambda}\theta'_\zeta|\mathcal{F}$. For each $\xi < \kappa$ we can find $q_\xi \in P$ such that $q_\xi \not\le p_\eta$ for any $\eta \le \xi$; because P is upwards-directed, we can suppose that $q_\xi \ge p_\xi$, so that $\{q_\xi : \xi < \kappa\}$ also is cofinal with P. Choose $f_\xi \in \prod_{\zeta<\lambda}\theta'_\zeta$ such that $f^\bullet_\xi = q_\xi$ for each ξ.

For each $\zeta < \lambda$, $\theta'_\zeta < \kappa$, so there is a countable set $M_\zeta \subseteq \theta'_\zeta$ such that $I_\zeta = \kappa \setminus \{\xi : f_\xi(\zeta) \in M_\zeta\}$ belongs to $\mathcal{I}$ (541E). As

$$\omega \le \lambda < \theta'_\zeta = \operatorname{cf}\theta'_\zeta,$$

we can find $g(\zeta)$ such that $M_\zeta \subseteq g(\zeta) < \theta'_\zeta$. Consider $g^\bullet$ in $\prod_{\zeta<\lambda}\theta'_\zeta|\mathcal{F}$. There is an $\eta < \kappa$ such that $g^\bullet \le p_\eta$, so that $f^\bullet_\xi \not\le g^\bullet$ for every $\xi \ge \eta$. On the other hand, $\eta \cup \bigcup_{\zeta<\lambda} I_\zeta$ belongs to $\mathcal{I}$, so there is a $\xi \ge \eta$ such that $f_\xi(\zeta) \in M_\zeta$ for every $\zeta < \lambda$; in which case $f_\xi \le g$, which is impossible. **X**

(b) ? Suppose, if possible, otherwise. Of course we can suppose that γ is infinite. For each $\xi<\kappa$ there must be a family $\langle\theta_{\xi\zeta}\rangle_{\zeta<\lambda_\xi}$ of regular cardinals less than α such that $\lambda_\xi<\gamma$, $\omega\le\lambda_\xi<\theta_{\xi\zeta}$ for every $\zeta<\lambda_\xi$ and $\mathrm{cf}(\prod_{\zeta<\lambda_\xi}\theta_{\xi\zeta})\ge\xi$. Let λ be such that $A=\{\xi:\xi<\kappa,\,\lambda_\xi=\lambda\}\notin\mathcal{I}$. By 541Ra, applied to the function $I\mapsto\{\theta_{\xi\zeta}:\xi\in A\cap I,\,\zeta<\lambda_\xi\}:[\kappa]^{<\omega}\to[\alpha]^{<\lambda^+}$, there are $C\in\mathcal{I}$ and $M\in[\alpha]^{\le\lambda}$ such that $\theta_{\xi\zeta}\in M$ whenever $\xi\in A\setminus C$ and $\zeta<\lambda$. Let $\langle\theta_\zeta\rangle_{\zeta<\lambda'}$ enumerate $\{\theta:\theta\in M$ is a regular cardinal, $\theta>\lambda\}$. By (a), there is a cofinal set $F\subseteq\prod_{\zeta<\lambda'}\theta_\zeta$ with $\#(F)<\kappa$. Let $\xi\in A\setminus C$ be such that $\xi>\#(F)$. For each $f\in F$ define $g_f\in\prod_{\zeta<\lambda}\theta_{\xi\zeta}$ by setting

$$g_f(\zeta)=f(\zeta')\text{ whenever }\theta_{\xi\zeta}=\theta_{\zeta'}.$$

Then $\{g_f:f\in F\}$ is cofinal with $\prod_{\zeta<\lambda}\theta_{\xi\zeta}$, because if $h\in\prod_{\zeta<\lambda}\theta_{\xi\zeta}$ there is an $f\in F$ such that

$$f(\zeta')\ge\sup\{h(\zeta):\zeta<\lambda,\,\theta_{\xi\zeta}=\theta_{\zeta'}\}$$

for every $\zeta'<\lambda'$, and in this case $h\le g_f$. So

$$\#(F)<\xi\le\mathrm{cf}(\textstyle\prod_{\zeta<\lambda}\theta_{\xi\zeta})\le\#(F),$$

which is absurd. **X**

(c) This is trivial if any of the cardinals α, β or γ is finite; let us take it that they are all infinite. Then

$$\mathrm{cov}_{\mathrm{Sh}}(\alpha,\beta,\gamma,\delta)\le\mathrm{cov}_{\mathrm{Sh}}(\alpha,\gamma,\gamma,\omega_1)\le\max(\omega,\alpha,\Theta(\alpha,\gamma))<\kappa$$

by 5A2D, 5A2G and (b) above.

(d) Because κ is an uncountable limit cardinal, $\kappa\le\Theta(\kappa,\kappa)$. (If $\omega\le\theta<\kappa$, then $\mathrm{cf}\,\theta^+\le\Theta(\kappa,\kappa)$.) On the other hand, let $\lambda<\kappa$ be an infinite cardinal and $\langle\theta_\zeta\rangle_{\zeta<\lambda}$ a family of infinite regular cardinals such that $\lambda<\theta_\zeta<\kappa$ for every $\zeta<\lambda$. Then $\alpha=\sup_{\zeta<\lambda}\theta_\zeta^+<\kappa$ and

$$\mathrm{cf}(\textstyle\prod_{\zeta<\lambda}\theta_\zeta)\le\Theta(\alpha,\lambda^+)<\kappa.$$

As $\langle\theta_\zeta\rangle_{\zeta<\lambda}$ is arbitrary, $\Theta(\kappa,\kappa)=\kappa$.

542E Theorem (Gitik & Shelah 93) If $\kappa\le\mathfrak{c}$ is a quasi-measurable cardinal, then

$$\{2^\gamma:\omega\le\gamma<\kappa\}$$

is finite.

proof ? Suppose, if possible, otherwise.

(a) Define a sequence $\langle\gamma_n\rangle_{n\in\mathbb{N}}$ of cardinals by setting

$$\gamma_0=\omega,\quad\gamma_{n+1}=\min\{\gamma:2^\gamma>2^{\gamma_n}\}\text{ for }n\in\mathbb{N}.$$

Then we are supposing that $\gamma_n<\kappa$ for every n, so by 541S, 5A2D, 5A2G and 5A2F γ_n is regular and

$$\begin{aligned}2^{\gamma_{n+1}}&=\underset{\mathrm{Sh}}{\mathrm{cov}}(2^{\gamma_n},\kappa,\gamma_{n+1}^+,\omega_1)\le\underset{\mathrm{Sh}}{\mathrm{cov}}(2^{\gamma_n},\gamma_{n+1}^+,\gamma_{n+1}^+,\omega_1)\\&\le\max(2^{\gamma_n},\Theta(2^{\gamma_n},\gamma_{n+1}^+))\le2^{\gamma_{n+1}}\end{aligned}$$

for every $n\in\mathbb{N}$.

(b) Now $\Theta(2^{\gamma_n},\gamma)=\Theta(\mathfrak{c},\gamma)$ whenever $n\in\mathbb{N}$ and γ is a regular cardinal with $\gamma_n<\gamma<\kappa$. **P** Induce on n. For $n=0$ we have $\mathfrak{c}=2^{\gamma_0}$. For the inductive step to $n+1$, if γ is regular and $\gamma_{n+1}<\gamma<\kappa$, then $\mathfrak{c}\ge\kappa>\Theta(\gamma,\gamma)$ (542Db), so

$$\Theta(2^{\gamma_{n+1}},\gamma)=\Theta(\Theta(2^{\gamma_n},\gamma_{n+1}^+),\gamma)$$

(by (a))

$$\le\Theta(\Theta(2^{\gamma_n},\gamma),\gamma)$$

(because $\gamma\ge\gamma_{n+1}^+$ and Θ is order-preserving)

$$= \Theta(\Theta(\mathfrak{c},\gamma),\gamma)$$

(by the inductive hypothesis)

$$\le \Theta(\mathfrak{c},\gamma)$$

(5A2H)

$$\le \Theta(2^{\gamma_{n+1}},\gamma)$$

(because $2^{\gamma_{n+1}} \ge \mathfrak{c}$). **Q** In particular,

$$2^{\gamma_{n+1}} = \Theta(2^{\gamma_n},\gamma_{n+1}^+) = \Theta(\mathfrak{c},\gamma_{n+1}^+)$$

for every $n \in \mathbb{N}$.

(c) For each $n \in \mathbb{N}$, let λ_n be the least infinite cardinal such that $\Theta(\lambda_n,\gamma_n^+) > \mathfrak{c}$. Then $\langle\lambda_n\rangle_{n\in\mathbb{N}}$ is non-increasing; also $\lambda_1 \le \mathfrak{c}$, because

$$\Theta(\mathfrak{c},\gamma_1^+) = 2^{\gamma_1} > \mathfrak{c},$$

so there are $n \ge 1$, $\lambda \le \mathfrak{c}$ such that $\lambda_m = \lambda$ for every $m \ge n$. Now for $m \ge n$ we have

$$\mathfrak{c} < \Theta(\lambda,\gamma_m^+) \le \max(\lambda,(\sup_{\lambda'<\lambda}\Theta(\lambda',\gamma_m^+))^{\operatorname{cf}\lambda})$$

(5A2I)

$$\le \max(\lambda,\mathfrak{c}^{\operatorname{cf}\lambda}) = 2^{\operatorname{cf}\lambda}.$$

Also we still have $\lambda \ge \kappa > \Theta(\gamma_n^+,\gamma_n^+)$ because $\Theta(\lambda',\gamma_n^+) < \kappa \le \mathfrak{c}$ for every $\lambda' < \kappa$. Using 5A2H again,

$$2^{\gamma_n} = \Theta(\mathfrak{c},\gamma_n^+) \le \Theta(\Theta(\lambda,\gamma_n^+),\gamma_n^+) \le \Theta(\lambda,\gamma_n^+) \le 2^{\operatorname{cf}\lambda};$$

consequently

$$2^{\gamma_n} < 2^{\gamma_{n+1}} \le 2^{\operatorname{cf}\lambda}$$

and $\operatorname{cf}\lambda > \gamma_n$. But 5A2Ia now tells us that

$$\Theta(\lambda,\gamma_n^+) \le \max(\lambda,\sup_{\lambda'<\lambda}\Theta(\lambda',\gamma_n^+)) \le \mathfrak{c},$$

which is absurd. **X**

This contradiction proves the theorem.

542F Corollary Let $\kappa \le \mathfrak{c}$ be a quasi-measurable cardinal.

(a) There is a regular infinite cardinal $\gamma < \kappa$ such that $2^\gamma = 2^\delta$ for every cardinal δ such that $\gamma \le \delta < \kappa$; that is, $\#([\kappa]^{<\kappa}) = 2^\gamma$.

(b) Let $\mathcal{I}$ be any proper ω_1-saturated κ-additive ideal of $\mathcal{P}\kappa$ containing singletons, and $\mathfrak{A} = \mathcal{P}\kappa/\mathcal{I}$. If γ is as in (a), then the cardinal power $\tau(\mathfrak{A})^\gamma$ is equal to 2^κ.

proof (a) By 542E, there is a first $\gamma < \kappa$ such that $2^\delta = 2^\gamma$ whenever $\gamma \le \delta < \kappa$. Of course γ is infinite; by 5A1Fh it is regular. Because κ is regular,

$$[\kappa]^{<\kappa} = \bigcup_{\xi<\kappa}\mathcal{P}\xi, \quad \#([\kappa]^{<\kappa}) = \max(\kappa,\sup_{\delta<\kappa}2^\delta) = \sup_{\delta<\kappa}2^\delta = 2^\gamma.$$

(b) Of course $\tau(\mathfrak{A}) \le \#(\mathfrak{A}) \le \#(\mathcal{P}\kappa)$, so $\tau(\mathfrak{A})^\gamma \le (2^\kappa)^\gamma = 2^\kappa$. In the other direction, we have an injective function $\phi_\xi : \mathcal{P}\xi \to \mathcal{P}\gamma$ for each $\xi < \kappa$. For $A \subseteq \kappa$ and $\eta < \gamma$ set

$$d_{A\eta} = \{\xi : \xi < \kappa,\ \eta \in \phi_\xi(A \cap \xi)\}^\bullet \in \mathfrak{A}.$$

If A, $B \subseteq \kappa$ are distinct then there is a $\zeta < \kappa$ such that $\phi_\xi(A\cap\xi) \ne \phi_\xi(B\cap\xi)$ for every $\xi \ge \zeta$, that is,

$$\bigcup_{\eta<\gamma}\{\xi : \eta \in \phi_\xi(A\cap\xi)\triangle\phi_\xi(B\cap\xi)\} \supseteq \kappa\setminus\zeta \notin \mathcal{I}.$$

Because $\mathcal{I}$ is κ-additive and $\gamma < \kappa$, there is an $\eta < \gamma$ such that $\{\xi : \eta \in \phi_\xi(A\cap\xi)\triangle\phi_\xi(B\cap\xi)\} \notin \mathcal{I}$, that is, $d_{A\eta} \ne d_{B\eta}$. Thus $A \mapsto \langle d_{A\eta}\rangle_{\eta<\gamma} : \mathcal{P}\kappa \to \mathfrak{A}^\gamma$ is injective, and $2^\kappa \le \#(\mathfrak{A})^\gamma$. But $\mathfrak{A}$ is ccc, so $\#(\mathfrak{A}) \le \max(4,\tau(\mathfrak{A})^\omega)$ (514De) and

$$2^\kappa \le (\tau(\mathfrak{A})^\omega)^\gamma = \tau(\mathfrak{A})^\gamma.$$

542G Corollary Suppose that κ is a quasi-measurable cardinal.

(a) If $\kappa \le \mathfrak{c} < \kappa^{(+\omega_1)}$ (notation: 5A1F(a-ii)), then $2^\lambda \le \mathfrak{c}$ for every cardinal $\lambda < \kappa$.

(b) Let $\mathcal{I}$ be any proper ω_1-saturated κ-additive ideal of $\mathcal{P}\kappa$ containing singletons, and $\mathfrak{A} = \mathcal{P}\kappa/\mathcal{I}$. If $2^\lambda \le \mathfrak{c}$ for every cardinal $\lambda < \kappa$, then $\#(\mathfrak{A}) = 2^\kappa$.

proof (a) ? Otherwise, taking γ as in 542Fa, $2^\gamma > \mathfrak{c}$. Let $\gamma_1 \le \gamma$ be the first cardinal such that $2^{\gamma_1} > \mathfrak{c}$; note that, using 542E and 5A1Fh, we can be sure that γ_1 is regular. Next,

$$\operatorname{cov}_{\mathrm{Sh}}(\kappa, \kappa, \gamma_1^+, \gamma_1) \le \operatorname{cf}[\kappa]^{<\kappa} = \kappa,$$

by 5A2Ea. So $\operatorname{cov}_{\mathrm{Sh}}(\alpha, \kappa, \gamma_1^+, \gamma_1) \le \alpha$ whenever $\kappa \le \alpha < \kappa^{(+\gamma_1)}$ (induce on α, using 5A2Eb). In particular, $\operatorname{cov}_{\mathrm{Sh}}(\mathfrak{c}, \kappa, \gamma_1^+, \gamma_1) \le \mathfrak{c}$. But 541S tells us that $\operatorname{cov}_{\mathrm{Sh}}(\mathfrak{c}, \kappa, \gamma_1^+, \gamma_1) = 2^{\gamma_1}$. **X**

(b) By 541B, $\mathfrak{A}$ is Dedekind complete; by 515L, $\#(\mathfrak{A})^\omega = \#(\mathfrak{A})$; so 542Fb tells us that

$$2^\kappa = \tau(\mathfrak{A})^\omega \le \#(\mathfrak{A}) \le \#(\mathcal{P}\kappa) = 2^\kappa.$$

542H Lemma Let κ be a quasi-measurable cardinal and $\langle \alpha_i \rangle_{i \in I}$ a countable family of ordinals less than κ and of cofinality at least ω_2. Then there is a set $F \subseteq P = \prod_{i \in I} \alpha_i$ such that

(i) F is cofinal with P;

(ii) if $\langle f_\xi \rangle_{\xi < \omega_1}$ is a non-decreasing family in F then $\sup_{\xi < \omega_1} f_\xi \in F$;

(iii) $\#(F) < \kappa$.

proof Note that $\operatorname{add} P = \min_{i \in I} \operatorname{cf} \alpha_i > \omega_1$ (I am passing over the trivial case $I = \emptyset$), so $\sup_{\xi < \omega_1} f_\xi$ is defined in P for every family $\langle f_\xi \rangle_{\xi < \omega_1}$ in P. We have

$$\operatorname{cf} P = \operatorname{cf}(\prod_{i \in I} \operatorname{cf} \alpha_i) \le \Theta(\sup_{i \in I} (\operatorname{cf} \alpha_i)^+, \omega_1) < \kappa,$$

by 542Db. So we can find a cofinal set $F_0 \subseteq P$ of cardinal less than κ. Now for $0 < \zeta \le \omega_2$ define F_ζ by saying that

$$F_{\zeta+1} = \{\sup_{\xi < \omega_1} f_\xi : \langle f_\xi \rangle_{\xi < \omega_1} \text{ is a non-decreasing family in } F_\zeta\},$$

$$F_\zeta = \bigcup_{\eta < \zeta} F_\eta \text{ for non-zero limit ordinals } \zeta \le \omega_2.$$

Then $\#(F_\zeta) < \kappa$ for every ζ. **P** Induce on ζ. For the inductive step to $\zeta + 1$, **?** suppose, if possible, that $\#(F_\zeta) < \kappa$ but $\#(F_{\zeta+1}) \ge \kappa$. Then there is a proper ω_1-saturated κ-additive $\mathcal{I} \lhd \mathcal{P}F_{\zeta+1}$ containing singletons. For each $h \in F_{\zeta+1}$ choose a non-decreasing family $\langle f_{h\xi} \rangle_{\xi < \omega_1}$ in F_ζ with supremum h. The set $h[I]$ of values of h is a countable subset of $Y = \bigcup_{i \in I} \alpha_i$, and $\#(Y) < \kappa$. By 541D, there is a set $H \subseteq F_{\zeta+1}$, with cardinal κ, such that $M = \bigcup_{h \in H} h[I]$ is countable. Now, for each $h \in H$, there is a $\gamma(h) < \omega_1$ such that whenever $i \in I$ and $\beta \in M$ then $h(i) > \beta$ iff $f_{h,\gamma(h)}(i) > \beta$. If g, $h \in H$ and $i \in I$ and $g(i) < h(i)$, then $f_{g,\gamma(g)}(i) \le g(i) < f_{h,\gamma(h)}(i)$, because $g(i) \in M$. Thus $h \mapsto f_{h,\gamma(h)} : H \to F_\zeta$ is injective; but $\#(F_\zeta) < \kappa = \#(H)$. **X**

Thus $\#(F_{\zeta+1}) < \kappa$ if $\#(F_\zeta) < \kappa$. At limit ordinals ζ the induction proceeds without difficulty because $\operatorname{cf} \kappa > \zeta$. **Q**

So $\#(F_{\omega_2}) < \kappa$ and we may take $F = F_{\omega_2}$.

542I Theorem (Shelah 96) Let κ be a quasi-measurable cardinal.

(a) For any cardinal θ, $\operatorname{cf}[\kappa]^{<\theta} \le \kappa$.

(b) For any cardinal $\lambda < \kappa$, and any θ, $\operatorname{cf}[\lambda]^{<\theta} < \kappa$.

proof (a)(i) Consider first the case $\theta = \omega_1$. Write G_1 for the set of ordinals less than κ of cofinality less than or equal to ω_1; for $\delta \in G_1$ let $\psi_\delta : \operatorname{cf} \delta \to \delta$ enumerate a cofinal subset of δ. Next, write G_2 for $\kappa \setminus G_1$, and for every countable set $A \subseteq G_2$ let $F(A) \subseteq \prod_{\alpha \in A} \alpha$ be a cofinal set, with cardinal less than κ, closed under suprema of non-decreasing families of length ω_1; such exists by 542H above.

(ii) It is worth observing at this point that if $\langle A_\zeta \rangle_{\zeta < \omega_1}$ is any family of countable subsets of G_2, $D = \bigcup_{\zeta < \omega_1} A_\zeta$, and $g \in \prod_{\alpha \in D} \alpha$, then there is an $f \in \prod_{\alpha \in D} \alpha$ such that $f \ge g$ and $f \restriction A_\zeta \in F(A_\zeta)$ for every $\zeta < \omega_1$. **P** Let $\langle \phi(\xi) \rangle_{\xi < \omega_1}$ run over ω_1 with cofinal repetitions. Choose a non-decreasing family $\langle f_\xi \rangle_{\xi < \omega_1}$ in $\prod_{\alpha \in D} \alpha$ in such a way that $f_0 = g$ and $f_{\xi+1} \restriction A_{\phi(\xi)} \in F(A_{\phi(\xi)})$ for every ξ; this is possible because

$\operatorname{add}(\prod_{\alpha\in D}) \ge \omega_2$ and $F(A)$ is cofinal with $\prod_{\alpha\in A}\alpha$ for every A. Set $f = \sup_{\xi<\omega_1} f_\xi$; this works because every $F(A)$ is closed under suprema of non-decreasing families of length ω_1. **Q**

(iii) We can now find a family $\mathcal{A}$ of countable subsets of κ such that

(α) $\{\alpha\} \in \mathcal{A}$ for every $\alpha < \kappa$;

(β) whenever A, $A' \in \mathcal{A}$ and $\zeta < \omega_1$ then $A \cup A'$, $A \cap G_2$ and $\{\psi_\alpha(\xi) : \alpha \in A \cap G_1, \xi < \min(\zeta, \operatorname{cf}\alpha)\}$ all belong to $\mathcal{A}$;

(γ) whenever $A \in \mathcal{A} \cap [G_2]^{\le\omega}$ and $f \in F(A)$ then $f[A] \in \mathcal{A}$;

(δ) $\#(\mathcal{A}) \le \kappa$.

(iv) ? Suppose, if possible, that $\operatorname{cf}[\kappa]^{\le\omega} > \kappa$. Because $[\kappa]^{\le\omega} = \bigcup_{\lambda<\kappa}[\lambda]^{\le\omega}$, there is a cardinal $\lambda < \kappa$ such that $\operatorname{cf}[\lambda]^{\le\omega} > \kappa$. We can therefore choose inductively a family $\langle a_\xi\rangle_{\xi<\kappa}$ of countable subsets of λ such that

$$a_\xi \not\subseteq \bigcup_{\eta\in A\cap\xi} a_\eta$$

whenever $\xi < \kappa$ and $A \in \mathcal{A}$. By 541D, there is a set $W \subseteq \kappa$, with cardinal κ, such that $\bigcup_{\xi\in W} a_\xi$ is countable. Let $\delta < \kappa$ be such that $W \cap \delta$ is cofinal with δ and of order type ω_1; then $\delta \in G_1$.

(v) I choose a family $\langle A_{k\zeta}\rangle_{\zeta<\omega_1,k\in\mathbb{N}}$ in $\mathcal{A}$ as follows. Start by setting $A_{0\zeta} = \psi_\delta[\zeta]$ for every $\zeta < \omega_1$; then $A_{0\zeta} \in \mathcal{A}$ by (iii)(α-β). Given $\langle A_{k\zeta}\rangle_{\zeta<\omega_1}$, set $A'_{k\zeta} = A_{k\zeta} \cap G_2$ for each $\zeta < \omega_1$. For $\alpha \in D_k = \bigcup_{\zeta<\omega_1} A'_{k\zeta}$, set $g_k(\alpha) = \sup(\alpha \cap W \cap \delta) < \alpha$; choose $f_k \in \prod_{\alpha\in D_k}\alpha$ such that $g_k \le f_k$ and $f_k{\restriction}A'_{k\zeta} \in F(A'_{k\zeta})$ for every ζ; this is possible by (ii) above. Set

$$A_{k+1,\zeta} = A_{k\zeta} \cup f_k[A'_{k\zeta}] \cup \{\psi_\alpha(\xi) : \alpha \in A_{k\zeta} \cap G_1, \xi < \min(\zeta, \operatorname{cf}\alpha)\} \in \mathcal{A}$$

for each $\zeta < \omega_1$, and continue. An easy induction on k shows that $\langle A_{k\zeta}\rangle_{\zeta<\omega_1}$ is non-decreasing for every k; also, every $A_{k\zeta}$ is a subset of δ.

(vi) Set $V_k = \bigcup_{\zeta<\omega_1} A_{k\zeta}$, $b_k = \bigcup\{a_\xi : \xi \in W \cap V_k\}$; then b_k is countable and there is a $\beta(k) < \omega_1$ such that $b_k = \bigcup\{a_\xi : \xi \in W \cap A_{k,\beta(k)}\}$. Now $\bigcup_{k\in\mathbb{N}} A_{k,\beta(k)}$ is a countable subset of δ, so there is a member γ of $W \cap \delta$ greater than its supremum. We have

$$a_\gamma \not\subseteq \bigcup\{a_\eta : \eta \in A_{k,\beta(k)}\},$$

so $a_\gamma \not\subseteq b_k$ and $\gamma \notin V_k$, for each k.

Set $V = \bigcup_{k\in\mathbb{N}} V_k$. We have just seen that $W \cap \delta \not\subseteq V$; set $\gamma_0 = \min(W \cap \delta \setminus V)$. Because $V_0 = \psi_\delta[\omega_1]$ is cofinal with δ, $V \setminus \gamma_0 \ne \emptyset$; let γ_1 be its least member. Then $\gamma_1 > \gamma_0$. There must be $k \in \mathbb{N}$ and $\zeta < \omega_1$ such that $\gamma_1 \in A_{k\zeta}$. Observe that if $\alpha \in V \cap G_1$ then $V \cap \alpha$ is cofinal with α; but $V \cap \gamma_1 \subseteq \gamma_0$, so $\gamma_1 \notin G_1$ and $\gamma_1 \in A'_{k\zeta} \subseteq D_k$. But now $f_k(\gamma_1) \in A_{k+1,\zeta} \subseteq V$ and $\gamma_0 \le g_k(\gamma_1) \le f_k(\gamma_1) < \gamma_1$, so $\gamma_1 \ne \min(V \setminus \gamma_0)$. **X**

(vii) This contradiction shows that $\operatorname{cf}[\kappa]^{\le\omega} \le \kappa$. Now consider $\operatorname{cf}[\kappa]^{\le\delta}$, where $\delta < \kappa$ is an infinite cardinal. Then

$$\underset{\mathrm{Sh}}{\operatorname{cov}}(\kappa, \delta^+, \delta^+, \omega_1) = \max(\kappa, \sup_{\lambda<\kappa} \underset{\mathrm{Sh}}{\operatorname{cov}}(\lambda, \delta^+, \delta^+, \omega_1))$$

(5A2Eb again)

$$\le \kappa$$

by 542Dc. So there is a family $\mathcal{B} \subseteq [\kappa]^{\le\delta}$, with cardinal at most κ, such that every member of $[\kappa]^{\le\delta}$ is covered by a sequence in $\mathcal{B}$. But now $\operatorname{cf}[\mathcal{B}]^{\le\omega} \le \kappa$, so there is a family $\mathfrak{C}$ of countable subsets of $\mathcal{B}$ which is cofinal with $[\mathcal{B}]^{\le\omega}$ and with cardinal at most κ; setting $\mathcal{D} = \{\bigcup\mathcal{C} : \mathcal{C} \in \mathfrak{C}\}$, we have $\mathcal{D}$ cofinal with $[\kappa]^{\le\delta}$ and with cardinal at most κ. So $\operatorname{cf}[\kappa]^{\le\delta} \le \kappa$.

Finally, of course, $[\kappa]^{<\theta} = \bigcup_{\delta<\theta}[\kappa]^{\le\delta}$, so

$$\operatorname{cf}[\kappa]^{<\theta} \le \sup_{\delta<\theta} \operatorname{cf}[\kappa]^{\le\delta} \le \kappa$$

whenever $\theta \le \kappa$. For $\theta > \kappa$ we have $\operatorname{cf}[\kappa]^{<\theta} = 1$, so $\operatorname{cf}[\kappa]^{<\theta} \le \kappa$ for every θ.

(b) If $\mathcal{A}$ is cofinal with $[\kappa]^{<\theta}$ then $\{A \cap \lambda : A \in \mathcal{A}\}$ is cofinal with $[\lambda]^{<\theta}$, so $\operatorname{cf}[\lambda]^{<\theta} \le \operatorname{cf}[\kappa]^\theta \le \kappa$, by (a).

? If $\operatorname{cf}[\lambda]^{<\theta} = \kappa$, there is a cofinal family $\langle A_\xi\rangle_{\xi<\kappa}$ in $[\lambda]^{<\theta}$ such that $A_\xi \not\subseteq A_\eta$ for any $\eta < \xi < \kappa$. Of course $\omega < \theta \le \lambda < \kappa = \operatorname{cf}\kappa$, so we may suppose that every A_ξ is infinite. So there is an infinite $\delta < \theta$ such that $E = \{\xi : \xi < \kappa,\, \#(A_\xi) = \delta\}$ has cardinal κ. Next, by 541D applied to a suitable ideal of subsets of E, there is a set $M \in [\lambda]^{\le\delta}$ such that $F = \{\xi : \xi \in E,\, A_\xi \subseteq M\}$ has cardinal κ. But now there must be an $\eta < \kappa$ such that $M \subseteq A_\eta$, and a $\xi \in F$ such that $\xi > \eta$; which is impossible. **X**

Thus $\operatorname{cf}[\lambda]^{<\theta} < \kappa$, as claimed.

542J Corollary Let κ be a quasi-measurable cardinal. Let $\langle P_\zeta\rangle_{\zeta<\lambda}$ be a family of partially ordered sets such that $\lambda < \operatorname{add} P_\zeta \le \operatorname{cf} P_\zeta < \kappa$ for every $\zeta < \lambda$. Then $\operatorname{cf}(\prod_{\zeta<\lambda} P_\zeta) < \kappa$.

proof For each $\zeta < \lambda$ let Q_ζ be a cofinal subset of P_ζ with cardinal less than κ. Set $P = \prod_{\zeta<\kappa} P_\zeta$, $Z = \bigcup_{\zeta<\lambda} Q_\zeta$; then $\#(Z) < \kappa$ so $\operatorname{cf}[Z]^{\le\lambda} < \kappa$, by 542Ib. Let $\mathcal{A}$ be a cofinal subset of $[Z]^{\le\lambda}$ with $\#(\mathcal{A}) < \kappa$. For each $A \in \mathcal{A}$ choose $f_A \in P$ such that $f_A(\zeta)$ is an upper bound for $A \cap P_\zeta$ for every ζ; this is possible because $\operatorname{add} P_\zeta > \#(A)$. Set $F = \{f_A : A \in \mathcal{A}\}$.

If $g \in P$, then there is an $h \in \prod_{\zeta<\lambda} Q_\zeta$ such that $g \le h$. Now $h[\lambda] \in [Z]^{\le\lambda}$ so there is an $A \in \mathcal{A}$ such that $h[\lambda] \subseteq A$. In this case $h \le f_A$. Accordingly F is cofinal with P and $\operatorname{cf} P \le \#(F) < \kappa$, as required.

542K For an application in §547 below, it will be useful to have the following variant of 542I.

Proposition Let κ be a quasi-measurable cardinal.

(a) For every cardinal $\theta < \kappa$ there is a family $\mathcal{D}_\theta$ of countable sets, with cardinal less than κ, which is stationary over θ in the sense of 5A1R.

(b) There is a family $\mathcal{A}$ of countable sets, with cardinal at most κ, which is stationary over κ.

proof Fix on a proper κ-additive ω_1-saturated ideal $\mathcal{I}$ of subsets of κ containing all singleton subsets of κ. Write $\mathcal{F}$ for $\{F : F \subseteq \kappa,\, \kappa \setminus F \in \mathcal{I}\}$.

(a)(i) By 542Ib, there is a cofinal subset $\mathcal{C}$ of $[\theta]^{\le\omega}$ with cardinal less than κ. Set

$$\mathcal{A} = \{\textstyle\bigcup_{\alpha<\omega_1} C_\alpha : \langle C_\alpha\rangle_{\alpha<\omega_1} \text{ is a non-decreasing family in } \mathcal{C}\}.$$

Note that $\mathcal{A} \subseteq [\theta]^{\le\omega_1}$. Now $\mathcal{A}$ is stationary over θ. **P** If $f : [\theta]^{<\omega} \to [\theta]^{\le\omega}$ is a function, choose $\langle C_\alpha\rangle_{\alpha<\omega_1}$ in $\mathcal{C}$ such that

$$C_\alpha \supseteq \textstyle\bigcup_{\beta<\alpha} C_\beta \cup \bigcup\{f(I) : I \in [\bigcup_{\beta<\alpha} C_\beta]^{<\omega}\}$$

for $\alpha < \omega_1$. Then $A = \bigcup_{\alpha<\omega_1} C_\alpha$ belongs to $\mathcal{A}$ and $f(I) \subseteq A$ for every $I \in [A]^{<\omega}$. As f is arbitrary, $\mathcal{A}$ is stationary over θ. **Q**

(ii) ? Suppose, if possible, that $\#(\mathcal{A}) \ge \kappa$. Let $\langle A_\xi\rangle_{\xi<\kappa}$ be a family of distinct elements of $\mathcal{A}$. For each $\xi < \kappa$ let $\langle C_{\xi\alpha}\rangle_{\alpha<\omega_1}$ be a non-decreasing family in $\mathcal{C}$ such that $A_\xi = \bigcup_{\alpha<\omega_1} C_{\xi\alpha}$. By 541D, there is a set $A \in [\theta]^{\le\omega_1}$ such that $D = \{\xi : \xi < \kappa,\, A_\xi \subseteq A\}$ belongs to $\mathcal{F}$. Let $\langle B_\alpha\rangle_{\alpha<\omega_1}$ be a non-decreasing family of countable sets with union A.

(iii) For each $\alpha < \omega_1$ there is a countable set $\mathcal{B}_\alpha$ such that $D_\alpha = \{\xi : \xi \in D,\, A_\xi \cap B_\alpha \in \mathcal{B}_\alpha\}$ belongs to $\mathcal{F}$. **P** For each $\xi \in D$ there is a $\gamma_\xi < \omega_1$ such that $A_\xi \cap B_\alpha = C_{\xi\gamma_\xi} \cap B_\alpha$. As $\#(\mathcal{C}) < \kappa$, there is a countable set $\mathcal{D}$ such that $D \setminus \{\xi : C_{\xi\gamma_\xi} \in \mathcal{D}\} \in \mathcal{I}$; set $\mathcal{B}_\alpha = \{C \cap B_\alpha : C \in \mathcal{D}\}$. **Q**

Set $D' = \bigcap_{\alpha<\omega_1} D_\alpha$; then $D' \in \mathcal{F}$.

(iv) Whenever ξ, $\eta \in D'$ are distinct there is an $\alpha < \omega_1$ such $A_\xi \cap B_\alpha \ne A_\eta \cap B_\alpha$. There is therefore, for each $\xi \in D'$, a $\beta_\xi < \omega_1$ such that $\{\eta : \eta \in D',\, A_\xi \cap B_{\beta_\xi} \ne A_\eta \cap B_{\beta_\xi}\} \in \mathcal{F}$. Next, there is a $\beta < \omega_1$ such that $D'' = \{\xi : \xi \in D',\, \beta_\xi \le \beta\}$ belongs to $\mathcal{I}$. However, because $\mathcal{B}_\beta$ is countable, there is a $B \in \mathcal{B}_\beta$ such that $E = \{\xi : \xi \in D'',\, A_\xi \cap B_\beta = B\}$ does not belong to $\mathcal{I}$. But now, if we take any $\xi \in E$, there must be an $\eta \in E$ such that $A_\xi \cap B_{\beta_\xi} \ne A_\eta \cap B_{\beta_\xi}$, in which case $A_\xi \cap B_\beta$ and $A_\eta \cap B_\beta$ are distinct and cannot both be equal to B. **X**

Thus $\#(\mathcal{A}) < \kappa$.

(v) Finally, observe that for each $A \in \mathcal{A}$ there is a family of countable sets, with cardinal at most ω_1, which is stationary over A. (If A is countable, we can take $\{A\}$; otherwise, use 5A1Sc.) Taking the union of these, we get a family of countable sets, with cardinal less than κ, which is stationary over θ by 5A1Sb.

(b) Because κ is a limit cardinal of uncountable cofinality, the set of cardinals less than κ is stationary over κ, by 5A1Sc. Putting this together with (a) and 5A1Sb, we see that there is a family of countable sets, with cardinal at most κ, which is stationary over κ.

542X Basic exercises (a) Let κ be a quasi-measurable cardinal, and θ a cardinal such that $2 \le \theta \le \kappa$. Show that $\mathrm{cf}[\kappa]^{<\theta} = \kappa$.

542Y Further exercises (a) Let X be a hereditarily weakly θ-refinable topological space such that there is no quasi-measurable cardinal less than or equal to the weight of X, and μ a totally finite Maharam submeasure on the Borel σ-algebra of X. (i) Show that μ is τ-subadditive in the sense that if whenever $\mathcal{G}$ is a non-empty upwards-directed family of open sets in X with union H, then $\inf_{G\in\mathcal{G}} \mu(H \setminus G) = 0$. (ii) Show that if X is Hausdorff and K-analytic, then the completion of μ is a Radon submeasure on X.

542 Notes and comments The arguments of this section have taken on a certain density, and I ought to explain what they are for. The cardinal arithmetic of 542E-542G is relevant to one of the most important questions in this chapter, to be treated in the next section: supposing that there is an extension of Lebesgue measure to a measure μ defined on all subsets of $\mathbb{R}$, what can we say about the Maharam type of μ? And 542I-542J will tell us something about the cofinalities of our favourite partially ordered sets under the same circumstances.

Let me draw your attention to a useful trick, used twice above. If κ is a quasi-measurable cardinal, and X is any set with cardinal at least κ, there is a non-trivial ω_1-saturated σ-ideal of subsets of X. This is the basis of the proof of 542H (taking $X = F_{\zeta+1}$ in the inductive step) and the final step in the proof of 542I (taking $X = E$). Exposed like this, the idea seems obvious. In the thickets of an argument it sometimes demands an imaginative jump.

Version of 11.11.13

543 The Gitik-Shelah theorem

I come now to the leading case at the centre of the work of the last two sections. If our ω_1-saturated σ-ideal of sets is the null ideal of a measure with domain $\mathcal{P}X$, it has some even more striking properties than those already discussed. I will go farther into these later in the chapter. But I will begin with what is known about one of the first questions I expect a reader of this book to ask: if $(X, \mathcal{P}X, \mu)$ is a probability space, what can, or must, its measure algebra be? There can, of course, be a purely atomic part; the interesting question relates to the atomless part, if any, always remembering that we need a special act of faith to believe that there can be an atomless case. Here we find that the Maharam type of an atomless probability defined on a power set must be greater than its additivity (543F), which must itself be 'large' (541L).

543A Definitions (a) A **real-valued-measurable cardinal** is an uncountable cardinal κ such that there is a κ-additive probability measure ν on κ, defined on every subset of κ, for which all singletons are negligible. In this context I will call ν a **witnessing probability**.

(b) If κ is a regular uncountable cardinal, a probability measure ν on κ with domain $\mathcal{P}\kappa$ is **normal** if its null ideal $\mathcal{N}(\nu)$ is normal. In this case, ν must be κ-additive (541H, 521Ad) and zero on singletons, so κ is real-valued-measurable, and I will say that ν is a **normal witnessing probability**.

(c) An **atomlessly-measurable cardinal** is a real-valued-measurable cardinal with an atomless witnessing probability.

543B Collecting ideas which have already appeared, some of them more than once, we have the following.

Proposition (a) Let $(X, \mathcal{P}X, \mu)$ be a totally finite measure space in which singletons are negligible and $\mu X > 0$. Then $\kappa = \mathrm{add}\,\mu$ is real-valued-measurable, and there are a non-negligible $Y \subseteq X$ and a function $g : Y \to \kappa$ such that the normalized image measure $B \mapsto \frac{1}{\mu Y}\mu g^{-1}[B]$ is a normal witnessing probability on κ.

(b) Every real-valued-measurable cardinal is quasi-measurable (definition: 542A) and has a normal witnessing probability; in particular, every real-valued-measurable cardinal is uncountable and regular.

(c) If $\kappa \le \mathfrak{c}$ is a real-valued-measurable cardinal, then κ is atomlessly-measurable, and every witnessing probability on κ is atomless.

(d) If $\kappa > \mathfrak{c}$ is a real-valued-measurable cardinal, then κ is two-valued-measurable, and every witnessing probability on κ is purely atomic.

(e) A cardinal λ is measure-free (definition: 438A) iff there is no real-valued-measurable cardinal $\kappa \le \lambda$; $\mathfrak{c}$ is measure-free iff there is no atomlessly-measurable cardinal.

(f) Again suppose that $(X, \mathcal{P}X, \mu)$ is a totally finite measure space.

(i) If μ is purely atomic, $\operatorname{add}\mu$ is either ∞ or a two-valued-measurable cardinal.

(ii) If μ is not purely atomic, $\operatorname{add}\mu$ is atomlessly-measurable.

proof (a) By 521Ad again, κ is the additivity of the null ideal $\mathcal{N}(\mu)$ of μ; because μ is σ-finite, $\mathcal{N}(\mu)$ is ω_1-saturated; and of course $\kappa \ge \omega_1$. By 541J, there are $Y \subseteq X$ and $g : Y \to \kappa$ such that $\mathcal{I} = \{B : B \subseteq \kappa, \mu g^{-1}[B] = 0\}$ is a normal ideal on κ. In particular, $\kappa \notin \mathcal{I}$ and Y is non-negligible. Set $\nu B = \frac{1}{\mu Y}\mu g^{-1}[B]$ for $B \subseteq \kappa$; then ν is a probability measure with domain $\mathcal{P}\kappa$. Its null ideal $\mathcal{N}(\nu) = \mathcal{I}$ is normal, so it is a normal measure and witnesses that κ is real-valued-measurable.

(b) If κ is real-valued-measurable, then (a) tells us that any witnessing probability on κ can be used to define a normal witnessing probability ν say. Since κ is the additivity of a σ-ideal, it must be uncountable and regular (513C(a-i)); also $\mathcal{N}(\nu)$ is an ω_1-saturated normal ideal, so κ is quasi-measurable.

(c)-(d) Apply 541P. If κ is real-valued-measurable, it is a regular uncountable cardinal, and if ν is a witnessing probability on κ then $\mathcal{N}(\nu)$ is a proper ω_1-saturated κ-additive ideal of subsets of κ. Taking $\mathfrak{A}$ to be the measure algebra of ν, then 541P tells us that either $\mathfrak{A}$ is atomless and $\kappa \le \mathfrak{c}$, or $\mathfrak{A}$ is purely atomic and κ is two-valued-measurable, in which case κ is surely greater than $\mathfrak{c}$ (541N). Turning this round, if $\kappa \le \mathfrak{c}$ then $\mathfrak{A}$ and ν must be atomless and κ is atomlessly-measurable, while if $\kappa > \mathfrak{c}$ then $\mathfrak{A}$ and ν are purely atomic and κ is two-valued-measurable.

(e) If $\kappa \le \lambda$ is real-valued-measurable, then any witnessing probability on κ extends to a probability measure with domain $\mathcal{P}\lambda$ which is zero on singletons, so λ is not measure-free. If λ is not measure-free, let μ be a probability measure with domain $\mathcal{P}\lambda$ which is zero on singletons; then (a) tells us that $\operatorname{add}\mu$ is real-valued-measurable, and $\operatorname{add}\mu \le \lambda$ because $\lambda = \bigcup \mathcal{N}(\mu)$.

If there is an atomlessly-measurable cardinal κ, then κ is real-valued-measurable and there is an atomless witnessing probability on κ, and $\kappa \le \mathfrak{c}$, by (d). So in this case $\mathfrak{c}$ is not measure-free. On the other hand, if $\mathfrak{c}$ is not measure-free, there is a real-valued-measurable cardinal $\kappa \le \mathfrak{c}$, which is atomlessly-measurable, by (c).

(f)(i) If μ is purely atomic, and $\operatorname{add}\mu$ is not ∞, set $\kappa = \operatorname{add}\mu$ and let $\langle A_\xi \rangle_{\xi<\kappa}$ be a family of negligible sets in X with non-negligible union A. Let $E \subseteq A$ be an atom for μ. Repeating the construction of (a), but starting from the subspace $(E, \mathcal{P}E, \mu_E)$, we see that the normalized image measure constructed on $\kappa = \operatorname{add}\mu_E$ can take only the two values 0 and 1, so that its null ideal is 2-saturated and witnesses that κ is two-valued-measurable.

(ii) If μ is not purely atomic, let E be the atomless part of X, so that μ_E is atomless and $\mu_{X\setminus E}$ is purely atomic. Singletons in E must be negligible, so (a) tells us that $\operatorname{add}\mu_E$ is a real-valued-measurable cardinal; also there is an inverse-measure-preserving function from E to $[0, \mu E]$ (343Cc), so E can be covered by $\mathfrak{c}$ negligible sets and $\operatorname{add}\mu_E \le \mathfrak{c}$ is atomlessly-measurable, by (c) here. Now (i) tells us that $\mathfrak{c} \le \operatorname{add}\mu_{X\setminus E}$, so $\operatorname{add}\mu = \min(\operatorname{add}\mu_E, \operatorname{add}\mu_{X\setminus E}) = \operatorname{add}\mu_E$ is atomlessly-measurable.

Remark 543Bc-543Bd are **Ulam's dichotomy**.

543C Theorem (see Kunen n70) Suppose that $(Y, \mathcal{P}Y, \nu)$ is a σ-finite measure space and that $(X, \mathfrak{T}, \Sigma, \mu)$ is a σ-finite quasi-Radon measure space with $w(X) < \operatorname{add}\nu$. Let $f : X \times Y \to [0, \infty]$ be any function. Then

$$\overline{\int}\Bigl(\int f(x,y)\nu(dy)\Bigr)\mu(dx) \le \int\Bigl(\overline{\int} f(x,y)\mu(dx)\Bigr)\nu(dy).$$

proof Because μ is σ-finite and effectively locally finite, there is a sequence of open sets of finite measure with conegligible union in X. Since none of the integrals are changed by deleting a negligible subset of X, and the weight of any subset of X is at most the weight of X, we may suppose that this conegligible union is X itself, so that μ is outer regular with respect to the open sets (412Wb). Set $\lambda = w(X) < \operatorname{add}\nu$; let $\langle G_\xi\rangle_{\xi<\lambda}$ enumerate a base for the topology of X. Fix $\epsilon > 0$. Because ν is σ-finite, we have a function $y \mapsto \epsilon_y : Y \to]0,\infty[$ such that $\int \epsilon_y \nu(dy) \le \epsilon$. For each $y \in Y$, let $h_y : X \to [0,\infty]$ be a lower semi-continuous function such that $f(x,y) \le h_y(x)$ for every $x \in X$ and

$$\int h_y(x)\mu(dx) \le \epsilon_y + \overline{\int} f(x,y)\mu(dx)$$

(412Wa). For $I \subseteq \lambda$, $x \in X$ and $y \in Y$, set

$$f_I(x,y) = \sup(\{0\} \cup \{s : \exists\, \xi \in I,\, x \in G_\xi,\, h_y(x') \ge s \;\forall\, x' \in G_\xi\}).$$

Then f_I is expressible as $\sup_{\xi\in I, s\in\mathbb{Q}^+} s\chi(G_\xi \times B_{\xi s})$, writing $\mathbb{Q}^+$ for the set of non-negative rational numbers and $B_{\xi s}$ for $\{y : h_y \ge s\chi G_\xi\}$. So f_I is $(\Sigma\widehat{\otimes}\mathcal{P}Y)$-measurable for all countable I, and for such I we shall have

$$\iint f_I(x,y)\mu(dx)\nu(dy) = \iint f_I(x,y)\nu(dy)\mu(dx),$$

by Fubini's theorem (252C). Next, for any $I \subseteq \lambda$, $x \mapsto f_I(x,y)$ is lower semi-continuous for each y, and

$$\sup_{I\in[\lambda]^{<\omega}} f_I(x,y) = h_y(x)$$

for all $x \in X$ and $y \in Y$, because each h_y is lower semi-continuous. So

$$\sup_{I\in[\lambda]^{<\omega}} \int f_I(x,y)\mu(dx) = \int h_y(x)\mu(dx)$$

for each $y \in Y$ (414Ba). Because $\lambda < \operatorname{add}\nu$, it follows that

$$\sup_{I\in[\lambda]^{<\omega}} \iint f_I(x,y)\mu(dx)\nu(dy) = \iint h_y(x)\mu(dx)\nu(dy)$$

(521B(d-i)). On the other hand, if we write

$$g_I(x) = \int f_I(x,y)\nu(dy)$$

for $x \in X$ and finite $I \subseteq \lambda$, then g_I also is lower semi-continuous. **P** If $x \in X$, set $J = \{\xi : \xi \in I, x \in G_\xi\}$ and $H = X \cap \bigcap_{\xi\in J} G_\xi$; then $f_I(x,y) \le f_I(x',y)$ whenever $x' \in H$ and $y \in Y$, so $g_I(x) \le g_I(x')$ for $x' \in H$, while $x \in \operatorname{int} H$. **Q** So $g = \sup_{I\in[\lambda]^{<\omega}} g_I$ is lower semi-continuous, and $\int g(x)\mu(dx) = \sup_{I\in[\lambda]^{<\omega}} \int g_I(x)\mu(dx)$. Also

$$g(x) = \sup_{I\in[\lambda]^{<\omega}} \int f_I(x,y)\nu(dy) = \int h_y(x)\nu(dy) \ge \int f(x,y)\nu(dy)$$

for every $x \in X$. So we have

$$\begin{aligned}\overline{\iint} f(x,y)\nu(dy)\mu(dx) &\le \int g(x)\mu(dx) = \sup_{I\in[\lambda]^{<\omega}} \int g_I(x)\mu(dx)\\ &= \sup_{I\in[\lambda]^{<\omega}} \iint f_I(x,y)\nu(dy)\mu(dx)\\ &= \sup_{I\in[\lambda]^{<\omega}} \iint f_I(x,y)\mu(dx)\nu(dy)\\ &= \iint h_y(x)\mu(dx)\nu(dy) \le \epsilon + \int\overline{\int} f(x,y)\mu(dx)\nu(dy).\end{aligned}$$

As ϵ is arbitrary, we have the result.

