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Chapter 53

Topologies and measures III

In this chapter I return to the concerns of earlier volumes, looking for results which can be expressed in the language so far developed in this volume. In Chapter 43 I examined relationships between measure-theoretic and topological properties. The concepts we now have available (in particular, the notion of 'precaliber') make it possible to extend this work in a new direction, seeking to understand the possible Maharam types of measures on a given topological space. §531 deals with general Radon measures; new patterns arise if we restrict ourselves to completion regular Radon measures (§532). In §533 I give a brief account of some further results depending on assumptions concerning the cardinals examined in Chapter 52, including notes on uniformly regular measures and a description of the cardinals κ for which $\mathbb{R}^\kappa$ is measure-compact (533J).

In §534 I set out the elementary theory of 'strong measure zero' ideals in uniform spaces, concentrating on aspects which can be studied in terms of concepts already introduced. Here there are some very natural questions which have not as far as I know been answered (534Z). In the same section I run through elementary properties of Hausdorff measures when examined in the light of the concepts in Chapter 52. In §535 I look at liftings and strong liftings, extending the results of §§341 and 453; in particular, asking which non-complete probability spaces have liftings. In §536 I run over what is known about Alexandra Bellow's problem concerning pointwise compact sets of continuous functions, mentioned in §463. With a little help from special axioms, there are some striking possibilities concerning repeated integrals, which I examine in §537. Moving into new territory, I devote a section (§538) to a study of special types of filter on $\mathbb{N}$ associated with measure-theoretic phenomena, and to medial limits. In §539, I complete my account of the result of B.Balcar, T.Jech and T.Pazák that it is consistent to suppose that every Dedekind complete ccc weakly (σ,∞)-distributive Boolean algebra is a Maharam algebra, and work through applications of the methods of Chapter 52 to Maharam submeasures and algebras.

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531 Maharam types of Radon measures

In the introduction to §434 I asked

What kinds of measures can arise on what kinds of topological space?

In §§434-435, and again in §438, I considered a variety of topological properties and their relations with measure-theoretic properties of Borel and Baire measures. I passed over, however, some natural questions concerning possible Maharam types, to which I now return. For a given Hausdorff space X, the possible measure algebras of totally finite Radon measures on X can be described in terms of the set $\mathrm{Mah}_{\mathrm{R}}(X)$ of Maharam types of Maharam-type-homogeneous Radon probability measures on X (531F). For $X \neq \emptyset$, $\mathrm{Mah}_{\mathrm{R}}(X)$ is of the form $\{0\} \cup [\omega, \kappa^*[$ for some infinite cardinal κ^* (531Ef). In 531E and 531G I give basic results from which $\mathrm{Mah}_{\mathrm{R}}(X)$ can often be determined; for obvious reasons we are primarily concerned with compact spaces X. In more abstract contexts, there are striking relationships between precalibers of measure algebras, the sets $\mathrm{Mah}_{\mathrm{R}}(X)$ and continuous surjections onto powers of $\{0,1\}$, which I examine in 531L-531M, 531T and 531V. Intertwined with these, we have results relating the character of X to $\mathrm{Mah}_{\mathrm{R}}(X)$ (531N-531O). The arguments here depend on an analysis of the structure of homogeneous measure algebras (531J, 531K, 531R).

531A Proposition Let $(X,\mathfrak{T},\Sigma,\mu)$ be a quasi-Radon measure space with measure algebra $(\mathfrak{A},\bar{\mu})$.
(a) $\tau(\mathfrak{A})$ is at most $w(X)$.

(b) $c(\mathfrak{A})$ is at most $\mathrm{hL}(X)$. If μ is locally finite, $c(\mathfrak{A})$ is at most $L(X)$.
(c) $\#(\{a : a \in \mathfrak{A},\ \bar{\mu}a < \infty\}) \le \max(1, w(X)^\omega)$.
(d) If X is Hausdorff and μ is a Radon measure, then $\tau(\mathfrak{A})$ is at most $\mathrm{nw}(X)$.

531B Proposition Let (X, Σ, μ) be a measure space, with measure algebra $\mathfrak{A}$, and $\mathfrak{T}$ a topology on X such that Σ includes a base for $\mathfrak{T}$ and μ is strictly positive.
(a) If X is regular, then $w(X) \le \#(\mathfrak{A})$.
(b) If X is Hausdorff, then $\#(X) \le 2^{\#(\mathfrak{A})}$.

531C Lemma Let $\langle X_i\rangle_{i\in I}$ be a family of topological spaces with product X, and μ a totally finite quasi-Radon measure on X with Maharam type κ. For each $i \in I$, let μ_i be the marginal measure on X_i, and κ_i its Maharam type. Then κ is at most the cardinal sum $\sum_{i\in I} \kappa_i$.

531D Definition If X is a Hausdorff space, I write $\mathrm{Mah}_{\mathrm{R}}(X)$ for the set of Maharam types of Maharam-type-homogeneous Radon probability measures on X. $0 \in \mathrm{Mah}_{\mathrm{R}}(X)$ iff X is non-empty, and any member of $\mathrm{Mah}_{\mathrm{R}}(X)$ is either 0 or an infinite cardinal.

531E Proposition Let X be a Hausdorff space.
(a) $\kappa \le w(X)$ for every $\kappa \in \mathrm{Mah}_{\mathrm{R}}(X)$.
(b) $\mathrm{Mah}_{\mathrm{R}}(Y) \subseteq \mathrm{Mah}_{\mathrm{R}}(X)$ for every $Y \subseteq X$.
(c) $\mathrm{Mah}_{\mathrm{R}}(X) = \bigcup\{\mathrm{Mah}_{\mathrm{R}}(K) : K \subseteq X \text{ is compact}\}$.
(d) If X is K-analytic (in particular, if X is compact) and Y is a continuous image of X, $\mathrm{Mah}_{\mathrm{R}}(Y) \subseteq \mathrm{Mah}_{\mathrm{R}}(X)$.
(e) $\omega \in \mathrm{Mah}_{\mathrm{R}}(X)$ iff X has a compact subset which is not scattered.
(f) If $\omega \le \kappa' \le \kappa \in \mathrm{Mah}_{\mathrm{R}}(X)$ then $\kappa' \in \mathrm{Mah}_{\mathrm{R}}(X)$.
(g) If Y is another Hausdorff space, and neither X nor Y is empty, then $\mathrm{Mah}_{\mathrm{R}}(X \times Y) = \mathrm{Mah}_{\mathrm{R}}(X) \cup \mathrm{Mah}_{\mathrm{R}}(Y)$; generally, for any non-empty finite family $\langle X_i\rangle_{i\in I}$ of non-empty Hausdorff spaces, $\mathrm{Mah}_{\mathrm{R}}(\prod_{i\in I} X_i) = \bigcup_{i\in I} \mathrm{Mah}_{\mathrm{R}}(X_i)$.

531F Proposition Let X be a Hausdorff space. Then a totally finite measure algebra $(\mathfrak{A}, \bar{\mu})$ is isomorphic to the measure algebra of a Radon measure on X iff (α) whenever $\mathfrak{A}_a$ is a non-trivial homogeneous principal ideal of $\mathfrak{A}$ then $\tau(\mathfrak{A}_a) \in \mathrm{Mah}_{\mathrm{R}}(X)$ (β) $c(\mathfrak{A}) \le \#(X)$.

531G Proposition Let $\langle X_i\rangle_{i\in I}$ be a family of non-empty Hausdorff spaces with product X. Then an infinite cardinal κ belongs to $\mathrm{Mah}_{\mathrm{R}}(X)$ iff *either* $\kappa \le \#(\{i : i \in I,\ \#(X_i) \ge 2\})$ *or* κ is expressible as $\sup_{i\in I} \kappa_i$ where $\kappa_i \in \mathrm{Mah}_{\mathrm{R}}(X_i)$ for every $i \in I$.

531I Notation For any set I, let ν_I be the usual measure on $\{0,1\}^I$, T_I its domain, $\mathcal{N}_I$ its null ideal and $(\mathfrak{B}_I, \bar{\nu}_I)$ its measure algebra. I will write $\langle e_i\rangle_{i\in I}$ for the standard generating family in $\mathfrak{B}_I$ (525A). For $J \subseteq I$ let $\mathfrak{C}_J$ be the closed subalgebra of $\mathfrak{B}_I$ generated by $\{e_i : i \in J\}$. For each $i \in I$, let $\phi_i : \mathfrak{B}_I \to \mathfrak{B}_I$ be the measure-preserving involution corresponding to reversal of the ith coordinate in $\{0,1\}^I$, that is, $\phi_i(e_i) = 1 \setminus e_i$ and $\phi_i(e_j) = e_j$ for $j \ne i$.

531J Lemma Let I be a set, and take $\mathfrak{B}_I$, $\mathfrak{C}_J$, for $J \subseteq I$, and ϕ_i, for $i \in I$, as in 531I.
(a) $\bigcup\{\mathfrak{C}_J : J \in [I]^{<\omega}\}$ is dense in $\mathfrak{B}_I$ for the measure-algebra topology of $\mathfrak{B}_I$.
(b) For every $a \in \mathfrak{B}_I$, there is a (unique) countable $J^*(a) \subseteq I$ such that, for $J \subseteq I$, $a \in \mathfrak{C}_J$ iff $J \supseteq J^*(a)$.
(c) $J^*(1 \setminus a) = J^*(a)$ for every $a \in \mathfrak{B}_I$.
(d) $\phi_i\phi_j = \phi_j\phi_i$ for all i, $j \in I$.
(e) If $J \subseteq I$, $a \in \mathfrak{C}_J$ and $i \in I$, then $a \cap \phi_i a$, $a \cup \phi_i a$ belong to $\mathfrak{C}_{J\setminus\{i\}}$.
(f) For $a \in \mathfrak{B}_I$ and $i \in I$ we have $\phi_i a = a$ iff $i \notin J^*(a)$.
(g) $\phi_i a \in \mathfrak{C}_J$ whenever $J \subseteq I$, $i \in I$ and $a \in \mathfrak{C}_J$.

531K Lemma Let $\kappa \ge \omega_2$ be a cardinal, and $\langle e_\xi \rangle_{\xi<\kappa}$ the standard generating family in $\mathfrak{B}_\kappa$. Suppose that we are given a family $\langle a_\xi \rangle_{\xi<\kappa}$ in $\mathfrak{B}_\kappa$. Then there are a set $\Gamma \in [\kappa]^\kappa$ and a family $\langle c_\xi \rangle_{\xi<\kappa}$ in $\mathfrak{B}_\kappa$ such that

$$c_\xi \subseteq a_\xi, \quad \bar{\nu}_\kappa c_\xi \ge 2\bar{\nu}_\kappa a_\xi - 1$$

for every ξ, and

$$\bar{\nu}_\kappa(\inf_{\xi \in I}(c_\xi \cap e_\xi) \cap \inf_{\eta \in J}(c_\eta \setminus e_\eta)) = \frac{1}{2^{\#(I\cup J)}}\bar{\nu}_\kappa(\inf_{\xi\in I\cup J} c_\xi)$$

whenever I, $J \subseteq \Gamma$ are disjoint finite sets.

531L Theorem Let X be a Hausdorff space.

(a) If $\omega \in \mathrm{Mah}_{\mathrm{R}}(X)$ then $\{0,1\}^\omega$ is a continuous image of a compact subset of X.

(b) If $\kappa \ge \omega_2$ belongs to $\mathrm{Mah}_{\mathrm{R}}(X)$ and $\lambda \le \kappa$ is an infinite cardinal such that (κ, λ) is a measure-precaliber pair of every probability algebra, then $\{0,1\}^\lambda$ is a continuous image of a compact subset of X.

531M Proposition If κ is an infinite cardinal and $\{0,1\}^\kappa$ is a continuous image of a closed subset of X whenever X is a compact Hausdorff space such that $\kappa \in \mathrm{Mah}_{\mathrm{R}}(X)$, then κ is a measure-precaliber of every probability algebra.

531N Proposition Let κ, κ' and λ be infinite cardinals such that (κ, κ') is not a measure-precaliber pair of $(\mathfrak{B}_\lambda, \bar{\nu}_\lambda)$. Then there is a compact Hausdorff space X such that $\kappa \in \mathrm{Mah}_{\mathrm{R}}(X)$ and $\chi(x, X) < \max(\kappa', \lambda^+)$ for every $x \in X$.

531O Proposition Let κ be a regular infinite cardinal. Then the following are equiveridical:

(i) κ is a measure-precaliber of every measurable algebra;

(ii) if X is a compact Hausdorff space such that $\kappa \in \mathrm{Mah}_{\mathrm{R}}(X)$, then $\chi(x, X) \ge \kappa$ for some $x \in X$.

531P Lemma Let Y be a zero-dimensional compact metrizable space, μ an atomless Radon probability measure on Y, $A \subseteq Y$ a μ-negligible set and $\mathcal{Q}$ a countable family of closed subsets of Y. Then there are closed sets K, $L \subseteq Y$, with union Y, such that

$K \cup L = Y$, $\quad K \cap L \cap A = \emptyset$, $\quad \mu(K \cap L) \ge \frac{1}{2}$,

$K \cap Q = \overline{Q \setminus L}$ and $L \cap Q = \overline{Q \setminus K}$ for every $Q \in \mathcal{Q}$.

531Q Proposition Suppose that $\mathrm{cf}\mathcal{N}_\omega = \omega_1$. Then there is a hereditarily separable perfectly normal compact Hausdorff space X, of weight ω_1, with a Radon probability measure of Maharam type ω_1 such that every negligible set is metrizable.

531R Lemma Let I be a set, and let $\mathfrak{B}_I$, $\langle e_i \rangle_{i\in I}$, $\langle \phi_i \rangle_{i \in I}$, $\langle \mathfrak{C}_K \rangle_{K \subseteq I}$ and $J^* : \mathfrak{B}_I \to [I]^{\le\omega}$ be as in 531I-531K. For $a \in \mathfrak{B}_I$ and $K \subseteq I$, set $S_K(a) = \mathrm{upr}(a, \mathfrak{C}_K) = \min\{c : a \subseteq c \in \mathfrak{C}_K\}$.

(a) For all $a \in \mathfrak{B}_I$, $i \in I$ and K, $L \subseteq I$,

(i) $S_I(a) = a$,

(ii) $S_L(a) \subseteq S_K(a)$ if $K \subseteq L$,

(iii) $J^* S_K(a) \subseteq J^*(a) \cap K$,

(iv) $S_{I\setminus\{i\}}(a) = a \cup \phi_i a$,

(v) $S_K S_L(a) = S_{K\cap L}(a)$.

(b) Whenever $a \in \mathfrak{B}_I$, $\epsilon > 0$ and $m \in \mathbb{N}$, there is a finite $L \subseteq I$ such that $\bar{\nu}_I(S_K(a) \setminus a) \le \epsilon$ whenever $L \subseteq K \subseteq I$ and $\#(I \setminus K) \le m$.

531S Lemma Suppose that $\omega_1 < \mathfrak{m}_{\mathrm{K}}$. Let $\langle e_\xi \rangle_{\xi<\omega_1}$ be the standard generating family in $\mathfrak{B}_{\omega_1}$, and $\langle a_\xi \rangle_{\xi<\omega_1}$ a family of elements of $\mathfrak{B}_{\omega_1}$ of measure greater than $\frac{1}{2}$. Then there is an uncountable set $\Gamma \subseteq \omega_1$ such that $\inf_{\xi\in I} a_\xi \cap e_\xi$ meets $\inf_{\eta\in J} a_\eta \setminus e_\eta$ whenever I, $J \subseteq \Gamma$ are finite and disjoint.

531T Theorem Suppose that $\omega \le \kappa < \mathfrak{m}_{\mathrm{K}}$. If X is a Hausdorff space and $\kappa \in \mathrm{Mah}_{\mathrm{R}}(X)$, then $\{0,1\}^\kappa$ is a continuous image of a compact subset of X.

531U Proposition Let X be a Hausdorff space.

(a) Give the space $P_{\mathrm{R}}(X)$ of Radon probability measures on X its narrow topology. If $\kappa \ge \omega_2$ belongs to $\mathrm{Mah}_{\mathrm{R}}(X)$, then $\{0,1\}^\kappa$ is a continuous image of a compact subset of $P_{\mathrm{R}}(X)$.

(b) Give the space $P_{\mathrm{R}}(X \times X)$ its narrow topology. Then its tightness $t(P_{\mathrm{R}}(X \times X))$ is at least $\sup \mathrm{Mah}_{\mathrm{R}}(X)$.

531V Proposition (a) Suppose that the continuum hypothesis is true. Then there is a compact Hausdorff space X such that $\omega_1 \in \mathrm{Mah}_{\mathrm{R}}(X)$ but $\{0,1\}^{\omega_1}$ is not a continuous image of a closed subset of $P_{\mathrm{R}}(X)$.

