Version of 30.8.14

Chapter 53

Topologies and measures III

In this chapter I return to the concerns of earlier volumes, looking for results which can be expressed in the language so far developed in this volume. In Chapter 43 I examined relationships between measure-theoretic and topological properties. The concepts we now have available (in particular, the notion of 'precaliber') make it possible to extend this work in a new direction, seeking to understand the possible Maharam types of measures on a given topological space. §531 deals with general Radon measures; new patterns arise if we restrict ourselves to completion regular Radon measures (§532). In §533 I give a brief account of some further results depending on assumptions concerning the cardinals examined in Chapter 52, including notes on uniformly regular measures and a description of the cardinals κ for which $\mathbb{R}^\kappa$ is measure-compact (533J).

In §534 I set out the elementary theory of 'strong measure zero' ideals in uniform spaces, concentrating on aspects which can be studied in terms of concepts already introduced. Here there are some very natural questions which have not as far as I know been answered (534Z). In the same section I run through elementary properties of Hausdorff measures when examined in the light of the concepts in Chapter 52. In §535 I look at liftings and strong liftings, extending the results of §§341 and 453; in particular, asking which non-complete probability spaces have liftings. In §536 I run over what is known about Alexandra Bellow's problem concerning pointwise compact sets of continuous functions, mentioned in §463. With a little help from special axioms, there are some striking possibilities concerning repeated integrals, which I examine in §537. Moving into new territory, I devote a section (§538) to a study of special types of filter on $\mathbb{N}$ associated with measure-theoretic phenomena, and to medial limits. In §539, I complete my account of the result of B.Balcar, T.Jech and T.Pazák that it is consistent to suppose that every Dedekind complete ccc weakly (σ,∞)-distributive Boolean algebra is a Maharam algebra, and work through applications of the methods of Chapter 52 to Maharam submeasures and algebras.

Version of 27.2.24

531 Maharam types of Radon measures

In the introduction to §434 I asked

What kinds of measures can arise on what kinds of topological space?

In §§434-435, and again in §438, I considered a variety of topological properties and their relations with measure-theoretic properties of Borel and Baire measures. I passed over, however, some natural questions concerning possible Maharam types, to which I now return. For a given Hausdorff space X, the possible measure algebras of totally finite Radon measures on X can be described in terms of the set $\mathrm{Mah}_{\mathrm{R}}(X)$ of Maharam types of Maharam-type-homogeneous Radon probability measures on X (531F). For $X \neq \emptyset$, $\mathrm{Mah}_{\mathrm{R}}(X)$ is of the form $\{0\} \cup [\omega, \kappa^*[$ for some infinite cardinal κ^* (531Ef). In 531E and 531G I give basic results from which $\mathrm{Mah}_{\mathrm{R}}(X)$ can often be determined; for obvious reasons we are primarily concerned with compact spaces X. In more abstract contexts, there are striking relationships between precalibers of measure algebras, the sets $\mathrm{Mah}_{\mathrm{R}}(X)$ and continuous surjections onto powers of $\{0,1\}$, which I examine in 531L-531M, 531T and 531V. Intertwined with these, we have results relating the character of X to $\mathrm{Mah}_{\mathrm{R}}(X)$ (531N-531O). The arguments here depend on an analysis of the structure of homogeneous measure algebras (531J, 531K, 531R).

531A Proposition Let $(X,\mathfrak{T},\Sigma,\mu)$ be a quasi-Radon measure space with measure algebra $(\mathfrak{A},\bar{\mu})$.

(a) The Maharam type $\tau(\mathfrak{A})$ of $\mathfrak{A}$ is at most the weight $w(X)$ of X.

(b) The cellularity $c(\mathfrak{A})$ of $\mathfrak{A}$ is at most the hereditary Lindelöf number $\mathrm{hL}(X)$ of X. If μ is locally finite, $c(\mathfrak{A})$ is at most the Lindelöf number $L(X)$ of X.

(c) $\#(\{a : a \in \mathfrak{A},\, \bar{\mu}a < \infty\}) \le \max(1, w(X)^\omega)$, where $w(X)^\omega$ is the cardinal power.

(d) If X is Hausdorff and μ is a Radon measure, then the Maharam type $\tau(\mathfrak{A})$ of $\mathfrak{A}$ is at most the network weight $\mathrm{nw}(X)$ of X.

proof (a) Let $\mathcal{U}$ be a base for $\mathfrak{T}$ with $\#(\mathcal{U}) = w(X)$. Set $B = \{U^\bullet : U \in \mathcal{U}\}$ and let $\mathfrak{B}$ be the order-closed subalgebra of $\mathfrak{A}$ generated by B; set $\mathrm{T} = \{E : E \in \Sigma,\, E^\bullet \in \mathfrak{B}\}$. Then T is a σ-subalgebra of Σ containing every negligible set.

If $G \subseteq X$ is open, then $G \in \mathrm{T}$. **P** By 414Aa, $G^\bullet = \sup\{U^\bullet : U \in \mathcal{U},\, U \subseteq G\}$ belongs to $\mathfrak{B}$. **Q** So every Borel set belongs to T. If $E \in \Sigma$ and $\mu E < \infty$, then, because μ is inner regular with respect to the Borel sets, there is a Borel subset F of E with the same measure, so F, $E \setminus F$ and E belong to T. Thus $\{a : a \in \mathfrak{A},\, \bar{\mu}a < \infty\} \subseteq \mathfrak{B}$; because μ is semi-finite, $\mathfrak{B} = \mathfrak{A}$ and $\tau(\mathfrak{A}) \le \#(B) \le \#(\mathcal{U}) = w(X)$.

(b)(i) If $L(X) = n$ is finite, and $F_0, \dots, F_n \subseteq X$ are disjoint closed sets, then at least one of them is empty. **P** For $i \le n$, set $G_i = X \setminus \bigcup_{j \le n, j \ne i} F_j$; then $\bigcup_{i \le n} G_i = X$, so there is some $k \le n$ such that $\bigcup_{i \ne k} G_i = X$, and now $F_k = \emptyset$. **Q** As μ is inner regular with respect to the closed sets, $c(\mathfrak{A}) \le n = L(X) \le \mathrm{hL}(X)$.

(ii) Suppose that $\omega \le L(X) \le \mathrm{hL}(X)$. Let $\mathcal{G}$ be the family of open subsets of X of finite measure. Then there is a set $\mathcal{H} \subseteq \mathcal{G}$, with cardinal at most $\mathrm{hL}(X)$, such that $\bigcup \mathcal{H} = \bigcup \mathcal{G}$ (5A4Bf). Now $\sup_{H \in \mathcal{H}} H^\bullet = 1$, because μ is effectively locally finite.

If $D \subseteq \mathfrak{A} \setminus \{0\}$ is disjoint, then for each $d \in D$ take $H_d \in \mathcal{H}$ such that $d \cap H_d^\bullet \ne 0$. If $H \in \mathcal{H}$, then $\{d : H_d = H\}$ must be countable, since $\mu H < \infty$. So $\#(D) \le \max(\omega, \#(\mathcal{H}))$; as D is arbitrary, $c(\mathfrak{A}) \le \max(\omega, \mathrm{hL}(X)) = \mathrm{hL}(X)$.

(iii) Finally, if $\omega \le L(X)$ and μ is locally finite, then in (ii) above we have $X = \bigcup \mathcal{G}$, so we can take $\mathcal{H}$ to have size at most $L(X)$, and continue as before, ending with $c(\mathfrak{A}) \le \max(\omega, \#(\mathcal{H})) = L(X)$.

(c) Again let $\mathcal{U}$ be a base for the topology of X with cardinal $w(X)$. Let T be the σ-subalgebra of Σ generated by $\mathcal{U}$. If $E \in \Sigma$ and $\mu E < \infty$, then for each $n \in \mathbb{N}$ we can find an open set G_n such that $\mu(G_n \triangle E) \le 2^{-n}$; now there is an open set H_n, a finite union of members of $\mathcal{U}$, such that $H_n \subseteq G_n$ and $\mu(G_n \setminus H_n) \le 2^{-n}$. Setting $F = \bigcup_{m \in \mathbb{N}} \bigcap_{n \ge m} H_n$, we see that $F \in \mathrm{T}$ and $E \triangle F$ is negligible. Thus $\{F^\bullet : F \in \mathrm{T}\} \supseteq \{a : \bar{\mu}a < \infty\}$ and

$$\#(\{a : \bar{\mu}a < \infty\}) \le \#(\mathrm{T}) \le \max(1, \#(\mathcal{U})^\omega) = \max(1, w(X)^\omega).$$

(d) If $a \in \mathfrak{A} \setminus \{0\}$ and the principal ideal $\mathfrak{A}_a$ is Maharam-type-homogeneous, then $\tau(\mathfrak{A}_a) \le \mathrm{nw}(X)$. **P** There is a compact set $K \subseteq X$ such that $0 \ne K^\bullet \subseteq \mathfrak{A}_a$; let μ_K be the subspace measure on K. Then

$$\tau(\mathfrak{A}_a) = \tau(\mu_K) \le w(K)$$

(by (a))

$$= \mathrm{nw}(K)$$

(5A4C(a-i))

$$\le \mathrm{nw}(X)$$

(5A4Bb). **Q**

By (b), $c(\mathfrak{A}) \le \#(\mathfrak{T}) \le 2^{\mathrm{nw}(X)}$ (5A4Ba); so 332S tells us that $\tau(\mathfrak{A}) \le \mathrm{nw}(X)$.

531B For strictly positive measures we have some easy inequalities in the other direction.

Proposition Let (X, Σ, μ) be a measure space, with measure algebra $\mathfrak{A}$, and $\mathfrak{T}$ a topology on X such that Σ includes a base for $\mathfrak{T}$ and μ is strictly positive.

(a) If X is regular, then $w(X) \le \#(\mathfrak{A})$.

(b) If X is Hausdorff, then $\#(X) \le 2^{\#(\mathfrak{A})}$.

proof Set $\mathcal{V} = \Sigma \cap \mathfrak{T}$, so that $\mathcal{V}$ is a base for $\mathfrak{T}$. If V, $W \in \mathcal{V}$ and $V^\bullet = W^\bullet$ in $\mathfrak{A}$, then $\operatorname{int} \overline{V} = \operatorname{int} \overline{W}$. **P** $\mu^*(V \setminus \overline{W}) \le \mu(V \setminus W) = 0$, so (because μ is strictly positive) $V \subseteq \overline{W}$ and $\overline{V} \subseteq \overline{W}$ and $\operatorname{int} \overline{V} \subseteq \operatorname{int} \overline{W}$. Similarly, $\operatorname{int} \overline{W} \subseteq \operatorname{int} \overline{V}$. **Q** So if we set $\mathcal{W} = \{\operatorname{int} \overline{V} : V \in \mathcal{V}\}$, $\#(\mathcal{W}) \le \#(\mathfrak{A})$.

(a) If $\mathfrak{T}$ is regular, $\mathcal{W}$ is a base for $\mathfrak{T}$, so $w(X) \le \#(\mathcal{W}) \le \#(\mathfrak{A})$.

(b) If $\mathfrak{T}$ is Hausdorff, then for any distinct x, $y \in X$, there is a $W \in \mathcal{W}$ containing x but not y. **P** Let G, H be disjoint open sets containing x, y respectively. Take $V \in \mathcal{V}$ such that $x \in V \subseteq G$, and set $W = \operatorname{int} \overline{V}$. **Q** So $\#(X) \le 2^{\#(\mathcal{W})} \le 2^{\#(\mathfrak{A})}$.

531C Lemma Let $\langle X_i \rangle_{i \in I}$ be a family of topological spaces with product X, and μ a totally finite quasi-Radon measure on X with Maharam type κ. For each $i \in I$, let μ_i be the marginal measure on X_i, and κ_i its Maharam type. Then κ is at most the cardinal sum $\sum_{i \in I} \kappa_i$.

proof For each $i \in I$, let $\langle E_{i\xi} \rangle_{\xi < \kappa_i}$ be a family in $\operatorname{dom} \mu_i$ such that $\{E_{i\xi}^\bullet : \xi < \kappa_i\}$ τ-generates the measure algebra of μ_i. Consider $\mathcal{W} = \{\pi_i^{-1}[E_{i\xi}] : i \in I, \xi < \kappa_i\}$, so that $\mathcal{W} \subseteq \operatorname{dom} \mu$ and $\#(\mathcal{W}) \le \sum_{i \in I} \kappa_i$. Let $\mathfrak{B}$ be the closed subalgebra of the measure algebra $\mathfrak{A}$ of μ generated by $\{W^\bullet : W \in \mathcal{W}\}$.

For each $i \in I$, the canonical map $\pi_i : X \to X_i$ induces a measure-preserving homomorphism ϕ_i from the measure algebra $\mathfrak{A}_i$ of μ_i to $\mathfrak{A}$ (324M). Now $\phi_i^{-1}[\mathfrak{B}]$ is a closed subalgebra of $\mathfrak{A}_i$ containing $E_{i\xi}^\bullet$ for every $\xi < \kappa_i$, so is the whole of $\mathfrak{A}_i$, that is, $\phi_i[\mathfrak{A}_i] \subseteq \mathfrak{B}$. In particular, if $G \subseteq X_i$ is open, $\pi_i^{-1}[G]^\bullet = \phi_i(G^\bullet)$ belongs to $\mathfrak{B}$. Now the family $\mathcal{V}$ of open sets $V \subseteq X$ such that $V^\bullet \in \mathfrak{B}$ is closed under finite intersections and contains $\pi_i^{-1}[G]$ whenever $i \in I$ and $G \subseteq X_i$ is open, so $\mathcal{V}$ is a base for the topology of X. But also $\mathcal{V}$ is closed under arbitrary unions, because $\mathfrak{B}$ is closed and μ is τ-additive (414Aa again). So $V^\bullet \in \mathfrak{B}$ for every open set $V \subseteq X$, and therefore for every Borel set $V \subseteq X$; as μ is inner regular with respect to the Borel sets, $\mathfrak{B} = \mathfrak{A}$.

Thus $\{W^\bullet : W \in \mathcal{W}\}$ witnesses that the Maharam type $\tau(\mathfrak{A})$ of μ is at most $\sum_{i \in I} \kappa_i$, as claimed.

531D Definition If X is a Hausdorff space, I write $\operatorname{Mah}_R(X)$ for the set of Maharam types of Maharam-type-homogeneous Radon probability measures on X. Note that $0 \in \operatorname{Mah}_R(X)$ iff X is non-empty, and that any member of $\operatorname{Mah}_R(X)$ is either 0 or an infinite cardinal.

531E Proposition Let X be a Hausdorff space.

(a) $\kappa \le w(X)$ for every $\kappa \in \operatorname{Mah}_R(X)$.

(b) $\operatorname{Mah}_R(Y) \subseteq \operatorname{Mah}_R(X)$ for every $Y \subseteq X$.

(c) $\operatorname{Mah}_R(X) = \bigcup\{\operatorname{Mah}_R(K) : K \subseteq X \text{ is compact}\}$.

(d) If X is K-analytic (in particular, if X is compact) and Y is a continuous image of X, $\operatorname{Mah}_R(Y) \subseteq \operatorname{Mah}_R(X)$.

(e) $\omega \in \operatorname{Mah}_R(X)$ iff X has a compact subset which is not scattered.

(f) (Haydon 77) If $\omega \le \kappa' \le \kappa \in \operatorname{Mah}_R(X)$ then $\kappa' \in \operatorname{Mah}_R(X)$.

(g) If Y is another Hausdorff space, and neither X nor Y is empty, then $\operatorname{Mah}_R(X \times Y) = \operatorname{Mah}_R(X) \cup \operatorname{Mah}_R(Y)$; generally, for any non-empty finite family $\langle X_i \rangle_{i \in I}$ of non-empty Hausdorff spaces, $\operatorname{Mah}_R(\prod_{i \in I} X_i) = \bigcup_{i \in I} \operatorname{Mah}_R(X_i)$.

proof (a) This is immediate from 531Aa.

(b) If $\kappa \in \operatorname{Mah}_R(Y)$, there is a Maharam-type-homogeneous Radon probability measure μ on Y with Maharam type κ. Set

$$\Sigma' = \{E : E \subseteq X, \mu \text{ measures } Y \cap E\},$$

$$\mu' E = \mu(Y \cap E) \text{ for } E \in \Sigma'\}.$$

It is easy to check that μ' is a Radon probability measure on X (see 416Xc and 418I), and that μ' and μ have isomorphic measure algebras (cf. 322J). So μ' is Maharam-type-homogeneous and has Maharam type κ, and $\kappa \in \operatorname{Mah}_R(X)$.

(c) By (b), $\operatorname{Mah}_R(K) \subseteq \operatorname{Mah}_R(X)$ for every compact set $K \subseteq X$. In the other direction, if $\kappa \in \operatorname{Mah}_R(X)$, there is a Maharam-type-homogeneous Radon probability measure μ on X with Maharam type κ. Let

$K \subseteq X$ be a compact set with $\mu K > 0$. Then the normalized subspace measure $\mu' = (\mu K)^{-1}\mu_K$ is a Radon probability measure on K, and its measure algebra is isomorphic to a principal ideal of the measure algebra of μ, so is Maharam-type-homogeneous with Maharam type κ. Accordingly $\kappa \in \mathrm{Mah_R}(K)$.

(d) Take $\kappa \in \mathrm{Mah_R}(Y)$. Then there is a Maharam-type-homogeneous Radon probability measure ν on Y with Maharam type κ. Let $f : X \to Y$ be a continuous surjection. By 432G, there is a Radon measure μ on X such that f is inverse-measure-preserving for μ and ν. Let $K \subseteq X$ be a compact set such that $\mu K > 0$. Then $f[K] \subseteq Y$ is compact and

$$\nu f[K] = \mu[f^{-1}[f[K]] \geq \mu K > 0.$$

Let $\nu_1 = \frac{1}{\nu f[K]}\nu_{f[K]}$ be the normalized subspace measure on $f[K]$. Then ν_1 is a Maharam-type-homogeneous Radon probability measure on $f[K]$ with Maharam type κ. By 418L, there is a Radon measure μ_1 on K such that $f{\restriction}K$ is inverse-measure-preserving for μ_1 and ν_1 and induces an isomorphism of their measure algebras. So μ_1 witnesses that $\kappa \in \mathrm{Mah_R}(K)$; by (b), $\kappa \in \mathrm{Mah_R}(X)$.

(e)(i) If X has a compact subset K which is not scattered, then there is a continuous surjection from K onto $[0,1]$ (4A2G(j-iv)). Of course Lebesgue measure witnesses that $\omega \in \mathrm{Mah_R}([0,1])$, so (d) and (b) tell us that $\omega \in \mathrm{Mah_R}(K) \subseteq \mathrm{Mah_R}(X)$.

(ii) If every compact subset of X is scattered and μ is a Maharam-type-homogeneous Radon probability measure on X, let K be a compact set of non-zero measure and $Z \subseteq K$ a closed self-supporting set. Then Z has an isolated point z say; in this case, $\mu\{z\} > 0$ so $\{z\}$ is an atom for μ and (because μ is Maharam-type-homogeneous) the Maharam type of μ is 0. As μ is arbitrary, $\omega \notin \mathrm{Mah_R}(X)$.

(f)(i) Suppose first that X is compact. Let μ be a Maharam-type-homogeneous Radon probability measure on X with Maharam type κ. Let $\langle E_\xi\rangle_{\xi<\kappa}$ be a stochastically independent family in $\operatorname{dom}\mu$ with $\mu E_\xi = \frac{1}{2}$ for every ξ. For each $\xi < \kappa'$ and $n \in \mathbb{N}$, let $f_{\xi n} \in C(X)$ be such that $\int |f_{\xi n} - \chi E_\xi| \leq 2^{-n}$ (416I). Define $f : X \to \mathbb{R}^{\kappa'\times\mathbb{N}}$ by setting $f(x)(\xi,n) = f_{\xi n}(x)$ for $x \in X$, $\xi < \kappa'$ and $n \in \mathbb{N}$. Then f is continuous, so by 418I the image measure $\nu = \mu f^{-1}$ on the compact set $f[X]$ is a Radon measure. For each $\xi < \kappa'$, the set

$$F_\xi = \{w : w \in f[X],\ \lim_{n\to\infty} w(\xi,n) = 1\}$$

is a Borel set, and $f^{-1}[F_\xi]\triangle E_\xi$ is μ-negligible; so $\langle F_\xi\rangle_{\xi<\kappa'}$ is a stochastically independent family of subsets of $f[X]$ with measure $\frac{1}{2}$. If $\mathfrak{B}$ is the measure algebra of ν, and $\mathfrak{C}$ the closed subalgebra of $\mathfrak{B}$ generated by $\{F_\xi^\bullet : \xi < \kappa'\}$, then $\mathfrak{C}$ is Maharam-type-homogeneous, with Maharam type κ'; at the same time,

$$\tau(\mathfrak{B}) \leq w(f[X]) \leq w(\mathbb{R}^{\kappa'\times\mathbb{N}}) = \kappa'.$$

By 332N, $\mathfrak{B}$ can be embedded in $\mathfrak{C}$; by 332Q, $\mathfrak{B}$ and $\mathfrak{C}$ are isomorphic, that is, $\mathfrak{B}$ is Maharam-type-homogeneous with Maharam type κ', and ν witnesses that $\kappa' \in \mathrm{Mah_R}(f[X])$. By (d), $\kappa' \in \mathrm{Mah_R}(X)$.

(ii) In general, (c) tells us that there is a compact set $K \subseteq X$ such that $\kappa \in \mathrm{Mah_R}(K)$, so $\kappa' \in \mathrm{Mah_R}(K) \subseteq \mathrm{Mah_R}(X)$.

(g) Because neither Y nor X is empty, both X and Y are homeomorphic to subspaces of $X \times Y$, so (b) tells us that $\mathrm{Mah_R}(X\times Y) \supseteq \mathrm{Mah_R}(X) \cup \mathrm{Mah_R}(Y)$. In the other direction, given a Maharam-type-homogeneous Radon probability measure μ on $X \times Y$, let μ_1, μ_2 be the marginal measures on X and Y respectively, so that each μ_k is a Radon probability measure (418I again). Let $\langle E_i\rangle_{i\in I}$, $\langle F_j\rangle_{j\in J}$ be countable partitions of X, Y into Borel sets such that all the subspace measures $(\mu_1)_{E_i}$ and $(\mu_2)_{F_j}$ are Maharam-type-homogeneous. Then there must be $i \in J$, $j \in J$ such that $\mu(E_i \times F_j) > 0$. Let μ' be the subspace measure $\mu_{E_i\times F_j}$; then the Maharam type of μ' is κ, because μ is Maharam-type-homogeneous. Let μ_1', μ_2' be the marginal measures of μ' on E_i and F_j respectively. Then μ_1' is an indefinite-integral measure over $(\mu_1)_{E_i}$ (415Oa), so its measure algebra is isomorphic to a principal ideal of the measure algebra of $(\mu_1)_{E_i}$ (322K), and has the same Maharam type κ_1 say. As in (b) above, $\kappa_1 \in \mathrm{Mah_R}(X)$. Similarly, the Maharam type κ_2 of μ_2' belongs to $\mathrm{Mah_R}(Y)$. Now 531C tells us that $\kappa \leq \kappa_1 + \kappa_2$. Since κ is either zero or infinite, it must be less than or equal to at least one of them, and belongs to $\mathrm{Mah_R}(X) \cup \mathrm{Mah_R}(Y)$ by (f) above.

The result for general finite products now follows easily by induction on $\#(I)$.

531F Proposition Let X be a Hausdorff space. Then a totally finite measure algebra $(\mathfrak{A},\bar{\mu})$ is isomorphic to the measure algebra of a Radon measure on X iff (α) whenever $\mathfrak{A}_a$ is a non-trivial homogeneous principal ideal of $\mathfrak{A}$ then $\tau(\mathfrak{A}_a)\in\mathrm{Mah}_{\mathrm{R}}(X)$ (β) $c(\mathfrak{A})\le\#(X)$.

proof (a) If μ is a totally finite Radon measure on X with measure algebra $\mathfrak{A}$ and the principal ideal $\mathfrak{A}_a$ generated by $a\in\mathfrak{A}\setminus\{0\}$ is homogeneous, then there is an $E\in\operatorname{dom}\mu$ such that $E^\bullet=a$. Let ν be the probability measure $(\mu F)^{-1}\mu\llcorner F$, that is, $\nu H=\mu(H\cap F)/\mu F$ whenever $H\subseteq X$ is such that μ measures $H\cap F$ (234M). Then ν is a Radon measure (416Sa), the measure algebra of ν is isomorphic to a principal ideal of $\mathfrak{A}_a$ (322K) so is homogeneous with the same Maharam type, and ν witnesses that $\tau(\mathfrak{A}_a)\in\mathrm{Mah}_{\mathrm{R}}(X)$. Thus $\mathfrak{A}$ satisfies (α). As for (β), if X is infinite this is trivial (because $(\mathfrak{A},\bar{\mu})$ is totally finite, so $\mathfrak{A}$ is ccc), and otherwise $\mathfrak{A}$ is finite, with

$$c(\mathfrak{A})=\#(\{a:a\in\mathfrak{A}\text{ is an atom}\})=\#(\{x:x\in X,\ \mu\{x\}>0\})\le\#(X).$$

(b) Now suppose that $(\mathfrak{A},\bar{\mu})$ is a totally finite measure algebra satisfying the conditions. Express it as the simple product of a countable family $\langle(\mathfrak{A}_i,\bar{\mu}'_i)\rangle_{i\in I}$ of non-zero homogeneous measure algebras (332B); we may suppose that $I\subseteq\mathbb{N}$. For $n\in I$, set $\kappa_n=\tau(\mathfrak{A}_n)$ and $\gamma_n=\bar{\mu}'_n1_{\mathfrak{A}_n}$. (β) tells us that $\#(I)\le\#(X)$; let $\langle x_n\rangle_{n\in I}$ be a family of distinct elements of X.

Set $J=\{n:n\in I,\ \kappa_n\ge\omega\}$. For each $n\in J$, (α) tells us that there is a Maharam-type-homogeneous Radon probability measure μ_n on X with Maharam type κ_n. Now there is a disjoint family $\langle E_n\rangle_{n\in\mathbb{N}}$ of Borel subsets of $X\setminus\{x_n:n\in I\}$ such that $\mu_nE_n>0$ for every $n\in J$. **P** Choose $\langle E_n\rangle_{n\in\mathbb{N}}$, $\langle F_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. $F_0=X\setminus\{x_n:n\in I\}$. Given that F_n is a Borel set and $\mu_jF_n>0$ for every $j\in J\setminus n$, then if $n\notin J$ set $E_n=\emptyset$ and $F_{n+1}=F_n$. Otherwise, for each $j\in J$ such that $j>n$, we can partition F_n into finitely many Borel sets of μ_n-measure less than $2^{-j}\mu_nF_n$, because μ_n is atomless; take one of these, G_{nj} say, such that $\mu_jG_{nj}>0$; now set $F_{n+1}=\bigcup_{j\in J,j>n}G_{nj}$ and $E_n=F_n\setminus F_{n+1}$. Continue. **Q** Now set

$$\mu E=\sum_{n\in I\setminus J,x_n\in E}\gamma_n+\sum_{n\in J}(\mu_nE_n)^{-1}\gamma_n\mu_n(E\cap E_n)$$

whenever $E\subseteq X$ is such that μ_n measures $E\cap E_n$ for every $n\in J$. Of course μ is a measure. Because every μ_n is a topological measure, so is μ; because every μ_n is inner regular with respect to the compact sets, so is μ; because every μ_n is complete, so is μ; thus μ is a Radon measure. Because every subspace measure $(\mu_n)_{E_n}$ is Maharam-type-homogeneous with Maharam type κ_n, the measure algebra of μ is isomorphic to $(\mathfrak{A},\bar{\mu})$.

531G Proposition Let $\langle X_i\rangle_{i\in I}$ be a family of non-empty Hausdorff spaces with product X. Then an infinite cardinal κ belongs to $\mathrm{Mah}_{\mathrm{R}}(X)$ iff *either* $\kappa\le\#(\{i:i\in I,\ \#(X_i)\ge2\})$ *or* κ is expressible as $\sup_{i\in I}\kappa_i$ where $\kappa_i\in\mathrm{Mah}_{\mathrm{R}}(X_i)$ for every $i\in I$.

proof (a)(i) Suppose that $\kappa=\sup_{i\in I}\kappa_i$ where $\kappa_i\in\mathrm{Mah}_{\mathrm{R}}(X_i)$ for each $i\in I$. For each i, let μ_i be a Maharam-type-homogeneous Radon probability measure on X_i with Maharam type κ_i and compact support (see the proof of 531Ec). Let λ be the ordinary product of the measures μ_i. By 325I, the measure algebra of λ can be identified with the probability algebra free product of the measure algebras of the μ_i. It is therefore isomorphic to the measure algebra of the usual measure on $\{0,1\}^{\kappa'}$, where κ' is the cardinal sum $\sum_{i\in I}\kappa_i$; in particular, it is homogeneous with Maharam type κ' (since we are supposing that $\kappa\ge\omega$). By 417E(b-i)[1], the measure algebra of the τ-additive product μ of $\langle\mu_i\rangle_{i\in I}$ can be identified with the measure algebra of λ, while μ is a Radon measure (417Q). So μ witnesses that $\kappa'\in\mathrm{Mah}_{\mathrm{R}}(X)$; by 531Ef, $\kappa\in\mathrm{Mah}_{\mathrm{R}}(X)$.

(ii) Suppose that $\omega\le\kappa\le\#(I')$ where $I'=\{i:i\in I,\ \#(X_i)\ge2\}$. For $i\in I'$, let x_i, y_i be distinct points of X_i and μ_i the point-supported probability measure on X_i such that $\mu_i\{x_i\}=\mu_i\{y_i\}=\frac{1}{2}$; for $i\in I\setminus I'$, let μ_i be the unique Radon probability measure on X_i. As in (i) above, the Radon measure product of $\langle\mu_i\rangle_{i\in I}$ is Maharam-type-homogeneous, with Maharam type $\#(I')$, so $\#(I')\in\mathrm{Mah}_{\mathrm{R}}(X)$; by 531Ef again, $\kappa\in\mathrm{Mah}_{\mathrm{R}}(X)$.

(b) Now suppose that $\omega\le\kappa\in\mathrm{Mah}_{\mathrm{R}}(X)$ and that $\kappa>\#(I')$. For each $i\in I$, let θ_i be the least cardinal greater than every member of $\mathrm{Mah}_{\mathrm{R}}(X_i)$. Note that $\kappa'\in\mathrm{Mah}_{\mathrm{R}}(X_i)$ whenever κ' is a cardinal and $\omega\le\kappa'<\theta_i$. Set

[1]Formerly 417E(ii).

$$I_1 = \{i : i \in I,\, \kappa < \theta_i\}, \quad Z_1 = \prod_{i \in I_1} X_i,$$

$$I_2 = \{i : i \in I,\, \theta_i \le \kappa,\, \operatorname{cf}\theta_i > \omega\}, \quad Z_2 = \prod_{i \in I_2} X_i,$$

$$I_3 = \{i : i \in I,\, \theta_i = \kappa,\, \operatorname{cf}\theta_i = \omega\}, \quad Z_3 = \prod_{i \in I_3} X_i,$$

$$I_4 = \{i : i \in I,\, \theta_i < \kappa,\, \operatorname{cf}\theta_i = \omega\}, \quad Z_4 = \prod_{i \in I_4} X_i,$$

$$I_5 = \{i : i \in I,\, \theta_i = 1,\, \#(X_i) > 1\}, \quad Z_5 = \prod_{i \in I_5} X_i,$$

$$I_6 = \{i : i \in I,\, \#(X_i) = 1\}, \quad Z_6 = \prod_{i \in I_6} X_i.$$

Then X can be identified with $\prod_{1 \le k \le 6} Z_k$, so 531Eg tells us that $\kappa \in \mathrm{Mah}_{\mathrm{R}}(Z_k)$ for some k. As Z_6 is a singleton, we actually have $\kappa \in \mathrm{Mah}_{\mathrm{R}}(Z_k)$ for some $k \le 5$.

case 1 Suppose $\kappa \in \mathrm{Mah}_{\mathrm{R}}(Z_1)$. Then, in particular, $I_1 \ne \emptyset$ and there is a $j \in I$ such that $\kappa < \theta_j$. In this case, $\kappa \in \mathrm{Mah}_{\mathrm{R}}(X_j)$, and we can set $\kappa_j = \kappa$, $\kappa_i = 0$ for $i \ne j$ to find a family in $\prod_{i \in I} \mathrm{Mah}_{\mathrm{R}}(X_i)$ with supremum κ.

case 2 Suppose that $\kappa \in \mathrm{Mah}_{\mathrm{R}}(Z_2)$. Let μ be a Radon probability measure on Z_2 with Maharam type κ. For each $i \in Z_2$, let μ'_i be the marginal measure on X_i, and κ'_i its Maharam type. By 531C,

$$\kappa \le \sum_{i \in I_2} \kappa'_i \le \max(\omega, \#(I_2), \sup_{i \in I} \kappa'_i)$$

(5A4F(b-i)); since $\emptyset \ne I_2 \subseteq I'$, $\#(I_2) < \kappa \le \sup_{i \in I_2} \max(\omega, \kappa'_i)$; since κ is infinite, it must be less than or equal to $\sup_{i \in I_2} \max(\omega, \kappa'_i)$. On the other hand, by 531F, each κ'_i is either finite or the supremum of some countable subset of $\mathrm{Mah}_{\mathrm{R}}(X_i)$; because $\operatorname{cf}\theta_i > \omega$, $\kappa'_i < \theta_i$ and $\max(\omega, \kappa'_i) \in \mathrm{Mah}_{\mathrm{R}}(X_i)$. Setting

$$\begin{aligned} \kappa_i &= \operatorname{med}(\kappa'_i, \omega, \kappa) \text{ for } i \in I_2, \\ &= 0 \text{ for } i \in I \setminus I_2, \end{aligned}$$

we have $\kappa_i \in \mathrm{Mah}_{\mathrm{R}}(X_i)$ for every $i \in I$ and $\kappa = \sup_{i \in I} \kappa_i$.

case 3 Suppose that $\kappa \in \mathrm{Mah}_{\mathrm{R}}(Z_3)$. Because $\kappa = \theta_i \notin \mathrm{Mah}_{\mathrm{R}}(X_i)$ for $i \in I_3$, 531Eg tells us that I_3 must be infinite. Let $\langle i_n \rangle_{n \in \mathbb{N}}$ be a sequence of distinct elements of I_3. Of course κ itself is uncountable and has countable cofinality, so we can find a sequence κ'_n of infinite cardinals less than κ with supremum κ. Setting $\kappa_{i_n} = \kappa'_n$, $\kappa_i = 0$ for $i \in I \setminus \{i_n : n \in \mathbb{N}\}$, we have $\kappa_i \in \mathrm{Mah}_{\mathrm{R}}(X_i)$ for every i and $\kappa = \sup_{i \in I} \kappa_i$.

case 4 Suppose that $\kappa \in \mathrm{Mah}_{\mathrm{R}}(Z_4)$. Following the scheme of case 2 above, let μ be a Radon probability measure on Z_4 with Maharam type κ, and for each $i \in I_4$ let μ'_i be the marginal measure on X_i and κ'_i its Maharam type. Then, as before, $\kappa \le \sup_{i \in I_4} \max(\omega, \kappa'_i)$. At the same time, $\kappa'_i \le \theta_i < \kappa$ for every i, so we must have $\kappa = \sup_{i \in I_4} \theta_i$. Set $\delta = \operatorname{cf}\kappa$. Then we can choose $\langle i_\xi \rangle_{\xi < \delta}$ inductively in I_4 so that $\theta_{i_\eta} < \theta_{i_\xi}$ whenever $\eta < \xi < \delta$ and $\sup_{\xi < \delta} \theta_{i_\xi} = \kappa$. Now define $\langle \kappa_i \rangle_{i \in I}$ by saying

$$\begin{aligned} \kappa_{i_{\xi+1}} &= \theta_{i_\xi} \text{ whenever } \xi < \delta, \\ \kappa_i &= 0 \text{ if } i \in I \setminus \{i_{\xi+1} : \xi < \delta\}. \end{aligned}$$

This gives $\kappa_i \in \mathrm{Mah}_{\mathrm{R}}(X_i)$ for every i and $\kappa = \sup_{i \in I} \kappa_i$.

case 5 ? Suppose, if possible, that $\kappa \in \mathrm{Mah}_{\mathrm{R}}(Z_5)$. Once again, we can find a Radon probability measure μ on Z_5 with Maharam type κ, and look at its marginal measures μ'_i for $i \in I_5$. This time, however, every μ'_i must be purely atomic and has Maharam type $\kappa'_i \le \omega$; also $\#(I_5) < \kappa$. So our formula $\kappa \le \sum_{i \in I_5} \kappa'_i$ becomes $\kappa \le \omega$. In this case I_5 must be finite and $\kappa \in \bigcup_{i \in I_5} \mathrm{Mah}_{\mathrm{R}}(X_i) = \{0\}$, which is absurd. **X**

Thus this case evaporates and the proof is complete.

531H Remarks The results above already enable us to calculate $\mathrm{Mah}_{\mathrm{R}}(X)$ for many spaces. Of course we begin with compact spaces (531Ec). If X is compact and Hausdorff, and $\{0,1\}^\kappa$ is a continuous image of a closed subset of X, where κ is an infinite cardinal, then $\kappa \in \mathrm{Mah}_{\mathrm{R}}(X)$ (531Ed); so if $\{0,1\}^{w(X)}$ is a continuous image of a closed subset of X, then $\mathrm{Mah}_{\mathrm{R}}(X)$ is completely specified, being $\{0\} \cup \{\kappa : \omega \le \kappa \le w(X)\}$

(531Ea, 531Ef). Of course it is not generally true that $w(X) \in \mathrm{Mah_R}(X)$ (531Xc). But it is quite often the case that $\{0,1\}^\kappa$ is a continuous image of a closed subset of X for every $\kappa \in \mathrm{Mah_R}(X)$, and I will now investigate this phenomenon.

531I Notation For the rest of the section, I will use the following notation, mostly familiar from earlier chapters of this volume. For any set I, let ν_I be the usual measure on $\{0,1\}^I$, T_I its domain, $\mathcal{N}_I$ its null ideal and $(\mathfrak{B}_I, \bar{\nu}_I)$ its measure algebra. In this context, I will write $\langle e_i\rangle_{i\in I}$ for the standard generating family in $\mathfrak{B}_I$ (525A). For $J \subseteq I$ let $\mathfrak{C}_J$ be the closed subalgebra of $\mathfrak{B}_I$ generated by $\{e_i : i \in J\}$. Now for a new idea. For each $i \in I$, let $\phi_i : \mathfrak{B}_I \to \mathfrak{B}_I$ be the measure-preserving involution corresponding to reversal of the ith coordinate in $\{0,1\}^I$, that is, $\phi_i(e_i) = 1 \setminus e_i$ and $\phi_i(e_j) = e_j$ for $j \neq i$.

531J Lemma Let I be a set, and take $\mathfrak{B}_I$, $\mathfrak{C}_J$, for $J \subseteq I$, and ϕ_i, for $i \in I$, as in 531I.

(a) $\bigcup\{\mathfrak{C}_J : J \in [I]^{<\omega}\}$ is dense in $\mathfrak{B}_I$ for the measure-algebra topology of $\mathfrak{B}_I$.

(b) For every $a \in \mathfrak{B}_I$, there is a (unique) countable $J^*(a) \subseteq I$ such that, for $J \subseteq I$, $a \in \mathfrak{C}_J$ iff $J \supseteq J^*(a)$.

(c) $J^*(1 \setminus a) = J^*(a)$ for every $a \in \mathfrak{B}_I$.

(d) $\phi_i\phi_j = \phi_j\phi_i$ for all i, $j \in I$.

(e) If $J \subseteq I$, $a \in \mathfrak{C}_J$ and $i \in I$, then $a \cap \phi_i a$, $a \cup \phi_i a$ belong to $\mathfrak{C}_{J\setminus\{i\}}$.

(f) For $a \in \mathfrak{B}_I$ and $i \in I$ we have $\phi_i a = a$ iff $i \notin J^*(a)$.

(g) $\phi_i a \in \mathfrak{C}_J$ whenever $J \subseteq I$, $i \in I$ and $a \in \mathfrak{C}_J$.

proof (a) See 254Fe.

(b) See 254Rd or 325Mb.

(c) For $J \subseteq I$, $1 \setminus a \in \mathfrak{C}_J$ iff $a \in \mathfrak{C}_J$.

(d) Because $\{e_k : k \in I\}$ τ-generates $\mathfrak{B}_I$, it is enough to check that $\phi_i\phi_j e_k = \phi_j\phi_i e_k$ for all i, j, $k \in I$, and this is easy.

(e) The subalgebra $\{(c \cap e_i) \cup (c' \setminus e_i) : c,\ c' \in \mathfrak{C}_{J\setminus\{i\}}\}$ generated by $\mathfrak{C}_{J\setminus\{i\}} \cup \{e_i\}$ is closed (323K), so includes $\mathfrak{C}_J$ and contains a. If c, $c' \in \mathfrak{C}_{J\setminus\{i\}}$ are such that $a = (c \cap e_i) \cup (c' \setminus e_i)$, then $\phi_i a = (c \setminus e_i) \cup (c' \cap e_i)$ and $a \cap \phi_i a = c \cap c'$, $a \cup \phi_i a = c \cup c'$ belong to $\mathfrak{C}_{J\setminus\{i\}}$.

(f) If $i \notin J^*(a)$ then $\phi_i a = a$ because $\phi_i(e_j) = e_j$ for every $j \neq i$. If $\phi_i a = a$ then $a = a \cap \phi_i a \in \mathfrak{C}_{I\setminus\{i\}}$, by (e), and $J^*(a) \subseteq I \setminus \{i\}$, that is, $i \notin J^*(a)$.

(g) $\mathfrak{C}_J$ is the closed subalgebra of $\mathfrak{B}_I$ generated by $\{e_j : j \in J\}$, so $\phi_i[\mathfrak{C}_J]$ is the closed subalgebra generated by $\{\phi_i e_j : j \in J\} \subseteq \mathfrak{C}_J$ (324L).

531K Lemma Let $\kappa \geq \omega_2$ be a cardinal, and $\langle e_\xi\rangle_{\xi<\kappa}$ the standard generating family in $\mathfrak{B}_\kappa$. Suppose that we are given a family $\langle a_\xi\rangle_{\xi<\kappa}$ in $\mathfrak{B}_\kappa$. Then there are a set $\Gamma \in [\kappa]^\kappa$ and a family $\langle c_\xi\rangle_{\xi<\kappa}$ in $\mathfrak{B}_\kappa$ such that

$$c_\xi \subseteq a_\xi, \quad \bar{\nu}_\kappa c_\xi \geq 2\bar{\nu}_\kappa a_\xi - 1$$

for every ξ, and

$$\bar{\nu}_\kappa(\inf_{\xi\in I}(c_\xi \cap e_\xi) \cap \inf_{\eta\in J}(c_\eta \setminus e_\eta)) = \tfrac{1}{2^{\#(I\cup J)}}\bar{\nu}_\kappa(\inf_{\xi\in I\cup J} c_\xi)$$

whenever I, $J \subseteq \Gamma$ are disjoint finite sets.

proof Let e_ξ, ϕ_ξ, for $\xi < \kappa$, $\mathfrak{C}_L$, for $L \subseteq \kappa$, and $J^*(a)$, for $a \in \mathfrak{B}_\kappa$, be as in 531I-531J. Set $L_\xi = J^*(a_\xi)$ and $c_\xi = a_\xi \cap \phi_\xi a_\xi$ for each ξ; then

$$\bar{\nu}_\kappa c_\xi = \bar{\nu}_\kappa a_\xi + \bar{\nu}_\kappa(\phi_\xi a_\xi) - \bar{\nu}_\kappa(a_\xi \cup \phi_\xi a_\xi) \geq 2\bar{\nu}_\kappa a_\xi - 1$$

and $c_\xi \in \mathfrak{C}_{L_\xi\setminus\{\xi\}}$ (531Je). By Hajnal's Free Set Theorem (5A1J(a-iii)), there is a set $\Gamma \in [\kappa]^\kappa$ such that $\xi \notin L_\eta$ whenever ξ, η are distinct members of Γ. (This is where we use the hypothesis that $\kappa \geq \omega_2$.) Now suppose that I, $J \subseteq \Gamma$ are finite and disjoint. Then $(L_\xi \setminus \{\xi\}) \cap (I \cup J) = \emptyset$, so $c_\xi \in \mathfrak{C}_{\kappa\setminus(I\cup J)}$, for every $\xi \in I \cup J$. Accordingly $c = \inf_{\xi\in I\cup J} c_\xi$ belongs to $\mathfrak{C}_{\kappa\setminus(I\cup J)}$. This means that c and the e_ξ, for $\xi \in I \cup J$, are stochastically independent, and

$$\bar\nu_\kappa(c\cap \inf_{\xi\in I} e_\xi \cap \inf_{\eta\in J}(1\setminus e_\eta)) = \bar\nu_\kappa c\cdot \prod_{\xi\in I}\bar\nu_\kappa e_\xi \cdot \prod_{\eta\in J}\bar\nu_\kappa(1\setminus e_\eta) = \frac{1}{2^{\#(I\cup J)}}\bar\nu_\kappa c,$$

as claimed.

531L Theorem Let X be a Hausdorff space.

(a) (Haydon 77) If $\omega\in\mathrm{Mah}_{\mathrm{R}}(X)$ then $\{0,1\}^\omega$ is a continuous image of a compact subset of X.

(b) (Haydon 77, Plebanek 97) If $\kappa\ge\omega_2$ belongs to $\mathrm{Mah}_{\mathrm{R}}(X)$ and $\lambda\le\kappa$ is an infinite cardinal such that (κ,λ) is a measure-precaliber pair of every probability algebra, then $\{0,1\}^\lambda$ is a continuous image of a compact subset of X.

proof (a) If $\omega\in\mathrm{Mah}_{\mathrm{R}}(X)$ then X has a compact subset K which is not scattered (531Ee) and there is a continuous surjection from K onto $[0,1]$ (4A2G(j-iv) again). As there is a continuous surjection from $[0,1]$ onto $[0,1]^\omega$ (5A4I(b-ii)), there is a continuous surjection $f:K\to[0,1]^\omega$. Setting $K'=f^{-1}[\{0,1\}^\omega]$, K' is a compact subset of X and $\{0,1\}^\omega$ is a continuous image of K'.

(b) Let μ be a Maharam-type-homogeneous Radon probability measure on X with Maharam type κ, Σ its domain, and $(\mathfrak{A},\bar\mu)$ its measure algebra, so that $(\mathfrak{A},\bar\mu)$ is isomorphic to the measure algebra $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$ as discussed in 531I-531K. Let $\langle e_\xi\rangle_{\xi<\kappa}$ be a stochastically independent τ-generating set of elements of measure $\frac12$ in $\mathfrak{A}$, so that $(\mathfrak{A},\langle e_\xi\rangle_{\xi<\kappa})$ is isomorphic to $\mathfrak{B}_\kappa$ with its standard generating family. For each $\xi<\kappa$, let $E_\xi\in\Sigma$ be such that $E_\xi^\bullet=e_\xi$ in $\mathfrak{A}$. Let $K'_\xi\subseteq E_\xi$, $K''_\xi\subseteq X\setminus E_\xi$ be compact sets of measure at least $\frac13$, and set $K_\xi=K'_\xi\cup K''_\xi$, $a_\xi=K_\xi^\bullet$ for $\xi<\kappa$. By 531K, copied into $\mathfrak{A}$, there are $\langle c_\xi\rangle_{\xi<\kappa}$ and $\Gamma_0\in[\kappa]^\kappa$ such that $c_\xi\subseteq a_\xi$ and $\bar\mu c_\xi\ge\frac13$ for each ξ, and

$$\bar\mu(\inf_{\xi\in I}(c_\xi\cap e_\xi)\cap\inf_{\eta\in J}(c_\eta\setminus e_\eta)) = \frac{1}{2^{\#(I\cup J)}}\bar\mu(\inf_{\xi\in I\cup J}c_\xi)$$

whenever I, $J\subseteq\Gamma_0$ are disjoint finite sets.

At this point, recall that (κ,λ) is supposed to be a measure-precaliber pair of every probability algebra. So there is a $\Gamma\in[\Gamma_0]^\lambda$ such that $\inf_{\xi\in I}c_\xi\ne0$ for every finite $I\subseteq\Gamma$. It follows at once that $\inf_{\xi\in I}(a_\xi\cap e_\xi)\cap\inf_{\eta\in J}(a_\eta\setminus e_\eta)$ is non-zero for all disjoint finite sets I, $J\subseteq\Gamma$. But this means that $X\cap\bigcap_{\xi\in I}K'_\xi\cap\bigcap_{\eta\in J}K''_\eta$ is non-negligible, therefore non-empty, for all disjoint finite I, $J\subseteq\Gamma$.

Set $K=\bigcap_{\xi\in\Gamma}K_\xi$, so that $K\subseteq X$ is compact. Then we have a continuous function $f:K\to\{0,1\}^\Gamma$ defined by setting

$$\begin{aligned} f(x)(\xi) &= 1 \text{ if } x\in K\cap E_\xi = K\cap K'_\xi,\\ &= 0 \text{ if } x\in K\setminus E_\xi = K\cap K''_\xi. \end{aligned}$$

Now f is surjective. **P** If $w\in\{0,1\}^\Gamma$ and $L\subseteq\Gamma$ is finite, then

$$\begin{aligned} F_L=\{x: x\in X,\ x\in K'_\xi &\text{ whenever } \xi\in L \text{ and } w(\xi)=1,\\ x\in K''_\xi &\text{ whenever } \xi\in L \text{ and } w(\xi)=0\} \end{aligned}$$

is a non-empty closed set. The family $\{F_L: L\in[\Gamma]^{<\omega}\}$ is downwards-directed, so has non-empty intersection; and if x is any point of the intersection, $x\in K$ and $f(x)=w$. **Q**

As $\#(\Gamma)=\lambda$, $\{0,1\}^\lambda$ is a continuous image of a compact subset of X.

531M Proposition (Plebanek 97) If κ is an infinite cardinal and $\{0,1\}^\kappa$ is a continuous image of a closed subset of X whenever X is a compact Hausdorff space such that $\kappa\in\mathrm{Mah}_{\mathrm{R}}(X)$, then κ is a measure-precaliber of every probability algebra.

proof It will be enough to show that κ is a measure-precaliber of $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$ (525I(a-i)). Let $\langle a_\xi\rangle_{\xi<\kappa}$ be a family in $\mathfrak{B}_\kappa$ such that $\inf_{\xi<\kappa}\bar\nu_\kappa a_\xi=\alpha>0$. Choose $\langle b_\xi\rangle_{\xi<\kappa}$ in $\mathfrak{B}_\kappa$ inductively, as follows. Given $\langle b_\eta\rangle_{\eta<\xi}$, let $\mathfrak{D}_\xi$ be the closed subalgebra of $\mathfrak{B}_\kappa$ generated by $\{b_\eta:\eta<\xi\}\cup\{a_\xi\}$. Because $\mathfrak{B}_\kappa$ is homogeneous with Maharam type $\kappa>\tau(\mathfrak{D}_\xi)$, it is relatively atomless over $\mathfrak{D}_\xi$, and there is a $b\in\mathfrak{B}_\kappa$ such that $\bar\nu_\kappa(b\cap c)=\frac12\bar\nu_\kappa c$ for every $c\in\mathfrak{D}_\xi$ (331B). Set $b_\xi=b\cap a_\xi$; then for any $\eta<\xi$ we have

$$\begin{aligned}\bar\nu_\kappa(b_\xi \triangle b_\eta) &= \bar\nu_\kappa b_\xi + \bar\nu_\kappa b_\eta - 2\bar\nu_\kappa(b_\xi \cap b_\eta)\\ &= \frac{1}{2}\bar\nu_\kappa a_\xi + \bar\nu_\kappa b_\eta - \bar\nu_\kappa(a_\xi \cap b_\eta) \ge \frac{1}{2}\bar\nu_\kappa a_\xi \ge \frac{\alpha}{2}.\end{aligned}$$

Continue.

Let $\mathfrak{C}$ be the subalgebra of $\mathfrak{B}_\kappa$ generated by $\{b_\xi : \xi < \kappa\}$, and X its Stone space. Then $\mathfrak{C}$ is isomorphic to the algebra of open-and-closed subsets of X, so we have a Radon measure μ on X defined by saying that $\mu\widehat{c} = \bar\nu_\kappa c$ for every $c \in \mathfrak{C}$, writing $\widehat{c}$ for the open-and-closed subset of X corresponding to c (416Qa). Now μ is strictly positive and we can identify $\mathfrak{C}$ with a topologically dense subalgebra of the measure algebra of μ. It follows that μ has a Maharam-type-homogeneous component of type at least κ. **P?** Otherwise, there would be a set $E \subseteq X$, of measure at least $1 - \frac{1}{4}\alpha$, such that the Maharam type of the subspace measure μ_E was less than κ. But

$$\mu(E \cap \widehat{b}_\xi \triangle \widehat{b}_\eta) \ge \bar\nu_\kappa(b_\xi \triangle b_\eta) - \frac{\alpha}{4} \ge \frac{\alpha}{4}$$

whenever $\eta < \xi < \kappa$, so the topological density of the measure algebra of μ_E is at least κ (5A4B(h-ii)) and the Maharam type of μ_E is at least κ (521E(a-ii)). **XQ** Thus $\kappa \in \mathrm{Mah}_{\mathrm{R}}(X)$.

Accordingly $\{0,1\}^\kappa$ is a continuous image of a closed subset of X. By 5A4C(d-iii), there is a non-empty closed subset K of X such that $\chi(x, K) \ge \kappa$ for every $x \in K$. Let $D \subseteq \kappa$ be a maximal set such that $\{K\} \cup \{\widehat{b}_\xi : \xi \in D\}$ has the finite intersection property. Set $Z = K \cap \bigcap_{\xi \in D} \widehat{b}_\xi$; then Z contains a point z say. Because $\{b_\xi : \xi \in D\}$ is centered, so is $\{a_\xi : \xi \in D\}$.

If $x \in X \setminus \{z\}$, then there is a $c \in \mathfrak{C}$ such that $x \in \widehat{c}$ and $z \notin \widehat{c}$; accordingly there is a $\zeta < \kappa$ such that one of x, z belongs to $\widehat{b}_\zeta$ and the other does not. If $\zeta \in D$ then $z \in \widehat{b}_\zeta$ and $x \notin \widehat{b}_\zeta$, so $x \notin Z$. If $\zeta \notin D$ then, by the maximality of D, $Z \cap \widehat{b}_\zeta = \emptyset$, so that $z \notin \widehat{b}_\zeta$, $x \in \widehat{b}_\zeta$ and again $x \notin Z$.

Thus $Z = \{z\}$, and $\{z\}$ can be expressed as the intersection of $\#(D)$ relatively open sets in K. By 4A2Gd, it follows that $\#(D) \ge \chi(z, K) \ge \kappa$, and we have already seen that $\{a_\xi : \xi \in D\}$ is centered. As $\langle a_\xi \rangle_{\xi<\kappa}$ is arbitrary, κ is a measure-precaliber of $\mathfrak{B}_\kappa$, as required.

531N In 531M we have a space X out of which there is no surjection onto $\{0,1\}^\kappa$ because every non-empty closed set has a point of character less than κ. From stronger properties of κ we can get compact spaces with stronger topological properties, as in the next two results.

Proposition Let κ, κ' and λ be infinite cardinals such that (κ, κ') is not a measure-precaliber pair of $(\mathfrak{B}_\lambda, \bar\nu_\lambda)$. Then there is a compact Hausdorff space X such that $\kappa \in \mathrm{Mah}_{\mathrm{R}}(X)$ and $\chi(x, X) < \max(\kappa', \lambda^+)$ for every $x \in X$.

proof Let $\langle a_\xi \rangle_{\xi<\kappa}$ be a family in $\mathfrak{B}_\lambda$, with no centered subfamily with cardinal κ', such that $\inf_{\xi<\kappa} \bar\mu a_\xi = \alpha > 0$. Let $\psi : \mathfrak{B}_\lambda \to \mathrm{T}_\lambda$ be a lifting; for each $\xi < \kappa$, let $K_\xi \subseteq \psi a_\xi$ be a compact set of measure at least $\frac{1}{2}\alpha$. If $D \subseteq \kappa$ and $\#(D) = \kappa'$, then there is a finite set $I \subseteq D$ such that $\inf_{\xi \in I} a_\xi = 0$, in which case $\bigcap_{\xi \in I} K_\xi \subseteq \bigcap_{\xi \in I} \psi a_\xi = \emptyset$. Thus $\{\xi : x \in K_\xi\}$ has cardinal less than κ' for every $x \in \{0,1\}^\lambda$.

Set

$$X = \bigcap_{\xi<\kappa'} \{(x, y) : x \in \{0,1\}^\lambda,\, y \in \{0,1\}^\kappa,\, x \in K_\xi \text{ or } y(\xi) = 0\},$$

so that X is a compact subset of $\{0,1\}^\lambda \times \{0,1\}^\kappa$. Now $\chi((x, y), X) < \max(\kappa', \lambda^+)$ for every $(x, y) \in X$. **P** Set $D = \{\xi : \xi < \kappa,\, x \in K_\xi\}$, so that $\#(D) < \kappa'$. For $I \in [\lambda]^{<\omega}$ and $J \in [D]^{<\omega}$ set

$$V_{IJ} = \{(x', y') : (x', y') \in X,\, x' \restriction I = x \restriction I,\, y' \restriction J = y \restriction J\},$$

so that $\mathcal{V} = \{V_{IJ} : I \in [\lambda]^{<\omega},\, J \in [D]^{<\omega}\}$ is a downwards-directed family of closed neighbourhoods of (x, y). If $(x', y') \in \bigcap \mathcal{V}$, then $x' = x$, so $x' \notin K_\xi$ for $\xi \in \kappa \setminus D$, and $y'(\xi) = y(\xi) = 0$ for $\xi \notin D$; also $y' \restriction D = y \restriction D$, so $(x', y') = (x, y)$. Thus $\bigcap \mathcal{V} = \{(x, y)\}$; by 4A2Gd again, $\mathcal{V}$ is a base of neighbourhoods of (x, y), and

$$\chi((x, y), X) \le \#(\mathcal{V}) \le \max(\#(D), \lambda) < \max(\kappa', \lambda^+). \text{ } \mathbf{Q}$$

Define $g : \{0,1\}^\lambda \times \{0,1\}^\kappa \to \{0,1\}^\kappa$ and $h : \{0,1\}^\lambda \times \{0,1\}^\kappa \to X$ by setting

$$\begin{aligned}g(x,y)(\xi) &= y(\xi) \text{ if } x \in K_\xi,\\ &= 0 \text{ otherwise},\\ h(x,y) &= (x, g(x,y)),\end{aligned}$$

for $\xi < \kappa$, $x \in \{0,1\}^\lambda$ and $y \in \{0,1\}^\kappa$. Write Σ for the domain of the product measure $\nu = \nu_\lambda \times \nu_\kappa$ on $\{0,1\}^\lambda \times \{0,1\}^\kappa$. Then the σ-algebra $\{F : F \subseteq X,\ h^{-1}[F] \in \Sigma\}$ contains all sets of the form $\{(x,y) : x(\eta) = 1\}$ and $\{(x,y) : y(\xi) = 1\}$, so includes a base for the topology of X and therefore contains every open-and-closed set. Accordingly we have an additive functional $U \mapsto \nu h^{-1}[U]$ on the algebra of open-and-closed subsets of X, which extends to a Radon probability measure μ on X (416Qa again). Set $F_\xi = \{(x,y) : (x,y) \in X,\ y(\xi) = 1\}$ for each $\xi < \kappa$; then for any $\eta < \xi < \kappa$,

$$\begin{aligned}\mu(F_\xi \setminus F_\eta) &= \nu h^{-1}[F_\xi \setminus F_\eta]\\ &\ge \nu\{(x,y) : x \in K_\xi,\ y(\xi) = 1,\ y(\eta) = 0\} = \frac{1}{4}\nu_\lambda K_\xi \ge \frac{1}{8}\alpha.\end{aligned}$$

As in the proof of 531M, this shows that the measure algebra of μ must have a homogeneous principal ideal with Maharam type at least κ, and $\kappa \in \mathrm{Mah}(X)$.

531O Putting these ideas together with 531L, we come to the following.

Proposition (Kunen & Mill 95, Plebanek 95) Let κ be a regular infinite cardinal. Then the following are equiveridical:

(i) κ is a measure-precaliber of every measurable algebra;

(ii) if X is a compact Hausdorff space such that $\kappa \in \mathrm{Mah}_{\mathrm{R}}(X)$, then $\chi(x, X) \ge \kappa$ for some $x \in X$.

proof (a) Consider first the case $\kappa \ge \omega_2$.

(i)⇒(ii) If $\kappa \in \mathrm{Mah}_{\mathrm{R}}(X)$, then $\{0,1\}^\kappa$ is a continuous image of a compact subset of X, by 531Lb. By 5A4C(d-iii) again and 5A4Bb, it follows at once that $\chi(x, X) \ge \kappa$ for many points $x \in X$.

not-(i)⇒not-(ii) By 525Ib there is a $\lambda < \kappa$ such that κ is not a precaliber of $\mathfrak{B}_\lambda$, and therefore not a measure-precaliber of $(\mathfrak{B}_\lambda, \bar\nu_\lambda)$ (525Db). Now 531N tells us that there is a compact Hausdorff space X such that $\kappa \in \mathrm{Mah}_{\mathrm{R}}(X)$ and $\chi(x, X) < \max(\kappa, \lambda^+) = \kappa$ for every $x \in X$.

(b) Now suppose that $\kappa = \omega_1$.

(i)⇒(ii) ? Suppose, if possible, that ω_1 is a precaliber of every probability algebra, but that there is a first-countable compact Hausdorff space X with $\omega_1 \in \mathrm{Mah}_{\mathrm{R}}(X)$. Let μ be a Maharam-type-homogeneous Radon probability measure on X with Maharam type ω_1, and $(\mathfrak{A}, \bar\mu)$ its measure algebra; let $\langle c_\xi \rangle_{\xi<\omega_1}$ be a τ-generating stochastically independent family of elements of measure $\frac{1}{2}$ in $\mathfrak{A}$. As in 531J, there is for each $a \in \mathfrak{A}$ a countable $J^*(a) \subseteq \omega_1$ such that a belongs to the closed subalgebra of $\mathfrak{A}$ generated by $\{c_\xi : \xi \in J^*(a)\}$.

For each $x \in X$, let $\mathcal{U}_x$ be a countable base of open neighbourhoods of x, and set $A_x = \{U^\bullet : U \in \mathcal{U}_x\}$, $J^\dagger(x) = \bigcup_{a \in A_x} J^*(a)$. Then $J^\dagger(x)$ is countable. For $\xi < \omega_1$, set $D_\xi = \{x : J^\dagger(x) \subseteq \xi\}$; then $\langle D_\xi \rangle_{\xi<\omega_1}$ is a non-decreasing family with union X. Now ω_1 is supposed to be a precaliber of $\mathfrak{A}$, so there must be a $\xi < \omega_1$ such that D_ξ has full outer measure (525Cc).

Let $G \subseteq X$ be open. Then $G^\bullet$ belongs to the closed subalgebra $\mathfrak{C}_\xi$ of $\mathfrak{A}$ generated by $\{c_\eta : \eta < \xi\}$. **P** For each $x \in G \cap D_\xi$, there is a $U_x \in \mathcal{U}_x$ such that $U_x \subseteq G$. Set $H = \bigcup\{U_x : x \in G \cap D_\xi\}$, so that $H \subseteq G$ is open and $G \cap D_\xi = H \cap D_\xi$; as D_ξ has full outer measure, $G \setminus H$ is negligible and $H^\bullet = G^\bullet$. But 414Aa once more tells us that $H^\bullet = \sup_{x \in D_\xi} U_x^\bullet$, and this belongs to $\mathfrak{C}_\xi$, because $J^\dagger(x) \subseteq \xi$ for every $x \in D_\xi$. **Q**

It follows at once that $F^\bullet \in \mathfrak{C}_\xi$ for every closed $F \subseteq X$. Because μ is inner regular with respect to the closed sets, $\mathfrak{C}_\xi$ is order-dense in $\mathfrak{A}$ and $\mathfrak{A} = \mathfrak{C}_\xi$ has Maharam type $\#(\xi) < \omega_1$. **X**

Thus (i)⇒(ii).

not-(i)⇒not-(ii) Suppose that (i) is false.

(α) By 525J, $\mathrm{cov}\,\mathcal{N}_{\omega_1} = \omega_1$ and there is a family $\langle A_\xi \rangle_{\xi<\omega_1}$ of negligible subsets of $\{0,1\}^{\omega_1}$ covering $\{0,1\}^{\omega_1}$. For each $\xi < \omega_1$, let $A'_\xi \supseteq A_\xi$ be a negligible set determined by coordinates in a countable set $J_\xi \subseteq \omega_1$; set $\tilde{A}_\xi = \bigcup\{A'_\eta : \eta < \xi,\ J_\eta \subseteq \xi\}$; then $\tilde{A}_\xi$ is determined by coordinates less than ξ. Set $H_\xi = \{y{\restriction}\xi : y \in \tilde{A}_\xi\}$, so that H_ξ is a ν_ξ-negligible subset of $\{0,1\}^\xi$.

We see that $\langle \tilde{A}_\xi \rangle_{\xi<\omega_1}$ is non-decreasing, and

$$\bigcup_{\xi<\omega_1} \tilde{A}_\xi = \bigcup_{\xi<\omega_1} A'_\xi = \{0,1\}^{\omega_1}.$$

Consequently $y{\restriction}\xi \in H_\xi$ whenever $\eta \le \xi < \omega_1$, $y \in \{0,1\}^{\omega_1}$ and $y{\restriction}\eta \in H_\eta$, while for every $y \in \{0,1\}^{\omega_1}$ there is a $\xi < \omega_1$ such that $y{\restriction}\xi \in H_\xi$.

(β) Set $Y = \{0\} \cup \{2^{-n} : n \in \mathbb{N}\} \subseteq [0,1]$. For $\xi \le \omega_1$ define $\phi_\xi : Y^\xi \to \{0,1\}^\xi$ by setting

$$\begin{aligned}\phi_\xi(x)(\eta) &= 0 \text{ if } \eta < \xi \text{ and } x(\eta) = 0,\\ &= 1 \text{ for other } \eta < \xi.\end{aligned}$$

Observe that ϕ_ξ is Borel measurable for every $\xi < \omega_1$. Choose $\langle X_\xi \rangle_{\xi<\omega_1}$, and $\langle K_{\xi n} \rangle_{\xi<\omega_1, n\in\mathbb{N}}$ inductively, as follows. The inductive hypothesis will be that X_ξ is a compact subset of Y^ξ, $\phi_\xi[X_\xi]$ is ν_ξ-conegligible in $\{0,1\}^\xi$, $\phi_\xi{\restriction}X_\xi$ is injective and $x{\restriction}\eta \in X_\eta$ whenever $x \in X_\xi$ and $\eta \le \xi < \omega_1$.

Start with $X_0 = Y^0 = \{\emptyset\}$ and $\phi_0 : X_0 \to \{0,1\}^0$ the identity map.

Given $\xi < \omega_1$ and $X_\xi \subseteq Y^\xi$, then 433D tells us that there is a Radon measure μ_ξ on X_ξ such that ν_ξ is the image measure $\mu_\xi \phi_\xi^{-1}$. Let $\langle K_{\xi n} \rangle_{n\in\mathbb{N}}$ be a disjoint sequence of compact subsets of $X_\xi \setminus \phi_\xi^{-1}[H_\xi]$ with μ_ξ-conegligible union. Set

$$\begin{aligned}X_{\xi+1} = \{x : x \in Y^{\xi+1},\ x{\restriction}\xi \in X_\xi,\ x(\xi) = 0\}&\\ \cup \bigcup_{n\in\mathbb{N}} \{x : x \in Y^{\xi+1},\ x{\restriction}\xi \in K_{\xi n},\ x(\xi) = 2^{-n}\}&.\end{aligned}$$

It is easy to see that $X_{\xi+1}$ is compact and $\phi_{\xi+1}{\restriction}X_{\xi+1}$ is injective, while surely $x{\restriction}\eta \in X_\eta$ whenever $x \in X_{\xi+1}$ and $\eta \le \xi + 1$, just because $x{\restriction}\xi \in X_\xi$. Also

$$\phi_{\xi+1}[X_{\xi+1}] \supseteq \{y : y \in \{0,1\}^{\xi+1},\ y{\restriction}\xi \in \bigcup_{n\in\mathbb{N}} \phi_\xi[K_{\xi n}]\}$$

is $\nu_{\xi+1}$-conegligible because $\phi_\xi[K_{\xi n}]$ must be analytic for every n and

$$\nu_\xi(\bigcup_{n\in\mathbb{N}} \phi_\xi[K_{\xi n}]) = \mu_\xi(\bigcup_{n\in\mathbb{N}} K_{\xi n}) = 1$$

because $\phi_\xi{\restriction}X_\xi$ is injective.

Given that X_η has been defined for $\eta < \xi$, where $\xi < \omega_1$ is a non-zero limit ordinal, set

$$X_\xi = \{x : x \in Y^\xi,\ x{\restriction}\eta \in X_\eta \text{ for every } \eta < \xi\}.$$

Of course X_ξ is compact and $\phi_\xi{\restriction}X_\xi$ is injective. To see that $\phi_\xi[X_\xi]$ is conegligible, observe that

$$W = \bigcap_{\eta<\xi} \{y : y \in \{0,1\}^\xi,\ y{\restriction}\eta \in \phi_\eta[X_\eta]\}$$

is conegligible. But if $y \in W$ and we choose $x_\eta \in X_\eta$ such that $\phi_\eta(x_\eta) = y{\restriction}\eta$ for each $\eta < \xi$, then we must have $x_\zeta = x_\eta{\restriction}\zeta$ whenever $\zeta \le \eta < \xi$, because $\phi_\zeta{\restriction}X_\zeta$ is injective; so there is an $x \in Y^\xi$ such that $x_\eta = x{\restriction}\eta$ for every $\eta < \xi$, in which case $x \in X_\xi$ and $\phi_\xi(x) = y$. Thus $\phi_\xi[X_\xi] \supseteq W$ is conegligible.

(γ) At the end of the induction, set

$$X = \{x : x \in Y^{\omega_1},\ x{\restriction}\xi \in X_\xi \text{ for every } \xi < \omega_1\}, \quad \phi = \phi_{\omega_1}{\restriction}X.$$

As in the limit stage of the construction in (β), we see that X is a closed subset of Y^{ω_1}, so with the subspace topology is a zero-dimensional compact Hausdorff space. This time, we do not expect that $\phi[X]$ should be conegligible in $\{0,1\}^{\omega_1}$, but we find that it has full outer measure. **P** If $K \subseteq \{0,1\}^{\omega_1}$ is a non-negligible closed G_δ set, there is a $\xi < \omega_1$ such that K is determined by coordinates less than ξ. Set $K' = \{y{\restriction}\xi : y \in K\}$; then $\nu_\xi K' = \nu_{\omega_1} K > 0$, so there is an $x_0 \in X_\xi$ such that $\phi_\xi(x_0) \in K'$. Extending x_0 to $x \in Y^{\omega_1}$ by setting $x(\eta) = 0$ for $\xi \le \eta < \omega_1$, we see by induction on ζ that $x{\restriction}\zeta \in X_\zeta$ for $\xi \le \zeta < \omega_1$, so $x \in X$; also $\phi(x){\restriction}\xi = \phi_\xi(x_0) \in K'$, so $\phi(x) \in K$ and K meets $\phi[X]$. As ν_{ω_1} is completion regular, $\phi[X]$ has full outer measure. **Q**

(δ) X is first-countable. **P** If $x \in X$, $\xi < \omega_1$ and $x(\xi) \ne 0$, then $x{\restriction}(\xi + 1)$ belongs to $X_{\xi+1}$, and there must be some $n \in \mathbb{N}$ such that $x(\xi) = 2^{-n}$ and $x{\restriction}\xi \in K_{\xi n}$; in which case $\phi_\xi(x{\restriction}\xi) \notin H_\xi$. Now take any $x \in X$. Then there is a $\xi < \omega_1$ such that $\phi(x) \in \tilde{A}_\xi$ and $\phi_\xi(x) = \phi(x)\restriction\xi$ belongs to H_ξ. In this case, $V = \{x' : x' \in X,\, x'\restriction\xi = x\restriction\xi\}$ is a G_δ subset of X containing x. But if $x' \in V$ then, for any $\eta \ge \xi$, $\phi_\eta(x'\restriction\eta) \in H_\eta$ and $x'(\eta) = 0$. Thus $V = \{x\}$. By 4A2Gd once more, x has a countable base of neighbourhoods in X; as x is arbitrary, X is first-countable. **Q**

(ϵ) By 234F, there is a measure λ on X such that ϕ is inverse-measure-preserving for λ and ν_{ω_1}. Of course λ is a probability measure. Now for any $\xi < \omega_1$ and $n \in \mathbb{N}$,

$$\{x : x \in X,\, x(\xi) = 0\} = \{x : \phi(x)(\xi) = 0\},$$

$$\begin{aligned}\{x : x \in X,\, x(\xi) = 2^{-n}\} &= \{x : \phi(x)(\xi) = 1,\, x\restriction\xi \in K_{\xi n}\}\\ &= \{x : \phi(x)(\xi) = 1,\, \phi_\xi(x\restriction\xi) \in \phi_\xi[K_{\xi n}]\}\\ &= \{x : \phi(x)(\xi) = 1,\, \phi(x)\restriction\xi \in \phi_\xi[K_{\xi n}]\}\end{aligned}$$

are measured by λ. So the domain of λ includes a base for the topology of the zero-dimensional compact Hausdorff space X. By 416Qa once more, there is a Radon measure μ on X agreeing with λ on the open-and-closed subsets of X; by the Monotone Class Theorem (136C), μ and λ agree on the σ-algebra generated by the open-and-closed sets, that is, the Baire σ-algebra of X (4A3Od). In particular, setting $E_\xi = \{x : x \in X,\, x(\xi) = 0\}$ for $\xi < \omega_1$,

$$\begin{aligned}\mu(E_\xi \cap E_\eta) = \lambda(E_\xi \cap E_\eta) &= \nu_{\omega_1}\{y : y \in \{0,1\}^{\omega_1},\, y(\xi) = y(\eta) = 0\}\\ &= \frac{1}{2} \text{ if } \xi = \eta < \omega_1,\\ &= \frac{1}{4} \text{ if } \xi,\, \eta < \omega_1 \text{ are different.}\end{aligned}$$

It follows that $\mu(E_\xi \triangle E_\eta) = \frac{1}{2}$ for all distinct ξ, $\eta < \omega_1$, so μ has uncountable Maharam type and $\omega_1 \in \mathrm{Mah}_{\mathrm{R}}(X)$. Thus X and μ witness that (ii) is false.

(c) Finally, if $\kappa = \omega$, both (i) and (ii) are true for elementary reasons (525Fa).

531P In 531O we saw that if ω_1 is not a precaliber of every measurable algebra then there is a first-countable compact Hausdorff space with a Radon measure with Maharam type ω_1. With a sharper hypothesis, and rather more work, we can get a stronger version, as follows.

Lemma Let Y be a zero-dimensional compact metrizable space, μ an atomless Radon probability measure on Y, $A \subseteq Y$ a μ-negligible set and $\mathcal{Q}$ a countable family of closed subsets of Y. Then there are closed sets K, $L \subseteq Y$, with union Y, such that

$K \cup L = Y$, $\quad K \cap L \cap A = \emptyset$, $\quad \mu(K \cap L) \ge \frac{1}{2}$,
$K \cap Q = \overline{Q \setminus L}$ and $L \cap Q = \overline{Q \setminus K}$ for every $Q \in \mathcal{Q}$.

proof We can of course suppose that $\emptyset \in \mathcal{Q}$. For each $Q \in \mathcal{Q}$ let D_Q be a countable dense subset of Q; let $S \subseteq Y \setminus (A \cup \bigcup_{Q\in\mathcal{Q}} D_Q)$ be a closed set of measure at least $\frac{1}{2}$. (This is where we need to know that μ is atomless, so that every D_Q is negligible.) Let $\mathcal{U}$ be a countable base for the topology of Y consisting of open-and-closed sets and let $\langle(U_n, Q_n)\rangle_{n\in\mathbb{N}}$ run over $\mathcal{U}\times\mathcal{Q}$. Choose inductively sequences $\langle G_n\rangle_{n\in\mathbb{N}}$, $\langle H_n\rangle_{n\in\mathbb{N}}$ of open-and-closed subsets of $Y \setminus S$, as follows. Start with $G_0 = H_0 = \emptyset$. Given that G_n and H_n are disjoint from each other and from S, then

— if $U_n \cap S = \emptyset$, take $G_{n+1} = G_n \cup (U_n \setminus H_n)$ and $H_{n+1} = H_n$;

— if $U_n \cap S \cap Q_n \ne \emptyset$, $U = U_n \setminus (G_n \cup H_n)$ is open and includes $U_n \cap S \cap Q_n$; as D_{Q_n} is dense in Q_n, $\overline{U \cap D_{Q_n}}$ includes $U \cap Q_n$ which meets S so cannot be included in D_{Q_n}, and $U \cap D_{Q_n}$ must be infinite; take two of its points y, y' say; neither belongs to S so we can enlarge G_n and H_n to disjoint open-and-closed subsets G_{n+1}, H_{n+1} of $Y \setminus S$ containing y, y' respectively, and therefore both meeting $U_n \cap Q_n$;

— otherwise, take $G_{n+1} = G_n$ and $H_{n+1} = H_n$.

At the end of the induction, set $G = \bigcup_{n\in\mathbb{N}} G_n$ and $H = \bigcup_{n\in\mathbb{N}} H_n$, so that G and H are disjoint open subsets of $Y \setminus S$. Now if y is any point of $Y \setminus S$, there must be some n such that $y \in U_n \subseteq Y \setminus S$, so that

$y \in G_{n+1} \cup H_n$; thus $Y = G \cup H \cup S$. Set $K = G \cup S = Y \setminus H$, $L = H \cup S = Y \setminus G$; then K and L are closed sets with union Y, and $K \cap L = S$ has measure at least $\frac{1}{2}$ and is disjoint from A.

If $Q \in \mathcal{Q}$, $y \in S \cap Q$ and U is any neighbourhood of y, there is an$\pi[Z \setminus W] \neq Q$ $n \in \mathbb{N}$ such that $Q_n = Q$ and $y \in U_n \subseteq U$. In this case, $U_n \cap S \cap Q_n \neq \emptyset$ and $G \cap Q \supseteq G_{n+1} \cap Q_n$, $H \cap Q \supseteq H_{n+1} \cap Q$ both meet $U_n \cap Q$. As U and y are arbitrary,

$$K \cap L \cap Q = S \cap Q \subseteq \overline{G \cap Q} \cap \overline{H \cap Q},$$

$$K \cap Q \subseteq (S \cap Q) \cup (G \cap Q) \subseteq \overline{G \cap Q} = \overline{Q \setminus L}$$

and similarly $L \cap Q \subseteq \overline{Q \setminus K}$. At the same time, $K \supseteq Q \setminus L$ and $L \supseteq Q \setminus K$, so $K \cap Q = \overline{Q \setminus L}$ and $L \cap Q = \overline{Q \setminus K}$. Thus K and L fulfil all the specifications.

531Q Proposition Suppose that $\mathrm{cf}\,\mathcal{N}_\omega = \omega_1$. Then there is a hereditarily separable perfectly normal compact Hausdorff space X, of weight ω_1, with a Radon probability measure of Maharam type ω_1 such that every negligible set is metrizable.

proof For $\eta \le \xi \le \omega_1$ and $x \in \{0,1\}^\xi$, set $\pi_{\eta\xi}(x) = x\restriction\eta$; write π_η for $\pi_{\eta\omega_1} : \{0,1\}^{\omega_1} \to \{0,1\}^\eta$. As in 531I, ν_ξ is to be the usual measure on $\{0,1\}^\xi$.

(a) Choose

$$\langle f_\xi \rangle_{\omega\le\xi\le\omega_1},\ \langle X_\xi \rangle_{\omega\le\xi\le\omega_1},\ \langle \mu_\xi \rangle_{\omega\le\xi<\omega_1},\ \langle K_\xi \rangle_{\omega\le\xi<\omega_1},\ \langle L_\xi \rangle_{\omega\le\xi<\omega_1},$$

$$\langle Q'_{\xi\theta} \rangle_{\omega\le\xi\le\theta<\omega_1},\ \langle Q_{\delta\xi} \rangle_{\omega\le\delta\le\xi<\omega_1},\ \langle Q_{\eta\delta\xi} \rangle_{\omega\le\eta\le\delta\le\xi<\omega_1},\ \langle A_{\xi\theta} \rangle_{\omega\le\xi\le\theta<\omega_1},\ \langle A_\xi \rangle_{\omega\le\xi<\omega_1}$$

inductively, as follows. Every X_ξ, K_ξ, L_ξ, $Q'_{\xi\theta}$, $Q_{\delta\xi}$ and $Q_{\eta\delta\xi}$ is to be a closed subset of $\{0,1\}^\xi$, every f_ξ will be a Baire measurable surjection from $\{0,1\}^\xi$ onto X_ξ, μ_ξ will always be the Radon probability measure $\nu_\xi f_\xi^{-1}$ on $\{0,1\}^\xi$, and A_ξ and $A_{\xi\theta}$ will always be μ_ξ-negligible subsets of $\{0,1\}^\xi$.

Given that $\omega \le \xi \le \omega_1$ and that K_η, L_η are closed subsets of $\{0,1\}^\eta$ covering $\{0,1\}^\eta$ whenever $\omega \le \eta < \xi$, then define $f_\xi(x)(\eta)$, for $x \in \{0,1\}^\xi$ and $\eta < \xi$, by setting

$$\begin{aligned} f_\xi(x)(\eta) &= 1 \text{ if } \eta \ge \omega \text{ and } x\restriction\eta \notin L_\eta,\\ &= 0 \text{ if } \eta \ge \omega \text{ and } x\restriction\eta \notin K_\eta,\\ &= x(\eta) \text{ otherwise.} \end{aligned}$$

(Thus the induction starts with $f_\omega(x) = x$ for $x \in \{0,1\}^\omega$.) Then $f_\xi : \{0,1\}^\xi \to \{0,1\}^\xi$ is Baire measurable (4A3Ne). Set

$$X_\xi = \bigcap_{\omega\le\eta<\xi} \{x : x \in \{0,1\}^\xi,\ x(\eta) = 1 \text{ or } x\restriction\eta \in L_\eta,\ x(\eta) = 0 \text{ or } x\restriction\eta \in K_\eta\};$$

then $X_\xi \subseteq \{0,1\}^\xi$ is compact, $f_\xi(x) \in X_\xi$ for every $x \in \{0,1\}^\xi$, and $f_\xi(x) = x$ for every $x \in X_\xi$. So $f_\xi[\{0,1\}^\xi] = X_\xi$.

If now $\xi < \omega_1$, $f_\xi : \{0,1\}^\xi \to \{0,1\}^\xi$ is Borel measurable; by 433E, f_ξ is ν_ξ-almost-continuous, and the image measure $\mu_\xi = \nu_\xi f_\xi^{-1}$ is a Radon measure on the compact metrizable space $\{0,1\}^\xi$ (418I). Of course $\mu_\xi X_\xi = 1$. Because $\{0,1\}^\xi$ has countable weight, or otherwise, μ_ξ has countable Maharam type (531Ad); by 524Pb, μ_ξ is inner regular with respect to a family with cardinal at most $\mathrm{cf}\,\mathcal{N}_\omega = \omega_1$, which we may suppose to consist of closed sets; let $\langle Q'_{\xi\theta} \rangle_{\xi\le\theta<\omega_1}$ run over such a family. Similarly, there is a family $\langle A_{\xi\theta} \rangle_{\xi\le\theta<\omega_1}$ running over a cofinal subset of the null ideal of μ_ξ (524Pf). Next, for $\omega \le \delta \le \xi$, let $Q_{\delta\xi} \subseteq \pi_{\delta\xi}^{-1}[Q'_{\delta\xi}]$ be the compact μ_ξ-self-supporting set of the same μ_ξ-measure as $\pi_{\delta\xi}^{-1}[Q'_{\delta\xi}]$ (414F). Note that $Q_{\delta\xi}$ will always be included in X_ξ, because $\mu_\xi X_\xi = 1$. Set $Q_{\eta\delta\xi} = X_\xi \cap \pi_{\delta\xi}^{-1}[Q_{\eta\delta}]$ for $\omega \le \eta \le \delta \le \xi$, and

$$\mathcal{A}_\xi = \{\pi_{\eta\xi}^{-1}[A_{\eta\delta}] : \omega \le \eta \le \delta \le \xi\}, \quad A_\xi = \bigcup\{A : A \in \mathcal{A}_\xi,\ \mu_\xi A = 0\};$$

because $\mathcal{A}_\xi$ is countable, A_ξ is μ_ξ-negligible. By 531P, we can find closed sets K_ξ, L_ξ covering $\{0,1\}^\xi$ such that $\mu_\xi(K_\xi \cap L_\xi) \ge \frac{1}{2}$, $K_\xi \cap L_\xi \cap A_\xi = \emptyset$, $K_\xi \cap Q_{\eta\delta\xi} = \overline{Q_{\eta\delta\xi} \setminus L_\xi}$ and $L_\xi \cap Q_{\eta\delta\xi} = \overline{Q_{\eta\delta\xi} \setminus K_\xi}$ whenever $\omega \le \eta \le \delta \le \xi$.

This deals with the inductive step to a successor ordinal $\xi+1$ when $\omega\le\xi<\omega_1$. For limit ordinals $\xi\in\,]\omega,\omega_1[$, f_ξ is defined by $\langle(K_\eta,L_\eta)\rangle_{\omega\le\eta<\xi}$, so the induction proceeds directly to ξ.

(b) At the end of the induction, write f for f_{ω_1} and X for $X_{\omega_1}=f[\{0,1\}^{\omega_1}]$. If $z\in\{0,1\}^{\omega_1}$ and $\xi<\omega_1$, the formula for f_ξ in (a) shows that $f(z)(\eta)=f_\xi(z\restriction\xi)(\eta)$ for every $\eta<\xi$, that is, that $f(z)\restriction\xi=f_\xi(z\restriction\xi)$. Next, $\nu_{\omega_1}f^{-1}$ measures every Baire subset of $\{0,1\}^{\omega_1}$ (use 4A3Na), so we have a Radon measure μ on $\{0,1\}^{\omega_1}$ defined by saying that $\mu V=\nu_{\omega_1}f^{-1}[V]$ for every Baire set $V\subseteq\{0,1\}^{\omega_1}$ (432F); of course $\mu V=0$ for every open-and-closed set V disjoint from X, so $\mu X=1$.

At the same time, it will be helpful to fill in the definition of $\langle Q_{\eta\delta\xi}\rangle_{\omega\le\eta\le\delta\le\xi<\omega_1}$ by taking $Q_{\eta\delta\omega_1}=X\cap\pi_\delta^{-1}[Q_{\eta\delta}]$ when $\omega\le\eta\le\delta<\omega_1$.

(c) Some simple facts.

(i) I have already observed that $\pi_\xi f=f_\xi\pi_\xi$ for $\omega\le\xi<\omega_1$; consequently

$$X_\xi=f_\xi[\{0,1\}^\xi]=f_\xi[\pi_\xi[\{0,1\}^{\omega_1}]]=\pi_\xi[f[\{0,1\}^{\omega_1}]]=\pi_\xi[X]$$

and

$$\begin{aligned}\mu_\xi V&=\nu_\xi f_\xi^{-1}[V]=(\nu_{\omega_1}\pi_\xi^{-1})f_\xi^{-1}[V]=\nu_{\omega_1}(f_\xi\pi_\xi)^{-1}[V]\\&=\nu_{\omega_1}(\pi_\xi f)^{-1}[V]=(\nu_{\omega_1}f^{-1})\pi_\xi^{-1}[V]=\mu\pi_\xi^{-1}[V]\end{aligned}$$

for every open-and-closed set $V\subseteq\{0,1\}^\xi$. Thus the Radon measures $\mu\pi_\xi^{-1}$ and μ_ξ are identical.

(ii) Equally, if $\omega\le\eta\le\xi<\omega_1$,

$$X_\eta=\pi_\eta[X]=\pi_{\eta\xi}[\pi_\xi[X]]=\pi_{\eta\xi}[X_\xi]$$

and

$$\mu_\eta=\mu\pi_\eta^{-1}=\mu(\pi_{\eta\xi}\pi_\xi)^{-1}=(\mu\pi_\xi^{-1})\pi_{\eta\xi}^{-1}=\mu_\xi\pi_{\eta\xi}^{-1}.$$

Accordingly, if $\omega\le\eta\le\delta\le\xi<\omega_1$,

$$\mu_\xi\pi_{\eta\xi}^{-1}[A_{\eta\delta}]=\mu_\eta A_{\eta\delta}=0.$$

Thus in fact $A_\xi=\bigcup\mathcal{A}_\xi$ and $K_\xi\cap L_\xi$ is disjoint from $\pi_{\eta\xi}^{-1}[A_{\eta\delta}]$ whenever $\omega\le\eta\le\delta\le\xi<\omega_1$.

(iii) If $\omega\le\eta\le\delta\le\zeta\le\xi\le\omega_1$ and $\delta<\omega_1$, then

$$\pi_{\zeta\xi}[Q_{\eta\delta\xi}]=\pi_{\zeta\xi}[X_\xi\cap\pi_{\delta\xi}^{-1}[Q_{\eta\delta}]]=\pi_{\zeta\xi}[X_\xi\cap\pi_{\zeta\xi}^{-1}[\pi_{\delta\zeta}^{-1}[Q_{\eta\delta}]]]=X_\zeta\cap\pi_{\delta\zeta}^{-1}[Q_{\eta\delta}]$$

(because $\pi_{\zeta\xi}[X_\xi]=X_\zeta$)

$$=Q_{\eta\delta\zeta}.$$

(iv) If z, $z'\in X$, $\omega\le\eta\le\delta<\omega_1$, $z\restriction\delta=z'\restriction\delta$ and $z\restriction\eta\in A_{\eta\delta}$, then $z'=z$. **P** Suppose that $\delta\le\xi<\omega_1$ and $z\restriction\xi=z'\restriction\xi$. Then $K_\xi\cap L_\xi$ does not meet $\pi_{\eta\xi}^{-1}[A_{\eta\delta}]$, so does not contain $z\restriction\xi$. Accordingly

$$z(\xi)=1\implies z\restriction\xi\in K_\xi\implies z\restriction\xi\notin L_\xi\implies z'\restriction\xi\notin L_\xi\implies z'(\xi)=1,$$

and similarly $z(\xi)=0\Rightarrow z'(\xi)=0$. So an easy induction on ξ shows that $z(\xi)=z'(\xi)$ whenever $\delta\le\xi<\omega_1$, and $z=z'$. **Q**

(d) We come now to the first key idea of this construction. If $\omega\le\eta\le\delta\le\xi\le\omega_1$ and $\delta<\omega_1$, then $g_{\eta\delta\xi}=\pi_{\delta\xi}\restriction Q_{\eta\delta\xi}$ is an irreducible continuous surjection onto $Q_{\eta\delta}$. **P** Of course $g_{\eta\delta\xi}$ is continuous, just because $\pi_{\delta\xi}:\{0,1\}^\xi\to\{0,1\}^\delta$ is continuous, and it is a surjection onto $Q_{\eta\delta}$ by (c-iii) just above. To see that it is irreducible, induce on ξ. At the start, $Q_{\eta\delta\delta}=Q_{\eta\delta}$ and $g_{\eta\delta\delta}$ is an identity function, so is certainly irreducible.

For the inductive step to $\xi+1$, given that $\eta\le\delta\le\xi<\omega_1$ and $g_{\eta\delta\xi}$ is irreducible, consider $h=\pi_{\xi,\xi+1}\restriction Q_{\eta,\delta,\xi+1}$. By (c-iii), $h[Q_{\eta,\delta,\xi+1}]=Q_{\eta\delta\xi}$. Note that $X_{\xi+1}$ can be identified with

$$\{(x,1):x\in X_\xi\cap K_\xi\}\cup\{(x,0):x\in X_\xi\cap L_\xi\}\subseteq\{0,1\}^\xi\times\{0,1\},$$

so that $Q_{\eta,\delta,\xi+1}$ is identified with

$$\{(x,1) : x \in Q_{\eta\delta\xi} \cap K_\xi\} \cup \{(x,0) : x \in Q_{\eta\delta\xi} \cap L_\xi\},$$

and that with this identification h becomes the first-coordinate projection from $Q_{\eta,\delta,\xi+1}$ onto $Q_{\eta\delta\xi}$, while $Q_{\eta\delta\xi} = (Q_{\eta\delta\xi} \cap K_\xi) \cup (Q_{\eta\delta\xi} \cap L_\xi)$. So 5A4L tells us that h is irreducible. But this means that $g_{\eta,\delta,\xi+1} = hg_{\eta\delta\xi}$ is a composition of irreducible continuous surjections and is irreducible, by 5A4C(d-iv)).

For the inductive step to a limit ordinal ξ such that $\delta < \xi \le \omega_1$, take a cylinder set $V \subseteq \{0,1\}^\xi$ meeting $Q_{\eta\delta\xi}$. This time, because ξ is a limit ordinal, there is a ζ such that $\delta \le \zeta < \xi$ and V is determined by coordinates less than ζ. Set $V' = \pi_{\zeta\xi}[V]$; then V' is a cylinder set in $\{0,1\}^\zeta$ meeting $\pi_{\zeta\xi}[Q_{\eta\delta\xi}] = Q_{\eta\delta\zeta}$. Now

$$\begin{aligned}\pi_{\delta\xi}[Q_{\eta\delta\xi} \setminus V] &= \pi_{\delta\zeta}[\pi_{\zeta\xi}[Q_{\eta\delta\xi} \setminus \pi_{\zeta\xi}^{-1}[V']]] \\ &= \pi_{\delta\zeta}[\pi_{\zeta\xi}[Q_{\eta\delta\xi}] \setminus V'] = \pi_{\delta\zeta}[Q_{\eta\delta\zeta} \setminus V'] \subset Q_{\eta\delta}\end{aligned}$$

because $g_{\eta\delta\zeta} : Q_{\eta\delta\zeta} \to Q_{\eta\delta}$ is irreducible. Thus the induction continues. **Q**

(e) It follows that if $H \subseteq X$ is closed, there is a $\xi < \omega_1$ such that $H = X \cap \pi_\xi^{-1}[\pi_\xi[H]]$ and $\pi_\xi \restriction H$ is irreducible. **P** For $\omega \le \eta \le \xi < \omega_1$,

$$\mu_\eta \pi_\eta[H] = \mu_\xi \pi_{\eta\xi}^{-1}[\pi_\eta[H]] = \mu_\xi \pi_{\eta\xi}^{-1}[\pi_{\eta\xi}[\pi_\xi[H]]] \ge \mu_\xi \pi_\xi[H].$$

So we have an $\eta < \omega_1$ such that $\mu_\eta \pi_\eta[H] = \mu_\xi \pi_\xi[H]$ whenever $\eta \le \xi < \omega_1$. Now recall that μ_η is inner regular with respect to $\{Q'_{\eta\delta} : \eta \le \delta < \omega_1\}$. So there is a countable set $I \subseteq \omega_1 \setminus \eta$ such that $\langle Q'_{\eta\delta}\rangle_{\delta\in I}$ is disjoint, $Q'_{\eta\delta} \subseteq \pi_\eta[H]$ for every $\delta \in I$ and $\sum_{\delta\in I} \mu_\eta Q'_{\eta\delta} = \mu_\eta \pi_\eta[H]$.

For each $\delta \in I$,

$$\begin{aligned}\mu_\delta(Q_{\eta\delta} \setminus \pi_\delta[H]) &\le \mu_\delta(\pi_{\eta\delta}^{-1}[Q'_{\eta\delta}] \setminus \pi_\delta[H]) \\ &\le \mu_\delta(\pi_{\eta\delta}^{-1}[\pi_\eta[H]] \setminus \pi_\delta[H]) \\ &= \mu_\delta(\pi_{\eta\delta}^{-1}[\pi_\eta[H]]) - \mu_\delta \pi_\delta[H]\end{aligned}$$

(because surely $\pi_\delta[H] \subseteq \pi_{\eta\delta}^{-1}[\pi_\eta[H]]$)

$$= \mu_\eta \pi_\eta[H] - \mu_\delta \pi_\delta[H] = 0$$

by (c-ii) and the choice of η. Because $Q_{\eta\delta}$ was μ_δ-self-supporting, and $\pi_\delta[H]$ is closed, $Q_{\eta\delta} \subseteq \pi_\delta[H]$. Because $\pi_\delta \restriction X \cap \pi_\delta^{-1}[Q_{\eta\delta}]$ is irreducible, $X \cap \pi_\delta^{-1}[Q_{\eta\delta}] \subseteq H$.

Set $\zeta = \sup(I \cup \{\eta\}) < \omega_1$. Since $Q_{\eta\delta} \subseteq \pi_{\eta\delta}^{-1}[Q'_{\eta\delta}]$, $\pi_{\delta\zeta}^{-1}[Q_{\eta\delta}] \subseteq \pi_{\eta\zeta}^{-1}[Q'_{\eta\delta}]$ for each $\delta \in I$; as $\langle Q'_{\eta\delta}\rangle_{\delta\in I}$ is disjoint, so is $\langle \pi_{\delta\zeta}^{-1}[Q_{\eta\delta}]\rangle_{\delta\in I}$; and

$$\begin{aligned}\sum_{\delta\in I} \mu_\zeta \pi_{\delta\zeta}^{-1}[Q_{\eta\delta}] &= \sum_{\delta\in I} \mu_\delta Q_{\eta\delta} = \sum_{\delta\in I} \mu_\delta \pi_{\eta\delta}^{-1}[Q'_{\eta\delta}] \\ &= \sum_{\delta\in I} \mu_\eta Q'_{\eta\delta} = \mu_\eta \pi_\eta[H] = \mu_\zeta \pi_\zeta[H].\end{aligned}$$

Because $X \cap \pi_\delta^{-1}[Q_{\eta\delta}] \subseteq H$, $\pi_\zeta[H] \supseteq X_\zeta \cap \pi_{\delta\zeta}^{-1}[Q_{\eta\delta}]$ for every $\delta \in I$. So $\pi_\zeta[H] \setminus \bigcup_{\delta\in I} \pi_{\delta\zeta}^{-1}[Q_{\eta\delta}]$ is μ_ζ-negligible and is included in $A_{\zeta\xi}$ for some $\xi \ge \zeta$. Repeating the arguments of the last two sentences at the new level, we see that

$$X_\xi \cap \bigcup_{\delta\in I} \pi_{\delta\xi}^{-1}[Q_{\eta\delta}] \subseteq \pi_\xi[H] \subseteq \bigcup_{\delta\in I} \pi_{\delta\xi}^{-1}[Q_{\eta\delta}] \cup \pi_{\zeta\xi}^{-1} A_{\zeta\xi}.$$

Now suppose that $V \subseteq \{0,1\}^{\omega_1}$ is an open set meeting H. Take $z \in V \cap H$. If $z \in \pi_\zeta^{-1}[A_{\zeta\xi}]$, then $z\restriction\xi \ne z'\restriction\xi$ for any other $z' \in X$, by (c-iv); so $z\restriction\xi \notin \pi_\xi[H \setminus V]$ and $\pi_\xi[H \setminus V] \ne \pi_\xi[H]$. Otherwise, there is a $\delta \in I$ such that $z\restriction\xi \in \pi_{\delta\xi}^{-1}[Q_{\eta\delta}]$, $z \in \pi_\delta^{-1}[Q_{\eta\delta}]$ and $V \cap \pi_\delta^{-1}[Q_{\eta\delta}]$ is not empty. Because $\pi_\delta \restriction X \cap \pi_\delta^{-1}[Q_{\eta\delta}]$ is irreducible, $\pi_\delta[H \setminus V]$ cannot cover $Q_{\eta\delta} \subseteq \pi_\delta[H]$. Thus $\pi_\delta[H \setminus V] \ne \pi_\delta[H]$; it follows at once that $\pi_\xi[H \setminus V] \ne \pi_\xi[H]$ in this case also. As V is arbitrary, $\pi_\xi \restriction H$ is irreducible.

I have still to check that $H = X \cap \pi_\xi^{-1}[\pi_\xi[H]]$. If z, $z' \in X$, $z \in H$ and $z'\restriction\xi = z\restriction\xi$, then if $z \in \pi_\zeta^{-1}[A_{\zeta\xi}]$ we have $z' = z \in H$. Otherwise, there is some $\delta \in I$ such that $z \in \pi_\delta^{-1}[Q_{\eta\delta}]$. In this case, $z'\restriction\delta = z\restriction\delta \in Q_{\eta\delta}$; but $X \cap \pi_\delta^{-1}[Q_{\eta\delta}] \subseteq H$, so again $z' \in H$. So we have the result. **Q**

(f) We are within sight of the end. From (e) we see, first, that if $H \subseteq X$ is closed then it is of the form $X \cap \pi_\xi^{-1}[\pi_\xi[H]]$ for some $\xi < \omega_1$, so is a zero set in X; accordingly X is perfectly normal, therefore first-countable (5A4Cb). Second, for any closed $H \subseteq X$, there is an irreducible continuous surjection from H onto a compact metrizable space $\pi_\xi[H]$; because $\pi_\xi[H]$ is separable, so is H (5A4C(d-i)). It follows that X is hereditarily separable. **P** If $A \subseteq X$, then $\overline{A}$ is separable; let $D \subseteq \overline{A}$ be a countable dense set. Because X is first-countable, each member of $\overline{A}$ is in the closure of a countable subset of A, and there is a countable $C \subseteq A$ such that $D \subseteq \overline{C}$. Now C is a countable dense subset of A. **Q**

(g) We need to check that $\omega_1 \in \mathrm{Mah}_{\mathrm{R}}(X)$. For $\omega \le \xi < \omega_1$, set $U_\xi = \{z : z \in \{0,1\}^{\omega_1},\ z{\restriction}\xi \in K_\xi \cap L_\xi,\ z(\xi) = 1\}$. Then $\mu(U_\xi \triangle E) \ge \frac{1}{4}$ whenever $E \subseteq \{0,1\}^{\omega_1}$ is a Baire set determined by coordinates less than ξ. **P** Set $E' = \pi_\xi[E]$, so that $E = \pi_\xi^{-1}[E']$ and E' is a Baire set. Then

$$\begin{aligned}\mu(U_\xi \setminus E) &= \nu_{\omega_1} f^{-1}[U_\xi \setminus E] = \nu_{\omega_1}\{z : f(z){\restriction}\xi \in K_\xi \cap L_\xi \setminus E',\ f(z)(\xi) = 1\}\\ &= \nu_{\omega_1}\{z : f_\xi(z{\restriction}\xi) \in K_\xi \cap L_\xi \setminus E',\ z(\xi) = 1\}\\ &= \frac{1}{2}\nu_{\omega_1}\{z : f_\xi(z{\restriction}\xi) \in K_\xi \cap L_\xi \setminus E'\} = \frac{1}{2}\mu_\xi(K_\xi \cap L_\xi \setminus E'),\end{aligned}$$

while

$$\begin{aligned}\mu(E \setminus U_\xi) &= \nu_{\omega_1} f^{-1}[E \setminus U_\xi] \ge \nu_{\omega_1} f^{-1}[E \cap \pi_\xi^{-1}[K_\xi \cap L_\xi] \setminus U_\xi]\\ &= \nu_{\omega_1}\{z : f(z){\restriction}\xi \in K_\xi \cap L_\xi \cap E',\ f(z)(\xi) = 0\}\\ &= \nu_{\omega_1}\{z : f_\xi(z{\restriction}\xi) \in K_\xi \cap L_\xi \cap E',\ z(\xi) = 0\}\\ &= \frac{1}{2}\nu_{\omega_1}\{z : f_\xi(z{\restriction}\xi) \in K_\xi \cap L_\xi \cap E'\} = \frac{1}{2}\mu_\xi(K_\xi \cap L_\xi \cap E').\end{aligned}$$

Putting these together,

$$\mu(E \triangle U_\xi) \ge \frac{1}{2}\mu_\xi(K_\xi \cap L_\xi) \ge \frac{1}{4}. \quad \mathbf{Q}$$

In particular, $\mu(U_\eta \triangle U_\xi) \ge \frac{1}{4}$ whenever $\omega \le \eta < \xi < \omega_1$. So μ has uncountable Maharam type. As $\mu X = 1$, the subspace measure μ_X on X also has uncountable Maharam type, and $\omega_1 \in \mathrm{Mah}_{\mathrm{R}}(X)$ (531Ef). Now we have

$$\omega_1 \le \tau(\mu_X) \le w(X)$$

(531Aa)

$$\le w(\{0,1\}^{\omega_1}) = \omega_1,$$

so $\tau(\mu_X) = w(X) = \omega_1$.

(h) Finally, I come to the metrizability of negligible subsets of X. Suppose that $A \subseteq X$ and $\mu_X A = 0$. Then we have a sequence $\langle H_n\rangle_{n\in\mathbb{N}}$ of closed subsets of $X \setminus A$ such that $\lim_{n\to\infty} \mu H_n = 1$. For each $n \in \mathbb{N}$ there is an $\eta_n < \omega_1$ such that $H_n = X \cap \pi_{\eta_n}^{-1}[\pi_{\eta_n}[H_n]]$, by (e); setting $\eta = \max(\omega, \sup_{n\in\mathbb{N}} \eta_n)$, $H_n = X \cap \pi_\eta^{-1}[\pi_\eta[H_n]]$ for every n, so $\pi_\eta^{-1}[\pi_\eta[A]]$ is disjoint from every H_n and is μ-negligible. As $\mu_\eta = \mu\pi_\eta^{-1}$, $\mu_\eta \pi_\eta[A] = 0$. There must therefore be a $\delta \ge \eta$ such that $\pi_\eta[A] \subseteq A_{\eta\delta}$. By (c-iv), $\pi_\delta{\restriction}A$ is injective.

Write B for $\pi_\delta[A]$ and h for the inverse function $(\pi_\delta{\restriction}A)^{-1} : B \to A$. Then $x \mapsto h(x)(\xi) : B \to \{0,1\}$ is continuous for every $\xi < \omega_1$. **P** Induce on ξ. For $\xi < \delta$ the result is trivial, as $x = h(x){\restriction}\delta$ for every $x \in B$. For the inductive step to $\xi \ge \delta$, we have

$$\begin{aligned}\{x : x \in B,\ h(x)(\xi) = 1\} &= \{x : x \in B,\ h(x){\restriction}\xi \notin L_\xi\}\\ &= \{x : x \in B,\ h(x){\restriction}\xi \in K_\xi\},\end{aligned}$$

but $x \mapsto h(x){\restriction}\xi$ is continuous, by the inductive hypothesis, so this is relatively open-and-closed in B. Thus $x \mapsto h(x)(\xi)$ is continuous and the induction continues. **Q**

Accordingly $\pi_\delta{\restriction}A$ and h are the two parts of a homeomorphism between A and $B \subseteq \{0,1\}^\delta$, and A is metrizable. So X and μ_X have all the properties claimed.

531R Returning to the ideas of 531K, we have the following construction.

Lemma Let I be a set, and let $\mathfrak{B}_I$, $\langle e_i\rangle_{i\in I}$, $\langle\phi_i\rangle_{i\in I}$, $\langle\mathfrak{C}_K\rangle_{K\subseteq I}$ and $J^*:\mathfrak{B}_I\to[I]^{\le\omega}$ be as in 531I-531K. For $a\in\mathfrak{B}_I$ and $K\subseteq I$, set $S_K(a)=\operatorname{upr}(a,\mathfrak{C}_K)=\min\{c:a\subseteq c\in\mathfrak{C}_K\}$, the upper envelope of a in $\mathfrak{C}_K$ (313S).

(a) For all $a\in\mathfrak{B}_I$, $i\in I$ and K, $L\subseteq I$,

(i) $S_I(a)=a$,

(ii) $S_L(a)\subseteq S_K(a)$ if $K\subseteq L$,

(iii) $J^*S_K(a)\subseteq J^*(a)\cap K$,

(iv) $S_{I\setminus\{i\}}(a)=a\cup\phi_i a$,

(v) $S_KS_L(a)=S_{K\cap L}(a)$.

(b) Whenever $a\in\mathfrak{B}_I$, $\epsilon>0$ and $m\in\mathbb{N}$, there is a finite $L\subseteq I$ such that $\bar\nu_I(S_K(a)\setminus a)\le\epsilon$ whenever $L\subseteq K\subseteq I$ and $\#(I\setminus K)\le m$.

proof (a)(i) $\mathfrak{C}_I=\mathfrak{B}_I$ contains a.

(ii) If $K\subseteq L$ then $\mathfrak{C}_L\supseteq\mathfrak{C}_K$ contains $S_K(a)$.

(iii) If $i\in I\setminus(J^*(a)\cap K)$ then $S_K(a)\in\mathfrak{C}_K$ so $\phi_iS_K(a)\in\mathfrak{C}_K$, by 531Kg. Also $\phi_iS_K(a)\supseteq a$. **P** If $i\notin K$ then $\phi_iS_K(a)=S_K(a)\supseteq a$, by 531Kf. If $i\notin J^*(a)$ then $\phi_iS_K(a)\supseteq\phi_i a=a$. **Q** So $\phi_iS_K(a)\supseteq S_K(a)$; but they have the same measure, so $\phi_iS_K(a)=S_K(a)$. As i is arbitrary, $J^*S_K(a)\subseteq J^*(a)\cap K$, by 531Kf in the other direction.

(iv) By 531Kf again, $S_{I\setminus\{i\}}(a)=\phi_iS_{I\setminus\{i\}}(a)\supseteq\phi_i a$; so $S_{I\setminus\{i\}}(a)\supseteq a\cup\phi_i a$. On the other hand, by 531Ke, $a\cup\phi_i(a)$ belongs to $\mathfrak{C}_{I\setminus\{i\}}$, so includes $S_{I\setminus\{i\}}(a)$.

(v) By (iii),

$$J^*S_KS_L(a)\subseteq J^*S_L(a)\cap K\subseteq J^*(a)\cap L\cap K,$$

and $S_KS_L(a)\in\mathfrak{C}_{K\cap L}$; since also $S_KS_L(a)\supseteq S_L(a)\supseteq a$, $S_KS_L(a)\supseteq S_{K\cap L}(a)$. On the other hand, $S_{K\cap L}(a)$ belongs to $\mathfrak{C}_K$ and includes $S_L(a)$, so includes $S_KS_L(a)$.

(b) Induce on m. For $m=0$ the result is immediate from (a-i). For the inductive step to $m+1$, take $L_0\in[I]^{<\omega}$ such that $\bar\nu_I(S_K(a)\setminus a)\le\frac{1}{3}\epsilon$ whenever $L_0\subseteq K$ and $\#(I\setminus K)\le m$. By 531Ja, there are a finite set $L_1\subseteq I$ and a $b\in\mathfrak{C}_{L_1}$ such that $\bar\nu_I(a\,\triangle\,b)\le\frac{1}{3}\epsilon$; set $L=L_0\cup L_1$. Suppose $L\subseteq J$ and $\#(I\setminus J)=m+1$; take $i\in I\setminus J$ and set $K=J\cup\{i\}$. Then

$$S_J(a)=S_{I\setminus\{i\}}S_K(a)=S_K(a)\cup\phi_iS_K(a)$$

by (a-v) and (a-iv). So

$$\begin{aligned}\bar\nu_I(S_J(a)\setminus a)&\le\bar\nu_I(S_K(a)\setminus a)+\bar\nu_I(\phi_iS_K(a)\setminus a)\\&\le\frac{\epsilon}{3}+\bar\nu_I\phi_i(S_K(a)\setminus a)+\bar\nu_I(\phi_i a\setminus a)\\&\le\frac{\epsilon}{3}+\bar\nu_I(S_K(a)\setminus a)+\bar\nu_I\phi_i(a\setminus b)+\bar\nu_I(\phi_i b\setminus b)+\bar\nu_I(b\setminus a)\\&\le\frac{2\epsilon}{3}+\bar\nu_I(a\setminus b)+\bar\nu_I(b\setminus a)\le\epsilon\end{aligned}$$

because ϕ_i is a measure-preserving Boolean homomorphism and $\phi_i b=b$. Thus the induction continues.

531S Moving on from hypotheses expressible as statements about measure algebras, we have a further result which can be used when Martin's axiom is true.

Lemma Suppose that $\omega_1<\mathfrak{m}_{\mathrm{K}}$ (definition: 517O). Let $\langle e_\xi\rangle_{\xi<\omega_1}$ be the standard generating family in $\mathfrak{B}_{\omega_1}$, and $\langle a_\xi\rangle_{\xi<\omega_1}$ a family of elements of $\mathfrak{B}_{\omega_1}$ of measure greater than $\frac{1}{2}$. Then there is an uncountable set $\Gamma\subseteq\omega_1$ such that $\inf_{\xi\in I}a_\xi\cap e_\xi$ meets $\inf_{\eta\in J}a_\eta\setminus e_\eta$ whenever I, $J\subseteq\Gamma$ are finite and disjoint.

proof (a) Define $J^*(a)$, for $a\in\mathfrak{B}_{\omega_1}$, and $S_I(a)$, for $a\in\mathfrak{B}_{\omega_1}$ and $I\subseteq\omega_1$, as in 531J and 531R. Let P be the set of pairs (c,I) where $I\subseteq\omega_1$ is finite, $0\ne c\subseteq\inf_{\xi\in I}a_\xi$ and $I\cap J^*(c)=\emptyset$. Order P by saying that

$(c,I)\le(c',I')$ if $I\subseteq I'$ and $c'\subseteq c$. Then P is a partially ordered set. For each $\xi<\omega_1$, $a_\xi\cap\phi_\xi a_\xi$ belongs to $\mathfrak{C}_{\kappa\setminus\{\xi\}}$ (531Je) and is non-zero, so $p_\xi=(a_\xi\cap\phi_\xi a_\xi,\{\xi\})$ belongs to P. The point of the proof is the following fact.

(b) P satisfies Knaster's condition upwards. **P** Let $\langle(c_\xi,I_\xi)\rangle_{\xi<\omega_1}$ be a family in P. Then there are an $\alpha>0$ and an uncountable $A_0\subseteq\omega_1$ such that $\bar\nu_{\omega_1}(c_\xi\cap c_\eta)\ge\alpha$ for all ξ, $\eta\in A_0$ (525Tc). Next, there is an uncountable $A_1\subseteq A_0$ such that $\langle I_\xi\rangle_{\xi\in A_1}$ is a Δ-system with root I say (4A1Db); let $m\in\mathbb{N}$ be such that $A_2=\{\xi:\xi\in A_1,\ \#(I_\xi\setminus I)=m\}$ is uncountable. Finally, because $J^*(c_\eta)$ is countable for each η, and $\langle I_\xi\setminus I\rangle_{\xi\in A_2}$ is disjoint, we can find an uncountable $A_3\subseteq A_2$ such that $J^*(c_\eta)\cap I_\xi\setminus I=\emptyset$ whenever η, $\xi\in A_3$ and $\eta<\xi$.

Take a strictly increasing sequence $\langle\eta_k\rangle_{k\in\mathbb{N}}$ in A_3 and a $\zeta\in A_3$ greater than every η_k. By 531Rb, there is a finite set $K\subseteq\omega_1$ such that $\bar\nu_{\omega_1}(S_J(1\setminus c_\zeta)\setminus(1\setminus c_\zeta))<\alpha$ whenever $K\subseteq J\subseteq\omega_1$ and $\#(\omega_1\setminus J)=m$. Let $k\in\mathbb{N}$ be such that $I_{\eta_k}\setminus I$ does not meet K. Set $c'_\zeta=S_{\kappa\setminus(I_{\eta_k}\setminus I)}(1\setminus c_\zeta)$. Then

$$\bar\nu_{\omega_1}(c'_\zeta\cap c_\zeta)\le\alpha<\bar\nu_{\omega_1}(c_\zeta\cap c_{\eta_k}),$$

so $c=c_{\eta_k}\setminus c'_\zeta$ is non-zero; as $c'_\zeta\supseteq 1\setminus c_\zeta$, $c\subseteq c_\zeta$. Set $L=I_{\eta_k}\cup I_\zeta$. Then $J^*(c_{\eta_k})$ is disjoint from I_{η_k} and from $I_\zeta\setminus I$, by the choice of A_3, while

$$J^*(c'_\zeta)\subseteq J^*(1\setminus c_\zeta)\setminus(I_{\eta_k}\setminus I)=J^*(c_\zeta)\setminus(I_{\eta_k}\setminus I)$$

(531R(a-iii)) is also disjoint from L; so $J^*(c)\subseteq J^*(c_{\eta_k})\cup J^*(c'_\zeta)$ is disjoint from L. Finally,

$$c\subseteq c_{\eta_k}\cap c_\zeta\subseteq\inf_{\xi\in I_{\eta_k}}a_\xi\cap\inf_{\xi\in I_\zeta}a_\xi=\inf_{\xi\in L}a_\xi,$$

so $(c,L)\in P$. Now (c,L) dominates both (c_{η_k},I_{η_k}) and (c_ζ,I_ζ).