Remark Compare 537O.

543D Corollary Let κ be a real-valued-measurable cardinal, with witnessing probability ν, and $(X,\mathfrak{T},\Sigma,\mu)$ a totally finite quasi-Radon measure space with $w(X) < \kappa$.

(a) If $C \subseteq X \times \kappa$ then

$$\overline{\int} \nu C[\{x\}]\mu(dx) \le \int \mu^* C^{-1}[\{\xi\}]\nu(d\xi).$$

(b) If $A \subseteq X$ and $\#(A) \le \kappa$, then there is a $B \subseteq A$ such that $\#(B) < \kappa$ and $\mu^* B = \mu^* A$.

(c) If $\langle C_\xi \rangle_{\xi<\kappa}$ is a family in $\mathcal{P}X \setminus \mathcal{N}(\mu)$ such that $\#(\bigcup_{\xi<\kappa} C_\xi) < \kappa$, then there are distinct ξ, $\eta < \kappa$ such that $\mu^*(C_\xi \cap C_\eta) > 0$.

(d) If we have a family $\langle h_\xi \rangle_{\xi<\kappa}$ of functions such that $\operatorname{dom} h_\xi$ is a non-negligible subset of X for each ξ and $\#(\bigcup_{\xi<\kappa} h_\xi) < \kappa$ (identifying each h_ξ with its graph), then there are distinct ξ, $\eta < \kappa$ such that

$$\mu^*\{x : x \in \operatorname{dom}(h_\xi) \cap \operatorname{dom}(h_\eta),\ h_\xi(x) = h_\eta(x)\} > 0.$$

proof (a) Apply 543C to $\chi C : X \times \kappa \to \mathbb{R}$.

(b)? Suppose, if possible, otherwise. Then surely $\#(A) = \kappa$; let $f : \kappa \to A$ be a bijection. Set

$$C = \{(f(\eta), \xi) : \eta \le \xi < \kappa\} \subseteq X \times \kappa.$$

If $x \in A$,

$$\nu C[\{x\}] = \nu\{\xi : f^{-1}(x) \le \xi < \kappa\} = 1,$$

so $\overline{\int} \nu C[\{x\}]\mu(dx) = \mu^* A$. If $\xi < \kappa$,

$$\mu^* C^{-1}[\{\xi\}] = \mu^*\{f(\eta) : \eta \le \xi\} < \mu^* A,$$

so $\int \mu^* C^{-1}[\{\xi\}]\nu(d\xi) < \mu^* A$. But this contradicts (a). **X**

(c) Let $\tilde{\nu}$ be the probability on $\kappa \times \kappa$ defined by writing

$$\tilde{\nu} A = \int \nu A[\{\xi\}]\, \nu(d\xi) \text{ for every } A \subseteq \kappa \times \kappa.$$

Then $\tilde{\nu}$ is κ-additive, by 521B(d-ii). Set

$$\begin{aligned} C &= \{(x, (\xi, \eta)) : \xi, \eta \text{ are distinct members of } \kappa,\ x \in C_\xi \cap C_\eta\} \\ &\subseteq X \times (\kappa \times \kappa). \end{aligned}$$

Set

$$E = \{x : x \in X,\ \nu\{\xi : x \in C_\xi\} = 0\}.$$

Because $\#(\bigcup_{\xi<\kappa} C_\xi) < \kappa$,

$$\{\xi : E \cap C_\xi \ne \emptyset\} = \bigcup\{\{\xi : x \in C_\xi\} : x \in E \cap \bigcup_{\eta<\kappa} C_\eta\}$$

is ν-negligible, and there is a $\xi < \kappa$ with $C_\xi \cap E = \emptyset$; thus $\mu^*(X \setminus E) > 0$. Now if $x \in X \setminus E$ then

$$\tilde{\nu}\{(\xi, \eta) : (x, (\xi, \eta)) \in C\} = (\nu\{\xi : x \in C_\xi\})^2 > 0.$$

So we have

$$\begin{aligned} 0 &< \overline{\int} \tilde{\nu}\{(\xi, \eta) : (x, (\xi, \eta)) \in C\}\mu(dx) \\ &\le \int \mu^*\{x : (x, (\xi, \eta)) \in C\}\tilde{\nu}(d(\xi, \eta)) \end{aligned}$$

by 543C, and there must be distinct ξ, $\eta < \kappa$ such that $\mu^*\{x : (x, (\xi, \eta)) \in C\} > 0$, as required.

(d) Set $Y = \bigcup_{\xi<\kappa} h_\xi[X]$. Give $X \times Y$ the measure $\tilde{\mu}$ and topology $\mathfrak{T}'$ defined as follows. The domain of $\tilde{\mu}$ is to be the family $\tilde{\Sigma}$ of subsets H of $X \times Y$ for which there are E, $E' \in \Sigma$ with $\mu(E' \setminus E) = 0$ and $E \times Y \subseteq H \subseteq E' \times Y$; and for such H, $\tilde{\mu} H$ is to be $\mu E = \mu E'$. The topology $\mathfrak{T}'$ is to be just the family $\{G \times Y : G \in \mathfrak{T}\}$. It is easy to check that $(X \times Y, \mathfrak{T}', \tilde{\Sigma}, \tilde{\mu})$ is a totally finite quasi-Radon measure space of weight less than κ, and that $\tilde{\mu}^* h_\xi = \mu^*(\operatorname{dom} h_\xi) > 0$ for each $\xi < \kappa$. So (c) gives the result.

543M Lemma Let $\kappa \le \mathfrak{c}$ be a quasi-measurable cardinal and $\lambda < \min(\kappa^{(+\omega)}, 2^\kappa)$ an infinite cardinal. Set $\zeta = \max(\lambda^+, \kappa^+)$.

(a) We have an infinite cardinal $\delta < \kappa$, a stationary set $S \subseteq \zeta$, and a family $\langle g_\alpha \rangle_{\alpha \in S}$ of functions from κ to 2^δ such that $g_\alpha[\kappa] \subseteq \alpha$ for every $\alpha \in S$ and $\#(g_\alpha \cap g_\beta) < \kappa$ for distinct α, $\beta \in S$. Moreover, we can arrange that

— if $\lambda < \mathrm{Tr}(\kappa)$ (definition: 5A1Mb), then $g_\alpha[\kappa] \subseteq \kappa$ for every $\alpha \in S$;

— if $\lambda \geq \mathrm{Tr}(\kappa)$, then $g_\alpha \restriction \gamma = g_\beta \restriction \gamma$ whenever $\gamma < \kappa$ is a limit ordinal and α, $\beta \in S$ are such that $g_\alpha(\gamma) = g_\beta(\gamma)$.

(b) Now suppose that $S_1 \subseteq S$ is stationary in ζ and $\langle \theta_\alpha \rangle_{\alpha \in S_1}$ is a family of limit ordinals less than κ. Then there are a $\theta < \kappa$ and a set $Y \in [2^\delta]^{<\kappa}$ such that $S_2 = \{\alpha : \alpha \in S_1,\, \theta_\alpha = \theta,\, g_\alpha[\theta] \subseteq Y\}$ is stationary in ζ.

proof (a) case 1 If $\lambda < \mathrm{Tr}(\kappa)$, then $\zeta \leq \mathrm{Tr}(\kappa)$ (5A1Na); as ζ is a successor cardinal, there is a family $\langle g_\alpha \rangle_{\alpha < \zeta}$ of functions from κ to κ such that $\#(g_\alpha \cap g_\beta) < \kappa$ for all distinct α, $\beta < \zeta$. Set $S = \zeta \setminus \kappa$, so that S is a stationary set in ζ and $g_\alpha[\kappa] \subseteq \alpha$ for every $\alpha \in S$. We know that $\kappa \leq \mathfrak{c}$; set $\delta = \omega$, so that $\delta < \kappa \leq 2^\delta$ and g_α is a function from κ to 2^δ for every α.

case 2 Suppose that $\lambda \geq \mathrm{Tr}(\kappa)$. Then

$$\kappa < \mathrm{Tr}(\kappa) < \lambda^+ = \zeta \leq \min(2^\kappa, \kappa^{(+\omega)}),$$

so $\sup_{\delta<\kappa} 2^\delta \geq \zeta$, by 5A1Nb; and as this supremum is attained (542E), there is a cardinal $\delta < \kappa$ such that $2^\delta \geq \zeta$. Because $\kappa < \lambda < \kappa^{(+\omega)}$, λ is regular, and of course $\lambda > \omega_1$. So 5A1P gives us the functions we need.

(b) Because $\zeta = \mathrm{cf}\,\zeta > \kappa$, there is a $\theta < \kappa$ such that

$$S_1' = \{\alpha : \alpha \in S_1,\, \theta_\alpha = \theta\}$$

is stationary in ζ, by the Pressing-Down Lemma, and of course θ is a limit ordinal.

case 1 If $\lambda < \mathrm{Tr}(\kappa)$, then $g_\alpha[\theta]$ is a subset of κ, and is therefore bounded above in κ, for each α. Let $\theta' < \kappa$ be such that

$$S_2 = \{\alpha : \alpha \in S_1',\, g_\alpha[\theta] \subseteq \theta'\}$$

is stationary in ζ. As $\kappa \leq \mathfrak{c} \leq 2^\delta$, $\theta' \in [2^\delta]^{<\kappa}$ and we can take $Y = \theta'$.

case 2 If $\lambda \geq \mathrm{Tr}(\kappa)$, then $g_\alpha(\theta) < \alpha$ for $\alpha \in S_1'$; so there is a $\theta' < \zeta$ such that

$$S_1'' = \{\alpha : \alpha \in S_1,\, g_\alpha(\theta) = \theta'\}$$

is stationary in ζ. Then $g_\alpha \restriction \theta = g_\beta \restriction \theta$ for all α, $\beta \in S_1''$; take Y to be the common value of $g_\alpha[\theta]$ for $\alpha \in S_1''$.

543E The Gitik-Shelah theorem (Gitik & Shelah 89, Gitik & Shelah 93) Let κ be an atomlessly-measurable cardinal, with witnessing probability ν. Then the Maharam type of ν is at least $\min(\kappa^{(+\omega)}, 2^\kappa)$.

proof (a) To begin with (down to the end of (g) below) let us suppose that ν is Maharam-type-homogeneous, with Maharam type λ; of course λ is infinite, because ν is atomless. Let $(\mathfrak{A}, \bar\nu)$ be the measure algebra of ν, ν_λ the usual measure of $\{0,1\}^\lambda$ and $\mathfrak{B}_\lambda$ the measure algebra of ν_λ; then there is a measure-preserving isomorphism $\phi : \mathfrak{B}_\lambda \to \mathfrak{A}$. Because ν_λ is a compact measure (342Jd), there is a function $f : \kappa \to \{0,1\}^\lambda$ such that $\phi(E^\bullet) = f^{-1}[E]^\bullet$ whenever ν_λ measures E (343B).

(b)? Suppose, if possible, that $\lambda < \min(\kappa^{(+\omega)}, 2^\kappa)$.

Of course κ is quasi-measurable and at most $\mathfrak{c}$. So 543Ma tells us that if we set $\zeta = \max(\lambda^+, \kappa^+)$, there will be an infinite cardinal $\delta < \kappa$, a stationary set $S \subseteq \zeta$, and a family $\langle g_\alpha \rangle_{\alpha \in S}$ of functions from κ to 2^δ such that $g_\alpha[\kappa] \subseteq \alpha$ for every $\alpha \in S$ and $\#(g_\alpha \cap g_\beta) < \kappa$ for distinct α, $\beta \in S$. Moreover, we can arrange that

— if $\lambda < \mathrm{Tr}(\kappa)$ (definition: 5A1Mb), then $g_\alpha[\kappa] \subseteq \kappa$ for every $\alpha \in S$;

— if $\lambda \geq \mathrm{Tr}(\kappa)$, then $g_\alpha \restriction \gamma = g_\beta \restriction \gamma$ whenever $\gamma < \kappa$ is a limit ordinal and α, $\beta \in S$ are such that $g_\alpha(\gamma) = g_\beta(\gamma)$.

(c) Fix an injective function $h : 2^\delta \to \{0,1\}^\delta$. For $\alpha \in S$, $\iota < \delta$ set

$$U_{\alpha\iota} = \{\xi : \xi < \kappa,\, (hg_\alpha(\xi))(\iota) = 1\},$$

and choose a Baire set $H_{\alpha\iota} \subseteq \{0,1\}^\lambda$ such that $\phi^{-1}(U_{\alpha\iota}^\bullet) = H_{\alpha\iota}^\bullet$ in $\mathfrak{B}_\lambda$. Define $\tilde g_\alpha : \{0,1\}^\lambda \to \{0,1\}^\delta$ by setting

$$(\tilde{g}_\alpha(x))(\iota) = 1 \text{ if } x \in H_{\alpha\iota},$$
$$= 0 \text{ otherwise.}$$

Then

$$\{\xi : \xi < \kappa,\, \tilde{g}_\alpha f(\xi) \ne hg_\alpha(\xi)\} = \bigcup_{\iota<\delta} \{\xi : (\tilde{g}_\alpha f(\xi))(\iota) \ne (hg_\alpha(\xi))(\iota)\}$$
$$= \bigcup_{\iota<\delta} U_{\alpha\iota} \triangle f^{-1}[H_{\alpha\iota}] \in \mathcal{N}(\nu)$$

because $\delta < \kappa = \operatorname{add}\mathcal{N}(\nu)$. Set $V_\alpha = \{\xi : \tilde{g}_\alpha f(\xi) = hg_\alpha(\xi)\}$, so that $\nu V_\alpha = 1$, for each $\alpha \in S$.

(d) Because every $H_{\alpha\iota}$ is a Baire set, there is for each $\alpha \in S$ a set $I_\alpha \subseteq \lambda$ such that $\#(I_\alpha) \le \delta$ and $H_{\alpha\iota}$ is determined by coordinates in I_α for every $\iota < \delta$, that is, $\tilde{g}_\alpha(x) = \tilde{g}_\alpha(y)$ whenever x, $y \in \{0,1\}^\lambda$ and $x{\restriction}I_\alpha = y{\restriction}I_\alpha$. Because $\lambda < \operatorname{cf}\zeta$, there is an $M \subseteq \lambda$ such that

$$S_1 = \{\alpha : \alpha \in S,\, I_\alpha \subseteq M\}$$

is stationary in ζ and $\operatorname{cf}(\#(M)) \le \delta$ (5A1K); because $\lambda < \kappa^{(+\omega)}$ and $\operatorname{cf}(\kappa) = \kappa > \delta$, $\#(M) < \kappa$. Set $\pi_M(z) = z{\restriction}M$ for $z \in \{0,1\}^\lambda$, and $f_M = \pi_M f$, so that $f_M : \kappa \to \{0,1\}^M$ is inverse-measure-preserving for ν and the usual measure ν_M of $\{0,1\}^M$. For $w \in \{0,1\}^M$ define $\psi(w) \in \{0,1\}^\lambda$ by setting

$$\psi(w)(\xi) = w(\xi) \text{ if } \xi \in M,$$
$$= 0 \text{ otherwise.}$$

If we set

$$g^*_\alpha = \tilde{g}_\alpha \psi : \{0,1\}^M \to \{0,1\}^\delta,$$

then g^*_α is Baire measurable in each coordinate, while $g^*_\alpha \pi_M = \tilde{g}_\alpha$ for $\alpha \in S_1$.

(e) For each $\alpha \in S_1$, there is a $\theta_\alpha < \kappa$ such that $\mu^*_M(f_M[V_\alpha \cap \theta_\alpha]) = 1$. **P** Apply 543Db to $f_M[V_\alpha] \subseteq \{0,1\}^M$. There must be a set $B \subseteq f_M[V_\alpha]$ such that $\#(B) < \kappa$ and $\mu^*_M B = \mu^*_M(f_M[V_\alpha])$; because κ is regular, there is a $\theta_\alpha < \kappa$ such that $B \subseteq f_M[V_\alpha \cap \theta_\alpha]$. On the other hand, because f_M is inverse-measure-preserving, $\mu^*_M(f_M[V_\alpha]) \ge \nu V_\alpha = 1$. **Q**

Evidently we may take it that every θ_α is a non-zero limit ordinal.

(f) Now 543Mb tells us that there are a $\theta < \kappa$ and a $Y \in [2^\delta]^{<\kappa}$ such that

$$S_2 = \{\alpha : \alpha \in S_1,\, \theta_\alpha = \theta,\, g_\alpha[\theta] \subseteq Y\}$$

is stationary in ζ.

(g) For each $\alpha \in S_2$, set

$$Q_\alpha = f_M[V_\alpha \cap \theta] = f_M[V_\alpha \cap \theta_\alpha],$$

so that $\mu^*_M Q_\alpha = 1$. If $y \in Q_\alpha$, take $\xi \in V_\alpha \cap \theta$ such that $f_M(\xi) = y$; then

$$g^*_\alpha(y) = g^*_\alpha \pi_M f(\xi) = \tilde{g}_\alpha f(\xi) = hg_\alpha(\xi) \in h[Y].$$

Thus $g^*_\alpha{\restriction}Q_\alpha \subseteq f_M[\theta] \times h[Y]$ for every $\alpha \in S_2$, and we can apply 543Dd to $X = \{0,1\}^M$, $\mu = \mu_M$ and the family $\langle g^*_\alpha{\restriction}Q_\alpha\rangle_{\alpha\in S'}$, where $S' \subseteq S_2$ is a set with cardinal κ, to see that there are distinct α, $\beta \in S_2$ such that $\mu^*_M\{y : y \in Q_\alpha \cap Q_\beta,\, g^*_\alpha(y) = g^*_\beta(y)\} > 0$. Now, however, consider

$$E = \{y : y \in \{0,1\}^M,\, g^*_\alpha(y) = g^*_\beta(y)\}.$$

Then $E = \bigcap_{\iota<\delta} E_\iota$, where

$$E_\iota = \{y : y \in \{0,1\}^M,\, g^*_\alpha(y)(\iota) = g^*_\beta(y)(\iota)\}$$

is a Baire subset of $\{0,1\}^M$ for each $\iota < \delta$. Because $\delta < \kappa$,

$$\nu f_M^{-1}[E] = \nu(\bigcap_{\iota<\delta} f_M^{-1}[E_\iota]) = \inf_{I\in[\delta]^{<\omega}} \nu(\bigcap_{\iota\in I} f_M^{-1}[E_\iota])$$
$$= \inf_{I\in[\delta]^{<\omega}} \mu_M(\bigcap_{\iota\in I} E_\iota) \ge \mu_M^* E > 0.$$

Consequently

$$0 < \nu f_M^{-1}[E] = \nu\{\xi : g_\alpha^*\pi_M f(\xi) = g_\beta^*\pi_M f(\xi)\}$$
$$= \nu\{\xi : \tilde g_\alpha f(\xi) = \tilde g_\beta f(\xi)\} = \nu\{\xi : \xi \in V_\alpha \cap V_\beta,\ \tilde g_\alpha f(\xi) = \tilde g_\beta f(\xi)\}$$
$$= \nu\{\xi : hg_\alpha(\xi) = hg_\beta(\xi)\} = \nu\{\xi : g_\alpha(\xi) = g_\beta(\xi)\}$$

(because h is injective). But this is absurd, because in (b) above we chose g_α, g_β in such a way that $\{\xi : g_\alpha(\xi) = g_\beta(\xi)\}$ would be bounded in κ. **X**

(h) Thus the result is true for Maharam-type-homogeneous witnessing probabilities on κ. In general, if ν is any witnessing probability on κ, there is a non-negligible $A \subseteq \kappa$ such that the subspace measure on A is Maharam-type-homogeneous; setting $\nu'C = \frac{1}{\nu A}\nu(A \cap C)$ for $C \subseteq \kappa$, we obtain a Maharam-type-homogeneous witnessing probability ν'. Now the Maharam type of ν is at least as great as the Maharam type of ν', so is at least $\min(2^\kappa, \kappa^{(+\omega)})$, as required.

543F Theorem Let $(X, \mathcal{P}X, \mu)$ be an atomless semi-finite measure space. Write $\kappa = \operatorname{add}\mu$. Then the Maharam type of $(X, \mathcal{P}X, \mu)$ is at least $\min(\kappa^{(+\omega)}, 2^\kappa)$, and in particular is greater than κ.

proof Let $\langle E_\xi\rangle_{\xi<\kappa}$ be a family in $\mathcal{N}(\mu)$ such that $E = \bigcup_{\xi<\kappa} E_\xi \notin \mathcal{N}(\mu)$. Let $F \subseteq E$ be a set of non-zero finite measure. Set $f(x) = \min\{\xi : x \in E_\xi\}$ for $x \in F$. Let μ_F be the subspace measure on F and $\mu' = (\mu F)^{-1}\mu_F$ the corresponding probability measure; of course $\operatorname{dom}\mu' = \mathcal{P}F$ and $\mu'(F \cap E_\xi) = 0$ for every $\xi < \kappa$. Note also that

$$\operatorname{add}\mu' = \operatorname{add}\mu_F \ge \operatorname{add}\mu \ge \kappa$$

(521Fc). Let ν be the image measure $\mu' f^{-1}$, so that $\operatorname{dom}\nu = \mathcal{P}\kappa$ and ν is κ-additive (521Hb). Also $\nu\{\xi\} \le \mu' E_\xi = 0$ for every ξ, so ν witnesses that κ is real-valued-measurable. Next, μ_F is atomless (214Ka), so μ' also is. There is therefore a function $g : F \to [0,1]$ which is inverse-measure-preserving for μ' and Lebesgue measure (343Cb), and F can be covered by $\mathfrak{c}$ negligible sets; accordingly $\operatorname{add}\mu' \le \mathfrak{c}$ so $\kappa \le \mathfrak{c}$ and ν must be atomless (543Bc).

Let $(\mathfrak{A}, \bar\mu)$, $(\mathfrak{A}', \bar\mu')$ and $(\mathfrak{B}, \bar\nu)$ be the measure algebras of μ, μ' and ν respectively. Then $\mathfrak{A}'$ is isomorphic to a principal ideal of $\mathfrak{A}$ (322I), so $\tau(\mathfrak{A}) \ge \tau(\mathfrak{A}')$ (514Ed). Next, $f : F \to \kappa$ induces a measure-preserving Boolean homomorphism from $\mathfrak{B}$ to $\mathfrak{A}'$, so that $\tau(\mathfrak{A}') \ge \tau(\mathfrak{B})$ (332Tb). Now 543E tells us that

$$\min(\kappa^{(+\omega)}, 2^\kappa) \le \tau(\mathfrak{B}) \le \tau(\mathfrak{A}),$$

as required.

543G Corollary Let $(X, \mathcal{P}X, \nu)$ be an atomless probability space, and $\kappa = \operatorname{add}\nu$. Let (Z, Σ, μ) be a compact probability space with Maharam type $\lambda \le \min(2^\kappa, \kappa^{(+\omega)})$ (e.g., $Z = \{0,1\}^\lambda$ with its usual measure). Then there is an inverse-measure-preserving function $f : X \to Z$.

proof Let $(\mathfrak{A}, \bar\mu)$ and $\mathfrak{B}, \bar\nu)$ be the measure algebras of μ, ν respectively. By 543F, the Maharam type of the subspace measure ν_C is at least λ whenever $C \subseteq X$ and $\nu C > 0$; that is, every non-zero principal ideal of $\mathfrak{B}$ has Maharam type at least λ. So there is a measure-preserving Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ (332P). Because μ is compact, this is represented by an inverse-measure-preserving function from X to Z (343B).

543H Corollary If κ is an atomlessly-measurable cardinal, and (Z, μ) is a compact probability space with Maharam type at most $\min(2^\kappa, \kappa^{(+\omega)})$, then there is an extension of μ to a κ-additive measure defined on $\mathcal{P}Z$.

proof Let ν be a witnessing probability on κ; by 543G, there is an inverse-measure-preserving function $f : X \to Z$; now the image measure νf^{-1} extends μ to $\mathcal{P}Z$.

543I Corollary If κ is an atomlessly-measurable cardinal, with witnessing probability ν, and $2^\kappa \le \kappa^{(+\omega)}$, then $(\kappa, \mathcal{P}\kappa, \nu)$ is Maharam-type-homogeneous, with Maharam type 2^κ.

proof If $C \in \mathcal{P}\kappa \setminus \mathcal{N}(\nu)$, then the Maharam type of the subspace measure on C is at least 2^κ, by 543F; but also it cannot be greater than $\#(\mathcal{P}C) = 2^\kappa$.

543J Proposition Let κ be an atomlessly-measurable cardinal, ν a witnessing probability on κ, and $\mathfrak{A}$ the measure algebra of ν. Then

(a) there is a $\gamma < \kappa$ such that $2^\gamma = 2^\delta$ for every cardinal δ such that $\gamma \le \delta < \kappa$;

(b) the cardinal power $\tau(\mathfrak{A})^\gamma$ is 2^κ;

(c) if $\mathfrak{c} < \kappa^{(+\omega_1)}$, then $\#(\mathfrak{A}) = \tau(\mathfrak{A})^\omega = 2^\kappa$.

proof Use 542F-542G. Because κ is quasi-measurable and $\kappa \le \mathfrak{c}$, 542Fa tells us that there is a γ as in (a); and now 542Fb and 515M tell us that

$$2^\kappa = \tau(\mathfrak{A})^\gamma = (\tau(\mathfrak{A})^\omega)^\gamma = \#(\mathfrak{A})^\gamma.$$

If $\mathfrak{c} < \kappa^{(+\omega_1)}$, then 542G tells us that $\#(\mathfrak{A}) = 2^\kappa$.

543K Proposition Let κ be an atomlessly-measurable cardinal. If there is a witnessing probability on κ with Maharam type λ, then there is a Maharam-type-homogeneous normal witnessing probability ν on κ with Maharam type at most λ.

proof Repeat the proof of 543Ba, with $X = \kappa$ and μ a witnessing probability on κ with Maharam type λ. Taking a non-negligible $Y \subseteq \kappa$ and $g : Y \to \kappa$ such that $\nu = \frac{1}{\mu Y}\mu_Y g^{-1}$ is normal, then g induces an embedding of the measure algebra of ν into a principal ideal of the measure algebra of μ, so the Maharam type of ν is at most λ. There is now an $E \in \mathcal{P}\kappa \setminus \mathcal{N}(\nu)$ such that the subspace measure ν_E is Maharam-type-homogeneous, and setting $\nu' A = \nu(A \cap E)/\nu E$ for $A \subseteq \kappa$ we obtain a Maharam-type-homogeneous probability measure ν' with Maharam type less than or equal to λ. Now ν' is again normal. **P** Let $\langle I_\xi \rangle_{\xi<\kappa}$ be any family in $\mathcal{N}(\nu')$, and set $I = \{\xi : \xi < \kappa,\ \xi \in \bigcup_{\eta<\xi} I_\eta\}$. Then $I_\xi \cap E \in \mathcal{N}(\nu)$ for every ξ, so $I \cap E = \{\xi : \xi < \kappa,\ \xi \in \bigcup_{\eta<\xi} I_\eta \cap E\}$ is ν-negligible and I is ν'-negligible. **Q** So we have an appropriate normal witnessing probability.

543L Proposition Suppose that ν is a Maharam-type-homogeneous witnessing probability on an atomlessly-measurable cardinal κ with Maharam type λ. Then there is a Maharam-type-homogeneous witnessing probability ν' on κ with Maharam type at least $\mathrm{Tr}_{\mathcal{N}(\nu)}(\kappa; \lambda)$.

proof Let ν_1 be the κ-additive probability on $\kappa \times \kappa$ given by

$$\nu_1 C = \int \nu C[\{\xi\}]\nu(d\xi) \text{ for every } C \subseteq \kappa \times \kappa.$$

Set $\theta = \mathrm{Tr}_{\mathcal{N}(\nu)}(\kappa; \lambda)$. By 541F there is a family $F \subseteq \lambda^\kappa$ such that $\#(F) = \theta$ and $\{\xi : f(\xi) = g(\xi)\} \in \mathcal{N}(\nu)$ for all distinct f, $g \in F$. Let $\langle E_\xi \rangle_{\xi<\lambda}$ be a ν-stochastically independent family of subsets of κ of ν-measure $\frac{1}{2}$. For each $f \in F$ set

$$C_f = \{(\xi, \eta) : \xi < \kappa,\ \eta \in E_{f(\xi)}\}.$$

Then for any non-empty finite subset I of F, $\nu(\bigcap_{f\in I} E_{f(\xi)}) = 2^{-\#(I)}$ for ν-almost every ξ, so that

$$\nu_1(\bigcap_{f\in I} C_f) = 2^{-\#(I)}.$$

Thus $\langle C_f \rangle_{f\in F}$ is stochastically independent for ν_1, and the Maharam type of the subspace measure $(\nu_1)_C$ is at least $\#(F) = \theta$ whenever $\nu_1 C > 0$. Once again, take $\nu_2 C = \nu_1(C \cap D)/\nu_1 D$ for some D for which $(\nu_1)_D$ is Maharam-type-homogeneous, to obtain a Maharam-type-homogeneous κ-additive probability ν_2 with Maharam type at least θ. Finally, of course, ν_2 can be copied onto a probability ν' on κ, as asked for.

543X Basic exercises (a) Let κ be an atomlessly-measurable cardinal. Show that the following are equiveridical: (i) every witnessing probability ν on κ is Maharam-type-homogeneous (ii) any two witnessing probabilities on κ have the same Maharam type.

(b) Let κ be an atomlessly-measurable cardinal. Show that the following are equiveridical: (i) every normal witnessing probability ν on κ is Maharam-type-homogeneous (ii) any two normal witnessing probabilities on κ have the same Maharam type.

(c) Suppose that $\mathfrak{c}$ is atomlessly-measurable. Show that there is a Maharam-type-homogeneous normal witnessing probability on $\mathfrak{c}$ with Maharam type $2^{\mathfrak{c}}$. (*Hint*: 542Ga, 5A1Nc.)

(d) Let μ be Lebesgue measure on $\mathbb{R}$, and $\theta = \frac{1}{2}(\mu^* + \mu_*)$ the outer measure described in 413Xd. Show that μ is the measure defined from θ by Carathéodory's method. (*Hint*: 438Ym.)

543Y Further exercises (a) Let ν be a witnessing probability on an atomlessly-measurable cardinal κ with Maharam type λ. Let F be the set of all functions $f \subseteq \kappa \times \lambda$ such that $\operatorname{dom} f \notin \mathcal{N}(\nu)$, and let θ be

$$\sup\{\#(F_0) : F_0 \subseteq F,\ \{\xi : \xi \in \operatorname{dom} f \cap \operatorname{dom} g,\ f(\xi) = g(\xi)\} \in \mathcal{N}(\nu)$$
$$\text{for all distinct } f,\ g \in F_0\}.$$

Show that there is a witnessing probability ν' on κ with Maharam type at least θ.

543Z Problems Let κ be an atomlessly-measurable cardinal.

(a) Must every witnessing probability ν on κ be Maharam-type-homogeneous? (See 555E.)

(b) Must every normal witnessing probability ν on κ be Maharam-type-homogeneous?

543 Notes and comments The results of 543I-543J leave a tantalizingly narrow gap; it seems possible that the Maharam type of a witnessing probability on an atomlessly-measurable cardinal κ is determined by κ (543Xa, 543Za). If so, there is at least a chance that there is a proof depending on no ideas more difficult than those above. To find a counter-example, however, we may need not only to make some strong assumptions about the potential existence of appropriate large cardinals, but also to find a new method of constructing models with atomlessly-measurable cardinals. Possibly we get a different question if we look at normal witnessing probabilities (543Zb). A positive answer to either part of 543Z would have implications for transversal numbers (543L, 543Ya).

Version of 31.12.13

544 Measure theory with an atomlessly-measurable cardinal

As is to be expected, a witnessing measure on a real-valued-measurable cardinal has some striking properties, especially if it is normal. What is less obvious is that the mere existence of such a cardinal can have implications for apparently unrelated questions in analysis. In 544J, for instance, we see that if there is any atomlessly-measurable cardinal then we have a version of Fubini's theorem, $\iint f(x,y)dxdy = \iint f(x,y)dydx$, for many functions f on $\mathbb{R}^2$ which are not jointly measurable. In this section I explore results of this kind. We find that, in the presence of an atomlessly-measurable cardinal, the covering number of the Lebesgue null ideal is large (544B) while its uniformity is small (544G-544H). There is a second inequality on repeated integrals (544C) to add to the one already given in 543C, and which tells us something about measure-precalibers (544D); I add a couple of variations (544I-544J). Next, I give a pair of theorems (544E-544F) on a measure-combinatorial property of the filter of conegligible sets of a normal witnessing measure. Revisiting the theory of Borel measures on metrizable spaces, discussed in §438 on the assumption that no real-valued-measurable cardinal was present, we find that there are some non-trivial arguments applicable to spaces with non-measure-free weight (544K-544L).

In §541 I briefly mentioned 'weakly compact' cardinals. Two-valued-measurable cardinals are always weakly compact; atomlessly-measurable cardinals never are; but atomlessly-measurable cardinals may or may not have a significant combinatorial property which can be regarded as a form of weak compactness (544M, 544Yc). Finally, I summarise what is known about the location of an atomlessly-measurable cardinal on Cichoń's diagram (544N).

544A Notation I repeat some of my notational conventions. For a measure μ, $\mathcal{N}(\mu)$ will be its null ideal. For any set I, ν_I will be the usual measure on $\{0,1\}^I$, $\mathcal{N}_I = \mathcal{N}(\nu_I)$ its null ideal and $(\mathfrak{B}_I, \bar{\nu}_I)$ its measure algebra.

544B Proposition Let κ be an atomlessly-measurable cardinal. If (X, Σ, μ) is any locally compact (definition: 342Ad) semi-finite measure space with $\mu X > 0$, then $\operatorname{cov} \mathcal{N}(\mu) \ge \kappa$.

proof By 521Lb, applied to any Maharam-type-homogeneous subspace of X with non-zero finite measure, it is enough to show that $\operatorname{cov} \mathcal{N}_\lambda \ge \kappa$ for every λ; by 523F, we need look only at the case $\lambda = \kappa$. Fix on an atomless κ-additive probability ν with domain $\mathcal{P}\kappa$. By 543G there is an inverse-measure-preserving function $f : \kappa \to \{0,1\}^\kappa$. So $\operatorname{cov} \mathcal{N}_\kappa \ge \operatorname{cov} \mathcal{N}(\nu) = \kappa$, by 521Ha.

544C Theorem (Kunen n70) Let κ be a real-valued-measurable cardinal and ν a normal witnessing probability on κ; let (X, μ) be a compact probability space and $f : X \times \kappa \to [0, \infty[$ any function. Then

$$\underline{\int}\Bigl(\int f(x,\xi)\nu(d\xi)\Bigr)\mu(dx) \le \int\Bigl(\overline{\int} f(x,\xi)\mu(dx)\Bigr)\nu(d\xi).$$

proof ? Suppose, if possible, otherwise.

(a) We are supposing that there is a μ-integrable function $g : X \to \mathbb{R}$ such that $0 \le g(x) \le \int f(x,\xi)\nu(d\xi)$ for every $x \in X$ and

$$\int g(x)\,\mu(dx) > \int \overline{\int} f(x,\xi)\mu(dx)\nu(d\xi).$$

We can suppose that g is a simple function; express it as $\sum_{i=0}^n t_i \chi F_i$ where $(F_0, \dots, F_n)$ is a partition of X into measurable sets. For any $\xi < \kappa$,

$$\overline{\int} f(x,\xi)\mu(dx) = \sum_{i=0}^n \overline{\int} f(x,\xi)\chi F_i(x)\mu(dx)$$

(133L). So there must be some $i \le n$ such that

$$t_i \mu F_i > \int \overline{\int} f(x,\xi)\chi F_i(x)\mu(dx)\nu(d\xi).$$

Set $Y = F_i$, $\mu_1 = (\mu F_i)^{-1}\mu \restriction \mathcal{P}F_i$, $t = t_i$; then (Y, μ_1) is a compact probability space (451Da) and

$$\int \overline{\int} f(y,\xi)\mu_1(dy)\nu(d\xi) = \frac{1}{\mu F_i} \int \overline{\int} f(x,\xi)\chi F_i(x)\mu(dx)\nu(d\xi)$$

(135Id)

$$< t \le \inf_{y \in Y} \int f(y,\xi)\nu(d\xi).$$

(b) Let $(\mathfrak{A}, \bar{\mu}_1)$ be the measure algebra of (Y, μ_1). Then there is a cardinal $\lambda \ge \kappa$ such that $(\mathfrak{A}, \bar{\mu}_1)$ can be embedded in $(\mathfrak{B}_\lambda, \bar{\nu}_\lambda)$, the measure algebra of ν_λ. Because μ_1 is compact, there is an inverse-measure-preserving function $\phi : \{0,1\}^\lambda \to Y$ (343B). By 235A, $\overline{\int} f(\phi(z),\xi)\nu_\lambda(dz) \le \overline{\int} f(y,\xi)\mu_1(dy)$ for every ξ, so $\int \overline{\int} f(\phi(z),\xi)\nu_\lambda(dz)\nu(d\xi) < t$.

For each $\xi < \kappa$ choose a Baire measurable function $h_\xi : \{0,1\}^\lambda \to \mathbb{R}$ such that $f(\phi(z),\xi) \le h_\xi(z)$ for every $z \in \{0,1\}^\lambda$ and $\int h_\xi(z)\nu_\lambda(dz) = \overline{\int} f(\phi(z),\xi)\nu_\lambda(dz)$; we can do this because ν_λ is the completion of its restriction to the Baire σ-algebra $\mathcal{B}\mathfrak{a}(\{0,1\}^\lambda)$ (see 4A3Of), so we can apply 133J(a-i) to the Baire measure $\nu_\lambda \restriction \mathcal{B}\mathfrak{a}(\{0,1\}^\lambda)$. For each ξ, there is a countable set $I_\xi \subseteq \lambda$ such that h_ξ is determined by coordinates in I_ξ, in the sense that $h_\xi(z) = h_\xi(z')$ whenever $z \restriction I_\xi = z' \restriction I_\xi$.

By 541Rb, there are $\Gamma \subseteq \kappa$ and a countable set $J \subseteq \lambda$ such that $\nu\Gamma = 1$ and $I_\xi \cap I_\eta \subseteq J$ whenever ξ, η are distinct members of Γ.

(c) For $u \in \{0,1\}^J$ and $u' \in \{0,1\}^{\lambda \setminus J}$ write $u \cup u'$ for their common extension to a member of $\{0,1\}^\lambda$. Set

$$f_1(u,\xi) = \int h_\xi(u \cup u')\nu_{\lambda \setminus J}(du')$$

for $u \in \{0,1\}^J$ and $\xi < \kappa$. Then, applying Fubini's theorem to $\{0,1\}^\lambda \cong \{0,1\}^J \times \{0,1\}^{\lambda\setminus J}$, we have

$$\int f_1(u,\xi)\nu_J(du) = \int h_\xi(z)\nu_\lambda(dz),$$

so that

$$\iint f_1(u,\xi)\nu_J(du)\nu(d\xi) = \int \overline{\int} f(\phi(z),\xi)\nu_\lambda(dz)\nu(d\xi) < t,$$

and

$$\overline{\int}\int f_1(u,\xi)\nu(d\xi)\nu_J(du) < t$$

by Theorem 543C. Accordingly there is a $u \in \{0,1\}^J$ such that $\int f_1(u,\xi)\nu(d\xi) < t$.

(d) For each $\xi \in \Gamma$ take $v_\xi \in \{0,1\}^{\lambda\setminus J}$ such that $h_\xi(u \cup v_\xi) \le f_1(u,\xi)$. Let $w \in \{0,1\}^\lambda$ be such that

$$w\restriction J = u, \quad w\restriction I_\xi \setminus J = v_\xi\restriction I_\xi \setminus J \text{ for every } \xi \in \Gamma;$$

such a w exists because if ξ, $\eta \in \Gamma$ and $\xi \ne \eta$ then $I_\xi \cap I_\eta \subseteq J$. Now

$$f(\phi(w),\xi) \le h_\xi(w) = h_\xi(u \cup v_\xi) \le f_1(u,\xi)$$

for every $\xi \in \Gamma$, so

$$\int f(\phi(w),\xi)\nu(d\xi) \le \int f_1(u,\xi)\nu(d\xi) < t,$$

contradicting the last sentence of (a) above. **X**

This completes the proof.

544D Corollary If κ is an atomlessly-measurable cardinal and $\omega \le \lambda \le \kappa$, then λ is a measure-precaliber of every probability algebra.

proof If $\lambda < \kappa$ this is a corollary of 544B and 525J. If $\lambda = \kappa$, we can use 544C and 525C. For let $\langle E_\xi \rangle_{\xi<\kappa}$ be a non-decreasing family in $\mathcal{N}_\kappa$ with union E. Let ν be a normal witnessing probability on κ. Set

$$C = \{(x,\xi) : \xi < \kappa,\, x \in E_\xi\} \subseteq \{0,1\}^\kappa \times \kappa.$$

Then

$$\underline{\int} \nu C[\{x\}]\nu_\kappa(dx) \ge \mu_* E, \quad \int \nu_\kappa^* C^{-1}[\{\xi\}]\nu(d\xi) = 0,$$

so 544C, applied to the indicator function of C, tells us that $\mu_* E = 0$; now 525Cc tells us that κ is a precaliber of $\mathfrak{B}_\kappa$, and therefore a measure-precaliber of every probability algebra, by 525Ia and 525Da.

544E Theorem (Kunen n70) Let κ be a real-valued-measurable cardinal and ν a normal witnessing probability on κ. If (X,μ) is a quasi-Radon probability space of weight strictly less than κ, and $f : [\kappa]^{<\omega} \to \mathcal{N}(\mu)$ is any function, then

$$\bigcap_{V\subseteq\kappa,\nu V=1} \bigcup_{I\in[V]^{<\omega}} f(I) \in \mathcal{N}(\mu).$$

proof Let $\mathcal{F}$ be the filter of ν-conegligible subsets of κ.

(a) I show by induction on $n \in \mathbb{N}$ that if $g : [\kappa]^{\le n} \to \mathcal{N}(\mu)$ is any function, then

$$E(g) = \bigcap_{V\in\mathcal{F}} \bigcup_{I\in[V]^{\le n}} g(I) \in \mathcal{N}(\mu).$$

P For $n = 0$ this is trivial; $E(g) = g(\emptyset) \in \mathcal{N}(\mu)$. For the inductive step to $n+1$, given $g : [\kappa]^{\le n+1} \to \mathcal{N}(\mu)$, then for each $\xi < \kappa$ define $g_\xi : [\kappa]^{\le n} \to \mathcal{N}(\mu)$ by setting $g_\xi(I) = g(I \cup \{\xi\})$ for each $I \in [\kappa]^{\le n}$. By the inductive hypothesis, $E(g_\xi) \in \mathcal{N}(\mu)$. Set

$$C = \{(x,\xi) : x \in E(g_\xi)\} \subseteq X \times \kappa.$$

Then

$$\int \mu^* C^{-1}[\{\xi\}]\nu(d\xi) = \int \mu^* E(g_\xi)\nu(d\xi) = 0,$$

so by 543C

$$\overline{\int} \nu C[\{x\}]\mu(dx) = 0,$$

and $\mu D = 0$, where $D = g(\emptyset) \cup \{x : \nu C[\{x\}] > 0\}$.

Take any $x \in X \setminus D$ and set $W = \kappa \setminus C[\{x\}] \in \mathcal{F}$. For each $\xi \in W$, $x \notin E(g_\xi)$, so there is a $V_\xi \in \mathcal{F}$ such that $\nu V_\xi = 1$ and $x \notin g_\xi(I)$ for every $I \in [V_\xi]^{\le n}$. Set

$$V = W \cap \{\xi : \xi \in V_\eta \text{ for every } \eta < \xi\}.$$

Then $V \in \mathcal{F}$. If $I \in [V]^{\le n+1}$, either $I = \emptyset$ and $x \notin g(I)$, or there is a least element ξ of I; in the latter case, $\xi \in W$ and $J = I \setminus \{\xi\} \subseteq V_\xi$ and $x \notin g_\xi(J) = g(I)$. So $x \notin \bigcup\{g(I) : I \in [V]^{\le n+1}\}$. As x is arbitrary, $E(g) \subseteq D \in \mathcal{N}(\mu)$ and the induction proceeeds. **Q**

(b) Now consider

$$G = \bigcup_{n\in\mathbb{N}} E(f\restriction[\kappa]^{\le n}) \in \mathcal{N}(\mu).$$

If $x \in X\setminus G$ then for each $n \in \mathbb{N}$ there is a $V_n \in \mathcal{F}$ such that $x \notin \bigcup\{f(I) : I \in [V_n]^{\le n}\}$. Set $V = \bigcap_{n\in\mathbb{N}} V_n \in \mathcal{F}$; then $x \notin \bigcup\{f(I) : I \in [V]^{<\omega}\}$. As x is arbitrary,

$$\bigcap_{V\subseteq\kappa,\nu V=1} \bigcup_{I\in[V]^{<\omega}} f(I) \subseteq G \in \mathcal{N}(\mu),$$

as required.

544F Theorem (Kunen n70) Let κ be a real-valued-measurable cardinal with a normal witnessing probability ν. If (X, μ) is a locally compact semi-finite measure space with $\mu X > 0$ and $f : [\kappa]^{<\omega} \to \mathcal{N}(\mu)$ is a function, then there is a ν-conegligible $V \subseteq \kappa$ such that $\bigcup\{f(I) : I \in [V]^{<\omega}\} \ne X$.

proof (a) Consider first the case $(X, \mu) = (\{0,1\}^\kappa, \nu_\kappa)$. For any $L \subseteq \kappa$ let $\pi_L : \{0,1\}^\kappa \to \{0,1\}^L$ be the restriction map. Let $\mathcal{F}$ be the conegligible filter on κ.

(i) For each $I \in [\kappa]^{<\omega}$, there is a countable set $g(I) \subseteq \kappa$ such that $\nu_{g(I)}(\pi_{g(I)}[f(I)]) = 0$ (254Od); enlarging $f(I)$ if necessary, we may suppose that $f(I) = \pi_{g(I)}^{-1}[\pi_{g(I)}[f(I)]]$. By 541Q there are a set $C \in \mathcal{F}$ and a function $h : [\kappa]^{<\omega} \to [\kappa]^{\le\omega}$ such that $g(I) \cap \eta \subseteq h(I \cap \eta)$ whenever $I \in [C]^{<\omega}$ and $\eta < \kappa$. Set

$$\Gamma = \{\gamma : \gamma < \kappa,\, h(I) \subseteq \gamma \text{ for every } I \in [\gamma]^{<\omega}\};$$

then Γ is a closed cofinal set in κ, because $\mathrm{cf}(\kappa) > \omega$. Let $\langle\gamma_\eta\rangle_{\eta\le\kappa}$ be the increasing enumeration of $\Gamma \cup \{0, \kappa\}$.