(b) Suppose that there is a family $\langle W_\xi\rangle_{\xi<\omega_1}$ in $\mathcal{N}_{\omega_1}$ such that every closed subset of $\{0,1\}^{\omega_1} \setminus \bigcup_{\xi<\omega_1} W_\xi$ is scattered. Then there is a compact Hausdorff space X such that $\omega_1 \in \mathrm{Mah}_{\mathrm{R}}(X)$ but $\{0,1\}^{\omega_1}$ is not a continuous image of a closed subset of X.

531Z Problems (a) Can there be a perfectly normal compact Hausdorff space X such that $\omega_2 \in \mathrm{Mah}_{\mathrm{R}}(X)$? (See 531Q, 554Xd.)

(b) Can there be a hereditarily separable compact Hausdorff space X such that $\omega_2 \in \mathrm{Mah}_{\mathrm{R}}(X)$?

Version of 1.6.13

532 Completion regular measures on $\{0,1\}^I$

As I remarked in the introduction to §434, the trouble with topological measure theory is that there are too many questions to ask. In §531 I looked at the problem of determining the possible Maharam types of Radon measures on a Hausdorff space X. But we can ask the same question for any of the other classes of topological measures listed in §411. It turns out that the very narrowly focused topic of completion regular Radon measures on powers of $\{0,1\}$ already leads us to some interesting arguments.

I define the classes $\mathrm{Mah}_{\mathrm{crR}}(X)$, corresponding to the $\mathrm{Mah}_{\mathrm{R}}(X)$ examined in §531, in 532A. They are less accessible, and I almost immediately specialize to the relation $\lambda \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$. This at least is more or less convex (532G, 532K), and can be characterized in terms of the measure algebras $\mathfrak{B}_\lambda$ (532I). On the way it is helpful to extend the treatment of completion regular measures given in §434 (532D, 532E, 532H). For fixed infinite λ, there is a critical cardinal $\kappa_0 \le (2^\lambda)^+$ such that $\lambda \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$ iff $\lambda \le \kappa < \kappa_0$; under certain conditions, when $\lambda = \omega$, we can locate κ_0 in terms of the cardinals of Cichoń's diagram (532P, 532Q). This depends on facts about the Lebesgue measure algebra (532M, 532O) which are of independent interest. Finally, for other λ of countable cofinality, the square principle and Chang's transfer principle are relevant (532R-532S).

532A Definition If X is a topological space, I write $\mathrm{Mah}_{\mathrm{cr}}(X)$ for the set of Maharam types of Maharam-type-homogeneous completion regular topological probability measures on X. If X is a Hausdorff space, I write $\mathrm{Mah}_{\mathrm{crR}}(X)$ for the set of Maharam types of Maharam-type-homogeneous completion regular Radon probability measures on X.

532B Proposition Let X be a Hausdorff space. Then a probability algebra $(\mathfrak{A},\bar\mu)$ is isomorphic to the measure algebra of a completion regular Radon probability measure on X iff (α) $\tau(\mathfrak{A}_a) \in \mathrm{Mah}_{\mathrm{crR}}(X)$ whenever $\mathfrak{A}_a$ is a non-zero homogeneous principal ideal of $\mathfrak{A}$ (β) the number of atoms of $\mathfrak{A}$ is not greater than the number of points $x \in X$ such that $\{x\}$ is a zero set.

532C Remarks If I is any set, ν_I will be the usual measure on $\{0,1\}^I$, $\mathfrak{B}_I$ its measure algebra and $\mathcal{N}_I$ its null ideal.

Let κ be an infinite cardinal. Then $\kappa \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$. $\lambda \le \kappa$ for every $\lambda \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$. $0 \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$ iff $\kappa = \omega$.

532D Theorem Let (X,μ_1) and (Y,μ_2) be effectively locally finite topological measure spaces of which X is quasi-dyadic, μ_1 is completion regular and μ_2 is τ-additive. Let μ be the c.l.d. product measure on $X\times Y$. Then μ is a τ-additive topological measure.

532E Corollary Let $\langle X_i\rangle_{i\in I}$ be a family of regular spaces with countable networks, and Y any topological space. Suppose that we are given a strictly positive topological probability measure μ_i on each X_i, and a τ-additive topological probability measure ν on Y. Let μ be the ordinary product measure on $Z=\prod_{i\in I}X_i\times Y$.

(a) μ is a topological measure.

(b) μ is τ-additive.

(c) If ν is completion regular, and every μ_i is inner regular with respect to the Borel sets, then μ is completion regular.

532F Corollary Let $\langle(X_i,\mu_i)\rangle_{i\in I}$ be a family of quasi-dyadic compact Hausdorff spaces with strictly positive completion regular Radon measures. Then the ordinary product measure μ on $\prod_{i\in I}X_i$ is a completion regular Radon measure.

532G Proposition Suppose that λ, λ' and κ are cardinals such that $\max(\omega,\lambda)\le\lambda'\le\kappa$ and $\lambda\in\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$. Then $\lambda'\in\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$.

532H Lemma Let $\langle X_i\rangle_{i\in I}$ be a family of separable metrizable spaces, and μ a totally finite completion regular topological measure on $X=\prod_{i\in I}X_i$. Then

(a) the support of μ is a zero set;

(b) μ is inner regular with respect to the self-supporting zero sets.

532I Theorem Let $\lambda\le\kappa$ be infinite cardinals. Then the following are equiveridical:

(i) $\lambda\in\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$;

(ii) there is a family $\langle X_\xi\rangle_{\xi<\kappa}$ of non-singleton separable metrizable spaces such that $\lambda\in\mathrm{Mah}_{\mathrm{cr}}(\prod_{\xi<\kappa}X_\xi)$;

(iii) there is a Boolean-independent family $\langle b_\xi\rangle_{\xi<\kappa}$ in $\mathfrak{B}_\lambda$ with the following property: for every $a\in\mathfrak{B}_\lambda$ there is a countable set $J\subseteq\kappa$ such that the subalgebras generated by $\{a\}\cup\{b_\xi:\xi\in J\}$ and $\{b_\eta:\eta\in\kappa\setminus J\}$ are Boolean-independent.

532J Corollary (a) Suppose that λ, κ are infinite cardinals and $\lambda\in\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$. Then κ is at most the cardinal power λ^ω.

(b) If κ is an infinite cardinal such that $\lambda^\omega<\kappa$ for every $\lambda<\kappa$, then $\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)=\{\kappa\}$.

532K Corollary If $\omega\le\lambda\le\kappa'\le\kappa$ and $\lambda\in\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$ then $\lambda\in\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^{\kappa'})$.

532L Corollary If $\omega\le\lambda\le\lambda'$ and $\mathrm{cf}[\lambda']^{\le\lambda}<\mathrm{cf}\,\kappa$ and $\lambda'\in\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$, then $\lambda\in\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$.

532M Proposition If $A\subseteq\mathfrak{B}_\omega\setminus\{0\}$ and $\#(A)<\mathfrak{d}$, then there is a $c\in\mathfrak{B}_\omega$ such that neither c nor $1\setminus c$ includes any member of A.

532N Lemma There is a Borel set $W\subseteq\{0,1\}^{\mathbb{N}}\times\{0,1\}^{\mathbb{N}}$ such that whenever E, $F\subseteq\{0,1\}^{\mathbb{N}}$ have positive measure for ν_ω then neither $(E\times F)\cap W$ nor $(E\times F)\setminus W$ is negligible for the product measure ν_ω^2 on $\{0,1\}^{\mathbb{N}}\times\{0,1\}^{\mathbb{N}}$.

532O Proposition If $A\subseteq\mathfrak{B}_\omega\setminus\{0\}$ and $\#(A)<\mathrm{cov}\,\mathcal{N}_\omega$, then there is a $c\in\mathfrak{B}_\omega$ such that neither c nor $1\setminus c$ includes any member of A.

532P Proposition Set $\kappa=\max(\mathfrak{d},\mathrm{cov}\,\mathcal{N}_\omega)$. If $\mathrm{FN}(\mathcal{P}\mathbb{N})=\omega_1$, then $\omega\in\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$. In particular, if $\mathfrak{c}=\omega_1$ then $\omega\in\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^{\omega_1})$.

532Q Proposition Suppose that $\operatorname{add}\mathcal{N}_\omega > \omega_1$.

(a) $\lambda \notin \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$ whenever $\lambda \geq \omega$ and $\max(\omega, \operatorname{cf}[\lambda]^{\leq\omega}) < \kappa$.

(b) If $\omega_1 \leq \kappa \leq \omega_\omega$ then $\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa) = \{\kappa\}$.

532R Proposition Suppose that λ is an uncountable cardinal with countable cofinality such that $\square_\lambda$ is true. Set $\kappa = \lambda^+$. Then $\lambda \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$.

532S Proposition Suppose that $\operatorname{add}\mathcal{N}_\omega > \omega_1$ and that λ is an infinite cardinal such that $\mathrm{CTP}(\lambda^+, \lambda)$ is true. Then $\lambda \notin \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$ for any $\kappa > \lambda$.

532Z Problems (a) In 532P, can we take $\kappa = \operatorname{cf}\mathcal{N}_\omega$?

(b) We have $\omega \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^{\omega_1})$ if $\mathrm{FN}(\mathcal{P}\mathbb{N}) = \omega_1$ and not if $\operatorname{add}\mathcal{N}_\omega > \omega_1$. Can we narrow the gap?

(c) For a Hausdorff space X let $\mathrm{Mah}_{\mathrm{spcrR}}(X)$ be the set of Maharam types of strictly positive Maharam homogeneous completion regular Radon measures on X. Describe the sets Γ of cardinals for which there are compact Hausdorff spaces X such that $\mathrm{Mah}_{\mathrm{spcrR}}(X) = \Gamma$.

Version of 4.1.14

533 Special topics

I present notes on certain questions which can be answered if we make particular assumptions concerning values of the cardinals considered in §§523-524. The first cluster (533A-533E) looks at Radon and quasi-Radon measures in contexts in which the additivity of Lebesgue measure is large compared with other cardinals of the structures considered. Developing ideas which arose in the course of §531, I discuss 'uniform regularity' in perfectly normal and first-countable spaces (533H). We also have a complete description of the cardinals κ for which $\mathbb{R}^\kappa$ is measure-compact (533J).

As previously, I write $\mathcal{N}(\mu)$ for the null ideal of a measure μ; ν_κ will be the usual measure on $\{0,1\}^\kappa$ and $\mathcal{N}_\kappa = \mathcal{N}(\nu_\kappa)$ its null ideal.

533A Lemma Let (X, Σ, μ) be a semi-finite measure space with measure algebra $(\mathfrak{A}, \bar\mu)$. If $\langle \mathcal{K}_\xi \rangle_{\xi<\kappa}$ is a family of ideals in Σ such that μ is inner regular with respect to every $\mathcal{K}_\xi$ and $\kappa < \min(\operatorname{add}\mathcal{N}(\mu), \operatorname{wdistr}(\mathfrak{A}))$, then μ is inner regular with respect to $\bigcap_{\xi<\kappa} \mathcal{K}_\xi$.

533B Corollary Let (X, Σ, μ) be a totally finite measure space with countable Maharam type. If $\mathcal{E} \subseteq \Sigma$, $\#(\mathcal{E}) < \min(\operatorname{add}\mathcal{N}_\omega, \operatorname{add}\mathcal{N}(\mu))$ and $\epsilon > 0$, there is a set $F \in \Sigma$ such that $\mu(X \setminus F) \leq \epsilon$ and $\{E \cap F : E \in \mathcal{E}\}$ is countable.

533C Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space with countable Maharam type.

(a) If $w(X) < \operatorname{add}\mathcal{N}_\omega$, then μ is inner regular with respect to the second-countable subsets of X; if moreover $\mathfrak{T}$ is regular and Hausdorff, then μ is inner regular with respect to the metrizable subsets of X.

(b) If Y is a topological space of weight less than $\operatorname{add}\mathcal{N}_\omega$, then any measurable function $f : X \to Y$ is almost continuous.

(c) If $\langle Y_i \rangle_{i\in I}$ is a family of topological spaces, with $\#(I) < \operatorname{add}\mathcal{N}_\omega$, and $f_i : X \to Y_i$ is almost continuous for every i, then $x \mapsto \langle f_i(x) \rangle_{i\in I} : X \to \prod_{i\in I} Y_i$ is almost continuous.

533D Proposition Let $(X, \mathfrak{T})$ be a first-countable compact Hausdorff space such that $\operatorname{cf}[w(X)]^{\leq\omega} < \operatorname{add}\mathcal{N}_\omega$, and μ a Radon measure on X with countable Maharam type. Then μ is inner regular with respect to the metrizable zero sets.

533E Corollary Suppose that $\operatorname{cov}\mathcal{N}_{\omega_1} > \omega_1$. Let $(X, \mathfrak{T})$ be a first-countable K-analytic Hausdorff space such that $\operatorname{cf}[w(X)]^{\leq\omega} < \operatorname{add}\mathcal{N}_\omega$. Then X is a Radon space.

533F Definition Let X be a topological space and μ a topological measure on X. I will say that μ is **uniformly regular** if there is a countable family $\mathcal{V}$ of open sets in X such that $G \setminus \bigcup\{V : V \in \mathcal{V}, V \subseteq G\}$ is negligible for every open set $G \subseteq X$.

533G Lemma Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a compact Radon measure space.

(a) The following are equiveridical:

(i) μ is uniformly regular;

(ii) there are a metrizable space Z and a continuous function $f : X \to Z$ such that $\mu f^{-1}[f[F]] = \mu F$ for every closed $F \subseteq X$;

(iii) there is a countable family $\mathcal{H}$ of cozero sets in X such that $\mu G = \sup\{\mu H : H \in \mathcal{H}, H \subseteq G\}$ for every open set $G \subseteq X$;

(iv) there is a countable family $\mathcal{E}$ of zero sets in X such that $\mu G = \sup\{\mu E : E \in \mathcal{E}, E \subseteq G\}$ for every open set $G \subseteq X$.

(b) If $\mathfrak{T}$ is perfectly normal, the following are equiveridical:

(i) μ is uniformly regular;

(ii) there are a metrizable space Z and a continuous function $f : X \to Z$ such that $\mu f^{-1}[f[E]] = \mu E$ for every $E \in \Sigma$;

(iii) there are a metrizable space Z and a continuous function $f : X \to Z$ such that $f[G] \neq f[X]$ whenever $G \subseteq X$ is open and $\mu G < \mu X$;

(iv) there is a countable family $\mathcal{E}$ of closed sets in X such that $\mu G = \sup\{\mu E : E \in \mathcal{E}, E \subseteq G\}$ for every open set $G \subseteq X$.

533H Theorem (a) Suppose that $\operatorname{cov}\mathcal{N}_{\omega_1} > \omega_1$. Let X be a perfectly normal compact Hausdorff space. Then every Radon measure on X is uniformly regular.

(b) Suppose that $\operatorname{cov}\mathcal{N}_{\omega_1} > \omega_1 = \operatorname{non}\mathcal{N}_\omega$. Let X be a first-countable compact Hausdorff space. Then every Radon measure on X is uniformly regular.

533I Definition A completely regular space X is **strongly measure-compact** if $\mu X = \sup\{\mu^* K : K \subseteq X \text{ is compact}\}$ for every totally finite Baire measure μ on X.

533J Theorem Let κ be a cardinal. Then the following are equiveridical:

(i) $\mathbb{R}^\kappa$ is measure-compact;

(ii) if $\langle X_\xi \rangle_{\xi<\kappa}$ is a family of strongly measure-compact completely regular Hausdorff spaces then $\prod_{\xi<\kappa} X_\xi$ is measure-compact;

(iii) whenever X is a compact Hausdorff space and $\langle G_\xi \rangle_{\xi<\kappa}$ is a family of cozero sets in X, then $X \cap \bigcap_{\xi<\kappa} G_\xi$ is measure-compact;

(iv) for any Radon measure, the union of κ or fewer closed negligible sets has inner measure zero;

(v) for any Radon measure, the union of κ or fewer negligible sets has inner measure zero;

(vi) $\kappa < \operatorname{cov}\mathcal{N}(\mu)$ for any Radon measure μ;

(vii) $\kappa < \operatorname{cov}\mathcal{N}_\kappa$;

(viii) $\kappa < \mathfrak{m}(\mathfrak{A})$ for every measurable algebra $\mathfrak{A}$.

533Z Problem For which cardinals κ is $\mathbb{R}^\kappa$ Borel-measure-compact?