What this shows is that if we write Q for

$$\{\{\eta,\zeta\}:\eta,\ \zeta\in A_3,\ (c_\eta,I_\eta)\text{ and }(c_\zeta,I_\zeta)\text{ are compatible upwards in }P\},$$

then whenever $\zeta\in A_3$ and $M\subseteq A_3\cap\zeta$ is infinite there is an $\eta\in M$ such that $\{\eta,\zeta\}\in Q$. By 5A1Hb, there is an uncountable $A_4\subseteq A_3$ such that $[A_4]^2\subseteq Q$, that is, $\langle(c_\xi,I_\xi)\rangle_{\xi\in A_4}$ is upwards-linked. As $\langle(c_\xi,I_\xi)\rangle_{\xi<\omega_1}$ is arbitrary, P satisfies Knaster's condition upwards. **Q**

(c) By 517S, there is a sequence $\langle R_n\rangle_{n\in\mathbb{N}}$ of upwards-directed subsets of P covering $\{p_\xi:\xi<\omega_1\}$; as ω_1 is uncountable, there must be some n such that $\Gamma=\{\xi:p_\xi\in R_n\}$ has cardinal ω_1. In this case, $\{p_\xi:\xi\in\Gamma\}$ is upwards-centered in P. If I, $J\subseteq\Gamma$ are finite and disjoint, then there must be a $(c,K)\in P$ which is an upper bound for $\{p_\xi:\xi\in I\cup J\}$; now $I\cup J\subseteq K$ does not meet $J^*(c)$, while $c\subseteq\inf_{\xi\in I\cup J}a_\xi$. But this means that

$$\bar\nu_{\omega_1}(c\cap\inf_{\xi\in I}e_\xi\cap\inf_{\eta\in J}(1\setminus e_\eta))=2^{-\#(I\cup J)}\bar\nu_{\omega_1}c>0,$$

and

$$0\ne c\cap\inf_{\xi\in I}e_\xi\cap\inf_{\eta\in J}(1\setminus e_\eta)\subseteq\inf_{\xi\in I}(a_\xi\cap e_\xi)\cap\inf_{\eta\in J}(a_\eta\setminus e_\eta).$$

So we have a set Γ of the kind required.

531T Theorem (Fremlin 97) Suppose that $\omega\le\kappa<\mathfrak{m}_{\mathrm{K}}$. If X is a Hausdorff space and $\kappa\in\mathrm{Mah}_{\mathrm{R}}(X)$, then $\{0,1\}^\kappa$ is a continuous image of a compact subset of X.

proof (a) Because $\kappa<\mathfrak{m}_{\mathrm{K}}\le\mathfrak{m}(\mathfrak{A})$ for every probability algebra $\mathfrak{A}$ (525Tb), κ is a measure-precaliber of all probability algebras (525Fb).

(b) If $\kappa=\omega$, X has a compact subset K which is not scattered (531E(e-ii), $[0,1]$ is a continuous image of K (4A2G(j-iv) again) and $\{0,1\}^{\mathbb{N}}$ is a continuous image of a closed subset of K (using 4A2Uc).

(c) If $\kappa=\omega_1$, let K be a compact subset of X such that $\omega_1\in\mathrm{Mah}_{\mathrm{R}}(K)$, μ a Maharam-type-homogeneous Radon probability measure on K with Maharam type ω_1, and $(\mathfrak{A},\bar\mu)$ its measure algebra. Let $\langle d_\xi\rangle_{\xi<\omega_1}$ be a τ-generating stochastically independent family of elements of $\mathfrak{A}$ of measure $\frac{1}{2}$. For $\xi<\omega_1$ let $E_\xi\in\operatorname{dom}\mu$ be such that $E_\xi^\bullet=d_\xi$, and $K'_\xi\subseteq E_\xi$, $K''_\xi\subseteq K\setminus E_\xi$ compact sets of measure greater than $\frac{1}{4}$; set $K_\xi=K'_\xi\cup K''_\xi$ and $a_\xi=K_\xi^\bullet$ in $\mathfrak{A}$. Because $(\mathfrak{A},\bar\mu,\langle d_\xi\rangle_{\xi<\omega_1})$ is isomorphic to $(\mathfrak{B}_{\omega_1},\bar\nu_{\omega_1},\langle e_\xi\rangle_{\xi<\omega_1})$, 531S tells us that there is an uncountable set $\Gamma\subseteq\omega_1$ such that

$$0 \neq \inf_{\xi\in I}(a_\xi \cap d_\xi) \cap \inf_{\eta\in J}(a_\eta \setminus d_\eta) = (K \cap \bigcap_{\xi\in I} K'_\xi \cap \bigcap_{\eta\in J} K''_\eta)^\bullet$$

whenever I, $J \subseteq \Gamma$ are finite. Just as in part (b) of the proof of 531L, it follows that there is a continuous surjection from $\bigcap_{\xi\in\Gamma} K_\xi$ onto $\{0,1\}^\Gamma \cong \{0,1\}^{\omega_1}$.

(d) If $\kappa \geq \omega_2$, then 531Lb gives the result.

531U If we are willing to settle for weaker conclusions, there are versions of 531L which do not call for any information on precalibers.

Proposition Let X be a Hausdorff space.

(a) Give the space $P_{\mathrm{R}}(X)$ of Radon probability measures on X its narrow topology (437J). If $\kappa \geq \omega_2$ belongs to $\mathrm{Mah}_{\mathrm{R}}(X)$, then $\{0,1\}^\kappa$ is a continuous image of a compact subset of $P_{\mathrm{R}}(X)$.

(b) Give the space $P_{\mathrm{R}}(X \times X)$ its narrow topology. Then its tightness $t(P_{\mathrm{R}}(X \times X))$ is at least $\sup \mathrm{Mah}_{\mathrm{R}}(X)$.

proof (a)(Plebanek 02)**(i)** The argument begins by copying part of the proof of 531Lb. By 531Ec, there is a compact set $K \subseteq X$ such that $\kappa \in \mathrm{Mah}_{\mathrm{R}}(K)$. Let μ be a Maharam-type-homogeneous Radon probability measure on K with Maharam type κ, Σ its domain, and $(\mathfrak{A}, \bar\mu)$ its measure algebra. Let $\langle e_\xi\rangle_{\xi<\kappa}$ be a stochastically independent τ-generating set of elements of measure $\frac{1}{2}$ in $\mathfrak{A}$. For each $\xi < \kappa$, let $E_\xi \in \Sigma$ be such that $E^\bullet_\xi = e_\xi$ in $\mathfrak{A}$; let $K'_\xi \subseteq E_\xi$, $K''_\xi \subseteq K \setminus E_\xi$ be compact sets of measure at least $\frac{5}{12}$. By 531K, copied into $\mathfrak{A}$, there are $\langle c_\xi\rangle_{\xi<\kappa}$ and $\Gamma \in [\kappa]^\kappa$ such that $c_\xi \subseteq (K'_\xi \cup K''_\xi)^\bullet$ and $\bar\mu c_\xi \geq \frac{2}{3}$ for each ξ, and

$$\bar\mu(\inf_{\xi\in I} c_\xi \cap e_\xi \cap \inf_{\eta\in J} c_\eta \setminus e_\eta) = \frac{1}{2^{\#(I\cup J)}}\bar\mu(\inf_{\xi\in I\cup J} c_\xi)$$

whenever I, $J \subseteq \Gamma$ are disjoint finite sets. In particular, $\inf_{\xi\in I} c_\xi \cap e_\xi$ meets $\inf_{\eta\in J} c_\eta \setminus e_\eta$ whenever I, $J \subseteq \Gamma$ are disjoint finite sets and $\inf_{\xi\in I\cup J} c_\xi \neq 0$. But as $c_\xi \cap e_\xi \subseteq (K'_\xi)^\bullet$ and $c_\xi \setminus e_\xi \subseteq (K''_\xi)^\bullet$ for every ξ, we see that $\bigcap_{\xi\in I} K'_\xi \cap \bigcap_{\xi\in J} K''_\xi \neq \emptyset$ whenever I, $J \subseteq \Gamma$ are disjoint finite sets and $\inf_{\xi\in I\cup J} c_\xi \neq 0$.

(ii) Now for the new idea. For each $I \subseteq \Gamma$, set

$$L_I = \{\nu : \nu \in P_{\mathrm{R}}(K),\, \nu K'_\xi \geq \frac{2}{3} \text{ for } \xi \in I \text{ and } \nu K''_\xi \geq \frac{2}{3} \text{ for } \xi \in \Gamma \setminus I\}.$$

Then $L_I \neq \emptyset$. **P** Consider the families $\{c_\xi : \xi \in \Gamma\} \subseteq \mathfrak{A}$ and $\{K'_\xi : \xi \in I\} \cup \{K''_\xi : \xi \in \Gamma \setminus I\} \subseteq \mathcal{P}X$. Because $\bar\mu : \mathfrak{A} \to [0,1]$ is additive and $\bar\mu c_\xi \geq \frac{2}{3}$ for every $\xi \in \Gamma$, the intersection number of $\{c_\xi : \xi \in \Gamma\}$ must be at least $\frac{2}{3}$ (391I). So if $\xi_0, \dots, \xi_n \in \Gamma$ there is a set $J \subseteq n+1$ such that $\#(J) \geq \frac{2}{3}(n+1)$ and $\inf_{j\in J} c_{\xi_j} \neq 0$. In this case, setting $J' = \{j : j \in J, \xi_j \in I\}$ and $J'' = J \setminus J'$ we have $\bigcap_{j\in J'} K'_{\xi_j} \cap \bigcap_{j\in J''} K''_{\xi_j} \neq \emptyset$. As $\xi_0, \dots, \xi_n$ are arbitrary, $\{K'_\xi : \xi \in I\} \cup \{K''_\xi : \xi \in \Gamma \setminus I\}$ has intersection number at least $\frac{2}{3}$.

By 391I in the other direction, there is an additive functional $\tilde\nu : \mathcal{P}K \to [0,1]$ such that $\tilde\nu K = 1$, $\tilde\nu K'_\xi \geq \frac{2}{3}$ for every $\xi \in I$ and $\tilde\nu K''_\xi \geq \frac{2}{3}$ for every $\xi \in \Gamma \setminus I$. By 416K, there is a Radon measure ν' on K such that $\nu' K'_\xi \geq \frac{2}{3}$ for every $\xi \in I$, $\nu' K''_\xi \geq \frac{2}{3}$ for every $\xi \in \Gamma \setminus I$, and $\nu' K \leq 1$. Setting $\nu = \frac{1}{\nu' K}\nu'$, we see that $\nu' \in L_I$. **Q**

(iii) Set

$$L = \bigcup_{I\subseteq\Gamma} L_I = \bigcap_{\xi\in\Gamma}(\{\nu : \nu \in P_{\mathrm{R}}(K),\, \nu K'_\xi \geq \frac{2}{3}\} \cup \{\nu : \nu \in P_{\mathrm{R}}(K),\, \nu K''_\xi \geq \frac{2}{3}\})$$

$$= P_{\mathrm{R}}(K) \setminus \bigcup_{\xi\in\Gamma}(\{\nu : \nu(K \setminus K'_\xi) > \frac{1}{3}\} \cap \{\nu : \nu(K \setminus K''_\xi) > \frac{1}{3}\}).$$

Then L is closed in $P_{\mathrm{R}}(K)$ for the narrow topology. Since all the sets $\{\nu : \nu K'_\xi \geq \frac{2}{3}\}$ and $\{\nu : \nu K''_\xi \geq \frac{2}{3}\}$ are closed in $P_{\mathrm{R}}(K)$, we have a continuous function $f : L \to \{0,1\}^\Gamma$ defined by saying that

$$f(\nu)(\xi) = 1 \text{ if } \nu K'_\xi \geq \frac{2}{3}$$

$$= 0 \text{ if } \nu K''_\xi \geq \frac{2}{3},$$

and (ii) tells us that this is surjective.

(iv) Recall now that the compact space $P_{\mathrm{R}}(K)$ can be identified with the subspace $\{\mu:\mu(X\setminus K)=0\}$ of $P_{\mathrm{R}}(X)$ (use 437Nb). So $\{0,1\}^\kappa\cong\{0,1\}^\Gamma$ is a continuous image of a compact subset of $P_{\mathrm{R}}(X)$.

(b) Take $\kappa\in\mathrm{Mah}_{\mathrm{R}}(X)$, and a Maharam homogeneous probability measure μ on X with Maharam type κ; write Σ for the domain of μ. I need to show that $\kappa\le t(P_{\mathrm{R}}(X\times X))$.

(i)(Plebanek & Sobota 15) To begin with (down to the end of (iii) below), suppose that X is compact and that $\kappa=\omega_1$. In $P_{\mathrm{R}}(X\times X)$ let L be the set of measures with both marginals equal to μ. By 437N(a-i), L is compact. If E, $F\in\Sigma$, $\nu\mapsto\nu(E\times F):L\to\mathbb{R}$ is continuous. **P** Take $\nu_0\in L$ and $\epsilon>0$. Then there are open $G\supseteq E$, $H\supseteq F$ such that

$$\begin{aligned}\epsilon&\ge\mu(G\setminus E)+\mu(H\setminus F)\\&=\nu((G\setminus E)\times X)+\nu(X\times(H\setminus F))\ge\nu((G\times H)\setminus(E\times F))\end{aligned}$$

for every $\nu\in L$. Now $U=\{\nu:\nu\in P_{\mathrm{R}}(X\times X),\ \nu(G\times H)>\nu_0(G\times H)-\epsilon\}$ is a neighbourhood of ν_0 (437Kd), and if $\nu\in U\cap L$ then

$$\nu(E\times F)\ge\nu(G\times H)-\epsilon\ge\nu_0(G\times H)-2\epsilon\ge\nu_0(E\times F)-2\epsilon.$$

Similarly, there are neighbourhoods V, W of ν_0 such that

$$\nu((X\setminus E)\times X)\ge\nu_0((X\setminus E)\times X)-2\epsilon\text{ for every }\nu\in V\cap L,$$

$$\nu(E\times(X\setminus F))\ge\nu_0(E\times(X\setminus F))-2\epsilon\text{ for every }\nu\in W\cap L,$$

But now we see that

$$\nu(E\times F)\le\nu_0(E\times F)+4\epsilon\text{ for every }\nu\in V\cap W\cap L,$$

$$|\nu(E\times F)-\nu_0(E\times F)|\le4\epsilon\text{ for every }\nu\in U\cap V\cap W\cap L.$$

As ν_0 and ϵ are arbitrary, $\nu\mapsto\nu(E\times F)$ is continuous on L. **Q**

(ii) Choose $\langle E_\xi\rangle_{\xi<\omega_1}$, $\langle G_\xi\rangle_{\xi<\omega_1}$ inductively in such a way that for each $\xi<\omega_1$

$$E_\xi\in\Sigma,\quad\mu E_\xi=\tfrac12,$$

$$E_\xi\text{ is independent of the }\sigma\text{-algebra generated by }\{G_\eta:\eta<\xi\}\cup\{E_\eta:\eta<\xi\},$$

$$G_\xi\subseteq X\text{ is open},\quad E_\xi\subseteq G_\xi,\quad\mu G_\xi\le\tfrac35.$$

Now whenever $\xi<\omega_1$ and $I\in[\omega_1\setminus\xi]^{<\omega}$ there is a $\nu_{\xi I}\in L$ such that $\nu_{\xi I}(G_\eta\times G_\eta)\le\frac{9}{25}$ for every $\eta<\xi$ and $\nu_{\xi I}(G_\zeta\times G_\zeta)\ge\frac12$ for every $\zeta\in I$. **P** Set $n=\#(I)$; let $\mathcal{A}$ be the set of atoms of the algebra generated by $\{E_\eta:\eta\in I\}$, $V=\bigcup_{A\in\mathcal{A}}A\times A$ and $\nu_{\xi I}=2^n\mu^2\llcorner V$ (234M) where μ^2 is the Radon product measure $\mu\times\mu$, so that $\mu A=2^{-n}$ for $A\in\mathcal{A}$, $\mu^2V=2^{-n}$ and $\nu_{\xi I}$ is a Radon probability measure on $X\times X$ (416Sa). If $F\in\operatorname{dom}\mu$ then

$$\begin{aligned}\nu_{\xi I}(F\times X)&=2^n\sum_{A\in\mathcal{A}}\mu^2(A^2\cap(F\times X))\\&=2^n\sum_{A\in\mathcal{A}}\mu(A\cap F)\cdot\mu A=\sum_{A\in\mathcal{A}}\mu(A\cap F)=\mu F\end{aligned}$$

and similarly $\nu_{\xi I}(X\times F)=\mu F$; thus $\nu_{\xi I}\in L$. If $\eta<\xi$ then

$$\nu_{\xi I}G_\eta^2=2^n\sum_{A\in\mathcal{A}}(\mu(A\cap G_\eta))^2=2^n\sum_{A\in\mathcal{A}}(\mu A\cdot\mu G_\eta)^2$$

(because G_η and $\langle E_\zeta\rangle_{\zeta\in I}$ are independent)

$$=(\mu G_\eta)^2\le\frac{9}{25}.$$

If $\zeta \in I$ then

$$\nu_{\xi I} G_\zeta^2 \geq \nu_{\xi I} E_\zeta^2 = 2^n \sum_{A \in \mathcal{A}} (\mu(A \cap E_\zeta))^2 = 2^n \sum_{A \in \mathcal{A}, A \subseteq E_\zeta} (\mu A)^2$$

$$= 2^{-n} \#(\{A : A \in \mathcal{A}, A \subseteq E_\zeta\}) = \frac{1}{2}.$$

So $\nu_{\xi I}$ works. **Q**

(iii) Now, for each $\xi < \omega_1$, we can choose $\nu_\xi \in \bigcap_{\xi \leq \zeta < \omega_1} \overline{\{\nu_{\xi I} : \zeta \in I \in [\omega_1 \setminus \xi]^{<\omega}\}}$. Because L is closed, $\nu_\xi \in L$; because $\nu \mapsto \nu E^2 : L \to \mathbb{R}$ is continuous whenever $E \in \Sigma$, by (i), $\nu_\xi G_\eta^2 \leq \frac{9}{25}$ for $\eta < \xi$ and $\nu_\xi G_\zeta^2 \geq \frac{1}{2}$ for $\zeta \geq \xi$. Next, there is a $\nu \in \bigcap_{\eta < \omega_1} \overline{\{\nu_\xi : \eta < \xi < \omega_1\}}$, and now $\nu G_\eta^2 \leq \frac{9}{25}$ for every $\eta < \omega_1$. Writing $D = \{\nu_\xi : \xi < \omega_1\} \subseteq L$, $\nu \in \overline{D}$; but any countable $C \subseteq D$ is included in $\{\nu_\xi : \xi \leq \zeta\}$ for some $\zeta < \omega_1$, so that

$$\sup\{\nu' G_\zeta^2 : \nu' \in C\} \leq \frac{9}{25} < \frac{1}{2} \leq \inf\{\nu_\xi G_\zeta^2 : \xi \leq \zeta\} \leq \nu G_{\zeta^2}$$

and $\nu \notin \overline{C}$. So D and ν witness that $t(P_{\mathrm{R}}(X \times X)) \geq \omega_1$.

(iv) Still supposing that X is compact, consider other possibilities for κ. If $\kappa = 0$ then of course $\kappa \leq t(P_{\mathrm{R}}(X \times X))$. If $\kappa = \omega$, then X has a compact subset which is not scattered (531Ee), so X has a point x which is not isolated; setting $A = X \setminus \{x\}$, $x \in \overline{A} \setminus \overline{B}$ whenever $B \in [A]^{<\omega}$, so $\omega \leq t(X) \leq t(X \times X) \leq t(P_{\mathrm{R}}(X \times X))$ by 5A4Bb and 437Jd. If $\kappa \geq \omega_2$, then

$$\kappa = t(\{0,1\}^\kappa)$$

(by 5A4I(b-iii))

$$\leq t(P_{\mathrm{R}}(X))$$

(by (a) above and 5A4Bb)

$$\leq t(P_{\mathrm{R}}(X \times X))$$

because $P_{\mathrm{R}}(X)$ is homeomorphic to a subspace of $P_{\mathrm{R}}(X \times X)$, by 437Nb.

(v) This deals with the case in which X is compact. For the general case, we see from 531Ec that there is a compact set $K \subseteq X$ such that $\kappa \in \mathrm{Mah}_{\mathrm{R}}(K)$, so (i)-(iv) tell us that $t(P_{\mathrm{R}}(K \times K)) \geq \kappa$. But $P_{\mathrm{R}}(K \times K)$ can be identified with a subset of $P_{\mathrm{R}}(X \times X)$ (437Nb) and $t(P_{\mathrm{R}}(X \times X)) \geq t(P_{\mathrm{R}}(K \times K)) \geq \kappa$.

531V 531Lb and 531U both depend on Hajnal's Free Set Theorem (5A1Ic), which here is useful when dealing with cardinals greater than or equal to ω_2. More elementary arguments, as in 531La, give similar results for ω, leaving ω_1 exposed as a special case. In fact it really is different in this context, as is shown by the following.

Proposition (a) Suppose that the continuum hypothesis is true. Then there is a compact Hausdorff space X such that $\omega_1 \in \mathrm{Mah}_{\mathrm{R}}(X)$ but $\{0,1\}^{\omega_1}$ is not a continuous image of a closed subset of $P_{\mathrm{R}}(X)$.

(b)(PLEBANEK 97) Suppose that there is a family $\langle W_\xi \rangle_{\xi < \omega_1}$ in $\mathcal{N}_{\omega_1}$ such that every closed subset of $\{0,1\}^{\omega_1} \setminus \bigcup_{\xi < \omega_1} W_\xi$ is scattered. Then there is a compact Hausdorff space X such that $\omega_1 \in \mathrm{Mah}_{\mathrm{R}}(X)$ but $\{0,1\}^{\omega_1}$ is not a continuous image of a closed subset of X.

proof (a)(i) In fact this is witnessed by the space X described in 531Q. (Since we are assuming that $\mathfrak{c} = \omega_1$, we certainly have $\mathrm{cf}\,\mathcal{N}_\omega = \omega_1$, so we can perform the construction in 531Q.) For the present argument, all we need to know is that X is a compact Hausdorff space of weight at most ω_1 carrying a Radon probability measure with uncountable Maharam type for which every negligible subset is separable and metrizable.

Let $\hat{\mu}$ be such a measure. Then

$$\omega_1 \leq \tau(\hat{\mu}) \leq w(X) \leq \omega_1$$

by 511Gc and 531Aa. Next, by 524Pf (or directly from the construction in 531Q), the cofinality of the null ideal $\mathcal{N}(\hat{\mu})$ is $\max(\mathrm{cf}\,\mathcal{N}_\omega, \mathrm{cf}[\omega_1]^{\leq\omega}) = \omega_1$; let $\langle H_\xi \rangle_{\xi < \omega_1}$ be a cofinal family in $\mathcal{N}(\hat{\mu})$ consisting of Borel sets.

(ii) Write $\mathcal{B}$ for the Borel σ-algebra of X, M for the set of totally finite Borel measures on X which are absolutely continuous with respect to $\mu = \hat{\mu}\restriction\mathcal{B}$, and for $\xi < \omega_1$ write M_ξ for the set of totally finite Borel measures on X for which $X \setminus H_\xi$ is negligible.

(α) $\#(M) \le \mathfrak{c}$. **P** For a non-negative μ-integrable real-valued function f write ν_f for the corresponding indefinite-integral measure, so that $\nu_f E = \int f \times \chi E \, d\mu$ for $E \in \mathcal{B}$. By the Radon-Nikodým theorem (232F), every member of M is expressible as ν_f for some f; moreover, if $f =_{\text{a.e.}} g$ then $\nu_f = \nu_g$. So $\#(M) \le \#(L^1(\mu))$. Now $L^1(\mu)$ can be identified with $L^1(\mathfrak{A}, \bar{\mu})$, where $(\mathfrak{A}, \bar{\mu})$ is the measure algebra of μ (365B); since $\hat{\mu}$ is the completion of μ, we can identify $(\mathfrak{A}, \bar{\mu})$ with the measure algebra of $\hat{\mu}$ (322Da) and apply 529Ba to see that the topological density of $L^1(\mathfrak{A}, \bar{\mu})$ is ω_1. Since $L^1(\mathfrak{A}, \bar{\mu})$ is metrizable,

$$\#(M) \le \#(L^1(\mu)) = \#(L^1(\mathfrak{A}, \bar{\mu})) \le d(L^1(\mathfrak{A}, \bar{\mu}))^\omega = \omega_1^\omega = \mathfrak{c}. \text{ } \mathbf{Q}$$

(β) For every $\xi < \omega_1$, $\#(M_\xi) \le \mathfrak{c}$. **P** A member of M_ξ is determined by its restriction to the Borel σ-algebra of H_ξ. Now H_ξ is separable and metrizable, therefore second-countable, and its topology has a countable base $\mathcal{U}_\xi$ containing H_ξ and closed under finite intersections. If ν, ν' are different totally finite Borel measures on H_ξ, then $\nu\restriction\mathcal{U}_\xi \ne \nu'\restriction\mathcal{U}_\xi$ by the Monotone Class Theorem (136C again), so the same is true if ν, ν' are different members of M_ξ, and

$$\#(M_\xi) \le \#(\mathbb{R}^{\mathcal{U}_\xi}) = \mathfrak{c}. \text{ } \mathbf{Q}$$

(γ) Every totally finite Borel measure ν on X can be expressed as a sum $\nu' + \nu''$ where $\nu' \in M$ and $\nu'' \in \bigcup_{\xi<\omega_1} M_\xi$. **P** By 232Ia, we can express ν as $\nu' + \nu''$ where ν', ν'' are countably additive, $\nu' \in M$ and ν'' is singular with respect to μ. There is a a Borel set F such that $\mu F = \nu''(X \setminus F) = 0$; and now there is a $\xi < \omega_1$ such that $F \subseteq H_\xi$, so that $\nu'' \in M_\xi$ and we have found a suitable expression for ν. **Q**

(iii) Of course every member of $P_{\text{R}}(X)$ is determined by its restriction to $\mathcal{B}$. We therefore have

$$\#(P_{\text{R}}(X)) \le \#(\bigcup_{\xi<\omega_1} M \times M_\xi) \le \max(\omega_1, \sup_{\xi<\omega_1} \#(M) \cdot \#(M_\xi))$$

(taking the cardinal products)

$$= \mathfrak{c} = \omega_1 < 2^{\omega_1} = \#(\{0,1\}^{\omega_1}).$$

So there cannot possibly be a continuous surjection from any subset of $P_{\text{R}}(X)$ onto $\{0,1\}^{\omega_1}$.

(b) For the second example we can use a variation in the construction in 531M.

(i) Set $E_\xi = \{z : z \in \{0,1\}^{\omega_1}, z(\xi) = 1\}$ for each $\xi < \omega_1$. Choose a family $\langle K_{\xi n}\rangle_{\xi<\omega_1, n\in\mathbb{N}}$ of compact sets in $\{0,1\}^{\omega_1}$ as follows. Given $\langle K_{\eta n}\rangle_{\eta<\xi, n\in\mathbb{N}}$, where $\xi < \omega_1$, such that $\bigcup_{n\in\mathbb{N}} K_{\eta n}$ is conegligible for every $\eta < \xi$, then for each $j \in \mathbb{N}$ we can find a family $\langle n(\xi, \eta, j)\rangle_{\eta<\xi}$ in $\mathbb{N}$ such that $L_{\xi j} = \bigcap_{\eta<\xi} \bigcup_{i\le n(\xi,\eta,j)} K_{\eta i}$ has measure at least $1 - 2^{-j-4}$. For $j \in \mathbb{N}$ choose a compact set $K'_{\xi j} \subseteq L_{\xi j} \setminus (W_\xi \cup \bigcup_{i<j} K'_{\xi i})$ of measure at least $1 - 2^{-j-3} - \nu_{\omega_1}(\bigcup_{i<j} K'_{\xi i})$. Set

$$K_{\xi,2i} = K'_{\xi i} \cap E_\xi, \quad K_{\xi,2i+1} = K'_{\xi i} \setminus E_\xi$$

for each $i \in \mathbb{N}$, and continue. Note that if $\xi < \omega_1$, $i \in \mathbb{N}$ and z, $z' \in K_{\xi i}$ then $z(\xi) = z'(\xi)$.

(ii) At the end of the induction, let $\mathfrak{C}$ be the algebra of subsets of $\{0,1\}^{\omega_1}$ generated by $\{K_{\xi i} : \xi < \omega_1, i \in \mathbb{N}\}$, and X its Stone space. Then we have a Radon probability measure μ on X defined by setting $\mu\widehat{C} = \nu_{\omega_1} C$ for every $C \in \mathfrak{C}$, where $\widehat{C}$ is the open-and-closed subset of X corresponding to C. For $\eta < \xi < \omega_1$, we have

$$\begin{aligned}
\mu(\widehat{K}_{\eta 0} \triangle \widehat{K}_{\xi 0}) &= \nu_{\omega_1}(K_{\eta 0} \triangle K_{\xi 0}) \\
&= \nu_{\omega_1}((E_\eta \cap K'_{\eta 0}) \triangle (E_\xi \cap K'_{\xi 0})) \\
&\ge \nu_{\omega_1}(E_\eta \triangle E_\xi) - \nu_{\omega_1}(E_\eta \setminus K'_{\eta 0}) - \nu_{\omega_1}(E_\xi \setminus K'_{\xi 0}) \\
&\ge \frac{1}{2} - \frac{1}{8} - \frac{1}{8} = \frac{1}{4},
\end{aligned}$$

so the Maharam type of μ is at least ω_1 and $\omega_1 \in \mathrm{Mah}_{\mathrm{R}}(X)$.

(iii) Let $F \subseteq X$ be a non-scattered closed set. Then there is a $\zeta < \omega_1$ such that $F \not\subseteq \bigcup_{i\in\mathbb{N}} \widehat{K}_{\zeta i}$. **P?** Otherwise, set

$$R = \bigcap_{\xi<\omega_1} \bigcup_{i\in\mathbb{N}} (F \cap \widehat{K}_{\xi i}) \times K_{\xi i} \subseteq X \times \{0,1\}^{\omega_1}.$$

Note that for each $\xi < \omega_1$ the $\widehat{K}_{\xi i}$ are disjoint open-and-compact sets covering the compact set F, so $\{i : F \cap \widehat{K}_{\xi i} \neq \emptyset\}$ is finite and $\bigcup_{i\in\mathbb{N}}(F \cap \widehat{K}_{\xi i}) \times K_{\xi i}$ is compact; thus R is compact. If (x,z) and $(x',z') \in R$ and $x \neq x'$, there must be some $C \in \mathfrak{C}$ such that $x \in \widehat{C}$ and $x' \notin \widehat{C}$, so there must be some $\xi < \omega_1$ and $i \in \mathbb{N}$ such that just one of x, x' belongs to $\widehat{K}_{\xi i}$; in this case, only the corresponding one of z, z' can belong to $K_{\xi i}$, and $z \neq z'$.

Conversely, if (x,z) and $(x',z') \in R$ and $z \neq z'$, there is some ξ such that $z(\xi) \neq z'(\xi)$. In this case, if i, $j \in \mathbb{N}$ are such that $(x,z) \in \widehat{K}_{\xi i} \times K_{\xi i}$ and $(x',z') \in \widehat{K}_{\xi j} \times K_{\xi j}$, $i \neq j$ and $x \neq x'$.

This shows that R is the graph of a bijection from F to $R[F]$. Because R is a compact subset of $F \times R[F]$, it is a homeomorphism, and $R[F]$ is not scattered. But, for each $\xi < \omega_1$, $R[F] \subseteq \bigcup_{i\in\mathbb{N}} K_{\xi i}$ is disjoint from W_ξ; and all compact subsets of $\{0,1\}^{\omega_1} \setminus \bigcup_{\xi<\omega_1} W_\xi$ are supposed to be scattered. **XQ**

(iv) Take $x \in F \setminus \bigcup_{i\in\mathbb{N}} \widehat{K}_{\zeta i}$. Then $\chi(x,X) \leq \omega$. **P** Consider the set

$$V = \bigcap_{\eta\leq\zeta, i\in\mathbb{N}} \{x' : x' \in X,\, x' \in \widehat{K}_{\eta i} \iff x \in \widehat{K}_{\eta i}\}.$$

This is a G_δ set containing x. **?** If there is an $x' \in V \setminus \{x\}$, there must be some $\xi < \omega_1$ and $j \in \mathbb{N}$ such that just one of x, x' belongs to $\widehat{K}_{\xi j}$. In this case, $\xi > \zeta$, so $K_{\xi j} \subseteq \bigcup_{i\leq k} K_{\zeta i}$ and $\widehat{K}_{\xi j} \subseteq \bigcup_{i\leq k} \widehat{K}_{\zeta i}$ for some $k \in \mathbb{N}$. But neither x nor x' belongs to $\bigcup_{i\leq k} \widehat{K}_{\zeta i}$. **X** Thus $V = \{x\}$; by 4A2Gd as usual, $\chi(x,X) \leq \omega$. **Q**

(v) Thus we see that whenever $F \subseteq X$ is a non-scattered closed set, there is an $x \in F$ such that $\chi(x,X)$ is countable. By 5A4C(d-iii), $\{0,1\}^{\omega_1}$ is not a continuous image of a closed subset of X.

531X Basic exercises (a) Show that there is a Hausdorff completely regular quasi-Radon probability space $(X,\mathfrak{T},\Sigma,\mu)$ with Maharam type greater than $\#(X)$. (*Hint*: 523Ib.)

(b) Give an example of a separable Radon measure space with magnitude $2^{\mathfrak{c}}$. (*Hint*: 4A2B(e-ii).)

(c) Let $I^{\|}$ be the split interval (343J, 419L). Show that $\mathrm{Mah}_{\mathrm{R}}(I) = \{0,\omega\}$.

(d) Let I be an infinite set, and βI the Stone-Čech compactification of the discrete space I. Show that $2^{\#(I)}$ is the greatest member of $\mathrm{Mah}_{\mathrm{R}}(\beta I)$. (*Hint*: 5A4Ia, 515I.)

(e) For a topological space X, write $\mathrm{Mah}_{\mathrm{qR}}(X)$ for the set of Maharam types of Maharam-type-homogeneous quasi-Radon probability measures on X. (i) Show that $\kappa \leq w(X)$ for every $\kappa \in \mathrm{Mah}_{\mathrm{qR}}(X)$. (ii) Show that $\mathrm{Mah}_{\mathrm{qR}}(Y) \subseteq \mathrm{Mah}_{\mathrm{qR}}(X)$ for every $Y \subseteq X$. (iii) Show that if Y is another topological space, and neither X nor Y is empty, then $\mathrm{Mah}_{\mathrm{qR}}(X \times Y) = \mathrm{Mah}_{\mathrm{qR}}(X) \cup \mathrm{Mah}_{\mathrm{qR}}(Y)$.

>(f) Let X be a Hausdorff topological group carrying Haar measures, and $\mathfrak{A}$ its Haar measure algebra (442H, 443A). Show that $w(X) = \max(c(\mathfrak{A}), \tau(\mathfrak{A}))$. (*Hint*: 443Gf, 529Ba.) Show that if X is σ-compact, locally compact, Hausdorff and not discrete then $w(X) \in \mathrm{Mah}_{\mathrm{R}}(X)$.

(g) Let X be a Hausdorff space such that $\mathrm{Mah}_{\mathrm{R}}(X) \subseteq \{0,\omega\}$, and $\mathcal{N}$ the null ideal of Lebesgue measure on $\mathbb{R}$. Show that the union of fewer than $\operatorname{add}\mathcal{N}$ universally Radon-measurable subsets of X is universally Radon-measurable.

(h) Let X be a completely regular Hausdorff space and κ an infinite cardinal. Suppose that whenever Y is a Hausdorff continuous image of X of weight κ then $\mathrm{Mah}_{\mathrm{R}}(Y) \subseteq \kappa$. Show that $\mathrm{Mah}_{\mathrm{R}}(X) \subseteq \kappa$.

(i) Let X be a Hausdorff space, and $\langle E_i \rangle_{i\in I}$ a family of universally Radon-measurable subsets of X such that $\#(I) < \operatorname{cov}\mathcal{N}_\kappa$ for every κ. Show that $\mathrm{Mah}_{\mathrm{R}}(\bigcup_{i\in I} E_i) = \bigcup_{i\in I} \mathrm{Mah}_{\mathrm{R}}(E_i)$.

(j) Let K be an Eberlein compactum. Show that $\mathrm{Mah_R}(K) \subseteq \{0, \omega\}$. (*Hint*: 467Xj.)

(k) Let X be a Hausdorff space and κ a cardinal. Show that there is a Radon probability measure on X with Maharam type κ iff *either* κ is finite and $2^\kappa \le 2\#(X)$ *or* $\kappa = \omega \le \#(X)$ *or* $\kappa \in \mathrm{Mah_R}(X)$ *or* $\mathrm{cf}\,\kappa = \omega$ and $\kappa = \sup \mathrm{Mah_R}(X)$.

>(l) Let X be a Hausdorff space and κ an infinite cardinal. (i) Show that $\{0,1\}^\kappa$ is a continuous image of a compact subset of X iff $[0,1]^\kappa$ is a continuous image of a compact subset of X, and that in this case $\{0,1\}^\kappa$ is a continuous image of a compact subset of $P_{\mathrm{R}}(X)$. (*Hint*: 437Xt.) (ii) Show that if X is normal and $\{0,1\}^\kappa$ is a continuous image of a closed subset of X then $[0,1]^\kappa$ is a continuous image of X. (*Hint*: 4A2F(d-ix).) (iii) Show that if X is completely regular and $\{0,1\}^\kappa$ is a continuous image of a compact subset of X then $[0,1]^\kappa$ is a continuous image of X. (*Hint*: 4A2F(h-iii).)

(m) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Hausdorff quasi-Radon probability space. Show that the Maharam type of μ is at most $\max(\omega, 2^{\chi(X)})$. (*Hint*: 5A4Ba, 5A4Bg.)

(n) In the language of 531R, show that if a, $b \in \mathfrak{B}_I$ and $K \subseteq I \setminus J^*(b)$ is finite, then $\bar\nu_I(a \mathbin{\triangle} S_{I\setminus K}(a)) \le 2^{\#(I)}\bar\nu_I(a \mathbin{\triangle} b)$.

(o) Show that if $\mathfrak{m}_{\mathrm{K}} > \omega_1$ and X is a countably tight compact Hausdorff space, then $\omega_1 \notin \mathrm{Mah_R}(X)$.

(p) Let X be an infinite compact Hausdorff space with a strictly positive Radon measure μ. Show that the topological density of $P_{\mathrm{R}}(X)$, with its narrow topology, is at most the Maharam type of μ. (*Hint*: the indefinite-integral measures over μ are dense in $P_{\mathrm{R}}(X)$.)

(q) Let $\mathcal{W} \subseteq \mathcal{N}_\omega$ be such that every compact subset of $\{0,1\}^\omega \setminus \bigcup\mathcal{W}$ is scattered. Show that there is a family $\mathcal{W}' \subseteq \mathcal{N}_{\omega_1}$ such that $\#(\mathcal{W}') = \#(\mathcal{W})$ and every compact subset of $\{0,1\}^{\omega_1} \setminus \bigcup\mathcal{W}'$ is scattered.

531Y Further exercises (a) Let κ be an infinite cardinal such that $\kappa = \kappa^{\mathfrak{c}}$. Show that there is a set $X \subseteq \{0,1\}^\kappa$, of full outer measure for ν_κ, such that every subset of X with cardinal $\mathfrak{c}$ is discrete. Show that $\mathrm{Mah_{qR}}(X)$ (531Xe) contains κ but not ω.

(b) Let X and Y be infinite compact Hausdorff spaces, and suppose that there is a norm-preserving linear isomorphism between the dual spaces $C(X)^*$ and $C(Y)^*$. Show that $\mathrm{Mah_R}(X) = \mathrm{Mah_R}(Y)$.

(c) Let μ be a τ-additive Borel probability measure on a topological space X, and κ a cardinal of uncountable cofinality such that (i) $\chi(x, X) < \mathrm{cf}\,\kappa$ for every $x \in X$ (ii) no non-negligible measurable set can be covered by $\mathrm{cf}\,\kappa$ negligible sets. Show that the Maharam type of μ cannot be κ.

(d) Let X be a completely regular Hausdorff space and $\kappa \ge \omega_2$ a cardinal. Show that if $\kappa \in \mathrm{Mah_R}(X)$ then the Banach space $\ell^1(\kappa)$ is isomorphic, as linear topological space, to a subspace of the Banach space $C_b(X)$.

(e) Let X be a locally compact Hausdorff space and κ an infinite cardinal such that $\ell^1(\kappa)$ is isomorphic, as linear topological space, to a subspace of $C_0(X)$ (definition: 436I). Show that $\kappa \in \mathrm{Mah_R}(X)$. (*Hint*: Reduce to the case in which X is compact. Show that if $\langle e_i \rangle_{i\in\mathbb{N}}$ is the standard generating family in ℓ^1, $n \in \mathbb{N}$ and $\langle \alpha_{ij} \rangle_{i<j\le n}$ is a family in $[0,\infty[$, then there is a family $\langle \epsilon_{ij} \rangle_{i<j\le n}$ in $\{-1,1\}$ such that $\|\sum_{i<j\le n} \epsilon_{ij}\alpha_{ij}(e_i - e_j)\|_1 \ge \sum_{i<j\le n} \alpha_{ij}$. See Pełczyński 68.)

531Z Problems (a) Can there be a perfectly normal compact Hausdorff space X such that $\omega_2 \in \mathrm{Mah_R}(X)$? (See 531Q, 554Xd.)

(b) Can there be a hereditarily separable compact Hausdorff space X such that $\omega_2 \in \mathrm{Mah_R}(X)$?

531 Notes and comments This section is directed to Radon measures, studying $\mathrm{Mah}_{\mathrm{R}}(X)$; of course we can look at Maharam types of quasi-Radon measures (531Xe, 531Ya), or Borel or Baire measures for that matter. In the next section I shall have something to say about completion regular measures. The function $X \mapsto \mathrm{Mah}_{\mathrm{R}}(X)$ has a much more satisfying list of basic properties (531E, 531G) than the others.

From 531L and 531T we see that there are many cardinals κ such that whenever X is a Hausdorff space and $\kappa \in \mathrm{Mah}_{\mathrm{R}}(X)$, then there is a continuous function from a closed subset of X onto $\{0,1\}^\kappa$. Such cardinals are said to have **Haydon's property**. From 531L, 531M and 531T we see that

ω has Haydon's property (531La);

if $\kappa \ge \omega_2$ and κ is a measure-precaliber of $\mathfrak{B}_\kappa$ then κ has Haydon's property (531Lb);

$\mathfrak{c}^{(+n)}$ has Haydon's property for $n \ge 1$ (525K);

if $\kappa \ge \omega$ is not a measure-precaliber of $\mathfrak{B}_\kappa$ then κ does not have Haydon's property (531M);

if $\omega_1 < \mathfrak{m}_{\mathrm{K}}$ then ω_1 has Haydon's property (531T).

(See also 544D.) Thus if $\mathfrak{m}_{\mathrm{K}} > \omega_1$, an infinite cardinal κ has Haydon's property iff it is a measure-precaliber of every probability algebra. ω_1 really is different; it is possible that ω_1 is a precaliber of every probability algebra but does not have Haydon's property. To check this, it is enough to find a model of set theory in which $\mathrm{cov}\,\mathcal{N}_{\omega_1} > \omega_1$ (525Gc) but there is a family $\langle W_\xi\rangle_{\xi<\omega_1}$ as in 531Vb; one is described in 553F.

You will observe that the key arguments of this section all depend on analysis of the measure algebras $\mathfrak{B}_\kappa$. We have already seen in §524 that many properties of a Radon measure can be determined from its measure algebra. Here we find that some important topological properties of compact Hausdorff spaces can be determined by the measure algebras of the Radon measures they carry. The results here largely depend for their applications on knowing enough about precalibers; I remind you that it seems to be still unknown whether it is possible that every infinite cardinal should be a measure-precaliber of every probability algebra.

The remarks above have concerned the existence of continuous surjections onto $\{0,1\}^\kappa$; a natural place to start, because measures of Maharam type κ arise immediately from such surjections. In 531N-531Q I look at different measures of the richness of a compact space X. Concerning characters, 531N-531O give us quite a lot of information, slightly irregular at the edges. I ought to offer a remark on the context of 531Q. In some set theories (for instance, when $\mathfrak{m} > \omega_1$), we find not only that ω_1 is a precaliber of every measurable algebra, but also that a compact Hausdorff space is hereditarily separable iff it is hereditarily Lindelöf (Fremlin 84a, 44H); so that, for instance, a hereditarily separable compact Hausdorff space must be first-countable, so cannot carry a Radon measure of uncountable Maharam type. Typically, the situation is very different if the continuum hypothesis or Jensen's $\diamondsuit$ is true, and 531Q is a descendant of the construction in Kunen 81 of a non-separable hereditarily Lindelöf compact Hausdorff space. See Džamonja & Kunen 93 for further exploration of these questions.

Following the lead of Haydon 77, more than half of this section is devoted to investigating properties of compact Hausdorff spaces carrying Radon measures of particular Maharam types. Most of the topological properties considered are very natural ones in this context. But in 531U I add an interesting pair of results concerning topological properties of $P_{\mathrm{R}}(X)$ or $P_{\mathrm{R}}(X \times X)$, less obviously connected to individual Radon measures on X.

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532 Completion regular measures on $\{0,1\}^I$

As I remarked in the introduction to §434, the trouble with topological measure theory is that there are too many questions to ask. In §531 I looked at the problem of determining the possible Maharam types of Radon measures on a Hausdorff space X. But we can ask the same question for any of the other classes of topological measures listed in §411. It turns out that the very narrowly focused topic of completion regular Radon measures on powers of $\{0,1\}$ already leads us to some interesting arguments.

I define the classes $\mathrm{Mah}_{\mathrm{crR}}(X)$, corresponding to the $\mathrm{Mah}_{\mathrm{R}}(X)$ examined in §531, in 532A. They are less accessible, and I almost immediately specialize to the relation $\lambda \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$. This at least is more or less convex (532G, 532K), and can be characterized in terms of the measure algebras $\mathfrak{B}_\lambda$ (532I). On the way it is helpful to extend the treatment of completion regular measures given in §434 (532D, 532E, 532H). For fixed infinite λ, there is a critical cardinal $\kappa_0 \le (2^\lambda)^+$ such that $\lambda \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$ iff $\lambda \le \kappa < \kappa_0$; under certain conditions, when $\lambda = \omega$, we can locate κ_0 in terms of the cardinals of Cichoń's diagram (532P, 532Q). This depends on facts about the Lebesgue measure algebra (532M, 532O) which are of independent interest. Finally, for other λ of countable cofinality, the square principle and Chang's transfer principle are relevant (532R-532S).

532A Definition If X is a topological space, I write $\mathrm{Mah}_{\mathrm{cr}}(X)$ for the set of Maharam types of Maharam-type-homogeneous completion regular topological probability measures on X. If X is a Hausdorff space, I write $\mathrm{Mah}_{\mathrm{crR}}(X)$ for the set of Maharam types of Maharam-type-homogeneous completion regular Radon probability measures on X.

532B Proposition Let X be a Hausdorff space. Then a probability algebra $(\mathfrak{A},\bar{\mu})$ is isomorphic to the measure algebra of a completion regular Radon probability measure on X iff (α) $\tau(\mathfrak{A}_a)\in\mathrm{Mah}_{\mathrm{crR}}(X)$ whenever $\mathfrak{A}_a$ is a non-zero homogeneous principal ideal of $\mathfrak{A}$ (β) the number of atoms of $\mathfrak{A}$ is not greater than the number of points $x\in X$ such that $\{x\}$ is a zero set.

proof (a) Suppose that μ is a completion regular Radon probability measure on X and $\mathfrak{A}_a$ is a non-zero homogeneous principal ideal of its measure algebra $\mathfrak{A}$. Let F be such that $F^\bullet=a$ and ν the indefinite-integral measure over μ defined by the function $\frac{1}{\mu F}\chi F$. Then ν is a Radon measure (416Sa), inner regular with respect to the zero sets (412Q); and its measure algebra is isomorphic, up to a scalar multiple, to $\mathfrak{A}_a$, so is homogeneous with Maharam type $\tau(\mathfrak{A}_a)$. So ν witnesses that $\tau(\mathfrak{A}_a)\in\mathrm{Mah}_{\mathrm{crR}}(X)$. This shows that $\mathfrak{A}$ satisfies condition (α).

As for condition (β), each atom of $\mathfrak{A}$ is of the form $\{x\}^\bullet$ for some $x\in X$ such that $\mu\{x\}>0$ (414G, or otherwise). In this case, because μ is completion regular, $\{x\}$ must be a zero set. So we have at least as many singleton zero sets as we have atoms in $\mathfrak{A}$.

(b) Now suppose that $(\mathfrak{A},\bar{\mu})$ satisfies the conditions. I copy the argment of 531F. Express $(\mathfrak{A},\bar{\mu})$ as the simple product of a countable family $\langle(\mathfrak{A}_i,\bar{\mu}_i')\rangle_{i\in I}$ of non-zero homogeneous measure algebras. For $i\in I$, set $\kappa_i=\tau(\mathfrak{A}_i)$ and $\gamma_i=\bar{\mu}_i'1_{\mathfrak{A}_i}$. Set $J=\{i:i\in I,\,\kappa_i\ge\omega\}$. (β) tells us that $\#(I\setminus J)$ is less than or equal to the number of singleton zero sets in X; let $\langle x_i\rangle_{i\in I\setminus J}$ be a family of distinct elements of X such that every $\{x_i\}$ is a zero set.

For each $i\in J$, (α) tells us that there is a completion regular Maharam-type-homogeneous Radon probability measure μ_i on X with Maharam type κ_i. Now there is a disjoint family $\langle E_i\rangle_{i\in J}$ of Baire subsets of X such that $\mu_iE_i>0$ for every $i\in J$. **P** We may suppose that $J\subseteq\mathbb{N}$. Choose $\langle E_i\rangle_{i\in\mathbb{N}}$, $\langle F_i\rangle_{i\in\mathbb{N}}$ inductively, as follows. $F_0=X\setminus\{x_i:i\in I\setminus J\}$. Given that F_i is a Baire set and $\mu_jF_i>0$ for every $j\in J\setminus i$, then if $i\notin J$ set $E_i=\emptyset$ and $F_{i+1}=F_i$; otherwise, because μ_i is atomless and completion regular, we can find, for each $j\in J$ such that $j>i$, a Baire set $G_{ij}\subseteq F_i$ such that $\mu_iG_{ij}<2^{-j}\mu_iF_i$ and $\mu_jG_{ij}>0$; set $F_{i+1}=\bigcup_{j\in J,j>i}G_{ij}$ and $E_i=F_i\setminus F_{i+1}$; continue. **Q** Now set

$$\mu E=\sum_{i\in I\setminus J,x_i\in E}\gamma_i+\sum_{i\in J}(\mu_iE_i)^{-1}\gamma_i\mu_i(E\cap E_i)$$

whenever $E\subseteq X$ is such that μ_i measures $E\cap E_i$ for every $i\in J$. Of course μ is a probability measure. Because every μ_i is a topological measure, so is μ; because every μ_i is inner regular with respect to the compact sets, so is μ; because every μ_i is complete, so is μ; so μ is a Radon measure. Because every subspace measure $(\mu_i)_{E_i}$ is Maharam-type-homogeneous with Maharam type κ_i, the measure algebra of μ is isomorphic to $(\mathfrak{A},\bar{\mu})$. Because all the $\{x_i\}$ are zero sets and all the μ_i are completion regular, μ is completion regular.

532C Remarks Nearly the whole of this section will be devoted to the usual measures on powers of $\{0,1\}$. Accordingly the following notation will be useful, as previously in this volume. If I is any set, ν_I will be the usual measure on $\{0,1\}^I$, $\mathfrak{B}_I$ its measure algebra and $\mathcal{N}_I$ its null ideal. In this context, $\langle e_i\rangle_{i\in I}$ will be the standard generating family in $\mathfrak{B}_I$ (525A), and for $J\subseteq I$, $\mathfrak{C}_J$ will be the closed subalgebra of $\mathfrak{B}_I$ generated by $\{e_i:i\in J\}$.

If X is a topological space, $\mathcal{B}(X)$ will be its Borel σ-algebra.

Let κ be an infinite cardinal. Then ν_κ is a completion regular Radon probability measure (416U), and $\mathfrak{B}_\kappa$ is homogeneous with Maharam type κ. So $\kappa\in\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$. Next, any Radon measure on $\{0,1\}^\kappa$ can have Maharam type at most $w(\{0,1\}^\kappa)$ (531Aa), so $\lambda \le \kappa$ for every $\lambda \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$. At the bottom end, $0 \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$ iff $\{0,1\}^\kappa$ has a singleton G_δ set, that is, iff $\kappa = \omega$.

From this we see already that we do not have direct equivalents of any of the results 531Eb-531Ef. However the class $\{(\lambda,\kappa) : \lambda \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)\}$ is convex in two senses (532G, 532K). For the first of these, it will be useful to have a result left over from §434.

532D Theorem (Fremlin & Grekas 95) Let (X,μ_1) and (Y,μ_2) be effectively locally finite topological measure spaces of which X is quasi-dyadic (definition: 434O), μ_1 is completion regular and μ_2 is τ-additive. Let μ be the c.l.d. product measure on $X \times Y$ as defined in §251. Then μ is a τ-additive topological measure.

proof (a) To begin with (down to the end of (e)) let us suppose that μ_1 and μ_2 are complete and totally finite and inner regular with respect to the Borel sets. Let $\langle X_i\rangle_{i\in I}$ be a family of separable metrizable spaces such that there is a continuous surjection $f : \prod_{i\in I} X_i \to X$. For each $i \in I$, let $\mathcal{U}_i$ be a countable base for the topology of X_i not containing $\emptyset$; for $J \subseteq I$, let $\mathcal{C}_J$ be the family of cylinder sets expressible in the form $\{z : z \in \prod_{i\in I} X_i,\ z(i) \in U_i \text{ for every } i \in K\}$ where $K \subseteq J$ is finite and $U_i \in \mathcal{U}_i$ for each $i \in K$.

(b) ? Suppose, if possible, that μ is not a topological measure. Let $W \subseteq X \times Y$ be a closed set which is not measured by μ. By 434Q, μ_1 is τ-additive; by 417C, there is a τ-additive topological measure $\tilde\mu$ extending μ, and $\mu^*W = \tilde\mu W$ (apply 417C(b-v-α) to the complement of W).

(c) If $J \subseteq I$ is countable, there are sets H, V, V' such that $H \subseteq Y$ is open, $V \in \mathcal{C}_J$, $V' \in \mathcal{C}_{I\setminus J}$, $f[V \cap V'] \times H$ is disjoint from W, and $\mu^*(W \cap (f[V] \times H)) > 0$. **P** For $V \in \mathcal{C}_J$, set

$$\mathcal{H}_V = \bigcup_{V'\in\mathcal{C}_{I\setminus J}} \{H : H \subseteq Y \text{ is open, } W \cap (f[V\cap V'] \times H) = \emptyset\},$$

$$H_V = \bigcup \mathcal{H}_V,$$

and choose a measurable envelope F_V of $f[V]$. As $\mathcal{C}_J$ is countable,

$$W_1 = (X \times Y) \setminus \bigcup_{V\in\mathcal{C}_J} F_V \times H_V$$

is measured by μ; also $W_1 \subseteq W$ because

$$\{f[V\cap V'] \times H : V \in \mathcal{C}_J,\ V' \in \mathcal{C}_{I\setminus J},\ H \subseteq Y \text{ is open}\}$$

is a network for the topology of $X \times Y$. So

$$\tilde\mu W_1 = \mu W_1 \le \mu_* W < \mu^* W = \tilde\mu W$$

and $\tilde\mu(W \setminus W_1) > 0$. There must therefore be a $V \in \mathcal{C}_J$ such that $\tilde\mu(W \cap (F_V \times H_V)) > 0$. Next, because μ_2 is τ-additive, there is a countable $\mathcal{H} \subseteq \mathcal{H}_V$ such that $\mu_2(H_V \setminus \bigcup\mathcal{H}) = 0$, and now $\tilde\mu(W \cap (F_V \times \bigcup\mathcal{H})) = \tilde\mu(W \cap (F_V \times H_V))$ is non-zero. Accordingly there is an $H \in \mathcal{H}$ such that $\tilde\mu(W \cap (F_V \times H)) > 0$. By 417G[2],

$$\int_{F_V} \mu_2(W[\{x\}] \cap H)\mu_1(dx) = \tilde\mu(W \cap (F_V \times H))$$

is greater than 0. But this means that $\mu_1\{x : x \in F_V,\ \mu_2(W[\{x\}] \cap H) > 0\} > 0$. (Recall that we are supposing that μ_1 is complete.) So $\{x : x \in f[V],\ \mu_2(W[\{x\}]\cap H) > 0\}$ is not μ_1-negligible, and $W \cap (f[V] \times H)$ is not μ-negligible. Finally, because $H \in \mathcal{H}_V$, there is a $V' \in \mathcal{C}_{I\setminus J}$ such that $W \cap (f[V \cap V'] \times H) = \emptyset$. **Q**

(d) We may therefore choose inductively families $\langle J_\xi\rangle_{\xi<\omega_1}$, $\langle H_\xi\rangle_{\xi<\omega_1}$, $\langle V_\xi\rangle_{\xi<\omega_1}$, $\langle V'_\xi\rangle_{\xi<\omega_1}$ in such a way that, for every $\xi < \omega_1$,

J_ξ is a countable subset of I,
H_ξ is an open subset of Y,
$V_\xi \in \mathcal{C}_{J_\xi}$, $V'_\xi \in \mathcal{C}_{I\setminus J_\xi}$,
$W \cap (f[V_\xi \cap V'_\xi] \times H_\xi) = \emptyset$,
$\mu^*(W \cap (f[V_\xi] \times H_\xi)) > 0$,
$\bigcup_{\eta<\xi} J_\eta \subseteq J_\xi$,
V_ξ, $V'_\xi \in \mathcal{C}_{J_{\xi+1}}$.

[2]Formerly 417H.

For each $\xi<\omega_1$, let K_ξ be a finite subset of $J_{\xi+1}$ such that V_ξ and V'_ξ are determined by coordinates in K_ξ. By the Δ-system Lemma (4A1Db), there is an uncountable set $A\subseteq\omega_1$ such that $\langle K_\xi\rangle_{\xi\in A}$ is a Δ-system with root K say. Set $\zeta_0=\min A$. Express each V_ξ as $\tilde V_\xi\cap\hat V_\xi$ where $\tilde V_\xi\in\mathcal{C}_K$ and $\hat V_\xi\in\mathcal{C}_{K_\xi\setminus K}$; because $\mathcal{C}_K$ is countable, there is a $\tilde V$ such that $B=\{\xi:\xi\in A,\,\xi>\zeta_0,\,\tilde V_\xi=\tilde V\}$ is uncountable. Note that $\mu_1^*f[\tilde V]>0$, because $\mu_1^*f[\tilde V]\ge\mu^*(W\cap(f[V_\xi]\times H_\xi))$ for any $\xi\in B$. Also

$$K\subseteq K_{\zeta_0}\subseteq J_{\zeta_0+1}\subseteq J_\xi,$$

so V'_ξ is determined by coordinates in $K_\xi\setminus J_\xi\subseteq K_\xi\setminus K$, for every $\xi\in B$.

(e) Set $H'_\xi=\bigcup_{\eta\in B\setminus\xi}H_\eta$ for each $\xi<\omega_1$. Then $\langle H'_\xi\rangle_{\xi<\omega_1}$ is non-increasing, so there is a $\zeta<\omega_1$ such that $\mu_2H'_\xi=\mu_2H'_\zeta$ whenever $\xi\ge\zeta$. Now consider $F=\{x:\mu_2(W[\{x\}]\cap H'_\zeta)>0\}$. Applying 417G to the indicator function of $W\cap(X\times H'_\zeta)$, and recalling once more that μ_1 is complete, we see that μ_1 measures F. Also $\mu_1^*(F\cap f[\tilde V])>0$. **P** Take any $\xi\in B\setminus\zeta$. Then

$$F\cap f[\tilde V]\supseteq\{x:x\in f[\tilde V_\xi],\,\mu_2(W[\{x\}]\cap H_\xi)>0\}$$

must be non-μ_1-negligible because $W\cap(f[\tilde V_\xi]\times H_\xi)$ is not μ-negligible. **Q**

At this point, recall that we are supposing that μ_1 is completion regular. So there is a zero set $Z\subseteq F$ such that $\mu_1Z>\mu_1F-\mu_1^*(F\cap f[\tilde V])$, and $Z\cap f[\tilde V]\ne\emptyset$, that is, $\tilde V\cap f^{-1}[Z]$ is not empty. $f^{-1}[Z]$ is a zero set (4A2C(b-iv)), so there is a countable set $J\subseteq I$ such that $f^{-1}[Z]$ is determined by coordinates in J (4A3Nc); we may suppose that $K\subseteq J$. Because $\langle K_\eta\setminus K\rangle_{\eta\in A}$ is disjoint, there is a $\xi\ge\zeta$ such that $J\cap K_\eta=K$ for every $\eta\in A\setminus\xi$.

Take any $w\in\tilde V\cap f^{-1}[Z]$ and modify it to produce $w'\in\prod_{i\in I}X_i$ such that $w'\restriction J=w\restriction J$ and $w'\in\hat V_\eta\cap V'_\eta$ for every $\eta\in B\setminus\xi$; this is possible because $\hat V_\eta\cap V'_\eta$ is determined by coordinates in $K_\eta\setminus K$ for each η, and J and the $K_\eta\setminus K$ are disjoint. Set $x=f(w')$; then $x\in Z\subseteq F$, so $\mu_2(W[\{x\}]\cap H'_\zeta)>0$.

$w'\in\tilde V$, because $w\in\tilde V$ and $\tilde V$ is determined by coordinates in $K\subseteq J$; so $w'\in\tilde V\cap\hat V_\eta\cap V'_\eta=V_\eta\cap V'_\eta$ for every $\eta\in B\setminus\xi$. Accordingly $x\in f[V_\eta\cap V'_\eta]$; as $W\cap(f[V_\eta\cap V'_\eta]\times H_\eta)=\emptyset$, $W[\{x\}]$ does not meet H_η. As η is arbitrary, $W[\{x\}]$ does not meet H'_ξ and $W[\{x\}]\cap H'_\zeta$ is μ_2-negligible. But this is impossible. **X**

(f) This contradiction shows that μ will be a topological measure, at least if μ_1 and μ_2 are complete, totally finite and inner regular with respect to the Borel sets. Now suppose just that μ_1 and μ_2 are totally finite. Let μ'_1 and μ'_2 be the completions of the Borel measures $\mu_1\restriction\mathcal{B}(X)$ and $\mu_2\restriction\mathcal{B}(Y)$, and μ' their c.l.d. product. Then $\mu_1\restriction\mathcal{B}(X)$ and μ'_1 are completion regular topological measures, while $\mu_2\restriction\mathcal{B}(Y)$ and μ'_2 are τ-additive. So (a)-(e) tell us that μ' measures every open set. Now the completions $\hat\mu_1$, $\hat\mu_2$ extend μ'_1 and μ'_2, and μ is the c.l.d. product of $\hat\mu_1$ and $\hat\mu_2$ (251T), so μ extends μ' (251L). Thus we again have a topological product measure μ.

(g) In the general case, let $W\subseteq X\times Y$ be an open set, $E\subseteq X$ a zero set of finite measure, and $F\subseteq Y$ any set of finite measure. Then μ measures $W\cap(E\times F)$. **P** Let $(\mu_1)_E$ and $(\mu_2)_F$ be the subspace measures. Then both are totally finite topological measures, $(\mu_1)_E$ is inner regular with respect to the zero sets (412Pd), E is quasi-dyadic (434Pc), and $(\mu_2)_F$ is τ-additive (414K). So the product $(\mu_1)_E\times(\mu_2)_F$ is a topological measure and measures $W\cap(E\times F)$. By 251Q, μ measures $W\cap(E\times F)$. **Q**

Let $\mathcal{K}$ be the family of zero sets of finite measure in X, $\mathcal{L}$ the family of Borel sets of finite measure in Y, and $\mathcal{M}$ the family of sets $M\subseteq X\times Y$ such that μ measures $W\cap M$. Because μ_1 is inner regular with respect to $\mathcal{K}$, μ_2 is inner regular with respect to $\mathcal{L}$, $E\times F\in\mathcal{M}$ for every $E\in\mathcal{K}$ and $F\in\mathcal{L}$, and $\mathcal{M}$ is a σ-algebra of sets, 412R tells us that μ is inner regular with respect to $\mathcal{M}$. As μ is complete and locally determined, it must measure W (412Ja). As W is arbitrary, μ is a topological measure.

(h) Finally, as noted in (b), μ_1 is τ-additive and there is a τ-additive topological measure $\tilde\mu$ on $X\times Y$ extending μ. (434Q and 417C still apply.) So μ too must be τ-additive.

532E Corollary Let $\langle X_i\rangle_{i\in I}$ be a family of regular spaces with countable networks, and Y any topological space. Suppose that we are given a strictly positive topological probability measure μ_i on each X_i, and a τ-additive topological probability measure ν on Y. Let μ be the ordinary product measure on $Z=\prod_{i\in I}X_i\times Y$.

(a) μ is a topological measure.

(b) μ is τ-additive.

(c) If ν is completion regular, and every μ_i is inner regular with respect to the Borel sets, then μ is completion regular.

proof (a) For each i, X_i is hereditarily Lindelöf (4A2Nb), so μ_i is τ-additive (414O); let μ_i' be the completion of the Borel measure $\mu_i\restriction\mathcal{B}(X_i)$. Then μ_i' is a quasi-Radon measure (415C). By 4A2Nb, X_i is perfectly normal, so μ_i' is completion regular. By 434Pb-434Pc, $\prod_{i\in I}X_i$ is quasi-dyadic. The product ν_1 of the μ_i' is a topological measure (453I) and inner regular with respect to the zero sets (412Ub); so the product μ' of ν_1 and ν is a topological measure, by 532D. Now μ' is also the product of the measures $\mu_i\restriction\mathcal{B}(X_i)$ and ν (254I, 254N), so μ extends μ' (254H) and μ also is a topological measure.

(b) Because every μ_i is τ-additive, as is ν, 417E tells us that there is a τ-additive measure extending μ, so μ itself must be τ-additive.

(c) For any $i\in I$, we know from (a) that μ_i' is inner regular with respect to the zero sets. Now every non-μ_i-negligible set includes a non-μ_i-negligible Borel set, which includes a non-μ_i-negligible zero set; accordingly μ_i is completion regular. By 412Ub again, μ is inner regular with respect to the zero sets, so is completion regular.

532F Corollary Let $\langle(X_i,\mu_i)\rangle_{i\in I}$ be a family of quasi-dyadic compact Hausdorff spaces with strictly positive completion regular Radon measures. Then the ordinary product measure μ on $\prod_{i\in I}X_i$ is a completion regular Radon measure.

proof By 532D, the ordinary product measure on $\prod_{i\in J}X_i$ is a topological measure, for every finite $J\subseteq I$. By 417Sc, μ is the τ-additive product measure on $\prod_{i\in I}X_i$, which by 417Q is a Radon measure. By 412Ub once more, μ is completion regular.

532G Proposition Suppose that λ, λ' and κ are cardinals such that $\max(\omega,\lambda)\le\lambda'\le\kappa$ and $\lambda\in\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$. Then $\lambda'\in\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$.

proof Let ν be a completion regular Maharam-type-homogeneous Radon probability measure on $\{0,1\}^\kappa$ with Maharam type λ, and consider the ordinary product measure μ of $\nu_{\lambda'}$ and ν on $X=\{0,1\}^{\lambda'}\times\{0,1\}^\kappa$. Applying 532E with $Y=\{0,1\}^\kappa$ and $X_\xi=\{0,1\}$ for $\xi<\lambda'$, we see that μ is a completion regular topological probability measure on a compact Hausdorff space, therefore (being complete) a Radon measure. By 334A, the Maharam type of μ is at most $\max(\omega,\lambda',\lambda)=\lambda'$, so the measure algebra $(\mathfrak{A},\bar\mu)$ of μ can be embedded in $\mathfrak{B}_{\lambda'}$. At the same time, the inverse-measure-preserving projection from X onto $\{0,1\}^{\lambda'}$ induces a measure-preserving embedding of $\mathfrak{B}_{\lambda'}$ into $\mathfrak{A}$. By 332Q, $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B}_{\lambda'},\bar\nu_{\lambda'})$ are isomorphic, that is, μ is Maharam-type-homogeneous with Maharam type λ'. So μ witnesses that $\lambda'\in\mathrm{Mah}_{\mathrm{crR}}(X)=\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$.

532H Lemma Let $\langle X_i\rangle_{i\in I}$ be a family of separable metrizable spaces, and μ a totally finite completion regular topological measure on $X=\prod_{i\in I}X_i$. Then

(a) the support of μ is a zero set;

(b) μ is inner regular with respect to the self-supporting zero sets.

proof (a) Recall from 434Q that μ is τ-additive, so has a support Z. Let $\langle K_n\rangle_{n\in\mathbb{N}}$ be a sequence of zero sets such that $K_n\subseteq Z$ and $\mu K_n\ge\mu X-2^{-n}$ for each n. Then there is a countable set $J\subseteq I$ such that every K_n is determined by coordinates in J (4A3Nc again). So $\bigcup_{n\in\mathbb{N}}K_n$ and $Z'=\overline{\bigcup_{n\in\mathbb{N}}K_n}$ are determined by coordinates in J (4A2B(g-i)), and Z' is a zero set, by 4A3Nc in the other direction. But $Z'\subseteq Z$ and $\mu Z'=\mu Z$ so $Z=Z'$ is a zero set.

(b) If $\mu E>\gamma$ then there is a zero set $K\subseteq E$ such that $\mu K\ge\gamma$. Now $\mu\llcorner K$ (234M) is a totally finite topological measure on X which is completion regular (412Q), so its support Z is a zero set, by (a); and $Z\subseteq K\subseteq E$ is self-supporting for μ with $\mu Z\ge\gamma$.

532I There is a useful general characterization of the sets $\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$ in terms of the measure algebras $\mathfrak{B}_\lambda$. At the same time, we can check that other products of separable metrizable spaces follow powers of $\{0,1\}$, as follows.

Theorem (Choksi & Fremlin 79) Let $\lambda\le\kappa$ be infinite cardinals. Then the following are equiveridical:

(i) $\lambda\in\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$;

(ii) there is a family $\langle X_\xi\rangle_{\xi<\kappa}$ of non-singleton separable metrizable spaces such that $\lambda\in\mathrm{Mah}_{\mathrm{cr}}(\prod_{\xi<\kappa}X_\xi)$;

(iii) there is a Boolean-independent family $\langle b_\xi\rangle_{\xi<\kappa}$ in $\mathfrak{B}_\lambda$ with the following property: for every $a\in\mathfrak{B}_\lambda$ there is a countable set $J\subseteq\kappa$ such that the subalgebras generated by $\{a\}\cup\{b_\xi:\xi\in J\}$ and $\{b_\eta:\eta\in\kappa\setminus J\}$ are Boolean-independent.

proof If $\kappa=\omega$ then $\lambda=\omega$ and (i)-(iii) are all true. So we may assume that κ is uncountable.

(i)⇒(ii) is trivial.

(ii)⇒(iii)(*α*) If $\lambda\in\mathrm{Mah}_{\mathrm{cr}}(X)$, where every X_ξ is a non-trivial separable metrizable space and $X=\prod_{\xi<\kappa}X_\xi$, let μ be a Maharam-type-homogeneous completion regular topological probability measure on X with Maharam type λ. By 532Ha and 4A3Nc, the support Z of μ is determined by coordinates in a countable subset L of κ.

(*β*) Let $\mathfrak{A}$ be the measure algebra of μ. For each $\xi<\kappa$, let $f_\xi:X_\xi\to[0,1]$ be a continuous function taking both values 0 and 1; let $t_\xi\in\,]0,1[$ be such that $\mu\{x:x\in X,\ f_\xi(x(\xi))=t_\xi\}=0$. Set $U_\xi=\{x:f_\xi(x(\xi))<t_\xi\}$, $V_\xi=\{x:f_\xi(x)>t_\xi\}$; then U_ξ and V_ξ are disjoint non-empty open sets in X, both determined by coordinates in $\{\xi\}$, and $\mu(U_\xi\cup V_\xi)=1$. Set $b_\xi=U_\xi^\bullet$ in $\mathfrak{A}$. Then $\langle b_\xi\rangle_{\xi<\kappa\setminus L}$ is Boolean-independent. **P** If I, $I'\subseteq\kappa\setminus L$ are disjoint finite sets, then $H=X\cap\bigcap_{\xi\in I}U_\xi\cap\bigcap_{\xi\in I'}V_\xi$ is a non-empty open set in X. As H is determined by coordinates in $I\cup I'$ and Z is determined by coordinates in L, $Z\cap H$ is non-empty and therefore non-negligible; so $\mu H>0$ and $\inf_{\xi\in I}b_\xi\setminus\sup_{\xi\in I'}b_\xi$ is non-zero in $\mathfrak{A}$. **Q**

(*γ*) If $a\in\mathfrak{A}$ let E be such that $E^\bullet=a$. By 532Hb, we can choose for each $n\in\mathbb{N}$ self-supporting zero sets $K_n\subseteq E$, $\tilde K_n\subseteq X\setminus E$ such that $\mu K_n+\mu\tilde K_n\ge1-2^{-n}$. Let $J\subseteq\kappa\setminus L$ be a countable set such that every K_n and every $\tilde K_n$ is determined by coordinates in $J\cup L$. Now the subalgebras $\mathfrak{D}_1$, $\mathfrak{D}_2$ generated by $\{a\}\cup\{b_\xi:\xi\in J\}$ and $\{b_\xi:\xi\in(\kappa\setminus L)\setminus J\}$ are Boolean-independent. **P** Take non-zero $d_1\in\mathfrak{D}_1$ and $d_2\in\mathfrak{D}_2$. Suppose for the moment that $d_1\cap a\ne0$. As in (β), there is an open set G, determined by coordinates in J, such that $0\ne a\cap G^\bullet\subseteq d_1$. There is also an open set H, determined by coordinates in $\kappa\setminus(J\cup L)$, such that $0\ne H^\bullet\subseteq d_2$. Next, as $a=\sup_{n\in\mathbb{N}}K_n^\bullet$, there is an $n\in\mathbb{N}$ such that $0\ne K_n^\bullet\cap G^\bullet$, that is, $K_n\cap G\ne\emptyset$. As $K_n\cap G$ is determined by coordinates in $J\cup L$ and H is determined by coordinates in $\kappa\setminus(J\cup L)$, $K_n\cap G\cap V\ne\emptyset$; as K_n is self-supporting,

$$0\ne(K_n\cap G\cap H)^\bullet\subseteq d_1\cap d_2.$$

In the same way, using K_n' in place of K_n, we see that $d_1\cap d_2\ne0$ if $d_1\setminus a\ne0$. As d_1 and d_2 are arbitrary, $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are Boolean-independent. **Q**

(*δ*) As $\#(\kappa\setminus L)=\kappa$ and $\mathfrak{A}\cong\mathfrak{B}_\lambda$, $\langle b_\xi\rangle_{\xi\in\kappa\setminus L}$, suitably reinterpreted, witnesses that (iii) is satisfied.

(iii)⇒(i) Now suppose that the conditions of (iii) are satisfied. Let (Z,ν) be the Stone space of $(\mathfrak{B}_\lambda,\bar\nu_\lambda)$. (See 411P for a summary of the properties of these spaces.) For $b\in\mathfrak{B}_\lambda$ write $\widehat{b}$ for the corresponding open-and-closed subset of Z. Define $\phi:Z\to\{0,1\}^\kappa$ by setting $\phi(z)=\langle\chi\widehat{b}_\xi(z)\rangle_{\xi<\kappa}$ for $z\in Z$. Then ϕ is continuous; let $\mu=\nu\phi^{-1}$ be the image Radon measure on $\{0,1\}^\kappa$ (418I). Now μ is completion regular. **P** Suppose that $K\subseteq\{0,1\}^\kappa$ is compact and self-supporting. Identifying $\mathfrak{B}_\lambda$ with the measure algebra of ν, we have a Boolean homomorphism $\psi:\mathrm{dom}\,\mu\to\mathfrak{B}_\lambda$ defined by setting $\psi E=(\phi^{-1}[E])^\bullet$ whenever μ measures E, and $\bar\nu_\lambda\psi E=\nu\phi^{-1}[E]=\mu E$ for every E; setting $E_\xi=\{x:x\in\{0,1\}^\kappa,\ x(\xi)=1\}$, $\psi E_\xi=b_\xi$. Set $a=\psi K$. Let $J\subseteq\kappa$ be a countable set such that the subalgebras $\mathfrak{D}_1$, $\mathfrak{D}_2$ generated by $\{a\}\cup\{b_\xi:\xi\in J\}$ and $\{b_\eta:\eta\in\kappa\setminus J\}$ are Boolean-independent. **?** If $x\in K$, $y\in\{0,1\}^\kappa\setminus K$ and $x\restriction J=y\restriction J$, let U be an open cylinder containing y and disjoint from K. Express U as $U'\cap U''$ where U' is determined by coordinates in J and U'' by coordinates in $\kappa\setminus J$. Then $\psi U'\in\mathfrak{D}_1$ and $\psi U''\in\mathfrak{D}_2$. As $\langle b_\xi\rangle_{\xi<\kappa}$ is Boolean-independent, $\psi U''\ne0$. Now K is self-supporting and $x\in K\cap U'$, so $\mu(K\cap U')>0$ and $\psi(K\cap U')=a\cap\psi U'$ is non-zero; also $a\cap\psi U'\in\mathfrak{D}_1$; because $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are Boolean-independent, $\psi(K\cap U)=a\cap\psi U'\cap\psi U''\ne0$ and $K\cap U$ cannot be empty, contrary to the choice of U. **X**

This shows that K is determined by coordinates in J and is a zero set (4A3Nc, in the other direction). As K is arbitrary, we see that all self-supporting compact sets are zero sets. But as μ is a Radon measure, it is inner regular with respect to the self-supporting compact sets, therefore with respect to the zero sets, and is completion regular. **Q**

The inverse-measure-preserving function ϕ (and, of course, the Boolean homomorphism ψ) correspond to an embedding of the measure algebra of μ into $\mathfrak{B}_\lambda$. So the Maharam type of μ is at most λ. There is therefore a $\lambda' \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$ such that $\lambda' \le \lambda$ (532B). By 532G, $\lambda \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$.

532J Corollary (a) Suppose that λ, κ are infinite cardinals and $\lambda \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$. Then κ is at most the cardinal power λ^ω.

(b) If κ is an infinite cardinal such that $\lambda^\omega < \kappa$ for every $\lambda < \kappa$ (e.g., $\kappa = \mathfrak{c}^+$), then $\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa) = \{\kappa\}$.

proof (a) By 532I, $\kappa \le \#(\mathcal{B}_\lambda)$; by 524Ma, $\#(\mathcal{B}_\lambda) \le \lambda^\omega$.

(b) By (a), no infinite cardinal less than κ can belong to $\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$. Also κ is uncountable, so the remarks in 532C tell us the rest of what we need.

532K Corollary If $\omega \le \lambda \le \kappa' \le \kappa$ and $\lambda \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$ then $\lambda \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^{\kappa'})$.

proof If $\langle b_\xi\rangle_{\xi<\kappa}$ witnesses the truth of 532I(iii) for λ and κ, then its subfamily $\langle b_\xi\rangle_{\xi<\kappa'}$ witnesses the truth of 532I(iii) for λ and κ'. **P** Of course $\langle b_\xi\rangle_{\xi<\kappa'}$ is Boolean-independent. If $a \in \mathfrak{B}_\lambda$, there is a countable set $J \subseteq \kappa$ such that the subalgebras generated by $\{a\} \cup \{b_\xi : \xi \in J\}$ and $\{b_\eta : \eta \in \kappa \setminus J\}$ are Boolean-independent. Now $J' = J \cap \kappa'$ is a countable subset of κ' and the subalgebras generated by $\{a\} \cup \{b_\xi : \xi \in J'\}$ and $\{b_\eta : \eta \in \kappa' \setminus J'\}$ are Boolean-independent. **Q**

532L Corollary If $\omega \le \lambda \le \lambda'$ and $\mathrm{cf}[\lambda']^{\le\lambda} < \mathrm{cf}\,\kappa$ and $\lambda' \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$, then $\lambda \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$.

proof Let $\langle b_\xi\rangle_{\xi<\kappa}$ be a family in $\mathfrak{B}_{\lambda'}$ satisfying (iii) of 532I. Let $\langle e_\eta\rangle_{\eta<\lambda'}$ be the standard generating family in $\mathfrak{B}_{\lambda'}$, and $\mathcal{J}$ a cofinal subset of $[\lambda']^\lambda$ with cardinal less than $\mathrm{cf}\,\kappa$. For each $\xi < \kappa$, there are a countable set $L \subseteq \lambda'$ such that b_ξ belongs to the closed subalgebra $\mathfrak{C}_L$ of $\mathfrak{B}_{\lambda'}$ generated by $\{e_\eta : \eta \in L\}$, and a $J_\xi \in \mathcal{J}$ such that $L \subseteq J_\xi$. Because $\#(J) < \mathrm{cf}\,\kappa$, there is a $J \in \mathcal{J}$ such that $A = \{\xi : \xi < \kappa,\ J_\xi = J\}$ has cardinal κ. Now the closed subalgebra $\mathfrak{C}_J$ of $\mathfrak{B}_{\lambda'}$ generated by $\{e_\eta : \eta \in J\}$ is isomorphic to $\mathfrak{B}_\lambda$, and the Boolean-independent $\langle b_\xi\rangle_{\xi\in A}$ in $\mathfrak{C}_J$ witnesses that 532I(iii) is true of λ and κ, as in the proof of 532K.

532M I turn now to the question of identifying those κ for which $\omega \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$. We know from 532C and 532Ja that they all lie between ω and $\mathfrak{c}$. To go farther we need to look at some of the cardinals from §522.

Proposition If $A \subseteq \mathfrak{B}_\omega \setminus \{0\}$ and $\#(A) < \mathfrak{d} = \mathrm{cf}(\mathbb{N}^\mathbb{N})$, then there is a $c \in \mathfrak{B}_\omega$ such that neither c nor $1 \setminus c$ includes any member of A.

proof Let $\langle e_n\rangle_{n\in\mathbb{N}}$ be the standard generating family in $\mathfrak{B}_\omega = \mathfrak{B}_\mathbb{N}$. For $a \in \mathfrak{A}$ and $n \in \mathbb{N}$ let $f_a(n) \in \mathbb{N}$ be such that there is a b in the subalgebra $\mathfrak{C}_{f_a(n)^2}$ generated by $\{e_i : i < f_a(n)^2\}$ such that $\bar\nu_\omega(b \triangle a) < 2^{-n-3}\bar\mu a$. Because $\#(A) < \mathfrak{d}$, there is an $f \in \mathbb{N}^\mathbb{N}$ such that $f \not\le f_a$ for every $a \in \mathfrak{A}$; we may suppose that f is strictly increasing and $f(0) > 0$. Note that

$$f(n)^2 + n + 1 < f(n)^2 + 2f(n) + 1 \le f(n+1)^2$$

for every n. For each $n \in \mathbb{N}$, set

$$I_n = f(n)^2 \subseteq \mathbb{N}, \quad I'_n = I_{n+1} \setminus I_n,$$

$$c'_n = \inf_{f(n)^2 \le i \le f(n)^2+n+1} e_i \in \mathfrak{C}_{I'_n};$$

then $\bar\nu_\omega c'_n = 2^{-n-2}$ for each n. Define $c_n \in \mathfrak{C}_{I_n}$, for $n \in \mathbb{N}$, by setting $c_0 = 0$ and $c_{n+1} = c_n \triangle c'_n$ for each n. Then $\bar\nu_\omega(c_{n+1} \triangle c_n) = 2^{-n-2}$ for every n, so $\langle c_n\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence for the measure metric on $\mathfrak{B}_\omega$, and has a limit c. Note that

$$\sum_{i=n}^{m-1} 2^{-i-3} \le \bar\nu_\omega(c_m \triangle c_n) \le \sum_{i=n}^{m-1} 2^{-i-2}$$

whenever $m \ge n$. **P** Induce on m. For $m = n$ the result is trivial (interpreting $\sum_{i=n}^{n-1}$ as zero). For the inductive step to $m+1$, $c'_m \in \mathfrak{C}_{I'_m}$ is stochastically independent of $c_m \triangle c_n \in \mathfrak{C}_{I_m}$, so

$$\begin{aligned}\bar{\nu}_\omega(c_{m+1} \triangle c_n) &= \bar{\nu}_\omega(c'_m \triangle c_m \triangle c_n)\\ &= \bar{\nu}_\omega c'_m + \bar{\nu}_\omega(c_m \triangle c_n) - 2\bar{\nu}_\omega(c'_m \cap (c_m \triangle c_n))\\ &= 2^{-m-2} + (1 - 2^{-m-1})\bar{\nu}_\omega(c_m \triangle c_n)\\ &\ge 2^{-m-2} + (1 - 2^{-m-1}) \sum_{i=n}^{m-1} 2^{-i-3}\end{aligned}$$

(by the inductive hypothesis)

$$= \sum_{i=n}^{m-1} 2^{-i-3} + 2^{-m-3}(2 - 4\sum_{i=n}^{m-1} 2^{-i-3}) \ge \sum_{i=n}^{m} 2^{-i-3};$$

on the other hand,

$$\bar{\nu}_\omega(c_{m+1} \triangle c_n) \le 2^{-m-2} + \bar{\nu}_\omega(c_m \triangle c_n) \le \textstyle\sum_{i=n}^{m} 2^{-i-2}.$$

So the induction proceeds. **Q** Taking the limit as $m \to \infty$, we see that $2^{-n-2} \le \bar{\nu}_\omega(c \triangle c_n) \le 2^{-n-1}$ for every $n \in \mathbb{N}$.

Take any $a \in A$. Let $n \in \mathbb{N}$ be such that $f_a(n) < f(n)$. Then there is a $b \in \mathfrak{C}_{I_n}$ such that $\bar{\nu}_\omega(a \triangle b) < 2^{-n-3}\bar{\mu}a$. Now $c \triangle c_n \in \mathfrak{C}_{\mathbb{N}\setminus I_n}$ is stochastically independent of both $b \setminus c_n$ and $b \cap c_n$, so

$$\begin{aligned}\bar{\nu}_\omega(b \setminus c) &= \bar{\nu}_\omega(((b \setminus c_n) \setminus (c \triangle c_n)) \cup ((b \cap c_n) \cap (c \triangle c_n)))\\ &= \bar{\nu}_\omega(b \setminus c_n)(1 - \bar{\nu}_\omega(c \triangle c_n)) + \bar{\nu}_\omega(b \cap c_n) \cdot \bar{\nu}_\omega(c \triangle c_n)\\ &\ge \bar{\nu}_\omega(b \setminus c_n)(1 - 2^{-n-1}) + 2^{-n-2}\bar{\nu}_\omega(b \cap c_n) \ge 2^{-n-2}\bar{\nu}_\omega b \ge 2^{-n-3}\bar{\nu}_\omega a.\end{aligned}$$

So

$$\bar{\nu}_\omega(a \setminus c) \ge 2^{n-3}\bar{\nu}_\omega a - \bar{\nu}_\omega(b \setminus a) > 0,$$

and $a \not\subseteq c$. Similarly,

$$\begin{aligned}\bar{\nu}_\omega(b \cap c) &= \bar{\nu}_\omega(b \cap c_n)(1 - \bar{\nu}_\omega(c \triangle c_n)) + \bar{\nu}_\omega(b \setminus c_n) \cdot \bar{\nu}_\omega(c \triangle c_n)\\ &\ge \bar{\nu}_\omega(b \cap c_n)(1 - 2^{-n-1}) + 2^{-n-2}\bar{\nu}_\omega(b \setminus c_n) \ge 2^{-n-2}\bar{\nu}_\omega b,\end{aligned}$$

and $\bar{\nu}_\omega(a \cap c) > 0$.

As a is arbitrary, we have found an appropriate c.

532N It will be useful to have a classic example relevant to a question already examined in 325F.

Lemma There is a Borel set $W \subseteq \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$ such that whenever E, $F \subseteq \{0,1\}^{\mathbb{N}}$ have positive measure for ν_ω then neither $(E \times F) \cap W$ nor $(E \times F) \setminus W$ is negligible for the product measure ν_ω^2 on $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$.

proof (a) (Cf. 134Jb.) There is a Borel set $H \subseteq \{0,1\}^{\mathbb{N}}$ such that both H and its complement meet every non-empty open set in a set of non-zero measure. **P** For $x \in \{0,1\}^{\mathbb{N}}$ set $I_x = \{n : x(i) = 0$ for $2^n \le i < 2^{n+1}\}$. Set $H = \{x : I_x$ is finite and not empty and $\max I_x$ is even$\}$. **Q**

(b) Let $+$ be the usual group operation on $\{0,1\}^{\mathbb{N}} \cong \mathbb{Z}_2^{\mathbb{N}}$. In this group, addition and subtraction are identical, as $x + x = 0$ for every x; but the formulae may be easier to read if I use the symbol $-$ when it seems appropriate. Set $W = \{(x,y) : x, y \in \{0,1\}^{\mathbb{N}}, x - y \in H\}$.

Let E, $F \subseteq \{0,1\}^{\mathbb{N}}$ be sets of positive measure. Then $\{z : z \in \{0,1\}^{\mathbb{N}}, \nu_\omega(E \cap (F + z)) > 0\}$ is open (443C) and not empty (443Da), so meets H in a set of positive measure. Now

$$\begin{aligned}\nu_\omega^2((E \times F) \cap W) &= \nu_\omega^2\{(x,y) : x \in E, y \in F, x - y \in H\}\\ &= \nu_\omega^2\{(x,z) : x \in E, x - z \in F, z \in H\}\end{aligned}$$

(because $(x,y) \mapsto (x, x - y)$ is a measure space automorphism for ν_ω^2, as in 255Ae or 443Xa)

$$= \nu_\omega^2\{(x,z) : x \in E,\, x \in F+z,\, z \in H\}$$
$$= \int_H \nu_\omega(E \cap (F+z))\nu_\omega(dz) > 0.$$

Applying the same argument with $\{0,1\}^{\mathbb{N}} \setminus H$ in the place of H, we see that the same is true of $(E \times F) \setminus W$.

532O Proposition If $A \subseteq \mathfrak{B}_\omega \setminus \{0\}$ and $\#(A) < \operatorname{cov}\mathcal{N}_\omega$, then there is a $c \in \mathfrak{B}_\omega$ such that neither c nor $1 \setminus c$ includes any member of A.

proof Take $W \subseteq \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$ as in 532N. For $x \in \{0,1\}^{\mathbb{N}}$, set $c_x = W[\{x\}]^\bullet$ in $\mathfrak{B}_\omega$. If $a \in A$, then $\{x : a \subseteq c_x\} \in \mathcal{N}_\omega$. **P** Let $F \in \mathrm{T}_\omega$ be such that $F^\bullet = a$, and set $E = \{x : a \subseteq c_x\}$. Because $x \mapsto c_x$ is measurable when $\mathfrak{B}_\omega$ is given its measure-algebra topology (418Ta), $E \in \mathrm{T}_\omega$. For every $x \in E$, $F \setminus W[\{x\}]$ is negligible, so $(E \times F) \setminus W$ is negligible, by Fubini's theorem (252D). But this means that at least one of E, F must be negligible; since $F^\bullet = a \neq 0$, $\nu_\omega E = 0$, as required. **Q**

Similarly, $\{x : a \cap c_x = 0\}$ is negligible. Since $\{0,1\}^{\mathbb{N}}$ cannot be covered by $\#(A)$ negligible sets, there is an $x \in \{0,1\}^{\mathbb{N}}$ such that c_x neither includes, nor is disjoint from, any member of A.

532P Proposition Set $\kappa = \max(\mathfrak{d}, \operatorname{cov}\mathcal{N}_\omega)$. If $\mathrm{FN}(\mathcal{P}\mathbb{N}) = \omega_1$, then $\omega \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$. In particular, if $\mathfrak{c} = \omega_1$ then $\omega \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^{\omega_1})$.

proof (a) By 524O(b-ii), $\mathrm{FN}(\mathfrak{B}_\omega) = \omega_1$; let $f : \mathfrak{B}_\omega \to [\mathfrak{B}_\omega]^{\le\omega}$ be a Freese-Nation function. By 532M (if $\kappa = \mathfrak{d}$) or 532O (if $\kappa = \operatorname{cov}\mathcal{N}_\omega$), we can choose inductively a family $\langle b_\xi \rangle_{\xi<\kappa}$ in $\mathfrak{B}_\omega$ such that neither b_ξ nor $1 \setminus b_\xi$ includes any nonzero member of $\mathfrak{D}_\xi$, where $\mathfrak{D}_\xi$ is the smallest subalgebra of $\mathfrak{B}_\omega$ including $\{b_\eta : \eta < \xi\}$ and such that $f(d) \subseteq \mathfrak{D}_\xi$ for every $d \in \mathfrak{D}_\xi$. Of course this implies that $\langle b_\xi \rangle_{\xi<\kappa}$ is Boolean-independent.

(b) For K, $L \subseteq \kappa$ set $d_{KL} = \inf_{\xi\in K} b_\xi \setminus \sup_{\xi\in L} b_\xi$. For $a \in \mathfrak{B}_\omega$, set $Q_a = \{(K,L) : K,\, L \in [\kappa]^{<\omega}$ are disjoint, $d_{KL} \subseteq a\}$, and let Q'_a be the set of minimal members of Q_a, taking $(K,L) \le (K',L')$ if $K \subseteq K'$ and $L \subseteq L'$. Of course Q_a is well-founded so Q'_a is coinitial with Q_a. Now $R_{an} = \{(K,L) : (K,L) \in Q'_a$, $\#(K \cup L) = n\}$ is countable for every $n \in \mathbb{N}$ and $a \in \mathfrak{B}_\omega$. **P** Induce on n. If $n = 0$ this is trivial. For the inductive step to $n+1$, set $R'_\zeta = \{(K,L) : K \cup L \subseteq \zeta,\, (K \cup \{\zeta\}, L) \in R_{a,n+1}\}$ for each $\zeta < \kappa$. For $(K,L) \in R'_\zeta$, $b_\zeta \cap d_{KL} = d_{K\cup\{\zeta\},L}$ is included in a, so there is a $c_{KL\zeta} \in f(d_{K\cup\{\zeta\},L}) \cap f(a)$ such that $d_{K\cup\{\zeta\},L} \subseteq c_{KL\zeta} \subseteq a$, in which case $b_\zeta \subseteq c_{KL\zeta} \cup (1 \setminus d_{KL})$. If $\zeta < \zeta' < \kappa$, $(K,L) \in R'_\zeta$ and $(K',L') \in R'_{\zeta'}$, then $d_{K'L'} \not\subseteq a$ (because $(K' \cup \{\zeta'\}, L')$ is a minimal member of Q_a), so $c_{KL\zeta} \cup (1 \setminus d_{K'L'}) \neq 1$; as $c_{KL\zeta}$ and $d_{K'L'}$ both belong to $\mathfrak{D}_{\zeta'}$, $b_{\zeta'} \not\subseteq c_{KL\zeta} \cup (1 \setminus d_{K'L'})$ and $c_{KL\zeta} \neq c_{K'L'\zeta'}$. As $f(a)$ is countable, $A = \{\zeta : R'_\zeta \neq \emptyset\}$ is countable. Next, for any $\zeta \in A$ and $(K,L) \in R'_\zeta$, we see that $d_{KL} \subseteq a \cup (1 \setminus b_\zeta)$, and indeed that $(K,L) \in Q'_{a\cup(1\setminus b_\zeta)}$, so that $(K,L) \in R_{a\cup(1\setminus b_\zeta),n}$. By the inductive hypothesis, R'_ζ is countable.

This shows that $\{(K,L,\zeta) : K \cup L \subseteq \zeta,\, (K \cup \{\zeta\}, L) \in R_{a,n+1}\}$ is countable. In the same way, applying the ideas above to $1 \setminus b_\zeta$ in place of b_ζ, $\{(K,L,\zeta) : K \cup L \subseteq \zeta,\, (K, L \cup \{\zeta\}) \in R_{a,n+1}\}$ is countable; so $R_{a,n+1}$ is countable and the induction proceeds. **Q**

It follows that Q'_a is countable for every $a \in \mathfrak{B}_\omega$.

(c) Now take any $a \in \mathfrak{B}_\omega$ and let $J \subseteq \kappa$ be a countable set such that $K \cup L \subseteq J$ whenever $(K,L) \in Q'_a \cup Q'_{1\setminus a}$. **?** Suppose, if possible, that the algebras $\mathfrak{C}_1$, $\mathfrak{C}_2$ generated by $\{a\} \cup \{b_\xi : \xi \in J\}$ and $\{b_\eta : \eta \in \kappa \setminus J\}$ are not Boolean-independent. Then there must be finite subsets K, L, K' and L' of κ such that $K \cup L \subseteq J$, $K' \cup L' \subseteq \kappa \setminus J$, $d_{K'L'} \neq 0$, and either

$$d_{KL} \cap a \neq 0,\ d_{K'L'} \cap d_{KL} \cap a = 0$$

or

$$d_{KL} \setminus a \neq 0,\ d_{K'L'} \cap d_{KL} \setminus a = 0.$$

Suppose the former. Then $(K \cup K', L \cup L') \in Q_{1\setminus a}$ so there is a $(K'', L'') \in Q'_{1\setminus a}$ such that $K'' \subseteq K \cup K'$ and $L'' \subseteq L \cup L'$; in which case $K'' \cup L'' \subseteq J$ so in fact $K'' \subseteq K$, $L'' \subseteq L$ and $d_{KL} \cap a \subseteq d_{K''L''} \cap a = 0$, which is impossible. Replacing a by $1 \setminus a$ we get a similar contradiction in the second case. **X** So $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are Boolean-independent.

(d) As a is arbitrary, (c) shows that $\langle b_\xi \rangle_{\xi<\kappa}$ satisfies the conditions of 532I(iii), so that ω belongs to $\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$, as claimed.

532Q Proposition Suppose that $\operatorname{add}\mathcal{N}_\omega > \omega_1$.
(a) $\lambda \notin \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$ whenever $\lambda \ge \omega$ and $\max(\omega, \operatorname{cf}[\lambda]^{\le\omega}) < \kappa$.
(b) If $\omega_1 \le \kappa \le \omega_\omega$ then $\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa) = \{\kappa\}$.

proof (a) ? If $\lambda \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$, set $\kappa' = (\max(\omega, \operatorname{cf}[\lambda]^{\le\omega}))^+$; then $\lambda < \kappa'$ so $\lambda \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^{\kappa'})$ (532K). As $\operatorname{cf}[\lambda]^{\le\omega} < \operatorname{cf}\kappa'$, ω belongs to $\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^{\lambda^+})$ (532L) and therefore to $\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^{\omega_1})$ (532K again).

Let $\langle b_\xi\rangle_{\xi<\omega_1}$ be a family in $\mathfrak{B}_\omega$ satisfying the conditions of 532I(iii). By 524Mb, $\omega_1 < \operatorname{wdistr}(\mathfrak{B}_\omega)$; by 514K, there is a countable $C \subseteq \mathfrak{B}_\omega \setminus \{0\}$ such that for every $\xi < \omega_1$ there is a $c \in C$ such that $c \subseteq b_\xi$. Let $a \in C$ be such that $\{\xi : \xi < \omega_1, a \subseteq b_\xi\}$ is uncountable. There is supposed to be a countable $J \subseteq \omega_1$ such that the subalgebras generated by $\{a\}$ and $\{b_\xi : \xi \in \omega_1 \setminus J\}$ are Boolean-independent; but then $\{\xi : a \subseteq b_\xi\} \subseteq J$, which is impossible. **X**

This shows that (a) is true.

(b) If $\omega \le \lambda < \kappa \le \omega_\omega$, then $\operatorname{cf}[\lambda]^{\le\omega} \le \lambda < \kappa$ (5A1F(e-iv)), so (a) tells us that $\lambda \notin \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$. From 532C we see that $\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$ must be $\{\kappa\}$ exactly.

532R Two combinatorial principles already used in 524O are relevant to the questions treated here.

Proposition Suppose that λ is an uncountable cardinal with countable cofinality such that $\square_\lambda$ (definition: 5A6D(a-ii)) is true. Set $\kappa = \lambda^+$. Then $\lambda \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$.

proof (a) Let $\langle I_\xi\rangle_{\xi<\kappa}$ be a family of countably infinite subsets of λ as in 5A6E. For each $\xi < \kappa$, let $\langle I_{\xi n}\rangle_{n\in\mathbb{N}}$, $\langle \alpha_{\xi n}\rangle_{n\in\mathbb{N}}$ be such that $\langle I_{\xi n}\rangle_{n\in\mathbb{N}}$ is a disjoint sequence of subsets of I_ξ with $\#(I_{\xi n}) = n$ for each n and $\langle \alpha_{\xi n}\rangle_{n\in\mathbb{N}}$ is a sequence of distinct points in $I_\xi \setminus \bigcup_{n\in\mathbb{N}} I_{\xi n}$. Set

$$U_{\xi n} = \{x : x \in \{0,1\}^\lambda,\ x(\eta) = 0 \text{ for every } \eta \in I_{\xi n}\},$$

$$V_{\xi n} = \{x : x \in U_{\xi n} \setminus \bigcup_{m>n} U_{\xi m},\ x(\alpha_{\xi n}) = 1\},$$

$$\tilde V_{\xi n} = \{x : x \in U_{\xi n} \setminus \bigcup_{m>n} U_{\xi m},\ x(\alpha_{\xi n}) = 0\}$$

for $n \in \mathbb{N}$. Note that as $\nu_\kappa U_{\xi m} = 2^{-m}$ for each n, $V_{\xi n}$ and $\tilde V_{\xi n}$ are non-negligible, while both are determined by coordinates in $\{\alpha_{\xi n}\} \cup \bigcup_{m\ge n} I_{\xi m} \subseteq I_\xi$. Set

$$F_\xi = \bigcup_{n\in\mathbb{N}} V_{\xi n}, \quad b_\xi = F_\xi^\bullet \in \mathfrak{B}_\lambda.$$

Note that $F_\xi \cap \tilde V_{\xi n} = \emptyset$ for every n.

(b) Take any $a \in \mathfrak{B}_\lambda$. Then we can express a as $E^\bullet$ where $E \subseteq \{0,1\}^\lambda$ is a Baire set; let $I \subseteq \lambda$ be a countable set such that E is determined by coordinates in I. By the choice of $\langle I_\xi\rangle_{\xi<\kappa}$ there is a countable set $J \subseteq \kappa$ such that $I \cap I_\xi$ is finite for every $\xi \in \kappa \setminus J$. Let $\mathfrak{D}_1$, $\mathfrak{D}_2$ be the subalgebras of $\mathfrak{B}_\lambda$ generated by $\{a\} \cup \{b_\xi : \xi \in J\}$ and $\{b_\xi : \xi \in \kappa \setminus J\}$ respectively. Then $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are Boolean-independent. **P** If $d_1 \in \mathfrak{D}_1$ and $d_2 \in \mathfrak{D}_2$ are non-zero, we can express d_1 as $H_1^\bullet$ where $H_1 \subseteq \{0,1\}^\lambda$ is a Baire set determined by coordinates in $L = I \cup \bigcup_{\xi\in K} I_\xi$ for some finite $K \subseteq J$. Next, we can find disjoint finite sets K', $K'' \subseteq \kappa \setminus J$ such that $d_2 \supseteq \inf_{\xi\in K'} b_\xi \setminus \sup_{\xi\in K''} b_\xi$. Because all the sets $I_\xi \cap I_\eta$, for distinct ξ, $\eta < \kappa$, and also the sets $I \cap I_\xi$, for $\xi \in \kappa \setminus J$, are finite, there is an $m \in \mathbb{N}$ such that all the sets $J_\xi = \{\alpha_{\xi m}\} \cup \bigcup_{n\ge m} I_{\xi n}$, for $\xi \in K' \cup K''$, are disjoint from each other and from I. Look at the sets $V_{\xi m}$, for $\xi \in K'$, and $\tilde V_{\xi m}$, for $\xi \in K''$. Set $H_2 = \{0,1\}^\lambda \cap \bigcap_{\xi\in K'} V_{\xi m} \cap \bigcap_{\xi\in K''} \tilde V_{\xi m}$. Then $H_2^\bullet \subseteq d_2$. But observe now that all the $V_{\xi m}$ and $\tilde V_{\xi m}$ are non-negligible and that $V_{\xi m}$, $\tilde V_{\xi m}$ are determined by coordinates in J_ξ for each $\xi \in K' \cup K''$. So the sets H_1, $V_{\xi m}$ (for $\xi \in K'$) and $\tilde V_{\xi m}$ (for $\xi \in K''$) are stochastically independent, and

$$\bar\nu_\lambda(d_1 \cap d_2) \ge \nu_\lambda(H_1 \cap H_2) = \nu_\lambda H_1 \cdot \prod_{\xi\in K'} \nu_\lambda V_{\xi m} \cdot \prod_{\xi\in K''} \nu_\lambda \tilde V_{\xi m} > 0.$$

Thus $d_1 \cap d_2 \ne 0$; as d_1 and d_2 are arbitrary, $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are stochastically independent. **Q**

(c) The argument of (b) works equally well with $I = \emptyset$ and J an arbitrary finite subset of κ to show that $\langle b_\xi\rangle_{\xi<\kappa}$ is Boolean-independent. So the conditions of 532I(iii) are satisfied and $\kappa \in \mathrm{Mah}_{\mathrm{crR}}(\lambda)$, as claimed.

532S Proposition Suppose that $\operatorname{add}\mathcal{N}_\omega > \omega_1$ and that λ is an infinite cardinal such that $\mathrm{CTP}(\lambda^+, \lambda)$ (definition: 5A6Fa) is true. Then $\lambda \notin \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$ for any $\kappa > \lambda$.

proof By 532K, it is enough to consider the case $\kappa = \lambda^+$. **?** Suppose, if possible, that there is a family $\langle b_\xi\rangle_{\xi<\kappa}$ in $\mathfrak{B}_\lambda$ satisfying the conditions of 532I(iii). Let $\langle e_\eta\rangle_{\eta<\lambda}$ be the standard generating family in $\mathfrak{B}_\lambda$. Then for each $\xi < \kappa$ we have a countable set $I_\xi \subseteq \lambda$ such that b_ξ belongs to the closed subalgebra of $\mathfrak{B}_\lambda$ generated by $\{e_\eta : \eta \in I_\xi\}$. Because $\mathrm{CTP}(\kappa, \lambda)$ is true, there is an uncountable set $A \subseteq \kappa$ such that $J = \bigcup_{\xi\in A} I_\xi$ is countable (5A6F(b-ii)). Now the closed subalgebra $\mathfrak{C}_J$ generated by $\{e_\eta : \eta \in J\}$ is isomorphic to $\mathfrak{B}_\omega$, so $\langle b_\xi\rangle_{\xi\in A}$ witnesses that $\omega \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^{\omega_1})$; but this contradicts 532Qa. **X**

532X Basic exercises (a) Let X be a normal Hausdorff space and $Y \subseteq X$ a zero set. Show that $\mathrm{Mah}_{\mathrm{crR}}(Y) \subseteq \mathrm{Mah}_{\mathrm{crR}}(X)$.

(b) Let $\beta\mathbb{N}$ be the Stone-Čech compactification of $\mathbb{N}$. (i) Show that $\mathrm{Mah}_{\mathrm{crR}}(\beta\mathbb{N}) = \{0\}$. (*Hint*: non-empty zero sets in $\beta\mathbb{N} \setminus \mathbb{N}$ are never ccc.) (ii) Give an example of a non-empty compact Hausdorff space X such that $\mathrm{Mah}_{\mathrm{crR}}(X) = \emptyset$.

(c) Let X and Y be compact Hausdorff spaces. Show that $\mathrm{Mah}_{\mathrm{crR}}(X \times Y) \subseteq \mathrm{Mah}_{\mathrm{crR}}(X) \cup \mathrm{Mah}_{\mathrm{crR}}(Y)$. (*Hint*: 434U.)

(d) Let λ and κ be infinite cardinals such that $\lambda \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$. (i) Show that there is a strictly positive Maharam-type-homogeneous completion regular Radon probability measure on $\{0,1\}^\kappa$ with Maharam type λ. (ii) Suppose that λ is uncountable and that $H \subseteq \{0,1\}^\kappa$ is a non-empty G_δ set. Show that $\lambda \in \mathrm{Mah}_{\mathrm{crR}}(H)$.

(e) Find a proof of 532E which does not rely on 532D. (*Hint*: 415E.)

(f) Let $\langle (X_i, \mu_i)\rangle_{i\in I}$ be a family of quasi-dyadic spaces with strictly positive completion regular topological probability measures. Show that the ordinary product measure on $\prod_{i\in I} X_i$ is a strictly positive completion regular τ-additive topological probability measure.

532Y Further exercises (a) Let Z be the Stone space of $\mathfrak{B}_\lambda$, where $\lambda \geq \omega$. (i) Show that if $F \subseteq Z$ is a non-empty nowhere dense zero set then it is not ccc. (ii) Show that $\mathrm{Mah}_{\mathrm{crR}}(Z) = \{\lambda\}$. (iii) Show that $\mathrm{Mah}_{\mathrm{crR}}(Z \times Z) = \emptyset$.

(b) Let $\langle X_i\rangle_{i\in I}$ be a family of topological spaces with countable networks, and Y any topological space. Suppose that we are given a strictly positive topological probability measure μ_i on each X_i, and a τ-additive topological probability measure ν on Y. Show that the ordinary product measure on $\prod_{i\in I} X_i \times Y$ is a topological measure.

(c) Suppose that $\mathrm{FN}(\mathcal{P}\mathbb{N}) = \omega_1$. Show that there are a Hausdorff space X and a completion regular Radon measure μ on X such that the Maharam type of μ is ω, but the Maharam type of $\mu\restriction\mathcal{B}(X)$ is ω_1. (*Hint*: 419C.)

532Z Problems (a) In 532P, can we take $\kappa = \operatorname{cf}\mathcal{N}_\omega$?

(b) We have $\omega \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^{\omega_1})$ if $\mathrm{FN}(\mathcal{P}\mathbb{N}) = \omega_1$ (532P, 532K) and not if $\operatorname{add}\mathcal{N}_\omega > \omega_1$ (532Q). Can we narrow the gap?

(c) For a Hausdorff space X let $\mathrm{Mah}_{\mathrm{spcrR}}(X)$ be the set of Maharam types of strictly positive Maharam homogeneous completion regular Radon measures on X. Describe the sets Γ of cardinals for which there are compact Hausdorff spaces X such that $\mathrm{Mah}_{\mathrm{spcrR}}(X) = \Gamma$.

532 Notes and comments I have spent a good many pages on a rather specialized topic. But I think the patterns here are instructive. When looking at $\mathrm{Mah}_{\mathrm{R}}(X)$, as in §531, we quickly come to feel that it is a measure of a certain kind of complexity; the richer the space X, the larger $\mathrm{Mah}_{\mathrm{R}}(X)$ will be. 531Eb and 531Ed are direct manifestations of this, and 531G develops the theme. $\mathrm{Mah}_{\mathrm{crR}}(X)$ can sometimes tell us more about X; knowing $\mathrm{Mah}_{\mathrm{crR}}(X)$ we may have a lower bound on the complexity of X as well as an upper bound. (On the other hand, $\mathrm{Mah}_{\mathrm{crR}}(X)$ can evaporate for non-trivial reasons, as in 532Xb and 532Ya, and leave us with very little idea of what X might be like.) In place of the straightforward facts in 531E, we have the relatively complex and partial results in 532G and 532K. As soon as we leave the constrained context of powers of $\{0,1\}$, the most natural questions seem to be obscure (532Zc).

However, if we follow the paths which are open, rather than those we might otherwise have chosen, we come to some interesting ideas, starting with 532I. Here, as happened in §531, we see that a proper understanding of the measure algebras $\mathfrak{B}_\lambda$ will take us a long way; and once again we find that this understanding has to be conditional on the model of set theory we are working in. Even to decide which powers of $\{0,1\}$ carry completion regular Radon measures with countable Maharam type we need to examine some new aspects of the Lebesgue measure algebra (532M-532O). Moreover, as well as the familiar cardinals of Cichoń's diagram, we have to look at the Freese-Nation number of $\mathcal{P}\mathbb{N}$ (532P). For larger Maharam types, in a way that we are becoming accustomed to, other combinatorial principles become relevant (532R, 532S).

Version of 4.1.14

533 Special topics

I present notes on certain questions which can be answered if we make particular assumptions concerning values of the cardinals considered in §§523-524. The first cluster (533A-533E) looks at Radon and quasi-Radon measures in contexts in which the additivity of Lebesgue measure is large compared with other cardinals of the structures considered. Developing ideas which arose in the course of §531, I discuss 'uniform regularity' in perfectly normal and first-countable spaces (533H). We also have a complete description of the cardinals κ for which $\mathbb{R}^\kappa$ is measure-compact (533J).

As previously, I write $\mathcal{N}(\mu)$ for the null ideal of a measure μ; ν_κ will be the usual measure on $\{0,1\}^\kappa$ and $\mathcal{N}_\kappa=\mathcal{N}(\nu_\kappa)$ its null ideal.

533A Lemma Let (X,Σ,μ) be a semi-finite measure space with measure algebra $(\mathfrak{A},\bar\mu)$. If $\langle\mathcal{K}_\xi\rangle_{\xi<\kappa}$ is a family of ideals in Σ such that μ is inner regular with respect to every $\mathcal{K}_\xi$ and $\kappa<\min(\operatorname{add}\mathcal{N}(\mu),\operatorname{wdistr}(\mathfrak{A}))$, then μ is inner regular with respect to $\bigcap_{\xi<\kappa}\mathcal{K}_\xi$.

proof Take $E\in\Sigma$ and $\gamma<\mu E$. Then there is an $E_1\in\Sigma$ such that $E_1\subseteq E$ and $\gamma<\mu E_1<\infty$. For $\xi<\kappa$, $D_\xi=\{K^\bullet:K\in\mathcal{K}_\xi\}$ is closed under finite unions and is order-dense in $\mathfrak{A}$, so includes a partition of unity A_ξ. Now there is a partition B of unity in $\mathfrak{A}$ such that $\{a:a\in A_\xi,\,a\cap b\ne 0\}$ is finite for every $b\in B$ and $\xi<\kappa$. Let $B'\subseteq B$ be a finite set such that $\bar\mu(E_1^\bullet\cap\sup B')\ge\gamma$, and let $E_2\subseteq E_1$ be such that $E_2^\bullet=E_1^\bullet\cap\sup B'$. For any $\xi<\kappa$,

$$A'_\xi=\{a:a\in A_\xi,\,a\cap E_2^\bullet\ne 0\}\subseteq\bigcup_{b\in B'}\{a:a\in A_\xi,\,a\cap b\ne 0\}$$

is finite, so $\sup A'_\xi$ belongs to D_ξ and can be expressed as $K_\xi^\bullet$ for some $K_\xi\in\mathcal{K}_\xi$. Now $E_2^\bullet\subseteq\sup A'_\xi$ so $E_2\setminus K_\xi$ is negligible. As $\kappa<\operatorname{add}\mathcal{N}(\mu)$, we have a negligible $H\in\Sigma$ including $\bigcup_{\xi<\kappa}E_2\setminus K_\xi$; now $E'=E_2\setminus H\subseteq E$, $\mu E'\ge\gamma$ and $E'\in\bigcap_{\xi<\kappa}\mathcal{K}_\xi$. As E and γ are arbitrary, μ is inner regular with respect to $\bigcap_{\xi<\kappa}\mathcal{K}_\xi$.