(ii) For $\eta < \kappa$, set $M(\eta) = \kappa \setminus \gamma_\eta$ and $L(\eta) = \gamma_{\eta+1} \setminus \gamma_\eta$; then $\nu_{M(\eta)}$ can be identified with the product measure $\nu_{L(\eta)} \times \nu_{M(\eta+1)}$. Choose $u_\eta \in \{0,1\}^{\gamma_\eta}$, $V_\eta \subseteq \kappa$ inductively, as follows. $u_0 \in \{0,1\}^0$ is the empty function. Given u_η, then for each $I \in [\kappa]^{<\omega}$ set

$$f'_\eta(I) = \{v : v \in \{0,1\}^{L(\eta)},\, \nu_{M(\eta+1)}\{w : u_\eta \cup v \cup w \in f(I)\} > 0\},$$

and

$$\begin{aligned} f_\eta(I) &= f'_\eta(I) \text{ if } \nu_{L(\eta)}(f'_\eta(I)) = 0, \\ &= \emptyset \text{ otherwise.} \end{aligned}$$

By 544E, applied to $J \mapsto f_\eta(K \cup J) \in \mathcal{N}_{L(\eta)}$, we can find for each $K \in [\gamma_{\eta+1}]^{<\omega}$ a set $E_{\eta K} \subseteq \{0,1\}^{L(\eta)}$ such that $\nu_{L(\eta)} E_{\eta K} = 1$ and for every $v \in E_{\eta K}$ there is a set $V \in \mathcal{F}$ such that $v \notin f_\eta(K \cup J)$ for any $J \in [V]^{<\omega}$. Choose $v_\eta \in \bigcap\{E_{\eta K} : K \in [\gamma_{\eta+1}]^{<\omega}\}$ (using 544B); for $K \in [\gamma_{\eta+1}]^{<\omega}$ choose $V_{\eta K} \in \mathcal{F}$ such that $v_\eta \notin f_\eta(K \cup J)$ for any $J \in [V_{\eta K}]^{<\omega}$. Set $V_\eta = \bigcap\{V_{\eta K} : K \in [\gamma_{\eta+1}]^{<\omega}\} \in \mathcal{F}$ and $u_{\eta+1} = u_\eta \cup v_\eta \in \{0,1\}^{\gamma_{\eta+1}}$.

At limit ordinals η with $0 < \eta \le \kappa$, set $u_\eta = \bigcup_{\xi<\eta} u_\xi \in \{0,1\}^{\gamma_\eta}$.

(iii) Now consider $u = u_\kappa \in \{0,1\}^\kappa$ and

$$V = \{\xi : \xi \in C,\, \xi \in V_\eta \text{ for every } \eta < \xi\} \in \mathcal{F}.$$

If $I \in [V]^{<\omega}$ then

$$\nu_{M(\eta)}\{w : u_\eta \cup w \in f(I)\} = 0$$

for every $\eta < \kappa$. **P** Induce on η. For $\eta = 0$ this says just that $\nu_\kappa f(I) = 0$, which was our hypothesis on f. For the inductive step to $\eta + 1$, we have

$$\nu_{M(\eta)}\{w : u_\eta \cup w \in f(I)\} = 0$$

by the inductive hypothesis, so Fubini's theorem tells us that

$$\nu_{L(\eta)}\{v : \nu_{M(\eta+1)}\{w : u_\eta \cup v \cup w \in f(I)\} > 0\} = 0,$$

that is, $\nu_{L(\eta)} f'_\eta(I) = 0$, so that $f_\eta(I) = f'_\eta(I)$. Now setting $K = I \cap \gamma_{\eta+1}$ and $J = I \setminus \gamma_{\eta+1}$, we see that $J \subseteq V_\eta$ (because of course $\eta < \gamma_{\eta+1}$), therefore $J \subseteq V_{\eta K}$ and $v_\eta \notin f_\eta(K \cup J) = f'_\eta(I)$; but this says just that

$$\nu_{M(\eta+1)}\{w : u_\eta \cup v_\eta \cup w \in f(I)\} = 0,$$

that is, that

$$\nu_{M(\eta+1)}\{w : u_{\eta+1} \cup w \in f(I)\} = 0,$$

so that the induction continues.

For the inductive step to a non-zero limit ordinal $\eta \le \kappa$, there is a non-zero $\zeta < \eta$ such that $I \cap \gamma_\eta \subseteq \gamma_\zeta$. Now

$$g(I) \cap \gamma_\eta \subseteq h(I \cap \gamma_\eta) = h(I \cap \gamma_\zeta) \subseteq \gamma_\zeta,$$

by the choice of Γ. As $f(I)$ is determined by coordinates in $g(I)$, this means that

$$\{w : w \in \{0,1\}^{M(\zeta)},\, u_\zeta \cup w \in f(I)\} = \{0,1\}^{\gamma_\eta \setminus \gamma_\zeta} \times \{w : w \in \{0,1\}^{M(\eta)},\, u_\eta \cup w \in f(I)\}.$$

By the inductive hypothesis,

$$\nu_{M(\zeta)}\{w : u_\zeta \cup w \in f(I)\} = 0,$$

so that

$$\nu_{M(\eta)}\{w : u_\eta \cup w \in f(I)\} = 0$$

and the induction continues. **Q**

(iv) But now, given $I \in [V]^{<\omega}$, there is surely some $\eta < \kappa$ such that $g(I) \subseteq \gamma_\eta$, and in this case $\{w : u_\eta \cup w \in f(I)\}$ is either $\emptyset$ or $\{0,1\}^{M(\eta)}$. As it is $\nu_{M(\eta)}$-negligible it must be empty, and $u \notin f(I)$.

Thus we have a point $u \notin \bigcup\{f(I) : I \in [V]^{<\omega}\}$, as required.

(b) If (X,μ) is a compact probability space, we have a $\lambda \ge \kappa$ and an inverse-measure-preserving function $\phi : \{0,1\}^\lambda \to X$. For each $I \in [\kappa]^{<\omega}$ let $J_I \in [\lambda]^{\le\omega}$ be such that $\nu_{J_I}\pi_{J_I}[\phi^{-1}[f(I)]] = 0$, where here π_{J_I} is interpreted as a map from $\{0,1\}^\lambda$ to $\{0,1\}^{J_I}$; set $J = \kappa \cup \bigcup_{I \in [\kappa]^{<\omega}} J_I$. Let $q : \{0,1\}^J \to \{0,1\}^\lambda$ be any function such that $\pi_J q$ is the identity on $\{0,1\}^J$, and set $\psi = \phi q$. For any $I \in [\kappa]^{<\omega}$,

$$\psi^{-1}[f(I)] = q^{-1}[\phi^{-1}[f(I)]] \subseteq \pi_J[\phi^{-1}[f(I)]] \subseteq \pi_J[\pi_{J_I}^{-1}[\pi_{J_I}[\phi^{-1}[f(I)]]]]$$

is ν_J-negligible because $\pi_{J_I}^{-1}[\pi_{J_I}[\phi^{-1}[f(I)]]]$ is ν_λ-neglgible and determined by coordinates in J.

Because $\#(J) = \kappa$, (a) tells us that there are $u \in \{0,1\}^J$ and a conegligible $V \subseteq \kappa$ such that $u \notin \psi^{-1}[f(I)]$ for every $I \in [V]^{<\omega}$; in which case $\psi(u) \notin f(I)$ for every $I \in [V]^{<\omega}$ and $\bigcup\{f(I) : I \in [V]^{<\omega}\} \ne X$.

(c) For the general case, take a subset E of X with non-zero finite measure, and apply (b) to the function $I \mapsto E \cap f(I)$ and the normalized subspace measure $\frac{1}{\mu E}\nu_E$.

544G Proposition Let κ be an atomlessly-measurable cardinal and $\omega_1 \le \lambda < \kappa$. If (X,μ) is an atomless locally compact semi-finite measure space of Maharam type less than κ, and $\mu X > 0$, then there is a Sierpiński set $A \subseteq X$ with cardinal λ.

proof (a) To begin with, suppose that $X = \{0,1\}^\theta$ and $\mu = \nu_\theta$ where $\theta < \kappa$. Let ν be an atomless κ-additive probability defined on $\mathcal{P}\kappa$. By 543G there is a function $f : \kappa \to (\{0,1\}^\theta)^\lambda$ which is inverse-measure-preserving for ν and the usual measure ν_θ^λ of $(\{0,1\}^\theta)^\lambda$, which we may think of either as the power of ν_θ, or as the Radon power of ν_θ, or as a copy of $\nu_{\theta\times\lambda}$. For $\xi < \kappa$, set

$$A_\xi = \{f(\xi)(\eta) : \eta < \lambda\} \subseteq \{0,1\}^\theta.$$

? Suppose, if possible, that for every $\xi < \kappa$ there is a set $J_\xi \subseteq \lambda$ such that $\#(J_\xi) = \omega_1$ but $E_\xi = f(\xi)[J_\xi]$ is ν_θ-negligible. For each ξ choose a countable set $I_\xi \subseteq \theta$ such that $E'_\xi = \pi_{I_\xi}^{-1}[\pi_{I_\xi}[E_\xi]$ is ν_θ-negligible, writing $\pi_{I_\xi}(x) = x{\restriction}I_\xi$ for $x \in \{0,1\}^\theta$. By 541D, there is a countable $I \subseteq \theta$ such that $V = \{\xi : I_\xi \subseteq I\}$ is

ν-conegligible. For $\xi \in V$ set $E^*_\xi = \pi_I[E_\xi] \subseteq \{0,1\}^I$, so that $\nu_I E^*_\xi = 0$. Fix a sequence $\langle U_m\rangle_{m\in\mathbb{N}}$ running over the open-and-closed subsets of $\{0,1\}^I$, and for each $\xi \in V$, $n \in \mathbb{N}$ choose an open set $G_{n\xi} \subseteq \{0,1\}^I$ such that $E^*_\xi \subseteq G_{n\xi}$ and $\nu_I(G_{n\xi}) \le 2^{-n}$. For $m, n \in \mathbb{N}$ set

$$D_{nm} = \{\xi : \xi \in V,\, U_m \subseteq G_{n\xi}\}.$$

For each $\alpha < \lambda$, set $f_\alpha(\xi) = \pi_I(f(\xi)(\alpha)) \in \{0,1\}^I$ for $\xi < \kappa$; then the functions f_α are all stochastically independent, in the sense that the σ-algebras $\Sigma_\alpha = \{f_\alpha^{-1}[H] : H \subseteq \{0,1\}^I \text{ is Borel}\}$ are independent. **P** Suppose that $\alpha_0, \dots, \alpha_n < \lambda$ are distinct and $H_0, \dots, H_n$ are Borel subsets of $\{0,1\}^I$. For each i, set

$$W_i = \{u : u \in (\{0,1\}^\theta)^\lambda,\, u(\alpha_i) \in \pi_I^{-1}[H_i]\}.$$

Then

$$\begin{aligned}\nu(\bigcap_{i\le n} f_{\alpha_i}^{-1}[H_i]) &= \nu f^{-1}[\bigcap_{i\le n} W_i] = \nu_\theta^\lambda(\bigcap_{i\le n} W_i)\\ &= \prod_{i\le n}\nu_\theta^\lambda W_i = \prod_{i\le n}\nu f_{\alpha_i}^{-1}[H_i]. \textbf{ Q}\end{aligned}$$

By 272Q, there is for each $\xi < \kappa$ an $\alpha(\xi) \in J_\xi$ such that $\Sigma_{\alpha(\xi)}$ is stochastically independent of the σ-algebra T generated by $\{D_{nm} : n,\, m \in \mathbb{N}\}$. Because $\lambda < \kappa$ and ν is κ-additive, there is a $\gamma < \lambda$ such that $B = \{\xi : \alpha(\xi) = \gamma\}$ has $\nu B > 0$. Take $n \in \mathbb{N}$ such that $\nu(B) > 2^{-n}$, and examine

$$C = \bigcup_{m\in\mathbb{N}}(D_{nm} \cap f_\gamma^{-1}[U_m]).$$

Then $\nu C = (\nu \times \nu_I)(\tilde C)$ where

$$\tilde C = \bigcup_{m\in\mathbb{N}}(D_{nm} \times U_m) \subseteq \kappa \times \{0,1\}^I$$

and $\nu \times \nu_I$ is the c.l.d. product measure on $\kappa \times \{0,1\}^I$. **P** Because T and Σ_γ are independent, and ν_I is the completion of its restriction to the Borel (or the Baire) σ-algebra of $\{0,1\}^I$, the map $\xi \mapsto (\xi, f_\gamma(\xi)) : \kappa \to \kappa \times \{0,1\}^I$ is inverse-measure-preserving for ν and $(\nu\restriction\mathrm{T}) \times \nu_I$ (cf. 272J). The inverse of $\tilde C$ under this map is just C, so

$$\begin{aligned}\nu C = ((\nu\restriction\mathrm{T}) \times \nu_I)(\tilde C) &= \int (\nu\restriction\mathrm{T})\tilde C^{-1}[\{u\}]\nu_I(du)\\ &= \int \nu\tilde C^{-1}[\{u\}]\nu_I(du) = (\nu \times \nu_I)(\tilde C). \textbf{ Q}\end{aligned}$$

But, for each $\xi < \kappa$, the vertical section $\tilde C[\{\xi\}]$ is just $\bigcup\{U_m : \xi \in D_{nm}\} = G_{n\xi}$, so

$$(\nu \times \nu_I)(\tilde C) = \int \nu_I(G_{n\xi})\nu(d\xi) \le 2^{-n}.$$

Accordingly $\nu C \le 2^{-n} < \nu B$ and there must be a $\xi \in B \cap V \setminus C$. But in this case $f(\xi)(\gamma) \in E_\xi$, because $\gamma = \alpha(\xi) \in J_\xi$, so $f_\gamma(\xi) = \pi_I(f(\xi)(\gamma)) \in E^*_\xi$. On the other hand, $f_\gamma(\xi) \notin G_{n\xi}$, because there is no m such that $f_\gamma(\xi) \in U_m \subseteq G_{n\xi}$; contrary to the choice of $G_{n\xi}$. **X**

So take some $\xi < \kappa$ such that $\nu_\theta^*(f(\xi)[J]) > 0$ for every uncountable $J \subseteq \lambda$. Evidently $f(\xi)$ is countable-to-one, so A_ξ must have cardinal λ, and will serve for A.

(b) Now suppose that (X, μ) is an atomless compact probability space with Maharam type $\theta < \kappa$. Then we have an inverse-measure-preserving map $h : \{0,1\}^\theta \to X$. Let $A \subseteq \{0,1\}^\theta$ be a Sierpiński set of cardinal λ; then $h[A]$ is a Sierpiński set with cardinal λ in X, by 537B(b-i).

(c) Finally, for the general case as stated, we can apply (b) to a normalized subspace measure, as usual.

544H Corollary Let κ be an atomlessly-measurable cardinal.
(a) $\operatorname{non}\mathcal{N}_\theta = \omega_1$ for $\omega \le \theta < \kappa$.
(b) $\operatorname{non}\mathcal{N}_\theta \le \kappa$ for $\theta \le \min(2^\kappa, \kappa^{(+\omega)})$.
(c) $\operatorname{non}\mathcal{N}_\theta < \theta$ for $\kappa < \theta < \kappa^{(+\omega)}$.

proof (a) Immediate from 544G.

(b) If ν is any witnessing probability on κ then we have an inverse-measure-preserving function $f : \kappa \to \{0,1\}^\theta$ (543G); now $f[\kappa]$ witnesses that $\operatorname{non}\mathcal{N}_\theta \le \kappa$.

(c) Induce on θ, using 523Ib.

Remark There may be more to be said; see 544Zc.

544I The following is an elementary corollary of Theorem 543C.

Proposition Let $(X,\mathfrak{T},\Sigma,\mu)$ be a totally finite quasi-Radon measure space and $(Y,\mathcal{P}Y,\nu)$ a probability space; suppose that $w(X) < \operatorname{add}\nu$. Let $f : X \times Y \to \mathbb{R}$ be a bounded function such that all the sections $x \mapsto f(x,y) : X \to \mathbb{R}$ are Σ-measurable. Then the repeated integrals $\iint f(x,y)\nu(dy)\mu(dx)$ and $\iint f(x,y)\mu(dx)\nu(dy)$ are defined and equal.

proof If $\mu X = 0$ this is trivial; otherwise, re-scaling μ if necessary, we may suppose that $\mu X = 1$. By 543C,

$$\overline{\int}\int f(x,y)\nu(dy)\mu(dx) \le \int\overline{\int} f(x,y)\mu(dx)\nu(dy) = \iint f(x,y)\mu(dx)\nu(dy).$$

Similarly

$$\overline{\int}\int(-f(x,y))\nu(dy)\mu(dx) \le \iint(-f(x,y))\mu(dx)\nu(dy),$$

so that

$$\underline{\int}\int f(x,y)\nu(dy)\mu(dx) \ge \iint f(x,y)\mu(dx)\nu(dy).$$

Putting these together we have the result.

544J Proposition (Zakrzewski 92) Let κ be an atomlessly-measurable cardinal and $(X,\mathfrak{T},\Sigma,\mu)$, $(Y,\mathfrak{S},\mathrm{T},\nu)$ Radon probability spaces both of weight less than κ; let $\mu \times \nu$ be the c.l.d. product measure on $X \times Y$, and Λ its domain. Let $f : X \times Y \to \mathbb{R}$ be a function such that all its horizontal and vertical sections

$$x \mapsto f(x,y^*) : X \to \mathbb{R}, \quad y \mapsto f(x^*,y) : Y \to \mathbb{R}$$

are measurable. Then

(a) if f is bounded, the repeated integrals

$$\iint f(x,y)\mu(dx)\nu(dy), \quad \iint f(x,y)\nu(dy)\mu(dx)$$

exist and are equal;

(b) in any case, there is a Λ-measurable function $g : X \times Y \to \mathbb{R}$ such that all the sections $\{x : g(x,y^*) \ne f(x,y^*)\}$, $\{y : g(x^*,y) \ne f(x^*,y)\}$ are negligible.

proof (a) By 543H there is a κ-additive measure $\tilde\nu$ on Y, with domain $\mathcal{P}Y$, extending ν. Now 544I tells us, among other things, that the function

$$x \mapsto \int f(x,y)\nu(dy) = \int f(x,y)\tilde\nu(dy) : X \to \mathbb{R}$$

is Σ-measurable. Similarly, $y \mapsto \int f(x,y)\mu(dx)$ is T-measurable. So returning to 544I we get

$$\begin{aligned}\iint f(x,y)\mu(dx)\nu(dy) &= \iint f(x,y)\mu(dx)\tilde\nu(dy)\\ &= \iint f(x,y)\tilde\nu(dy)\mu(dx) = \iint f(x,y)\nu(dy)\mu(dx).\end{aligned}$$

(b)(i) Suppose first that f is bounded. By (a), we can define a measure θ on $X \times Y$ by saying that

$$\theta G = \int \nu G[\{x\}]\mu(dx) = \int \mu G^{-1}[\{y\}]\nu(dy)$$

whenever $G \subseteq X \times Y$ is such that $G[\{x\}] \in \mathrm{T}$ for almost every $x \in X$ and $G^{-1}[\{y\}] \in \Sigma$ for almost every $y \in Y$. This θ extends $\mu \times \nu$; so the Radon-Nikodým theorem (232G) tells us that there is a Λ-measurable function $h : X \times Y \to \mathbb{R}$ such that $\int_G f(x,y)\theta(dxdy) = \int_G h(x,y)\theta(dxdy)$ for every $G \in \Lambda$.

Let $\mathcal{U}$ be a base for the topology $\mathfrak{T}$, closed under finite intersections, with $\#(\mathcal{U}) < \kappa$. For any $U \in \mathcal{U}$ consider

$$V_U = \{y : \int_U f(x,y)\mu(dx) > \int_U h(x,y)\mu(dx)\}.$$

The argument of (a) shows that $y \mapsto \int_U f(x,y)\mu(dx)$ is T-measurable, so $V_U \in \mathrm{T}$, and

$$\begin{aligned}\int_{V_U}\int_U f(x,y)\mu(dx)\nu(dy) &= \int_{U\times V_U} f(x,y)\theta(dxdy)\\ &= \int_{U\times V_U} h(x,y)\theta(dxdy) = \int_{V_U}\int_U h(x,y)\mu(dx)\nu(dy),\end{aligned}$$

so $\nu V_U = 0$. Similarly

$$\nu\{y : \int_U f(x,y)\mu(dx) < \int_U h(x,y)\mu(dx)\} = 0.$$

Because $\#(\mathcal{U}) < \kappa$, and no non-negligible measurable set in Y can be covered by fewer than κ negligible sets (544B), we must have

$$\nu^*\{y : \int_U f(x,y)\mu(dx) = \int_U h(x,y)\mu(dx) \text{ for every } U \in \mathcal{U}\} = 1.$$

But because $\mathcal{U}$ is a base for the topology of X closed under finite intersections, we see that

$$\nu^*\{y : f(x,y) = h(x,y) \text{ for } \mu\text{-almost every } x\} = 1.$$

(For each y such that $\int_U f(x,y)\mu(dx) = \int_U h(x,y)\mu(dx)$ for every $U \in \mathcal{U}$, apply 415H(v) to the indefinite-integral measures over μ defined by the functions $x \mapsto f(x,y)$, $x \mapsto h(x,y)$; these are quasi-Radon by 415Ob.) Again using (a), we know that the the repeated integral $\iint |f(x,y) - h(x,y)|\mu(dx)\nu(dy)$ exists, and it must be 0. Thus

$$\nu\{y : f(x,y) = h(x,y) \text{ for } \mu\text{-almost every } x\} = 1.$$

Similarly,

$$\mu\{x : f(x,y) = h(x,y) \text{ for } \nu\text{-almost every } y\} = 1.$$

But now, changing h on a set of the form $(E\times Y)\cup(X\times F)$ where $\mu E = \nu F = 0$, we can get a function g, still Λ-measurable, such that $\{(x,y) : f(x,y) \ne g(x,y)\}$ has all its horizontal and vertical sections negligible.

(ii) This deals with bounded f. But for general f we can look at the truncates $(x,y) \mapsto \mathrm{med}(-n, f(x,y), n)$ for each n to get a sequence $\langle g_n\rangle_{n\in\mathbb{N}}$ of functions which will converge at an adequate number of points to provide a suitable g.

544K Proposition If X is a metrizable space and μ is a σ-finite Borel measure on X, then $\mathrm{add}\,\mathcal{N}(\mu) \ge \mathrm{add}\,\mathcal{N}_\omega$.

proof (a) If there is an atomlessly-measurable cardinal then

$$\mathrm{add}\,\mathcal{N}_\omega \le \mathrm{non}\,\mathcal{N}_\omega = \omega_1$$

(544Ha), so the result is immediate. So let us henceforth suppose otherwise.

(b) Because there is a totally finite measure with the same domain and the same null ideal as μ (215B(vii)), we can suppose that μ itself is totally finite. Let $(\mathfrak{A}, \bar\mu)$ be the measure algebra of μ and $\mathcal{U}$ a σ-disjoint base for the topology of X (4A2Lg); express $\mathcal{U}$ as $\bigcup_{n\in\mathbb{N}}\mathcal{U}_n$ where each $\mathcal{U}_n$ is disjoint. For $\mathcal{V} \subseteq \mathcal{U}_n$ set $\nu_n\mathcal{V} = \mu(\bigcup\mathcal{V})$. Then ν_n is a totally finite measure with domain $\mathcal{P}\mathcal{U}_n$. Because there is no atomlessly-measurable cardinal, $\mathrm{add}\,\nu_n$ is either ∞ or a two-valued-measurable cardinal; in either case, ν_n is $\mathfrak{c}$-additive and purely atomic (438Ce, 543B).

(c) μ has countable Maharam type. **P** Because ν_n is purely atomic, there is a sequence $\langle\mathcal{U}_{ni}\rangle_{i\in\mathbb{N}}$ of subsets of $\mathcal{U}_n$ such that for every $\mathcal{V} \subseteq \mathcal{U}_n$ there is a $J \subseteq \mathbb{N}$ such that $\nu_n(\mathcal{V}\triangle\bigcup_{i\in J}\mathcal{U}_{ni}) = 0$. Set $W_{ni} = \bigcup\mathcal{U}_{ni}$ for each i. Let $\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{W_{ni}^\bullet : n, i \in \mathbb{N}\}$.

If $G \subseteq X$ is open, set $\mathcal{V}_n = \{U : U \in \mathcal{U}_n, U \subseteq G\}$ and $G_n = \bigcup\mathcal{V}_n$ for each n. Then we have $J_n \subseteq \mathbb{N}$ such that

$$0 = \nu_n(\mathcal{V}_n\triangle\bigcup_{i\in J_n}\mathcal{U}_{ni}) = \mu(G_n\triangle\bigcup_{i\in J_n}W_{ni}),$$

so $G_n^\bullet = \sup_{i\in J_n} W_{ni}^\bullet \in \mathfrak{B}$. Now $G = \bigcup_{n\in\mathbb{N}} G_n$ so $G^\bullet = \sup_{n\in\mathbb{N}} G_n^\bullet$ belongs to $\mathfrak{B}$.

The set $\Sigma = \{E : E \subseteq X$ is Borel, $E^\bullet \in \mathfrak{B}\}$ is a σ-algebra of subsets of X, and we have just seen that it contains every open set; so Σ is the whole Borel σ-algebra and $\mathfrak{A} = \mathfrak{B}$ has countable Maharam type. **Q**

(d) Next, if $\langle G_\xi \rangle_{\xi<\kappa}$ is a family of open sets where $\kappa < \mathfrak{c}$, and $G = \bigcup_{\xi<\kappa} G_\xi$, then $G^\bullet = \sup_{\xi<\kappa} G_\xi^\bullet$ in $\mathfrak{A}$. **P** Look at $\mathcal{V}_{n\xi} = \{U : U \in \mathcal{U}_n,\, U \subseteq G_\xi\}$, $\mathcal{V}_n = \bigcup_{\xi<\kappa} \mathcal{V}_{n\xi}$ for each n. Because ν_n is $\mathfrak{c}$-additive,

$$\mu(\bigcup \mathcal{V}_n) = \nu \mathcal{V}_n = \sup_{J \subseteq \kappa \text{ is finite}} \nu(\bigcup_{\xi \in J} \mathcal{V}_{n\xi})$$

and there is a countable set $J_n \subseteq \kappa$ such that $\mu(\bigcup \mathcal{V}_n) = \mu(\bigcup_{\xi\in J_n} \bigcup \mathcal{V}_{n\xi})$. Now

$$G^\bullet = \sup_{n\in\mathbb{N}} (\bigcup \mathcal{V}_n)^\bullet = \sup_{n\in\mathbb{N}} \sup_{\xi \in J_n} (\bigcup \mathcal{V}_{n\xi})^\bullet \subseteq \sup_{\xi<\kappa} G_\xi^\bullet \subseteq G^\bullet. \textbf{ Q}$$

(e) Let $\langle E_\xi \rangle_{\xi<\kappa}$ be a family in $\mathcal{N}(\mu)$ where $\kappa < \operatorname{add}\mathcal{N}_\omega$. Because μ is inner regular with respect to the closed sets (412D), we can find closed sets $F_{\xi n} \subseteq X \setminus E_\xi$ such that $\mu F_{\xi n} \ge \mu X - 2^{-n}$ for $\xi < \kappa$ and $n \in \mathbb{N}$. By 524Mb and (c) above, $\operatorname{wdistr}(\mathfrak{A}) \ge \operatorname{add}\mathcal{N}_\omega$, so there is a countable $C \subseteq \mathfrak{A}$ such that $F_{\xi n}^\bullet = \sup\{c : c \in C,\, c \subseteq F_{\xi n}^\bullet\}$ for every $n \in \mathbb{N}$ and $\xi < \kappa$ (514K). Again because μ is inner regular with respect to the closed sets, there is a sequence$\langle F_m \rangle_{m\in\mathbb{N}}$ of closed sets such that whenever $C' \subseteq C$ is finite then $\bar\mu(\sup C') = \sup\{\mu F_m : m \in \mathbb{N},\, F_m^\bullet \subseteq \sup C'\}$. Consequently

$$\mu F_{\xi n} = \sup\{\mu F_m : m \in \mathbb{N},\, F_m^\bullet \subseteq F_{\xi n}^\bullet\}$$

for every $\xi < \kappa$ and $n \in \mathbb{N}$. Set

$$H_m = X \cap \bigcap\{F_{\xi n} : n \in \mathbb{N},\, \xi < \kappa,\, F_m^\bullet \subseteq F_{\xi n}^\bullet\}.$$

Applying (d) to $\{X \setminus F_{\xi n} : F_m^\bullet \subseteq F_{\xi n}^\bullet\}$, we see that $H_m^\bullet \supseteq F_m^\bullet$, that is, $F_m \setminus H_m$ is negligible, for each m.

Each H_m is closed; let $f_m : X \to [0,1]$ be a continuous function such that $H_m = f_m^{-1}[\{0\}]$. Set $f(x) = \langle f_m(x) \rangle_{m\in\mathbb{N}} \in [0,1]^{\mathbb{N}}$ for $x \in X$, and let ν be the restriction of the image measure μf^{-1} to the Borel σ-algebra of $[0,1]^{\mathbb{N}}$. Then $\operatorname{add}\mathcal{N}(\nu) \ge \operatorname{add}\mathcal{N}_\omega$ (apply 522W(a-i) to the atomless part of ν). For each $\xi < \kappa$ and $n \in \mathbb{N}$, there is an $m \in \mathbb{N}$ such that $F_m^\bullet \subseteq F_{\xi n}^\bullet$ and $\mu F_m \ge \mu F_{\xi n} - 2^{-n} \ge \mu X - 2^{-n+1}$; now $H_m \subseteq F_{\xi n}$ is disjoint from E_ξ and $\mu H_m \ge \mu X - 2^{-n+1}$. So $\{z : z \in [0,1]^{\mathbb{N}},\, z(m) > 0\}$ includes $f[E_\xi]$ and has measure at most 2^{-n+1}. Accordingly $f[E_\xi]$ is ν-negligible.

As $\operatorname{add}\mathcal{N}(\nu) > \kappa$, $\bigcup_{\xi<\kappa} f[E_\xi]$ is ν-negligible; as f is inverse-measure-preserving, $\bigcup_{\xi<\kappa} E_\xi$ is μ-negligible; as $\langle E_\xi \rangle_{\xi<\kappa}$ is arbitrary, $\operatorname{add}\mathcal{N}(\mu) \ge \operatorname{add}\mathcal{N}_\omega$.

544L Corollary Let X be a metrizable space.

(a) If $\mathcal{U}\mathfrak{n}$ is the σ-ideal of universally negligible subsets of X, then $\operatorname{add}\mathcal{U}\mathfrak{n} \ge \operatorname{add}\mathcal{N}_\omega$.

(b) If Σ_{um} is the σ-algebra of universally measurable subsets of X, then $\bigcup\mathcal{E} \in \Sigma_{\mathrm{um}}$ whenever $\mathcal{E} \subseteq \Sigma_{\mathrm{um}}$ and $\#(\mathcal{E}) < \operatorname{add}\mathcal{N}_\omega$.

proof (a) Let $\mathcal{E} \subseteq \mathcal{U}\mathfrak{n}$ be a set with cardinal less than $\operatorname{add}\mathcal{N}_\omega$, and μ a Borel probability measure on X such that $\mu\{x\} = 0$ for every $x \in X$. Then $\mathcal{E} \subseteq \mathcal{N}(\mu)$; by 544K, $\bigcup\mathcal{E} \in \mathcal{N}(\mu)$; as μ is arbitrary, $\bigcup\mathcal{E} \in \mathcal{U}\mathfrak{n}$; as $\mathcal{E}$ is arbitrary, $\operatorname{add}\mathcal{U}\mathfrak{n} \ge \operatorname{add}\mathcal{N}_\omega$.

(b) Let μ be a totally finite Borel measure on X and $\hat\mu$ its completion. By 521Ad and 544K,

$$\operatorname{add}\hat\mu = \operatorname{add}\mathcal{N}(\hat\mu) = \operatorname{add}\mathcal{N}(\mu) \ge \operatorname{add}\mathcal{N}_\omega > \#(\mathcal{E}).$$

Since $\hat\mu$ measures every member of $\mathcal{E}$, it also measures $\bigcup\mathcal{E}$ (521Aa); as μ is arbitrary, $\bigcup\mathcal{E} \in \Sigma_{\mathrm{um}}$.

544M Theorem Let κ be an atomlessly-measurable cardinal. Then the following are equiveridical:

(i) for every family $\langle f_\xi \rangle_{\xi<\kappa}$ of regressive functions defined on $\kappa \setminus \{0\}$ there is a family $\langle \alpha_\xi \rangle_{\xi<\kappa}$ in κ such that

$$\{\kappa \setminus \zeta : \zeta < \kappa\} \cup \{f_\xi^{-1}[\{\alpha_\xi\}] : \xi < \kappa\}$$

has the finite intersection property;

(ii) for every family $\langle f_\xi \rangle_{\xi<\kappa}$ in $\mathbb{N}^\kappa$ there is a family $\langle m_\xi \rangle_{\xi<\kappa}$ in $\mathbb{N}$ such that

$$\{\kappa \setminus \zeta : \zeta < \kappa\} \cup \{f_\xi^{-1}[\{m_\xi\}] : \xi < \kappa\}$$

has the finite intersection property;

(iii) $\operatorname{cov}\mathcal{N}_\kappa > \kappa$;

(iv) $\operatorname{cov}\mathcal{N}(\mu) > \kappa$ whenever (X,μ) is a locally compact semi-finite measure space and $\mu X > 0$.

proof Let ν be a normal witnessing probability on κ.

(i)⇒(ii) Given a family $\langle f_\xi\rangle_{\xi<\kappa}$ as in (ii), apply (i) to $\langle f'_\xi\rangle_{\xi<\kappa}$ where $f'_\xi(\eta) = 0$ if $0 < \eta < \omega$, $f_\xi(\eta)$ if $\omega \le \eta < \kappa$.

(ii)⇒(iii) Let $\langle A_\alpha\rangle_{\alpha<\kappa}$ be a family in $\mathcal{N}_\kappa$. For each $\alpha < \kappa$ let $\langle F_{\alpha n}\rangle_{n\in\mathbb{N}}$ be a disjoint sequence of compact subsets of $\{0,1\}^\kappa \setminus A_\alpha$ such that $\nu_\kappa(\bigcup_{n\in\mathbb{N}} F_{\alpha n}) = 1$. By 543G there is a function $h : \kappa \to \{0,1\}^\kappa$ which is inverse-measure-preserving for ν and ν_κ. Set $H_\alpha = h^{-1}(\bigcup_{n\in\mathbb{N}} F_{\alpha n})$; then $\nu H_\alpha = 1$. Let H be the diagonal intersection of $\langle H_\alpha\rangle_{\alpha<\kappa}$, so that $\nu H = 1$. Let $\langle\gamma_\xi\rangle_{\xi<\kappa}$ be the increasing enumeration of H.

For α, $\xi < \kappa$ set

$$\begin{aligned} f_\alpha(\xi) &= n \text{ if } h(\gamma_\xi) \in F_{\alpha n},\\ &= 0 \text{ if } \gamma_\xi \notin H_\alpha. \end{aligned}$$

Then there is a family $\langle m_\alpha\rangle_{\alpha<\kappa}$ in $\mathbb{N}$ such that $\mathcal{E} = \{\kappa\setminus\zeta : \zeta < \kappa\} \cup \{f_\alpha^{-1}[\{m_\alpha\}] : \alpha < \kappa\}$ has the finite intersection property. Let $\mathcal{F} \supseteq \mathcal{E}$ be an ultrafilter. For any $\alpha < \kappa$ we have $H \setminus H_\alpha \subseteq \alpha + 1$, so that $\{\xi : \gamma_\xi \notin H_\alpha\}$ is bounded above in κ and cannot belong to $\mathcal{F}$. Consequently $\{\xi : h(\gamma_\xi) \in F_{\alpha,m_\alpha}\} \in \mathcal{F}$. But this implies at once that $\langle F_{\alpha,m_\alpha}\rangle_{\alpha<\kappa}$ has the finite intersection property; because all the $F_{\alpha n}$ are compact, there is a $y \in \bigcap_{\alpha<\kappa} F_{\alpha,m_\alpha}$, and now $y \notin \bigcup_{\alpha<\kappa} A_\alpha$.

Because $\langle A_\alpha\rangle_{\alpha<\kappa}$ was arbitrary, $\operatorname{cov}\mathcal{N}_\kappa > \kappa$.

(iii)⇒(iv) As in 544B, this follows from 523F and 521Lb.

(iv)⇒(i) Let $(Z,\tilde\nu)$ be the Stone space of the measure algebra $\mathfrak{A}$ of ν; for $A \subseteq \kappa$ let A^* be the open-and-closed subset of Z corresponding to the image $A^\bullet$ of A in $\mathfrak{A}$.

Now let $\langle f_\xi\rangle_{\xi<\kappa}$ be a family of regressive functions defined on $\kappa\setminus\{0\}$. Because $\mathcal{N}(\nu)$ is normal and ω_1-saturated and f_ξ is regressive, there is for each $\xi < \kappa$ a countable set $D_\xi \subseteq \kappa$ such that $\nu f_\xi^{-1}[D_\xi] = 1$ (541Ka). For ξ, $\eta < \kappa$ set $A_{\xi\eta} = f_\xi^{-1}[\{\eta\}]$; then $\nu(\bigcup_{\eta\in D_\xi} A_{\xi\eta}) = 1$ so (because D_ξ is countable) $\sup_{\eta\in D_\xi} A^\bullet_{\xi\eta} = 1$ in $\mathfrak{A}$ and $\tilde\nu(\bigcup_{\eta\in D_\xi} A^*_{\xi\eta}) = 1$. Set $E_\xi = Z \setminus \bigcup_{\eta\in D_\xi} A^*_{\xi\eta} \in \mathcal{N}(\tilde\nu)$. By (iv), $Z \ne \bigcup_{\xi<\kappa} E_\xi$; take $z \in Z \setminus \bigcup_{\xi<\kappa} E_\xi$. Then for every $\xi < \kappa$ there must be an $\alpha_\xi \in D_\xi$ such that $z \in A^*_{\xi,\alpha_\xi}$. But this implies that $\{A^*_{\xi,\alpha_\xi} : \xi < \kappa\}$ is a centered family of open subsets of Z. It follows that $\{A^\bullet_{\xi,\alpha_\xi} : \xi < \kappa\}$ is centered in $\mathfrak{A}$. Since $\nu\zeta = 0$ for every $\zeta < \kappa$, $\{A_{\xi,\alpha_\xi} : \xi < \kappa\} \cup \{\kappa\setminus\zeta : \zeta < \kappa\}$ must have the finite intersection property, as required.

544N Cichoń's diagram and other cardinals (a) Returning to the concerns of Chapter 52, we see from the results above that any atomlessly-measurable cardinal κ is necessarily connected with the structures there. By 544B, $\kappa \le \operatorname{cov}\mathcal{N}_\lambda$ for every λ; by 544G, $\operatorname{non}\mathcal{N}_\omega = \omega_1$, so all the cardinals on the bottom line of Cichoń's diagram (522B), and therefore the Martin numbers $\mathfrak{m}$, $\mathfrak{p}$ etc. (522T), must be ω_1, while all the cardinals on the top line must be at least κ. From 522Ub we see also that $\mathrm{FN}(\mathcal{P}\mathbb{N})$ must be at least κ. Concerning $\mathfrak{b}$ and $\mathfrak{d}$, the position is more complicated.

(b) If κ is an atomlessly-measurable cardinal, then $\mathfrak{b} < \kappa$. **P?** Otherwise, we can choose inductively a family $\langle f_\xi\rangle_{\xi<\kappa}$ in $\mathbb{N}^\mathbb{N}$ such that $\{n : f_\xi(n) \le f_\eta(n)\}$ is finite whenever $\eta < \xi < \kappa$. Let ν be a witnessing probability measure on κ. For m, $i \in \mathbb{N}$ set $D_{mi} = \{\xi : \xi < \kappa,\ f_\xi(m) = i\}$. Then

$$\begin{aligned} W = \{(\xi,\eta) : \eta < \xi < \kappa\} &= \bigcup_{n\in\mathbb{N}} \bigcap_{m\ge n} \{(\xi,\eta) : f_\eta(m) < f_\xi(m)\}\\ &= \bigcup_{n\in\mathbb{N}} \bigcap_{m\ge n} \bigcup_{i<j} D_{mi} \times D_{mj} \end{aligned}$$

belongs to $\mathcal{P}\kappa\widehat{\otimes}\mathcal{P}\kappa$. But also

$$\int \nu W[\{\xi\}]\nu(d\xi) = 0 < 1 = \int \nu W^{-1}[\{\eta\}]\nu(d\eta),$$

so this contradicts Fubini's theorem. **XQ**

(c) If κ is an atomlessly-measurable cardinal, then $\operatorname{cf}\mathfrak{d} \ne \kappa$. **P?** Otherwise, let $A \subseteq \mathbb{N}^{\mathbb{N}}$ be a cofinal set with cardinal $\mathfrak{d}$, and express A as $\bigcup_{\xi<\kappa} A_\xi$ where $\langle A_\xi\rangle_{\xi<\kappa}$ is non-decreasing and $\#(A_\xi) < \mathfrak{d}$ for every $\xi < \kappa$. For each $\xi < \kappa$, we have an $f_\xi \in \mathbb{N}^{\mathbb{N}}$ such that $f_\xi \not\le g$ for any $g \in A_\xi$. Let ν be a witnessing probability on κ. Then for each $n \in \mathbb{N}$ we have an $h(n) \in \mathbb{N}$ such that $\nu\{\xi : f_\xi(n) \ge h(n)\} \le 2^{-n-2}$. This defines a function $h \in \mathbb{N}^{\mathbb{N}}$. There must be a $g \in A$ such that $h \le g$; let $\zeta < \kappa$ be such that $g \in A_\zeta$. The set $\{\xi : f_\xi \le h\}$ has measure at least $\frac{1}{2}$, so there is some $\xi \ge \zeta$ such that

$$f_\xi \le h \le g \in A_\zeta \subseteq A_\xi,$$

contrary to the choice of f_ξ. **XQ**

(d) As for the cardinals studied in §523, I have already noted that $\operatorname{cov}\mathcal{N}_\lambda \ge \kappa$ for any atomlessly-measurable cardinal κ and any λ, and we can say something about the possibility that $\operatorname{cov}\mathcal{N}_\lambda = \kappa$ (544M). Recall that $\operatorname{cf}[\kappa]^{\le\omega} = \kappa$ (542Ia), so that $\operatorname{cf}\mathcal{N}_\kappa = \max(\kappa, \operatorname{cf}\mathcal{N}_\omega)$ for any quasi-measurable cardinal κ.

544X Basic exercises (a) Let κ be an atomlessly-measurable cardinal, and ν a witnessing probability on κ. Show that there is a set $C \subseteq \{0,1\}^\kappa \times \kappa$ such that $\nu_\kappa C^{-1}[\{\xi\}] = 0$ for every $\xi < \kappa$, but $\nu_\kappa^*\{x : \nu C[\{x\}] = 1\} = 1$.

(b) Suppose that κ is an atomlessly-measurable cardinal. Show that $\mathbb{R}^\lambda$ is measure-compact for every $\lambda < \kappa$. (*Hint*: 533J.)

(c) Let κ be a two-valued-measurable cardinal, $\mathcal{I}$ a normal maximal ideal of $\mathcal{P}\kappa$, (X,μ) a quasi-Radon probability space of weight strictly less than κ, and $f : [\kappa]^{<\omega} \to \mathcal{N}(\mu)$ a function. Show that there is a $V \in \mathcal{I}$ such that $\bigcup\{f(I) : I \in [\kappa \setminus V]^{<\omega}\}$ is μ-negligible. (*Hint*: 541Xf.)

(d) In 544F, show that if the magnitude of μ is less than κ and the augmented shrinking number $\operatorname{shr}^+(\mu)$ is at most κ then there is a ν-conegligible $V \subseteq \kappa$ such that $\mu_*(\bigcup_{I\in[V]^{<\omega}} f(I)) = 0$.

(e) Suppose that there is an atomlessly-measurable cardinal. Show that every Radon measure on a first-countable compact Hausdorff space is uniformly regular. (*Hint*: 533Hb.)

(f) Suppose that κ is an atomlessly-measurable cardinal and that $2^\kappa = \kappa^{(+n+1)}$. Show that $\operatorname{non}\mathcal{N}_{2^{\kappa^+}} \le \kappa^+$. (*Hint*: 523I(a-iv).)

(g) Suppose that κ is an atomlessly-measurable cardinal and that (X,ρ) is a metric space. Show that no subset of X with strong measure zero can have cardinal κ.

(h) Let (X,Σ,μ) be a σ-finite measure space such that every subset of X^2 is measured by the c.l.d. product measure $\mu\times\mu$. Show that there is a countable subset of X with full outer measure. (*Hint*: if singletons are negligible, consider a well-ordering of X as a subset of X^2.)

(i) Let κ be an atomlessly-measurable cardinal, and G a group of permutations of κ such that $\#(G) < \kappa$. Show that there is a non-zero strictly localizable atomless G-invariant measure with domain $\mathcal{P}\kappa$ and magnitude at most $\#(G)$. (*Hint*: start with G countable.)

544Y Further exercises (a) Let κ be a real-valued-measurable cardinal with witnessing probability ν. Give κ its discrete topology, so that ν is a Borel measure and $\kappa^{\mathbb{N}}$ is metrizable. Let λ be the Borel measure on $\kappa^{\mathbb{N}}$ constructed from ν by the method of 434Ym. (i) Show that if κ is atomlessly-measurable then $\operatorname{add}\mathcal{N}(\lambda) = \omega_1$. (ii) Show that if κ is two-valued-measurable then $\operatorname{add}\mathcal{N}(\lambda) = \kappa$.

(b) Show that $\mathfrak{c}$ does not have the property of 544M(ii).

(c) Show that a cardinal κ is weakly compact iff it is strongly inaccessible and has the property (i) of 544M.

544Z Problems (a) In 543C, can we replace '$w(X) < \operatorname{add}\nu$' with '$\tau(\mu) < \operatorname{add}\nu$'? More concretely, suppose that (Z,λ) is the Stone space of $(\mathfrak{B}_\omega,\bar{\nu}_\omega)$, κ is an atomlessly-measurable cardinal and ν a normal witnessing probability on κ, so that

$$\tau(\lambda) = \omega < \operatorname{add}\nu \le \mathfrak{c} = w(Z).$$

Let $C \subseteq \kappa \times Z$ be such that $\lambda C[\{\xi\}] = 0$ for every $\xi < \kappa$. By 544C, we know that $\{z : \nu C^{-1}[\{z\}] > 0\}$ has inner measure zero. But does it have to be negligible?

(b) Suppose that κ is an atomlessly-measurable cardinal. Must there be a Sierpiński set $A \subseteq \{0,1\}^\omega$ with cardinal κ? (See 552E.)

(c) Suppose that κ is an atomlessly-measurable cardinal. Can $\operatorname{non}\mathcal{N}_\kappa$ be greater than ω_1? What if $\kappa = \mathfrak{c}$? (See 552H.)

(d) Can there be an atomlessly-measurable cardinal less than $\mathfrak{d}$? (See the notes to §555.)

(e) Can there be an atomlessly-measurable cardinal less than or equal to $\operatorname{shr}\mathcal{N}_\omega$? (See 555Yd.)

(f) Suppose that there is an atomlessly-measurable cardinal. Does it follow that $\operatorname{cov}\mathcal{N}_\omega = \mathfrak{c}$? (See 552Gc.)