Version of 27.6.22

534 Hausdorff measures, strong measure zero and Rothberger's property

In this section I look at constructions which are primarily metric rather than topological. I start with a note on Hausdorff measures, spelling out connexions between Hausdorff r-dimensional measure on a separable metric space and the basic σ-ideal $\mathcal{N}$ (534B).

The main part of the section section is a brief introduction to a class of ideals which are of great interest in set-theoretic analysis. While the most important ones are based on separable metric spaces, some of the ideas can be expressed in more general contexts, and I give a definition of 'strong measure zero' in terms of uniformities (534Ca). An associated topological notion is what I call 'Rothberger's property' (534Cb). A famous characterization of sets of strong measure zero in $\mathbb{R}$ in terms of translations of meager sets can also be represented as a theorem about σ-compact groups (534K). There are few elementary results describing the cardinal functions of strong measure zero ideals, but I give some information on their additivities (534M) and uniformities (534Q). There seem to be some interesting questions concerning spaces with isomorphic strong measure zero ideals, which I consider in 534N-534P. A particularly important question, from the very beginning of the topic in Borel 1919, concerns the possible cardinals of sets of strong measure zero; in 534Q-534S I give some sample facts and illustrative examples.

534A Lemma Let (X,ρ) be a separable metric space. Then there is a countable family $\mathcal{C}$ of subsets of X such that whenever $A\subseteq X$ has finite diameter and $\eta>0$ then there is a $C\in\mathcal{C}$ such that $A\subseteq C$ and $\operatorname{diam}C\le\eta+2\operatorname{diam}A$.

534B Hausdorff measures: Theorem Let (X,ρ) be a metric space and $r>0$. Write μ_{Hr} for r-dimensional Hausdorff measure on X, $\mathcal{N}(\mu_{Hr})$ for its null ideal, $\mathcal{N}$ for the null ideal of Lebesgue measure on $\mathbb{R}$ and $\mathcal{M}$ for the ideal of meager subsets of $\mathbb{R}$.

(a) $\operatorname{add}\mu_{Hr}=\operatorname{add}\mathcal{N}(\mu_{Hr})$.

(b) If X is separable, $\mathcal{N}(\mu_{Hr})\preccurlyeq_{\mathrm{T}}\mathcal{N}$, so that $\operatorname{add}\mu_{Hr}\ge\operatorname{add}\mathcal{N}$ and $\operatorname{cf}\mathcal{N}(\mu_{Hr})\le\operatorname{cf}\mathcal{N}$.

(c) If X is separable, $(X,\in,\mathcal{N}(\mu_{Hr}))\preccurlyeq_{\mathrm{GT}}(\mathcal{M},\not\ni,\mathbb{R})$, so that $\operatorname{cov}\mathcal{N}(\mu_{Hr})\le\operatorname{non}\mathcal{M}$ and $\mathfrak{m}_{\mathrm{countable}}\le\operatorname{non}\mathcal{N}(\mu_{Hr})$.

(d) If X is analytic and $\mu_{Hr}X>0$, then $\operatorname{add}\mu_{Hr}=\operatorname{add}\mathcal{N}$, $\operatorname{cf}\mathcal{N}(\mu_{Hr})=\operatorname{cf}\mathcal{N}$, $\operatorname{non}\mathcal{N}(\mu_{Hr})\le\operatorname{non}\mathcal{N}$ and $\operatorname{cov}\mathcal{N}(\mu_{Hr})\ge\operatorname{cov}\mathcal{N}$.

534C Strong measure zero and Rothberger's property (a) Let $(X,\mathcal{W})$ be a uniform space and $A\subseteq X$. I say that A has **strong measure zero** or **property C** in X if for any sequence $\langle W_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{W}$ there is a cover $\langle A_n\rangle_{n\in\mathbb{N}}$ of A such that $A_n\times A_n\subseteq W_n$ for every $n\in\mathbb{N}$. If (X,ρ) is a metric space, a subset A of X has strong measure zero in X if it has strong measure zero for the uniformity defined by the metric.

I will write $\mathcal{S}\mathfrak{mz}(X,\mathcal{W})$ or $\mathcal{S}\mathfrak{mz}(X,\rho)$ for the family of sets of strong measure zero in a uniform space $(X,\mathcal{W})$ or a metric space (X,ρ).

(b) If X is a topological space and A is a subset of X, I will say that A has **Rothberger's property in** X if for every sequence $\langle\mathcal{G}_n\rangle_{n\in\mathbb{N}}$ of non-empty open covers of X there is a sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ such that $G_n\in\mathcal{G}_n$ for every $n\in\mathbb{N}$ and $A\subseteq\bigcup_{n\in\mathbb{N}}G_n$. I will write $\mathcal{R}\mathfrak{bg}(X)$ for the family of subsets of X with Rothberger's property in X.

534D Proposition (a)(i) If $(X,\mathcal{W})$ is a uniform space and $A\subseteq X$, then A has strong measure zero in X iff it has strong measure zero in itself when it is given its subspace uniformity.

(ii) If $(X,\mathcal{W})$ is a uniform space, then $\mathcal{S}\mathfrak{mz}(X,\mathcal{W})$ is a σ-ideal containing all the countable subsets of X.

(iii) If $(X,\mathcal{W})$ and $(Y,\mathcal{V})$ are uniform spaces and $f:X\to Y$ is uniformly continuous, then $f[A]\in\mathcal{S}\mathfrak{mz}(Y,\mathcal{V})$ whenever $A\in\mathcal{S}\mathfrak{mz}(X,\mathcal{W})$.

(iv) Let $(X,\mathcal{W})$ be a uniform space and $A\subseteq X$. Then $A\in\mathcal{S}\mathfrak{mz}(X,\mathcal{W})$ iff $f[A]\in\mathcal{S}\mathfrak{mz}(Y,\rho)$ whenever (Y,ρ) is a metric space and $f:X\to Y$ is uniformly continuous.

(v) Let $(X,\mathcal{W})$ be a uniform space and $A\in\mathcal{S}\mathfrak{mz}(X,\mathcal{W})$. If $B\subseteq X$ is such that $B\setminus G\in\mathcal{S}\mathfrak{mz}(X,\mathcal{W})$ whenever G is an open set including A, then $B\in\mathcal{S}\mathfrak{mz}(X,\mathcal{W})$.

(b) Let X be a topological space.

(i) $\mathcal{R}\mathfrak{bg}(X)$ is a σ-ideal containing all the countable subsets of X.

(ii) If Y is another topological space, $f:X\to Y$ is continuous and $A\in\mathcal{R}\mathfrak{bg}(X)$, then $f[A]\in\mathcal{R}\mathfrak{bg}(Y)$.

(iii) If $A\in\mathcal{R}\mathfrak{bg}(X)$ and $B\subseteq X$ is such that $B\setminus G\in\mathcal{R}\mathfrak{bg}(X)$ whenever G is an open set including A, then $B\in\mathcal{R}\mathfrak{bg}(X)$.

(iv) If $F\subseteq X$ is closed, then $\mathcal{R}\mathfrak{bg}(F)=\{A:A\in\mathcal{R}\mathfrak{bg}(X),\ A\subseteq F\}$.

534E Proposition Let $(X, \mathcal{W})$ be a uniform space, and give X the topology induced by $\mathcal{W}$.
(a) $\mathcal{R}\mathrm{bg}(X) \subseteq \mathcal{S}\mathrm{mz}(X, \mathcal{W})$.
(b) If X is σ-compact, $\mathcal{R}\mathrm{bg}(X) = \mathcal{S}\mathrm{mz}(X, \mathcal{W})$.

534F Proposition Let X be a regular paracompact space, and $\mathcal{W}$ the uniformity on X defined by the family of all continuous pseudometrics on X. Then

$$\begin{aligned}\mathcal{R}\mathrm{bg}(X) &= \mathcal{S}\mathrm{mz}(X, \mathcal{W}) \\ &= \{A : A \subseteq X,\ f[A] \in \mathcal{S}\mathrm{mz}(Y, \rho) \text{ whenever } (Y, \rho) \text{ is a metric space} \\ &\qquad\qquad \text{and } f : X \to Y \text{ is continuous}\}.\end{aligned}$$

534G Remarks We see from 534E that in Euclidean space, the context of the original investigation of these ideas, what I call Rothberger's property and strong measure zero coincide; and as the latter phrase is more commonly used and has a more generally accepted meaning, it is tempting to prefer it. But in the framework of this treatise, devoted as it is to maximal convenient generality, the concepts diverge. Strong measure zero has an obvious interpretation in any metric space, and can readily be applied in general uniform spaces; while Rothberger's property is a topological notion. They have very different natures as soon as we leave the area of σ-compact spaces. In particular, the Polish space $\mathbb{N}^{\mathbb{N}}$, topologically identifiable with $\mathbb{R} \setminus \mathbb{Q}$, has a wide variety of compatible uniformities, giving rise to potentially very different strong measure zero ideals. So we find ourselves with the possibility that $\mathcal{R}\mathrm{bg}(\mathbb{R} \setminus \mathbb{Q})$ may be much smaller than the trace of $\mathcal{R}\mathrm{bg}(\mathbb{R})$ on the subset $\mathbb{R} \setminus \mathbb{Q}$, even though $\mathbb{Q} \in \mathcal{R}\mathrm{bg}(\mathbb{R})$ (534Sb). Strong measure zero, of course, is much more manageable on subsets (534D(a-i)).

534H Proposition If (X, ρ) is a metric space and $A \in \mathcal{S}\mathrm{mz}(X, \rho)$, then A is separable, zero-dimensional and universally negligible, and all compact subsets of A are countable.

534I Let X be a regular topological space. Then X has Rothberger's property in itself iff it is Lindelöf and zero-dimensional and $f[X] \in \mathcal{R}\mathrm{bg}(\mathbb{R} \setminus \mathbb{Q})$ whenever $f : X \to \mathbb{R} \setminus \mathbb{Q}$ is continuous.

534J Proposition Let X be a Hausdorff space, and K a compact subset of X. Then K belongs to $\mathcal{R}\mathrm{bg}(X)$ iff it is scattered.

534K Theorem Let X be a σ-compact locally compact Hausdorff topological group and A a subset of X. Then the following are equiveridical:
(i) $A \in \mathcal{R}\mathrm{bg}(X)$;
(ii) for any sequence $\langle U_n \rangle_{n \in \mathbb{N}}$ of neighbourhoods of the identity e of X, there is a sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ in X such that $A \subseteq \bigcup_{n \in \mathbb{N}} U_n x_n$;
(iii) $FA \neq X$ for any nowhere dense set $F \subseteq X$;
(iv) $EA \neq X$ for any meager set $E \subseteq X$;
(v) $AF \neq X$ for any nowhere dense set $F \subseteq X$;
(vi) $AE \neq X$ for any meager set $E \subseteq X$.

534L Proposition Let (X, ρ) be a separable metric space. Then $\mathcal{S}\mathrm{mz}(X, \rho) \preccurlyeq_{\mathrm{T}} \mathcal{N}^{\mathfrak{d}}$.

534M Corollary (a) If $(X, \mathcal{W})$ is a Lindelöf uniform space, then $\mathrm{add}\,\mathcal{S}\mathrm{mz}(X, \mathcal{W}) \geq \mathrm{add}\,\mathcal{N}$, where $\mathcal{N}$ is the null ideal of Lebesgue measure on $\mathbb{R}$.
(b) If X is a Lindelöf regular topological space, then $\mathrm{add}\,\mathcal{R}\mathrm{bg}(X) \geq \mathrm{add}\,\mathcal{N}$.

534N $\mathcal{S}\mathrm{mz}$-equivalence (a) If $(X, \mathcal{V})$ and $(Y, \mathcal{W})$ are uniform spaces, they are **$\mathcal{S}\mathrm{mz}$-equivalent** if there is a bijection $f : X \to Y$ such that a set $A \subseteq X$ has strong measure zero in X iff $f[A]$ has strong measure zero in Y.

(b) If $(X,\mathcal{V})$ and $(Y,\mathcal{W})$ are uniform spaces, X is **$\mathcal{S}mz$-embeddable** in Y if it is $\mathcal{S}mz$-equivalent to a subspace of Y. Evidently this is transitive in the sense that if X is $\mathcal{S}mz$-embeddable in Y and Y is $\mathcal{S}mz$-embeddable in Z then X is $\mathcal{S}mz$-embeddable in Z.

534O Lemma (a) Suppose that $(X,\mathcal{W})$ and $(Y,\mathcal{V})$ are uniform spaces and that $\langle X_n\rangle_{n\in\mathbb{N}}$, $\langle Y_n\rangle_{n\in\mathbb{N}}$ are partitions of X, Y respectively such that X_n is $\mathcal{S}mz$-equivalent to Y_n for every n. Then X is $\mathcal{S}mz$-equivalent to Y.

(b) Suppose that $(X,\mathcal{W})$ and $(Y,\mathcal{V})$ are uniform spaces such that X is $\mathcal{S}mz$-embeddable in Y and Y is $\mathcal{S}mz$-embeddable in X. Then $(X,\mathcal{W})$ and $(Y,\mathcal{V})$ are $\mathcal{S}mz$-equivalent.

534P Proposition $\mathbb{R}^r$, $]0,1[^r$, $[0,1]^r$ and $\{0,1\}^{\mathbb{N}}$ are $\mathcal{S}mz$-equivalent for every integer $r\ge 1$.

534Q Large sets with strong measure zero: Proposition (a) Let X be a Lindelöf space. Then $\operatorname{non}\mathcal{R}bg(X)\ge\mathfrak{m}_{\text{countable}}$.

(b) Give $\mathbb{N}^{\mathbb{N}}$ the metric ρ defined by setting $\rho(x,y)=\inf\{2^{-n}:n\in\mathbb{N},\ x\restriction n=y\restriction n\}$ for x, $y\in\mathbb{N}^{\mathbb{N}}$. Then $\operatorname{non}\mathcal{S}mz(\mathbb{N}^{\mathbb{N}},\rho)=\operatorname{non}\mathcal{R}bg(\mathbb{N}^{\mathbb{N}})=\mathfrak{m}_{\text{countable}}$.

534R Proposition (a) If (X,ρ) is a separable metric space and $A\subseteq X$ has cardinal less than $\mathfrak{c}$, there is a Lipschitz function $f:X\to\mathbb{R}$ such that $f\restriction A$ is injective.

(b) If $\kappa<\mathfrak{c}$ is a cardinal and there is any separable metric space with a set with cardinal κ which is of strong measure zero, then there is a subset of $\mathbb{R}$ with cardinal κ which has Rothberger's property in $\mathbb{R}$.

(c)(i) If $\operatorname{cf}(\mathfrak{m}_{\text{countable}})=\mathfrak{b}$ there is a subset of $\mathbb{R}$ with cardinal $\mathfrak{m}_{\text{countable}}$ which has Rothberger's property in itself.

(ii) If $\mathfrak{b}=\omega_1$ there is a subset of $\mathbb{R}$ with cardinal ω_1 which has Rothberger's property in itself.

(iii) If $\mathfrak{m}_{\text{countable}}=\mathfrak{d}$ there is a subset of $\mathbb{R}$ with cardinal $\mathfrak{m}_{\text{countable}}$ which has Rothberger's property in itself.

534S Example Suppose that $\mathfrak{m}_{\text{countable}}=\mathfrak{c}$.

(a) There is a set $A\subseteq[0,1]\setminus\mathbb{Q}$ such that

(α) $\#(A\cap K)<\mathfrak{c}$ for every compact $K\subseteq[0,1]\setminus\mathbb{Q}$,

(β) there is a continuous function $f:[0,1]\setminus\mathbb{Q}\to[0,1]$ such that $f[A]=[0,1]$,

(γ) $A+A\supseteq\,]0,1]$.

(b) Now $A\cup\mathbb{Q}$ has Rothberger's property in itself, $A\in\mathcal{R}bg(\mathbb{R})$, A is not meager, $A\notin\mathcal{R}bg(\mathbb{R}\setminus\mathbb{Q})$ and $A\times A\notin\mathcal{R}bg(\mathbb{R}^2)$.

534Z Problems (a) Let $\mu_{H1}^{(2)}$ be one-dimensional Hausdorff measure on $\mathbb{R}^2$. Is the covering number $\operatorname{cov}\mathcal{N}(\mu_{H1}^{(2)})$ necessarily equal to $\operatorname{cov}\mathcal{N}$?

(b) Can $\operatorname{cf}\mathcal{R}bg(\mathbb{R})$ be ω_1?

(c) How many types of complete separable metric spaces under $\mathcal{S}mz$-equivalence can there be? If we give $\mathbb{N}^{\mathbb{N}}$ the metric of 534Qb, can it fail to be $\mathcal{S}mz$-equivalent to $[0,1]^{\mathbb{N}}$ with the metric $(x,y)\mapsto\sup_{n\in\mathbb{N}}2^{-n}|x(n)-y(n)|$?