Remark Of course this result is covered by 412Ac unless $\operatorname{wdistr}(\mathfrak{A})>\omega_1$, which nearly forces $\mathfrak{A}$ to have countable Maharam type (524Mb).

533B Corollary Let (X,Σ,μ) be a totally finite measure space with countable Maharam type. If $\mathcal{E}\subseteq\Sigma$, $\#(\mathcal{E})<\min(\operatorname{add}\mathcal{N}_\omega,\operatorname{add}\mathcal{N}(\mu))$ and $\epsilon>0$, there is a set $F\in\Sigma$ such that $\mu(X\setminus F)\le\epsilon$ and $\{E\cap F:E\in\mathcal{E}\}$ is countable.

proof Let $(\mathfrak{A},\bar\mu)$ be the measure algebra of μ. Then $\mathfrak{A}$ is separable in its measure-algebra topology (521Ea). Let $\mathcal{H}\subseteq\Sigma$ be a countable set such that $\{H^\bullet:H\in\mathcal{H}\}$ is dense in $\mathfrak{A}$. For $E\in\mathcal{E}$ and $n\in\mathbb{N}$

choose $H_{En} \in \mathcal{H}$ such that $\mu(E \triangle H_{En}) \le 2^{-n}$; let $\mathcal{K}_E$ be the family of measurable sets K such that K is disjoint from $\bigcup_{i \ge n} E \triangle H_{Ei}$ for some n. Then μ is inner regular with respect to $\mathcal{K}_E$. Because $\#(\mathcal{E}) < \min(\mathrm{wdistr}(\mathfrak{A}), \mathrm{add}\,\mathcal{N}(\mu))$ (524Mb), μ is inner regular with respect to $\bigcap_{E \in \mathcal{E}} \mathcal{K}_E$ (533A) and there is an $F \in \bigcap \mathcal{K}_E$ such that $\mu F \ge \mu X - \epsilon$. If $E \in \mathcal{E}$, there is an $n \in \mathbb{N}$ such that $F \cap (E \triangle H_{En}) = \emptyset$, that is, $F \cap E = F \cap H_{En}$; so $\{F \cap E : E \in \mathcal{E}\} \subseteq \{F \cap H : H \in \mathcal{H}\}$ is countable.

533C Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space with countable Maharam type.

(a) If $w(X) < \mathrm{add}\,\mathcal{N}_\omega$, then μ is inner regular with respect to the second-countable subsets of X; if moreover $\mathfrak{T}$ is regular and Hausdorff, then μ is inner regular with respect to the metrizable subsets of X.

(b) If Y is a topological space of weight less than $\mathrm{add}\,\mathcal{N}_\omega$, then any measurable function $f : X \to Y$ is almost continuous.

(c) If $\langle Y_i \rangle_{i \in I}$ is a family of topological spaces, with $\#(I) < \mathrm{add}\,\mathcal{N}_\omega$, and $f_i : X \to Y_i$ is almost continuous for every i, then $x \mapsto f(x) = \langle f_i(x) \rangle_{i \in I} : X \to \prod_{i \in I} Y_i$ is almost continuous.

proof Note first that $\mathrm{add}\,\mathcal{N}(\mu) \ge \mathrm{add}\,\mathcal{N}_\omega$, by 524Ta.

(a) Let $\mathcal{U}$ be a base for $\mathfrak{T}$ with $\#(\mathcal{U}) < \mathrm{add}\,\mathcal{N}_\omega$. Set

$$\mathcal{F} = \{F : F \subseteq X,\ \{F \cap U : U \in \mathcal{U}\} \text{ is countable}\}.$$

Then μ is inner regular with respect to $\mathcal{F}$. **P** If $E \in \Sigma$ and $\gamma < \mu E$, let $H \in \Sigma$ be such that $H \subseteq E$ and $\gamma < \mu H < \infty$. Then the subspace measure μ_H still has countable Maharam type (use 322I and 514Ed) and

$$\mathrm{add}\,\mathcal{N}(\mu_H) \ge \mathrm{add}\,\mathcal{N}(\mu) \ge \mathrm{add}\,\mathcal{N}_\omega > \#(\{H \cap U : U \in \mathcal{U}\}).$$

By 533B, there is an $F \in \mathrm{dom}\,\mu_H$ such that $\mu_H F \ge \gamma$ and $\{F \cap H \cap U : U \in \mathcal{U}\}$ is countable; now $F \in \mathcal{F}$, $F \subseteq E$ and $\mu F \ge \gamma$. **Q** But every member of $\mathcal{F}$ is second-countable (use 4A2B(a-vi)). If $\mathfrak{T}$ is regular and Hausdorff, then every member of $\mathcal{F}$ is separable and metrizable (4A2Pb).

(b) If $f : X \to Y$ is measurable, let $\mathcal{V}$ be a base for the topology of Y with $\#(\mathcal{V}) < \mathrm{add}\,\mathcal{N}_\omega$. Suppose that $E \in \Sigma$ and $\gamma < \mu E$. By 533B, there is an $F \in \Sigma$ such that $F \subseteq E$, $\gamma < \mu F < \infty$ and $\{F \cap f^{-1}[V] : V \in \mathcal{V}\}$ is countable. It follows that $\{f[F] \cap V : V \in \mathcal{V}\}$ is countable, so that the subspace topology on $f[F]$ is second-countable (4A2B(a-vi) again). Giving F its subspace topology $\mathfrak{T}_F$ and measure μ_F, μ_F is inner regular with respect to the closed sets (412Pc). If $H \subseteq f[F]$ is relatively open in $f[F]$, it is of the form $G \cap f[F]$ where G is an open subset of Y, so that $(f{\restriction}F)^{-1}[H] = F \cap f^{-1}[G]$ is measured by μ_F; thus $f{\restriction}F : F \to f[F]$ is measurable. By 418J, $f{\restriction}F$ is almost continuous, and there is a $K \in \Sigma$ such that $K \subseteq F$, $\mu K \ge \gamma$ and $f{\restriction}K$ is continuous.

As E and γ are arbitrary, f is almost continuous.

(c) For each $i \in I$, set $\mathcal{K}_i = \{K : K \in \Sigma,\ f_i{\restriction}K \text{ is continuous}\}$. Then $\mathcal{K}_i$ is an ideal in Σ and μ is inner regular with respect to $\mathcal{K}_i$. Also, as in 533B, $\#(I) < \mathrm{wdistr}(\mathfrak{A})$, where $\mathfrak{A}$ is the measure algebra of μ. So μ is inner regular with respect to $\mathcal{K} = \bigcap_{i \in I} \mathcal{K}_i$, by 533A. But $f{\restriction}K$ is continuous for every $K \in \mathcal{K}$, so f is almost continuous.

533D Proposition Let $(X, \mathfrak{T})$ be a first-countable compact Hausdorff space such that $\mathrm{cf}[w(X)]^{\le\omega} < \mathrm{add}\,\mathcal{N}_\omega$, and μ a Radon measure on X with countable Maharam type. Then μ is inner regular with respect to the metrizable zero sets.

proof Set $\kappa = w(X)$. Then there is an injective continuous function $f : X \to [0,1]^\kappa$ (5A4Cc). Let $\mathcal{I}$ be a cofinal subset of $[\kappa]^{\le\omega}$ with $\#(\mathcal{I}) < \mathrm{add}\,\mathcal{N}_\omega$. By 524Pa, $\mathrm{add}\,\mu \ge \mathrm{add}\,\mathcal{N}_\omega$.

For $I \in \mathcal{I}$ and $x \in X$ set $f_I(x) = f(x){\restriction}I$. We need to know that for every $x \in X$ there is an $I \in \mathcal{I}$ such that $\{x\} = f_I^{-1}[f_I[\{x\}]]$. **P** Set $F_I = f_I^{-1}[f_I[\{x\}]]$ for each I. Because $\mathcal{I}$ is upwards-directed, $\langle F_I \rangle_{I \in \mathcal{I}}$ is downwards-directed. Because f is injective and $\bigcup \mathcal{I} = \kappa$, $\bigcap_{I \in \mathcal{I}} F_I = \{x\}$. Let $\mathcal{V}$ be a countable base of open neighbourhoods of x. For each $V \in \mathcal{V}$ there is an $I_V \in \mathcal{I}$ such that $F_{I_V} \cap (X \setminus V) = \emptyset$. Let $I \in \mathcal{I}$ be such that $\bigcup_{V \in \mathcal{V}} I_V \subseteq I$; then $F_I = \{x\}$. **Q**

For $I \in \mathcal{I}$, let λ_I be the image measure μf_I^{-1} on $[0,1]^I$; note that λ_I is a Radon measure (418I). Of course $\mathrm{add}\,\lambda_I$ is also at least $\mathrm{add}\,\mathcal{N}_\omega$, and in particular is greater than κ. If $G \subseteq X$ is open, then G and $f_I[G]$ are expressible as unions of at most κ compact sets, so λ_I measures $f_I[G]$.

There is an $I \in \mathcal{I}$ such that $\mu f_I^{-1}[f_I[G]] = \mu G$ for every open set $G \subseteq X$. **P?** Suppose, if possible, otherwise. For each $I \in \mathcal{I}$ choose an open set $G_I \subseteq X$ such that $E_I = f_I^{-1}[f_I[G_I]] \setminus G_I$ is non-negligible; because λ_I measures $f_I[G_I]$, μ measures E_I. Set $E'_I = \bigcup_{J\in\mathcal{I},J\supseteq I} E_J$ for each $I \in \mathcal{I}$; because $\#(\mathcal{I}) < \operatorname{add}\mu$, μ measures E'_I. Note that $E'_I \subseteq E'_J$ whenever $J \subseteq I$ in $\mathcal{I}$; moreover, any sequence in $\mathcal{I}$ has an upper bound in $\mathcal{I}$. There is therefore an $M \in \mathcal{I}$ such that $E'_M \setminus E'_I$ is negligible for every $I \in \mathcal{I}$. Again because $\#(\mathcal{I}) < \operatorname{add}\mu$, $E'_M \setminus \bigcap_{I\in\mathcal{I}} E'_I$ is negligible; as E'_M is not negligible, there is an $x \in \bigcap_{I\in\mathcal{I}} E'_I$. But there is an $I \in \mathcal{I}$ such that $\{x\} = f_I^{-1}[f_I[\{x\}]]$, so $x \notin E_J$ for any $J \supseteq I$. **XQ**

Let $\mathcal{U}$ be a base for the topology of X with $\#(\mathcal{U}) = \kappa$. Then $\bigcup_{U\in\mathcal{U}} f_I^{-1}[f_I[U]] \setminus U$ is μ-negligible; let Y be its complement. If $x \in X$ and $y \in Y$ and $x \neq y$, there is a $U \in \mathcal{U}$ containing x but not y, so $f_I^{-1}[f_I[U]]$ contains x and not y and $f(x) \neq f(y)$. If $F \subseteq Y$ is compact, then F is homeomorphic to the metrizable $f_I[F]$, so is metrizable, and $F = f_I^{-1}[f_I[F]]$ is a zero set. As μ is surely inner regular with respect to the compact subsets of the conegligible set Y, it is inner regular with respect to the metrizable zero sets.

533E Corollary Suppose that $\operatorname{cov}\mathcal{N}_{\omega_1} > \omega_1$. Let $(X, \mathfrak{T})$ be a first-countable K-analytic Hausdorff space such that $\operatorname{cf}[w(X)]^{\le\omega} < \operatorname{add}\mathcal{N}_\omega$. Then X is a Radon space.

proof Let μ be a totally finite Borel measure on X, $E \subseteq X$ a Borel set and $\gamma < \mu E$. Because X is K-analytic, there is a compact set $K \subseteq X$ such that $\mu(E \cap K) > \gamma$ (apply 432B to the measure $\mu \llcorner E$). Let λ be the Radon measure on K defined by saying that $\int f d\lambda = \int_K f d\mu$ for every $f \in C(K)$ (using the Riesz Representation Theorem, 436J/436K). Because $\operatorname{cov}\mathcal{N}_{\omega_1} > \omega_1$, ω_1 is a precaliber of every measurable algebra (525J); as K is first-countable, $\omega_1 \notin \mathrm{Mah}_{\mathrm{R}}(K)$ (531O) and λ must have countable Maharam type (531Ef). By 533D, λ is completion regular. But if $F \subseteq K$ is a zero set (for the subspace topology of K), there is a non-increasing sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in $C(K)$ with infimum χF, so

$$\lambda F = \lim_{n\to\infty} \int f_n d\lambda = \lim_{n\to\infty} \int_K f_n d\mu = \mu F.$$

Accordingly

$$\lambda H = \sup\{\lambda F : F \subseteq H \text{ is a zero set}\} = \sup\{\mu F : F \subseteq H \text{ is a zero set}\} \le \mu H$$

for every Borel set $H \subseteq K$. As $\lambda K = \mu K$, λ agrees with μ on the Borel subsets of K. In particular, $\lambda(E \cap K) > \gamma$; now there is a compact set $L \subseteq E \cap K$ such that $\gamma \le \lambda L = \mu L$.

As E and γ are arbitrary, μ is tight; as μ is arbitrary, X is a Radon space.

533F Definition Let X be a topological space and μ a topological measure on X. I will say that μ is **uniformly regular** if there is a countable family $\mathcal{V}$ of open sets in X such that $G \setminus \bigcup\{V : V \in \mathcal{V}, V \subseteq G\}$ is negligible for every open set $G \subseteq X$.

533G Lemma Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a compact Radon measure space.

(a) The following are equiveridical:

(i) μ is uniformly regular;

(ii) there are a metrizable space Z and a continuous function $f : X \to Z$ such that $\mu f^{-1}[f[F]] = \mu F$ for every closed $F \subseteq X$;

(iii) there is a countable family $\mathcal{H}$ of cozero sets in X such that $\mu G = \sup\{\mu H : H \in \mathcal{H}, H \subseteq G\}$ for every open set $G \subseteq X$;

(iv) there is a countable family $\mathcal{E}$ of zero sets in X such that $\mu G = \sup\{\mu E : E \in \mathcal{E}, E \subseteq G\}$ for every open set $G \subseteq X$.

(b) If $\mathfrak{T}$ is perfectly normal, the following are equiveridical:

(i) μ is uniformly regular;

(ii) there are a metrizable space Z and a continuous function $f : X \to Z$ such that $\mu f^{-1}[f[E]] = \mu E$ for every $E \in \Sigma$;

(iii) there are a metrizable space Z and a continuous function $f : X \to Z$ such that $f[G] \neq f[X]$ whenever $G \subseteq X$ is open and $\mu G < \mu X$;

(iv) there is a countable family $\mathcal{E}$ of closed sets in X such that $\mu G = \sup\{\mu E : E \in \mathcal{E}, E \subseteq G\}$ for every open set $G \subseteq X$.

proof (a)(i)⇒(iii) Given $\mathcal{V}$ as in 533F, then for each $V \in \mathcal{V}$ there is a cozero set $H_V \subseteq V$ of the same measure. **P** $\mathfrak{T}$ is completely regular, so $\mathcal{H}_V = \{H : H \subseteq V \text{ is a cozero set}\}$ has union V; μ is τ-additive, so there is a sequence $\langle H_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{H}_V$ such that $\mu V = \mu(\bigcup_{n\in\mathbb{N}} H_n)$; set $H_V = \bigcup_{n\in\mathbb{N}} H_n$; by 4A2C(b-iii), H_V is a cozero set. **Q** Now $\mathcal{H} = \{H_V : V \in \mathcal{V}\}$ witnesses that (iii) is true.

(iii)⇒(iv) Given $\mathcal{H}$ as in (iii), then for each $H \in \mathcal{H}$ let $\langle F_n(H)\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of zero sets with union H (4A2C(b-vi)). Set $\mathcal{E} = \{F_n(H) : H \in \mathcal{H},\, n \in \mathbb{N}\}$, so that $\mathcal{E}$ is a countable family of zero sets. If $G \subseteq X$ is open,

$$\mu G = \sup_{H\in\mathcal{H},H\subseteq G} \mu H = \sup_{H\in\mathcal{H},H\subseteq G,n\in\mathbb{N}} \mu F_n(H) \le \sup_{E\in\mathcal{E},E\subseteq G} \mu E \le \mu G,$$

so $\mathcal{E}$ witnesses that (iv) is true.

(iv)⇒(ii) Given $\mathcal{E}$ as in (iv), then for each $E \in \mathcal{E}$ choose a continuous $f_E : X \to \mathbb{R}$ such that $E = f_E^{-1}[\{0\}]$, and set $f(x) = \langle f_E(x)\rangle_{E\in\mathcal{E}}$ for $x \in X$. Then $f : X \to Z = \mathbb{R}^{\mathcal{E}}$ is continuous and Z is metrizable and $f^{-1}[f[E]] = E$ for every $E \in \mathcal{E}$. If $F \subseteq X$ is closed, set $\mathcal{E}_0 = \{E : E \in \mathcal{E},\, E \cap F = \emptyset\}$. Then $\bigcup \mathcal{E}_0$ has the same measure as $X \setminus F$ and does not meet $f^{-1}[f[F]]$, so $\mu f^{-1}[f[F]] = \mu F$. As F is arbitrary, f and Z witness that μ satisfies (ii).

(ii)⇒(i) Take Z and $f : X \to Z$ as in (ii). Replacing Z by $f[X]$ if necessary, we may suppose that f is surjective, so that Z is compact, therefore second-countable (4A2P(a-ii)). Let $\mathcal{U}$ be a countable base for the topology of Z closed under finite unions, and set $\mathcal{V} = \{f^{-1}[U] : U \in \mathcal{U}\}$, so that $\mathcal{V}$ is a countable family of open sets in X. If $G \subseteq X$ is open, set $F = X \setminus G$, $\mathcal{U}_0 = \{U : U \in \mathcal{U},\, U \cap f[F] = \emptyset\}$, $\mathcal{V}_0 = \{f^{-1}[U] : U \in \mathcal{U}_0\}$. Then $Z \setminus f[F] = \bigcup \mathcal{U}_0$ so $X \setminus f^{-1}[f[F]] = \bigcup \mathcal{V}_0$ and (because $\mathcal{U}_0$ and $\mathcal{V}_0$ are closed under finite unions)

$$\begin{aligned}\sup\{\mu V : V \in \mathcal{V},\, V \subseteq G\} &\ge \sup_{V\in\mathcal{V}_0} \mu V = \mu(X \setminus f^{-1}[f[F]])\\ &= \mu X - \mu f^{-1}[f[F]] = \mu X - \mu F = \mu G.\end{aligned}$$

Thus $\mathcal{V}$ witnesses that μ is uniformly regular.

(b)(i)⇒(iii) If μ is uniformly regular, then by (a-ii) there are a metrizable space Z and a continuous function $f : X \to Z$ such that $\mu f^{-1}[f[F]] = \mu F$ for every closed $F \subseteq X$. If now $G \subseteq X$ is open and $\mu G < \mu X$, there is a sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ of closed sets with union G, because $\mathfrak{T}$ is perfectly normal. In this case $f^{-1}[f[G]] = \bigcup_{n\in\mathbb{N}} f^{-1}[f[F_n]]$ has the same measure as G, so is not the whole of X, and $f[G] \ne f[X]$. Thus f and Z witness that (iii) is true.

(iii)⇒(ii) Take Z and f from (iii). Let ν be the image measure μf^{-1} on Z; then μ is a Radon measure (418I again). **?** If $E \in \Sigma$ and $\mu^* f^{-1}[f[E]] > \mu E$, let $E' \supseteq E$ be a Borel set such that $\mu E' = \mu E$. Because X is perfectly normal, E' belongs to the Baire σ-algebra of X (4A3Kb), so is Souslin-F (421L), therefore K-analytic (422Hb); consequently $f[E']$ is K-analytic (422Gd) therefore measured by ν (432A). This means that $f^{-1}[f[E']] \in \Sigma$, and of course

$$\mu f^{-1}[f[E']] \ge \mu^* f^{-1}[f[E]] > \mu E = \mu E'.$$

We can therefore find open sets $G \supseteq E'$ and $G' \supseteq X \setminus f^{-1}[f[E']]$ such that $\mu G + \mu G' < \mu X$. But now $G \cup G'$ is an open set of measure less than μX and $f[G \cup G'] = f[X]$, which is supposed to be impossible. **X**

Thus, for any $E \in \Sigma$, we have $\mu^* f^{-1}[f[E]] = \mu E$; of course it follows at once that $f^{-1}[f[E]]$ is measurable, with the same measure as E, as required by (ii).

(ii)⇒(i)⇔(iv) These follow immediately from (a), because all closed sets in X are zero sets.

533H Theorem (a) Suppose that $\operatorname{cov} \mathcal{N}_{\omega_1} > \omega_1$. Let X be a perfectly normal compact Hausdorff space. Then every Radon measure on X is uniformly regular.

(b) (Plebanek 00) Suppose that $\operatorname{cov} \mathcal{N}_{\omega_1} > \omega_1 = \operatorname{non} \mathcal{N}_{\omega}$. Let X be a first-countable compact Hausdorff space. Then every Radon measure on X is uniformly regular.

proof (a) Let μ be a Radon measure on X. **?** If μ is not uniformly regular, then we can choose $\langle g_\xi\rangle_{\xi<\omega_1}$ and $\langle G_\xi\rangle_{\xi<\omega_1}$ inductively, as follows. Given that $g_\eta : X \to \mathbb{R}$ is continuous for every $\eta < \xi$, set $f_\xi(x) = \langle g_\eta(x)\rangle_{\eta<\xi}$ for $x \in X$, so that $f_\xi : X \to \mathbb{R}^\xi$ is continuous. By 533G(b-iii), there is an open set G_ξ such that $\mu G_\xi < \mu X$

and $f_\xi[G_\xi] = f_\xi[X]$; now G_ξ is a cozero set and there is a continuous function $g_\xi : X \to \mathbb{R}$ such that $G_\xi = \{x : g_\xi(x) \neq 0\}$. Continue.

At the end of the induction, we have a continuous function $f_{\omega_1} : X \to \mathbb{R}^{\omega_1}$, setting $f_{\omega_1}(x) = \langle g_\xi(x)\rangle_{\xi<\omega_1}$ for each x. Now ω_1 is a precaliber of every measurable algebra (525J again), and $\mu(X \setminus G_\xi) > 0$ for each ξ, so there is an $x \in X$ such that $A = \{\xi : x \notin G_\xi\}$ is uncountable (525Ca). Set $H = \{y : f_{\omega_1}(y) \neq f_{\omega_1}(x)\}$; then H is an open set, so expressible as $\bigcup_{n\in\mathbb{N}} K_n$ where each K_n is compact. For each $\xi \in A$ there is an $x_\xi \in G_\xi$ such that $f_\xi(x_\xi) = f_\xi(x)$. As $g_\xi(x_\xi) \neq 0 = g_\xi(x)$, $x_\xi \in H$. Let $n \in \mathbb{N}$ be such that $A' = \{\xi : \xi \in A, x_\xi \in K_n\}$ is uncountable. Then

$$f_{\omega_1}(x) \in \overline{\{f_{\omega_1}(x_\xi) : \xi \in A'\}} \subseteq f_{\omega_1}[K_n];$$

but this is impossible, because $K_n \subseteq H$. **X**

So μ must be uniformly regular, as required.

(b) Let μ be a Radon measure on X. If $\mu X = 0$ then of course μ is uniformly regular; suppose $\mu X > 0$. As in (a) and the proof of 533E, the Maharam type of μ is countable. Let $\mathfrak{A}$ be the measure algebra of μ; then $d(\mathfrak{A}) \leq \operatorname{non}\mathcal{N}_\omega$ (524Me), so there is a set $A \subseteq X$, of full outer measure, with $\#(A) \leq \omega_1$ (521Lc). For each $x \in X$, let $\langle V_{xn}\rangle_{n\in\mathbb{N}}$ run over a base of neighbourhoods of x. Let $\mathcal{H}$ be the family of sets expressible as finite unions of V_{xn} for $x \in A$ and $n \in \mathbb{N}$, so that $\mathcal{H}$ is a family of open sets in X and $\#(\mathcal{H}) \leq \omega_1$.

For any open $G \subseteq X$, $\mu G = \sup\{\mu H : H \in \mathcal{H}, H \subseteq G\}$. **P** Set $H^* = \bigcup\{H : H \in \mathcal{H}, H \subseteq G\}$. For any $x \in A \cap G$, there is an $n \in \mathbb{N}$ such that $V_{xn} \subseteq G$, and now $V_{xn} \in \mathcal{H}$, so $x \in H^*$. Thus $G \setminus H^*$ does not meet A; as A has full outer measure,

$$\mu G = \mu H^* = \sup\{\mu H : H \in \mathcal{H}, H \subseteq G\}$$

because $\{H : H \in \mathcal{H}, H \subseteq G\}$ is closed under finite unions. **Q** So there is a countable $\mathcal{H}' \subseteq \{H : H \in \mathcal{H}, H \subseteq G\}$ such that $\mu G = \sup_{H\in\mathcal{H}'} \mu H$.

Let $\langle H_\xi\rangle_{\xi<\omega_1}$ run over $\mathcal{H}$. For $\xi < \omega_1$, set

$$\mathcal{G}_\xi = \{G : G \subseteq X \text{ is open}, \mu G = \sup\{\mu H_\eta : \eta \leq \xi, H_\eta \subseteq G\}\}.$$

Then $\bigcup_{\xi<\omega_1} \mathcal{G}_\xi = \mathfrak{T}$. For each $\xi < \omega_1$, set

$$Y_\xi = \{y : y \in X, V_{yn} \in \mathcal{G}_\xi \text{ for every } n \in \mathbb{N}\};$$

then $X = \bigcup_{\xi<\omega_1} Y_\xi$. Now there is a $\xi < \omega_1$ such that Y_ξ has full outer measure. **P** Let ξ be such that $\mu^* Y_\xi = \mu^* Y_\eta$ for every $\eta \geq \xi$. **?** If $\mu^* Y_\xi < \mu X$, let $K \subseteq X \setminus Y_\xi$ be a non-negligible measurable set. Then the subspace measure μ_K is a Radon measure with countable Maharam type, so

$$\operatorname{cov}\mathcal{N}(\mu_K) \geq \operatorname{cov}\mathcal{N}_\omega \geq \operatorname{cov}\mathcal{N}_{\omega_1} > \omega_1.$$

Since $K \subseteq \bigcup_{\eta<\omega_1} Y_\eta$, there must be some $\eta < \omega_1$ such that $\mu_K^*(K \cap Y_\eta) > 0$; but now $\mu^*(K \cap Y_\eta) > 0$ and $\eta > \xi$ and

$$\mu^* Y_\eta = \mu^*(Y_\eta \setminus K) + \mu^*(Y_\eta \cap K) > \mu^* Y_\xi. \text{ **X**}$$

So Y_ξ has full outer measure. **Q**

Set $\mathcal{H}_\xi = \{H_\eta : \eta \leq \xi\}$. If $G \subseteq X$ is open, and $H^* = \bigcup\{H : H \in \mathcal{H}_\xi, H \subseteq G\}$, then $G \setminus H^*$ is negligible. **P** Set $\mathcal{V} = \{V_{yn} : y \in Y_\xi, n \in \mathbb{N}, V_{yn} \subseteq G\}$, $H_1^* = \bigcup \mathcal{V}$. Then Y_ξ does not meet $G \setminus H_1^*$, so $\mu H_1^* = \mu G$. Let $\mathcal{V}_0 \subseteq \mathcal{V}$ be a countable set such that $\mu(\bigcup \mathcal{V}_0) = \mu G$. If $V \in \mathcal{V}_0$, then $V \in \mathcal{G}_\xi$ and $V \subseteq G$ so $V \setminus H^*$ is negligible. Accordingly

$$G \setminus H^* \subseteq (G \setminus \bigcup \mathcal{V}_0) \cup \bigcup_{V\in\mathcal{V}_0}(V \setminus H^*)$$

is negligible. **Q** So if we take $\mathcal{H}'$ to be the set of finite unions of members of $\mathcal{H}_\xi$, $\mathcal{H}'$ will be a countable family of open sets and $\mu G = \sup\{\mu H : H \in \mathcal{H}', H \subseteq G\}$ for every open $G \subseteq X$. Thus μ is uniformly regular.

533I We know from 435Fb/435H and 439P that $\mathbb{R}^{\mathbb{N}}$ is measure-compact and $\mathbb{R}^{\mathfrak{c}}$ is not. It turns out that we already have a language in which to express a necessary and sufficient condition for $\mathbb{R}^\kappa$ to be measure-compact. To give the result in its full strength I repeat a definition from 435Xk.

Definition A completely regular space X is **strongly measure-compact** if $\mu X = \sup\{\mu^* K : K \subseteq X$ is compact$\}$ for every totally finite Baire measure μ on X.

Remark For the elementary properties of these spaces, see 435Xk. I repeat one here: a completely regular space X is strongly measure-compact iff it is measure-compact and pre-Radon. **P(i)** Suppose that X is measure-compact and pre-Radon and that μ is a totally finite Baire measure on X. Because X is measure-compact, μ has an extension to a quasi-Radon measure $\tilde{\mu}$ (435D); because X is pre-Radon, $\tilde{\mu}$ is Radon (434Jb) and

$$\mu X = \tilde{\mu} X = \sup_{K \subseteq X \text{ is compact}} \tilde{\mu} K$$
$$= \sup_{K \subseteq X \text{ is compact}} \tilde{\mu}^* K \le \sup_{K \subseteq X \text{ is compact}} \mu^* K \le \mu X.$$

As μ is arbitrary, X is strongly measure-compact. **(ii)** Suppose that X is strongly measure-compact. (α) Let μ be a Baire probability measure on X. Then there is a non-negligible compact set, so X cannot be covered by the negligible open sets; by 435Fa, this is enough to ensure that X is measure-compact. (β) Now let μ be a totally finite τ-additive Borel measure on X. Write ν for the restriction of μ to the Baire σ-algebra of X. Then there is a compact set $K \subseteq X$ which is not ν-negligible. **?** If $\mu(X \setminus K) = \mu X$, then, because μ is τ-additive and X is regular, there is a closed set $F \subseteq X \setminus K$ such that $\mu F + \nu^* K > \mu X$. Because X is completely regular, there is a zero set G including K and disjoint from F, in which case $\nu^* K > \mu G = \nu G$, which is impossible. **X** So $\mu K > 0$; by 434J(a-iii), this tells us that X is pre-Radon. **Q**

533J Theorem (see FREMLIN 77) Let κ be a cardinal. Then the following are equiveridical:

(i) $\mathbb{R}^\kappa$ is measure-compact;

(ii) if $\langle X_\xi \rangle_{\xi<\kappa}$ is a family of strongly measure-compact completely regular Hausdorff spaces then $\prod_{\xi<\kappa} X_\xi$ is measure-compact;

(iii) whenever X is a compact Hausdorff space and $\langle G_\xi \rangle_{\xi<\kappa}$ is a family of cozero sets in X, then $X \cap \bigcap_{\xi<\kappa} G_\xi$ is measure-compact;

(iv) for any Radon measure, the union of κ or fewer closed negligible sets has inner measure zero;

(v) for any Radon measure, the union of κ or fewer negligible sets has inner measure zero;

(vi) $\kappa < \operatorname{cov} \mathcal{N}(\mu)$ for any Radon measure μ;

(vii) $\kappa < \operatorname{cov} \mathcal{N}_\kappa$;

(viii) $\kappa < \mathfrak{m}(\mathfrak{A})$ for every measurable algebra $\mathfrak{A}$.

proof not-(iv)⇒not-(i) Suppose that X is a Hausdorff space, μ is a Radon measure on X and $\langle F_\xi \rangle_{\xi<\kappa}$ is a family of closed μ-negligible subsets of X such that $\mu_*(\bigcup_{\xi<\kappa} F_\xi) > 0$. Then there is a compact set $K \subseteq \bigcup_{\xi<\kappa} F_\xi$ such that $\mu K > 0$.

For each $\xi < \kappa$, there is a continuous $g_\xi : K \to [0,1[$ such that $g_\xi(z) = 0$ for $z \in K \cap F_\xi$ and $g_\xi^{-1}[\{0\}]$ is negligible. **P** For each $n \in \mathbb{N}$, there is a compact set $L_n \subseteq K \setminus F_\xi$ such that $\mu L_n \ge \mu K - 2^{-n}$; there is a continuous $f_n : K \to [0,1]$ such that $f_n(z) = 0$ for $z \in K \cap F_\xi$, 1 for $z \in L_n$; set $g_\xi = \sum_{n=0}^\infty 2^{-n-2} f_n$. **Q** Set $g(z) = \langle g_\xi(z) \rangle_{\xi<\kappa}$ for $z \in K$, so that $g : K \to [0,1[^\kappa$ is continuous.

Let ν be the Baire measure on $[0,1]^\kappa$ defined by setting $\nu H = \mu g^{-1}[H]$ for every Baire set $H \subseteq [0,1]^\kappa$. Then $]0,1[^\kappa$ has full outer measure for ν. **P** If $H \subseteq [0,1]^\kappa$ is a Baire set including $]0,1[^\kappa$, then H is determined by coordinates in some countable subset I of κ (4A3Mb). If $z \in K$ and $g_\xi(z) > 0$ for every $\xi \in I$, then $g(z) \restriction I \in]0,1[^I$ is equal to $w \restriction I$ for some $w \in H$, so $g(z) \in H$. Thus $g^{-1}[H]$ includes $\{z : z \in K$, $g_\xi(z) > 0$ for every $\xi \in I\}$ and

$$\nu H = \mu g^{-1}[H] \ge \mu\{z : g_\xi(z) > 0 \text{ for every } \xi \in I\} = \mu K = \nu [0,1]^\kappa. \text{ } \mathbf{Q}$$

On the other hand, every point y of $]0,1[^\kappa$ belongs to a ν-negligible cozero set. **P** $g[K]$ is a compact set not containing y, so there is a cozero set W containing y and disjoint from $g[K]$, and now $\nu W = 0$. **Q**

Let ν_0 be the subspace measure on $]0,1[^\kappa$. By 4A3Nd, ν_0 is a Baire measure on $]0,1[^\kappa$. If $y \in]0,1[^\kappa$ it belongs to a ν-negligible cozero set $W \subseteq [0,1]^\kappa$, and now $W \cap]0,1[^\kappa$ is a ν_0-negligible cozero set in $]0,1[^\kappa$ containing y. At the same time,

$$\nu_0]0,1[^\kappa = \nu [0,1]^\kappa = \mu K > 0.$$

So ν_0 witnesses that $]0,1[^\kappa$ is not measure-compact; as $\mathbb{R}^\kappa$ is homeomorphic to $]0,1[^\kappa$, it also is not measure-compact.

(iv)⇒(iii) Suppose that (iv) is true and that we have X and $\langle G_\xi\rangle_{\xi<\kappa}$, as in (iii), with a Baire probability measure μ on $Y = X\cap\bigcap_{\xi<\kappa}G_\xi$. Let ν be the Radon probability measure on X defined by saying that $\int f d\nu = \int(f\restriction Y)d\mu$ for every $f\in C(X)$ (436J/436K again). Then $\nu G_\xi = 1$ for each $\xi<\kappa$. **P** Let $f : X\to\mathbb{R}$ be a continuous function such that $G_\xi = \{x : x\in X,\ f(x)\ne 0\}$. Set $f_n = n|f|\wedge\chi X$ for each n. Then $\lim_{n\to\infty} f_n = \chi G_\xi$, so

$$\nu G_\xi = \lim_{n\to\infty}\int f_n d\nu = \lim_{n\to\infty}\int(f_n\restriction Y)d\mu = \mu Y = 1.\ \mathbf{Q}$$

By (iv), $\nu_*(\bigcup_{\xi<\kappa}(X\setminus G_\xi)) = 0$, that is, Y has full outer measure. In particular, Y must meet the support of ν; take any z in the intersection. If U is a cozero set in Y containing z, there is an open set $G\subseteq X$ such that $U = G\cap Y$; now there is a continuous $f : X\to[0,1]$ such that $f(z) = 1$ and $f(x) = 0$ for $x\in X\setminus G$; in this case

$$\mu U \ge \int(f\restriction Y)d\mu = \int f d\nu > 0$$

because $\{x : f(x) > 0\}$ is an open set meeting the support of ν. This shows that Y is not covered by the μ-negligible relatively cozero sets; as μ is arbitrary, Y is measure-compact (435Fa).

(iii)⇒(i) We can express $\mathbb{R}^\kappa$ in the form of (iii) by taking $X = [-\infty,\infty]^\kappa$ and $G_\xi = \{x : x(\xi)$ is finite$\}$ for each ξ.

(iv)⇒(vii) Let Z be the Stone space of the measure algebra of ν_κ, and λ its usual measure. If $\langle E_\xi\rangle_{\xi<\kappa}$ is a family of λ-negligible sets, then, because λ is inner regular with respect to the open-and-closed sets, we can find negligible zero sets $F_\xi\supseteq E_\xi$ for each ξ. By (iv), $\{F_\xi : \xi<\kappa\}$ cannot cover Z, so the same is true of $\{E_\xi : \xi<\kappa\}$. Thus $\operatorname{cov}\mathcal{N}(\lambda) > \kappa$. By 524Jb, $\operatorname{cov}\mathcal{N}_\kappa > \kappa$.

(vii)⇒(vi) Let θ be $\min\{\operatorname{cov}\mathcal{N}(\nu) : \nu$ is a non-zero Radon measure$\}$. By 524Pc, there is an infinite cardinal κ' such that $\theta = \operatorname{cov}\mathcal{N}_{\kappa'}$; by 523F, $\theta = \operatorname{cov}\mathcal{N}_\theta$. **?** If $\theta\le\kappa$, then 523B tells us that

$$\kappa < \operatorname{cov}\mathcal{N}_\kappa \le \operatorname{cov}\mathcal{N}_\theta = \theta.\ \mathbf{X}$$

So $\theta > \kappa$, as required.

(vi)⇒(v) If (vi) is true, (X,μ) is a Radon measure space, $\langle F_\xi\rangle_{\xi<\kappa}$ is a family of negligible sets, and $E\subseteq\bigcup_{\xi<\kappa}F_\xi$ is a measurable set, then the subspace measure μ_E is a Radon measure (416Rb), while E can be covered by κ negligible sets; by (vi), $\mu E = 0$; as E is arbitrary, $\mu_*(\bigcup_{\xi<\kappa}F_\xi) = 0$.

(v)⇒(ii) Suppose that (v) is true, that $\langle X_\xi\rangle_{\xi<\kappa}$ is a family of strongly measure-compact completely regular Hausdorff spaces with product X, and that μ is a Baire probability measure on X. For each $\xi<\kappa$ let Z_ξ be the Stone-Čech compactification of X_ξ; set $Z = \prod_{\xi<\kappa}Z_\xi$, and $\pi_\xi(z) = z(\xi)$ for $z\in Z$, $\xi<\kappa$. Then we have a Radon probability measure λ on Z defined by saying that $\int g\,d\lambda = \int_X(g\restriction X)d\mu$ for every $g\in C(Z)$. Note that if $W\subseteq Z$ is a zero set, there is a non-increasing sequence $\langle g_n\rangle_{n\in\mathbb{N}}$ in $C(Z)$ with infimum χW, so that

$$\lambda W = \inf_{n\in\mathbb{N}}\int g_n d\lambda = \inf_{n\in\mathbb{N}}\int_X(g_n\restriction X)d\mu = \mu(W\cap X).$$

Now $\lambda\pi_\xi^{-1}[X_\xi] = 1$ for each ξ. **P** Let $\epsilon>0$. We have a Baire probability measure μ_ξ on X_ξ defined by setting $\mu_\xi E = \mu(X\cap\pi_\xi^{-1}[E])$ for every Baire set $E\subseteq X_\xi$, and a Radon measure $\lambda_\xi = \lambda\pi_\xi^{-1}$ on Z_ξ. Because X_ξ is strongly measure-compact, there is a compact set $K\subseteq X_\xi$ such that $\mu_\xi^*K\ge 1-\epsilon$. Now K is still compact when regarded as a subset of Z_ξ, so there is a zero set $F\subseteq Z_\xi$, including K, such that $\lambda_\xi F = \lambda_\xi K$. In this case, $F\cap X_\xi$ is a zero set in X_ξ including K, so

$$\begin{aligned}\lambda_*\pi_\xi^{-1}[X_\xi] &\ge \lambda\pi_\xi^{-1}[K] = \lambda_\xi K = \lambda_\xi F = \lambda\pi_\xi^{-1}[F]\\ &= \mu(X\cap\pi_\xi^{-1}[F]) = \mu_\xi(F\cap X_\xi) \ge \mu_\xi^*K \ge 1-\epsilon.\end{aligned}$$

As ϵ is arbitrary, we have the result. **Q**

By (v), $X = Z\cap\bigcap_{\xi<\kappa}\pi_\xi^{-1}[X_\xi]$ has full outer measure for λ. Let $\mathcal{G}$ be the family of μ-negligible cozero sets in X and $\mathcal{H}$ the family of λ-negligible open sets in Z. If $x\in G\in\mathcal{G}$, then there is a continuous function $g : Z \to [0,1]$ such that $g(x) = 1$ and $H = \{y : y \in X,\ g(y) > 0\}$ is included in G; now $\int g\,d\lambda = \int(g{\restriction}X)d\mu = 0$, so $\lambda H = 0$. This shows that $\bigcup\mathcal{G} \subseteq \bigcup\mathcal{H}$ is λ-negligible, and, in particular, is not the whole of X. By 435Fa as usual, this is enough to show that X is measure-compact, as required.

(ii)⇒(i) is elementary, because $\mathbb{R}$ is certainly strongly measure-compact.

(vi)⇒(viii)⇒(vii) are immediate from 524Md.

533X Basic exercises (a) Describe a family $\langle\mathcal{K}_t\rangle_{t\in\mathbb{R}}$ such that every $\mathcal{K}_t$ consists of compact sets, Lebesgue measure on $\mathbb{R}$ is inner regular with respect to every $\mathcal{K}_t$, but $\bigcap_{t\in\mathbb{R}}\mathcal{K}_t = \emptyset$.

(b) Let μ be a uniformly regular topological measure on a topological space X. (i) Show that if $A \subseteq X$ then the subspace measure on A is uniformly regular. (ii) Show that any indefinite-integral measure over μ is uniformly regular. (iii) Show that if Y is another topological space and $f : X \to Y$ is a continuous open map, then the image measure μf^{-1} is uniformly regular.

(c) Show that any Radon measure on the split interval is uniformly regular. (*Hint*: 419L.)

(d) (Babiker 76) Let X and Y be compact Hausdorff spaces, μ a Radon measure on X, $f : X \to Y$ a continuous surjection and $\nu = \mu f^{-1}$ the image measure on Y. Show that the following are equiveridical: (i) $\nu f[F] = \mu F$ for every closed $F \subseteq X$; (ii) $\int g\,d\mu = \inf\{\int h\,d\nu : h \in C(Y),\ hf \ge g\}$ for every $g \in C(X)$; (iii) for every $g \in C(X)$, $\{y : g$ is constant on $f^{-1}[\{y\}]\}$ is ν-conegligible.

(e) Show that any uniformly regular Borel measure has countable Maharam type.

(f) Let $\langle X_i\rangle_{i\in I}$ be a countable family of topological spaces with product X, and μ a τ-additive topological measure on X. Suppose that the marginal measure of μ on X_i is uniformly regular for every $i \in I$. Show that μ is uniformly regular.

(g) Let X be $[0,1] \times \{0,1\}$ with the topology generated by

$$\{G \times \{0,1\} : G \subseteq [0,1] \text{ is relatively open for the usual topology}\}$$
$$\cup \{\{(t,1)\} : t \in [0,1]\} \cup \{X \setminus \{(t,1)\} : t \in [0,1]\}.$$

Show that X is compact and Hausdorff. Let μ be the Radon measure on X which is the image of Lebesgue measure on $[0,1]$ under the map $t \mapsto (t,0)$. Show that μ is uniformly regular but not completion regular.

(h) Let X be a topological space and μ a uniformly regular topological probability measure on X. Show that there is an equidistributed sequence in X.

(i) Show that there is a first-countable compact Hausdorff space with a uniformly regular topological probability measure, inner regular with respect to the closed sets, which is not τ-additive. (*Hint*: 439K.)

533Y Further exercises (a) (Pol 82) Let X be a compact Hausdorff space and μ a uniformly regular Radon measure on X. Show that if we give the space M_{R}^+ of Radon measures on X its narrow topology (437Jd) then $\chi(\mu, M_{\mathrm{R}}^+) \le \omega$.

(b) For a topological measure μ on a space X, write $\mathrm{ureg}(\mu)$ for the smallest size of any family $\mathcal{V}$ of open subsets of X such that $G \setminus \bigcup\{V : V \in \mathcal{V},\ V \subseteq G\}$ is negligible for every open $G \subseteq X$. (i) Show that if μ is inner regular with respect to the Borel sets then the Maharam type $\tau(\mu)$ of μ is at most $\mathrm{ureg}(\mu)$. (ii) Show that if X is compact and Hausdorff and μ is a Radon measure, then $\mathrm{ureg}(\mu) \le \max(\mathrm{non}\,\mathcal{N}_{\tau(\mu)}, \chi(X))$. (iii) Show that if X is compact and Hausdorff, μ is a Radon probability measure and $\mathrm{cov}\,\mathcal{N}_{\tau(\mu)} > \mathrm{ureg}(\mu)$, then μ has an equidistributed sequence.

(c) (Plebanek 00) Suppose that κ is a regular infinite cardinal such that $\mathrm{non}\,\mathcal{N}_\kappa < \mathrm{cov}\,\mathcal{N}_\kappa = \kappa$. Let (X,μ) be a Radon probability space such that $\chi(X) < \kappa$. Show that μ has an equidistributed sequence.

(d) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon measure space with countable Maharam type, $\mathcal{A}\subseteq\Sigma$ a set with cardinal less than $\operatorname{add}\mathcal{N}_\omega$, and $\mathfrak{S}$ the topology on X generated by $\mathfrak{T}\cup\mathcal{A}$. Show that μ is $\mathfrak{S}$-Radon.

533Z Problem For which cardinals κ is $\mathbb{R}^\kappa$ Borel-measure-compact?

533 Notes and comments I suppose that from the standpoint of measure theory the most fundamental of all the properties of ω is the fact that the union of countably many Lebesgue negligible sets is again Lebesgue negligible; this is of course shared by every $\kappa<\operatorname{add}\mathcal{N}_\omega$ (which is in effect the definition of $\operatorname{add}\mathcal{N}_\omega$). In 533A-533E and 533J we have results showing that uncountable cardinals can be 'almost countable' in other ways. In each case the fact that ω has the property examined is either trivial (as in 533B) or a basic result from Volume 4 (as in 533Cb, 533Cc and 533E). Similarly, the fact that $\mathfrak{c}$ does *not* have any of these properties is attested by classical examples. If you are familiar with Martin's axiom you will not be surprised to observe that everything here is sorted out if we assume that $\mathfrak{m}=\mathfrak{c}$.

533H does not quite fit this pattern, and the hypothesis in 533Hb definitely contradicts Martin's axiom. 'Uniformly regular' measures got squeezed out of §434 by shortage of space; in the exercises 533Xb-533Xi I sketch some of what was missed. Here I mention them just to show that there is more to say on the subject of first-countable and perfectly normal spaces than I put into 531O and 531Q. Another phenomenon of interest is the occurrence of measures which are inner regular with respect to a family of compact metrizable sets (462J, 533Ca, 533D).

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534 Hausdorff measures, strong measure zero and Rothberger's property

In this section I look at constructions which are primarily metric rather than topological. I start with a note on Hausdorff measures, spelling out connexions between Hausdorff r-dimensional measure on a separable metric space and the basic σ-ideal $\mathcal{N}$ (534B).

The main part of the section section is a brief introduction to a class of ideals which are of great interest in set-theoretic analysis. While the most important ones are based on separable metric spaces, some of the ideas can be expressed in more general contexts, and I give a definition of 'strong measure zero' in terms of uniformities (534Ca). An associated topological notion is what I call 'Rothberger's property' (534Cb). A famous characterization of sets of strong measure zero in $\mathbb{R}$ in terms of translations of meager sets can also be represented as a theorem about σ-compact groups (534K). There are few elementary results describing the cardinal functions of strong measure zero ideals, but I give some information on their additivities (534M) and uniformities (534Q). There seem to be some interesting questions concerning spaces with isomorphic strong measure zero ideals, which I consider in 534N-534P. A particularly important question, from the very beginning of the topic in Borel 1919, concerns the possible cardinals of sets of strong measure zero; in 534Q-534S I give some sample facts and illustrative examples.

534A An elementary lemma will be useful.

Lemma Let (X,ρ) be a separable metric space. Then there is a countable family $\mathcal{C}$ of subsets of X such that whenever $A\subseteq X$ has finite diameter and $\eta>0$ then there is a $C\in\mathcal{C}$ such that $A\subseteq C$ and $\operatorname{diam}C\le\eta+2\operatorname{diam}A$.

proof Let D be a countable dense subset of X and set $\mathcal{C}=\{\emptyset\}\cup\{B(x,q):x\in D,\ q\in\mathbb{Q},\ q\ge 0\}$. If $A\subseteq X$ has finite diameter and $\eta>0$, then if $A=\emptyset$ we can take $C=\emptyset$. Otherwise, take $y\in A$ and $q\in\mathbb{Q}$ such that $\operatorname{diam}A+\frac{1}{4}\eta\le q\le\operatorname{diam}A+\frac{1}{2}\eta$. Let $x\in D$ be such that $\rho(x,y)\le\frac{1}{4}\eta$; then $C=B(x,q)\in\mathcal{C}$, $A\subseteq B(y,\operatorname{diam}A)\subseteq C$ and $\operatorname{diam}C\le 2q\le\eta+2\operatorname{diam}A$.

534B Hausdorff measures There are difficult questions concerning the cardinals associated with even the most familiar Hausdorff measures. However we do have some easy results.

Theorem Let (X,ρ) be a metric space and $r>0$. Write μ_{Hr} for r-dimensional Hausdorff measure on X, $\mathcal{N}(\mu_{Hr})$ for its null ideal, $\mathcal{N}$ for the null ideal of Lebesgue measure on $\mathbb{R}$ and $\mathcal{M}$ for the ideal of meager subsets of $\mathbb{R}$.

(a) $\operatorname{add}\mu_{Hr}=\operatorname{add}\mathcal{N}(\mu_{Hr})$.

(b) If X is separable, $\mathcal{N}(\mu_{Hr})\preccurlyeq_{\mathrm{T}}\mathcal{N}$, so that $\operatorname{add}\mu_{Hr}\ge\operatorname{add}\mathcal{N}$ and $\operatorname{cf}\mathcal{N}(\mu_{Hr})\le\operatorname{cf}\mathcal{N}$.

(c) If X is separable, $(X,\in,\mathcal{N}(\mu_{Hr}))\preccurlyeq_{\mathrm{GT}}(\mathcal{M},\not\ni,\mathbb{R})$, so that $\operatorname{cov}\mathcal{N}(\mu_{Hr})\le\operatorname{non}\mathcal{M}$ and $\mathfrak{m}_{\mathrm{countable}}\le\operatorname{non}\mathcal{N}(\mu_{Hr})$.

(d) If X is analytic and $\mu_{Hr}X>0$, then $\operatorname{add}\mu_{Hr}=\operatorname{add}\mathcal{N}$, $\operatorname{cf}\mathcal{N}(\mu_{Hr})=\operatorname{cf}\mathcal{N}$, $\operatorname{non}\mathcal{N}(\mu_{Hr})\le\operatorname{non}\mathcal{N}$ and $\operatorname{cov}\mathcal{N}(\mu_{Hr})\ge\operatorname{cov}\mathcal{N}$.

proof (a) 521Ac.

(b)(i) Let $\mathcal{C}$ be a countable family of subsets of X such that whenever $A\subseteq X$ has finite diameter and $\eta>0$ there is a $C\in\mathcal{C}$ such that $A\subseteq C$ and $\operatorname{diam}C\le\eta+2\operatorname{diam}A$ (534A).

If $A\subseteq X$, then $A\in\mathcal{N}(\mu_{Hr})$ iff for every $\epsilon>0$ there is a sequence $\langle C_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{C}$ such that $A\subseteq\bigcup_{n\in\mathbb{N}}C_n$ and $\sum_{n=0}^\infty(\operatorname{diam}C_n)^r\le\epsilon$. **P** If A is negligible and $\epsilon>0$, then (by the definition in 471A) there must be a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of subsets of X such that $A\subseteq\bigcup_{n\in\mathbb{N}}A_n$ and $\sum_{n=0}^\infty(\operatorname{diam}A_n)^r<2^{-r}\epsilon$. Let $\langle\eta_n\rangle_{n\in\mathbb{N}}$ be a sequence of strictly positive real numbers such that $\sum_{n=0}^\infty(\eta_n+2\operatorname{diam}A_n)^r\le\epsilon$. For each n we can find $C_n\in\mathcal{C}_n$ such that $A_n\subseteq C_n$ and $\operatorname{diam}C_n\le\eta_n+2\operatorname{diam}A_n$, so that $\sum_{n=0}^\infty(\operatorname{diam}C_n)^r\le\epsilon$, while $A\subseteq\bigcup_{n\in\mathbb{N}}C_n$.

On the other hand, if A satisfies the condition, then for every ϵ, $\delta>0$ there is a sequence $\langle C_n\rangle_{n\in\mathbb{N}}$ of subsets of X such that $A\subseteq\bigcup_{n\in\mathbb{N}}C_n$ and $\sum_{n=0}^\infty(\operatorname{diam}C_n)^r\le\min(\epsilon,\delta^r)$. In this case, $\operatorname{diam}C_n\le\delta$ for every n, so $\theta_{r\delta}A$, as defined in 471A, is at most ϵ. As ϵ is arbitrary, $\theta_{r\delta}A=0$; as δ is arbitrary, A is μ_{Hr}-negligible. **Q**

(ii) It follows that $(\mathcal{N}(\mu_{Hr}),\subseteq,\mathcal{N}(\mu_{Hr}))\preccurlyeq_{\mathrm{GT}}(\mathbb{N}^{\mathbb{N}},\subseteq^*,\mathcal{S})$, where $(\mathbb{N}^{\mathbb{N}},\subseteq^*,\mathcal{S})$ is the $\mathbb{N}$-localization relation (522K).

P$(\boldsymbol{\alpha})$ For each $n\in\mathbb{N}$, let $\mathcal{I}_n$ be the family of finite subsets I of $\mathcal{C}$ such that $\sum_{C\in I}(\operatorname{diam}C)^r\le 4^{-n}$. Let $\langle I_{nj}\rangle_{j\in\mathbb{N}}$ be a sequence running over $\mathcal{I}_n$. Now, given $A\in\mathcal{N}(\mu_{Hr})$, then for each $n\in\mathbb{N}$ let $\langle C_{ni}\rangle_{i\in\mathbb{N}}$ be a sequence in $\mathcal{C}$, covering A, such that $\sum_{i=0}^\infty(\operatorname{diam}C_{ni})^r\le 2^{-n-1}$. Let $\langle C_i\rangle_{i\in\mathbb{N}}$ be a re-indexing of the family $\langle C_{ni}\rangle_{n,i\in\mathbb{N}}$, so that $\langle C_i\rangle_{i\in\mathbb{N}}$ is a sequence in $\mathcal{C}$, $\sum_{i=0}^\infty(\operatorname{diam}C_i)^r\le 1$, and $A\subseteq\bigcap_{m\in\mathbb{N}}\bigcup_{i\ge m}C_i$. Let $\langle k(n)\rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence in $\mathbb{N}$ such that $k(0)=0$ and $\sum_{i=k(n)}^\infty(\operatorname{diam}C_i)^r\le 4^{-n}$ for every n. Now, for $n\in\mathbb{N}$, let $\phi(A)(n)$ be such that $\{C_i:k(n)\le i<k(n+1)\}=I_{n,\phi(A)(n)}$.

This process defines a function $\phi:\mathcal{N}(\mu_{Hr})\to\mathbb{N}^{\mathbb{N}}$ such that

$$A\subseteq\bigcap_{m\in\mathbb{N}}\bigcup_{n\ge m}\bigcup I_{n,\phi(A)(n)}$$

for every $A\in\mathcal{N}(\mu_{Hr})$.

$(\boldsymbol{\beta})$ For $S\in\mathcal{S}$, set

$$\psi(S)=\bigcap_{m\in\mathbb{N}}\bigcup_{n\ge m}\bigcup_{i\in S[\{n\}]}\bigcup I_{ni}\subseteq X.$$

If $n\in\mathbb{N}$, then

$$\sum\{(\operatorname{diam}C)^r:C\in\bigcup_{i\in S[\{n\}]}I_{ni}\}\le 2^n\cdot 4^{-n}=2^{-n},$$

because $\#(S[\{n\}])\le 2^n$ and $\sum\{(\operatorname{diam}C)^r:C\in I_{ni}\}\le 4^{-n}$ for every i. But this means that, for any $m\in\mathbb{N}$,

$$\sum\{(\operatorname{diam}C)^r:C\in\bigcup_{n\ge m}\bigcup_{i\in S[\{n\}]}I_{ni}\}\le 2^{-m+1},$$

while

$$\psi(S)\subseteq\bigcup(\bigcup_{n\ge m}\bigcup_{i\in S[\{n\}]}I_{ni}).$$

So $\psi(S)\in\mathcal{N}(\mu_{Hr})$ for every $S\in\mathcal{S}$.

$(\boldsymbol{\gamma})$ Suppose that $A\in\mathcal{N}(\mu_{Hr})$ and $\phi(A)\subseteq^* S\in\mathcal{S}$. Then there is some $m_0\in\mathbb{N}$ such that $(n,\phi(A)(n))\in S$ for every $n\ge m_0$. Now, for any $m\in\mathbb{N}$, we have

$$A \subseteq \bigcup_{n \geq \max(m,m_0)} \bigcup I_{n,\phi(A)(n)}$$
$$\subseteq \bigcup_{n \geq \max(m,m_0)} \bigcup_{i \in S[\{n\}]} \bigcup I_{ni} \subseteq \bigcup_{n \geq m} \bigcup_{i \in S[\{n\}]} \bigcup I_{ni},$$

so $A \subseteq \psi(S)$. This shows that (ϕ, ψ) is a Galois-Tukey connection from $(\mathcal{N}(\mu_{Hr}), \subseteq, \mathcal{N}(\mu_{Hr}))$ to $(\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S})$, and $(\mathcal{N}(\mu_{Hr}), \subseteq, \mathcal{N}(\mu_{Hr})) \preccurlyeq_{\mathrm{GT}} (\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S})$. **Q**

(iii) Since $(\mathcal{N}, \subseteq, \mathcal{N}) \equiv_{\mathrm{GT}} (\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S})$ (522M), $(\mathcal{N}(\mu_{Hr}), \subseteq, \mathcal{N}(\mu_{Hr})) \preccurlyeq_{\mathrm{GT}} (\mathcal{N}, \subseteq, \mathcal{N})$, that is, $\mathcal{N}(\mu_{Hr}) \preccurlyeq_{\mathrm{T}} \mathcal{N}$.

(iv) By 513Ee, as usual, we can conclude that $\operatorname{add} \mathcal{N}(\mu_{Hr}) \geq \operatorname{add} \mathcal{N}$ and $\operatorname{cf} \mathcal{N}(\mu_{Hr}) \leq \operatorname{cf} \mathcal{N}$.

(c)(i) If $\mu_{Hr} X = 0$, the result is trivial. **P** Set $\phi(x) = \emptyset$ for $x \in X$, $\psi(t) = X$ for $t \in \mathbb{R}$; then (ϕ, ψ) is a Galois-Tukey connection from $(X, \in, \mathcal{N}(\mu_{Hr}))$ to $(\mathcal{M}, \not\ni, \mathbb{R})$. **Q** So let us suppose that X is infinite.

(ii) Let F be the set of 1-Lipschitz functions $f : X \to [0,1]$. Define $T : X \to \ell^\infty(F)$ by setting $(Tx)(f) = f(x)$ for $f \in F$ and $x \in X$. Then

$$\|Tx - Ty\|_\infty = \sup_{f \in F} |f(x) - f(y)| = \min(1, \rho(x,y))$$

for all x, $y \in X$. **P** Of course $\sup_{f \in F} |f(x) - f(y)| \leq \min(1, \rho(x,y))$, by the definition of F. On the other hand, we can set $f(z) = \min(1, \rho(z,x))$ for every $z \in X$; then $f \in F$ and $|f(x) - f(y)| = \min(1, \rho(x,y))$. So we have equality. **Q** Thus T is 1-Lipschitz for ρ and the usual metric on $\ell^\infty(F)$, and $T[X]$ is a separable subset of $\ell^\infty(F)$ (4A2B(e-iii)). Let V be the closed linear subspace of $\ell^\infty(F)$ generated by $T[X]$; then V is separable (4A4Bg). Being a closed subset of the complete metric space $\ell^\infty(F)$, V is a Polish space. Since X has more than one point, and T is injective, V is non-empty and has no isolated points.

Let $\langle v_n \rangle_{n \in \mathbb{N}}$ enumerate a dense subset of V. Set

$$E = \bigcap_{n \in \mathbb{N}} \bigcup_{i \geq n} U(v_i, 2^{-i-1})$$

where $U(v, \delta) = \{u : u \in V, \|u - v\|_\infty < \delta\}$ for $v \in V$, $\delta > 0$. Then E is the intersection of a sequence of dense open sets in V, so is comeager, and $M = V \setminus E$ belongs to the ideal $\mathcal{M}(V)$ of meager subsets of V. For any $v \in V$, the map $u \mapsto u - v : V \to V$ is a homeomorphism, so $M - v \in \mathcal{M}(V)$. Define $\phi : X \to \mathcal{M}(V)$ by setting $\phi(x) = M - Tx$ for $x \in X$.

In the other direction, define $\psi : V \to \mathcal{P}X$ by setting $\psi(v) = T^{-1}[E - v]$ for $v \in V$. Then $\psi(v) \in \mathcal{N}(\mu_{Hr})$ for every $v \in V$. **P** If $v \in V$ and $\delta \leq \frac{1}{2}$, then $\|u - u'\|_\infty < 1$ for all u, $u' \in U(v, \delta)$, so $\rho(x, x') \leq \|Tx - Tx'\|_\infty$ whenever x, $x' \in T^{-1}[U(v,\delta)]$. Accordingly $\operatorname{diam} T^{-1}[U(v_i - v, 2^{-i-1})] \leq 2^{-i}$ for every $i \in \mathbb{N}$. This means that

$$\mu^*_{Hr} T^{-1}[E - v] = \mu^*_{Hr}(\bigcap_{n \in \mathbb{N}} \bigcup_{i \geq n} T^{-1}[U(v_i - v, 2^{-i-1})]$$
$$\leq \inf_{n \in \mathbb{N}} \sum_{i=n}^{\infty} (2^{-i})^r = 0. \textbf{ Q}$$

So ψ is a function from V to $\mathcal{N}(\mu_{Hr})$. We now see that

$$\phi(x) \not\ni v \Longrightarrow v \notin M - Tx \Longrightarrow Tx \notin M - v$$
$$\Longrightarrow Tx \in E - v \Longrightarrow x \in \psi(v).$$

Thus (ϕ, ψ) is a Galois-Tukey connection from $(X, \in, \mathcal{N}(\mu_{Hr}))$ to $(\mathcal{M}(V), \not\ni, V)$ and

$$(X, \in, \mathcal{N}(\mu_{Hr})) \preccurlyeq_{\mathrm{GT}} (\mathcal{M}(V), \not\ni, V) \cong (\mathcal{M}, \not\ni, \mathbb{R})$$

(522Wb).

(iii) Now

$$\operatorname{cov} \mathcal{N}(\mu_{Hr}) = \operatorname{cov}(X, \in, \mathcal{N}(\mu_{Hr})) \leq \operatorname{cov}(\mathcal{M}, \not\ni, \mathbb{R}) = \operatorname{non} \mathcal{M},$$

$$\mathrm{non}\,\mathcal{N}(\mu_{Hr}) = \mathrm{add}(X, \in, \mathcal{N}(\mu_{Hr})) \ge \mathrm{add}(\mathcal{M}, \not\ni, \mathbb{R}) = \mathrm{cov}\,\mathcal{M} = \mathfrak{m}_{\mathrm{countable}}$$

(512D, 512Ed, 522Sa).

(d) If X is analytic and $\mu_{Hr}X > 0$, then by Howroyd's theorem (471S) there is a compact set $K \subseteq X$ such that $0 < \mu_{Hr}K < \infty$. Now the subspace measure $\mu_{Hr}^{(K)}$ on K is an atomless Radon measure (471E, 471Dg, 471F) on a compact metric space, so

$$\mathrm{add}\,\mathcal{N} \le \mathrm{add}\,\mathcal{N}(\mu_{Hr}) \le \mathrm{add}\,\mathcal{N}(\mu_{Hr}^{(K)}) = \mathrm{add}\,\mathcal{N},$$

$$\mathrm{cf}\,\mathcal{N} \ge \mathrm{cf}\,\mathcal{N}(\mu_{Hr}) \ge \mathrm{cf}\,\mathcal{N}(\mu_{Hr}^{(K)}) = \mathrm{cf}\,\mathcal{N},$$

$$\mathrm{non}\,\mathcal{N}(\mu_{Hr}) \le \mathrm{non}\,\mathcal{N}(\mu_{Hr}^{(K)}) = \mathrm{non}\,\mathcal{N},$$

$$\mathrm{cov}(X, \mathcal{N}(\mu_{Hr})) \ge \mathrm{cov}(K, \mathcal{N}(\mu_{Hr}^{(K)})) = \mathrm{cov}\,\mathcal{N}$$

by (b) above, 521F and 522Wa.

534C Strong measure zero and Rothberger's property **(a)** Let $(X, \mathcal{W})$ be a uniform space and $A \subseteq X$. I say that A has **strong measure zero** or **property C** in X if for any sequence $\langle W_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{W}$ there is a cover $\langle A_n\rangle_{n\in\mathbb{N}}$ of A such that $A_n \times A_n \subseteq W_n$ for every $n \in \mathbb{N}$. If (X, ρ) is a metric space, a subset A of X has strong measure zero in X if it has strong measure zero for the uniformity defined by the metric (3A4B), that is, for any sequence $\langle \epsilon_n\rangle_{n\in\mathbb{N}}$ of strictly positive real numbers there is a cover $\langle A_n\rangle_{n\in\mathbb{N}}$ of X such that $\mathrm{diam}\, A_n \le \epsilon_n$ for every $n \in \mathbb{N}$.

I will write $\mathcal{S}\mathfrak{mz}(X, \mathcal{W})$ or $\mathcal{S}\mathfrak{mz}(X, \rho)$ for the family of sets of strong measure zero in a uniform space $(X, \mathcal{W})$ or a metric space (X, ρ).

(b) If X is a topological space and A is a subset of X, I will say that A has **Rothberger's property in** X if for every sequence $\langle \mathcal{G}_n\rangle_{n\in\mathbb{N}}$ of non-empty open covers of X there is a sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ such that $G_n \in \mathcal{G}_n$ for every $n \in \mathbb{N}$ and $A \subseteq \bigcup_{n\in\mathbb{N}} G_n$. I will write $\mathcal{R}\mathfrak{bg}(X)$ for the family of subsets of X with Rothberger's property in X.

534D Proposition (a)(i) If $(X, \mathcal{W})$ is a uniform space and $A \subseteq X$, then A has strong measure zero in X iff it has strong measure zero in itself when it is given its subspace uniformity.

(ii) If $(X, \mathcal{W})$ is a uniform space, then $\mathcal{S}\mathfrak{mz}(X, \mathcal{W})$ is a σ-ideal containing all the countable subsets of X.

(iii) If $(X, \mathcal{W})$ and $(Y, \mathcal{V})$ are uniform spaces and $f : X \to Y$ is uniformly continuous, then $f[A] \in \mathcal{S}\mathfrak{mz}(Y, \mathcal{V})$ whenever $A \in \mathcal{S}\mathfrak{mz}(X, \mathcal{W})$.

(iv) Let $(X, \mathcal{W})$ be a uniform space and $A \subseteq X$. Then $A \in \mathcal{S}\mathfrak{mz}(X, \mathcal{W})$ iff $f[A] \in \mathcal{S}\mathfrak{mz}(Y, \rho)$ whenever (Y, ρ) is a metric space and $f : X \to Y$ is uniformly continuous.

(v) Let $(X, \mathcal{W})$ be a uniform space and $A \in \mathcal{S}\mathfrak{mz}(X, \mathcal{W})$. If $B \subseteq X$ is such that $B \setminus G \in \mathcal{S}\mathfrak{mz}(X, \mathcal{W})$ whenever G is an open set including A, then $B \in \mathcal{S}\mathfrak{mz}(X, \mathcal{W})$.

(b) Let X be a topological space.

(i) $\mathcal{R}\mathfrak{bg}(X)$ is a σ-ideal containing all the countable subsets of X.

(ii) If Y is another topological space, $f : X \to Y$ is continuous and $A \in \mathcal{R}\mathfrak{bg}(X)$, then $f[A] \in \mathcal{R}\mathfrak{bg}(Y)$.

(iii) If $A \in \mathcal{R}\mathfrak{bg}(X)$ and $B \subseteq X$ is such that $B \setminus G \in \mathcal{R}\mathfrak{bg}(X)$ whenever G is an open set including A, then $B \in \mathcal{R}\mathfrak{bg}(X)$.

(iv) If $F \subseteq X$ is closed, then $\mathcal{R}\mathfrak{bg}(F) = \{A : A \in \mathcal{R}\mathfrak{bg}(X),\ A \subseteq F\}$.

proof (a)(i) Recall that the subspace uniformity on A is just $\mathcal{W}_A = \{W \cap (A \times A) : W \in \mathcal{W}\}$ (3A4D). If $A \in \mathcal{S}\mathfrak{mz}(A, \mathcal{W}_A)$ and $\langle W_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{W}$, then $\langle W_n \cap (A \times A)\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{W}_A$, so we have a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of sets covering A with $A_n \times A_n \subseteq W_n \cap (A \times A) \subseteq W_n$ for every n; as $\langle W_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $A \in \mathcal{S}\mathfrak{mz}(X, \mathcal{W})$. If $A \in \mathcal{S}\mathfrak{mz}(X, \mathcal{W})$ and $\langle V_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{W}_A$, we can choose for each n a $W_n \in \mathcal{W}$ such that $V_n = W_n \cap (A \times A)$; now we have a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of sets covering A with $A_n \times A_n \subseteq W_n$ for every n, in which case $\langle A_n \cap A\rangle_{n\in\mathbb{N}}$ covers A and $(A_n \cap A) \times (A_n \cap A) \subseteq V_n$ for every n; as $\langle V_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $A \in \mathcal{S}\mathfrak{mz}(A, \mathcal{A})$.

(ii) It is immediate from the definition that any subset of a set in $\mathcal{S}\mathrm{mz}(X,\mathcal{W})$ belongs to $\mathcal{S}\mathrm{mz}(X,\mathcal{W})$, and so does any countable set. Now suppose that $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{S}\mathrm{mz}(X,\mathcal{W})$. Let $\langle W_n\rangle_{n\in\mathbb{N}}$ be any sequence in $\mathcal{W}$. For each $k\in\mathbb{N}$, $\langle W_{2^k(2i+1)}\rangle_{i\in\mathbb{N}}$ is a sequence in $\mathcal{W}$, so there is a sequence $\langle A_{ki}\rangle_{i\in\mathbb{N}}$, covering A_k, such that $A_{ki}\times A_{ki}\subseteq W_{2^k(2i+1)}$ for every i. Set $B_0=\emptyset$ and $B_n=A_{ki}$ if $n=2^k(2i+1)$ where k, $i\in\mathbb{N}$; then $A\subseteq\bigcup_{n\in\mathbb{N}}B_n$ and $B_n\times B_n\subseteq W_n$ for every n. As $\langle W_n\rangle_{n\in\mathbb{N}}$ is arbitrary, A has strong measure zero; as $\langle A_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathcal{S}\mathrm{mz}(X,\mathcal{W})$ is a σ-ideal.

(iii) Let $\langle V_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{V}$. For each $n\in\mathbb{N}$, there is a $W_n\in\mathcal{W}$ such that $(f(x),f(x'))\in V_n$ whenever $(x,x')\in W_n$. Because $A\in\mathcal{S}\mathrm{mz}(X,\mathcal{W})$, there is a cover $\langle A_n\rangle_{n\in\mathbb{N}}$ of A such that $A_n\times A_n\subseteq W_n$ for every n; now $f[A_n]\times f[A_n]\subseteq V_n$ for every n and $\bigcup_{n\in\mathbb{N}}f[A_n]=f[A]$. As $\langle V_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $f[A]\in\mathcal{S}\mathrm{mz}(Y,\mathcal{V})$.

(iv) If A has strong measure zero, then of course $f[A]$ has strong measure zero for any uniformly continuous function f from X to a metric space, by (iii). Now suppose that A satisfies the condition, and that $\langle W_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{W}$. Then there is a pseudometric ρ on X, compatible with the uniformity in the sense that $\{(x,y):\rho(x,y)\le\epsilon\}\in\mathcal{W}$ for every $\epsilon>0$, such that $\{(x,y):\rho(x,y)<2^{-n}\}\subseteq W_n$ for every n (4A2Ja). Set $\sim\,=\{(x,y):\rho(x,y)=0\}$. Then $\sim$ is an equivalence relation on X. If Y is the set of equivalence classes, we have a metric $\tilde\rho$ on Y defined by setting $\tilde\rho(x^\bullet,y^\bullet)=\rho(x,y)$ for all x, $y\in X$. Setting $f(x)=x^\bullet$ for $x\in X$, $f:X\to Y$ is uniformly continuous. So $f[A]\in\mathcal{S}\mathrm{mz}(Y,\tilde\rho)$. Let $\langle B_n\rangle_{n\in\mathbb{N}}$ be a cover of $f[A]$ such that $\operatorname{diam}B_n\le 2^{-n-1}$ for every n, and set $A_n=f^{-1}[B_n]$ for each n. Then $\langle A_n\rangle_{n\in\mathbb{N}}$ is a cover of A. If $n\in\mathbb{N}$ and x, $y\in A_n$, then $\rho(x,y)=\tilde\rho(f(x),f(y))\le 2^{-n-1}$, so $(x,y)\in W_n$. Thus $A_n\times A_n\subseteq W_n$. As $\langle W_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $A\in\mathcal{S}\mathrm{mz}(X,\mathcal{W})$.

(v) Let $\langle W_n\rangle_{n\in\mathbb{N}}$ be any sequence in $\mathcal{W}$. For each $n\in\mathbb{N}$, let $V_n\in\mathcal{W}$ be such that $V_n\circ V_n\circ V_n^{-1}\subseteq W_{2n}$. Then there is a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$, covering A, such that $A_n\times A_n\subseteq V_n$ for every n. Set $B_{2n}=\operatorname{int}V_n[A_n]$ for each n, and $G=\bigcup_{n\in\mathbb{N}}B_{2n}$; then $B_{2n}\times B_{2n}\subseteq W_{2n}$ for every n and G is an open set including A. Accordingly $B\setminus G\in\mathcal{S}\mathrm{mz}(X,\mathcal{W})$ and there is a sequence $\langle B_{2n+1}\rangle_{n\in\mathbb{N}}$, covering $B\setminus G$, such that $B_{2n+1}\times B_{2n+1}\subseteq W_{2n+1}$ for every n. Now $\langle B_n\rangle_{n\in\mathbb{N}}$ covers B and $B_n\times B_n\subseteq W_n$ for every n. As $\langle W_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $B\in\mathcal{S}\mathrm{mz}(X,\mathcal{W})$.

(b)(i) We can copy the argument of (a-ii). As before, it is immediate from the definition that any subset of a set in $\mathcal{R}\mathrm{bg}(X)$, and any countable subset of X, belong to $\mathcal{R}\mathrm{bg}(X)$. Now suppose that $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{R}\mathrm{bg}(X)$, with union A. Let $\langle\mathcal{G}_n\rangle_{n\in\mathbb{N}}$ be any sequence of non-empty open covers of X. For each $k\in\mathbb{N}$, $\langle\mathcal{G}_{2^k(2i+1)}\rangle_{i\in\mathbb{N}}$ is a sequence of open covers of X, so there is a sequence $\langle G_{ki}\rangle_{i\in\mathbb{N}}$, covering A_k, such that $G_{ki}\in\mathcal{G}_{2^k(2i+1)}$ for every i. Take G_0 to be any member of $\mathcal{G}_0$, and set $G_n=G_{ki}$ if $n=2^k(2i+1)$ where k, $i\in\mathbb{N}$; then $A\subseteq\bigcup_{n\in\mathbb{N}}G_n$ and $G_n\in\mathcal{G}_n$ for every n. As $\langle\mathcal{G}_n\rangle_{n\in\mathbb{N}}$ is arbitrary, A has Rothberger's property in X.

(ii) This uses the idea of (a-iii). Let $\langle\mathcal{H}_n\rangle_{n\in\mathbb{N}}$ be a sequence of non-empty open covers of Y. For each $n\in\mathbb{N}$, set $\mathcal{G}_n=\{f^{-1}[H]:H\in\mathcal{H}_n\}$; then $\mathcal{G}_n$ is a non-empty open cover of X. Because $A\in\mathcal{R}\mathrm{bg}(X)$, there is a cover $\langle G_n\rangle_{n\in\mathbb{N}}$ of A such that $G_n\in\mathcal{G}_n$ for every $n\in\mathbb{N}$. Expressing G_n as $f^{-1}[H_n]$ where $H_n\in\mathcal{H}_n$ for each $n\in\mathbb{N}$, $f[A]\subseteq\bigcup_{n\in\mathbb{N}}H_n$. As $\langle\mathcal{H}_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $f[A]$ has Rothberger's property in Y.

(iii) And here we can copy from (a-v). Let $\langle\mathcal{G}_n\rangle_{n\in\mathbb{N}}$ be any sequence of open covers of X. Then there is a sequence $\langle G_{2n}\rangle_{n\in\mathbb{N}}$, covering A, such that $G_{2n}\in\mathcal{G}_{2n}$ for every n. Set $H=\bigcup_{n\in\mathbb{N}}G_{2n}$; then there is a sequence $\langle G_{2n+1}\rangle_{n\in\mathbb{N}}$, covering $B\setminus H$, such that $G_{2n+1}\in\mathcal{G}_{2n+1}$ for each n. Putting these together, we have a sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ covering B such that $G_n\in\mathcal{G}_n$ for every n. As $\langle\mathcal{G}_n\rangle_{n\in\mathbb{N}}$ is arbitrary, B has Rothberger's property in X.

(iv) If $A\in\mathcal{R}\mathrm{bg}(F)$ then $A\in\mathcal{R}\mathrm{bg}(X)$ by (ii), because the identity map from F to X is continuous. Conversely, if $A\subseteq F$ and $A\in\mathcal{R}\mathrm{bg}(X)$, let $\langle\mathcal{G}_n\rangle_{n\in\mathbb{N}}$ be a sequence of non-empty relatively open covers of F. For $n\in\mathbb{N}$ set

$$\mathcal{H}_n=\{H:H\subseteq X\text{ is open and there is a }G\in\mathcal{G}_n\text{ such that }H\cap F\subseteq G\}.$$

Then $\mathcal{H}_n$ is a non-empty open cover of X because $\mathcal{G}_n$ covers F and $X\setminus F\in\mathcal{H}_n$. Because $A\in\mathcal{R}\mathrm{bg}(X)$, there is a sequence $\langle H_n\rangle_{n\in\mathbb{N}}$ such that $H_k\in\mathcal{H}_n$ for every $n\in\mathbb{N}$ and $A\subseteq\bigcup_{n\in\mathbb{N}}H_n$. For $n\in\mathbb{N}$ choose $G_n\in\mathcal{G}_n$ such that $H_n\cap F\subseteq G_n$; then $A\subseteq\bigcup_{n\in\mathbb{N}}G_n$. As $\langle\mathcal{G}_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $A\in\mathcal{R}\mathrm{bg}(F)$.

534E Proposition Let $(X, \mathcal{W})$ be a uniform space, and give X the topology induced by $\mathcal{W}$.
(a) $\mathcal{R}\mathsf{bg}(X) \subseteq \mathcal{S}\mathsf{mz}(X, \mathcal{W})$.
(b) If X is σ-compact, $\mathcal{R}\mathsf{bg}(X) = \mathcal{S}\mathsf{mz}(X, \mathcal{W})$.

proof (a) Suppose that $A \in \mathcal{R}\mathsf{bg}(X)$, and that $\langle W_n \rangle_{n\in\mathbb{N}}$ is any sequence in $\mathcal{W}$. For each $n \in \mathbb{N}$, set $\mathcal{G}_n = \{G : G \subseteq X \text{ is open}, G \times G \subseteq W_n\}$; then $\mathcal{G}_n$ is a non-empty open cover of X. So we can find a cover $\langle G_n \rangle_{n\in\mathbb{N}}$ of A such that $G_n \in \mathcal{G}_n$, that is, $G_n \times G_n \subseteq W_n$, for each n. As $\langle W_n \rangle_{n\in\mathbb{N}}$ is arbitrary, $A \in \mathcal{S}\mathsf{mz}(X, \mathcal{W})$.

(b)(i) Let $K \subseteq X$ be compact and $\mathcal{G}$ an open cover of X. Then there is a $W \in \mathcal{W}$ such that whenever $x \in K$ there is a $G \in \mathcal{G}$ such that $W[\{x\}] \subseteq G$. **P** (Cf. 2A2Ed.) Set

$$Q = \{(x, V) : x \in X,\, V \in \mathcal{W},\, V[V[\{x\}]] \subseteq G \text{ for some } G \in \mathcal{G}\}.$$

Then for every $x \in X$ there are a $G \in \mathcal{G}$ such that $x \in G$ and a $V \in \mathcal{W}$ such that $V[V[\{x\}]] \subseteq G$, and in this case $(x, V) \in Q$ and $x \in \operatorname{int} V[\{x\}]$. So $\{\operatorname{int} V[\{x\}] : (x, V) \in Q\}$ is an open cover of X and there is a finite set $Q_0 \subseteq Q$ such that $K \subseteq \bigcup\{\operatorname{int} V[\{x\}] : (x, V) \in Q_0\}$. Let $W \in \mathcal{W}$ be such that $W \subseteq V$ whenever $(x, V) \in Q_0$. If $x \in K$, there is an $(x', V) \in Q_0$ such that $x \in V[\{x'\}]$; and now there is a $G \in \mathcal{G}$ including $V[V[\{x'\}]] \supseteq W[\{x\}]$. **Q**

(ii) Suppose that $K \subseteq X$ is compact and $A \in \mathcal{S}\mathsf{mz}(X, \mathcal{W})$. Then $A \cap K \in \mathcal{R}\mathsf{bg}(X)$. **P** Let $\langle \mathcal{G}_n \rangle_{n\in\mathbb{N}}$ be a sequence of non-empty open covers of X. For each $n \in \mathbb{N}$ let $W_n \in \mathcal{W}$ be such that $\{W_n[\{x\}] : x \in K\}$ refines $\mathcal{G}_n$. Let $\langle A_n \rangle_{n\in\mathbb{N}}$ be a cover of A such that $A_n \times A_n \subseteq W_n$ for every n. If $n \in \mathbb{N}$ and $A_n \cap K = \emptyset$, take any $G_n \in \mathcal{G}_n$. Otherwise, take $x_n \in A_n \cap K$ and $G_n \in \mathcal{G}_n$ such that $W_n[\{x_n\}] \subseteq G_n$. If $x \in A \cap K$, there is an $n \in \mathbb{N}$ such that $x \in A_n$; now $(x_n, x) \in A_n \times A_n \subseteq W_n$ and $x \in W_n[\{x_n\}] \subseteq G_n$. As x is arbitrary, $A \cap K \subseteq \bigcup_{n\in\mathbb{N}} G_n$. As $\langle \mathcal{G}_n \rangle_{n\in\mathbb{N}}$ is arbitrary, $A \cap K$ has Rothberger's property in X. **Q**

(iii) $\mathcal{S}\mathsf{mz}(X, \mathcal{W}) \subseteq \mathcal{R}\mathsf{bg}(X)$. **P** If $A \in \mathcal{S}\mathsf{mz}(X, \mathcal{W})$, let $\langle K_n \rangle_{n\in\mathbb{N}}$ be a sequence of compact subsets of X covering X. By (ii) here, $A \cap K_n \in \mathcal{R}\mathsf{bg}(X)$ for each n; by 534D(b-i), $A \in \mathcal{R}\mathsf{bg}(X)$. **Q** Putting this together with (a), we see that $\mathcal{R}\mathsf{bg}(X) = \mathcal{S}\mathsf{mz}(X, \mathcal{W})$.

534F Another case in which Rothberger's property and strong measure zero coincide is the following.

Proposition Let X be a regular paracompact space, and $\mathcal{W}$ the uniformity on X defined by the family of all continuous pseudometrics on X. Then

$$\begin{aligned}\mathcal{R}\mathsf{bg}(X) &= \mathcal{S}\mathsf{mz}(X, \mathcal{W}) \\ &= \{A : A \subseteq X,\ f[A] \in \mathcal{S}\mathsf{mz}(Y, \rho) \text{ whenever } (Y, \rho) \text{ is a metric space} \\ &\qquad\qquad \text{and } f : X \to Y \text{ is continuous}\}.\end{aligned}$$

proof X is normal, therefore completely regular (4A2Ge), so $\mathcal{W}$ induces its topology (4A2J(g-i)), and $\mathcal{R}\mathsf{bg}(X) \subseteq \mathcal{S}\mathsf{mz}(X, \mathcal{W})$ by 534Eb. If $A \in \mathcal{S}\mathsf{mz}(X, \mathcal{W})$, (Y, ρ) is a metric space and $f : X \to Y$ is continuous, then $(x, y) \mapsto \rho(f(x), f(y))$ is a continuous pseudometric on X so is one of the pseudometrics defining $\mathcal{W}$, and f is uniformly continuous; now $f[A] \in \mathcal{S}\mathsf{mz}(Y, \rho)$ by 534D(b-ii).

Now suppose that $A \subseteq X$ is such that $f[A] \in \mathcal{S}\mathsf{mz}(Y, \rho)$ whenever (Y, ρ) is a metric space and $f : X \to Y$ is continuous, and let $\langle \mathcal{G}_n \rangle_{n\in\mathbb{N}}$ be a sequence of open covers of X. By 5A4Fb there is for each $n \in \mathbb{N}$ a continuous pseudometric σ_n on X such that every subset of X of σ_n-diameter at most 1 is included in a member of $\mathcal{G}_n$. Set

$$\sigma(x, y) = \sum_{n=0}^{\infty} 2^{-n} \min(2, \sigma_n(x, y))$$

for x, $y \in X$. Then σ is a continuous pseudometric on X. Let $\sim$ be the corresponding equivalence relation $\{(x, y) : \sigma(x, y) = 0\}$ and $Y = X/\sim$ the set of equivalence classes; then Y has a metric ρ defined by saying that $\rho(x^\bullet, y^\bullet) = \sigma(x, y)$ for x, $y \in X$, and $x \mapsto x^\bullet : X \to Y$ is continuous. Accordingly $f[A] = \{x^\bullet : x \in A\}$ belongs to $\mathcal{S}\mathsf{mz}(Y, \rho)$, and there is a sequence $\langle B_n \rangle_{n\in\mathbb{N}}$ of subsets of Y, covering $f[A]$, such that the ρ-diameter of B_n is at most 2^{-n} for each n. Setting $A_n = f^{-1}[B_n]$ for each n, $A \subseteq \bigcup_{n\in\mathbb{N}} A_n$. If $n \in \mathbb{N}$ and x, $y \in A_n$, then $\sigma(x, y) \le 2^{-n}$ so $\sigma_n(x, y) \le 1$; by the choice of σ_n, there is a set $G_n \in \mathcal{G}_n$ including A_n. But now we see that $A \subseteq \bigcup_{n\in\mathbb{N}} \mathcal{G}_n$. As $\langle \mathcal{G}_n \rangle_{n\in\mathbb{N}}$ is arbitrary, $A \in \mathcal{R}\mathsf{bg}(X)$.

534G Remarks We see from 534E that in Euclidean space, the context of the original investigation of these ideas, what I call Rothberger's property and strong measure zero coincide; and as the latter phrase is more commonly used and has a more generally accepted meaning, it is tempting to prefer it. But in the framework of this treatise, devoted as it is to maximal convenient generality, the concepts diverge. Strong measure zero has an obvious interpretation in any metric space, and can readily be applied in general uniform spaces; while Rothberger's property is a topological notion. They have very different natures as soon as we leave the area of σ-compact spaces. In particular, the Polish space $\mathbb{N}^{\mathbb{N}}$, topologically identifiable with $\mathbb{R}\setminus\mathbb{Q}$, has a wide variety of compatible uniformities, giving rise to potentially very different strong measure zero ideals. So we find ourselves with the possibility that $\mathcal{R}\mathsf{bg}(\mathbb{R}\setminus\mathbb{Q})$ may be much smaller than the trace of $\mathcal{R}\mathsf{bg}(\mathbb{R})$ on the subset $\mathbb{R}\setminus\mathbb{Q}$, even though $\mathbb{Q}\in\mathcal{R}\mathsf{bg}(\mathbb{R})$ (534Sb). Strong measure zero, of course, is much more manageable on subsets (534D(a-i)).

534H Of course sets with strong measure zero or Rothberger's property are necessarily small in other ways.

Proposition If (X,ρ) is a metric space and $A\in\mathcal{S}\mathrm{mz}(X,\rho)$, then A is separable, zero-dimensional and universally negligible, and all compact subsets of A are countable.

proof (a) ? If A is not separable, there is an uncountable $B\subseteq A$ such that $\epsilon=\inf_{x,y\in B,x\ne y}\rho(x,y)$ is greater than 0 (5A4B(h-iii)). Now there can be no cover $\langle A_n\rangle_{n\in\mathbb{N}}$ of B by sets of diameter less than ϵ. **X**
Thus A is separable.

(b) Now suppose that μ is a Borel probability measure on A. Then there is a $\delta>0$ such that for every $n\in\mathbb{N}$ there is a relatively Borel set $E_n\subseteq A$ with $\operatorname{diam}E_n\le 2^{-n}$ and $\mu E_n\ge\delta$. **P?** Otherwise, we can find for each $n\in\mathbb{N}$ an $\epsilon_n>0$ such that $\mu E\le 2^{-n-2}$ whenever $E\subseteq A$ is a relatively Borel set and $\operatorname{diam}E\le\epsilon_n$. Let $\langle A_n\rangle_{n\in\mathbb{N}}$ be a cover of A such that $\operatorname{diam}A_n\le\epsilon_n$ for every n; then $\operatorname{diam}\overline{A}_n\le\epsilon_n$, so $\mu(A\cap\overline{A}_n)\le 2^{-n-2}$ for every n, and

$$\mu A\le\sum_{n=0}^{\infty}\mu(A\cap\overline{A}_n)<1. \text{ **XQ**}$$

Now consider $E=\bigcap_{n\in\mathbb{N}}\bigcup_{m\ge n}E_m$. Since $\mu E\ge\delta>0$, there is an $x\in E$. For any $n\in\mathbb{N}$, there is an $m\ge n$ such that

$$x\in E_m\subseteq B(x,2^{-m})\subseteq B(x,2^{-n}),$$

so

$$\mu\{x\}=\inf_{n\in\mathbb{N}}\mu(A\cap B(x,2^{-n}))\ge\delta>0.$$

As μ is arbitrary, this shows that A is universally negligible.

(c) In particular, $[0,1]$, with its usual metric, is not of strong measure zero. Now if $G\subseteq X$ is open and $x\in G$, let $\delta>0$ be such that $B(x,\delta)\subseteq G$, and set $f(y)=\max(0,1-\frac{1}{\delta}\rho(y,x))$ for $y\in X$; then $f:X\to[0,1]$ is uniformly continuous, so $f[X]$ has strong measure zero (534D(a-iii)) and cannot be the whole of $[0,1]$. As $f(x)=1$, there is an $\alpha\in[0,1[\setminus f[X]$, and $f^{-1}[[\alpha,1]]=f^{-1}[]\alpha,1]]$ is an open-and-closed neighbourhood of x included in G. As x and G are arbitrary, X is zero-dimensional.

(d) If $K\subseteq X$ is compact, it must be scattered (439C(a-v)); because it is first-countable, it must be countable (4A2G(j-vi)).

534I Let X be a regular topological space. Then X has Rothberger's property in itself iff it is Lindelöf and zero-dimensional and $f[X]\in\mathcal{R}\mathsf{bg}(\mathbb{R}\setminus\mathbb{Q})$ whenever $f:X\to\mathbb{R}\setminus\mathbb{Q}$ is continuous.

proof (a) Suppose that X has Rothberger's property in itself. Let $\mathcal{G}$ be an open cover of X. If X is empty then $\emptyset$ is a countable subset of $\mathcal{G}$ covering X. Otherwise, setting $\mathcal{G}_n=\mathcal{G}$ for each n, we have a sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ covering X such that $G_n\in\mathcal{G}_n$ for every n, and $\{G_n:n\in\mathbb{N}\}$ is a countable subcover of $\mathcal{G}$. So X is Lindelöf.

Thus X is Lindelöf and regular, therefore normal and completely regular (4A2H(b-i)). If $G\subseteq X$ is open and $x\in G$, there is a continuous $f:X\to[0,1]$ such that $f(x)=1$ and $f(y)=0$ for every $y\in X\setminus G$. Since $f[X]\in\mathcal{R}\mathsf{bg}([0,1])$ (534D(b-ii)), $f[X]\ne[0,1]$; taking $\alpha\in[0,1]\setminus X$, $f^{-1}[]\alpha,1]]=f^{-1}[[\alpha,1]]$ is an open-and-closed subset of G containing x. Thus X is zero-dimensional. And of course $f[X] \in \mathcal{R}\mathsf{bg}(\mathbb{R} \setminus \mathbb{Q})$ for every continuous $f : X \to \mathbb{R} \setminus \mathbb{Q}$, by by 534D(b-ii) again.

(b) Suppose that X has the given properties.

(i) If Z is a zero-dimensional Polish space and $f : X \to Z$ is continuous then $f[X] \in \mathcal{R}\mathsf{bg}(Z)$. **P** By 5A4If, we can suppose that Z is a closed subspace of $\mathbb{N}^{\mathbb{N}}$. In this case, $f[X] \in \mathcal{R}\mathsf{bg}(\mathbb{N}^{\mathbb{N}})$, by hypothesis; by 534D(b-iv), $f[X] \in \mathcal{R}\mathsf{bg}(Z)$.

(ii) Now take a sequence $\langle \mathcal{G}_n \rangle_{n\in\mathbb{N}}$ of non-empty open covers of X. For each $n \in \mathbb{N}$, let $\mathcal{G}'_n$ be the family of open-and-closed subsets of X included in members of $\mathcal{G}_n$; as X is zero-dimensional, each $\mathcal{G}'_n$ is a non-empty open cover of X. As X is Lindelöf, there is for each $n \in \mathbb{N}$ a sequence $\langle G_{ni} \rangle_{i\in\mathbb{N}}$ in $\mathcal{G}'_n$ such that $X = \bigcup_{i\in\mathbb{N}} G_{in}$. Define $f : X \to Z = \{0,1\}^{\mathbb{N}\times\mathbb{N}}$ by setting $f(x)(n,i) = \chi G_{ni}$ for $x \in X$ and n, $i \in \mathbb{N}$; as every G_{ni} is open-and-closed, f is continuous. For n, $i \in \mathbb{N}$, set $H_{ni} = \{z : z \in Z, \, z(n,i) = 1\}$, so that $G_{ni} = f^{-1}[H_{ni}$. Set $E = \bigcap_{n\in\mathbb{N}} \bigcup_{i\in\mathbb{N}} H_{ni}$, so that E is a G_δ subset of Z including $f[X]$. By 4A2Qd, E is Polish, so $f[X] \in \mathcal{R}\mathsf{bg}(E)$, by (i) above.

For each $n \in \mathbb{N}$, $\{E \cap H_{ni} : i \in \mathbb{N}\}$ is a relatively open cover of E. There is therefore a sequence $\langle i_n \rangle_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that $f[X] \subseteq \bigcup_{n\in\mathbb{N}} H_{ni_n}$ and $X = \bigcup_{n\in\mathbb{N}} G_{ni_n}$. Finally there is for each $n \in \mathbb{N}$ a $G_n \in \mathcal{G}_n$ such that $G_{ni_n} \subseteq G_n$, so that $X = \bigcup_{n\in\mathbb{N}} G_n$. As $\langle \mathcal{G}_n \rangle_{n\in\mathbb{N}}$ is arbitrary, X has Rothberger's property in itself.

534J Proposition Let X be a Hausdorff space, and K a compact subset of X. Then K belongs to $\mathcal{R}\mathsf{bg}(X)$ iff it is scattered.

proof (a) Set

$$\mathcal{F} = \{F : F \subseteq K \text{ is closed, } L \in \mathcal{R}\mathsf{bg}(X) \text{ for every closed } L \subseteq K \setminus F\}.$$

(b) $F_1 \cap F_2 \in \mathcal{F}$ whenever F_1, $F_2 \in \mathcal{F}$. **P** If $L \subseteq K \setminus (F_1 \cap F_2)$ is closed then $L \cap F_1$, $L \cap F_2$ are disjoint compact subsets of the Hausdorff space X, so there are disjoint open subsets G_1, G_2 of X such that $L \cap F_1 \subseteq G_1$ and $L \cap F_2 \subseteq G_2$ (4A2Fh). Now $L \setminus G_2$ is a closed subset of X disjoint from F_1, so belongs to $\mathcal{R}\mathsf{bg}(X)$, and similarly $L \setminus G_1 \in \mathcal{R}\mathsf{bg}(X)$, so $L = (L \setminus G_1) \cup (L \setminus G_2)$ belongs to $\mathcal{R}\mathsf{bg}(X)$. As L is arbitrary, $F_1 \cap F_2 \in \mathcal{F}$. **Q**

(c) $K^* = \bigcap \mathcal{F}$ belongs to $\mathcal{F}$. **P** Since $K \in \mathcal{F}$, $K^* \subseteq K$ and K^* is closed. If $L \subseteq K \setminus K^*$ is closed, therefore compact, there must be a finite subset $\mathcal{F}_0$ of $\mathcal{F}$ such that $L \cap \bigcap \mathcal{F}_0$ is empty; we can take it that $K \in \mathcal{F}_0$, and now (b) assures us that $\bigcap \mathcal{F}_0 \in \mathcal{F}$ so $L \in \mathcal{R}\mathsf{bg}(X)$. As L is arbitrary, $K^* \in \mathcal{F}$. **Q**

(d) K^* has no isolated point. **P?** If $x \in K^*$ is an isolated point of K^*, set $F = K^* \setminus \{x\}$. Then F is a closed subset of K not belonging to $\mathcal{F}$, so there is a closed set $L \subseteq K \setminus F$ which does not belong to $\mathcal{R}\mathsf{bg}(X)$. Now $\{x\}$ certainly belongs to $\mathcal{R}\mathsf{bg}(X)$, so by 534D(b-iii) there is an open set H containing x such that $L \setminus H \notin \mathcal{R}\mathsf{bg}(X)$. But $L \setminus H$ is a closed subset of $K \setminus K^*$ and $K^* \in \mathcal{F}$, by (c). **XQ**

(e) If K is scattered, then K^* must be empty, $K \subseteq K \setminus K^*$ and $K \in \mathcal{R}\mathsf{bg}(X)$.

(f) Finally, if K is not scattered then there is a continuous surjection from K to $[0,1]$ (4A2G(j-iv)); now $[0,1] \notin \mathcal{S}\mathsf{mz}(\mathbb{R}, \rho)$, where ρ is the usual metric on $\mathbb{R}$, by 534H, so $[0,1] \notin \mathcal{R}\mathsf{bg}(\mathbb{R})$ (534Sa), $K \notin \mathcal{R}\mathsf{bg}(K)$ (534D(b-ii)) and $K \notin \mathcal{R}\mathsf{bg}(X$ (534D(b-iv)).

534K Theorem Let X be a σ-compact locally compact Hausdorff topological group and A a subset of X. Then the following are equiveridical:

(i) $A \in \mathcal{R}\mathsf{bg}(X)$;

(ii) for any sequence $\langle U_n \rangle_{n\in\mathbb{N}}$ of neighbourhoods of the identity e of X, there is a sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ in X such that $A \subseteq \bigcup_{n\in\mathbb{N}} U_n x_n$;

(iii) $FA \neq X$ for any nowhere dense set $F \subseteq X$;

(iv) $EA \neq X$ for any meager set $E \subseteq X$;

(v) $AF \neq X$ for any nowhere dense set $F \subseteq X$;

(vi) $AE \neq X$ for any meager set $E \subseteq X$.

Remark For the general theory of topological groups see §4A5 and Chapter 44. Readers unfamiliar with this theory, or impatient with the extra discipline needed to deal with non-commutative groups, may prefer to start by assuming that $X=\mathbb{R}^2$, so that every xU becomes $x+U$, every $V^{-1}V$ becomes $V-V$, and the right uniformity is the Euclidean metric uniformity.

proof (i)⇒(ii) Suppose that (i) is true, and that $\langle U_n\rangle_{n\in\mathbb{N}}$ is any sequence of neighbourhoods of e. Then $\{\operatorname{int}U_nx:x\in X\}$ is an open cover of X for each n, so there is a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ such that $A\subseteq\bigcup_{n\in\mathbb{N}}U_nx_n$.

(ii)⇒(i) Suppose that (ii) is true, and that $\langle W_n\rangle_{n\in\mathbb{N}}$ is any sequence in the right uniformity $\mathcal{W}$ of X (4A5Ha). Then for each $n\in\mathbb{N}$ there is a neighbourhood U_n of e such that $W_n\supseteq\{(x,y):xy^{-1}\in U_n\}$; let V_n be a neighbourhood of e such that $V_nV_n^{-1}\subseteq U_n$. By (ii), there is a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ such that $A\subseteq\bigcup_{n\in\mathbb{N}}V_nx_n$. Set $A_n=V_nx_n$ for each n. Then $A_nA_n^{-1}=V_nV_n^{-1}\subseteq U_n$, so $A_n\times A_n\subseteq W_n$, for each n, while $\langle A_n\rangle_{n\in\mathbb{N}}$ covers A. As $\langle W_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $A\in\mathcal{S}\mathrm{mz}(X,\mathcal{W})$. By 534Eb, $A\in\mathcal{R}\mathrm{bg}(X)$.

(ii)⇒(iv) Suppose that A satisfies (ii), and that $E\subseteq X$ is meager.

(α) If $K\subseteq X$ is compact and nowhere dense, then there is a sequence $\langle U_n\rangle_{n\in\mathbb{N}}$ of neighbourhoods of e such that $K'=\bigcap_{n\in\mathbb{N}}U_nK$ is still compact and nowhere dense. **P** By 443N(ii), there is a nowhere dense zero set $F\supseteq K$. Now F is a G_δ set; suppose that $F=\bigcap_{n\in\mathbb{N}}G_n$ where G_n is open for each n. As $K\subseteq G_n$, the open set $U_n'=\{x:xK\subseteq G_n\}$ (4A5Ei) contains e; let U_n be a compact neighbourhood of e included in U_n'. Then $U_nK\subseteq G_n$ for every n, so $K'=\bigcap_{n\in\mathbb{N}}U_nK\subseteq F$ is nowhere dense, while K' is compact (use 4A5Ef). **Q**

(β) Let $K\subseteq X$ be compact and nowhere dense and U a neighbourhood of e. Then there is a neighbourhood V of e such that for every $x\in X$ there is an $x'\in Ux$ such that $Vx'\cap K=\emptyset$. **P** Let $\langle U_n\rangle_{n\in\mathbb{N}}$ be a sequence of neighbourhoods of e such that $K'=\bigcap_{n\in\mathbb{N}}U_nK$ is compact and nowhere dense ((α) above). Choose a sequence $\langle V_n\rangle_{n\in\mathbb{N}}$ of compact neighbourhoods of e such that $V_0\subseteq U$ and $V_{n+1}V_{n+1}^{-1}\subseteq U_n\cap V_n$ for each $n\in\mathbb{N}$. Then $Y=\bigcap_{n\in\mathbb{N}}V_n$ is a compact subgroup of X (see the proof of 4A5S), and $YK=\bigcap_{n\in\mathbb{N}}V_nK$ (4A5Eh). **?** If for every $n\in\mathbb{N}$ there is an $x_n\in X$ such that $V_n^{-1}x'\cap K\ne\emptyset$ for every $x'\in Ux_n$, then, in particular, $V_n^{-1}x_n\cap K\ne\emptyset$, so $x_n\in V_nK$. Since $\langle V_nK\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of compact sets, $\langle x_n\rangle_{n\in\mathbb{N}}$ has a cluster point

$$x^*\in\bigcap_{n\in\mathbb{N}}V_nK=YK\subseteq K'.$$

Because K' is nowhere dense, $V_1x^*\not\subseteq K'$; take $x\in V_1x^*\setminus K'$. Let W be an open neighbourhood of e such that $Wx\cap K'=\emptyset$. Then Wx is disjoint from $YK=Y^{-1}YK$ so $YWx\cap YK=\emptyset$. Now YW is an open set including $Y=\bigcap_{n\in\mathbb{N}}V_n$, and all the V_n are compact, so there is an $m\ge1$ such that $V_m\subseteq YW$ and $V_mx\cap YK=\emptyset$.

But observe that there is an $n>m$ such that $x_n\in V_1x^*$, so that

$$x\in V_1V_1^{-1}x_n\subseteq V_0x_n\subseteq Ux_n,$$

while $V_n^{-1}x\cap K\subseteq V_mx\cap YK$ is empty. **X**

Thus we can take $V=V_n^{-1}$ for some n. **Q**

(γ) Because X is σ-compact, any F_σ set in X is actually K_σ, and there is a sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ of nowhere dense compact sets covering E; we can suppose that $\langle K_n\rangle_{n\in\mathbb{N}}$ is non-decreasing. Choose inductively sequences $\langle U_n\rangle_{n\in\mathbb{N}}$, $\langle V_n\rangle_{n\in\mathbb{N}}$, $\langle V_n'\rangle_{n\in\mathbb{N}}$ and $\langle V_n''\rangle_{n\in\mathbb{N}}$ of neighbourhoods of e such that

U_0 is any compact neighbourhood of e,

given U_n, V_n is to be a neighbourhood of e such that $V_nV_n\subseteq U_n$,

given V_n, V_n' is to be a neighbourhood of e such that for every $y\in X$ there is a $z\in V_ny$ such that $V_n'z\cap K_{n+1}=\emptyset$

(using (β)),

given V_n', V_n'' is to be an open neighbourhood of e such that $(V_n'')^{-1}V_n''\subseteq V_n'$,

given V_n'', U_{n+1} is to be a compact neighbourhood of e, included in $V_n\cap V_n''$, such that $K_{n+1}U_{n+1}\subseteq V_n''K_{n+1}$.

(This last is possible by 4A5Ei, because $V_n''K_{n+1}$ is an open set including K_{n+1}, so $\{x:K_{n+1}x\subseteq V_n''K_{n+1}\}$ is an open set containing e.)

(δ) For each $k\in\mathbb{N}$, $\langle U_{2^k(2i+1)}\rangle_{i\in\mathbb{N}}$ is a sequence of neighbourhoods of e, so there must be a sequence $\langle x_{ki}\rangle_{i\in\mathbb{N}}$ such that $A\subseteq\bigcup_{i\in\mathbb{N}}U_{2^k(2i+1)}x_{ki}$. Set $x_0=e$ and $x_n=x_{ki}$ if $n=2^k(2i+1)$. For any $k\in\mathbb{N}$,

$$A \subseteq \bigcup_{i\in\mathbb{N}} A_{ki} \subseteq \bigcup_{i\in\mathbb{N}} U_{2^k(2i+1)}x_{ki} \subseteq \bigcup_{n\ge 2^k} U_n x_n \subseteq \bigcup_{n\ge k} U_n x_n.$$

This means that $EA \subseteq \bigcup_{n\ge1} K_n U_n x_n$. **P** If $z \in EA$, we can express it as xy where $x \in E$ and $y \in A$. There are a $k \ge 1$ such that $x \in K_k$ and an $n \ge k$ such that $y \in U_n x_n$, in which case $z \in K_n U_n x_n$. **Q**

(ϵ) Now choose $\langle y_n\rangle_{n\in\mathbb{N}}$, $\langle z_n\rangle_{n\in\mathbb{N}}$ as follows. Start from $y_0 = e$. Given y_n, let $z_n \in V_n y_n x_{n+1}^{-1}$ be such that $V'_n z_n \cap K_{n+1} = \emptyset$; this is possible by the choice of V'_n. Now set $y_{n+1} = z_n x_{n+1}$, and continue.

For each n,

$$U_{n+1}y_{n+1} \subseteq V_n y_{n+1}$$

(by the choice of U_{n+1})

$$= V_n z_n x_{n+1} \subseteq V_n V_n y_n x_{n+1}^{-1} x_{n+1}$$

(by the choice of z_n)

$$\subseteq U_n y_n$$

by the choice of V_n. Consequently, $U_{n+1}y_{n+1} \cap K_{n+1}U_{n+1}x_{n+1} = \emptyset$. **P** We chose z_n such that $V'_n z_n \cap K_{n+1} = \emptyset$. Because $(V''_n)^{-1}V''_n \subseteq V'_n$, $V''_n z_n \cap V''_n K_{n+1} = \emptyset$. Because $K_{n+1}U_{n+1} \subseteq V''_n K_{n+1}$ and $U_{n+1} \subseteq V''_n$, $U_{n+1}z_n \cap K_{n+1}U_{n+1} = \emptyset$, that is, $U_{n+1}y_{n+1} \cap K_{n+1}U_{n+1}x_{n+1} = \emptyset$. **Q**

(ζ) From (ϵ) we see that $\langle U_n y_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of compact sets, so has non-empty intersection. Take any $x \in \bigcap_{n\in\mathbb{N}} U_n y_n$. Then $x \notin K_{n+1}U_{n+1}x_{n+1}$ for any n, so $x \notin \bigcup_{n\ge1} K_n U_n x_n \supseteq EA$. Thus $EA \ne X$. As E is arbitrary, (iv) is true.

(iv)$\Rightarrow$(iii) is trivial.

(iii)$\Rightarrow$(ii) Suppose that (iii) is true. Let $\langle U_n\rangle_{n\in\mathbb{N}}$ be any sequence of open neighbourhoods of e. Then there is a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in X such that $G = \bigcup_{n\in\mathbb{N}} x_n U_n^{-1}$ is dense. **P** Let $\langle V_n\rangle_{n\in\mathbb{N}}$ be a sequence of neighbourhoods of e such that $V_{n+1}V_{n+1}^{-1} \subseteq V_n \cap U_n^{-1}$ for every $n \in \mathbb{N}$. Then there is a compact normal subgroup Y of X such that $Y \subseteq \bigcap_{n\in\mathbb{N}} V_n$ and X/Y is metrizable (4A5S). The canonical map $x \mapsto x^\bullet : X \to X/Y$ is continuous, so X/Y is σ-compact, therefore separable (4A2P(a-ii)). Let $\langle x_n\rangle_{n\in\mathbb{N}}$ be a sequence in X such that $\{x_n^\bullet : n \in \mathbb{N}\}$ is dense in X/Y. Set $G_0 = \bigcup_{n\in\mathbb{N}} x_n V_{n+1} Y$. **?** If $H = X \setminus \overline{G}_0$ is non-empty, then $\{x^\bullet : x \in H\}$ is open (4A5Ja) so contains $x_n^\bullet$ for some n. But $x_n Y \subseteq x_n V_{n+1} Y \subseteq G_0$, so there can be no $x \in H$ such that $x^\bullet = x_n^\bullet$. **X** Thus G_0 is dense. But, for any $n \in \mathbb{N}$, $Y \subseteq V_{n+1}^{-1}$ so $V_{n+1}Y \subseteq U_n^{-1}$, and $G = \bigcup_{n\in\mathbb{N}} x_n U_n^{-1}$ includes G_0. Thus G is dense, as required. **Q**

Accordingly $F = X \setminus G$ is nowhere dense, and $FA \ne X$; suppose $x \in X \setminus FA$. Then $F \cap xA^{-1} = \emptyset$, that is, $xA^{-1} \subseteq \bigcup_{n\in\mathbb{N}} x_n U_n^{-1}$, that is, $A^{-1} \subseteq \bigcup_{n\in\mathbb{N}} x^{-1}x_n U_n^{-1}$, that is, $A \subseteq \bigcup_{n\in\mathbb{N}} U_n x_n^{-1} x$. As $\langle U_n\rangle_{n\in\mathbb{N}}$ is arbitrary, (ii) is true.

(i)$\Leftrightarrow$(v)$\Leftrightarrow$(vi) Because $x \mapsto x^{-1}$ is a homeomorphism,

$$\begin{aligned} A \in \mathcal{R}\mathrm{bg}(X) &\Longrightarrow A^{-1} \in \mathcal{R}\mathrm{bg}(X) \\ &\Longrightarrow EA^{-1} \ne X \text{ whenever } E \subseteq X \text{ is meager} \\ &\Longrightarrow E^{-1}A^{-1} \ne X \text{ whenever } E \subseteq X \text{ is meager} \end{aligned}$$

(because E^{-1} is meager if E is)

$$\begin{aligned} &\Longleftrightarrow AE \ne X \text{ whenever } E \subseteq X \text{ is meager} \\ &\Longrightarrow AF^{-1} \ne X \text{ whenever } F \subseteq X \text{ is nowhere dense} \end{aligned}$$

(because F^{-1} is nowhere dense if F is)

$$\begin{aligned} &\Longrightarrow FA^{-1} \ne X \text{ whenever } F \subseteq X \text{ is nowhere dense} \\ &\Longrightarrow A^{-1} \in \mathcal{R}\mathrm{bg}(X) \\ &\Longrightarrow A \in \mathcal{R}\mathrm{bg}(X). \end{aligned}$$

Remark The case $X = \mathbb{R}$ is due to Galvin Mycielski & Solovay 79.

534L Proposition (Fremlin 91) Let (X,ρ) be a separable metric space. Then $\mathcal{S}mz(X,\rho) \preccurlyeq_{\mathrm{T}} \mathcal{N}^{\mathfrak{d}}$, where $\mathcal{N}$ is the null ideal of Lebesgue measure on $\mathbb{R}$ and $\mathfrak{d}$ is the dominating number (522A).

proof (a) By 534A, there is a countable family $\mathcal{C}$ of subsets of X such that whenever $A \subseteq X$ has finite diameter and $\eta > 0$, there is a $C \in \mathcal{C}$ such that $A \subseteq C$ and $\operatorname{diam} C \le \eta + 2\operatorname{diam} A$. For each $i \in \mathbb{N}$, let $\langle C_{ij}\rangle_{j\in\mathbb{N}}$ be a sequence running over $\{C : C \in \mathcal{C},\ \operatorname{diam} C \le 2^{-i}\}$. Let $(\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S})$ be the $\mathbb{N}$-localization relation.

(b) Let $D \subseteq \mathbb{N}^{\mathbb{N}}$ be a cofinal set with cardinal $\mathfrak{d}$. For each $d \in D$ we can find a function $\phi_d : \mathcal{S}mz(X,\rho) \to \mathbb{N}^{\mathbb{N}}$ such that $A \subseteq \bigcap_{n\in\mathbb{N}}\bigcup_{i\ge n} C_{d(i),\phi_d(A)(i)}$ for every $A \in \mathcal{S}mz(X,\rho)$. **P** For $A \in \mathcal{S}mz(X,\rho)$ and $k \in \mathbb{N}$, choose a sequence $\langle A_{ki}\rangle_{i\in\mathbb{N}}$ of sets covering A such that $2\operatorname{diam} A_{ki} < 2^{-d(2^k(2i+1))}$ for every $i \in \mathbb{N}$. For $n = 2^k(2i+1)$, let $A_n \in \mathcal{C}$ be such that $A_{ki} \subseteq A_n$ and $\operatorname{diam} A_n \le 2^{-d(n)}$; choose $\phi_d(A)(n)$ such that $A_n = C_{d(n),\phi_d(A)(n)}$. **Q** Define $\phi : \mathcal{S}mz(X,\rho) \to (\mathbb{N}^{\mathbb{N}})^D$ by setting $\phi(A) = \langle\phi_d(A)\rangle_{d\in D}$ for $A \in \mathcal{S}mz(X,\rho)$.