544 Notes and comments The vocabulary of this section ('locally compact semi-finite measure space', 'quasi-Radon probability space of weight less than κ', 'compact probability space with Maharam type less than κ') makes significant demands on the reader, especially the reader who really wants to know only what happens to Lebesgue measure. But the formulations I have chosen are not there just on the off-chance that someone may wish to apply the results in unexpected contexts. I have tried to use the concepts established earlier in this treatise to signal the nature of the arguments used at each stage. Thus in 543C we had an argument which depended on topological ideas, and could work only on a space with a base which was small compared with the atomlessly-measurable cardinal in hand; in 544C, the argument depends on an inverse-measure-preserving function from some power $\{0,1\}^\lambda$, so requires a compact measure, but then finds a Δ-nebula with a countable root-cover J, so that 543C can be applied to the usual measure on $\{0,1\}^J$, irrespective of the size of λ. Similar, but to my mind rather deeper, ideas lead from 544E to 544F. In both cases, there is a price to be paid for moving to spaces X of arbitrary complexity; in one, an inequality $\overline{\int}\int \le \int\overline{\int}$ becomes the weaker $\underline{\int}\int \le \int\overline{\int}$; in the other, a negligible set turns into a set of inner measure zero (544Xd).

Another way to classify the results here is to ask which of them depend on the Gitik-Shelah theorem. The formulae in 544H betray such a dependence; but it seems that the Gitik-Shelah theorem is also needed for the full strength of 544B, 544G, 544J and 544M as written. Historically this is significant, because the idea behind 544G was worked out by K.Prikry and R.M.Solovay before it was known for sure that a witnessing measure on an atomlessly-measurable cardinal could not have countable Maharam type. However 544B and 544J, for instance, can be proved for Lebesgue measure without using the Gitik-Shelah theorem.

In the next chapter I will present a description of measure theory in random real models. Those already familiar with random real forcing may recognise some of the theorems of this section (544G, 544N) as versions of characteristic results from this theory (552E, 552C).

544M is something different. It was recognised in the 1960s that some of the ways in which two-valued-measurable cardinals are astonishing is that they are 'weakly Π^1_1-indescribable' (and, moreover, have many weakly Π^1_1-indescribable cardinals below them; see Fremlin 93, 4K). I do not give the 'proper' definition of weak Π^1_1-indescribability, which relies on concepts from model theory; you may find it in Levy 71, Baumgartner Taylor & Wagon 77 or Fremlin 93; for our purposes here, the equivalent combinatorial definition in 544M(i) will I think suffice. For strongly inaccessible cardinals, it is the same thing as weak compactness (544Yc). Here I mention it only because it turns out to be related to one of the standard questions I have been asking in this volume (544M(iii)). Of course the arguments above beg the question, whether an atomlessly-measurable cardinal can be weakly Π^1_1-indescribable, especially in view of 544Yb; see Fremlin 93, 4R.

In 544K-544L I look at a question which seems to belong in Chapter 52, or perhaps with the corresponding result in Hausdorff measures (534Bb). But unless I am missing something, the facts here depend on the Gitik-Shelah theorem via 544Ha.

This section has a longer list of problems than most. In the last four sections I have tried to show something of the richness of the structures associated with any atomlessly-measurable cardinal; I remain quite uncertain how much more we can hope to glean from the combinatorial and measure-theoretic arguments available. The problems of this chapter mostly have a special status. They are of course vacuous unless we suppose that there is an atomlessly-measurable cardinal; but there is something else. There is a well-understood process, 'Solovay's method', for building models of set theory with atomlessly-measurable cardinals from models with two-valued-measurable cardinals (§555). In most cases, the problems have been solved for such models, and perhaps they should be regarded as challenges to develop new forcing techniques.

Version of 10.2.14

545 PMEA and NMA

One of the reasons for supposing that it is consistent to assume that there are measurable cardinals is that very much stronger axioms have been studied at length without any contradiction appearing. Here I mention two such axioms which have obvious consequences in measure theory.

545A Theorem The following are equiveridical:

(i) for every cardinal λ, there is a probability space $(X,\mathcal{P}X,\mu)$ with $\tau(\mu)\ge\lambda$ and $\operatorname{add}\mu\ge\mathfrak{c}$;

(ii) for every cardinal λ, there is an extension of the usual measure ν_λ on $\{0,1\}^\lambda$ to a $\mathfrak{c}$-additive probability measure with domain $\mathcal{P}(\{0,1\}^\lambda)$;

(iii) for every semi-finite locally compact measure space (X,Σ,μ) (definition: 342Ad), there is an extension of μ to a $\mathfrak{c}$-additive measure with domain $\mathcal{P}X$.

proof (i)$\Rightarrow$(ii) Assume (i). Let λ be a cardinal; of course (ii) is surely true for finite λ, so we may take it that $\lambda\ge\omega$. Let $(X,\mathcal{P}X,\mu)$ be a probability space with Maharam type at least λ^+ and with $\operatorname{add}\mu\ge\mathfrak{c}$. Taking $\mathfrak{A}$ to be the measure algebra of μ, there is an $a\in\mathfrak{A}$ such that the principal ideal $\mathfrak{A}_a$ it generates is homogeneous with Maharam type at least λ (332S). Let $E\in\mathcal{P}X$ be such that $E^\bullet=a$, so that the subspace measure μ_E is Maharam-type-homogeneous with Maharam type at least λ. Setting $\mu'A=\mu A/\mu E$ for $A\subseteq E$, $(E,\mathcal{P}E,\mu')$ is a Maharam-type-homogeneous probability space with Maharam type at least λ, and $\operatorname{add}\mu'\ge\operatorname{add}\mu\ge\mathfrak{c}$. By 343Ca, there is a function $f:E\to\{0,1\}^\lambda$ which is inverse-measure-preserving for μ' and ν_λ. Now the image measure $\nu=\mu'f^{-1}$ is a $\mathfrak{c}$-additive extension of ν_λ to $\mathcal{P}(\{0,1\}^\lambda)$.

(ii)$\Rightarrow$(iii) Assume (ii).

(α) Suppose that (X,Σ,μ) is a compact probability space. Set $\lambda=\max(\omega,\tau(\mu))$. Then there is an inverse-measure-preserving function $f:\{0,1\}^\lambda\to X$ (343Cd). If ν is a $\mathfrak{c}$-additive extension of ν_λ to $\mathcal{P}(\{0,1\}^\lambda)$, then νf^{-1} is a $\mathfrak{c}$-additive extension of μ to $\mathcal{P}X$.

(β) Let (X,Σ,μ) be any semi-finite locally compact measure space. Set $\Sigma^{f+}=\{E:E\in\Sigma,\ 0<\mu E<\infty\}$ and let $\mathcal{E}\subseteq\Sigma^{f+}$ be maximal subject to $E\cap F$ being negligible for all distinct E, $F\in\Sigma$. If $H\in\Sigma$ and $\mu H<\infty$, then $\mathcal{H}=\{E:E\in\mathcal{E},\ \mu(E\cap H)>0\}$ is countable and $E\setminus\bigcup\mathcal{H}$ is negligible, so $\mu H=\sum_{E\in\mathcal{E}}\mu(E\cap H)$; because μ is semi-finite, $\mu H=\sum_{E\in\mathcal{E}}\mu(E\cap H)$ for every $H\in\Sigma$.

For each $E\in\mathcal{E}$, the subspace measure μ_E is compact; applying (α) to a normalization of μ_E, we have an extension μ'_E of μ_E to a $\mathfrak{c}$-additive measure with domain $\mathcal{P}E$. Set $\mu'A=\sum_{E\in\mathcal{E}}\mu'_E(A\cap E)$ for $A\subseteq X$; then $\mu':\mathcal{P}X\to[0,\infty]$ is a $\mathfrak{c}$-additive measure extending μ.

(iii)$\Rightarrow$(ii) and **(ii)$\Rightarrow$(i)** are trivial.

545B Definition PMEA (the '**product measure extension axiom**') is the assertion that the statements (i)-(iii) of 545A are true.

545C Proposition If PMEA is true, then $\mathfrak{c}$ is atomlessly-measurable.

proof By 545A(ii) we have an extension of the usual measure on $\{0,1\}^\omega$ to a $\mathfrak{c}$-additive measure μ with domain $\mathcal{P}(\{0,1\}^\omega)$. Since μ is zero on singletons, $\operatorname{add}\mu = \mathfrak{c}$ exactly, so 543Ba and 543Bc tell us that $\mathfrak{c}$ is real-valued-measurable, therefore atomlessly-measurable.

545D Definition NMA (the '**normal measure axiom**') is the statement

For every set I there is a $\mathfrak{c}$-additive probability measure ν on $S = [I]^{<\mathfrak{c}}$, with domain $\mathcal{P}S$, such that

(α) $\nu\{s : i \in s \in S\} = 1$ for every $i \in I$,

(β) if $A \subseteq S$, $\nu A > 0$ and $f : A \to I$ is such that $f(s) \in s$ for every $s \in A$, then there is an $i \in I$ such that $\nu\{s : s \in A,\, f(s) = i\} > 0$.

545E Proposition NMA implies PMEA.

proof Assume NMA. Let λ be any cardinal. Let κ be a regular infinite cardinal greater than the cardinal power λ^ω, and ν a $\mathfrak{c}$-additive probability on $[\kappa]^{<\mathfrak{c}}$ as in 545D. For $\xi < \kappa$ define $f_\xi : [\kappa]^{<\mathfrak{c}} \to \mathfrak{c}$ by setting $f_\xi(s) = \operatorname{otp}(s \cap \xi)$ for every $s \in [\kappa]^{<\mathfrak{c}}$. Then if $\xi < \eta < \kappa$ we have $f_\xi(s) < f_\eta(s)$ whenever $\xi \in s$, that is, for ν-almost every s.

Let $g : \mathfrak{c} \to \mathcal{P}\mathbb{N}$ be any injection. For $\xi < \kappa$ and $n \in \mathbb{N}$ let $a_{\xi n}$ be the equivalence class $\{s : n \in g(f_\xi(s))\}^\bullet$ in the measure algebra $\mathfrak{A}$ of ν. If $\xi < \eta < \kappa$ then $g(f_\xi(s)) \neq g(f_\eta(s))$ for ν-almost every s, so $\sup_{n\in\mathbb{N}} a_{\xi n} \mathbin{\triangle} a_{\eta n} = 1$ in $\mathfrak{A}$ and there is an $n \in \mathbb{N}$ such that $a_{\xi n} \neq a_{\eta n}$. Accordingly $\#(\mathfrak{A})^\omega \geq \kappa > \lambda^\omega$ and $\#(\mathfrak{A}) > \lambda^\omega$. As $\#(\mathfrak{A}) \leq \max(4, \tau(\mathfrak{A})^\omega)$ (4A1O/514De), $\tau(\mathfrak{A}) > \lambda$. So ν witnesses that 545A(i) is true of λ.

545F Proposition Suppose that NMA is true. Let $\mathfrak{A}$ be a Boolean algebra such that whenever $s \in [\mathfrak{A}]^{<\mathfrak{c}}$ there is a subalgebra $\mathfrak{B} \subseteq \mathfrak{A}$, including s, with a strictly positive countably additive functional. Then there is a strictly positive countably additive functional on $\mathfrak{A}$.

Remark For the definition and elementary properties of countably additive functionals on arbitrary Boolean algebras, see §326.

proof Of course we can suppose that $\mathfrak{A} \neq \{0\}$. Let ν be a $\mathfrak{c}$-additive probability on $S = [\mathfrak{A}]^{<\mathfrak{c}}$ as in 545D. For each $s \in S$, let $\mathfrak{B}_s$ be a subalgebra of $\mathfrak{A}$ including s with a strictly positive countably additive functional μ_s. Normalizing μ_s if necessary, we may suppose that $\mu_s 1 = 1$. Now, for $a \in \mathfrak{A}$, set $\mu(a) = \int \mu_s(a)\nu(ds)$; because $a \in s \subseteq \mathfrak{B}_s = \operatorname{dom}\mu_s$ for ν-almost every s, the integral is well-defined. Because every μ_s is additive, so is μ; because every μ_s is strictly positive, so is μ. If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0, then $\lim_{n\to\infty} \mu_s(a_n) = 0$ whenever $s \in [\mathfrak{A}]^{<\mathfrak{c}}$ contains every a_n, that is, for ν-almost every s; so $\lim_{n\to\infty} \mu a_n = 0$. As $\langle a_n\rangle_{n\in\mathbb{N}}$ is arbitrary, μ is countably additive (326Ka).

545G Corollary Suppose that NMA is true. Let $\mathfrak{A}$ be a Boolean algebra such that every $s \in [\mathfrak{A}]^{<\mathfrak{c}}$ is included in a subalgebra of $\mathfrak{A}$ which is, in itself, a measurable algebra. Then $\mathfrak{A}$ is a measurable algebra.

proof (a) Because $\mathfrak{c}$ is atomlessly-measurable, it is surely greater than ω_1 (419G/438Cd/542C). So a family in $\mathfrak{A} \setminus \{0\}$ with cardinal ω_1 lies within some measurable subalgebra of $\mathfrak{A}$ and cannot be disjoint. Thus $\mathfrak{A}$ is ccc.

(b) If $A \subseteq \mathfrak{A}$, set

$$D_1 = \{d : d \in \mathfrak{A},\, d \subseteq a \text{ for some } a \in A\},$$

$$D_2 = \{d : d \in \mathfrak{A},\, d \cap a = 0 \text{ for every } a \in A\}.$$

Then $D_1 \cup D_2$ is order-dense in $\mathfrak{A}$ so includes a partition D of unity in $\mathfrak{A}$. By (a), D is countable, so lies within a measurable subalgebra $\mathfrak{B}$ of $\mathfrak{A}$. Now $D \cap D_1$ has a supremum b in $\mathfrak{B}$ which is disjoint from every member of $D \cap D_2$. But this means that b is the supremum of A in $\mathfrak{A}$. As A is arbitrary, $\mathfrak{A}$ is Dedekind complete.

(c) By 545F, $\mathfrak{A}$ has a strictly positive countably additive functional μ; but now $(\mathfrak{A}, \mu)$ is a totally finite measure algebra.

545X Basic exercises (a) Suppose that I is a set, and that ν is a $\mathfrak{c}$-additive probability measure with domain $\mathcal{P}([I]^{<\mathfrak{c}})$ satisfying the conditions of 545D. Suppose that $A \subseteq [I]^{<\mathfrak{c}}$ and $f : A \to I$ are such that $f(s) \in s$ for every $s \in A$. Show that there is a countable set $D \subseteq I$ such that $f(s) \in D$ for ν-almost every $s \in A$.

545Y Further exercises (a) Suppose that I is a set, and that ν is a $\mathfrak{c}$-additive probability measure with domain $\mathcal{P}S$, where $S = [I]^{<\mathfrak{c}}$, satisfying the conditions of 545D. Suppose that $f : [I]^{<\omega} \to S$ is any function. Show that $\nu\{s : s \in S,\ f(J) \subseteq s \text{ for every } J \in [s]^{<\omega}\} = 1$.

(b) Suppose that NMA is true. Show that $\square_\lambda$ is false for every $\lambda \geq \mathfrak{c}$. (Cf. 555Yf below.)

545 Notes and comments I have given the sketchiest of accounts here. The main interest of PMEA and NMA has so far been in their remarkable consequences in general topology and (for NMA) its associated reflection principles; see Fremlin 93 and the references there. 545G is such a reflection principle. Note that the measurable subalgebras declared to exist need not be regularly embedded in the given algebra. For K.Prikry's theorem that it is consistent to assume NMA if it is consistent to suppose that there is a supercompact cardinal, see 555N below.

Version of 3.2.21

546 Power set σ-quotient algebras

One way of interpreting the Gitik-Shelah theorem (543E) is to say that it shows that 'simple' atomless probability algebras cannot be of the form $\mathcal{P}X/\mathcal{N}(\mu)$. Similarly, the results of §541-§542 show that any ccc Boolean algebra expressible as the quotient of a power set by a non-trivial σ-ideal involves us in dramatic complexities, though it is not clear when these must appear in the quotient algebra itself. In the next section I will present further examples of algebras which cannot appear in this way. To prepare for these I collect some general facts about quotients of power set algebras.

546A(a) Definition A **power set σ-quotient algebra** is a Boolean algebra which is isomorphic to an algebra of the form $\mathcal{P}X/\mathcal{I}$ where X is a set and $\mathcal{I}$ is a σ-ideal of subsets of X.

(b) A **normal power set σ-quotient algebra** is a Boolean algebra which is isomorphic to an algebra of the form $\mathcal{P}\kappa/\mathcal{I}$ where κ is a regular uncountable cardinal and $\mathcal{I}$ is a normal ideal of $\mathcal{P}\kappa$.

(c) I recall some notation which I will use in this section. As in §522, I will write $\operatorname{non}\mathcal{M}$ for the uniformity of the meager ideal of $\mathbb{R}$, $\operatorname{non}\mathcal{N}$ for the uniformity of the Lebesgue null ideal and $\operatorname{cov}\mathcal{N}$ for the covering number of the Lebesgue null ideal. If κ is a cardinal, ν_κ will be the usual measure on $\{0,1\}^\kappa$. $\mathcal{N}_\kappa$ its null ideal and $\mathfrak{B}_\kappa$ its measure algebra; $\mathfrak{G}_\kappa$ will be the category algebra of $\{0,1\}^\kappa$ (4A3R-4A3S[1]). Note that we know that the covering number and uniformity of $\mathcal{N}_\omega$ are $\operatorname{cov}\mathcal{N}$ and $\operatorname{non}\mathcal{N}$ respectively (522W(a-i)), while $\operatorname{non}\mathcal{M}$ is the uniformity of the meager ideal $\mathcal{M}_\omega$ of $\{0,1\}^{\mathbb{N}} = \{0,1\}^\omega$.

546B Proposition (a) Any power set σ-quotient algebra is Dedekind σ-complete.

(b) If $\mathfrak{A}$ is a power set σ-quotient algebra, $\mathfrak{B}$ is a Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{B}$ is a surjective sequentially order-continuous Boolean homomorphism, then $\mathfrak{B}$ is a power set σ-quotient algebra. In particular, any principal ideal of a power set σ-quotient algebra is a power set σ-quotient algebra.

(c) The simple product of any family of power set σ-quotient algebras is a power set σ-quotient algebra.

proof (a) 314C.

(b) Observe that by 313Qb a Boolean algebra $\mathfrak{A}$ is a power set σ-quotient algebra iff there are a set X and a surjective sequentially order-continuous Boolean homomorphism from $\mathcal{P}X$ onto $\mathcal{A}$. It follows immediately that an image of such an algebra under a sequentially order-continuous homomorphism is again a power set

[1]Formerly 4A3Q-4A3R.

σ-quotient algebra. And of course a principal ideal of $\mathfrak{A}$ is an image of $\mathfrak{A}$ under a homomorphism $a \mapsto a \cap c$ which is actually order-continuous.

(c) If $\langle \mathfrak{A}_i \rangle_{i \in I}$ is a family of power set σ-quotient algebras, then for each $i \in I$ we have a set X_i and a surjective sequentially order-continuous Boolean homomorphism $\phi_i : \mathcal{P}X_i \to \mathfrak{A}_i$. We can arrange that the X_i are disjoint; set $X = \bigcup_{i \in I} X_i$. Now $A \mapsto \langle \phi_i(A \cap X_i) \rangle_{i \in I} : \mathcal{P}X \to \prod_{i \in I} \mathfrak{A}_i$ is a surjective sequentially order-continuous Boolean homomorphism, so $\prod_{i \in I} \mathfrak{A}_i$ is a power set σ-quotient algebra.

546C Proposition A non-zero principal ideal of a normal power set σ-quotient algebra is a normal power set σ-quotient algebra.

proof Let κ be a regular uncountable cardinal, $\mathcal{I}$ a normal ideal on κ and a a non-zero element of $\mathfrak{A} = \mathcal{P}\kappa/\mathcal{I}$. Let $A \subseteq \kappa$ be such that $A^\bullet = a$, and set $\mathcal{J} = \mathcal{I} \cap \mathcal{P}A$. Then $\mathcal{J}$ is a proper ideal of subsets of A and the principal ideal $\mathfrak{A}_a$ is isomorphic to $\mathcal{P}A/\mathcal{J}$. Because κ is regular and $\sup A = \kappa$, $\operatorname{otp} A = \kappa$ and we have an order-isomorphism $h : \kappa \to A$. If $S \in \mathcal{P}A \setminus \mathcal{J}$ and $f : S \to A$ is regressive, then $S \in \mathcal{P}\kappa \setminus \mathcal{I}$ so there is a $\beta < \kappa$ such that $f^{-1}[\{\beta\}] \notin \mathcal{I}$ (541H(iii)); now $\beta \in A$ and $f^{-1}[\{\beta\}] \notin \mathcal{J}$. By 541H in the other direction, $\mathcal{J}^* = \{h^{-1}[J] : J \in \mathcal{J}\}$ is a normal ideal on κ, and

$$\mathfrak{A}_a \cong \mathcal{P}A/\mathcal{J} \cong \mathcal{P}\kappa/\mathcal{J}^*$$

is a normal power set σ-quotient algebra.

546D You will observe that I do not claim that an order-closed subalgebra of a power set σ-quotient algebra is a power set σ-quotient algebra. See 546Xa. However, our power set σ-quotient algebras will often possess important power set σ-quotient subalgebras, as in the following.

Lemma Let X be a set, $\mathcal{I}$ a proper σ-ideal of subsets of X containing singletons, and $\mathfrak{A}$ the quotient algebra $\mathcal{P}X/\mathcal{I}$. Write κ for $\operatorname{add}\mathcal{I}$.

(a) There are an $a \in \mathfrak{A} \setminus \{0\}$, a σ-subalgebra $\mathfrak{C}$ of the principal ideal $\mathfrak{A}_a$ and a κ-additive ideal $\mathcal{J}$ of $\mathcal{P}\kappa$, containing singletons, such that $\mathfrak{C} \cong \mathcal{P}\kappa/\mathcal{J}$.

(b) If $\mathfrak{A}$ is atomless and ccc then $\kappa \le \mathfrak{c}$ is quasi-measurable, $\mathfrak{C}$ is atomless and we can arrange that $\mathcal{J}$ should be a normal ideal, so that $\mathfrak{C}$ is a normal power set σ-quotient algebra.

proof (a) Let $\langle Y_\xi \rangle_{\xi < \kappa}$ be a disjoint family in $\mathcal{I}$ with union $Y \notin \mathcal{I}$. Define $g : Y \to \kappa$ by setting $g(x) = \xi$ when $x \in Y_\xi$. Set

$$\mathcal{J} = \{B : B \subseteq \kappa,\, g^{-1}[B] \in \mathcal{I}\}.$$

Then $\mathcal{J}$ is a proper κ-additive ideal of subsets of κ containing singletons. The map $B \mapsto g^{-1}[B] : \mathcal{P}\kappa \to \mathcal{P}Y$ induces an injective sequentially order-continuous Boolean homomorphism from $\mathcal{P}\kappa/\mathcal{J}$ to the principal ideal of $\mathcal{P}X/\mathcal{I}$ generated by $a = Y^\bullet$, so we have a sequentially order-continuous embedding of $\mathcal{P}\kappa/\mathcal{J}$ into $\mathfrak{A}_a$; of course $a \ne 0$. By 314F(b-i), the image of $\mathcal{P}\kappa/\mathcal{J}$ is a σ-subalgebra of $\mathfrak{A}_a$.

(b) Of course κ is uncountable (because $\mathcal{I}$ is a σ-ideal) and not ∞ (because $X = \bigcup \mathcal{I} \notin \mathcal{I}$), so κ is regular (513C(a-i)). As $\mathfrak{A}$ is ccc, $\mathcal{I}$ is ω_1-saturated, by the definition in 541A, so κ is quasi-measurable (542B); as $\mathfrak{A}$ is atomless, $\kappa \le \mathfrak{c}$ (541O). So $\mathfrak{C} \cong \mathcal{P}\kappa/\mathcal{J}$ will be atomless, by 541P.

Returning to the argument of (a), as $\mathfrak{A}$ is ccc we have the option of using 541J to give us a function $g : A \to \kappa$ such that $\mathcal{J} = \{B : B \subseteq \kappa,\, g^{-1}[B] \in \mathcal{I}\}$ is a normal ideal, and then proceeding as before.

546E Proposition The measure algebra of Lebesgue measure on $\mathbb{R}$ is not a power set σ-quotient algebra.

proof This is really a very special case of 543E-543F. Let $\mathfrak{A}$ be the measure algebra of Lebesgue measure on $\mathbb{R}$. Then there is a functional $\bar{\mu}$ such that $(\mathfrak{A}, \bar{\mu})$ is a probability algebra. **?** If $\mathfrak{A}$ is a power set σ-quotient algebra, let X, $\mathcal{I}$ be such that $\mathcal{I}$ is a σ-ideal of $\mathcal{P}X$ and there is an isomorphism $\pi : \mathcal{P}X \to \mathfrak{A}$. Set $\mu E = \bar{\mu}(\pi E^\bullet)$ for $E \subseteq X$, so that $(X, \mathcal{P}X, \mu)$ is a probability space with null ideal $\mathcal{I}$ and measure algebra isomorphic to $\mathfrak{A}$. As $\mathfrak{A}$ is atomless, so is μ (322Bg). Write κ for $\operatorname{add}\mu$. By 543B(f-ii), κ is atomlessly-measurable. By 543F, the Maharam type $\tau(\mu)$ of μ is at least $\min(\kappa^{(+\omega)}, 2^\kappa)$; but $\tau(\mu)$ is defined to be the Maharam type of $\mathfrak{A}$, which is ω (331P[2], 331Xd). As there is surely no atomlessly-measurable cardinal less than ω, we have a contradiction. **X**

[2]Later editions only.

546F Definition Let $\mathfrak{A}$ be a Boolean algebra. I will say that an **e-h family** in $\mathfrak{A}$ is a double sequence $\langle e_{ij}\rangle_{i,j\in\mathbb{N}}$ in $\mathfrak{A}$ such that

$$\langle e_{ij}\rangle_{j\in\mathbb{N}} \text{ is disjoint for every } i\in\mathbb{N}$$

and

$$\sup_{i\in\mathbb{N}} e_{i,f(i)} = 1$$

for every $f\in\mathbb{N}^{\mathbb{N}}$.

Remark I hope you will find this definition simple enough to be manageable without any particular motivation. It is here for the sake of Lemma 546Ib, and will reappear in 547P. See also 546Xc-546Xd.

546G Lemma Let $\mathfrak{A}$ be a Boolean algebra and $\langle e_{ij}\rangle_{i,j\in\mathbb{N}}$ an e-h family in $\mathfrak{A}$. Then $\sup_{i\ge n} e_{i,f(i)} = 1$ for every $f\in\mathbb{N}^{\mathbb{N}}$ and $n\in\mathbb{N}$.

proof ? Otherwise, take $f\in\mathbb{N}^{\mathbb{N}}$, $n\in\mathbb{N}$ and $a\in\mathfrak{A}\setminus\{0\}$ such that $a\cap e_{i,f(i)}=0$ for every $i\ge n$. Choose $\langle a_i\rangle_{i\le n}$, $\langle g(i)\rangle_{i<n}$ such that $a_0=a$, $g(i)\in\mathbb{N}$ and $a_{i+1}=a_i\setminus e_{i,g(i)}$ is non-zero for every $i<n$. Set $g(i)=f(i)$ for $i\ge n$; then $g\in\mathbb{N}^{\mathbb{N}}$ and $a_n\cap e_{i,g(i)}=0$ for every $i\in\mathbb{N}$, which is supposed to be impossible. **X**

546H Free products and completed free products We shall need the following ideas from Volume 3. For Boolean algebras $\mathfrak{A}$, $\mathfrak{B}$ I write $\mathfrak{A}\otimes\mathfrak{B}$ for their free product (315N) and $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$ for its Dedekind completion (314U); for $a\in\mathfrak{A}$ and $b\in\mathfrak{B}$ I write $a\otimes b$ for the intersection of their canonical images in $\mathfrak{A}\otimes\mathfrak{B}$ (315N again). If $\mathfrak{A}_0$, $\mathfrak{B}_0$ are order-dense subalgebras of $\mathfrak{A}$ and $\mathfrak{B}$ respectively, then $\mathfrak{A}_0\otimes\mathfrak{B}_0$ is order-dense in $\mathfrak{A}\otimes\mathfrak{B}$ (315Kb) and therefore in the Dedekind complete Boolean algebra $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$, so $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$ can be identified with $\mathfrak{A}_0\widehat{\otimes}\mathfrak{B}_0$ (314Ub). Next, if $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ are Boolean algebras, then $(\mathfrak{A}\otimes\mathfrak{B})\otimes\mathfrak{C}\cong\mathfrak{A}\otimes(\mathfrak{B}\otimes\mathfrak{C})$ (315L), so $(\mathfrak{A}\widehat{\otimes}\mathfrak{B})\widehat{\otimes}\mathfrak{C}\cong\mathfrak{A}\widehat{\otimes}(\mathfrak{B}\widehat{\otimes}\mathfrak{C})$. Finally, for any set I, the algebra $\mathcal{E}_I$ of open-and-closed subsets of $\{0,1\}^I$ is order-dense in the regular open algebra $\mathfrak{G}_I$ of $\{0,1\}^I$ (314Uc[2]), while if I and J are disjoint then $\{0,1\}^{I\cup J}\cong\{0,1\}^I\times\{0,1\}^J$, so $\mathcal{E}_{I\cup J}\cong\mathcal{E}_I\otimes\mathcal{E}_J$ (315Ia) and $\mathfrak{G}_{I\cup J}\cong\mathfrak{G}_I\widehat{\otimes}\mathfrak{G}_J$; consequently $\mathfrak{G}_\kappa\cong\mathfrak{G}_\kappa\widehat{\otimes}\mathfrak{G}_\omega$ for any infinite κ.

546I In §544 I discussed the consequences of supposing that a non-trivial measurable algebra is a power set σ-quotient algebra, so that there is an atomlessly-measurable cardinal. Other types of power set σ-quotient algebra also entail facts about the cardinals $\operatorname{non}\mathcal{M}$, $\operatorname{non}\mathcal{N}$ and $\operatorname{cov}\mathcal{N}$.

In the following, I will write Z for $\{0,1\}^{\mathbb{N}}$, S_2 for $\bigcup_{n\in\mathbb{N}}\{0,1\}^n$ and for $\sigma\in S_2$ I will set $I_\sigma=\{z:\sigma\subseteq z\in Z\}$.

Lemma Let X be a set, $\mathcal{I}$ a proper σ-ideal of subsets of X, and $\mathfrak{A}$ the quotient algebra $\mathcal{P}X/\mathcal{I}$. Write κ for $\operatorname{add}\mathcal{I}$.

(a)(i) If $\mathfrak{A}$ has an atomless order-closed subalgebra which is a measurable algebra, then

(α) $\kappa\le\operatorname{non}\mathcal{M}$,

(β) $\operatorname{non}\mathcal{N}\le\#(X)$,

(γ) $\operatorname{cov}\mathcal{N}\ge\operatorname{cov}\mathcal{I}\ge\kappa$.

(ii) If $\mathfrak{G}_\omega$ can be regularly embedded in $\mathfrak{A}$, then $\kappa\le\operatorname{non}\mathcal{N}$.

(b) If $\mathfrak{A}$ has a non-trivial principal ideal with an e-h family, then $\operatorname{non}\mathcal{M}\le\#(X)$.

(c) Suppose that $\kappa\le\operatorname{non}\mathcal{M}$. Then there is an $a\in\mathfrak{A}\setminus\{0\}$ such that for every family $\langle a_{\xi n}\rangle_{\xi<\kappa,n\in\mathbb{N}}$ in $\mathfrak{A}$ there is a family $\langle c_{m\sigma}\rangle_{m\in\mathbb{N},\sigma\in S_2}$ in $\mathfrak{A}$ such that

$$\inf_{\sigma\in S_2} c_{m,\tau^\frown\sigma}=c_{m\tau},\quad \sup_{\sigma\in S_2} c_{m,\tau^\frown\sigma}=1$$

for every $\tau\in S_2$ and $m\in\mathbb{N}$, and

$$a\cap\inf_{m\in\mathbb{N}}\sup_{\sigma\in S_2}\bigl(c_{m\sigma}\cap\inf_{\substack{i<\#(\sigma)\\ \sigma(i)=1}}a_{\xi i}\setminus\sup_{\substack{i<\#(\sigma)\\ \sigma(i)=0}}a_{\xi i}\bigr)=0$$

for every $\xi<\kappa$.

(d) Suppose that $\kappa=\operatorname{non}\mathcal{N}$. Then $\mathfrak{A}$ has a countably generated order-closed subalgebra which is not a measurable algebra.

(e) Suppose that $\mathfrak{A}$ is isomorphic to $\mathfrak{C}\widehat{\otimes}\mathfrak{G}_\omega$ where $\mathfrak{C}$ is a ccc Boolean algebra. Then $\operatorname{cov}\mathcal{N} = \omega_1$.

proof (a)(i) Any non-trivial atomless measurable algebra includes a closed subalgebra which is isomorphic, as measure algebra, to $\mathfrak{B}_\omega$, so we have a regular embedding $\pi : \mathfrak{B}_\omega \to \mathfrak{A}$. Let $\langle e_i\rangle_{i\in\mathbb{N}}$ be the standard generating family in $\mathfrak{B}_\omega$ (525A); for $i \in I$ choose $E_i \subseteq X$ such that $\pi e_i = E_i^\bullet$ in $\mathfrak{A}$. Define $f : X \to Z$ by setting $f(x) = \langle\chi E_i(x)\rangle_{i\in\mathbb{N}}$ for $x \in X$. Then the set

$$\{V : V \subseteq Z \text{ is Borel, } f^{-1}[V]^\bullet = \pi V^\bullet\}$$

is a σ-algebra of sets containing every E_i, so is the Borel σ-algebra of Z.

($\boldsymbol{\alpha}$) Fix on a ν_ω-conegligible meager Borel subset V_0 of Z. Let $D \subseteq Z$ be any set with cardinal less than κ. For $z \in D$ set $A_z = f^{-1}[z + V_0]$, where $+$ here is the usual group operation of Z. Because ν_ω is the Haar probability measure on the compact Hausdorff group $(Z, +)$ (254Jd, 416U), $\nu_\omega(z + V_0) = 1$, $(z + V_0)^\bullet = 1$ in $\mathfrak{B}_\omega$ and $A_z^\bullet = 1$ in $\mathfrak{A}$, that is, $X \setminus A_z \in \mathcal{I}$. Because $\#(D) < \operatorname{add}\mathcal{I}$, $\bigcup_{z\in D} X \setminus A_z \in \mathcal{I}$ and $\bigcap_{z\in D} A_z \neq 0$. Take $x \in \bigcap_{z\in D} A_z$. If $z \in D$ then $f(x) \in z + V_0$ so $z \in f(x) + V_0$; thus $D \subseteq f(x) + V_0$. But $z \mapsto f(x) + z : Z \to Z$ is a homeomorphism so $f(x) + V_0$, like V_0, is meager, and D is meager. As D is arbitrary, $\kappa \le \operatorname{non}\mathcal{M}_\omega = \operatorname{non}\mathcal{M}$.

($\boldsymbol{\beta}$) On the other hand, if $W \subseteq Z$ is a ν_ω-negligible Borel set, $f^{-1}[W]^\bullet = 0$ and there is an $x \in X$ such that $f(x) \notin W$. So $f[X] \notin \mathcal{N}_\omega$ and $\operatorname{non}\mathcal{N} = \operatorname{non}\mathcal{N}_\omega \le \#(X)$.

($\boldsymbol{\gamma}$) If $\lambda < \operatorname{cov}\mathcal{I}$ and $\langle H_\xi\rangle_{\xi<\lambda}$ is a family of negligible Borel sets in Z, $f^{-1}[H_\xi]^\bullet = \pi H_\xi^\bullet = 0$ and $f^{-1}[H_\xi] \in \mathcal{I}$ for every $\xi < \lambda$. There is therefore an $x \in X$ such that $f(x) \notin H_\xi$ for every ξ. As $\langle H_\xi\rangle_{\xi<\lambda}$ is arbitrary,

$$\operatorname{cov}\mathcal{N} = \operatorname{cov}\mathcal{N}_\omega \ge \operatorname{cov}\mathcal{I} \ge \operatorname{add}\mathcal{I} \ge \kappa.$$

(ii) We can use essentially the same argument as in (i-α). This time, we have a regular embedding $\pi : \mathfrak{G}_\omega \to \mathfrak{A}$. For $i \in \mathbb{N}$ set $H_i = \{z : z \in Z,\ z(i) = 1\}$, as before, but now take e_i to be the equivalence class $H_i^\bullet \in \mathfrak{G}_\omega$; as before, choose $E_i \subseteq X$ such that $\pi e_i = E_i^\bullet$ in $\mathfrak{A}$. Again define $f : X \to Z$ by setting $f(x) = \langle\chi E_i(x)\rangle_{i\in\mathbb{N}}$ for $x \in X$. Once again, $f^{-1}[V]^\bullet = \pi V^\bullet\}$ for every Borel set $V \subseteq Z$. Now take a comeager ν_ω-negligible Borel subset V_0' of Z (e.g., the complement of the set V_0 chosen in (i-α)).

Once again, let $D \subseteq Z$ be any set with cardinal less than κ, and for $z \in D$ set $A_z = f^{-1}[z + V_0']$. As before, $A_z^\bullet = 1$ in $\mathfrak{A}$ for every $z \in D$, so there is an $x \in \bigcap_{z\in D} A_z$. Now $D \subseteq f(x)+V_0$, which is ν_ω-negligible. As D is arbitrary, $\operatorname{non}\mathcal{N} = \operatorname{non}\mathcal{N}_\omega \ge \kappa$.

(b) Let $\langle e_{ij}\rangle_{i,j\in\mathbb{N}}$ be an e-h family in $\mathfrak{A}_a$, where $a \in \mathfrak{A} \setminus \{0\}$. For each $i \in \mathbb{N}$ let $\langle E_{ij}\rangle_{j\in\mathbb{N}}$ be a disjoint sequence of subsets of X such that $E_{ij}^\bullet = e_{ij}$ for every $j \in \mathbb{N}$. For $x \in X$ set $z_x(i) = j$ if $j \in \mathbb{N}$ and $x \in E_{ij}$, 0 if $x \in X \setminus \bigcup_{j\in\mathbb{N}} E_{ij}$. Then $D = \{z_x : x \in X\}$ is a subset of $\mathbb{N}^\mathbb{N}$. If $g \in \mathbb{N}^\mathbb{N}$ and $E = \bigcap_{n\in\mathbb{N}}\bigcup_{i\ge n} E_{i,g(i)}$ then $E^\bullet = \inf_{n\in\mathbb{N}}\sup_{i\ge n} e_{i,f(i)} = 1$ (546G), so $E \neq \emptyset$. Take any $x \in E$; then $\{i : z_x(i) = g(i)\} = \{i : x \in E_{i,g(i)}\}$ is infinite.

Thus for every $g \in \mathbb{N}^\mathbb{N}$ there is an $f \in D$ such that $f \cap g$ is infinite. By 522Sc, $\operatorname{non}\mathcal{M} \le \#(D) \le \#(X)$.

(c) Fix a disjoint family $\langle Y_\xi\rangle_{\xi<\kappa}$ in $\mathcal{I}$ such that $Y = \bigcup_{\xi<\kappa} Y_\xi$ does not belong to $\mathcal{I}$. Set $a = Y^\bullet \in \mathfrak{A}\setminus\{0\}$. Given a family $\langle a_{\xi n}\rangle_{\xi<\kappa,n\in\mathbb{N}}$, choose for each $\xi < \kappa$ and $n \in \mathbb{N}$ a set $A_{\xi n} \subseteq X$ such that $A_{\xi n}^\bullet = a_{\xi n}$; for $\xi < \kappa$, define $f_\xi : X \to Z$ by setting $f_\xi(x) = \langle\chi A_{\xi n}(x)\rangle_{n\in\mathbb{N}}$ for every $x \in X$.

For $x \in X$, set

$$\begin{aligned} B_x &= \{f_\eta(x) : \eta < \xi\} \text{ if } \xi < \kappa \text{ and } x \in Y_\xi, \\ &= \emptyset \text{ otherwise.} \end{aligned}$$

Because $\kappa \le \operatorname{non}\mathcal{M} = \operatorname{non}\mathcal{M}_\omega$, B_x is meager in Z. We can therefore find a sequence $\langle G_{mx}\rangle_{m\in\mathbb{N}}$ of dense open subsets of Z such that $B_x \cap \bigcap_{m\in\mathbb{N}} G_{mx} = \emptyset$.

For $\sigma \in S_2$ and $m \in \mathbb{N}$, set

$$C_{m\sigma} = \{x : I_\sigma \subseteq G_{mx}\}, \quad c_{m\sigma} = C_{m\sigma}^\bullet$$

in $\mathfrak{A}$. If τ, $\sigma \in S_2$ and $m \in \mathbb{N}$, then $I_\tau \supseteq I_{\tau^\frown\sigma}$, $C_{m\tau} \subseteq C_{m,\tau^\frown\sigma}$ and $c_{m\tau} \subseteq c_{m,\tau^\frown\sigma}$. If $\tau \in S_2$, $m \in \mathbb{N}$ and $x \in X$ then there must be a $\sigma \in S_2$ such that $G_{mx} \supseteq I_{\tau^\frown\sigma}$, because G_{mx} is dense and open; so $\bigcup_{\sigma\in S_2} C_{m,\tau^\frown\sigma} = X$ and $\sup_{\sigma\in S_2} c_{m,\tau^\frown\sigma} = 1$, while $\inf_{\sigma\in S_2} c_{m,\tau^\frown\sigma} = c_{m\tau}$.

Now take any $\xi < \kappa$, $x \in Y \setminus \bigcup_{\eta\le\xi} Y_\eta$ and $\sigma \in S_2$. Then $f_\xi(x) \in B_x$ and there is an $m \in \mathbb{N}$ such that $f_\xi(x) \notin G_{mx}$. But this means that if $x \in C_{m\sigma}$ then $\sigma \not\subseteq f_\xi(x)$, that is, there is an $i < \#(\sigma)$ such that $\sigma(i) \ne f_\xi(x)(i) = \chi A_{\xi i}(x)$, that is,

$$x \notin C_{m\sigma} \cap \bigcap_{\substack{i<\#(\sigma)\\ \sigma(i)=1}} A_{\xi i} \setminus \bigcup_{\substack{i<\#(\sigma)\\ \sigma(i)=0}} A_{\xi i}.$$

Thus $Y \setminus \bigcup_{\eta\le\xi} Y_\eta$ is disjoint from

$$\bigcap_{m\in\mathbb{N}} \bigcup_{\sigma\in S_2} (C_{m\sigma} \cap \bigcap_{\substack{i<\#(\sigma)\\ \sigma(i)=1}} A_{\xi i} \setminus \bigcup_{\substack{i<\#(\sigma)\\ \sigma(i)=0}} A_{\xi i}).$$

Because every Y_η belongs to $\mathcal{I}$ and $\operatorname{add}\mathcal{I} = \kappa$, $(Y \setminus \bigcup_{\eta\le\xi} Y_\eta)^\bullet = a$, so

$$a \cap \inf_{m\in\mathbb{N}} \sup_{\sigma\in S_2} \bigl(c_{m\sigma} \cap \inf_{\substack{i<\#(\sigma)\\ \sigma(i)=1}} a_{\xi i} \setminus \sup_{\substack{i<\#(\sigma)\\ \sigma(i)=0}} a_{\xi i}\bigr) = 0,$$

as required.

(d) Take a non-negligible subset of Z of size κ, and enumerate it as $\langle z_\xi\rangle_{\xi<\kappa}$. For each $\xi < \kappa$, let V_ξ be a ν_ω-negligible G_δ set including $\{z_\eta : \eta \le \xi\}$; express V_ξ as $\bigcap_{n\in\mathbb{N}} \bigcup_{\sigma\in L_{n\xi}} I_\sigma$, where $L_{n\xi} \subseteq S_2$ for each n; we can do this in such a way that $\sum_{\sigma\in L_{n\xi}} 2^{-\#(\sigma)} \le 2^{-n}$ for every $n \in \mathbb{N}$.

Again let $\langle Y_\xi\rangle_{\xi<\kappa}$ be a disjoint family in $\mathcal{I}$ such that $Y = \bigcup_{\xi<\kappa} Y_\xi$ does not belong to $\mathcal{I}$. For $n \in \mathbb{N}$ and $\sigma \in S_2$, set $A_{n\sigma} = \bigcup\{Y_\xi : \xi < \kappa,\ \sigma \in L_{n\xi}\}$. Let $\mathfrak{B}$ be the order-closed subalgebra of $\mathfrak{A}$ generated by $\{A^\bullet_{n\sigma} : n \in \mathbb{N},\ \sigma \in S_2\}$. Of course $\mathfrak{B}$ is countably generated.

? Suppose, if possible, that $\mathfrak{B}$ is a measurable algebra; let $\bar\mu$ be such that $(\mathfrak{B}, \bar\mu)$ is a probability algebra. Set $\Sigma = \{E : E \subseteq X,\ E^\bullet \in \mathfrak{B}\}$ and $\mu E = \bar\mu E^\bullet$ for $E \in \Sigma$. Then (X, Σ, μ) is a probability space. In the product space $X \times Z$ consider the set

$$W = \bigcap_{n\in\mathbb{N}} \bigcup_{\sigma\in S_2} A_{n\sigma} \times I_\sigma \subseteq Y \times Z.$$

Then W is measured by the product measure $\mu \times \nu_\omega$.

If $\xi \le \zeta < \kappa$ and $n \in \mathbb{N}$, $z_\xi \in V_\zeta$ so there is a $\sigma \in L_{n\zeta}$ such that $z_\xi \in I_\sigma$; now $\{z_\xi\} \times Y_\zeta \subseteq I_\sigma \times A_{n\sigma}$. So for any $\xi < \kappa$,

$$W^{-1}[\{z_\xi\}] = \bigcap_{n\in\mathbb{N}} \bigcup_{\sigma\in S_2, z_\xi\in I_\sigma} A_{n\sigma} \supseteq \bigcup_{\xi\le\zeta<\kappa} Y_\zeta = Y \setminus \bigcup_{\eta<\xi} Y_\eta$$

has measure at least $\mu^*Y > 0$, because $Y^\bullet \ne 0$ and $(\bigcup_{\eta<\xi} Y_\eta)^\bullet = 0$ in $\mathfrak{A}$. Since $\{z_\xi : \xi < \kappa\} \notin \mathcal{N}_\omega$, W cannot be $(\mu \times \nu_\omega)$-negligible and there must be an $x \in X$ such that $\mu W[\{x\}] > 0$. As $W[\{x\}] \ne \emptyset$, $x \in Y$; let $\xi < \kappa$ be such that $x \in Y_\xi$. Then

$$W[\{x\}] = \bigcap_{n\in\mathbb{N}} \bigcup_{\sigma\in S_2, x\in A_{n\sigma}} I_\sigma = \bigcap_{n\in\mathbb{N}} \bigcup_{\sigma\in L_{n\xi}} I_\sigma = V_\xi;$$

but V_ξ is ν_ω-negligible. **X**

So we have a countably generated order-closed subalgebra of $\mathfrak{A}$ which is not measurable, as required.

(e) (see Bartoszyński & Judah 95, 3.3.12) As the regular open algebra $\mathfrak{G}$ of $\mathbb{N}^\mathbb{N}$ is isomorphic to $\mathfrak{G}_\omega$ (use 515Oa), we have a surjective sequentially order-continuous Boolean homomorphism $\pi : \mathcal{P}X \to \mathfrak{C}\widehat{\otimes}\mathfrak{G}$ with kernel $\mathcal{I}$. For each $n \in \mathbb{N}$ let $\langle H_{ni}\rangle_{i\in\mathbb{N}}$ enumerate the open-and-closed subsets of Z of measure at most 2^{-n}; let $\langle E_{ni}\rangle_{i\in\mathbb{N}}$ be a partition of X such that $1_\mathfrak{C} \otimes \{\beta : \beta \in \mathbb{N}^\mathbb{N},\ \beta(n) = i\} \in \mathfrak{C} \otimes \mathfrak{G}$ is equal to πE_{ni} for each i. For $x \in X$ set $H_x = \bigcap_{n\in\mathbb{N}} \bigcup_{i\ge n} H_{i,f_x(i)}$ where $f_x(n) = i$ if $x \in E_{ni}$. Then $H_x \in \mathcal{N}_\omega$. Write S for $\bigcup_{m\in\mathbb{N}} \mathbb{N}^m$, and for $\tau \in S$ write G_τ for $\{\beta : \tau \subseteq \beta \in \mathbb{N}^\mathbb{N}\} \in \mathfrak{G}$. Let $D \subseteq Z$ be a set with cardinal ω_1.