(d) Suppose that there is a separable metric space with cardinal $\mathfrak{c}$ with strong measure zero. Must there be a subset of $\mathbb{R}$ with cardinal $\mathfrak{c}$ with Rothberger's property in $\mathbb{R}$?

(e) On $\mathbb{R}$, let $\mathfrak{T}$ be the usual topology and $\mathfrak{S}$ the right-facing Sorgenfrey topology (415Xc). Must $\mathcal{R}bg(\mathbb{R},\mathfrak{S})$ and $\mathcal{R}bg(\mathbb{R},\mathfrak{T})$ be the same?

Version of 28.4.25

535 Liftings

I introduced the Lifting Theorem as one of the fundamental facts about complete strictly localizable measure spaces. Of course we can always complete a measure space and thereby in effect obtain a lifting for any σ-finite measure. For the applications of the Lifting Theorem in §§452-453 this procedure is natural and effective; and generally in this treatise I have taken the view that one should work with completed measures unless there is some strong reason not to. But I have also embraced the principle of maximal convenient generality, seeking formulations which will exhibit the full force of each idea in the context appropriate to that idea, uncluttered by the special features of intended applications. So the question of when, and why, liftings for incomplete measures can be found is one which automatically arises. It turns out to be a fruitful question, in the sense that it leads us to new arguments, even though the answers so far available are unsatisfying.

As usual, much of what we want to know depends on the behaviour of the usual measures on powers of $\{0,1\}$ (535B). An old argument relying on the continuum hypothesis shows that Lebesgue measure can have a Borel lifting; this has been usefully refined, and I give a strong version in 535D-535E. We know that we cannot expect to have translation-invariant Borel liftings, but strong Borel liftings are possible (535H-535I), and in some cases can be built from Borel liftings (535J-535N).

For certain applications in functional analysis, we are more interested in liftings for L^∞ spaces than in liftings for measure algebras; and it is sometimes sufficient to have a 'linear lifting', not necessarily corresponding to a lifting in the strict sense. I give a couple of paragraphs to linear liftings (535O-535R) because in some ways they are easier to handle and it is conceivable that they are relevant to the main outstanding problem (535Za).

535A Notation (a) If (X,Σ,μ) is a measure space and $\mathfrak{T}$ a topology on X, a **Borel lifting** of μ is a lifting which takes values in the Borel σ-algebra $\mathcal{B}(X)$ of X. Similarly, a **Baire lifting** of μ is a lifting which takes values in the Baire σ-algebra $\mathcal{B}\mathfrak{a}(X)$ of X.

(b) I remark at once that if $(X,\mathfrak{T},\Sigma,\mu)$ is a topological measure space and $\phi:\Sigma\to\mathcal{B}(X)$ is a Borel lifting for μ, then $\phi\restriction\mathcal{B}(X)$ is a lifting for the Borel measure $\mu\restriction\mathcal{B}(X)$. Conversely, if $\phi':\mathcal{B}(X)\to\mathcal{B}(X)$ is a lifting for $\mu\restriction\mathcal{B}(X)$, and if for every $E\in\Sigma$ there is a Borel set E' such that $E\triangle E'$ is negligible, then ϕ' extends uniquely to a Borel lifting ϕ of μ.

In the same way, any Baire lifting for a measure μ which measures every zero set will give a lifting for $\mu\restriction\mathcal{B}\mathfrak{a}(X)$; and a lifting for $\mu\restriction\mathcal{B}\mathfrak{a}(X)$ will correspond to a Baire lifting for μ if μ is completion regular.

(c) I will say that, for any set I, ν_I is the usual measure on $\{0,1\}^I$ and $\mathfrak{B}_I$ its measure algebra.

535B Proposition Let (X,Σ,μ) be a strictly localizable measure space with non-zero measure. Suppose that ν_κ has a Baire lifting for every infinite cardinal κ such that the Maharam-type-κ component of the measure algebra of μ is non-zero. Then μ has a lifting.

535C Proposition If λ and κ are cardinals with $\lambda=\lambda^\omega\le\kappa$, and ν_κ has a Baire lifting, then ν_λ has a Baire lifting.

535D Theorem Let (X,Σ,μ) be a measure space such that $\mu X>0$, and suppose that its measure algebra is tightly ω_1-filtered. Then μ has a lifting.

535E Proposition Suppose that $\mathfrak{c}\le\omega_2$ and the Freese-Nation number $\mathrm{FN}(\mathcal{P}\mathbb{N})$ is ω_1.

(a) If $\mathfrak{A}$ is a measurable algebra with cardinal at most ω_2, it is tightly ω_1-filtered.

(b) (Mokobodzki 7?) Let (X,Σ,μ) be a σ-finite measure space with non-zero measure and Maharam type at most ω_2.

(i) μ has a lifting.

(ii) If $\mathfrak{T}$ is a topology on X such that μ is inner regular with respect to the Borel sets, then μ has a Borel lifting.

(iii) If $\mathfrak{T}$ is a topology on X such that μ is inner regular with respect to the zero sets, then μ has a Baire lifting.

535F Proposition Let (X, Σ, μ) be a measure space such that $\mu X > 0$ and $\#(\mathfrak{A}) \le \operatorname{add}\mu$, where $\mathfrak{A}$ is the measure algebra of μ, and suppose that $\underline{\theta} : \mathfrak{A} \to \Sigma$ is such that

$$\underline{\theta}0 = \emptyset, \quad \underline{\theta}(a \cap b) = \underline{\theta}a \cap \underline{\theta}b \text{ for all } a,\, b \in \mathfrak{A}, \quad (\underline{\theta}a)^\bullet \subseteq a \text{ for every } a \in \mathfrak{A}.$$

Then μ has a lifting $\theta : \mathfrak{A} \to \Sigma$ such that $\theta a \supseteq \underline{\theta}a$ for every $a \in \mathfrak{A}$.

535G Corollary Suppose that $\mathfrak{c} = \omega_1$. Then for any integer $r \ge 1$ there is a Borel lifting θ of Lebesgue measure on $\mathbb{R}^r$ such that $x \in \theta E^\bullet$ whenever $E \subseteq \mathbb{R}^r$ is a Borel set and x is a density point of E.

535H Theorem Let $(X, \mathfrak{T}, \Sigma, \mu)$ be an effectively locally finite τ-additive topological measure space with measure algebra $\mathfrak{A}$. If $\#(\mathfrak{A}) \le \operatorname{add}\mu$ and μ is strictly positive, then μ has a strong lifting.

535I Corollary Suppose that $\mathfrak{c} = \omega_1$. Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a strictly positive σ-finite quasi-Radon measure space with Maharam type at most $\mathfrak{c}$. Then μ has a strong Borel lifting.

535J Lemma Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a completely regular totally finite topological measure space with a Borel lifting ϕ. Suppose that $K \subseteq X$ is a self-supporting set of non-zero measure, homeomorphic to $\{0,1\}^{\mathbb{N}}$, such that $K \cap G \subseteq \phi G$ for every open set $G \subseteq X$. Then the subspace measure μ_K has a strong Borel lifting.

535K Lemma Let X be a metrizable space, μ an atomless Radon measure on X and ν an atomless strictly positive Radon measure on $\{0,1\}^{\mathbb{N}}$. Let $\mathcal{K}$ be the family of those subsets K of X such that K, with the subspace topology and measure, is isomorphic to $\{0,1\}^{\mathbb{N}}$ with its usual topology and a scalar multiple of ν. Then μ is inner regular with respect to $\mathcal{K}$.

535L Lemma (a) If $(X, \mathfrak{T})$ is a separable metrizable space, there is a zero-dimensional separable metrizable topology $\mathfrak{S}$ on X, finer than $\mathfrak{T}$, with the same Borel sets as $\mathfrak{T}$, such that $\mathfrak{T}$ is a π-base for $\mathfrak{S}$.

(b) If X is a non-empty zero-dimensional separable metrizable space without isolated points, it is homeomorphic to a dense subset of $\{0,1\}^{\mathbb{N}}$.

(c) Any completely regular space with cardinal less than $\mathfrak{c}$ is zero-dimensional.

535M Lemma Suppose that there is a Borel probability measure on $\{0,1\}^{\mathbb{N}}$ with a strong lifting. Then whenever X is a separable metrizable space and $D \subseteq X$ is a dense set, there is a Boolean homomorphism ϕ from $\mathcal{P}D$ to the Borel σ-algebra $\mathcal{B}(X)$ of X such that $\phi A \subseteq \overline{A}$ for every $A \subseteq D$.

535N Theorem Suppose there is a metrizable space X with a non-zero atomless semi-finite tight Borel measure μ which has a lifting. Then whenever Y is a metrizable space and ν is a strictly positive σ-finite Borel measure on Y, ν has a strong lifting.

535O Linear liftings Let (X, Σ, μ) be a measure space, with measure algebra $\mathfrak{A}$. Write $\mathcal{L}^\infty(\Sigma)$ for the space of bounded Σ-measurable real-valued functions on X. A **linear lifting** for μ is

either a positive linear operator $T : L^\infty(\mu) \to \mathcal{L}^\infty(\Sigma)$ such that $T(\chi X^\bullet) = \chi X$ and $(Tu)^\bullet = u$ for every $u \in L^\infty(\mu)$

or a positive linear operator $S : \mathcal{L}^\infty(\Sigma) \to \mathcal{L}^\infty(\Sigma)$ such that $S(\chi X) = \chi X$, $Sf = 0$ whenever $f = 0$ a.e. and $Sf =_{\text{a.e.}} f$ for every $f \in \mathcal{L}^\infty(\Sigma)$.

If $\theta : \mathfrak{A} \to \Sigma$ is a lifting for μ, then we have a corresponding Riesz homomorphism $T : L^\infty(\mathfrak{A}) \to \mathcal{L}^\infty(\Sigma)$ such that $T(\chi a) = \chi(\theta a)$ for every $a \in \mathfrak{A}$. Identifying $L^\infty(\mathfrak{A})$ with $L^\infty(\mu)$, T can be regarded as a linear lifting.

I will say that a **Borel linear lifting** is a linear lifting such that all its values are Borel measurable functions; a **Baire linear lifting** is a linear lifting such that all its values are Baire measurable functions.

535P Proposition Let (X, Σ, μ) be a countably compact measure space such that Σ is countably generated, (Y, T, ν) a σ-finite measure space with a linear lifting, and $f : X \to Y$ an inverse-measure-preserving function. Then there is a disintegration $\langle \mu_y \rangle_{y \in Y}$ of μ over ν, consistent with f, such that $y \mapsto \mu_y E$ is a T-measurable function for every $E \in \Sigma$.

535Q Proposition Let (X,Σ,μ) and (Y,T,ν) be probability spaces, and λ the c.l.d. product measure on $X\times Y$. Suppose that $\lambda\restriction\Sigma\widehat{\otimes}\mathrm{T}$ has a linear lifting. Then μ has a linear lifting.

535R Proposition Write ν_ω^2 for the usual measure on $(\{0,1\}^\omega)^2$, and $\mathrm{T}_\omega^{(2)}$ for its domain. Suppose that ν_κ has a Baire linear lifting for some $\kappa\ge\mathfrak{c}^{++}$. Then there is a Borel linear lifting S for ν_ω^2 which respects coordinates in the sense that if $f\in\mathcal{L}^\infty(\mathrm{T}_\omega^{(2)})$ is determined by a single coordinate, then Sf is determined by the same coordinate.

535Z Problems (a) Can it be that every probability space has a lifting?

(b) Suppose that $\mathfrak{c}\ge\omega_3$. Does ν_ω have a Borel lifting?

(c) (A.H.Stone) Can there be a countable ordinal ζ and a lifting ϕ of ν_ω such that ϕE is a Borel set, with Baire class at most ζ, for every Borel set $E\subseteq\{0,1\}^\omega$?

(d) Can there be a strictly positive Radon probability measure of countable Maharam type which does not have a strong lifting?

(e) Is there a probability space which has a linear lifting but no lifting?

(f) Can there be a Borel linear lifting for the usual measure on $(\{0,1\}^\omega)^2$ which respects coordinates in the sense of 535R?

Version of 20.2.12

536 Alexandra Bellow's problem

In 463Za I mentioned a curious problem concerning pointwise compact sets of continuous functions. This problem is known to be soluble if we are allowed to assume the continuum hypothesis, for instance. Here I present the relevant arguments, with supplementary remarks on 'stable' sets of measurable functions (536E-536F).

536A The problem Let (X,Σ,μ) be a measure space, and $\mathcal{L}^0$ the space of all Σ-measurable functions from X to $\mathbb{R}$. On $\mathcal{L}^0$ we have the linear space topologies $\mathfrak{T}_p$ and $\mathfrak{T}_m$ of pointwise convergence and convergence in measure. $\mathfrak{T}_p$ is Hausdorff and locally convex; if μ is σ-finite, $\mathfrak{T}_m$ is pseudometrizable. The question is this: suppose that $K\subseteq\mathcal{L}^0$ is compact for $\mathfrak{T}_p$, and that $\mathfrak{T}_m$ is Hausdorff on K. Does it follow that $\mathfrak{T}_p$ and $\mathfrak{T}_m$ agree on K?

536B Known cases Let (X,Σ,μ) be a σ-finite measure space. Given that $K\subseteq\mathcal{L}^0$ is compact for $\mathfrak{T}_p$, and $\mathfrak{T}_m$ is Hausdorff on K, and

either K is sequentially compact for $\mathfrak{T}_p$

or K is countably tight for $\mathfrak{T}_p$

or K is convex

or X has a topology for which $K\subseteq C(X)$, μ is a strictly positive topological measure, and every function $h\in\mathbb{R}^X$ which is continuous on every relatively countably compact set is continuous

or μ is perfect

or K is stable,

then K is metrizable for $\mathfrak{T}_p$, and $\mathfrak{T}_p$ and $\mathfrak{T}_m$ agree on K.

536C Proposition Let (X,Σ,μ) be a probability space such that the π-weight $\pi(\mu)$ of μ is at most $\mathfrak{p}$. If $K\subseteq\mathcal{L}^0$ is $\mathfrak{T}_p$-compact then it is $\mathfrak{T}_m$-compact.

536D Theorem Let (X,Σ,μ) be a probability space, and $\mathcal{L}^0$ the space of Σ-measurable real-valued functions on X. Write $\mathfrak{T}_p$, $\mathfrak{T}_m$ for the topologies of pointwise convergence and convergence in measure on

$\mathcal{L}^0$. Suppose that $K \subseteq \mathcal{L}^0$ is $\mathfrak{T}_p$-compact and that $\mu\{x : f(x) \neq g(x)\} > 0$ for any distinct f, $g \in K$, but that K is not $\mathfrak{T}_p$-metrizable.

(a) Every infinite Hausdorff space which is a continuous image of a closed subset of K has a non-trivial convergent sequence.

(b) There is a continuous surjection from a closed subset of K onto $\{0,1\}^{\omega_1}$.

(c) Every infinite compact Hausdorff space of weight at most ω_1 has a non-trivial convergent sequence.

(d) $\mathfrak{c} > \omega_1$.

(e) The Maharam type of μ is at least 2^{ω_1}.

(f) There is a non-negligible measurable set in Σ which can be covered by ω_1 negligible sets.

(g) $\pi(\mu) > \mathfrak{p}$.

(h) $\mathfrak{m}_{\text{countable}} = \omega_1$.

536E Proposition Let (X, Σ, μ) be a semi-finite measure space, with null ideal $\mathcal{N}(\mu)$. For $E \in \Sigma$ let μ_E be the subspace measure on E. Suppose that $\pi(\mu_E) \le \operatorname{cov}(E, \mathcal{N}(\mu))$ whenever $E \in \Sigma$ and $0 < \mu E < \infty$. Then every $\mathfrak{T}_p$-separable $\mathfrak{T}_p$-compact subset of $\mathcal{L}^0 = \mathcal{L}^0(\Sigma)$ is stable.

536F Proposition Suppose that $\operatorname{cov}\mathcal{N} = \operatorname{cf}\mathcal{N}$, where $\mathcal{N}$ is the null ideal of Lebesgue measure on $\mathbb{R}$. Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space. Then every $\mathfrak{T}_p$-separable $\mathfrak{T}_p$-compact subset of $\mathcal{L}^0(\mu)$ is stable.