(c) For $S \in \mathcal{S}$ and $d \in D$, define

$$\psi_d(S) = \bigcap_{n\in\mathbb{N}}\bigcup_{i\ge n}\bigcup_{j\in S[\{i\}]} C_{d(i),j} \subseteq X.$$

For $\langle S_d\rangle_{d\in D} \in \mathcal{S}^D$ set $\psi(\langle S_d\rangle_{d\in D}) = \bigcap_{d\in D}\psi_d(S_d)$. Then $A = \psi(\langle S_d\rangle_{d\in D})$ has strong measure zero. **P** Let $\langle\epsilon_i\rangle_{i\in\mathbb{N}}$ be any family of strictly positive real numbers. Let $d \in D$ be such that $2^{-d(k)} \le \epsilon_i$ whenever $k \in \mathbb{N}$ and $i < 2^{k+1}$. For each $k \in \mathbb{N}$, $\#(S_d[\{k\}]) \le 2^k$, so we can find a sequence $\langle A_i\rangle_{i\in\mathbb{N}}$ such that $\langle A_i\rangle_{2^k\le i<2^{k+1}}$ is a re-enumeration of $\langle C_{d(k),j}\rangle_{j\in S[\{k\}]}$ supplemented by empty sets if necessary. This will ensure that if $2^k \le i < 2^{k+1}$ then $\operatorname{diam} A_i \le 2^{-d(k)} \le \epsilon_i$, while

$$A \subseteq \psi_d(S_d) \subseteq \bigcup_{k\in\mathbb{N}}\bigcup_{j\in S_d[\{k\}]} C_{d(k),j} = \bigcup_{(k,j)\in S_d} C_{d(k),j} = \bigcup_{i\in\mathbb{N}} A_i.$$

As $\langle\epsilon_i\rangle_{i\in\mathbb{N}}$ is arbitrary, $A \in \mathcal{S}mz(X,\rho)$. **Q**

(d) Taking $(\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S})$ to be the $\mathbb{N}$-localization relation, as in the proof of 534B, (ϕ,ψ) is a Galois-Tukey connection from $(\mathcal{S}mz(X,\rho), \subseteq, \mathcal{S}mz(X,\rho))$ to $(\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S})^D$, that is, $((\mathbb{N}^{\mathbb{N}})^D, T, \mathcal{S}^D)$, where T is the simple product relation as defined in 512H. **P** $\phi : \mathcal{S}mz(X,\rho) \to (\mathbb{N}^{\mathbb{N}})^D$ and $\psi : \mathcal{S}^D \to \mathcal{S}mz(X,\rho)$ are functions. Suppose that $A \in \mathcal{S}mz(X,\rho)$ and $\langle S_d\rangle_{d\in D}$ are such that $(\phi(A), \langle S_d\rangle_{d\in D}) \in T$, that is, $\phi_d(A) \subseteq^* S_d$ for every d. Fix $d \in D$ for the moment. Then there is an $n \in \mathbb{N}$ such that $(i, \phi_d(A)(i)) \in S_d$ for every $i \ge n$. Now, for any $m \ge n$,

$$A \subseteq \bigcup_{i\ge m} C_{d(i),\phi_d(A)(i)} \subseteq \bigcup_{i\ge m}\bigcup_{j\in S_d[\{i\}]} C_{d(i),j}.$$

Thus

$$A \subseteq \bigcap_{m\in\mathbb{N}}\bigcup_{i\ge m}\bigcup_{j\in S_d[\{i\}]} C_{d(i),j)} = \psi_d(S_d).$$

This is true for every d, so $A \subseteq \psi(\langle S_d\rangle_{d\in D})$. As A and $\langle S_d\rangle_{d\in D}$ are arbitrary, (ϕ,ψ) is a Galois-Tukey connection. **Q**

(e) Thus $(\mathcal{S}mz(X,\rho), \subseteq, \mathcal{S}mz(X,\rho)) \preccurlyeq_{\mathrm{GT}} (\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S})^D$. But $(\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S}) \equiv_{\mathrm{GT}} (\mathcal{N}, \subseteq, \mathcal{N})$ (522M), so $(\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S})^D \equiv_{\mathrm{GT}} (\mathcal{N}, \subseteq, \mathcal{N})^D$ (512Hb) and

$$(\mathcal{S}mz(X,\rho), \subseteq, \mathcal{S}mz(X,\rho)) \preccurlyeq_{\mathrm{GT}} (\mathcal{N}, \subseteq, \mathcal{N})^D = (\mathcal{N}^D, \le, \mathcal{N}^D)$$

where $\le$ is the natural partial order of the product partially ordered set $\mathcal{N}^D$. Accordingly $\mathcal{S}mz(X,\rho) \preccurlyeq_{\mathrm{T}} \mathcal{N}^D \cong \mathcal{N}^{\mathfrak{d}}$, as claimed.

534M Corollary (a) If $(X, \mathcal{W})$ is a Lindelöf uniform space, then $\operatorname{add}\mathcal{S}mz(X,\mathcal{W}) \ge \operatorname{add}\mathcal{N}$, where $\mathcal{N}$ is the null ideal of Lebesgue measure on $\mathbb{R}$.

(b) If X is a Lindelöf regular topological space, then $\operatorname{add}\mathcal{R}bg(X) \ge \operatorname{add}\mathcal{N}$.

proof (a)(i) If (X,ρ) is a separable metric space, then 534L tells us that $\mathcal{S}mz(X,\rho) \preccurlyeq_{\mathrm{T}} \mathcal{N}^{\mathfrak{d}}$, so $\operatorname{add}\mathcal{S}mz(X,\rho) \ge \operatorname{add}\mathcal{N}^{\mathfrak{d}} = \operatorname{add}\mathcal{N}$ (513E(e-ii), 511Hg).

(ii) In general, if $\mathcal{A} \subseteq \mathcal{S}mz(X,\mathcal{W})$ and $\#(\mathcal{A}) < \operatorname{add}\mathcal{N}$, take any metric space (Y,ρ) and uniformly continuous $f : X \to Y$. Then $f[X]$ is Lindelöf (5A4Bc), therefore separable (4A2Pc), and $f[A]$ has strong measure zero in $f[X]$ for every $A \in \mathcal{A}$ (534D(a-iii)), so $f[\bigcup\mathcal{A}] = \bigcup_{A\in\mathcal{A}} f[A]$ has strong measure zero, by (i). As f is arbitrary, $\bigcup\mathcal{A}$ has strong measure zero, by 534D(a-iv); as $\mathcal{A}$ is arbitrary, $\operatorname{add}\mathcal{S}mz(X,\mathcal{W}) \ge \operatorname{add}\mathcal{N}$.

(b) Being Lindelöf and regular, X is paracompact and normal (4A2H(b-i)), so there is a uniformity $\mathcal{W}$ on X, inducing its topology, with $\mathcal{R}\mathrm{bg}(X) = \mathcal{S}\mathrm{mz}(X, \mathcal{W})$ (534F); so $\operatorname{add} \mathcal{R}\mathrm{bg}(X) = \operatorname{add} \mathcal{S}\mathrm{mz}(X, \mathcal{W}) \ge \operatorname{add} \mathcal{N}$, by (a).

534N $\mathcal{S}\mathrm{mz}$-equivalence (a) If $(X, \mathcal{V})$ and $(Y, \mathcal{W})$ are uniform spaces, I say that they are **$\mathcal{S}\mathrm{mz}$-equivalent** if there is a bijection $f : X \to Y$ such that a set $A \subseteq X$ has strong measure zero in X iff $f[A]$ has strong measure zero in Y. Of course this is an equivalence relation on the class of uniform spaces.

(b) If $(X, \mathcal{V})$ and $(Y, \mathcal{W})$ are uniform spaces, I say that X is **$\mathcal{S}\mathrm{mz}$-embeddable** in Y if it is $\mathcal{S}\mathrm{mz}$-equivalent to a subspace of Y (with the subspace uniformity, of course). Evidently this is transitive in the sense that if X is $\mathcal{S}\mathrm{mz}$-embeddable in Y and Y is $\mathcal{S}\mathrm{mz}$-embeddable in Z then X is $\mathcal{S}\mathrm{mz}$-embeddable in Z.

534O Lemma (a) Suppose that $(X, \mathcal{W})$ and $(Y, \mathcal{V})$ are uniform spaces and that $\langle X_n \rangle_{n \in \mathbb{N}}$, $\langle Y_n \rangle_{n \in \mathbb{N}}$ are partitions of X, Y respectively such that X_n is $\mathcal{S}\mathrm{mz}$-equivalent to Y_n for every n. Then X is $\mathcal{S}\mathrm{mz}$-equivalent to Y.

(b) Suppose that $(X, \mathcal{W})$ and $(Y, \mathcal{V})$ are uniform spaces such that X is $\mathcal{S}\mathrm{mz}$-embeddable in Y and Y is $\mathcal{S}\mathrm{mz}$-embeddable in X. Then $(X, \mathcal{W})$ and $(Y, \mathcal{V})$ are $\mathcal{S}\mathrm{mz}$-equivalent.

proof (a) For each $n \in \mathbb{N}$, let $f_n : X_n \to Y_n$ be a bijection identifying the ideals of sets with strong measure zero. Then $f = \bigcup_{n \in \mathbb{N}} f_n$ is a bijection identifying $\mathcal{S}\mathrm{mz}(X, \mathcal{W})$ and $\mathcal{S}\mathrm{mz}(Y, \mathcal{V})$.

(b) (Compare 344D.) Let $X_1 \subseteq X$ and $Y_1 \subseteq Y$ be $\mathcal{S}\mathrm{mz}$-equivalent to Y, X respectively; let $f : X \to Y_1$ and $g : Y \to X_1$ be bijections identifying the ideals of strong measure zero in each pair. Set $X_0 = X$, $Y_0 = Y$, $X_{n+1} = g[Y_n]$ and $Y_{n+1} = f[X_n]$ for each $n \ge 1$; then $\langle X_n \rangle_{n \in \mathbb{N}}$ is a non-increasing sequence of subsets of X and $\langle Y_n \rangle_{n \in \mathbb{N}}$ is a non-increasing sequence of subsets of Y. Set $X_\infty = \bigcap_{n \in \mathbb{N}} X_n$, $Y_\infty = \bigcap_{n \in \mathbb{N}} Y_n$. Then $f \restriction X_{2k} \setminus X_{2k+1}$ is an $\mathcal{S}\mathrm{mz}$-equivalence between $X_{2k} \setminus X_{2k+1}$ and $Y_{2k+1} \setminus Y_{2k+2}$, while $g \restriction Y_{2k} \setminus Y_{2k+1}$ is an $\mathcal{S}\mathrm{mz}$-equivalence between $Y_{2k} \setminus Y_{2k+1}$ and $X_{2k+1} \setminus X_{2k+2}$; and $g \restriction Y_\infty$ is an $\mathcal{S}\mathrm{mz}$-equivalence between Y_∞ and X_∞. So (a) gives the required $\mathcal{S}\mathrm{mz}$-equivalence between X and Y.

534P Proposition $\mathbb{R}^r$, $]0,1[^r$, $[0,1]^r$ and $\{0,1\}^{\mathbb{N}}$ are $\mathcal{S}\mathrm{mz}$-equivalent for every integer $r \ge 1$.

proof As these spaces are σ-compact and completely regular, we do not have to specify the uniformities we are thinking of, by 534Eb; in each case, the sets with strong measure zero are the sets with Rothberger's property.

(a) Give $\mathbb{R}$ its usual metric ρ. Of course the identity maps are $\mathcal{S}\mathrm{mz}$-embeddings of $]0,1[$ in $[0,1]$ and $[0,1]$ in $\mathbb{R}$. To complete the circuit, use 534Eb; any homeomorphism between $\mathbb{R}$ and $]0,1[$ matches $\mathcal{R}\mathrm{bg}(\mathbb{R}) = \mathcal{S}\mathrm{mz}(\mathbb{R}, \rho)$ with $\mathcal{R}\mathrm{bg}(]0,1[) = \mathcal{S}\mathrm{mz}(]0,1[, \rho)$. By 534Ob, $\mathbb{R}$ and $[0,1]$ and $]0,1[$ are $\mathcal{S}\mathrm{mz}$-equivalent.

(b) Give $\{0,1\}^{\mathbb{N}}$ the metric ρ defined by saying that

$$\rho(x, y) = \inf\{2^{-n} : n \in \mathbb{N},\, x \restriction n = y \restriction n\}$$

for x, $y \in \{0,1\}^{\mathbb{N}}$. Define $f : \{0,1\}^{\mathbb{N}} \to [0,1]$ by setting $f(x) = \sum_{n=0}^{\infty} 2^{-n-1} x(n)$ for $x \in \{0,1\}^{\mathbb{N}}$. Then f is continuous, therefore uniformly continuous, so $f[A]$ has strong measure zero in $[0,1]$ whenever $A \subseteq \{0,1\}^{\mathbb{N}}$ has strong measure zero in $\{0,1\}^{\mathbb{N}}$. It is also the case that $f^{-1}[B]$ has strong measure zero whenever $B \subseteq [0,1]$ does. **P** Let $\langle \epsilon_n \rangle_{n \in \mathbb{N}}$ be any sequence of strictly positive numbers. Then there is a sequence $\langle B_n \rangle_{n \in \mathbb{N}}$, covering B, such that $\operatorname{diam} B_n < \frac{1}{2} \min(1, \epsilon_{2n}, \epsilon_{2n+1})$ for every n. Fix n for the moment and consider $f^{-1}[B_n]$. If k is such that $2^{-k-1} \le \operatorname{diam} B_n < 2^{-k}$, then B_n can meet at most two intervals of the type $I_{ki} = [2^{-k} i, 2^{-k}(i+1)]$. So $f^{-1}[B_n]$ can meet at most two sets of the type $\{x : x \restriction k = z\}$, and we can express it as $A_{2n} \cup A_{2n+1}$ where

$$\max(\operatorname{diam} A_{2n}, \operatorname{diam} A_{2n+1}) \le 2^{-k} \le 2 \operatorname{diam} B_n \le \min(\epsilon_{2n}, \epsilon_{2n+1}).$$

Putting these together, we have a cover $\langle A_n \rangle_{n \in \mathbb{N}}$ of $\bigcup_{n \in \mathbb{N}} f^{-1}[B_n] \supseteq f^{-1}[B]$ such that $\operatorname{diam} A_n \le \epsilon_n$ for every n; as $\langle \epsilon_n \rangle_{n \in \mathbb{N}}$ is arbitrary, $f^{-1}[B]$ has strong measure zero. **Q**

Of course f is not a bijection, so it is not in itself an $\mathcal{S}\mathrm{mz}$-equivalence. But if we set

$$D_1 = \{x : x \in \{0,1\}^{\mathbb{N}},\, x \text{ is eventually constant}\},$$

$$D_2 = \{2^{-k}i : k \in \mathbb{N},\, i \le 2^k\},$$

then $D_1 \subseteq \{0,1\}^{\mathbb{N}}$ and $D_2 \subseteq [0,1]$ are countably infinite, and $f{\restriction}\{0,1\}^{\mathbb{N}} \setminus D_1$ is an $\mathcal{S}\mathfrak{mz}$-equivalence between $\{0,1\}^{\mathbb{N}} \setminus D_1$ and $[0,1] \setminus D_2$. Putting this together with any bijection between D_1 and D_2, we have an $\mathcal{S}\mathfrak{mz}$-equivalence between $\{0,1\}^{\mathbb{N}}$ and $[0,1]$.

(c)(i) I show by induction on r that $[0,1]^r$ is $\mathcal{S}\mathfrak{mz}$-equivalent to $\mathbb{R}$ and therefore to $[0,1]$. The case $r = 1$ is covered by (a). For the inductive step to $r \ge 2$, I adapt the method of (b). Give $\{0,1\}^{\mathbb{N}\times r}$ the metric ρ defined by setting

$$\rho(x,y) = \inf\{2^{-n} : n \in \mathbb{N},\, x{\restriction}(n \times r) = y{\restriction}(n \times r)\}$$

for x, $y \in \{0,1\}^{\mathbb{N}\times r}$. Define $f : \{0,1\}^{\mathbb{N}\times r} \to [0,1]^r$ by setting

$$f(x) = \langle \textstyle\sum_{i=0}^{\infty} 2^{-i-1}x(i,j)\rangle_{j<r}$$

for $x \in \{0,1\}^{\mathbb{N}\times r}$. Then f is uniformly continuous, so $f[A]$ has strong measure zero in $[0,1]^r$ whenever A has strong measure zero in $\{0,1\}^{\mathbb{N}\times r}$. Moreover, we find once again that $f^{-1}[B]$ has strong measure zero whenever $B \subseteq [0,1]^r$ has strong measure zero. **P** Let $\langle \epsilon_n\rangle_{n\in\mathbb{N}}$ be a sequence of strictly positive real numbers. This time, set $m = 2^r$ and let $\langle B_n\rangle_{n\in\mathbb{N}}$ be a cover of B such that $\operatorname{diam} B_n < \frac{1}{2}\min(1, \inf_{mn\le i<mn+m}\epsilon_i)$ for every n. (For definiteness, let me say that I am giving $[0,1]^r$ its Euclidean metric.) In this case, if $2^{-k-1} \le \operatorname{diam} B_n < 2^{-k}$, B_n can meet at most 2^r intervals of the form $[2^{-k}\boldsymbol{n}, 2^{-k}(\boldsymbol{n}+\boldsymbol{1})]$ where $\boldsymbol{n} \in \mathbb{N}^r$ and $\boldsymbol{1} = (1,\dots,1)$. So $f^{-1}[B_n]$ can meet at most $2^r = m$ sets of the form $\{x : x{\restriction}(k\times r) = z\}$, and can be covered by m sets $\langle A_j\rangle_{mn\le j<mn+m}$ where

$$\operatorname{diam} A_j \le 2^{-k} \le 2\operatorname{diam} B_n \le \epsilon_j$$

for every j. Putting these together, we have a cover $\langle A_j\rangle_{j\in\mathbb{N}}$ of $f^{-1}[B]$ such that $\operatorname{diam} A_j \le \epsilon_j$ for every j; as $\langle \epsilon_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $f^{-1}[B]$ has strong measure zero. **Q**

The function f here is very far from being one-to-one. But if we set

$$D_1^* = \textstyle\bigcup_{j<r}\{x : x \in \{0,1\}^{\mathbb{N}\times r},\, \langle x(i,j)\rangle_{i\in\mathbb{N}} \in D_1\},$$

$$D_2^* = \textstyle\bigcup_{j<r}\{z : z \in [0,1]^r,\, z(j) \in D_2\},$$

where $D_1 \subseteq \{0,1\}^{\mathbb{N}}$, $D_2 \subseteq [0,1]$ are defined as in the proof of (b), then f is a bijection between $\{0,1\}^{\mathbb{N}\times r} \setminus D_1^*$ and $[0,1]^r \setminus D_2^*$, so is an $\mathcal{S}\mathfrak{mz}$-equivalence between these. Accordingly $[0,1]^r \setminus D_2^*$ is $\mathcal{S}\mathfrak{mz}$-embeddable in $\{0,1\}^{\mathbb{N}\times r}$, which is homeomorphic, therefore uniformly equivalent, to $\{0,1\}^{\mathbb{N}}$, which is in turn $\mathcal{S}\mathfrak{mz}$-equivalent to $]0,1[$; so $[0,1]^r \setminus D_2^*$ is $\mathcal{S}\mathfrak{mz}$-embeddable in $]0,1[$.

Now consider D_2^*. This is a countable union of sets which are isometric, therefore $\mathcal{S}\mathfrak{mz}$-equivalent, to $[0,1]^{r-1}$ and therefore to $]0,1[$, by the inductive hypothesis. We can therefore express D_2^* as $\bigcup_{n\in\mathbb{N}} X_n$ where $\langle X_n\rangle_{n\in\mathbb{N}}$ is disjoint and every X_n is $\mathcal{S}\mathfrak{mz}$-embeddable in $]0,1[$ and therefore in $]n+1, n+2[$. Assembling these with the $\mathcal{S}\mathfrak{mz}$-equivalence between $[0,1]^r \setminus D_2^*$ and $]0,1[$ we have already found, we have an $\mathcal{S}\mathfrak{mz}$-embedding from $[0,1]^r$ to $\mathbb{R}$. In the other direction, we certainly have an isometric embedding of $[0,1]$ in $[0,1]^r$ and therefore a $\mathcal{S}\mathfrak{mz}$-embedding of $\mathbb{R}$ in $[0,1]^r$; so $\mathbb{R}$ and $[0,1]^r$ are $\mathcal{S}\mathfrak{mz}$-equivalent. Thus the induction proceeds.

(ii) As for $\mathbb{R}^r$, we have a homeomorphism between $\mathbb{R}^r$ and $]0,1[^r$, which (because these again are σ-compact) is an $\mathcal{S}\mathfrak{mz}$-equivalence and therefore an $\mathcal{S}\mathfrak{mz}$-embedding of $\mathbb{R}^r$ in $[0,1]^r$. So 534Ob, once more, tells us that $\mathbb{R}^r$ and $[0,1]^r$ and $[0,1]$ are $\mathcal{S}\mathfrak{mz}$-equivalent.

(d) Thus $\mathbb{R}^r$, $]0,1[^r$, $[0,1]^r$ and $\{0,1\}^{\mathbb{N}}$ are $\mathcal{S}\mathfrak{mz}$-equivalent, for any uniformities inducing their usual topologies.

534Q Large sets with strong measure zero It is a remarkable fact that it is relatively consistent with ZFC to suppose that the only subsets of $\mathbb{R}$ with strong measure zero are the countable sets (Laver 76, Ihoda 88 or Bartoszyński & Judah 95, §8.3). We therefore find ourselves investigating constructions of non-trivial sets with strong measure zero under special axioms.

Proposition (a) Let X be a Lindelöf space. Then $\operatorname{non}\mathcal{R}\mathsf{bg}(X) \ge \mathfrak{m}_{\text{countable}}$.

(b) (see Fremlin & Miller 88) Give $\mathbb{N}^{\mathbb{N}}$ the metric ρ defined by setting $\rho(x,y) = \inf\{2^{-n} : n \in \mathbb{N}, x{\restriction}n = y{\restriction}n\}$ for x, $y \in \mathbb{N}^{\mathbb{N}}$. Then $\operatorname{non}\mathcal{S}\mathfrak{mz}(\mathbb{N}^{\mathbb{N}},\rho) = \operatorname{non}\mathcal{R}\mathfrak{bg}(\mathbb{N}^{\mathbb{N}}) = \mathfrak{m}_{\text{countable}}$.

proof (a) Suppose that $A \subseteq X$ and $\#(A) < \mathfrak{m}_{\text{countable}}$. Let $\langle \mathcal{G}_n\rangle_{n\in\mathbb{N}}$ be a sequence of non-empty open covers of X. Because X is Lindelöf, we can choose for each n a non-empty countable $\mathcal{G}'_n \subseteq \mathcal{G}_n$ covering X. Let P be the set of finite sequences $p = \langle p(i)\rangle_{i<n}$ such that $p(i) \in \mathcal{G}'_i$ for every $i < n$; say that $p \le q$ in P if q extends P. Then P is a countable partially ordered set. For each $x \in A$, the set $Q_x = \{p : x \in p(i)$ for some $i < \#(p)\}$ is cofinal with P. **P** Given $p \in P$, set $n = \#(p)$; let $G \in \mathcal{G}'_n$ be such that $x \in G$; set $q = p \cup \{(n, G)\}$; then $p \le q \in Q_x$. **Q**

Because $\#(A) < \mathfrak{m}_{\text{countable}} \le \mathfrak{m}^{\uparrow}(P)$ (517Pc), there is an upwards-directed family $R \subseteq P$ meeting every Q_x (517B(iv)). Now $p^* = \bigcup R$ is a function; $A \subseteq \bigcup_{i\in\operatorname{dom}p^*} p^*(i)$ and $p^*(i) \in \mathcal{G}_i$ for every $i \in \operatorname{dom}p^*$. As $\langle \mathcal{G}_n\rangle_{n\in\mathbb{N}}$ is arbitrary, A has Rothberger's property in X; as A is arbitrary, $\operatorname{non}\mathcal{R}\mathfrak{bg}(X) \ge \mathfrak{m}_{\text{countable}}$.

(b) By 522Sb, there is a set $A \subseteq \mathbb{N}^{\mathbb{N}}$, with cardinal $\mathfrak{m}_{\text{countable}}$, such that for every $y \in \mathbb{N}^{\mathbb{N}}$ there is an $x \in A$ such that $x(n) \ne y(n)$ for every n. **?** If $A \in \mathcal{S}\mathfrak{mz}(\mathbb{N}^{\mathbb{N}},\rho)$, take a sequence $\langle y_n\rangle_{n\in\mathbb{N}}$ in $\mathbb{N}^{\mathbb{N}}$ such that $A \subseteq \bigcup_{n\in\mathbb{N}} B(y_n, 2^{-n-1})$. Set $y(n) = y_n(n)$ for every n. Then there is an $x \in A$ such that $x(n) \ne y(n)$ for every n. But in this case $x(n) \ne y_n(n)$ and $x{\restriction}n+1 \ne y_n{\restriction}n+1$ and $x \notin B(y_n, 2^{-n-1})$ for every n. **X**

Thus A witnesses that $\operatorname{non}\mathcal{S}\mathfrak{mz}(\mathbb{N}^{\mathbb{N}},\rho) \le \mathfrak{m}_{\text{countable}}$. But we know from (a) that $\operatorname{non}\mathcal{R}\mathfrak{bg}(\mathbb{N}^{\mathbb{N}}) \ge \mathfrak{m}_{\text{countable}}$ and from 534Ea that $\operatorname{non}\mathcal{S}\mathfrak{mz}(\mathbb{N}^{\mathbb{N}},\rho) \ge \operatorname{non}\mathcal{R}\mathfrak{bg}(\mathbb{N}^{\mathbb{N}})$, so the three cardinals are equal.

534R Proposition (a) If (X,ρ) is a separable metric space and $A \subseteq X$ has cardinal less than $\mathfrak{c}$, there is a Lipschitz function $f : X \to \mathbb{R}$ such that $f{\restriction}A$ is injective.

(b) (Carlson 93) If $\kappa < \mathfrak{c}$ is a cardinal and there is any separable metric space with a set with cardinal κ which is of strong measure zero, then there is a subset of $\mathbb{R}$ with cardinal κ which has Rothberger's property in $\mathbb{R}$.

(c)(i) If $\operatorname{cf}(\mathfrak{m}_{\text{countable}}) = \mathfrak{b}$ there is a subset of $\mathbb{R}$ with cardinal $\mathfrak{m}_{\text{countable}}$ which has Rothberger's property in itself.

(ii) (Rothberger 1941) If $\mathfrak{b} = \omega_1$ there is a subset of $\mathbb{R}$ with cardinal ω_1 which has Rothberger's property in itself.

(iii) If $\mathfrak{m}_{\text{countable}} = \mathfrak{d}$ there is a subset of $\mathbb{R}$ with cardinal $\mathfrak{m}_{\text{countable}}$ which has Rothberger's property in itself.

proof (a) If $X = \emptyset$ this is trivial. Otherwise, let $\langle x_n\rangle_{n\in\mathbb{N}}$ run over a dense sequence in X, and for $x \in X$ define $g_x : \mathbb{R} \to \mathbb{R}$ by setting

$$g_x(t) = \sum_{n=0}^{\infty} \frac{\min(1,\rho(x,x_n))}{n!} t^n$$

for $t \in \mathbb{R}$. Then g_x is a real-entire function (5A5A). If x, $y \in X$ are distinct, then there must be some n such that $\min(1,\rho(x,x_n)) \ne \min(1,\rho(y,x_n))$, so that one of the coefficients of $g_x - g_y$ is non-zero, and $\{t : g_x(t) = g_y(t)\}$ is countable (5A5A). So if $A \subseteq X$ and $\#(A) < \mathfrak{c}$, we can find a $t \ge 0$ such that $g_x(t) \ne g_y(t)$ for all distinct x, $y \in A$. Set $f(x) = g_x(t)$ for $x \in X$; then $f : X \to \mathbb{R}$ is a function such that $f{\restriction}A$ is injective. If x, $y \in X$ then

$$\begin{aligned}|f(x) - f(y)| &= |\sum_{n=0}^{\infty}(\min(1,\rho(x,x_n)) - \min(1,\rho(y,y_n))\frac{t^n}{n!}|\\ &\le e^t \sup_{n\in\mathbb{N}} |\rho(x,x_n) - \rho(y,y_n)| \le e^t\rho(x,y),\end{aligned}$$

so that f is Lipschitz.

(b) Let (X,ρ) be a separable metric space with a set $A \in [X]^{\kappa}$ of strong measure zero. Then (a) tells us that we have a uniformly continuous function $f : X \to \mathbb{R}$ which is injective on A, so that $f[A] \in [\mathbb{R}]^{\kappa}$ has strong measure zero in $\mathbb{R}$ (534D(a-iii)).

(c)(i) Let $\langle x_\xi\rangle_{\xi<\mathfrak{b}}$ be a family in $\mathbb{N}^{\mathbb{N}}$ which is increasing and unbounded for the pre-order $\le^*$ of 522C(i). Let $C \subseteq \mathfrak{m}_{\text{countable}}$ be a closed cofinal set with cardinal $\mathfrak{b}$ (5A1Ae), and $\langle \zeta_\xi\rangle_{\xi<\mathfrak{b}}$ the increasing enumeration of C; let $\langle y_\eta\rangle_{\eta<\mathfrak{m}_{\text{countable}}}$ be a family of distinct elements of $\mathbb{N}^{\mathbb{N}}$ such that $y_\eta \ge x_\xi$ whenever $\xi < \mathfrak{b}$ and $\zeta_\xi \le \eta < \zeta_{\xi+1}$.

If $K \subseteq \mathbb{N}^{\mathbb{N}}$ is compact, then $\{\eta : y_\eta \in K\}$ has cardinal strictly less than $\mathfrak{m}_{\text{countable}}$. **P** Set $x(n) = \sup_{y \in K} y(n)$ for each $n \in \mathbb{N}$ (I pass over the trivial case $K = \emptyset$). Then there is a $\xi < \mathfrak{b}$ such that $x_\xi \not\leq^* x$. If $\zeta_\xi \leq \eta < \mathfrak{m}_{\text{countable}}$, there is a $\xi' \geq \xi$ such that $\zeta_{\xi'} \leq \eta < \zeta_{\xi'+1}$ (this is where we need to know that C is closed), and now

$$y_\eta \geq x_{\xi'} \geq^* x_\xi, \quad y_\eta \not\leq x, \quad y_\eta \notin K.$$

So $\{\eta : y_\eta \in K\} \subseteq \zeta_\xi$ has cardinal less than $\mathfrak{m}_{\text{countable}}$. **Q**

Let $f : \mathbb{N}^{\mathbb{N}} \to [0,1] \setminus \mathbb{Q}$ be any homeomorphism (4A2Ub), and consider $A = \{f(y_\eta) : \eta < \mathfrak{m}_{\text{countable}}\} \cup \mathbb{Q}$. Then $\#(A) = \mathfrak{m}_{\text{countable}}$. Also A has Rothberger's property in A. **P** Of course $\mathbb{Q}$, being countable, has Rothberger's property in A. Let $G \subseteq \mathbb{R}$ be an open set including $\mathbb{Q}$. Then $[0,1] \setminus G$ and $K = f^{-1}[\,[0,1] \setminus G]$ are compact. Now

$$\#(A \setminus G) = \#(\{\eta : y_\eta \in K\}) < \mathfrak{m}_{\text{countable}}$$

so $A \setminus G$ has Rothberger's property in A, by 534Qa. By 534D(b-iii), this is enough to show that A has Rothberger's property in itself. **Q**

Thus we have a set of the required kind.

(ii) This follows immediately if $\mathfrak{m}_{\text{countable}} = \omega_1$, and otherwise we can take any subset of $\mathbb{R}$ of cardinal ω_1.

(iii) The argument is similar to that in (i). This time, let $\langle x_\xi \rangle_{\xi < \mathfrak{d}}$ be a cofinal family in $\mathbb{N}^{\mathbb{N}}$. For each $\xi < \mathfrak{d}$, let $y_\xi \in \mathbb{N}^{\mathbb{N}}$ be such that $y_\xi \geq x_\xi$ and $y_\xi \not\leq x_\eta$ for any $\eta < \xi$. Again, if $K \subseteq \mathbb{N}^{\mathbb{N}}$ is compact, then $\{\eta : y_\eta \in K\}$ has cardinal strictly less than $\mathfrak{m}_{\text{countable}}$. **P** Taking $x = \sup K$ as before, there is a $\xi < \mathfrak{d} = \mathfrak{m}_{\text{countable}}$ such that $x \leq x_\xi$; now for any $\eta > \xi$ we know that $y_\eta \not\leq x_\xi$ so $y_\eta \notin K$. **Q** The rest of the proof proceeds as before. (The set $\{y_\eta : \eta < \mathfrak{d}\}$ has cardinal $\mathfrak{d}$ because it is cofinal with $\mathbb{N}^{\mathbb{N}}$.)

534S Subject to the continuum hypothesis we have many ways of building sets with strong measure zero, in addition to those in the proof of 534R. I give one example to suggest what can be done with a weak form of Martin's axiom.

Example Suppose that $\mathfrak{m}_{\text{countable}} = \mathfrak{c}$.

(a) There is a set $A \subseteq [0,1] \setminus \mathbb{Q}$ such that

(α) $\#(A \cap K) < \mathfrak{c}$ for every compact $K \subseteq [0,1] \setminus \mathbb{Q}$,

(β) there is a continuous function $f : [0,1] \setminus \mathbb{Q} \to [0,1]$ such that $f[A] = [0,1]$,

(γ) $A + A \supseteq \,]0,1]$.

(b) Now $A \cup \mathbb{Q}$ has Rothberger's property in itself, $A \in \mathcal{R}\mathsf{bg}(\mathbb{R})$, A is not meager, $A \notin \mathcal{R}\mathsf{bg}(\mathbb{R} \setminus \mathbb{Q})$ and $A \times A \notin \mathcal{R}\mathsf{bg}(\mathbb{R}^2)$.

proof (a)(i) For $x \in \mathbb{N}^{\mathbb{N}}$, define $\psi(x) \in \{0,1\}^{\mathbb{N}}$ by setting $\psi(x)(n) = 0$ if $x(n)$ is even, 1 if $x(n)$ is odd. Then $\psi : \mathbb{N}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ is a continuous surjection. Let $\phi : \mathbb{N}^{\mathbb{N}} \to [0,1] \setminus \mathbb{Q}$ be a homeomorphism (4A2Ub again). Enumerate $\mathbb{N}^{\mathbb{N}}$ as $\langle x_\xi \rangle_{\xi < \mathfrak{c}}$ and $]0,1]$ as $\langle t_\xi \rangle_{\xi < \mathfrak{c}}$. For $\xi \leq \mathfrak{c}$, set $K_\xi = \{x : x \in \mathbb{N}^{\mathbb{N}}, x \leq x_\xi\}$, so that K_ξ is compact and $\phi[K_\xi]$ is a compact subset of $[0,1] \setminus \mathbb{Q}$, therefore nowhere dense in $\mathbb{R}$. Write $\mathcal{M}$ for the ideal of meager subsets of $\mathbb{R}$, as in §522.

Choose $\langle a_\xi \rangle_{\xi < \mathfrak{c}}$, $\langle b_\xi \rangle_{\xi < \mathfrak{c}}$ and $\langle c_\xi \rangle_{\xi < \mathfrak{c}}$ as follows. For each $\xi < \mathfrak{c}$, $\{x_\eta : \eta < \xi\}$ is not cofinal with $\mathbb{N}^{\mathbb{N}}$, because

$$\operatorname{cf} \mathbb{N}^{\mathbb{N}} = \mathfrak{d} \geq \operatorname{cov} \mathcal{M} = \mathfrak{m}_{\text{countable}} = \mathfrak{c}$$

(522I, 522Sa again), so we can find a $y_\xi \in \mathbb{N}^{\mathbb{N}}$ such that $y_\xi \not\leq x_\eta$ for any $\eta < \xi$; raising y_ξ if need be, we can arrange that $\psi(y_\xi) = \psi(x_\xi)$. Set $a_\xi = \phi(y_\xi)$. Consider

$$\mathcal{E}_\xi = \{\phi[K_\eta] : \eta < \xi\} \cup \{t_\xi - \phi[K_\eta] : \eta < \xi\} \cup \{\mathbb{Q}\} \cup \{t_\xi - \mathbb{Q}\}.$$

This is a family of fewer than $\mathfrak{c} = \mathfrak{m}_{\text{countable}}$ meager subsets of $\mathbb{R}$, so does not cover $]0, t_\xi[$ (522Sa once more). Take any $b_\xi \in \,]0, t_\xi[\setminus \bigcup \mathcal{E}_\xi$; then neither b_ξ nor $c_\xi = t_\xi - b_\xi$ belongs to $\mathbb{Q} \cup \bigcup_{\eta < \xi} \phi[K_\eta]$.

At the end of the process, set

$$A = \{a_\xi : \xi < \mathfrak{c}\} \cup \{b_\xi : \xi < \mathfrak{c}\} \cup \{c_\xi : \xi < \mathfrak{c}\}.$$

(ii)(α) If $K \subseteq [0,1] \setminus \mathbb{Q}$ is compact, then $\phi^{-1}[K] \subseteq \mathbb{N}^{\mathbb{N}}$ is compact, so there is an $\eta < \mathfrak{c}$ such that $\phi^{-1}[K] \subseteq K_\eta$ and $K \subseteq \phi[K_\eta]$. If $\eta < \xi < \mathfrak{c}$, $y_\xi \notin K_\eta$ so $a_\xi \notin K$, while neither b_ξ nor c_ξ belongs to $\phi[K_\eta] \supseteq K$. So $A \cap K \subseteq \{a_\xi : \xi \le \eta\} \cup \{b_\xi : \xi \le \eta\} \cup \{c_\xi : \xi \le \eta\}$ has cardinal less than $\mathfrak{c}$.

(β) For $x \in \{0,1\}^{\mathbb{N}}$ set $h(x) = \frac{1}{2}\sum_{i=0}^{\infty} 2^{-i}x(i)$, so that $h : \{0,1\}^{\mathbb{N}} \to [0,1]$ is a continuous surjection. Set $f = h\psi\phi^{-1} : [0,1] \setminus \mathbb{Q} \to [0,1]$. Then f is continuous. Since $\psi\phi^{-1}(a_\xi) = \psi(y_\xi) = \psi(x_\xi)$ for every $\xi < \mathfrak{c}$, $\psi\phi^{-1}[A] = \{0,1\}^{\mathbb{N}}$ and $f[A] = [0,1]$. So $f{\restriction}A$ is a surjection from A onto $[0,1]$.

(γ) Since $t_\xi = b_\xi + c_\xi \in A + A$ for every $\xi < \mathfrak{c}$, $A + A \supseteq\,]0,1]$.

(b)(i) Let $H \subseteq A \cup \mathbb{Q}$ be a relatively open set including $\mathbb{Q}$, and take an open set $G \subseteq \mathbb{R}$ such that $H = G \cap (A \cup \mathbb{Q})$. Then $K = [0,1] \setminus G$ is a compact subset of $[0,1] \setminus \mathbb{Q}$ and $\phi^{-1}[K]$ is a compact subset of $\mathbb{N}^{\mathbb{N}}$. There is therefore an $\eta < \mathfrak{c}$ such that $\phi^{-1}[K]$ is bounded above by x_η, that is, $\phi^{-1}[K] \subseteq K_\eta$ and $K \subseteq \phi[K_\eta]$. So neither a_ξ nor b_ξ nor c_ξ can belong to K for any $\xi > \eta$, and $\#(A \cap K) < \mathfrak{c} = \mathfrak{m}_{\text{countable}}$. By 534Qa, $(A \cup \mathbb{Q}) \setminus H = (A \cup \mathbb{Q}) \setminus G = A \cap K$ belongs to $\mathcal{R}\mathsf{bg}(A \cup \mathbb{Q})$; as $\mathbb{Q} \in \mathcal{R}\mathsf{bg}(A \cup \mathbb{Q})$ and H is arbitrary, $A \in \mathcal{R}\mathsf{bg}(A \cup \mathbb{Q})$ (534D(b-iii)).

(ii) As the embedding $A \cup \mathbb{Q} \subsetneqq \mathbb{R}$ is continuous, $A \cup \mathbb{Q} \in \mathcal{R}\mathsf{bg}(\mathbb{R})$ (534D(b-ii)) and $A \in \mathcal{R}\mathsf{bg}(\mathbb{R})$ (534D(b-i)).

(iii) By 534Ea, A is of strong measure zero for the usual metric on $\mathbb{R}$. Setting $B = A + \mathbb{Z}$, B is the union of a sequence of sets isometric to A, so is of strong measure zero. As $A + A \supseteq\,]0,1]$, $B + A = \mathbb{R}$; by 534K, A is not meager.

(iv) Of course $[0,1]$ is not of strong measure zero for its usual metric (534H) so does not belong to $\mathcal{R}\mathsf{bg}([0,1])$ (534Ea); now (a-β) here and 534D(b-ii) tell us that A cannot belong to $\mathcal{R}\mathsf{bg}([0,1] \setminus \mathbb{Q})$. But $[0,1] \setminus \mathbb{Q}$ is relatively closed in $\mathbb{R} \setminus \mathbb{Q}$, so A cannot belong to $\mathcal{R}\mathsf{bg}(\mathbb{R} \setminus \mathbb{Q})$, by 534D(b-iv).

(v) If we give $\mathbb{R}$ and $\mathbb{R}^2$ their usual metrics, addition is a uniformly continuous function from $\mathbb{R}^2$ to $\mathbb{R}$, while $A + A \supseteq\,]0,1]$ is not of strong measure zero. So $A \times A$ is not of strong measure zero (534D(a-iii)) and cannot belong to $\mathcal{R}\mathsf{bg}(\mathbb{R}^2)$.

534X Basic exercises (a)(i) Let (X,ρ) be a metric space, $r > 0$ and $A \subseteq X$ a set with strong measure zero. Show that A has zero Hausdorff r-dimensional measure. (ii) Find a subset of $\mathbb{R}^2$ which is universally negligible but does not have strong measure zero (for the usual metric on $\mathbb{R}^2$). (*Hint*: 439G.) (iii) Find a subset of $\{0,1\}^{\mathbb{N}}$ which is universally negligible but does not have strong measure zero for the metric of 534Qb.

(b) Let r, $s \ge 1$ be integers. Let $A \subseteq \mathbb{R}^r$ be a set with strong measure zero, and $f : A \to \mathbb{R}^s$ a function which is differentiable relative to its domain at every point of A. Show that $f[A]$ has strong measure zero. (*Hint*: 262N.)

(c) Let $(X, \mathcal{W})$ and $(Y, \mathcal{V})$ be uniform spaces and $f : X \to Y$ a continuous function. Suppose that $A \in \mathcal{S}\mathsf{mz}(X, \mathcal{W})$ is covered by a sequence of compact subsets of X. Show that $f[A] \in \mathcal{S}\mathsf{mz}(Y, \mathcal{V})$.

(d) Let X be a σ-compact topological space which is either Hausdorff or regular, and $A \subseteq X$. Show that $A \in \mathcal{R}\mathsf{bg}(X)$ iff for every sequence $\langle \mathcal{G}_n \rangle_{n\in\mathbb{N}}$ of finite open covers of X, there is a sequence $\langle G_n \rangle_{n\in\mathbb{N}}$, covering A, such that $G_n \in \mathcal{G}_n$ for every n.

(e) Let $(X, \mathcal{W})$ be a Hausdorff uniform space with strong measure zero. Show that X is universally negligible iff it is a Radon space.

(f)(i) Let $(X, \mathcal{W})$ be a Hausdorff uniform space. Show that if X has strong measure zero then it is universally τ-negligible. (ii) Let X be a Hausdorff topological space. Show that if $A \in \mathcal{R}\mathsf{bg}(X)$ then A is universally τ-negligible (definition: 439Xh).

(g) Give $\omega_1 + 1$ its order topology. Show that it has Rothberger's property in itself but is not universally negligible.

(h) Give ω_1+1 its order topology. Show that ω_1 has Rothberger's property in ω_1+1 but not in itself.

(i) Let X be a locally compact Hausdorff topological group. Show that a subset of X has Rothberger's property in X iff it has strong measure zero for the right uniformity of X iff it has strong measure zero for the bilateral uniformity of X.

(j)(i) Let $(X,\mathcal{W})$ be a Lindelöf uniform space. Show that there is some κ such that $\mathcal{S}\mathfrak{m}\mathfrak{z}(X,\mathcal{W})\preccurlyeq_{\mathrm{T}}\mathcal{N}^\kappa$, where $\mathcal{N}$ is the null ideal of Lebesgue measure on $\mathbb{R}$. (ii) Let X be a regular Lindelöf space. Show that there is some κ such that $\mathcal{R}\mathfrak{b}\mathfrak{g}(X)\preccurlyeq_{\mathrm{T}}\mathcal{N}^\kappa$.

(k) Show that every separable metric space (X,ρ) is uniformly equivalent to a subspace of $[0,1]^{\mathbb{N}}$ and is therefore $\mathcal{S}\mathfrak{m}\mathfrak{z}$-embeddable in $[0,1]^{\mathbb{N}}$.

(l) Let $\langle I_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of finite sets covering $\mathbb{Z}$. For x, $y\in\mathbb{Z}^{\mathbb{N}}$ set $\rho(x,y)=\inf\{2^{-n}:n\in\mathbb{N},\ x\restriction I_n=y\restriction I_n\}$. Show that ρ is a metric on $\mathbb{Z}^{\mathbb{N}}$ inducing its topological group uniformity (4A5He), and that $\operatorname{non}\mathcal{S}\mathfrak{m}\mathfrak{z}(\mathbb{Z}^{\mathbb{N}},\rho)=\mathfrak{m}_{\text{countable}}$.

(m)(i) Show that no cofinal subset of $\mathbb{N}^{\mathbb{N}}$ has strong measure zero for the metric ρ of 534Qb. (ii) Suppose that $\mathfrak{m}_{\text{countable}}=\mathfrak{d}$. Show that there are a subset A of $\mathbb{R}\setminus\mathbb{Q}$ and a metric ρ' on $\mathbb{R}\setminus\mathbb{Q}$ inducing the usual topology of $\mathbb{R}\setminus\mathbb{Q}$ such that A has strong measure zero for the usual metric on $\mathbb{R}$ but not for ρ'.

(n) Let A be the set constructed in 534Sa on the assumption that $\mathfrak{m}_{\text{countable}}=\mathfrak{c}$. Show that A has strong measure zero for the usual metric of $\mathbb{R}$, and describe a metric on $[0,1]\setminus\mathbb{Q}$, inducing the usual topology on $[0,1]\setminus\mathbb{Q}$, for which A does not have strong measure zero. (See also 534Ye.)

(o) [In this exercise, I will say that a topological space which has Rothberger's property in itself has 'property C$'$'.] (i) Show that any Lindelöf space with cardinal less than $\mathfrak{m}_{\text{countable}}$ has property C$'$. (ii) Show that if X is a topological space expressible as the union of a sequence of subspaces with property C$'$, then X has property C$'$. (iii) Show that if X is a regular Lindelöf space expressible as the union of fewer than $\operatorname{add}\mathcal{N}$ subspaces with property C$'$, then X has property C$'$. (iv) Show that a continuous image of a space with property C$'$ has property C$'$. (v) Show that a closed subset of a space with property C$'$ has property C$'$. (vi) Show that if X is a topological space, $A\subseteq X$ has property C$'$ and every closed subset of $X\setminus A$ has property C$'$, then X has property C$'$.

534Y Further exercises (a) Let (X,ρ) be an analytic metric space and μ_{Hr} Hausdorff r-dimensional measure on X, where $r>0$; suppose that $\mu_{Hr}X>0$. Let $\mathcal{I}$ be the σ-ideal of subsets of X generated by $\{A:\mu_{Hr}^*A<\infty\}$. Show that

$$\begin{aligned}\operatorname{non}\mathcal{N}(\mu_{Hr})=\min(\operatorname{non}\mathcal{N},\operatorname{non}\mathcal{I})&=\operatorname{non}\mathcal{N}\text{ if }\mu_{Hr}\text{ is }\sigma\text{-finite},\\&=\operatorname{non}\mathcal{I}\text{ otherwise.}\end{aligned}$$

(b)(i) Set $\mathcal{I}=\{[4^{-m}i,4^{-m}(i+1)[\,:m\in\mathbb{N},\ i\in\mathbb{Z}\}$. For $A\subseteq\mathbb{R}$ set $\theta(A)=\inf\{\sum_{I\in\mathcal{I}'}\sqrt{\operatorname{diam}I}:\mathcal{I}'\subseteq\mathcal{I}$ covers $A\}$. Show that if $\mu_{H,1/2}^{(1)}$ is Hausdorff $\frac{1}{2}$-dimensional measure on $\mathbb{R}$, then $\mu_{H,1/2}^{(1)}(A)=0$ iff $\theta(A)=0$. (ii) Set $\mathcal{J}=\{[2^{-m}i,2^{-m}(i+1)[\,\times[2^{-m}j,2^{-m}(j+1)[\,:m\in\mathbb{N},\ i,\ j\in\mathbb{Z}\}$, and for $A\subseteq\mathbb{R}^2$ set $\theta'(A)=\inf\{\sum_{J\in\mathcal{J}'}\operatorname{diam}J:\mathcal{J}'\subseteq\mathcal{J}$ covers $A\}$. Show that if $\mu_{H1}^{(2)}$ is Hausdorff 1-dimensional measure on $\mathbb{R}^2$, then $\mu_{H1}^{(2)}(A)=0$ iff $\theta'(A)=0$. (iii) Show that the null ideals $\mathcal{N}(\mu_{H,1/2}^{(1)})$ and $\mathcal{N}(\mu_{H1}^{(2)})$ are isomorphic.

(c) Show that if *either* $\operatorname{non}\mathcal{N}=\operatorname{cf}\mathcal{N}$ *or* $\operatorname{non}\mathcal{N}<\operatorname{cov}\mathcal{N}$, where $\mathcal{N}$ is the null ideal of Lebesgue measure on $\mathbb{R}$, then Hausdorff one-dimensional measure on $\mathbb{R}^2$ does not have the measurable envelope property.

(e) Suppose that $\mathfrak{m}_{\text{countable}}=\mathfrak{c}$. Let X be the group of all permutations of $\mathbb{N}$, regarded as the isometry group of $\mathbb{N}$ with its $\{0,1\}$-valued metric, so that X is a Polish group (441Xq). Show that there is a subset A of X such that A has strong measure zero for the right uniformity of X but A^{-1} does not.

(d) Let $\mathfrak{G}$ be a collection of families of sets. Let us say that a set A has the **$\mathfrak{G}$-Rothberger property** if for every sequence $\langle \mathcal{G}_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{G}$ there is a cover $\langle G_n \rangle_{n\in\mathbb{N}}$ of A such that $G_n \in \mathcal{G}_n$ for every $n \in \mathbb{N}$. (i) Show that the family $\mathcal{I}$ of sets with the $\mathfrak{G}$-Rothberger property is a σ-ideal of sets containing every countable subset of $\bigcap_{\mathcal{G}\in\mathfrak{G}} \bigcup \mathcal{G}$. (ii) Show that if $\mathfrak{H}$ is another collection of families of sets, and f is a function such that for every $\mathcal{H} \in \mathfrak{H}$ there is a member of $\mathfrak{G}$ refining $\{f^{-1}[H] : H \in \mathcal{H}\}$, then $f[A]$ has the $\mathfrak{H}$-Rothberger property whenever $A \in \mathcal{I}$. (iii) Suppose that $\mathfrak{G}$ is a collection of families of open subsets of a topological space X, that $A \in \mathcal{I}$ has the $\mathfrak{G}$-Rothberger property, and that $B \subseteq X$ is such that $B \setminus G \in \mathcal{I}$ for every open set $G \supseteq A$. Show that $B \in \mathcal{I}$. (iv) Suppose that $X = \bigcup \mathcal{G}$ for every $\mathcal{G} \in \mathfrak{G}$, and that every member of $\mathfrak{G}$ is countable. Show that $\mathrm{non}(\mathcal{I}, X) \geq \mathfrak{m}_{\mathrm{countable}}$.

(e) Suppose that $\mathfrak{m}_{\mathrm{countable}} = \mathfrak{d}$. Show that there are two complete metrics ρ, ρ' on $\mathbb{N}^{\mathbb{N}}$, both inducing the usual topology of $\mathbb{N}^{\mathbb{N}}$, such that $\mathcal{S}\mathfrak{mz}(\mathbb{N}^{\mathbb{N}}, \rho) \neq \mathcal{S}\mathfrak{mz}(\mathbb{N}^{\mathbb{N}}, \rho')$.

534Z Problems (a) Let $\mu_{H1}^{(2)}$ be one-dimensional Hausdorff measure on $\mathbb{R}^2$. Is the covering number $\mathrm{cov}\,\mathcal{N}(\mu_{H1}^{(2)})$ necessarily equal to $\mathrm{cov}\,\mathcal{N}$? As observed in 534Bc-534Bd, we have $\mathrm{cov}\,\mathcal{N} \leq \mathrm{cov}\,\mathcal{N}(\mu_{H1}^{(2)}) \leq \mathrm{non}\,\mathcal{M}$. We can ask the same question for r-dimensional Hausdorff measure on $\mathbb{R}^n$ whenever $0 < r < n$; in particular, for r-dimensional Hausdorff measure on $[0,1]$, where $0 < r < 1$, and these questions are strongly connected (534Yb). Shelah & Steprāns 05 show that $\mathrm{non}\,\mathcal{N}(\mu_{H1}^{(2)})$ can be less than $\mathrm{non}\,\mathcal{N}$; of course this is possible only because $\mu_{H1}^{(2)}$ is not semi-finite (439H, 521Xg).

(b) Can $\mathrm{cf}\,\mathcal{R}\mathfrak{bg}(\mathbb{R})$ be ω_1?

(c) How many types of complete separable metric spaces under $\mathcal{S}\mathfrak{mz}$-equivalence can there be? If we give $\mathbb{N}^{\mathbb{N}}$ the metric of 534Qb, can it fail to be $\mathcal{S}\mathfrak{mz}$-equivalent to $[0,1]^{\mathbb{N}}$ with the metric $(x, y) \mapsto \sup_{n\in\mathbb{N}} 2^{-n}|x(n) - y(n)|$?

(d) Suppose that there is a separable metric space with cardinal $\mathfrak{c}$ with strong measure zero. Must there be a subset of $\mathbb{R}$ with cardinal $\mathfrak{c}$ with Rothberger's property in $\mathbb{R}$?

(e) On $\mathbb{R}$, let $\mathfrak{T}$ be the usual topology and $\mathfrak{S}$ the right-facing Sorgenfrey topology (415Xc). Must $\mathcal{R}\mathfrak{bg}(\mathbb{R}, \mathfrak{S})$ and $\mathcal{R}\mathfrak{bg}(\mathbb{R}, \mathfrak{T})$ be the same?

534 Notes and comments I have very little to say about Hausdorff measures, and 534B is here only because it would seem even lonelier in a section by itself. All I have tried to do is to run through the obvious questions connecting §471 with Chapter 52. But at the next level there is surely much more to be done (534Za).

'Strong measure zero' has attracted a great deal of attention, starting with the work of E.Borel, who suggested that every subset of $\mathbb{R}$ with strong measure zero must be countable; this is the **Borel conjecture**. It turns out that this is undecidable in ZFC (see the preamble to 534Q), and that if the Borel conjecture is true then there are no uncountable sets of strong measure zero in any separable metric space (534Rb). So we have some questions of a new kind: in the ideals $\mathcal{S}\mathfrak{mz}(X, \mathcal{W})$ of sets of strong measure zero, in addition to the standard cardinals add, non, cov and cf, we find ourselves asking for the possible cardinals of sets belonging to the ideal.

The next point is that strong measure zero is not (or rather, not always) either a topological property or a metric property; it is a property of uniform spaces. We must therefore be prepared to examine uniformities, even if we are happy to stay with metrizable ones. In 534Xm we see that we can have a set which has strong measure zero for one of two equivalent metrics and not for the other. Goldstern Judah & Shelah 93 describe a model in which $\mathfrak{m}_{\mathrm{countable}} = \omega_1$, $\mathrm{add}\,\mathcal{R}\mathfrak{bg}(\mathbb{R}) = \mathfrak{c} = \omega_2$ and there is a subset of $\mathbb{R}$ of cardinal ω_2 with strong measure zero. So in this case $\mathbb{N}^{\mathbb{N}}$, with the metric described in 534Qb, is not even $\mathcal{S}\mathfrak{mz}$-embeddable in $\mathbb{R}$ with its usual metric. Of course in models of set theory in which the Borel conjecture is true we do have a topologically determined structure on any separable metrizable space.

Note that for any uncountable complete separable metric space (X, ρ), there is a subset of X homeomorphic to $\{0,1\}^{\mathbb{N}}$ (423Ba, 423K[3]), and the homeomorphism must be a uniform equivalence; so that $\{0,1\}^{\mathbb{N}}$

[3]Formerly 423J.

and its companions $[0,1]^r$, $\mathbb{R}^r$ (534P) must be $\mathcal{S}\mathfrak{mz}$-embeddable in X. In this sense they are the 'simplest' uncountableomplete metric spaces. In the same sense, $[0,1]^{\mathbb{N}}$ is the most complex separable metric space (534Xk).

For σ-compact spaces, strong measure zero becomes a topological property (534Eb), corresponding to what I call 'Rothberger's property' (534Cb). Rothberger 1938b investigated subsets of $\mathbb{R}$ which have Rothberger's property in themselves, under the name 'property C′'. The ideas of 534Da and 534L-534Ma can be re-presented as theorems about Rothberger's property (534Db, 534Mb, 534Xj); the machinery of 534Yd is supposed to suggest a reason for this. It is natural to be attracted to a topological concept, but there is a difficulty in that Rothberger's property is not hereditary in the usual way (534Xh, 534Xm, 534Xn). I note that while 534P can be stated in terms of $\mathcal{R}\mathfrak{bg}$-equivalence, isomorphism of the ideals of sets with the appropriate Rothberger's property, the concept of strong measure zero seems to be necessary in the Schröder-Bernstein arguments based on 534O. Of course the spaces here are paracompact and normal, so 534F gives us an alternative approach to this issue.

For a fuller discussion of strong measure zero in $\mathbb{R}$, see Bartoszyński & Judah 95, chap. 8, from which many of the ideas of this section are taken.

Version of 1.6.11

535 Liftings

I introduced the Lifting Theorem (§341) as one of the fundamental facts about complete strictly localizable measure spaces. Of course we can always complete a measure space and thereby in effect obtain a lifting for any σ-finite measure. For the applications of the Lifting Theorem in §§452-453 this procedure is natural and effective; and generally in this treatise I have taken the view that one should work with completed measures unless there is some strong reason not to. But I have also embraced the principle of maximal convenient generality, seeking formulations which will exhibit the full force of each idea in the context appropriate to that idea, uncluttered by the special features of intended applications. So the question of when, and why, liftings for incomplete measures can be found is one which automatically arises. It turns out to be a fruitful question, in the sense that it leads us to new arguments, even though the answers so far available are unsatisfying.

As usual, much of what we want to know depends on the behaviour of the usual measures on powers of $\{0,1\}$ (535B). An old argument relying on the continuum hypothesis shows that Lebesgue measure can have a Borel lifting; this has been usefully refined, and I give a strong version in 535D-535E. We know that we cannot expect to have translation-invariant Borel liftings (345F), but strong Borel liftings are possible (535H-535I), and in some cases can be built from Borel liftings (535J-535N).

For certain applications in functional analysis, we are more interested in liftings for L^∞ spaces than in liftings for measure algebras; and it is sometimes sufficient to have a 'linear lifting', not necessarily corresponding to a lifting in the strict sense (535O, 535P). I give a couple of paragraphs to linear liftings because in some ways they are easier to handle and it is conceivable that they are relevant to the main outstanding problem (535Zf).

535A Notation (a) The most interesting questions to be examined in this section can be phrased in the following language. If (X,Σ,μ) is a measure space and $\mathfrak{T}$ a topology on X, I will say that a **Borel lifting** of μ is a lifting which takes values in the Borel σ-algebra $\mathcal{B}(X)$ of X. (As usual, I will use the word 'lifting' indifferently for homomorphisms from Σ to itself, or from $\mathfrak{A}$ to Σ, where $\mathfrak{A}$ is the measure algebra of μ. Of course a homomorphism $\theta:\mathfrak{A}\to\Sigma$ is a Borel lifting iff the corresponding homomorphism $E\mapsto\theta E^\bullet:\Sigma\to\Sigma$ is a Borel lifting.) Similarly, a **Baire lifting** of μ is a lifting which takes values in the Baire σ-algebra $\mathcal{B}\mathfrak{a}(X)$ of X.

(b) I remark at once that if $(X,\mathfrak{T},\Sigma,\mu)$ is a topological measure space and $\phi:\Sigma\to\mathcal{B}(X)$ is a Borel lifting for μ, then $\phi\restriction\mathcal{B}(X)$ is a lifting for the Borel measure $\mu\restriction\mathcal{B}(X)$. Conversely, if $\phi':\mathcal{B}(X)\to\mathcal{B}(X)$ is a lifting for $\mu\restriction\mathcal{B}(X)$, and if for every $E\in\Sigma$ there is a Borel set E' such that $E\triangle E'$ is negligible, then ϕ' extends uniquely to a Borel lifting ϕ of μ.

In the same way, any Baire lifting for a measure μ which measures every zero set will give us a lifting for $\mu\restriction\mathcal{B}\mathfrak{a}(X)$; and a lifting for $\mu\restriction\mathcal{B}\mathfrak{a}(X)$ will correspond to a Baire lifting for μ if, for instance, μ is completion regular, as in 535B below.

(c) As in Chapter 52, I will say that, for any set I, ν_I is the usual measure on $\{0,1\}^I$ and $\mathfrak{B}_I$ its measure algebra.

535B Proposition Let (X,Σ,μ) be a strictly localizable measure space with non-zero measure. Suppose that ν_κ has a Baire lifting (that is, $\nu_\kappa\restriction\mathcal{B}\mathfrak{a}(\{0,1\}^\kappa)$ has a lifting) for every infinite cardinal κ such that the Maharam-type-κ component of the measure algebra of μ is non-zero. Then μ has a lifting.

proof Write $(\mathfrak{A},\bar\mu)$ for the measure algebra of μ.

(a) Suppose first that μ is a Maharam-type-homogeneous probability measure. In this case $\mathfrak{A}$ is either $\{0,1\}$ or isomorphic to $\mathfrak{B}_\kappa$ for some infinite κ. The case $\mathfrak{A}=\{0,1\}$ is trivial, as we can set $\phi E=\emptyset$ if $E\in\Sigma$ is negligible, $\phi E=X$ if $E\in\Sigma$ is conegligible. Otherwise, $\mathfrak{A}$ is τ-generated by a stochastically independent family $\langle e_\xi\rangle_{\xi<\kappa}$ of elements of measure $\frac{1}{2}$. For each $\xi<\kappa$, choose $E_\xi\in\Sigma$ such that $E_\xi^\bullet=e_\xi$, and define $f:X\to\{0,1\}^\kappa$ by setting $f(x)(\xi)=\chi E_\xi(x)$ for $x\in X$ and $\xi<\kappa$. Then $\{F:F\subseteq\{0,1\}^\kappa,\ \nu F$ and $\mu f^{-1}[F]$ are defined and equal$\}$ is a Dynkin class containing all the measurable cylinders in $\{0,1\}^\kappa$, so includes $\mathcal{B}\mathfrak{a}_\kappa=\mathcal{B}\mathfrak{a}(\{0,1\}^\kappa)$, and f is inverse-measure-preserving for μ and $\nu'_\kappa=\nu_\kappa\restriction\mathcal{B}\mathfrak{a}_\kappa$. Note that $\mathfrak{B}_\kappa$ can be identified with the measure algebra of ν'_κ (put 415E and 322Da together, or see 415Xs[4]). So we have an induced measure-preserving Boolean homomorphism $\pi:\mathfrak{B}_\kappa\to\mathfrak{A}$ defined by setting $\pi F^\bullet=f^{-1}[F]^\bullet$ for every $F\in\mathcal{B}\mathfrak{a}_\kappa$. Since $\pi[\mathfrak{B}_\kappa]$ is an order-closed subalgebra of $\mathfrak{A}$ (324Kb) containing every e_ξ, it is the whole of $\mathfrak{A}$.

We are supposing that there is a lifting $\theta:\mathfrak{B}_\kappa\to\mathcal{B}\mathfrak{a}_\kappa$ of ν_κ. Define $\theta_1:\mathfrak{A}\to\Sigma$ by setting $\theta_1 a=f^{-1}[\theta\pi^{-1}a]$ for every $a\in\mathfrak{A}$; then θ_1 is a Boolean homomorphism because θ and π^{-1} are, and

$$(\theta_1 a)^\bullet=\pi((\theta\pi^{-1}a)^\bullet)=\pi\pi^{-1}a=a$$

for every $a\in\mathfrak{A}$, so θ_1 is a lifting for μ.

(b) It follows at once that if μ is any non-zero totally finite Maharam-type-homogeneous measure, then it will have a lifting, as we can apply (a) to a scalar multiple of μ. Now consider the general case. Let $\mathcal{K}$ be the family of measurable subsets K of X such that the subspace measure μ_K is non-zero, totally finite and Maharam-type-homogeneous. Then μ is inner regular with respect to $\mathcal{K}$, by Maharam's theorem (332B). By 412Ia, there is a decomposition $\langle X_i\rangle_{i\in I}$ of X such that at most one X_i does not belong to $\mathcal{K}$, and that exceptional one, if any, is negligible; adding a trivial element $X_k=\emptyset$ if necessary, we may suppose that there is exactly one $k\in I$ such that $\mu X_k=\emptyset$. For each $i\in I\setminus\{k\}$, let μ_i be the subspace measure on X_i, and Σ_i its domain; then μ_i has a lifting $\phi_i:\Sigma_i\to\Sigma_i$. (The point is that if the Maharam type κ of μ_i is infinite, then the Maharam-type-κ component of $\mathfrak{A}$ includes $X_i^\bullet$ and is non-zero, so our hypothesis tells us that ν_κ has a Baire lifting.) At this point, recall that we are also supposing that $\mu X>0$, so there is some $j\in I\setminus\{k\}$; fix $z\in X_j$, and define $\phi:\Sigma\to\mathcal{P}X$ by setting

$$\begin{aligned}\phi E&=\bigcup_{i\in I\setminus\{k\}}\phi_i(E\cap X_i)\text{ if }z\notin\phi_j(E\cap X_j),\\&=X_k\cup\bigcup_{i\in I\setminus\{k\}}\phi_i(E\cap X_i)\text{ if }z\in\phi_j(E\cap X_j).\end{aligned}$$

Then ϕ is a lifting for μ. **P** It is a Boolean homomorphism because every ϕ_i is. If $E\in\Sigma$, then $X_i\cap\phi E=\phi_i(E\cap X_i)$ if $i\in I\setminus\{k\}$, and is either X_k or $\emptyset$ if $i=k$; in any case, it belongs to Σ_i; as $\langle X_i\rangle_{i\in I}$ is a decomposition for μ, $\phi E\in\Sigma$. Also

$$\mu(E\triangle\phi E)\le\mu X_k+\textstyle\sum_{i\in I\setminus\{k\}}\mu_i((E\cap X_i)\triangle\phi_i(E\cap X_i))=0.$$

Finally, if $\mu E=0$, then $\mu_i(E\cap X_i)=0$ and $\phi_i(E\cap X_i)=\emptyset$ for every $i\in I\setminus\{k\}$, so $\phi E=\emptyset$. **Q**

535C Proposition If λ and κ are cardinals with $\lambda=\lambda^\omega\le\kappa$, and ν_κ has a Baire lifting, then ν_λ has a Baire lifting.

proof If λ is finite, the result is trivial, so we may suppose that $\lambda\ge\omega$ (and therefore that $\lambda\ge\mathfrak{c}$). For $I\subseteq\kappa$, write $\mathcal{B}\mathfrak{a}_I$ for the Baire σ-algebra of $\{0,1\}^I$ and T_I for the family of those $E\in\mathcal{B}\mathfrak{a}_\kappa$ which are

[4] Formerly 415Xp.

determined by coordinates in I. Set $\pi_I(x) = x\restriction I$ for every $x \in \{0,1\}^\kappa$; then $H \mapsto \pi_I^{-1}[H]$ is a Boolean isomorphism between $\mathcal{B}\mathfrak{a}_I$ and T_I, with inverse $E \mapsto \pi_I[E]$. **P** Because π_I is continuous, $\pi_I^{-1}[H] \in \mathcal{B}\mathfrak{a}_\kappa$ for every $H \in \mathcal{B}\mathfrak{a}_I$. Of course $H \mapsto \pi_I^{-1}[H]$ is a Boolean homomorphism, and it is injective because π_I is surjective. Identifying $\{0,1\}^\kappa$ with $\{0,1\}^I \times \{0,1\}^{\kappa\setminus I}$, we have a function $h : \{0,1\}^I \to \{0,1\}^\kappa$ defined by setting $h(v) = (v, \mathbf{0})$ for $v \in \{0,1\}^I$. This is continuous, therefore $(\mathcal{B}\mathfrak{a}_I, \mathcal{B}\mathfrak{a}_\kappa)$-measurable. If $E \in \mathrm{T}_I$, then $E = \pi_I^{-1}[\pi_I[E]] = \pi_I^{-1}[h^{-1}[E]]$; so $H \mapsto \pi_I^{-1}[H]$ is surjective and is an isomorphism. **Q**

Consequently $\#(\mathrm{T}_I) \le \mathfrak{c}$ for every countable $I \subseteq \kappa$ (4A1O, because $\mathcal{B}\mathfrak{a}_I$ is σ-generated by the cylinder sets, by 4A3Na). For any I, $\mathrm{T}_I = \bigcup_{J\in[I]^{\le\omega}} \mathrm{T}_J$, because every member of $\mathcal{B}\mathfrak{a}_I$ is determined by coordinates in a countable set (4A3Nb). So $\#(\mathrm{T}_I) \le \max(\mathfrak{c}, \#(I)^\omega) = \lambda$ whenever $I \subseteq \kappa$ and $\#(I) = \lambda$.

Let ϕ be a Baire lifting for ν_κ. Choose a non-decreasing family $\langle J_\xi\rangle_{\xi<\omega_1}$ in $[\kappa]^\lambda$ such that $J_0 = \lambda$ and $\phi E \in \mathrm{T}_{J_{\xi+1}}$ whenever $\xi < \omega_1$ and $E \in \mathrm{T}_{J_\xi}$. Set $J = \bigcup_{\xi<\omega_1} J_\xi$; then $\mathrm{T}_J = \bigcup_{\xi<\omega_1} \mathrm{T}_{J_\xi}$, so $\phi E \in \mathrm{T}_J$ for every $E \in \mathrm{T}_J$.

We therefore have a Boolean homomorphism $\phi_1 : \mathcal{B}\mathfrak{a}_J \to \mathcal{B}\mathfrak{a}_J$ defined by setting $\phi_1 H = \pi_J[\phi(\pi_J^{-1}[H])]$ for every $H \in \mathcal{B}\mathfrak{a}_J$. If $\nu_J H = 0$, then $\nu_\kappa \pi_J^{-1}[H] = 0$ and $\phi_1 H = \phi(\pi_J^{-1}[H]) = 0$. For any $H \in \mathcal{B}\mathfrak{a}_J$,

$$\pi_J^{-1}[H \triangle \phi_1 H] = \pi_J^{-1}[H] \triangle \phi(\pi_J^{-1}[H])$$

is ν_κ-negligible, so $H\triangle\phi_1 H$ is ν_J-negligible. Thus ϕ_1 is a lifting for $\nu_J\restriction\mathcal{B}\mathfrak{a}_J$. As $\nu_J\restriction\mathcal{B}\mathfrak{a}_J$ is isomorphic to $\nu_\lambda\restriction\mathcal{B}\mathfrak{a}_\lambda$, the latter also has a lifting. As ν_λ is completion regular (416U), the measure algebra of $\nu_\lambda\restriction\mathcal{B}\mathfrak{a}_\lambda$ can be identified with $\mathfrak{B}_\lambda$, and we can interpret a lifting for $\nu_\lambda\restriction\mathcal{B}\mathfrak{a}_\lambda$ as a Baire lifting for its completion ν_λ.

535D The following result covers most of the cases in which non-complete probability measures are known to have liftings.

Theorem Let (X, Σ, μ) be a measure space such that $\mu X > 0$, and suppose that its measure algebra is tightly ω_1-filtered (definition: 511Di). Then μ has a lifting.

proof This is a special case of 518L.

535E Proposition Suppose that $\mathfrak{c} \le \omega_2$ and the Freese-Nation number $\mathrm{FN}(\mathcal{P}\mathbb{N})$ is ω_1.

(a) If $\mathfrak{A}$ is a measurable algebra with cardinal at most ω_2, it is tightly ω_1-filtered.

(b) (Mokobodzki 7?) Let (X, Σ, μ) be a σ-finite measure space with non-zero measure and Maharam type at most ω_2.

(i) μ has a lifting.

(ii) If $\mathfrak{T}$ is a topology on X such that μ is inner regular with respect to the Borel sets, then μ has a Borel lifting.

(iii) If $\mathfrak{T}$ is a topology on X such that μ is inner regular with respect to the zero sets, then μ has a Baire lifting.

proof (a) By 524O(b-iii), $\mathrm{FN}(\mathfrak{A}) \le \omega_1$, so 518M gives the result.

(b)(i) By 514De, the measure algebra of μ has cardinal at most

$$\omega_2^\omega = \max(\mathfrak{c}, \omega_2) \le \omega_2$$

(5A1F(e-iii)). So we can put (a) and 535D together.

(ii) Because μ is σ-finite and inner regular with respect to the Borel sets, every measurable set can be expressed as the union of a Borel set and a negligible set. By (i), $\mu\restriction\mathcal{B}(X)$ has a lifting, which can be interpreted as a Borel lifting for μ, as in 535Ab.

(iii) As (ii), but with $\mathcal{B}\mathfrak{a}(X)$ in place of $\mathcal{B}(X)$.

535F Using the continuum hypothesis, we can go a little farther with ideas from 341J.

Proposition Let (X, Σ, μ) be a measure space such that $\mu X > 0$ and $\#(\mathfrak{A}) \le \omega_1$, where $\mathfrak{A}$ is the measure algebra of μ, and suppose that $\underline{\theta} : \mathfrak{A} \to \Sigma$ is such that

$$\underline{\theta}0 = \emptyset, \quad \underline{\theta}(a \cap b) = \underline{\theta}a \cap \underline{\theta}b \text{ for all } a,\, b \in \mathfrak{A}, \quad (\underline{\theta}a)^\bullet \subseteq a \text{ for every } a \in \mathfrak{A}.$$

Then μ has a lifting $\theta : \mathfrak{A} \to \Sigma$ such that $\theta E^\bullet \supseteq \underline{\theta}E$ for every $E \in \Sigma$.

proof (a) Adjusting $\underline{\theta}1$ if necessary, we can suppose that $\underline{\theta}1 = X$. Note that $\underline{\theta}a \subseteq \underline{\theta}b$ whenever $a \subseteq b$ in $\mathfrak{A}$. Let $\langle a_\xi \rangle_{\xi<\omega_1}$ be a family running over $\mathfrak{A}$, and for $\alpha \le \omega_1$ let $\mathfrak{C}_\alpha$ be the subalgebra of $\mathfrak{A}$ generated by $\{a_\xi : \xi < \alpha\}$. Define Boolean homomorphisms $\theta_\alpha : \mathfrak{C}_\alpha \to \Sigma$ inductively, as follows. The inductive hypothesis will be that $(\theta_\alpha c)^\bullet = c$ and $\theta_\alpha c \supseteq \underline{\theta}c$ for every $c \in \mathfrak{C}_\alpha$, while θ_α extends θ_β for every $\beta \le \alpha$. Start with $\theta_0 0 = \emptyset$, $\theta_0 1 = X$.

(b) Given θ_α, where $\alpha < \omega_1$, set

$$F = \bigcup\{\underline{\theta}(c \cup a_\alpha) \setminus \theta_\alpha c : c \in \mathfrak{C}_\alpha\},$$

$$G = \bigcup\{\underline{\theta}(c \cup (1 \setminus a_\alpha)) \setminus \theta_\alpha c : c \in \mathfrak{C}_\alpha\}.$$

Because $\mathfrak{C}_\alpha$ is countable, F and G belong to Σ. If $c \in \mathfrak{C}_\alpha$, then

$$(\underline{\theta}(c \cup a_\alpha) \setminus \theta_\alpha c)^\bullet = \underline{\theta}(c \cup a_\alpha)^\bullet \setminus c \subseteq (c \cup a_\alpha) \setminus c \subseteq a_\alpha,$$

so $F^\bullet \subseteq a_\alpha$; similarly, $G^\bullet \subseteq 1 \setminus a_\alpha$. Next, $F \cap G = \emptyset$. **P** If b, $c \in \mathfrak{C}_\alpha$, then

$$\begin{aligned}(\underline{\theta}(b \cup a_\alpha) \setminus \theta_\alpha b) \cap (\underline{\theta}(c \cup (1 \setminus a_\alpha)) \setminus \theta_\alpha c) &= \underline{\theta}((b \cup a_\alpha) \cap (c \cup (1 \setminus a_\alpha))) \setminus (\theta_\alpha b \cup \theta_\alpha c) \\ &\subseteq \underline{\theta}(b \cup c) \setminus \theta_\alpha(b \cup c) = \emptyset. \ \mathbf{Q}\end{aligned}$$

Choose any $E \in \Sigma$ such that $E^\bullet = a_\alpha$ and set $E_\alpha = (E \cup F) \setminus G$; then $E_\alpha^\bullet = a_\alpha$, $F \subseteq E_\alpha$ and $G \cap E_\alpha = \emptyset$.

If $c \in \mathfrak{C}_\alpha$ and $c \subseteq a_\alpha$, then $\underline{\theta}((1 \setminus c) \cup a_\alpha) = \underline{\theta}1 = X$, so

$$\theta_\alpha c = \underline{\theta}((1 \setminus c) \cup a_\alpha) \setminus \theta_\alpha(1 \setminus c) \subseteq F \subseteq E_\alpha.$$

Similarly, if $c \in \mathfrak{C}_\alpha$ and $c \cap a_\alpha = 0$, then

$$\theta_\alpha c = \underline{\theta}((1 \setminus c) \cup (1 \setminus a_\alpha)) \setminus \theta_\alpha(1 \setminus c) \subseteq G$$

is disjoint from E_α. We can therefore define a Boolean homomorphism $\theta_{\alpha+1} : \mathfrak{C}_{\alpha+1} \to \Sigma$ by setting

$$\theta_{\alpha+1}((b \cap a_\alpha) \cup (c \setminus a_\alpha)) = (\theta_\alpha b \cap E_\alpha) \cup (\theta_\alpha c \setminus E_\alpha)$$

for all b, $c \in \mathfrak{C}_\alpha$ (312O), and $\theta_{\alpha+1}$ will extend θ_β for every $\beta \le \alpha + 1$. Because $(\theta_{\alpha+1} a_\alpha)^\bullet = E_\alpha^\bullet = a_\alpha$ and $\theta_{\alpha+1} c = \theta_\alpha c$ for every $c \in \mathfrak{C}_\alpha$, $(\theta_{\alpha+1} a)^\bullet = a$ for every $a \in \mathfrak{C}_{\alpha+1}$.

I have still to check the other part of the inductive hypothesis. If b, $c \in \mathfrak{C}_\alpha$, then

$$\begin{aligned}\underline{\theta}((b \cap a_\alpha) \cup (c \setminus a_\alpha)) &= \underline{\theta}((b \cup c) \cap (c \cup a_\alpha) \cap (b \cup (1 \setminus a_\alpha))) \\ &= \underline{\theta}(b \cup c) \cap \underline{\theta}(c \cup a_\alpha) \cap \underline{\theta}(b \cup (1 \setminus a_\alpha)) \\ &\subseteq \theta_\alpha(b \cup c) \cap (F \cup \theta_\alpha c) \cap (G \cup \theta_\alpha b) \\ &\subseteq \theta_{\alpha+1}(b \cup c) \cap (\theta_{\alpha+1} a_\alpha \cup \theta_{\alpha+1} c) \cap (\theta_{\alpha+1}(1 \setminus a_\alpha) \cup \theta_{\alpha+1} b) \\ &= \theta_{\alpha+1}((b \cap a_\alpha) \cup (c \setminus a_\alpha)),\end{aligned}$$

which is what we need to know.

(c) For non-zero limit ordinals $\alpha \le \omega_1$, we have $\mathfrak{C}_\alpha = \bigcup_{\beta<\alpha} \mathfrak{C}_\beta$ so we can, and must, take $\theta_\alpha = \bigcup_{\beta<\alpha} \theta_\beta$. At the end of the induction, $\theta_{\omega_1} : \mathfrak{A} \to \Sigma$ is an appropriate lifting.

535G Corollary (see NEUMANN 1931) Suppose that $\mathfrak{c} = \omega_1$. Then for any integer $r \ge 1$ there is a Borel lifting θ of Lebesgue measure on $\mathbb{R}^r$ such that $x \in \theta E^\bullet$ whenever $E \subseteq \mathbb{R}^r$ is a Borel set and x is a density point of E.

proof In 535F, let $\underline{\theta}$ be lower Lebesgue density (341E), interpreted as a function from the Lebesgue measure algebra to the Borel σ-algebra. We need to check that $\underline{\theta}E^\bullet$ is indeed always a Borel set; this is because

$$\underline{\theta}E^\bullet = \text{int}^*E = \{x : \lim_{n\to\infty} \tfrac{\mu(E \cap B(x,2^{-n}))}{\mu B(x,2^{-n})} = 1\}$$

and the functions $x \mapsto \mu(E \cap B(x, 2^{-n}))$ are all continuous (use 443B).

535H Again using the continuum hypothesis, we have some results on 'strong' liftings, as described in §453.

Theorem Let $(X,\mathfrak{T},\Sigma,\mu)$ be an effectively locally finite τ-additive topological measure space with measure algebra $\mathfrak{A}$. If $\#(\mathfrak{A})\le\operatorname{add}\mu$ and μ is strictly positive, then μ has a strong lifting.

proof (a) For each $a\in\mathfrak{A}$, set

$$\overline{a}=\bigcap\{F:F\subseteq X\text{ is closed, }F^\bullet\supseteq a\}.$$

Then $\overline{a}$ is closed and $\overline{a}^\bullet\supseteq a$ (414Ac). If a, $b\in\mathfrak{A}$, then $\overline{a\cup b}=\overline{a}\cup\overline{b}$. **P** Of course $\overline{a\cup b}\supseteq\overline{a}\cup\overline{b}$, because the operation $^-$ is order-preserving. On the other hand, $\overline{a}\cup\overline{b}$ is a closed set and $(\overline{a}\cup\overline{b})^\bullet\supseteq a\cup b$, so $\overline{a}\cup\overline{b}\supseteq\overline{a\cup b}$. **Q**

For a subalgebra $\mathfrak{B}$ of $\mathfrak{A}$, say that a function $\theta:\mathfrak{B}\to\Sigma$ is 'potentially a strong lifting' if it is a Boolean homomorphism and $(\theta b)^\bullet=b$ and $\theta b\subseteq\overline{b}$ for every $b\in\mathfrak{B}$.

(b) (The key.) Suppose that $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$, with cardinal less than $\operatorname{add}\mu$, and $c\in\mathfrak{A}$; let $\mathfrak{B}_1$ be the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{B}\cup\{c\}$. If $\theta:\mathfrak{B}\to\Sigma$ is potentially a strong lifting, then it has an extension $\theta_1:\mathfrak{B}_1\to\Sigma$ which is also potentially a strong lifting.

P Set

$$C_0=\bigcup\{\theta a:a\in\mathfrak{B},\,a\subseteq c\},$$

$$D_0=\bigcap\{\theta b:b\in\mathfrak{B},\,c\subseteq b\},$$

$$C_1=\bigcup\{\theta a\setminus\overline{a\setminus c}:a\in\mathfrak{B}\},$$

$$D_1=\bigcap\{(X\setminus\theta b)\cup\overline{b\cap c}:b\in\mathfrak{B}\}.$$

Fix $E_0\in\Sigma$ such that $E_0^\bullet=c$.

If a, a', b, $b'\in\mathfrak{B}$ and $a'\subseteq c\subseteq b'$, then

$$a'\subseteq b',\text{ so }\theta a'\subseteq\theta b';$$

$$\theta a'\cap\theta b=\theta(a'\cap b)\subseteq\overline{a'\cap b}\subseteq\overline{b\cap c},\text{ so }\theta a'\subseteq(X\setminus\theta b)\cup\overline{b\cap c};$$

$$\theta a\setminus\theta b'=\theta(a\setminus b')\subseteq\overline{a\setminus b'}\subseteq\overline{a\setminus c},\text{ so }\theta a\setminus\overline{a\setminus c}\subseteq\theta b';$$

$$\theta a\cap\theta b=\theta(a\cap b)\subseteq\overline{a\cap b}=\overline{a\cap b\cap c}\cup\overline{a\cap b\setminus c}\subseteq\overline{a\setminus c}\cup\overline{b\cap c},$$

so

$$\theta a\setminus\overline{a\setminus c}\subseteq(X\setminus\theta b)\cup\overline{b\cap c}.$$

This shows that $C_0\cup C_1\subseteq D_0\cap D_1$. At the same time,

$$E_0^\bullet=c\supseteq a',\text{ so }\theta a'\setminus E_0\text{ is negligible;}$$

$$E_0^\bullet=c\subseteq b',\text{ so }E_0\setminus\theta b'\text{ is negligible;}$$

$$(E_0\cup\overline{a\setminus c})^\bullet\supseteq c\cup(a\setminus c)\supseteq a=(\theta a)^\bullet$$

so $(\theta a\setminus\overline{a\cap c})\setminus E_0$ is negligible;

$$E_0^\bullet=c\subseteq(1\setminus b)\cup(b\cap c)\subseteq(X\setminus\theta b)^\bullet\cup\overline{b\cap c}^\bullet,$$

so $E_0\setminus((X\setminus\theta b)\cup\overline{b\cap c})$ is negligible. Because $\#(\mathfrak{B})<\operatorname{add}\mu$, $(C_0\cup C_1)\setminus E_0$ and $E_0\setminus(D_0\cap D_1)$ are measurable and negligible.

If we set

$$E=(E_0\cup C_0\cup C_1)\cap(D_0\cap D_1),$$

then $E\in\Sigma$, $E^\bullet=c$ and $C_0\cup C_1\subseteq E_0\subseteq D_0\cap D_1$. So we can set $\theta_1c=E$ to define a homomorphism from $\mathfrak{B}_1$ to Σ (312O again), and we shall have $(\theta_1d)^\bullet=d$ for every $d\in\mathfrak{B}_1$.

We must check that $\theta_1d\subseteq\overline{d}$ for every $d\in\mathfrak{B}_1$. Now d is expressible as $(b\cap c)\cup(a\setminus c)$ for some a, $b\in\mathfrak{B}$, and in this case

$$\theta b\cap E\subseteq\theta b\cap((X\setminus\theta b)\cup\overline{b\cap c})\subseteq\overline{b\cap c},$$

$$\theta a \setminus E \subseteq \theta a \setminus (\theta a \setminus \overline{a \setminus c}) \subseteq \overline{a \setminus c},$$

so

$$\theta_1 d = (\theta b \cap E) \cup (\theta a \setminus E) \subseteq \overline{b \cap c} \cup \overline{a \setminus c} = \overline{d}.$$

So θ_1 is a potential strong lifting, as required. **Q**

(c) Enumerate $\mathfrak{A}$ as $\langle a_\xi \rangle_{\xi \in \kappa}$ where $\kappa \le \operatorname{add}\mu$, and for $\alpha \le \kappa$ let $\mathfrak{B}_\alpha$ be the subalgebra of $\mathfrak{A}$ generated by $\{a_\xi : \xi < \alpha\}$. Then (b) tells us that we can choose inductively a family $\langle \theta_\alpha \rangle_{\alpha < \kappa}$ such that $\theta_\alpha : \mathfrak{B}_\alpha \to \Sigma$ is a potential strong lifting and $\theta_{\alpha+1}$ extends θ_α for each $\alpha < \kappa$. (At non-zero limit ordinals α, $\mathfrak{B}_\alpha = \bigcup_{\xi<\alpha} \mathfrak{B}_\xi$ so we can take θ_α to be the common extension of $\bigcup_{\xi<\alpha} \theta_\xi$. We need to know that μ is strictly positive in order to be sure that $\overline{1} = X$, so that we can take $\theta_0 1 = X$.) In this way we obtain a lifting $\theta = \theta_\kappa$ of μ. Also $\theta a \subseteq \overline{a}$ for every $a \in \mathfrak{A}$. Looking at this from the other side, if $F \subseteq X$ is closed then $\overline{F^\bullet} \subseteq F$ so $\theta(F^\bullet) \subseteq F$, and θ is a strong lifting.

535I Corollary (see MOKOBODZKI 75) Suppose that $\mathfrak{c} = \omega_1$. Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a strictly positive σ-finite quasi-Radon measure space with Maharam type at most $\omega_1 = \mathfrak{c}$. Then μ has a strong Borel lifting.

proof Because μ is σ-finite, its measure algebra $\mathfrak{A}$ is ccc, and has size at most $\mathfrak{c}^\omega = \omega_1$; so we can apply 535H to $\mu \restriction \mathcal{B}(X)$.

535J Under certain conditions, we can deduce the existence of a strong lifting from the existence of a lifting. The basic case is the following.

Lemma Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a completely regular totally finite topological measure space with a Borel lifting ϕ. Suppose that $K \subseteq X$ is a self-supporting set of non-zero measure, homeomorphic to $\{0,1\}^{\mathbb{N}}$, such that $K \cap G \subseteq \phi G$ for every open set $G \subseteq X$. Then the subspace measure μ_K has a strong Borel lifting.

proof (a) Taking $\mathcal{E}$ to be the algebra of relatively open-and-closed subsets of K, we have a Boolean homomorphism $\psi_0 : \mathcal{E} \to \mathcal{B}(X)$ such that $E \subseteq \operatorname{int} \psi_0 E$ for every $E \in \mathcal{E}$. **P** We have a Boolean-independent sequence $\langle E_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{E}$ which generates $\mathcal{E}$ and separates the points of K. Because every member of $\mathcal{E}$ is compact, we can choose for each $n \in \mathbb{N}$ an open $H_n \subseteq X$ such that $E_n = K \cap H_n = K \cap \overline{H}_n$. Define $h : X \to K$ by saying that, for every $n \in \mathbb{N}$ and $x \in X$, $h(x) \in E_n$ iff $x \in H_n$. Define $\psi_0 : \mathcal{E} \to \mathcal{B}(X)$ by setting $\psi_0 E = h^{-1}[E]$ for $E \in \mathcal{E}$. Then ψ_0 is a Boolean homomorphism. The set

$$\{E : E \in \mathcal{E},\, E \subseteq \operatorname{int} \psi_0 E,\, K \setminus E \subseteq \operatorname{int} \psi_0(K \setminus E)\}$$

is a subalgebra of $\mathcal{E}$ containing every E_n, so is the whole of $\mathcal{E}$, and ψ_0 has the required property. **Q**

(b) Let $\mathfrak{A}$ be the measure algebra of μ, and $\theta : \mathfrak{A} \to \mathcal{B}(X)$ the lifting corresponding to ϕ. Set $\psi_1 E = (\psi_0 E)^\bullet$ for $E \in \mathcal{E}$, so that $\psi_1 : \mathcal{E} \to \mathfrak{A}$ is a Boolean homomorphism. Let $\mathcal{I}$ be the null ideal of μ_K. Because K is self-supporting, $\mathcal{E} \cap \mathcal{I} = \{\emptyset\}$. Taking $\mathcal{E}' = \{E \triangle F : E \in \mathcal{E},\, F \in \mathcal{I}\}$, $\mathcal{E}'$ is a subalgebra of $\mathcal{P}K$, and we have a Boolean homomorphism $\psi' : \mathcal{E}' \to \mathcal{E}$ defined by setting $\psi'(E \triangle F) = E$ whenever $E \in \mathcal{E}$ and $F \in \mathcal{I}$; set $\psi_1' = \psi_1 \psi'$, so that $\psi_1' : \mathcal{E}' \to \mathfrak{A}$ is a Boolean homomorphism extending ψ_1, and $\psi_1' F = 0$ whenever $F \in \mathcal{I}$. Because μ is totally finite, $\mathfrak{A}$ is Dedekind complete, and there is a Boolean homomorphism $\tilde{\psi}_1 : \mathcal{P}K \to \mathfrak{A}$ extending ψ_1' (314K). Now set

$$\phi_1 E = K \cap (\phi E \cup (\theta \tilde{\psi}_1 E \setminus \phi K))$$

for every measurable $E \subseteq K$. Then ϕ_1 is a strong lifting for μ_K. **P** $\phi \restriction \Sigma_K$ is a Boolean homomorphism from the domain Σ_K of μ_K to $\mathcal{B}(\phi K)$, while $E \mapsto \theta \tilde{\psi}_1 E \setminus \phi K$ is a Boolean homomorphism from Σ_K to $\mathcal{B}(X \setminus \phi K)$; putting these together, ϕ_1 is a Boolean homomorphism from Σ_K to $\mathcal{B}(K)$. If $E \in \Sigma_K$, then $E \triangle (K \cap \phi E)$ and $K \setminus \phi K$ are negligible, so $E \triangle \phi_1 E$ is negligible. If $E \in \Sigma_K$ is negligible, then $\phi E = \emptyset$, $\psi_1' E = 0$ and $\phi_1 E$ is empty. Thus ϕ_1 is a lifting for μ_K. Morover, if $E \in \mathcal{E}$, set $G = \operatorname{int} \psi_0 E$, so that $E = K \cap G$. In this case,

$$E \subseteq \phi G = \theta G^\bullet \subseteq \theta(\psi_0 E)^\bullet = \theta \psi_1 E = \theta \tilde{\psi}_1 E,$$

while

$$E \cap \phi K \subseteq \phi G \cap \phi K = \phi E;$$

so $E \subseteq \phi_1 E$. So if $V \subseteq K$ is relatively open,

$$V=\bigcup\{E:E\in\mathcal{E},\,E\subseteq V\}\subseteq\bigcup\{\phi_1E:E\in\mathcal{E},\,E\subseteq V\}\subseteq\phi_1V.$$

Thus ϕ_1 is strong. **Q**

535K Lemma Let X be a metrizable space, μ an atomless Radon measure on X and ν an atomless strictly positive Radon measure on $\{0,1\}^{\mathbb{N}}$. Let $\mathcal{K}$ be the family of those subsets K of X such that K, with the subspace topology and measure, is isomorphic to $\{0,1\}^{\mathbb{N}}$ with its usual topology and a scalar multiple of ν. Then μ is inner regular with respect to $\mathcal{K}$.

proof (a) It will be helpful to note that if $E\in\operatorname{dom}\mu$ and $\gamma<\mu E$ there is a compact set $K\subseteq E$ such that $\mu K=\gamma$. **P** Let $\langle\gamma_n\rangle_{n\in\mathbb{N}}$ be a strictly decreasing sequence with $\gamma_0<\mu E$ and $\inf_{n\in\mathbb{N}}\gamma_n=\gamma$. Choose $\langle K_n\rangle_{n\in\mathbb{N}}$, $\langle E_n\rangle_{n\in\mathbb{N}}$ inductively as follows. $E_0=E$. Given that $\mu E_n>\gamma_n$, let $K_n\subseteq E_n$ be a compact set such that $\mu K_n\ge\gamma_n$; now let E_{n+1} be a measurable set with measure γ_n (215D, because μ is atomless). At the end of the induction, set $K=\bigcap_{n\in\mathbb{N}}K_n$. **Q**

(b) Now for the main argument. Suppose that $0\le\gamma<\mu E$. Let $\langle\gamma_n\rangle_{n\in\mathbb{N}}$ be a strictly decreasing sequence with $\gamma_0<\mu E$ and $\inf_{n\in\mathbb{N}}\gamma_n=\gamma$. Set $\gamma'_n=\frac{1}{2}(\gamma_n+\gamma_{n+1})$ for each n. For $\sigma\in\bigcup_{n\in\mathbb{N}}\{0,1\}^n$, set $I_\sigma=\{z:\sigma\subseteq z\in\{0,1\}^{\mathbb{N}}\}$. Let K_0 be a compact subset of E of measure γ_0; because X is metrizable, K_0 is second-countable; let $\langle V_n\rangle_{n\in\mathbb{N}}$ run over a base for the topology of K_0. Choose $\langle m(n)\rangle_{n\in\mathbb{N}}$ and L_σ, for $\sigma\in\{0,1\}^{m(n)}$, as follows. Start with $m(0)=0$ and $L_\emptyset=K_0$. Given that $\langle L_\sigma\rangle_{\sigma\in\{0,1\}^{m(n)}}$ is a disjoint family of compact subsets of X with $\mu L_\sigma=\gamma_n\nu I_\sigma$ for every $\sigma\in\{0,1\}^{m(n)}$, let $m(n+1)>m(n)$ be so large that $\gamma_{n+1}\nu I_\tau<(\gamma_n-\gamma_{n+1})\nu I_\sigma$ whenever $\sigma\in\{0,1\}^{m(n)}$ and $\tau\in\{0,1\}^{m(n+1)}$. (This is where we need to know that ν is atomless and strictly positive.) Now, for each $\sigma\in\{0,1\}^{m(n)}$, enumerate $\{\tau:\sigma\subseteq\tau\in\{0,1\}^{m(n+1)}\}$ as $\langle\tau(\sigma,i)\rangle_{i<2^{m(n+1)-m(n)}}$. Choose inductively disjoint compact sets $L_{\tau(\sigma,i)}\subseteq L_\sigma$, for $i<2^{m(n+1)-m(n)}$, in such a way that $\mu L_{\tau(\sigma,i)}=\gamma_{n+1}\nu I_{\tau(\sigma,i)}$ and $L_{\tau(\sigma,i)}$ is always either included in V_n or disjoint from it; this will be possible because when we come to choose $L_{\tau(\sigma,i)}$, the measure of the set $F=L_\sigma\setminus\bigcup_{j<i}L_{\tau(\sigma,j)}$ available will be

$$\begin{aligned}\gamma_n\nu I_\sigma-\sum_{j<i}\gamma_{n+1}\nu I_{\tau(\sigma,j)}&\ge(\gamma_n-\gamma_{n+1})\nu I_\sigma+\gamma_{n+1}\nu I_{\tau(\sigma,i)}\\&>2\gamma_{n+1}\nu I_{\tau(\sigma,i)},\end{aligned}$$

so at least one of $F\cap V_n$, $F\setminus V_n$ will be of measure greater than $\gamma_{n+1}\nu I_{\tau(\sigma,i)}$. Continue.

Set $K_n=\bigcup\{L_\sigma:\sigma\in\{0,1\}^{m(n)}\}$ for each $n\in\mathbb{N}$, and $K=\bigcap_{n\in\mathbb{N}}K_n$. The construction ensures that whenever $n\le k$, $\sigma\in\{0,1\}^{m(n)}$, $\tau\in\{0,1\}^{m(k)}$ and $\sigma\subseteq\tau$, then $L_\tau\subseteq L_\sigma$. We therefore have a function $f:K\to\{0,1\}^{\mathbb{N}}$ defined by saying that $f(x)\restriction m(n)=\sigma$ whenever $n\in\mathbb{N}$, $\sigma\in\{0,1\}^{m(n)}$ and $x\in K\cap L_\sigma$. Because all the L_σ are compact, f is continuous. But it is also injective. **P** If x, $y\in K$ are different, there is an $n\in\mathbb{N}$ such that $x\in V_n$ and $y\notin V_n$; now $f(x)\restriction m(n+1)\ne f(y)\restriction m(n+1)$. **Q**

For any $n\in\mathbb{N}$, $\sigma\in\{0,1\}^{m(n)}$ and $k\ge n$,

$$\mu(\bigcup\{L_\tau:\sigma\subseteq\tau\in\{0,1\}^{m(k)}\})=\textstyle\sum_{\sigma\subseteq\tau\in\{0,1\}^{m(k)}}\gamma_k\nu I_\tau=\gamma_k\nu I_\sigma.$$

So

$$\mu(f^{-1}[I_\sigma])=\inf_{k\ge n}\gamma_k\nu I_\sigma=\gamma\nu I_\sigma.$$

Thus the Radon measure μf^{-1} on $\{0,1\}^{\mathbb{N}}$ agrees with the Radon measure $\gamma\nu$ on $\{I_\sigma:\sigma\in\bigcup_{n\in\mathbb{N}}\{0,1\}^{m(n)}\}$; as this is a base for the topology of $\{0,1\}^{\mathbb{N}}$ closed under finite intersections, μf^{-1} and $\gamma\nu$ are identical (415H(v)). Once again because ν is strictly positive, f is surjective and is a homeomorphism. So f witnesses that $K\in\mathcal{K}$. As E and γ are arbitrary, μ is inner regular with respect to $\mathcal{K}$.

535L Lemma (a) If $(X,\mathfrak{T})$ is a separable metrizable space, there is a zero-dimensional separable metrizable topology $\mathfrak{S}$ on X, finer than $\mathfrak{T}$, with the same Borel sets as $\mathfrak{T}$, such that $\mathfrak{T}$ is a π-base for $\mathfrak{S}$.

(b) If X is a non-empty zero-dimensional separable metrizable space without isolated points, it is homeomorphic to a dense subset of $\{0,1\}^{\mathbb{N}}$.

(c) Any completely regular space with cardinal less than $\mathfrak{c}$ is zero-dimensional.

proof (a) Enumerate a countable base for $\mathfrak{T}$ as $\langle U_n\rangle_{n\in\mathbb{N}}$. Define a sequence $\langle\mathfrak{S}_n\rangle_{n\in\mathbb{N}}$ of topologies on X by saying that $\mathfrak{S}_0=\mathfrak{T}$ and that $\mathfrak{S}_{n+1}$ is the topology on X generated by $\mathfrak{S}_n\cup\{V_n\}$, where V_n is the closure of U_n for $\mathfrak{S}_n$. Inducing on n, we see that $\mathfrak{S}_n$ is second-countable and has the same Borel sets as $\mathfrak{T}$, for every n. So taking $\mathfrak{S}$ to be the topology generated by $\bigcup_{n\in\mathbb{N}}\mathfrak{S}_n$ (that is, the topology generated by $\{U_n:n\in\mathbb{N}\}\cup\{V_n:n\in\mathbb{N}\}$), this also is second-countable and has the same Borel sets as $\mathfrak{T}$. Each V_n is open for $\mathfrak{S}_{n+1}$ and closed for $\mathfrak{S}_n$, so is open-and-closed for $\mathfrak{S}$. Moreover, since

$$U_n=\bigcup\{U_m:m\in\mathbb{N},\,\overline{U}_m^{\mathfrak{T}}\subseteq U_n\}=\bigcup\{V_m:m\in\mathbb{N},\,V_m\subseteq U_n\}$$

for each n, $\{V_n:n\in\mathbb{N}\}$ is a base for $\mathfrak{S}$ consisting of open-and-closed sets for $\mathfrak{S}$, and $\mathfrak{S}$ is zero-dimensional. Finally, observe that if V_n is not empty, then $V_n\supseteq U_n\neq\emptyset$, so $\mathfrak{T}\supseteq\{U_n:n\in\mathbb{N}\}$ is a π-base for $\mathfrak{S}$.

(b) The family $\mathcal{E}_0$ of open-and-closed subsets of X is a base for the topology of X, so includes a countable base $\mathcal{U}$ (4A2P(a-iii)). Because X has no isolated points, the subalgebra $\mathcal{E}_1$ of $\mathcal{E}_0$ generated by $\mathcal{U}$ is countable, atomless and non-trivial, and must be isomorphic to the algebra $\mathcal{E}$ of open-and-closed subsets of $\{0,1\}^{\mathbb{N}}$ (316M). Let $\pi:\mathcal{E}\to\mathcal{E}_1$ be an isomorphism. Then we have a function $f:X\to\{0,1\}^{\mathbb{N}}$ defined by saying that, for $E\in\mathcal{E}$, $f(x)\in E$ iff $x\in\pi E$. Because $\pi E\neq\emptyset$ for every non-empty $E\in\mathcal{E}$, $f[X]$ is dense in $\{0,1\}^{\mathbb{N}}$. Because $\{f^{-1}[E]:E\in\mathcal{E}\}=\mathcal{E}_1\supseteq\mathcal{U}$ is a base for the topology of X, f is a homeomorphism between X and $f[X]$.

(c) If X is a completely regular space and $\#(X)<\mathfrak{c}$, $G\subseteq X$ is open and $x\in G$, let $f:X\to[0,1]$ be a continuous function such that $f(x)=1$ and $f(y)=0$ for $y\in X\setminus G$. Because $\#(X)<\mathfrak{c}$, there is an $\alpha\in[0,1]\setminus f[X]$, and now $\{y:f(x)>\alpha\}=\{y:f(x)\geq\alpha\}$ is an open-and-closed set containing x and included in G. As x and G are arbitrary, X is zero-dimensional.

535M Lemma Suppose that there is a Borel probability measure on $\{0,1\}^{\mathbb{N}}$ with a strong lifting. Then whenever X is a separable metrizable space and $D\subseteq X$ is a dense set, there is a Boolean homomorphism ϕ from $\mathcal{P}D$ to the Borel σ-algebra $\mathcal{B}(X)$ of X such that $\phi A\subseteq\overline{A}$ for every $A\subseteq D$.

proof case 1 Suppose that X is countable. Then it is zero-dimensional (535Lc), so has a base $\mathcal{U}$ consisting of open-and-closed sets; let $\mathcal{E}$ be the algebra of sets generated by $\mathcal{U}$. For $E\in\mathcal{E}$ set $\pi E=E\cap D$; then π is an isomorphism between $\mathcal{E}$ and a subalgebra $\mathcal{E}'$ of $\mathcal{P}D$. Because $\mathcal{B}(X)=\mathcal{P}X$ is Dedekind complete, the Boolean homomorphism $\pi^{-1}:\mathcal{E}'\to\mathcal{E}$ extends to a Boolean homomorphism $\phi:\mathcal{P}D\to\mathcal{P}X=\mathcal{B}(X)$ (314K again). If $A\subseteq D$ and $x\in X\setminus\overline{A}$, then there is a $U\in\mathcal{U}$ such that $x\in U$ and $A\cap U=\emptyset$, in which case

$$\phi A\subseteq\pi^{-1}(D\setminus U)=X\setminus U$$

does not contain x. As x is arbitrary, $\phi A\subseteq\overline{A}$; as A is arbitrary, ϕ has the required property.

case 2 Suppose that X is zero-dimensional and has no isolated points. If X is empty the result is trivial; otherwise, by 535Lb, we may suppose that X is a dense subset of $\{0,1\}^{\mathbb{N}}$. This time, let $\mathcal{E}$ be the algebra of open-and-closed subsets of $\{0,1\}^{\mathbb{N}}$. For $E\in\mathcal{E}$, set $\pi E=E\cap D$. Because D is dense in X and therefore in $\{0,1\}^{\mathbb{N}}$, π is an isomorphism between $\mathcal{E}$ and a subalgebra $\mathcal{E}'$ of $\mathcal{P}D$. Fix a Borel probability measure μ on $\{0,1\}^{\mathbb{N}}$ with a strong lifting θ, and let $\mathfrak{A}$ be the measure algebra of μ. Then $A\mapsto(\pi^{-1}A)^\bullet$ is a Boolean homomorphism from $\mathcal{E}'$ to $\mathfrak{A}$; because $\mathfrak{A}$ is Dedekind complete, it extends to a Boolean homomorphism $\psi:\mathcal{P}D\to\mathfrak{A}$. For $E\subseteq\{0,1\}^{\mathbb{N}}$, set $\tilde{\pi}E=E\cap X$. Then $\phi=\tilde{\pi}\theta\psi$ is a Boolean homomorphism from $\mathcal{P}D$ to $\mathcal{B}(X)$. If $A\subseteq D$ and $x\in X\setminus\overline{A}$, then there is an $E\in\mathcal{E}$ such that $x\in E$ and $A\cap E=\emptyset$, in which case

$$\phi A\subseteq\theta\psi A\subseteq\theta\psi(D\setminus E)=\theta(\{0,1\}^{\mathbb{N}}\setminus E)^\bullet=\{0,1\}^{\mathbb{N}}\setminus E,$$

and $x\notin\phi A$. As x and A are arbitrary, ϕ is a suitable homomorphism.

case 3 Suppose that X has no isolated points. Write $\mathfrak{T}$ for the given topology on X. By 535La, there is a finer zero-dimensional separable metrizable topology $\mathfrak{S}$ on X, with the same Borel sets, such that $\mathfrak{T}$ is a π-base for $\mathfrak{S}$. If $V\in\mathfrak{S}$ is non-empty, there is a non-empty $U\in\mathfrak{T}$ such that $U\subseteq V$, and $D\cap V\supseteq D\cap U$ is non-empty; so D is $\mathfrak{S}$-dense. By case 2, there is a Boolean homomorphism $\phi:\mathcal{P}D\to\mathcal{B}(X,\mathfrak{S})$ such that $\phi A\subseteq\overline{A}^{\mathfrak{S}}$ for every $A\subseteq D$. As $\mathcal{B}(X,\mathfrak{S})=\mathcal{B}(X,\mathfrak{T})$, and $\overline{A}^{\mathfrak{S}}\subseteq\overline{A}^{\mathfrak{T}}$ for every $A\subseteq X$, this ϕ satisfies the conditions required.

general case In general, let $\mathcal{G}$ be the family of countable open subsets of X, and $G_0=\bigcup\mathcal{G}$; because X is separable and metrizable, therefore hereditarily Lindelöf, G_0 is countable. Set $Z=X\setminus G_0$, and let

D_0 be a countable dense subset of Z; set $Y = D \cup G_0 \cup D_0$. By case 1, there is a Boolean homomorphism $\phi_0 : \mathcal{P}D \to \mathcal{P}Y$ such that $\phi_0 A \subseteq \overline{A}$ for every $A \subseteq D$. By case 3, there is a Boolean homomorphism $\phi_1 : \mathcal{P}(Y \cap Z) \to \mathcal{B}(Z)$ such that $\phi_1 B \subseteq \overline{B}$ for every $B \subseteq Y \cap Z$. Now set

$$\phi A = (\phi_0 A \setminus Z) \cup \phi_1(Z \cap \phi_0 A)$$

for every $A \subseteq D$. Then ϕ is a Boolean homomorphism from $\mathcal{P}D$ to $\mathcal{B}(X)$; and if $A \subseteq D$, then

$$\phi A \subseteq \phi_0 A \cup \phi_1(Z \cap \phi_0 A) \subseteq \overline{A} \cup \overline{Z \cap \overline{A}} = \overline{A},$$

so in this case also we have a homomorphism of the kind we need.

535N Theorem Suppose there is a metrizable space X with a non-zero atomless semi-finite tight Borel measure μ which has a lifting. Then whenever Y is a metrizable space and ν is a strictly positive σ-finite Borel measure on Y, ν has a strong lifting.

proof (a) Let ϕ be a lifting for μ. Then there is a Borel set $E \subseteq X$, of non-zero finite measure, such that $E \cap G \subseteq \phi G$ for every open $G \subseteq X$. **P** Let $L_0 \subseteq X$ be a compact set of non-zero measure; then L_0 has a countable base $\mathcal{U}$; set $E = L_0 \cap \phi L_0 \setminus \bigcup_{U \in \mathcal{U}}(U \triangle \phi U)$, so that $\mu E = \mu L_0 \in \,]0, \infty[$. If $G \subseteq X$ is open and $x \in E \cap G$, then there is a $U \in \mathcal{U}$ such that $x \in U \subseteq G$. Since $x \in E \cap U$, $x \in \phi U \subseteq \phi G$. As x and G are arbitrary, we have an appropriate E. **Q**

(b) Let λ be any strictly positive atomless Radon measure on $\{0,1\}^{\mathbb{N}}$. There is a compact set $K \subseteq E$ such that K, with its induced topology and measure, is isomorphic to $\{0,1\}^{\mathbb{N}}$ with its usual topology and a non-zero multiple of λ, by 535K. In particular, K is self-supporting. By 535J, the subspace measure on K has a strong Borel lifting. It follows at once that λ has a strong Borel lifting.

(c) Refining (b) slightly, we see that if $Y \subseteq \{0,1\}^{\mathbb{N}}$ is a dense set and λ is a strictly positive atomless totally finite Borel measure on Y, then λ has a strong lifting. **P** There is a Radon measure ν on $\{0,1\}^{\mathbb{N}}$ such that $\nu E = \lambda(Y \cap E)$ for every Borel set $E \subseteq \{0,1\}^{\mathbb{N}}$ (416F); because λ is atomless, so is ν; because λ is strictly positive and Y is dense, ν is strictly positive. So ν has a strong Borel lifting ψ_0 say. If E, $F \in \mathcal{B}(\{0,1\}^{\mathbb{N}})$ and $E \cap Y = F \cap Y$, then $\nu(E \triangle F) = 0$ and $\psi_0 E = \psi_0 F$; we therefore have a Boolean homomorphism $\psi : \mathcal{B}(Y) \to \mathcal{B}(Y)$ defined by setting $\psi(E \cap Y) = Y \cap \psi_0 E$ for every Borel set $E \subseteq \{0,1\}^{\mathbb{N}}$. It is easy to check that ψ is a lifting for λ, and it is strong because if $G \subseteq \{0,1\}^{\mathbb{N}}$ is open then $\psi(Y \cap G) = Y \cap \psi_0 G \subseteq Y \cap G$. **Q**

(d) If $(Y, \mathfrak{S})$ is a separable metrizable space with a strictly positive atomless totally finite Borel measure ν, then ν has a strong lifting. **P** If $Y = \emptyset$ the result is trivial. Otherwise, by 535La, there is a finer separable metrizable topology $\mathfrak{S}'$ on Y with the same Borel sets such that $\mathfrak{S}$ is a π-base for $\mathfrak{S}'$. Because $\mathfrak{S}$ and $\mathfrak{S}'$ have the same Borel sets, ν is a Borel measure for $\mathfrak{S}'$; because every non-empty $\mathfrak{S}'$-open set includes a non-empty $\mathfrak{S}$-open set, ν is strictly positive for $\mathfrak{S}'$; because ν is atomless, Y has no $\mathfrak{S}'$-isolated points. By 535Lb, $(Y, \mathfrak{S}')$ is homeomorphic to a dense subset of $\{0,1\}^{\mathbb{N}}$; by (c) above, ν has a lifting ϕ which is strong with respect to the topology $\mathfrak{S}'$. But now ϕ is still strong with respect to the coarser topology $\mathfrak{S}$. **Q**

(e) Now suppose that Y is a separable metrizable space with a strictly positive totally finite Borel measure ν. Then ν has a strong lifting. **P** The set $D = \{y : \nu\{y\} > 0\}$ is countable. If D is empty, then the result is immediate from (d) applied to a scalar multiple of ν. (If $\nu Y = 0$ then $Y = \emptyset$ and the result is trivial.) Otherwise, let $\nu_{Y \setminus D}$ be the subspace measure; then $\nu_{Y \setminus D}$ is a totally finite Borel measure on $Y \setminus D$, and is zero on singletons, so must be atomless. Because $Y \setminus D$ is hereditarily Lindelöf, $\nu_{Y \setminus D}$ is τ-additive; let Z be its support, and ν_Z the subspace measure on Z. Then ν_Z has a strong Borel lifting ψ_0, by (d) again. Next, Z is relatively closed in $Y \setminus D$, so is expressible as $F \setminus D$ for some closed set $F \subseteq Y$. If $x \in Y \setminus F$ and G is an open set containing x, then $G' = G \setminus F$ is a non-empty open set, so has non-zero measure, while $\nu_{Y \setminus D}(G' \setminus D) = 0$; accordingly $G' \cap D \neq \emptyset$. This shows that $Y \setminus F \subseteq \overline{D}$ so D is dense in $Y \setminus Z$. Now 535M (with (b) above) tells us that there is a Boolean homomorphism $\psi_1 : \mathcal{P}D \to \mathcal{B}(Y \setminus Z)$ such that $\psi_1 A \subseteq \overline{A}$ for every $A \subseteq D$. Define $\psi : \mathcal{B}(Y) \to \mathcal{B}(Y)$ by setting

$$\psi E = \psi_0(E \cap Z) \cup (E \cap D) \cup (\psi_1(E \cap D) \setminus D)$$

for every Borel set $E \subseteq Y$. ψ is a Boolean homomorphism because ψ_0 and ψ_1 are. If $\nu E = 0$, then $\nu_Z(E \cap Z) = 0$ and $E \cap D = \emptyset$, so $\psi E = \emptyset$. For any $E \in \mathcal{B}(Y)$, $\psi_0(E \cap Z) \triangle (E \cap Z)$ and $Y \setminus (D \cup Z)$ are negligible, so $E \triangle \psi E$ is negligible. Thus ψ is a lifting for ν. Finally, for any E,

$$\psi E \subseteq \psi_0(E \cap Z) \cup (E \cap D) \cup \psi_1(E \cap D) \subseteq \overline{E},$$

so ψ is a strong lifting. **Q**

(f) Finally, if Y is a metrizable space and ν is a strictly positive σ-finite Borel measure on Y, then Y must be ccc, therefore separable; and there is a totally finite Borel measure ν' with the same null ideal as ν, so that ν' has a strong lifting, by (e), which is also a strong lifting for ν.