? If $D + H_x \ne Z$ for every $x \in X$, take $\alpha_x \in Z \setminus (D + H_x)$ for each x. For each $d \in D$, $\alpha_x + d \notin H_x$ for every x, so there is an $n_d \in \mathbb{N}$ such that $A_d = \{x : \alpha_x + d \notin \bigcup_{i\ge n_d} H_{i,f_x(i)}\}$ does not belong to $\mathcal{I}$; let

$a_d \in \mathfrak{C}\setminus\{0\}$, $\tau_d \in S$ be such that $a_d \otimes G_{\tau_d} \subseteq \pi A_d$. Let $n \in \mathbb{N}$, $m \in \mathbb{N}$, $\tau \in \mathbb{N}^m$ be such that $D' = \{d : d \in D, n_d = n, \tau_d \subseteq \tau\}$ is uncountable. Extending τ if necessary, we can arrange that $m \ge n$. Because $\mathfrak{C}$ is ccc, there is an infinite $B \subseteq D'$ such that $\inf_{d\in I} a_d \ne 0$ for every finite $I \subseteq B$ (this is trivial if $\mathfrak{C}$ is finite, and otherwise follows from 516Ld).

Let k be so large that $\frac{1+\ln k}{k} \le 2^{-m}$. Let $r \ge m$ be such that there is a set $F \subseteq \{d{\restriction}r : d \in B\}$ with $\#(F) = k$; let $I \in [B]^k$ be such that $F = \{d{\restriction}r : d \in I\}$. Consider the set

$$R = \{(u, v) : u,\ v \in \{0,1\}^r,\ u + v \in F\}$$

where $+$ here is the group operation on $\{0,1\}^r = \mathbb{Z}_2^r$. Then $\#(R[\{u\}]) = \#(R^{-1}[\{v\}]) = k$ for every u, $v \in \{0,1\}^r$. By 5A1Q, there is a $K \subseteq \{0,1\}^r$ such that $R[K] = \{0,1\}^r$ and $\#(K) \le \frac{2^r(1+\ln k)}{k} \le 2^{r-m}$. Now $\nu_\omega\{z : z{\restriction}r \in K\} = 2^{-r}\#(K) \le 2^{-m}$, so there is a $j \in \mathbb{N}$ such that $H_{mj} = \{z : z{\restriction}r \in K\}$. But this means that $I + H_{mj} = Z$. **P** If $\alpha \in Z$, $\alpha{\restriction}r \in R[K]$ and there are $u \in K$, $v \in F$ such that $\alpha{\restriction}r = u + v$. Let $d \in I$ be such that $v = d{\restriction}r$; then

$$(\alpha + d){\restriction}r = (u + v) + v = u \in K$$

so $\alpha + d \in H_{mj}$. **Q**

By the choice of B, $a = \inf_{d\in I} a_d$ is non-zero, so $a \otimes \{\beta : \beta(m) = j\} \ne 0$ and $A = \bigcap_{d\in I} A_d \cap E_{mj}$ does not belong to $\mathcal{I}$. We therefore have an $x \in A$, and $f_x(m) = j$. In this case, because $m \ge n = n_d$, $\alpha_x \notin d + H_{m,f_x(m)} = d + H_{mj}$ for every $d \in I$. But $I + H_{mj} = Z$. **X**

So there is an $x \in X$ such that $D + H_x = Z$. As $d + H_x \in \mathcal{N}_\omega$ for every $d \in D$, $\omega_1 = \operatorname{cov}\mathcal{N}_\omega = \operatorname{cov}\mathcal{N}$.

546J Theorem Let $\mathfrak{A}$ be a Boolean algebra such that $\mathfrak{B}_\omega$ can be regularly embedded in $\mathfrak{A}$ and $\mathfrak{A} \cong \mathfrak{C}\widehat{\otimes}\mathfrak{G}_\omega$ for some ccc Boolean algebra $\mathfrak{C}$. Then $\mathfrak{A}$ is not a power set σ-quotient algebra.

proof Note first that $\mathfrak{C} \otimes \mathfrak{G}_\omega$ is ccc because $\mathfrak{C}$ is ccc and $\mathfrak{G}_\omega$ satisfies Knaster's condition (516U), so $\mathfrak{A}$ is ccc (514Ee). Also $\mathfrak{A}$ is atomless (316Rb[3]). **?** If $\mathcal{I}$ is a σ-ideal of subsets of X and $\mathcal{P}X/\mathcal{I} \cong \mathfrak{A}$, then $\mathcal{I}$ is ω_1-saturated and must contain all singleton subsets of X, so $\operatorname{add}\mathcal{I}$ is quasi-measurable (542B again) and greater than ω_1 (542C), and $\operatorname{cov}\mathcal{I} > \omega_1$. Because $\mathfrak{B}_\omega$ can be regularly embedded in $\mathfrak{A}$, $\operatorname{cov}\mathcal{N} > \omega_1$ (546I(a-i-γ)). But this contradicts 546Ie. **X**

546K Corollary (A.Kumar, private communication, January 2016) If λ, κ are infinite cardinals then $\mathfrak{A} = \mathfrak{B}_\lambda\widehat{\otimes}\mathfrak{G}_\kappa$ is not a power set σ-quotient algebra.

proof $\mathfrak{B}_\omega$ can be regularly embedded in $\mathfrak{B}_\lambda$ which can be regularly embedded in $\mathfrak{A}$ (315Kc), so $\mathfrak{B}_\omega$ can be regularly embedded in $\mathfrak{A}$ (313N). On the other hand, $\mathfrak{A}$ is ccc (being the Dedekind completion of the free product of Boolean algebras satisfying Knaster's condition) and $\mathfrak{G}_\kappa \cong \mathfrak{G}_\kappa\widehat{\otimes}\mathfrak{G}_\omega$. Accordingly

$$\mathfrak{A} = \mathfrak{B}_\lambda\widehat{\otimes}\mathfrak{G}_\kappa \cong \mathfrak{B}_\lambda\widehat{\otimes}(\mathfrak{G}_\kappa\widehat{\otimes}\mathfrak{G}_\omega) \cong (\mathfrak{B}_\lambda\widehat{\otimes}\mathfrak{G}_\kappa)\widehat{\otimes}\mathfrak{G}_\omega = \mathfrak{A}\widehat{\otimes}\mathfrak{G}_\omega$$

and we can apply 546J.

546X Basic exercises (a) Let $\mathfrak{A}$ be any Dedekind complete ccc Boolean algebra. Show that it can be regularly embedded in a power set σ-quotient algebra. (*Hint*: 314M.)

(b) Show that the measure algebra of the usual measure on $\{0,1\}^{\mathfrak{b}}$ is not a power set σ-quotient algebra. (*Hint*: 544N.)

>(c) Show that $\mathfrak{G}_\omega$ has an e-h family. (*Hint*: for i, $j \in \mathbb{N}$ set $E_{ij} = \{x : x \in \{0,1\}^{\mathbb{N}}, x(i+k) = 0$ for $k < j$, $x(i+j) = 1\}$.)

(d) Let $\mathfrak{A}$ be a weakly (σ,∞)-distributive Boolean algebra, not $\{0\}$. Show that there is no e-h family in $\mathfrak{A}$.

[3]Formerly 316Xi.

(e) In 546I, show that if $\kappa = \operatorname{non}\mathcal{M}$ then $\mathfrak{A}$ has a countably generated order-closed subalgebra which is not isomorphic to $\mathfrak{G}_\omega$.

546Y Further exercises (a) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, and set $X = \mathbb{N}^{\mathbb{N}}$. Let $\mathbb{P}$ be the forcing notion $(\mathfrak{A} \setminus \{0\}, \subseteq, 1, \downarrow)$ (5A3M). Show that $\mathfrak{A}$ has an e-h family iff

$$\Vdash_{\mathbb{P}} \text{ there is an } \alpha \in \mathbb{N}^{\mathbb{N}} \text{ such that } \alpha \cap \beta \text{ is infinite for every } \beta \in \check{X}.$$

(b) Suppose that $\mathfrak{A}$ is a Boolean algebra, not $\{0\}$, and that κ is a cardinal such that for every family $\langle a_{\xi n}\rangle_{\xi<\kappa,n\in\mathbb{N}}$ in $\mathfrak{A}$ there is a family $\langle c_{m\sigma}\rangle_{m\in\mathbb{N},\sigma\in S_2}$ in $\mathfrak{A}$ such that

$$c_{m\tau} \subseteq c_{m,\tau^\frown\sigma} \text{ for every } \sigma \in S_2, \quad \sup_{\sigma\in S_2} c_{m,\tau^\frown\sigma} = 1$$

for every $\tau \in S_2$ and $m \in \mathbb{N}$, and

$$\inf_{m\in\mathbb{N}} \sup_{\sigma\in S_2} \bigl(c_{m\sigma} \cap \inf_{\substack{i<\#(\sigma)\\ \sigma(i)=1}} a_{\xi i} \setminus \sup_{\substack{i<\#(\sigma)\\ \sigma(i)=0}} a_{\xi i}\bigr) = 0$$

for every $\xi < \kappa$. Let $\mathbb{P}$ be the forcing notion $(\mathfrak{A} \setminus \{0\}, \subseteq, 1, \downarrow)$. Show that $\Vdash_{\mathbb{P}} \operatorname{non}\mathcal{M} > \check{\kappa}$.

(c) Let $\mathfrak{A}$ be an atomless ccc Dedekind complete Boolean algebra, not $\{0\}$. Suppose that every order-closed subalgebra of $\mathfrak{A}$ with countable Maharam type is purely atomic. (See 539P.) Show that $\mathfrak{A}$ is not a power set σ-quotient algebra.

546 Notes and comments The substantial ideas of this section are all in 546I. The results there were developed in the process of understanding the forcing universes associated with power set σ-quotient algebras; I have spelt out two of the translations involved in 546Ya-546Yb. I don't present them in this form in the main exposition because none of the principal theorems of the present chapter rely on the Forcing Theorem, and it is therefore not strictly necessary to know anything about forcing to follow the proofs as written. But it is hard to imagine that anyone would have come to the formulae in 546Ic without having first considered whether 546Yb could be true. Let me emphasize that the methods of 546I are very special to quotient algebras of power sets.

An obvious question arises immediately from the definition in 546Aa. For a given power set σ-quotient algebra $\mathfrak{A}$, how much scope for variation is there in the pairs $(X, \mathcal{I})$ such that $\mathfrak{A} \cong \mathcal{P}X/\mathcal{I}$? In view of 546I, it seems natural to begin with the cardinals $\#(X)$ and $\operatorname{add}\mathcal{I}$. But I do not have even a well-formed problem to pose here. It may make a difference if we restrict our attention to normal power set σ-quotient algebras. I introduce these because we shall see in the next section that the category algebra $\mathfrak{G}_{\omega_2}$ is not a normal power set σ-quotient algebra (547F), but conceivably is a power set σ-quotient algebra.

Version of 24.10.20

547 Cohen algebras and σ-measurable algebras

I examine the conditions under which two classes of algebra can be power set σ-quotient algebras. In the first, shorter, part of the section (547B-547G) I look at Cohen algebras. I then turn to 'σ-measurable' algebras (547H-547S).

547A Notation If I is a set, $\mathfrak{G}_I$ will be the category algebra of $\{0,1\}^I$. If $\mathcal{I}$ is an ideal of subsets of X and $\mathcal{J}$ an ideal of subsets of Y, then

$$\mathcal{I} \ltimes \mathcal{J} = \{W : W \subseteq X \times Y,\ \{x : W[\{x\}] \notin \mathcal{J}\} \in \mathcal{I}\},$$

$$\mathcal{I} \rtimes \mathcal{J} = \{W : W \subseteq X \times Y,\ \{y : W^{-1}[\{y\}] \notin \mathcal{I}\} \in \mathcal{J}\}$$

(527B). If X is a topological space, $\mathcal{M}(X)$ will be its meager ideal, $\mathcal{B}\mathfrak{a}(X)$ its Borel σ-algebra and $\widehat{\mathcal{B}}(X)$ its Baire-property algebra. If $\mathfrak{A}$ is a Boolean algebra, $\mathfrak{C}$ is a subalgebra of $\mathfrak{A}$ and $a \in \mathfrak{A}$, then $\operatorname{upr}(a, \mathfrak{C}) = \min\{c : a \subseteq c \in \mathfrak{C}\}$ if this is defined (313S).

I will write S_2 for $\bigcup_{n\in\mathbb{N}}\{0,1\}^n$, ordered by $\subseteq$. For $\sigma \in S_2$, set $I_\sigma = \{z : \sigma \subseteq z \in \{0,1\}^{\mathbb{N}}\}$; then σ, $\tau \in S_2$ are (upwards) incompatible iff neither extends the other iff $I_\sigma \cap I_\tau = \emptyset$, while $\{I_\sigma : \sigma \in S_2\}$ is a base for the usual topology of $\{0,1\}^{\mathbb{N}}$.

547B In 527M I introduced 'harmless' algebras. Here we need to know a little about harmless power set σ-quotient algebras.

Lemma Suppose that κ is a regular uncountable cardinal and $\mathcal{I}$ is a κ-additive ideal of subsets of κ such that $\mathcal{P}\kappa/\mathcal{I}$ is harmless. Then $\mathfrak{A} = \mathcal{P}(\kappa \times \kappa)/\mathcal{I} \ltimes \mathcal{I}$ is harmless.

proof (a) If $\kappa = \omega_1$ then $\mathcal{I}$ is of the form $\{I : A \cap I = \emptyset\}$ for some countable $A \subseteq \kappa$. **P** Set $A = \{\xi : \xi < \omega_1, \{\xi\} \notin \mathcal{I}\}$. Because $\mathcal{P}\omega_1/\mathcal{I}$ is harmless, it is ccc, so A is countable. Set $\mathcal{J} = \mathcal{I} \cap \mathcal{P}(\omega_1 \setminus A)$, so that $\mathcal{J}$ is a σ-ideal of subsets of $\omega_1 \setminus A$ containing singletons, and $\mathcal{P}(\omega_1 \setminus A)/\mathcal{J}$ can be identified with the principal ideal of $\mathcal{P}\omega_1/\mathcal{I}$ generated by $(\omega_1 \setminus A)^\bullet$, so is ccc. Since ω_1 is certainly not weakly inaccessible, $\mathcal{J}$ is not a proper ideal, by 541L, and $\omega_1 \setminus A \in \mathcal{I}$. It follows that $\mathcal{I} = \mathcal{P}(\omega_1 \setminus A)$, as stated. **Q**

In this case, $\mathfrak{A} \cong \mathcal{P}(A \times A)$ has a countable π-base and is harmless, by 527Nd. So let us suppose from now on that $\kappa > \omega_1$.

(b) By 527Bb, $\mathcal{I} \ltimes \mathcal{I}$ is κ-additive; in particular, it is a σ-ideal. Next, it is ω_1-saturated. **P** Let $\langle V_\alpha\rangle_{\alpha<\omega_1}$ be a disjoint family in $\mathcal{P}(\kappa \times \kappa)$. For each $\xi < \kappa$, there is an $\alpha_\xi < \omega_1$ such that $V_\alpha[\{\xi\}] \in \mathcal{I}$ for every $\alpha \geq \alpha_\xi$. Because $\mathcal{I}$ is ω_2-additive and ω_1-saturated, there is an $\alpha^* < \omega_1$ such that $\{\xi : \alpha_\xi > \alpha^*\} \in \mathcal{I}$ (541E). But now $V_\alpha \in \mathcal{I} \ltimes \mathcal{I}$ for every $\alpha > \alpha^*$. **Q**

Consequently $\mathfrak{A}$ is Dedekind complete (541B).

(c) Let $\mathfrak{C}$ be an order-closed subalgebra of $\mathfrak{A}$ with countable Maharam type; let $\langle C_n\rangle_{n\in\mathbb{N}}$ be a sequence of subsets of $\kappa \times \kappa$ such that $\mathfrak{C}$ is the order-closed subalgebra of $\mathfrak{A}$ generated by $\{C_n^\bullet : n \in \mathbb{N}\}$. For each $\xi < \kappa$, let $\mathfrak{B}_\xi$ be the order-closed subalgebra of $\mathcal{P}\kappa/\mathcal{I}$ generated by $\{C_n[\{\xi\}]^\bullet : n \in \mathbb{N}\}$. By 527Nb, $\mathfrak{B}_\xi$ has countable π-weight; let $\mathcal{E}_\xi$ be a countable subset of $\mathcal{P}\kappa$ such that $\{E^\bullet : E \in \mathcal{E}_\xi\}$ is an order-dense subset of $\mathfrak{B}_\xi$; let $\langle E_{\xi n}\rangle_{n\in\mathbb{N}}$ run over $\mathcal{E}_\xi$. We can of course suppose that $C_n[\{\xi\}] \in \mathcal{E}_\xi$ and that $E_{\xi,2n} = C_n[\{\xi\}]$ for each n. For $n \in \mathbb{N}$ set $E_n = \{(\xi, \eta) : \xi < \kappa, \eta \in E_{\xi n}\}$; we have $E_{2n} = C_n$ for each n. Let $\mathfrak{B}$ be the order-closed subalgebra of $\mathcal{P}\kappa/\mathcal{I}$ generated by $\{\{\xi : E_{\xi m} \cap E_{\xi n} \in \mathcal{I}\}^\bullet : m, n \in \mathbb{N}\}$, and $\mathcal{F}$ a countable subset of $\mathcal{P}\kappa$, containing κ and $\emptyset$, such that $\{F^\bullet : F \in \mathcal{F}\}$ is an order-dense set in $\mathfrak{B}$. Let $\mathcal{A}$ be the countable set $\{\emptyset\} \cup \{E_m \cap (F \times \kappa) : m \in \mathbb{N}, F \in \mathcal{F}\}$.

(d) If $\langle A_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\mathcal{A}$, there is a sequence $\langle \hat{A}_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{A}$ such that $\sup_{n\in\mathbb{N}} \hat{A}_n^\bullet$ is the complement of $\sup_{n\in\mathbb{N}} A_n^\bullet$ in $\mathfrak{A}$. **P** Express A_n as $E_{m_n} \cap (F_n \times \kappa)$ where $m_n \in \mathbb{N}$ and $F_n \in \mathcal{F}$ for each n. Set $W = \bigcup_{n\in\mathbb{N}} A_n$. If $\xi < \kappa$ then $W[\{\xi\}] = \bigcup_{n\in\mathbb{N},\xi\in F_n} E_{\xi m_n}$, so $W[\{\xi\}]^\bullet \in \mathfrak{B}_\xi$; set $J_\xi = \{j : W[\{\xi\}] \cap E_{\xi j} \in \mathcal{I}\}$. If $G_j = \{\xi : \xi < \kappa, j \in J_\xi\}$ for each j and $\hat{W} = \bigcup_{j\in\mathbb{N}} E_j \cap (G_j \times \kappa)$, $W[\{\xi\}]^\bullet$ and $\hat{W}[\{\xi\}]^\bullet$ are complementary elements of $\mathfrak{B}_\xi$ for each ξ, so $W^\bullet$ and $\hat{W}^\bullet$ are complementary elements of $\mathfrak{A}$.

Now

$$G_j = \{\xi : W[\{\xi\}] \cap E_{\xi j} \in \mathcal{I}\} = \bigcap_{n\in\mathbb{N}}\{\xi : \xi \notin F_n \text{ or } E_{\xi m_n} \cap E_{\xi j} \in \mathcal{I}\},$$

so $G_j^\bullet \in \mathfrak{B}$ and there is an $\mathcal{F}_j \subseteq \mathcal{F}$ such that $G_j^\bullet = \sup_{F\in\mathcal{F}_j} F^\bullet$. Taking $\langle \hat{A}_n\rangle_{n\in\mathbb{N}}$ to run over $\{\emptyset\} \cup \{E_j \cap (F \times \kappa) : j \in \mathbb{N}, F \in \mathcal{F}_j\}$, we get a sequence in $\mathcal{A}$ such that $\sup_{n\in\mathbb{N}} \hat{A}_n^\bullet = \hat{W}^\bullet$, as required. **Q**

(e) It follows that if we take $\mathfrak{D}$ to be the set of those $a \in \mathfrak{A}$ expressible in the form $\sup_{n\in\mathbb{N}} A_n^\bullet$ for some sequence in $\mathcal{A}$, the complement of an element of $\mathfrak{D}$ belongs to $\mathfrak{D}$; as $\mathfrak{D}$ is certainly closed under countable suprema, it is a σ-subalgebra of $\mathfrak{A}$, therefore order-closed, because $\mathfrak{A}$ is ccc. And $\{A_n^\bullet : n \in \mathbb{N}\}$ witnesses that $\pi(\mathfrak{D}) \leq \omega$.

As $C_n = E_{2n} \cap (\kappa \times \kappa) \in \mathcal{A}$ for each n, $\mathfrak{C} \subseteq \mathfrak{D}$. So $\omega \geq \pi(\mathfrak{D}) \geq \pi(\mathfrak{C})$, by 514Eb.

As $\mathfrak{C}$ is an arbitrary countably generated order-closed subalgebra of $\mathfrak{A}$, $\mathfrak{A}$ is harmless, by 527Nb in the other direction.

547C I wish to follow the lines of the argument in 543C-543E to prove a similar theorem in which 'measure' is replaced by 'category'. The lemma just proved corresponds to the definition of $\tilde{\nu}$ in part (c) of the proof of 543D. The next result will play the role previously taken by 543C.

Proposition Suppose that κ is a regular uncountable cardinal and $\mathcal{I}$ is a κ-additive ideal of subsets of κ such that $\mathcal{P}\kappa/\mathcal{I}$ is harmless. Let X be a ccc topological space of π-weight less than κ. Then $\mathcal{M}(X) \rtimes \mathcal{I} \subseteq \mathcal{M}(X) \ltimes \mathcal{I}$.

proof (a) Take $C \in \mathcal{M}(X) \rtimes \mathcal{I}$. Set $A = \{\xi : \xi < \kappa,\, C^{-1}[\{\xi\}] \notin \mathcal{M}(X)\}$ and $B = \{x : x \in X,\, C[\{x\}] \notin \mathcal{I}\}$, so that $A \in \mathcal{I}$ and I need to show that $B \in \mathcal{M}(X)$. For $\xi \in \kappa \setminus A$, let $\langle F_{\xi n}\rangle_{n\in\mathbb{N}}$ be a sequence of nowhere dense sets in X with union $C^{-1}[\{\xi\}]$; for $\xi \in A$ set $F_{\xi n} = \emptyset$ for every n. For each n, set $C_n = \{(x,\xi) : \xi < \kappa,\, x \in F_{\xi n}\}$ and $B_n = \{x : C_n[\{x\}] \notin \mathcal{I}\}$, so that $B = \bigcup_{n\in\mathbb{N}} B_n$ and it will be enough to show that every B_n is meager. Fix $n \in \mathbb{N}$.

(b) Let $\langle G_\alpha\rangle_{\alpha<\pi(X)}$ enumerate a π-base for the topology of X, and for $\alpha < \pi(X)$ let D_α be the set of those $\xi < \kappa$ such that $G_\alpha \cap F_{\xi n} = \emptyset$. Then $W = \bigcup_{\alpha<\pi(X)} G_\alpha \times D_\alpha$ is disjoint from C_n. For each $\xi < \kappa$, set $I_\xi = \{\alpha : \alpha < \pi(X),\, \xi \in D_\alpha\}$; then $\bigcup_{\alpha\in I_\xi} G_\alpha$ is dense in X. Because X is ccc, there is a countable $J_\xi \subseteq I_\xi$ such that $\bigcup_{\alpha\in J_\xi} G_\alpha$ is dense (5A4Bd). Now $\mathcal{I}$ is ω_1-saturated and $\pi(X) < \operatorname{add}\mathcal{I}$, so there is a countable $I \subseteq \pi(X)$ such that $A' = \{\xi : \xi < \kappa,\, J_\xi \not\subseteq I\}$ belongs to $\mathcal{I}$ (541D).

(c) Let $\mathfrak{B}$ be the order-closed subalgebra of $\mathcal{P}\kappa/\mathcal{I}$ generated by $\{D_\alpha^\bullet : \alpha \in I\}$. Because $\mathcal{P}\kappa/\mathcal{I}$ is harmless, $\pi(\mathfrak{B}) \le \omega$ (527Nb again); let $\langle F_i\rangle_{i\in\mathbb{N}}$ be a sequence in $\mathcal{P}\kappa$ such that $\{F_i^\bullet : i \in \mathbb{N}\}$ is order-dense in $\mathfrak{B}$. Let $\mathcal{E}$ be the countable subalgebra of $\mathcal{P}\kappa$ generated by $\{F_i : i \in \mathbb{N}\} \cup \{D_\alpha : \alpha \in I\}$, and set $V = \bigcup(\mathcal{E} \cap \mathcal{I})$, so that $V \in \mathcal{I}$. Give $Y = \kappa \setminus V$ the second-countable topology generated by $\{E \setminus V : E \in \mathcal{E}\}$.

If H is a dense open set in Y, then $\kappa \setminus H \in \mathcal{I}$. **P** Setting $\mathcal{E}' = \{E : E \in \mathcal{E},\, E \setminus V \subseteq H\}$, $H = (\bigcup \mathcal{E}') \setminus V$, so $H^\bullet = \sup_{E\in\mathcal{E}'} E^\bullet$ in $\mathcal{P}\kappa/\mathcal{I}$, and $H^\bullet \in \mathfrak{B}$. **?** If $\kappa \setminus H \notin \mathcal{I}$, then $H^\bullet \ne 1$ and there is an $i \in \mathbb{N}$ such that $F_i^\bullet$ is non-zero and disjoint from $H^\bullet$. In this case, $F_i \cap E \in \mathcal{I}$ for every $E \in \mathcal{E}'$, so $F_i \setminus V$ is disjoint from H; but $F_i \setminus V$ is a non-empty open subset of Y. **XQ**

Consequently $\mathcal{M}(Y) \subseteq \mathcal{I}$.

(d) Set $W_0 = \bigcup_{\alpha\in I} G_\alpha \times (D_\alpha \setminus V)$, so that W_0 is an open set in $X \times Y$. Then W_0 is dense in $X \times Y$. **P?** Otherwise, we have a non-empty open set $G \subseteq X$ and a non-empty open set $U \subseteq Y$ such that $I = I' \cup I''$, where $I' = \{\alpha : \alpha \in I,\, G \cap G_\alpha = \emptyset\}$ and $I'' = \{\alpha : \alpha \in I,\, U \cap D_\alpha = \emptyset\}$. As U includes some non-empty set of the form $E \setminus V$ where $E \in \mathcal{E}$, $U \notin \mathcal{I}$. So there must be a $\xi \in U \setminus A'$. In this case, $J_\xi \subseteq I$ while $\bigcup_{\alpha\in J_\xi} G_\alpha$ is dense and meets G; there is therefore an $\alpha \in J_\xi \setminus I'$. But now $\alpha \in I_\xi$ so $\xi \in D_\alpha$, while also $\alpha \in I \setminus I' \subseteq I''$, so $U \cap D_\alpha = \emptyset$ and $\xi \notin D_\alpha$. **XQ**

(e) $W' = (X \times Y) \setminus W$ is therefore meager in $X \times Y$ and belongs to $\mathcal{M}(X) \ltimes \mathcal{M}(Y)$, by 527Db. If $x \in B_n$, then $C_n[\{x\}] \notin \mathcal{I}$; but $C_n[\{x\}] \setminus V \subseteq W'[\{x\}]$, so $W'[\{x\}] \notin \mathcal{I}$ and $W'[\{x\}] \notin \mathcal{M}(Y)$. Accordingly

$$B_n \subseteq \{x : W'[\{x\}] \notin \mathcal{M}(Y)\} \in \mathcal{M}(X),$$

as required.

547D Corollary Suppose that κ is a regular uncountable cardinal and $\mathcal{I}$ is a proper κ-additive ideal of subsets of κ, containing singletons, such that $\mathcal{P}\kappa/\mathcal{I}$ is harmless. Let X be a ccc topological space of π-weight less than κ.

(a) Suppose that $\langle A_\xi\rangle_{\xi<\kappa}$ is a non-decreasing family of subsets of X with union A. Then there is a $\theta < \kappa$ such that $E \cap A_\theta$ is non-meager whenever $E \subseteq X$ is a set with the Baire property and $E \cap A$ is not meager.

(b) If $\langle A_\xi\rangle_{\xi<\kappa}$ is a family in $\mathcal{P}X \setminus \mathcal{M}(X)$ such that $\#(\bigcup_{\xi<\kappa} A_\xi) < \kappa$, then there are distinct ξ, $\eta < \kappa$ such that $A_\xi \cap A_\eta \notin \mathcal{M}(X)$.

(c) If we have a family $\langle h_\xi\rangle_{\xi<\kappa}$ of functions such that $\operatorname{dom} h_\xi$ is a non-meager subset of X for each ξ and $\#(\bigcup_{\xi<\kappa} h_\xi) < \kappa$ (identifying each h_ξ with its graph), then there are distinct ξ, $\eta < \kappa$ such that $\{x : h_\xi(x)$ and $h_\eta(x)$ are defined and equal$\}$ is non-meager.

proof (a) Let $\mathfrak{G} = \widehat{\mathcal{B}}(X)/\mathcal{M}(X)$ be the category algebra of X; for $B \subseteq X$ set $\psi(B) = \inf\{E^\bullet : B \subseteq E \in \widehat{\mathcal{B}}(X)\}$, as in 514Ie. Because $\mathfrak{G}$ is ccc (514Ja), there is a sequence $\langle \xi_n\rangle_{n\in\mathbb{N}}$ in κ such that $\sup_{n\in\mathbb{N}} \psi(A_{\xi_n}) =$

$\sup_{\xi<\kappa}\psi(A_\xi)$; setting $\theta=\sup_{n\in\mathbb{N}}\xi_n$, we see that $\theta<\kappa$ (because $\mathrm{cf}\,\kappa>\omega$) and that $\psi(A_\xi)\subseteq\psi(A_\theta)$ for every $\xi<\kappa$.

? Suppose, if possible, that there is a set $E\in\widehat{\mathcal{B}}(X)$ such that $E\cap A_\theta$ is meager but $E\cap A$ is non-meager. Replacing E by $E\setminus A_\theta$ if necessary, we may suppose that $E\cap A_\theta$ is empty. If $\xi<\kappa$, then

$$\psi(E\cap A_\xi)\subseteq E^\bullet\cap\psi(A_\theta)\subseteq E^\bullet\cap(X\setminus E)^\bullet=0,$$

so $E\cap A_\xi$ is meager.

Define $f:E\cap A\to\kappa$ by setting $f(x)=\min\{\xi:x\in A_\xi\}$ for $x\in E$. Consider the set

$$C=\{(x,\xi):f(x)\le\xi<\kappa\}\subseteq(E\cap A)\times\kappa.$$

If $\xi<\kappa$, then

$$C^{-1}[\{\xi\}]=\{x:x\in E,\,f(x)\le\xi\}\subseteq E\cap A_\xi\in\mathcal{M}(X);$$

thus $C\in\mathcal{M}(X)\rtimes\mathcal{I}$. By 547C, $C\in\mathcal{M}(X)\ltimes\mathcal{I}$. As $E\cap A$ is not meager, there is an $x\in E\cap A$ such that $C[\{x\}]\in\mathcal{I}$. But $C[\{x\}]=\{\xi:f(x)\le\xi<\kappa\}\notin\mathcal{I}$. **X**

So θ has the required property.

(b) Write $\mathcal{J}=\mathcal{I}\ltimes\mathcal{I}\lhd\mathcal{P}(\kappa\times\kappa)$. By 547B, $\mathcal{P}(\kappa\times\kappa)/\mathcal{J}$ is harmless. Set

$$W=\{(x,\xi,\eta):\xi,\,\eta<\kappa,\,\xi\ne\eta,\,x\in A_\xi\cap A_\eta\}.$$

? If $A_\xi\cap A_\eta\in\mathcal{M}(X)$ for all distinct ξ, $\eta<\kappa$, then W, regarded as a subset of $X\times(\kappa\times\kappa)$, belongs to $\mathcal{M}(X)\rtimes\mathcal{J}$; by 547C, $W\in\mathcal{M}(X)\ltimes\mathcal{J}$. For $x\in X$ set $C_x=\{\xi:\xi<\kappa,\,x\in A_\xi\}$. Then $W[\{x\}]=C_x^2\setminus\Delta$, where $\Delta=\{(\xi,\xi):\xi<\kappa\}$. So $W[\{x\}]\in\mathcal{J}$ iff $C_x\in\mathcal{I}$, and $E=\{x:C_x\notin\mathcal{I}\}$ is meager. Next, $A=\bigcup_{\xi<\kappa}A_\xi$ is supposed to have cardinal less than κ, so $\bigcup_{x\in A\setminus E}C_x\in\mathcal{I}$ and there is some $\zeta\in\kappa\setminus\bigcup_{x\in A\setminus E}C_x$. But in this case $A_\zeta\subseteq E$ is meager. **X** So we have the result.

(c)(i) For each $\xi<\kappa$, set $A_\xi=\mathrm{dom}\,h_\xi$ and let H_ξ be the regular open set in X such that $A_\xi\setminus H_\xi$ is meager and $G\cap H_\xi$ is empty whenever G is open and $G\cap A_\xi$ is meager (4A3Sa[4]). Set $h'_\xi=h_\xi\restriction H_\xi$ and $Y=\bigcup_{\xi<\kappa}h'_\xi$; let $\pi_1:Y\to X$ be the first-coordinate projection. Give Y the topology $\mathfrak{S}=\{\pi_1^{-1}[G]:G\in\mathfrak{T}\}$, where $\mathfrak{T}$ is the topology of X.

(ii) If $\mathcal{U}$ is any π-base for $\mathfrak{T}$, then $\mathcal{V}=\{\pi_1^{-1}[U]:U\in\mathcal{U}\}$ is a π-base for $\mathfrak{S}$. **P** If $H\subseteq Y$ is open and not empty, take $G\in\mathfrak{T}$ such that $H=\pi_1^{-1}[G]$ and a $\xi<\kappa$ such that $H\cap h'_\xi\ne\emptyset$. Then $G\cap H_\xi\cap A_\xi=G\cap\mathrm{dom}\,h'_\xi$ is non-empty; by the choice of H_ξ, $G\cap H_\xi\cap A_\xi$ is non-meager. Set $\mathcal{U}'=\{U:U\in\mathcal{U},\,U\cap G\cap H_\xi\cap A_\xi=\emptyset\}$. Then $\bigcup\mathcal{U}'$ cannot be dense and there is a non-empty $U\in\mathcal{U}$ disjoint from $\bigcup\mathcal{U}'$. But now $U\cap G\ne\emptyset$, so there is a non-empty $U'\in\mathcal{U}$ with $U'\subseteq U\cap G$; in which case $V=\pi_1^{-1}[U']$ belongs to $\mathcal{V}$, is included in H and meets h'_ξ, so is not empty. As H is arbitrary, $\mathcal{V}$ is a π-base for $\mathfrak{S}$. **Q**

(iii) It follows at once that $\pi(Y)\le\pi(X)<\kappa$. We see also that if $A\subseteq X$ is nowhere dense, then $\{G:G\in\mathfrak{T},\,G\cap A=\emptyset\}$ is a π-base for $\mathfrak{T}$,

$$\{\pi_1^{-1}[G]:G\in\mathfrak{T},\,G\cap A=\emptyset\}=\{H:H\in\mathfrak{S},\,H\cap\pi_1^{-1}[A]=\emptyset\}$$

is a π-base for $\mathfrak{S}$ and $\pi_1^{-1}[A]$ is nowhere dense in Y. Accordingly $\pi_1^{-1}[A]\in\mathcal{M}(Y)$ for every $A\in\mathcal{M}(X)$.

(iv) If $B\subseteq Y$ is nowhere dense in Y then $\pi_1[B]$ is nowhere dense in X. **P** If $G\subseteq X$ is a non-empty open set, then either $\pi_1^{-1}[G]$ is empty and $G\cap\pi_1[B]=\emptyset$, or $\pi_1^{-1}[G]$ is a non-empty open subset of Y. In the latter case, $\pi_1^{-1}[G]\setminus\overline{B}$ must be of the form $\pi_1^{-1}[G']$ for some open set $G'\subseteq X$, and $G'\cap G$ is a non-empty open subset of G disjoint from $\pi_1[B]$. **Q** It follows at once that $\pi_1[B]\in\mathcal{M}(X)$ whenever $B\in\mathcal{M}(Y)$.

(v) Since $\pi_1[h'_\xi]=A_\xi\cap H_\xi$ is non-meager in X, h'_ξ is non-meager in Y, for every ξ. So (b) here tells us that there are distinct ξ, $\eta<\kappa$ such that $h'_\xi\cap h'_\eta$ is non-meager in Y. In this case, setting $A=\{x:h_\xi(x)$ and $h_\xi(y)$ are defined and equal$\}$, $\pi_1^{-1}[A]$ includes $h'_\xi\cap h'_\eta$ so is non-meager, and A is non-meager, by (iii).

547E I separate an elementary fact from the argument of the next theorem.

[4]Formerly 4A3Ra.

Lemma Suppose that X, I are sets, $\mathcal{I}$ is a σ-ideal of subsets of X and ϕ is a sequentially order-continuous Boolean homomorphism from $\mathfrak{G}_I$ to $\mathcal{P}X/\mathcal{I}$. Then there is a function $f : X \to \{0,1\}^I$ such that $f^{-1}[E]^\bullet = \phi E^\bullet$ in $\mathfrak{A}$ for every E in the Baire-property algebra $\widehat{\mathcal{B}}$ of $\{0,1\}^I$.

proof Write $\mathcal{M}$ for the meager ideal of $\{0,1\}^I$. For $i \in I$ set $H_i = \{y : y \in \{0,1\}^I, y(i) = 1\} \in \widehat{\mathcal{B}}$ and choose $F_i \subseteq X$ such that $F_i^\bullet = \phi H_i^\bullet$ in $\mathfrak{G} = \widehat{\mathcal{B}}/\mathcal{M}$. Set $f(x) = \langle \chi F_i(x)\rangle_{i\in I}$ for $x \in X$; then $f^{-1}[H_i]^\bullet = F_i^\bullet = \phi H_i^\bullet$ for every i. Since $H \mapsto f^{-1}[H]^\bullet$ and $H \mapsto \phi H^\bullet$ are both sequentially order-continuous Boolean homomorphisms, they must agree on the σ-algebra of subsets of $\{0,1\}^I$ generated by $\{H_i : i \in I\}$, which is the Baire σ-algebra $\mathcal{B}\mathfrak{a}$ of $\{0,1\}^I$ (4A3Of). Next, as $\{0,1\}^I$ is completely regular and ccc, $\mathcal{M}$ is generated by $\mathcal{B}\mathfrak{a} \cap \mathcal{M}$ (5A4E(c-ii)) and $f^{-1}[H] \in \mathcal{I}$ for every $H \in \mathcal{M}$. Now if $E \in \widehat{\mathcal{B}}$ it is of the form $G \triangle M$ where G is a cozero set and M is meager (5A4E(c-iii)), so

$$f^{-1}[E]^\bullet = f^{-1}[G]^\bullet \triangle f^{-1}[M]^\bullet = \phi G^\bullet = \phi E^\bullet.$$

547F The Gitik-Shelah theorem for Cohen algebras I come now to a companion result to the Gitik-Shelah Theorem in 543E. I follow the proof I gave before as closely as I can.

Theorem (Gitik & Shelah 89, Gitik & Shelah 93) Let κ be a regular uncountable cardinal and $\mathcal{I}$ a κ-additive ideal of subsets of κ such that $\mathfrak{A} = \mathcal{P}\kappa/\mathcal{I}$ is isomorphic to $\mathfrak{G}_\lambda$ for some infinite cardinal λ. Then $\lambda \geq \min(\kappa^{(+\omega)}, 2^\kappa)$.

proof (a) Let $\phi : \mathfrak{G}_\lambda \to \mathfrak{A}$ be an isomorphism. By 547E, we have a function $f : \kappa \to X$ such that $f^{-1}[E]^\bullet = \phi E^\bullet$ in $\mathfrak{A}$ for every $E \in \widehat{\mathcal{B}}(X)$.

(b)? Suppose, if possible, that $\lambda < \min(\kappa^{(+\omega)}, 2^\kappa)$.

As $\mathfrak{A}$ is atomless, ccc and not $\{0\}$, κ is quasi-measurable (542B) and at most $\mathfrak{c}$ (541P). So 543Ma tells us that if we set $\zeta = \max(\lambda^+, \kappa^+)$, there are an infinite cardinal $\delta < \kappa$, a stationary set $S \subseteq \zeta$, and a family $\langle g_\alpha\rangle_{\alpha\in S}$ of functions from κ to 2^δ such that $g_\alpha[\kappa] \subseteq \alpha$ for every $\alpha \in S$ and $\#(g_\alpha \cap g_\beta) < \kappa$ for distinct α, $\beta \in S$. Moreover, we can arrange that

— if $\lambda < \mathrm{Tr}(\kappa)$, then $g_\alpha[\kappa] \subseteq \kappa$ for every $\alpha \in S$;

— if $\lambda \geq \mathrm{Tr}(\kappa)$, than $g_\alpha\restriction\gamma = g_\beta\restriction\gamma$ whenever $\gamma < \kappa$ is a limit ordinal and α, $\beta \in S$ are such that $g_\alpha(\gamma) = g_\beta(\gamma)$.

(c) Fix an injective function $h : 2^\delta \to \{0,1\}^\delta$. For $\alpha \in S$ and $\iota < \delta$ set

$$U_{\alpha\iota} = \{\xi : \xi < \kappa, (hg_\alpha(\xi))(\iota) = 1\},$$

and let $H_{\alpha\iota} \in \mathcal{B}\mathfrak{a}(X)$ be such that $H_{\alpha\iota}^\bullet = \phi^{-1}(U_{\alpha\iota}^\bullet)$ in $\mathfrak{G}_\lambda$; then $U_{\alpha\iota} \triangle f^{-1}[H_{\alpha\iota}] \in \mathcal{I}$. Define $\tilde{g}_\alpha : X \to \{0,1\}^\delta$ by setting

$$\begin{aligned}(\tilde{g}_\alpha(x))(\iota) &= 1 \text{ if } x \in H_{\alpha\iota},\\ &= 0 \text{ otherwise.}\end{aligned}$$

Then

$$\begin{aligned}\{\xi : \xi < \kappa, \tilde{g}_\alpha f(\xi) \neq hg_\alpha(\xi)\} &= \bigcup_{\iota<\delta}\{\xi : (\tilde{g}_\alpha f(\xi))(\iota) \neq (hg_\alpha(\xi))(\iota)\}\\ &= \bigcup_{\iota<\delta} U_{\alpha\iota} \triangle f^{-1}[H_{\alpha\iota}] \in \mathcal{I}\end{aligned}$$

because $\delta < \kappa = \operatorname{add}\mathcal{I}$. Set $V_\alpha = \{\xi : \tilde{g}_\alpha f(\xi) = hg_\alpha(\xi)\}$, so that $\kappa \setminus V_\alpha \in \mathcal{I}$, for each $\alpha \in S$.

(d) Because every $H_{\alpha\iota}$ is determined by coordinates in a countable set, there is for each $\alpha \in S$ a set $I_\alpha \subseteq \lambda$ such that $\#(I_\alpha) \leq \delta$ and $H_{\alpha\iota}$ is determined by coordinates in I_α for every $\iota < \delta$. By 5A1K there is an $M \subseteq \lambda$ such that $S_1 = \{\alpha : \alpha \in S, I_\alpha \subseteq M\}$ is stationary in ζ and $\mathrm{cf}(\#(M)) \leq \delta$; because $\lambda < \kappa^{(+\omega)}$ and $\mathrm{cf}(\kappa) = \kappa > \delta$, $\#(M) < \kappa$. Set $\pi_M(z) = z\restriction M$ for $z \in X$, and $f_M = \pi_M f : \kappa \to \{0,1\}^M$.

If $E \subseteq \{0,1\}^M$ has the Baire property, then $\pi_M^{-1}[E]$ has the Baire property in X and $\pi_M^{-1}[E]$ is meager iff E is (5A4E(b-iii), applied to $\{0,1\}^M \times \{0,1\}^{\lambda\setminus M}$). So $f_M^{-1}[E] \in \mathcal{I}$ iff E is meager.

(e) For each $\alpha \in S_1$, there is a $\theta_\alpha < \kappa$ such that $f_M[V_\alpha \cap \theta_\alpha]$ meets every non-empty open subset of $\{0,1\}^M$ in a non-meager set. **P** Apply 547Da to $\langle f_M[V_\alpha \cap \xi]\rangle_{\xi<\kappa}$. $\bigcup \mathcal{I} = \kappa$ because $\mathfrak{A}$ is atomless, and $\kappa \notin \mathcal{I}$ because $\mathfrak{A} \neq \{0\}$; while $\{0,1\}^M$ is certainly ccc, and has π-weight at most $\max(\omega, \#(M)) < \kappa$. There is therefore a $\theta_\alpha < \kappa$ such that $E \cap f_M[V_\alpha \cap \theta]$ is non-meager whenever $E \subseteq \{0,1\}^M$ has the Baire property and $E \cap f_M[V_\alpha]$ is non-meager. If $G \subseteq \{0,1\}^M$ is a non-empty open set, then

$$f_M^{-1}[G \setminus f_M[V_\alpha]] \subseteq \kappa \setminus V_\alpha \in \mathcal{I},$$

so either $G \setminus f_M[V_\alpha]$ is meager or it does not have the Baire property; in either case, $G \cap f_M[V_\alpha]$ is non-meager so $G \cap f_M[V_\alpha \cap \theta_\alpha]$ is non-meager. **Q**

Evidently we may take it that every θ_α is a non-zero limit ordinal.

(f) By 543Mb, there are a $\theta < \kappa$ and a $Y \in [2^\delta]^{<\kappa}$ such that $S_2 = \{\alpha : \alpha \in S_1,\ \theta_\alpha = \theta\},\ g_\alpha[\theta] \subseteq Y\}$ is stationary in ζ.

(g) For each $\alpha \in S_2$, set

$$Q_\alpha = f_M[V_\alpha \cap \theta] = f_M[V_\alpha \cap \theta_\alpha],$$

so that Q_α meets every non-empty open subset of $\{0,1\}^M$ in a non-meager set. Now every $H_{\alpha\iota}$ is determined by coordinates in $I_\alpha \subseteq M$, so we can express $\tilde{g}_\alpha$ as $g_\alpha^* \pi_M$, where $g_\alpha^* : \{0,1\}^M \to \{0,1\}^\delta$ is Baire measurable in each coordinate. If $y \in Q_\alpha$, take $\xi \in V_\alpha \cap \theta$ such that $f_M(\xi) = y$; then

$$g_\alpha^*(y) = g_\alpha^* \pi_M f(\xi) = \tilde{g}_\alpha f(\xi) = h g_\alpha(\xi) \in h[Y].$$

Thus $g_\alpha^* \restriction Q_\alpha \subseteq f_M[\theta] \times h[Y]$ for every $\alpha \in S_2$, and we may apply 547Dc to $\{0,1\}^M$ and the family $\langle g_\alpha^* \restriction Q_\alpha \rangle_{\alpha \in S^*}$, where $S^* \subseteq S_2$ is a set with cardinal κ, to see that there are distinct α, $\beta \in S_2$ such that $\{y : y \in Q_\alpha \cap Q_\beta,\ g_\alpha^*(y) = g_\beta^*(y)\}$ is non-meager. Now, however, consider

$$E = \{y : y \in \{0,1\}^M,\ g_\alpha^*(y) = g_\beta^*(y)\}.$$

Then $E = \bigcap_{\iota<\delta} E_\iota$, where

$$E_\iota = \{y : y \in \{0,1\}^M,\ g_\alpha^*(y)(\iota) = g_\beta^*(y)(\iota)\}$$

is a Baire subset of $\{0,1\}^M$ for each $\iota < \delta$. Because $\delta < \kappa$ and $\mathcal{I}$ is κ-additive and ω_1-saturated,

$$\begin{aligned} f_M^{-1}[E]^\bullet &= (\bigcap_{\iota<\delta} f_M^{-1}[E_\iota])^\bullet = \inf_{\iota<\delta} f_M^{-1}[E_\iota]^\bullet \\ &= \inf_{\iota \in K} f_M^{-1}[E_\iota]^\bullet = f_M^{-1}[\bigcap_{\iota \in K} E_\iota]^\bullet \end{aligned}$$

for some countable $K \subseteq \delta$. In this case, $E' = \bigcap_{\iota \in K} E_\iota$ is a Baire set including E, and $f_M^{-1}[E' \setminus E] \in \mathcal{I}$; since E' includes the non-meager set $\{y : y \in Q_\alpha \cap Q_\beta,\ g_\alpha^*(y) = g_\beta^*(y)\}$, E' is non-meager and $f_M^{-1}[E'] \notin \mathcal{I}$, by (d) above; accordingly $f_M^{-1}[E] \notin \mathcal{I}$.