Version of 12.8.13

537 Sierpiński sets, shrinking numbers and strong Fubini theorems

W.Sierpiński observed that if the continuum hypothesis is true then there are uncountable subsets of $\mathbb{R}$ which have no uncountable negligible subsets, and that such sets lead to curious phenomena; he also observed that, again assuming the continuum hypothesis, there would be a (non-measurable) function $f : [0,1]^2 \to \{0,1\}$ for which Fubini's theorem failed radically, in the sense that

$$\iint f(x,y)dxdy = 0, \quad \iint f(x,y)dydx = 1.$$

In this section I set out to explore these two insights in the light of the concepts introduced in Chapter 52. I start with definitions of 'Sierpiński' and 'strongly Sierpiński' set (537A), with elementary facts and an excursion into the theory of 'entangled' sets (537C-537G). Turning to repeated integration, I look at three interesting cases in which, for different reasons, some form of separate measurability is enough to ensure equality of repeated integrals (537I, 537L, 537S). Working a bit harder, we find that there can be valid non-trivial inequalities of the form $\overline{\int}\underline{\int} dxdy \le \overline{\int}\overline{\int} dydx$ (537N-537Q).

As elsewhere, I will write $\mathcal{N}(\mu)$ for the null ideal of a measure μ.

537A Definitions (a) If (X, Σ, μ) is a measure space, a subset of X is a **Sierpiński set** if it is uncountable but meets every negligible set in a countable set.

(b) If (X, Σ, μ) is a measure space, a subset A of X is a **strongly Sierpiński set** if it is uncountable and for every $n \ge 1$ and for every set $W \subseteq X^n$ which is negligible for the product measure on X^n, the set $\{u : u \in A^n \cap W,\ u(i) \neq u(j) \text{ for } i < j < n\}$ is countable.

537B Proposition (a) Let (X, Σ, μ) be a measure space and $A \subseteq X$ a Sierpiński set.

(i) $\operatorname{add}\mathcal{N}(\mu) = \operatorname{non}\mathcal{N}(\mu) = \omega_1$ and $\operatorname{cov}\mathcal{N}(\mu) \ge \#(A)$.

(ii) If $\{x\}$ is negligible for every $x \in A$, then $\operatorname{cf}\mathcal{N}(\mu) \ge \operatorname{cf}([\#(A)]^{\le\omega})$.

(b) Suppose that (X, Σ, μ) and (Y, T, ν) are measure spaces such that singleton subsets of Y are negligible. Let $f : X \to Y$ be an inverse-measure-preserving function.

(i) If $A \subseteq X$ is a Sierpiński set, then $f[A]$ is a Sierpiński set in Y and $\#(f[A]) = \#(A)$.

(ii) Now suppose that ν is σ-finite. If $A \subseteq X$ is a strongly Sierpiński set, then $f[A]$ is a strongly Sierpiński set in Y.

(c) Suppose that λ and κ are infinite cardinals and that (X,Σ,μ) is a locally compact semi-finite measure space of Maharam type at most λ in which singletons are negligible and $\mu X>0$. Give $\{0,1\}^\lambda$ its usual measure.

(i) If $\{0,1\}^\lambda$ has a Sierpiński subset with cardinal κ, then X has a Sierpiński subset with cardinal κ.

(ii) If $\{0,1\}^\lambda$ has a strongly Sierpiński subset with cardinal κ, then X has a strongly Sierpiński subset with cardinal κ.

537C Entangled sets (a) Definition If X is a totally ordered set, then X is ω_1**-entangled** if whenever $n\ge 1$, $I\subseteq n$ and $\langle x_{\xi i}\rangle_{\xi<\omega_1,i<n}$ is a family of distinct elements of X, then there are distinct ξ, $\eta<\omega_1$ such that $I=\{i:i<n,\,x_{\xi i}\le x_{\eta i}\}$.

(b) Give $\{0,1\}^{\mathbb{N}}$ its lexicographic ordering, that is,

$$x\le y \text{ iff either } x=y \text{ or there is an } n\in\mathbb{N} \text{ such that } x{\restriction}n=y{\restriction}n \text{ and } x(n)<y(n).$$

Then any ω_1-entangled subset of $\{0,1\}^{\mathbb{N}}$ can be transferred to an ω_1-entangled subset of $\mathbb{R}$.

537D Lemma Let X be an ω_1-entangled totally ordered set.

(a) There is a countable set $D\subseteq X$ which meets $[x,y]$ whenever $x<y$ in X.

(b) Whenever $n\ge 1$, $I\subseteq n$ and $\langle x_{\xi i}\rangle_{\xi<\omega_1,i<n}$ is a family of distinct elements of X, there are $\xi<\eta<\omega_1$ such that $I=\{i:i<n,\,x_{\xi i}\le x_{\eta i}\}$.

537E Lemma Suppose that $n\ge 1$, $I\subseteq n$ and that $A\subseteq(\{0,1\}^{\mathbb{N}})^n$ is non-negligible for the usual product measure $\nu_{\mathbb{N}}^n$ on $(\{0,1\}^{\mathbb{N}})^n$. Let $\le$ be the lexicographic ordering of $\{0,1\}^{\mathbb{N}}$. Then there are v, $w\in A$ such that $v(i)\ne w(i)$ for every $i<n$ and $\{i:i<n,\,v(i)\le w(i)\}=I$.

537F Corollary Suppose that $A\subseteq\{0,1\}^{\mathbb{N}}$ is strongly Sierpiński for the usual measure on $\{0,1\}^{\mathbb{N}}$. Then A is ω_1-entangled for the lexicographic ordering of $\{0,1\}^{\mathbb{N}}$.

537G Theorem Suppose that there is an ω_1-entangled totally ordered set X of size $\kappa\ge\omega_1$. Then there are two upwards-ccc partially ordered sets P, Q such that $c^\uparrow(P\times Q)\ge\kappa$.

537H Scalarly measurable functions (a) Definition Let X be a set, Σ a σ-algebra of subsets of X and U a linear topological space. A function $\phi:X\to U$ is **scalarly (Σ-)measurable** if $f\phi:X\to\mathbb{R}$ is (Σ-)measurable for every $f\in U^*$.

(b) If $\phi:X\to U$ is scalarly measurable, V is another linear topological space and $T:U\to V$ is a continuous linear operator, then $T\phi:X\to V$ is scalarly measurable.

(c) If U is a separable metrizable locally convex space and $\phi:X\to U$ is scalarly measurable, then it is measurable.

537I Proposition Let (X,Σ,μ) and (Y,T,ν) be probability spaces and U a reflexive Banach space. Suppose that $x\mapsto u_x:X\to U$ and $y\mapsto f_y:Y\to U^*$ are bounded scalarly measurable functions. Then $\iint f_y(u_x)\mu(dx)\nu(dy)$ and $\iint f_y(u_x)\nu(dy)\mu(dx)$ are defined and equal.

537J Corollary Let (X,Σ,μ), (Y,T,ν) and (Z,Λ,σ) be probability spaces. Let $x\mapsto U_x:X\to\Lambda$ and $y\mapsto V_y:Y\to\Lambda$ be functions such that

$$x\mapsto\sigma(U_x\cap W),\quad y\mapsto\sigma(V_y\cap W)$$

are measurable for every $W\in\Lambda$. Then $\iint\sigma(U_x\cap V_y)\mu(dx)\nu(dy)$ and $\iint\sigma(U_x\cap V_y)\nu(dy)\mu(dx)$ are defined and equal.

537K Theorem Let $\langle(X_j,\Sigma_j,\mu_j)\rangle_{j\le m}$ be a finite sequence of probability spaces and $\langle\kappa_j\rangle_{j\le m}$ a sequence of cardinals such that $X_j^{\mathbb{N}}$, with its product measure $\mu_j^{\mathbb{N}}$, has a subset with cardinal κ_j which is not covered by κ_{j-1} negligible sets (if $j \ge 1$) and is not negligible (if $j = 0$). Let $f : \prod_{j \le m} X_j \to \mathbb{R}$ be a bounded function, and suppose that $\sigma : m+1 \to m+1$ and $\tau : m+1 \to m+1$ are permutations. Set

$$I = \underline{\int} \ldots \underline{\int} f(x_0, \ldots, x_m) dx_{\sigma(m)} \ldots dx_{\sigma(0)},$$

$$I' = \overline{\int} \ldots \overline{\int} f(x_0, \ldots, x_m) dx_{\tau(m)} \ldots dx_{\tau(0)}.$$

Then $I \le I'$.

537L Corollary Let $\langle (X_j, \Sigma_j, \mu_j) \rangle_{j \le m}$ be a finite sequence of probability spaces such that $X_j^{\mathbb{N}}$, with its product measure $\mu_j^{\mathbb{N}}$, has a Sierpiński set with cardinal ω_{j+1} for each $j \le m$. Let $f : \prod_{j \le m} X_j \to \mathbb{R}$ be a bounded function, and suppose that $\sigma : m+1 \to m+1$ and $\tau : m+1 \to m+1$ are permutations such that the two repeated integrals

$$I = \int \ldots \int f(x_0, \ldots, x_m) dx_{\sigma(m)} \ldots dx_{\sigma(0)},$$

$$I' = \int \ldots \int f(x_0, \ldots, x_m) dx_{\tau(m)} \ldots dx_{\tau(0)},$$

are both defined. Then $I = I'$.

537M Lemma Suppose that (X, Σ, μ) is a totally finite measure space and f is a $[0, \infty]$-valued function defined almost everywhere in X.

(a) If $\gamma < \overline{\int} f$, then there is a measurable integrable function $g : X \to [0, \infty[$ such that $\int g \ge \gamma$ and $\{x : x \in \operatorname{dom} f,\ g(x) \le f(x)\}$ has full outer measure in X.

(b) If $\underline{\int} f < \gamma$, then there is a measurable integrable function $g : X \to [0, \infty[$ such that $\int g \le \gamma$ and $\{x : x \in \operatorname{dom} f,\ f(x) \le g(x)\}$ has full outer measure in X.

537N Proposition Let (X, Σ, μ) be a semi-finite measure space, (Y, T, ν) a probability space, and $\nu^{\mathbb{N}}$ the product measure on $Y^{\mathbb{N}}$. If $\operatorname{non}(E, \mathcal{N}(\mu)) < \operatorname{cov} \mathcal{N}(\nu^{\mathbb{N}})$ for every $E \in \Sigma \setminus \mathcal{N}(\mu)$, then

$$\underline{\int} \underline{\int} f(x, y) \nu(dy) \mu(dx) \le \overline{\int} \overline{\int} f(x, y) \mu(dx) \nu(dy)$$

for every function $f : X \times Y \to [0, \infty]$.

537O Corollary Let (X, Σ, μ) and (Y, T, ν) be probability spaces, and $\nu^{\mathbb{N}}$ the product measure on $Y^{\mathbb{N}}$. If $\operatorname{shr}^+ \mathcal{N}(\mu) \le \operatorname{cov} \mathcal{N}(\nu^{\mathbb{N}})$ then

$$\overline{\int} \underline{\int} f(x, y) \nu(dy) \mu(dx) \le \overline{\int} \overline{\int} f(x, y) \mu(dx) \nu(dy)$$

for every function $f : X \times Y \to [0, \infty[$.

537P Corollary Let (X, Σ, μ) and (Y, T, ν) be probability spaces, and $\nu^{\mathbb{N}}$ the product measure on $Y^{\mathbb{N}}$; suppose that $\operatorname{shr}^+ \mathcal{N}(\mu) \le \operatorname{cov} \mathcal{N}(\nu^{\mathbb{N}})$, and that $f : X \times Y \to \mathbb{R}$ is bounded.

(a)

$$\overline{\int} \underline{\int} f(x, y) \nu(dy) \mu(dx) \le \overline{\int} \overline{\int} f(x, y) \mu(dx) \nu(dy),$$

$$\underline{\int} \underline{\int} f(x, y) \mu(dx) \nu(dy) \le \underline{\int} \overline{\int} f(x, y) \nu(dy) \mu(dx).$$

(b) If $\iint f(x, y) \mu(dx) \nu(dy)$ is defined, and $\int f(x, y) \nu(dy)$ is defined for almost every x, then the other repeated integral $\iint f(x, y) \nu(dy) \mu(dx)$ is defined and equal to $\iint f(x, y) \mu(dx) \nu(dy)$.

537Q Proposition Let (X, Σ, ν) and (Y, T, μ) be probability spaces, and $\mu^{\mathbb{N}}$, $\nu^{\mathbb{N}}$ the product measures on $X^{\mathbb{N}}$, $Y^{\mathbb{N}}$ respectively. If $\operatorname{shr}^+ \mathcal{N}(\mu^{\mathbb{N}}) \le \operatorname{cov} \mathcal{N}(\nu^{\mathbb{N}})$ then $\underline{\int} \underline{\int} f(x, y) \mu(dx) \nu(dy) \le \underline{\int} \overline{\int} f(x, y) \nu(dy) \mu(dx)$ for every function $f : X \times Y \to [0, \infty[$.

537R Lemma Let (X,Σ,μ) be a complete probability space and (Y,T,ν) a probability space such that $\mathrm{shr}^+\mathcal{N}(\mu) \le \mathrm{cov}\,\mathcal{N}(\nu^{\mathbb{N}})$, where $\nu^{\mathbb{N}}$ is the product measure on $Y^{\mathbb{N}}$. Let $f : X \times Y \to \mathbb{R}$ be a bounded function which is measurable in each variable separately, and set $u(x) = \int f(x,y)\nu(dy)$ for $x \in X$. Then $u : X \to \mathbb{R}$ is measurable.

537S Proposition Let (X,Σ,μ) and (Y,T,ν) be probability spaces such that

$$\mathrm{shr}^+\mathcal{N}(\mu) \le \mathrm{cov}\,\mathcal{N}(\nu^{\mathbb{N}}),$$

where $\nu^{\mathbb{N}}$ is the product measure on $Y^{\mathbb{N}}$, and

$$\mathrm{cf}([\tau(\nu)]^{\le\omega}) < \mathrm{cov}(E,\mathcal{N}(\mu)) \text{ for every } E \in \Sigma \setminus \mathcal{N}(\mu),$$

where $\tau(\nu)$ is the Maharam type of ν. Let $f : X \times Y \to [0,\infty[$ be a function which is measurable in each variable separately. Then $\iint f(x,y)\mu(dx)\nu(dy)$ and $\iint f(x,y)\nu(dy)\mu(dx)$ exist and are equal.

537Z Problems (a) Is it relatively consistent with ZFC to suppose that $\mathbb{R}$, with Lebesgue measure, has a Sierpiński subset but no strongly Sierpiński subset?

(b) Is it relatively consistent with ZFC to suppose that there is a probability space (X,μ) such that (X,μ) has a Sierpiński set but its power $(X^{\mathbb{N}},\mu^{\mathbb{N}})$ does not?

Version of 18.2.14

538 Filters and limits

A great many special types of filter have been studied. In this section I look at some which are particularly interesting from the point of view of measure theory: Ramsey ultrafilters, measure-converging filters and filters with the Fatou property. About half the section is directed towards Benedikt's theorem (538M) on extensions of perfect probability measures; on the way we need to look at measure-centering ultrafilters (538G-538K) and iterated products of filters (538E, 538L). The second major topic here is a study of 'medial limits' (538P-538S); these are Banach limits of a very special type. In between, the measure-converging property (538N) and the Fatou property (538O) offer some intriguing patterns.

538A Filters: Definitions Let $\mathcal{F}$ be a filter on $\mathbb{N}$.

(a) $\mathcal{F}$ is **free** if it includes the Fréchet filter.

(b) $\mathcal{F}$ is a **p-point filter** if it is free and for every sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{F}$ there is an $A \in \mathcal{F}$ such that $A \setminus A_n$ is finite for every $n \in \mathbb{N}$.

(c) $\mathcal{F}$ is **Ramsey** or **selective** if it is free and for every $f : [\mathbb{N}]^2 \to \{0,1\}$ there is an $A \in \mathcal{F}$ such that f is constant on $[A]^2$.

(d) $\mathcal{F}$ is **rapid** if it is free and for every sequence $\langle t_n\rangle_{n\in\mathbb{N}}$ of real numbers which converges to 0, there is an $A \in \mathcal{F}$ such that $\sum_{n\in A}|t_n|$ is finite. Note that a free filter $\mathcal{F}$ on $\mathbb{N}$ is rapid iff for every $f \in \mathbb{N}^{\mathbb{N}}$ there is an $A \in \mathcal{F}$ such that $\#(A \cap f(k)) \le k$ for every $k \in \mathbb{N}$.

(e) $\mathcal{F}$ is **nowhere dense** if for every sequence $\langle t_n\rangle_{n\in\mathbb{N}}$ in $\mathbb{R}$ there is an $A \in \mathcal{F}$ such that $\{t_n : n \in A\}$ is nowhere dense.

(f) $\mathcal{F}$ is **measure-centering** or has **property M** if whenever $\mathfrak{A}$ is a Boolean algebra, $\nu : \mathfrak{A} \to [0,\infty[$ is an additive functional, and $\langle a_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}$ such that $\inf_{n\in\mathbb{N}} \nu a_n > 0$, there is an $A \in \mathcal{F}$ such that $\{a_n : n \in A\}$ is centered.