535O Linear liftings Let (X, Σ, μ) be a measure space, with measure algebra $\mathfrak{A}$. Write $\mathcal{L}^\infty(\Sigma)$ for the space of bounded Σ-measurable real-valued functions on X. A **linear lifting** for μ is

either a positive linear operator $T : L^\infty(\mu) \to \mathcal{L}^\infty(\Sigma)$ such that $T(\chi X^\bullet) = \chi X$ and $(Tu)^\bullet = u$ for every $u \in L^\infty(\mu)$

or a positive linear operator $S : \mathcal{L}^\infty(\Sigma) \to \mathcal{L}^\infty(\Sigma)$ such that $S(\chi X) = \chi X$, $Sf = 0$ whenever $f = 0$ a.e. and $Sf =_{\text{a.e.}} f$ for every $f \in \mathcal{L}^\infty(\Sigma)$.

As with liftings (see 341A-341B) we have a direct correspondence between the two kinds of linear operator; given T as in the first formulation, we can set $Sf = T(f^\bullet)$ for every $f \in \mathcal{L}^\infty(\Sigma)$; given S as in the second formulation, we can set $T(f^\bullet) = Sf$ for every $f \in \mathcal{L}^\infty(\Sigma)$.

If $\theta : \mathfrak{A} \to \Sigma$ is a lifting for μ, then we have a corresponding Riesz homomorphism $T : L^\infty(\mathfrak{A}) \to \mathcal{L}^\infty(\Sigma)$ such that $T(\chi a) = \chi(\theta a)$ for every $a \in \mathfrak{A}$, by 363F. Identifying $L^\infty(\mathfrak{A})$ with $L^\infty(\mu)$ as in 363I, we see that T can be regarded as a linear lifting. Of course the associated linear operator from $\mathcal{L}^\infty(\Sigma)$ to itself is the operator derived by the process of 363F from the Boolean homomorphism $E \mapsto \theta E^\bullet : \Sigma \to \Sigma$.

As in 535Aa, I will say that a **Borel linear lifting** is a linear lifting such that all its values are Borel measurable functions; similarly, a **Baire linear lifting** is a linear lifting such that all its values are Baire measurable functions.

535P I give a sample result to show that for some purposes linear liftings are adequate.

Proposition Let (X, Σ, μ) be a countably compact measure space such that Σ is countably generated, (Y, T, ν) a σ-finite measure space with a linear lifting, and $f : X \to Y$ an inverse-measure-preserving function. Then there is a disintegration $\langle \mu_y \rangle_{y \in Y}$ of μ over ν, consistent with f, such that $y \mapsto \mu_y E$ is a T-measurable function for every $E \in \Sigma$.

proof I use the method of 452H-452I.

(a) Suppose first that μ and ν are probability measures. Let $S : L^\infty(\nu) \to \mathcal{L}^\infty(\mathrm{T})$ be a linear lifting for ν. Let $T : L^\infty(\mu) \to L^\infty(\nu)$ be the positive linear operator defined by saying that $\int_F Tu = \int_{f^{-1}[F]} u$ whenever $u \in L^\infty(\mu)$ and $F \in \mathrm{T}$ (as in part (a) of the proof of 452I). For $y \in Y$ and $E \in \Sigma$, set

$$\psi_y E = (ST(\chi E^\bullet))(y)$$

as in part (b) of the proof of 452H. Because μ is countably compact, we can use the argument of 452H to see that we have a family $\langle \mu'_y \rangle_{y \in Y}$ of totally finite measures on X such that, for any $E \in \Sigma$, $\mu'_y E = \psi_y E$ for almost every $y \in Y$.

Let $\mathcal{H}$ be a countable subalgebra of Σ such that Σ is the σ-algebra of sets generated by $\mathcal{H}$. Set $Y_0 = \{y : \mu'_y H = \psi_y H$ for every $H \in \mathcal{H}\}$, so that Y_0 is conegligible; let $Y_1 \subseteq Y_0$ be a measurable conegligible set; set $\mu_y = \mu'_y$ for $y \in Y_1$, and take μ_y to be the zero measure on X for $y \in Y \setminus Y_1$. If $H \in \mathcal{H}$, then

$$\mu_y H = \psi_y H = ST(\chi H^\bullet)(y)$$

for every $y \in Y_1$, so $y \mapsto \mu_y H$ is T-measurable; also, of course,

$$\int_F \mu_y H \nu(dy) = \int_F ST(\chi H^\bullet) d\nu = \int_F T(\chi H^\bullet) = \int_{f^{-1}[F]} \chi H^\bullet = \mu(H \cap f^{-1}[F]).$$

Now consider the family $\mathcal{E}$ of those $E \in \Sigma$ such that $y \mapsto \mu_y E$ is T-measurable and $\int_F \mu_y E \nu(dy) = \mu(E \cap f^{-1}[F])$ for every $F \in \mathrm{T}$. This is a Dynkin class including $\mathcal{H}$, so is the whole of Σ; which is what we need to know.

(b) In general, if $\nu Y = 0$, the result is trivial. Otherwise, apply (a) to a suitable pair of indefinite-integral measures over μ and ν, as in part (c) of the proof of 452I.

535Q Proposition Let (X, Σ, μ) and (Y, T, ν) be probability spaces, and λ the c.l.d. product measure on $X \times Y$. Suppose that $\lambda \restriction \Sigma\widehat{\otimes}\mathrm{T}$ has a linear lifting. Then μ has a linear lifting.

proof Let $S : \mathcal{L}^\infty(\Sigma\widehat{\otimes}\mathrm{T}) \to \mathcal{L}^\infty(\Sigma\widehat{\otimes}\mathrm{T})$ be a linear lifting for $\lambda \restriction \Sigma\widehat{\otimes}\mathrm{T}$. For $h \in \mathcal{L}^\infty(\Sigma\widehat{\otimes}\mathrm{T})$, set $(Uh)(x) = \int h(x, y)\nu(dy)$ for every $x \in X$; by 252P, Uh is well-defined and is Σ-measurable. Now U is a positive linear operator from $\mathcal{L}^\infty(\Sigma\widehat{\otimes}\mathrm{T})$ to $\mathcal{L}^\infty(\Sigma)$, and $U(\chi(X \times Y)) = \chi X$, because $\nu Y = 1$. Note that

$$\int |Uh| d\mu \le \int U|h| d\mu = \iint |h(x, y)|\nu(dy)\mu(dx) = \int |h| d\lambda$$

for every $h \in \mathcal{L}^\infty(\Sigma\widehat{\otimes}\mathrm{T})$ (252P again). Next, for $f \in \mathcal{L}^\infty(\Sigma)$ set $(V f)(x, y) = f(x)$ for every $x \in X$ and $y \in Y$, so that V is a positive linear operator from $\mathcal{L}^\infty(\Sigma)$ to $\mathcal{L}^\infty(\Sigma\widehat{\otimes}\mathrm{T})$.

Consider $S_1 = USV : \mathcal{L}^\infty(\Sigma) \to \mathcal{L}^\infty(\Sigma)$. This is a positive linear operator and $S_1(\chi X) = \chi X$. If $f \in \mathcal{L}^\infty(\Sigma)$ and $f = 0$ μ-a.e., then $V f = 0$ λ-a.e. and $SV f = 0$, so $S_1 f = 0$. For any $f \in \mathcal{L}^\infty(\Sigma)$,

$$\int |f - S_1 f| d\mu = \int |f - USV f| d\mu = \int |UV f - USV f| d\mu \le \int |V f - SV f| d\lambda = 0,$$

so $f =_{\text{a.e.}} S_1 f$; thus S_1 is a linear lifting for μ.

535R Proposition Write ν_ω^2 for the usual measure on $(\{0, 1\}^\omega)^2$, and $\mathrm{T}_\omega^{(2)}$ for its domain. Suppose that ν_κ has a Baire linear lifting for some $\kappa \ge \mathfrak{c}^{++}$. Then there is a Borel linear lifting S for ν_ω^2 which respects coordinates in the sense that if $f \in \mathcal{L}^\infty(\mathrm{T}_\omega^{(2)})$ is determined by a single coordinate, then Sf is determined by the same coordinate.

proof Because $(\{0, 1\}^\kappa, \nu_\kappa)$ is isomorphic, as topological measure space, to $(\{0, 1\}^{\kappa\times\omega}, \nu_{\kappa\times\omega})$, the latter has a Baire linear lifting S_0 say. For $I \subseteq \kappa$, let T_I be the σ-algebra of Baire subsets of $\{0, 1\}^{\kappa\times\omega}$ determined by coordinates in $I \times \omega$. Then $\#(\mathrm{T}_I) \le \mathfrak{c}$ whenever $\#(I) \le \mathfrak{c}$. Also $\mathcal{B}\mathfrak{a}(\{0, 1\}^{\kappa\times\omega}) = \bigcup\{\mathrm{T}_I : I \in [\kappa]^{\le\omega}\}$ (4A3N). It follows that for every $\xi < \kappa$ there is a set $I_\xi \subseteq \kappa$, with cardinal at most $\mathfrak{c}$, such that $\xi \in I_\xi$ and $S_0(\chi E)$ is T_{I_ξ}-measurable whenever $E \in \mathrm{T}_{I_\xi}$; so that $S_0 f$ is T_{I_ξ}-measurable whenever $f : \{0, 1\}^{\kappa\times\omega} \to \mathbb{R}$ is bounded and T_{I_ξ}-measurable.

Because $\kappa \ge \mathfrak{c}^{++}$, there are ξ, $\eta < \kappa$ such that $\xi \notin I_\eta$ and $\eta \notin I_\xi$ (5A1J(a-iii)). Set $J = \{\xi\} \times \omega$, $K = \{\eta\} \times \omega$ and $L = (\kappa \times \omega) \setminus (J \cup K)$, so that $\{0, 1\}^{\kappa\times\omega}$ can be identified with $\{0, 1\}^{J\cup K} \times \{0, 1\}^L$ and $\mathcal{B}\mathfrak{a}(\{0, 1\}^{\kappa\times\omega})$ with $\mathcal{B}\mathfrak{a}(\{0, 1\}^{J\cup K})\widehat{\otimes}\mathcal{B}\mathfrak{a}(\{0, 1\}^L)$. Set $(V f)(w, z) = f(w)$ when $f : \{0, 1\}^{J\cup K} \to \mathbb{R}$ is a function, $w \in \{0, 1\}^{J\cup K}$ and $z \in \{0, 1\}^L$; and $(Uh)(w) = \int h(w, z)\nu_L(dz)$ when $h : \{0, 1\}^{\kappa\times\omega} \to \mathbb{R}$ is a bounded Baire measurable function and $w \in \{0, 1\}^{J\cup K}$. Then $S_1 = US_0V$ is a Baire linear lifting for $\nu_{J\cup K}$, just as in 535Q. Moreover, if $f : \{0, 1\}^{J\cup K} \to \mathbb{R}$ is a bounded Baire measurable function determined by coordinates in J, in the sense that $f(x, y) = f(x, y')$ whenever $x \in \{0, 1\}^J$ and y, $y' \in \{0, 1\}^K$, then $S_1 f$ is determined by coordinates in J. **P** $V f$ is determined by coordinates in J, so $S_0V f$ is determined by coordinates in $I_\xi \times \omega$; since $K \cap (I_\xi \times \omega)$ is empty, $S_0V f(x, y, z) = S_0V f(x, y', z)$ for all $x \in \{0, 1\}^J$, $z \in \{0, 1\}^L$ and y, $y' \in \{0, 1\}^K$. It follows at once that

$$S_1 f(x, y) = \int S_0V f(x, y, z)\nu_L(dz) = \int S_0V f(x, y', z)\nu_L(dz) = S_1 f(x, y')$$

whenever $x \in \{0, 1\}^J$ and y, $y' \in \{0, 1\}^K$. **Q** Similarly, if $f : \{0, 1\}^{J\cup K} \to \mathbb{R}$ is a bounded Baire measurable function determined by coordinates in K, then $S_1 f$ is determined by coordinates in K.

Now we can transfer S_1 from $\{0, 1\}^{J\cup K} \cong \{0, 1\}^J \times \{0, 1\}^K$ to $(\{0, 1\}^\omega)^2$, and we shall obtain a Baire (or Borel) linear lifting S for ν_ω^2 which respects coordinates.

535X Basic exercises (a) Let (X, Σ, μ) be a measure space with a lifting, and A any subset of X. Show that if A has a measurable envelope then the subspace measure μ_A has a lifting. (*Hint*: 322I.)

(b) Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of measure spaces, with $\mu_i X_i > 0$ for every $i \in I$, and (X, Σ, μ) their direct sum. Show that μ has a lifting iff every μ_i has a lifting.

(c) Let $\mathfrak{A}$ be a Boolean algebra and I a proper ideal of $\mathfrak{A}$. Suppose that $\sup A$ is defined in $\mathfrak{A}$ and belongs to I whenever $A \subseteq I$ and $\#(A) < \#(\mathfrak{A})$. Show that there is a Boolean homomorphism $\theta : \mathfrak{A}/I \to \mathfrak{A}$ such that $(\theta b)^\bullet = b$ for every $b \in \mathfrak{A}/I$. (*Hint*: enumerate $\mathfrak{A}$ as $\{a_\xi : \xi < \kappa\}$; let $\mathfrak{C}_\xi$ be the subalgebra of $\mathfrak{A}/I$ generated by $\{a_\eta^\bullet : \eta < \xi\}$; construct $\theta \restriction \mathfrak{C}_\xi$ inductively by choosing $\theta a_\xi^\bullet$ appropriately.)

(d) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and I a proper ideal of $\mathfrak{A}$. Show that if the quotient Boolean algebra $\mathfrak{A}/I$ is tightly ω_1-filtered, then there is a Boolean homomorphism $\theta : \mathfrak{A}/I \to \mathfrak{A}$ such that $(\theta b)^\bullet = b$ for every $b \in \mathfrak{A}/I$.

(e) Let $\mathfrak{A}$ be a tightly ω_1-filtered Boolean algebra, $\mathfrak{B}$ a Dedekind σ-complete Boolean algebra and $\mathfrak{A}_0$ a countable subalgebra of $\mathfrak{A}$. Show that every Boolean homomorphism from $\mathfrak{A}_0$ to $\mathfrak{B}$ extends to a Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

(f) Let $\mathfrak{A}$, $\mathfrak{B}$ be Boolean algebras such that $\sup A$ is defined in $\mathfrak{A}$ whenever $A \subseteq \mathfrak{A}$ and $\#(A) < \#(\mathfrak{B})$, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a surjective Boolean homomorphism. Suppose that $\underline{\theta} : \mathfrak{B} \to \mathfrak{A}$ is such that $\underline{\theta}0 = 0$, $\pi\underline{\theta}b \subseteq b$ for every $b \in \mathfrak{B}$ and $\underline{\theta}(b \cap c) = \underline{\theta}b \cap \underline{\theta}c$ for all b, $c \in \mathfrak{B}$. Show that there is a Boolean homomorphism $\theta : \mathfrak{B} \to \mathfrak{A}$ such that $\underline{\theta}b \subseteq \theta b$ and $\pi\theta b = b$ for every $b \in \mathfrak{B}$.

(g) Suppose that $\mathfrak{c} \le \omega_2$ and $\mathrm{FN}(\mathcal{P}\mathbb{N}) = \omega_1$. Show that ν_κ has a strong Baire lifting whenever $\kappa \le \omega_2$. (*Hint*: let $\langle e_\xi \rangle_{\xi<\kappa}$ be the standard generating family for $\mathfrak{B}_\kappa$. Show that there is a tight ω_1-filtration $\langle a_\eta \rangle_{\eta<\zeta}$ of $\mathfrak{B}_\kappa$ such that for every $\xi < \kappa$ there is an $\eta < \zeta$ such that the closed subalgebras generated by $\{e_\delta : \delta < \xi\}$ and $\{a_\delta : \delta < \eta\}$ are the same and $e_\xi = a_\eta$.)

(h) Suppose that $\mathfrak{c} \le \omega_2$ and $\mathrm{FN}(\mathcal{P}\mathbb{N}) = \omega_1$. Show that whenever X is a separable metrizable space and $D \subseteq X$ is a dense set, there is a Boolean homomorphism $\phi : \mathcal{P}D \to \mathcal{B}(X)$ such that $\phi A \subseteq \overline{A}$ for every $A \subseteq D$.

(i) Let (X, Σ, μ) be a measure space. Show that a linear lifting $S : \mathcal{L}^\infty(\Sigma) \to \mathcal{L}^\infty(\Sigma)$ of μ corresponds to a lifting iff it is 'multiplicative', that is, $S(f \times g) = Sf \times Sg$ for all f, $g \in \mathcal{L}^\infty(\Sigma)$.

(j) Let (X, Σ, μ) be a strictly localizable measure space with non-zero measure. Suppose that ν_κ has a Baire linear lifting for every infinite cardinal κ such that the Maharam-type-κ component of the measure algebra of μ is non-zero. Show that μ has a linear lifting.

(k) Let (X, Σ, μ) be a probability space such that whenever $\mathcal{E} \subseteq \Sigma$, $\#(\mathcal{E}) \le \mathfrak{c}$ and $\bigcup\mathcal{E}$ is negligible, then $\bigcup\mathcal{E} \in \Sigma$. Show that μ has a linear lifting. (*Hint*: 363Yf.)

(l) Let (Y, T, ν) be a σ-finite measure space with a linear lifting, Z a set, Υ a countably generated σ-algebra of subsets of Z, and μ a measure with domain $\mathrm{T}\widehat{\otimes}\Upsilon$ such that ν is the marginal measure of μ on Y and the marginal measure of μ on Z is countably compact. Show that there is a family $\langle \mu_y \rangle_{y \in Y}$ of measures with domain Υ such that $y \mapsto \mu_y H$ is a T-measurable function for every $H \in \Upsilon$ and $\mu W = \int \mu_y W[\{y\}]\nu(dy)$ for every $W \in \mathrm{T}\widehat{\otimes}\Upsilon$.

(m) Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be τ-additive topological probability spaces, and λ the τ-additive product measure on $X \times Y$ (417F[5]). Suppose that λ has a Borel linear lifting and that μ is inner regular with respect to the Borel sets. Show that μ has a Borel linear lifting.

535Y Further exercises (a) Suppose that we are provided with a bijection between $\mathcal{B}(\mathbb{R})$ and ω_1, but are otherwise not permitted to use the axiom of choice. Show that we can construct a Borel lifting for Lebesgue measure.

(b) Suppose that for every cardinal κ there is a Baire linear lifting for ν_κ. Show that for every $n \in \mathbb{N}$ there is a Borel linear lifting S for Lebesgue measure on $[0,1]^n$ which (α) respects coordinates in the sense that if $f : [0,1]^n \to \mathbb{R}$ is a bounded measurable function determined by coordinates in $I \subseteq n$, then Sf also is determined by coordinates in I (β) is symmetric in the sense that if $\rho : n \to n$ is any permutation and $(\hat\rho f)(x) = f(x\rho)$ for $x \in [0,1]^n$ and $f : [0,1]^n \to \mathbb{R}$, then S commutes with $\hat\rho$. (*Hint*: 5A1Jb.)

(c) Let (X, Σ, μ) be a countably compact measure space, (Y, T, ν) a σ-finite measure space with a linear lifting, and $f : X \to Y$ an inverse-measure-preserving function. Suppose there is a family $\mathcal{H} \subseteq \Sigma$ such that Σ is the σ-algebra of sets generated by $\mathcal{H}$ and $\#(\mathcal{H}) < \operatorname{add}\nu$. Show that there is a disintegration $\langle \mu_y \rangle_{y \in Y}$ of μ over ν, consistent with f, such that $y \mapsto \mu_y E$ is a T-measurable function for every $E \in \Sigma$.

[5]Formerly 417G.

(d) (Törnquist 11) Let (X, Σ, μ) be a countably separated perfect complete strictly localizable measure space, $\mathfrak{A}$ its measure algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$ of cardinal at most $\min(\operatorname{add}\mathcal{N}, \mathfrak{p})$, where $\mathcal{N}$ is the null ideal of Lebesgue measure on $\mathbb{R}$. Show that there is an action $\bullet$ of G on X such that $\pi \bullet E = \{\pi \bullet x : x \in E\}$ belongs to Σ and $(\pi \bullet E)^\bullet = \pi(E^\bullet)$ whenever $\pi \in G$ and $E \in \Sigma$. (*Hint*: 344C, 425Ya.)

535Z Problems (a) Can it be that every probability space has a lifting?

By 535B, it is enough to consider $(\{0,1\}^\kappa, \mathcal{B}\mathfrak{a}(\{0,1\}^\kappa), \nu_\kappa \restriction \mathcal{B}\mathfrak{a}(\{0,1\}^\kappa))$ where κ is a cardinal. Since Mokobodzki's theorem (535Eb) deals with $\kappa \le \omega_2$ when $\mathfrak{c} = \omega_1$, the key case to consider seems to be $\kappa = \omega_3$.

(b) Suppose that $\mathfrak{c} \ge \omega_3$. Does ν_ω have a Borel lifting?

It is known to be relatively consistent with ZFC to suppose that $\mathfrak{c} = \omega_2$ and that $\mathrm{FN}(\mathcal{P}\mathbb{N}) = \omega_1$ (554G-554H). In this case ν_ω has a Borel lifting (535E(b-ii)). But if $\mathfrak{c} \ge \omega_3$ then $\mathfrak{B}_\omega$ is not tightly ω_1-filtered (518S).

(c) (A.H.Stone) Can there be a countable ordinal ζ and a lifting ϕ of ν_ω such that ϕE is a Borel set, with Baire class at most ζ, for every Borel set $E \subseteq \{0,1\}^\omega$?

The point of this question is that while, subject to the continuum hypothesis, we can almost write down a formula for a Borel lifting for Lebesgue measure (535Ya), the method gives no control over the Baire classes of the sets constructed.

(d) Can there be a strictly positive Radon probability measure of countable Maharam type which does not have a strong lifting? (See 453G, 453N, 535I, 535Xg.)

(e) Is there a probability space which has a linear lifting but no lifting?

(f) Can there be a Borel linear lifting for the usual measure on $(\{0,1\}^\omega)^2$ which respects coordinates in the sense of 535R?

It seems possible that there is a proof in ZFC that there is no such lifting; in which case 535R shows that we should have a negative answer to (a).

535 Notes and comments For a fuller account of this topic, see Burke 93.

Neumann & Stone 1935 used a direct construction along the lines of 535Xc to show that if the continuum hypothesis is true then Lebesgue measure has a Borel lifting. The method works equally well for ν_{ω_1}, but for ν_{ω_2} we need a further idea from Mokobodzki 7?; the version I give here is based on Geschke 02, itself derived at some remove from Carlson Frankiewicz & Zbierski 94, who showed that we could have a Borel lifting for Lebesgue measure in a model in which the continuum hypothesis is false (554I).

It is not a surprise that there should be a model of set theory in which Lebesgue measure has no Borel lifting. Nor is it a surprise that the first such model should have been found by S.Shelah (Shelah 83). What does remain surprising is that in most of the vast number of models of set theory which have been studied, we do not know whether there is such a lifting. Only in the familiar case $\mathfrak{c} = \omega_1$, the special combination $\mathfrak{c} = \omega_2 = \mathrm{FN}(\mathcal{P}\mathbb{N})^+$ (535E), and in variations of Shelah's model, do we have definite information. It remains possible that in any model in which $\mathfrak{m} > \omega_1$ or $\mathfrak{c} = \omega_3$ there is no Borel lifting for Lebesgue measure. When we leave the real line, the position is even more open; conceivably it is relatively consistent with ZFC to suppose that every probability space has a lifting, and at least equally believably it is a theorem of ZFC that ν_{ω_3} does not have a Baire lifting.

From 535I we see that ω_2 appears in Losert's example (453N) for a good reason. Once again, it seems to be unknown whether it is consistent to suppose that there is a (completed) strictly positive Radon probability measure with countable Maharam type which has no strong lifting (535Zd). When we come to look for strong Borel liftings, we have some useful information in the separable metrizable case (535N). The result is natural enough. We are used to supposing that Polish spaces are all very much the same, and that point-supported measures are trivial. But because the concept of 'strong' lifting is topological, and cannot easily be reduced to the Borel structure, we have to work a bit; and it seems also that point-supported measures need care (535M).

'Linear liftings' (535O-535R) remain poor relations. I give them house room here partly for completeness and partly because of a slender hope that they will lead us to a solution of 535Za. Of course the match between ω_3 in 535Za and $\mathfrak{c}^{++}$ in 535R may show only a temporarily coincidental frontier of ignorance. Burke & Shelah 92 have shown that it is relatively consistent with ZFC to suppose that ν_ω has no Borel linear lifting.

Version of 20.2.12

536 Alexandra Bellow's problem

In 463Za I mentioned a curious problem concerning pointwise compact sets of continuous functions. This problem is known to be soluble if we are allowed to assume the continuum hypothesis, for instance. Here I present the relevant arguments, with supplementary remarks on 'stable' sets of measurable functions (536E-536F).

536A The problem I recall some ideas from §463. Let (X,Σ,μ) be a measure space, and $\mathcal{L}^0=\mathcal{L}^0(\Sigma)$ the space of all Σ-measurable functions from X to $\mathbb{R}$, so that $\mathcal{L}^0$ is a linear subspace of $\mathbb{R}^X$. On $\mathcal{L}^0$ we have the linear space topologies $\mathfrak{T}_p$ and $\mathfrak{T}_m$ of pointwise convergence and convergence in measure (462Ab, 245Ab). $\mathfrak{T}_p$ is Hausdorff and locally convex; if μ is σ-finite, $\mathfrak{T}_m$ is pseudometrizable. The question, already asked in 463Za, is this: suppose that $K\subseteq\mathcal{L}^0$ is compact for $\mathfrak{T}_p$, and that $\mathfrak{T}_m$ is Hausdorff on K. Does it follow that $\mathfrak{T}_p$ and $\mathfrak{T}_m$ agree on K?

536B Known cases Let (X,Σ,μ) be a σ-finite measure space. Given that $K\subseteq\mathcal{L}^0$ is compact for $\mathfrak{T}_p$, and $\mathfrak{T}_m$ is Hausdorff on K, and

either K is sequentially compact for $\mathfrak{T}_p$

or K is countably tight for $\mathfrak{T}_p$

or K is convex

or X has a topology for which $K\subseteq C(X)$, μ is a strictly positive topological measure, and every function $h\in\mathbb{R}^X$ which is continuous on every relatively countably compact set is continuous

or μ is perfect

or K is stable, in the sense of 465A,

then K is metrizable for $\mathfrak{T}_p$, and $\mathfrak{T}_p$ and $\mathfrak{T}_m$ agree on K (463Cd, 463F, 463G, 463H, 463Lc, 465G).

Now for the new results.

536C Proposition (see Talagrand 84, 9-3-3.) Let (X,Σ,μ) be a probability space such that the π-weight $\pi(\mu)$ of μ is at most $\mathfrak{p}$. If $K\subseteq\mathcal{L}^0$ is $\mathfrak{T}_p$-compact then it is $\mathfrak{T}_m$-compact.

proof (a) For the time being (down to the end of (d) below), suppose that $|f|\le\chi X$ for every $f\in K$. Let $\langle f_i\rangle_{i\in\mathbb{N}}$ be any sequence in K.

(b) For $I\in[\mathbb{N}]^\omega$, write $\limsup_{i\to I}f_i$ for $\inf_{n\in\mathbb{N}}\sup_{i\in I\setminus n}f_i$ and $\liminf_{i\to I}f_i$ for $\sup_{n\in\mathbb{N}}\inf_{i\in I\setminus n}f_i$. Then there is an $I\in[\mathbb{N}]^\omega$ such that $\liminf_{i\to J}f_i=_{\text{a.e.}}\liminf_{i\to I}f_i$ and $\limsup_{i\to J}f_i=_{\text{a.e.}}\limsup_{i\to I}f_i$ for every $J\in[I]^\omega$. **P** (See the proof of 463D.) For I, $J\in[\mathbb{N}]^\omega$ set $\Delta(I)=\int\limsup_{i\to I}f_i-\liminf_{i\to I}f_i$ and say that $J\preccurlyeq I$ if either $J\subseteq I$ or $J\setminus I$ is finite and $I\setminus J$ is infinite. Then $\Delta(J)\le\Delta(I)$ whenever $J\preccurlyeq I$, and any non-increasing sequence in $[\mathbb{N}]^\omega$ has a $\preccurlyeq$-lower bound in $[\mathbb{N}]^\omega$. By 513P, inverted, there is an $I\in[\mathbb{N}]^\omega$ such that $\Delta(J)=\Delta(I)$ whenever $J\preccurlyeq I$, and this I will serve. **Q**

Set $g=\liminf_{i\to I}f_i$ and $h=\limsup_{i\to I}f_i$.

(c) ? Suppose, if possible, that $E=\{x:g(x)<h(x)\}$ is not negligible. Let $\mathcal{H}$ be a coinitial subset of $\Sigma\setminus\mathcal{N}(\mu)$, where $\mathcal{N}(\mu)$ is the null ideal of μ, with cardinal $\pi(\mu)\le\mathfrak{p}$, and $\langle H_\xi\rangle_{\xi<\mathfrak{p}}$ a family running over $\{H:H\in\mathcal{H},\,H\subseteq E\}$. Choose $\langle I_\xi\rangle_{\xi<\mathfrak{p}}$, $\langle x_\xi\rangle_{\xi<\mathfrak{p}}$ and $\langle y_\xi\rangle_{\xi<\mathfrak{p}}$ inductively, as follows. The inductive hypothesis will be that, for any $\xi<\mathfrak{p}$, $\langle I_\eta\rangle_{\eta<\xi}$ is a family of infinite subsets of $\mathbb{N}$ such that $I_\eta\setminus I_\zeta$ is finite whenever $\zeta\le\eta<\xi$. Start with $I_0=I$. For the inductive step to $\xi+1$, where $\xi<\mathfrak{p}$, since $g=_{\text{a.e.}}\liminf_{i\to I_\xi}f_i$, there must be an $x_\xi\in H_\xi\cap E$ such that $g(x_\xi)=\liminf_{i\to I_\xi}f_i(x)$. Let $J\in[I_\xi]^\omega$ be such that $\lim_{i\to J}f_i(x_\xi)=g(x_\xi)$. Now $\limsup_{i\to J}f_i=_{\text{a.e.}}h$, so we can find a $y_\xi\in E\cap H_\xi$ such that $\limsup_{i\to J}f_i(y_\xi)=h(y_\xi)$ and an $I_{\xi+1}\in[J]^{<\omega}$ such that $\lim_{i\to I_{\xi+1}}f_i(y_\xi)=h(y_\xi)$.

For non-zero limit ordinals $\xi<\mathfrak{p}$, let I_ξ be an infinite subset of I such that $I_\xi\setminus I_\eta$ is finite for every $\eta<\xi$.

At the end of the induction, there will be a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$ containing I_ξ for every $\xi < \mathfrak{p}$. Set $f = \lim_{i\to\mathcal{F}} f_i$. Because K is $\mathfrak{T}_p$-compact, $f \in K \subseteq \mathcal{L}^0$. So at least one of the measurable sets $E' = \{x : x \in E,\, g(x) < f(x)\}$ and $E'' = \{x : x \in E,\, f(x) < h(x)\}$ is non-negligible and contains H_ξ for some $\xi < \mathfrak{p}$. Now $I_{\xi+1} \in \mathcal{F}$, so $f(x_\xi) = \lim_{i\to I_{\xi+1}} f_i(x_\xi) = g(x_\xi)$ and $f(y_\xi) = h(y_\xi)$. But this means that $x_\xi \in H_\xi \setminus E''$ and $y_\xi \in H_\xi \setminus E'$, so H_ξ cannot be included in either E' or E''. **X**

(d) So $g =_{\text{a.e.}} h$ and $\{x : g(x) = \lim_{i\to I} f_i(x)\}$ includes the conegligible set $\{x : g(x) = h(x)\}$. We also have a $g_0 \in K$ which is a $\mathfrak{T}_p$-cluster point of $\langle f_i\rangle_{i\in I}$. Of course $g \le g_0 \le h$, and all three must be equal μ-a.e. But this means that $\langle f_i\rangle_{i\in I}$ converges almost everywhere to g_0, and therefore converges in measure to g_0 (245Ec). Now recall that $\langle f_i\rangle_{i\in\mathbb{N}}$ was an arbitrary sequence in K. So we see that every sequence in K has a subsequence which is $\mathfrak{T}_m$-convergent to a point of K. As $\mathfrak{T}_m$ is pseudometrizable, K is $\mathfrak{T}_m$-compact (4A2Le).

(e) This concludes the proof when $|f| \le \chi X$ for every $f \in K$. For the general case, let $\phi : \mathbb{R} \to \,]-1,1[$ be a homeomorphism, and consider $K' = \{\phi f : f \in K\}$. Since $f \mapsto \phi f$ is a $\mathfrak{T}_p$-continuous function from $\mathcal{L}^0$ to itself, K' is $\mathfrak{T}_p$-compact, therefore $\mathfrak{T}_m$-compact, by (a)-(c). Next, $f \mapsto \phi^{-1} f : K' \to K$ is $\mathfrak{T}_m$-continuous. **P** If $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence in K' which is $\mathfrak{T}_m$-convergent to $f \in K'$, and $\langle g_n\rangle_{n\in\mathbb{N}}$ is a subsequence of $\langle f_n\rangle_{n\in\mathbb{N}}$, then $\langle g_n\rangle_{n\in\mathbb{N}}$ has a sub-subsequence $\langle h_n\rangle_{n\in\mathbb{N}}$ converging a.e. to f (245Ka); now $\phi^{-1}h_n$ converges a.e. to $\phi^{-1}f \in K$, so converges in measure to $\phi^{-1}f$. As $\langle g_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\langle \phi^{-1}f_n\rangle_{n\in\mathbb{N}}$ converges in measure to $\phi^{-1}f$. Thus $f \mapsto \phi^{-1}f$ is sequentially continuous for $\mathfrak{T}_m$, therefore continuous (4A2Ld). **Q** So $K = \{\phi^{-1}f : f \in K'\}$ is $\mathfrak{T}_m$-compact, as claimed.

536D Theorem Let (X,Σ,μ) be a probability space, and $\mathcal{L}^0$ the space of Σ-measurable real-valued functions on X. Write $\mathfrak{T}_p$, $\mathfrak{T}_m$ for the topologies of pointwise convergence and convergence in measure on $\mathcal{L}^0$. Suppose that $K \subseteq \mathcal{L}^0$ is $\mathfrak{T}_p$-compact and that $\mu\{x : f(x) \ne g(x)\} > 0$ for any distinct f, $g \in K$, but that K is not $\mathfrak{T}_p$-metrizable.

(a) Every infinite Hausdorff space which is a continuous image of a closed subset of K has a non-trivial convergent sequence.

(b) There is a continuous surjection from a closed subset of K onto $\{0,1\}^{\omega_1}$.

(c) Every infinite compact Hausdorff space of weight at most ω_1 has a non-trivial convergent sequence.

(d) $\mathfrak{c} > \omega_1$.

(e) The Maharam type of μ is at least 2^{ω_1}.

(f) There is a non-negligible measurable set in Σ which can be covered by ω_1 negligible sets.

(g) $\pi(\mu) > \mathfrak{p}$.

(h) $\mathfrak{m}_{\text{countable}} = \omega_1$.

proof For f, $g \in \mathcal{L}^0$ set $\rho(f,g) = \int \min(1,|f-g|)$; then ρ is a pseudometric on $\mathcal{L}^0$ defining $\mathfrak{T}_m$, and $\rho\restriction K\times K$ is a metric on K. Set $\Delta(\emptyset) = 0$, and for non-empty $A \subseteq \mathcal{L}^0$ set $\Delta(A) = \sup\{\rho(\inf L, \sup L) : \emptyset \ne L \in [A]^{<\omega}\}$. Note that if $A \subseteq K$ has more than one member then $\Delta(A) > 0$, and that $\Delta(A) \le \Delta(B)$ whenever $A \subseteq B$.

(a)(i) ? Suppose, if possible, that Z is an infinite Hausdorff space, $K_0 \subseteq K$ is closed, $\phi : K_0 \to Z$ is a continuous surjection and there is no non-trivial convergent sequence in Z. Write $\mathcal{L}$ for the family of closed subsets L of K_0 such that $\phi[L]$ is infinite. Then $L = \bigcap_{n\in\mathbb{N}} L_n$ belongs to $\mathcal{L}$ for every non-increasing sequence $\langle L_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{L}$. **P** $\langle \phi[L_n]\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of infinite closed subsets of Z; because Z is supposed to have no non-trivial convergent sequence, $M = \bigcap_{n\in\mathbb{N}} \phi[L_n]$ is infinite (4A2G(h-i)). Since $\phi[L] = M$ (5A4Cf), $L \in \mathcal{L}$. **Q** By 513P again, there is a $K_1 \in \mathcal{L}$ such that $\Delta(L) = \Delta(K_1)$ for every $L \in \mathcal{L}$ such that $L \subseteq K_1$.

(ii) Now there is no non-trivial convergent sequence in $\phi[K_1]$, so $\phi[K_1]$ cannot be scattered (4A2G(h-ii)), and there is a continuous surjection $\psi : \phi[K_1] \to [0,1]$ (4A2G(j-iv)). Let $M \subseteq \phi[K_1]$ be a closed set such that $\psi[M] = [0,1]$ and $\psi\restriction M$ is irreducible (4A2G(i-i)). Then M is infinite, has a countable π-base and no isolated points (4A2G(i-ii)). Let $K_2 \subseteq \phi^{-1}[M]$ be a closed set such that $\phi[K_2] = M$ and $\phi\restriction K_2$ is irreducible. Then K_2 has a countable π-base, and $\phi[K_2]$ is infinite, so $\Delta[K_2] = \Delta[K_1]$.

Let $\mathcal{V}$ be a countable π-base for the topology of K_2, not containing $\emptyset$. For each $V \in \mathcal{V}$, choose $h_V \in V$. Set $g_0 = \inf_{V\in\mathcal{V}} h_V$, $g_1 = \sup_{V\in\mathcal{V}} h_V$ in $\mathbb{R}^X$. Then g_0 and g_1 are measurable, and

$$\int g_1 - g_0 \ge \Delta(K_2) = \Delta(K_1) > 0.$$

Set $g(x) = \max(\frac{1}{2}(g_0(x) + g_1(x)), g_1(x) - \frac{1}{2})$ for $x \in X$, and

$$E = \{x : g_0(x) < g_1(x)\} = \{x : g(x) < g_1(x)\} = \{x : g_0(x) < g(x)\},$$

so that $\mu E > 0$. For $x \in E$, the set $F_x = \{f : f \in K_2, f(x) \le g(x)\}$ is a proper closed subset of K_2, so there is some $V \in \mathcal{V}$ such that $V \cap F_x = \emptyset$. Because $\mathcal{V}$ is countable, there is a $V \in \mathcal{V}$ such that $D = \{x : x \in E, V \cap F_x = \emptyset\}$ is non-negligible. But now observe that $f(x) > g(x)$ whenever $f \in V$ and $x \in D$, so $h_U(x) > g(x)$ whenever $U \in \mathcal{V}$, $U \subseteq V$ and $x \in D$. Set $\mathcal{V}' = \{U : U \in \mathcal{V}, U \subseteq V\}$, $g_0' = \inf_{U \in \mathcal{V}'} h_U$ and $L = \{f : f \in K_2, g_0' \le f \le g_1\}$. Then $g \le g_0'$ and

$$\{x : x \in X, g_1(x) - g_0'(x) < \min(1, g_1(x) - g_0(x))\} \supseteq D$$

is non-negligible, so

$$\Delta(L) \le \int \min(1, g_1 - g_0') < \int \min(1, g_1 - g_0) = \Delta(K_1).$$

On the other hand, L meets every member of $\mathcal{V}'$, so $L \cap V$ is dense in V and L includes V. Because $\phi \restriction K_2$ is irreducible, $\phi[K_2 \setminus V] \ne M$ and $\phi[L]$ includes the non-empty open subset $M \setminus \phi[K_2 \setminus V]$ of M, which is infinite because M has no isolated points. So $\Delta(L)$ ought to be equal to $\Delta(K_1)$, by the choice of K_1. **X**

Thus (a) is true.

(b) If $\langle f_n \rangle_{n \in \mathbb{N}}$ is a sequence in K which converges at almost every point of X, then any two $\mathfrak{T}_p$-cluster points of $\langle f_n \rangle_{n \in \mathbb{N}}$ must be equal a.e. and therefore equal, so $\langle f_n \rangle_{n \in \mathbb{N}}$ is $\mathfrak{T}_p$-convergent (5A4Ce).

? Suppose, if possible, that there is no continuous surjection from a closed subset of K onto $\{0,1\}^{\omega_1}$. Then 463D tells us that every sequence in K has a subsequence which is convergent almost everywhere, therefore convergent. So K is sequentially compact, which is impossible, as noted in 536B. **X**

(c) Since $[0,1]$ is a continuous image of $\{0,1\}^{\mathbb{N}}$, $[0,1]^{\omega_1}$ is a continuous image of $\{0,1\}^{\omega_1 \times \mathbb{N}} \cong \{0,1\}^{\omega_1}$ and therefore of a closed subset of K. If Z is an infinite compact Hausdorff space of weight at most ω_1, it is homeomorphic to a closed subset of $[0,1]^{\omega_1}$ (5A4Cc) and therefore to a continuous image of a closed subset of K. By (a), Z must have a non-trivial convergent sequence.

(d) Since $\beta\mathbb{N}$ has weight $\mathfrak{c}$ (5A4Ia), is infinite, but has no non-trivial convergent subsequence (4A2I(b-v)), we must have $\omega_1 < \mathfrak{c}$.

(e)(i) If F_1, F_2 are disjoint non-empty $\mathfrak{T}_p$-closed subsets of K, then $\rho(F_1, F_2) > 0$. **P?** Otherwise, there are sequences $\langle f_n \rangle_{n \in \mathbb{N}}$ in F_1, $\langle g_n \rangle_{n \in \mathbb{N}}$ in F_2 such that $\rho(f_n, g_n) \le 2^{-n}$ for every $n \in \mathbb{N}$. Let $\mathcal{F}$ be any non-principal ultrafilter on $\mathbb{N}$ and set $f = \lim_{n \to \mathcal{F}} f_n$, $g = \lim_{n \to \mathcal{F}} g_n$, taking the limits in K for the topology $\mathfrak{T}_p$. Then, for any $n \in \mathbb{N}$,

$$\{x : |f(x) - g(x)| > 2^{-n}\} \subseteq \bigcup_{i \ge 2n} \{x : |f_i(x) - g_i(x)| > 2^{-n}\}$$

has measure at most $\sum_{i=2n}^{\infty} 2^{-i+n} = 2^{-n+1}$, so $f =_{\text{a.e.}} g$ and $f = g$; but $f \in F_1$ and $g \in F_2$, so this is impossible. **XQ**

(ii) By (b), there are a closed subset K_0 of K and a continuous surjection $\psi : K_0 \to \{0,1\}^{\omega_1}$. For $\xi < \omega_1$, set $F_\xi = \{f : f \in K_0, \psi(f)(\xi) = 0\}$, $F_\xi' = \{f : f \in K_0, \psi(f)(\xi) = 1\}$; then $\rho(F_\xi, F_\xi') > 0$. There must therefore be a $\delta > 0$ such that $C = \{\xi : \rho(F_\xi, F_\xi') \ge \delta\}$ is uncountable. For each $D \subseteq C$, choose $h_D \in K_0$ such that $\psi(h_D) \restriction C = \chi D$. Then $\rho(h_D, h_{D'}) \ge \delta$ for all distinct D, $D' \subseteq C$. Thus $A = \{h_D^\bullet : D \subseteq C\}$ is a subset of $L^0 = L^0(\mu)$, of cardinal 2^{ω_1}, such that any two members of A are distance at least δ apart for the metric on L^0 corresponding to ρ. Accordingly the cellularity and topological density of L^0 are at least 2^{ω_1}; by 529Bb, the Maharam type of μ is at least 2^{ω_1}.

(f)(i) By (b), there is a continuous surjection $\psi : K_0 \to \{0,1\}^{\omega_1}$ where $K_0 \subseteq K$ is closed. Let Q be the set of pairs (F, C) such that $F \subseteq K_0$ is closed, $C \subseteq \omega_1$ is closed and cofinal and $\{\psi(f) \restriction C : f \in F\} = \{0,1\}^C$. If $\langle (F_n, C_n) \rangle_{n \in \mathbb{N}}$ is a non-increasing sequence in Q, then it has a lower bound in Q. **P** Set $F = \bigcap_{n \in \mathbb{N}} F_n$ and $C = \bigcap_{n \in \mathbb{N}} C_n$. Then for any $z \in \{0,1\}^C$ and $n \in \mathbb{N}$ there is an $f_n \in F_n$ such that $\psi(f_n) \restriction C = z$; now take a $\mathfrak{T}_p$-cluster point f of $\langle f_n \rangle_{n \in \mathbb{N}}$, and see that $f \in F$ and that $\psi(f) \restriction C = z$. As z is arbitrary, $(F, C) \in Q$. **Q** By 513P once more, there is a member (K_1, C^*) of Q such that $\Delta(F) = \Delta(K_1)$ whenever $(F, C) \in Q$, $F \subseteq K_1$ and $C \subseteq C^*$. Now C^* is order-isomorphic to ω_1 and its order topology agrees with the subspace topology induced by the order topology of ω_1 (4A2Rm). Let $\theta : \omega_1 \to C^*$ be an order-isomorphism and set

$\psi_1(f) = \psi(f)\theta$ for $f \in K_1$. Then $\psi_1 : K_1 \to \{0,1\}^{\omega_1}$ is a continuous surjection, and if $F \subseteq K_1$ is closed, $C \subseteq \omega_1$ is closed and cofinal and $\{\psi_1(f)\restriction C : f \in F\} = \{0,1\}^C$, then $(F, \theta[C]) \in Q$ so $\Delta(F) = \Delta(K_1)$.

(ii) Let $K_2 \subseteq K_1$ be a compact set such that $\psi_1\restriction K_2$ is an irreducible surjection onto $\{0,1\}^{\omega_1}$ (4A2G(i-i) again). Because $\{0,1\}^{\omega_1}$ is separable (4A2B(e-ii)), so is K_2 (5A4C(d-i)). Let $\langle f_n\rangle_{n\in\mathbb{N}}$ enumerate a dense subset of K_2. Because K_2 is compact in $\mathbb{R}^X$, $h_1 = \sup_{n\in\mathbb{N}} f_n$ and $h_0 = \inf_{n\in\mathbb{N}} f_n$ are defined in $\mathbb{R}^X$, and of course they belong to $\mathcal{L}^0$. If $f \in K_2$, then

$$f(x) \in \overline{\{f_n(x) : n \in \mathbb{N}\}} \subseteq [h_0(x), h_1(x)]$$

for every x, and $h_0 \le f \le h_1$. Accordingly we have

$$\Delta(K_2) \le \rho(h_0, h_1) = \sup_{n\in\mathbb{N}} \rho(\inf_{i\le n} f_i, \sup_{i\le n} f_i) \le \Delta(K_2).$$

Let $\mathcal{U}$ be the family of non-empty cylinder sets in $\{0,1\}^{\omega_1}$. For $U \in \mathcal{U}$ set $I_U = \{n : n \in \mathbb{N},\ \psi_1(f_n) \in U\}$ and $g_U = \inf\{f_n : n \in I_U\}$. Observe that $F_U = \{f : f \in K_2,\ g_U \le f \le h_1\}$ is a closed subset of K_1 and that $F_U \cap \psi_1^{-1}[U]$ is dense in $\psi_1^{-1}[U]$, so $U \cap \psi_1[F_U]$ must be dense in U and $U \subseteq \psi_1[F_U]$. There is a finite set $I \subseteq \omega_1$ such that U is determined by coordinates in I; in this case, $C = \omega_1 \setminus I$ is closed and cofinal in ω_1, and $\{z\restriction C : z \in U\} = \{0,1\}^C$. By the choice of K_1, $\Delta(F_U) = \Delta(K_1)$. As $F_U \subseteq [g_U, h_1]$ in $\mathcal{L}^0$, $\rho(g_U, h_1) = \Delta(K_1) = \rho(h_0, h_1)$, and $\min(1, h_1 - g_U) =_{\text{a.e.}} \min(1, h_1 - h_0)$.

Set $h(x) = \max(\frac{1}{2}(h_0(x) + h_1(x)), h_1(x) - \frac{1}{2})$ for $x \in X$, and $E = \{x : h_0(x) < h_1(x)\} = \{x : h(x) < h_1(x)\}$, so that E is measurable and not negligible. If $U \in \mathcal{U}$, then

$$\begin{aligned} E_U &= \{x : x \in E,\ h(x) \le g_U(x)\} \\ &\subseteq \{x : x \in E,\ h_1(x) - g_U(x) < \min(1, h_1(x) - h_0(x))\} \end{aligned}$$

is negligible.

For every $x \in E$, $F'_x = \{f : f \in K_2,\ f(x) \le h(x)\}$ is a proper closed subset of K_2, so $\psi_1[F'_x] \ne \{0,1\}^{\omega_1}$ and there is some $U \in \mathcal{U}$ such that $U \cap \psi_1[F'_x] = \emptyset$. In this case $f_n \notin F'_x$, that is, $f_n(x) > h(x)$, for every $n \in I_U$, so $g_U(x) \ge h(x)$. Thus $E = \bigcup_{U\in\mathcal{U}} E_U$ is a non-negligible measurable set covered by ω_1 negligible sets.

(g) This is immediate from 536C, since we already know that K cannot be stable.

(h) Continuing the argument from (f), define $\phi : X \to \mathbb{R}^{\mathbb{N}}$ by setting $\phi(x) = \langle f_n(x)\rangle_{n\in\mathbb{N}}$ for $x \in X$. Then ϕ is measurable (418Bd), so we have a non-zero totally finite Borel measure ν on $\mathbb{R}^{\mathbb{N}}$ defined by setting $\nu H = \mu(E \cap \phi^{-1}[H])$ for every Borel set $H \subseteq \mathbb{R}^{\mathbb{N}}$. Note that $\phi[X] \subseteq \ell^\infty$ and that $\ell^\infty = \bigcup_{n\in\mathbb{N}}\bigcap_{i\in\mathbb{N}}\{w : |w(i)| \le n\}$ is an F_σ set in $\mathbb{R}^{\mathbb{N}}$. Now set

$$h'_1(w) = \sup_{n\in\mathbb{N}} w(n), \quad h'_0(w) = \inf_{n\in\mathbb{N}} w(n),$$

$$h'(w) = \max(\tfrac{1}{2}(h'_0(w) + h'_1(w)), h'_1(w) - \tfrac{1}{2})$$

for $w \in \ell^\infty$, so that $h_1 = h'_1\phi$, $h_0 = h'_0\phi$ and $h = h'\phi$; for $U \in \mathcal{U}$, set

$$E'_U = \{w : w \in \ell^\infty,\ h'(w) \le \inf_{n\in I_U} w(n)\}$$

so that E'_U is an F_σ set and $E_U = E \cap \phi^{-1}[E'_U]$; accordingly $\nu E'_U = 0$. Because $E \subseteq \bigcup_{U\in\mathcal{U}} E_U$, $\phi[E] \subseteq \bigcup_{U\in\mathcal{U}} E'_U$.

Thus we have a non-negligible subset of $\mathbb{R}^{\mathbb{N}}$ which is covered by ω_1 negligible F_σ sets and therefore by ω_1 closed negligible sets. By 526M, $\mathfrak{m}_{\text{countable}} = \omega_1$.

536E The discussion of stable sets in §465 emphasized their connection with pointwise compactness. In 465D and 465G we saw that stable sets are relatively pointwise compact and that on a stable set $\mathfrak{T}_m$ is coarser than $\mathfrak{T}_p$. The question of when we might be able to be sure that a pointwise compact set is stable was left open (but see 465Xj and 465Xn). We now have the concepts to take another step in this direction, which fits fairly naturally here, though it is not obviously connected with the question in 536A.

Proposition Let (X, Σ, μ) be a semi-finite measure space, with null ideal $\mathcal{N}(\mu)$. For $E \in \Sigma$ let μ_E be the subspace measure on E. Suppose that $\pi(\mu_E) \le \mathrm{cov}(E, \mathcal{N}(\mu))$ whenever $E \in \Sigma$ and $0 < \mu E < \infty$. Then every $\mathfrak{T}_p$-separable $\mathfrak{T}_p$-compact subset of $\mathcal{L}^0 = \mathcal{L}^0(\Sigma)$ is stable.

proof (a) ? Suppose that K is a $\mathfrak{T}_p$-separable $\mathfrak{T}_p$-compact subset of $\mathcal{L}^0$ which is not stable. Let A be a countable $\mathfrak{T}_p$-dense subset of K. By 465C(a-ii), A is not stable. So there are a set $E \in \Sigma$ and $\alpha < \beta$ in $\mathbb{R}$ such that $0 < \mu E < \infty$ and, in the language of 465A, $(\mu^{2k})^* D_k(A, E, \alpha, \beta) = (\mu E)^{2k}$ for every $k \geq 1$. Because A is countable,

$$D_k(A, E, \alpha, \beta) = \bigcup_{f \in A} \{w : w \in E^{2k}, f(w(2i)) \leq \alpha,$$

$$f(w(2i+1)) \geq \beta \text{ for } i < k\}.$$

is measured by the product measure μ^{2k} for every k, so that $E^{2k} \setminus D_k(A, E, \alpha, \beta)$ is μ^{2k}-negligible for every k.

(b) For sets I, $J \subseteq E$ set

$$A_{IJ} = \{f : f \in A,\ f(x) \leq \alpha \text{ for } x \in I,\ f(x) \geq \beta \text{ for } i \in J\}.$$

Let Q be the family of pairs (I, J) of finite subsets of E such that $E^{2k} \setminus D_k(A_{IJ}, E, \alpha, \beta)$ is μ^{2k}-negligible for every k. Then whenever $(I, J) \in Q$, the set $\{(x, y) : x,\ y \in E,\ (I \cup \{x\}, J \cup \{y\}) \notin Q\}$ is μ^2-negligible. **P** For any $k \geq 1$, if we identify E^{2k+2} with $E^{2k} \times E^2$,

$$\begin{aligned} D_{k+1}(A_{IJ}, E, \alpha, \beta) = \bigcup_{f \in A_{IJ}} &\{(w, (x, y)) : w \in E^{2k},\ x,\ y \in E,\ f(x) \leq \alpha,\ f(y) \geq \beta, \\ &f(w(2i)) \leq \alpha,\ f(w(2i+1)) \geq \beta \text{ for } i < k\} \\ = \{&(w, (x, y)) : x,\ y \in E,\ w \in D_k(A_{I \cup \{x\}, J \cup \{y\}}, E, \alpha, \beta)\}. \end{aligned}$$

Let F_k be the set of those $(x, y) \in E^2$ such that $E^{2k} \setminus D_k(A_{I \cup \{x\}, J \cup \{y\}}, E, \alpha, \beta)$ is not μ^{2k}-negligible. As $E^{2k+2} \setminus D_{k+1}(A_{IJ}, E, \alpha, \beta)$ is μ^{2k+2}-negligible, F_k is μ^2-negligible (252D). As k is arbitrary.

$$\{(x, y) : x,\ y \in E,\ (I \cup \{x\}, J \cup \{y\}) \notin Q\} = \textstyle\bigcup_{k \geq 1} F_k$$

is μ^2-negligible. **Q**

(c) Set $\kappa = \pi(\mu_E)$; then $\kappa \geq \mathrm{cov}(E, \mathcal{N}(\mu))$ is infinite. Let $\langle H_\xi \rangle_{\xi < \kappa}$ run over a coinitial set in $\{H : H \in \Sigma \setminus \mathcal{N}(\mu),\ H \subseteq E\}$. Then we can choose $\langle (x_\xi, y_\xi) \rangle_{\xi < \kappa}$ in such a way that, for each $\xi < \kappa$,

$$x_\xi,\ y_\xi \in H_\xi, \quad (\{x_\eta : \eta \in I\}, \{y_\eta : \eta \in I\}) \in Q \text{ for every finite } I \subseteq \xi,$$

P When we come to choose (x_ξ, y_ξ) we shall need to find a point (x, y) of H_ξ^2 such that

$$(\{x\} \cup \{x_\eta : \eta \in I\}, \{y\} \cup \{y_\eta : \eta \in I\})$$

belongs to Q for every finite $I \subseteq \xi$. By (b) and the inductive hypothesis, the forbidden set

$$H_\xi^2 \cap \textstyle\bigcup_{I \in [\xi]^{<\omega}} \{(x, y) : (\{x\} \cup \{x_\eta : \eta \in I\}, \{y\} \cup \{y_\eta : \eta \in I\}) \notin Q\}$$

is the union of fewer than κ μ^2-negligible subsets of H_ξ^2 and cannot cover H_ξ^2, by 521Jd, since $\kappa \geq \mathrm{cov}(H_\xi, \mathcal{N}(\mu))$. We therefore have a candidate eligible to be (x_ξ, y_ξ), and the induction can proceed. **Q**

(d) At the end of the induction, we see that

$$C_I = A_{\{x_\eta : \eta \in I\}, \{y_\eta : \eta \in I\}}$$

is non-empty for every finite $I \subseteq \kappa$. Let $\mathcal{F}$ be the filter on K generated by $\{C_I : I \in [\kappa]^{<\omega}\}$. Because K is $\mathfrak{T}_p$-compact, $\mathcal{F}$ has a $\mathfrak{T}_p$-cluster point $f \in K \subseteq \mathcal{L}^0$. Now one of $\{x : x \in E,\ f(x) < \beta\}$ and $\{x : x \in E,\ f(x) > \alpha\}$ must belong to $\Sigma \setminus \mathcal{N}(\mu)$ and include some H_ξ; but $x_\xi,\ y_\xi \in H_\xi$, while $f(x_\xi) \leq \alpha$ and $f(y_\xi) \geq \beta$. **X**

(e) Thus every pointwise separable-and-compact subset of $\mathcal{L}^0$ must be stable, as claimed.

536F Proposition Suppose that $\mathrm{cov}\,\mathcal{N} = \mathrm{cf}\,\mathcal{N}$, where $\mathcal{N}$ is the null ideal of Lebesgue measure on $\mathbb{R}$. Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space. Then every $\mathfrak{T}_p$-separable $\mathfrak{T}_p$-compact subset of $\mathcal{L}^0(\mu)$ is stable.

proof (a) To begin with (down to the end of (c) below), suppose that μ is totally finite. Let $K \subseteq \mathcal{L}^0(\mu)$ be $\mathfrak{T}_p$-separable and $\mathfrak{T}_p$-compact. If K is empty, it is surely stable and we can stop. Otherwise, let $\langle f_n \rangle_{n\in\mathbb{N}}$ be a sequence running over a $\mathfrak{T}_p$-dense subset of K. Define $\phi : X \to \mathbb{R}^{\mathbb{N}}$ by setting $\phi(x) = \langle f_n(x) \rangle_{n\in\mathbb{N}}$ for $n \in \mathbb{N}$. Then ϕ is measurable (418Bd again), therefore almost continuous (418J). Set $Z = \phi[X]$, and let ν be the image measure $\mu\phi^{-1}$ on Z; then ν is a Radon measure (418I). (This is where it helps to assume that μ is totally finite.)

(b) Consider the set $L = \{g : g \in \mathbb{R}^Z,\ g\phi \in K\}$.

(i) $L \subseteq \mathcal{L}^0(\nu)$. **P** If $g \in L$ and $\alpha > 0$, then

$$\phi^{-1}[\{z : z \in Z,\ g(z) > \alpha\}] = \{x : x \in X,\ g\phi(x) > \alpha\}$$

is measured by μ so $\{z : g(z) > \alpha\}$ is measured by ν. **Q**

(ii) L is $\mathfrak{T}_p$-separable. **P** Set $g_n(z) = z(n)$ for $n \in \mathbb{N}$ and $z \in Z$. Then $g_n\phi = f_n \in K$ so $g_n \in L$. If $g \in L$, there is a filter $\mathcal{F}$ on $\mathbb{N}$ such that $g\phi$ is the $\mathfrak{T}_p$-limit $\lim_{n\in\mathcal{F}} f_n$, that is,

$$g\phi(x) = \lim_{n\to\mathcal{F}} f_n(x) = \lim_{n\to\mathcal{F}} g_n\phi(x)$$

for every x. But now $g(z) = \lim_{n\to\mathcal{F}} g_n(z)$ for every $z \in Z$ and $g = \lim_{n\to\mathcal{F}} g_n$ belongs to the $\mathfrak{T}_p$-closure of $\{g_n : n \in \mathbb{N}\}$. So the countable set $\{g_n : n \in \mathbb{N}\}$ witnesses that L is $\mathfrak{T}_p$-separable. **Q**

(iii) L is $\mathfrak{T}_p$-compact. **P** Note first that if $z \in Z$ there is an $x \in \phi^{-1}[\{z\}]$, and now

$$\sup_{g\in L} |g(z)| = \sup_{g\in L} |g\phi(x)| \le\le \sup_{f\in K} |f(x)|$$

is finite. As g is arbitrary, L is relatively $\mathfrak{T}_p$-compact in $\mathbb{R}^Z$; write $\overline{L}$ for its $\mathfrak{T}_p$-closure. The map $g \mapsto g\phi : \mathbb{R}^Z \to \mathbb{R}^X$ is continuous for the pointwise topologies and $g\phi \in K$ for every $g \in L$, so $g\phi \in \overline{K} = K$ for every $g \in \overline{L}$, and $L = \overline{L}$ is $\mathfrak{T}_p$-compect. **Q**

(iv) $K = \{g\phi : g \in L\}$. **P** As the function $g \mapsto g\phi$ is continuous, $K' = \{g\phi : g \in L\}$ is $\mathfrak{T}_p$-compact, therefore $\mathfrak{T}_p$-closed; since it contains $f_n = g_n\phi$ for every n, it includes K. By the definition of L, $K' \subseteq K$ and they are equal. **Q**

(c) Now note that ν is a Radon measure on a separable metrizable space. So $\tau(\nu) \le \omega$ (531Ad), $\pi(\nu) \le \mathrm{cf}\,\mathcal{N}$ (524Pb) and $\mathrm{cov}\,\mathcal{N}(\nu_F) \ge \mathrm{cov}\,\mathcal{N}$ for every non-negligible set $F \in \mathrm{dom}\,\nu$ (524Pc). We are supposing that $\mathrm{cov}\,\mathcal{N} = \mathrm{cf}\,\mathcal{N}$, so 563E assures us that L is stable. Since ϕ is inverse-measure-preserving, $K = \{g\phi : g \in L\}$ is stable (465Cd[6]).

(d) This deals with the case of totally finite μ. For the general case, take any $E \in \Sigma$ such that $\mu E < \infty$. Then $A_E = \{f{\restriction}E : f \in A\}$ is included in $\mathcal{L}^0(\mathrm{dom}\,\mu_E)$, and it is $\mathfrak{T}_p$-separable and $\mathfrak{T}_p$-compact because the map $f \mapsto f{\restriction}E$ is pointwise continuous. Also μ_E is a Radon measure, by 416Rb. So A_E is stable, by (a)-(c). As E is arbitrary, A is stable (456C(c-iv)).

536X Basic exercises (a) Let (X, Σ, μ) be a complete measure space, with null ideal $\mathcal{N}(\mu)$. Suppose that $\mathrm{add}\,\mathcal{N}(\mu) = \mathrm{cov}\,\mathcal{N}(\mu)$. Show that there is a $\mathfrak{T}_p$-compact $\mathfrak{T}_m$-compact $K \subseteq \mathcal{L}^0(\Sigma)$ such that the identity map on K is not $(\mathfrak{T}_p, \mathfrak{T}_m)$-continuous.

(b) Let (X, Σ, μ) be a perfect measure space. Suppose that $\mathrm{non}(E, \mathcal{N}(\mu)) < \mathrm{cov}(E, \mathcal{N}(\mu))$ for every non-negligible measurable set E of finite measure. Show that if $K \subseteq \mathcal{L}^0(\Sigma)$ is $\mathfrak{T}_p$-compact, then the identity map on K is $(\mathfrak{T}_p, \mathfrak{T}_m)$-continuous.

536Y Further exercises (a) Suppose that the additivity and covering number of the Lebesgue null ideal are equal. Find a strictly localizable perfect measure space (X, Σ, μ) and a $\mathfrak{T}_p$-compact $K \subseteq \mathcal{L}^0(\Sigma)$ such that $\mathfrak{T}_m$ is Hausdorff on K but K is not $\mathfrak{T}_m$-compact.

[6] Formerly 465Xe.

536 Notes and comments The methods of 536C-536D are derived from ideas of M.Talagrand. They seem frustratingly close to delivering an answer to the original question. But it seems clear that even if a positive answer – every $\mathfrak{T}_p$-compact $\mathfrak{T}_m$-separated set is metrizable – is true in ZFC, some further idea will be needed in the proof. On the other side, while it may well be that in some familiar model of set theory there is a negative answer, parts (c), (d) and (g) of 536D give simple tests to rule out many candidates.

Version of 12.8.13

537 Sierpiński sets, shrinking numbers and strong Fubini theorems

W.Sierpiński observed that if the continuum hypothesis is true then there are uncountable subsets of $\mathbb{R}$ which have no uncountable negligible subsets, and that such sets lead to curious phenomena; he also observed that, again assuming the continuum hypothesis, there would be a (non-measurable) function $f : [0,1]^2 \to \{0,1\}$ for which Fubini's theorem failed radically, in the sense that

$$\iint f(x,y)dxdy = 0, \quad \iint f(x,y)dydx = 1.$$

In this section I set out to explore these two insights in the light of the concepts introduced in Chapter 52. I start with definitions of 'Sierpiński' and 'strongly Sierpiński' set (537A), with elementary facts and an excursion into the theory of 'entangled' sets (537C-537G). Turning to repeated integration, I look at three interesting cases in which, for different reasons, some form of separate measurability is enough to ensure equality of repeated integrals (537I, 537L, 537S). Working a bit harder, we find that there can be valid non-trivial inequalities of the form $\overline{\int}\underline{\int}dxdy \le \overline{\int}\,\overline{\int}dydx$ (537N-537Q).

As elsewhere, I will write $\mathcal{N}(\mu)$ for the null ideal of a measure μ.

537A Definitions (a) If (X,Σ,μ) is a measure space, a subset of X is a **Sierpiński set** if it is uncountable but meets every negligible set in a countable set.

(b) If (X,Σ,μ) is a measure space, a subset A of X is a **strongly Sierpiński set** if it is uncountable and for every $n \ge 1$ and for every set $W \subseteq X^n$ which is negligible for the (c.l.d.) product measure on X^n, the set $\{u : u \in A^n \cap W,\ u(i) \ne u(j) \text{ for } i < j < n\}$ is countable.

537B Proposition (a) Let (X,Σ,μ) be a measure space and $A \subseteq X$ a Sierpiński set.

(i) $\operatorname{add}\mathcal{N}(\mu) = \operatorname{non}\mathcal{N}(\mu) = \omega_1$ and $\operatorname{cov}\mathcal{N}(\mu) \ge \#(A)$.

(ii) If $\{x\}$ is negligible for every $x \in A$, then $\operatorname{cf}\mathcal{N}(\mu) \ge \operatorname{cf}([\#(A)]^{\le\omega})$.

(b) Suppose that (X,Σ,μ) and (Y,T,ν) are measure spaces such that singleton subsets of Y are negligible. Let $f : X \to Y$ be an inverse-measure-preserving function.

(i) If $A \subseteq X$ is a Sierpiński set, then $f[A]$ is a Sierpiński set in Y and $\#(f[A]) = \#(A)$.

(ii) Now suppose that ν is σ-finite. If $A \subseteq X$ is a strongly Sierpiński set, then $f[A]$ is a strongly Sierpiński set in Y.

(c) Suppose that λ and κ are infinite cardinals and that (X,Σ,μ) is a locally compact semi-finite measure space of Maharam type at most λ in which singletons are negligible and $\mu X > 0$. Give $\{0,1\}^\lambda$ its usual measure.

(i) If $\{0,1\}^\lambda$ has a Sierpiński subset with cardinal κ, then X has a Sierpiński subset with cardinal κ.

(ii) If $\{0,1\}^\lambda$ has a strongly Sierpiński subset with cardinal κ, then X has a strongly Sierpiński subset with cardinal κ.

proof (a)(i) We are told that A is uncountable; now any subset of A with ω_1 members witnesses that $\operatorname{non}\mathcal{N}(\nu) \le \omega_1$. On the other hand, if $\mathcal{E}$ is a family of negligible sets covering X, then $\#(A) \le \max(\omega,\#(\mathcal{E}))$, so $\#(\mathcal{E}) \ge \#(A)$; as $\mathcal{E}$ is arbitrary, $\operatorname{cov}\mathcal{N}(\mu) \ge \#(A)$.

(ii) If $\{x\}$ is negligible for every $x \in A$, then $[A]^{\le\omega} \subseteq \mathcal{N}(\mu)$, and the identity function is a Tukey function from $[A]^{\le\omega}$ to $\mathcal{N}(\mu)$; so $\operatorname{cf}[A]^{\le\omega} \le \operatorname{cf}\mathcal{N}(\mu)$.

(b)(i) If $y \in Y$, then $\{y\}$ and $f^{-1}[\{y\}]$ are negligible, so $A \cap f^{-1}[\{y\}]$ is countable; consequently $\#(A) \le \max(\omega, \#(f[A]))$ and $\#(f[A]) = \#(A)$. If $F \subseteq Y$ is negligible, then $f^{-1}[F]$ is negligible so $A \cap f^{-1}[F]$ and $f[A] \cap F$ are countable. So $f[A]$ is a Sierpiński set.

(ii) Let $W \subseteq Y^n$ be a negligible set for the product measure λ' on Y^n, where $n \ge 1$. Define $\boldsymbol{f} : X^n \to Y^n$ by saying that $\boldsymbol{f}(x_0, \dots, x_{n-1}) = (f(x_0), \dots, f(x_{n-1}))$ for $x_0, \dots, x_{n-1} \in X$. Because ν is σ-finite, $\boldsymbol{f}$ is inverse-measure-preserving for λ and λ' (251Wp). If W is λ'-negligible, then $\boldsymbol{f}^{-1}[W]$ is λ-negligible, and $B = \{u : u \in A^n \cap \boldsymbol{f}^{-1}[W],\, u(i) \ne u(j) \text{ for } i < j < n\}$ is countable. Consequently

$$\{v : v \in f[A]^n \cap W,\, v(i) \ne v(j) \text{ for } i < j < n\} \subseteq \boldsymbol{f}[B]$$

is countable.

(c) Take any set $E \subseteq X$ of non-zero finite measure, and give E its normalized subspace measure $\mu'_E = (\mu E)^{-1}\mu_E$. Then there is an $f : \{0,1\}^\lambda \to E$ which is inverse-measure-preserving for ν_λ and μ'_E (343Cd). So (b) tells us that E has a subset A with cardinal κ which is Sierpiński or strongly Sierpiński for μ'_E. But now A is still Sierpiński or strongly Sierpiński for μ.

537C Entangled sets (a) Definition If X is a totally ordered set, then X is **ω_1-entangled** if whenever $n \ge 1$, $I \subseteq n$ and $\langle x_{\xi i}\rangle_{\xi<\omega_1, i<n}$ is a family of distinct elements of X, then there are distinct ξ, $\eta < \omega_1$ such that $I = \{i : i < n,\, x_{\xi i} \le x_{\eta i}\}$.

(b) Give $\{0,1\}^{\mathbb{N}}$ its lexicographic ordering, that is,

$$x \le y \text{ iff either } x = y \text{ or there is an } n \in \mathbb{N} \text{ such that } x\restriction n = y\restriction n \text{ and } x(n) < y(n).$$

Then the map $x \mapsto \frac{2}{3}\sum_{n=0}^{\infty} 3^{-n}x(n) : \{0,1\}^{\mathbb{N}} \to \mathbb{R}$ is an order-isomorphism between $\{0,1\}^{\mathbb{N}}$ and the Cantor set, so any ω_1-entangled subset of $\{0,1\}^{\mathbb{N}}$ can be transferred to an ω_1-entangled subset of $\mathbb{R}$.

537D Lemma Let X be an ω_1-entangled totally ordered set.

(a) There is a countable set $D \subseteq X$ which meets $[x, y]$ whenever $x < y$ in X.

(b) Whenever $n \ge 1$, $I \subseteq n$ and $\langle x_{\xi i}\rangle_{\xi<\omega_1, i<n}$ is a family of distinct elements of X, there are $\xi < \eta < \omega_1$ such that $I = \{i : i < n,\, x_{\xi i} \le x_{\eta i}\}$.

proof (a)(i) There is a countable set $D_0 \subseteq X$ which meets $[x, z]$ whenever $x < y < z$ in X. **P?** Otherwise, choose $\langle x_{\xi i}\rangle_{\xi<\omega_1, i<3}$ inductively so that $x_{\xi 0} < x_{\xi 1} < x_{\xi 2}$ and $[x_{\xi 0}, x_{\xi 2}]$ does not meet $\{x_{\eta i} : \eta < \xi,\, i < 3\}$. Now, if ξ, $\eta < \omega_1$ are different, we cannot have

$$x_{\xi 0} \le x_{\eta 0}, \quad x_{\xi 1} > x_{\eta 1}, \quad x_{\xi 2} \le x_{\eta 2}.$$

So $\langle x_{\xi i}\rangle_{\xi<\omega_1, i<3}$ witnesses that X is not ω_1-entangled. **XQ**

(ii) Set $A = \{(x, y) : x < y,\, [x, y] \cap D_0 = \emptyset\}$. Note that if (x, y), $(x', y') \in A$ are distinct, then $[x, y] \cap [x', y'] = \emptyset$, since otherwise $[\min(x, x'), \max(y, y')]$ would be an interval disjoint from D_0 with at least three elements. It follows that A is countable. **P?** Otherwise, let $\langle (x_{\xi 0}, x_{\xi 1})\rangle_{\xi<\omega_1}$ be a family of distinct elements of A. Then all the $x_{\xi i}$ are distinct. But if ξ, $\eta < \omega_1$ are different, we cannot have

$$x_{\xi 0} \le x_{\eta 0}, \quad x_{\xi 1} > x_{\eta 1}.$$

So $\langle x_{\xi i}\rangle_{\xi<\omega_1, i<2}$ witnesses that X is not entangled. **XQ**

(iii) So if we set $D = D_0 \cup \{x : (x, y) \in A\}$ we shall have a suitable countable set.

(b) For $i < n$ write $\le_i = \le$ if $i \in I$, $\le_i = \ge$ if $i \in n \setminus I$; we are seeking $\xi < \eta$ such that $x_{\xi i} \le_i x_{\eta i}$ for every $i < n$. For each family $\boldsymbol{d} = \langle d_i\rangle_{i<n}$ in D, set $A_{\boldsymbol{d}} = \{\xi : x_{\xi i} \le_i d_i \text{ for each } i < n\}$. Let $\zeta < \omega_1$ be such that $A_{\boldsymbol{d}} \cap \zeta \ne \emptyset$ whenever $\boldsymbol{d} \in D^n$ and $A_{\boldsymbol{d}} \ne \emptyset$. Now there are distinct ξ', $\eta \in \omega_1 \setminus \zeta$ such that $x_{\xi' i} \le_i x_{\eta i}$ for every $i < n$. For each $i < n$, there is a $d_i \in D$ such that $x_{\xi' i} \le_i d_i \le_i x_{\eta i}$. Set $\boldsymbol{d} = \langle d_i\rangle_{i<n}$; then $\xi' \in A_{\boldsymbol{d}}$ so there is a $\xi \in \zeta \cap A_{\boldsymbol{d}}$. Now $\xi < \eta$ and $x_{\xi i} \le_i x_{\eta i}$ for every i, as required.

537E Lemma Suppose that $n \ge 1$, $I \subseteq n$ and that $A \subseteq (\{0,1\}^{\mathbb{N}})^n$ is non-negligible for the usual product measure $\nu_{\mathbb{N}}^n$ on $(\{0,1\}^{\mathbb{N}})^n$. Let $\le$ be the lexicographic ordering of $\{0,1\}^{\mathbb{N}}$. Then there are v, $w \in A$ such that $v(i) \ne w(i)$ for every $i < n$ and $\{i : i < n,\, v(i) \le w(i)\} = I$.

proof For each $k \in \mathbb{N}$ let Σ_k be the algebra of subsets of $X = (\{0,1\}^{\mathbb{N}})^n$ generated by sets of the form $\{v : v \in X,\, v(i)(j) = 1\}$ for $i < n$ and $j < k$. Then $\langle \Sigma_k \rangle_{k\in\mathbb{N}}$ is a non-decreasing sequence of finite algebras and the σ-algebra generated by $\bigcup_{k\in\mathbb{N}} \Sigma_k$ is the Borel σ-algebra $\mathcal{B}(X)$ of X. Let $E \in \mathcal{B}(X)$ be a measurable envelope of A for $\nu_{\mathbb{N}}^n$. For each $k \in \mathbb{N}$, let f_k be the conditional expectation of χE on Σ_k, that is,

$$f_k(u) = 2^{kn}\nu_{\mathbb{N}}^n\{v : v \in E,\, v(i){\restriction}k = u(i){\restriction}k \text{ for every } i < n\}$$

for $u \in X$. By Lévy's martingale theorem (275I), $\chi E =_{\text{a.e.}} \lim_{k\to\infty} f_k$. In particular, there are a $u \in A$ and a $k \in \mathbb{N}$ such that $f_k(u) > 1 - 2^{-n}$. But this means that

$$F = \{v : v \in E,\, v(i){\restriction}k = u(i){\restriction}k \text{ for every } i < n\}$$

has measure greater than $2^{-kn}(1 - 2^{-n})$, and both the sets

$$F' = \{v : v \in F,\, v(i)(k) = 0 \text{ for } i \in I,\, v(i)(k) = 1 \text{ for } i \in n \setminus I\},$$

$$F'' = \{w : w \in F,\, w(i)(k) = 1 \text{ for } i \in I,\, w(i)(k) = 0 \text{ for } i \in n \setminus I\},$$

must have positive measure. Accordingly we can find $v \in A \cap F'$ and $w \in A \cap F''$, and these will serve.

537F Corollary Suppose that $A \subseteq \{0,1\}^{\mathbb{N}}$ is strongly Sierpiński for the usual measure on $\{0,1\}^{\mathbb{N}}$. Then A is ω_1-entangled for the lexicographic ordering of $\{0,1\}^{\mathbb{N}}$.

proof Let $\langle x_{\xi i} \rangle_{\xi<\omega_1, i<n}$ be a family of distinct points in A, where $n \geq 1$, and I a subset of n. Then $x_\xi = \langle x_{\xi i} \rangle_{i<n}$ belongs to A^n, and has no two coordinates the same, for every $\xi < \omega_1$. So $D = \{x_\xi : \xi < \omega_1\}$ cannot be negligible. By 537E, there are distinct ξ, $\eta < \omega_1$ such that $I = \{i : x_{\xi i} \leq x_{\eta i}\}$.

537G Theorem (Todorčević 85) Suppose that there is an ω_1-entangled totally ordered set X of size $\kappa \geq \omega_1$. Then there are two upwards-ccc partially ordered sets P, Q such that $c^{\uparrow}(P \times Q) \geq \kappa$.

proof (a) Let $Y \subseteq X$ be a set such that $\#(Y) = \#(X \setminus Y) = \kappa$, and $f : Y \to X \setminus Y$ an injective function. Set

$$P = \{I : I \in [Y]^{<\omega},\, f{\restriction}I \text{ is order-preserving}\},$$

$$Q = \{I : I \in [Y]^{<\omega},\, f{\restriction}I \text{ is order-reversing}\},$$

both ordered by $\subseteq$. Then $\{(\{y\},\{y\}) : y \in Y\}$ is an up-antichain in $P \times Q$, so $c^{\uparrow}(P \times Q) \geq \kappa$.

(b) P is upwards-ccc. **P** Let $\langle I_\alpha \rangle_{\alpha<\omega_1}$ be a family in P. By the Δ-system Lemma (4A1Db), there is an uncountable set $A \subseteq \omega_1$ such that $\langle I_\alpha \rangle_{\alpha\in A}$ is a Δ-system with root I say; now there is an $n \in \mathbb{N}$ such that $B = \{\alpha : \alpha \in A,\, \#(I_\alpha \setminus I) = n\}$ is uncountable. If $n = 0$ then $I_\alpha = I_\beta$ are upwards-compatible for any α, $\beta \in B$ and we can stop.

If $n \geq 1$, enumerate $I_\alpha \setminus I$ in increasing order as $\langle x_{\alpha i} \rangle_{i<n}$, for each $\alpha \in B$. Let $D \subseteq X$ be a countable set such that D meets every interval in X with more than one member (537Da). For $i < j < n$ and $\alpha \in B$ let $d_{\alpha ij}$, $d'_{\alpha ij} \in D$ be such that $x_{\alpha i} \leq d_{\alpha ij} \leq x_{\alpha j}$ and $f(x_{\alpha i}) \leq d'_{\alpha ij} \leq f(x_{\alpha j})$. (Because $I_\alpha \in P$, $f{\restriction}I_\alpha$ is order-preserving so $f(x_{\alpha i}) < f(x_{\alpha j})$.) Let $\langle d_{ij} \rangle_{i<j<n}$, $\langle d'_{ij} \rangle_{i<j<n}$ be such that

$$C = \{\alpha : \alpha \in B,\, d_{\alpha ij} = d_{ij} \text{ and } d'_{\alpha ij} = d'_{ij} \text{ whenever } i < j < n\}$$

is uncountable.

Consider the family $\langle y_{\alpha i} \rangle_{\alpha\in C, i<2n}$ where $y_{\alpha i} = x_{\alpha i}$ and $y_{\alpha,i+n} = f(x_{\alpha i})$ if $i < n$. Because X is entangled, there must be distinct α, $\beta \in C$ such that $y_{\alpha i} \leq y_{\beta i}$ for every $i < 2n$, that is, $x_{\alpha i} \leq x_{\beta i}$ and $f(x_{\alpha i}) \leq f(x_{\beta i})$ for every $i < n$. But now examine $I = I_\alpha \cup I_\beta$. If x, $x' \in I$ and $x \leq x'$,

either both x and x' belong to I_α and $f(x) \leq f(x')$ because $I_\alpha \in P$,
or both x and x' belong to I_β and $f(x) \leq f(x')$,
or $x = x_{\alpha i}$ and $x' = x_{\beta j}$ where $i < j < n$, so that

$$f(x) = f(x_{\alpha i}) \leq d'_{ij} \leq f(x_{\beta_j}) = f(x'),$$

or $x = x_{\beta i}$ and $x' = x_{\alpha j}$ where $i < j < n$, so that $f(x) \leq f(x')$,
or $x = x_{\alpha i}$ and $x' = x_{\beta i}$ where $i < n$, so that $f(x) = f(x_{\alpha i}) \leq f(x_{\beta i}) = f(x')$.

(Note that we cannot have $x = x_{\alpha i}$ and $x' = x_{\beta j}$ with $j < i$, because in this case $x_{\beta j} \le d_{ji} \le x_{\alpha i}$ while $x_{\beta j} \ne x_{\alpha i}$; nor can we have $x = x_{\beta i} < x' = x_{\alpha i}$ with $i < n$.) So $f{\restriction}I$ is order-preserving and $I \in P$ witnesses that I_α and I_β are upwards-compatible in P. As $\langle I_\alpha\rangle_{\alpha<\omega_1}$ is arbitrary, P is upwards-ccc. **Q**

(c) Similarly, Q is upwards-ccc. **P** The principal changes needed in the argument above are

—— in the choice of the $d'_{\alpha ij}$, we need to write '$f(x_{\alpha i}) \ge d'_{\alpha ij} \ge f(x_{\alpha j})$';

—— in the choice of particular α and β in the set C, we need to write '$y_{\alpha i} \le y_{\beta i}$ for $i < n$ and $y_{\alpha i} \ge y_{\beta i}$ for $n \le i < 2n$'. **Q**

So P and Q satisfy our requirements.

537H Scalarly measurable functions (a) Definition Let X be a set, Σ a σ-algebra of subsets of X and U a linear topological space. A function $\phi : X \to U$ is **scalarly (Σ-)measurable** if $f\phi : X \to \mathbb{R}$ is (Σ-)measurable for every $f \in U^*$.

(b) If $\phi : X \to U$ is scalarly measurable, V is another linear topological space and $T : U \to V$ is a continuous linear operator, then $T\phi : X \to V$ is scalarly measurable, because $hT \in U^*$ for every $h \in V^*$.

(c) If U is a separable metrizable locally convex space and $\phi : X \to U$ is scalarly measurable, then it is measurable. **P** $\mathrm{T} = \{F : F \subseteq U, \phi^{-1}[F] \in \Sigma\}$ includes the cylindrical σ-algebra of U (4A3U[7]), which is the Borel σ-algebra (4A3W[8]). **Q**

537I Proposition Let (X, Σ, μ) and (Y, T, ν) be probability spaces and U a reflexive Banach space. Suppose that $x \mapsto u_x : X \to U$ and $y \mapsto f_y : Y \to U^*$ are bounded scalarly measurable functions. Then $\iint f_y(u_x)\mu(dx)\nu(dy)$ and $\iint f_y(u_x)\nu(dy)\mu(dx)$ are defined and equal.

proof (a)(i) Recall from 467Hc that if $V \subseteq U$ and $W \subseteq U^*$ are closed linear subspaces, I call them a 'projection pair' if $U = V \oplus W^\circ$ and $\|v + v'\| \ge \|v\|$ for all $v \in V$ and $v' \in W^\circ$. We need to know that this is symmetric; that is, that in this case

$$U^* = W \oplus V^\circ, \quad \|g + g'\| \ge \|g\| \text{ for all } g \in W,\ g' \in V^\circ.$$

P Note first that if $g \in W \cap V^\circ$, then $g(u) = 0$ for every $u \in W^\circ + V$, that is, $g = 0$. Now take any $f \in U^*$. Define $g : U \to \mathbb{R}$ by saying that $g(v + v') = f(v)$ for $v \in V$, $v' \in W^\circ$. Then g is linear and continuous and $\|g\| \le \|f\|$. Now $g(v') = 0$ for every $v' \in W^\circ$, that is, $g \in W^{\circ\circ}$, which is the weak*-closure of W (4A4Eg); but as U and U^* are reflexive, this is just the norm-closure of W, which is equal to W. Set $g' = f - g$. Then $g' \in V^\circ$. This shows that $f \in W + V^\circ$; as f is arbitrary, $U^* = W \oplus V^\circ$. Finally, I remarked in the course of the argument that $\|g\| \le \|f\|$, which is what we need to know to check that $\|g\| \le \|g + g'\|$ whenever $g \in W$ and $g' \in V^\circ$. **Q**

(ii) Because U is reflexive, its unit ball is weakly compact, so U is surely weakly compactly generated, therefore weakly K-countably determined (467M). Now turn to Lemma 467J. This tells us that there is a family $\mathcal{M}$ of subsets of $U \cup U^*$ such that

for every $B \subseteq X \cup X^*$ there is an $M \in \mathcal{M}$ such that $B \subseteq M$ and $\#(M) \le \max(\omega, \#(B))$;

whenever $\mathcal{M}' \subseteq \mathcal{M}$ is upwards-directed, then $\bigcup \mathcal{M}' \in \mathcal{M}$;

whenever $M \in \mathcal{M}$ then (V_M, W_M) is a projection pair of subspaces of U and U^*,

where I write $V_M = \overline{M \cap U}$ and $W_M = \overline{M \cap U^*}$. For $M \in \mathcal{M}$,

$$U = V_M \oplus W_M^\circ, \quad U^* = W_M \oplus V_M^\circ;$$

let $P_M : U \to V_M$ and $Q_M : U^* \to W_M$ be the corresponding projections. Since $\|v\| \le \|v + v'\|$ whenever $v \in V_M$ and $v' \in W_M^\circ$, $\|P_M\| \le 1$; similarly, $\|Q_M\| \le 1$.

If $u \in U$, $f \in U^*$ and $M \in \mathcal{M}$, then

$$f(P_M u) = (Q_M f)(u) = (Q_M f)(P_M u).$$

P Express u as $v + v'$ and f as $g + g'$, where $v \in V_M$, $v' \in W_M^\circ$, $g \in W_M$ and $g' \in V_M^\circ$. Then

[7] Formerly 4A3T.
[8] Formerly 4A3V.

$$f(v)=g(v)=g(u),$$

that is,

$$f(P_Mu)=(Q_Mf)(P_Mu)=(Q_Mf)(u).\ \mathbf{Q}$$

(iii) If M_0, $M_1\in\mathcal{M}$ and $M_0\subseteq M_1$ then $P_{M_0}=P_{M_0}P_{M_1}=P_{M_1}P_{M_0}$. **P** If $u\in U$, express it as v_0+v_0' where $v_0\in V_{M_0}$ and $v_0'\in W^\circ_{M_0}$; now express v_0' as v_1+v_1' where $v_1\in V_{M_1}$ and $v_1'\in W^\circ_{M_1}$. Then

$$P_{M_0}u=v_0\in V_{M_1},$$

so $P_{M_1}P_{M_0}u=P_{M_0}u$. On the other hand, $u=v_0+v_1+v_1'$ where $v_0+v_1\in V_{M_1}$ and $v_1'\in W^\circ_{M_1}$, so $P_{M_1}u=v_0+v_1$; and as $v_1=v_0'-v_1'$ belongs to $W^\circ_{M_0}$, $P_{M_0}P_{M_1}u=v_0=P_{M_0}u$. **Q**

(iv) If $\langle M_\xi\rangle_{\xi<\zeta}$ is a non-decreasing family in $\mathcal{M}$, where ζ is a non-zero limit ordinal, then we know that $M=\bigcup_{\xi<\zeta}M_\xi$ belongs to $\mathcal{M}$. Now

$$P_Mu=\lim_{\xi\uparrow\zeta}P_{M_\xi}u$$

for every $u\in U$, the limit being for the norm topology on U. **P** Let $\epsilon>0$. We know that $P_Mu\in V_M=\overline{M\cap U}$, so there is a $u'\in M\cap U$ such that $\|u'-P_Mu\|\le\frac{1}{2}\epsilon$. Let $\xi<\zeta$ be such that $u'\in M_\xi$. If $\xi\le\eta<\zeta$, then

$$\|P_{M_\eta}u-P_Mu\|=\|P_{M_\eta}(P_Mu-u')+P_M(u'-P_Mu)\|$$

(because $u'\in V_{M_\eta}$, so $P_Mu'=P_{M_\eta}u'=u'$)

$$\le 2\|P_Mu-u'\|\le\epsilon.\ \mathbf{Q}$$

(v) Similarly,

$$Q_{M_0}=Q_{M_0}Q_{M_1}=Q_{M_1}Q_{M_0}$$

whenever M_0, $M_1\in\mathcal{M}$ and $M_0\subseteq M_1$, and

$$Q_Mf=\lim_{\xi\uparrow\zeta}Q_{M_\xi}f$$

whenever $f\in U^*$ and ζ is a non-zero limit ordinal and $\langle M_\xi\rangle_{\xi<\zeta}$ is a non-decreasing family in $\mathcal{M}$ with union M.

(b) Now let $\mathcal{M}_0$ be $\{M:M\in\mathcal{M},\,\#(M)\le\omega\}$. Then there is an $M_0\in\mathcal{M}_0$ such that

$$P_{M_0}(u_x)=P_M(u_x)\ \mu\text{-a.e.}(x)$$

whenever $M_0\subseteq M\in\mathcal{M}_0$.

P? Suppose, if possible, otherwise. Then we can choose inductively an increasing family $\langle M_\xi\rangle_{\xi<\omega_1}$ in $\mathcal{M}_0$ such that

$$\mu\{x:P_{M_{\xi+1}}(u_x)\ne P_{M_\xi}(u_x)\}>0\text{ for every }\xi<\omega_1,$$

$$M_\xi=\bigcup_{\eta<\xi}M_\eta\text{ whenever }\xi<\omega_1\text{ is a non-zero countable limit ordinal.}$$

(The set of x for which $P_{M_{\xi+1}}(u_x)\ne P_{M_\xi}(u_x)$ is necessarily measurable because $x\mapsto P_{M_{\xi+1}}u_x-P_{M_\xi}u_x$ is scalarly measurable, by 537Hb, therefore measurable for the norm topology, by 537Hc, since $V_{M_{\xi+1}}$ is separable.) Now there must be a $\delta>0$ such that

$$A=\{\xi:\xi<\omega_1,\,\mu E_\xi\ge\delta\}$$

is infinite, where

$$E_\xi=\{x:\|P_{M_{\xi+1}}(u_x)-P_{M_\xi}(u_x)\|\ge\delta\}$$

for each $\xi<\omega_1$. But in this case there must be an $x\in X$ such that

$$A'=\{\xi:\xi\in A,\,x\in E_\xi\}$$

is infinite. (Take a sequence $\langle\xi_n\rangle_{n\in\mathbb{N}}$ of distinct points in A, and $x\in\bigcap_{n\in\mathbb{N}}\bigcup_{m\ge n}E_{\xi_m}$.) Let ζ be any cluster point of A' in ω_1. Then

$$P_{M_\zeta}(u_x) = \lim_{\xi\uparrow\zeta} P_{M_\xi}(u_x)$$

((a-iv) above), which is impossible. **XQ**

(c) Similarly, there is an $M_1 \in \mathcal{M}_0$ such that $M_1 \supseteq M_0$ and

$$P_{M_1}(f_y) = P_M(f_y) \ \nu\text{-a.e.}(y)$$

whenever $M_1 \subseteq M \in \mathcal{M}_0$. Because $x \mapsto P_{M_1}(u_x)$ and $y \mapsto Q_{M_1}(f_y)$ are scalarly measurable maps to norm-separable spaces, they are norm-measurable; again because V_{M_1} and W_{M_1} are separable, $(x,y) \to (P_{M_1}u_x, Q_{M_1}f_y) : X \times Y \to V_{M_1} \times W_{M_1}$ is $\Sigma\widehat{\otimes}\mathrm{T}$-measurable (418Bd). Because $(f,x) \mapsto f(x) : U^* \times U \to \mathbb{R}$ is norm-continuous, $(x,y) \mapsto (Q_{M_1}f_y)(P_{M_1}u_x)$ is $\Sigma\widehat{\otimes}\mathrm{T}$-measurable, and

$$\iint (Q_{M_1}f_y)(P_{M_1}u_x)\mu(dx)\nu(dy) = \iint (Q_{M_1}f_y)(P_{M_1}u_x)\nu(dy)\mu(dx)$$

by Fubini's theorem (252C).

Now observe that if $y \in Y$ there is an $M \in \mathcal{M}_0$ such that $M_1 \subseteq M$ and $f_y \in M$. So

$$\begin{aligned}\int f_y(u_x)\mu(dx) &= \int (Q_M f_y)(u_x)\mu(dx) = \int f_y(P_M u_x)\mu(dx)\\ &= \int f_y(P_{M_1}u_x)\mu(dx) = \int (Q_{M_1}f_y)(P_{M_1}u_x)\mu(dx).\end{aligned}$$

This is true for every y. So $\iint f_y(u_x)\mu(dx)\nu(dy)$ is defined and equal to $\iint (Q_{M_1}f_y)(P_{M_1}u_x)\mu(dx)\nu(dy)$. Similarly,

$$\iint f_y(u_x)\nu(dy)\mu(dx) = \iint (Q_{M_1}f_y)(P_{M_1}u_x)\nu(dy)\mu(dx).$$

Putting these together, we have the result.

537J Corollary Let (X,Σ,μ), (Y,T,ν) and (Z,Λ,σ) be probability spaces. Let $x \mapsto U_x : X \to \Lambda$ and $y \mapsto V_y : Y \to \Lambda$ be functions such that

$$x \mapsto \sigma(U_x \cap W), \quad y \mapsto \sigma(V_y \cap W)$$

are measurable for every $W \in \Lambda$. Then $\iint \sigma(U_x \cap V_y)\mu(dx)\nu(dy)$ and $\iint \sigma(U_x \cap V_y)\nu(dy)\mu(dx)$ are defined and equal.

proof (a) For $x \in X$ set $u_x = (\chi U_x)^\bullet$ in $L^2(\sigma)$. Then $x \mapsto u_x$ is scalarly measurable. **P** If $f \in U^*$, there is a $v \in L^2(\sigma)$ such that $f(u) = \int u \times v$ for every $u \in L^2(\sigma)$ (244K). Let $\epsilon > 0$. Then there are $W_0, \dots, W_n \in \Lambda$ and $\alpha_0, \dots, \alpha_n \in \mathbb{R}$ such that $\|v - \sum_{i=0}^n \alpha_i(\chi W_i)^\bullet\|_2 \le \epsilon$ (244Ha), so that

$$|f(u_x) - \sum_{i=0}^n \alpha_i\sigma(U_x \cap W_i)| = |\int u_x \times v - \int u_x \times \sum_{i=0}^n \alpha_i(\chi W_i)^\bullet| \le \epsilon\|u_x\|_2 \le \epsilon$$

for every $x \in X$. Now the function $x \mapsto \sum_{i=0}^n \alpha_i\sigma(U_x \cap W_i)$ is Σ-measurable. So we see that the function $x \mapsto f(u_x)$ is uniformly approximated by Σ-measurable functions and is itself Σ-measurable. As f is arbitrary, $x \mapsto u_x$ is scalarly measurable. **Q**

(b) Similarly, setting $v_y = (\chi V_y)^\bullet$ for $y \in Y$, $y \mapsto v_y : Y \to L^2(\sigma)$ is scalarly measurable. Identifying $L^2(\sigma)$ with its dual, 537I tells us that

$$\iint (u_x|v_y)\mu(dx)\nu(dy) = \iint (u_x|v_y)\nu(dy)\mu(dx),$$

that is, that

$$\iint \sigma(U_x \cap V_y)\mu(dx)\nu(dy) = \iint \sigma(U_x \cap V_y)\nu(dy)\mu(dx).$$

537K The next few paragraphs will be concerned with upper and lower integrals. For the basic theory of these, see §133 and 214J.

Theorem (FREILING 86, SHIPMAN 90) Let $\langle(X_j,\Sigma_j,\mu_j)\rangle_{j\le m}$ be a finite sequence of probability spaces and $\langle\kappa_j\rangle_{j\le m}$ a sequence of cardinals such that $X_j^{\mathbb{N}}$, with its product measure $\mu_j^{\mathbb{N}}$, has a subset with cardinal κ_j which is not covered by κ_{j-1} negligible sets (if $j \ge 1$) and is not negligible (if $j = 0$). Let $f : \prod_{j\le m} X_j \to \mathbb{R}$ be a bounded function, and suppose that $\sigma : m+1 \to m+1$ and $\tau : m+1 \to m+1$ are permutations. Set

$$I=\underline{\int}\ldots\underline{\int}f(x_0,\ldots,x_m)dx_{\sigma(m)}\ldots dx_{\sigma(0)},$$

$$I'=\overline{\int}\ldots\overline{\int}f(x_0,\ldots,x_m)dx_{\tau(m)}\ldots dx_{\tau(0)}.$$

Then $I\le I'$.

proof Let $M\ge 0$ be such that $|f(x_0,\ldots,x_m)|\le M$ for all $x_0,\ldots,x_m$.

(a) Set $Z=\prod_{j\le m}X_j^{\mathbb{N}}$. The key fact is that we can find negligible sets $W(\boldsymbol{u})\subseteq X_k^{\mathbb{N}}$, for $k\le m$ and $\boldsymbol{u}\in\prod_{j\le m,j\ne k}X_j^{\mathbb{N}}$, such that

$$I\le\liminf_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n f(t_{0i},\ldots,t_{mi})$$

whenever $\langle t_j\rangle_{j\le m}=\langle\langle t_{ji}\rangle_{i\in\mathbb{N}}\rangle_{j\le m}$ is such that $t_k\notin W(t_0,\ldots,t_{k-1},t_{k+1},\ldots,t_m)$ for every k. **P** Because the formula

$$\liminf_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n f(t_{0i},\ldots,t_{mi})$$

is tolerant of permutations of the coordinates $0,\ldots,m$, it is enough to consider the case $\sigma(j)=j$ for $j\le m$, so that

$$I=\underline{\int}\ldots\underline{\int}f(x_0,\ldots,x_m)dx_m\ldots dx_0.$$

(i) Define $D_0,\ldots,D_{m+1}$ as follows. $D_0=\{\emptyset\}=\prod_{j<0}X_j^{\mathbb{N}}$. For $0<k\le m$ let D_k be the set of those $(t_0,\ldots,t_{k-1})\in\prod_{j<k}X_j^{\mathbb{N}}$ such that

$$I\le\liminf_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n\underline{\int}\ldots\underline{\int}f(t_{0i},\ldots,t_{k-1,i},x_k,\ldots,x_m)dx_m\ldots dx_k,$$

where $t_j=\langle t_{ji}\rangle_{i\in\mathbb{N}}$ for $j<k$. For $k<m$ and $\boldsymbol{u}=(u_0,\ldots,u_{k-1},u_{k+1},\ldots,u_m)$ in $\prod_{j\le m,j\ne k}X_j^{\mathbb{N}}$, set

$$\begin{aligned}W(\boldsymbol{u})&=\emptyset\text{ if }(u_0,\ldots,u_{k-1})\notin D_k,\\&=\{t:t\in X_k^{\mathbb{N}},\,(u_0,\ldots,u_{k-1},t)\notin D_{k+1}\}\text{ otherwise.}\end{aligned}$$

(ii) $W(\boldsymbol{u})\subseteq X_k^{\mathbb{N}}$ is negligible. To see this, we need consider only the case in which $(u_0,\ldots,u_{k-1})$ belongs to D_k. Express u_j as $\langle u_{ji}\rangle_{i\in\mathbb{N}}$ for $j<k$, and for $i\in\mathbb{N}$ define $h_i:X_k\to\mathbb{R}$ by setting

$$h_i(x)=\underline{\int}\ldots\underline{\int}f(u_{0i},\ldots,u_{k-1,i},x,x_{k+1},\ldots,x_m)dx_m\ldots dx_{k+1}$$

for $x\in X_k$. Now the definition of D_k tells us just that

$$I\le\liminf_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n\underline{\int}\ldots\underline{\int}f(u_{0i},\ldots,u_{k-1,i},x_k,\ldots,x_m)dx_m\ldots dx_k,$$

that is, that

$$I\le\liminf_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n\underline{\int}h_i(x)dx.$$

For each $i\in\mathbb{N}$ let $g_i:X_k\to[-M,M]$ be a measurable function such that $g_i(x)\le h_i(x)$ for every x and $\int g_id\mu_k=\underline{\int}h_id\mu_k$. Now consider the functions $\tilde g_i:X_k^{\mathbb{N}}\to\mathbb{R}$ defined by setting $\tilde g_i(t)=g_i(t_i)$ for $t=\langle t_i\rangle_{i\in\mathbb{N}}\in X_k^{\mathbb{N}}$. We have $\int\tilde g_id\mu_k^{\mathbb{N}}=\underline{\int}h_id\mu_k$ for each i, while $\langle\tilde g_i\rangle_{i\in\mathbb{N}}$ is a uniformly bounded independent sequence of random variables. By the strong law of large numbers in the form 273H,

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n(\tilde g_i(t)-\int\tilde g_id\mu_k^{\mathbb{N}})=0$$

for almost every $t\in X_k^{\mathbb{N}}$. Since

$$\liminf_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n\int\tilde g_id\mu_k^{\mathbb{N}}=\liminf_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n\underline{\int}h_id\mu_k\ge I,$$

we have

$$I \le \liminf_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} \tilde g_i(t) \le \liminf_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} h_i(t_i)$$
$$= \liminf_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} \underline{\int} \dots \underline{\int} f(u_{0i},\dots,u_{k-1,i},t_i,x_{k+1},\dots,x_m)dx_m \dots dx_{k+1}$$

for almost every $t = \langle t_i\rangle_{i\in\mathbb{N}} \in X_k^{\mathbb{N}}$, that is, $(u_0,\dots,u_{k-1},t) \in D_{k+1}$ for almost every $t \in X_k^{\mathbb{N}}$, that is, $W(\boldsymbol{u})$ is negligible, as required.

(iii) Suppose that $\boldsymbol{t} = (t_0,\dots,t_m) \in Z$ is such that $t_k \notin W(t_0,\dots,t_{k-1},t_{k+1},\dots,t_m)$ for every $k < m$. Then $(t_0,\dots,t_k) \in D_{k+1}$ for every k; in particular, $\boldsymbol{t} \in D_{m+1}$ and, writing $t_j = \langle t_{ji}\rangle_{i\in\mathbb{N}}$ for each j,

$$I \le \liminf_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(t_{0i},\dots,t_{mi}). \quad \mathbf{Q}$$

(b) Similarly, or applying the argument above to $-f$, we have negligible sets $W'(\boldsymbol{u}) \subseteq X_k^{\mathbb{N}}$, for $k \le m$ and $\boldsymbol{u} \in \prod_{j\le m, j\ne k} X_j^{\mathbb{N}}$, such that

$$I' \ge \limsup_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(t_{0i},\dots,t_{mi})$$

whenever $\langle t_j\rangle_{j\le m} = \langle\langle t_{ji}\rangle_{i\in\mathbb{N}}\rangle_{j\le m}$ is such that $t_k \notin W'(t_0,\dots,t_{k-1},t_{k+1},\dots,t_m)$ for every k. Enlarging the $W'(\boldsymbol{u})$ if necessary, we may suppose that $W'(\boldsymbol{u}) \supseteq W(\boldsymbol{u})$ for every $\boldsymbol{u}$.

(c) Now the point of the construction is that we can find a $\boldsymbol{t} = (t_0,\dots,t_m) \in Z$ such that $t_k \notin W'(t_0,\dots,t_{k-1},t_{k+1},\dots,t_m)$ for every k. **P** For each $k \le m$ let $A_k \subseteq X_k^{\mathbb{N}}$ be a non-negligible set with cardinal κ_k which (if $k \ge 1$) cannot be covered by κ_{k-1} negligible sets. Choose $t_m, t_{m-1},\dots,t_0$ in such a way that

$$t_k \in A_k, \quad t_k \notin W(\boldsymbol{u}) \text{ whenever } \boldsymbol{u} \in \prod_{j<k} A_j \times \prod_{k<j\le m}\{t_j\};$$

this is always possible because $\#(A_0 \times \dots \times A_{k-1}) = \kappa_{k-1}$ if $k \ge 1$. **Q**

So we get

$$I \le \liminf_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(t_{0i},\dots,t_{mi})$$
$$\le \limsup_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(t_{0i},\dots,t_{mi}) \le I',$$

as claimed.

537L Corollary Let $\langle(X_j,\Sigma_j,\mu_j)\rangle_{j\le m}$ be a finite sequence of probability spaces such that $X_j^{\mathbb{N}}$, with its product measure $\mu_j^{\mathbb{N}}$, has a Sierpiński set with cardinal ω_{j+1} for each $j \le m$. Let $f : \prod_{j\le m} X_j \to \mathbb{R}$ be a bounded function, and suppose that $\sigma : m+1 \to m+1$ and $\tau : m+1 \to m+1$ are permutations such that the two repeated integrals

$$I = \int \dots \int f(x_0,\dots,x_m)dx_{\sigma(m)} \dots dx_{\sigma(0)},$$
$$I' = \int \dots \int f(x_0,\dots,x_m)dx_{\tau(m)} \dots dx_{\tau(0)},$$

are both defined. Then $I = I'$.

proof Apply 537K in both directions.

537M A pair of simple facts which I never got round to spelling out will be useful below.

Lemma Suppose that (X,Σ,μ) is a totally finite measure space and f is a $[0,\infty]$-valued function defined almost everywhere in X.

(a) If $\gamma < \overline{\int} f$, then there is a measurable integrable function $g : X \to [0, \infty[$ such that $\int g \ge \gamma$ and $\{x : x \in \operatorname{dom} f,\ g(x) \le f(x)\}$ has full outer measure in X.

(b) If $\underline{\int} f < \gamma$, then there is a measurable integrable function $g : X \to [0, \infty[$ such that $\int g \le \gamma$ and $\{x : x \in \operatorname{dom} f,\ f(x) \le g(x)\}$ has full outer measure in X.

proof (a) By 135H(b-i),

$$\overline{\int} f = \sup_{k\in\mathbb{N}} \overline{\int} \min(f(x), k)\mu(dx);$$

let $k \in \mathbb{N}$ be such that $\overline{\int} f_k > \gamma$, where $f_k(x) = \min(f(x), k)$ for $x \in \operatorname{dom} f$. Because $\mu X < \infty$, $\overline{\int} f_k$ is finite. By 133J(a-i), there is an integrable h such that $\int h = \overline{\int} f_k$ and $f_k \le_{\text{a.e.}} h$; adjusting h on a negligible set if necessary, we can arrange that h is defined (and finite) everywhere on X and is measurable. Set $\epsilon = (\int h - \gamma)/(1 + \mu X)$, and $g = h - \epsilon\chi X$; then by the last part of 133J(a-i),

$$\{x : x \in \operatorname{dom} f,\ g(x) \le f(x)\} = \{x : x \in \operatorname{dom} f,\ h(x) \le f(x) + \epsilon\}$$

has full outer measure in X, while $\int g \ge \gamma$.

(b) By 135Ha, there is a measurable $h : X \to [0, \infty]$ such that $h \le_{\text{a.e.}} f$ and $\int h = \underline{\int} f$; as $\int h$ is finite, h is finite a.e. and can be adjusted to be finite everywhere. Set $\epsilon = (\gamma - \int h)/(1 + \mu X)$, and $g = h + \epsilon\chi X$; then $\int g \le \gamma$ and $\{x : f(x) \le g(x)\}$ has full outer measure.

537N For ordinary two-variable repeated integrals we can squeeze a little bit more out than is given by 537K.

Proposition Let (X, Σ, μ) be a semi-finite measure space, (Y, T, ν) a probability space, and $\nu^{\mathbb{N}}$ the product measure on $Y^{\mathbb{N}}$. If $\operatorname{non}(E, \mathcal{N}(\mu)) < \operatorname{cov} \mathcal{N}(\nu^{\mathbb{N}})$ for every $E \in \Sigma \setminus \mathcal{N}(\mu)$, then

$$\underline{\int} \underline{\int} f(x, y)\nu(dy)\mu(dx) \le \overline{\int} \overline{\int} f(x, y)\mu(dx)\nu(dy)$$

for every function $f : X \times Y \to [0, \infty]$.

proof (a) To begin with, suppose that $\mu X < \infty$ and $\#(X) < \operatorname{cov} \mathcal{N}(\nu^{\mathbb{N}})$. For each $y \in Y$, let $h_y : X \to [0, \infty]$ be a measurable function such that $f(x, y) \le h_y(x)$ for every $x \in X$ and $\int h_y d\mu = \overline{\int} f(x, y)\mu(dx)$; let $v : Y \to [0, \infty]$ be a measurable function such that $\int h_y d\mu \le v(y)$ for every $y \in Y$ and $\int v\, d\nu = \overline{\int} \overline{\int} f(x, y)\mu(dx)\nu(dy)$. If this is infinite, we can stop. Otherwise, for each $x \in X$ let $g_x : Y \to [0, \infty]$ be a measurable function such that $g_x(y) \le f(x, y)$ for every $y \in Y$ and $\int g_x d\nu = \underline{\int} f(x, y)\nu(dy)$, and let $u : X \to [0, \infty]$ be a measurable function such that $u(x) \le \int g_x\, d\nu$ for every x and $\int u\, d\mu = \underline{\int} \underline{\int} f(x, y)\nu(dy)\mu(dx)$.

As $\#(X) < \operatorname{cov} \mathcal{N}(\nu^{\mathbb{N}})$, we can find a sequence $\langle y_i \rangle_{i\in\mathbb{N}}$ in Y such that

$$\int v\, d\nu = \lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} v(y_i)$$

and

$$\int g_x d\nu = \lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} g_x(y_i)$$

for every $x \in X$. (For by 273J, the set of such sequences is the intersection of fewer than $\operatorname{cov} \mathcal{N}(\nu^{\mathbb{N}})$ conegligible sets in $Y^{\mathbb{N}}$, and cannot be empty.) If $x \in X$, then

$$u(x) \le \int g_x d\nu = \liminf_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} g_x(y_i) \le \liminf_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} h_{y_i}(x).$$

So

$$\underline{\int} \underline{\int} f(x, y)\nu(dy)\mu(dx) = \int u\, d\mu \le \liminf_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} \int h_{y_i} d\mu$$

(by Fatou's Lemma)

$$\le \liminf_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} v(y_i) = \int v\,d\nu$$

$$= \overline{\int}\,\overline{\int} f(x,y)\mu(dx)\nu(dy),$$

as required.

(b) Now suppose that μ is totally finite and that X has a subset A of full outer measure with $\#(A) < \operatorname{cov}\mathcal{N}(\nu^{\mathbb{N}})$. Let μ_A be the subspace measure on A. Then for any $q : X \to [0,\infty]$ we have

$$\underline{\int} q\,d\mu \le \underline{\int} (q{\restriction}A)d\mu_A \le \overline{\int} (q{\restriction}A)d\mu_A \le \overline{\int} q\,d\mu$$

(214J). So, writing f_A for the restriction of f to $A \times Y$,

$$\underline{\int}\,\underline{\int} f(x,y)\nu(dy)\mu(dx) \le \underline{\int}\,\underline{\int} f_A(x,y)\nu(dy)\mu_A(dx)$$

$$\le \overline{\int}\,\overline{\int} f_A(x,y)\mu_A(dx)\nu(dy)$$

(by (a))

$$\le \overline{\int}\,\overline{\int} f(x,y)\mu(dx)\nu(dy).$$

(c) For the general case, let $u : X \to [0,\infty]$ be a measurable function such that $u(x) \le \underline{\int} f(x,y)\nu(dy)$ for every $x \in X$ and $\int u\,d\mu = \underline{\int}\,\underline{\int} f(x,y)\nu(dy)\mu(dx)$. Take any $\gamma < \int u\,d\mu$. Because μ is semi-finite, there is a non-empty set $F \in \Sigma$ of finite measure such that $\int_F u\,d\mu > \gamma$. Now let $\mathcal{E}$ be the family of measurable sets $E \subseteq F$ of finite measure for which there is a non-empty set $A \subseteq E$, with cardinal less than $\operatorname{cov}\mathcal{N}(\nu^{\mathbb{N}})$, such that $\mu^*A = \mu E$, that is, A has full outer measure for the subspace measure μ_E, that is, E is a measurable envelope of A. Then $\mathcal{E}$ is closed under finite unions and every non-empty member of Σ includes a member of $\mathcal{E}$. So there is a non-decreasing sequence $\langle E_k\rangle_{k\in\mathbb{N}}$ in $\mathcal{E}$ such that $\bigcup_{k\in\mathbb{N}} E_k \subseteq F$ and $F \setminus \bigcup_{k\in\mathbb{N}} E_k$ is negligible. In this case, $\gamma < \int_F u\,d\mu = \lim_{k\to\infty} \int_{E_k} u\,d\mu$, so there is a $k \in \mathbb{N}$ such that $\gamma \le \int_{E_k} u\,d\mu$. Set $E = E_k$.

Consider the restriction f_E of f to $E \times Y$ and the subspace measure μ_E on E. We have

$$\gamma \le \int_E u\,d\mu = \int (u{\restriction}E)d\mu_E \le \underline{\int}\,\underline{\int} f_E(x,y)\nu(dy)\mu_E(dx)$$

$$\le \overline{\int}\,\overline{\int} f_E(x,y)\mu_E(dx)\nu(dy)$$

(because $E \in \mathcal{E}$, so we can use (b))

$$\le \overline{\int}\,\overline{\int} f(x,y)\mu(dx)\nu(dy)$$

because $\overline{\int} f_E(x,y)\mu_E(dx) \le \overline{\int} f(x,y)\mu(dx)$ for every y, by 214Ja or otherwise. Since γ is arbitrary,

$$\underline{\int}\,\underline{\int} f(x,y)\nu(dy)\mu(dx) \le \overline{\int}\,\overline{\int} f(x,y)\mu(dx)\nu(dy)$$

in this case also.