Consequently

$$\begin{aligned} \{\xi : g_\alpha(\xi) = g_\beta(\xi)\}^\bullet &= \{\xi : h g_\alpha(\xi) = h g_\beta(\xi)\}^\bullet \\ &= \{\xi : \xi \in V_\alpha \cap V_\beta,\ \tilde{g}_\alpha f(\xi) = \tilde{g}_\beta f(\xi)\}^\bullet \\ &= \{\xi : g_\alpha^* \pi_M f(\xi) = g_\beta^* \pi_M f(\xi)\}^\bullet = f_M^{-1}[E]^\bullet \neq 0 \end{aligned}$$

in $\mathfrak{A}$. But this is absurd, because in (b) above we chose g_α, g_β in such a way that $\{\xi : g_\alpha(\xi) = g_\beta(\xi)\}$ would be bounded in κ. **X**

Thus we have the required contradiction, and the theorem is true.

547G Corollary (a) $\mathfrak{G}_\omega$ is not a power set σ-quotient algebra.

(b) $\mathfrak{G}_{\omega_1}$ is not a power set σ-quotient algebra.

proof I shall be applying results from the last part of §515, where I spoke of regular open algebras; so we shall need the fact that $\mathfrak{G}_I$ is isomorphic to the regular open algebra of $\{0,1\}^I$ for every set I (515Na).

(a) **?** Suppose that $\mathfrak{G}_\omega$ is a power set σ-quotient algebra. By 546Db there are a regular uncountable cardinal κ and a normal ideal $\mathcal{J}$ on κ such that $\mathcal{P}\kappa/\mathcal{J}$ is isomorphic to an atomless σ-subalgebra $\mathfrak{D}$ of a principal ideal $(\mathfrak{G}_\omega)_c$ of $\mathfrak{G}_\omega$. By 515Nc and 515Ob, $(\mathfrak{G}_\omega)_c$ and $\mathfrak{D}$ are isomorphic to $\mathfrak{G}_\omega$. But by 547F this is impossible. **X**

(b) We can argue similarly, but using 515Q in place of 515Ob. **?** Suppose that $\mathfrak{G}_{\omega_1}$ is a power set σ-quotient algebra. Then there are a regular uncountable cardinal κ and a normal ideal $\mathcal{J}$ on κ such that $\mathcal{P}\kappa/\mathcal{J}$ is isomorphic to an atomless σ-subalgebra $\mathfrak{D}$ of a principal ideal $(\mathfrak{G}_{\omega_1})_c$ of $\mathfrak{G}_{\omega_1}$. By 515Nc, $(\mathfrak{G}_{\omega_1})_c$ is isomorphic to $\mathfrak{G}_{\omega_1}$. By 515Q, $\mathfrak{D}$ is isomorphic to either $\mathfrak{G}_\omega$ or $\mathfrak{G}_{\omega_1}$ or their simple product. But this means that at least one of $\mathfrak{G}_\omega$, $\mathfrak{G}_{\omega_1}$ is isomorphic to a principal ideal of the normal power set σ-quotient algebra $\mathfrak{D}$, and is therefore itself a normal power set σ-quotient algebra (546C); and both are ruled out by 547F. **X**

547H Definitions Let $\mathfrak{A}$ be a Boolean algebra.

(a) I will say that $\mathfrak{A}$ is **σ-measurable**, with **witnessing sequence** $\langle\mathfrak{B}_n\rangle_{n\in\mathbb{N}}$, if $\mathfrak{A}$ is Dedekind complete, every $\mathfrak{B}_n$ is an order-closed subalgebra of $\mathfrak{A}$ which is, in itself, a measurable algebra in the sense of §391, and $\bigcup_{n\in\mathbb{N}}\mathfrak{B}_n$ is order-dense in $\mathfrak{A}$ in the sense of 313J.

(b) If $\mathfrak{A}$ is σ-measurable algebra, I will say that

$$\tau_{\sigma\text{-m}}(\mathfrak{A})=\min\{\sum_{n=0}^\infty\tau(\mathfrak{B}_n):\langle\mathfrak{B}_n\rangle_{n\in\mathbb{N}}\text{ is a witnessing sequence for }\mathfrak{A}\},$$

where the sums here are cardinal sums (5A1F(b-i)).

547I Examples (a) Every measurable algebra is σ-measurable.

(b) If $\mathfrak{A}$ is Dedekind complete and has countable π-weight, it is σ-measurable. (We can take every $\mathfrak{B}_n$ to be of the form $\{0,b,1\setminus b,1\}$.) In particular, $\mathfrak{G}_\omega$ is σ-measurable (515Oa).

(c) If $\mathfrak{A}$ is a measurable algebra and $\mathfrak{B}$ is a Boolean algebra with countable π-weight, then the Dedekind completion $\mathfrak{C}$ of the free product $\mathfrak{A}\otimes\mathfrak{B}$ (315I, 314U) is σ-measurable, with $\tau_{\sigma\text{-m}}(\mathfrak{C})\le\max(\omega,\tau(\mathfrak{A}))$. **P** If $\mathfrak{B}=\{0\}$ then $\mathfrak{C}=\{0\}$ and the result is trivial. Otherwise, let $\langle e_n\rangle_{n\in\mathbb{N}}$ run over an order-dense subset of $\mathfrak{B}\setminus\{0\}$; set $c_n=1\otimes e_n\in\mathfrak{C}$ for each n. For $n\in\mathbb{N}$ set

$$\mathfrak{D}_n=\{a\otimes e_n:a\in\mathfrak{A}\}=\{(a\otimes 1)\cap c_n:a\in\mathfrak{A}\},$$

$$\mathfrak{B}_n=\{b:b\in\mathfrak{C},\ b\cap c_n\in\mathfrak{D}_n,\ b\setminus c_n\in\{0,1\setminus c_n\}\}.$$

As $a\mapsto a\otimes e_n$ is injective (315K(e-ii)), $\mathfrak{D}_n$, regarded as a subalgebra of the principal ideal $\mathfrak{C}_{c_n}$ of $\mathfrak{C}$, is a Boolean algebra isomorphic to $\mathfrak{A}$, so is in itself a measurable algebra. Next, $a\mapsto a\otimes 1$ is an order-continuous Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{A}\otimes\mathfrak{B}$ (315Kc) and therefore from $\mathfrak{A}$ to $\mathfrak{C}$ (314Ta), while $c\mapsto c\cap c_n$ is an order-continuous Boolean homomorphism from $\mathfrak{C}$ to $\mathfrak{C}_{c_n}$ (313Xi), $a\mapsto a\otimes e_n$ is an order-continuous Boolean surjection from $\mathfrak{A}$ onto $\mathfrak{D}_n$, and $\mathfrak{D}_n$ is order-closed in $\mathfrak{C}_{c_n}$. Next, $\mathfrak{B}_n$ is isomorphic to the simple product $\mathfrak{D}_n\times\{0,1\setminus c_n\}$, so again is a measurable algebra; and as $\mathfrak{D}_n\times\{0,1\setminus c_n\}$ is order-closed in $\mathfrak{C}_{c_n}\times\mathfrak{C}_{1\setminus c_n}\cong\mathfrak{C}$, $\mathfrak{B}_n$ is order-closed in $\mathfrak{C}$.

Thus $\mathfrak{C}$ is σ-measurable. To estimate $\tau_{\sigma\text{-m}}(\mathfrak{C})$ we have only to note that, for each n, $\tau(\mathfrak{D}_n)=\tau(\mathfrak{A})$, and if $D\subseteq\mathfrak{D}_n$ τ-generates $\mathfrak{D}_n$ then $D\cup\{c_n\}$ τ-generates $\mathfrak{B}_n$. So $\tau(\mathfrak{B}_n)\le\tau(\mathfrak{A})+1$ and $\tau_{\sigma\text{-m}}(\mathfrak{C})$ is at most the cardinal product $\omega\times(\tau(\mathfrak{A})+1)=\max(\omega,\tau(\mathfrak{A}))$. **Q**

547J Since we are looking at measurable subalgebras of Boolean algebras, it will be helpful to have an elementary fact out in the open.

Lemma Let $\mathfrak{A}$ be a Boolean algebra, and $\mathfrak{B}$ a regularly embedded subalgebra of $\mathfrak{A}$ which is a measurable algebra. Then $\mathfrak{B}_a=\{b\cap a:b\in\mathfrak{B}\}$ is a measurable algebra, with $\tau(\mathfrak{B}_a)\le\tau(\mathfrak{B})$, for every $a\in\mathfrak{A}$.

proof Because $\mathfrak{B}$ is regularly embedded, the identity map from $\mathfrak{B}$ to $\mathfrak{A}$ is order-continuous; the map $d\mapsto d\cap a:\mathfrak{A}\to\mathfrak{A}_a$ is an order-continuous Boolean homomorphism; so $b\mapsto a\cap b:\mathfrak{B}\to\mathfrak{B}_a$ is an order-continuous Boolean homomorphism. By 391Lc[5], $\mathfrak{B}_a$ is a measurable algebra, isomorphic to a principal ideal of $\mathfrak{B}$; so $\tau(\mathfrak{B}_a)\le\tau(\mathfrak{B})$, by 331Hc or 514Ed.

[5]Later editions only.

547K I give some more or less straightforward properties of σ-measurable algebras.

Proposition Let $\mathfrak{A}$ be a σ-measurable Boolean algebra with witnessing sequence $\langle\mathfrak{B}_n\rangle_{n\in\mathbb{N}}$.

(a) $\mathfrak{A}$ satisfies Knaster's condition, so is ccc.

(b) $a=\inf_{n\in\mathbb{N}}\operatorname{upr}(a,\mathfrak{B}_n)$ for every $a\in\mathfrak{A}$.

(c) $\tau(\mathfrak{A})\le\tau_{\sigma\text{-m}}(\mathfrak{A})$.

(d) If $a\in\mathfrak{A}$ then the principal ideal $\mathfrak{A}_a$ generated by a is σ-measurable, with $\tau_{\sigma\text{-m}}(\mathfrak{A}_a)\le\tau_{\sigma\text{-m}}(\mathfrak{A})$.

(e) If $\mathfrak{A}$ is actually measurable, then $\tau_{\sigma\text{-m}}(\mathfrak{A})=\tau(\mathfrak{A})$.

(f) $\pi(\mathfrak{A})\le\sum_{n=0}^{\infty}\pi(\mathfrak{B}_n)$.

proof (a) Let $\langle a_\xi\rangle_{\xi<\omega_1}$ be a family in $\mathfrak{A}\setminus\{0\}$. For each $\xi<\omega_1$ choose a non-zero $b_\xi\in\bigcup_{n\in\mathbb{N}}\mathfrak{B}_n$ such that $b_\xi\subseteq a_\xi$. Then there is an $n\in\mathbb{N}$ such that $D=\{\xi:\xi<\omega_1,\ b_\xi\in\mathfrak{B}_n\}$ is uncountable. Now $\mathfrak{B}_n$ satisfies Knaster's condition (525Tb), so there is an uncountable $C\subseteq D$ such that $\langle b_\xi\rangle_{\xi\in C}$ is linked, in which case $\langle a_\xi\rangle_{\xi\in C}$ will be linked. As $\langle a_\xi\rangle_{\xi<\omega_1}$ is arbitrary, $\mathfrak{A}$ satisfies Knaster's condition and is ccc (511Ef).

(b) ? Otherwise, $\inf_{n\in\mathbb{N}}\operatorname{upr}(a,\mathfrak{B}_n)\setminus a$ is non-zero and includes a non-zero element b of $\bigcup_{n\in\mathbb{N}}\mathfrak{B}_n$. Let k be such that $b\in\mathfrak{B}_k$; then $a\subseteq 1\setminus b\in\mathfrak{B}_k$ so $\operatorname{upr}(a,\mathfrak{B}_k)$ is included in $1\setminus b$ and cannot include b. **X**

(c) We can suppose that $\langle\mathfrak{B}_n\rangle_{n\in\mathbb{N}}$ has been chosen such that $\sum_{n=0}^{\infty}\tau(\mathfrak{B}_n)=\tau_{\sigma\text{-m}}(\mathfrak{A})$. For each $n\in\mathbb{N}$ let $B_n\subseteq\mathfrak{B}_n$ be a set with cardinal $\tau(\mathfrak{B}_n)$ which τ-generates $\mathfrak{B}_n$. Set $B=\bigcup_{n\in\mathbb{N}}B_n$. The order-closed subalgebra of $\mathfrak{A}$ generated by B includes every $\mathfrak{B}_n$, so is order-dense in $\mathfrak{A}$, and therefore actually equal to $\mathfrak{A}$. Thus

$$\tau(\mathfrak{A})\le\#(B)\le\sum_{n=0}^{\infty}\tau(\mathfrak{B}_n)=\tau_{\sigma\text{-m}}(\mathfrak{A}).$$

(d) Again suppose that $\langle\mathfrak{B}_n\rangle_{n\in\mathbb{N}}$ was chosen such that $\sum_{n=0}^{\infty}\tau(\mathfrak{B}_n)=\tau_{\sigma\text{-m}}(\mathfrak{A})$. For each $n\in\mathbb{N}$ set $\mathfrak{B}'_n=\{b\cap a:b\in\mathfrak{B}_n\}$. Then $\mathfrak{B}'_n$ is an order-closed subalgebra of $\mathfrak{A}_a$ (314F(a-i)) and is a measurable algebra with Maharam type at most $\tau(\mathfrak{B}_n)$ (547J). If $c\in\mathfrak{A}_a$ is non-zero, there is a $b\in\bigcup_{n\in\mathbb{N}}\mathfrak{B}_n$ such that $0\ne b\subseteq c$, and now $b\in\bigcup_{n\in\mathbb{N}}\mathfrak{B}'_n$. So $\langle\mathfrak{B}'_n\rangle_{n\in\mathbb{N}}$ witnesses that $\mathfrak{A}_a$ is σ-measurable and that

$$\tau_{\sigma\text{-m}}(\mathfrak{A}_a)\le\sum_{n=0}^{\infty}\tau(\mathfrak{B}'_n)\le\sum_{n=0}^{\infty}\tau(\mathfrak{B}_n)=\tau_{\sigma\text{-m}}(\mathfrak{A}).$$

(e) We already know from (c) that $\tau(\mathfrak{A})\le\tau_{\sigma\text{-m}}(\mathfrak{A})$. But as $\mathfrak{A}$ is itself measurable, $(\mathfrak{A},\{0,1\},\{0,1\},\dots)$ is a witnessing sequence for $\mathfrak{A}$ so $\tau_{\sigma\text{-m}}(\mathfrak{A})\le\tau(\mathfrak{A})$.

(f) For each $n\in\mathbb{N}$ there is an order-dense subset B_n of $\mathfrak{B}_n$ with $\#(B_n)=\pi(\mathfrak{B}_n)$. Now $\bigcup_{n\in\mathbb{N}}B_n$ is order-dense in $\mathfrak{A}$, so

$$\pi(\mathfrak{A})\le\#(\bigcup_{n\in\mathbb{N}}B_n)\le\sum_{n=0}^{\infty}\#(B_n)=\sum_{n=0}^{\infty}\pi(\mathfrak{B}_n).$$

547L It will be useful to have an alternative definition of σ-measurable algebra.

Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\langle\mathfrak{C}_n\rangle_{n\in\mathbb{N}}$ a sequence of σ-subalgebras of $\mathfrak{A}$. For each $n\in\mathbb{N}$ let $\nu_n:\mathfrak{C}_n\to[0,\infty[$ be a countably additive functional. Suppose that for every $a\in\mathfrak{A}$ there are an $n\in\mathbb{N}$ and a $c\in\mathfrak{C}_n$ such that $c\subseteq a$ and $\nu_nc>0$. Then $\mathfrak{A}$ is a σ-measurable algebra and $\tau_{\sigma\text{-m}}(\mathfrak{A})\le\max(\omega,\sup_{n\in\mathbb{N}}\tau(\mathfrak{C}_n/\nu_n^{-1}[\{0\}]))$.

proof (a) $\mathfrak{A}$ is ccc. **P** Suppose that $A\subseteq\mathfrak{A}\setminus\{0\}$ is disjoint. For each $a\in A$ choose $n_A\in\mathbb{N}$ and $c_a\in\mathfrak{C}_{n_a}$ such that $c_a\subseteq a$ and $\nu_{n_a}c_a>0$. For $n\in\mathbb{N}$ set $A_n=\{a:a\in A,\ n_a=n\}$. Then $\{c_a:a\in A_n\}$ is a disjoint family in $\mathfrak{C}_n$, so $\sum_{a\in A_n}\nu_nc_a\le\nu_n1$ is finite, and A_n must be countable. Accordingly $A=\bigcup_{n\in\mathbb{N}}A_n$ is countable. As A is arbitrary, $\mathfrak{A}$ is ccc. **Q**

Consequently $\mathfrak{A}$ is Dedekind complete and every $\mathfrak{C}_n$ is order-closed in $\mathfrak{A}$ (316Fb).

(b) For each $n\in\mathbb{N}$, $\mathfrak{C}_n$ is ccc and Dedekind complete, and ν_n is completely additive (326P). Set $c_n=\sup\{c:c\in\mathfrak{C}_n,\ \nu_nc=0\}$; then $\nu_nc_n=0$. Set $\mathfrak{B}_n=\{a:a\in\mathfrak{A},\ a\setminus c_n\in\mathfrak{C}_n,\ a\cap c_n\in\{0,c_n\}\}$. Then $\mathfrak{B}_n$ is an order-closed subalgebra of $\mathfrak{A}$ and the restriction of ν_n to its principal ideal $(\mathfrak{B}_n)_{1\setminus c_n}$ is a strictly positive countably additive functional, so $(\mathfrak{B}_n)_{1\setminus c_n}$ is a measurable algebra. On the other hand, the complementary principal ideal $(\mathfrak{B}_n)_{c_n}=\{0,c_n\}$ is also a measurable algebra. So $\mathfrak{B}_n$ is measurable. Note also that $(\mathfrak{B}_n)_{1\setminus c_n}\cong\mathfrak{C}_n/\nu_n^{-1}[\{0\}]$, so $\tau(\mathfrak{B}_n)$ is at most the cardinal sum $\tau(\mathfrak{C}_n/\nu_n^{-1}[\{0\}])+1$.

(c) If $a \in \mathfrak{A}$ there are an $n \in \mathbb{N}$ and a $c \in \mathfrak{C}_n$ such that $c \subseteq a$ and $\nu_n c > 0$. In this case, $c \setminus c_n \in \mathfrak{B}_n$ and $0 \neq c \setminus c_n \subseteq a$. Thus $\bigcup_{n\in\mathbb{N}} \mathfrak{B}_n$ is order-dense in $\mathfrak{A}$, so $\mathfrak{A}$ is σ-measurable, with

$$\tau_{\sigma\text{-m}}(\mathfrak{A}) \leq \sum_{n=0}^{\infty} \tau(\mathfrak{B}_n) \leq \max(\omega, \sup_{n\in\mathbb{N}} \tau(\mathfrak{C}_n/\nu_n^{-1}[\{0\}])),$$

as claimed.

547M Turning to nowhere measurable algebras (391Bc), the facts we need are the following.

Proposition Let $\mathfrak{A}$ be a Boolean algebra.

(a) If $\mathfrak{A}$ is ccc and Dedekind complete and is not a measurable algebra, then it has a non-zero principal ideal which is nowhere measurable.

(b) If $\mathfrak{A}$ is nowhere measurable and Dedekind complete and $\mathfrak{B}$ is an order-closed subalgebra of $\mathfrak{A}$ which is a measurable algebra, then for any $a \in \mathfrak{A}$ there is a $d \subseteq a$ such that $\operatorname{upr}(d, \mathfrak{B}) = \operatorname{upr}(a \setminus d, \mathfrak{B}) = \operatorname{upr}(a, \mathfrak{B})$.

(c) Suppose that $\mathfrak{A}$ is nowhere measurable and Dedekind complete, $\nu : \mathfrak{A} \to [0, \infty[$ is a submeasure (definition: 392A) and $\mathfrak{B}_0, \dots, \mathfrak{B}_n$ are order-closed subalgebras of $\mathfrak{A}$ which are all measurable algebras. Then for any $a \in \mathfrak{A}$ there is a $d \subseteq a$ such that $\operatorname{upr}(a \setminus d, \mathfrak{B}_i) = \operatorname{upr}(a, \mathfrak{B}_i)$ for every $i \leq n$ and $\nu d \geq 2^{-n-1}\nu a$.

(d) If $\mathfrak{A}$ is nowhere measurable and Dedekind complete and $\mathfrak{B}_0, \dots, \mathfrak{B}_n$ are order-closed subalgebras of $\mathfrak{A}$ which are all measurable algebras, then there are disjoint $a_0, \dots, a_n \in \mathfrak{A}$ such that $\operatorname{upr}(a_i, \mathfrak{B}_i) = 1$ for every $i \leq n$.

(e) If $\mathfrak{A}$ is nowhere measurable and Dedekind complete, $\mathfrak{B}_0, \dots, \mathfrak{B}_n$ are order-closed subalgebras of $\mathfrak{A}$ which are all measurable algebras and $a \in \mathfrak{A}$, then there is a $d \subseteq a$ such that $\operatorname{upr}(d, \mathfrak{B}_i) = \operatorname{upr}(a \setminus d, \mathfrak{B}_i) = \operatorname{upr}(a, \mathfrak{B}_i)$ for every $i \leq n$.

proof (a) Set $A = \{a : a \in \mathfrak{A}, \mathfrak{A}_a$ is a measurable algebra$\}$. Then A is a σ-ideal of $\mathfrak{A}$ so $a^* = \sup A$ belongs to A. As $\mathfrak{A}$ is not itself measurable, $a^* \neq 1$, and $\mathfrak{A}_{1\setminus a^*}$ is a non-zero nowhere measurable principal ideal of $\mathfrak{A}$.

(b) Note first that if $c \in \mathfrak{A} \setminus \{0\}$ then $\mathfrak{B}_c$ is a measurable algebra, by 547J, but $\mathfrak{A}_c$ is not, because we are supposing that $\mathfrak{A}$ is nowhere measurable. So $\mathfrak{A}_c \neq \mathfrak{B}_c$.

Let B be the set of those $b \in \mathfrak{B}$ for which there is a $d \subseteq a$ such that

$$\operatorname{upr}(d, \mathfrak{B}) \cap \operatorname{upr}(a \setminus d, \mathfrak{B}) \supseteq b.$$

Let $B' \subseteq B$ be a maximal disjoint set. **?** If $a \not\subseteq \sup B'$, set $c = a \setminus \sup B'$. Then there is a $d \subseteq c$ such that d is not of the form $b \cap c$ for any $b \in \mathfrak{B}$. Consider $b = \operatorname{upr}(d, \mathfrak{B}) \cap \operatorname{upr}(a \setminus d, \mathfrak{B})$. By the maximality of B', $b = 0$. But this means that $\operatorname{upr}(d, \mathfrak{B}) \cap a \setminus d = 0$ and $d = c \cap \operatorname{upr}(d, \mathfrak{B})$. **X**

Thus $a \subseteq \sup B'$. For each $b \in B'$ choose $d_b \subseteq a$ such that $\operatorname{upr}(d_b, \mathfrak{B}) \cap \operatorname{upr}(a \setminus d_b, \mathfrak{B})$ includes b. Set $d = \sup_{b\in B'} b \cap d_b$. Then

$$\operatorname{upr}(d, \mathfrak{B}) = \sup_{b\in B'} \operatorname{upr}(b \cap d_b, \mathfrak{B}) = \sup_{b\in B'} b \cap \operatorname{upr}(d_b, \mathfrak{B}) = \sup_{b\in B'} b \supseteq a$$

so $\operatorname{upr}(a, \mathfrak{B}) \subseteq \operatorname{upr}(d, \mathfrak{B})$; as $d \subseteq a$, $\operatorname{upr}(a, \mathfrak{B}) = \operatorname{upr}(d, \mathfrak{B})$. Similarly

$$\operatorname{upr}(a \setminus d, \mathfrak{B}) = \sup_{b\in B'} \operatorname{upr}(b \cap a \setminus d, \mathfrak{B}) = \sup_{b\in B'} \operatorname{upr}(b \cap a \setminus d_b, \mathfrak{B})$$

(because B' is disjoint)

$$= \sup_{b\in B'} b \cap \operatorname{upr}(a \setminus d_b, \mathfrak{B}) = \sup_{b\in B'} b = \operatorname{upr}(a, \mathfrak{B}).$$

So this d serves.

(c) By (b), there is for each $i \leq n$ an $a_i \subseteq a$ such that $\operatorname{upr}(a_i, \mathfrak{B}_i) = \operatorname{upr}(a \setminus a_i, \mathfrak{B}_i) = \operatorname{upr}(a, \mathfrak{B}_i)$. For $I \subseteq n+1$ set $c_I = a \cap \inf_{i\in I} a_i \setminus \sup_{i\leq n, i\notin I} a_i$. Then $a = \sup_{I\subseteq n+1} c_I$; as ν is subadditive, there is an $I \subseteq n+1$ such that $\nu c_I \geq 2^{-n-1}\nu a$. If $i \leq n$ then one of a_i, $a \setminus a_i$ is included in $a \setminus c_I$, so $\operatorname{upr}(a \setminus c_I, \mathfrak{B}_i) = \operatorname{upr}(a, \mathfrak{B}_i)$. Accordingly we can take $d = c_I$.

(d) If $\mathfrak{A} = \{0\}$, this is trivial; suppose otherwise. For each $i \leq n$ let $\bar{\mu}_i$ be such that $(\mathfrak{B}_i, \bar{\mu}_i)$ is a probability algebra; set $\bar{\mu}_i^* a = \bar{\mu}_i(\operatorname{upr}(a, \mathfrak{B}_i))$ for $a \in \mathfrak{A}$. Then $\bar{\mu}_i^*$ is a submeasure. **P** $\bar{\mu}_i^* 0 = 0$ because

$\operatorname{upr}(0,\mathfrak{B}_i)=0$. If $a\subseteq a'$ then $\operatorname{upr}(a,\mathfrak{B}_i)\subseteq\operatorname{upr}(a',\mathfrak{B}_i)$ so $\bar\mu_i^*a\le\bar\mu_i^*a'$. For any a, $a'\in\mathfrak{A}$, $\operatorname{upr}(a\cup a',\mathfrak{B}_i)=\operatorname{upr}(a,\mathfrak{B}_i)\cup\operatorname{upr}(a',\mathfrak{B}_i)$, so $\bar\mu_i^*(a\cup a')\le\bar\mu_i^*a+\bar\mu_i^*a'$. **Q**

Furthermore, if $a\in\mathfrak{A}$ and $b\in\mathfrak{B}_i$,

$$\begin{aligned}\bar\mu_i^*a&=\bar\mu_i(\operatorname{upr}(a,\mathfrak{B}_i)\cap b)+\bar\mu_i(\operatorname{upr}(a,\mathfrak{B}_i)\setminus b)\\&=\bar\mu_i(\operatorname{upr}(a\cap b,\mathfrak{B}_i))+\bar\mu_i(\operatorname{upr}(a\setminus b,\mathfrak{B}_i))=\bar\mu_i^*(a\cap b)+\bar\mu^*(a\setminus b).\end{aligned}$$

In addition, if $\langle a_k\rangle_{k\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{A}$ with supremum a, $\langle\operatorname{upr}(a_k,\mathfrak{B}_i))\rangle_{k\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{B}_i$ with supremum $\operatorname{upr}(a,\mathfrak{B}_i)$, so $\bar\mu_i^*a=\sup_{k\in\mathbb{N}}\bar\mu_i^*a_k$.

Choose $\langle d_m\rangle_{m\in\mathbb{N}}$, $\langle a_{mi}\rangle_{m\in\mathbb{N},i\le n}$ inductively, as follows. The inductive hypothesis will be that $\operatorname{upr}(d_m,\mathfrak{B}_i)=1$ for every $i\le n$. Start with $d_0=1$ and $a_{0i}=0$ for every $i\le n$. For the inductive step to $m+1$, take $j\le n$ such that $\bar\mu_j^*a_{mj}$ is minimal. Set $c=\operatorname{upr}(a_{mj},\mathfrak{B}_j)$ and $\nu a=\bar\mu_j^*(a\setminus c)$ for $a\in\mathfrak{A}$. Then $\nu:\mathfrak{A}\to[0,1]$ is subadditive and $\nu d_m=1-\bar\mu_jc$. By (d), there is a $d_{m+1}\subseteq d_m$ such that $\operatorname{upr}(d_{m+1},\mathfrak{B}_i)=\operatorname{upr}(d_m\setminus d_{m+1},\mathfrak{B}_i)=1$ for every $i\le n$ and $\nu(d_m\setminus d_{m+1})\ge 2^{-n-1}\nu d_m$; set

$$a_{m+1,j}=a_{mj}\cup((d_m\setminus d_{m+1})\setminus c),\quad a_{m+1,i}=a_{mi}\text{ for }i\ne j.$$

Note that

$$\begin{aligned}\bar\mu_j^*a_{m+1,j}&=\bar\mu_j^*(a_{m+1,j}\cap c)+\bar\mu_j^*(a_{m+1,j}\setminus c)=\bar\mu_j^*a_{mj}+\bar\mu_j^*(d_m\setminus d_{m+1})\setminus c)\\&=\bar\mu_j^*a_{mj}+\nu(d_m\setminus d_{m+1})\ge\bar\mu_j^*a_{mj}+2^{-n-1}\nu d_m\\&=\bar\mu_j^*a_{mj}+2^{-n-1}(1-\bar\mu_jc)\ge\bar\mu_j^*a_{mj}+2^{-n-1}(1-\tfrac{1}{n+1}\sum_{i=0}^n\bar\mu_i^*a_{mi})\end{aligned}$$

because $\bar\mu_jc=\bar\mu_j^*a_{mj}$ was minimal.

Continue. At the end of the induction, if we set $\gamma_m=\sum_{i=0}^n\bar\mu_i^*a_{mi}$ for each m, we have

$$\gamma_{m+1}\ge\gamma_m+2^{-n-1}(1-\tfrac{\gamma_m}{n+1})$$

and

$$n+1-\gamma_{m+1}\le(n+1-\gamma_m)(1-\tfrac{2^{-n-1}}{n+1})$$

for every m. As surely $\gamma_m\le n+1$ for every n, $\lim_{m\to\infty}\gamma_m=n+1$.

Set $a_i=\sup_{m\in\mathbb{N}}a_{mi}$ for each i. Then $\bar\mu_i^*a_i=\sup_{m\in\mathbb{N}}\bar\mu_i^*a_{mi}$ for each i, so $\sum_{i=0}^n\bar\mu_i^*a_i\ge n+1$. As $\bar\mu_i^*1=1$ for each i, we must have $\bar\mu_i^*a_i=1$ for each i, that is, $\operatorname{upr}(a_i,\mathfrak{B}_i)=1$ for each i.

Finally, because $\langle d_m\rangle_{m\in\mathbb{N}}$ is non-increasing and for each m $\langle a_{m+1,i}\setminus a_{mi}\rangle_{i\le n}$ is a disjoint family with $\sup_{i\le n}a_{m+1,i}\setminus a_{mi}\subseteq d_m\setminus d_{m+1}$, we have $a_{mi}\cap a_{mj}=0$ whenever $m\in\mathbb{N}$ and $i\ne j$. So $\langle a_i\rangle_{i\le n}$ is disjoint. Thus we have found a suitable family.

(e) For $i\le n$ set $\mathfrak{C}_i=(\mathfrak{B}_i)_a=\{a\cap b:b\in\mathfrak{B}_i\}$, so that $\mathfrak{C}_i$ is an order-closed subalgebra of $\mathfrak{A}_a$ and a measurable algebra, We need to know that if $d\subseteq a$ then

$$\begin{aligned}\operatorname{upr}(d,\mathfrak{C}_i)&=\inf\{c:c\in\mathfrak{C}_i,\,d\subseteq c\}=\inf\{a\cap b:b\in\mathfrak{B}_i,\,d\subseteq b\}\\&=a\cap\inf\{b:b\in\mathfrak{B}_i,\,d\subseteq b\}=a\cap\operatorname{upr}(d,\mathfrak{B}_i).\end{aligned}$$

Of course $\mathfrak{A}_a$ is nowhere measurable. By (d), we have disjoint $a_0,\dots,a_n$ in $\mathfrak{A}_a$ such that $\operatorname{upr}(a_i,\mathfrak{C}_i)=a$ for each i. So

$$a=a\cap\operatorname{upr}(a_i,\mathfrak{B}_i)\subseteq\operatorname{upr}(a_i,\mathfrak{B}_i)\subseteq\operatorname{upr}(a,\mathfrak{B}_i)$$

and $\operatorname{upr}(a,\mathfrak{B}_i)=\operatorname{upr}(a_i,\mathfrak{B}_i)$.

Next, for each i we have a $d_i\subseteq a$ such that $\operatorname{upr}(d_i,\mathfrak{B}_i)=\operatorname{upr}(a_i\setminus d_i,\mathfrak{B}_i)=\operatorname{upr}(a,\mathfrak{B}_i)$, by (b). Set $d=\sup_{i\le n}d_i$. Then, for any $i\le n$, $d_i\subseteq d$ and $a_i\setminus d\subseteq a\setminus d$. So

$$\operatorname{upr}(a,\mathfrak{B}_i)=\operatorname{upr}(a_i,\mathfrak{B}_i)=\operatorname{upr}(d_i,\mathfrak{B}_i)\subseteq\operatorname{upr}(d,\mathfrak{B}_i)\subseteq\operatorname{upr}(a,\mathfrak{B}_i),$$

$$\operatorname{upr}(a,\mathfrak{B}_i)=\operatorname{upr}(a_i,\mathfrak{B}_i)=\operatorname{upr}(a_i\setminus d_i,\mathfrak{B}_i)\subseteq\operatorname{upr}(a\setminus d,\mathfrak{B}_i)\subseteq\operatorname{upr}(a,\mathfrak{B}_i)$$

and $\operatorname{upr}(d,\mathfrak{B}_i)=\operatorname{upr}(a\setminus d,\mathfrak{B}_i)=\operatorname{upr}(a,\mathfrak{B}_i)$, as required.

547N Putting the concepts of σ-measurable algebra and nowhere measurable algebra together, we have the following.

Proposition If $\mathfrak{A}$ is a non-trivial nowhere measurable σ-measurable algebra, then $\mathfrak{G}_\omega$ can be regularly embedded in $\mathfrak{A}$.

proof (a) Let $\langle\mathfrak{B}_n\rangle_{n\in\mathbb{N}}$ be a witnessing sequence of subalgebras of $\mathfrak{A}$. Choose $\langle a_\sigma\rangle_{\sigma\in S_2}$ inductively as follows. Start with $a_\emptyset = 1$. Given a_σ where $\sigma \in \{0,1\}^n$, use 547Me to find $d \subseteq a_\sigma$ such that $\mathrm{upr}(d,\mathfrak{B}_i) = \mathrm{upr}(a_\sigma \setminus d,\mathfrak{B}_i) = \mathrm{upr}(a_\sigma,\mathfrak{B}_i)$ for every $i \le n$; set $a_{\sigma^\frown<0>} = d$ and $a_{\sigma^\frown<1>} = a_\sigma \setminus d$.

Observe that $\mathrm{upr}(a_\sigma,\mathfrak{B}_i) = \mathrm{upr}(a_\tau,\mathfrak{B}_i)$ whenever $\sigma \subseteq \tau$ in S_2 and $i \le \#(\sigma)$.

(b) Write $\mathcal{E}$ for the algebra of open-and-closed subsets of $\{0,1\}^{\mathbb{N}}$. Because $a_\emptyset = 1$ and we always have $a_{\sigma^\frown<0>}$, $a_{\sigma^\frown<1>}$ disjoint with supremum a_σ, we have a Boolean homomorphism $\pi : \mathcal{E} \to \mathfrak{A}$ defined by setting $\pi I_\sigma = a_\sigma$ for $\sigma \in S_2$. Because $a_\sigma \ne 0$ for every σ, π is injective. The point is that π is order-continuous. **P** Suppose that $\mathcal{G} \subseteq \mathcal{E}$ has supremum $1_{\mathcal{E}} = \{0,1\}^{\mathbb{N}}$. Then $\bigcup\mathcal{G}$ is a dense open subset of $\{0,1\}^{\mathbb{N}}$. **?** If $\sup_{G\in\mathcal{G}} \pi G \ne 1_{\mathfrak{A}}$, there are an $n \in \mathbb{N}$ and a non-zero $b \in \mathfrak{B}_n$ such that $b \cap \pi G = 0$ for every $G \in \mathcal{G}$. Now there is a $\sigma \in \{0,1\}^n$ such that $b \cap a_\sigma \ne 0$. There must be a $G \in \mathcal{G}$ such that $G \cap I_\sigma \ne \emptyset$, and a τ extending σ such that $I_\tau \subseteq G$. In this case, $a_\tau \subseteq \pi G$ and $b \cap a_\tau = 0$. It follows that b is disjoint from $\mathrm{upr}(a_\tau,\mathfrak{B}_n) = \mathrm{upr}(a_\sigma,\mathfrak{B}_n)$ and $b \cap a_\sigma = 0$; contrary to the choice of σ. **X**

Thus $\sup \pi[\mathcal{G}] = 1_{\mathfrak{A}}$ whenever $\sup\mathcal{G} = 1_{\mathcal{E}}$. By 313Lb, π is order-continuous. **Q**

(c) Now $\mathcal{E}$ is an order-dense subalgebra of the regular open algebra $\mathrm{RO}(\{0,1\}^{\mathbb{N}})$ (314Uc[6]). Because $\mathfrak{A}$ is Dedekind complete, we have an extension of π to an order-continuous Boolean homomorphism from $\mathrm{RO}(\{0,1\}^{\mathbb{N}})$ to $\mathfrak{A}$ (314Tb) which is still injective (313Xs[6]). Thus $\mathrm{RO}(\{0,1\}^{\mathbb{N}})$ is regularly embedded in $\mathfrak{A}$. But $\mathfrak{G}_\omega \cong \mathrm{RO}(\{0,1\}^{\mathbb{N}})$ (515Oa), so $\mathfrak{G}_\omega$ is regularly embedded in $\mathfrak{A}$, as claimed.

547O I come now to some less obvious properties of σ-measurable algebras (547P, 547Q).

Lemma Let $(\mathfrak{A},\bar\mu)$ be a probability algebra. Suppose that $\langle a_\sigma\rangle_{\sigma\in S_2}$ is a family in $\mathfrak{A}$ such that

$$a_\sigma = a_{\sigma^\frown<0>} \cup a_{\sigma^\frown<1>} = \sup_{\tau\in S_2,\tau\supseteq\sigma} a_{\tau^\frown<0>} \cap a_{\tau^\frown<1>}$$

for every $\sigma \in S_2$. For $A \subseteq S_2$ set $c_A = \sup_{\sigma\in A} a_\sigma$.

(a) For every $k \ge 1$ and $\epsilon > 0$ there is an $m \in \mathbb{N}$ such that

$$\bar\mu(\sup_{I\in[\{0,1\}^m]^k} \inf_{\sigma\in I} a_\sigma) \ge \bar\mu a_\emptyset - \epsilon.$$

(b) For A, $B \subseteq S_2$ I will say that $A \perp B$ if σ and τ are incompatible whenever $\sigma \in A$ and $\tau \in B$. Now for any $\epsilon > 0$ there are finite A, $B \subseteq S_2$ such that $A \perp B$ and $\bar\mu c_A$, $\bar\mu c_B$ are both at least $\bar\mu a_\emptyset - \epsilon$.

(c) For any $\epsilon > 0$ there is a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of finite subsets of S_2 such that $\bar\mu(\inf_{n\in\mathbb{N}} c_{A_n}) \ge \bar\mu a_\emptyset - \epsilon$ and $A_m \perp A_n$ whenever $m < n$ in $\mathbb{N}$.

proof (a)(i) Note first that if $m \in \mathbb{N}$ and $\sigma \in \{0,1\}^m$, then $a_\sigma = \sup\{a_\tau : \sigma \subseteq \tau \in \{0,1\}^n\}$ for every $n \ge m$. (Induce on n.) So if we set

$$b_{\sigma n} = \sup\{a_\tau \cap a_{\tau'} : \tau,\ \tau' \in \{0,1\}^n \text{ are different extensions of } \sigma\},$$

then $b_{\sigma n} \subseteq a_\sigma$ and

$$\begin{aligned} b_{\sigma n} &= \sup\{a_\tau \cap a_{\tau'} : \tau,\ \tau' \in \{0,1\}^n \text{ are different extensions of } \sigma\} \\ &= \sup\{a_\upsilon \cap a_{\upsilon'} : \upsilon,\ \upsilon' \in \{0,1\}^{n+1} \text{ and } \upsilon\restriction n,\ \upsilon'\restriction n \text{ are different extensions of } \sigma\} \\ &\subseteq \sup\{a_\upsilon \cap a_{\upsilon'} : \upsilon,\ \upsilon' \in \{0,1\}^{n+1} \text{ are different extensions of } \sigma\} \\ &= b_{\sigma,n+1} \end{aligned}$$

for every $n \in \mathbb{N}$. As

$$a_\sigma = \sup_{\tau\supseteq\sigma} a_{\tau^\frown<0>} \cap a_{\tau^\frown<1>} = \sup_{n\in\mathbb{N}} \sup_{\substack{\tau\supseteq\sigma \\ \#(\tau)=n}} a_{\tau^\frown<0>} \cap a_{\tau^\frown<1>} \subseteq \sup_{n\in\mathbb{N}} b_{\sigma,n+1},$$

[6] Later editions only.

$\bar{\mu}a_\sigma = \sup_{n\in\mathbb{N}} \bar{\mu}b_{\sigma n}$.

(ii) Induce on k. For $k = 1$ we can take $m = 0$. For the inductive step to $k+1 \ge 2$, take l such that $d_0 = \sup_{I\in[\{0,1\}^l]^k} \inf_{\sigma\in I} a_\sigma)$ has measure at least $\bar{\mu}a_\emptyset - \frac{1}{2}\epsilon$. For each $\sigma \in \{0,1\}^l$ let $m_\sigma \ge l$ be such that

$$\bar{\mu}(b_{\sigma m_\sigma}) \ge \bar{\mu}a_\sigma - 2^{-l-2}\epsilon.$$

Set $m = \sup_{\sigma\in\{0,1\}^l} m_\sigma$. Setting $d_1 = \sup_{\sigma\in\{0,1\}^l} b_{\sigma m}$,

$$\bar{\mu}(a_\emptyset \setminus d_1) \le \sum_{\sigma\in\{0,1\}^l} \bar{\mu}(a_\sigma \setminus b_{\sigma m}) \le \frac{1}{2}\epsilon.$$

Next, if we take any atom e of the subalgebra of $\mathfrak{A}$ generated by $\{a_\tau : \tau \in \{0,1\}^m\}$ which is included in $d_0 \cap d_1$,

$$A = \{\sigma : \sigma \in \{0,1\}^l,\ e \subseteq a_\sigma\} = \{\sigma : \sigma \in \{0,1\}^l,\ e \cap a_\sigma \ne 0\}$$

has at least k members, and for each $\sigma \in A$ there are distinct τ, $\tau' \in \{0,1\}^m$ extending σ such that $e \cap a_\tau \cap a_{\tau'} \ne 0$ and $e \subseteq a_\tau \cap a_{\tau'}$, so $\{\tau : \tau \in \{0,1\}^m,\ e \subseteq a_\tau\}$ has at least $2k$ members, and $e \subseteq \inf_{\tau\in I} a_\tau$ for some $I \in [\{0,1\}^m]^{2k}$. As e is arbitrary,

$$\sup_{I\in[\{0,1\}^m]^{k+1}} \inf_{\tau\in I} a_\tau \supseteq d_0 \cap d_1$$

has measure at least $\bar{\mu}a_\emptyset - \epsilon$, and the induction continues.

(b) If $a_\emptyset = 0$ or $\epsilon > 1$ the result is trivial, so suppose otherwise. Set $\eta = \frac{1}{3}\epsilon > 0$. Let $k \ge 1$ be so large that $2^{-k} \le \eta$. By (a), there is an $m \in \mathbb{N}$ such that $\bar{\mu}d \ge \bar{\mu}a_\emptyset - \eta$, where

$$d = \sup_{I\in[\{0,1\}^m]^k} \inf_{\sigma\in I} a_\sigma.$$

Of course $d \subseteq a_\emptyset$ and $\bar{\mu}(a_\emptyset \setminus d) \le \eta$. Give $\{0,1\}^m$ its usual probability measure ν (254J), so that singleton sets have measure 2^{-m}. Let E be the set of atoms of the subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ generated by $\{a_\sigma : \sigma \in \{0,1\}^m\}$; note that $d \in \mathfrak{B}$ and $c_A \in \mathfrak{B}$ for every $A \subseteq \{0,1\}^m$. We see that if $e \in E$ and $e \subseteq d$ then there is an $I \in [\{0,1\}^m]^k$ such that $e \cap \inf_{\sigma\in I} a_\sigma \ne 0$, in which case

$$\nu\{A : A \subseteq \{0,1\}^m,\ e \cap c_A = 0\} \le \nu\{A : A \cap I = \emptyset\} \le 2^{-k} \le \eta$$

so

$$\begin{aligned}\int \bar{\mu}(d \setminus c_A)\nu(dA) &= \sum_{e\in E, e\subseteq d} \bar{\mu}e \cdot \nu\{A : e \cap c_A = 0\} \\ &\le \sum_{e\in E, e\subseteq d} \eta\bar{\mu}e = \eta\bar{\mu}d \le \eta\end{aligned}$$

and $\nu\{A : \bar{\mu}(d \setminus c_A) > 2\eta\} < \frac{1}{2}$. There is therefore an $A \subseteq \{0,1\}^m$ such that, setting $B = \{0,1\}^m \setminus A$, both $\bar{\mu}(d \setminus c_A)$ and $\bar{\mu}(d \setminus c_B)$ are at most 2η. But now $A \perp B$ and both $\bar{\mu}c_A$ and $\bar{\mu}c_B$ are at least $\bar{\mu}d - 2\eta \ge \bar{\mu}a_\emptyset - \epsilon$.

(c)(i) If $\epsilon > 0$ and $A \subseteq S_2$ is finite there are finite B, $B' \subseteq S_2$ such that every member of $B \cup B'$ extends a member of A, $B \perp B'$ and $\min(\bar{\mu}c_B, \bar{\mu}c_{B'}) \ge \bar{\mu}c_A - \epsilon$. **P** Let A_0 be the set of minimal members of A; because S_2 is a tree, every member of A extends a member of A_0, so $c_{A_0} = c_A$, while no two members of A_0 are compatible. For each $\tau \in A_0$, we can apply (b) to $\langle a_{\tau^\frown\sigma}\rangle_{\sigma\in S_2}$ to see that there are finite B_τ, $B'_\tau \subseteq S_2$ such that every member of $B_\tau \cup B'_\tau$ extends τ, $B_\tau \perp B'_\tau$ and

$$\min(\bar{\mu}c_{B_\tau}, \bar{\mu}c_{B'_\tau}) \ge \bar{\mu}a_\tau - \frac{\epsilon}{1+\#(A)},$$

that is,

$$\max(\bar{\mu}(a_\tau \setminus c_{B_\tau}), \bar{\mu}(a_\tau \setminus c_{B'_\tau})) \le \frac{\epsilon}{1+\#(A)}.$$

Set $B = \bigcup_{\tau\in A_0} B_\tau$, $B' = \bigcup_{\tau\in A_0} B'_\tau$; these work. **Q**

(ii) Now choose sequences $\langle B_n\rangle_{n\in\mathbb{N}}$, $\langle A_n\rangle_{n\in\mathbb{N}}$ of finite subsets of S_2 inductively such that $B_0=\{\emptyset\}$ and, for each $n\in\mathbb{N}$,

$$\text{every member of } A_n\cup B_{n+1} \text{ extends a member of } B_n,\quad A_n\perp B_{n+1},$$

$$\bar\mu c_{A_n} \text{ and } \bar\mu c_{B_{n+1}} \text{ are both at least } \bar\mu c_{B_n}-2^{-n-2}\epsilon.$$

Then for each $n\ge 0$

$$\begin{aligned}\bar\mu(c_{B_{n+1}}\cap\inf_{i\le n}c_{A_i})&\ge\bar\mu(c_{B_n}\cap\inf_{i<n}c_{A_i})-(\bar\mu c_{B_n}-\bar\mu c_{B_{n+1}})-(\bar\mu c_{B_n}-\bar\mu c_{A_n})\\&\ge\bar\mu(c_{B_n}\cap\inf_{i<n}c_{A_i})-2^{-n-1}\epsilon,\end{aligned}$$

so

$$\bar\mu(\inf_{i\in\mathbb{N}}c_{A_i})\ge\bar\mu a_\emptyset-\sum_{n=0}^\infty 2^{-n-1}\epsilon=\bar\mu a_\emptyset-\epsilon.$$

Also an easy induction shows that $B_n\perp A_i$ whenever $i<n$ in $\mathbb{N}$ and therefore that $A_n\perp A_i$ whenever $i<n$.