(g) $\mathcal{F}$ is **measure-converging** if whenever (X,Σ,μ) is a probability space, $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in Σ, and $\lim_{n\to\infty} \mu E_n = 1$, then $\bigcup_{A\in\mathcal{F}}\bigcap_{n\in A} E_n$ is conegligible.

(h) $\mathcal{F}$ has the **Fatou property** if whenever (X,Σ,μ) is a probability space, $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in Σ, and $X=\bigcup_{A\in\mathcal{F}}\bigcap_{n\in A}E_n$, then $\lim_{n\to\mathcal{F}}\mu E_n$ is defined and equal to 1.

(i) For any countably infinite set I, I will say that a filter $\mathcal{F}$ on I is free, or a p-point filter, or Ramsey, etc., if it is isomorphic to such a filter on $\mathbb{N}$. For 'rapid' and 'measure-converging' filters, we need an appropriate translation of 'sequence converging to 0'; the corresponding notion on an arbitrary index set I is a function $u\in\boldsymbol{c}_0(I)$, that is, a real-valued function u on I such that $\{i:i\in I,\,|u(i)|\ge\epsilon\}$ is finite for every $\epsilon>0$.

538B The Rudin-Keisler ordering If $\mathcal{F}$, $\mathcal{G}$ are filters on sets I, J respectively, I will say that $\mathcal{F}\le_{\mathrm{RK}}\mathcal{G}$ if there is a function $f:J\to I$ such that

$$\mathcal{F}=f[[\mathcal{G}]]=\{A:A\subseteq I,\,f^{-1}[A]\in\mathcal{G}\},$$

the filter on I generated by $\{f[B]:B\in\mathcal{G}\}$. $\le_{\mathrm{RK}}$ is reflexive and transitive. If $\mathcal{F}\le_{\mathrm{RK}}\mathcal{G}$ and $\mathcal{G}$ is an ultrafilter, then $\mathcal{F}$ is an ultrafilter. If $\mathcal{F}$ is a principal ultrafilter then $\mathcal{F}\le_{\mathrm{RK}}\mathcal{G}$ for every filter $\mathcal{G}$.

538C Lemma (a) If I is a set, $\mathcal{F}$ is an ultrafilter on I and $f:I\to I$ is a function such that $f[[\mathcal{F}]]=\mathcal{F}$, then $\{i:f(i)=i\}\in\mathcal{F}$.

(b) If I is a set, $\mathcal{F}$ and $\mathcal{G}$ are ultrafilters on I, $\mathcal{F}\le_{\mathrm{RK}}\mathcal{G}$ and $\mathcal{G}\le_{\mathrm{RK}}\mathcal{F}$, then there is a permutation $h:I\to I$ such that $h[[\mathcal{F}]]=\mathcal{G}$; that is, $\mathcal{F}$ and $\mathcal{G}$ are isomorphic.

538D Finite products of filters (a) Suppose that $\mathcal{F}$, $\mathcal{G}$ are filters on sets I, J respectively. I will write $\mathcal{F}\ltimes\mathcal{G}$ for

$$\{A:A\subseteq I\times J,\,\{i:A[\{i\}]\in\mathcal{G}\}\in\mathcal{F}\}.$$

$\mathcal{F}\ltimes\mathcal{G}$ is a filter.

(b) If $\mathcal{F}$ and $\mathcal{G}$ are ultrafilters, so is $\mathcal{F}\ltimes\mathcal{G}$.

(c) If $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ are filters on I, J, K respectively, then the natural bijection between $(I\times J)\times K$ and $I\times(J\times K)$ is an isomorphism between $(\mathcal{F}\ltimes\mathcal{G})\ltimes\mathcal{H}$ and $\mathcal{F}\ltimes(\mathcal{G}\ltimes\mathcal{H})$.

(d) It follows that we can define a product $\mathcal{F}_0\ltimes\ldots\ltimes\mathcal{F}_n$ of any finite string $\mathcal{F}_0,\ldots,\mathcal{F}_n$ of filters, and under the natural identifications of the base sets we shall have $(\mathcal{F}_0\ltimes\ldots\ltimes\mathcal{F}_n)\ltimes(\mathcal{F}_{n+1}\ltimes\ldots\ltimes\mathcal{F}_m)$ identified with $\mathcal{F}_0\ltimes\ldots\ltimes\mathcal{F}_m$ whenever $\mathcal{F}_0,\ldots,\mathcal{F}_n,\ldots,\mathcal{F}_m$ are filters.

(e) $\mathcal{F}_m\le_{\mathrm{RK}}\mathcal{F}_0\ltimes\ldots\ltimes\mathcal{F}_n$ whenever $\mathcal{F}_0,\ldots,\mathcal{F}_n$ are filters and $m\le n$.

(f) If $\mathcal{F}$, $\mathcal{F}'$, $\mathcal{G}$ and $\mathcal{G}'$ are filters, with $\mathcal{F}\le_{\mathrm{RK}}\mathcal{F}'$ and $\mathcal{G}\le_{\mathrm{RK}}\mathcal{G}'$, then $\mathcal{F}\ltimes\mathcal{G}\le_{\mathrm{RK}}\mathcal{F}'\ltimes\mathcal{G}'$.

$\mathcal{F}_0\ltimes\ldots\ltimes\mathcal{F}_n\le_{\mathrm{RK}}\mathcal{G}_0\ltimes\ldots\ltimes\mathcal{G}_n$ whenever $\mathcal{F}_i\le_{\mathrm{RK}}\mathcal{G}_i$ for every $i\le n$.

(g) It follows that if $\mathcal{F}_0,\ldots,\mathcal{F}_n$ are filters and $k_0<\ldots<k_m\le n$, then $\mathcal{F}_{k_0}\ltimes\ldots\ltimes\mathcal{F}_{k_m}\le_{\mathrm{RK}}\mathcal{F}_0\ltimes\ldots\ltimes\mathcal{F}_n$.

538E Iterated products of filters (a) Set $S=\bigcup_{i\in\mathbb{N}}\mathbb{N}^i$. Fix on a family $\langle\theta(\xi,k)\rangle_{1\le\xi<\omega_1,k\in\mathbb{N}}$ such that each $\langle\theta(\xi,k)\rangle_{k\in\mathbb{N}}$ is a non-decreasing sequence running over a cofinal subset of ξ.

(b) Now suppose that ζ is a non-zero countable ordinal. Let $\langle\mathcal{F}_\xi\rangle_{1\le\xi\le\zeta}$ be a family of filters on $\mathbb{N}$. For $\xi\le\zeta$, define $\mathcal{G}_\xi\subseteq\mathcal{P}S$ as follows. Start by taking $\mathcal{G}_0$ to be the principal filter generated by $\{\emptyset\}$. For $1\le\xi\le\zeta$, set

$$\mathcal{G}_\xi=\{A:A\subseteq S,\,\{k:k\in\mathbb{N},\,\{\tau:<\!k\!>^\frown\tau\in A\}\in\mathcal{G}_{\theta(\xi,k)}\}\in\mathcal{F}_\xi\}.$$

(See 5A1C for the notation here.) It is elementary to check that every $\mathcal{G}_\xi$ is a filter, and that if every $\mathcal{F}_\xi$ is free, so is every $\mathcal{G}_\xi$. Moreover, if every $\mathcal{F}_\xi$ is an ultrafilter, so is every $\mathcal{G}_\xi$.

(c) $\mathcal{F}_\xi\le_{\mathrm{RK}}\mathcal{G}_\xi$ whenever $1\le\xi\le\zeta$ and $\mathcal{G}_\eta\le_{\mathrm{RK}}\mathcal{G}_\xi$ whenever $0\le\eta\le\xi\le\zeta$.

(d) $\mathcal{F}_{\xi_n} \ltimes \ldots \ltimes \mathcal{F}_{\xi_0} \le_{\mathrm{RK}} \mathcal{G}_\zeta$ whenever $1 \le \xi_0 < \ldots < \xi_n \le \zeta$.

(e) Suppose that we are given $A_\xi \in \mathcal{F}_\xi$ for each $\xi \in [1,\zeta]$. Define $T \subseteq S$ and $\alpha : T \to [0,\zeta]$ as follows. Start by saying that $\emptyset \in T$ and $\alpha(\emptyset) = \zeta$. Having determined $T \cap \mathbb{N}^n$ and $\alpha\restriction T \cap \mathbb{N}^n$, where $n \in \mathbb{N}$, then for $\tau \in \mathbb{N}^{n+1}$ say that $\tau \in T$ iff τ is of the form $\sigma^\frown{<}k{>}$ where

$$\sigma \in T \cap \mathbb{N}^n, \quad \alpha(\sigma) > 0, \quad k \in A_{\alpha(\sigma)}, \quad \sigma(i) < k \text{ for every } i < n,$$

and in this case set $\alpha(\tau) = \theta(\alpha(\sigma), k)$. Continue. Observe that $\alpha(\tau) < \alpha(\sigma)$ whenever σ, $\tau \in T$ and τ properly extends σ.

Suppose that $D \in \bigcap_{1 \le \xi \le \zeta} \mathcal{F}_\xi$. Then $T_D^* = \{\tau : \tau \in T \cap \bigcup_{n\in\mathbb{N}} D^n,\ \alpha(\tau) = 0\}$ belongs to $\mathcal{G}_\zeta$.

538F Ramsey filters: Proposition (a) A Ramsey filter on $\mathbb{N}$ is a rapid p-point ultrafilter.

(b) If $\mathcal{F}$ is a Ramsey ultrafilter on $\mathbb{N}$, $\mathcal{G}$ is a non-principal ultrafilter on $\mathbb{N}$, and $\mathcal{G} \le_{\mathrm{RK}} \mathcal{F}$, then $\mathcal{F}$ and $\mathcal{G}$ are isomorphic and $\mathcal{G}$ is a Ramsey ultrafilter.

(c) Let $\mathcal{F}$ be a Ramsey filter on $\mathbb{N}$. Suppose that $\langle A_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\mathcal{F}$. Then there is an $A \in \mathcal{F}$ such that $n \in A_m$ whenever m, $n \in A$ and $m < n$.

(d) Let $\mathcal{F}$ be a Ramsey filter on $\mathbb{N}$. Let $\mathcal{S} \subseteq [\mathbb{N}]^{<\omega}$ be such that $\emptyset \in \mathcal{S}$ and $\{n : I \cup \{n\} \in \mathcal{S}\} \in \mathcal{F}$ for every $I \in \mathcal{S}$. Then there is an $A \in \mathcal{F}$ such that $[A]^{<\omega} \subseteq \mathcal{S}$.

(e) If $\mathfrak{F}$ is a countable family of distinct Ramsey filters on $\mathbb{N}$, there is a disjoint family $\langle A_{\mathcal{F}}\rangle_{\mathcal{F}\in\mathfrak{F}}$ of subsets of $\mathbb{N}$ such that $A_{\mathcal{F}} \in \mathcal{F}$ for every $\mathcal{F} \in \mathfrak{F}$.

(f) Let $\mathfrak{F}$ be a countable family of non-isomorphic Ramsey ultrafilters on $\mathbb{N}$, and $\mathfrak{h} : \mathbb{N} \to [\mathfrak{F}]^{<\omega}$ a function. Suppose that we are given an $A_{\mathcal{F}} \in \mathcal{F}$ for each $\mathcal{F} \in \mathfrak{F}$. Then there is an $A \in \bigcap\mathfrak{F}$ such that whenever i, $j \in A$, $\mathcal{F} \in \mathfrak{h}(i)$ and $i < j$, there is a $k \in A_{\mathcal{F}}$ such that $i < k < j$.

(g) If $\mathfrak{m}_{\mathrm{countable}} = \mathfrak{c}$, there is a Ramsey ultrafilter on $\mathbb{N}$.

538G Measure-centering filters: Theorem Let $\mathcal{F}$ be a free filter on $\mathbb{N}$. Write ν_ω for the usual measure on $\{0,1\}^{\mathbb{N}}$, T_ω for its domain and $(\mathfrak{B}_\omega, \bar\nu_\omega)$ for its measure algebra. Then the following are equiveridical:

(i) $\mathcal{F}$ is measure-centering;

(ii) whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{B}_\omega$ such that $\inf_{n\in\mathbb{N}} \bar\nu_\omega a_n > 0$, there is an $A \in \mathcal{F}$ such that $\{a_n : n \in A\}$ is centered;

(iii) whenever $\langle F_n\rangle_{n\in\mathbb{N}}$ is a sequence in T_ω such that $\inf_{n\in\mathbb{N}} \nu_\omega F_n > 0$, there is an $A \in \mathcal{F}$ such that $\bigcap_{n\in A} F_n \ne \emptyset$;

(iv) whenever (X,Σ,μ) is a perfect totally finite measure space and $\langle F_n\rangle_{n\in\mathbb{N}}$ is a sequence in Σ, $\mu^*(\bigcup_{A\in\mathcal{F}} \bigcap_{n\in A} F_n) \ge \liminf_{n\to\mathcal{F}} \mu F_n$;

(v) whenever μ is a Radon probability measure on $\mathcal{P}\mathbb{N}$, then $\mu^*\mathcal{F} \ge \liminf_{n\to\mathcal{F}} \mu E_n$, where $E_n = \{a : n \in a \subseteq \mathbb{N}\}$ for each n.

538H Proposition (a) Any measure-centering filter on $\mathbb{N}$ is an ultrafilter.

(b) If $\mathcal{F}$ is a measure-centering ultrafilter on $\mathbb{N}$ and $\mathcal{G}$ is a filter on $\mathbb{N}$ such that $\mathcal{G} \le_{\mathrm{RK}} \mathcal{F}$, then $\mathcal{G}$ is measure-centering.

(c) Every Ramsey ultrafilter on $\mathbb{N}$ is measure-centering.

(d) Every measure-centering ultrafilter on $\mathbb{N}$ is a nowhere dense ultrafilter.

(e) If $\operatorname{cov}\mathcal{N} = \mathfrak{c}$, where $\mathcal{N}$ is the Lebesgue null ideal, then there is a measure-centering ultrafilter on $\mathbb{N}$.

538I Theorem Suppose that $\mathcal{F}$ is a measure-centering ultrafilter on $\mathbb{N}$, and that (X,Σ,μ) is a perfect probability space. Let $\mathcal{A}$ be the family of all sets of the form $\lim_{n\to\mathcal{F}} E_n$ where $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in Σ. Then there is a unique complete measure λ on X such that λ is inner regular with respect to $\mathcal{A}$ and $\lambda(\lim_{n\to\mathcal{F}} E_n) = \lim_{n\to\mathcal{F}} \mu E_n$ for every sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in Σ; and λ extends μ.

Remark By '$\lim_{n\to\mathcal{F}} E_n$' I mean the limit in the compact Hausdorff space $\mathcal{P}X$, that is,

$$\{x : \{n : x \in E_n\} \in \mathcal{F}\} = \bigcup_{A\in\mathcal{F}} \bigcap_{n\in A} E_n = \bigcap_{A\in\mathcal{F}} \bigcup_{n\in A} E_n.$$

Notation In this context, I will call λ the **$\mathcal{F}$-extension** of μ.

538J Proposition Let $\mathcal{F}$ be a measure-centering ultrafilter on $\mathbb{N}$ and (X,Σ,μ) a perfect probability space; let λ be the $\mathcal{F}$-extension of μ as defined in 538I.

(a) Let $(\mathfrak{A},\bar\mu)$ be the measure algebra of μ, $(\mathfrak{B},\bar\lambda)$ the measure algebra of λ, and $(\mathfrak{C},\bar\nu)$ the probability algebra reduced power $(\mathfrak{A},\bar\mu)^{\mathbb{N}}|\mathcal{F}$ (328C). Then we have a measure-preserving isomorphism $\pi:\mathfrak{B}\to\mathfrak{C}$ defined by saying that

$$\pi((\lim_{n\to\mathcal{F}}E_n)^\bullet)=\langle E_n^\bullet\rangle_{n\in\mathbb{N}}^\bullet$$

for every sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in Σ.

(b) Let (X',Σ',μ') be another perfect probability space, and $\phi:X\to X'$ an inverse-measure-preserving function. Let λ' be the $\mathcal{F}$-extension of μ'. Then ϕ is inverse-measure-preserving for λ and λ'.

(c) Let $\mathcal{F}'$ be a filter on $\mathbb{N}$ such that $\mathcal{F}'\le_{\mathrm{RK}}\mathcal{F}$, and λ' the $\mathcal{F}'$-extension of μ. Then λ extends λ'.

538K Proposition Let (X,Σ,μ) be a perfect probability space, $\mathcal{F}$ a measure-centering ultrafilter on $\mathbb{N}$ and λ the $\mathcal{F}$-extension of μ.