537O Corollary Let (X,Σ,μ) and (Y,T,ν) be probability spaces, and $\nu^{\mathbb{N}}$ the product measure on $Y^{\mathbb{N}}$. If $\operatorname{shr}^+\mathcal{N}(\mu) \le \operatorname{cov}\mathcal{N}(\nu^{\mathbb{N}})$ then

$$\overline{\int}\,\underline{\int} f(x,y)\nu(dy)\mu(dx) \le \overline{\int}\,\overline{\int} f(x,y)\mu(dx)\nu(dy)$$

for every function $f : X \times Y \to [0,\infty[$.

proof Take any $\gamma < \overline{\int}\underline{\int} f(x,y)\nu(dy)\mu(dx)$. By 537Ma, there are a measurable function $u : X \to [0,\infty[$ and a set A of full outer measure in X such that $\int u\,d\mu \ge \gamma$ and $u(x) \le \underline{\int} f(x,y)\nu(dy)\mu(dx)$ for every $x \in A$. Let μ_A be the subspace measure on A, and f_A the restriction of f to $A \times Y$. If $B \subseteq A$ is any non-negligible relatively measurable set, there is a non-negligible $D \subseteq B$ such that $\#(D) < \mathrm{shr}^+\mathcal{N}(\mu)$, so

$$\mathrm{non}(B, \mathcal{N}(\mu_A)) = \mathrm{non}(B, \mathcal{N}(\mu)) \le \#(D) < \mathrm{cov}\,\mathcal{N}(\nu^{\mathbb{N}}).$$

So

$$\gamma \le \int u\,d\mu = \int (u{\restriction}A)d\mu_A \le \underline{\int}\underline{\int} f_A(x,y)\nu(dy)\mu_A(dx)$$

(because $u{\restriction}A$ is measurable and $(u{\restriction}A)(x) \le \underline{\int} f_A(x,y)\nu(dy)$ for every $x \in A$)

$$\le \overline{\int}\overline{\int} f_A(x,y)\mu_A(dx)\nu(dy)$$

(by 537N)

$$\le \overline{\int}\overline{\int} f(x,y)\mu(dx)\nu(dy)$$

because $\overline{\int} f_A(x,y)\mu_A(dx) \le \overline{\int} f(x,y)\mu(dx)$ for every y, by 214J again. As γ is arbitrary, we have the result.

Remark There is a similar inequality, under different hypotheses, in 543C below.

537P Corollary Let (X,Σ,μ) and (Y,T,ν) be probability spaces, and $\nu^{\mathbb{N}}$ the product measure on $Y^{\mathbb{N}}$; suppose that $\mathrm{shr}^+\mathcal{N}(\mu) \le \mathrm{cov}\,\mathcal{N}(\nu^{\mathbb{N}})$, and that $f : X \times Y \to \mathbb{R}$ is bounded.

(a)

$$\overline{\int}\underline{\int} f(x,y)\nu(dy)\mu(dx) \le \overline{\int}\overline{\int} f(x,y)\mu(dx)\nu(dy),$$

$$\underline{\int}\underline{\int} f(x,y)\mu(dx)\nu(dy) \le \underline{\int}\overline{\int} f(x,y)\nu(dy)\mu(dx).$$

(b) If $\iint f(x,y)\mu(dx)\nu(dy)$ is defined, and $\int f(x,y)\nu(dy)$ is defined for almost every x, then the other repeated integral $\iint f(x,y)\nu(dy)\mu(dx)$ is defined and equal to $\iint f(x,y)\mu(dx)\nu(dy)$.

proof (a) Apply 537O to the functions $(x,y) \mapsto M + f(x,y)$, $(x,y) \mapsto M - f(x,y)$ for suitable M.

(b) By (a),

$$\iint f(x,y)\mu(dx)\nu(dy) \le \underline{\int}\int f(x,y)\nu(dy)\mu(dx)$$
$$\le \overline{\int}\int f(x,y)\nu(dy)\mu(dx) \le \iint f(x,y)\mu(dx)\nu(dy).$$

537Q We can extend the second part of 537Pa, as well as the first, to unbounded functions, if we strengthen the set-theoretic hypothesis.

Proposition (Humke & Laczkovich 05) Let (X,Σ,ν) and (Y,T,μ) be probability spaces, and $\mu^{\mathbb{N}}$, $\nu^{\mathbb{N}}$ the product measures on $X^{\mathbb{N}}$, $Y^{\mathbb{N}}$ respectively. If $\mathrm{shr}^+\mathcal{N}(\mu^{\mathbb{N}}) \le \mathrm{cov}\,\mathcal{N}(\nu^{\mathbb{N}})$ then $\underline{\int}\underline{\int} f(x,y)\mu(dx)\nu(dy) \le \underline{\int}\overline{\int} f(x,y)\nu(dy)\mu(dx)$ for every function $f : X \times Y \to [0,\infty[$.

proof ? Suppose, if possible, otherwise.

(a) There is a measurable function $u : Y \to [0,\infty[$ such that

$$u(y) \le \underline{\int} f(x,y)\mu(dx) \text{ for every } y, \quad \underline{\int}\overline{\int} f(x,y)\nu(dy)\mu(dx) < \int u\,d\nu.$$

Since $\int u\,d\nu$ is the supremum of the integrals of the non-negative simple functions dominated by u, we may suppose that u itself is a simple function; express it as $\sum_{j=0}^{m}\alpha_j\chi F_j$ where $\alpha_j \ge 0$ for each i and $(F_0,\dots,F_m)$ is a partition of Y into measurable sets. Now

$$\sum_{j=0}^{m}\underline{\int}\overline{\int} f(x,y)\chi F_j(y)\nu(dy)\mu(dx) \le \underline{\int}\sum_{j=0}^{m}\overline{\int} f(x,y)\chi F_j(y)\nu(dy)\mu(dx)$$

(133J(b-v))

$$\le \underline{\int}\overline{\int} f(x,y)\nu(dy)\mu(dx)$$

(because if $x\in X$ and $q : Y \to [0,\infty]$ is measurable and $f(x,y) \le q(y)$ for every y, then the sum $\sum_{j=0}^{m}\overline{\int} f(x,y)\chi F_j(y)\nu(dy)$ is at most $\sum_{j=0}^{m}\int q\times\chi F_j d\nu = \int q\,d\nu$)

$$< \int u\,d\nu = \sum_{j=0}^{m}\alpha_j\nu F_j.$$

There are therefore a $j\le m$ and a $\gamma<1$ such that

$$\underline{\int}\overline{\int} f(x,y)\chi F_j(y)\nu(dy)\mu(dx) < \gamma\alpha_j\nu F_j.$$

Now there is a measurable function $v : X \to [0,\infty[$ such that $\int v\,d\mu \le \gamma\alpha_j\nu F_j$ and

$$D = \{x : x\in X,\ \overline{\int} f(x,y)\chi F_j(y)\nu(dy) \le v(x)\}$$

has full outer measure in X, by 537Mb.

(b) For $y\in Y$ and $\boldsymbol{x} = \langle x_i\rangle_{i\in\mathbb{N}} \in X^{\mathbb{N}}$, set $h(\boldsymbol{x},y) = \liminf_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^{n} f(x_i,y)$. If $y\in Y$, then $\underline{\int} f(x,y)\mu(dx) \le h(\boldsymbol{x},y)$ for $\mu^{\mathbb{N}}$-almost every $\boldsymbol{x}$. **P** We have a measurable function $q : X\to[0,\infty[$ such that $q(x)\le f(x,y)$ for every x and

$$\underline{\int} f(x,y)\mu(dx) = \int q\,d\mu \le \liminf_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^{n} q(x_i)$$

$$\le \liminf_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^{n} f(x_i,y) = h(\boldsymbol{x},y)$$

for almost every $\boldsymbol{x} = \langle x_i\rangle_{i\in\mathbb{N}}$. **Q** At the same time,

$$V = \{\langle x_i\rangle_{i\in\mathbb{N}} : \liminf_{n\to\infty}\tfrac{1}{n+1}\textstyle\sum_{i=0}^{n} v(x_i) \le \gamma\alpha_j\nu F_j\}$$

is conegligible in $X^{\mathbb{N}}$, because $\int v\,d\mu \le \gamma\alpha_j\nu F_j$.

(c) Set

$$W = \{(\boldsymbol{x},y) : \boldsymbol{x}\in X^{\mathbb{N}},\ y\in F_j,\ h(\boldsymbol{x},y)\ge\alpha_j\}$$

and consider the function $\chi W : X^{\mathbb{N}}\times Y \to \{0,1\}$. If $y\in F_j$ then $\underline{\int} f(x,y)\mu(dx) \ge \alpha_j$ so $W^{-1}[\{y\}]$ is conegligible in $X^{\mathbb{N}}$. On the other hand, if $\boldsymbol{x} = \langle x_i\rangle_{i\in\mathbb{N}}$ belongs to $V\cap D^{\mathbb{N}}$,

$$\overline{\int}\alpha_j\chi W(\boldsymbol{x},y)\nu(dy) \le \overline{\int} h(\boldsymbol{x},y)\chi F_j(y)\nu(dy)$$

$$= \overline{\int}\liminf_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^{n} f(x_i,y)\chi F_j(y)\nu(dy)$$

$$\le \liminf_{n\to\infty}\overline{\int}\frac{1}{n+1}\sum_{i=0}^{n} f(x_i,y)\chi F_j(y)\nu(dy)$$

(133Kb)

$$\leq \liminf_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} \overline{\int} f(x_i, y)\chi F_j(y)\nu(dy)$$

(133J(b-ii))

$$\leq \liminf_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} v(x_i) \leq \gamma\alpha_j \nu F_j.$$

(d) As V is conegligible and $D^{\mathbb{N}}$ has full outer measure (254Lb),

$$\underline{\int}\overline{\int} \chi W(\boldsymbol{x}, y)\nu(dy)\mu^{\mathbb{N}}(d\boldsymbol{x}) \leq \gamma\nu F_j < \nu F_j = \iint \chi W(\boldsymbol{x}, y)\mu^{\mathbb{N}}(d\boldsymbol{x})\nu(dy)$$

$$= \underline{\int}\,\underline{\int} \chi W(\boldsymbol{x}, y)\mu^{\mathbb{N}}(d\boldsymbol{x})\nu(dy).$$

But we are supposing that $\mathrm{shr}^+\mathcal{N}(\mu^{\mathbb{N}}) \leq \mathrm{cov}\,\mathcal{N}(\nu^{\mathbb{N}})$, so this contradicts 537P. **X**

So we have the result.

537R Lemma Let (X, Σ, μ) be a complete probability space and (Y, T, ν) a probability space such that $\mathrm{shr}^+\mathcal{N}(\mu) \leq \mathrm{cov}\,\mathcal{N}(\nu^{\mathbb{N}})$, where $\nu^{\mathbb{N}}$ is the product measure on $Y^{\mathbb{N}}$. Let $f : X \times Y \to \mathbb{R}$ be a bounded function which is measurable in each variable separately, and set $u(x) = \int f(x, y)\nu(dy)$ for $x \in X$. Then $u : X \to \mathbb{R}$ is measurable.

proof ? Otherwise, there are a non-negligible measurable set $E \subseteq X$ and α, $\beta \in \mathbb{R}$ such that $\alpha < \beta$ and

$$\mu^*\{x : x \in E,\, u(x) \leq \alpha\} = \mu^*\{x : x \in E,\, u(x) \geq \beta\} = \mu E$$

(413G). Let $A \subseteq \{x : x \in E,\, u(x) \leq \alpha\}$ and $B \subseteq \{x : x \in E,\, u(x) \geq \beta\}$ be sets with cardinal less than $\mathrm{shr}^+\mathcal{N}(\mu)$ and outer measure greater than $\frac{1}{2}\mu E$ (521Ca). Let $\langle y_i \rangle_{i\in\mathbb{N}}$ be a sequence in Y such that

$$u(x) = \lim_{n\to\infty} \tfrac{1}{n+1}\textstyle\sum_{i=0}^{n} f(x, y_i)$$

for every $x \in A \cup B$. Because $x \mapsto f(x, y_i)$ is measurable for each i, $u\restriction A \cup B$ is measurable; but this means that A and B can be separated by measurable sets, which is impossible, because $\mu^*A + \mu^*B > \mu E$. **X**

537S Proposition Let (X, Σ, μ) and (Y, T, ν) be probability spaces such that

$$\mathrm{shr}^+\mathcal{N}(\mu) \leq \mathrm{cov}\,\mathcal{N}(\nu^{\mathbb{N}}),$$

where $\nu^{\mathbb{N}}$ is the product measure on $Y^{\mathbb{N}}$, and

$$\mathrm{cf}([\tau(\nu)]^{\leq\omega}) < \mathrm{cov}(E, \mathcal{N}(\mu)) \text{ for every } E \in \Sigma \setminus \mathcal{N}(\mu),$$

where $\tau(\nu)$ is the Maharam type of ν. Let $f : X \times Y \to [0, \infty[$ be a function which is measurable in each variable separately. Then $\iint f(x, y)\mu(dx)\nu(dy)$ and $\iint f(x, y)\nu(dy)\mu(dx)$ exist and are equal.

proof (a) Let $\tilde{\Lambda} \supseteq \Sigma\widehat{\otimes}\mathrm{T}$ be the σ-algebra of sets $W \subseteq X \times Y$ such that all the vertical and horizontal sections of W are measurable. If $W \in \tilde{\Lambda}$, then $x \mapsto \nu W[\{x\}] : X \to [0, 1]$ is measurable, by 537R. If $W \in \tilde{\Lambda}$ and almost every horizontal section of W is negligible, then

$$\overline{\int} \nu W[\{x\}]\mu(dx) = \overline{\int}\,\underline{\int} \chi W(x, y)\nu(dy)\mu(dx)$$

$$\leq \overline{\int}\,\overline{\int} \chi W(x, y)\mu(dx)\nu(dy) = 0$$

by 537Pa, so almost every vertical section of W is negligible.

(b) Let $(\mathfrak{B}, \bar{\nu})$ be the measure algebra of (Y, T, ν). If $W \in \tilde{\Lambda}$ and there is a metrically separable subalgebra $\mathfrak{C}$ of $\mathfrak{B}$ containing $W[\{x\}]^\bullet$ for every $x \in X$, then there is a $W' \in \Sigma\widehat{\otimes}\mathrm{T}$ such that $W[\{x\}]\triangle W'[\{x\}]$ is negligible for almost every x. **P** Note first that for every $F \in \mathrm{T}$ the map

$$x \mapsto \nu(W[\{x\}] \triangle F) = \nu((W \triangle (X \times F))[\{x\}]$$

is measurable, by (a). So $x \mapsto W[\{x\}]^\bullet : X \to \mathfrak{C}$ is measurable, by 418Bc. By 418T(b-ii), there is a $W' \in \Sigma \widehat{\otimes} \mathrm{T}$ such that $W[\{x\}]^\bullet = W'[\{x\}]^\bullet$ for almost every x. **Q**

(c) In fact we find that for any $W \in \tilde{\Lambda}$ there is a $W' \in \Sigma \widehat{\otimes} \mathrm{T}$ such that $W[\{x\}] \triangle W'[\{x\}]$ is negligible for almost every x. **P** Set $\kappa = \tau(\nu) = \tau(\mathfrak{B})$, and let $\langle e_\xi \rangle_{\xi<\kappa}$ generate $\mathfrak{B}$. Let $\mathcal{K} \subseteq [\kappa]^{\le\omega}$ be a cofinal set with cardinal $\mathrm{cf}[\kappa]^{\le\omega}$. For $K \in \mathcal{K}$, let $\mathfrak{B}_K$ be the closed subalgebra of $\mathfrak{B}$ generated by $\{e_\xi : \xi \in K\}$ and A_K the set $\{x : x \in X,\ W[\{x\}]^\bullet \in \mathfrak{B}_K\}$. Note that $K \mapsto A_K$ is non-decreasing and that the union of any sequence in $\mathcal{K}$ is included in a member of $\mathcal{K}$. So there is a $K_0 \in \mathcal{K}$ such that $\mu^* A_{K_0} = \sup_{K \in \mathcal{K}} \mu^* A_K$.

If E is a measurable envelope of A_{K_0}, then $\{A_K \setminus E : K \in \mathcal{K}\}$ is a cover of $X \setminus E$ by negligible sets. So $\mathrm{cov}(X \setminus E, \mathcal{N}(\mu)) \le \mathrm{cf}[\kappa]^{\le\omega}$ and $X \setminus E$ must be negligible, that is, A_{K_0} has full outer measure.

Taking a sequence $\langle F_n \rangle_{n\in\mathbb{N}}$ in T such that $\{F_n^\bullet : n \in \mathbb{N}\}$ is dense in $\mathfrak{B}_{K_0}$, we see from (a) that $x \mapsto \inf_{n\in\mathbb{N}} \nu(W[\{x\}] \triangle F_n)$ is measurable, while it is zero on A_{K_0}. So $W[\{x\}]^\bullet \in \mathfrak{B}_{K_0}$ for almost every $x \in X$, that is, A_{K_0} is actually conegligible. Taking a measurable conegligible set $E' \subseteq A_{K_0}$ and applying (b) to $W \cap (E' \times Y)$, we see that there is a $W' \in \Sigma \widehat{\otimes} \mathrm{T}$ such that $W[\{x\}] \triangle W'[\{x\}]$ is negligible for almost every $x \in X$. **Q**

(d) Now turn to the function f under consideration. For $q \in \mathbb{Q}$ set $W_q = \{(x,y) : f(x,y) \ge q\} \in \tilde{\Lambda}$. By (c), we have $V_q \in \Sigma \widehat{\otimes} \mathrm{T}$ such that $V_q[\{x\}] \triangle W_q[\{x\}]$ is ν-negligible for μ-almost every x, and therefore $W_q^{-1}[\{y\}] \triangle V_q^{-1}[\{y\}]$ is μ-negligible for ν-almost every y, by (a). If $q \le q'$ then $W_{q'} \setminus W_q$ is empty, so $V_{q'}[\{x\}] \setminus V_q[\{x\}]$ is ν-negligible for μ-almost every x, and $V_{q'} \setminus V_q$ is $(\mu \times \nu)$-negligible, where $\mu \times \nu$ is the product measure on $X \times Y$. Similarly, $\bigcap_{q'<q} V_{q'} \setminus V_q$ is negligible for every q. Moreover, writing V_∞ for $\bigcap_{q\in\mathbb{Q}} V_q$, $V_\infty[\{x\}]$ is ν-negligible for μ-almost every x, so $(\mu \times \nu)V_\infty = 0$; similarly, $(\mu \times \nu)V_0 = 1$. There is therefore a $\Sigma \widehat{\otimes} \mathrm{T}$-measurable $g : X \times Y \to [0,\infty[$ such that $V_q \triangle \{(x,y) : g(x,y) \ge q\}$ is $(\mu \times \nu)$-negligible for every $q \in \mathbb{Q}$. In this case,

$$\{x : f(x,y) \ne g(x,y)\} \text{ is } \mu\text{-negligible for } \nu\text{-almost every } y,$$

$$\{y : f(x,y) \ne g(x,y)\} \text{ is } \nu\text{-negligible for } \mu\text{-almost every } x,$$

and

$$\begin{aligned} \iint f(x,y)\mu(dx)\nu(dy) &= \iint g(x,y)\mu(dx)\nu(dy) \\ &= \iint g(x,y)\nu(dy)\mu(dx) = \iint f(x,y)\nu(dy)\mu(dx) \end{aligned}$$

by 252H.

(e) Finally, if f is unbounded, set $f_k(x,y) = \min(f(x,y), k)$ for each $k \in \mathbb{N}$. Then

$$\begin{aligned} \iint f(x,y)\mu(dx)\nu(dy) &= \lim_{k\to\infty} \iint f_k(x,y)\mu(dx)\nu(dy) \\ &= \lim_{k\to\infty} \iint f_k(x,y)\nu(dy)\mu(dx) = \iint f(x,y)\nu(dy)\mu(dx). \end{aligned}$$

537X Basic exercises (a)(i) Let (X, Σ, μ) be a measure space such that singletons are negligible and $\mathrm{cf}\,\mathcal{N}(\mu) = \omega_1$. Show that there is a Sierpiński subset of X. (ii) Show that if μ is Lebesgue measure on $\mathbb{R}$ and $\mathrm{cf}\,\mathcal{N}(\mu) = \omega_1$, then there is a strongly Sierpiński subset of $\mathbb{R}$.

(b) Show that for any uncountable cardinal κ there is a purely atomic probability space with a strongly Sierpiński set with cardinal κ.

(c) Let (X, Σ, μ) be a measure space. Show that the union of any sequence of Sierpiński sets in X is again a Sierpiński set in X.

(d) Let (X,Σ,μ) be a measure space and Y any subspace of X. Show that a subset of Y is a Sierpiński set for the subspace measure on Y iff it is a Sierpiński set for μ.

(e) Suppose that λ is an infinite cardinal and the usual measure ν_λ on $\{0,1\}^\lambda$ has a Sierpiński set with cardinal κ. Show that ν_λ has a Sierpiński set A such that $\#(A\cap E)=\kappa$ whenever $\nu_\lambda E>0$.

(f) Let (X,ρ) be a non-separable metric space with r-dimensional Hausdorff measure, where $r>0$. Show that X has a Sierpiński subset with cardinal equal to the topological density of X.

>(g) Suppose that $\operatorname{non}\mathcal{N}<\operatorname{cov}\mathcal{N}$, where $\mathcal{N}$ is the null ideal of Lebesgue measure on $\mathbb{R}$. Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be Radon probability spaces of countable Maharam type, and $f:X\times Y\to[0,\infty[$ a function such that $I=\iint f(x,y)\mu(dx)\nu(dy)$ and $I'=\iint f(x,y)\nu(dy)\mu(dx)$ are both defined. Show that $I=I'$.

>(h) Let (X,Σ,μ) be a probability space in which there is a well-ordered family in $\mathcal{N}(\mu)$ with union X; e.g., because $\operatorname{non}\mathcal{N}(\mu)=\#(X)$ or $\operatorname{add}\mathcal{N}(\mu)=\operatorname{cov}\mathcal{N}(\mu)$. Show that there is a function $f:X\times X\to[0,1]$ such that $\int f(x,y)\mu(dx)=0$ for every $y\in X$ and $\int f(x,y)\mu(dy)=1$ for every $x\in X$.

>(i) (In this exercise, all integrals are to be taken with respect to one-dimensional Lebesgue measure μ.) (i) Find a function $f:[0,1]^2\to\{0,1\}$ such that $\int\overline{\int}f(x,y)dxdy=1$ but $\iint f(x,y)dydx=0$. (*Hint*: there is a disjoint family $\langle A_y\rangle_{y\in[0,1]}$ of sets of full outer measure.) (ii) Find a function $f:[0,1]^2\to\{0,1\}$ such that $\iint f(x,y)dxdy=1$ but $\int\underline{\int}f(x,y)dydx=0$. (iii) Find a function $f:[0,1]^2\to\{0,1\}$ such that $\overline{\int}\int f(x,y)dxdy=1$ but $\underline{\int}\int f(x,y)dydx=0$. (*Hint*: enumerate $[0,1]$ as $\langle x_\xi\rangle_{\xi<\mathfrak{c}}$ in such a way that $\{x_\xi:\xi<\operatorname{non}\mathcal{N}(\mu)\}$ has full outer measure; set $f(x_\xi,x_\eta)=1$ if $\eta<\xi$.)

537Y Further exercises (a) Let $\langle(X_j,\Sigma_j,\mu_j)\rangle_{j\le m}$ be a finite sequence of probability spaces and $\langle\kappa_j\rangle_{j\le m}$ a sequence of cardinals such that X_j has a subset with cardinal κ_j which is not covered by κ_{j-1} negligible sets (if $j\ge1$) and is not negligible (if $j=0$). Set $X=\prod_{j\le m}X_j$, and for $k\le m$ write Z_k for $\prod_{j\le m,j\ne k}X_j$. Suppose that for each $k\le m$ we have a set $A_k\subseteq X$ such that, identifying X with $X_k\times Z_k$, $\{z:(x,z)\in A_k\}\subseteq Z_k$ is negligible for the product measure on Z_k whenever $x\in X_k$. Show that $\bigcup_{k\le m}A_k\ne X$.

537Z Problems (a) Is it relatively consistent with ZFC to suppose that $\mathbb{R}$, with Lebesgue measure, has a Sierpiński subset but no strongly Sierpiński subset?

(b) Is it relatively consistent with ZFC to suppose that there is a probability space (X,μ) such that (X,μ) has a Sierpiński set but its power $(X^{\mathbb{N}},\mu^{\mathbb{N}})$ does not?

537 Notes and comments It is easy to see that if $\mathfrak{c}=\omega_1$ then there is a strongly Sierpiński set with cardinal ω_1 for Lebesgue measure (537Xa). Countable-cocountable measures have strongly Sierpiński sets for trivial reasons. To eliminate all Sierpiński sets (on the definition of 537A) from atomless complete locally determined measure spaces, it is enough to ensure that the uniformity of Lebesgue measure is greater than ω_1 (537Bb). For the simplest models with non-trivial Sierpiński sets with cardinal greater than ω_1, see 552E below.

The 'entangled sets' of 537C-537G belong rather to combinatorics than to measure theory; I go as far as I do into this theory because it is interesting in view of 552E. But it includes a proof that if the continuum hypothesis is true then there are two ccc partially ordered sets whose product is not ccc, which in its own context is of great importance.

Fubini's theorem is so important in measure theory that exploration of its boundaries has been a perennial challenge. I gave elementary examples in 252Xf-252Xg to show that as soon as we abandon the requirement that $\iint|f(x,y)|dxdy<\infty$ our repeated integrals can be expected to be unreliable. But for non-negative functions f on σ-finite spaces, measurability is enough to ensure that repeated integrals are equal (252H). In this section I look for results which will be valid for non-measurable functions. In 537I-537J we have a rather esoteric example – or, some would say, an example from a topic which I have neglected in this book – which is unusual in that it is a theorem of ZFC; for a note on its ancestry see Fremlin 93, 5L. In 537K-537L we see that, in the presence of a sufficient supply of Sierpiński sets, for instance, we must have

$\iint f(x,y)dxdy = \iint f(x,y)dydx$ for ordinary bounded real-valued functions on the product of probability spaces, as long as both repeated integrals are defined. The argument here depends on using the strong law of large numbers to replace an integral $\int f(x,y)dx$ by the limit of a sequence of averages of values $f(x_i,y)$. This is why the Sierpiński sets must be available not in the original probability spaces $X_0,\ldots,X_m$ but in their powers $X_j^{\mathbb{N}}$. Of course for our favourite spaces, starting with $[0,1]$, $(X^{\mathbb{N}},\mu^{\mathbb{N}})$ is isomorphic to (X,μ), so this does not seem too large a step; but it begs an obvious question (537Zb). For any result of this kind we certainly need some special axiom (537Xh).

In 537L the hypothesis includes strong 'separate measurability' conditions; we need not only separate measurability, but measurability of the functions $x\mapsto\int f(x,y)dy$ and $y\mapsto\int f(y,x)dx$. With a different set-theoretic hypothesis we can relax these (537S). I approach this form through ideas from Humke & Laczkovich 05, where there is a careful analysis of repeated integrals of the form $\underline{\int}\overline{\int}$, etc. My own version is in 537N-537Q. At every step there are ZFC examples to show that we cannot change the formulae involving $\underline{\int}$, $\overline{\int}$ without disaster (537Xi); but it is not so clear that the set-theoretic hypotheses offered are unimprovable.

Version of 18.2.14

538 Filters and limits

A great many special types of filter have been studied. In this section I look at some which are particularly interesting from the point of view of measure theory: Ramsey ultrafilters, measure-converging filters and filters with the Fatou property. About half the section is directed towards Benedikt's theorem (538M) on extensions of perfect probability measures; on the way we need to look at measure-centering ultrafilters (538G-538K) and iterated products of filters (538E, 538L). The second major topic here is a study of 'medial limits' (538P-538S); these are Banach limits of a very special type. In between, the measure-converging property (538N) and the Fatou property (538O) offer some intriguing patterns.

538A Filters For ease of reference, I begin the section with a list of the special types of filter on $\mathbb{N}$ which we shall be looking at later.

Definitions Let $\mathcal{F}$ be a filter on $\mathbb{N}$.

(a) $\mathcal{F}$ is **free** if it contains every cofinite subset of $\mathbb{N}$, that is, includes the Fréchet filter.

(b) $\mathcal{F}$ is a **p-point filter** if it is free and for every sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{F}$ there is an $A\in\mathcal{F}$ such that $A\setminus A_n$ is finite for every $n\in\mathbb{N}$. (Compare 5A6Ga.)

(c) $\mathcal{F}$ is **Ramsey** or **selective** if it is free and for every $f:[\mathbb{N}]^2\to\{0,1\}$ there is an $A\in\mathcal{F}$ such that f is constant on $[A]^2$.

(d) $\mathcal{F}$ is **rapid** if it is free and for every sequence $\langle t_n\rangle_{n\in\mathbb{N}}$ of real numbers which converges to 0, there is an $A\in\mathcal{F}$ such that $\sum_{n\in A}|t_n|$ is finite. Note that a free filter $\mathcal{F}$ on $\mathbb{N}$ is rapid iff for every $f\in\mathbb{N}^{\mathbb{N}}$ there is an $A\in\mathcal{F}$ such that $\#(A\cap f(k))\le k$ for every $k\in\mathbb{N}$. **P** (i) If $\mathcal{F}$ is rapid and $f\in\mathbb{N}^{\mathbb{N}}$, let $g\in\mathbb{N}^{\mathbb{N}}$ be a strictly increasing sequence such that $f\le g$. Set $t_i=2$ if $i<g(0)$, $\frac{1}{k+1}$ if $g(k)\le i<g(k+1)$; then there is an $A\in\mathcal{F}$ such that $\sum_{i\in A}t_i$ is finite; as $\mathcal{F}$ is free, there is an $A\in\mathcal{F}$ such that $\sum_{i\in A}t_i\le 1$, in which case $\#(A\cap f(k))\le\#(A\cap g(k))\le k$ for every $k\in\mathbb{N}$. (ii) If $\mathcal{F}$ satisfies the condition and $\langle t_i\rangle_{i\in\mathbb{N}}\to 0$, take a strictly increasing $f\in\mathbb{N}^{\mathbb{N}}$ such that $|t_i|\le 2^{-k}$ whenever $k\in\mathbb{N}$ and $i\ge f(k)$; let $A\in\mathcal{F}$ be such that $\#(A\cap f(k))\le k$ for every k; then $\sum_{i\in A}|t_i|\le\sum_{k=0}^{\infty}2^{-k}\#(A\cap f(k+1)\setminus f(k))$ is finite. **Q**

(e) $\mathcal{F}$ is **nowhere dense** if for every sequence $\langle t_n\rangle_{n\in\mathbb{N}}$ in $\mathbb{R}$ there is an $A\in\mathcal{F}$ such that $\{t_n:n\in A\}$ is nowhere dense.

(f) $\mathcal{F}$ is **measure-centering** or has **property M** if whenever $\mathfrak{A}$ is a Boolean algebra, $\nu:\mathfrak{A}\to[0,\infty[$ is an additive functional, and $\langle a_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}$ such that $\inf_{n\in\mathbb{N}}\nu a_n>0$, there is an $A\in\mathcal{F}$ such that $\{a_n:n\in A\}$ is centered.

(g) $\mathcal{F}$ is **measure-converging** if whenever (X,Σ,μ) is a probability space, $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in Σ, and $\lim_{n\to\infty}\mu E_n=1$, then $\bigcup_{A\in\mathcal{F}}\bigcap_{n\in A}E_n$ is conegligible.

(h) $\mathcal{F}$ has the **Fatou property** if whenever (X,Σ,μ) is a probability space, $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in Σ, and $X=\bigcup_{A\in\mathcal{F}}\bigcap_{n\in A}E_n$, then $\lim_{n\to\mathcal{F}}\mu E_n$ is defined and equal to 1.

(i) For any countably infinite set I, I will say that a filter $\mathcal{F}$ on I is free, or a p-point filter, or Ramsey, etc., if it is isomorphic to such a filter on $\mathbb{N}$. Of course this usage is possible only because every property here is invariant under permutations of $\mathbb{N}$. For 'rapid' and 'measure-converging' filters, we need an appropriate translation of 'sequence converging to 0'; the corresponding notion on an arbitrary index set I is a function $u\in\boldsymbol{c}_0(I)$, that is, a real-valued function u on I such that $\{i:i\in I,\,|u(i)|\ge\epsilon\}$ is finite for every $\epsilon>0$; if we give I its discrete topology, $c_0(I)$ is $C_0(I)$ as defined in 436I.

538B We need a number of basic ideas which can profitably be examined in a rather more general context. I start with a fundamental pre-order on the class of all filters.

The Rudin-Keisler ordering If $\mathcal{F}$, $\mathcal{G}$ are filters on sets I, J respectively, I will say that $\mathcal{F}\le_{\mathrm{RK}}\mathcal{G}$ if there is a function $f:J\to I$ such that

$$\mathcal{F}=f[[\mathcal{G}]]=\{A:A\subseteq I,\,f^{-1}[A]\in\mathcal{G}\},$$

the filter on I generated by $\{f[B]:B\in\mathcal{G}\}$. (I ought to remark that while this is a standard idea for ultrafilters, in the case of general filters the terminology is not well established.) Of course $\le_{\mathrm{RK}}$ is reflexive and transitive. If $\mathcal{F}\le_{\mathrm{RK}}\mathcal{G}$ and $\mathcal{G}$ is an ultrafilter, then $\mathcal{F}$ is an ultrafilter (2A1N). If $\mathcal{F}$ is a principal ultrafilter then $\mathcal{F}\le_{\mathrm{RK}}\mathcal{G}$ for every filter $\mathcal{G}$.

538C Lemma (a) If I is a set, $\mathcal{F}$ is an ultrafilter on I and $f:I\to I$ is a function such that $f[[\mathcal{F}]]=\mathcal{F}$, then $\{i:f(i)=i\}\in\mathcal{F}$.

(b) If I is a set, $\mathcal{F}$ and $\mathcal{G}$ are ultrafilters on I, $\mathcal{F}\le_{\mathrm{RK}}\mathcal{G}$ and $\mathcal{G}\le_{\mathrm{RK}}\mathcal{F}$, then there is a permutation $h:I\to I$ such that $h[[\mathcal{F}]]=\mathcal{G}$; that is, $\mathcal{F}$ and $\mathcal{G}$ are isomorphic.

proof (a) It is enough to consider the case in which $I=\kappa$ is a cardinal.

(i) $\{\xi:\xi<\kappa,\,\xi\le f(\xi)\}\in\mathcal{F}$. **P** Define $\langle D_n\rangle_{n\in\mathbb{N}}$, $\langle E_n\rangle_{n\in\mathbb{N}}$ by saying that

$$D_0=\kappa,\quad D_{n+1}=\{\xi:\xi\in D_n,\,f(\xi)\in D_n,\,f(\xi)<\xi\},\quad E_n=D_n\setminus D_{n+1}$$

for $n\in\mathbb{N}$. If $\xi\in D_n$ then $\xi>f(\xi)>\ldots>f^n(\xi)$, so $\bigcap_{n\in\mathbb{N}}D_n=\emptyset$ and $\langle E_n\rangle_{n\in\mathbb{N}}$ is a partition of κ. If $\xi\in E_{n+1}$ then $f^{n+1}(\xi)<f^n(\xi)<\ldots<\xi$, $f^{n+1}(\xi)\le f^{n+2}(\xi)$, so $f(\xi)\in E_n$. Set $E=\bigcup_{n\ge1}E_{2n}$, $E'=\bigcup_{n\in\mathbb{N}}E_{2n+1}$; then $f[E]\subseteq E'$ is disjoint from E, so $E\notin\mathcal{F}$. Also $f[E']\subseteq E\cup E_0$ is disjoint from E', so $E'\notin\mathcal{F}$. Because $\mathcal{F}$ is an ultrafilter, $E_0\in\mathcal{F}$, as claimed. **Q**

(ii) If $A\subseteq I$ and $A\notin\mathcal{F}$ then $B=\bigcup_{n\in\mathbb{N}}(f^n)^{-1}[A]$ does not belong to $\mathcal{F}$. **P** For $\xi\in B$ set $m(\xi)=\min\{n:n\in\mathbb{N},\,f^n(\xi)\in A\}$. If $m(\xi)>0$ then $m(f(\xi))=m(\xi)-1$. So setting $C=\{\xi:m(\xi)$ is even and not $0\}$, $C'=\{\xi:m(\xi)$ is odd$\}$ we have $f[C]\cap C=\emptyset$, $f[C']\cap C'=\emptyset$ and $B\subseteq A\cup C\cup C'$; so $B\notin\mathcal{F}$. **Q**

Turning this round, if $A\in\mathcal{F}$ then $\bigcup_{n\in\mathbb{N}}(f^n)^{-1}[\kappa\setminus A]\notin\mathcal{F}$ and $\bigcap_{n\in\mathbb{N}}(f^n)^{-1}[A]\in\mathcal{F}$.

(iii) For $\xi<\kappa$ set

$$g(\xi)=\min\{\zeta:\text{there is some }n\in\mathbb{N}\text{ such that }f^n(\zeta)=\xi\}.$$

Then $g[[\mathcal{F}]]=\mathcal{F}$. **P** If $A\in\mathcal{F}$ then $\mathcal{F}$ contains $\bigcap_{n\in\mathbb{N}}(f^n)^{-1}[A]\subseteq g^{-1}[A]$, so $g^{-1}[A]\in\mathcal{F}$. Thus $\mathcal{F}\subseteq g[[\mathcal{F}]]$; as $\mathcal{F}$ is an ultrafilter, $\mathcal{F}=g[[\mathcal{F}]]$. **Q**

Now $g(\xi)\le\xi$ for every $\xi<\kappa$; applying (i) to g, we see that $G=\{\xi:g(\xi)=\xi\}\in\mathcal{F}$. But consider $H=\{\xi:\xi<f(\xi)\}$. Then $g(\eta)<\eta$ for every $\eta\in f[H]$, so $f[H]\notin\mathcal{F}$ and $H\notin\mathcal{F}$. Since we already know that $\{\xi:\xi\le f(\xi)\}\in\mathcal{F}$, we see that $\{\xi:f(\xi)=\xi\}$ belongs to $\mathcal{F}$, as claimed.

(b) Let f, $g:I\to I$ be such that $f[[\mathcal{F}]]=\mathcal{G}$ and $g[[\mathcal{G}]]=\mathcal{F}$. Then $(gf)[[\mathcal{F}]]=g[[f[[\mathcal{F}]]\,]]=\mathcal{F}$, so $J_0=\{i:g(f(i))=i\}\in\mathcal{F}$, by (a). Similarly, $J_1=\{i:f(g(i))=i\}$ belongs to $\mathcal{G}$. Set $J=J_0\cap f^{-1}[J_1]\in\mathcal{F}$; then $g(f(i))=i$ for every $i\in J$ and $f(g(j))=j$ for every $j\in f[J]$, so $f{\restriction}J$ and $g{\restriction}f[J]$ are inverse bijections between $J\in\mathcal{F}$ and $f[J]\in\mathcal{G}$. If J is finite, then certainly $\#(I\setminus J)=\#(I\setminus f[J])$ and there is an extension of $f\restriction J$ to a permutation of I. If J is infinite, let $J'\subseteq J$ be a set such that $\#(J')=\#(J\setminus J')=\#(J)$ and $J'\in\mathcal{F}$; then $\#(I\setminus J')=\#(I\setminus f[J'])=\#(I)$ so there is an extension of $f\restriction J'$ to a permutation of I.

Thus in either case we have a permutation $h:I\to I$ and a $K\in\mathcal{F}$ such that $K\subseteq J$ and $h\restriction K=f\restriction K$. But now $h[[\mathcal{F}]]=\mathcal{G}$ and h is an isomorphism between $(I,\mathcal{F})$ and $(I,\mathcal{G})$.

538D Finite products of filters (a) Suppose that $\mathcal{F}$, $\mathcal{G}$ are filters on sets I, J respectively. I will write $\mathcal{F}\ltimes\mathcal{G}$ for

$$\{A:A\subseteq I\times J,\,\{i:A[\{i\}]\in\mathcal{G}\}\in\mathcal{F}\}.$$

It is easy to check that $\mathcal{F}\ltimes\mathcal{G}$ is a filter. (Compare the skew product $\mathcal{I}\ltimes\mathcal{J}$ of ideals defined in 527Ba.)

(b) If $\mathcal{F}$ and $\mathcal{G}$ are ultrafilters, so is $\mathcal{F}\ltimes\mathcal{G}$. **P** If $A\subseteq I\times J$ and $A\notin\mathcal{F}\ltimes\mathcal{G}$, then $\{i:A[\{i\}]\in\mathcal{G}\}\notin\mathcal{F}$ and

$$\{i:((I\times J)\setminus A)[\{i\}]\in\mathcal{G}\}=\{i:i\in I,\,J\setminus A[\{i\}]\in\mathcal{G}\}=I\setminus\{i:A[\{i\}]\in\mathcal{G}\}\in\mathcal{F},$$

so $(I\times J)\setminus A\in\mathcal{F}\ltimes\mathcal{G}$. **Q**

(c) If $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ are filters on I, J, K respectively, then the natural bijection between $(I\times J)\times K$ and $I\times(J\times K)$ is an isomorphism between $(\mathcal{F}\ltimes\mathcal{G})\ltimes\mathcal{H}$ and $\mathcal{F}\ltimes(\mathcal{G}\ltimes\mathcal{H})$. **P** If $A\subseteq I\times(J\times K)$ and $B=\{((i,j),k):(i,(j,k))\in A\}$, then

$$\begin{aligned}A\in\mathcal{F}\ltimes(\mathcal{G}\ltimes\mathcal{H})&\iff\{i:A[\{i\}]\in\mathcal{G}\times\mathcal{H}\}\in\mathcal{F}\\&\iff\{i:\{j:(A[\{i\}])[\{j\}]\in\mathcal{H}\}\in\mathcal{G}\}\in\mathcal{F}\\&\iff\{(i,j):(A[\{i\}])[\{j\}]\in\mathcal{H}\}\in\mathcal{F}\ltimes\mathcal{G}\\&\iff\{(i,j):B[\{(i,j)\}]\in\mathcal{H}\}\in\mathcal{F}\ltimes\mathcal{G}\\&\iff B\in(\mathcal{F}\ltimes\mathcal{G})\ltimes\mathcal{H}.\ \mathbf{Q}\end{aligned}$$

(d) It follows that we can define a product $\mathcal{F}_0\ltimes\ldots\ltimes\mathcal{F}_n$ of any finite string $\mathcal{F}_0,\ldots,\mathcal{F}_n$ of filters, and under the natural identifications of the base sets we shall have $(\mathcal{F}_0\ltimes\ldots\ltimes\mathcal{F}_n)\ltimes(\mathcal{F}_{n+1}\ltimes\ldots\ltimes\mathcal{F}_m)$ identified with $\mathcal{F}_0\ltimes\ldots\ltimes\mathcal{F}_m$ whenever $\mathcal{F}_0,\ldots,\mathcal{F}_n,\ldots,\mathcal{F}_m$ are filters.

(e) For any filters $\mathcal{F}$ and $\mathcal{G}$, $\mathcal{F}\le_{\mathrm{RK}}\mathcal{F}\ltimes\mathcal{G}$ and $\mathcal{G}\le_{\mathrm{RK}}\mathcal{F}\ltimes\mathcal{G}$. **P** Taking the base sets to be I, J respectively and $f(i,j)=i$, $g(i,j)=j$ for $i\in I$ and $j\in J$, we have $\mathcal{F}=f[[\mathcal{F}\ltimes\mathcal{G}]]$ and $\mathcal{G}=g[[\mathcal{F}\ltimes\mathcal{G}]]$. **Q**

Inducing on n, we see that $\mathcal{F}_n\le_{\mathrm{RK}}\mathcal{F}_0\ltimes\ldots\ltimes\mathcal{F}_n$ whenever $\mathcal{F}_0,\ldots,\mathcal{F}_n$ are filters; consequently $\mathcal{F}_m\le_{\mathrm{RK}}\mathcal{F}_0\ltimes\ldots\ltimes\mathcal{F}_n$ whenever $\mathcal{F}_0,\ldots,\mathcal{F}_n$ are filters and $m\le n$.

(f) If $\mathcal{F}$, $\mathcal{F}'$, $\mathcal{G}$ and $\mathcal{G}'$ are filters, with $\mathcal{F}\le_{\mathrm{RK}}\mathcal{F}'$ and $\mathcal{G}\le_{\mathrm{RK}}\mathcal{G}'$, then $\mathcal{F}\ltimes\mathcal{G}\le_{\mathrm{RK}}\mathcal{F}'\ltimes\mathcal{G}'$. **P** Let the base sets of the filters be I, I', J and J', and let $f:I'\to I$ and $g:J'\to J$ be such that $\mathcal{F}=f[[\mathcal{F}']]$ and $\mathcal{G}=g[[\mathcal{G}']]$. Set $h(i,j)=(f(i),g(j))$ for $i\in I$ and $j\in J$. If $A\subseteq I\times J$, then

$$\begin{aligned}h^{-1}[A]\in\mathcal{F}'\ltimes\mathcal{G}'&\iff\{i:(h^{-1}[A])[\{i\}]\in\mathcal{G}'\}\in\mathcal{F}'\\&\iff\{i:g^{-1}[A[\{f(i)\}]]\in\mathcal{G}'\}\in\mathcal{F}'\\&\iff\{i:A[\{f(i)\}]\in\mathcal{G}\}\in\mathcal{F}'\\&\iff\{i:A[\{i\}]\in\mathcal{G}\}\in\mathcal{F}\iff A\in\mathcal{F}\ltimes\mathcal{G}.\end{aligned}$$

So $\mathcal{F}\ltimes\mathcal{G}=h[[\mathcal{F}'\ltimes\mathcal{G}']]$ and $\mathcal{F}\ltimes\mathcal{G}\le_{\mathrm{RK}}\mathcal{F}'\ltimes\mathcal{G}'$. **Q**

Accordingly $\mathcal{F}_0\ltimes\ldots\ltimes\mathcal{F}_n\le_{\mathrm{RK}}\mathcal{G}_0\ltimes\ldots\ltimes\mathcal{G}_n$ whenever $\mathcal{F}_i\le_{\mathrm{RK}}\mathcal{G}_i$ for every $i\le n$.

(g) It follows that if $\mathcal{F}_0,\ldots,\mathcal{F}_n$ are filters and $k_0<\ldots<k_m\le n$, then $\mathcal{F}_{k_0}\ltimes\ldots\ltimes\mathcal{F}_{k_m}\le_{\mathrm{RK}}\mathcal{F}_0\ltimes\ldots\ltimes\mathcal{F}_n$. **P** Induce on m to see that $\mathcal{F}_{k_0}\ltimes\ldots\ltimes\mathcal{F}_{k_m}\le_{\mathrm{RK}}\mathcal{F}_0\ltimes\ldots\ltimes\mathcal{F}_{k_m}$. **Q**

538E There are many variations on the construction here. A fairly elaborate extension will be needed in 538L below.

Iterated products of filters (a) First, a scrap of notation for the rest of the first half of this section (down to 538M). Set $S = \bigcup_{i\in\mathbb{N}} \mathbb{N}^i$. Fix on a family $\langle\theta(\xi,k)\rangle_{1\le\xi<\omega_1,k\in\mathbb{N}}$ such that each $\langle\theta(\xi,k)\rangle_{k\in\mathbb{N}}$ is a non-decreasing sequence running over a cofinal subset of ξ. (You will probably prefer to suppose that when $\xi=\eta+1$ is a successor ordinal, then $\theta(\xi,k)=\eta$ for every $k\in\mathbb{N}$.)

(b) Now suppose that ζ is a non-zero countable ordinal. Let $\langle\mathcal{F}_\xi\rangle_{1\le\xi\le\zeta}$ be a family of filters on $\mathbb{N}$. For $\xi\le\zeta$, define $\mathcal{G}_\xi\subseteq\mathcal{P}S$ as follows. Start by taking $\mathcal{G}_0$ to be the principal filter generated by $\{\emptyset\}$. For $1\le\xi\le\zeta$, set

$$\mathcal{G}_\xi = \{A : A\subseteq S,\ \{k : k\in\mathbb{N},\ \{\tau : {<}k{>}^\frown\tau\in A\}\in\mathcal{G}_{\theta(\xi,k)}\}\in\mathcal{F}_\xi\}.$$

(See 5A1C for the notation here.) It is elementary to check that every $\mathcal{G}_\xi$ is a filter, and that if every $\mathcal{F}_\xi$ is free, so is every $\mathcal{G}_\xi$. Moreover, if every $\mathcal{F}_\xi$ is an ultrafilter, so is every $\mathcal{G}_\xi$.

(c) Continuing from (b), we find that $\mathcal{F}_\xi\le_{\mathrm{RK}}\mathcal{G}_\xi$ whenever $1\le\xi\le\zeta$ and $\mathcal{G}_\eta\le_{\mathrm{RK}}\mathcal{G}_\xi$ whenever $0\le\eta\le\xi\le\zeta$. **P** Induce on ξ. (i) If $\xi\ge1$, define $f:S\to\mathbb{N}$ by setting $f(\tau)=\tau(0)$ if $\tau\ne\emptyset$, $f(\emptyset)=0$. Then, for $B\subseteq\mathbb{N}$,

$$\begin{aligned} f^{-1}[B]\in\mathcal{G}_\xi &\iff \{k : \{\tau : {<}k{>}^\frown\tau\in f^{-1}[B]\}\in\mathcal{G}_{\theta(\xi,k)}\}\in\mathcal{F}_\xi\\ &\iff B\in\mathcal{F}_\xi, \end{aligned}$$

so $\mathcal{F}_\xi = f[[\mathcal{G}_\xi]]\le_{\mathrm{RK}}\mathcal{G}_\xi$. (ii) If $\eta=\xi\le\zeta$ then of course $\mathcal{G}_\eta\le_{\mathrm{RK}}\mathcal{G}_\xi$. (iii) If $0\le\eta<\xi$ then there is a k_0 such that $\eta\le\theta(\xi,k)$ for $k\ge k_0$. For $k\ge k_0$, $\mathcal{G}_\eta\le_{\mathrm{RK}}\mathcal{G}_{\theta(\xi,k)}$ by the inductive hypothesis; let $g_k:S\to S$ be such that $\mathcal{G}_\eta = g_k[[\mathcal{G}_{\theta(\xi,k)}]]$. Now define $g:S\to S$ by setting

$$\begin{aligned} g(\tau) &= g_k(\sigma) \text{ if } k\ge k_0 \text{ and } \tau={<}k{>}^\frown\sigma,\\ &=\emptyset \text{ otherwise.} \end{aligned}$$

For $B\subseteq S$,

$$\begin{aligned} g^{-1}[B]\in\mathcal{G}_\xi &\iff \{k : \{\sigma : {<}k{>}^\frown\sigma\in g^{-1}[B]\}\in\mathcal{G}_{\theta(\xi,k)}\}\in\mathcal{F}_\xi\\ &\iff \{k : k\ge k_0,\ \{\sigma : g({<}k{>}^\frown\sigma)\in B\}\in\mathcal{G}_{\theta(\xi,k)}\}\in\mathcal{F}_\xi \end{aligned}$$

(because $\mathcal{F}_\xi$ is free)

$$\begin{aligned} &\iff \{k : k\ge k_0,\ \{\sigma : g_k(\sigma)\in B\}\in\mathcal{G}_{\theta(\xi,k)}\}\in\mathcal{F}_\xi\\ &\iff \{k : k\ge k_0,\ B\in\mathcal{G}_\eta\}\in\mathcal{F}_\xi \iff B\in\mathcal{G}_\eta, \end{aligned}$$

so $\mathcal{G}_\eta = g[[\mathcal{G}_\xi]]\le_{\mathrm{RK}}\mathcal{G}_\xi$. **Q**

(d) It follows that if $1\le\xi_0<\ldots<\xi_n\le\zeta$ then $\mathcal{F}_{\xi_n}\ltimes\ldots\ltimes\mathcal{F}_{\xi_0}\le_{\mathrm{RK}}\mathcal{G}_{\xi_n}$. **P** Induce on the pair (ξ_n,n). If $\xi_n=1$ then $n=0$ and we just have $\mathcal{F}_1\le_{\mathrm{RK}}\mathcal{G}_1$, as in part (i) of the proof of (c). For the inductive step to $\xi_n=\xi>1$, if $n=0$ then again we need only note that $\mathcal{F}_{\xi_0}=\mathcal{F}_\xi\le_{\mathrm{RK}}\mathcal{G}_\xi$. If $n>0$, let $k_0\ge1$ be such that $\xi_{n-1}\le\theta(\xi,k)$ for every $k\ge k_0$. For $k\ge k_0$,

$$\mathcal{F}_{\xi_{n-1}}\ltimes\ldots\ltimes\mathcal{F}_{\xi_0}\le_{\mathrm{RK}}\mathcal{G}_{\xi_{n-1}}\le\mathcal{G}_{\theta(\xi,k)}$$

by the inductive hypothesis, so we have a function $g_k:S\to\mathbb{N}^n$ such that $\mathcal{F}_{\xi_{n-1}}\ltimes\ldots\ltimes\mathcal{F}_{\xi_0}=g_k[[\mathcal{G}_{\theta(\xi,k)}]]$. Define $g:S\to\mathbb{N}^{n+1}$ by setting

$$\begin{aligned} g(\tau) &= {<}k{>}^\frown g_k(\sigma) \text{ if } k\ge k_0 \text{ and } \tau={<}k{>}^\frown\sigma,\\ &= \text{the constant function with value 0 otherwise.} \end{aligned}$$

Then, for $B\subseteq\mathbb{N}^{n+1}$,

$$\begin{aligned} g^{-1}[B] \in \mathcal{G}_\xi &\iff \{k : \{\sigma : g(<k>^\frown\sigma) \in B\} \in \mathcal{G}_{\theta(\xi,k)}\} \in \mathcal{F}_\xi \\ &\iff \{k : k \ge k_0,\, \{\sigma : <k>^\frown g_k(\sigma) \in B\} \in \mathcal{G}_{\theta(\xi,k)}\} \in \mathcal{F}_\xi \\ &\iff \{k : k \ge k_0,\, \{\sigma : g_k(\sigma) \in B_k\} \in \mathcal{G}_{\theta(\xi,k)}\} \in \mathcal{F}_\xi \end{aligned}$$

(writing $B_k = \{\sigma : <k>^\frown\sigma \in B\} \subseteq \mathbb{N}^n$ for $k \in \mathbb{N}$)

$$\begin{aligned} &\iff \{k : k \ge k_0,\, B_k \in \mathcal{F}_{\xi_{n-1}} \ltimes \ldots \ltimes \mathcal{F}_{\xi_0}\} \in \mathcal{F}_\xi \\ &\iff \{k : k \in \mathbb{N},\, B_k \in \mathcal{F}_{\xi_{n-1}} \ltimes \ldots \ltimes \mathcal{F}_{\xi_0}\} \in \mathcal{F}_\xi \\ &\iff B \in \mathcal{F}_{\xi_n} \ltimes \ldots \ltimes \mathcal{F}_{\xi_0}. \end{aligned}$$

Thus g witnesses that $\mathcal{F}_{\xi_n} \ltimes \ldots \ltimes \mathcal{F}_{\xi_0} \le_{\mathrm{RK}} \mathcal{G}_{\xi_n}$, and the induction proceeds. **Q**

Consequently $\mathcal{F}_{\xi_n} \ltimes \ldots \ltimes \mathcal{F}_{\xi_0} \le_{\mathrm{RK}} \mathcal{G}_\zeta$ whenever $1 \le \xi_0 < \ldots < \xi_n \le \zeta$.

(e) The following special remark will be useful in Theorem 538L. Suppose that we are given $A_\xi \in \mathcal{F}_\xi$ for each $\xi \in [1,\zeta]$. Define $T \subseteq S$ and $\alpha : T \to [0,\zeta]$ as follows. Start by saying that $\emptyset \in T$ and $\alpha(\emptyset) = \zeta$. Having determined $T \cap \mathbb{N}^n$ and $\alpha\restriction T \cap \mathbb{N}^n$, where $n \in \mathbb{N}$, then for $\tau \in \mathbb{N}^{n+1}$ say that $\tau \in T$ iff τ is of the form $\sigma^\frown<k>$ where

$$\sigma \in T \cap \mathbb{N}^n, \quad \alpha(\sigma) > 0, \quad k \in A_{\alpha(\sigma)}, \quad \sigma(i) < k \text{ for every } i < n,$$

and in this case set $\alpha(\tau) = \theta(\alpha(\sigma), k)$. Continue. Observe that $\alpha(\tau) < \alpha(\sigma)$ whenever σ, $\tau \in T$ and τ properly extends σ.

Suppose that $D \in \bigcap_{1\le\xi\le\zeta} \mathcal{F}_\xi$. Then $T^*_D = \{\tau : \tau \in T \cap \bigcup_{n\in\mathbb{N}} D^n,\, \alpha(\tau) = 0\}$ belongs to $\mathcal{G}_\zeta$. **P** I aim to show by induction on ξ that if $\tau \in T \cap \bigcup_{n\in\mathbb{N}} D^n$ and $\alpha(\tau) = \xi$ then $\{\sigma : \tau^\frown\sigma \in T^*_D\}$ belongs to $\mathcal{G}_\xi$. If $\xi = 0$ then of course $\{\sigma : \tau^\frown\sigma \in T^*_D\} = \{\emptyset\} \in \mathcal{G}_0$. For the inductive step to $\xi > 0$,

$$\begin{aligned} &\{k : \{\sigma : \tau^\frown<k>^\frown\sigma \in T^*_D\} \in \mathcal{G}_{\theta(\xi,k)}\} \\ &\qquad \supseteq \{k : k \in D,\, \tau^\frown<k> \in T,\, \alpha(\tau^\frown<k>) = \theta(\xi,k)\} \end{aligned}$$

(by the inductive hypothesis)

$$\begin{aligned} &\qquad \supseteq \{k : k \in A_\xi \cap D,\, \tau(i) < k \text{ for every } i < \operatorname{dom}\tau\} \\ &\qquad \in \mathcal{F}_\xi, \end{aligned}$$

so $\{\sigma : \tau^\frown\sigma \in T^*_D\} \in \mathcal{G}_\xi$. At the end of the induction, we can apply this to $\tau = \emptyset$ and $\xi = \zeta$. **Q**

538F Ramsey filters There is an extensive and fascinating theory of Ramsey filters; see, for instance, Comfort & Negrepontis 74. Here, however, I will give only those fragments which are directly relevant to the other work of this section.

Proposition (a) A Ramsey filter on $\mathbb{N}$ is a rapid p-point ultrafilter.

(b) If $\mathcal{F}$ is a Ramsey ultrafilter on $\mathbb{N}$, $\mathcal{G}$ is a non-principal ultrafilter on $\mathbb{N}$, and $\mathcal{G} \le_{\mathrm{RK}} \mathcal{F}$, then $\mathcal{F}$ and $\mathcal{G}$ are isomorphic and $\mathcal{G}$ is a Ramsey ultrafilter.

(c) Let $\mathcal{F}$ be a Ramsey filter on $\mathbb{N}$. Suppose that $\langle A_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\mathcal{F}$. Then there is an $A \in \mathcal{F}$ such that $n \in A_m$ whenever m, $n \in A$ and $m < n$.

(d) Let $\mathcal{F}$ be a Ramsey filter on $\mathbb{N}$. Let $\mathcal{S} \subseteq [\mathbb{N}]^{<\omega}$ be such that $\emptyset \in \mathcal{S}$ and $\{n : I \cup \{n\} \in \mathcal{S}\} \in \mathcal{F}$ for every $I \in \mathcal{S}$. Then there is an $A \in \mathcal{F}$ such that $[A]^{<\omega} \subseteq \mathcal{S}$.

(e) If $\mathfrak{F}$ is a countable family of distinct Ramsey filters on $\mathbb{N}$, there is a disjoint family $\langle A_\mathcal{F}\rangle_{\mathcal{F}\in\mathfrak{F}}$ of subsets of $\mathbb{N}$ such that $A_\mathcal{F} \in \mathcal{F}$ for every $\mathcal{F} \in \mathfrak{F}$.

(f) Let $\mathfrak{F}$ be a countable family of non-isomorphic Ramsey ultrafilters on $\mathbb{N}$, and $\mathfrak{h} : \mathbb{N} \to [\mathfrak{F}]^{<\omega}$ a function. Suppose that we are given an $A_\mathcal{F} \in \mathcal{F}$ for each $\mathcal{F} \in \mathfrak{F}$. Then there is an $A \in \bigcap\mathfrak{F}$ such that whenever i, $j \in A$, $\mathcal{F} \in \mathfrak{h}(i)$ and $i < j$, there is a $k \in A_\mathcal{F}$ such that $i < k < j$.

(g) If $\mathfrak{m}_{\mathrm{countable}} = \mathfrak{c}$, there is a Ramsey ultrafilter on $\mathbb{N}$.

proof (a) Let $\mathcal{F}$ be a Ramsey filter on $\mathbb{N}$.

(i) $\mathcal{F}$ is an ultrafilter. **P** Let A be any subset of $\mathbb{N}$. Define $f:[\mathbb{N}]^2\to\{0,1\}$ by setting $f(I)=1$ if $\#(I\cap A)=1$, 0 otherwise. Then we have an $I\in\mathcal{F}$ such that f is constant on $[I]^2$. As $\mathcal{F}$ is free, $\#(I)\ge 3$ and the constant value of f cannot be 1. So either $I\subseteq A$ and $A\in\mathcal{F}$, or $I\cap A=\emptyset$ and $\mathbb{N}\setminus A\in\mathcal{F}$. As A is arbitrary, $\mathcal{F}$ is an ultrafilter. **Q**

(ii) $\mathcal{F}$ is a p-point filter. **P** Let $\langle I_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{F}$. Set $K_n=(\mathbb{N}\setminus n)\cap\bigcap_{i<n}I_i$, $J_n=K_n\setminus K_{n+1}$ for each n; then $\langle J_n\rangle_{n\in\mathbb{N}}$ is a partition of $\mathbb{N}$. Define $f:[\mathbb{N}]^2\to\{0,1\}$ by setting $f(a)=0$ if there is an $n\in\mathbb{N}$ such that $a\subseteq J_n$, 1 otherwise. Let $I\in\mathcal{F}$ be such that f is constant on $[I]^2$.

Since $\mathbb{N}\setminus J_n\in\mathcal{F}$ for every n, there must be two points in I belonging to different J_n; so that the constant value of f must be 1, and no two points of I belong to the same J_n. In particular, $I\cap J_n$ is always finite, and $I\setminus I_n\subseteq\bigcup_{i\le n}I\cap J_i$ is always finite. As $\langle I_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathcal{F}$ is a p-point filter. **Q**

(iii) $\mathcal{F}$ is rapid. **P** Let $\langle t_n\rangle_{n\in\mathbb{N}}$ be a sequence converging to 0. For each n, set $I_n=\{i:|t_i|\le 2^{-n}\}$; as $\mathcal{F}$ is free, $I_n\in\mathcal{F}$. Looking again at the proof of (ii) above, we see that the construction there gives us an $I\in\mathcal{F}$ such that $\#(I\setminus I_n)\le n+1$ for every n. We can therefore enumerate I as $\langle k_n\rangle_{n\in\mathbb{N}}$ in such a way that $k_{n+1}\in I_n$ for every n. But this means that

$$\sum_{i\in I}|t_i|=\sum_{n=0}^{\infty}|t_{k_n}|\le|t_{k_0}|+\sum_{n=1}^{\infty}2^{-n+1}<\infty.$$

As $\langle t_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathcal{F}$ is rapid. **Q**

(b) Let $f:\mathbb{N}\to\mathbb{N}$ be such that $f[[\mathcal{F}]]=\mathcal{G}$. For $K\in[\mathbb{N}]^2$, set $h(K)=0$ if $f{\restriction}K$ is constant, 1 otherwise. Then there is an $A\in\mathcal{F}$ such that h is constant on $[A]^2$, that is, f is either constant or injective on A. Since $f[A]\in\mathcal{G}$, $f[A]$ is infinite, so f is injective on A. Let $g:\mathbb{N}\to\mathbb{N}$ be any function extending $(f{\restriction}A)^{-1}$; then $gf(n)=n$ for every $n\in A$, so

$$(gf)[[\mathcal{F}]]=\{I:(gf)^{-1}[I]\in\mathcal{F}\}=\{I:A\cap(gf)^{-1}[I]\in\mathcal{F}\}=\{I:A\cap I\in\mathcal{F}\}=\mathcal{F}.$$

But this means that $g[[\mathcal{G}]]=\mathcal{F}$ and $\mathcal{F}\le_{\mathrm{RK}}\mathcal{G}$.

By 538Cb, $\mathcal{F}$ and $\mathcal{G}$ are isomorphic, so $\mathcal{G}$ also must be a Ramsey ultrafilter.

(c) For $m<n$ in $\mathbb{N}$, set $h(\{m,n\})=1$ if $n\in A_m$, 0 otherwise. Then there is an $A\in\mathcal{F}$ such that $h{\restriction}[A]^2$ is constant. Setting $k=\min A$, A meets $A_k\setminus(k+1)$, so h takes the value 1 on $[A]^2$; consequently $n\in A_m$ whenever m, $n\in A$ and $m<n$.

(d) For $n\in\mathbb{N}$, set

$$A_n=\{i:I\cup\{i\}\in\mathcal{S}\text{ whenever }I\subseteq n+1\text{ and }I\in\mathcal{S}\}\in\mathcal{F}.$$

By (c), there is an $A\in\mathcal{F}$ such that $n\in A_m$ whenever m, $n\in A$ and $m<n$; and we can suppose that $A\subseteq A_0$, so that $\{n\}\in\mathcal{S}$ for every $n\in A$. Now an easy induction on n shows that $\mathcal{P}(A\cap n)\subseteq\mathcal{S}$ for every n, so $[A]^{<\omega}\subseteq\mathcal{S}$.

(e) Enumerate $\mathfrak{F}$ as $\langle\mathcal{F}_n\rangle_{n<\#(\mathfrak{F})}$. For distinct m, $n<\#(\mathfrak{F})$ there is a $B_{mn}\in\mathcal{F}_m\setminus\mathcal{F}_n$. **P** We know that there is a set in $\mathcal{F}_m\triangle\mathcal{F}_n$; now either this set or its complement will serve for B_{mn}. **Q** Because every member of $\mathfrak{F}$ is a p-point filter ((a) above), we can find for each $n<\#(\mathfrak{F})$ a set $C_n\in\mathcal{F}_n$ such that $C_n\setminus(B_{nm}\setminus B_{mn})$ is finite for every $m<\#(\mathfrak{F})$ such that $m\ne n$. Set $A_{\mathcal{F}_n}=C_n\setminus\bigcup_{m<n}C_m$ for $n<\#(\mathfrak{F})$; then $\langle A_{\mathcal{F}}\rangle_{\mathcal{F}\in\mathfrak{F}}$ is disjoint. Since

$$C_m\cap C_n\subseteq(C_m\setminus B_{mn})\cup(C_n\cap B_{mn})$$

is finite whenever $m\ne n$, $C_n\setminus A_{\mathcal{F}_n}$ is finite and $A_{\mathcal{F}_n}\in\mathcal{F}_n$ for each $n<\#(\mathfrak{F})$.

(f)(i) We can suppose that $\mathfrak{h}(i)\subseteq\mathfrak{h}(j)$ whenever $i\le j$, and that $\mathfrak{F}=\bigcup_{i\in\mathbb{N}}\mathfrak{h}(i)$. Let $g:\mathbb{N}\to\mathbb{N}$ be a strictly increasing function such that $g(0)>0$ and whenever $i\in\mathbb{N}$ and $\mathcal{F}\in\mathfrak{h}(i)$, there is a $k\in A_{\mathcal{F}}$ such that $i<k<g(i)$. Set $l_m=g^m(0)$ and $J_m=l_{m+1}\setminus l_m$ for each m, so that $\langle J_m\rangle_{m\in\mathbb{N}}$ is a partition of $\mathbb{N}$. Let $\langle a_\xi\rangle_{\xi<\omega_1}$ be a family of infinite subsets of $\mathbb{N}$, all containing 0, such that $a_\xi\cap a_\eta$ is finite for all distinct ξ, $\eta<\omega_1$ (5A1Ga), and set $M_\xi=\bigcup_{m\in a_\xi}J_m$ for each ξ; then $M_\xi\cap M_\eta$ is finite for all distinct ξ, $\eta<\omega_1$. It follows that each member of $\mathfrak{F}$ can contain at most one M_ξ, and there is a $\xi<\omega_1$ such that M_ξ does not belong to any member of $\mathfrak{F}$, that is, $M=\mathbb{N}\setminus M_\xi$ belongs to $\bigcap\mathfrak{F}$.

(ii) Define $f:\mathbb{N}\to\mathbb{N}$ by setting $f(n)=\max\{m:m\in a_\xi,\ l_m\le n\}$ for $n\in\mathbb{N}$. For each $\mathcal{F}\in\mathfrak{F}$, $f[[\mathcal{F}]]$ is isomorphic to $\mathcal{F}$, by (b). It follows that if $\mathcal{F}$, $\mathcal{F}'$ are distinct members of $\mathfrak{F}$, $f[[\mathcal{F}]]\ne f[[\mathcal{F}']]$. Because $\mathfrak{F}$

is countable, there is a disjoint family $\langle K_{\mathcal{F}}\rangle_{\mathcal{F}\in\mathfrak{F}}$ of sets such that $K_{\mathcal{F}} \in f[[\mathcal{F}]]$ for every $\mathcal{F}\in\mathfrak{F}$ ((e) above). Set $L_{\mathcal{F}} = f^{-1}[K_{\mathcal{F}}] \in \mathcal{F}$ for each $\mathcal{F}\in\mathfrak{F}$.

(iii) For $i<j$ in $\mathbb{N}$, set $h(\{i,j\}) = 1$ if $j < g(i)$, 0 otherwise. $\mathcal{F}\in\mathfrak{F}$, there is an $L'_{\mathcal{F}} \in \mathcal{F}$ such that $L'_{\mathcal{F}} \subseteq L_{\mathcal{F}}$ and h is constant on $[L'_{\mathcal{F}}]^2$. As $L'_{\mathcal{F}}$ is infinite, the constant value cannot be 1 and must be 0, that is, $g(i) \le j$ whenever i, $j \in L'_{\mathcal{F}}$ and $i<j$.

(iv) Consider $A = \bigcup_{\mathcal{F}\in\mathfrak{F}} L'_{\mathcal{F}} \cap M$. Then $A \in \bigcap\mathfrak{F}$. Suppose that i, $j\in A$ and $i<j$; then $g(i)\le j$. **P** Let $\mathcal{F}$, $\mathcal{F}'\in\mathfrak{F}$ be such that $i \in L'_{\mathcal{F}}$ and $j\in L'_{\mathcal{F}'}$.

case 1 If $\mathcal{F}=\mathcal{F}'$, then both i and j belong to $L'_{\mathcal{F}}$, so $g(i)\le j$ by (iii).

case 2 If $\mathcal{F}\ne\mathcal{F}'$, then $i\in L_{\mathcal{F}}$ and $j\in L_{\mathcal{F}'}$, so $f(i)\in K_{\mathcal{F}}$ and $f(j)\in K_{\mathcal{F}'}$ and $f(i)\ne f(j)$. Let m, $n\in\mathbb{N}$ be such that $i\in J_m$ and $j\in J_n$; since $j\notin M_\xi$, $n\notin a_\xi$ and $f(j)<n$. As $K_{\mathcal{F}}$ and $K_{\mathcal{F}'}$ are disjoint, $f(i)<f(j)$. It follows that $m<f(j)<n$, so

$$g(i) \le g(l_{m+1}) \le g(l_{f(j)}) \le l_n \le j$$

and $g(i)\le j$ in this case also. **Q**

By the choice of g, this means that if $\mathcal{F}\in\mathfrak{h}(i)$ there must be a $k\in A_{\mathcal{F}}$ such that $i<k<j$, as required.

(g)(i) Suppose that $\mathcal{E}\subseteq\mathcal{P}\mathbb{N}$ is a filter base, containing $\mathbb{N}\setminus n$ for every $n\in\mathbb{N}$, and with cardinal less than $\mathfrak{m}_{\text{countable}}$. Let $f:[\mathbb{N}]^2\to\{0,1\}$ be a function. Then there is an $F\subseteq\mathbb{N}$ such that f is constant on $[F]^2$ and F meets every member of $\mathcal{E}$. **P** Set

$$\mathcal{E}^+ = \{J : J\subseteq\mathbb{N},\, J\cap E\ne\emptyset \text{ for every } E\in\mathcal{E}\},$$

$$S_n = \{n\}\cup\{i : i\in\mathbb{N}\setminus\{n\},\, f(\{i,n\})=1\},$$

$$S'_n = \{n\}\cup\{i : i\in\mathbb{N}\setminus\{n\},\, f(\{i,n\})=0\}$$

for $n\in\mathbb{N}$.

case 1 Suppose that $\{n : n\in J,\, J\cap S_n\in\mathcal{E}^+\}$ belongs to $\mathcal{E}^+$ for every $J\in\mathcal{E}^+$. Set

$$\mathcal{I} = \{I : I\in[\mathbb{N}]^{<\omega},\, f(K)=1 \text{ for every } K\in[I]^2,\, \mathbb{N}\cap\textstyle\bigcap_{i\in I} S_i\in\mathcal{E}^+\}.$$

If $I\in\mathcal{I}$, $J=\mathbb{N}\cap\bigcap_{i\in I}S_i$ and $E\in\mathcal{E}$, then $J\in\mathcal{E}^+$; because $\mathcal{E}$ is a filter base, $J\cap E\in\mathcal{E}^+$; by hypothesis, $\{n : n\in J\cap E,\, J\cap E\cap S_n\in\mathcal{E}^+\}$ belongs to $\mathcal{E}^+$ and is not empty. There is therefore some $n\in J\cap E$ such that $J\cap S_n\in\mathcal{E}^+$, in which case $I\cup\{n\}\in\mathcal{I}$.

In particular, there is some $k\in\mathbb{N}$ such that $\{k\}\in\mathcal{I}$. Set

$$C = \{\alpha : \alpha\in\mathbb{N}^{\mathbb{N}},\, \{\alpha(i) : i<m\}\in\mathcal{I} \text{ for every } m\in\mathbb{N}\}.$$

Then C is compact, and it is non-empty because the constant function with value k belongs to C. Moreover, if $\alpha\in C$ and $m\in\mathbb{N}$ and $E\in\mathcal{E}$, there is an $n\in E$ such that $\{\alpha(i) : i<m\}\cup\{n\}\in\mathcal{I}$, so there is a $\beta\in C$ such that $\beta(i)=\alpha(i)$ for $i<m$ and $\beta(m)=n$. Thus $\{\beta : \beta\in C,\, E\cap\beta[\mathbb{N}]\ne\emptyset\}$ is a dense open subset of C. Writing $\mathcal{M}(C)$ for the ideal of meager subsets of C, $\operatorname{cov}\mathcal{M}(C)$ is either ∞ (if C has an isolated point) or $\operatorname{cov}\mathcal{M}(\mathbb{R})=\mathfrak{m}_{\text{countable}}$, by 522Wb and 522Sa; in either case, it is greater than $\#(\mathcal{E})$. There is therefore some $\alpha\in C$ such that $F=\alpha[\mathbb{N}]$ meets every member of $\mathcal{E}$; in this case, f is equal to 1 everywhere in $[F]^2$, so we have an appropriate F.

case 2 Otherwise, there is a $K\in\mathcal{E}^+$ such that $\{n : n\in K,\, K\cap S_n\in\mathcal{E}^+\}$ does not belong to $\mathcal{E}^+$. Let $E_0\in\mathcal{E}$ be disjoint from $\{n : n\in K,\, K\cap S_n\in\mathcal{E}^+\}$. Set $\mathcal{G}=\mathcal{E}\cup\{K\cap E : E\in\mathcal{E}\}$, so that $\mathcal{G}$ is a filter base and $\#(\mathcal{G})<\mathfrak{m}_{\text{countable}}$. If $n\in E_0$ then there is an $E'_n\in\mathcal{E}$ disjoint from $K\cap S_n$. So if $J\in\mathcal{G}^+$, $J\cap S'_n\supseteq(J\cap K\cap E'_n)\setminus\{n\}$ belongs to $\mathcal{G}^+$ for every $n\in E_0$; accordingly $\{n : n\in J,\, J\cap S'_n\in\mathcal{G}^+\}\supseteq J\cap E_0$ belongs to $\mathcal{G}^+$.

We can therefore apply the argument of case 1 to $\mathcal{G}$ and the function $1-f$ to see that there is an $F\subseteq\mathbb{N}$, meeting every member of $\mathcal{G}\supseteq\mathcal{E}$, such that $f=0$ on $[F]^2$. **Q**

(ii) Enumerate the set of functions from $[\mathbb{N}]^2$ to $\{0,1\}$ as $\langle f_\xi\rangle_{\xi<\mathfrak{c}}$. Choose a non-decreasing family $\langle\mathcal{E}_\xi\rangle_{\xi\le\mathfrak{c}}$ inductively, as follows; the inductive hypothesis will be that $\mathcal{E}_\xi\subseteq\mathcal{P}\mathbb{N}$ is a filter base with cardinal at most $\max(\omega,\#(\xi))$. Start with $\mathcal{E}_0=\{\mathbb{N}\setminus n : n\in\mathbb{N}\}$. Given $\mathcal{E}_\xi$, where $\xi<\mathfrak{c}=\mathfrak{m}_{\text{countable}}$, use (i) to find a set F_ξ, meeting every member of $\mathcal{E}_\xi$, such that f_ξ is constant on $[F_\xi]^2$; take $\mathcal{E}_{\xi+1} = \mathcal{E}_\xi \cup \{E \cap F_\xi : E \in \mathcal{E}_\xi\}$. Given $\langle \mathcal{E}_\eta \rangle_{\eta<\xi}$, where $\xi \le \mathfrak{c}$ is a non-zero limit ordinal, set $\mathcal{E}_\xi = \bigcup_{\eta<\xi} \mathcal{E}_\eta$.

At the end of the induction, let $\mathcal{F}$ be the filter generated by $\mathcal{E}_\mathfrak{c}$; then $\mathcal{F}$ is a Ramsey filter.

538G Measure-centering filters: Theorem Let $\mathcal{F}$ be a free filter on $\mathbb{N}$. Write ν_ω for the usual measure on $\{0,1\}^{\mathbb{N}}$, T_ω for its domain and $(\mathfrak{B}_\omega, \bar\nu_\omega)$ for its measure algebra. Then the following are equiveridical:

(i) $\mathcal{F}$ is measure-centering;

(ii) whenever $\langle a_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{B}_\omega$ such that $\inf_{n\in\mathbb{N}} \bar\nu_\omega a_n > 0$, there is an $A \in \mathcal{F}$ such that $\{a_n : n \in A\}$ is centered;

(iii) whenever $\langle F_n \rangle_{n\in\mathbb{N}}$ is a sequence in T_ω such that $\inf_{n\in\mathbb{N}} \nu_\omega F_n > 0$, there is an $A \in \mathcal{F}$ such that $\bigcap_{n\in A} F_n \ne \emptyset$;

(iv) whenever (X, Σ, μ) is a perfect totally finite measure space and $\langle F_n \rangle_{n\in\mathbb{N}}$ is a sequence in Σ, $\mu^*(\bigcup_{A\in\mathcal{F}} \bigcap_{n\in A} F_n) \ge \liminf_{n\to\mathcal{F}} \mu F_n$;

(v) whenever μ is a Radon probability measure on $\mathcal{P}\mathbb{N}$, then $\mu^*\mathcal{F} \ge \liminf_{n\to\mathcal{F}} \mu E_n$, where $E_n = \{a : n \in a \subseteq \mathbb{N}\}$ for each n.

proof (i)⇒(ii) is trivial.

not-(iv)⇒not-(ii) Suppose there are a perfect totally finite measure space (X, Σ, μ) and a sequence $\langle F_n \rangle_{n\in\mathbb{N}}$ in Σ such that $\liminf_{n\in\mathbb{N}} \mu F_n > \mu^*(\bigcup_{A\in\mathcal{F}} \bigcap_{n\in A} F_n)$. Let F be a measurable envelope of $\bigcup_{A\in\mathcal{F}} \bigcap_{n\in A} F_n$. Let T be the σ-subalgebra of Σ generated by $\{F\} \cup \{F_n : n \in \mathbb{N}\}$; then $\mu{\restriction}\mathrm{T}$ is a compact measure (451F). Let ν be its normalization $\frac{1}{\mu X}\mu{\restriction}\mathrm{T}$; then ν is a compact probability measure. We see that $\liminf_{n\to\mathcal{F}} \nu F_n > \nu F$; take γ such that $\nu F < \gamma < \liminf_{n\to\mathcal{F}} \nu F_n$, and set $C = \{n : \nu F_n > \gamma\}$, so that $C \in \mathcal{F}$.

Let $\mathcal{K}$ be a compact class such that ν is inner regular with respect to $\mathcal{K}$. For $n \in C$, let $K_n \in \mathcal{K} \cap \mathrm{T}$ be such that $K_n \subseteq F_n \setminus F$ and $\nu K_n \ge \gamma - \nu F$; for $n \in \mathbb{N} \setminus C$ set $K_n = X$.

The measure algebra $(\mathfrak{B}, \bar\nu)$ of ν is a probability algebra with countable Maharam type, so there is a measure-preserving Boolean homomorphism $\pi : \mathfrak{B} \to \mathfrak{B}_\omega$ (332P or 333D). Set $a_n = \pi K_n^\bullet$ for each n. Then

$$\bar\nu_\omega a_n = \nu K_n \ge \gamma - \nu F > 0$$

for every n. On the other hand, if $A \in \mathcal{F}$, then $A \cap C \in \mathcal{F}$ so $\bigcap_{n\in A\cap C} K_n \subseteq \bigcap_{n\in A\cap C} F_n \setminus F$ is empty. As K_n belongs to the compact class $\mathcal{K}$ for every $n \in A \cap C$, there must be a finite set $I \subseteq A \cap C$ such that $\bigcap_{n\in I} K_n = \emptyset$, in which case $\inf_{n\in I} a_n = \pi(\bigcap_{n\in I} K_n)^\bullet = 0$. This shows that $\{a_n : n \in A\}$ is not centered. So $\langle a_n \rangle_{n\in\mathbb{N}}$ witnesses that (ii) is false.

(iv)⇒(i) Suppose that (iv) is true. Take a Boolean algebra $\mathfrak{A}$, an additive functional $\nu : \mathfrak{A} \to [0, \infty[$ and a sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $\inf_{n\in\mathbb{N}} \nu a_n > 0$. By 311E and 311H, we can suppose that $\mathfrak{A}$ is the algebra of open-and-closed subsets of a compact zero-dimensional space Z. In this case, there is a Radon measure μ on Z extending ν (416Qa). Of course μ is perfect (416Wa), and $\liminf_{n\to\mathcal{F}} \mu a_n \ge \inf_{n\in\mathbb{N}} \nu a_n > 0$, so (iv) tells us that there is an $A \in \mathcal{F}$ such that $\bigcap_{n\in A} a_n \ne \emptyset$, in which case $\{a_n : n \in \mathbb{N}\}$ is centered in $\mathfrak{A}$. As $\mathfrak{A}$, ν and $\langle a_n \rangle_{n\in\mathbb{N}}$ are arbitrary, $\mathcal{F}$ is measure-centering.

(iv)⇒(v) The point is simply that μ is perfect (416Wa again) and that

$$\bigcup_{A\in\mathcal{F}} \bigcap_{n\in A} E_n = \bigcup_{A\in\mathcal{F}} \{a : A \subseteq a \subseteq \mathbb{N}\} = \mathcal{F}.$$

(v)⇒(iii) Suppose that (v) is true, and that $\langle F_n \rangle_{n\in\mathbb{N}}$ is a sequence in T_ω such that $\inf_{n\in\mathbb{N}} \nu_\omega F_n > 0$. Define $\phi : \{0,1\}^{\mathbb{N}} \to \mathcal{P}\mathbb{N}$ by setting $\phi(x) = \{n : x \in F_n\}$ for each n. Then ϕ is almost continuous (418J), so the image measure $\mu = \nu_\omega \phi^{-1}$ is a Radon probability measure on $\mathcal{P}\mathbb{N}$ (418I). Defining E_n as in (v), we have

$$\mu E_n = \nu_\omega \phi^{-1}[E_n] = \nu_\omega F_n$$

for every $n \in \mathbb{N}$, so

$$0 < \inf_{n\in\mathbb{N}} \bar\nu_\omega a_n \le \liminf_{n\to\mathcal{F}} \mu E_n \le \mu^*\mathcal{F} = \nu_\omega^* \phi^{-1}[\mathcal{F}]$$

(451Pc). In particular, there must be an $x \in \phi^{-1}[\mathcal{F}]$, so that $A = \{n : x \in F_n\}$ belongs to $\mathcal{F}$, and $\bigcap_{n\in A} F_n$ is non-empty.

(iii)⇒(ii) Assume (iii). Let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{B}_\omega$ such that $\inf_{n\in\mathbb{N}}\bar\nu_\omega a_n > 0$. Let $\theta : \mathfrak{B}_\omega \to \mathrm{T}_\omega$ be a lifting (341K), and set $F_n = \theta a_n$ for each n. Then $\nu_\omega F_n = \bar\nu_\omega a_n$ for every n, so (iii) tells us that there is an $A \in \mathcal{F}$ such that $\bigcap_{n\in A} F_n \neq \emptyset$. In this case, $\theta(\inf_{n\in I} a_n) = \bigcap_{n\in I} F_n \neq \emptyset$ for every non-empty finite $I \subseteq A$, so $\{a_n : n \in A\}$ is centered.

538H Proposition (a) Any measure-centering filter on $\mathbb{N}$ is an ultrafilter.

(b) If $\mathcal{F}$ is a measure-centering ultrafilter on $\mathbb{N}$ and $\mathcal{G}$ is a filter on $\mathbb{N}$ such that $\mathcal{G} \le_{\mathrm{RK}} \mathcal{F}$, then $\mathcal{G}$ is measure-centering.

(c) Every Ramsey ultrafilter on $\mathbb{N}$ is measure-centering.

(d) (Shelah 98b) Every measure-centering ultrafilter on $\mathbb{N}$ is a nowhere dense ultrafilter.

(e) (Benedikt 99) If $\operatorname{cov}\mathcal{N} = \mathfrak{c}$, where $\mathcal{N}$ is the Lebesgue null ideal, then there is a measure-centering ultrafilter on $\mathbb{N}$.

proof (a) Let a, b be disjoint non-zero elements of $\mathfrak{B}_\omega$, where $(\mathfrak{B}_\omega, \bar\nu_\omega)$ is the measure algebra of the usual measure on $\{0,1\}^{\mathbb{N}}$, as in 538G. Given $I \subseteq \mathbb{N}$, set $a_n = a$ if $n \in I$, b if $n \in \mathbb{N}\setminus I$. Then $\inf_{n\in\mathbb{N}}\bar\nu_\omega a_n > 0$, so there is a $J \in \mathcal{F}$ such that $\{a_n : n \in J\}$ is centered, in which case either $J \subseteq I$ or $J \cap I = \emptyset$; so that one of I, $\mathbb{N}\setminus I$ must belong to $\mathcal{F}$.

(b) Let $f : \mathbb{N} \to \mathbb{N}$ be such that $f[[\mathcal{F}]] = \mathcal{G}$. Let $(\mathfrak{A}, \bar\mu)$ be a totally finite measure algebra and $\langle a_n\rangle_{n\in\mathbb{N}}$ a sequence in $\mathfrak{A}$ with $\inf_{n\in\mathbb{N}}\bar\mu a_n > 0$. Then $\langle a_{f(n)}\rangle_{n\in\mathbb{N}}$ has the same property, so there is an $A \in \mathcal{F}$ such that $\{a_{f(n)} : n \in A\}$ is centered. Now $f[A] \in \mathcal{G}$ and $\{a_m : m \in f[A]\}$ is centered.

(c) Let $\mathcal{F}$ be a Ramsey ultrafilter and $\langle b_n\rangle_{n\in\mathbb{N}}$ a sequence in $\mathfrak{B}_\omega$ such that $\gamma = \inf_{n\in\mathbb{N}}\bar\nu_\omega b_n$ is greater than 0. Set $b = \inf_{A\in\mathcal{F}}\sup_{n\in A} b_n$; then $\bar\nu_\omega b \ge \gamma$. Set $\mathcal{S} = \{I : I \in [\mathbb{N}]^{<\omega},\ b \cap \inf_{n\in I} b_n \neq 0\}$. Then $\emptyset \in \mathcal{S}$. If $I \in \mathcal{S}$, set $c = b \cap \inf_{n\in I} b_n$ and $C = \{n : c \cap b_n = 0\}$. Then $\sup_{n\in C} b_n$ does not meet c so does not include b, and $C \notin \mathcal{F}$. Accordingly

$$\{n : I \cup \{n\} \in \mathcal{S}\} = \mathbb{N}\setminus C \in \mathcal{F}.$$

By 538Fd, there is an $A \in \mathcal{F}$ such that $[A]^{<\omega} \subseteq \mathcal{S}$, in which case $\{b_n : n \in A\}$ is centered. As $\langle b_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathcal{F}$ is measure-centering.

(d) Let $\mathcal{F}$ be a measure-centering ultrafilter, and $\langle t_n\rangle_{n\in\mathbb{N}}$ a sequence in $\mathbb{R}$. Let $F \subseteq [0,1[$ be a nowhere dense set with non-zero Lebesgue measure, and set $H = \bigcup_{k\in\mathbb{Z}} F + k$, so that H is nowhere dense in $\mathbb{R}$; let μ be Lebesgue measure on $[0,1]$. For $n \in \mathbb{N}$ set

$$E_n = \{x : x \in [0,1],\ x + t_n \in H\} = [0,1] \cap \bigcup_{k\in\mathbb{Z}} F - t_n + k,$$

so that $\mu E_n = \mu F > 0$. By 538G(iv), there is an $A \in \mathcal{F}$ such that $\bigcap_{n\in A} E_n$ is non-empty; take $x \in \bigcap_{n\in A} E_n$, so that $t_n \in H - x$ for every $n \in A$, and $\{t_n : n \in A\}$ is nowhere dense. As $\langle t_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathcal{F}$ is a nowhere dense filter.

(e)(i) Let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{B}_\omega$ such that $\inf_{n\in\mathbb{N}}\bar\nu_\omega a_n > 0$, and $\mathcal{C} \subseteq \mathcal{P}\mathbb{N}$ a filter base such that $\#(\mathcal{C}) < \operatorname{cov}\mathcal{N}$. Then there is an $A \subseteq \mathbb{N}$ such that A meets every member of C and $\{a_n : n \in A\}$ is centered. **P** Set $\epsilon = \inf_{n\in\mathbb{N}}\bar\nu_\omega a_n$. For $C \in \mathcal{C}$ set $b_C = \sup_{n\in C} a_n$; because $C \neq \emptyset$, $\bar\nu_\omega b_C \ge \epsilon$. Set $b = \inf_{C\in\mathcal{C}} b_C$; because $\mathcal{C}$ is downwards-directed, $\bar\nu_\omega b \ge \epsilon$ (321F) and $b \neq 0$.

Let $\theta : \mathfrak{B}_\omega \to \mathrm{T}_\omega$ be a lifting (341K). For $C \in \mathcal{C}$, set $F_C = \bigcup_{n\in C}\theta a_n$; then

$$F_C^\bullet = b_C \supseteq b,$$

so $\theta b \setminus F_C$ is negligible. Because $b \neq 0$, θb is not negligible; because $\#(\mathcal{C}) < \operatorname{cov}\mathcal{N}$, $\theta b \cap \bigcap_{C\in\mathcal{C}} F_C$ is non-empty (apply 522Wa to the subspace measure on θb). Take any x in the intersection, and set $A = \{n : x \in \theta a_n\}$. For every $C \in \mathcal{C}$, there is an $n \in C$ such that $x \in \theta a_n$, so $A \cap C \neq \emptyset$. If $I \subseteq A$ is finite and not empty, then $\theta(\inf_{n\in I} a_n) = \bigcap_{n\in I}\theta a_n$ contains x, so $\inf_{n\in I} a_n \neq 0$; thus $\{a_n : n \in A\}$ is centered. **Q**

(ii) Since $\#(\mathfrak{B}_\omega) = \mathfrak{c}$ (524Ma), we can enumerate as $\langle\langle a_{\xi n}\rangle_{n\in\mathbb{N}}\rangle_{\xi<\mathfrak{c}}$ the family of all sequences $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{B}_\omega$ such that $\inf_{n\in\mathbb{N}}\bar\nu_\omega a_n > 0$. Choose $\langle\mathcal{C}_\xi\rangle_{\xi<\mathfrak{c}}$ inductively, as follows. The inductive hypothesis will be that $\mathcal{C}_\xi \subseteq \mathcal{P}\mathbb{N}$ is a filter base and $\#(\mathcal{C}_\xi) \le \max(\omega, \#(\xi))$. Start with $\mathcal{C}_0 = \{\mathbb{N}\setminus n : n \in \mathbb{N}\}$. Given $\mathcal{C}_\xi$, where $\xi < \mathfrak{c}$, such that

$$\#(\mathcal{C}_\xi) \le \max(\omega, \#(\xi)) < \mathfrak{c} = \operatorname{cov}\mathcal{N},$$

(i) tells us that there is an $A_\xi \subseteq \mathbb{N}$, meeting every member of $\mathcal{C}_\xi$, such that $\{a_{\xi n} : n \in A_\xi\}$ is centered; set

$$\mathcal{C}_{\xi+1} = \mathcal{C}_\xi \cup \{C \cap A_\xi : C \in \mathcal{C}_\xi\}.$$

For a non-zero limit ordinal $\xi \le \mathfrak{c}$, set $\mathcal{C}_\xi = \bigcup_{\eta<\xi} \mathcal{C}_\eta$. Let $\mathcal{F}$ be the filter generated by $\mathcal{C}_\mathfrak{c}$; then $\mathcal{F}$ is a free filter satisfying 538G(ii), so is measure-centering.

538I Theorem Suppose that $\mathcal{F}$ is a measure-centering ultrafilter on $\mathbb{N}$, and that (X,Σ,μ) is a perfect probability space. Let $\mathcal{A}$ be the family of all sets of the form $\lim_{n\to\mathcal{F}} E_n$ where $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in Σ. Then there is a unique complete measure λ on X such that λ is inner regular with respect to $\mathcal{A}$ and $\lambda(\lim_{n\to\mathcal{F}} E_n) = \lim_{n\to\mathcal{F}} \mu E_n$ for every sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in Σ; and λ extends μ.

Remark By '$\lim_{n\to\mathcal{F}} E_n$' I mean the limit in the compact Hausdorff space $\mathcal{P}X$, that is,

$$\{x : \{n : x \in E_n\} \in \mathcal{F}\} = \bigcup_{A\in\mathcal{F}} \bigcap_{n\in A} E_n = \bigcap_{A\in\mathcal{F}} \bigcup_{n\in A} E_n.$$

proof (a) $\mathcal{A}$ is an algebra of subsets of X. **P** If $\langle E_n\rangle_{n\in\mathbb{N}}$, $\langle F_n\rangle_{n\in\mathbb{N}}$ are sequences in Σ, then

$$\lim_{n\to\mathcal{F}}(E_n \cap F_n) = (\lim_{n\to\mathcal{F}} E_n) \cap (\lim_{n\to\mathcal{F}} F_n),$$

$$\lim_{n\to\mathcal{F}}(E_n \triangle F_n) = (\lim_{n\to\mathcal{F}} E_n) \triangle (\lim_{n\to\mathcal{F}} F_n)$$

because $\mathcal{F}$ is an ultrafilter. **Q** Of course $\Sigma \subseteq \mathcal{A}$, because if $E_n = E$ for every n then $\lim_{n\to\mathcal{F}} E_n = E$.

(b) If $\langle E_n\rangle_{n\in\mathbb{N}}$ and $\langle F_n\rangle_{n\in\mathbb{N}}$ are sequences in Σ and $\lim_{n\to\mathcal{F}} E_n = \lim_{n\to\mathcal{F}} F_n$, then $\lim_{n\to\mathcal{F}} \mu E_n = \lim_{n\to\mathcal{F}} \mu F_n$. **P**

$$|\lim_{n\to\mathcal{F}} \mu E_n - \lim_{n\to\mathcal{F}} \mu F_n| = \lim_{n\to\mathcal{F}} |\mu E_n - \mu F_n| \le \lim_{n\to\mathcal{F}} \mu(E_n \triangle F_n) \le \mu^*(\lim_{n\to\mathcal{F}} E_n \triangle F_n)$$

(538G(iv))

$$= \mu^*(\lim_{n\to\mathcal{F}} E_n \triangle \lim_{n\to\mathcal{F}} F_n) = \mu^*\emptyset = 0. \text{ **Q**}$$

(c) We therefore have a functional $\phi : \mathcal{A} \to [0,1]$ defined by setting $\phi(\lim_{n\to\mathcal{F}} E_n) = \lim_{n\to\mathcal{F}} \mu E_n$ for every sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in Σ. Clearly ϕ extends μ. Also ϕ is additive. **P** If $\langle E_n\rangle_{n\in\mathbb{N}}$, $\langle F_n\rangle_{n\in\mathbb{N}}$ are sequences in Σ such that $\lim_{n\to\mathcal{F}} E_n$ and $\lim_{n\to\mathcal{F}} F_n$ are disjoint, then

$$\begin{aligned}
\phi(\lim_{n\to\mathcal{F}} E_n \cup \lim_{n\to\mathcal{F}} F_n) &= \phi(\lim_{n\to\mathcal{F}} E_n \cup F_n) = \lim_{n\to\mathcal{F}} \mu(E_n \cup F_n) \\
&= \lim_{n\to\mathcal{F}} \mu E_n + \mu F_n - \mu(E_n \cap F_n) \\
&= \lim_{n\to\mathcal{F}} \mu E_n + \lim_{n\to\mathcal{F}} \mu F_n - \lim_{n\to\mathcal{F}} \mu(E_n \cap F_n) \\
&= \phi(\lim_{n\to\mathcal{F}} E_n) + \phi(\lim_{n\to\mathcal{F}} F_n) - \phi(\lim_{n\to\mathcal{F}} E_n \cap F_n) \\
&= \phi(\lim_{n\to\mathcal{F}} E_n) + \phi(\lim_{n\to\mathcal{F}} F_n). \text{ **Q**}
\end{aligned}$$

(d) Next, if $\langle A_m\rangle_{m\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{A}$, and $0 \le \gamma < \inf_{m\in\mathbb{N}} \phi A_m$, there is an $A \in \mathcal{A}$ such that $A \subseteq \bigcap_{m\in\mathbb{N}} A_m$ and $\phi A \ge \gamma$. **P** We can suppose that $A_0 = X$. For each $m \in \mathbb{N}$, let $\langle E_{mn}\rangle_{n\in\mathbb{N}}$ be a sequence in Σ such that $A_m = \lim_{n\to\mathcal{F}} E_{mn}$, starting with $E_{0n} = X$ for every n. For $m \in \mathbb{N}$, set $E'_{mn} = \bigcap_{i\le m} E_{in}$ for $n \in \mathbb{N}$; then

$$A_m = \bigcap_{i\le m} A_i = \lim_{n\to\mathcal{F}} E'_{mn};$$

set $I_m = \{n : n \in \mathbb{N},\ \mu E'_{mn} \ge \gamma\}$. Since $\lim_{n\to\mathcal{F}} \mu E'_{mn} = \phi A_m > \gamma$, $I_m \in \mathcal{F}$. For $n \in \mathbb{N}$, set $F_n = \bigcap\{E'_{mn} : m \in \mathbb{N},\ \mu E'_{mn} \ge \gamma\}$; set $A = \lim_{n\to\mathcal{F}} F_n$. Then $\mu F_n \ge \gamma$ for every n, so $\phi A \ge \gamma$. Also, for $m \in \mathbb{N}$, $F_n \subseteq E'_{mn}$ whenever $n \in I_m$, so $A \subseteq \lim_{n\to\mathcal{F}} E'_{mn} = A_m$. **Q**

(e) In particular, $\inf_{m\in\mathbb{N}} \phi A_m$ must be 0 whenever $\langle A_m\rangle_{m\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{A}$ with empty intersection. By 413K, there is a complete measure λ on X extending ϕ and inner regular with respect to

$\mathcal{A}_\delta$, the set of intersections of sequences in $\mathcal{A}$. But $\lambda C = \sup\{\lambda A : A \in \mathcal{A},\ A \subseteq C\}$ for every $C \in \mathcal{A}_\delta$. **P** Suppose that $0 \le \gamma < \lambda C$. There is a sequence $\langle A_m\rangle_{m\in\mathbb{N}}$ in $\mathcal{A}$ with intersection C; because $\mathcal{A}$ is an algebra of sets, we can suppose that $\langle A_m\rangle_{m\in\mathbb{N}}$ is non-increasing. Now

$$\gamma < \lambda C = \inf_{m\in\mathbb{N}} \lambda A_m = \inf_{m\in\mathbb{N}} \phi A_m,$$

so (d) tells us that there is an $A \in \mathcal{A}$ such that $A \subseteq C$ and $\gamma \le \phi A = \lambda A$. **Q** It follows at once that λ is inner regular with respect to $\mathcal{A}$.

(f) If $E \in \Sigma$ and we set $E_n = E$ for every $n \in \mathbb{N}$, then $E = \lim_{n\to\mathcal{F}} E_n$ belongs to $\mathcal{A}$ and

$$\lambda E = \phi E = \lim_{n\to\mathcal{F}} \mu E_n = \mu E.$$

So λ extends μ. Finally, we see from 412Mb, as usual, that λ is uniquely defined.

Notation In this context, I will call λ the **$\mathcal{F}$-extension** of μ.

538J Proposition Let $\mathcal{F}$ be a measure-centering ultrafilter on $\mathbb{N}$ and (X, Σ, μ) a perfect probability space; let λ be the $\mathcal{F}$-extension of μ as defined in 538I.

(a) Let $(\mathfrak{A}, \bar\mu)$ be the measure algebra of μ, $(\mathfrak{B}, \bar\lambda)$ the measure algebra of λ, and $(\mathfrak{C}, \bar\nu)$ the probability algebra reduced power $(\mathfrak{A}, \bar\mu)^{\mathbb{N}}|\mathcal{F}$ (328C). Then we have a measure-preserving isomorphism $\pi : \mathfrak{B} \to \mathfrak{C}$ defined by saying that

$$\pi((\lim_{n\to\mathcal{F}} E_n)^\bullet) = \langle E_n^\bullet\rangle_{n\in\mathbb{N}}^\bullet$$

for every sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in Σ.

(b) Let (X', Σ', μ') be another perfect probability space, and $\phi : X \to X'$ an inverse-measure-preserving function. Let λ' be the $\mathcal{F}$-extension of μ'. Then ϕ is inverse-measure-preserving for λ and λ'.

(c) Let $\mathcal{F}'$ be a filter on $\mathbb{N}$ such that $\mathcal{F}' \le_{\mathrm{RK}} \mathcal{F}$, and λ' the $\mathcal{F}'$-extension of μ. Then λ extends λ'.

proof (a)(i) I had better check first that the formula for π defines a function. If $\langle E_n\rangle_{n\in\mathbb{N}}$, $\langle F_n\rangle_{n\in\mathbb{N}}$ are sequences in Σ such that $(\lim_{n\to\mathcal{F}} E_n)^\bullet = (\lim_{n\to\mathcal{F}} F_n)^\bullet$ in $\mathfrak{B}$, then

$$\begin{aligned} 0 = \lambda(\lim_{n\to\mathcal{F}} E_n \triangle \lim_{n\to\mathcal{F}} F_n) &= \lim_{n\to\mathcal{F}} \mu(E_n \triangle F_n) \\ &= \lim_{n\to\mathcal{F}} \bar\mu(E_n^\bullet \triangle F_n^\bullet) = \bar\nu(\langle E_n^\bullet\rangle_{n\in\mathbb{N}}^\bullet \triangle \langle F_n^\bullet\rangle_{n\in\mathbb{N}}^\bullet), \end{aligned}$$

so $\langle E_n^\bullet\rangle_{n\in\mathbb{N}}^\bullet = \langle F_n^\bullet\rangle_{n\in\mathbb{N}}^\bullet$ in $\mathfrak{C}$.

(ii) Setting $\mathfrak{B}_0 = \{E^\bullet : E \in \mathcal{A}\}$, where $\mathcal{A}$ is as in 538I, it is now routine to check that $\pi : \mathfrak{B}_0 \to \mathfrak{C}$ is a surjective measure-preserving Boolean homomorphism. (Recall that $\mathfrak{C}$ is, by definition, the quotient of $\mathfrak{A}^{\mathbb{N}}$ by the ideal $\{\langle a_n\rangle_{n\in\mathbb{N}} : \lim_{n\to\mathcal{F}} \bar\mu a_n = 0\}$.) But of course this means that $\mathfrak{B}_0$ is isomorphic to $\mathfrak{C}$, therefore Dedekind complete. Since λ is inner regular with respect to $\mathcal{A}$ (538I), $\mathfrak{B}_0$ is order-dense in $\mathfrak{B}$, and must be the whole of $\mathfrak{B}$.

(b) Setting

$$\mathcal{A} = \{\lim_{n\to\mathcal{F}} E_n : E_n \in \Sigma\ \forall\, n \in \mathbb{N}\}, \quad \mathcal{A}' = \{\lim_{n\to\mathcal{F}} F_n : F_n \in \Sigma'\ \forall\, n \in \mathbb{N}\}$$

as in 538I, $\phi^{-1}[C] \in \mathcal{A}$ and $\lambda\phi^{-1}[C] = \lambda' C$ for every $C \in \mathcal{A}'$. **P** Let $\langle F_n\rangle_{n\in\mathbb{N}}$ be a sequence in Σ' such that $C = \lim_{n\to\mathcal{F}} F_n$; then

$$\begin{aligned} \lambda\phi^{-1}[C] = \lambda\phi^{-1}[\lim_{n\to\mathcal{F}} F_n] &= \lambda(\lim_{n\to\mathcal{F}} \phi^{-1}[F_n]) \\ &= \lim_{n\to\mathcal{F}} \mu\phi^{-1}[F_n] = \lim_{n\to\mathcal{F}} \mu' F_n = \lambda' C. \ \textbf{Q} \end{aligned}$$

By 412K, ϕ is inverse-measure-preserving for λ and λ'.

(c) By 538Hb, $\mathcal{F}'$ is measure-centering. Let $f : \mathbb{N} \to \mathbb{N}$ be such that $\mathcal{F}' = f[[\mathcal{F}]]$. Setting

$$\mathcal{A} = \{\lim_{n\to\mathcal{F}} E_n : E_n \in \Sigma\ \forall\, n \in \mathbb{N}\}, \quad \mathcal{A}' = \{\lim_{n\to\mathcal{F}'} E_n : E_n \in \Sigma\ \forall\, n \in \mathbb{N}\},$$

$\mathcal{A}' \subseteq \mathcal{A}$ and $\lambda A = \lambda' A$ for every $A \in \mathcal{A}'$. **P** Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a sequence in Σ such that $A = \lim_{n\to\mathcal{F}'} E_n$; then $A = \lim_{n\to\mathcal{F}} E_{f(n)}$, so

$$\lambda A = \lim_{n\to\mathcal{F}} \mu E_{f(n)} = \lim_{n\to\mathcal{F}'} \mu E_n = \lambda' A. \text{ **Q**}$$

By 412K again, the identity map from X to itself is inverse-measure-preserving for λ and λ', that is, λ extends λ'.

538K Having identified the measure algebra of a measure-centering-ultrafilter extension λ as a probability algebra reduced product (538Ja), we are in a position to apply the results of §377.

Proposition Let (X,Σ,μ) be a perfect probability space, $\mathcal{F}$ a measure-centering ultrafilter on $\mathbb{N}$ and λ the $\mathcal{F}$-extension of μ as constructed in 538I.

(a)(i) Let $\langle f_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{L}^0(\mu)$ such that $\{f_n^\bullet : n\in\mathbb{N}\}$ is bounded in the linear topological space $L^0(\mu)$. Then

(α) $f(x) = \lim_{n\to\mathcal{F}} f_n(x)$ is defined in $\mathbb{R}$ for λ-almost every $x\in X$;

(β) $f\in\mathcal{L}^0(\lambda)$.

(ii) For every $f\in\mathcal{L}^0(\lambda)$ there is a sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{L}^0(\mu)$, bounded in the sense of (i), such that $f = \lim_{n\to\mathcal{F}} f_n$ λ-a.e.

(b) Suppose that $1\le p\le\infty$, and $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}^p(\mu)$ such that $\sup_{n\in\mathbb{N}}\|f_n\|_p$ is finite. Set $f(x) = \lim_{n\to\mathcal{F}} f_n(x)$ whenever this is defined in $\mathbb{R}$.

(i)(α) $f\in\mathcal{L}^p(\lambda)$;

(β) $\|f\|_p \le \lim_{n\to\mathcal{F}}\|f_n\|_p$.

(ii) Let g be a conditional expectation of f on Σ.

(α) If $p=1$ and $\{f_n : n\in\mathbb{N}\}$ is uniformly integrable, then then $\|f\|_1 = \lim_{n\to\mathcal{F}}\|f_n\|_1$ and $g^\bullet = \lim_{n\to\mathcal{F}} f_n^\bullet$ for the weak topology of $L^1(\mu)$.

(β) If $1<p<\infty$, then $g^\bullet = \lim_{n\to\mathcal{F}} f_n^\bullet$ for the weak topology of $L^p(\mu)$.

(c) Suppose that $1\le p\le\infty$ and $f\in\mathcal{L}^p(\lambda)$.

(i) There is a sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{L}^p(\mu)$ such that $f = \lim_{n\to\mathcal{F}} f_n$ λ-a.e. and $\|f\|_p = \sup_{n\in\mathbb{N}}\|f_n\|_p$.

(ii) If $p=1$, we can arrange that $\langle f_n\rangle_{n\in\mathbb{N}}$ should be uniformly integrable.

(d) Let (X',Σ',μ') be another perfect measure space, and λ' the $\mathcal{F}$-extension of μ'. Let $S : L^1(\mu)\to L^1(\mu')$ be a bounded linear operator.

(i) There is a unique bounded linear operator $\hat S : L^1(\lambda)\to L^1(\lambda')$ such that $\hat S f^\bullet = g^\bullet$ whenever $\langle f_n\rangle_{n\in\mathbb{N}}$, $\langle g_n\rangle_{n\in\mathbb{N}}$ are uniformly integrable sequences in $\mathcal{L}^1(\mu)$, $\mathcal{L}^1(\nu)$ respectively, $f = \lim_{n\to\mathcal{F}} f_n$ λ-a.e., $g = \lim_{n\to\mathcal{F}} g_n$ λ'-a.e., and $g_n^\bullet = S f_n^\bullet$ for every $n\in\mathbb{N}$.

(ii) The map $S\mapsto\hat S : \mathsf{B}(L^1(\mu);L^1(\mu'))\to\mathsf{B}(L^1(\lambda);L^1(\lambda'))$ is a norm-preserving Riesz homomorphism.

proof We shall find that most of the work for this result has been done in §377. The only new step is in (a)(i), but we shall have some checking to do.

(a)(i) Let $\langle\tilde f_n\rangle_{n\in\mathbb{N}}$ be a sequence of Σ-measurable functions from X to $\mathbb{R}$ such that $\tilde f_n = f_n$ μ-a.e. for every $n\in\mathbb{N}$.

(α) Let $\epsilon>0$. Applying 367Rd[9] to $\{\tilde f_n^\bullet : n\in\mathbb{N}\} = \{f_n^\bullet : n\in\mathbb{N}\}$, there is a $\gamma>0$ such that $\mu E_n\le\epsilon$ for every $n\in\mathbb{N}$, where $E_n = \{x : |\tilde f_n(x)|\ge\gamma\}$. Set $E = \lim_{n\to\mathcal{F}} E_n$, so that $\lambda E\le\epsilon$. For $x\in X\setminus E$, $\{n : |\tilde f_n(x)|\le\gamma\}\in\mathcal{F}$, so $\lim_{n\to\mathcal{F}}\tilde f_n(x)$ is defined in $\mathbb{R}$. As ϵ is arbitrary, $\lim_{n\to\mathcal{F}}\tilde f_n(x)$ is defined in $\mathbb{R}$ for λ-almost every x. Since

$$\{x : x\in\operatorname{dom} f_n \text{ and } f_n(x) = \tilde f_n(x) \text{ for every } n\in\mathbb{N}\}$$

is μ-conegligible, therefore λ-conegligible, $\lim_{n\to\mathcal{F}} f_n$ is defined in $\mathbb{R}$ λ-a.e.

(β) For any $\alpha\in\mathbb{R}$,

$$\{x : \lim_{n\to\mathcal{F}}\tilde f_n(x) > \alpha\} = \bigcup_{k\in\mathbb{N}}\lim_{n\to\mathcal{F}}\{x : f_n(x)\ge\alpha+2^{-k}\}\in\operatorname{dom}\lambda.$$

So $f =_{\text{a.e.}} \lim_{n\to\mathcal{F}}\tilde f_n$ belongs to $\mathcal{L}^0(\lambda)$.

(ii) At this point I seek to import the machinery of §377.

(α) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\lambda)$ be the measure algebras of μ, λ respectively; recall that we can identify $L^0(\mu)$ and $L^0(\lambda)$ with $L^0(\mathfrak{A})$ and $L^0(\mathfrak{B})$ (364Ic). Write $(\mathfrak{C},\bar\nu)$ for the probability algebra reduced power

[9] Later editions only.

$(\mathfrak{A},\bar\mu)^{\mathbb{N}}|\mathcal{F}$; let $\phi:\mathfrak{A}^{\mathbb{N}}\to\mathfrak{C}$ be the canonical surjection, and $\pi:\mathfrak{B}\to\mathfrak{C}$ the isomorphism of 538Ja; set $\psi=\pi^{-1}\phi:\mathfrak{A}^{\mathbb{N}}\to\mathfrak{B}$. Then $\bar\lambda\psi(\langle a_n\rangle_{n\in\mathbb{N}})=\lim_{n\to\mathcal{F}}\bar\mu a_n$ for every sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$, and ψ is surjective.

(β) Let $W_0\subseteq L^0(\mathfrak{A})^{\mathbb{N}}$ be the set of sequences bounded for the topology of convergence in measure, and $\mathcal{W}_0\subseteq\mathcal{L}^0(\mu)^{\mathbb{N}}$ the set of sequences $\langle f_n\rangle_{n\in\mathbb{N}}$ such that $\langle f_n^\bullet\rangle_{n\in\mathbb{N}}\in W_0$. Then we have a Riesz homomorphism $T:W^0\to L^0(\mathfrak{B})$ defined by saying that $T(\langle f_n^\bullet\rangle_{n\in\mathbb{N}})=(\lim_{n\to\mathcal{F}}f_n)^\bullet$ whenever $\langle f_n\rangle_{n\in\mathbb{N}}\in\mathcal{W}_0$. **P** We know from (i) that $(\lim_{n\to\mathcal{F}}f_n)^\bullet$ is defined in $L^0(\lambda)\cong L^0(\mathfrak{B})$ whenever $\langle f_n\rangle_{n\in\mathbb{N}}\in\mathcal{W}_0$. (I am taking the domain of $\lim_{n\to\mathcal{F}}f_n$ to be $\{x:\lim_{n\to\mathcal{F}}f_n(x)$ is defined in $\mathbb{R}\}$.) Since

$$\lim_{n\to\mathcal{F}}f_n=_{\text{a.e.}}\lim_{n\to\mathcal{F}}g_n$$

whenever $f_n=_{\text{a.e.}}g_n$ for every n, T is well-defined. Since

$$\lim_{n\to\mathcal{F}}f_n+g_n=_{\text{a.e.}}\lim_{n\to\mathcal{F}}f_n+\lim_{n\to\mathcal{F}}g_n,$$

$$\lim_{n\to\mathcal{F}}\alpha f_n=_{\text{a.e.}}\alpha\lim_{n\to\mathcal{F}}f_n,\quad \lim_{n\to\mathcal{F}}|f_n|=_{\text{a.e.}}|\lim_{n\to\mathcal{F}}f_n|$$

whenever $\langle f_n\rangle_{n\in\mathbb{N}},\langle g_n\rangle_{n\in\mathbb{N}}\in\mathcal{W}_0$ and $\alpha\in\mathbb{R}$, T is a Riesz homomorphism. **Q**

(γ) If $\langle a_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\mathfrak{A}$, $T(\langle\chi a_n\rangle_{n\in\mathbb{N}})=\chi\psi(\langle a_n\rangle_{n\in\mathbb{N}})$. **P** Express each a_n as $E_n^\bullet$, where $E_n\in\Sigma$, and set $F=\lim_{n\to\mathcal{F}}E_n$. In the language of 538Ja,

$$\psi(\langle a_n\rangle_{n\in\mathbb{N}})=\pi^{-1}\phi(\langle a_n\rangle_{n\in\mathbb{N}})=\pi^{-1}(\langle a_n\rangle_{n\in\mathbb{N}}^\bullet)=F^\bullet,$$

so

$$T(\langle\chi a_n\rangle_{n\in\mathbb{N}})=(\lim_{n\to\mathcal{F}}\chi E_n)^\bullet=(\chi F)^\bullet=\chi(F^\bullet)=\chi\psi(\langle a_n\rangle_{n\in\mathbb{N}}).\ \mathbf{Q}$$

(δ) Recalling that W_0 is just the family of sequences $\langle u_n\rangle_{n\in\mathbb{N}}$ in L^0 such that $\inf_{k\in\mathbb{N}}\sup_{n\in\mathbb{N}}\bar\mu[\![|u_n|>k]\!]=0$ (367Rd again), (γ) means that we can identify $T:W_0\to L^0(\mathfrak{B})$ with the Riesz homomorphism described in 377B. By 377D(d-i), $T[W_0]=L^0(\mathfrak{B})$, which is what we need to prove the immediate result here.