547P Proposition (see KUMAR & SHELAH 17, 4.2) Let $\mathfrak{A}$ be a σ-measurable Boolean algebra. If $\mathfrak{C}$ is an order-closed subalgebra of $\mathfrak{A}$ of countable Maharam type, there is a $c\in\mathfrak{C}$ such that the principal ideal $\mathfrak{C}_c$ has an e-h family (definition: 546F) and its complement $\mathfrak{C}_{1\setminus c}$ is a measurable algebra.

proof (a) The result is trivial if $\mathfrak{A}=\{0\}$, so suppose otherwise. Let $\langle\mathfrak{B}_n\rangle_{n\in\mathbb{N}}$ be a witnessing sequence of measurable subalgebras of $\mathfrak{A}$. For $n\in\mathbb{N}$ let $\bar\mu_n$ be such that $(\mathfrak{B}_n,\bar\mu_n)$ is a probability algebra. Take a maximal disjoint set $D\subseteq\mathfrak{C}^+$ such that $\mathfrak{C}_d$ is measurable for every $d\in D$. Set $d^*=\sup D$. By 547Ka, D is countable; consequently $\mathfrak{C}_{d^*}$ is a measurable algebra (391C, 391Xl). Set $c=1\setminus d^*$.

(b) If $c=0$ then of course $\mathfrak{C}_c=\{0\}$ has an e-h family in which every term is zero. So suppose otherwise. By the maximality of the set D in (a) above, $\mathfrak{C}_c$ has no non-trivial measurable principal ideal; in particular, it is atomless. Since $\mathfrak{C}_c$ also has countable Maharam type (514Ed), there is a family $\langle c_\sigma\rangle_{\sigma\in S_2}$ in $\mathfrak{C}$ such that $c_\emptyset=c$, $\mathfrak{C}_c$ is generated by $\{c_\sigma:\sigma\in S_2\}$ and

$$c_{\sigma^\frown<0>}\cup c_{\sigma^\frown<1>}=c_\sigma,\quad c_{\sigma^\frown<0>}\cap c_{\sigma^\frown<1>}=0$$

for every $\sigma\in S_2$.

For $A\subseteq S_2$ set $d_A=\sup_{\sigma\in A}c_\sigma$. Then $d_A\cap d_B=0$ whenever A, $B\subseteq S_2$ and $A\perp B$ in the sense of 547O.

(c) For $n\in\mathbb{N}$ and $\sigma\in S_2$ set $\tilde c_\sigma^{(n)}=\operatorname{upr}(c_\sigma,\mathfrak{B}_n)$. Then $\tilde c_\sigma^{(n)}=\tilde c_{\sigma^\frown<0>}^{(n)}\cup\tilde c_{\sigma^\frown<1>}^{(n)}$ for every $\sigma\in S_2$ (313Sb). Also

$$\tilde c_\sigma^{(n)}\subseteq\sup_{\tau\in S_2,\tau\supseteq\sigma}\tilde c_{\tau^\frown<0>}^{(n)}\cap\tilde c_{\tau^\frown<1>}^{(n)}$$

for every $\sigma\in S_2$. **P?** Otherwise, since $\sup_{\tau\supseteq\sigma}\tilde c_{\tau^\frown<0>}^{(n)}\cap\tilde c_{\tau^\frown<1>}^{(n)}\in\mathfrak{B}_n$,

$$e=c_\sigma\setminus\sup_{\tau\supseteq\sigma}\tilde c_{\tau^\frown<0>}^{(n)}\cap\tilde c_{\tau^\frown<1>}^{(n)}$$

is non-zero. If $\tau\supseteq\sigma$ and $e\cap\tilde c_\tau^{(n)}\subseteq c_\tau$, then

$$e\cap\tilde c_{\tau^\frown<0>}^{(n)}=e\cap\tilde c_\tau^{(n)}\setminus\tilde c_{\tau^\frown<1>}^{(n)}$$

(because $e\cap\tilde c_{\tau^\frown<0>}^{(n)}\cap\tilde c_{\tau^\frown<1>}^{(n)}=0$)

$$\subseteq e\cap c_\tau\setminus c_{\tau^\frown<1>}=e\cap c_{\tau^\frown<0>}$$

and similarly $e\cap\tilde c_{\tau^\frown<1>}^{(n)}\subseteq e\cap c_{\tau^\frown<1>}$. So we see by induction on $\#(\tau)$ that $e\cap\tilde c_\tau^{(n)}=e\cap c_\tau$ whenever $\tau\supseteq\sigma$.

Now $\mathfrak{D}_0=\{e\cap b:b\in\mathfrak{B}_n\}$ is an order-closed subalgebra of $\mathfrak{A}_e$ (314F(a-i) again) containing $e\cap\tilde c_\tau^{(n)}$ whenever $\sigma\subseteq\tau\in S_2$; similarly, $\mathfrak{D}_1=\{e\cap c':c'\in\mathfrak{C}\}$ is the order-closed subalgebra of $\mathfrak{A}_e$ generated by

$$\{e\cap c_\tau:\tau\in S_2\}=\{e\cap c_\sigma\cap c_\tau:\tau\in S_2\}\subseteq\{e\cap c_\tau:\sigma\subseteq\tau\in S_2\}\cup\{0\}$$
$$\subseteq\{e\cap\tilde c_\tau^{(n)}:\sigma\subseteq\tau\in S_2\}\cup\{0\}\subseteq\mathfrak D_0$$

(see 313Mc). But this means that $\mathfrak D_1$ is included in $\mathfrak D_0$ and is accordingly an order-closed subalgebra of $\mathfrak D_0$. Since $\mathfrak D_0$ is the image of the measurable algebra $\mathfrak B_n$ under an order-continuous homomorphism, it is measurable (391Lc), so $\mathfrak D_1$ also is measurable (391La). Finally, $\mathfrak D_1$ must be isomorphic to the principal ideal of $\mathfrak C$ generated by $\operatorname{upr}(e,\mathfrak C)$. So $\mathfrak C_c$ has a non-trivial measurable principal ideal, which we know is not the case. **XQ**

(d) For n, $k\in\mathbb N$ we can choose a partition $\langle d_{nkj}\rangle_{j\in\mathbb N}$ of unity in $\mathfrak C_c$ such that $b\cap d_{nkj}\ne0$ whenever $b\in\mathfrak B_n$, $b\subseteq c$ and $\bar\mu_nb>2^{-k}$. **P** Set

$$a=1\setminus\operatorname{upr}(1\setminus c,\mathfrak B_n)=\sup\{b:b\in\mathfrak B_n,\ b\subseteq c\}.$$

For $\sigma\in S_2$, set $a_\sigma=a\cap\tilde c_\sigma^{(n)}=\operatorname{upr}(a\cap c_\sigma,\mathfrak B_n)$ (313Sc). Using (c) we see that

$$a_\sigma=a_{\sigma^\frown<0>}\cup a_{\sigma^\frown<1>}=\sup_{\tau\in S_2,\tau\supseteq\sigma}a_{\tau^\frown<0>}\cap a_{\tau^\frown<1>}$$

for every $\sigma\in S_2$. For $A\subseteq S_2$ set

$$d_A=\sup_{\sigma\in A}c_\sigma,\quad \tilde d_A=a\cap\operatorname{upr}(d_A,\mathfrak B_n)=\sup_{\sigma\in A}\tilde c_\sigma^{(n)}$$

(313Sb again). By 547Oc, applied in $\mathfrak B_n$, we have a sequence $\langle A_j\rangle_{j\in\mathbb N}$ of subsets of S_2 such that $A_j\perp A_{j'}$ whenever $j\ne j'$ and $\bar\mu_n\tilde d_{A_j}\ge\bar\mu_na-2^{-k}$ for every $j\in\mathbb N$. Set

$$d_{nkj}=d_{A_j}\text{ for }j\ge1,\quad d_{nk0}=c\setminus\sup_{j\ge1}d_{A_j}\supseteq d_{A_0}.$$

Then $\langle d_{nkj}\rangle_{j\in\mathbb N}$ is a partition of unity in $\mathfrak C_c$. If $b\in\mathfrak B_n$, $j\in\mathbb N$, $b\subseteq c$ and $\bar\mu_nb>2^{-k}$, then $b\cap\operatorname{upr}(a\cap d_{A_j},\mathfrak B_n)\ne0$ so $b\cap d_{nkj}\ne0$. **Q**

(e) $\mathfrak C_c$ has an e-h family. **P** Take $\langle d_{nkj}\rangle_{n,k,j\in\mathbb N}$ from (d) above. Suppose that $f:\mathbb N\times\mathbb N\to\mathbb N$ is a function and that $e\in\mathfrak C_c$ is non-zero. Take $n\in\mathbb N$, $b\in\mathfrak B_n$ and $k\in\mathbb N$ such that $0\ne b\subseteq e$ and $2^{-k}<\bar\mu_nb$. Then $0\ne b\cap d_{n,k,f(n,k)}\subseteq e\cap d_{n,k,f(n,k)}$. As e is arbitrary, $c=\sup\{d_{n,k,f(n,k)}:(n,k)\in\mathbb N\times\mathbb N\}$. As f is arbitrary, $\langle\langle d_{nkj}\rangle_{j\in\mathbb N}\rangle_{n,k\in\mathbb N}$, suitably re-enumerated, is an e-h family in $\mathfrak C_c$. **Q**

547Q Lemma (see Kumar & Shelah 17, 5.1) Let $\mathfrak A$ be a σ-measurable algebra. Set $\lambda=\max(\omega,\tau_{\sigma\text{-m}}(\mathfrak A))$ and $\kappa=\operatorname{non}\mathcal M(\lambda^{\mathbb N})$, where in this product λ is given its discrete topology. Then there is a family $\langle a_{\xi n}\rangle_{\xi<\kappa,n\in\mathbb N}$ in $\mathfrak A$ such that

whenever $\langle d_{m\sigma}\rangle_{m\in\mathbb N,\sigma\in S_2}$ is a family in $\mathfrak A$ such that

$$d_{m\tau}\subseteq d_{m,\tau^\frown\sigma}\text{ for every }\sigma\in S_2,\quad\sup_{\sigma\in S_2}d_{m,\tau^\frown\sigma}=1$$

for every $\tau\in S_2$ and $m\in\mathbb N$, there is a $\xi<\kappa$ such that

$$\sup_{\sigma\in S_2}\bigl(d_{m\sigma}\cap\inf_{\substack{i<\#(\sigma)\\ \sigma(i)=1}}a_{\xi i}\setminus\sup_{\substack{i<\#(\sigma)\\ \sigma(i)=0}}a_{\xi i}\bigr)=1$$

for every $m\in\mathbb N$.

proof (a) Let $\langle\mathfrak B_n\rangle_{n\in\mathbb N}$ be a witnessing sequence for $\mathfrak A$ with $\sum_{n=0}^\infty\tau(\mathfrak B_n)=\lambda$. Set $X=\{0,1\}^{\mathbb N}$. For $i\in\mathbb N$, set $E_i=\{x:x\in X,\ x(i)=1\}$.

For each $n\in\mathbb N$ such that $\mathfrak B_n\ne\{0\}$ let μ_n be a Radon probability measure on $Y_n=\{0,1\}^{\max(1,\tau(\mathfrak B_n))}$ with measure algebra isomorphic to $\mathfrak B_n$ (524U); when $\mathfrak B_n=\{0\}$ take μ_n to be the zero measure on $Y_n=\{0,1\}^\emptyset=\{\emptyset\}$. In either case let $\bar\mu_n$ be the associated measure on $\mathfrak B_n$. For $n\in\mathbb N$ and $E\in\operatorname{dom}\mu_n$, write π_nE for the corresponding element of $\mathfrak B_n$. Let $\mathcal E$ be the algebra of open-and-closed subsets of $Y=\prod_{n\in\mathbb N}Y_n\cong\{0,1\}^\lambda$; as λ is infinite, $\#(\mathcal E)=\lambda$. Then we have a Boolean homomorphism $\pi:\mathcal E\to\mathfrak A$ defined by setting $\pi(\{z:z(n)\in F\})=\pi_nF$ whenever $n\in\mathbb N$ and $F\subseteq Y_n$ is open-and-closed (cf. 315I-315J).

Set $Z=C(Y;X)$ with its compact-open topology (5A4I). Giving $\mathcal E$ its discrete topology, Z is homeomorphic to $\mathcal E^{\mathbb N}\cong\lambda^{\mathbb N}$ (5A4Ib-5A4Ic), and there is a non-meager family $\langle g_\xi\rangle_{\xi<\kappa}$ in Z. Write $\mathcal J$ for the proper σ-ideal

$$\{A : A \subseteq \kappa,\ \{g_\xi : \xi \in A\} \text{ is meager in } Z\}$$

of subsets of κ. Set

$$a_{\xi i} = \pi(g_\xi^{-1}[E_i])$$

for $\xi < \kappa$ and $i \in \mathbb{N}$.

(b) Let $\langle d_{m\sigma}\rangle_{m\in\mathbb{N},\sigma\in S_2}$ be a family in $\mathfrak{A}$ such that

$$d_{m\tau} \subseteq d_{m,\tau^\frown\sigma} \text{ for every } \sigma \in S_2, \quad \sup_{\sigma\in S_2} d_{m,\tau^\frown\sigma} = 1$$

for every $\tau \in S_2$ and $m \in \mathbb{N}$. For $\xi < \kappa$ and $m \in \mathbb{N}$, set

$$e_{m\xi} = \sup_{\sigma\in S_2} \bigl(d_{m\sigma} \cap \inf_{\substack{i<\#(\sigma)\\ \sigma(i)=1}} a_{\xi i} \setminus \sup_{\substack{i<\#(\sigma)\\ \sigma(i)=0}} a_{\xi i}\bigr).$$

(c) We are approaching the heart of the argument. **?** Suppose, if possible, that

$$D_0 = \{\xi : \xi < \kappa,\ \inf_{m\in\mathbb{N}} e_{m\xi} \ne 1\}$$

does not belong to $\mathcal{J}$.

(i) There is an $m \in \mathbb{N}$ such that

$$D_1 = \{\xi : e_{m\xi} \ne 1\}$$

does not belong to $\mathcal{J}$. For $j \in \mathbb{N}$, set

$$c_{j\sigma} = \inf_{\tau\in\{0,1\}^j} d_{m,\tau^\frown\sigma}$$

for $\sigma \in S_2$. Then $\sup_{\sigma\in S_2} c_{j\sigma} = 1$. **P** Enumerate $\{0,1\}^j$ as $\langle\tau_i\rangle_{i<k}$. Take $a \in \mathfrak{A} \setminus \{0\}$. Choose $\langle a_i\rangle_{i<k}$ and $\langle\sigma_i\rangle_{i\le k}$ such that

$$a_0 = a, \quad \sigma_0 = \emptyset,$$

given that $i < k$, $a_i \in \mathfrak{A} \setminus \{0\}$ and $\sigma_i \in S_2$, $\sigma_{i+1} \in S_2$ is to be an extension of σ_i such that $a_{i+1} = a_i \cap d_{m,\tau_i^\frown\sigma_{i+1}}$ is non-zero; such an extension exists because $\sup_{\sigma\in S_2} d_{m,\tau_i^\frown\sigma_i^\frown\sigma} = 1$.

At the end of the induction,

$$0 \ne a_k \subseteq a_{i+1} \subseteq d_{m,\tau_i^\frown\sigma_{i+1}} \subseteq d_{m,\tau_i^\frown\sigma_k}$$

for every $i < k$, so $a_k \subseteq c_{j\sigma_k}$; as $a_k \subseteq a$, $a \cap c_{j\sigma_k} \ne 0$. As a is arbitrary, $\sup_{\sigma\in S_2} c_{j\sigma} = 1$. **Q**

(ii) For $\xi \in D_1$ choose $b_\xi \in \bigcup_{n\in\mathbb{N}} \mathfrak{B}_n$ such that $0 \ne b_\xi \subseteq 1 \setminus e_{m\xi}$; let $n \in \mathbb{N}$ and $\epsilon > 0$ be such that $D_2 = \{\xi : \xi \in D_1,\ b_\xi \in \mathfrak{B}_n,\ \bar\mu_n b_\xi > \epsilon\}$ does not belong to $\mathcal{J}$. For $j \in \mathbb{N}$ and $\sigma \in S_2$, write $\tilde c_{j\sigma}$ for $\operatorname{upr}(c_{j\sigma}, \mathfrak{B}_n)$, and choose $H_{j\sigma} \in \operatorname{dom}\mu_n$ such that $\pi_n H_{j\sigma} = \tilde c_{j\sigma}$. As $\sup_{\sigma\in S_2} \tilde c_{j\sigma} = 1$ in $\mathfrak{B}_n$, $\bigcup_{\sigma\in S_2} H_{j\sigma}$ is μ_n-conegligible. Because S_2 is countable, there is a set $K \subseteq Y_n$ such that

$$K \subseteq \bigcup_{\sigma\in S_2} H_{j\sigma} \text{ for every } j \in \mathbb{N},$$

$$K \cap H_{j\sigma},\ K \setminus H_{j\sigma} \text{ are compact for every } j \in \mathbb{N} \text{ and } \sigma \in S_2,$$

$$\mu_n K \ge 1 - \epsilon.$$

As K is compact and $H_{j\sigma} \cap K$ is relatively open in K for all j and σ, we see that for every $j \in \mathbb{N}$ there is a finite set $J_j \subseteq S_2$ such that $K \subseteq \bigcup_{\sigma\in J_j} H_{j\sigma}$. Again because $K \cap H_{j\sigma}$ is relatively open in the compact set K for each $\sigma \in J_j$, we have a partition $\langle W_{j\sigma}\rangle_{\sigma\in J_j}$ of Y_n into open-and-closed sets such that $K \cap W_{j\sigma} \subseteq H_{j\sigma}$ for every $\sigma \in J_j$.

(iii) For $j \in \mathbb{N}$, set

$$\begin{aligned} G_j = \{g : g \in Z,\ & g(z)(j+i) = \sigma(i) \text{ whenever } \sigma \in J_j, \\ & i < \#(\sigma),\ z \in Y \text{ and } z(n) \in W_{j\sigma}\}. \end{aligned}$$

Because every $W_{j\sigma}$ is compact, G_j is open in Z. Accordingly $G = \bigcup_{j\in\mathbb{N}} G_j$ is open. Moreover, G is dense in Z. **P** Suppose that $U \subseteq Z$ is a non-empty open set. Take $h \in U$. Then there is a $j \in \mathbb{N}$ such that

$$U_1 = \{g : g^{-1}[E_i] = h^{-1}[E_i] \text{ for every } i < j\}$$

is an open neighbourhood of h included in U (5A4Ib-5A4Ic again). So if we define $g \in Z$ by saying that, for $z \in Y$ and $i \in \mathbb{N}$,

$$\begin{aligned} g(z)(i) &= \sigma(i-j) \text{ if } \sigma \in J_j,\ j \le i < j + \#(\sigma) \text{ and } z(n) \in W_{j\sigma}, \\ &= h(z)(i) \text{ otherwise,} \end{aligned}$$

we have a member of $G_j \cap U_1 \subseteq G \cap U$. **Q**

(iv) As $D_2 \notin \mathcal{J}$, $\{g_\xi : \xi \in D_2\}$ cannot be nowhere dense and there is a $\xi \in D_2$ such that $g_\xi \in G$. Let $j \in \mathbb{N}$ be such that $g_\xi \in G_j$. For $\tau \in \{0,1\}^j$, $\sigma \in S_2$ set

$$a'_\tau = \inf_{\substack{i<j \\ \tau(i)=1}} a_{\xi i} \setminus \sup_{\substack{i<j \\ \tau(i)=0}} a_{\xi i}, \quad b'_\sigma = \inf_{\substack{i<\#(\sigma) \\ \sigma(i)=1}} a_{\xi,j+i} \setminus \sup_{\substack{i<\#(\sigma) \\ \sigma(i)=0}} a_{\xi,j+i}.$$

Of course $\sup_{\tau\in\{0,1\}^j} a'_\tau = 1$.

Because $\bar\mu_n b_\xi + \bar\mu_n \pi_n K > 1$, $b_\xi \cap \pi_n K \ne 0$. Now $K \subseteq \bigcup_{\sigma\in J_j} W_{j\sigma}$ so there is a $\sigma \in J_j$ such that $b = b_\xi \cap \pi_n[K \cap W_{j\sigma}] \ne 0$. Note that $b \in \mathfrak{B}_n$. As $K \cap W_{j\sigma} \subseteq H_{j\sigma}$,

$$b \subseteq \pi_n H_{j\sigma} = \tilde c_{j\sigma} = \operatorname{upr}(c_{j\sigma}, \mathfrak{B}_n)$$

and $b \cap c_{j\sigma} \ne 0$.

As $g_\xi \in G_j$, $g_\xi(z)(j+i) = \sigma(i)$ whenever $z(n) \in W_{j\sigma}$ and $i < \#(\sigma)$, that is,

whenever $i < \#(\sigma)$ and $\sigma(i) = 1$, $g_\xi^{-1}[E_{j+i}] \supseteq \{z : z(n) \in W_{j\sigma}\}$ and $a_{\xi,j+i} \supseteq \pi_n W_{j\sigma}$,

whenever $i < \#(\sigma)$ and $\sigma(i) = 0$, $g_\xi^{-1}[E_{j+i}] \cap \{z : z(n) \in W_{j\sigma}\} = \emptyset$ and $a_{\xi,j+i} \cap \pi_n W_{j\sigma} = 0$,

so $\pi_n W_{j\sigma} \subseteq b'_\sigma$ and $b \subseteq b'_\sigma$. Thus $b_\xi \cap b'_\sigma \cap c_{j\sigma} \ne 0$.

We have

$$\begin{aligned} e_{m\xi} &= \sup_{\upsilon\in S_2} \bigl(d_{m\upsilon} \cap \inf_{\substack{i<\#(\upsilon) \\ \upsilon(i)=1}} a_{\xi i} \setminus \sup_{\substack{i<\#(\upsilon) \\ \upsilon(i)=0}} a_{\xi i}\bigr) \\ &\supseteq \sup_{\tau\in\{0,1\}^j} d_{m,\tau^\frown\sigma} \cap \bigl(\inf_{\substack{i<j \\ \tau(i)=1}} a_{\xi i} \cap \inf_{\substack{i<\#(\sigma) \\ \sigma(i)=1}} a_{\xi,j+i}\bigr) \setminus \bigl(\sup_{\substack{i<j \\ \tau(i)=0}} a_{\xi i} \cup \sup_{\substack{i<\#(\sigma)) \\ \sigma(i)=0}} a_{\xi,j+i}\bigr) \\ &= \sup_{\tau\in\{0,1\}^j} d_{m,\tau^\frown\sigma} \cap a'_\tau \cap b'_\sigma \supseteq \sup_{\tau\in\{0,1\}^j} c_{j\sigma} \cap a'_\tau \cap b'_\sigma = c_{j\sigma} \cap b'_\sigma. \end{aligned}$$

So $b_\xi \cap e_{m\xi} \ne 0$. But we chose b_ξ to be disjoint from $e_{m\xi}$. **X**

(v) We conclude that $D_0 \in \mathcal{J}$.

(d) In particular, $D_0 \ne \kappa$ and there is a $\xi < \kappa$ such that $e_{m\xi} = 1$ for every ξ, as required.

547R Theorem Suppose that X is a set and $\mathcal{I}$ is a proper σ-ideal of subsets of X such that the quotient algebra $\mathfrak{A} = \mathcal{P}X/\mathcal{I}$ is atomless and σ-measurable. Then $\tau_{\sigma\text{-m}}(\mathfrak{A}) > \operatorname{add}\mathcal{I}$.

proof (a) Write κ for $\operatorname{add}\mathcal{I}$ and λ for $\max(\omega, \tau_{\sigma\text{-m}}(\mathfrak{A}))$. As κ is surely infinite, it will be enough to show that $\kappa < \lambda$. For the time being (down to the end of (g) below), suppose that $\mathfrak{A}$ is nowhere measurable and that there is a disjoint family $\langle Y_\xi\rangle_{\xi<\kappa}$ in $\mathcal{I}$ with union X. Set $\mathcal{J} = \{J : J \subseteq \kappa, \bigcup_{\xi\in J} Y_\xi \in \mathcal{I}\}$; then $\mathcal{J}$ is a proper ω_1-saturated σ-ideal of $\mathcal{P}\kappa$ containing singletons and $\operatorname{add}\mathcal{J} = \kappa$. Accordingly κ is quasi-measurable. Set $\mathfrak{C} = \{(\bigcup_{\xi\in J} Y_\xi)^\bullet : J \subseteq \kappa\}$; then $\mathfrak{C}$ is a σ-subalgebra of $\mathfrak{A}$, so is order-closed in $\mathfrak{A}$, because $\mathfrak{A}$ is ccc. Also $\mathfrak{C} \cong \mathcal{P}\kappa/\mathcal{J}$.

(b) $\kappa \le \operatorname{non}\mathcal{N}$. **P** By 547N, $\mathfrak{G}_\omega$ can be regularly embedded in $\mathfrak{A}$, so 546I(a-ii) gives the result. **Q** It follows that $\mathfrak{C} \cong \mathcal{P}\kappa/\mathcal{J}$ is atomless (541P).

(c) $\mathfrak{C}$ has a countably generated order-closed subalgebra $\mathfrak{D}$ which is not measurable. **P** Because $\mathfrak{A}$ satisfies Knaster's condition, so does $\mathfrak{C}$ (516Sa). By 516V, $\mathfrak{C}$ has an atomless countably generated order-closed subalgebra $\mathfrak{E}$ say. If $\mathfrak{E}$ is not measurable, we can stop. Otherwise, $\operatorname{non}\mathcal{N} \le \kappa$ (546I(a-i-β), applied to κ and $\mathcal{J}$). So $\kappa = \operatorname{non}\mathcal{N}$ and 546Id tells us that $\mathfrak{C}$ has a countably generated order-closed subalgebra which is *not* measurable. **Q**

(d) $\operatorname{non}\mathcal{M} \le \kappa$. **P** $\mathfrak{D}$ is a countably generated order-closed subalgebra of $\mathfrak{A}$ which is not measurable. By 547P, there is a non-zero $d \in \mathfrak{D}$ such that the corresponding principal ideal of $\mathfrak{D}$ has an e-h family, which is still an e-h family in the principal ideal $\mathfrak{C}_d$ of $\mathfrak{C}$. By 546Ib, applied to κ and $\mathcal{J}$, $\operatorname{non}\mathcal{M} \le \kappa$. **Q**

(e) $\pi(\mathfrak{A})$ is uncountable. **P** $\mathfrak{A}$ was set up as a power set σ-quotient algebra, so cannot be isomorphic to $\mathfrak{G}_\omega$, by 547G. But $\mathfrak{A}$ is atomless, Dedekind complete and not $\{0\}$, so by 515Oa once more it cannot have countable π-weight. **Q**

(f) $\kappa \le \operatorname{non}\mathcal{M}$. **P** Let $\langle\mathfrak{B}_n\rangle_{n\in\mathbb{N}}$ be a sequence witnessing that $\mathfrak{A}$ is σ-measurable. Because $\pi(\mathfrak{A}) > \omega$, there is an $n \in \mathbb{N}$ such that $\pi(\mathfrak{B}_n)$ is uncountable (547Kf) and $\mathfrak{B}_n$ is not purely atomic; let $b \in \mathfrak{B}_n$ be such that the corresponding principal ideal is atomless. Now the principal ideal $\mathfrak{A}_b$ of $\mathfrak{A}$ is σ-measurable (547Kd) and has an order-closed subalgebra $\mathfrak{B}_n \cap \mathfrak{A}_b$ which is atomless and measurable. At the same time, if $E \subseteq X$ is such that $E^\bullet = b$, $\mathfrak{A}_b \cong \mathcal{P}E/\mathcal{I}\cap\mathcal{P}E$, so 546I(a-i-$\alpha$) tells us that $\operatorname{non}\mathcal{M} \ge \operatorname{add}(\mathcal{I}\cap\mathcal{P}E) \ge \kappa$. **Q**

Thus $\kappa = \operatorname{non}\mathcal{M}$.

(g) **?** Suppose, if possible, that $\lambda \le \kappa$. Because κ is quasi-measurable, there is a family $\mathcal{A}$ of countable sets which is stationary over λ and has cardinal at most λ (542K). Set $\theta = \operatorname{non}\mathcal{M}(\lambda^{\mathbb{N}})$, where λ here is given its discrete topology. By 5A4J,

$$\theta \le \max(\lambda, \operatorname{non}\mathcal{M}) = \kappa.$$

By 547Q, there is a family $\langle a_{\xi n}\rangle_{\xi<\theta,n\in\mathbb{N}}$ such that

whenever $\langle d_{m\sigma}\rangle_{m\in\mathbb{N},\sigma\in S_2}$ is a family in $\mathfrak{A}$ such that

$$d_{m\tau} \subseteq d_{m,\tau^\frown\sigma} \text{ for every } \sigma \in S_2, \quad \sup_{\sigma\in S_2} d_{m,\tau^\frown\sigma} = 1$$

for every $\tau \in S_2$ and $m \in \mathbb{N}$, there is a $\xi < \theta$ such that

$$\sup_{\sigma\in S_2}\bigl(d_{m\sigma} \cap \inf_{\substack{i<\#(\sigma)\\ \sigma(i)=1}} a_{\xi i} \setminus \sup_{\substack{i<\#(\sigma)\\ \sigma(i)=0}} a_{\xi i}\bigr) = 1$$

for every $m \in \mathbb{N}$.

But this contradicts 546Ic, because $\theta \le \operatorname{non}\mathcal{M}$. **X**

Accordingly $\kappa < \lambda$, as required.

(h) Thus we have the result when X is itself a union of κ members of $\mathcal{I}$ and $\mathfrak{A}$ is nowhere measurable. If $\mathfrak{A}$ is not nowhere measurable, that is, there is an $a \in \mathfrak{A}\setminus\{0\}$ such that $\mathfrak{A}_a$ is measurable, take $A \subseteq X$ such that $A^\bullet = a$, and consider $\mathcal{I}_A = \mathcal{I}\cap\mathcal{P}A$. Then $\mathcal{P}A/\mathcal{I}_A \cong \mathfrak{A}_a$ is an atomless measurable algebra. By the Gitik-Shelah theorem (543E), $\tau(\mathfrak{A}_a) > \operatorname{add}\mathcal{I}_A$. But now

$$\kappa \le \operatorname{add}\mathcal{I}_A < \tau(\mathfrak{A}_a) = \tau_{\sigma\text{-m}}(\mathfrak{A}_a) \le \tau_{\sigma\text{-m}}(\mathfrak{A}) \le \lambda$$

(547Kd, 547Kc), so $\kappa < \lambda$ in this case.

(i) So we need consider only the case in which $\mathfrak{A}$ is nowhere measurable, but perhaps X is not itself a union of κ members of $\mathcal{I}$.

Again let $\langle Y_\xi\rangle_{\xi<\kappa}$ be a family in $\mathcal{I}$ such that $Y = \bigcup_{\xi<\kappa} Y_\xi$ does not belong to $\mathcal{I}$, and set $b = Y^\bullet \in \mathfrak{A}$. Then $\mathcal{I}_Y = \mathcal{I}\cap\mathcal{P}Y$ is a proper ω_1-saturated σ-ideal of subsets of Y and $\operatorname{add}\mathcal{I}_Y = \kappa$, while $\mathfrak{A}' = \mathcal{P}Y/\mathcal{I}_Y$ is isomorphic to $\mathfrak{A}_b$. So $\mathfrak{A}'$ is non-trivial, atomless, σ-measurable and nowhere measurable with $\tau_{\sigma\text{-m}}(\mathfrak{A}') \le \tau_{\sigma\text{-m}}(\mathfrak{A})$ (547Kd again). But now (a)-(g) tell us that $\operatorname{add}\mathcal{I}_Y < \tau_{\sigma\text{-m}}(\mathfrak{A}')$ so that $\kappa < \lambda$ in this case also. This completes the proof.

547S Corollary If a non-zero σ-measurable algebra $\mathfrak{A}$ is also an atomless power set σ-quotient algebra, then there is a quasi-measurable cardinal less than $\tau_{\sigma\text{-m}}(\mathfrak{A})$.

proof Let X be a set and $\mathcal{I}$ a σ-ideal of subsets of X such that $\mathfrak{A} \cong \mathcal{P}X/\mathcal{I}$. Then $\mathcal{I}$ must be a proper ω_1-saturated ideal (547Ka), so $\operatorname{add}\mathcal{I}$ is quasi-measurable (542B); but 547R tells us that $\operatorname{add}\mathcal{I} < \tau_{\sigma\text{-m}}(\mathfrak{A})$.

547X Basic exercises (a) Show that a power set σ-quotient algebra with countable π-weight is purely atomic.

(b) Show that the simple product of a countable family of σ-measurable algebras is σ-measurable.

(c) Let $\mathfrak{A}$ be a σ-measurable algebra. Show that if a cardinal κ is a precaliber of every probability algebra it is a precaliber of $\mathfrak{A}$.

(d) Let $\mathfrak{A}$ be a σ-measurable algebra. Show that $\tau_{\sigma\text{-m}}(\mathfrak{A})$ is at most $\pi(\mathfrak{A})$.

(e)(i) Show that every σ-measurable algebra is chargeable (definition: 391Bb). (ii) Show that a weakly (σ,∞)-distributive σ-measurable algebra is measurable.

(f) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, and suppose that there is a finite family $\mathbb{B}$ of order-closed subalgebras of $\mathfrak{A}$ such that every member of $\mathbb{B}$ is a measurable algebra and $\bigcup\mathbb{B}$ is order-dense in $\mathfrak{A}$. Show that $\mathfrak{A}$ is a measurable algebra. (*Hint*: show that $\mathfrak{A}$ is weakly (σ,∞)-distributive.)

(g) Show that a σ-measurable Maharam algebra is a measurable algebra.

(h) Suppose that T is a Souslin tree (5A1Ed). Show that its regular open algebra $\mathrm{RO}^{\uparrow}(T)$ is not σ-measurable.

(i) Let $\mathfrak{A}$ be a σ-measurable algebra and $\mathfrak{B}$ an order-closed subalgebra of $\mathfrak{A}$ which is σ-measurable. Show that $\tau_{\sigma\text{-m}}(\mathfrak{B}) \le \tau_{\sigma\text{-m}}(\mathfrak{A})$. (*Hint*: reduce to case in which $\mathfrak{A}$ is infinite and $\mathfrak{B}$ is a homogeneous measure algebra with standard generating family $\langle e_\xi\rangle_{\xi<\kappa}$; take a witnessing sequence $\langle\mathfrak{B}_n\rangle_{n\in\mathbb{N}}$ for $\mathfrak{A}$ and for $\xi<\kappa$ choose $d_\xi \in \bigcup_{n\in\mathbb{N}}\mathfrak{B}_n$ with $0 \ne d_\xi \subseteq e_\xi$; show that $\#(\{\xi : d_\xi \in \mathfrak{B}_n\})$ is at most the topological weight of the metric space $\mathfrak{B}_n$.)

547Y Further exercises (a) Let $\mathfrak{A}$ be a measurable algebra, $\mathfrak{B}$ a Boolean algebra, $\mathfrak{C}$ the Dedekind completion of $\mathfrak{A}\otimes\mathfrak{B}$, and $\mathfrak{D}$ an order-closed subalgebra of $\mathfrak{C}$. Show that $\tau(\mathfrak{D}) \le \max(\omega,\tau(\mathfrak{A}),\pi(\mathfrak{B}))$.

(b) Let $\mathfrak{A}$ be a σ-measurable algebra. Show that there is a unique $a \in \mathfrak{A}$ such that the principal ideal $\mathfrak{A}_a$ has an e-h family and every countably generated order-closed subalgebra of $\mathfrak{A}_{1\setminus a}$ is a measurable algebra. (See 539O.)

(c) Show that if κ is an infinite cardinal less than ω_ω, and we give κ its discrete topology, then $\mathrm{non}(\mathcal{M}(\kappa^{\mathbb{N}})) = \max(\kappa, \mathrm{non}\,\mathcal{M})$. (See 5A4J.)

(d) Suppose that X is a set, $\mathcal{I}$ is a proper σ-ideal of subsets of X and $\mathcal{P}X/\mathcal{I}$ is weakly (σ,∞)-distributive. (i) Show that $\mathrm{add}\,\mathcal{I} \ne \mathfrak{b}$. (ii) Show that $\mathrm{add}\,\mathcal{I} \ne \mathrm{cf}\,\mathfrak{d}$. (See 544N.)

547Z Problems (a) Can $\mathfrak{G}_{\omega_2}$ be a power set σ-quotient algebra?

(b) Can there be a power set σ-quotient algebra $\mathfrak{A}$ such that $c(\mathfrak{A}) = \omega$ and $\pi(\mathfrak{A}) = \omega_1$? Note that there seems to be no obstacle to an atomless ccc power set σ-quotient algebra having Maharam type ω (555K). Similarly, it is generally supposed that it is possible for a power set σ-quotient algebra to have π-weight ω_1 (Foreman 10, Theorem 7.60).

(c) Let $\mathfrak{A}$ be a non-purely-atomic Maharam algebra with countable Maharam type. Can $\mathfrak{A}$ be a power set σ-quotient algebra? What about algebras constructed as in 394B-394M?

(d) Find a σ-measurable algebra with an order-closed subalgebra which is not σ-measurable.

(e) Is there a σ-measurable algebra $\mathfrak{A}$ such that $\tau(\mathfrak{A}) < \tau_{\sigma\text{-m}}(\mathfrak{A})$?

547 Notes and comments The general message of the work here seems to be that 'standard' algebras cannot be power set σ-quotient algebras (though see 555K). It is true that the algebras we are most interested in can be expressed as order-closed subalgebras of power set σ-quotient algebras (546Xa), but this is too easy to be useful. If there are no quasi-measurable cardinals, of course, this section becomes redundant. The point is that 547G, like 546E, is a theorem of ZFC; and, as I explained in the notes to §541, I think it worth exploring worlds in which there are quasi-measurable cardinals.

All the results as stated in 543F, 547G and 547R depend on moving from a power set σ-quotient algebra to an order-closed subalgebra which is a normal power set σ-quotient algebra. If we start from a measurable algebra, the subalgebra will again be measurable, with Maharam type no greater than that of the original algebra; this is why 543E implies 543F. For category algebras this doesn't work in the same way, because an order-closed subalgebra of $\mathfrak{G}_{\omega_2}$, for instance, can be very different in character. (For examples see Koppelberg & Shelah 96 and Balcar Jech & Zapletal 97.) So only 547G can be directly deduced from 547F.

I have every hope that there is a great deal more to be said about the questions here, but I have to confess that I have no idea which way to go. The problems in 547Z seem to me natural ones, but I don't know whether they will turn out to be useful. Looking at 546I and the argument of 547R, it seems that it might be worth asking whether, for our favourite ccc power set σ-quotient algebras $\mathfrak{A}$, we can describe the types – for instance, the combinatorial properties – of quasi-measurable cardinals which can appear as values of $\operatorname{add}\mathcal{I}$ arising in an expression of $\mathfrak{A}$ as $\mathcal{P}X/\mathcal{I}$. (See 547Yd.) Note that as it seems that there can be a proper class of two-valued-measurable cardinals, and as every two-valued-measurable cardinal corresponds to a representation of the algebra $\{0,1\}$ as a normal power set σ-quotient algebra, we do not expect any upper bounds on the complexity of the quasi-measurable cardinals arising in this way.

Version of 30.6.21

548 Selectors and disjoint refinements

We come now to a remarkable result (548C) which is a minor extension of the principal theorem of Kumar & Shelah 17. This leads directly to 548E, which is a corresponding elaboration of the main theorem of Gitik & Shelah 01. Both of these results apply to spaces whose Maharam types are not too large, so give interesting facts about Lebesgue measure not dependent on special axioms (548F). A similar restriction on shrinking number leads to further results of this kind (548G-548H) which are not necessarily applicable to Lebesgue measure. If we choose an easier target other methods are available (548I-548K).

Notation will follow that of §§546-547; in particular, I will speak of σ-measurable algebras and the associated cardinal function $\tau_{\sigma\text{-m}}$ (547H). As usual, I write $\mathcal{N}(\mu)$ for the null ideal of a measure μ.

548A Lemma Let X be a set, $\mathcal{I}$ a σ-ideal of subsets of X, $\langle(Y_n,\mathrm{T}_n,\nu_n)\rangle_{n\in\mathbb{N}}$ a sequence of totally finite measure spaces and $\langle f_n\rangle_{n\in\mathbb{N}}$ a sequence of functions, each f_n being an injective function from a subset of X to Y_n. Suppose that for every $A\in\mathcal{P}X\setminus\mathcal{I}$ there are an $n\in\mathbb{N}$ and an $F\in\mathrm{T}_n$ such that $\nu_nF>0$ and $F\subseteq f_n[A]$. Then $\mathfrak{A}=\mathcal{P}X/\mathcal{I}$ is σ-measurable and $\tau_{\sigma\text{-m}}(\mathfrak{A})\le\max(\omega,\sup_{n\in\mathbb{N}}\tau(\nu_n))$.

proof (a) Because $\mathcal{I}$ is a σ-ideal of $\mathcal{P}X$, $\mathfrak{A}$ is Dedekind σ-complete and the quotient map $A\mapsto A^\bullet:\mathcal{P}X\to\mathfrak{A}$ is sequentially order-continuous (313Qb).

(b) For each $n\in\mathbb{N}$ set $\mathcal{E}_n=\{F:F\in\mathrm{T}_n,\ f_n^{-1}[F]\in\mathcal{I}\}$. Then $\mathcal{E}_n$ is closed under countable unions so there is an $F_n\in\mathcal{E}_n$ such that $\nu_n(F\setminus F_n)=0$ for every $F\in\mathcal{E}_n$ (215Ac). Set $\nu'_nF=\nu_n(F\setminus F_n)$ for $F\in\mathrm{T}_n$ and

$$\mathfrak{C}_n=\{(f_n^{-1}[F])^\bullet:F\in\mathrm{T}_n\}\subseteq\mathfrak{A},$$

so that ν'_n is a totally finite measure with domain T_n and $\mathfrak{C}_n$ is a σ-subalgebra of $\mathfrak{A}$.

The map $F\mapsto(f_n^{-1}[F])^\bullet:\mathrm{T}_n\to\mathfrak{C}_n$ is a surjective sequentially order-continuous Boolean homomorphism, and ν'_n is zero on its kernel. We therefore have a functional $\hat\nu_n:\mathfrak{C}_n\to[0,\infty[$ defined by saying that $\hat\nu_nc=\nu'_nF$ whenever $c\in\mathfrak{C}_n$, $F\in\mathrm{T}_n$ and $c=(f_n^{-1}[F])^\bullet$, and $\hat\nu_n$ is σ-additive because ν'_n is.

(c) If $a \in \mathfrak{A} \setminus \{0\}$, let $A \in \mathcal{P}X \setminus \mathcal{I}$ be such that $a = A^\bullet$. Then $A_1 = A \setminus \bigcup_{n\in\mathbb{N}} f_n^{-1}[F_n]$ does not belong to $\mathcal{I}$, so there are an $n \in \mathbb{N}$ and an $F \in \mathrm{T}_n$ such that $\nu_n F > 0$ and $F \subseteq f_n[A_1]$. As f_n is injective, $f_n^{-1}[F] \subseteq A_1$; as $F \cap F_n = \emptyset$, $\nu'_n F > 0$. Consider $c = (f_n^{-1}[F])^\bullet$. Then $\hat{\nu}_n c = \nu'_n F > 0$ while

$$c \subseteq A_1^\bullet = A^\bullet = a.$$

Thus all the conditions of 547L are satisfied by $\mathfrak{A}$ and $\langle(\mathfrak{C}_n, \hat{\nu}_n\rangle_{n\in\mathbb{N}})$, and $\mathfrak{A}$ is σ-measurable, with $\tau_{\sigma\text{-m}}(\mathfrak{A}) \le \max(\omega, \sup_{n\in\mathbb{N}} \tau(\mathfrak{C}_n/\hat{\nu}_n^{-1}[\{0\}]))$.

(d) Next, $\tau(\mathfrak{C}_n/\hat{\nu}_n^{-1}[\{0\}]) \le \tau(\nu_n)$ for every $n \in \mathbb{N}$. **P** Write $\mathfrak{B}_n$ for the measure algebra of ν_n, and let $B \subseteq \mathfrak{B}_n$ be a τ-generating set with cardinal $\tau(\mathfrak{B}_n) = \tau(\nu_n)$. Now we have a set $\mathcal{B} \subseteq \mathrm{T}_n$, with cardinality $\tau(\nu_n)$, such that $B = \{F^\bullet : F \in \mathcal{B}\}$. Let T' be the σ-subalgebra of T_n generated by $\mathcal{B}$; then $\{F^\bullet : F \in \mathrm{T}'\}$ is a σ-subalgebra of $\mathfrak{B}_n$ including B. Because $\mathfrak{B}_n$ is ccc, $\{F^\bullet : F \in \mathrm{T}'\}$ is order-closed and is the whole of $\mathfrak{B}_n$. Thus for any $F \in \mathrm{T}_n$ there is an $F' \in \mathrm{T}'$ such that $\nu_n(F \triangle F') = 0$.

Now consider $\mathfrak{D} = \{(f_n^{-1}[F])^\bullet : F \in \mathrm{T}'\}$. This is a σ-subalgebra of $\mathfrak{C}_n$. If $c \in \mathfrak{C}_n$ there is an $F \in \mathrm{T}_n$ such that $c = (f_n^{-1}[F])^\bullet$ and an $F' \in \mathrm{T}'$ such that $\nu_n(F \triangle F') = 0$. Of course $\nu'_n(F \triangle F') = 0$ so $\hat{\nu}_n(c \triangle c') = 0$ where $c' = (f_n^{-1}[F])^\bullet \in \mathfrak{D}$. This means that the image of $\mathfrak{D}$ in $\mathfrak{C}_n/\hat{\nu}_n^{-1}[\{0\}]$ is the whole of $\mathfrak{C}_n/\hat{\nu}_n^{-1}[\{0\}]$.

Because T' is the σ-subalgebra of itself generated by $\mathcal{B}$, $\mathfrak{D}$ must be the σ-subalgebra of itself generated by $D = (f_n^{-1}[F])^\bullet : F \in \mathcal{B}\}$. But now $\mathfrak{C}_n/\hat{\nu}_n^{-1}[\{0\}]$ is the σ-subalgebra of itself generated by the image of D, and

$$\tau(\mathfrak{C}_n/\hat{\nu}_n^{-1}[\{0\}]) \le \#(D) \le \#(\mathcal{B}) = \tau(\nu_n). \ \mathbf{Q}$$

(e) It follows that

$$\tau_{\sigma\text{-m}}(\mathfrak{A}) \le \max(\omega, \sup_{n\in\mathbb{N}} \tau(\mathfrak{C}_n/\hat{\nu}_n^{-1}[\{0\}])) \le \max(\omega, \sup_{n\in\mathbb{N}} \tau(\nu_n)),$$

as claimed.

548B The next argument will go more smoothly if we know the following not-quite-trivial fact.

Lemma Lat (X, Σ, μ) be a probability space and $\langle X_i\rangle_{i\in\mathbb{N}}$ a partition of X. Then μ can be extended to a probability measure ν measuring every X_i and with Maharam type $\tau(\nu) \le \max(\omega, \tau(\mu))$.

proof By 214P, there is a measure λ on X, extending μ, such that $\lambda(E \cap \bigcup_{i\le n} X_i)$ is defined and equal to $\mu^*(E \cap \bigcup_{i\le n} X_i)$ for every $E \in \Sigma$ and $n \in \mathbb{N}$. In particular, $\bigcup_{i\le n} X_i$ is measured by λ for every n, so X_n also is. Let ν be the restriction of λ to the σ-algebra T of subsets of X generated by $\Sigma \cup \{X_i : i \in \mathbb{N}\}$; then ν is a probability measure extending μ and measuring every X_i.