(a)(i) Let $\langle f_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{L}^0(\mu)$ such that $\{f_n^\bullet:n\in\mathbb{N}\}$ is bounded in the linear topological space $L^0(\mu)$. Then

(α) $f(x)=\lim_{n\to\mathcal{F}}f_n(x)$ is defined in $\mathbb{R}$ for λ-almost every $x\in X$;

(β) $f\in\mathcal{L}^0(\lambda)$.

(ii) For every $f\in\mathcal{L}^0(\lambda)$ there is a sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{L}^0(\mu)$, bounded in the sense of (i), such that $f=\lim_{n\to\mathcal{F}}f_n$ λ-a.e.

(b) Suppose that $1\le p\le\infty$, and $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}^p(\mu)$ such that $\sup_{n\in\mathbb{N}}\|f_n\|_p$ is finite. Set $f(x)=\lim_{n\to\mathcal{F}}f_n(x)$ whenever this is defined in $\mathbb{R}$.

(i)(α) $f\in\mathcal{L}^p(\lambda)$;

(β) $\|f\|_p\le\lim_{n\to\mathcal{F}}\|f_n\|_p$.

(ii) Let g be a conditional expectation of f on Σ.

(α) If $p=1$ and $\{f_n:n\in\mathbb{N}\}$ is uniformly integrable, then then $\|f\|_1=\lim_{n\to\mathcal{F}}\|f_n\|_1$ and $g^\bullet=\lim_{n\to\mathcal{F}}f_n^\bullet$ for the weak topology of $L^1(\mu)$.

(β) If $1<p<\infty$, then $g^\bullet=\lim_{n\to\mathcal{F}}f_n^\bullet$ for the weak topology of $L^p(\mu)$.

(c) Suppose that $1\le p\le\infty$ and $f\in\mathcal{L}^p(\lambda)$.

(i) There is a sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{L}^p(\mu)$ such that $f=\lim_{n\to\mathcal{F}}f_n$ λ-a.e. and $\|f\|_p=\sup_{n\in\mathbb{N}}\|f_n\|_p$.

(ii) If $p=1$, we can arrange that $\langle f_n\rangle_{n\in\mathbb{N}}$ should be uniformly integrable.

(d) Let (X',Σ',μ') be another perfect measure space, and λ' the $\mathcal{F}$-extension of μ'. Let $S:L^1(\mu)\to L^1(\mu')$ be a bounded linear operator.

(i) There is a unique bounded linear operator $\hat S:L^1(\lambda)\to L^1(\lambda')$ such that $\hat Sf^\bullet=g^\bullet$ whenever $\langle f_n\rangle_{n\in\mathbb{N}}$, $\langle g_n\rangle_{n\in\mathbb{N}}$ are uniformly integrable sequences in $\mathcal{L}^1(\mu)$, $\mathcal{L}^1(\nu)$ respectively, $f=\lim_{n\to\mathcal{F}}f_n$ λ-a.e., $g=\lim_{n\to\mathcal{F}}g_n$ λ'-a.e., and $g_n^\bullet=Sf_n^\bullet$ for every $n\in\mathbb{N}$.

(ii) The map $S\mapsto\hat S:\mathsf{B}(L^1(\mu);L^1(\mu'))\to\mathsf{B}(L^1(\lambda);L^1(\lambda'))$ is a norm-preserving Riesz homomorphism.

538L Theorem Suppose that ζ is a non-zero countable ordinal and $\langle\mathcal{F}_\xi\rangle_{1\le\xi\le\zeta}$ is a family of Ramsey ultrafilters on $\mathbb{N}$, no two isomorphic. Let $\langle\mathcal{G}_\xi\rangle_{\xi\le\zeta}$ be the corresponding iterated product system, as described in 538E. Then $\mathcal{G}_\zeta$ is a measure-centering ultrafilter.

538M Benedikt's theorem Let (X,Σ,μ) be a perfect probability space. Then there is a measure λ on X, extending μ, such that $\lambda(\lim_{n\to\mathcal{F}}E_n)$ is defined and equal to $\lim_{n\to\mathcal{F}}\mu E_n$ for every sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in Σ and every Ramsey filter $\mathcal{F}$ on $\mathbb{N}$.

538N Measure-converging filters: Proposition (a) Let $\mathcal{F}$ be a free filter on $\mathbb{N}$. Let ν_ω be the usual measure on $\{0,1\}^{\mathbb{N}}$, and T_ω its domain. Then the following are equiveridical:

(i) $\mathcal{F}$ is measure-converging;

(ii) whenever $\langle F_n\rangle_{n\in\mathbb{N}}$ is a sequence in T_ω and $\lim_{n\to\infty}\nu_\omega F_n=1$, then $\bigcup_{A\in\mathcal{F}}\bigcap_{n\in A}F_n$ is conegligible;

(iii) whenever (X,Σ,μ) is a measure space with locally determined negligible sets, and $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}^0$ which converges in measure to $f\in\mathcal{L}^0$, then $\lim_{n\to\mathcal{F}}f_n=_{\text{a.e.}}f$;

(iv) whenever μ is a Radon measure on $\mathcal{P}\mathbb{N}$ such that $\lim_{n\to\infty}\mu E_n = 1$, where $E_n = \{a : n \in a \subseteq \mathbb{N}\}$ for each n, then $\mu\mathcal{F} = 1$.

(b) Every measure-converging filter is free.

(c) Suppose that $\mathcal{F}$ is a measure-converging filter.

(i) If $\mathcal{G}$ is a filter on $\mathbb{N}$ including $\mathcal{F}$, then $\mathcal{G}$ is measure-converging.

(ii) If $\mathcal{G}$ is a filter on $\mathbb{N}$ and $\mathcal{G} \le_{\mathrm{RB}} \mathcal{F}$, then $\mathcal{G}$ is measure-converging.

(d) Every rapid filter is measure-converging.

(e) If there is a measure-converging filter, there is a measure-converging filter which is not rapid.

(f) Let $\mathcal{F}$ be a measure-converging filter on $\mathbb{N}$ and $\mathcal{G}$ any filter on $\mathbb{N}$. Then $\mathcal{G} \ltimes \mathcal{F}$ is measure-converging.

(g) If $\mathfrak{m}_{\mathrm{countable}} = \mathfrak{d}$, there is a rapid filter.

538O The Fatou property: Proposition (a) Let $\mathcal{F}$ be a filter on $\mathbb{N}$. Let ν_ω be the usual measure on $\{0,1\}^{\mathbb{N}}$, and T_ω its domain. Then the following are equiveridical:

(i) $\mathcal{F}$ has the Fatou property;

(ii) whenever $\langle F_n\rangle_{n\in\mathbb{N}}$ is a sequence in T_ω and $\nu_\omega^*(\bigcup_{A\in\mathcal{F}}\bigcap_{n\in A} F_n) = 1$, then $\lim_{n\to\mathcal{F}}\nu_\omega F_n = 1$;

(iii) whenever (X,Σ,μ) is a measure space and $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence of non-negative functions in $\mathcal{L}^0(\mu)$, then $\overline{\int}\liminf_{n\to\mathcal{F}} f_n d\mu \le \liminf_{n\to\mathcal{F}}\int f_n d\mu$;

(iv) whenever μ is a Radon probability measure on $\mathcal{P}\mathbb{N}$, then $\mu^*\mathcal{F} \le \liminf_{n\to\mathcal{F}}\mu E_n$, where $E_n = \{a : n \in a \subseteq \mathbb{N}\}$ for each $n \in \mathbb{N}$.

(b) If $\mathcal{F}$ and $\mathcal{G}$ are filters on $\mathbb{N}$, $\mathcal{G} \le_{\mathrm{RK}} \mathcal{F}$ and $\mathcal{F}$ has the Fatou property, then $\mathcal{G}$ has the Fatou property.

(c) If $\mathcal{F}$ and $\mathcal{G}$ are filters on $\mathbb{N}$ with the Fatou property, then $\mathcal{F} \ltimes \mathcal{G}$ has the Fatou property.

538P Theorem Let $\nu : \mathcal{P}\mathbb{N} \to \mathbb{R}$ be a bounded finitely additive functional. Write $⨍ \ldots d\nu$ for the associated linear functional on ℓ^∞, and set $E_n = \{a : n \in a \subseteq \mathbb{N}\}$ for each $n \in \mathbb{N}$. Then the following are equiveridical:

(i) whenever μ is a Radon probability measure on $\mathcal{P}\mathbb{N}$, $\int \nu(a)\mu(da)$ is defined and equal to $⨍ \mu E_n \nu(dn)$;

(ii) whenever μ is a Radon probability measure on $[0,1]^{\mathbb{N}}$, $\int⨍ x\, d\nu\, \mu(dx)$ is defined and equal to $⨍\int x(n)\mu(dx)\nu(dn)$;

(iii) whenever (X,Σ,μ) is a probability space and $\langle f_n\rangle_{n\in\mathbb{N}}$ is a uniformly bounded sequence of measurable real-valued functions on X, then $\int⨍ f_n(x)\nu(dn)\mu(dx)$ is defined and equal to $⨍\int f_n d\mu\,\nu(dn)$;

(iv) whenever $\langle F_n\rangle_{n\in\mathbb{N}}$ is a sequence of Borel subsets of $\{0,1\}^{\mathbb{N}}$, $\int⨍ \chi F_n(x)\nu(dn)\nu_\omega(dx)$ is defined and equal to $⨍ \nu_\omega F_n \nu(dn)$, where ν_ω is the usual measure on $\{0,1\}^{\mathbb{N}}$.

538Q Definition I will say that a bounded finitely additive functional ν satisfying (i)-(iv) of 538P is a **medial functional**; if, in addition, ν is non-negative, $\nu a = 0$ for every finite set $a \subseteq \mathbb{N}$ and $\nu\mathbb{N} = 1$, I will call ν a **medial limit**.

538P(i) tells us that a medial limit is universally Radon-measurable, therefore universally measurable.

538R Proposition Let M be the L-space of bounded finitely additive functionals on $\mathcal{P}\mathbb{N}$, and $M_{\mathrm{med}} \subseteq M$ the set of medial functionals.

(a) M_{med} is a band in M, and if $T \in L^\times(\ell^\infty;\ell^\infty)$ and $T' : M \to M$ is its adjoint, then $T'\nu \in M_{\mathrm{med}}$ for every $\nu \in M_{\mathrm{med}}$.

(b) Taking M_τ to be the band of completely additive functionals on $\mathcal{P}\mathbb{N}$ and M_{m} the band of measurable functionals, as described in §464, $M_\tau \subseteq M_{\mathrm{med}} \subseteq M_{\mathrm{m}}$.

(c) Suppose that $\langle\nu_k\rangle_{k\in\mathbb{N}}$ is a norm-bounded sequence in M_{med}, and that $\nu \in M_{\mathrm{med}}$. Set $\tilde\nu(a) = ⨍ \nu_k(a)\nu(dk)$ for $a \subseteq \mathbb{N}$. Then $\tilde\nu \in M_{\mathrm{med}}$.

(d) Suppose that $\nu \in M$ is a medial limit, and set $\mathcal{F} = \{a : a \subseteq \mathbb{N},\ \nu(a) = 1\}$. Then $\mathcal{F}$ is a measure-converging filter with the Fatou property.

(e) Let (X,Σ,μ) and (Y,T,λ) be probability spaces, and $T \in L^\times(L^\infty(\mu);L^\infty(\lambda))$. Let $\langle f_n\rangle_{n\in\mathbb{N}}$, $\langle g_n\rangle_{n\in\mathbb{N}}$ be sequences in $\mathcal{L}^\infty(\mu)$, $\mathcal{L}^\infty(\nu)$ respectively such that $Tf_n^\bullet = g_n^\bullet$ for every n and $\langle f_n^\bullet\rangle_{n\in\mathbb{N}}$ is norm-bounded in $L^\infty(\mu)$. Let $\nu \in M$ be a medial functional. Then $f(x) = ⨍ f_n(x)\nu(dn)$ and $g(y) = ⨍ g_n(y)\nu(dn)$ are defined for almost every $x \in X$ and $y \in Y$; moreover, $f \in \mathcal{L}^\infty(\mu)$, $g \in \mathcal{L}^\infty(\lambda)$ and $Tf^\bullet = g^\bullet$.

538S Theorem (a) If $\mathfrak{m}_{\mathrm{countable}} = \mathfrak{c}$, there is a medial limit.

(b) Suppose that the filter dichotomy is true. If I is any set and ν is a finitely additive real-valued functional on $\mathcal{P}I$ which is universally measurable for the usual topology on $\mathcal{P}I$, then ν is completely additive. Consequently there is no medial limit.

538Z Problem Show that it is relatively consistent with ZFC to suppose that there are no measure-converging filters on $\mathbb{N}$.

Version of 24.5.14/30.8.18

539 Maharam submeasures

Continuing the work of §§392-394 and 496, I return to Maharam submeasures and the forms taken by the ideas of the present volume in this context. At least for countably generated algebras, and in some cases more generally, many of the methods of Chapter 52 can be applied (539B-539K). In 539L-539N I give the main result of Balcar Jech & Pazak 05 and Veličković 05: it is consistent to suppose that every Dedekind complete ccc weakly (σ,∞)-distributive Boolean algebra is a Maharam algebra. In 539R-539U I introduce the idea of 'exhaustivity rank' of an exhaustive submeasure.

539B Proposition Let $\mathfrak{A}$ be a Maharam algebra, $\tau(\mathfrak{A})$ its Maharam type and $d_{\mathfrak{T}}(\mathfrak{A})$ its topological density for its Maharam-algebra topology. Then $\tau(\mathfrak{A}) \le d_{\mathfrak{T}}(\mathfrak{A}) \le \max(\omega, \tau(\mathfrak{A}))$.

539C Theorem Let $\mathfrak{A}$ be a Maharam algebra.

(a)

$$(\mathfrak{A}^+, \supseteq', [\mathfrak{A}^+]^{\le\max(\omega,\tau(\mathfrak{A}))}) \preccurlyeq_{\mathrm{GT}} (\mathrm{Pou}(\mathfrak{A}), \sqsubseteq^*, \mathrm{Pou}(\mathfrak{A})).$$

(b) $\mathrm{Pou}(\mathfrak{A}) \preccurlyeq_{\mathrm{T}} \mathcal{N}_{\tau(\mathfrak{A})}$.

539D Corollary Let $\mathfrak{A}$ be a Maharam algebra.

(a) $\pi(\mathfrak{A}) \le \max(\mathrm{cf}[\tau(\mathfrak{A})]^{\le\omega}, \mathrm{cf}\,\mathcal{N})$.

(b) If $\tau(\mathfrak{A}) \le \omega$, then $\mathrm{wdistr}(\mathfrak{A}) \ge \mathrm{add}\,\mathcal{N}$.

539E Proposition If $\mathfrak{A}$ is an atomless Maharam algebra, not $\{0\}$, there is a sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $\sup_{n\in I} a_n = 1$ and $\inf_{n\in I} a_n = 0$ for every infinite $I \subseteq \mathbb{N}$.

539F Definition The **splitting number** $\mathfrak{s}$ is the least cardinal of any family $\mathcal{A} \subseteq \mathcal{P}\mathbb{N}$ such that for every infinite $I \subseteq \mathbb{N}$ there is an $A \in \mathcal{A}$ such that $I \cap A$ and $I \setminus A$ are both infinite.

539G Proposition Let X be a set, Σ a σ-algebra of subsets of X, and ν an atomless Maharam submeasure on Σ. Let $\mathcal{M}$ be the ideal of meager subsets of $\mathbb{R}$.

(a) $\mathrm{non}\,\mathcal{N}(\nu) \ge \max(\mathfrak{s}, \mathfrak{m}_{\mathrm{countable}})$.

(b) $\mathrm{cov}\,\mathcal{N}(\nu) \le \mathrm{non}\,\mathcal{M}$.

539H Corollary Let $\mathfrak{A}$ be an atomless Maharam algebra, not $\{0\}$. Then $d(\mathfrak{A}) \ge \max(\mathfrak{s}, \mathfrak{m}_{\mathrm{countable}})$.

539I Corollary Suppose that $\#(X) < \max(\mathfrak{s}, \mathfrak{m}_{\mathrm{countable}})$, where $\mathfrak{s}$ is the splitting number. Let Σ be a σ-algebra of subsets of X such that (X, Σ) is countably separated, and $\mathcal{I}$ a σ-ideal of Σ containing singletons. Then there is no non-zero Maharam submeasure on $\Sigma/\mathcal{I}$.

539J Theorem (a) Let ν be a totally finite Radon submeasure on a Hausdorff space X and $\mathfrak{A}$ its Maharam algebra. Then $\mathcal{N}(\nu) \preccurlyeq_{\mathrm{T}} \mathrm{Pou}(\mathfrak{A})$.