(b)(i) As in part (a) of the proof of 377C, we see that a $\|\;\|_p$-bounded sequence in $\mathcal{L}^p(\mu)$ will belong to $\mathcal{W}_0$. So we can use 377Db.

(ii) Use 377Ec.

(c) Use 377Dd.

(d) Use 377F.

538L Theorem Suppose that ζ is a non-zero countable ordinal and $\langle\mathcal{F}_\xi\rangle_{1\le\xi\le\zeta}$ is a family of Ramsey ultrafilters on $\mathbb{N}$, no two isomorphic. Let $\langle\mathcal{G}_\xi\rangle_{\xi\le\zeta}$ be the corresponding iterated product system, as described in 538E. Then $\mathcal{G}_\zeta$ is a measure-centering ultrafilter.

proof (a) Define $\langle(\mathfrak{A}_\xi,\bar\mu_\xi)\rangle_{\xi\le\zeta}$ inductively, as follows. $(\mathfrak{A}_0,\bar\mu_0)=(\mathfrak{B}_\omega,\bar\nu_\omega)$ is to be the measure algebra of the usual measure on $\{0,1\}^{\mathbb{N}}$. Given $\langle(\mathfrak{A}_\eta,\bar\mu_\eta)\rangle_{\eta<\xi}$, where $0<\xi\le\zeta$, let $(\mathfrak{A}_\xi,\bar\mu_\xi)$ be the probability algebra reduced product $\prod_{k\in\mathbb{N}}(\mathfrak{A}_{\theta(\xi,k)},\bar\mu_{\theta(\xi,k)})|\mathcal{F}_\xi$, as described in 328A-328C. At the end of the induction, write $(\mathfrak{C},\bar\nu)$ for $(\mathfrak{A}_\zeta,\bar\mu_\zeta)$.

(b) We have a family $\langle\phi_{\xi\eta}\rangle_{\eta\le\xi\le\zeta}$ defined by induction on ξ, as follows. The inductive hypothesis will be that $\phi_{\eta'\eta}$ is a measure-preserving Boolean homomorphism from $\mathfrak{A}_\eta$ to $\mathfrak{A}_{\eta'}$, and that $\phi_{\eta''\eta}=\phi_{\eta''\eta'}\phi_{\eta'\eta}$ whenever $\eta\le\eta'\le\eta''<\xi$. For the inductive step to ξ, take $\phi_{\xi\xi}$ to be the identity map on $\mathfrak{A}_\xi$. If $\xi>0$, set $\tilde\phi_{kj}=\phi_{\theta(\xi,k),\theta(\xi,j)}$ for $j\le k$ in $\mathbb{N}$; then 328Ea tells us that we have measure-preserving Boolean homomorphisms $\tilde\phi_k:\mathfrak{A}_{\theta(\xi,k)}\to\mathfrak{A}_\xi$ such that $\tilde\phi_j=\tilde\phi_k\tilde\phi_{kj}$ for $j\le k$. If $j\le k$ and $\eta\le\theta(\xi,j)$, then

$$\tilde\phi_k\phi_{\theta(\xi,k),\eta}=\tilde\phi_k\tilde\phi_{kj}\phi_{\theta(\xi,j),\eta}=\tilde\phi_j\phi_{\theta(\xi,j),\eta}$$

whenever $k\ge j$; so we can take this common value for $\phi_{\xi\eta}$. If $\eta\le\eta'<\xi$, then take k such that $\eta'\le\theta(\xi,k)$, and see that

$$\phi_{\xi\eta'}\phi_{\eta'\eta}=\tilde\phi_k\phi_{\theta(\xi,k),\eta'}\phi_{\eta'\eta}=\tilde\phi_k\phi_{\theta(\xi,k),\eta}=\phi_{\xi\eta},$$

so the induction proceeds.

For each $\xi \le \zeta$, write π_ξ for $\phi_{\zeta\xi} : \mathfrak{A}_\xi \to \mathfrak{C}$, and $\mathfrak{C}_\xi$ for the subalgebra $\pi_\xi[\mathfrak{A}_\xi]$. Of course $\pi_\xi\phi_{\xi\eta} = \pi_\eta$, so that $\mathfrak{C}_\eta \subseteq \mathfrak{C}_\xi$, whenever $\eta \le \xi \le \zeta$.

(c) For each $\xi > 0$, we have a canonical map $\langle a_k\rangle_{k\in\mathbb{N}} \to \langle a_k\rangle^\bullet_{k\in\mathbb{N}} : \prod_{k\in\mathbb{N}} \mathfrak{A}_{\theta(\xi,k)} \to \mathfrak{A}_\xi$. Since every $\pi_\xi : \mathfrak{A}_\xi \to \mathfrak{C}_\xi$ is a measure-preserving isomorphism, we have a corresponding map $\psi_\xi : \prod_{k\in\mathbb{N}} \mathfrak{C}_{\theta(\xi,k)} \to \mathfrak{C}_\xi$. Reading off the basic facts of 328Ab and 328Eb, we see that

— $\bar\nu\psi_\xi(\langle c_k\rangle_{k\in\mathbb{N}}) = \lim_{k\to\mathcal{F}_\xi} \bar\nu c_k$ for every sequence $\langle c_k\rangle_{k\in\mathbb{N}} \in \prod_{k\in\mathbb{N}} \mathfrak{C}_{\theta(\xi,k)}$,

— $\psi_\xi(\langle c_k\rangle_{k\in\mathbb{N}}) \subseteq \sup_{k\in A} c_k$ whenever $\langle c_k\rangle_{k\in\mathbb{N}} \in \prod_{k\in\mathbb{N}} \mathfrak{C}_{\theta(\xi,k)}$ and $A \in \mathcal{F}_\xi$

(we can take the supremum in $\mathfrak{C}$ because $\mathfrak{C}_\xi$ is regularly embedded in $\mathfrak{C}$, as noted in 314Ga).

(d) Let $\langle a_\tau\rangle_{\tau\in S}$ be a family in $\mathfrak{A}_0 = \mathfrak{B}_\omega$ such that $\gamma = \inf_{\tau\in S} \bar\mu a_\tau$ is non-zero. By 538Fe, we can find a disjoint family $\langle A_\xi\rangle_{1\le\xi\le\zeta}$ of subsets of $\mathbb{N}$ such that $A_\xi \in \mathcal{F}_\xi$ for every ξ. Use these to define $T \subseteq S$ and $\alpha : T \to [0,\zeta]$ as in 538Ee. Set $c_\tau = 0$ for $\tau \in S \setminus T$. For $\tau \in T$ define $c_\tau \in \mathfrak{C}_{\alpha(\tau)}$ by induction on $\alpha(\tau)$, as follows. If $\alpha(\tau) = 0$, set $c_\tau = \pi_0 a_\tau$. For the inductive step to $\alpha(\tau) = \xi > 0$, we know that $\tau^\frown{<}k{>} \in T$ and $\alpha(\tau^\frown{<}k{>}) = \theta(\xi,k)$ whenever $k \in A_\xi$ and $\tau(i) < k$ for every $i < \operatorname{dom}\tau$; for other k, $\tau^\frown{<}k{>} \notin T$ so $c_{\tau^\frown<k>} = 0 \in \mathfrak{C}_{\theta(\xi,k)}$. Thus $c_{\tau^\frown<k>} \in \mathfrak{C}_{\theta(\xi,k)}$ for every k, and $\psi_\xi(\langle c_{\tau^\frown<k>}\rangle_{k\in\mathbb{N}}) \in \mathfrak{C}_\xi$; take this for c_τ. Note that

$$\bar\nu c_\tau = \lim_{k\to\mathcal{F}_\xi} \bar\nu c_{\tau^\frown<k>} \ge \inf\{\bar\nu c_{\tau^\frown<k>} : k \in \mathbb{N},\, \tau^\frown{<}k{>} \in T\}.$$

Inducing on $\alpha(\tau)$, we see that $\bar\nu c_\tau \ge \gamma$ for every $\tau \in T$. In particular, $\bar\nu c_\emptyset \ge \gamma$.

(e) For $I \subseteq \mathbb{N}$, set $T_I = T \cap \bigcup_{n\in\mathbb{N}} I^n$ and $e_I = \inf_{\tau\in T_I} c_\tau$; let $\mathcal{S}$ be the family of those finite subsets I of $\mathbb{N}$ such that $e_I \ne \emptyset$. Then $T_\emptyset = \{\emptyset\}$, $e_\emptyset = c_\emptyset$ and $\emptyset \in \mathcal{S}$. Moreover, if $I \in \mathcal{S}$ and $1 \le \xi \le \zeta$, then $\{k : I \cup \{k\} \in \mathcal{S}\} \in \mathcal{F}_\xi$. **P** Set $k_0 = \sup I + 1$. If $k \in A_\xi$ and $k \ge k_0$, set

$$d_k = \inf\{c_{\tau^\frown<k>} : \tau \in T_I,\, \alpha(\tau) = \xi\}.$$

Set $B = \{k : k \in A_\xi,\, k \ge k_0,\, d_k \cap e_I \ne 0\}$. If $k \in B$, then

$$T_{I\cup\{k\}} = T_I \cup \{\tau^\frown{<}k{>} : \tau \in T_I,\, \alpha(\tau) = \xi\},$$

because every member of T is strictly increasing and $\tau^\frown{<}k{>}$ can belong to T only when $k \in A_{\alpha(\tau)}$, that is, $\alpha(\tau) = \xi$. So $e_{I\cup\{k\}} = d_k \cap e_I \ne 0$ and $I \cup \{k\} \in \mathcal{S}$.

? If $B \notin \mathcal{F}_\xi$, then $B' = \{k : k \in A_\xi,\, k \ge k_0,\, d_k \cap e_I = 0\}$ belongs to $\mathcal{F}_\xi$. So

$$\begin{aligned} e_I &\subseteq \inf\{c_\tau : \tau \in T_I,\, \alpha(\tau) = \xi\} \\ &= \inf_{\substack{\tau\in T_I \\ \alpha(\tau)=\xi}} \psi_\xi(\langle c_{\tau^\frown<k>}\rangle_{k\in\mathbb{N}}) = \psi_\xi(\langle \inf_{\substack{\tau\in T_I \\ \alpha(\tau)=\xi}} c_{\tau^\frown<k>}\rangle_{k\in\mathbb{N}}) \end{aligned}$$

(because ψ_ξ is a Boolean homomorphism and T_I is finite)

$$\subseteq \sup_{k\in B'} \inf_{\substack{\tau\in T_I \\ \alpha(\tau)=\xi}} c_{\tau^\frown<k>}$$

(by (c))

$$= \sup_{k\in B'} d_k.$$

But $e_I \cap d_k = 0$ for every $k \in B'$ and $e_I \ne 0$. **X**

Thus $\{k : I \cup \{k\} \in \mathcal{S}\} \supseteq B \in \mathcal{F}_\xi$. **Q**

(f) For $i \in \mathbb{N}$ set

$$C_i = \{k : I \cup \{k\} \in \mathcal{S} \text{ whenever } I \in \mathcal{S} \text{ and } I \subseteq i\},$$

so that $C_i \in \mathcal{F}_\xi$ for every $\xi \in [1,\zeta]$. At this point, recall that every $\mathcal{F}_\xi$ is supposed to be a Ramsey ultrafilter. So for each $\xi \in [1,\zeta]$ we have an $A'_\xi \in \mathcal{F}_\xi$ such that $A'_\xi \subseteq A_\xi \cap C_0$ and $j \in C_i$ whenever $i, j \in A'_\xi$ and $i < j$ (538Fc). Next, for $i \in \mathbb{N}$ set $M_i = \{\alpha(\tau) : \tau \in T,\, \tau(j) \le i$ whenever $j < \operatorname{dom}\tau\}$; then M_i is finite, so there is a $D \in \bigcap_{1\le\xi\le\zeta} \mathcal{F}_\xi$ such that whenever $i, j \in D$, $i < j$ and $\xi \in M_i$, there is a $k \in A'_\xi$ such that $i < k < j$ (538Ff). Of course we can suppose that $D \subseteq \bigcup_{1\le\xi\le\zeta} A'_\xi$, so that $D \cap A_\xi = D \cap A'_\xi$ for every ξ.

(g) $J \in \mathcal{S}$ for every finite subset J of D. **P** Induce on $\#(J)$. We know that $\emptyset \in \mathcal{S}$. If $i \in D$, then $\{i\} \in \mathcal{S}$ because $D \subseteq C_0$. For the inductive step to $\#(J) \geq 2$, set $j = \max J$, $I = J \setminus \{j\}$ and $i = \max I$. Then $I \in \mathcal{S}$, by the inductive hypothesis; so if $T_J = T_I$, we certainly have $J \in \mathcal{S}$. Otherwise, there is a member of $T_J \setminus T_I$, and this must be of the form $\tau^\frown{<}j{>}$ where $\tau \in T_I$ and $j \in A_{\alpha(\tau)}$; as $j \in D$, $j \in A'_{\alpha(\tau)}$. But this means that $\alpha(\tau) \in M_i$ and there is a $k \in A'_{\alpha(\tau)}$ such that $i < k < j$. In this case, $j \in C_k$ and $I \subseteq k$, so $J = I \cup \{k\}$ belongs to $\mathcal{S}$, and the induction proceeds. **Q**

(h) Thus $\{c_\tau : \tau \in T_D\}$ is centered; setting $T^*_D = \{\tau : \tau \in T_D, \alpha(\tau) = 0\}$, $\{c_\tau : \tau \in T^*_D\}$ and therefore $\{a_\tau : \tau \in T^*_D\}$ are centered. But T^*_D belongs to $\mathcal{G}_\zeta$, by 538Ee.

Since $\langle a_\tau \rangle_{\tau \in S}$ was chosen arbitrarily in (d) above, $\mathcal{G}_\zeta$ satisfies the condition of 538G(ii), translated to the countably infinite set S, and is measure-centering.

538M Benedikt's theorem (Benedikt 98) Let (X, Σ, μ) be a perfect probability space. Then there is a measure λ on X, extending μ, such that $\lambda(\lim_{n\to\mathcal{F}} E_n)$ is defined and equal to $\lim_{n\to\mathcal{F}} \mu E_n$ for every sequence $\langle E_n \rangle_{n\in\mathbb{N}}$ in Σ and every Ramsey filter $\mathcal{F}$ on $\mathbb{N}$.

proof (a) If there are no Ramsey filters, we can take $\lambda = \mu$ and stop; so let us suppose that there is at least one Ramsey filter. Let $\mathfrak{F}$ be a family of Ramsey filters consisting of just one member of each isomorphism class, so that every Ramsey filter is isomorphic to some member of $\mathfrak{F}$, and no two members of $\mathfrak{F}$ are isomorphic. Fix a well-ordering $\preccurlyeq$ of $\mathfrak{F}$ with a greatest member $\mathcal{F}^*$ and a family $\langle \theta(\xi, k) \rangle_{1\leq\xi<\omega_1, k\in\mathbb{N}}$ such that $\langle \theta(\xi, k) \rangle_{k\in\mathbb{N}}$ is always a non-decreasing sequence of ordinals less than ξ such that $\{\theta(\xi, k) : k \in \mathbb{N}\}$ is cofinal with ξ.

(b)(i) For any non-empty countable set $D \subseteq \mathfrak{F}$ containing $\mathcal{F}^*$, enumerate it in $\preccurlyeq$-increasing order as $\langle \mathcal{F}_\xi \rangle_{1\leq\xi\leq\zeta}$, and let $\mathcal{G}_D$ be the measure-centering ultrafilter constructed from $\langle \mathcal{F}_\xi \rangle_{1\leq\xi\leq\zeta}$ and $\langle \theta(\xi, k) \rangle_{1\leq\xi\leq\zeta, k\in\mathbb{N}}$ by the method of 538E-538L; let λ_D be the $\mathcal{G}_D$-extension of μ as defined in 538I.

(ii) For any non-empty finite set $I \subseteq \mathfrak{F}$, list it in $\preccurlyeq$-increasing order as $\mathcal{F}_0, \dots, \mathcal{F}_n$, and set $\mathcal{H}_I = \mathcal{F}_n \ltimes \dots \ltimes \mathcal{F}_0$ as defined in 538D. By 538Ed, or otherwise, $\mathcal{H}_I \leq_{\mathrm{RK}} \mathcal{G}_I$, so $\mathcal{H}_I$ is measure-centering (538Hb); let λ'_I be the $\mathcal{H}_I$-extension of μ.

(c) If $\emptyset \neq I \subseteq J \in [\mathfrak{F}]^{<\omega}$, then $\mathcal{H}_I \leq_{\mathrm{RK}} \mathcal{H}_J$, by 538Dg, and λ'_J extends λ'_I, by 538Jc. Thus $\langle \lambda'_I \rangle_{\emptyset\neq I\in[\mathfrak{F}]^{<\omega}}$ is an upwards-directed family of probability measures on X.

If $\mathcal{I} \subseteq [\mathfrak{F}]^{<\omega} \setminus \{\emptyset\}$ is countable, we have a non-empty countable set $D \subseteq \mathfrak{F}$ including $\{\mathcal{F}^*\} \cup \bigcup \mathcal{I}$. Now 538Ed tells us that $\mathcal{H}_I \leq_{\mathrm{RK}} \mathcal{G}_D$ for every $I \in \mathcal{I}$, so that λ_D extends λ'_I for every $I \in \mathcal{I}$ (538Jc again). Thus for every countable subset of $\{\lambda'_I : I \in [\mathfrak{F}]^{<\omega} \setminus \{\emptyset\}\}$ there is a measure on X extending them all. By 457G, there is a measure λ on X extending every λ'_I.

(d) If $\mathcal{F}$ is a Ramsey ultrafilter and $\langle E_n \rangle_{n\in\mathbb{N}}$ is a sequence in Σ, there is an $\mathcal{F}' \in \mathfrak{F}$ such that $\mathcal{F}$ and $\mathcal{F}'$ are isomorphic. In particular, $\mathcal{F} \leq_{\mathrm{RK}} \mathcal{F}'$, so $\tilde{\lambda}_{\mathcal{F}'}$ extends $\tilde{\lambda}_{\mathcal{F}}$, where $\tilde{\lambda}_{\mathcal{F}}$, $\tilde{\lambda}_{\mathcal{F}'}$ are the $\mathcal{F}$-extension and $\mathcal{F}'$-extension of μ. But λ extends $\lambda'_{\{\mathcal{F}'\}} = \tilde{\lambda}_{\mathcal{F}'}$ and therefore extends $\tilde{\lambda}_{\mathcal{F}}$. Accordingly $\lambda(\lim_{n\to\mathcal{F}} E_n)$ is defined and equal to $\tilde{\lambda}_{\mathcal{F}}(\lim_{n\to\mathcal{F}} E_n) = \lim_{n\to\mathcal{F}} \mu E_n$, as required.

538N Measure-converging filters: Proposition (a) Let $\mathcal{F}$ be a free filter on $\mathbb{N}$. Let ν_ω be the usual measure on $\{0,1\}^{\mathbb{N}}$, and T_ω its domain. Then the following are equiveridical:

(i) $\mathcal{F}$ is measure-converging;

(ii) whenever $\langle F_n \rangle_{n\in\mathbb{N}}$ is a sequence in T_ω and $\lim_{n\to\infty} \nu_\omega F_n = 1$, then $\bigcup_{A\in\mathcal{F}} \bigcap_{n\in A} F_n$ is conegligible;

(iii) whenever (X, Σ, μ) is a measure space with locally determined negligible sets (definition: 213I), and $\langle f_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}^0 = \mathcal{L}^0(\mu)$ which converges in measure to $f \in \mathcal{L}^0$, then $\lim_{n\to\mathcal{F}} f_n =_{\text{a.e.}} f$;

(iv) whenever μ is a Radon measure on $\mathcal{P}\mathbb{N}$ such that $\lim_{n\to\infty} \mu E_n = 1$, where $E_n = \{a : n \in a \subseteq \mathbb{N}\}$ for each n, then $\mu\mathcal{F} = 1$.

(b) Every measure-converging filter is free.

(c) Suppose that $\mathcal{F}$ is a measure-converging filter.

(i) If $\mathcal{G}$ is a filter on $\mathbb{N}$ including $\mathcal{F}$, then $\mathcal{G}$ is measure-converging.

(ii) If $\mathcal{G}$ is a filter on $\mathbb{N}$ and $\mathcal{G} \leq_{\mathrm{RB}} \mathcal{F}$ (definition: 5A6Ic), then $\mathcal{G}$ is measure-converging.

(d) (M.Foreman) Every rapid filter is measure-converging.

(e) (M.Talagrand) If there is a measure-converging filter, there is a measure-converging filter which is not rapid.

(f) Let $\mathcal{F}$ be a measure-converging filter on $\mathbb{N}$ and $\mathcal{G}$ any filter on $\mathbb{N}$. Then $\mathcal{G} \ltimes \mathcal{F}$ is measure-converging.

(g) If $\mathfrak{m}_{\mathrm{countable}} = \mathfrak{d}$, there is a rapid filter.

proof (a)(i)⇒(iii) Suppose that $\mathcal{F}$ is measure-converging, and that (X, Σ, μ), $\langle f_n \rangle_{n\in\mathbb{N}}$ and f are as in (iii). Let $H \in \Sigma$ be a conegligible set such that $H \subseteq \operatorname{dom} f \cap \operatorname{dom} f_n$ and $f{\restriction}H$ and $f_n{\restriction}H$ are measurable for every $n \in \mathbb{N}$. Let $k \in \mathbb{N}$; set $H_k = \{x : x \in H, \limsup_{n\to\mathcal{F}} |f_n(x) - f(x)| > 2^{-k}\}$. Then $H_k \cap F$ is negligible whenever $F \in \Sigma$ and $\mu F < \infty$. **P** If $\mu F = 0$ this is trivial. Otherwise, let $\nu = \frac{1}{\mu F}\mu_F$ be the normalized subspace measure on F. For each $n \in \mathbb{N}$, set $F_n = \{x : x \in F \cap H, |f_n(x) - f(x)| \leq 2^{-k}\}$. Then

$$\lim_{n\to\infty} \nu(F \setminus F_n) \leq \frac{2^k}{\mu F} \lim_{n\to\infty} \int \min(|f_n - f|, \chi F) d\mu = 0$$

because $\langle f_n \rangle_{n\in\mathbb{N}} \to f$ in measure. So $\lim_{n\to\infty} \nu F_n = 1$ and $H' = \bigcup_{A\in\mathcal{F}} \bigcap_{n\in A} F_n$ is ν-conegligible. But $H' \cap H_k = \emptyset$, so $\mu^*(H_k \cap F) = \nu^*(H_k \cap F) = 0$. **Q**

Since μ has locally determined negligible sets, H_k is negligible. This is true for every $k \in \mathbb{N}$, so $H \setminus \bigcup_{k\in\mathbb{N}} H_k$ is conegligible; and $\lim_{n\to\mathcal{F}} f_n(x) = f(x)$ for every $x \in H \setminus \bigcup_{k\in\mathbb{N}} H_k$, so $\lim_{n\to\mathcal{F}} f_n = f$ a.e., as required.

(iii)⇒(iv) Assuming (iii), let μ and $\langle E_n \rangle_{n\in\mathbb{N}}$ be as in (iv). Set $f_n = \chi(\mathcal{P}\mathbb{N} \setminus E_n)$ for each n; then $\lim_{n\to\infty} \int f_n d\mu = 0$, so $\langle f_n \rangle_{n\in\mathbb{N}} \to 0$ in measure, and $H = \{a : \lim_{n\to\mathcal{F}} f_n(a) = 0\}$ is conegligible. But for any $a \in H$,

$$a = \{n : a \in E_n\} = \{n : f_n(x) \leq \tfrac{1}{2}\}$$

belongs to $\mathcal{F}$, so $H \subseteq \mathcal{F}$ and $\mu\mathcal{F} = 1$.

(iv)⇒(ii) Assume (iv), and let $\langle F_n \rangle_{n\in\mathbb{N}}$ be as in (ii). Define $\phi : \{0,1\}^{\mathbb{N}} \to \mathcal{P}\mathbb{N}$ by setting $\phi(x) = \{n : x \in F_n\}$ for $x \in \{0,1\}^{\mathbb{N}}$. Then ϕ is almost continuous (418J), so the image measure $\mu = \nu_\omega \phi^{-1}$ on $\mathcal{P}\mathbb{N}$ is a Radon measure (418I). Since $F_n = \phi^{-1}[E_n]$ for each n, $\lim_{n\to\infty} \mu E_n = 1$ and $1 = \mu\mathcal{F} = \nu_\omega \phi^{-1}[\mathcal{F}]$. But now

$$\bigcup_{a\in\mathcal{F}} \bigcap_{n\in a} F_n = \bigcup_{a\in\mathcal{F}} \{x : a \subseteq \phi(x)\} = \phi^{-1}[\mathcal{F}]$$

is ν_ω-conegligible, as required.

(ii)⇒(i) Assume (ii), and take a probability space (X, Σ, μ) and a sequence $\langle H_n \rangle_{n\in\mathbb{N}}$ in Σ such that $\lim_{n\to\infty} \mu H_n = 1$; set $G = \bigcup_{A\in\mathcal{F}} \bigcap_{n\in A} H_n$.

Let λ be the c.l.d. product measure on $X \times \{0,1\}^{\mathbb{N}}$, and set

$$W_n = H_n \times \{0,1\}^{\mathbb{N}}, \quad V_n = \{(x, y) : x \in X,\, y \in \{0,1\}^{\mathbb{N}},\, y(n) = 1\}$$

for $n \in \mathbb{N}$. Let Λ_1 be the σ-algebra of subsets of $X \times \{0,1\}^{\mathbb{N}}$ generated by $\{W_n : n \in \mathbb{N}\} \cup \{V_n : n \in \mathbb{N}\}$, and λ_1 the completion of the restriction $\lambda{\restriction}\Lambda_1$. Note that as the identity map from $X \times \{0,1\}^{\mathbb{N}}$ is inverse-measure-preserving for λ and $\lambda{\restriction}\Lambda_1$, it is inverse-measure-preserving for their completions (234Ba); but λ is complete, so this just means that λ extends λ_1. Then λ_1 is a complete atomless probability measure with countable Maharam type. Its measure algebra $\mathfrak{C}$ is therefore isomorphic, as measure algebra, to the measure algebra $\mathfrak{B}_\omega$ of ν_ω; let $\pi : \mathfrak{B}_\omega \to \mathfrak{C}$ be a measure-preserving isomorphism. By 343B, or otherwise, there is a realization $\phi : X \times \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$, inverse-measure-preserving for λ_1 and ν_ω, such that $\phi^{-1}[F]^\bullet = \pi F^\bullet$ in $\mathfrak{C}$ for every $F \in \mathrm{T}_\omega$. Because π is surjective, there is for each $n \in \mathbb{N}$ an $F_n \in \mathrm{T}_\omega$ such that $\phi^{-1}[F_n] \triangle W_n$ is λ_1-negligible.

Since

$$\lim_{n\to\infty} \nu_\omega F_n = \lim_{n\to\infty} \lambda_1 W_n = \lim_{n\to\infty} \lambda W_n = \lim_{n\to\infty} \mu H_n = 1,$$

$F = \bigcup_{A\in\mathcal{F}} \bigcap_{n\in A} F_n$ is ν_ω-conegligible, and $\phi^{-1}[F] = \bigcup_{A\in\mathcal{F}} \bigcap_{n\in A} \phi^{-1}[F_n]$ is λ_1-conegligible. We have $G \times \{0,1\}^{\mathbb{N}} = \bigcup_{A\in\mathcal{F}} \bigcap_{n\in A} W_n$, so

$$(G \times \{0,1\}^{\mathbb{N}}) \triangle \phi^{-1}[F] \subseteq \bigcup_{n\in\mathbb{N}} W_n \triangle \phi^{-1}[F_n]$$

is λ_1-negligible. Thus $G\times\{0,1\}^{\mathbb N}$ is λ_1-conegligible, therefore λ-conegligible. But this means that G is μ-conegligible, by 252D applied to $G\times\{0,1\}^{\mathbb N}$; and this is what we needed to know.

(b) Let $\mathcal F$ be a measure-converging filter and $m\in\mathbb N$. Take a singleton set $X=\{x\}$ and the probability measure μ on X; set $E_i=\emptyset$ for $i<n$, X for $i\ge n$. Then $\lim_{i\to\infty}\mu E_i=1$, so there is an $A\in\mathcal F$ such that $\bigcap_{i\in A}E_i$ is non-empty. Now $\mathbb N\setminus n\supseteq A$ belongs to $\mathcal F$; as n is arbitrary, $\mathcal F$ is free.

(c)(i) Immediate from the definition in 538Ag.

(ii) Let $f:\mathbb N\to\mathbb N$ be a finite-to-one function such that $\mathcal G=f[[\mathcal F]]$. Let (X,Σ,μ) be a probability space and $\langle E_n\rangle_{n\in\mathbb N}$ a sequence in Σ such that $\lim_{n\to\infty}\mu E_n=1$. Set $F_n=E_{f(n)}$ for $n\in\mathbb N$; because f is finite-to-one, $\lim_{n\to\infty}\mu F_n=1$. So $H=\bigcup_{A\in\mathcal F}\bigcap_{n\in A}F_n$ is conegligible. If $x\in H$, set $A_x=\{n:x\in E_n\}$; then

$$f^{-1}[A_x]=\{n:x\in E_{f(n)}\}=\{n:x\in F_n\}$$

belongs to $\mathcal F$ so $A_x\in f[[\mathcal F]]$ and $x\in\bigcup_{B\in f[[\mathcal F]]}\bigcap_{n\in B}E_n$. Thus $\bigcup_{B\in f[[\mathcal F]]}\bigcap_{n\in B}E_n\supseteq H$ is conegligible. As (X,Σ,μ) and $\langle E_n\rangle_{n\in\mathbb N}$ are arbitrary, $f[[\mathcal F]]$ is measure-converging.

(d) Let $\mathcal F$ be a rapid filter on $\mathbb N$, and $\langle H_n\rangle_{n\in\mathbb N}$ a sequence in $\mathrm T_\omega$ such that $\lim_{n\to\infty}\nu_\omega H_n=1$. Set $G=\bigcup_{A\in\mathcal F}\bigcap_{n\in A}H_n$. Since $\lim_{n\to\infty}(1-\nu_\omega H_n)=0$, there is an $A\in\mathcal F$ such that $\sum_{n\in A}1-\nu_\omega H_n<\infty$; set $H=\bigcup_{m\in\mathbb N}\bigcap_{n\in A\setminus m}H_n\subseteq G$. Then

$$\nu_\omega H\ge\sup_{m\in\mathbb N}1-\textstyle\sum_{n\in A\setminus m}(1-\nu_\omega H_n)=1,$$

so G is conegligible. Thus $\mathcal F$ satisfies (a-ii) and is measure-converging.

(e) Let $\mathcal F$ be a measure-converging filter. Let $\langle I_n\rangle_{n\in\mathbb N}$ be a sequence of non-empty finite subsets of $\mathbb N$ such that $\lim_{n\to\infty}\#(I_n)=\infty$. Let $\mathcal G$ be

$$\{A:A\subseteq\mathbb N,\ \lim_{n\to\mathcal F}\frac{1}{\#(I_n)}\#(A\cap I_n)=1\}.$$

Then $\mathcal G$ is a filter on $\mathbb N$.

(i) $\mathcal G$ is measure-converging. **P** Let $\langle H_i\rangle_{i\in\mathbb N}$ be a sequence in $\mathrm T_\omega$ such that $\lim_{i\to\infty}\nu_\omega H_i=1$, and set $G=\bigcup_{A\in\mathcal G}\bigcap_{i\in A}H_i$. Set $g_n=\frac{1}{\#(I_n)}\sum_{i\in I_n}\chi H_i$ for each n; then

$$\lim_{n\to\infty}\int g_n=\lim_{n\to\infty}\frac{1}{\#(I_n)}\textstyle\sum_{i\in I_n}\nu_\omega H_i=1$$

because $\lim_{n\to\infty}\#(I_n)=\infty$ and $\lim_{i\to\infty}\nu_\omega H_i=1$. Since $0\le g_n\le\chi\{0,1\}^{\mathbb N}$ for every n, $\langle g_n\rangle_{n\in\mathbb N}\to\chi\{0,1\}^{\mathbb N}$ in measure. By (a-iii) above, $H=\{x:\lim_{n\to\mathcal F}g_n(x)=1\}$ is conegligible.

For $x\in H$, set $A_x=\{i:x\in H_i\}$. Then

$$\frac{1}{\#(I_n)}\#(I_n\cap A_x)=g_n(x)\to1$$

as $n\to\mathcal F$, so $A_x\in\mathcal G$ and $x\in G$. Accordingly $G\supseteq H$ is conegligible. As $\langle H_i\rangle_{i\in\mathbb N}$ is arbitrary, $\mathcal G$ is measure-converging. **Q**

(ii) $\mathcal G$ is not rapid. **P** Define $\langle t_i\rangle_{i\in\mathbb N}$ by saying that

$$t_i=\sup\{\frac{1}{\#(I_n)}:n\in\mathbb N,\ i\in I_n\}$$

for $i\in\mathbb N$, counting $\sup\emptyset$ as 0. Then $\lim_{i\to\infty}t_i=0$. If $A\in\mathcal G$ and $m\in\mathbb N$, then $B=\{n:\#(A\cap I_n)\ge\frac23\#(I_n)\}$ belongs to $\mathcal F$, and must be infinite, by (b) above. So there is an $n\in B$ such that $\#(I_n)\ge3m$, and now

$$\textstyle\sum_{i\in A\setminus m}t_i\ge\#(A\cap I_n\setminus m)\cdot\frac{1}{\#(I_n)}\ge\frac13.$$

As m is arbitrary, $\sum_{i\in A}t_i=\infty$; as A is arbitrary, $\mathcal G$ is not rapid. **Q**

(f) Let $\langle E_{ij}\rangle_{i,j\in\mathbb N}$ be a family in $\mathrm T_\omega$ such that $\langle\nu_\omega E_{i_nj_n}\rangle_{n\in\mathbb N}\to1$ for some, or any, enumeration $\langle(i_n,j_n)\rangle_{n\in\mathbb N}$ of $\mathbb N\times\mathbb N$. Set $G=\bigcup_{C\in\mathcal G\ltimes\mathcal F}\bigcap_{(i,j)\in C}E_{ij}$. For each $i\in\mathbb N$, $\lim_{j\to\infty}\nu_\omega E_{ij}=1$, so $G_i=$

$\bigcup_{A\in\mathcal{F}}\bigcap_{j\in A}E_{ij}$ is conegligible; set $H=\bigcap_{i\in\mathbb{N}}G_i$. For $x\in H$, set $A_x=\{(i,j):x\in E_{ij}\}$. As $x\in G_i$, $A_x[\{i\}]\in\mathcal{F}$ for every $i\in\mathbb{N}$; thus $A_x\in\mathcal{G}\ltimes\mathcal{F}$ and $x\in G$. So G includes the conegligible set H, and is itself conegligible. As $\langle E_{ij}\rangle_{i,j\in\mathbb{N}}$ is arbitrary, $\mathcal{G}$ is measure-converging.

(g)(i) Suppose that $\mathcal{E}\subseteq[\mathbb{N}]^\omega$ is a family with $\#(\mathcal{E})<\mathfrak{m}_{\text{countable}}$, and that $f\in\mathbb{N}^\mathbb{N}$ is non-decreasing. Then there is an $A\subseteq\mathbb{N}$, meeting every member of $\mathcal{E}$, such that $\#(A\cap f(n))\le n$ for every $n\in\mathbb{N}$. **P** Consider $X=\prod_{n\in\mathbb{N}}\mathbb{N}\setminus f(n)$. Then X is a closed subset of $\mathbb{N}^\mathbb{N}$, homeomorphic to $\mathbb{N}^\mathbb{N}$. For $E\in\mathcal{E}$, set

$$G_E=\{x:x\in X,\,E\cap x[\mathbb{N}]\ne\emptyset\};$$

then G_E is a dense open subset of X. Writing $\mathcal{M}(X)$ for the ideal of meager subsets of X, $\#(\mathcal{E})<\mathfrak{m}_{\text{countable}}=\operatorname{cov}\mathcal{M}(X)$, so there is an $x\in X\cap\bigcap_{E\in\mathcal{E}}G_E$; set $A=x[\mathbb{N}]$. **Q**

(ii) Let $\langle f_\xi\rangle_{\xi<\mathfrak{d}}$ be a cofinal family in $\mathbb{N}^\mathbb{N}$; we may suppose that every f_ξ is strictly increasing. Choose a non-decreasing family $\langle\mathcal{E}_\xi\rangle_{\xi\le\mathfrak{d}}$ inductively, as follows. $\mathcal{E}_0=\{\mathbb{N}\setminus n:n\in\mathbb{N}\}$. Given that $\xi<\mathfrak{d}=\mathfrak{m}_{\text{countable}}$ and that $\mathcal{E}_\xi\subseteq[\mathbb{N}]^{<\omega}$ is a filter base with cardinal at most $\max(\omega,\#(\xi))$, use (i) to find a set $A_\xi\subseteq\mathbb{N}$, meeting every member of $\mathcal{E}_\xi$, such that $\#(A_\xi\cap f_\xi(n))\le n$ for every n; set

$$\mathcal{E}_{\xi+1}=\mathcal{E}_\xi\cup\{A_\xi\cap E:E\in\mathcal{E}_\xi\}.$$

For non-zero limit ordinals $\xi\le\mathfrak{d}$ set $\mathcal{E}_\xi=\bigcup_{\eta<\xi}\mathcal{E}_\eta$.

At the end of the induction, let $\mathcal{F}$ be the filter on $\mathbb{N}$ generated by $\mathcal{E}_\mathfrak{d}$. Then $\mathcal{F}$ is rapid. **P** It is free because $\mathcal{E}_0\subseteq\mathcal{F}$. If $\langle t_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}$ converging to 0, let $f\in\mathbb{N}^\mathbb{N}$ be such that $|t_i|\le 2^{-n}$ whenever $n\in\mathbb{N}$ and $i\ge f(n)$, and let $\xi<\mathfrak{d}$ be such that $f\le f_\xi$. Then $A_\xi\in\mathcal{F}$ and

$$\sum_{i\in A_\xi}|t_i|\le\sum_{n=0}^\infty 2^{-n}\#(A_\xi\cap f_\xi(n+1)\setminus f_\xi(n))\le\sum_{n=0}^\infty 2^{-n}(n+1)$$

is finite. **Q**

538O The Fatou property: Proposition (a) Let $\mathcal{F}$ be a filter on $\mathbb{N}$. Let ν_ω be the usual measure on $\{0,1\}^\mathbb{N}$, and T_ω its domain. Then the following are equiveridical:

(i) $\mathcal{F}$ has the Fatou property;

(ii) whenever $\langle F_n\rangle_{n\in\mathbb{N}}$ is a sequence in T_ω and $\nu_\omega^*(\bigcup_{A\in\mathcal{F}}\bigcap_{n\in A}F_n)=1$, then $\lim_{n\to\mathcal{F}}\nu_\omega F_n=1$;

(iii) whenever (X,Σ,μ) is a measure space and $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence of non-negative functions in $\mathcal{L}^0(\mu)$, then $\overline{\int}\liminf_{n\to\mathcal{F}}f_nd\mu\le\liminf_{n\to\mathcal{F}}\int f_nd\mu$;

(iv) whenever μ is a Radon probability measure on $\mathcal{P}\mathbb{N}$, then $\mu^*\mathcal{F}\le\liminf_{n\to\mathcal{F}}\mu E_n$, where $E_n=\{a:n\in a\subseteq\mathbb{N}\}$ for each $n\in\mathbb{N}$.

(b) If $\mathcal{F}$ and $\mathcal{G}$ are filters on $\mathbb{N}$, $\mathcal{G}\le_{\text{RK}}\mathcal{F}$ and $\mathcal{F}$ has the Fatou property, then $\mathcal{G}$ has the Fatou property.

(c) If $\mathcal{F}$ and $\mathcal{G}$ are filters on $\mathbb{N}$ with the Fatou property, then $\mathcal{F}\ltimes\mathcal{G}$ has the Fatou property.

proof (a) not-(iii)⇒not-(i) Suppose that (X,Σ,μ) is a measure space and $\langle f_n\rangle_{n\in\mathbb{N}}$ a sequence of non-negative functions in $\mathcal{L}^0$ such that $\overline{\int}\liminf_{n\to\mathcal{F}}f_nd\mu>\liminf_{n\to\mathcal{F}}\int f_nd\mu$. Changing the f_n on negligible sets does not change either $\overline{\int}\liminf_{n\to\mathcal{F}}f_nd\mu$ or $\overline{\int}\liminf_{n\to\mathcal{F}}f_nd\mu$, so we may assume that every f_n is defined everywhere in X and is Σ-measurable. Take α such that $\overline{\int}\liminf_{n\to\mathcal{F}}f_nd\mu>\alpha>\liminf_{n\to\mathcal{F}}\int f_nd\mu$; set $A=\{n:\int f_nd\mu\le\alpha\}$; then A meets every member of $\mathcal{F}$. Since f_n is integrable for every $n\in A$, the set $G=\{x:\sup_{n\in A}f_n(x)>0\}$ is a countable union of sets of finite measure.

Let λ be the c.l.d. product measure on $G\times\mathbb{R}$, and consider the ordinate sets $W_n=\{(x,\beta):x\in G,\,0\le\beta<f_n(x)\}$ for $n\in A$. Set $W=\bigcup_{C\in\mathcal{F}}\bigcap_{n\in C\cap A}W_n$; writing g for $\liminf_{n\to\mathcal{F}}f_n$,

$$\{(x,\beta):x\in G,\,0\le\beta<g(x)\}\subseteq W.$$

Since λ is a product of two σ-finite measures it is σ-finite, and W has a measurable envelope $\tilde W$ say. Now $\lambda^*W>\alpha$. **P?** Otherwise, $\lambda\tilde W\le\alpha$. Writing μ_L for Lebesgue measure on $\mathbb{R}$,

$$\alpha\ge\lambda\tilde W=\int_G\mu_L\tilde W[\{x\}]\mu(dx)$$

(252D)

$$\geq \overline{\int}_G g\,d\mu > \alpha. \text{ XQ}$$

There is therefore a set $V \subseteq \tilde{W}$ such that $\alpha < \lambda V < \infty$, and now $\lambda^*(V \cap W) > \alpha$. Let ν be the subspace measure on $V \cap W$. Set

$$V_n = V \cap W \cap W_n \text{ if } n \in A,$$
$$= V \cap W \text{ if } n \in \mathbb{N} \setminus A.$$

Then

$$\liminf_{n \to \mathcal{F}} \nu V_n = \sup_{C \in \mathcal{F}} \inf_{n \in C} \nu V_n \leq \sup_{n \in A} \nu V_n$$
$$\leq \sup_{n \in A} \lambda W_n = \sup_{n \in A} \int f_n d\mu \leq \alpha.$$

On the other hand,

$$\bigcup_{C \in \mathcal{F}} \bigcap_{n \in C} V_n = \bigcup_{C \in \mathcal{F}} \bigcap_{n \in C \cap A} V \cap W \cap W_n = V \cap W$$

and $\nu(V \cap W) = \lambda^*(V \cap W) > \alpha$. Moving to a normalization of ν, we see that (i) is false.

(iii)⇒(iv) If $\mathcal{F}$ satisfies (iii) and μ is a Radon probability measure on $\mathcal{P}\mathbb{N}$, set $g = \liminf_{n \to \mathcal{F}} \chi E_n$. If $a \in \mathcal{F}$, then $\{n : \chi E_n(a) = 1\} = a \in \mathcal{F}$, so $g(a) = 1$; thus

$$\mu^* \mathcal{F} = \overline{\int} \chi \mathcal{F} d\mu$$

(133Je)

$$\leq \overline{\int} g\,d\mu \leq \liminf_{n \to \mathcal{F}} \int \chi E_n = \liminf_{n \to \mathcal{F}} \mu E_n,$$

as required.

(iv)⇒(ii) Given (iv), suppose that $\langle F_n \rangle_{n \in \mathbb{N}}$ is a sequence in T_ω and $\nu_\omega^*(\bigcup_{A \in \mathcal{F}} \bigcap_{n \in A} F_n) = 1$. As in the corresponding part of the argument for 538Na, define $\phi : \{0,1\}^{\mathbb{N}} \to \mathcal{P}\mathbb{N}$ by setting $\phi(x) = \{n : x \in F_n\}$, and let μ be the Radon measure $\nu_\omega \phi^{-1}$. Then

$$\mu^* \mathcal{F} = \nu_\omega^* \phi^{-1}[\mathcal{F}] = \nu_\omega^*(\bigcup_{A \in \mathcal{F}} \bigcap_{n \in A} F_n) = 1$$

(using 451Pc again for the first equality), so $\lim_{n \to \mathcal{F}} \nu_\omega F_n = \lim_{n \to \mathcal{F}} \mu E_n = 1$.

(ii)⇒(i) Assume (ii), and take a probability space (X, Σ, μ) and a sequence $\langle H_n \rangle_{n \in \mathbb{N}}$ in Σ such that $X = \bigcup_{A \in \mathcal{F}} \bigcap_{n \in A} H_n$.

As in the corresponding part of the argument for 538Na, let λ be the c.l.d. product measure on $X \times \{0,1\}^{\mathbb{N}}$, and set

$$W_n = H_n \times \{0,1\}^{\mathbb{N}}, \quad V_n = \{(x,y) : x \in X,\ y \in \{0,1\}^{\mathbb{N}},\ y(n) = 1\}$$

for $n \in \mathbb{N}$. Let Λ_1 be the σ-algebra of subsets of $X \times \{0,1\}^{\mathbb{N}}$ generated by $\{W_n : n \in \mathbb{N}\} \cup \{V_n : n \in \mathbb{N}\}$, and λ_1 the completion of the restriction $\lambda \restriction \Lambda_1$. As before, there is a function $\phi : X \times \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$, inverse-measure-preserving for λ_1 and ν_ω, such that there is for each $n \in \mathbb{N}$ an $F_n \in \mathrm{T}_\omega$ such that $\phi^{-1}[F_n] \triangle W_n$ is λ_1-negligible. Set $G = \bigcup_{A \in \mathcal{F}} \bigcap_{n \in A} F_n$.

Since $X = \bigcup_{A \in \mathcal{F}} \bigcap_{n \in A} H_n$, $X \times \{0,1\}^{\mathbb{N}} = \bigcup_{A \in \mathcal{F}} \bigcap_{n \in A} W_n$ and

$$(X \times \{0,1\}^{\mathbb{N}}) \setminus \phi^{-1}[G] \subseteq \bigcup_{n \in \mathbb{N}} \phi^{-1}[F_n] \triangle W_n$$

is λ_1-negligible. By 413Eh,

$$\nu_\omega^* G \geq \lambda_1 \phi^{-1}[G] = 1.$$

By (ii), $\lim_{n \to \mathcal{F}} \nu_\omega F_n = 1$. But

$$\nu_\omega F_n = \lambda_1\phi^{-1}[F_n] = \lambda_1 W_n = \lambda W_n = \mu H_n$$

for each n, so $\lim_{n\to\mathcal{F}}\mu H_n = 1$. As (X,Σ,μ) and $\langle H_n\rangle_{n\in\mathbb{N}}$ are arbitrary, $\mathcal{F}$ has the Fatou property.

(b) Let $h:\mathbb{N}\to\mathbb{N}$ be such that $\mathcal{G} = h[[\mathcal{F}]]$. Let (X,Σ,μ) be a probability space and $\langle H_n\rangle_{n\in\mathbb{N}}$ a sequence in Σ such that

$$\begin{aligned} X &= \bigcup_{A\in\mathcal{G}}\bigcap_{n\in A} H_n \\ &= \bigcup_{A\subseteq\mathbb{N},h^{-1}[A]\in\mathcal{F}}\bigcap_{n\in A} H_n = \bigcup_{A\in\mathcal{F}}\bigcap_{n\in A} H_{h(n)}. \end{aligned}$$

Then

$$\begin{aligned} 1 &= \liminf_{n\to\mathcal{F}}\mu H_{h(n)} = \sup_{A\in\mathcal{F}}\inf_{n\in A}\mu H_{h(n)} \\ &\le \sup_{A\in\mathcal{G}}\inf_{n\in A}\mu H_n = \liminf_{n\to\mathcal{G}}\mu H_n. \end{aligned}$$

As (X,Σ,μ) and $\langle H_n\rangle_{n\in\mathbb{N}}$ are arbitrary, $\mathcal{G}$ has the Fatou property.

(c) Let (X,Σ,μ) be a probability space and $\langle E_{ij}\rangle_{i,j\in\mathbb{N}}$ a family in Σ such that $X = \bigcup_{C\in\mathcal{F}\ltimes\mathcal{G}}\bigcap_{(i,j)\in C} E_{ij}$. For each $i\in\mathbb{N}$, set $F_i = \bigcup_{B\in\mathcal{G}}\bigcap_{j\in B} E_{ij}$, and let $G_i\in\Sigma$ be a measurable envelope of F_i. Then $\bigcup_{A\in\mathcal{F}}\bigcap_{i\in A} G_i = X$. **P** If $x\in X$, there is a $C\in\mathcal{F}\ltimes\mathcal{G}$ such that $x\in E_{ij}$ whenever $(i,j)\in C$. Set $A = \{i : C[\{i\}]\in\mathcal{G}\}\in\mathcal{F}$. If $i\in A$, then

$$x\in\bigcap\nolimits_{j\in C[\{i\}]} E_{ij}\subseteq F_i\subseteq G_i,$$

so $x\in\bigcap_{i\in A} G_i$. **Q**

Accordingly $\lim_{i\to\mathcal{F}}\mu G_i = 1$. Take $\epsilon > 0$; then $A = \{i : \mu G_i\ge 1-\epsilon\}$ belongs to $\mathcal{F}$. For each $i\in A$,

$$1-\epsilon\le\mu G_i = \mu^* F_i = \overline{\int}\chi F_i = \overline{\int}\liminf_{j\to\mathcal{G}}\chi E_{ij}\le\liminf_{j\to\mathcal{G}}\int\chi E_{ij}$$

(by (a-iii) above)

$$= \liminf_{j\to\mathcal{G}}\mu E_{ij},$$

so $\{j : \mu E_{ij}\ge 1-2\epsilon\}\in\mathcal{G}$. But this means that $\{(i,j) : \mu E_{ij}\ge 1-2\epsilon\}\in\mathcal{F}\ltimes\mathcal{G}$. As ϵ is arbitrary, $\lim_{(i,j)\to\mathcal{F}\ltimes\mathcal{G}}\mu E_{ij} = 1$. As (X,Σ,μ) and $\langle E_{ij}\rangle_{i,j\in\mathbb{N}}$ are arbitrary, $\mathcal{F}\ltimes\mathcal{G}$ has the Fatou property.

538P Theorem Let $\nu:\mathcal{P}\mathbb{N}\to\mathbb{R}$ be a bounded finitely additive functional. Write $⨍\ldots d\nu$ for the associated linear functional on ℓ^∞ (see 363L), and set $E_n = \{a : n\in a\subseteq\mathbb{N}\}$ for each $n\in\mathbb{N}$. Then the following are equiveridical:

(i) whenever μ is a Radon probability measure on $\mathcal{P}\mathbb{N}$, $\int\nu(a)\mu(da)$ is defined and equal to $⨍\mu E_n\nu(dn)$;

(ii) whenever μ is a Radon probability measure on $[0,1]^{\mathbb{N}}$, $\int⨍x\,d\nu\,\mu(dx)$ is defined and equal to $⨍\int x(n)\mu(dx)\nu(dn)$;

(iii) whenever (X,Σ,μ) is a probability space and $\langle f_n\rangle_{n\in\mathbb{N}}$ is a uniformly bounded sequence of measurable real-valued functions on X, then $\int⨍f_n(x)\nu(dn)\mu(dx)$ is defined and equal to $⨍\int f_n d\mu\,\nu(dn)$;

(iv) whenever $\langle F_n\rangle_{n\in\mathbb{N}}$ is a sequence of Borel subsets of $\{0,1\}^{\mathbb{N}}$, $\int⨍\chi F_n(x)\nu(dn)\nu_\omega(dx)$ is defined and equal to $⨍\nu_\omega F_n\nu(dn)$, where ν_ω is the usual measure on $\{0,1\}^{\mathbb{N}}$.

proof (i)⇒(ii)(α) For $t\in[0,1]$ define $h_t:[0,1]^{\mathbb{N}}\to\mathcal{P}\mathbb{N}$ by setting $h_t(x) = \{n : x(n)\ge t\}$ for $x\in[0,1]^{\mathbb{N}}$, and let $\mu_t = \mu h_t^{-1}$ be the image measure on $\mathcal{P}\mathbb{N}$. Then μ_t is a Radon measure for each t. **P** Because h_t is Borel measurable and $\mathcal{P}\mathbb{N}$ is metrizable, h_t is almost continuous (418J), so μ_t is a Radon measure (418I). **Q**

(β) For $m\in\mathbb{N}$ define $v_m\in[0,1]^{\mathbb{N}}$ by setting

$$v_m(n) = 2^{-m}\sum\nolimits_{k=1}^{2^m}\mu\{x : x(n)\ge 2^{-m}k\}.$$

Then $\|v_{m+1}-v_m\|_\infty\le 2^{-m-1}$. **P** For any $n\in\mathbb{N}$,

$$\begin{aligned}v_m(n)-v_{m+1}(n)&=2^{-m}\sum_{k=1}^{2^m}\mu\{x:x(n)\ge 2^{-m}k\}-2^{-m-1}\sum_{k=1}^{2^{m+1}}\mu\{x:x(n)\ge 2^{-m-1}k\}\\&=2^{-m-1}\sum_{k=1}^{2^m}(2\mu\{x:x(n)\ge 2^{-m}k\}-\mu\{x:x(n)\ge 2^{-m}k\}\\&\qquad\qquad-\mu\{x:x(n)\ge 2^{-m-1}(2k+1)\})\\&=2^{-m-1}\sum_{k=1}^{2^m}\mu\{x:2^{-m}k\le x(n)<2^{-m-1}(2k+1)\}\le 2^{-m-1}.\ \mathbf{Q}\end{aligned}$$

So $v=\lim_{m\to\infty}v_m$ is defined in ℓ^∞ and $⨍ v\,d\nu=\lim_{m\to\infty}⨍ v_m d\nu$. Also $v(n)=\int x(n)\mu(dx)$ for every $n\in\mathbb{N}$, so $⨍\int x(n)\mu(dx)\nu(dn)=⨍ v\,d\nu$.

(γ) Set

$$f(t)=⨍\mu_t E_n\nu(dn)=\int\nu(a)\mu_t(da)$$

for each $t\in[0,1]$ (using (i)). Then, for any $m\in\mathbb{N}$,

$$\begin{aligned}⨍ v_m d\nu&=2^{-m}\sum_{k=1}^{2^m}⨍\mu\{x:x(n)\ge 2^{-m}k\}\nu(dn)\\&=2^{-m}\sum_{k=1}^{2^m}⨍\mu\{x:h_{2^{-m}k}(x)\in E_n\}\nu(dn)\\&=2^{-m}\sum_{k=1}^{2^m}⨍\mu_{2^{-m}k}E_n\nu(dn)=2^{-m}\sum_{k=1}^{2^m}f(2^{-m}k).\end{aligned}$$

(δ) Next, for $m\in\mathbb{N}$ and $x\in[0,1]^{\mathbb{N}}$, set $q_m(x)=2^{-m}\sum_{k=1}^{2^m}\chi h_{2^{-m}k}(x)$, so that $\langle q_m(x)\rangle_{m\in\mathbb{N}}$ is non-decreasing and $\|x-q_m(x)\|_\infty\le 2^{-m}$ for each m, while $q_m:[0,1]^{\mathbb{N}}\to[0,1]^{\mathbb{N}}$ is Borel measurable. Now

$$\begin{aligned}\int⨍ q_m(x)d\nu\,\mu(dx)&=2^{-m}\sum_{k=1}^{2^m}\int\nu(h_{2^{-m}k}(x))\mu(dx)\\&=2^{-m}\sum_{k=1}^{2^m}\int\nu(a)\mu_{2^{-m}k}(da)=2^{-m}\sum_{k=1}^{2^m}f(2^{-m}k).\end{aligned}$$

Also $\langle ⨍ q_m(x)d\nu\rangle_{m\in\mathbb{N}}\to⨍ x\,d\nu$ uniformly for $x\in[0,1]^{\mathbb{N}}$, so $\int⨍ x\,d\nu\,\mu(dx)$ is defined and equal to

$$\begin{aligned}\lim_{m\to\infty}\int⨍ q_m(x)d\nu\,\mu(dx)&=\lim_{m\to\infty}2^{-m}\sum_{k=1}^{2^m}f(2^{-m}k)=\lim_{m\to\infty}⨍ v_m d\nu\\&=⨍ v\,d\nu=⨍\int x(n)\mu(dx)\nu(dn).\end{aligned}$$

As μ is arbitrary, (ii) is true.

(ii)$\Rightarrow$(iii) Assume (ii), and let (X,Σ,μ) be a probability space and $\langle f_n\rangle_{n\in\mathbb{N}}$ a uniformly bounded sequence of measurable real-valued functions on X. As completing μ does not affect the integral $\int\ldots d\mu$ (212Fb), we may suppose that μ is complete. Let $\gamma>0$ be such that $|f_n(x)|\le\gamma$ for every $n\in\mathbb{N}$ and $x\in X$, and set $q(x)(n)=\frac{1}{2\gamma}(\gamma+f_n(x))$ for all n and x. Then $q:X\to[0,1]^{\mathbb{N}}$ is measurable, so there is a Radon probability measure λ on $[0,1]^{\mathbb{N}}$ such that q is inverse-measure-preserving for μ and λ. **P** Taking $\lambda_0E=\mu q^{-1}[E]$ for

Borel sets $E \subseteq [0,1]^{\mathbb{N}}$, q is inverse-measure-preserving for μ and λ_0; taking λ to be the completion of λ_0, q is inverse-measure-preserving for μ and λ, by 234Ba; and λ is a Radon measure by 433Cb. **Q** Now

$$\fint\int f_n d\mu\,\nu(dn) = 2\gamma\fint\int q(x)(n)\mu(dx)\nu(dn) - \gamma$$

$$= 2\gamma\fint\int z(n)\lambda(dz)\nu(dn) - \gamma$$

(235Gc)

$$= 2\gamma\int\fint z(n)\nu(dn)\lambda(dz) - \gamma$$

(by (ii))

$$= 2\gamma\int\fint q(x)(n)\nu(dn)\mu(dx) - \gamma = \int\fint f_n(x)\nu(dn)\mu(dx).$$

As μ and $\langle f_n\rangle_{n\in\mathbb{N}}$ are arbitrary, (iii) is true.

(iii)⇒(iv) is elementary, taking $f_n = \chi F_n$ and $\mu = \nu_\omega$.

(iv)⇒(i) If (iv) is true and μ is a Radon probability measure on $\mathcal{P}\mathbb{N}$, there is an inverse-measure-preserving function ϕ from $(\{0,1\}^{\mathbb{N}}, \nu_\omega)$ to $(\mathcal{P}\mathbb{N}, \mu)$ (343Cd). For each $n \in \mathbb{N}$, set $F_n = \phi^{-1}[E_n]$ for each n and choose a Borel set $F'_n \subseteq \{0,1\}^{\mathbb{N}}$ such that $\nu_\omega(F'_n \triangle F_n) = 0$. Then $\int\fint \chi F'_n(x)\nu(dn)\nu_\omega(dx)$ is defined and equal to

$$\fint \nu_\omega F'_n \nu(dn) = \fint \nu_\omega F_n \nu(dn) = \fint \mu E_n \nu(dn).$$

Now

$$\fint \mu E_n \nu(dn) = \int\fint \chi F'_n(x)\nu(dn)\nu_\omega(dx) = \int\fint \chi F_n(x)\nu(dn)\nu_\omega(dx)$$

(because for almost every x, $\chi F_n(x) = \chi F'_n(x)$ for every n)

$$= \int\fint \chi E_n(\phi(x))\nu(dn)\nu_\omega(dx) = \int\fint \chi E_n(a)\nu(dn)\mu(da)$$

(235Gc again)

$$= \int\fint \chi a(n)\nu(dn)\mu(da) = \int \nu(a)\,\mu(da).$$

As μ is arbitrary, (i) is true.

538Q Definition I will say that a bounded finitely additive functional ν satisfying (i)-(iv) of 538P is a **medial functional**; if, in addition, ν is non-negative, $\nu a = 0$ for every finite set $a \subseteq \mathbb{N}$ and $\nu\mathbb{N} = 1$, I will call ν a **medial limit**. I should remark that the term 'medial limit' is normally used for the associated linear functional $\fint \ldots d\nu$ on ℓ^∞, rather than the additive functional ν on $\mathcal{P}\mathbb{N}$; thus $h \in (\ell^\infty)^*$ is a medial limit if $h \ge 0$, $h(w) = \lim_{n\to\infty} w(n)$ for every convergent sequence $w \in \ell^\infty$ and $\int h(\langle f_n(x)\rangle_{n\in\mathbb{N}})\mu(dx)$ is defined and equal to $h(\langle\int f_n d\mu\rangle_{n\in\mathbb{N}})$ whenever (X, Σ, μ) is a probability space and $\langle f_n\rangle_{n\in\mathbb{N}}$ is a uniformly bounded sequence of measurable real-valued functions on X.

Note that 538P(i) tells us that a medial limit $\nu : \mathcal{P}\mathbb{N} \to \mathbb{R}$ is universally Radon-measurable (definition: 434Ec), therefore universally measurable (434Fc).

538R Proposition Let $M \cong (\ell^\infty)^*$ be the L-space of bounded finitely additive functionals on $\mathcal{P}\mathbb{N}$, and $M_{\mathrm{med}} \subseteq M$ the set of medial functionals.

(a) M_{med} is a band in M, and if $T \in L^\times(\ell^\infty;\ell^\infty)$ (definition: 355G) and $T' : M \to M$ is its adjoint, then $T'\nu \in M_{\mathrm{med}}$ for every $\nu \in M_{\mathrm{med}}$.

(b) Taking M_τ to be the band of completely additive functionals on $\mathcal{P}\mathbb{N}$ and M_{m} the band of measurable functionals, as described in §464, $M_\tau \subseteq M_{\mathrm{med}} \subseteq M_{\mathrm{m}}$.

(c) Suppose that $\langle\nu_k\rangle_{k\in\mathbb{N}}$ is a norm-bounded sequence in M_{med}, and that $\nu\in M_{\mathrm{med}}$. Set $\tilde\nu(a)=\fint\nu_k(a)\nu(dk)$ for $a\subseteq\mathbb{N}$. Then $\tilde\nu\in M_{\mathrm{med}}$.

(d) Suppose that $\nu\in M$ is a medial limit, and set $\mathcal{F}=\{a:a\subseteq\mathbb{N},\ \nu(a)=1\}$. Then $\mathcal{F}$ is a measure-converging filter with the Fatou property.

(e) Let (X,Σ,μ) and (Y,T,λ) be probability spaces, and $T\in L^\times(L^\infty(\mu);L^\infty(\lambda))$. Let $\langle f_n\rangle_{n\in\mathbb{N}}$, $\langle g_n\rangle_{n\in\mathbb{N}}$ be sequences in $\mathcal{L}^\infty(\mu)$, $\mathcal{L}^\infty(\nu)$ respectively such that $Tf_n^\bullet=g_n^\bullet$ for every n and $\langle f_n^\bullet\rangle_{n\in\mathbb{N}}$ is norm-bounded in $L^\infty(\mu)$. Let $\nu\in M$ be a medial functional. Then $f(x)=\fint f_n(x)\nu(dn)$ and $g(y)=\fint g_n(y)\nu(dn)$ are defined for almost every $x\in X$ and $y\in Y$; moreover, $f\in\mathcal{L}^\infty(\mu)$, $g\in\mathcal{L}^\infty(\lambda)$ and $Tf^\bullet=g^\bullet$.

proof (a)(i) Any of the four conditions of 538P makes it clear that M_{med} is a linear subspace of M.

We see also that M_{med} is norm-closed in M. **P** Let $\langle\nu_n\rangle_{n\in\mathbb{N}}$ be a sequence in M_{med} which is norm-convergent to $\nu\in M$. If μ is a Radon probability measure on $[0,1]^{\mathbb{N}}$, then $\langle\fint x\,d\nu_n\rangle_{n\in\mathbb{N}}\to\fint x\,d\nu$ uniformly for $x\in[0,1]^{\mathbb{N}}$, so

$$\int\!\!\fint x\,d\nu\,\mu(dx)=\lim_{n\to\infty}\int\!\!\fint x\,d\nu_n\mu(dx)$$
$$=\lim_{n\to\infty}\fint\!\!\int x(i)\mu(dx)\nu_n(di)=\fint\!\!\int x(i)\,\mu(dx)\nu(di).$$

As μ is arbitrary, $\nu\in M_{\mathrm{med}}$. **Q**

(ii) Before completing the proof that M_{med} is a band, I deal with the second clause of (a).

(α) Recall from §355 that $L^\times(\ell^\infty;\ell^\infty)$ is the set of differences of order-continuous positive linear operators from ℓ^∞ to itself. Since M can be identified with $(\ell^\infty)^*$, any $T\in L^\times(\ell^\infty;\ell^\infty)$ has an adjoint $T':M\to M$ defined by saying that $(T'\nu)(a)=\fint T(\chi a)d\nu$ for every $a\subseteq\mathbb{N}$. Since $x\mapsto\fint Tx\,d\nu$ and $x\mapsto\fint x\,d(T'\nu)$ both belong to $(\ell^\infty)^*$ and agree on $\{\chi a:a\subseteq\mathbb{N}\}$, they are equal, that is, $\fint Tx\,d\nu=\fint x\,d(T'\nu)$ for every $x\in\ell^\infty$.

(β) If $T:\ell^\infty\to\ell^\infty$ is an order-continuous positive linear operator, it is a norm-bounded linear operator (355C), and all the functionals $x\mapsto(Tx)(n)$ are order-continuous, therefore represented by members of ℓ^1; that is, we have a family $\langle\alpha_{ni}\rangle_{n,i\in\mathbb{N}}$ in $[0,\infty[$ such that

$$(Tx)(n)=\textstyle\sum_{i=0}^\infty\alpha_{ni}x(i)\text{ whenever }x\in\ell^\infty\text{ and }n\in\mathbb{N},$$

$$\textstyle\sup_{n\in\mathbb{N}}\sum_{i=0}^\infty\alpha_{ni}=\|T\|\text{ is finite.}$$

In this case, if $\nu\in M$ and $\nu'=T'\nu$ in M,

$$\fint x\,d\nu'=\fint(Tx)(n)\nu(dn)=\fint\textstyle\sum_{i=0}^\infty\alpha_{ni}x(i)\nu(dn)$$

for every $x\in\ell^\infty$.

Now suppose that that $\|T\|\le 1$, so that $\sum_{i=0}^\infty\alpha_{ni}\le 1$ for every n. Consider the function $\phi=T\restriction[0,1]^{\mathbb{N}}$. This is a function from $[0,1]^{\mathbb{N}}$ to itself, and it is continuous for the product topology on $\mathbb{N}$.

Take any $\nu\in M$ and Radon probability measure μ on $[0,1]^{\mathbb{N}}$; then the image measure $\mu_1=\mu\phi^{-1}$ on $[0,1]^{\mathbb{N}}$ is a Radon probability measure (418I), and $\int f(\phi(x))\mu(dx)=\int f(x)\mu_1(dx)$ for any μ_1-integrable function f. In particular, setting $f(x)=\fint x\,d\nu$,

$$\textstyle\int\!\fint\phi(x)\,d\nu\,\mu(dx)=\int\!\fint x\,d\nu\,\mu_1(dx)=\fint\!\int x(n)\mu_1(dx)\nu(dn)$$

because $\nu\in M_{\mathrm{med}}$.

Set $\nu'=T'\nu$. Then we can calculate

$$\fint\!\!\int x(n)\mu(dx)\nu'(dn)=\fint\sum_{i=0}^\infty\alpha_{ni}\int x(i)\mu(dx)\nu(dn)=\fint\!\!\int\sum_{i=0}^\infty\alpha_{ni}x(i)\mu(dx)\nu(dn)$$

(the inner integral is with respect to a genuine σ-additive measure, so we have B.Levi's theorem)

$$=\fint\!\!\int\phi(x)(n)\mu(dx)\nu(dn)=\fint\!\!\int x(n)\mu_1(dx)\nu(dn)$$
$$=\int\!\!\fint\phi(x)\,d\nu\,\mu(dx)=\int\!\!\fint Tx\,d\nu\,\mu(dx)=\int\!\!\fint x\,d\nu'\,\mu(dx).$$

As μ is arbitrary, ν' satisfies 538P(ii), and is a medial functional.

(γ) Thus $T'\nu \in M_{\mathrm{med}}$ whenever $\nu \in M_{\mathrm{med}}$ and $T : \ell^\infty \to \ell^\infty$ is positive, order-continuous and of norm at most 1. As M_{med} is a linear subspace of M, the same is true for every positive order-continuous T and for differences of such operators, that is, for every $T \in L^\times(\ell^\infty; \ell^\infty)$, as claimed.

(iii) I now return to the question of showing that M_{med} is a band. The point is that if ν is a medial functional and $b \subseteq \mathbb{N}$, then ν_b is a medial functional, where $\nu_b(a) = \nu(a \cap b)$ for every $a \subseteq \mathbb{N}$. **P** Define $T : \ell^\infty \to \ell^\infty$ by setting $Tx = x \times \chi b$ for $x \in \ell^\infty$. Then T is a positive order-continuous operator, and $T'\nu \in M_{\mathrm{med}}$, by (iii) above. But

$$(T'\nu)(a) = ⨍ T(\chi a) d\nu = ⨍ \chi(a \cap b) d\nu = \nu(a \cap b) = \nu_b(a)$$

for every $a \subseteq \mathbb{N}$, so $\nu_b = T'\nu$ is a medial functional. **Q**

By 436M, this is enough to ensure that M_{med} is a band in M.

(b)(i) Recall that an additive functional on $\mathcal{P}\mathbb{N}$ is completely additive iff it corresponds to an element of ℓ^1, that is, belongs to the band generated by the elementary functionals δ_k where $\delta_k(a) = \chi a(k)$ for $k \in \mathbb{N}$ and $a \subseteq \mathbb{N}$. To see that δ_k belongs to M_{med}, all we have to do is to note that $\delta_k = \chi E_k$ where E_k is defined as in 538P; so if μ is a Radon probability measure on $\mathcal{P}\mathbb{N}$, we shall have

$$\int \delta_k d\mu = \mu E_k = ⨍ \mu E_n \delta_k(dn).$$

Since M_{med} is a band, it must include M_τ.

(ii) On the other side, 538P(i) tells us that every member of M_{med} is universally measurable, and therefore belongs to M_{m}, which is just the set of bounded additive functionals which are Σ-measurable, where Σ is the domain of the usual measure on $\mathcal{P}\mathbb{N}$.

(c)(i) Because $\langle \nu_k \rangle_{k \in \mathbb{N}}$ is norm-bounded, $\tilde{\nu}$ is well-defined and additive; also it is bounded. **P** If γ is such that $\|\nu\| \le \gamma$ and $\|\nu_k\| \le \gamma$ for every k, then

$$|\tilde{\nu}(a)| \le \gamma \sup_{k \in \mathbb{N}} |\nu_k(a)| \le \gamma^2$$

for every $a \subseteq \mathbb{N}$. **Q**

Note that

$$⨍ \chi a \, d\tilde{\nu} = \tilde{\nu}(a) = ⨍ \nu_k(a) \nu(dk) = ⨍⨍ \chi a \, d\nu_k \nu(dk)$$

for every $a \subseteq \mathbb{N}$, so that

$$⨍ x \, d\tilde{\nu} = ⨍ x(n) \, \tilde{\nu}(dn) = ⨍⨍ x(n) \, \nu_k(dn) \nu(dk) = ⨍⨍ x \, d\nu_k \nu(dk)$$

whenever $x \in \ell^\infty$ is a linear combination of indicator functions, and therefore for every $x \in \ell^\infty$.

(ii) Now suppose that (X, Σ, μ) is a probability space and that $\langle f_n \rangle_{n \in \mathbb{N}}$ is a uniformly bounded sequence of measurable real-valued functions on X. Let $(X, \hat{\Sigma}, \hat{\mu})$ be the completion of (X, Σ, μ). For $k \in \mathbb{N}$ and $x \in X$ set $g_k(x) = ⨍ f_n(x) \nu_k(dn)$; because ν_k is a medial functional, we know that $\int g_k d\mu = ⨍\int f_n(x) \mu(dx) \nu_k(dn)$ is defined, so g_k is $\hat{\Sigma}$-measurable. Consequently $\int⨍ g_k(x) \nu(dk) \hat{\mu}(dx)$ is defined and equal to $⨍\int g_k(x) \hat{\mu}(dx) \nu(dk)$. It follows that

$$\begin{aligned} ⨍\int f_n(x)\mu(dx)\tilde{\nu}(dn) &= ⨍⨍\int f_n(x)\mu(dx)\nu_k(dn)\nu(dk) \\ &= ⨍\int⨍ f_n(x)\nu_k(dn)\mu(dx)\nu(dk) = ⨍\int g_k(x)\hat{\mu}(dx)\nu(dk) \\ &= \int⨍ g_k(x)\nu(dk)\hat{\mu}(dx) = \int⨍⨍ f_n(x)\nu_k(dn)\nu(dk)\hat{\mu}(dx) \\ &= \int⨍ f_n(x)\tilde{\nu}(dn)\hat{\mu}(dx) = \int⨍ f_n(x)\tilde{\nu}(dn)\mu(dx). \end{aligned}$$

(Recall that μ and $\hat{\mu}$ give rise to the same integrals, by 212Fb again.) As (X, Σ, μ) and $\langle f_n \rangle_{n \in \mathbb{N}}$ are arbitrary, $\tilde{\nu} \in M_{\mathrm{med}}$.

(d) Of course $\mathcal{F} = \{\mathbb{N} \setminus a : \nu(a) = 0\}$ is a filter.

(i) If (X, Σ, μ) is a probability space, $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in Σ, and $\lim_{n\to\infty} \mu E_n = 1$, then

$$\int\!\!⨍ \chi E_n(x)\nu(dn)\mu(dx) = ⨍\!\!\int \chi E_n d\mu\,\nu(dn) = ⨍ \mu E_n \nu(dn) = 1.$$

So $E = \{x : ⨍\chi E_n(x)\nu(dn) = 1\}$ is μ-conegligible. But if $x \in E$ and $a = \{n : x \in E_n\}$, then $\nu a = ⨍\chi E_n(x)\nu(dn) = 1$ and $a \in \mathcal{F}$ and $x \in \bigcap_{n\in a} E_n$. Thus $\bigcup_{a\in\mathcal{F}} \bigcap_{n\in a} E_n \supseteq E$ is conegligible. As (X, Σ, μ) and $\langle E_n\rangle_{n\in\mathbb{N}}$ are arbitrary, $\mathcal{F}$ is measure-converging.

(ii) If (X, Σ, μ) is a probability space, $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in Σ, and $X = \bigcup_{A\in\mathcal{F}} \bigcap_{n\in A} E_n$, then $\{n : x \in E_n\} \in \mathcal{F}$ for every $x \in X$, and

$$⨍ \mu E_n \nu(dn) = \int\!\!⨍ \chi E_n(x)\nu(dn)\mu(dx) = \int \nu\{n : x \in E_n\}\mu(dx) = 1.$$

So for any $\epsilon > 0$, $\nu\{n : \mu E_n \le 1 - \epsilon\} = 0$ and $\{n : \mu E_n \ge 1 - \epsilon\} \in \mathcal{F}$; accordingly $\lim_{n\to\mathcal{F}} \mu E_n = 1$. As (X, Σ, μ) and $\langle E_n\rangle_{n\in\mathbb{N}}$ are arbitrary, $\mathcal{F}$ has the Fatou property.

(e)(i) For each $n \in \mathbb{N}$, we can find a Σ-measurable function $f'_n : X \to \mathbb{R}$, equal almost everywhere to f_n, and such that $\sup_{x\in X} |f'_n(x)| = \operatorname{ess\,sup} |f_n|$. Now $\langle f'_n\rangle_{n\in\mathbb{N}}$ is uniformly bounded, so $f'(x) = ⨍ f'_n(x)\nu(dn)$ is defined for every $x \in X$; and $f(x)$ is defined and equal to $f'(x)$ for μ-almost every x. Since f' is integrable, f' and f are μ-virtually measurable and essentially bounded, and $f \in \mathcal{L}^\infty(\mu)$. Similarly, $g \in \mathcal{L}^\infty(\lambda)$.

(ii) If $h \in \mathcal{L}^1(\mu)$, then $\int f \times h\,d\mu = ⨍\!\int f_n \times h\,d\mu\,\nu(dn)$. **P** (α) If h is defined everywhere, measurable and bounded, then, taking f'_n and f' as in (i), $(f' \times h)(x) = ⨍ f'_n(x)h(x)\nu(dn)$ for every $x \in X$, so

$$\begin{aligned}\int f \times h\,d\mu = \int f' \times h\,d\mu &= \int\!\!⨍ (f'_n \times h)(x)\nu(dn)\mu(dx) \\ &= ⨍\!\!\int f'_n \times h\,d\mu\,\nu(dn) = ⨍\!\!\int f_n \times h\,d\mu\,\nu(dn).\end{aligned}$$

(β) In general, set $\gamma = \sup_{n\in\mathbb{N}} \operatorname{ess\,sup} f_n$. Given $\epsilon > 0$, there is a simple function h' such that $\|h - h'\|_1 \le \epsilon$, and now

$$\begin{aligned}&|\int f \times h\,d\mu - ⨍\!\!\int f_n \times h\,d\mu\,\nu(dn)| \\ &\quad\le |\int f \times h\,d\mu - \int f \times h'\,d\mu| + |\int f \times h'\,d\mu - ⨍\!\!\int f_n \times h'\,d\mu\,\nu(dn)| \\ &\qquad\qquad + |⨍\!\!\int f_n \times h'\,d\mu\,\nu(dn) - ⨍\!\!\int f_n \times h\,d\mu\,\nu(dn)| \\ &\quad\le \|f\|_\infty \|h - h'\|_1 + \sup_{n\in\mathbb{N}} |\int f_n \times h'\,d\mu - \int f_n \times h\,d\mu| \le 2\epsilon\gamma.\end{aligned}$$

As ϵ is arbitrary, $\int f \times h\,d\mu = ⨍\!\int f_n \times h\,d\mu\,\nu(dn)$. **Q**

Similarly, $\int g \times h\,d\lambda = ⨍\!\int g_n \times h\,d\lambda\,\nu(dn)$ for every λ-integrable h.

(iii) If $h \in \mathcal{L}^1(\lambda)$ there is an $\tilde h \in \mathcal{L}^1(\mu)$ such that $\int \tilde h^\bullet \times v = \int h^\bullet \times Tv$ for every $v \in L^\infty(\mu)$. **P** Recall that $L^1(\mu)$, $L^1(\lambda)$ can be identified with $L^\infty(\mu)^\times$ and $L^\infty(\nu)^\times$ (365Lb[10]); perhaps I should remark that the formulae $\int \tilde h^\bullet \times v$, $\int h^\bullet \times Tv$ represent abstract integrals taken in $L^1(\mu)$, $L^1(\lambda)$ respectively (242B). Setting $\phi(w) = \int h^\bullet \times w$ for $w \in L^\infty(\lambda)$, $\phi \in L^\infty(\lambda)^\times$, so $\phi T \in L^\infty(\mu)^\times$ (355G) and there is an $\tilde h \in \mathcal{L}^1(\mu)$ such that

$$\int \tilde h^\bullet \times v = \phi(Tv) = \int h^\bullet \times Tv$$

for every $v \in L^\infty(\mu)$. **Q**

(iv) Take h and $\tilde h$ as in (iii), and consider

$$\int h^\bullet \times g^\bullet = \int h \times g\,d\lambda = ⨍\!\!\int h \times g_n\,d\lambda\,\nu(dn)$$

(by (ii))

[10]Formerly 365Mb.

$$= ⨍(\int h^\bullet \times g_n^\bullet)\nu(dn) = ⨍(\int h^\bullet \times Tf_n^\bullet)\nu(dn)$$

$$= ⨍(\int \tilde{h}^\bullet \times f_n^\bullet)\nu(dn) = ⨍\int \tilde{h} \times f_n d\mu\,\nu(dn) = \int \tilde{h} \times f d\mu$$

(by (ii) again)

$$= \int \tilde{h}^\bullet \times f^\bullet = \int h^\bullet \times Tf^\bullet.$$

As h is arbitrary, and the duality between $L^\infty(\mu)$ and $L^1(\lambda)$ is separating, $Tf^\bullet = g^\bullet$, as required.

538S Theorem (a) If $\mathfrak{m}_{\text{countable}} = \mathfrak{c}$, there is a medial limit.

(b) (Larson 09) Suppose that the filter dichotomy (5A6Id) is true. If I is any set and ν is a finitely additive real-valued functional on $\mathcal{P}I$ which is universally measurable for the usual topology on $\mathcal{P}I$, then ν is completely additive.[11] Consequently there is no medial limit.

proof (a)(i) Let M be the L-space of bounded additive functionals on $\mathcal{P}\mathbb{N}$. Let us say that a subset C of M is **rationally convex** if $\alpha\nu + (1-\alpha)\nu' \in C$ whenever ν, $\nu' \in C$ and $\alpha \in [0,1] \cap \mathbb{Q}$; for $A \subseteq M$, write $\Gamma_{\mathbb{Q}}(A)$ for the smallest rationally convex set including A. Set $Q = \Gamma_{\mathbb{Q}}(\{\delta_n : n \in \mathbb{N}\})$ where $\delta_n(a) = \chi a(n)$ for $a \subseteq \mathbb{N}$ and $n \in \mathbb{N}$. In the language of 538Rb, $Q \subseteq M_\tau \subseteq M_{\text{med}}$, so 538P(i) tells us that $\int \nu\, d\mu = ⨍ \mu E_n \nu(dn)$ for every $\nu \in Q$, where $E_n = \{a : n \in a \subseteq \mathbb{N}\}$ as usual.

(ii) Suppose that $\mathcal{F}$ is a filter base on Q, consisting of rationally convex sets, with cardinal less than $\mathfrak{m}_{\text{countable}}$. Let μ be a Radon probability measure on $\mathcal{P}\mathbb{N}$. Then there is a sequence $\langle\nu_k\rangle_{k\in\mathbb{N}}$ in Q such that

$\sum_{k=0}^\infty \int |\nu_{k+1}(a) - \nu_k(a)|\mu(da) < \infty$,

$\{k : k \in \mathbb{N},\, \nu_k \in F\}$ is infinite for every $F \in \mathcal{F}$.

P Each $\nu \in Q$ is a bounded Borel measurable real-valued function on $\mathcal{P}\mathbb{N}$; let $u \in L^2 = L^2(\mu)$ be a $\mathfrak{T}_s(L^2, L^2)$-cluster point of $\langle\nu^\bullet\rangle_{\nu\in Q}$ along the filter generated by $\mathcal{F}$. For any $F \in \mathcal{F}$, the $\|\,\|_2$-closure of the rationally convex set $\{\nu^\bullet : \nu \in F\} \subseteq L^2$ is convex, so includes the weak closure of $\{\nu^\bullet : \nu \in F\}$ and therefore contains u. So $\{\nu^\bullet : \nu \in F\}$ meets $\{v : v \in L^2,\, \|v - u\|_2 \le \epsilon\}$ for every $\epsilon > 0$.

Set $H_k = \{\nu : \nu \in Q,\, \|\nu^\bullet - u\|_2 \le 2^{-k}\}$ for each $k \in \mathbb{N}$; then every H_k meets every member of $\mathcal{F}$. If we give each H_k its discrete topology, and take H to be the product $\prod_{k\in\mathbb{N}} H_k$, then H is homeomorphic to $\mathbb{N}^\mathbb{N}$. Writing $\mathcal{M}(H)$ for the ideal of meager subsets of H, $\operatorname{cov}\mathcal{M}(H) = \mathfrak{m}_{\text{countable}} > \#(\mathcal{F})$, while

$$\bigcup_{k\ge n}\{\alpha : \alpha \in H,\, \alpha(k) \in F\}$$

is a dense open subset of H for every $F \in \mathcal{F}$ and $n \in \mathbb{N}$. There is therefore an $\alpha \in H$ such that $\{k : \alpha(k) \in F\}$ is infinite for every $F \in \mathcal{F}$; take $\nu_k = \alpha(k)$ for each k. Since μ is a probability measure,

$$\int |\nu_{k+1} - \nu_k| d\mu \le \|\nu_{k+1}^\bullet - \nu_k^\bullet\|_2$$

(244E; see 244Xd)

$$\le 2^{-k-1} + 2^{-k}$$

for every k, and $\sum_{k=0}^\infty \int |\nu_{k+1} - \nu_k| d\mu$ is finite. **Q**

(iii) Because a Radon probability measure on $\mathcal{P}\mathbb{N}$ is defined by its values on the countable algebra $\mathfrak{B}$ of open-and-closed sets, the number of such measures is at most $\#(\mathbb{R}^\mathfrak{B}) = \mathfrak{c}$. Enumerate them as $\langle\mu_\xi\rangle_{\xi<\mathfrak{c}}$. Choose a non-decreasing family $\langle\mathcal{F}_\xi\rangle_{\xi\le\mathfrak{c}}$ of filter bases on Q, as follows. The inductive hypothesis will be that $\mathcal{F}_\xi$ has cardinal at most $\max(\omega, \#(\xi))$ and consists of rationally convex sets. Start with $\mathcal{F}_0 = \{F_n : n \in \mathbb{N}\}$ where $F_n = \Gamma_{\mathbb{Q}}(\{\delta_i : i \ge n\})$ for each n. Given $\mathcal{F}_\xi$ where $\xi < \mathfrak{c}$, apply (ii) with $\mu = \mu_\xi$ to see that there is a sequence $\langle\nu_{\xi k}\rangle_{k\in\mathbb{N}}$ in Q such that

$\sum_{k\in\mathbb{N}} \int |\nu_{\xi,k+1} - \nu_{\xi k}| d\mu_\xi < \infty$,

$\{k : \nu_{\xi k} \in F\}$ is infinite for every $F \in \mathcal{F}_\xi$.

[11] The result developed into this form in the course of correspondence with J.Pachl.

Let $\mathcal{F}_{\xi+1}$ be

$$\mathcal{F}_\xi \cup \{F \cap \Gamma_{\mathbb{Q}}(\{\nu_{\xi k} : k \geq l\}) : F \in \mathcal{F}_\xi,\ l \in \mathbb{N}\}.$$

For non-zero limit ordinals $\xi \leq \mathfrak{c}$, set $\mathcal{F}_\xi = \bigcup_{\eta<\xi} \mathcal{F}_\eta$.

(iv) At the end of the induction, let $\mathcal{F}$ be the filter on $M \cong (\ell^\infty)^*$ generated by $\mathcal{F}_\mathfrak{c}$, and let θ be a cluster point of $\mathcal{F}$ for the weak* topology of $(\ell^\infty)^*$. Then θ is a medial limit. **P** If μ is a Radon probability measure on $\mathcal{P}\mathbb{N}$, take $\xi < \mathfrak{c}$ such that $\mu = \mu_\xi$. Because $\Gamma_{\mathbb{Q}}(\{\nu_{\xi k} : k \geq l\})$ belongs to $\mathcal{F}$ for every $l \in \mathbb{N}$, $⨍ u(n)\theta(dn) = \lim_{k\to\infty} ⨍ u(n)\nu_{\xi k}(dn)$ for every $u \in \ell^\infty$ for which the limit is defined. In particular, $\theta(a) = \lim_{k\to\infty} \nu_{\xi k}(a)$ whenever $a \subseteq \mathbb{N}$ is such that the limit is defined. Because $\sum_{k\in\mathbb{N}} \int |\nu_{\xi,k+1} - \nu_{\xi k}| d\mu$ is finite, this is the case for μ-almost every a, so

$$\int \theta(a)\mu(da) = \lim_{k\to\infty} \int \nu_{\xi k}(a)\mu(da) = \lim_{k\to\infty} ⨍ \mu E_n \nu_{\xi k}(dn);$$

and because the latter limit is defined it is equal to $⨍ \mu E_n \theta(dn)$. As μ is arbitrary, θ satisfies condition (i) of 538P, and is a medial functional; because $Q \in \mathcal{F}$, $\theta\mathbb{N} = 1$; and because $\mathcal{F}_0 \subseteq \mathcal{F}$, $\theta(a) = 0$ for every finite $a \subseteq \mathbb{N}$, so θ is a medial limit. **Q**

(b)(i) The key is the following. Suppose that $\nu : \mathcal{P}I \to \mathbb{R}$ is a universally measurable additive functional.

(α) For every set J and function $\phi : I \to J$, $\nu\phi^{-1}$ is universally measurable, where $(\nu\phi^{-1})(b) = \nu(\phi^{-1}[b])$ for every $b \subseteq J$. **P** We have only to observe that $b \mapsto \phi^{-1}[b] : \mathcal{P}J \to \mathcal{P}I$ is continuous, and apply 434Df. **Q**

(β) ν is bounded. **P?** Otherwise, there is a disjoint sequence $\langle c_k \rangle_{k\in\mathbb{N}}$ of subsets of I such that $\lim_{k\to\infty} |\nu c_k| = \infty$ (326D(ii)). Enlarging c_0 if necessary, we can suppose that $\bigcup_{k\in\mathbb{N}} c_k = I$. Set $\phi(i) = k$ for $k \in \mathbb{N}$ and $i \in c_k$. Then $\nu\phi^{-1}[\{k\}] \to \infty$ as $k \to \infty$. But $\nu' = \nu\phi^{-1}$ is universally measurable, therefore $\mathrm{T}_{\mathbb{N}}$-measurable, where $\mathrm{T}_{\mathbb{N}}$ is the domain of the usual measure $\lambda_{\mathbb{N}}$ on $\mathcal{P}\mathbb{N}$. Let M be such that $\lambda_{\mathbb{N}} E > 0$ where $E = \{a : |\nu' a| \leq M\}$. Then there are an $n \in \mathbb{N}$ such that for every $k \geq n$ there are a, $b \in E$ such that $a \triangle b = \{k\}$ (345E; recall that the natural bijection $a \to \chi a : \mathcal{P}\mathbb{N} \to \{0,1\}^{\mathbb{N}}$ identifies $\lambda_{\mathbb{N}}$ with the usual measure on $\{0,1\}^{\mathbb{N}}$). In this case, k belongs to exactly one of a, b; suppose that $k \in a \setminus b$; then $|\nu'\{k\}| = |\nu a - \nu' b| \leq 2M$. This is supposed to be true for every $k \geq n$, so $\limsup_{k\to\infty} |\nu'\{k\}| \leq 2M$. **XQ**

(γ) $|\nu|$ is universally measurable. **P** As in part (b-i) of the proof of 464K, there is a sequence $\langle c_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{P}I$ such that $\nu^+ a = \lim_{n\to\infty} \nu(a \cap c_n)$ for every $a \subseteq I$. Since all the functions $a \mapsto a \cap c_n$ are continuous, $a \mapsto \nu(a \cap c_n)$ is universally measurable for every n, and ν^+ is universally measurable (use 418C). Consequently $|\nu| = 2\nu^+ - \nu$ is universally measurable. **Q**

(ii) If $\nu : \mathcal{P}\mathbb{N} \to [0,\infty[$ is a universally measurable additive functional and $\nu\{n\} = 0$ for every $n \in \mathbb{N}$, then $\nu = 0$. **P?** Otherwise, consider $\mathcal{F} = \{a : \nu a = \nu\mathbb{N}\}$. This is a filter on $\mathbb{N}$ containing every cofinite set. Let $\phi : \mathbb{N} \to \mathbb{N}$ be finite-to-one, and write ν' for $\nu\phi^{-1}$. Setting $\mathcal{I} = \{a : \nu' a = 0\}$, we have a strictly positive additive functional on the quotient algebra $\mathcal{P}\mathbb{N}/\mathcal{I}$, so $\mathcal{P}\mathbb{N}/\mathcal{I}$ is ccc and $\mathcal{I}$ cannot be $[\mathbb{N}]^{<\omega}$, that is, $\phi[[\mathcal{F}]]$ is not the Fréchet filter. On the other hand, ν' is universally measurable, by (i-α), so

$$\phi[[\mathcal{F}]] = \{a : \phi^{-1}[a] \in \mathcal{F}\} = \{a : \nu' a = \nu'\mathbb{N}\}$$

is a universally measurable subset of $\mathcal{P}\mathbb{N}$, and cannot be an ultrafilter (464Ca). Thus $\mathcal{F}$ witnesses that the filter dichotomy is false. **XQ**

(iii) Returning to the general case of a universally measurable additive functional $\nu : \mathcal{P}I \to \mathbb{R}$, set $\gamma_i = \nu\{i\}$ for $i \in I$. By (i-β), $\sup_{J\in[I]^{<\omega}} |\sum_{j\in J} \gamma_j| = \sup_{J\in[I]^{<\omega}} |\nu J|$ is finite, so $\sum_{i\in I} |\gamma_i| < \infty$, and we have a functional $\nu_1 : \mathcal{P}I \to \mathbb{R}$ defined by setting $\nu_1 a = \sum_{i\in a} \gamma_i$ for every $a \subseteq I$. ν_1 is continuous for the topology of $\mathcal{P}I$, so $\nu_2 = \nu - \nu_1$ is universally measurable, and $\nu' = |\nu_2|$ is universally measurable, by (i-γ).

$\nu' J = 0$ for every countable set $J \subseteq I$. **P** If J is finite, this is trivial, because

$$|\nu_2|\{i\} = |\nu_2\{i\}| = |\nu\{i\} - \nu_1\{i\}| = |\gamma_i - \gamma_i| = 0$$

for every $i \in I$. If J is countably infinite, then the embedding $\mathcal{P}J \subsetneq \mathcal{P}I$ is continuous, so $\nu' \restriction \mathcal{P}J$ is universally measurable for the usual topology on $\mathcal{P}J$; also it is still zero on singletons, so (ii) tells us that it is zero on the whole of $\mathcal{P}J$. **Q**

It follows that ν' is zero everywhere. **P** Take $c \subseteq I$ and $\epsilon > 0$. ν' must be T_I-measurable, where T_I is the domain of the usual measure λ_I on $\mathcal{P}I$. Since λ_I is a completion regular Radon measure (416U), there must be a non-negligible zero set $K \subseteq \mathcal{P}I$ such that $|\nu' a - \nu' b| \le \epsilon$ for all a, $b \in K$; and there is a countable set $J \subseteq I$ such that K is determined by coordinates in J (4A3Nc, applied to $\{0,1\}^I \cong \mathcal{P}I$). Take any $a \in K$. Then $c_1 = (c \setminus J) \cup (a \cap J)$ and $a \cap J$ both belong to K. But as $\nu'(c \cap J) = 0$,

$$|\nu' c| = |\nu' c_1 - \nu'(a \cap J)| \le \epsilon.$$

As c and ϵ are arbitrary, $\nu' = 0$. **Q**

Accordingly $\nu_2 = 0$ and $\nu = \nu_1$. But of course ν_1 is completely additive.

(iv) Finally, a medial limit would be a non-zero additive functional from $\mathcal{P}\mathbb{N}$ to $[0,1]$ which was universally measurable, as noted in 538Q, and zero on singletons; and this has already been ruled out by (ii).

Remark It is possible to have medial limits when $\mathfrak{m}_{\text{countable}} \ll \mathfrak{c}$; see 553N.

538X Basic exercises (a) Let $\mathcal{F}$ be a filter on $\mathbb{N}$, and I an infinite subset of $\mathbb{N}$ such that $\mathbb{N} \setminus I \notin \mathcal{F}$; write $\mathcal{F}\lceil I$ for the filter $\{A \cap I : A \in \mathcal{F}\}$. Show that if $\mathcal{F}$ is free, or a p-point filter, or Ramsey, or rapid, or nowhere dense, or measure-centering, or measure-converging, or with the Fatou property, then so is $\mathcal{F}\lceil I$.

(b) For $A \in [\mathbb{N}]^\omega$ let $f_A : \mathbb{N} \to A$ be the increasing enumeration of A. Let $\mathcal{F}$ be a free filter on $\mathbb{N}$. Show that $\mathcal{F}$ is rapid iff $\{f_A : A \in \mathcal{F}\}$ is cofinal with $\mathbb{N}^\mathbb{N}$.

(c) Let $\mathcal{F}$ be a filter which is universally measurable (regarded as a subset of $\mathcal{P}(\bigcup \mathcal{F})$ with its usual topology), and $\mathcal{G}$ another filter such that $\mathcal{G} \le_{\text{RK}} \mathcal{F}$. Show that $\mathcal{G}$ is universally measurable.

(d) Let $\mathcal{F}_{\text{Fr}}$ be the Fréchet filter and $\mathcal{F}_d$ the asymptotic density filter, the filter of subsets of $\mathbb{N}$ with asymptotic density 1. (i) Show that $\mathcal{F}_{\text{Fr}}$ and $\mathcal{F}_d$ are p-point filters. (ii) Show that $\mathcal{F}_{\text{Fr}} \le_{\text{RB}} \mathcal{F}_d$ but that $\mathcal{F}_{\text{Fr}} \ltimes \mathcal{F}_{\text{Fr}} \not\le_{\text{RK}} \mathcal{F}_d$.

(e)(i) Let $\langle \mathcal{F}_n \rangle_{n \in \mathbb{N}}$ be a sequence of filters on $\mathbb{N}$, and $\mathcal{F}$ a filter on $\mathbb{N}$. Write $\lim_{n \to \mathcal{F}} \mathcal{F}_n$ for the filter $\{A : A \subseteq \mathbb{N},\ \{n : n \in \mathbb{N},\ A \in \mathcal{F}_n\} \in \mathcal{F}\}$. Show that if every $\mathcal{F}_n$ is rapid, then $\lim_{n \to \mathcal{F}} \mathcal{F}_n$ is rapid. (ii) Let $\mathcal{F}$ be a rapid filter, and $\mathcal{G}$ any filter on $\mathbb{N}$. Show that $\mathcal{G} \ltimes \mathcal{F}$ is rapid. (iii) In 538E, suppose that $\mathcal{F}_1$ is rapid. Show that $\mathcal{G}_\xi$ is rapid for every $\xi \ge 1$.

(f)(i) Let $\mathcal{F}$ be a nowhere dense filter, and $\mathcal{G}$ a filter on $\mathbb{N}$ such that $\mathcal{G} \le_{\text{RK}} \mathcal{F}$. Show that $\mathcal{G}$ is nowhere dense. (ii) Show that a p-point ultrafilter is nowhere dense. (iii) In 538E, show that if every $\mathcal{F}_\xi$ is a nowhere dense ultrafilter, then $\mathcal{G}_\zeta$ is a nowhere dense ultrafilter.

>(g) Let $\mathcal{F}$ be a free filter on $\mathbb{N}$. Show that the following are equiveridical: (i) $\mathcal{F}$ is a Ramsey filter; (ii) whenever K is finite, $k \in \mathbb{N}$ and $f : [\mathbb{N}]^k \to K$ is a function, there is an $F \in \mathcal{F}$ such that f is constant on $[F]^k$; (iii) $\mathcal{F}$ is a p-point filter and whenever $\langle E_n \rangle_{n \in \mathbb{N}}$ is a disjoint sequence in $[\mathbb{N}]^{<\omega}$, there is an $F \in \mathcal{F}$ such that $\#(F \cap E_n) \le 1$ for every n; (iv) whenever $\langle E_n \rangle_{n \in \mathbb{N}}$ is a disjoint sequence in $\mathcal{P}\mathbb{N} \setminus \mathcal{F}$, there is an $F \in \mathcal{F}$ such that $\#(F \cap E_n) \le 1$ for every n.

(h) Let $\mathfrak{F}$ be a countable family of distinct p-point ultrafilters on $\mathbb{N}$. Show that there is a disjoint family $\langle A_\mathcal{F} \rangle_{\mathcal{F} \in \mathfrak{F}}$ of subsets of $\mathbb{N}$ such that $A_\mathcal{F} \in \mathcal{F}$ for every $\mathcal{F} \in \mathfrak{F}$.

(i) Let (X, Σ, μ) be a complete perfect probability space, $(Y, \mathfrak{S})$ a perfectly normal compact Hausdorff space, $\langle f_n \rangle_{n \in \mathbb{N}}$ a sequence of measurable functions from X to Y, $\mathcal{F}$ a measure-centering ultrafilter on $\mathbb{N}$ and λ the $\mathcal{F}$-extension of μ. (i) Setting $f(x) = \lim_{n \to \mathcal{F}} f_n(x)$ for $x \in X$, show that f is $\operatorname{dom} \lambda$-measurable. (ii) For each $n \in \mathbb{N}$, show that there is a unique Radon measure ν_n on Y such that f_n is inverse-measure-preserving for μ and ν_n. (iii) Let ν be the limit $\lim_{n \to \mathcal{F}} \nu_n$ for the narrow topology on the space of Radon probability measures on Y (437Jd). Show that f is inverse-measure-preserving for λ and ν. (*Hint*: look at the Radon measure associated with the image measure λf^{-1}. You may prefer to begin with metrizable Y.)

(j) Let $\langle(\mathfrak{A}_i,\bar{\mu}_i)\rangle_{i\in I}$ be a family of probability algebras, $\mathcal{F}$ an ultrafilter on I, and $(\mathfrak{A},\bar{\mu})$ the probability algebra reduced product of $\prod_{i\in I}(\mathfrak{A}_i,\bar{\mu}_i)|\mathcal{F}$. For each $i\in I$, let $\subseteq_i$ be the order relation on $\mathfrak{A}_i$; set $P=\prod_{i\in I}\mathfrak{A}_i$ and let $P|\mathcal{F}$ be the partial order reduced product of $\langle(\mathfrak{A}_i,\subseteq_i)\rangle_{i\in I}$ modulo $\mathcal{F}$ as defined in 5A2A. Describe a canonical order-preserving map from $P|\mathcal{F}$ to $\mathfrak{A}$.

(k)(i) Let $(\mathfrak{A},\bar{\mu})$ be a homogeneous probability algebra with Maharam type κ, I a non-empty set, $\mathcal{F}$ an ultrafilter on I and $(\mathfrak{C},\bar{\nu})$ the probability algebra reduced power $(\mathfrak{A},\bar{\mu})^I|\mathcal{F}$. Show that $\mathfrak{C}$ is homogeneous, with Maharam type the transversal number $\operatorname{Tr}_{\mathcal{I}}(I;\kappa)$ (definition: 5A1M), where $\mathcal{I}=\{I\setminus A:A\in\mathcal{F}\}$. (*Hint*: 5A1Nd, 521Eb.) (ii) Show that if $(\mathfrak{A},\bar{\mu})$ is any probability algebra and $\mathcal{F}$ and $\mathcal{G}$ are non-principal ultrafilters on $\mathbb{N}$, then the probability algebra reduced powers $(\mathfrak{A},\bar{\mu})^{\mathbb{N}}|\mathcal{F}$ and $(\mathfrak{A},\bar{\mu})^{\mathbb{N}}|\mathcal{G}$ are isomorphic.

(l) Let (X,Σ,μ) be a perfect probability space and μ' an indefinite-integral measure over μ which is also a probability measure. Let $\mathcal{F}$ be a measure-centering ultrafilter on $\mathbb{N}$ and λ, λ' the $\mathcal{F}$-extensions of μ and μ'. Show that λ' is an indefinite-integral measure over λ.

>(m) (Benedikt 98) (i) Let $\mathcal{F}$ be any free filter on $\mathbb{N}$. Show that $\mathcal{F}\ltimes\mathcal{F}$ is not measure-centering. (*Hint*: let $\langle e_n\rangle_{n\in\mathbb{N}}$ be the standard generating family in $\mathfrak{B}_\omega$, and consider $a_{mn}=e_m\setminus e_n$ if $m<n$, 1 otherwise.) (ii) Let $\mathcal{F}$ be a measure-centering ultrafilter on $\mathbb{N}$. Show that if f, $g\in\mathbb{N}^{\mathbb{N}}$ and $\{n:f(n)\ne g(n)\}\in\mathcal{F}$, then $f[[\mathcal{F}]]\ne g[[\mathcal{F}]]$. (*Hint*: consider $a_n=e_{f(n)}\setminus e_{g(n)}$ if $f(n)\ne g(n)$.)

(n) Let X be a locally compact Hausdorff topological group, and μ a left Haar measure on X. Show that there is a complete locally determined left-translation-invariant measure λ on X such that $\lambda(\lim_{n\to\mathcal{F}}E_n)$ is defined and equal to $\sup_{K\subseteq X\text{ is compact}}\lim_{n\to\mathcal{F}}\mu(E_n\cap K)$ whenever $\mathcal{F}$ is a Ramsey ultrafilter on $\mathbb{N}$ and $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence of Haar measurable subsets of X.

(o)(i) Let $\langle\mathcal{F}_n\rangle_{n\in\mathbb{N}}$ be a sequence of measure-converging filters on $\mathbb{N}$. Show that $\bigcap_{n\in\mathbb{N}}\mathcal{F}_n$ is measure-converging, so that $\lim_{n\to\mathcal{F}}\mathcal{F}_n$ (538Xe) is measure-converging for any filter $\mathcal{F}$ on $\mathbb{N}$. (ii) In 538E, suppose that $\mathcal{F}_1$ is measure-converging. Show that $\mathcal{G}_\xi$ is measure-converging for every $\xi\in[1,\zeta]$.

(p) Suppose that $\langle\mathcal{F}_\xi\rangle_{\xi<\kappa}$ is a family of measure-converging filters, where κ is non-zero and less than the additivity $\operatorname{add}\mathcal{N}$ of Lebesgue measure. Show that $\bigcap_{\xi<\kappa}\mathcal{F}_\xi$ is measure-converging.

(q)(i) Let $\mathcal{F}$ be a filter on $\mathbb{N}$. Show that $\mathcal{F}$ has the Fatou property iff $\int f d\mu$ and $\lim_{n\to\mathcal{F}}\int f_n d\mu$ are defined and equal whenever (X,Σ,μ) is a measure space, $g:X\to[0,\infty[$ is an integrable function and $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence of measurable functions on X such that $|f_n|\le_{\text{a.e.}}g$ for every n and $\lim_{n\to\mathcal{F}}f_n=_{\text{a.e.}}f$. (ii) Show that a non-principal ultrafilter on $\mathbb{N}$ cannot have the Fatou property. (*Hint*: 464Ca.)

(r) Show that the asymptotic density filter (538Xd) has the Fatou property.

(s)(i) Let $\langle\mathcal{F}_n\rangle_{n\in\mathbb{N}}$ be a sequence of filters with the Fatou property, and $\mathcal{F}$ a filter with the Fatou property. Show that $\lim_{n\to\mathcal{F}}\mathcal{F}_n$ (538Xe) has the Fatou property. (ii) In 538E, suppose that $\mathcal{F}_\xi$ has the Fatou property for every $\xi\in[1,\zeta]$. Show that $\mathcal{G}_\xi$ has the Fatou property for every $\xi\le\zeta$.

(t) Let $\nu:\mathcal{P}\mathbb{N}\to\mathbb{R}$ be a bounded additive functional. (i) Show that ν is a medial functional iff $\int\nu\{n:x\in E_n\}\mu(dx)$ is defined and equal to $⨍\mu E_n\nu(dn)$ whenever (X,Σ,μ) is a probability space and $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in Σ. (ii) Show that in this case $a\mapsto\nu\phi^{-1}[a]$ is a medial functional for any $\phi:\mathbb{N}\to\mathbb{N}$.

>(u) Let (X,Σ,μ) be a probability space, and T a σ-subalgebra of Σ. Let $\langle f_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{L}^\infty(\mu)$ such that $\sup_{n\in\mathbb{N}}\operatorname{ess\,sup}|f_n|$ is finite, and for each $n\in\mathbb{N}$ let g_n be a conditional expectation of f_n on T. Suppose that ν is a medial functional. Show that $f(x)=⨍f_n(x)\nu(dn)$ and $g(x)=⨍g_n(x)\nu(dn)$ are defined for almost every x, that $f\in\mathcal{L}^\infty(\mu)$, and that g is a conditional expectation of f on T.

(v) (V.Bergelson) Show that there are a probability algebra $(\mathfrak{A},\bar{\mu})$ and a sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $\inf_{n\in\mathbb{N}}\bar{\mu}a_n>0$ but $a_m\cap a_n\cap a_{m+n}=0$ whenever m, $n>0$. (*Hint*: for $n\ge1$, set $E_n=\{x:x\in[0,1],\ \lfloor 3nx\rfloor\equiv1\mod 3\}$.)

538Y Further exercises (a) Show that if $\mathcal{F}$ and $\mathcal{G}$ are filters and $\mathcal{F} \leq_{\mathrm{RK}} \mathcal{G}$, then, in the language of 512A, $(\mathcal{F}, \supseteq, \mathcal{F}) \preccurlyeq_{\mathrm{GT}} (\mathcal{G}, \supseteq, \mathcal{G})$, so that $\mathrm{ci}\,\mathcal{F} \leq \mathrm{ci}\,\mathcal{G}$ and $\mathcal{F}$ is κ-complete whenever κ is a cardinal and $\mathcal{G}$ is κ-complete.

(b) Let $\mathcal{F}$ be a free ultrafilter on $\mathbb{N}$, and suppose that whenever $\mathcal{G}$ is a free filter on $\mathbb{N}$ and $\mathcal{G} \leq_{\mathrm{RK}} \mathcal{F}$, then $\mathcal{F} \leq_{\mathrm{RK}} \mathcal{G}$. Show that $\mathcal{F}$ is a Ramsey ultrafilter. (*Hint*: Comfort & Negrepontis 74.)

(c) Show that if $\mathfrak{p} = \mathfrak{c}$ then there are $2^{\mathfrak{c}}$ Ramsey ultrafilters on $\mathbb{N}$, and therefore $2^{\mathfrak{c}}$ isomorphism classes of Ramsey ultrafilters.

(d) Let $\mathcal{F}$ be an ultrafilter on $\mathbb{N}$. Show that $\mathcal{F}$ is measure-centering iff whenever $\mathfrak{A}$ is a Boolean algebra, $D \subseteq \mathfrak{A} \setminus \{0\}$ has intersection number greater than 0 (definition: 391H) and $\langle a_n \rangle_{n\in\mathbb{N}}$ is a sequence in D, then there is an $A \in \mathcal{F}$ such that $\{a_n : n \in A\}$ is centered.

(e)(i) Show that if $\mathrm{cov}\,\mathcal{N} = \mathfrak{c}$, there is a measure-centering ultrafilter on $\mathbb{N}$ including the asymptotic density filter (538Xd). (ii) Show that an ultrafilter on $\mathbb{N}$ including the asymptotic density filter cannot be a p-point filter. (iii) Show that a filter on $\mathbb{N}$ including the asymptotic density filter cannot be a rapid filter.

(f)(i) Let $\mathcal{F}$, $\mathcal{G}$ be free filters on $\mathbb{N}$ such that $\mathcal{F} \ltimes \mathcal{G}$ is measure-centering. Show that there is no free filter $\mathcal{H}$ such that $\mathcal{H} \leq_{\mathrm{RK}} \mathcal{F}$ and $\mathcal{H} \leq_{\mathrm{RK}} \mathcal{G}$. (ii) Show that if there are two non-isomorphic Ramsey ultrafilters on $\mathbb{N}$, then there are two non-isomorphic measure-centering ultrafilters $\mathcal{F}$, $\mathcal{G}$ on $\mathbb{N}$ such that $\mathcal{F} \ltimes \mathcal{G}$ is not measure-centering.

(g) For an uncountable set I, let us say that a filter $\mathcal{F}$ on I is **uniform and measure-centering** if $\#(A) = \#(I)$ for every $A \in \mathcal{F}$ and whenever $\mathfrak{A}$ is a Boolean algebra, $\nu : \mathfrak{A} \to [0, \infty[$ is an additive functional, and $\langle a_i \rangle_{i\in I}$ is a family in $\mathfrak{A}$ with $\inf_{i\in I} \nu a_i > 0$, there is an $A \in \mathcal{F}$ such that $\{a_i : i \in A\}$ is centered. (i) State and prove a result corresponding to 538G for such filters. (*Hint*: in the part corresponding to 538G(iv), use 'compact' measures rather than 'perfect' measures.) (ii) State and prove a result corresponding to 538H. (*Hint*: set $\kappa = \#(I)$. In the part corresponding to 538Hc, suppose that you have a κ-complete ultrafilter on I, rather than a Ramsey ultrafilter; see 4A1L. In the part corresponding to 538He, suppose that κ is regular and that $\mathrm{cov}\,\mathcal{N}_\kappa = 2^\kappa$, where $\mathcal{N}_\kappa$ is the null ideal of the usual measure on $\{0,1\}^\kappa$.) (iii) State and prove results corresponding to 538I-538K. (iv) State and prove results corresponding to 538L-538M, but with 'normal ultrafilters' in place of 'Ramsey ultrafilters'.

(h) Show that if $\mathcal{F}$ and $\mathcal{G}$ are filters on $\mathbb{N}$, $\mathcal{F}$ is rapid and $\mathcal{G} \leq_{\mathrm{RB}} \mathcal{F}$, then $\mathcal{G}$ is rapid.

(i) Give an example of filters $\mathcal{F}$, $\mathcal{G}$ on $\mathbb{N}$ such that $\mathcal{F}$ has the Fatou property, $\mathcal{G} \subseteq \mathcal{F}$ and $\mathcal{G}$ does not have the Fatou property.

(j)(i) Let $\mathcal{F}$ be a nowhere dense filter on $\mathbb{N}$, and $\mathcal{I}$ the ideal $\{\mathbb{N} \setminus A : A \in \mathcal{F}\}$. Show that $\mathcal{P}\mathbb{N}/\mathcal{I}$ is finite. (ii) Show that a free filter with the Fatou property cannot be nowhere dense.

(k) Let (X, Σ, μ) be a probability space and $\langle f_m \rangle_{m\in\mathbb{N}}$, $\langle g_n \rangle_{n\in\mathbb{N}}$ two uniformly bounded sequences of real-valued measurable functions defined on X. Let ν, $\nu' : \mathcal{P}\mathbb{N} \to \mathbb{R}$ be bounded additive functionals. Show that $⨍⨍\int f_m \times g_n d\mu\, \nu(dm)\nu'(dn) = ⨍⨍\int f_m \times g_n d\mu\, \nu'(dn)\nu(dm)$.