Let $\mathfrak{A}$, $\mathfrak{B}$ be the measure algebras of μ and ν, and for $E \in \Sigma$, $F \in \mathrm{T}$ write $E^\bullet$, F° for their images in $\mathfrak{A}$, $\mathfrak{B}$ respectively. There is a subset B of the measure algebra $\mathfrak{A}$ of μ such that $\#(B) = \tau(\mu)$ and B τ-generates $\mathfrak{A}$. Choose $\mathcal{B} \subseteq \Sigma$ such that $B = \{E^\bullet : E \in \mathcal{B}\}$ and $\#(\mathcal{B}) = \#(B)$, and take T' to be the σ-algebra of subsets of X generated by $\mathcal{B} \cup \{X_i : i \in \mathbb{N}\}$. Then $\mathrm{T}' \cap \Sigma$ is a σ-subalgebra of Σ so its image $\mathfrak{A}' \subseteq \mathfrak{A}$ is a σ-subalgebra of $\mathfrak{A}$ including $\mathfrak{B}$. Because $\mathfrak{A}$ is ccc, $\mathfrak{A}'$ is a closed subalgebra of $\mathfrak{A}$ and must be the whole of $\mathfrak{A}$. But this means that for every $E \in \Sigma$ there is an $E' \in \mathrm{T}' \cap \Sigma$ such that $0 = \mu(E \triangle E') = \nu(E \triangle E')$. Moving now to $\mathfrak{B}$, we see that $\mathfrak{B}' = \{F^\circ : F \in \mathrm{T}'\}$ is the σ-subalgebra of $\mathfrak{B}$ generated by $\{F^\circ : F \in \mathcal{B}\} \cup \{X_i^\circ : i \in \mathbb{N}\}$. As $\mathfrak{B}'$ contains E° for every $E \in \Sigma$, it is in fact the whole of $\mathfrak{B}$, so

$$\tau(\nu) = \tau(\mathfrak{B}) \le \#(\mathcal{B} \cup \{X_i : i \in \mathbb{N}\}) \le \max(\omega, \#(\mathcal{B})) = \max(\omega, \tau(\mu)).$$

548C Theorem (see KUMAR & SHELAH 17) Suppose that (X, Σ, μ) is an atomless σ-finite measure space and that there is no quasi-measurable cardinal less than the Maharam type $\tau(\mu)$ of μ. Let $R \subseteq X \times X$ be an equivalence relation with countable equivalence classes. Then there is an R-free set (definition: 5A1Jd) which has full outer measure in X.

proof (a) If $\mu X = 0$ this is trivial, as we can take the empty set; suppose otherwise. As μ is atomless, $\tau(\mu)$ is infinite. There is a probability measure on X with the same measurable sets and the same negligible sets as μ (215B); this will have the same quotient algebra $\Sigma/\mathcal{N}$ and the same Maharam type, so we may take it that μ itself is a probability measure. By 5A1J(e-i) there is a partition $\langle X_i\rangle_{i\in\mathbb{N}}$ of X into R-free sets. By 548B there is a probability measure on X extending μ, measuring every X_i and with the same

Maharam type; as extending the measure never increases the outer measure of a set, we can suppose that this extension has been done, and that every X_i is measured by μ. Completing μ does not change the sets of full outer measure (212Eb) or the measure algebra (322Da), so we may suppose that μ is complete.

For each $i \in \mathbb{N}$, set $f_i = R \cap (X \times X_i)$; then $f_i : R[X_i] \to X_i$ is a function which is injective on every X_k.

(b) Suppose that we are given Y, $A \subseteq X$ such that $\mu^* A > 0$. Then there is a $B \subseteq A$ such that $\mu^* B > 0$ and $\mu^*(Y \setminus R[B]) = \mu^* Y$. **P?** Otherwise, take ν_i to be the subspace measure on $Y \cap X_i$ for each $i \in \mathbb{N}$, and set $\mathcal{I} = \{B : B \subseteq A,\ \mu^* B = 0\}$, the null ideal of the subspace measure on A. Then $\mathcal{I}$ is a proper σ-ideal of $\mathcal{P}A$. Because μ is atomless, $\mathcal{I}$ contains every singleton subset of A.

Take any $B \in \mathcal{P}A \setminus \mathcal{I}$. We are supposing that $\mu^*(Y \setminus R[B]) < \mu^* Y$. If G is a measurable envelope of $Y \setminus R[B]$, $\mu^*(Y \setminus G) > 0$ and there is an $i \in \mathbb{N}$ such that $\mu^*(Y \cap X_i \setminus G) > 0$, in which case $F = Y \cap X_i \setminus G$ belongs to the domain of ν_i and $\nu_i F > 0$. At the same time, $F \subseteq R[B] \cap X_i$, so if $y \in F$ there is an $x \in B$ such that $(x, y) \in R$ and $f_i(x) = y$; that is, $F \subseteq f_i[B]$.

Accordingly A, $\mathcal{I}$, $\langle (Y_i, \nu_i) \rangle_{i \in \mathbb{N}}$ and $\langle f_i \restriction A \rangle_{i \in \mathbb{N}}$ satisfy the conditions of 548A and $\mathfrak{A} = \mathcal{P}A/\mathcal{I}$ is σ-measurable, with

$$\tau_{\sigma\text{-m}}(\mathfrak{A}) \le \max(\omega, \sup_{i \in \mathbb{N}} \tau(\nu_i)) \le \max(\omega, \tau(\mu)) = \tau(\mu).$$

(For the second inequality, note that as μ is a probability measure ν_i is totally finite and $\tau(\nu_i) \le \tau(\mu)$ for every $i \in \mathbb{N}$ (521Ff).) Also $\mathcal{P}A/\mathcal{I}$ is atomless, because if $C \in \mathcal{P}A \setminus \mathcal{I}$ the subspace measure on C is totally finite and atomless (214Q), so there is a $C' \subseteq C$ such that neither C' nor $C \setminus C'$ is μ-negligible. So 547R tells us that $\tau_{\sigma\text{-m}}(\mathfrak{A}) > \operatorname{add} \mathcal{I}$. But $\mathcal{P}A/\mathcal{I}$, being σ-measurable, is ccc, so $\mathcal{I}$ is ω_1-saturated, and $\operatorname{add} \mathcal{I}$ is quasi-measurable and less than $\tau(\mu)$, contrary to hypothesis. **XQ**

(c) Now suppose that Y, $A \subseteq X$ and A is R-free. Then there is a $B \subseteq A$ such that $\mu^* B \ge \frac{1}{2} \mu^* A$ and $\mu^*(Y \setminus R[B]) = \mu^* Y$. **P?** Otherwise, let $\mathcal{B}$ be the family of those sets $B \subseteq A$ such that $\mu^*(Y \setminus R[B]) = \mu^* Y$, and choose $\langle B_n \rangle_{n \in \mathbb{N}}$, $\langle \gamma_n \rangle_{n \in \mathbb{N}}$ inductively, as follows. $B_0 = \emptyset$. Given that $B_n \in \mathcal{B}$, set $\gamma_n = \sup\{\mu^* B : B_n \subseteq B \in \mathcal{B}\}$ and choose B_{n+1} such that $B_n \subseteq B_{n+1} \in \mathcal{B}$ and $\mu^* B_{n+1} \ge \gamma_n - 2^{-n}$. Continue.

At the end of the induction, set $C = \bigcup_{n \in \mathbb{N}} B_n$. As $B_n \in \mathcal{B}$, our counter-hypothesis declares that $\mu^* B_n < \frac{1}{2} \mu^* A$ for every n; as $\langle B_n \rangle_{n \in \mathbb{N}}$ is non-decreasing, $\mu^* C < \mu^* A$. Let $E \in \Sigma$ be a measurable envelope of C; then $A \setminus E$ is non-negligible. Set $Y' = Y \setminus R[C]$. Applying (b) to Y and $A \setminus E$, we see that there is a $B \subseteq A \setminus E$ such that $\mu^* B > 0$ and $\mu^*(Y' \setminus R[B]) = \mu^* Y'$. As A is R-free and $B \cap C = \emptyset$, $R[B] \cap R[C] = \emptyset$ (5A1J(e-ii)).

There must be an $n \in \mathbb{N}$ such that $\mu^* B_{n+1} + 2^{-n} < \mu^* B_n + \mu^* B$. Now

$$\begin{aligned}\mu^*(B_n \cup B) &= \mu^*((B_n \cup B) \cap E) + \mu^*((B_n \cup B) \setminus E) \\ &= \mu^* B_n + \mu^* B > \mu^* B_{n+1} + 2^{-n} \ge \gamma_n.\end{aligned}$$

So $B_n \cup B \notin \mathcal{B}$ and $\mu^*(Y \setminus R[B_n \cup B]) < \mu^* Y$. Let $F \in \Sigma$ be the complement of a measurable envelope of $Y \setminus R[B_n \cup B]$, so that $Y \cap F$ is non-negligible and included in $R[B_n \cup B] = R[B_n] \cup R[B]$. As $R[B_n] \subseteq R[C]$, $Y' \cap F \subseteq R[B]$. Since $\mu^*(Y' \setminus R[B]) = \mu^* Y'$, $Y' \cap F$ must be negligible. We are supposing that μ is complete, so $F' = F \setminus Y'$ belongs to Σ and $\mu(Y \cap F') > 0$. But

$$Y \cap F' \subseteq (R[B_n] \cup R[B]) \cap R[C] = R[B_n],$$

so $\mu^*(Y \setminus R[B_n]) < \mu^* Y$; which is not so, because $B_n \in \mathcal{B}$. **XQ**

(d) Fix on a sequence $\langle k_n \rangle_{n \in \mathbb{N}}$ running over $\mathbb{N}$ with cofinal repetitions, and choose a sequence $\langle A_n \rangle_{n \in \mathbb{N}}$ of subsets of X inductively, as follows. The inductive hypothesis will be that $\mu^*(X \setminus R[A_n]) = 1$ and that A_n is R-free. Start with $A_0 = \emptyset$. Given A_n, choose a measurable envelope G for $X_{k_n} \cap A_n$ and apply (c) to find a $B \subseteq X_{k_n} \setminus (G \cup R[A_n])$ such that $\mu^* B \ge \frac{1}{2} \mu^*(X_{k_n} \setminus (G \cup R[A_n]))$ and

$$\mu^*((X \setminus R[A_n]) \setminus R[B]) = \mu^*(X \setminus R[A_n]) = 1.$$

Set $A_{n+1} = A_n \cup B$. As A_n and $B \subseteq X_{k_n}$ are R-free and B is disjoint from $R[A_n]$, A_{n+1} is R-free, and the induction continues.

We also see that, whenever k, $n \in \mathbb{N}$ and $k_n = k$, there is a measurable envelope G of $X_k \cap A_n$ such that $A_{n+1} \setminus A_n \subseteq X_k \setminus G$ and $\mu^*(A_{n+1} \setminus A_n) \ge \frac{1}{2} \mu^*(X_k \setminus (G \cup R[A_n]))$. So

$$\begin{aligned}\mu^*(A_{n+1}\cap X_k) &= \mu^*(A_{n+1}\cap X_k\cap G)+\mu^*(A_{n+1}\cap X_k\setminus G)\\ &= \mu^*(A_n\cap X_k)+\mu^*(A_{n+1}\setminus A_n)\\ &\ge \mu^*(A_n\cap X_k)+\frac{1}{2}\mu^*((X_k\setminus R[A_n])\setminus G)\\ &= \mu^*(A_n\cap X_k)+\frac{1}{2}\mu(X_k\setminus G)\end{aligned}$$

(because $X\setminus R[A_n]$ has full outer measure and $X_k\setminus G$ is measurable)

$$= \mu^*(A_n\cap X_k)+\frac{1}{2}(\mu X_k-\mu^*(X_k\cap A_n))=\frac{1}{2}(\mu X_k+\mu^*(A_n\cap X_k)).$$

But since $\{n:k_n=k\}$ is infinite, we see that $\lim_{n\to\infty}\mu^*(A_n\cap X_k)=\mu X_k$; and this is true for every $k\in\mathbb{N}$.

(d) At the end of the induction, $A=\bigcup_{n\in\mathbb{N}}A_n$ is the union of an upwards-directed family of R-free sets so is R-free. Also, because $\langle X_k\rangle_{k\in\mathbb{N}}$ is a partition of X into measurable sets,

$$\mu^*A=\sum_{k=0}^{\infty}\mu^*(A\cap X_k)=\sum_{k=0}^{\infty}\lim_{n\to\infty}\mu^*(A_n\cap X_k)=\sum_{k=0}^{\infty}\mu X_k=1.$$

So we have found an R-free set of full outer measure.

548D Lemma Let (X,Σ,μ) be a measure space. Then the following are equiveridical:

(i) whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of subsets of X there is a disjoint sequence $\langle A'_n\rangle_{n\in\mathbb{N}}$ of sets such that $A'_n\subseteq A_n$ and $\mu^*(A'_n)=\mu^*(A_n)$ for every $n\in\mathbb{N}$;

(ii) whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of subsets of X there is a set $D\subseteq X$ such that $\mu^*(A_n\cap D)=\mu^*(A_n\setminus D)=\mu^*A_n$ for every $n\in\mathbb{N}$.

proof (a) If (i) is true and $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of subsets of X, set $B_{2n}=B_{2n+1}=A_n$ for every $n\in\mathbb{N}$, and take a disjoint sequence $\langle B'_n\rangle_{n\in\mathbb{N}}$ of sets such that $B'_n\subseteq B_n$ and $\mu^*B'_n=\mu^*B_n$ for every n. Set $D=\bigcup_{n\in\mathbb{N}}B'_n$; then $\mu^*(A_n\cap D)\ge\mu^*B'_{2n}=\mu^*A_n$ and $\mu^*(A_n\setminus D)\ge\mu^*B'_{2n+1}=\mu^*A_n$, so then $\mu^*(A_n\cap D)=\mu^*(A_n\setminus D)=\mu^*A_n$, for every $n\in\mathbb{N}$.

(b) If (ii) is true and $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of subsets of X, choose $\langle D_n\rangle_{n\in\mathbb{N}}$ inductively so that

$$\mu^*(A_m\cap D_n\setminus\bigcup_{i<n}D_i)=\mu^*(A_m\setminus\bigcup_{i\le n}D_i)=\mu^*(A_m\setminus\bigcup_{i<n}D_i)$$

for every m, $n\in\mathbb{N}$. Then in fact $\mu^*(A_m\cap D_n\setminus\bigcup_{i<n}D_i)=\mu^*A_m$ for all m and n, so if we set $A'_n=A_n\cap D_n\setminus\bigcup_{i<n}D_i$ for each n, we see that $\langle A'_n\rangle_{n\in\mathbb{N}}$ is disjoint and $\mu^*A'_n=\mu^*A_n$ for every $n\in\mathbb{N}$.

548E Theorem (see Gitik & Shelah 01) Let (X,Σ,μ) be an atomless σ-finite measure space such that there is no quasi-measurable cardinal less than the Maharam type of μ. Then for any sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of subsets of X there is a disjoint sequence $\langle A'_n\rangle_{n\in\mathbb{N}}$ such that $A'_n\subseteq A_n$ and $\mu^*A'_n=\mu^*A_n$ for every $n\in\mathbb{N}$.

proof For each $n\in\mathbb{N}$ let ν_n be the subspace measure on A_n; set $Y=\bigcup_{n\in\mathbb{N}}A_n\times\{n\}$ and give Y the σ-finite direct sum measure ν, so that $\nu E=\sum_{n=0}^{\infty}\nu_n(E^{-1}[\{n\}])$ whenever $E\subseteq Y$ is such that $E^{-1}[\{n\}]$ is measured by ν_n for every n. Then the Maharam type of ν is at most $\sum_{n=0}^{\infty}\tau(\nu_n))\le\max(\omega,\tau(\mu))$ (521G, 521Ff), so there is no quasi-measurable cardinal less than $\tau(\nu)$. Also every ν_n is atomless (214Q again) so ν is atomless (214Xh).

Let R be the equivalence relation

$$\{((x,m),(x,n)):m,\,n\in\mathbb{N},\,x\in A_m\cap A_n\}\subseteq Y\times Y.$$

By 548C, there is an R-free set $A\subseteq Y$ of full outer measure in Y. Set $A'_n=\{x:(x,n)\in A\}$ for each n. Then $\langle A'_n\rangle_{n\in\mathbb{N}}$ is disjoint and

$$\mu^*A'_n=\nu^*_nA'_n=\nu^*(A\cap(A_n\times\{n\}))=\nu(A_n\times\{n\})=\nu_nA_n=\mu^*A_n$$

for every n, as required.

548F Corollary Let μ be Lebesgue measure on $\mathbb{R}$.

(a) Let X be a subset of $\mathbb{R}$ and $\sim$ an equivalence relation on X with countable equivalence classes. Then there is a subset of X, with full outer measure for the subspace measure on X, which meets each equivalence class in at most one point.

(b) Let $\langle A_n\rangle_{n\in\mathbb{N}}$ be any sequence of subsets of $\mathbb{R}$. Then there is a disjoint sequence $\langle A'_n\rangle_{n\in\mathbb{N}}$ such that A'_n is a subset of A_n, with the same outer measure as A_n, for every n.

(c) Let $\langle A_n\rangle_{n\in\mathbb{N}}$ be any sequence of subsets of $\mathbb{R}$. Then there is a $D\subseteq\mathbb{R}$ such that $\mu^*(A_n\cap D) = \mu^*(A_n\setminus D) = \mu^*A_n$ for every $n\in\mathbb{N}$.

proof For (a)-(b), apply 548C and 548E, noting that the measures here have countable Maharam type and that there is surely no quasi-measurable cardinal less than ω. For (c), use 548D.

548G The point of the formulation of 548C-548E in terms of Maharam types is that we get non-trivial corollaries which are valid in ZFC, just as we did in 547G. With other hypotheses involving quasi-measurable cardinals we can get further results, as in the following.

Lemma Let (X,Σ,μ) be a totally finite measure space in which singleton sets are negligible and suppose that there is no quasi-measurable cardinal less than or equal to the shrinking number $\operatorname{shr}\mathcal{N}(\mu)$. Then for any $A\subseteq X$ there is a disjoint family $\langle A_\xi\rangle_{\xi<\omega_1}$ of subsets of A such that $\mu^*A_\xi=\mu^*A$ for every $\xi<\omega_1$.

Remark 548C-548E refer to atomless measure spaces rather than those in which singletons are negligible. Of course singletons are negligible in any atomless totally finite measure space (215E).

proof (a) Let $\mathcal{E}$ be the set of those $F\in\Sigma$ such that $\mu F>0$ and there is a disjoint family $\langle B_\xi\rangle_{\xi<\omega_1}$ of subsets of $F\cap A$ such that $\mu^*B_\xi=\mu F$ for every $\xi<\omega_1$. Then whenever $E\in\Sigma$ and $\mu^*(E\cap A)>0$ there is an $F\in\mathcal{E}$ included in E. **P** There is a $B\subseteq E\cap A$ such that $\#(B)\le\operatorname{shr}\mathcal{N}(\mu)$ and $\mu^*B>0$. Consider the ideal $\mathcal{N}$ of negligible subsets of B. This is a σ-ideal containing every singleton set; as $\#(B)$ is not quasi-measurable, $\mathcal{N}$ cannot be ω_1-saturated and there is a disjoint family $\langle C_\xi\rangle_{\xi<\omega_1}$ of non-negligible subsets of B. For each $\xi<\omega_1$ let $E_\xi\subseteq E$ be a measurable envelope of C_ξ and $a_\xi=E^\bullet_\xi$ the corresponding element in the measure algebra $\mathfrak{A}$ of μ. For $\xi<\omega_1$ set $b_\xi=\sup_{\eta\ge\xi}a_\eta$; then $\langle b_\xi\rangle_{\xi<\omega_1}$ is non-increasing. As $\mathfrak{A}$ is ccc, there is a $\zeta<\omega_1$ such that $b_\xi=b_\zeta$ whenever $\zeta\le\xi<\omega_1$. Write b for b_ζ; of course $b\supseteq a_\zeta$ is non-zero, and also $b\subseteq E^\bullet$.

If $\xi<\omega_1$ there is an $\eta_\xi<\omega_1$ such that $\sup_{\xi\le\eta<\eta_\xi}a_\eta=b_\xi\supseteq b$. Let $f:\omega_1\to\omega_1$ be a strictly increasing function such that $f(\xi+1)=\eta_{f(\xi)}$ for every $\xi<\omega_1$. Consider $B_\xi=\bigcup_{f(\xi)\le\eta<f(\xi+1)}C_\eta$, $F_\xi=\bigcup_{f(\xi)\le\eta<f(\xi+1)}E_\eta$ for $\xi<\omega_1$. Then $\langle B_\xi\rangle_{\xi<\omega_1}$ is disjoint, F_ξ is a measurable envelope of B_ξ for each ξ (132Ed), and $F^\bullet_\xi=\sup_{f(\xi)\le\eta<f(\xi+1)}a_\eta\supseteq b$. So if we take $F\in\Sigma$ such that $F\subseteq E$ and $F^\bullet=b$, $\langle F\cap B_\xi\rangle_{\xi<\omega_1}$ is a disjoint family of subsets of $F\cap A$ witnessing that $F\in\mathcal{E}$. **Q**

(b) Let $\langle F_i\rangle_{i\in I}$ be a maximal disjoint family in $\mathcal{E}$; set $G=X\setminus\bigcup_{i\in I}F_i$. Then I is countable and $G\in\Sigma$. By (a), $G\cap A$ must be negligible. For each $i\in I$, choose a disjoint family $\langle B_{i\xi}\rangle_{\xi<\omega_1}$ of subsets of $F_i\cap A$ such that $\mu^*B_{i\xi}=\mu F_i$ for every ξ. Set $A_\xi=\bigcup_{i\in I}B_{i\xi}$ for each ξ. Then $\langle A_\xi\rangle_{\xi<\omega_1}$ is a disjoint family of subsets of A. Now, for each $\xi<\omega_1$,

$$\mu^*A\le\sum_{i\in I}\mu F_i=\sum_{i\in I}\mu^*B_{i\xi}=\sum_{i\in I}\mu^*(A_\xi\cap F_i)=\sum_{i\in I}\mu_{A_\xi}(A_\xi\cap F_i)$$

(where μ_{A_ξ} is the subspace measure on A_ξ)

$$\le\mu_{A_\xi}A_\xi=\mu^*A_\xi\le\mu^*A$$

and we have $\mu^*A=\mu^*A_\xi$, as required.

Remark For cases in which one of the hypotheses

'there is no quasi-measurable cardinal less than $\tau(\mu)$',

'there is no quasi-measurable cardinal less than or equal to $\operatorname{shr}\mathcal{N}(\mu)$'

is satisfied and the other is not, see 548Ya and 555Yg.

548H Proposition Let (X,Σ,μ) be a totally finite measure space. Suppose that for every $A\subseteq X$ there is a partition $\langle A_\xi\rangle_{\xi<\omega_1}$ of A such that $\mu^*A_\xi=\mu^*A$ for every $\xi<\omega_1$.

(a) If R is an equivalence relation on X with countable equivalence classes, there is an R-free set with full outer measure.

(b) For any sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of subsets of X there is a disjoint sequence $\langle A'_n\rangle_{n\in\mathbb{N}}$ such that $A'_n\subseteq A_n$ and $\mu^*A'_n=\mu^*A_n$ for every $n\in\mathbb{N}$.

(c) For any sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of subsets of X, there is a $D\subseteq X$ such that $\mu^*(A_n\cap D)=\mu^*(A_n\setminus D)=\mu^*A_n$ for every $n\in\mathbb{N}$.

proof (a) By 5A1J(e-i), there is a partition $\langle X_n\rangle_{n\in\mathbb{N}}$ of X into R-free sets. Now we can choose inductively a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of sets such that $A_n\subseteq X_n\setminus\bigcup_{i<n}R[A_i]$ and $\mu^*A_n=\mu^*X_n$ for every n, while $\mu^*(X_j\setminus\bigcup_{i<n}R[A_i])=\mu^*X_j$ whenever $j\ge n$. **P** Given $\langle A_i\rangle_{i<n}$ write $B_{jn}=X_j\setminus\bigcup_{i<n}R[A_i]$ for $j\in\mathbb{N}$ and choose a partition $\langle C_\xi\rangle_{\xi<\omega_1}$ of B_{nn} such that $\mu^*C_\xi=\mu^*B_{nn}$ for every $\xi<\omega_1$; by the inductive hypothesis, $\mu^*C_\xi=\mu^*X_n$ for every ξ. Now for any $j\in\mathbb{N}$, $\langle R[C_\xi]\rangle_{\xi<\omega_1}$ is disjoint (5A1J(e-ii)), so $\mu^*(B_{jn}\setminus R[C_\xi])=\mu^*B_{jn}$ for all but countably many ξ (521Od). There is therefore a $\xi<\omega_1$ such that $\mu^*(B_{jn}\setminus R[C_\xi])=\mu^*B_{jn}$ for every $j\in\mathbb{N}$. If $j>n$, then

$$\mu^*(X_j\setminus(R[C_\xi]\cup\bigcup_{i<n}R[A_i]))=\mu^*(B_{jn}\setminus R[C_\xi])=\mu^*B_{jn}=\mu^*X_j$$

by the inductive hypothesis; so if we set $A_n=C_\xi$ the induction will continue. **Q**

At the end of the induction, we see that A_n is R-free and $A_n\cap\bigcup_{i<n}R[A_i]=\emptyset$ for every n, so $A=\bigcup_{n\in\mathbb{N}}A_n$ is R-free. And because $X=\bigcup_{n\in\mathbb{N}}X_n$ and $\mu^*A_n=\mu^*X_n$ for every n, A has full outer measure. **P** If $F\in\Sigma$ and $\mu F>0$, there is an $n\in\mathbb{N}$ such that $\mu^*(F\cap X_n)>0$; moving to the subspace measure μ_{X_n} on X_n, $\mu_{X_n}(F\cap X_n)>0$ so

$$0<\mu^*_{X_n}(F\cap A_n)=\mu^*(F\cap A_n)$$

(214Cd)

$$\le\mu^*(F\cap A).\ \mathbf{Q}$$

(b) As in the proof of 548E, we can apply (a) to the direct sum measure ν on $Y=\bigcup_{n\in\mathbb{N}}A_n\times\{n\}$, though this time we take the measure on A_n to be $\nu_n=2^{-n}\mu_{A_n}$ in order to ensure that ν is totally finite. Of course we have to check that the hypothesis of this proposition is satisfied by (Y,ν), but this is easy. Now we can use the same equivalence relation R on Y as in 548E, and if $A\subseteq Y$ is an R-free set of full outer measure the last line will read

$$\mu^*A'_n=2^n\nu^*_nA'_n=2^n\nu^*(A\cap(A_n\times\{n\})=2^n\nu(A_n\times\{n\})=2^n\nu_nA_n=\mu^*A_n$$

for every n.

(c) This now follows from (b) by 548D.

548I I do not know how far we can hope to extend 548H(b-c) to uncountable families in place of $\langle A_n\rangle_{n\in\mathbb{N}}$. If in place of

$$\mu^*(A_n\cap D)=\mu^*(A_n\setminus D)=\mu^*A_n$$

we ask rather for

$$\min(\mu^*(A_n\cap D),\mu^*(A_n\setminus D))>0$$

we are led to rather different patterns, as follows.

Proposition Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and κ a cardinal. Then the following are equiveridical:

(i) $\kappa<\pi(\mathfrak{A}_d)$ for every $d\in\mathfrak{A}^+=\mathfrak{A}\setminus\{0\}$, writing $\mathfrak{A}_d$ for the principal ideal generated by d;

(ii) whenever $A\subseteq\mathfrak{A}^+$ and $\#(A)\le\kappa$ there is a $b\in\mathfrak{A}$ such that $a\cap b$ and $a\setminus b$ are both non-zero for every $a\in A$.

proof (a) Suppose that (i) is false; that there are a non-zero $d\in\mathfrak{A}$ and an order-dense set $A\subseteq\mathfrak{A}^+_d$ such that $\#(A)\le\kappa$. If $b\in\mathfrak{A}$ and $b\cap d=0$ then $a\cap b=0$ for every $a\in A$; if $b\cap d\ne0$ then there is an $a\in A$ such that $a\subseteq b\cap d$ and $a\setminus b=0$. So A witnesses that (ii) is false.

For the rest of the proof, therefore, I suppose that (i) is true and seek to prove (ii).

(b) We need an elementary calculation. Let $\langle c_i\rangle_{i<n}$ be a stochastically independent family in $\mathfrak{A}$ such that $\bar\mu c_i = \gamma$ for every $i < n$, where $0 < \gamma < 1$. Suppose that $a \in \mathfrak{A}$ and that

$$\sup_{i<n} \bar\mu(a \cap c_i) \le \beta\gamma\bar\mu a$$

where $\beta < 1$. Then $n \le \frac{1}{(1-\beta)^2\gamma\bar\mu a}$.

P In $L^2(\mathfrak{A}, \bar\mu)$ set

$$e_i = \sqrt{\tfrac{\gamma}{1-\gamma}}\chi(1 \setminus c_i) - \sqrt{\tfrac{1-\gamma}{\gamma}}\chi c_i$$

for each $i < n$. An easy calculation shows that $\langle e_i\rangle_{i<n}$ is orthonormal. Next, for each i,

$$\begin{aligned}(e_i|\chi a) &= \sqrt{\frac{\gamma}{1-\gamma}}\bar\mu(a \setminus c_i) - \sqrt{\frac{1-\gamma}{\gamma}}\bar\mu(a \cap c_i)\\ &\ge \sqrt{\frac{\gamma}{1-\gamma}}(\bar\mu a - \beta\gamma\bar\mu a) - \sqrt{\frac{1-\gamma}{\gamma}}\beta\gamma\bar\mu a\\ &= \sqrt{\frac{1}{\gamma(1-\gamma)}}(\gamma(\bar\mu a - \beta\gamma\bar\mu a) - (1-\gamma)\beta\gamma\bar\mu a)\\ &= \gamma\bar\mu a\sqrt{\frac{1}{\gamma(1-\gamma)}}(1-\beta) \ge (1-\beta)\bar\mu a\sqrt{\gamma}.\end{aligned}$$

By 4A4Ji,

$$\bar\mu a = \|\chi a\|_2^2 \ge \sum_{i<n} |(e_i|\chi a)|^2 \ge n\gamma(1-\beta)^2(\bar\mu a)^2$$

and

$$n \le \frac{1}{(1-\beta)^2\gamma\bar\mu a}$$

as claimed. **Q**

(c) If $\mathfrak{A}$ has an atom, then $\kappa = 0$ and there is nothing to prove. So we may suppose henceforth that $\mathfrak{A}$ is atomless. Set $\lambda = \min\{\tau(\mathfrak{A}_d) : d \in \mathfrak{A}^+\}$. Then the measure algebra $(\mathfrak{B}_\lambda, \bar\nu_\lambda)$ can be embedded in $(\mathfrak{A}, \bar\mu)$ (332P). In particular, we can find, for each n, a stochastically independent family $\langle c_{n\xi}\rangle_{\xi<\lambda}$ of elements of $\mathfrak{A}$ with $\bar\mu c_{n\xi} = \frac{1}{n!}$ for every ξ.

Set $q(n) = 4n((2n)! + (2n+1)!)$ for each $n \in \mathbb{N}$, and let $(\lambda^{\mathbb{N}}, \subseteq^*, \mathcal{S}_\lambda^{(q)})$ be the corresponding version of the λ-localization relation as described in 522L. For $a \in \mathfrak{A}^+$ let $n_a \in \mathbb{N}$ be such that $n_a\bar\mu a \ge 1$ and set

$$\begin{aligned}S_a = \{(n,\xi) : n \ge n_a \text{ and either } \bar\mu(a \cap c_{2n,\xi}) \le \frac{\bar\mu a}{2(2n)!} \text{ or } \bar\mu(a \cap c_{2n+1,\xi}) \le \frac{\bar\mu a}{2(2n+1)!}\}\\ \subseteq \mathbb{N} \times \lambda.\end{aligned}$$

Then $S_a \in \mathcal{S}_\lambda^{(q)}$. **P** If $n \ge n_a$ then (b), with $\beta = \frac{1}{2}$, tells us that

$$\#(\{\xi : \bar\mu(a \cap c_{2n,\xi}) \le \frac{\bar\mu a}{2(2n)!}\}) \le \frac{4(2n)!}{\bar\mu a} \le 4n(2n)!,$$

$$\#(\{\xi : \bar\mu(a \cap c_{2n+1,\xi}) \le \frac{\bar\mu a}{2(2n+1)!}\}) \le \frac{4(2n+1)!}{\bar\mu a} \le 4n(2n+1)!,$$

so $\#(S_a[\{n\}]) \le 4n((2n)! + (2n+1)!) = q(n)$. **Q**

(d) Now observe that

$$\min\{\pi(\mathfrak{A}_d) : d \in \mathfrak{A}^+\} = \pi(\mathfrak{B}_\lambda)$$

(see 524Mc)

$$= \mathrm{ci}(\mathfrak{B}_\lambda^+) = \mathrm{cov}(\mathfrak{B}_\lambda^+, \supseteq, \mathfrak{B}_\lambda^+) = \mathrm{cov}(\mathfrak{B}_\lambda^+, \supseteq', [\mathfrak{B}_\lambda^+]^{\le\omega})$$

(512Gf)

$$= \mathrm{cov}(\lambda^{\mathbb{N}}, \subseteq^*, \mathcal{S}_\lambda)$$

(where $(\lambda^{\mathbb{N}}, \subseteq^*, \mathcal{S}_\lambda)$ is the ordinary λ-localization relation, by 524H and 512Da)

$$= \mathrm{cov}(\lambda^{\mathbb{N}}, \subseteq^*, \mathcal{S}_\lambda^{(q)})$$

by 522L.

(e) Let $A \subseteq \mathfrak{A}^+$ be a set with cardinal less than $\min\{\pi(\mathfrak{A}_d) : d \in \mathfrak{A}^+\}$. Then $\#(A) < \mathrm{cov}(\lambda^{\mathbb{N}}, \subseteq^*, \mathcal{S}_\lambda^{(q)})$, so there must be an $f \in \lambda^{\mathbb{N}}$ such that $f \not\subseteq^* S_a$ for any $a \in A$. Set

$$b_{2n} = c_{2n,f(n)}, \quad b_{2n+1} = c_{2n+1,f(n)}, \quad b'_n = b_n \setminus \sup_{i>n} b_i \text{ for } n \in \mathbb{N},$$

$$b = \sup_{n\in\mathbb{N}} b'_{2n}.$$

Then $a \cap b$ and $a \setminus b$ are both non-zero for every $a \in A$. **P** There is an $n \ge n_a$ such that $(n, f(n)) \notin S_a$, so that

$$\bar\mu(a \cap b_{2n}) = \bar\mu(a \cap c_{2n,f(n)}) > \frac{\bar\mu a}{2(2n)!} \ge \frac{1}{2n(2n)!},$$

$$\bar\mu(a \cap b_{2n+1}) = \bar\mu(a \cap c_{2n+1,f(n)}) > \frac{\bar\mu a}{2(2n+1)!} \ge \frac{1}{2n(2n+1)!}.$$

But

$$\bar\mu(b_{2n} \setminus b'_{2n}) \le \sum_{i=2n+1}^\infty \bar\mu b_i = \sum_{i=2n+1}^\infty \frac{1}{i!}$$

$$\le \frac{1}{(2n)!}\sum_{j=1}^\infty \frac{1}{(2n+1)^j} = \frac{1}{2n(2n)!} < \bar\mu(a \cap b_{2n}),$$

$$\bar\mu(b_{2n+1} \setminus b'_{2n+1}) \le \frac{1}{(2n+1)(2n+1)!} < \bar\mu(a \cap b_{2n+1}),$$

so $a \cap b \supseteq a \cap b'_{2n}$ and $a \setminus b \supseteq a \cap b'_{2n+1}$ are both non-zero. **Q**

(f) As A is arbitrary, (ii) is true.

548J Proposition Let (X, Σ, μ) be a strictly localizable measure space with null ideal $\mathcal{N}(\mu)$, and κ a cardinal such that

(*) whenever $\mathcal{E} \in [\Sigma \setminus \mathcal{N}(\mu)]^{\le\kappa}$ and $F \in \Sigma \setminus \mathcal{N}(\mu)$, there is a non-negligible measurable $G \subseteq F$ such that $E \setminus G$ is non-negligible for every $E \in \mathcal{E}$.

Then whenever $\langle A_\xi\rangle_{\xi<\kappa}$ is a family of non-negligible subsets of X, there is a $G \in \Sigma$ such that $A_\xi \cap G$ and $A_\xi \setminus G$ are non-negligible for every $\xi < \kappa$.

proof (a) Suppose to begin with that $\mu X = 1$. Let $\mathfrak{A}$ be the measure algebra of μ. Then (*) says just that $\kappa < \pi(\mathfrak{A}_d)$ for every $d \in \mathfrak{A}^+$. For each $\xi < \kappa$ let E_ξ be a measurable envelope of A_ξ and set $a_\xi = E_\xi^\bullet$ in $\mathfrak{A}$. By 548I, there is a $b \in \mathfrak{A}$ such that $a_\xi \cap b$ and $a_\xi \setminus b$ are non-zero for every $\xi < \kappa$. Let $G \in \Sigma$ be such that $b = G^\bullet$; then $E_\xi \cap G$ and $E_\xi \setminus G$ are non-negligible for every ξ. But this means that $A_\xi \cap G$ and $A_\xi \setminus G$ are non-negligible for every ξ.

(b) If $\mu X = 0$ the result is trivial. For other totally finite μ, we get the result from (a) if we replace μ by a suitable scalar multiple.

(c) For the general case, let $\langle X_i\rangle_{i\in I}$ be a decomposition of X and for $i \in I$ set $J_i = \{\xi : \xi < \kappa,\ A_\xi \cap X_i$ is not negligible$\}$. By (b), applied to the subspace measure on X_i, there is a measurable $G_i \subseteq X_i$ such that $A_\xi \cap G_i$ and $A_\xi \cap X_i \setminus G_i$ are non-negligible for every $\xi \in J_i$. Set $G = \bigcup_{i\in I} G_i$; this works.

548K Corollary Let (X,Σ,μ) be an atomless quasi-Radon measure space and $\langle A_\xi\rangle_{\xi<\omega_1}$ a family of non-negligible subsets of X. Then there is a $D\subseteq X$ such that $A_\xi\cap D$ and $A_\xi\setminus D$ are non-negligible for every $\xi<\omega_1$.

proof (a) Suppose to begin with that μ is a Maharam-type-homogeneous probability measure. Let $\mathfrak{A}$ be the measure algebra of μ. If $\pi(\mathfrak{A})>\omega_1$ we can use 548J. Otherwise, the π-weight $\pi(\mu)$ of μ is ω_1 (524Tb); let $\langle E_\xi\rangle_{\xi<\omega_1}$ be a coinitial family in $\Sigma\setminus\mathcal{N}(\mu)$. Then we can choose $x_{\xi\eta}$ and $y_{\xi\eta}$, for ξ, $\eta<\omega_1$, so that

all the $x_{\xi\eta}$, $y_{\xi\eta}$ are different,

if $A_\xi\cap E_\eta\notin\mathcal{N}(\mu)$ then $x_{\xi\eta}$ and $y_{\xi\eta}$ belong to $A_\xi\cap E_\eta$.

Set $D=\{x_{\xi\eta}:\xi,\,\eta<\omega_1\}$; then $\mu^*(A_\xi\cap D)=\mu^*A_\xi$ for every ξ. **P?** Otherwise, let E be a measurable envelope of A_ξ and F a measurable envelope of $A_\xi\cap D$. We have $\mu(E\setminus F)>0$, so there is an $\eta<\omega_1$ such that $E_\eta\subseteq E\setminus F$, in which case

$$x_{\xi\eta}\in A_\xi\cap E_\eta\cap D\subseteq F.\ \textbf{XQ}$$

Similarly, $\mu^*(A_\xi\setminus D)=\mu^*A_\xi$ for every ξ, and we have a suitable set D.

(b) In general, X has a decomposition into Maharam-type-homogeneous subspaces (as in the proofs of 524J and 524P), so the full result follows as in (b)-(c) of the proof of 548J.

548X Basic exercises (a) Suppose that there is a real-valued-measurable cardinal. Show that there are an atomless probability space X and an equivalence relation $\sim$ on X with countable equivalence classes such that no set of full outer measure in X can meet every equivalence class in a finite set.

(b) (Kumar 13) Write μ for Lebesgue measure on $\mathbb{R}$. Show that for any $X\subseteq\mathbb{R}$ there is an $A\subseteq X$ such that $\mu^*A=\mu^*X$ and $|x-y|\notin\mathbb{Q}$ for any distinct x, $y\in A$.

(c) (A.Kumar) Let μ be Lebesgue measure on $\mathbb{R}$. Show that if $\mathcal{A}$ is any partition of $\mathbb{R}$ into countable sets, there are disjoint subsets $\mathcal{A}_0$, $\mathcal{A}_1$ of $\mathcal{A}$ such that $\bigcup\mathcal{A}_0$ and $\bigcup\mathcal{A}_1$ both have full outer measure in $\mathbb{R}$.

(d) Suppose that there is an atomlessly-measurable cardinal. Show that there are an atomless probability space X and a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of subsets of X such that whenever $A'_n\subseteq A_n$ is a set of full outer measure in A_n for each n, $\langle A'_n\rangle_{n\in\mathbb{N}}$ is not disjoint.

(e) Let (X,Σ,μ) be a complete semi-finite measure space with the measurable envelope property (definition: 213Xl), and suppose that μ is nowhere all-measuring in the sense of 214Yd, that is, that whenever $A\subseteq X$ is not μ-negligible there is a subset of A which is not measured by the subspace measure on A. (i) Show that for any $A\subseteq X$ there is a $D\subseteq A$ such that $\mu^*D=\mu^*(A\setminus D)=\mu^*A$. (ii) Show that whenever $\langle A_i\rangle_{i\in I}$ is a finite family of subsets of X, there is a disjoint family $\langle A'_i\rangle_{i\in I}$ such that $A'_i\subseteq A_i$ and $\mu^*A'_i=\mu^*A_i$ for every $i\in I$.

(f) Show that the following are equiveridical: (i) there is no quasi-measurable cardinal; (ii) if (X,Σ,μ) is a probability space such that $\mu\{x\}=0$ for every $x\in X$ then there is a disjoint family $\langle D_\xi\rangle_{\xi<\omega_1}$ of subsets of X such that $\mu^*D_\xi=1$ for every $\xi<\omega_1$.

(g) Let (X,Σ,μ) be a measure space such that $\operatorname{non}\mathcal{N}(\mu)=\pi(\mu)=\kappa\ge\omega$. Show that (α) if R is an equivalence relation on X and all its equivalence classes have size less than κ then there is an R-free set of full outer measure (β) whenever $\langle A_\xi\rangle_{\xi<\kappa}$ is a family of subsets of X there is a disjoint family $\langle A'_\xi\rangle_{\xi<\kappa}$ such that $A'_\xi\subseteq A_\xi$ and $\mu^*A'_\xi=\mu^*A_\xi$ for every $\xi<\kappa$.

(h) Let $\langle E_\xi\rangle_{\xi<\mathfrak{c}}$ be a family of Lebesgue measurable subsets of $\mathbb{R}$. Show that there is a disjoint family $\langle A_\xi\rangle_{\xi<\mathfrak{c}}$ of sets such that $A_\xi\subseteq E_\xi$ and $\mu^*A_\xi=\mu E_\xi$ for every $\xi<\mathfrak{c}$, where μ is Lebesgue measure on $\mathbb{R}$. (*Hint*: 419I.)

(i) (M.R.Burke) Let $\mathcal{N}$ be the null ideal of Lebesgue measure μ on $\mathbb{R}$. Show that if $2^{\operatorname{non}\mathcal{N}}=\mathfrak{c}$ then there is a family $\langle A_\xi\rangle_{\xi<\mathfrak{c}}$ of subsets of $\mathbb{R}$ such that for every $D\subseteq\mathbb{R}$ there is some $\xi<\mathfrak{c}$ such that $\min(\mu^*(A_\xi\cap D),\mu^*(A_\xi\setminus D))<\mu^*A_\xi$.

(j) Let $\mathfrak{A}$ be a Boolean algebra. (i) Show that the following are equiveridical: (α) $\pi(\mathfrak{A}_a) > \omega$ for every $a \in \mathfrak{A}^+$, where $\mathfrak{A}_a$ is the principal ideal generated by a (β) for every sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}^+$ there is a disjoint sequence $\langle b_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}^+$ such that $b_n \subseteq a_n$ for every n. (ii) Show that if $\mathfrak{A}$ has the σ-interpolation property then we can add (γ) whenever $A \subseteq \mathfrak{A}^+$ is countable, there is a $b \in \mathfrak{A}$ such that $a \cap b$ and $a \setminus b$ are both non-zero for every $a \in A$.

548Y Further exercises (a) Suppose that there is a quasi-measurable cardinal. Show that there is a probability space (X, μ) such that there is a quasi-measurable cardinal less than $\operatorname{shr}\mathcal{N}(\mu)$ but no quasi-measurable cardinal less than $\tau(\mu)$.

548Z Problems (a) Suppose that $\langle A_\xi \rangle_{\xi<\omega_1}$ is a family of subsets of $[0,1]$. Must there be a set $D \subseteq [0,1]$ such that $\mu^*(A_\xi \cap D) = \mu^*(A_\xi \setminus D) = \mu^* A_\xi$ for every $\xi < \omega_1$, where μ is Lebesgue measure on $[0,1]$?

(b) Suppose that there is no quasi-measurable cardinal. Let (X, Σ, μ) be an atomless probability space and $\langle A_\xi \rangle_{\xi<\omega_1}$ a family of subsets of X. Must there be a disjoint family $\langle A'_\xi \rangle_{\xi<\omega_1}$ such that $A'_\xi \subseteq A_\xi$ and $\mu^* A'_\xi = \mu^* A_\xi$ for every $\xi < \omega_1$?

(c) (P.Komjath) Suppose that $X \subseteq \mathbb{R}^2$. Must there be a set $A \subseteq X$, of the same Lebesgue outer measure as X, such that $\|x - y\| \notin \mathbb{Q}$ whenever x, $y \in A$ are distinct? (See 548Xb.)

548 Notes and comments Of course 548F is much the most important case of 548C-548E, with facts about Lebesgue measure provable in ZFC, whether or not there are quasi-measurable cardinals or special relationships between the cardinals of §522. As far as I know there is no real simplification available for this special case if we wish to avoid special axioms. In many models of set theory, of course, there are other approaches, as in 548G and 548Xg; and I note that it makes a difference that we start with not-necessarily-measurable sets A_n in 548Fb (548Xh).

The arguments here leave many obvious questions open. The first group concerns possible extensions of 548Fc or 548E to uncountable families of sets, as in 548Z. I remark that Shelah 03 describes a model in which there is a set $A \in \mathcal{P}\mathbb{R} \setminus \mathcal{N}$ such that $\mathcal{P}A \cap \mathcal{N}$ is ω_1-saturated in $\mathcal{P}A$, where $\mathcal{N}$ is the null ideal of Lebesgue measure. Elsewhere we can ask, in 548C and 548E, whether the hypotheses involving quasi-measurable cardinals could be rewritten with atomlessly-measurable cardinals. Only in 548G is it clear that non-atomlessly-measurable quasi-measurable cardinals are relevant (548Xf).

The questions tackled in this section can be re-phrased as questions about structures $(\mathcal{P}X/\mathcal{I}, \mathfrak{A})$ where $\mathcal{I}$ is a σ-ideal of subsets of X and $\mathfrak{A}$ is a σ-subalgebra of the power set σ-quotient algebra $\mathcal{P}X/\mathcal{I}$; a requirement of the form '$\mu^*(A \cap D) = \mu^* A$' becomes (in the context of a totally finite measure μ) '$\operatorname{upr}(a \cap d, \mathfrak{A}) = \operatorname{upr}(a, \mathfrak{A})$', where $\operatorname{upr}(a, \mathfrak{A})$ is the upper envelope of a in $\mathfrak{A}$ (313S).

I include 548I-548K to show that if we are less ambitious then there are quite different, and rather easier, arguments available. The condition (*) of 548J is exact if we are looking for a measurable splitting set G. But I am not at all sure that 548K is in the right form.

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