(b) Let ν be a totally finite Radon submeasure on a Hausdorff space X and $\mathfrak{A}$ its Maharam algebra.

(i) $\mathrm{wdistr}(\mathfrak{A}) \le \mathrm{add}\,\mathcal{N}(\nu)$.

(ii) $\tau(\mathfrak{A}) \le w(X)$.

(iii) $\mathrm{cf}\,\mathcal{N}(\nu) \le \max(\mathrm{cf}[\tau(\mathfrak{A})]^{\le\omega}, \mathrm{cf}\,\mathcal{N})$.

(iv) If $\tau(\mathfrak{A}) \le \omega$, then $\mathrm{add}\,\mathcal{N}(\nu) \ge \mathrm{add}\,\mathcal{N}$ and $\mathrm{cf}\,\mathcal{N}(\nu) \le \mathrm{cf}\,\mathcal{N}$.

539K Proposition Let $\mathfrak{A}$ be a Boolean algebra and ν an exhaustive submeasure on $\mathfrak{A}$.

(a) Let $\langle a_i\rangle_{i\in\mathbb{N}}$ be a sequence in $\mathfrak{A}$ such that $\inf_{i\in\mathbb{N}}\nu a_i>0$.

(i) There is an infinite $I\subseteq\mathbb{N}$ such that $\{a_i:i\in I\}$ is centered.

(ii) For every $k\in\mathbb{N}$ there are an $S\in[\mathbb{N}]^\omega$ and a $\delta>0$ such that $\nu(\inf_{i\in J}a_i)\ge\delta$ for every $J\in[S]^k$.

(b) Suppose that $\langle a_\xi\rangle_{\xi<\kappa}$ is a family in $\mathfrak{A}$ such that $\inf_{\xi<\kappa}\nu a_\xi>0$, where κ is a regular uncountable cardinal. Then for every $k\in\mathbb{N}$ there are a stationary set $S\subseteq\kappa$ and a $\delta>0$ such that $\nu(\inf_{i\in J}a_i)\ge\delta$ for every $J\in[S]^k$.

(c) If ν is strictly positive, then (κ,κ,k) is a precaliber triple of $\mathfrak{A}$ for every regular uncountable cardinal κ and every $k\in\mathbb{N}$; in particular, $\mathfrak{A}$ satisfies Knaster's condition.

539L Lemma Let $\mathfrak{A}$ be a Boolean algebra, and $\mathcal{I}$ the family of countable subsets I of $\mathfrak{A}$ for which there is a partition C of unity such that $\{a:a\in I,\,a\cap c\ne 0\}$ is finite for every $c\in C$.

(a) $\mathcal{I}$ is an ideal of $\mathcal{P}\mathfrak{A}$ including $[\mathfrak{A}]^{<\omega}$.

(b) If $A\subseteq\mathfrak{A}^+$ is such that $A\cap I$ is finite for every $I\in\mathcal{I}$, and $B=\{b:b\supseteq a$ for some $a\in A\}$, then $B\cap I$ is finite for every $I\in\mathcal{I}$.

(c) If $\mathfrak{A}$ is ccc, then there is no uncountable $B\subseteq\mathfrak{A}$ such that $[B]^{\le\omega}\subseteq\mathcal{I}$.

(d) If $\mathfrak{A}$ is ccc and weakly (σ,∞)-distributive, $\mathcal{I}$ is a p-ideal.

Remark $\mathcal{I}$ is called **Quickert's ideal**.

539M Lemma Let $\mathfrak{A}$ be a weakly (σ,∞)-distributive ccc Dedekind σ-complete Boolean algebra, and suppose that $\mathfrak{A}^+$ is expressible as $\bigcup_{k\in\mathbb{N}}D_k$ where no infinite subset of any D_k belongs to Quickert's ideal $\mathcal{I}$. Then $\mathfrak{A}$ is a Maharam algebra.

539N Theorem (Balcar Jech & Pazák 05, Veličković 05) Suppose that Todorčević's p-ideal dichotomy is true. Then every Dedekind σ-complete ccc weakly (σ,∞)-distributive Boolean algebra is a Maharam algebra.

539O Corollary Suppose that Todorčević's p-ideal dichotomy is true. Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra such that every countably generated order-closed subalgebra of $\mathfrak{A}$ is a measurable algebra. Then $\mathfrak{A}$ is a measurable algebra.

539P Souslin algebras: Proposition Suppose that T is a well-pruned Souslin tree, and set $\mathfrak{A}=\mathrm{RO}^{\uparrow}(T)$.

(a) $\mathfrak{A}$ is Dedekind complete, ccc and weakly (σ,∞)-distributive.

(b) If $\mathfrak{B}$ is an order-closed subalgebra of $\mathfrak{A}$ and $\tau(\mathfrak{B})\le\omega$, then $\mathfrak{B}\cong\mathcal{P}I$ for some countable set I; in particular, $\mathfrak{B}$ is a measurable algebra.

(c) The only Maharam submeasure on $\mathfrak{A}$ is identically zero.

539Q Reflection principles (a) If $\mathfrak{A}$ is a Boolean algebra and every subset of $\mathfrak{A}$ of cardinal ω_1 is included in a ccc subalgebra of $\mathfrak{A}$, then $\mathfrak{A}$ is ccc.

(b) If $\mathfrak{A}$ is ccc and every countable subset of $\mathfrak{A}$ is included in a weakly (σ,∞)-distributive subalgebra of $\mathfrak{A}$, then $\mathfrak{A}$ is weakly (σ,∞)-distributive.

(c) If every countable subset of $\mathfrak{A}$ is included in a subalgebra of $\mathfrak{A}$ with the σ-interpolation property, then $\mathfrak{A}$ has the σ-interpolation property.

(d) If $\mathfrak{A}$ is a Maharam algebra and every countably generated closed subalgebra of $\mathfrak{A}$ is a measurable algebra, then $\mathfrak{A}$ is measurable.

(e) Suppose that Todorčević's p-ideal dichotomy is true. Let $\mathfrak{A}$ be a Boolean algebra such that every subset of $\mathfrak{A}$ of cardinal at most ω_1 is included in a subalgebra of $\mathfrak{A}$ which is a Maharam algebra. Then $\mathfrak{A}$ is a Maharam algebra.

(f) Suppose that Todorčević's p-ideal dichotomy is true. Let $\mathfrak{A}$ be a Boolean algebra such that every subset of $\mathfrak{A}$ of cardinal at most $\mathfrak{c}$ is included in a subalgebra of $\mathfrak{A}$ which is a measurable algebra. Then $\mathfrak{A}$ is measurable.

(g) On the other hand, if κ is an infinite cardinal such that $2^\kappa = \kappa^+$, $\square_\kappa$ is true and the cardinal power κ^ω is equal to κ, then there is a Dedekind complete Boolean algebra $\mathfrak{A}$, with cardinal κ^+, such that every order-closed subalgebra of $\mathfrak{A}$ with cardinal at most κ is a measurable algebra, but $\mathfrak{A}$ is not a measurable algebra (and therefore is not a Maharam algebra). In particular, this can be the case with $\kappa = \mathfrak{c}$.

539R Exhaustivity rank: Definitions Suppose that $\mathfrak{A}$ is a Boolean algebra and ν an exhaustive submeasure on $\mathfrak{A}$. For $\epsilon > 0$, say that $a \preccurlyeq_\epsilon b$ if either $a = b$ or $a \subseteq b$ and $\nu(b \setminus a) > \epsilon$. Then $\preccurlyeq_\epsilon$ is a well-founded partial order on $\mathfrak{A}$. Let $r_{\nu\epsilon} : \mathfrak{A} \to \mathrm{On}$ be the corresponding rank function. Now the **exhaustivity rank** of ν is $\sup_{\epsilon>0} r_{\nu\epsilon}(1)$.

539S Elementary facts Let $\mathfrak{A}$ be a Boolean algebra with an exhaustive submeasure ν and associated rank functions $r_{\nu\epsilon}$ for $\epsilon > 0$.

(a) $r_{\nu\delta}(a) \le r_{\nu\epsilon}(b)$ whenever $\nu(a \setminus b) \le \delta - \epsilon$.

$$r_{\nu\epsilon}(a) \le r_{\nu\epsilon}(b) \text{ if } a \subseteq b, \quad r_{\nu\delta}(a) \le r_{\nu\epsilon}(a) \text{ if } \epsilon \le \delta.$$

(b) If a, $b \in \mathfrak{A}$ are disjoint and $\epsilon > 0$, then $r_{\nu\epsilon}(a \cup b)$ is at least the ordinal sum $r_{\nu\epsilon}(a) + r_{\nu\epsilon}(b)$.

539T The rank of a Maharam algebra (a) Note that the rank function $r_{\nu\epsilon}$ associated with an exhaustive submeasure ν depends only on the set $\{a : \nu a > \epsilon\}$. In particular, if ν and ν' are exhaustive submeasures on a Boolean algebra $\mathfrak{A}$ and $\nu a \le \epsilon$ whenever $\nu' a \le \delta$, then $r_{\nu\epsilon}(a) \le r_{\nu'\delta}(a)$ for every $a \in \mathfrak{A}$. If $\mathfrak{A}$ is a Maharam algebra, then any two Maharam submeasures on $\mathfrak{A}$ are mutually absolutely continuous , so have the same exhaustivity rank; I will call this the **Maharam submeasure rank** of $\mathfrak{A}$, $\mathrm{Mhsr}(\mathfrak{A})$. Note that if $a \in \mathfrak{A}$ then $\mathrm{Mhsr}(\mathfrak{A}_a) \le \mathrm{Mhsr}(\mathfrak{A})$.

(b) If $\mathfrak{A}$ is a measurable algebra, $\mathrm{Mhsr}(\mathfrak{A}) \le \omega$. More generally, for any uniformly exhaustive submeasure ν and $\epsilon > 0$, $r_{\nu\epsilon}(a)$ is finite, being the maximal size of any disjoint set consisting of elements, included in a, of submeasure greater than ϵ.

(c)(i) Suppose that $\mathfrak{A}$ is a Maharam algebra with a strictly positive Maharam submeasure ν, and that $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$ which is dense for the Maharam-algebra topology of $\mathfrak{A}$. For $\epsilon > 0$, write $r_\epsilon = r_{\nu\epsilon}$ for the corresponding rank function on $\mathfrak{A}$, and $r'_\epsilon = r_{\nu\restriction\mathfrak{B},\epsilon}$ for the rank function on $\mathfrak{B}$ corresponding to the exhaustive submeasure $\nu\restriction\mathfrak{B}$. If $0 < \delta < \epsilon$, $a \in \mathfrak{A}$, $b \in \mathfrak{B}$, $\xi \in \mathrm{On}$, $\nu(a \mathbin{\triangle} b) < \epsilon - \delta$ and $r_\epsilon(a) \ge \xi$, then $r'_\delta(b) \ge \xi$.

(ii) It follows that if $\mathfrak{A}$ is an infinite Maharam algebra, then $\mathrm{Mhsr}(\mathfrak{A}) < \tau(\mathfrak{A})^+$.

(d) The Maharam algebras described in §394 are all defined from exhaustive submeasures with domain the countable algebra $\mathfrak{B}$ of open-and-closed subsets of a compact metrizable space. By (c), such algebras must have Maharam submeasure rank less than ω_1.

539U Theorem Suppose that $\mathfrak{A}$ is a non-measurable Maharam algebra. Then $\mathrm{Mhsr}(\mathfrak{A})$ is at least the ordinal power ω^ω.

539V PV norms and exhaustivity (a) If we construct a submeasure ν on an algebra $\mathfrak{B}$ from a PV norm $\| \, \|$ on $[\mathbb{N}]^{<\omega}$ and sequences $\langle T_n\rangle_{n\in\mathbb{N}}$, $\langle \alpha_k\rangle_{k\in\mathbb{N}}$ and $\langle N_k\rangle_{k\in\mathbb{N}}$ as in 394B and 394H, we can relate the exhaustivity rank of ν to $\| \, \|$, as follows. Note first that the set $\mathcal{L} = \{L : L \in [\mathbb{N}]^{<\omega}, \nu L \le 1\}$, ordered by $\subseteq$, is a tree with no infinite branches, by the last clause of 394Aa. For $\mathcal{K} \subseteq [\mathbb{N}]^{<\omega}$, set $\partial\mathcal{K} = \{K \setminus \{\max K\} : \emptyset \ne K \in \mathcal{K}\}$; iterating as in 421N, set

$$\partial^0\mathcal{L} = \mathcal{L}, \quad \partial^\xi\mathcal{L} = \partial(\textstyle\bigcap_{\eta<\xi} \partial^\eta\mathcal{L})$$

for ordinals $\xi > 0$. Now observe that if $L \subset L' \in \mathcal{L}$ and $z \in \prod_{r\in L'} T_r$, then (at least if every T_r has at least two members) $Y_{z\restriction L} \setminus Y_z$ includes some $Y_{z'}$ where $z' \in \prod_{r\in L'} T_r$, so $\nu(Y_{z\restriction L} \setminus Y_z) \ge 8$ (394G) and $r_{\nu 1}(Y_{z\restriction L}) > r_{\nu 1}(Y_z)$. An easy induction now shows that $r_{\nu 1}(Y_z) \ge \xi$ whenever $L \in \partial^\xi \mathcal{L}$ and $z \in \prod_{r\in L} T_r$. So if $\emptyset \in \partial^\xi \mathcal{L}$ then $r_{\nu 1}(X) \ge \xi$.

(b) Moving to the Maharam algebra $\mathfrak{A} = \widehat{\mathfrak{B}}$ defined from ν, as in 394Nc, we see that $\mathfrak{A}$ has a strictly positive Maharam submeasure $\hat\nu$ extending ν, so that the same formulae, interpreted in $\mathfrak{A}$, tell us that $\mathrm{Mhsr}(\mathfrak{A}) \ge \xi$ whenever $\emptyset \in \partial^\xi \mathcal{L}$.

(c) The next step is to understand which families $\mathcal{L} \subseteq [\mathbb{N}]^{<\omega}$ can be expressed as $\{L : \|L\| \le 1\}$ for some PV norm $\|\,\|$. Looking through the definition in 394Aa, we see that we shall need, at least,

— $\{n\} \in \mathcal{L}$ for every $n \in \mathbb{N}$,

— $I \in \mathcal{L}$ whenever $J \in \mathcal{L}$ and $\#(I \cap n) \le \#(J \cap n)$ for every n,

— for every infinite $A \subseteq \mathbb{N}$ there is an $n \in \mathbb{N}$ such that $A \cap n \notin \mathcal{L}$.

Following Perović & Veličković 18, I will say that a family satisfying these three conditions is **admissible**. The point is that they are sufficient as well as necessary.

(d) For every $\xi < \omega_1$ there is an admissible family $\mathcal{L}_\xi \subseteq [\mathbb{N}]^{<\omega}$ such that $\emptyset \in \partial^\eta \mathcal{L}_\xi$ for every $\eta < \xi$.

(e) Putting these together, we see that if $\xi < \omega_1$ we have an admissible family $\mathcal{L}_{\xi+1}$ such that we can define a PV norm $\|\,\|_\xi$ from $\mathcal{L}_{\xi+1}$ as in (c), a submeasure ν_ξ on a countable atomless algebra $\mathfrak{B}$ from $\|\,\|_\xi$ as in 394H, and a Maharam algebra $\mathfrak{A}_\xi$ from ν_ξ as in 394Nc, in such a way that the exhaustivity rank of ν_ξ is at least ξ and $\mathrm{Mhsr}(\mathfrak{A}_\xi) \ge \xi$.

539W The set of exhaustive submeasures: Theorem Let $\mathfrak{C}$ be a countable atomless Boolean algebra, not $\{0\}$. Write M_{sm} for the set of totally finite submeasures on $\mathfrak{C}$, regarded as a subset of $[0,\infty[^{\mathfrak{C}}$, and M_{esm} for the set of exhaustive totally finite submeasures on $\mathfrak{C}$. Then M_{sm} is Polish, and $M_{\mathrm{esm}} \subseteq M_{\mathrm{sm}}$ is coanalytic and not Borel. Setting

$$F_\xi = \{\nu : \nu \in M_{\mathrm{esm}} \text{ has exhaustivity rank at most } \xi\}$$

for $\xi < \omega_1$, every F_ξ is a Borel subset of M_{sm} and every analytic subset of M_{esm} is included in some F_ξ.

539Z Problems (a) Let ν be a non-zero totally finite Radon submeasure on a Hausdorff space X. Must there be a lifting for ν?

(b) Is there a Maharam algebra with uncountable Maharam submeasure rank?