(l) (Meyer 73) Let ν be a medial limit. Write U for the set of sequences $u \in \mathbb{R}^{\mathbb{N}}$ such that $\sup\{⨍ v\, d\nu : v \in \ell^\infty, v \leq |u|\}$ is finite; for $u \in U$, write $⨍ u\, d\nu$ for $\lim_{m\to\infty} ⨍ \mathrm{med}(-m, u(n), m)\nu(dn)$ (see 364Xj). Suppose that (X, Σ, μ) is a probability space and $\langle f_n \rangle_{n\in\mathbb{N}}$ a sequence of μ-integrable real-valued functions on X such that $\langle \int |f_n| d\mu \rangle_{n\in\mathbb{N}} \in U$. (i) Show that $\langle f_n(x) \rangle_{n\in\mathbb{N}} \in U$ for μ-almost every $x \in X$. Set $f(x) = ⨍ f_n(x)\nu(dn)$ whenever $\langle f_n(x) \rangle_{n\in\mathbb{N}} \in U$. (ii) Show that if every f_n is non-negative then $\int f d\mu \leq ⨍\int f_n d\mu\, \nu(dn)$. (iii) Show that if $\{f_n : n \in \mathbb{N}\}$ is uniformly integrable then $\int f d\mu = ⨍\int f_n d\mu\, \nu(dn)$. (iv) Show that if $\langle f_n^\bullet \rangle_{n\in\mathbb{N}}$ is weakly convergent to 0 in $L^1(\mu)$, then $f =_{\mathrm{a.e.}} 0$. (v) Suppose that $\langle f_n \rangle_{n\in\mathbb{N}}$ is uniformly integrable. Let T be a σ-subalgebra of Σ, and for each $n \in \mathbb{N}$ let g_n be a conditional expectation of f_n on T; set $g(x) = ⨍ g_n(x)\nu(dn)$ whenever $\langle g_n(x) \rangle_{n\in\mathbb{N}} \in U$. Show that g is a conditional expectation of f on T.

(m) Suppose that $\mathcal{F}$ is a filter on $\mathbb{N}$ with the Fatou property, and $\langle\nu_n\rangle_{n\in\mathbb{N}}$ a sequence of medial limits. Set $\mathcal{G}=\{A:A\subseteq\mathbb{N},\,\lim_{n\to\mathcal{F}}\nu_nA=1\}$. Show that $\mathcal{G}$ is a filter with the Fatou property.

(n) Show that $\mathfrak{u}\ge\mathfrak{r}(\omega,\omega)\ge\max(\operatorname{cov}\mathcal{N},\mathfrak{m}_{\text{countable}})$ (definitions: 5A6Ia, 529G).

(o)(i) Show that if $\mathcal{F}$ is a rapid filter on $\mathbb{N}$, then $\operatorname{ci}\mathcal{F}\ge\mathfrak{d}$. (ii) Show that $\mathfrak{d}\ge\mathfrak{g}$ (definition: 5A6I(b-ii)). (iii) Show that if $\mathfrak{u}<\mathfrak{g}$ there are no rapid filters on $\mathbb{N}$, and if there is a measure-converging filter there is a measure-converging ultrafilter with coinitiality $\mathfrak{u}$.

(p) Suppose that the filter dichotomy is true. (i) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Show that if $\nu:\mathfrak{A}\to\mathbb{R}$ is an additive functional which is universally measurable for the order-sequential topology of $\mathfrak{A}$, then ν is countably additive. (ii) Let $(\mathfrak{A},\bar\mu)$ be a localizable measure algebra. Show that if $\nu:\mathfrak{A}\to\mathbb{R}$ is an additive functional which is universally measurable for the measure-algebra topology on $\mathfrak{A}$, then it is continuous.

(q)(i) Show that there is a semigroup operation $\dot+$ on the set $\beta\mathbb{N}$ of ultrafilters on $\mathbb{N}$ defined by saying that $\mathcal{F}\dot+\mathcal{G}=+[[\mathcal{F}\ltimes\mathcal{G}]]$ for all $\mathcal{F},\mathcal{G}\in\beta\mathbb{N}$, where $+:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ is addition. (ii) Show that if we identify $\beta\mathbb{N}$ with the Stone-Čech compactification of $\mathbb{N}$ (4A2I(b-i)), then $\dot+$ is continuous in the first variable. (iii) Show that there is a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$ which is **idempotent**, that is, $\mathcal{F}\dot+\mathcal{F}=\mathcal{F}$. (*Hint*: consider a minimal closed sub-semigroup of the set of non-principal ultrafilters.) (iv) For any function $f\in\mathbb{N}^{\mathbb{N}}$, write $\mathrm{FS}(f)$ for $\{\sum_{n\in K}f(n):K\in[\mathbb{N}]^{<\omega}\}$; say a **finite sum set** is a set of the form $\mathrm{FS}(f)$ for some strictly increasing function $f\in\mathbb{N}^{\mathbb{N}}$. Show that if $\mathcal{F}$ is a non-principal idempotent ultrafilter on $\mathbb{N}$ and $I\in\mathcal{F}$, then I includes a finite sum set. (This is a version of **Hindman's theorem**.) (v) Show that if $I\subseteq\mathbb{N}$ is a finite sum set there is an idempotent ultrafilter containing I. (vi) Suppose that $(\mathfrak{A},\bar\mu)$ is a probability algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ is a measure-preserving Boolean homomorphism. (α) Show that if $\mathcal{F}$ is an idempotent ultrafilter on $\mathbb{N}$, then $\lim_{n\to\mathcal{F}}\mu(a\cap\pi^na)\ge(\mu a)^2$ for every $a\in\mathfrak{A}$ (β) Show that there is a finite sum set $I\subseteq\mathbb{N}$ such that $\{\pi^na:n\in I\}$ is centered. (vii) Show that no idempotent ultrafilter is measure-centering. (*Hint*: 538Xv.) (viii) Show that if $\mathcal{F}$ is a p-point ultrafilter then $\mathcal{F}\dot+\mathcal{F}$ is isomorphic to $\mathcal{F}\ltimes\mathcal{F}$ and is not measure-centering. (ix) Repeat, as far as possible, for semigroups other than $(\mathbb{N},+)$.

(r) (V.Bergelson-M.Talagrand) Show that there are a probability algebra $(\mathfrak{A},\bar\mu)$ and a sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $\bar\mu a_n=\frac{1}{2}$ for every $n\in\mathbb{N}$ but $\inf_{m,n\in I}\bar\mu(a_m\cap a_n)=0$ whenever $I\subseteq\mathbb{N}$ does not have asymptotic density 0.

538Z Problem Show that it is relatively consistent with ZFC to suppose that there are no measure-converging filters on $\mathbb{N}$.

538 Notes and comments This is a long section, and rather a lot of ideas are crowded into it, starting with the list in 538A. If you have looked at ultrafilters on $\mathbb{N}$ at all, you are likely to have encountered 'p-point', 'rapid' and 'Ramsey' ultrafilters, and most of 538B-538D and 538F will probably be familiar. The 'iterated products' of 538E will also be a matter of adapting known concepts to my particular formulation.

Some of the slightly contorted language of 538Fe and 538Ff (with references to '$\#(\mathfrak{F})$') is there because we do not know how many isomorphism classes of Ramsey filters there are. If there are none (as in random real models, see 553H), or one (Shelah 82, §VI.5), then things are very simple. If there are infinitely many then we could rephrase 538Ff in terms of sequences of non-isomorphic filters. But it is possible that there should be two, or seventeen (Shelah 98a, p. 335).

In 538H-538M I try to set out, and expand, some of the principal ideas of Benedikt 98. The starting point is the observation that a Ramsey ultrafilter gives us an extension of Lebesgue measure on $[0,1]$, indeed of any perfect probability measure. Observing that this property is preserved by iterations, we are led to 'measure-centering' ultrafilters. Once we have the idea of measure-centering-ultrafilter extension of a perfect probability measure, we can set out to look at its properties in terms of the (by now very extensive) general theory of this treatise. The first step has to be the identification of its measure algebra (538Ja, 538Xk), followed, if possible, by the identification of the corresponding Banach function spaces. It turns out that these can be reached by an alternative route *not* involving special properties of the ultrafilter or the probability space, which I have expressed in general forms in §§328 and 377. This gives a long list of facts, which I have written out in 538Ja and 538K. Minor variations of the measure and the filter are straightforward (538Jb, 538Jc, 538Xl). For iterated products of filters we have more work to do (538L), especially if we are to express them in a form adequate for the objective, the universal-extension result of 538M.

You will have noticed that in the statement of 538G I speak of '$\bigcup_{A\in\mathcal{F}}\bigcap_{n\in A}F_n$' and '$\liminf_{n\to\mathcal{F}}\mu F_n$'. Something of the sort is necessary since in that theorem I do not insist from the outset that $\mathcal{F}$ should be an ultrafilter. Of course only ultrafilters are of interest in this context, by 538Ha, and for these we have $\bigcup_{A\in\mathcal{F}}\bigcap_{n\in A}F_n=\lim_{n\to\mathcal{F}}F_n$ and $\liminf_{n\to\mathcal{F}}\mu F_n=\lim_{n\to\mathcal{F}}\mu F_n$, as in 538I.

For most of this section I have kept firmly to the study of filters on $\mathbb{N}$. For measure-centering filters, at least, there are interesting extensions to filters on uncountable sets, which I mention in 538Yg. We can do a good deal with the ideas of 538G-538K on cardinals less than $\mathfrak{c}$ in the presence of (for instance) Martin's axiom; but for anything corresponding to 538L-538M it seems that we must use a two-valued-measurable cardinal (541M below).

Measure-converging filters (538N) and filters with the Fatou property (538O) form an oddly complementary pair. I have tried to emphasize the correspondence in the characterizations 538Na and 538Oa (compare 538G(v), 538Na(iv) and 538Oa(iv)), but after this they seem to diverge. The phrase 'Fatou property' comes from 538O(a-iii); if you like, Fatou's Lemma says that the Fréchet filter has the Fatou property. From 538Xq(i) I see that I could just as well have called it the 'Lebesgue property'. Note that any filter larger than a measure-converging filter is again measure-converging, so that if there is a measure-converging filter there is a measure-converging ultrafilter; but that no non-principal ultrafilter can have the Fatou property (538Xq(ii)). On the other hand, there are many free filters with the Fatou property, but it is not known for sure whether there have to be measure-converging filters. It is possible for a measure-converging filter to have the Fatou property (538Rd).

In the last part of the section I look at a different kind of limit. A 'Banach limit' is an extension to ℓ^∞ of the ordinary limit regarded as a linear functional on the closed subspace of convergent sequences; a 'medial limit' is a Banach limit which commutes with integration in appropriate settings. To study these I use the formulae of repeated integration to do some surprising things. In 363L I tried to explain what I meant by the formula '$\fint\ldots d\nu$' for a *finitely* additive functional ν. This defines linear functionals which are positive for non-negative ν. In 'repeated integrals' like $\fint\int f_n(x)\mu(dx)\nu(dn)$ (538P(iii)), we must interpret the formula as $\fint\bigl(\int f_n(x)\mu(dx)\bigr)\nu(dn)$; the 'inner integral' is an ordinary integral with respect to the countably additive measure μ, and the 'outer integral' is a name for a linear functional. In the integral $\fint\ldots d\nu$ we have no problem with measurability, though we must check that the integrand $n\mapsto\int f_nd\mu$ is bounded (or, at least, satisfies the condition in 538Yl); but when we look at the other repeated integrals, $\int\nu(a)\mu(da)$ or $\int\fint x\,d\nu\,\mu(dx)$ or $\int\fint f_n(x)\nu(dn)\mu(dx)$, the conditions of 538P must explicitly assert that the outer integrals are defined.

Because we don't need to consider measurability, the 'finitely additive integrals' here are in some ways easy to deal with; 'disintegrations' like $\tilde\nu=\fint\nu_k\,\nu(dk)$ (538Rc) slide past all the usual questions. However we must always be vigilant against the temptations of limiting processes. As with the Riemann integral, of course, we can integrate the limit of a uniformly convergent sequence of functions. But see the manoeuvres of part (a-iii) of the proof of 538R, where the sums $\sum_{i=0}^\infty\alpha_{ni}\ldots$ demand different treatments at different points. And Fubini's theorem nearly disappears; the point of 'medial functionals' is that something extraordinary has to happen before we can expect to change the order of integration.

I have used the language of Volume 3 to express 538Re in a general form. Of course by far the most important example is when the operator T is a conditional expectation operator (538Xu). For more examples of operators in $L^\times(L^\infty;L^\infty)$, see §§373-374.

For most of the classes of filter here, there is a question concerning their existence. Subject to the continuum hypothesis, there are many Ramsey ultrafilters, and refining the argument we find that the same is true if $\mathfrak{p}=\mathfrak{c}$ (538Yc). There are many ways of forcing non-existence of Ramsey ultrafilters, of which one of the simplest is in 553H below. With more difficulty, we can eliminate p-point ultrafilters (Wimmers 82) or rapid filters (Miller 80) or nowhere dense filters and therefore measure-centering ultrafilters (538Hd, Shelah 98b). It is not known for sure that we can eliminate measure-converging filters (538Z).

Version of 24.5.14/30.8.18

539 Maharam submeasures

Continuing the work of §§392-394 and 496, I return to Maharam submeasures and the forms taken by the ideas of the present volume in this context. At least for countably generated algebras, and in some cases more generally, many of the methods of Chapter 52 can be applied (539B-539K). In 539L-539N I give the main result of BALCAR JECH & PAZAK 05 and VELIČKOVIĆ 05: it is consistent to suppose that every Dedekind complete ccc weakly (σ,∞)-distributive Boolean algebra is a Maharam algebra. In 539R-539U I introduce the idea of 'exhaustivity rank' of an exhaustive submeasure.

539A The story so far As submeasures have hardly appeared before in this volume, I begin by repeating some of the essential ideas.

(a) If $\mathfrak{A}$ is a Boolean algebra, a **submeasure** on $\mathfrak{A}$ is a functional $\nu:\mathfrak{A}\to[0,\infty]$ such that $\nu 0=0$, $\nu a\le\nu b$ whenever $a\subseteq b$, and $\nu(a\cup b)\le\nu a+\nu b$ for all a, $b\in\mathfrak{B}$ (392A); it is **totally finite** if $\nu 1<\infty$. If ν is a submeasure defined on an algebra of subsets of a set X, I say that the **null ideal** of ν is the ideal $\mathcal{N}(\nu)$ of subsets of X generated by $\{E:\nu E=0\}$ (496Bc). A submeasure ν on a Boolean algebra $\mathfrak{A}$ is **exhaustive** if $\lim_{n\to\infty}\nu a_n=0$ for every disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$; it is **uniformly exhaustive** if for every $\epsilon>0$ there is an $n\in\mathbb{N}$ such that there is no disjoint family $a_0,\ldots,a_n$ with $\nu a_i\ge\epsilon$ for every $i\le n$ (392Bc). A **Maharam submeasure** is a totally finite sequentially order-continuous submeasure (393A); a Maharam submeasure on a Dedekind σ-complete Boolean algebra is exhaustive (393Bc).

(b) A **Maharam algebra** is a Dedekind σ-complete Boolean algebra with a strictly positive Maharam submeasure. Any Maharam algebra is ccc and weakly (σ,∞)-distributive (393Eb). A Maharam algebra is measurable iff it carries a strictly positive uniformly exhaustive submeasure (393D). If ν is any Maharam submeasure on a Dedekind σ-complete Boolean algebra $\mathfrak{A}$, its Maharam algebra is the quotient $\mathfrak{A}/\{a:\nu a=0\}$ (496Ba).

(c) If ν is any strictly positive totally finite submeasure on a Boolean algebra $\mathfrak{A}$, there is an associated metric $(a,b)\mapsto\nu(a\triangle b)$ on $\mathfrak{A}$; the completion $\widehat{\mathfrak{A}}$ of $\mathfrak{A}$ under this metric is a Boolean algebra (392Hc). If ν is exhaustive, then $\widehat{\mathfrak{A}}$ is a Maharam algebra (393H). If ν and ν' are both strictly positive Maharam submeasures on the same Maharam algebra $\mathfrak{A}$, ν is absolutely continuous with respect to ν' (393F). Consequently the associated metrics are uniformly equivalent, and $\mathfrak{A}$ has a canonical topology and uniformity, its **Maharam-algebra topology** and **Maharam-algebra uniformity** (393G).

(d) Let $\mathfrak{A}$ be a Boolean algebra.

(i) A sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ order*-converges to $a\in\mathfrak{A}$ (definition: 367A) iff there is a partition B of unity in $\mathfrak{A}$ such that $\{n:b\cap(a_n\triangle a)\ne0\}$ is finite for every $b\in B$ (393Ma).

(ii) The **order-sequential topology** on $\mathfrak{A}$ is the topology for which the closed sets are just the sets closed under order*-convergence (393L).

(iii) If $\mathfrak{A}$ is ccc and Dedekind σ-complete, a subalgebra of $\mathfrak{A}$ is order-closed iff it is closed for the order-sequential topology (393O).

(iv) If $\mathfrak{A}$ is ccc and weakly (σ,∞)-distributive, then the closure of a set $A\subseteq\mathfrak{A}$ for the order-sequential topology is the set of order*-limits of sequences in A (393Pb).

(v) If $\mathfrak{A}$ is a Maharam algebra, then its Maharam-algebra topology is its order-sequential topology (393N).

(vi) If $\mathfrak{A}$ is a Dedekind σ-complete ccc weakly (σ,∞)-distributive Boolean algebra, and $\{0\}$ is a G_δ set for the order-sequential topology, then $\mathfrak{A}$ is a Maharam algebra (393Q).

(e) It was a long-outstanding problem (the 'Control Measure Problem') whether every Maharam algebra is in fact a measurable algebra; this was solved by a counterexample in TALAGRAND 08, described in §394.

(f) If X is a Hausdorff space, a **totally finite Radon submeasure** on X is a totally finite submeasure ν defined on a σ-algebra Σ of subsets of X such that (i) if $E \subseteq F \in \Sigma$ and $\nu F = 0$ then $E \in \Sigma$ (ii) every open set belongs to Σ (iii) if $E \in \Sigma$ and $\epsilon > 0$ there is a compact set $K \subseteq E$ such that $\nu(E \setminus K) \le \epsilon$ (496C). Every totally finite Radon submeasure is a Maharam submeasure (496Da). If X is a Hausdorff space and ν is a totally finite Radon submeasure on X, a set $E \in \operatorname{dom}\nu$ is **self-supporting** if $\nu(E \cap G) > 0$ whenever $G \subseteq X$ is an open set meeting E. If $E \in \operatorname{dom}\nu$ and $\epsilon > 0$, there is a compact self-supporting $K \subseteq E$ such that $\nu(E \setminus K) \le \epsilon$ (496Dd).

Let ν be a strictly positive Maharam submeasure on a Dedekind σ-complete Boolean algebra $\mathfrak{A}$. Let Z be the Stone space of $\mathfrak{A}$, and write $\widehat{a}$ for the open-and-closed subset of Z corresponding to each $a \in \mathfrak{A}$. Then there is a unique totally finite Radon submeasure ν' on Z such that $\nu'\widehat{a} = \nu a$ for every $a \in \mathfrak{A}$; the null ideal of ν' is the nowhere dense ideal of Z (496G).

(g) For a cardinal κ, I write $\mathcal{N}_\kappa$ for the null ideal of the usual measure on $\{0,1\}^\kappa$; $\mathcal{N} \cong \mathcal{N}_\omega$ will be the null ideal of Lebesgue measure on $\mathbb{R}$, and $\mathcal{M}$ the meager ideal of $\mathbb{R}$.

539B Proposition Let $\mathfrak{A}$ be a Maharam algebra, $\tau(\mathfrak{A})$ its Maharam type and $d_{\mathfrak{T}}(\mathfrak{A})$ its topological density for its Maharam-algebra topology. Then $\tau(\mathfrak{A}) \le d_{\mathfrak{T}}(\mathfrak{A}) \le \max(\omega, \tau(\mathfrak{A}))$.

proof Recall that the Maharam-algebra topology is the order-sequential topology (539A(d-v)). $\mathfrak{A}$ is ccc and weakly (σ,∞)-distributive (539Ab), so if $D \subseteq \mathfrak{A}$ is topologically dense, then every element of $\mathfrak{A}$ is expressible as the order*-limit $\inf_{n\in\mathbb{N}} \sup_{m\ge n} a_m$ of some sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in D (539A(d-iv)). In this case D τ-generates $\mathfrak{A}$ and $\tau(\mathfrak{A}) \le \#(D)$; accordingly $\tau(\mathfrak{A}) \le d_{\mathfrak{T}}(\mathfrak{A})$. If $D \subseteq \mathfrak{A}$ τ-generates $\mathfrak{A}$, let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}$ generated by D and $\overline{\mathfrak{B}}$ its topological closure. Then $\overline{\mathfrak{B}}$ is order-closed (because $\mathfrak{A}$ is ccc), so is the whole of $\mathfrak{A}$, and $d_{\mathfrak{T}}(\mathfrak{A}) \le \#(\mathfrak{B}) \le \max(\omega, \#(D))$; accordingly $d_{\mathfrak{T}}(\mathfrak{A}) \le \max(\omega, \tau(\mathfrak{A}))$.

539C Theorem Let $\mathfrak{A}$ be a Maharam algebra.

(a)

$$(\mathfrak{A}^+, \supseteq', [\mathfrak{A}^+]^{\le\max(\omega,\tau(\mathfrak{A}))}) \preccurlyeq_{\mathrm{GT}} (\mathrm{Pou}(\mathfrak{A}), \sqsubseteq^*, \mathrm{Pou}(\mathfrak{A})),$$

where $\mathfrak{A}^+ = \mathfrak{A} \setminus \{0\}$, $(\mathfrak{A}^+, \supseteq', [\mathfrak{A}^+]^{\le\kappa})$ is defined as in 512F, and $(\mathrm{Pou}(\mathfrak{A}), \sqsubseteq^*)$ as in 512Ee.

(b) $\mathrm{Pou}(\mathfrak{A}) \preccurlyeq_{\mathrm{T}} \mathcal{N}_{\tau(\mathfrak{A})}$.

proof If $\mathfrak{A} = \{0\}$ these are both trivial; suppose otherwise. Fix a strictly positive Maharam submeasure ν on $\mathfrak{A}$ such that $\nu 1 = 1$. Let $\mathfrak{B}$ be a subalgebra of $\mathfrak{A}$ which is dense in $\mathfrak{A}$ for the metric $(a,b) \mapsto \nu(a \triangle b)$ and has cardinal at most $\kappa = \max(\omega, \tau(\mathfrak{A}))$ (539B).

(a)(i) For $a \in \mathfrak{A}^+$ choose $\phi(a) \in \mathrm{Pou}(\mathfrak{A})$ as follows. Start by taking $d_n \in \mathfrak{B}$, for $n \in \mathbb{N}$, such that $\nu(d_n \triangle (1 \setminus a)) \le 2^{-n-2}\nu a$ for each n; set $b_n = d_n \setminus \sup_{i<n} b_i$ for $n \in \mathbb{N}$, $a' = 1 \setminus \sup_{n\in\mathbb{N}} b_n = 1 \setminus \sup_{n\in\mathbb{N}} d_n$; then every b_n belongs to $\mathfrak{B}$,

$$\nu(a' \setminus a) \le \inf_{n\in\mathbb{N}} \nu((1 \setminus d_n) \setminus a) \le \inf_{n\in\mathbb{N}} \nu(d_n \triangle (1 \setminus a)) = 0,$$

$$\nu(a \setminus a') \le \sum_{n=0}^{\infty} \nu(a \cap d_n) < \nu a,$$

so $0 \ne a' \subseteq a$. Now set $\phi(a) = \{a'\} \cup \{b_n : n \in \mathbb{N}\}$.

(ii) For $C \in \mathrm{Pou}(\mathfrak{A})$, set

$$\psi(C) = \{c \cap b : c \in C,\, b \in \mathfrak{B}\} \setminus \{0\} \in [\mathfrak{A}^+]^{\le\kappa}.$$

(iii) Suppose that $a \in \mathfrak{A}^+$, $C \in \mathrm{Pou}(\mathfrak{A})$ and $\phi(a) \sqsubseteq^* C$. Then there is a $b \in \psi(C)$ such that $b \subseteq a$. **P** Let $c \in C$ be such that $c \cap a' \ne 0$, where a' is defined as in (i) above. Then $B = \{b : b \in \phi(a) \setminus \{a'\},\ c \cap b \ne 0\}$ is a finite subset of $\mathfrak{B}$, so $\sup B \in \mathfrak{B}$ and $c \setminus \sup B \in \psi(C)$. But $c \setminus \sup B = c \cap a' \subseteq a$. **Q** Thus $a \supseteq' \psi(C)$.

As a is arbitrary, (ϕ, ψ) is a Galois-Tukey connection from $(\mathfrak{A}^+, \supseteq', [\mathfrak{A}^+]^{\le\kappa})$ to $(\mathrm{Pou}(\mathfrak{A}), \sqsubseteq^*, \mathrm{Pou}(\mathfrak{A})$, and $(\mathfrak{A}^+, \supseteq', [\mathfrak{A}^+]^{\le\kappa}) \preccurlyeq_{\mathrm{GT}} (\mathrm{Pou}(\mathfrak{A}), \sqsubseteq^*, \mathrm{Pou}(\mathfrak{A})$.

(b)(i) If $\tau(\mathfrak{A})$ is finite, then $\mathfrak{A}$ is purely atomic and $\mathrm{Pou}(\mathfrak{A})$ has an upper bound in itself, as does $\mathcal{N}_\kappa$; so the result is trivial. Accordingly we may suppose henceforth that $\tau(\mathfrak{A})=\kappa$ is infinite.

(ii) If $C\in\mathrm{Pou}(\mathfrak{A})$, there is a sequence $\langle b_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{B}$ such that $\nu b_n\le 4^{-n}$ for every $n\in\mathbb{N}$ and $\{c:c\in C$, $c\not\subseteq\sup_{i\ge n}b_i\}$ is finite for every $n\in\mathbb{N}$. **P** If C is finite this is trivial. Otherwise, set $\epsilon_n=4^{-n}/(n+2)$ for each $n\in\mathbb{N}$, and enumerate C as $\langle c_n\rangle_{n\in\mathbb{N}}$. Let $\langle k(n)\rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence such that $\nu c'_n\le\epsilon_n$ for every n, where $c'_n=\sup_{i\ge k(n)}c_i$; choose $\langle b_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{B}$ inductively so that

$$\nu(b_n\triangle\sup_{j\le n}(c'_j\setminus\sup_{j\le i<n}b_i))\le\epsilon_{n+1}$$

for each $n\in\mathbb{N}$. Then we see by induction on n that

$$\nu(c'_j\setminus\sup_{j\le i<n}b_i)\le\epsilon_n$$

whenever $j\le n$ in $\mathbb{N}$, and therefore that

$$\nu b_n\le\epsilon_{n+1}+(n+1)\epsilon_n\le 4^{-n}$$

for each n; while $c'_j\subseteq\sup_{i\ge j}b_i$ for every j, so

$$1\setminus\sup_{i\ge n}b_i\subseteq 1\setminus c'_n=\sup_{i<k(n)}c_i$$

meets only finitely many members of C, for every n. **Q**

(iii) Now fix on an enumeration $\langle b_\xi\rangle_{\xi<\kappa}$ of $\mathfrak{B}$. Consider the κ-localization relation $(\kappa^{\mathbb{N}},\subseteq^*,\mathcal{S}_\kappa)$ (522K). We see from (ii) that we can find a function $\phi:\mathrm{Pou}(\mathfrak{A})\to\kappa^{\mathbb{N}}$ such that

$$\nu b_{\phi(C)(n)}\le 4^{-n}\text{ for every }n\in\mathbb{N},$$

$$1\setminus\sup_{i\ge n}b_{\phi(C)(i)}\text{ meets only finitely many members of }C\text{, for every }n\in\mathbb{N}.$$

Next, define $\psi:\mathcal{S}_\kappa\to\mathrm{Pou}(\mathfrak{A})$ as follows. Given $S\in\mathcal{S}_\kappa$, set $a_0(S)=1$,

$$a_{n+1}(S)=\sup_{m\ge n}\sup\{b_\xi:(m,\xi)\in S,\ \nu b_\xi\le 4^{-m}\}$$

for each n; then $\nu a_{n+1}(S)\le\sum_{m=n}^\infty 2^{-m}=2^{-n+1}$ for every n, so $\psi(S)=\{a_n(S)\setminus a_{n+1}(S):n\in\mathbb{N}\}$ is a partition of unity in $\mathfrak{A}$.

(iv) Suppose that $C\in\mathrm{Pou}(\mathfrak{A})$ and $S\in\mathcal{S}_\kappa$ are such that $\phi(C)\subseteq^*S$. In this case there is an $m\in\mathbb{N}$ such that $(n,\phi(C)(n))\in S$ for every $n\ge m$. Since $\nu b_{\phi(C)(n)}\le 4^{-n}$ for every n, $\sup_{i\ge n}b_{\phi(C)(i)}\subseteq a_{n+1}(S)$ and $1\setminus a_{n+1}(S)$ meets only finitely many members of C, for every $n\ge m$. Thus every member of $\psi(S)$ meets only finitely many members of C, and $C\sqsubseteq^*\psi(S)$.

This shows that (ϕ,ψ) is a Galois-Tukey connection from $(\mathrm{Pou}(\mathfrak{A}),\sqsubseteq^*,\mathrm{Pou}(\mathfrak{A}))$ to $(\kappa^{\mathbb{N}},\subseteq^*,\mathcal{S}_\kappa)$, and $(\mathrm{Pou}(\mathfrak{A}),\sqsubseteq^*,\mathrm{Pou}(\mathfrak{A}))\preccurlyeq_{\mathrm{GT}}(\kappa^{\mathbb{N}},\subseteq^*,\mathcal{S}_\kappa)$. On the other side, we know already that $(\kappa^{\mathbb{N}},\subseteq^*,\mathcal{S}_\kappa)\preccurlyeq_{\mathrm{GT}}(\mathcal{N}_\kappa,\subseteq,\mathcal{N}_\kappa)$ (524G); so $(\mathrm{Pou}(\mathfrak{A}),\sqsubseteq^*,\mathrm{Pou}(\mathfrak{A}))\preccurlyeq_{\mathrm{GT}}(\mathcal{N}_\kappa,\subseteq,\mathcal{N}_\kappa)$, that is, $\mathrm{Pou}(\mathfrak{A})\preccurlyeq_{\mathrm{T}}\mathcal{N}_\kappa$.

539D Corollary Let $\mathfrak{A}$ be a Maharam algebra.
(a) $\pi(\mathfrak{A})\le\max(\mathrm{cf}[\tau(\mathfrak{A})]^{\le\omega},\mathrm{cf}\,\mathcal{N})$.
(b) If $\tau(\mathfrak{A})\le\omega$, then $\mathrm{wdistr}(\mathfrak{A})\ge\mathrm{add}\,\mathcal{N}$.

proof Set $\kappa=\tau(\mathfrak{A})$.

(a) If $\pi(\mathfrak{A})$ is countable, or $\pi(\mathfrak{A})\le\mathrm{cf}[\kappa]^{\le\omega}$, we can stop. Otherwise, κ is infinite and

$$\begin{aligned}\max(\omega,\kappa)&\le\max(\omega,\mathrm{cf}[\kappa]^{\le\omega})<\pi(\mathfrak{A})\\&=\mathrm{cov}(\mathfrak{A}^+,\supseteq,\mathfrak{A}^+)\le\max(\omega,\kappa,\mathrm{cov}(\mathfrak{A}^+,\supseteq',[\mathfrak{A}^+]^{\le\kappa}))\end{aligned}$$

(512Gf), so

$$\pi(\mathfrak{A})\le\mathrm{cov}(\mathfrak{A}^+,\supseteq',[\mathfrak{A}^+]^{\le\kappa})\le\mathrm{cov}(\mathrm{Pou}(\mathfrak{A}),\sqsubseteq^*,\mathrm{Pou}(\mathfrak{A}))$$

(539Ca, 512Da)

$$=\mathrm{cf}\,\mathrm{Pou}(\mathfrak{A})\le\mathrm{cf}\,\mathcal{N}_\kappa$$

(539Cb, 513E(e-i))

$$= \max(\mathrm{cf}[\kappa]^{\le\omega}, \mathrm{cf}\,\mathcal{N})$$

(523N).

(b) If κ is finite, $\mathrm{wdistr}(\mathfrak{A}) = \infty$ and we can stop. Otherwise, $\kappa = \omega$ and

$$\mathrm{wdistr}(\mathfrak{A}) = \mathrm{add}\,\mathrm{Pou}(\mathfrak{A})$$

(512Ee)

$$\ge \mathrm{add}\,\mathcal{N}_\kappa$$

(539Cb, 513E(e-ii))

$$= \mathrm{add}\,\mathcal{N}.$$

539E Proposition (Veličković 05, Balcar Jech & Pazák 05) If $\mathfrak{A}$ is an atomless Maharam algebra, not $\{0\}$, there is a sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $\sup_{n\in I} a_n = 1$ and $\inf_{n\in I} a_n = 0$ for every infinite $I \subseteq \mathbb{N}$.

proof Fix a strictly positive Maharam submeasure ν on $\mathfrak{A}$.

(a) If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}$ such that $\delta = \inf_{n\in\mathbb{N}} \nu a_n$ is greater than 0, there are a non-zero $d \in \mathfrak{A}$ and an infinite $I \subseteq \mathbb{N}$ such that $d \subseteq \sup_{i\in J} a_i$ for every infinite $J \subseteq I$. **P?** Otherwise, set $b_J = \sup_{i\in J} a_i$ for $J \subseteq \mathbb{N}$. Choose $\langle I_\xi\rangle_{\xi<\omega_1}$, $\langle c_\xi\rangle_{\xi<\omega_1}$ and $\langle d_\xi\rangle_{\xi<\omega_1}$ inductively, as follows. $I_0 = \mathbb{N}$. The inductive hypothesis will be that I_ξ is an infinite subset of $\mathbb{N}$, $I_\xi \setminus I_\eta$ is finite whenever $\eta \le \xi$, and $c_\xi \cap b_{I_{\xi+1}} = 0$ for every $\xi < \omega_1$. Given $\langle I_\eta\rangle_{\eta\le\xi}$ where $\xi < \omega_1$, set $d_\xi = \inf_{n\in\mathbb{N}} b_{I_\xi\setminus n}$. Since $\nu b_J \ge \delta$ for every non-empty $J \subseteq \mathbb{N}$, $\nu d_\xi \ge \delta$ and $d_\xi \ne 0$. By hypothesis, there is an infinite $I_{\xi+1} \subseteq I_\xi$ such that $c_\xi = d_\xi \setminus b_{I_{\xi+1}}$ is non-zero. Given $\langle I_\eta\rangle_{\eta<\xi}$ where $\xi < \omega_1$ is a non-zero limit ordinal, let I_ξ be an infinite set such that $I_\xi \setminus I_\eta$ is finite for every $\eta < \xi$, and continue.

Now observe that if $\eta < \xi < \omega_1$, $I_\xi \setminus I_\eta$ is finite, so that there is an $n \in \mathbb{N}$ such that $I_\xi \setminus n \subseteq I_{\eta+1}$, and

$$c_\xi \subseteq d_\xi \subseteq b_{I_\xi\setminus n} \subseteq b_{I_{\eta+1}}$$

is disjoint from c_η. But this means that $\langle c_\xi\rangle_{\xi<\omega_1}$ is disjoint, which is impossible, because $\mathfrak{A}$ is ccc. **XQ**

(b) Let us say that a Boolean algebra $\mathfrak{B}$ **splits reals** if there is a sequence $\langle b_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{B}$ such that $\sup_{n\in I} b_n = 1$ and $\inf_{n\in I} b_n = 0$ for every infinite $I \subseteq \mathbb{N}$. Now the set of those $d \in \mathfrak{A}$ such that the principal ideal $\mathfrak{A}_d$ generated by d splits reals is order-dense in $\mathfrak{A}$. **P** Let $a \in \mathfrak{A}^+$.

case 1 If $\nu\restriction\mathfrak{A}_a$ is uniformly exhaustive, then $\mathfrak{A}_a$ is measurable (539Ab). Let $\bar\mu$ be a probability measure on $\mathfrak{A}_a$; because $\mathfrak{A}_a$, like $\mathfrak{A}$, is atomless, there is a stochastically independent family $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}_a$ with $\bar\mu a_n = \frac{1}{2}$ for every n, and now $\langle a_n\rangle_{n\in\mathbb{N}}$ witnesses that $\mathfrak{A}_a$ splits reals.

case 2 If $\nu\restriction\mathfrak{A}_a$ is not uniformly exhaustive, let $\langle b_{ni}\rangle_{i\le n\in\mathbb{N}}$ be a family of elements of $\mathfrak{A}_a$ such that $\langle b_{ni}\rangle_{i\le n}$ is disjoint for each n and $\epsilon = \inf_{i\le n\in\mathbb{N}} \nu b_{ni}$ is greater than 0. There is a family $\langle f_\xi\rangle_{\xi<\omega_1}$ in $\prod_{n\in\mathbb{N}}\{0,\dots,n\}$ such that $\{n : f_\xi(n) = f_\eta(n)\}$ is finite whenever $\eta < \xi < \omega_1$. (For each $\xi < \omega_1$ let $\theta_\xi : \xi \to \mathbb{N}$ be injective. Now define $\langle f_\xi\rangle_{\xi<\omega_1}$ inductively by saying that

$$f_\xi(n) = \min(\mathbb{N} \setminus \{f_\eta(n) : \eta < \xi,\, \theta_\xi(\eta) < n\})$$

for every $\xi < \omega_1$ and $n \in \mathbb{N}$.)

? If for every $\xi < \omega_1$ and $I \in [\mathbb{N}]^\omega$ there is a $J \in [I]^\omega$ such that $\inf_{i\in J} b_{i,f_\xi(i)} \ne 0$, choose $\langle I_\xi\rangle_{\xi<\omega_1}$ inductively so that $I_\xi \in [\mathbb{N}]^\omega$, $I_\xi \setminus I_\eta$ is finite for every $\eta < \xi$, and $c_\xi = \inf_{i\in I_\xi} b_{i,f_\xi(i)}$ is non-zero for every $\xi < \omega_1$. Then whenever $\eta < \xi$ the set $I_\xi \cap I_\eta$ is infinite, so there is an $i \in I_\xi \cap I_\eta$ such that $f_\xi(i) \ne f_\eta(i)$; now $c_\xi \cap c_\eta \subseteq b_{i,f_\xi(i)} \cap b_{i,f_\eta(i)} = 0$. But this means that we have an uncountable disjoint family in $\mathfrak{A}_a$, which is impossible, because $\mathfrak{A}$ is ccc. **X**

Thus we have a $\xi < \omega_1$ and an infinite $I \subseteq \mathbb{N}$ such that $\inf_{i\in J} d_i = 0$ for every infinite $J \subseteq I$, where $d_i = b_{i,f_\xi(i)}$ for $i \in I$. Next, applying (a) to $\langle d_i\rangle_{i\in I}$, we have an infinite $K \subseteq I$ and a $d \ne 0$ such that $d = \sup_{i\in J} d_i$ for every infinite $J \subseteq K$. But this means that $\langle d \cap d_i\rangle_{i\in K}$ witnesses that $\mathfrak{A}_d$ splits reals; while $d \subseteq a$.

As a is arbitrary, we have the result. **Q**

(c) By 313K, there is a partition D of unity in $\mathfrak{A}$ such that $\mathfrak{A}_d$ splits reals for every $d \in D$; choose a sequence $\langle a_{dn}\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}_d$ witnessing this for each $d \in D$. Set $a_n = \sup_{d\in D} a_{dn}$ for each n. If $I \subseteq \mathbb{N}$ is infinite, then

$$\sup_{n\in I} a_n = \sup_{d\in D} \sup_{n\in I} a_{dn} = \sup D = 1,$$

while

$$d \cap \inf_{n\in I} a_n = \inf_{n\in I} a_{dn} = 0$$

for every $d \in D$, so $\inf_{n\in I} a_n = 0$. Thus $\langle a_n\rangle_{n\in\mathbb{N}}$ witnesses that $\mathfrak{A}$ splits reals, as claimed.

539F Definition For the next result I need a name for one more cardinal between ω_1 and $\mathfrak{c}$. The **splitting number** $\mathfrak{s}$ is the least cardinal of any family $\mathcal{A} \subseteq \mathcal{P}\mathbb{N}$ such that for every infinite $I \subseteq \mathbb{N}$ there is an $A \in \mathcal{A}$ such that $I \cap A$ and $I \setminus A$ are both infinite.

539G Proposition Let X be a set, Σ a σ-algebra of subsets of X, and ν an atomless Maharam submeasure on Σ. Let $\mathcal{M}$ be the ideal of meager subsets of $\mathbb{R}$.

(a) $\operatorname{non}\mathcal{N}(\nu) \geq \max(\mathfrak{s}, \mathfrak{m}_{\text{countable}})$.

(b) $\operatorname{cov}\mathcal{N}(\nu) \leq \operatorname{non}\mathcal{M}$.

proof If $\nu X = 0$, these are both trivial; suppose otherwise.

(a)(i) Suppose that $D \subseteq X$ and $\#(D) < \mathfrak{m}_{\text{countable}}$. For any $\epsilon > 0$, there is an $F \in \Sigma$ such that $D \subseteq F$ and $\nu F \leq \epsilon$. **P** By 393I, there is for each $n \in \mathbb{N}$ a finite partition $\mathcal{E}_n$ of X into members of Σ such that $\nu E \leq 2^{-n-1}\epsilon$ for each $E \in \mathcal{E}_n$. Express each $\mathcal{E}_n$ as $\{E_{ni} : i < k(n)\}$. For $x \in D$, let $f_x \in \prod_{n\in\mathbb{N}} k(n)$ be such that $x \in E_{n,f_x(n)}$ for every n. Because $\#(D) < \mathfrak{m}_{\text{countable}}$, there is an $f \in \mathbb{N}^{\mathbb{N}}$ such that $f \cap f_x \neq \emptyset$ for every $x \in D$ (522Sb); we may suppose that $f(n) < k(n)$ for every n. Set $F = \bigcup_{n\in\mathbb{N}} E_{n,f(n)}$; this works. **Q**

Applying this repeatedly, we get a sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in Σ such that $D \subseteq F_n$ and $\nu F_n \leq 2^{-n}$ for every n; now $F = \bigcap_{n\in\mathbb{N}} F_n$ includes D and belongs to $\mathcal{N}(\nu)$. As D is arbitrary, $\operatorname{non}\mathcal{N}(\nu) \geq \mathfrak{m}_{\text{countable}}$.

(ii) Set $\mathfrak{A} = \Sigma/\Sigma\cap\mathcal{N}(\nu)$, and define $\bar\nu : \mathfrak{A} \to [0,\infty[$ by setting $\bar\nu E^\bullet = \nu E$ for every $E \in \Sigma$. Then $\bar\nu$ is a strictly positive atomless Maharam submeasure on $\mathfrak{A}$. By 539E, there is a sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $\sup_{n\in I} a_n = 1$ and $\inf_{n\in I} a_n = 0$ for every infinite $I \subseteq \mathbb{N}$. For each $n \in \mathbb{N}$, let $E_n \in \Sigma$ be such that $E_n^\bullet = a_n$.

Suppose that $D \subseteq X$ and $\#(D) < \mathfrak{s}$. For $x \in D$, set $A_x = \{n : x \in E_n\}$. Because $\#(D) < \mathfrak{s}$, there is an infinite $I \subseteq \mathbb{N}$ such that one of $I \cap A_x$, $I \setminus A_x$ is finite for every $x \in D$. Set

$$F = \bigcup_{m\in\mathbb{N}}\bigl((X \setminus \bigcup_{n\in I\setminus m} E_n) \cup (\bigcap_{n\in I\setminus m} E_n)\bigr);$$

then

$$F^\bullet = \sup_{m\in\mathbb{N}}\bigl((1 \setminus \sup_{n\in I\setminus m} a_n) \cup (\inf_{n\in I\setminus m} a_n)\bigr) = 0,$$

so $F \in \mathcal{N}(\nu)$, while $D \subseteq F$. As D is arbitrary, $\operatorname{non}\mathcal{N}(\nu) \geq \mathfrak{s}$.

(b) Let $\langle k(n)\rangle_{n\in\mathbb{N}}$, $\langle E_{ni}\rangle_{i<k(n)}$ and $\langle f_x\rangle_{x\in X}$ be as in (a-i) above, with $\epsilon = 1$. Give $Z = \prod_{n\in\mathbb{N}} k(n)$ its compact metrizable product topology. By 522Wb, there is a family $\langle g_\xi\rangle_{\xi<\operatorname{non}\mathcal{M}}$ in Z such that $\{g_\xi : \xi < \operatorname{non}\mathcal{M}\}$ is non-meager. For each $f \in Z$, the set

$$H(f) = \bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} \{g : g \in Z,\ g(n) = f(n)\}$$

is comeager in Z, so contains some g_ξ; turning this round, $Z = \bigcup_{\xi<\operatorname{non}\mathcal{M}} H(g_\xi)$. Consider the sets $F_\xi = \{x : x \in X,\ f_x \in H(g_\xi)\}$; then $X = \bigcup_{\xi<\operatorname{non}\mathcal{M}} F_\xi$, while

$$\nu F_\xi \leq \inf_{m\in\mathbb{N}} \sum_{n=m}^{\infty} \nu E_{n,g_\xi(n)} = 0$$

for every ξ. So $\operatorname{cov}\mathcal{N}(\nu) \leq \operatorname{non}\mathcal{M}$.

539H Corollary Let $\mathfrak{A}$ be an atomless Maharam algebra, not $\{0\}$. Then $d(\mathfrak{A}) \geq \max(\mathfrak{s}, \mathfrak{m}_{\text{countable}})$.

proof Let Z be the Stone space of $\mathfrak{A}$ and ν' the totally finite Radon submeasure on Z corresponding to a strictly positive Maharam submeasure ν on $\mathfrak{A}$ (539Af), so that $\mathcal{N}(\nu')$ is the ideal of meager subsets of Z. Note that the meager sets of Z are all nowhere dense, because $\mathfrak{A}$ is weakly (σ,∞)-distributive (316I).

Because $\mathfrak{A}$ is atomless, so are ν and ν'. As every meager subset of Z is nowhere dense (and $Z \neq \emptyset$), no dense set can be meager, and

$$d(\mathfrak{A}) = d(Z)$$

(514Bd)

$$\geq \operatorname{non}\mathcal{N}(\nu') \geq \max(\mathfrak{s}, \mathfrak{m}_{\text{countable}})$$

by 539Ga.

539I Corollary Suppose that $\#(X) < \max(\mathfrak{s}, \mathfrak{m}_{\text{countable}})$, where $\mathfrak{s}$ is the splitting number. Let Σ be a σ-algebra of subsets of X such that (X, Σ) is countably separated, in the sense that there is a sequence in Σ separating the points of X, and $\mathcal{I}$ a σ-ideal of Σ containing singletons. Then there is no non-zero Maharam submeasure on $\Sigma/\mathcal{I}$.

proof (a) Let μ be a Maharam submeasure on $\Sigma/\mathcal{I}$. Then we have a Maharam submeasure ν on Σ defined by setting $\nu E = \mu E^\bullet$ for every $E \in \Sigma$, and $\nu\{x\} = 0$ for every $x \in X$.

(b) ν is atomless. **P** Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a sequence in Σ separating the points of X, and $F \in \Sigma$ such that $\nu F > 0$. Choose $\langle F_n\rangle_{n\in\mathbb{N}}$ inductively so that $F_0 = F$ and, given that $\nu F_n > 0$, F_{n+1} is either $F_n \cap E_n$ or $F_n \setminus E_n$ and $\nu F_{n+1} > 0$. Then $\bigcap_{n\in\mathbb{N}} F_n$ has at most one member, so $\lim_{n\to\infty} \nu F_n = 0$, and there is an n such that $\nu F_n = \nu(F \cap F_n)$ and $\nu(F \setminus F_n)$ are non-zero. **Q**

(c) By 539Ga,

$$\operatorname{non}\mathcal{N}(\nu) \geq \max(\mathfrak{s}, \mathfrak{m}_{\text{countable}}) > \#(X)$$

and $\nu X = 0$, so μ is identically 0.

539J Theorem (a) Let ν be a totally finite Radon submeasure on a Hausdorff space X (539Af) and $\mathfrak{A}$ its Maharam algebra. Then $\mathcal{N}(\nu) \preccurlyeq_{\mathrm{T}} \operatorname{Pou}(\mathfrak{A})$.

(b) Let ν be a totally finite Radon submeasure on a Hausdorff space X and $\mathfrak{A}$ its Maharam algebra.

(i) $\operatorname{wdistr}(\mathfrak{A}) \leq \operatorname{add}\mathcal{N}(\nu)$.

(ii) $\tau(\mathfrak{A}) \leq w(X)$.

(iii) $\operatorname{cf}\mathcal{N}(\nu) \leq \max(\operatorname{cf}[\tau(\mathfrak{A})]^{\leq\omega}, \operatorname{cf}\mathcal{N})$.

(iv) If $\tau(\mathfrak{A}) \leq \omega$ (e.g., because X is second-countable), then $\operatorname{add}\mathcal{N}(\nu) \geq \operatorname{add}\mathcal{N}$ and $\operatorname{cf}\mathcal{N}(\nu) \leq \operatorname{cf}\mathcal{N}$.

proof (a) For $E \in \mathcal{N}(\nu)$, let $\mathcal{K}_E$ be a maximal disjoint family of compact sets of non-zero submeasure disjoint from E, and set $C_E = \{K^\bullet : K \in \mathcal{K}_E\}$. Because ν is inner regular with respect to the compact sets, $C_E \in \operatorname{Pou}(\mathfrak{A})$. Now $E \mapsto C_E : \mathcal{N}(\nu) \to \operatorname{Pou}(\mathfrak{A})$ is a Tukey function. **P** Suppose that $\mathcal{E} \subseteq \mathcal{N}(\nu)$ and $D \in \operatorname{Pou}(\mathfrak{A})$ are such that $C_E \sqsubseteq^* D$ for every $E \in \mathcal{E}$; take any $\epsilon > 0$. Because D is countable, we have a countable partition $\mathcal{H}$ of X into measurable sets such that $D = \{H^\bullet : H \in \mathcal{H}\}$. Because ν is inner regular with respect to the self-supporting compact sets (539Af), we can find a self-supporting compact set $K \subseteq X$ such that $\nu(X \setminus K) \leq \epsilon$ and K is covered by finitely many members of $\mathcal{H}$; consequently $K^\bullet$ meets only finitely many members of D.

If $E \in \mathcal{E}$, then $K^\bullet$ meets only finitely many members of C_E, so there is a finite set $\mathcal{K}'_E \subseteq \mathcal{K}_E$ such that $K \setminus K_E$ is negligible, where $K_E = \bigcup \mathcal{K}'_E$. But K_E is compact and K is self-supporting, so $K \subseteq K_E$ and $K \cap E = \emptyset$.

This means that $\bigcup\mathcal{E} \subseteq X \setminus K$ is included in an open set of submeasure at most ϵ. This is true for every $\epsilon > 0$, so $\bigcup\mathcal{E}$ is included in a negligible G_δ set and belongs to $\mathcal{N}(\nu)$; that is, $\mathcal{E}$ is bounded above in $\mathcal{N}(\nu)$. As $\mathcal{E}$ is arbitrary, $E \mapsto C_E$ is a Tukey function. **Q**

(b)(i) Putting (a) and 513E(e-ii) together,

$$\operatorname{wdistr}(\mathfrak{A}) = \operatorname{add}\operatorname{Pou}(\mathfrak{A}) \leq \operatorname{add}\mathcal{N}(\nu).$$

(ii) If $\mathcal{U}$ is a base for the topology of X with $\#(\mathcal{U}) = w(X)$, consider $D = \{U^\bullet : U \in \mathcal{U}\}$ and the order-closed subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ generated by D; note that $\mathfrak{B}$ is closed for the order-sequential (or Maharam-algebra) topology of $\mathfrak{A}$ (539Ad). Let $\mathcal{E}$ be the algebra of sets generated by $\mathcal{U}$. If $F \in \operatorname{dom}\nu$ and $\epsilon > 0$, there are compact sets $K \subseteq F$, $L \subseteq X \setminus F$ such that $\nu(X \setminus (K \cup L)) \le \epsilon$. There is an $E \in \mathcal{E}$ such that $K \subseteq E \subseteq X \setminus L$, so $\nu(E \triangle F) \le \epsilon$. Now $E^\bullet \in \mathfrak{B}$ and $\bar{\nu}(F^\bullet \triangle E^\bullet) \le \epsilon$; as ϵ is arbitrary, $F^\bullet \in \mathfrak{B}$; as F is arbitrary, $\mathfrak{B} = \mathfrak{A}$ and $\mathfrak{A}$ is τ-generated by D. This means that $\tau(\mathfrak{A}) \le \#(D) \le w(X)$, as required.

(iii) Setting $\kappa = \tau(\mathfrak{A})$, (a) and 539Cb tell us that $\mathcal{N}(\nu) \preccurlyeq_{\mathrm{T}} \mathcal{N}_\kappa$, where $\mathcal{N}_\kappa$ is the null ideal of the usual measure on $\{0,1\}^\kappa$. So $\operatorname{add}\mathcal{N}(\nu) \ge \operatorname{add}\mathcal{N}_\kappa$ and

$$\operatorname{cf}\mathcal{N}(\nu) \le \operatorname{cf}\mathcal{N}_\kappa \le \max(\operatorname{cf}[\kappa]^{\le\omega}, \operatorname{cf}\mathcal{N})$$

(513E(e-i), 523N).

(iv) If $\kappa \le \omega$ then $\mathcal{N}_\kappa \preccurlyeq_{\mathrm{T}} \mathcal{N}$ so $\operatorname{add}\mathcal{N}(\nu) \ge \operatorname{add}\mathcal{N}$ and $\operatorname{cf}\mathcal{N}(\nu) \le \operatorname{cf}\mathcal{N}$.

539K We can approach precalibers by some of the same combinatorial methods as before.

Proposition Let $\mathfrak{A}$ be a Boolean algebra and ν an exhaustive submeasure on $\mathfrak{A}$.

(a) Let $\langle a_i \rangle_{i\in\mathbb{N}}$ be a sequence in $\mathfrak{A}$ such that $\inf_{i\in\mathbb{N}} \nu a_i > 0$.

(i) There is an infinite $I \subseteq \mathbb{N}$ such that $\{a_i : i \in I\}$ is centered.

(ii) For every $k \in \mathbb{N}$ there are an $S \in [\mathbb{N}]^\omega$ and a $\delta > 0$ such that $\nu(\inf_{i\in J} a_i) \ge \delta$ for every $J \in [S]^k$.

(b) Suppose that $\langle a_\xi \rangle_{\xi<\kappa}$ is a family in $\mathfrak{A}$ such that $\inf_{\xi<\kappa} \nu a_\xi > 0$, where κ is a regular uncountable cardinal. Then for every $k \in \mathbb{N}$ there are a stationary set $S \subseteq \kappa$ and a $\delta > 0$ such that $\nu(\inf_{i\in J} a_i) \ge \delta$ for every $J \in [S]^k$.

(c) If ν is strictly positive, then (κ, κ, k) is a precaliber triple of $\mathfrak{A}$ for every regular uncountable cardinal κ and every $k \in \mathbb{N}$; in particular, $\mathfrak{A}$ satisfies Knaster's condition.

proof (a)(i) This is 392J.

(ii) Induce on k. The cases $k = 0$, $k = 1$ are trivial. For the inductive step to $k+1$, let $M \in [\mathbb{N}]^\omega$ and $\delta > 0$ be such that $\nu(\inf_{i\in J} a_i) \ge \delta$ for every $J \in [M]^k$. **?** Suppose, if possible, that for every $S \in [M]^\omega$ and $\gamma > 0$ there is a $J \in [S]^{k+1}$ such that $\nu(\inf_{i\in J} a_i) < \gamma$. Using Ramsey's theorem (4A1G) repeatedly, we can find $\langle I_n \rangle_{n\in\mathbb{N}}$ such that $I_0 \in [M]^\omega$, $I_{n+1} \in [I_n]^\omega$, $r_n = \min I_n \notin I_{n+1}$ and $\nu(\inf_{i\in J} a_i) \le 2^{-n-2}\delta$ for every $n \in \mathbb{N}$ and $J \in [I_n]^{k+1}$. Set $S = \{r_n : n \in \mathbb{N}\}$. If $J \in [S]^k$ and $\min J = r_n$, then $J \cup \{r_m\} \in [I_m]^{k+1}$, so $\nu(\inf_{i\in J} a_i \cap a_{r_m}) \le 2^{-m-2}\delta$, for every $m < n$. It follows that $\nu(\inf_{i\in J} a_i \cap \sup_{m<n} a_{r_m}) \le \frac{1}{2}\delta$ and $\nu(\inf_{i\in J} a_i \setminus \sup_{m<n} a_{r_m}) \ge \frac{1}{2}\delta$. But this means that $\nu c_n \ge \frac{1}{2}\delta$ where $c_n = a_{r_n} \setminus \sup_{m<n} a_{r_m}$ for each n. As $\langle c_n \rangle_{n\in\mathbb{N}}$ is disjoint, this is impossible. **X**

Thus we can find $\gamma > 0$ and $S \in [M]^\omega$ such that $\nu(\inf_{i\in J} a_i) \ge \gamma$ for every $J \in [S]^{k+1}$, and the induction continues.

(b) Again induce on k. The cases $k = 0$, $k = 1$ are trivial. For the inductive step to $k+1 \ge 2$, write $c_J = \inf_{i\in J} a_i$ for $J \in [\kappa]^{<\omega}$. We know from the inductive hypothesis that there are a stationary set $S \subseteq \kappa$ and a $\delta > 0$ such that $\nu c_J \ge 3\delta$ for every $J \in [S]^k$. For each $\xi \in S$, choose $m(\xi) \in \mathbb{N}$ and $\langle J_{\xi i} \rangle_{i<m(\xi)}$ as follows. Given $\langle J_{\xi i} \rangle_{i<j}$, where $j \in \mathbb{N}$, choose, if possible, $J_{\xi j} \in [S \cap \xi]^k$ such that $\nu(c_{J_{\xi j}} \cap c_{J_{\xi i}}) \le 2^{-i}\delta$ for every $i < j$ and $\nu(a_\xi \cap c_{J_{\xi j}}) \le 2^{-j}\delta$; if this is not possible, set $m(\xi) = j$ and stop. Now the point is that we always do have to stop. **P?** Otherwise, set $d_i = c_{J_{\xi i}}$ for each $i \in \mathbb{N}$. Because $J_{\xi i} \in [S]^k$, $\nu d_i \ge 3\delta$ for each i; also $\nu(d_i \cap d_j) \le 2^{-i}\delta$ for $i < j$; so $\nu d'_j \ge \delta$, where $d'_j = d_j \setminus \sup_{i<j} d_i$ for each j. But now $\langle d'_j \rangle_{j\in\mathbb{N}}$ is disjoint and ν is not exhaustive. **XQ**

At the end of the process, we have $m(\xi)$ and $\langle J_{\xi i} \rangle_{i<m(\xi)}$ for each $\xi \in S$. By the Pressing-Down Lemma (4A1Cc), there are $\tilde{m}$ and $\langle \tilde{J}_i \rangle_{i<\tilde{m}}$ such that $S' = \{\xi : \xi \in S,\ m(\xi) = \tilde{m},\ J_{\xi i} = \tilde{J}_i$ for every $i < \tilde{m}\}$ is stationary in κ. **?** Suppose, if possible, that $I \in [S']^{k+1}$ and $\nu c_I \le 2^{-\tilde{m}}\delta$. Set $\xi = \max I$, $J = I \setminus \{\xi\}$, $\eta = \min I \in J$. Then $J \in [S \cap \xi]^k$. For each $i < \tilde{m} = m(\xi)$,

$$\nu(c_J \cap c_{J_{\xi i}}) \le \nu(a_\eta \cap c_{J_{\xi i}}) = \nu(a_\eta \cap c_{J_{\eta i}}) \le 2^{-i}\delta,$$

while

$$\nu(a_\xi \cap c_J) = \nu c_I \le 2^{-\tilde{m}}\delta.$$

But this means that we could have extended the sequence $\langle J_{\xi i} \rangle_{i<\tilde{m}}$ by setting $J_{\xi\tilde{m}} = J$. **X**

So S' and $2^{-\tilde{m}}\delta$ provide the next step in the induction.

(c) This is now immediate from (b).

539L I come now to the work of BALCAR JECH & PAZÁK 05, based on the characterizations of Maharam algebras set out in §393.

Lemma (QUICKERT 02) Let $\mathfrak{A}$ be a Boolean algebra, and $\mathcal{I}$ the family of countable subsets I of $\mathfrak{A}$ for which there is a partition C of unity such that $\{a : a \in I,\, a \cap c \neq 0\}$ is finite for every $c \in C$.

(a) $\mathcal{I}$ is an ideal of $\mathcal{P}\mathfrak{A}$ including $[\mathfrak{A}]^{<\omega}$.

(b) If $A \subseteq \mathfrak{A}^+$ is such that $A \cap I$ is finite for every $I \in \mathcal{I}$, and $B = \{b : b \supseteq a$ for some $a \in A\}$, then $B \cap I$ is finite for every $I \in \mathcal{I}$.

(c) If $\mathfrak{A}$ is ccc, then there is no uncountable $B \subseteq \mathfrak{A}$ such that $[B]^{\leq\omega} \subseteq \mathcal{I}$.

(d) If $\mathfrak{A}$ is ccc and weakly (σ,∞)-distributive, $\mathcal{I}$ is a p-ideal (definition: 5A6Ga).

proof (a) Of course every finite subset of $\mathfrak{A}$ belongs to $\mathcal{I}$. If I_0, $I_1 \in \mathcal{I}$ and $J \subseteq I_0 \cup I_1$, then $J \in [\mathfrak{A}]^{\leq\omega}$. For each j, we have a partition C_j of unity in $\mathfrak{A}$ such that $\{a : a \in I_j,\, a \cap c \neq 0\}$ is finite for every $c \in C_j$. Set $C = \{c_0 \cap c_1 : c_0 \in C_0,\, c_1 \in C_1\}$; then C is a partition of unity in $\mathfrak{A}$ and $\{a : a \in J,\, a \cap c \neq 0\}$ is finite for every $c \in C$.

(b) Take $I \in \mathcal{I}$. Set $J = B \cap I$. For each $b \in J$, let $a_b \in A$ be such that $a_b \subseteq b$. Let C be a partition of unity such that $\{b : b \in I,\, b \cap c \neq 0\}$ is finite for every $c \in C$; then $\{a_b : b \in J,\, a_b \cap c \neq 0\}$ is finite for every $c \in C$, so $\{a_b : b \in J\}$ belongs to $\mathcal{I}$ and must be finite. **?** If J is infinite, there is an $a \in A$ such that $K = \{b : b \in J,\, a = a_b\}$ is infinite; but in this case there is a $c \in C$ such that $a \cap c \neq 0$ and $b \cap c \neq 0$ for every $b \in K$. **X** So J is finite, as claimed.

(c) Let $\widehat{\mathfrak{A}}$ be the Dedekind completion of $\mathfrak{A}$ (314U). Let $B \subseteq \mathfrak{A}$ be an uncountable set, and $\langle b_\xi \rangle_{\xi<\omega_1}$ a family of distinct elements of B. Set $d = \inf_{\xi<\omega_1} \sup_{\xi\leq\eta<\omega_1} b_\eta$, taken in $\widehat{\mathfrak{A}}$. Then (because $\widehat{\mathfrak{A}}$ is ccc, by 514Ee) $d = \sup_{\xi\leq\eta<\omega_1} b_\eta$ for some ξ (316E); in particular, $d \neq 0$. Next, we can find a strictly increasing sequence $\langle \xi_n \rangle_{n\in\mathbb{N}}$ in ω_1 such that $d \subseteq \sup_{\xi_n\leq\eta<\xi_{n+1}} b_\eta$ for every $n \in \mathbb{N}$. Set $I = \{b_\eta : \eta < \sup_{n\in\mathbb{N}} \xi_n\} \in [B]^{\leq\omega}$. If C is any partition of unity in $\mathfrak{A}$, there must be some $c \in C$ such that $c \cap d \neq 0$, and now $\{a : a \in I,\, a \cap c \neq 0\}$ is infinite. So $I \notin \mathcal{I}$.

(d) Let $\langle I_n \rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{I}$. For each $n \in \mathbb{N}$, let C_n be a partition of unity such that $\{a : a \in I_n,\, a \cap c \neq 0\}$ is finite for every $c \in C_n$. Let D be a partition of unity such that $\{c : c \in C_n,\, c \cap d \neq 0\}$ is finite for every $d \in D$ and $n \in \mathbb{N}$. Then

$$\{a : a \in I_n,\, a \cap d \neq 0\} \subseteq \bigcup_{c\in C_n, c\cap d\neq 0} \{a : a \in I_n,\, a \cap c \neq 0\}$$

is finite for every $d \in D$ and $n \in \mathbb{N}$. Let $\langle d_n \rangle_{n\in\mathbb{N}}$ be a sequence running over $D \cup \{\emptyset\}$ and set $I = \bigcup_{n\in\mathbb{N}} \{a : a \in I_n,\, a \cap d_i = 0$ for every $i \leq n\}$. Then

$$I_n \setminus I \subseteq \bigcup_{i\leq n} \{a : a \in I_n,\, a \cap d_i \neq \emptyset\}$$

is finite for each n. Also

$$\{a : a \in I,\, a \cap d_n \neq 0\} \subseteq \bigcup_{i<n} \{a : a \in I_i,\, a \cap d_n \neq 0\}$$

is finite for each n, so $I \in \mathcal{I}$.

Remark In this context, $\mathcal{I}$ is called **Quickert's ideal**.

539M Lemma Let $\mathfrak{A}$ be a weakly (σ,∞)-distributive ccc Dedekind σ-complete Boolean algebra, and suppose that $\mathfrak{A}^+$ is expressible as $\bigcup_{k\in\mathbb{N}} D_k$ where no infinite subset of any D_k belongs to Quickert's ideal $\mathcal{I}$. Then $\mathfrak{A}$ is a Maharam algebra.

proof The point is that if $\langle a_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}$ which order*-converges to 0, then $\{a_n : n \in \mathbb{N}\} \in \mathcal{I}$ (539A(d-i)). So no sequence in any D_k can order*-converge to 0. Because $\mathfrak{A}$ is weakly (σ,∞)-distributive and ccc, 0 does not belong to the closure $\overline{D}_k$ of D_k for the order-sequential topology on $\mathfrak{A}$ (539A(d-iv)). So $\mathfrak{A}^+ = \bigcup_{k\in\mathbb{N}} \overline{D}_k$ is F_σ and $\{0\}$ is G_δ for the order-sequential topology. It follows that $\mathfrak{A}$ is a Maharam algebra (539A(d-vi)).

539N Theorem (BALCAR JECH & PAZÁK 05, VELIČKOVIĆ 05) Suppose that Todorčević's p-ideal dichotomy (5A6Gb) is true. Then every Dedekind σ-complete ccc weakly (σ,∞)-distributive Boolean algebra is a Maharam algebra.

proof Let $\mathfrak{A}$ be a Dedekind σ-complete ccc weakly (σ,∞)-distributive Boolean algebra. Let $\mathcal{I}$ be Quickert's ideal on $\mathfrak{A}$; then $\mathcal{I}$ is a p-ideal (539Ld). By 539Lc, there is no $B \in [\mathfrak{A}]^{\omega_1}$ such that $[B]^{\le\omega} \subseteq \mathcal{I}$. We are assuming that Todorčević's p-ideal dichotomy is true; so $\mathfrak{A}$ must be expressible as $\bigcup_{n\in\mathbb{N}} D_n$ where no infinite subset of any D_n belongs to $\mathcal{I}$. By 539M, $\mathfrak{A}$ is a Maharam algebra.

539O Corollary Suppose that Todorčević's p-ideal dichotomy is true. Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra such that every countably generated order-closed subalgebra of $\mathfrak{A}$ is a measurable algebra. Then $\mathfrak{A}$ is a measurable algebra.

proof (a) $\mathfrak{A}$ is ccc. **P?** Otherwise, let $\langle a_\xi\rangle_{\xi<\omega_1}$ be a disjoint family of non-zero elements of $\mathfrak{A}$. Let $f:\omega_1\to\{0,1\}^{\mathbb{N}}$ be an injective function, and set $b_n=\sup\{a_\xi:\xi<\omega_1,\ f_\xi(n)=1\}$ for each n; let $\mathfrak{B}$ be the order-closed subalgebra of $\mathfrak{A}$ generated by $\{b_n:n\in\mathbb{N}\}\cup\{\sup_{\xi<\omega_1}a_\xi\}$. Then $a_\xi\in\mathfrak{B}$ for every $\xi<\omega_1$, so $\mathfrak{B}$ is not ccc; but $\mathfrak{B}$ is supposed to be measurable. **XQ**

(b) $\mathfrak{A}$ is weakly (σ,∞)-distributive. **P** Let $\langle C_n\rangle_{n\in\mathbb{N}}$ be a sequence of partitions of unity in $\mathfrak{A}$. As $\mathfrak{A}$ is ccc, every C_n is countable; let $\mathfrak{B}$ be the order-closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{n\in\mathbb{N}}C_n$. Then $\mathfrak{B}$ is measurable, therefore weakly (σ,∞)-distributive, and there is a partition D of unity in $\mathfrak{B}$ such that $\{c:c\in C_n,\ c\cap d\ne 0\}$ is finite for every $n\in\mathbb{N}$ and $d\in D$. As $\mathfrak{B}$ is order-closed, D is still a partition of unity in $\mathfrak{A}$. As $\langle C_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathfrak{A}$ is weakly (σ,∞)-distributive. **Q**

(c) By 539N, $\mathfrak{A}$ is a Maharam algebra; let ν be a strictly positive Maharam submeasure on $\mathfrak{A}$. Now ν is uniformly exhaustive. **P?** Otherwise, there are $\epsilon>0$ and a family $\langle a_{ni}\rangle_{i\le n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $\langle a_{ni}\rangle_{i\le n}$ is disjoint for every $n\in\mathbb{N}$ and $\nu a_{ni}\ge\epsilon$ whenever $i\le n\in\mathbb{N}$. Let $\mathfrak{B}$ be the order-closed subalgebra of $\mathfrak{A}$ generated by $\{a_{ni}:i\le n\in\mathbb{N}\}$. Then $\mathfrak{B}$ is a measurable algebra; let $\bar\mu$ be a functional such that $(\mathfrak{B},\bar\mu)$ is a totally finite measure algebra. Since $\bar\mu$ and $\nu\restriction B$ are both strictly positive Maharam submeasures on $\mathfrak{B}$, ν is absolutely continuous with respect to $\bar\mu$ (539Ac). But $\nu a_{ni}\ge\epsilon$ for every n and i, while $\inf_{i\le n\in\mathbb{N}}\bar\mu a_{ni}$ must be zero. **XQ**

(d) So $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra with a strictly positive uniformly exhaustive Maharam submeasure, and is a measurable algebra (539Ab).

539P I should say at once that 539N-539O really do need some special axiom. In fact the following example was found at the very beginning of the study of Maharam algebras.

Souslin algebras: Proposition Suppose that T is a well-pruned Souslin tree (554Yc, 5A1Ed), and set $\mathfrak{A}=\mathrm{RO}^{\uparrow}(T)$.

(a) $\mathfrak{A}$ is Dedekind complete, ccc and weakly (σ,∞)-distributive.

(b) If $\mathfrak{B}$ is an order-closed subalgebra of $\mathfrak{A}$ and $\tau(\mathfrak{B})\le\omega$, then $\mathfrak{B}\cong\mathcal{P}I$ for some countable set I; in particular, $\mathfrak{B}$ is a measurable algebra.

(c) (Maharam 1947) The only Maharam submeasure on $\mathfrak{A}$ is identically zero.

proof (a)(i) $\mathfrak{A}$ is Dedekind complete just because it is a regular open algebra.

(ii) T is upwards-ccc, so $\mathfrak{A}$ is ccc, by 514Nc.

(iii) For $t\in T$, set $\widehat{t}=\operatorname{int}\overline{[t,\infty[}\in\mathfrak{A}$; then $\{\widehat{t}:t\in T\}$ is order-dense in $\mathfrak{A}$. Let $r:T\to\mathrm{On}$ be the rank function of T (5A1Ea). For each $\xi<\omega_1$, $A_\xi=\{\widehat{t}:t\in T,\ r(t)=\xi\}$ is a partition of unity in $\mathfrak{A}$. **P** If $r(t)=r(t')$ and $t\ne t'$ then $[t,\infty[\,\cap\,[t',\infty[\,=\emptyset$ so $\widehat{t}\cap\widehat{t'}=0$ in $\mathfrak{A}$; thus A_ξ is disjoint. If $a\in\mathfrak{A}\setminus\{0\}$, there is an $s\in T$ such that $\widehat{s}\subseteq a$; if $r(s)\ge\xi$, there is a $t\le s$ such that $r(t)=\xi$, and $a\cap\widehat{t}\ne 0$; if $r(s)<\xi$, there is a $t\ge s$ such that $r(t)=\xi$ (because T is well-pruned), and $\widehat{t}\subseteq a$. Thus $\sup A_\xi=1$ in $\mathfrak{A}$. **Q**

If $A\subseteq\mathfrak{A}$ is a partition of unity, there is a $\xi<\omega_1$ such that A_ξ refines A in the sense that every member of A_ξ is included in some member of A (see 311Ge). **P** $B=\{\widehat{t}:t\in T,\ \widehat{t}\subseteq a$ for some $a\in A\}$ is order-dense in $\mathfrak{A}$, so there is a partition C of unity included in B; C is countable; let $D\subseteq T$ be a countable set such that $C=\{\widehat{t}:t\in D\}$; set $\xi=\sup_{t\in D}r(t)$. **Q**

Of course A_η refines A_ξ whenever $\xi\le\eta<\omega_1$. So if $\langle C_n\rangle_{n\in\mathbb{N}}$ is a sequence of partitions of unity in $\mathfrak{A}$, there is a $\xi<\omega_1$ such that A_ξ refines C_n for every $n\in\mathbb{N}$, and then $\{c:c\in C_n,\ a\cap c\ne 0\}$ has just one member for every $a\in A_\xi$ and $n\in\mathbb{N}$. As $\langle C_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathfrak{A}$ is weakly (σ,∞)-distributive.

(b) If $B \subseteq \mathfrak{A}$ is a countable set τ-generating $\mathfrak{B}$, there is a countable set $D \subseteq T$ such that $b = \sup\{\widehat{t} : t \in D, \widehat{t} \subseteq b\}$ for every $b \in B$. Now $\xi = \sup\{r(t) : t \in D\}$ is countable, and $b = \sup\{a : a \in A_\xi, a \subseteq b\}$ for every $b \in B$, so $\mathfrak{B}$ is included in the order-closed subalgebra $\mathfrak{C}$ of $\mathfrak{A}$ generated by A_ξ. Of course A_ξ is order-dense in $\mathfrak{C}$. For $a \in A_\xi$, set $b_a = \inf\{b : b \in \mathfrak{B}, b \supseteq a\}$; then every b_a is an atom in $\mathfrak{B}$ and $\{b_a : a \in A_\xi\}$ is order-dense in $\mathfrak{B}$, so $\mathfrak{B}$ is purely atomic. As $\mathfrak{B}$ is ccc, the set I of its atoms is countable; being Dedekind complete, $\mathfrak{B}$ is isomophic to $\mathcal{P}I$.

(c) Let ν be a Maharam submeasure on $\mathfrak{A}$. Then for every $\epsilon > 0$ there is a $\xi < \omega_1$ such that $\nu a \le \epsilon$ for every $a \in A_\xi$. **P** Set

$$T' = \{t : \nu\widehat{t} \ge \epsilon\}.$$

Then T' is a subtree of T and $\{t : t \in T', r(t) = \xi\}$ is finite for every $\xi < \omega_1$, because ν is exhaustive. Also T', like T, can have no uncountable branches. It follows that the height of T' is countable (5A1E(b-i)), that is, that there is a $\xi < \omega_1$ such that $r(t) < \xi$ for every $t \in T'$ and $\nu a \le \epsilon$ for every $a \in A_\xi$. **Q**

As this is true for every $\epsilon > 0$, there is actually a $\xi < \omega_1$ such that $\nu a = 0$ for every $a \in A_\xi$. But as A_ξ is a countable partition of unity and ν is a Maharam submeasure, $\nu 1 = 0$ and ν is identically zero.

539Q Reflection principles In 539O, we have a theorem of the type 'if every small subalgebra of $\mathfrak{A}$ is ..., then $\mathfrak{A}$ is ...'. There was a similar result in 518I, and we shall have another in 545G. Here I collect some simple facts which are relevant to the present discussion.

(a) If $\mathfrak{A}$ is a Boolean algebra and every subset of $\mathfrak{A}$ of cardinal ω_1 is included in a ccc subalgebra of $\mathfrak{A}$, then $\mathfrak{A}$ is ccc. (For there can be no disjoint set with cardinal ω_1.)

(b) If $\mathfrak{A}$ is ccc and every countable subset of $\mathfrak{A}$ is included in a weakly (σ,∞)-distributive subalgebra of $\mathfrak{A}$, then $\mathfrak{A}$ is weakly (σ,∞)-distributive. **P** If C_n is a partition of unity in $\mathfrak{A}$ for every n, set

$$D = \{d : \{c : c \in C_n, c \cap d \ne 0\} \text{ is finite for every } n \in \mathbb{N}\}.$$

? If D is not order-dense in $\mathfrak{A}$, take $a \in \mathfrak{A}^+$ such that $d \not\subseteq a$ for every $d \in D$. Let $\mathfrak{B}$ be a weakly (σ,∞)-distributive subalgebra of $\mathfrak{A}$ including $\{a\} \cup \bigcup_{n\in\mathbb{N}} C_n$. Then every C_n is a partition of unity in $\mathfrak{B}$, so there is a partition B of unity in $\mathfrak{B}$ such that $B \subseteq D$. But now $a \in \mathfrak{B}^+$ so there is a $b \in B$ such that $a \cap b \ne 0$ and $a \cap b \in D$. **X**

So D is order-dense in $\mathfrak{A}$ and includes a partition of unity in $\mathfrak{A}$. As $\langle C_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathfrak{A}$ is weakly (σ,∞)-distributive. **Q**

(c) If every countable subset of $\mathfrak{A}$ is included in a subalgebra of $\mathfrak{A}$ with the σ-interpolation property, then $\mathfrak{A}$ has the σ-interpolation property. **P** If A, $B \subseteq \mathfrak{A}$ are countable and $a \subseteq b$ whenever $a \in A$ and $b \in B$, let $\mathfrak{B}$ be a subalgebra of $\mathfrak{A}$, including $A \cup B$, with the σ-interpolation property; then there is a $c \in \mathfrak{B}$ such that $a \subseteq c \subseteq b$ for every $a \in A$ and $b \in B$. **Q**

(d) If $\mathfrak{A}$ is a Maharam algebra and every countably generated closed subalgebra of $\mathfrak{A}$ is a measurable algebra, then $\mathfrak{A}$ is measurable. (This is part (c) of the proof of 539O.)

(e) Suppose that Todorčević's p-ideal dichotomy is true. Let $\mathfrak{A}$ be a Boolean algebra such that every subset of $\mathfrak{A}$ of cardinal at most ω_1 is included in a subalgebra of $\mathfrak{A}$ which is a Maharam algebra. Then $\mathfrak{A}$ is a Maharam algebra. **P** By (a), $\mathfrak{A}$ is ccc; by (c), $\mathfrak{A}$ is Dedekind complete; by (b), $\mathfrak{A}$ is weakly (σ,∞)-distributive; by 539N, $\mathfrak{A}$ is a Maharam algebra. **Q**

(f) Suppose that Todorčević's p-ideal dichotomy is true. Let $\mathfrak{A}$ be a Boolean algebra such that every subset of $\mathfrak{A}$ of cardinal at most $\mathfrak{c}$ is included in a subalgebra of $\mathfrak{A}$ which is a measurable algebra. Then $\mathfrak{A}$ is measurable. **P** By (a), $\mathfrak{A}$ is ccc. So if $\mathfrak{B}$ is a countably generated order-closed subalgebra, it has cardinal $\mathfrak{c}$, and is included in a measurable subalgebra $\mathfrak{C}$ of $\mathfrak{A}$. Now $\mathfrak{B}$ is order-closed in $\mathfrak{C}$, so is itself a measurable algebra. By 539O, $\mathfrak{A}$ also is measurable. **Q**

(g) On the other hand, FARAH & VELIČKOVIĆ 06 show that if κ is an infinite cardinal such that $2^\kappa = \kappa^+$, $\square_\kappa$ (5A6D) is true and the cardinal power κ^ω is equal to κ, then there is a Dedekind complete Boolean algebra $\mathfrak{A}$, with cardinal κ^+, such that every order-closed subalgebra of $\mathfrak{A}$ with cardinal at most κ is a measurable algebra, but $\mathfrak{A}$ is not a measurable algebra (and therefore is not a Maharam algebra, by (d) above). In particular, this can easily be the case with $\kappa = \mathfrak{c}$.

539R Exhaustivity rank While we now know that there are non-measurable Maharam algebras, we know practically nothing about their structure. The following idea is one tool for investigation.

Definitions Suppose that $\mathfrak{A}$ is a Boolean algebra and ν an exhaustive submeasure on $\mathfrak{A}$. For $\epsilon > 0$, say that $a \preccurlyeq_\epsilon b$ if either $a = b$ or $a \subseteq b$ and $\nu(b \setminus a) > \epsilon$. Then $\preccurlyeq_\epsilon$ is a well-founded partial order on $\mathfrak{A}$ (use 5A1Dc; if $\langle a_n\rangle_{n\in\mathbb{N}}$ were strictly decreasing for $\preccurlyeq_\epsilon$, then $\langle a_n \setminus a_{n+1}\rangle_{n\in\mathbb{N}}$ would be disjoint, with $\nu(a_n \setminus a_{n+1}) \ge \epsilon$ for every n). Let $r_{\nu\epsilon} : \mathfrak{A} \to \mathrm{On}$ be the corresponding rank function, so that

$$r_{\nu\epsilon}(a) = \sup\{r_{\nu\epsilon}(b) + 1 : b \subseteq a,\, \nu(a \setminus b) > \epsilon\}$$

for every $a \in \mathfrak{A}$ (5A1Db). Now the **exhaustivity rank** of ν is $\sup_{\epsilon>0} r_{\nu\epsilon}(1)$.

539S Elementary facts Let $\mathfrak{A}$ be a Boolean algebra with an exhaustive submeasure ν and associated rank functions $r_{\nu\epsilon}$ for $\epsilon > 0$.

(a) $r_{\nu\delta}(a) \le r_{\nu\epsilon}(b)$ whenever $\nu(a \setminus b) \le \delta - \epsilon$. **P** Induce on $r_{\nu\epsilon}(b)$. If $r_{\nu\epsilon}(b) = 0$, then $\nu b \le \epsilon$ so $\nu a \le \delta$ and $r_{\nu\delta}(a) = 0$. For the inductive step to $r_{\nu\epsilon}(b) = \xi$, if $c \subseteq a$ and $\nu(a \setminus c) > \delta$ then $\nu(b \setminus c) > \epsilon$ and $r_{\nu\epsilon}(b \cap c) < \xi$. Also $\nu(c \setminus b) \le \delta - \epsilon$ so, by the inductive hypothesis, $r_{\nu\delta}(c) \le r_{\nu\delta}(b \cap c) < \xi$; as c is arbitrary, $r_{\nu\delta}(a) \le \xi$ and the induction continues. **Q** In particular,

$$r_{\nu\epsilon}(a) \le r_{\nu\epsilon}(b) \text{ if } a \subseteq b, \quad r_{\nu\delta}(a) \le r_{\nu\epsilon}(a) \text{ if } \epsilon \le \delta.$$

(b) If a, $b \in \mathfrak{A}$ are disjoint and $\epsilon > 0$, then $r_{\nu\epsilon}(a \cup b)$ is at least the ordinal sum $r_{\nu\epsilon}(a) + r_{\nu\epsilon}(b)$. **P** Induce on $r_{\nu\epsilon}(b)$. If $r_{\nu\epsilon}(b) = 0$, the result is immediate from (a) above. For the inductive step to $r_{\nu\epsilon}(b) = \xi$, we have for any $\eta < \xi$ a $c \subseteq b$ such that $\nu(b \setminus c) > \epsilon$ and $\eta \le r_{\nu\epsilon}(c) < \xi$. Now $r_{\nu\epsilon}(a \cup c) \ge r_{\nu\epsilon}(a) + \eta$, by the inductive hypothesis, and $\nu((a \cup b) \setminus (a \cup c)) > \epsilon$, so $r_{\nu\epsilon}(a \cup b) > r_{\nu\epsilon}(a) + \eta$; as η is arbitrary, $r_{\nu\epsilon}(a \cup b) \ge r_{\nu\epsilon}(a) + \xi$ and the induction continues. **Q**

539T The rank of a Maharam algebra (a) Note that the rank function $r_{\nu\epsilon}$ associated with an exhaustive submeasure ν depends only on the set $\{a : \nu a > \epsilon\}$. In particular, if ν and ν' are exhaustive submeasures on a Boolean algebra $\mathfrak{A}$ and $\nu a \le \epsilon$ whenever $\nu' a \le \delta$, then $r_{\nu\epsilon}(a) \le r_{\nu'\delta}(a)$ for every $a \in \mathfrak{A}$. If $\mathfrak{A}$ is a Maharam algebra, then any two Maharam submeasures on $\mathfrak{A}$ are mutually absolutely continuous (539Ac), so have the same exhaustivity rank; I will call this the **Maharam submeasure rank** of $\mathfrak{A}$, $\mathrm{Mhsr}(\mathfrak{A})$. Note that if $a \in \mathfrak{A}$ then $\mathrm{Mhsr}(\mathfrak{A}_a) \le \mathrm{Mhsr}(\mathfrak{A})$.

(b) If $\mathfrak{A}$ is a measurable algebra, $\mathrm{Mhsr}(\mathfrak{A}) \le \omega$, because if μ is an additive functional and $\epsilon > 0$, then $\mu a > \epsilon r_{\mu\epsilon}(a)$ for every $a \in \mathfrak{A}$. More generally, for any uniformly exhaustive submeasure ν and $\epsilon > 0$, $r_{\nu\epsilon}(a)$ is finite, being the maximal size of any disjoint set consisting of elements, included in a, of submeasure greater than ϵ.

(c)(i) Suppose that $\mathfrak{A}$ is a Maharam algebra with a strictly positive Maharam submeasure ν, and that $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$ which is dense for the Maharam-algebra topology of $\mathfrak{A}$. For $\epsilon > 0$, write $r_\epsilon = r_{\nu\epsilon}$ for the corresponding rank function on $\mathfrak{A}$, and $r'_\epsilon = r_{\nu\restriction\mathfrak{B},\epsilon}$ for the rank function on $\mathfrak{B}$ corresponding to the exhaustive submeasure $\nu\restriction\mathfrak{B}$. If $0 < \delta < \epsilon$, $a \in \mathfrak{A}$, $b \in \mathfrak{B}$, $\xi \in \mathrm{On}$, $\nu(a \mathbin{\triangle} b) < \epsilon - \delta$ and $r_\epsilon(a) \ge \xi$, then $r'_\delta(b) \ge \xi$. **P** Induce on ξ. If $\xi = 0$ the result is trivial. For the inductive step to $\xi > 0$, take any $\eta < \xi$. Then we have an $a' \subseteq a$ such that $\nu(a \setminus a') > \epsilon$ and $r_\epsilon(a') > \eta$. Let $b' \in \mathfrak{B}$ be such that $\nu(a' \mathbin{\triangle} b') < \epsilon - \delta - \nu(a \mathbin{\triangle} b)$ and consider $b \cap b'$. We have

$$\nu(a' \mathbin{\triangle} (b \cap b')) = \nu((a \cap a') \mathbin{\triangle} (b \cap b')) \le \nu(a \mathbin{\triangle} b) + \nu(a' \mathbin{\triangle} b') < \epsilon - \delta$$

so $r'_\delta(b \cap b') > \eta$, by the inductive hypothesis. Moreover,

$$\begin{aligned}\nu(b \setminus (b \cap b')) = \nu(b \setminus b') &\ge \nu(a \setminus a') - \nu(a \setminus b) - \nu(b' \setminus b)\\ &> \epsilon - \nu(a \mathbin{\triangle} b) - \nu(a' \mathbin{\triangle} b') > \delta\end{aligned}$$

so $r'_\delta(b) \ge \eta + 1$. This is true for every $\eta < \xi$, so $r'_\delta(b) \ge \xi$. **Q**

(ii) It follows that if $\mathfrak{A}$ is an infinite Maharam algebra, then $\mathrm{Mhsr}(\mathfrak{A}) < \tau(\mathfrak{A})^+$. **P** $\mathfrak{A}$ has a dense subalgebra $\mathfrak{B}$ with cardinal $\tau = \tau(\mathfrak{A})$ (539B). If ν is a strictly positive Maharam submeasure on $\mathfrak{A}$, then

$\nu{\restriction}\mathfrak{B}$ is an exhaustive submeasure on $\mathfrak{B}$, so $r_{\nu\restriction\mathfrak{B},\delta}(1) < \tau^+$ for every $\delta > 0$, by 5A1Dd. By (i) here, $r_{\nu\epsilon}(1) < \tau^+$ for every $\epsilon > 0$. Since $\operatorname{cf}\tau^+ > \omega$, $\operatorname{Mhsr}(\mathfrak{A}) = \sup_{n\in\mathbb{N}} r_{\nu,2^{-n}}(1)$ is less than τ^+. **Q**

(d) The Maharam algebras described in §394 are all defined from exhaustive submeasures with domain the countable algebra $\mathfrak{B}$ of open-and-closed subsets of a compact metrizable space. By (c), such algebras must have Maharam submeasure rank less than ω_1.

539U Theorem Suppose that $\mathfrak{A}$ is a non-measurable Maharam algebra. Then $\operatorname{Mhsr}(\mathfrak{A})$ is at least the ordinal power ω^ω.

proof Let ν be a strictly positive Maharam submeasure on $\mathfrak{A}$.

(a) For the time being (down to the end of (d) below), assume that $\mathfrak{A}$ is nowhere measurable (definition: 391Bc). For $a \in \mathfrak{A}$, set

$$\check{\nu}a = \inf_{n\in\mathbb{N}} \sup\{\min_{i\le n} \nu a_i : a_0, \dots , a_n \subseteq a \text{ are disjoint}\}.$$

Then $\check{\nu}$ is a Maharam submeasure. **P** Of course $\check{\nu}0 = 0$ and $\check{\nu}a \le \check{\nu}b$ whenever $a \subseteq b$. If a, $b \in \mathfrak{A}$ and $\epsilon > 0$, then there are n_0, $n_1 \in \mathbb{N}$ such that whenever $\langle c_i\rangle_{i\in I}$ is a disjoint family in $\mathfrak{A}$, then $\#(\{i : \nu(c_i \cap a) \ge \check{\nu}a + \epsilon\}) \le n_0$ and $\#(\{i : \nu(c_i \cap b) \ge \check{\nu}b + \epsilon\}) \le n_1$. So

$$\#(\{i : \nu(c_i \cap (a \cup b)) \ge \check{\nu}a + \check{\nu}b + 2\epsilon\}) \le n_0 + n_1.$$

It follows that $\check{\nu}(a \cup b) \le \check{\nu}a + \check{\nu}b + 2\epsilon$; as ϵ, a and b are arbitrary, $\check{\nu}$ is a submeasure. Because $\check{\nu} \le \nu$, $\check{\nu}$ is a Maharam submeasure. **Q**

(b) Because $\mathfrak{A}$ is nowhere measurable, $\check{\nu}$ is strictly positive. **P** If $a \in \mathfrak{A} \setminus \{0\}$, the principal ideal $\mathfrak{A}_a$ is not measurable, so the Maharam submeasure $\nu{\restriction}\mathfrak{A}_a$ cannot be uniformly exhaustive; that is, there is an $\epsilon > 0$ such that there are arbitrarily long disjoint strings $\langle a_i\rangle_{i\le n}$ in $\mathfrak{A}_a$ with $\nu a_i \ge \epsilon$ for every $i \le n$. But this means that $\check{\nu}a \ge \epsilon > 0$. **Q**

(c) Let $r_{\nu\epsilon}$, $r_{\check{\nu}\epsilon}$ be the rank functions associated with ν and $\check{\nu}$. Then $r_{\nu\epsilon}(a)$ is at least the ordinal product $\omega \cdot r_{\check{\nu}\epsilon}(a)$ whenever $a \in \mathfrak{A}$ and $\epsilon > 0$. **P** Induce on $r_{\check{\nu}\epsilon}(a)$. If $r_{\check{\nu}\epsilon}(a) = 0$, the result is trivial. For the inductive step to $r_{\check{\nu}\epsilon}(a) = \xi + 1$, take $b \subseteq a$ such that $\check{\nu}b > \epsilon$ and $r_{\check{\nu}\epsilon}(a \setminus b) = \xi$. Then for every $n \in \mathbb{N}$ there are disjoint $b_0, \dots , b_n \subseteq b$ such that $\nu b_i > \epsilon$ for every i, and $r_{\nu\epsilon}(b) \ge \omega$; by the inductive hypothesis, $r_{\nu\epsilon}(a \setminus b) \ge \omega \cdot \xi$; by 539Sb, $r_{\nu\epsilon}(a) \ge \omega \cdot \xi + \omega = \omega \cdot (\xi + 1)$, and the induction proceeds. The inductive step to non-zero limit ξ is elementary. **Q**

(d) Now

$$\operatorname{Mhsr}(\mathfrak{A}) = \sup_{\epsilon>0} r_{\nu\epsilon}(1) \ge \sup_{\epsilon>0} \omega \cdot r_{\check{\nu}\epsilon}(1) = \omega \cdot \sup_{\epsilon>0} r_{\check{\nu}\epsilon}(1)$$

(5A1Bb)

$$= \omega \cdot \operatorname{Mhsr}(\mathfrak{A});$$

as $\operatorname{Mhsr}(\mathfrak{A}) > 0$, $\operatorname{Mhsr}(\mathfrak{A}) \ge \omega^\omega$ (5A1Bc).

(e) For the general case, let $a \in \mathfrak{A}^+$ be such that the principal ideal $\mathfrak{A}_a$ is nowhere measurable. Then $\operatorname{Mhsr}(\mathfrak{A}) \ge \operatorname{Mhsr}(\mathfrak{A}_a) \ge \omega^\omega$.

539V PV norms and exhaustivity (a) If we construct a submeasure ν on an algebra $\mathfrak{B}$ from a PV norm $\| \ \|$ on $[\mathbb{N}]^{<\omega}$ and sequences $\langle T_n\rangle_{n\in\mathbb{N}}$, $\langle \alpha_k\rangle_{k\in\mathbb{N}}$ and $\langle N_k\rangle_{k\in\mathbb{N}}$ as in 394B and 394H, we can relate the exhaustivity rank of ν to $\| \ \|$, as follows. Note first that the set $\mathcal{L} = \{L : L \in [\mathbb{N}]^{<\omega}, \nu L \le 1\}$, ordered by $\subseteq$, is a tree with no infinite branches, by the last clause of 394Aa. For $\mathcal{K} \subseteq [\mathbb{N}]^{<\omega}$, set $\partial\mathcal{K} = \{K \setminus \{\max K\} : \emptyset \ne K \in \mathcal{K}\}$; iterating as in 421N, set

$$\partial^0\mathcal{L} = \mathcal{L}, \quad \partial^\xi\mathcal{L} = \partial(\textstyle\bigcap_{\eta<\xi} \partial^\eta\mathcal{L})$$

for ordinals $\xi > 0$. Now observe that if $L \subset L' \in \mathcal{L}$ and $z \in \prod_{r\in L'} T_r$, then (at least if every T_r has at least two members) $Y_{z\restriction L} \setminus Y_z$ includes some $Y_{z'}$ where $z' \in \prod_{r\in L'} T_r$, so $\nu(Y_{z\restriction L} \setminus Y_z) \ge 8$ (394G) and $r_{\nu 1}(Y_{z\restriction L}) > r_{\nu 1}(Y_z)$. An easy induction now shows that $r_{\nu 1}(Y_z) \ge \xi$ whenever $L \in \partial^\xi\mathcal{L}$ and $z \in \prod_{r\in L} T_r$. So if $\emptyset \in \partial^\xi\mathcal{L}$ then $r_{\nu 1}(X) \ge \xi$.

(b) Moving to the Maharam algebra $\mathfrak{A} = \widehat{\mathfrak{B}}$ defined from ν, as in 394Nc, we see that $\mathfrak{A}$ has a strictly positive Maharam submeasure $\hat{\nu}$ extending ν, so that the same formulae, interpreted in $\mathfrak{A}$, tell us that $\mathrm{Mhsr}(\mathfrak{A}) \ge \xi$ whenever $\emptyset \in \partial^\xi \mathcal{L}$.

(c) The next step is to understand which families $\mathcal{L} \subseteq [\mathbb{N}]^{<\omega}$ can be expressed as $\{L : \|L\| \le 1\}$ for some PV norm $\| \ \|$. Looking through the definition in 394Aa, we see that we shall need, at least,

— $\{n\} \in \mathcal{L}$ for every $n \in \mathbb{N}$,

— $I \in \mathcal{L}$ whenever $J \in \mathcal{L}$ and $\#(I \cap n) \le \#(J \cap n)$ for every n,

— for every infinite $A \subseteq \mathbb{N}$ there is an $n \in \mathbb{N}$ such that $A \cap n \notin \mathcal{L}$.

Following Perović & Veličković 18, I will say that a family satisfying these three conditions is **admissible**. The point is that they are sufficient as well as necessary. **P** Given an admissible family $\mathcal{L} \subseteq [\mathbb{N}]^{<\omega}$, set

$$\|I\| = \min\{\#(\mathcal{L}_0) : \mathcal{L}_0 \subseteq \mathcal{L},\, I \subseteq \bigcup \mathcal{L}_0\}$$

for $I \in [\mathbb{N}]^{<\omega}$. Because $\mathcal{L}$ contains all singletons, $\|I\| \le \#(I)$ is always finite. $\|I\| = 0$ iff $I \subseteq \bigcup \emptyset$ iff $I = \emptyset$. If $\#(I) = 1$ then $I \in \mathcal{L}$ so $\|I\| \le 1$. Of course $\| \ \|$ is subadditive. If I, $J \in [\mathbb{N}]^{<\omega}$ and $\#(I \cap n) \le \#(J \cap n)$ for every n, there is an injective function $f : I \to J$ such that $f(i) \le i$ for every $i \in I$ (set $f(i) = \min(J \setminus f[I \cap i])$ for $i \in I$); now if $\mathcal{L}_0 \subseteq \mathcal{L}$ and $J \subseteq \bigcup \mathcal{L}_0$ then $f^{-1}[L] \in \mathcal{L}$ for every $L \in \mathcal{L}_0$ and $I \subseteq \bigcup_{L \in \mathcal{L}_0} f^{-1}[L]$. So $\|I\| \le \|J\|$. Finally, if $A \subseteq \mathbb{N}$ and $\|A \cap n\| \le m$ for every $n \in \mathbb{N}$, let $\langle L_{ni} \rangle_{n \in \mathbb{N}, i < m}$ be a family in $\mathcal{L}$ such that $A \cap n \subseteq \bigcup_{i<m} L_{ni}$ for every n. For $j \in A$ let $g_j : \mathbb{N} \to m$ be such that $j \in L_{n,g_j(n)}$ whenever $j < n$. Let $h : \mathbb{N} \to m$ be such that for every $k \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that $g_j(n) = h(j)$ for every $j \in A \cap k$, so that $j \in L_{n,h(j)}$ for $j \in A \cap k$, and $h^{-1}[\{l\}] \cap A \cap k$ is included in L_{nl} and belongs to $\mathcal{L}$. As k is arbitrary, $h^{-1}[\{l\}] \cap A$ must be finite; as l is arbitrary, A is finite.

Now we see that $\|I\| \le 1$ iff there is an $L \in \mathcal{L}$ including I, that is, iff $I \in \mathcal{L}$. So we have expressed $\mathcal{L}$ in the required form. **Q**

(d) For every $\xi < \omega_1$ there is an admissible family $\mathcal{L}_\xi \subseteq [\mathbb{N}]^{<\omega}$ such that $\emptyset \in \partial^\eta \mathcal{L}_\xi$ for every $\eta < \xi$. **P** Recall from 5A1Tb that there is a sequence $\langle \le_n \rangle_{n \in \mathbb{N}}$ of partial orders on ω_1 such that

$\langle \le_n \rangle_{n \in \mathbb{N}}$ is non-decreasing and $\bigcup_{n \in \mathbb{N}} \le_n$ is the usual ordering of ω_1,

if $\xi < \omega_1$ and $n \in \mathbb{N}$ then $\{\eta : \eta \le_n \xi\}$ is finite.

Define $\langle \mathcal{L}_\xi \rangle_{\xi < \omega_1}$ inductively by saying that $\mathcal{L}_0 = \{I : I \subseteq \mathbb{N},\, \#(I) \le 1\}$ and

$$\mathcal{L}_\xi = \mathcal{L}_0 \cup \bigcup_{\eta<\xi} \{I : I \in [\mathbb{N}]^{<\omega},\, \#(I) \ge 2,\, \eta \le_{\min I} \xi,\, I \setminus \{\min I\} \in \mathcal{L}_\eta\}$$

for $0 < \xi < \omega_1$. We see at once that $\mathcal{L}_0$ is admissible. Supposing that $\mathcal{L}_\eta$ is admissible and $\emptyset \in \partial^\zeta \mathcal{L}_\eta$ whenever $\zeta < \eta < \xi$, we need to check the following.

(i) $\{n\} \in \mathcal{L}_\xi$ for every $n \in \mathbb{N}$, because $\{n\} \in \mathcal{L}_0$.

(ii) If $J \in \mathcal{L}_\xi$ and $\#(I \cap n) \le \#(J \cap n)$ for every $n \in \mathbb{N}$, either $\#(I) \le 1$ and certainly $I \in \mathcal{L}_\xi$, or $\#(I) > 1$, $\#(J) > 1$ and $\min J \le \min I$. In this case, $\#((I \setminus \{\min I\}) \cap n) \le \#((J \setminus \{\min J\}) \cap n)$ for every $n \in \mathbb{N}$. Now there is an $\eta < \xi$ such that $\eta \le_{\min J} \xi$ and $J \setminus \{\min J\} \in \mathcal{L}_\eta$. Because $\mathcal{L}_\eta$ is admissible, $I \setminus \{\min I\} \in \mathcal{L}_\eta$; because $\min J \le \min I$, $\eta \le_{\min I} \xi$ and $I \in \mathcal{L}_\xi$.

(iii) If $A \subseteq \mathbb{N}$ is infinite, then $D = \{\eta : \eta \le_{\min A} \xi\}$ is finite. For each $\eta \in D$ there is an $n_\eta \in \mathbb{N}$ such that $(A \setminus \{\min A\}) \cap n_\eta \notin \mathcal{L}_\eta$. Setting $n = \max(\{1 + \min(A \setminus \{\min A\})\} \cup \{n_\eta : \eta \in D\}$, we see that $\#(A \cap n) \ge 2$ and $(A \cap n) \setminus \{\min(A \cap n)\} \notin \mathcal{L}_\eta$ for any $\eta \in D$, so $A \cap n \notin \mathcal{L}_\xi$.

(iv) Thus $\mathcal{L}_\xi$ is admissible. Now suppose that $\eta < \xi$.

(α) If $\eta \le 1$, we have $\{\emptyset, \{0\}\} \subseteq \mathcal{L}_\xi$ so $\emptyset \in \partial \mathcal{L}_\xi \subseteq \partial^\eta \mathcal{L}_\xi$.

(β) If $\eta \ge 2$ let n be such that $\eta \le_n \xi$ and consider $\mathcal{L} = \{\{n\} \cup (I + n + 1) : I \in \mathcal{L}_\eta\}$, where I write $I + n + 1$ for $\{i + n + 1 : i \in I\}$. Then $\mathcal{L} \subseteq \mathcal{L}_\xi$. An easy induction on ζ shows that

$$\partial^\zeta \mathcal{L}_\xi \supseteq \partial^\zeta \mathcal{L} \supseteq \{\{n\} \cup (I + n + 1) : I \in \partial^\zeta \mathcal{L}_\eta\}$$

for every ζ such that $\emptyset \in \partial^\zeta \mathcal{L}_\eta$, and in particular for every $\zeta < \eta$. So $\{n\} \in \bigcap_{\zeta<\eta} \partial^\zeta \mathcal{L}_\xi$ and $\emptyset \in \partial^\eta \mathcal{L}_\xi$.

(v) Inducing on ξ, we see that $\emptyset \in \partial^\eta \mathcal{L}_\xi$ whenever $\eta < \xi < \omega_1$. **Q**

(e) Putting these together, we see that if $\xi < \omega_1$ we have an admissible family $\mathcal{L}_{\xi+1}$ such that we can define a PV norm $\| \|_\xi$ from $\mathcal{L}_{\xi+1}$ as in (c), a submeasure ν_ξ on a countable atomless algebra $\mathfrak{B}$ from $\| \|_\xi$ as in 394H, and a Maharam algebra $\mathfrak{A}_\xi$ from ν_ξ as in 394Nc, in such a way that the exhaustivity rank of ν_ξ is at least ξ and $\mathrm{Mhsr}(\mathfrak{A}_\xi) \ge \xi$.

539W The set of exhaustive submeasures: Theorem Let $\mathfrak{C}$ be a countable atomless Boolean algebra, not $\{0\}$. Write M_{sm} for the set of totally finite submeasures on $\mathfrak{C}$, regarded as a subset of $[0,\infty[^{\mathfrak{C}}$, and M_{esm} for the set of exhaustive totally finite submeasures on $\mathfrak{C}$. Then M_{sm} is Polish, and $M_{\mathrm{esm}} \subseteq M_{\mathrm{sm}}$ is coanalytic and not Borel. Setting

$$F_\xi = \{\nu : \nu \in M_{\mathrm{esm}} \text{ has exhaustivity rank at most } \xi\}$$

for $\xi < \omega_1$, every F_ξ is a Borel subset of M_{sm} and every analytic subset of M_{esm} is included in some F_ξ.

proof (a) Directly from the definition in 539Aa, we see that M_{sm} is a closed subset of the Polish space $[0,\infty[^{\mathfrak{C}}$, and is itself Polish. Writing $D \subseteq \mathfrak{C}^{\mathbb{N}}$ for the set of infinite disjoint sequences in $\mathfrak{C}$, we see that

$$\{(\nu, d) : \nu d(n) \ge \epsilon \text{ for every } n \in \mathbb{N}\}$$

is closed in $M_{\mathrm{sm}} \times \mathfrak{C}^{\mathbb{N}}$ (if we give $\mathfrak{C}$ its discrete topology) for every ϵ, so that

$$\{\nu : \text{ there is some } d \in D \text{ such that } \nu(d(n)) \ge \epsilon \text{ for every } n \in \mathbb{N}\}$$

is analytic in M_{sm} for every ϵ (423B), and

$$\begin{aligned}\{\nu : \nu \in M_{\mathrm{sm}},\ \nu \text{ is not exhaustive}\}\\ = \bigcup_{k\in\mathbb{N}} \{\nu : \text{ there is some } d \in D \text{ such that } \nu(d(n)) \ge 2^{-k} \text{ for every } n \in \mathbb{N}\}\end{aligned}$$

is analytic (423B, 423E). Accordingly its complement in M_{sm}, the set of M_{esm} of exhaustive totally finite submeasures, is coanalytic.

(b) Define $\langle E_{a\epsilon\xi}\rangle_{a\in\mathfrak{C},\epsilon>0,\xi<\omega_1}$ by saying that

$$E_{a\epsilon 0} = \{\nu : \nu \in M_{\mathrm{sm}},\ \nu a \le \epsilon\},$$

$$E_{a\epsilon\xi} = \{\nu : \nu \in M_{\mathrm{sm}},\ \nu \in \textstyle\bigcup_{\eta<\xi} E_{b\epsilon\eta} \text{ whenever } b \subseteq a \text{ and } \nu(a \setminus b) > \epsilon\}$$

for $a \in \mathfrak{C}$, $\epsilon > 0$ and $0 < \xi < \omega_1$. Then every $E_{a\epsilon\xi}$ is a Borel subset of M_{sm}. **P** For $\xi = 0$ this is just because $\nu \mapsto \nu a : M_{\mathrm{sm}} \to [0,\infty[$ is continuous. For $\xi > 0$ we have

$$E_{a\epsilon\xi} = \bigcap_{\substack{b\in\mathfrak{C}\\ b\subseteq a}} \bigcup_{\eta<\xi} \{\nu : \nu(a \setminus b) \le \epsilon \text{ or } \nu \in E_{b\epsilon\eta}\}$$

which is Borel because $\mathfrak{C}$ is countable. **Q** Observe also that $E_{a\delta\xi} \subseteq E_{a\epsilon\xi}$ whenever $a \in \mathfrak{C}$, $0 < \delta \le \epsilon$ and $\xi < \omega_1$.

(c)(i) If $\nu \in M_{\mathrm{esm}}$, $a \in \mathfrak{C}$, $\epsilon > 0$ and $\xi < \omega_1$, then $r_{\nu\epsilon}(a) \le \xi$ iff $\nu \in E_{a\epsilon\xi}$. **P** Induce on ξ. For $\xi = 0$ we have

$$r_{\nu\epsilon}(a) = 0 \iff \nu a \le \epsilon \iff \nu \in E_{a\epsilon 0}.$$

For the inductive step to $\xi > 0$,

$$\begin{aligned} r_{\nu\epsilon}(a) \le \xi &\iff r_{\nu\epsilon}(b) < \xi \text{ whenever } b \subseteq a \text{ and } \nu(a \setminus b) > \epsilon\\ &\iff \nu \in \bigcup_{\eta<\xi} E_{b\epsilon\eta} \text{ whenever } b \subseteq a \text{ and } \nu(a \setminus b) > \epsilon\\ &\iff \nu \in E_{a\epsilon\xi}. \textbf{ Q}\end{aligned}$$

So for $\nu \in M_{\mathrm{esm}}$ and $\xi \le \omega_1$,

$$\nu \text{ has exhaustivity rank at most } \xi \iff \nu \in E_{1\epsilon\xi} \text{ for every } \epsilon > 0.$$

(ii) Next, if $\nu \in M_{\text{sm}}$ and $\xi < \omega_1$ are such that $\nu \in E_{1\epsilon\xi}$ for every $\epsilon > 0$, then ν is exhaustive. **P** **?** Otherwise, there are an $\epsilon > 0$ and a non-increasing sequence $\langle a_i \rangle_{i\in\mathbb{N}}$ such that $\nu(a_i \setminus a_{i+1}) > \epsilon$ for every $i \in \mathbb{N}$. Of course we can suppose that $a_0 = 0$. But now we find, inducing on η, that $\nu \notin E_{a_i\epsilon\eta}$ for every $i \in \mathbb{N}$ and $\eta < \omega_1$, which is impossible. **XQ**

(iii) Setting

$$F_\xi = \bigcap_{\epsilon>0} E_{1\epsilon\xi} = \bigcap_{k\in\mathbb{N}} E_{1,2^{-k},\xi}$$

for $\xi < \omega_1$, we see that F_ξ is a Borel subset of M_{sm} and is precisely the set of exhaustive submeasures on $\mathfrak{C}$ with exhaustivity rank at most ξ.

(d)(i) For $\epsilon > 0$, write W_ϵ for the set of triples (ν, ν', H) such that

ν, $\nu' \in M_{\text{sm}}$ and $H \subseteq \mathfrak{C}^2$.

$(1, 1) \in H$,

whenever $(a, b) \in H$ there is a $b' \subseteq b$ such that $\nu'(b \setminus b') > \epsilon$ and $(a', b') \in H$ whenever $a' \subseteq a$ and $\nu(a \setminus a') > \epsilon$.

Then W_ϵ is a Borel subset of $M_{\text{sm}} \times M_{\text{sm}} \times \mathcal{P}(\mathfrak{C}^2)$, where the power set $\mathcal{P}(\mathfrak{C}^2)$ is given its usual compact metrizable topology (4A2Ud). So $V_\epsilon = \{(\nu, \nu') :$ there is an H such that $(\nu, \nu', H) \in W_\epsilon\}$ is an analytic subset of M_{sm}^2.

(ii) If $\epsilon > 0$, ν, $\nu' \in M_{\text{esm}}$ and $(\nu, \nu', H) \in W_\epsilon$ then $r_{\nu\epsilon}(a) < r_{\nu'\epsilon}(b)$ whenever $(a, b) \in H$. **P** I show by induction on ξ that if $(a, b) \in H$ and $r_{\nu\epsilon}(a) \ge \xi$ then $r_{\nu'\epsilon}(b) > \xi$. **P** Induce on ξ. If $\xi = 0$ we know that there is a $b' \subseteq b$ such that $\nu'(b \setminus b) > \epsilon$ so $r_{\nu'\epsilon}(b) > 0$. For the inductive step to $\xi > 0$, we know that there is a $b' \subseteq b$ such that $\nu'(b \setminus b') > \epsilon$ and $(a', b') \in H$ whenever $a' \subseteq a$ and $\nu(a \setminus a') > \epsilon$. If $\eta < \xi$ then there is an $a' \subseteq a$ such that $\nu(a \setminus a') > \epsilon$ and $r_{\nu\epsilon}(a') \ge \eta$; now $(a', b') \in H$ so $r_{\nu'\epsilon}(b') > \eta$, by the inductive hypothesis. As η is arbitrary, $\xi \le r_{\nu'\epsilon}(b') < r_{\nu'\epsilon}(b)$. Thus the induction continues. **Q**

(iii) If $\epsilon > 0$ and ν, $\nu' \in M_{\text{esm}}$ then $(\nu, \nu') \in V_\epsilon$ iff $r_{\nu\epsilon}(1) < r_{\nu'\epsilon}(1)$. **P** If $(\nu, \nu') \in V_\epsilon$ there is an H such that $(\nu, \nu', H) \in W_\epsilon$; now $(1, 1) \in H$ so (ii) tells us that $r_{\nu\epsilon}(1) < r_{\nu'\epsilon}(1)$. If $r_{\nu\epsilon}(1) < r_{nu'\epsilon}(1)$ set $H = \{(a, b) : a, b \in \mathfrak{C}, r_{\nu\epsilon}(a) < r_{\nu'\epsilon}(b)\}$; then it is easy to check that $(\nu, \nu', H) \in W_\epsilon$, so $(\nu, \nu') \in V_\epsilon$. **Q**

(e) Now suppose that $A \subseteq M_{\text{esm}}$ is an analytic set, and that $\epsilon > 0$. Consider the relation $\preccurlyeq_\epsilon$ on A defined by saying that $\nu \preccurlyeq_\epsilon \nu'$ if either $\nu = \nu'$ or $r_{\nu\epsilon}(1) < r_{\nu'\epsilon}(1)$. This is a partial ordering, and it is well-founded because if $B \subseteq A$ is well-founded and $\min_{\nu\in B} r_{\nu\epsilon}(1) = \xi$ then any $\nu \in B$ such that $r_{\nu\epsilon}(1) = \xi$ is minimal in B. Now $\{(\nu, \nu') : \nu \prec_\epsilon \nu'\} = A^2 \cap V_\epsilon$, so by the Kunen-Martin theorem (5A1De) $\preccurlyeq_\epsilon$ has countable height. Since $\nu \prec_\epsilon \nu'$ whenever ν, $\nu' \in A$ and $r_{\nu\epsilon}(1) < r_{\nu'\epsilon}(1)$, $\{r_{\nu\epsilon}(1) : \nu \in A\}$ must be countable.

This is true for every $\epsilon > 0$, so $\zeta = \sup_{\nu\in A, k\in\mathbb{N}} r_{\nu,2^{-k}}(1)$ is less than ω_1. But now $A \subseteq F_\zeta$.

(f) Finally, no F_ζ can be the whole of M_{esm}. **P** We know from 539Ve that there are a countable atomless Boolean algebra $\mathfrak{B}$ and a totally finite exhaustive submeasure on $\mathfrak{B}$ with exhaustivity rank at least $\zeta + 1$. But $\mathfrak{C}$ and $\mathfrak{B}$ are isomorphic (316M) so the same is true of $\mathfrak{A}$, that is, $M_{\text{esm}} \setminus F_\zeta \ne \emptyset$. **Q** By (e) here, M_{esm} cannot be analytic, so cannot be a Borel subset of the Polish space M_{sm}.

Remark In the language of 423S, $\langle F_\xi \rangle_{\xi<\omega_1}$ is a family of Borel constituents of M_{ens}.

539X Basic exercises (a) Let $\mathfrak{A}$ be a Maharam algebra. Show that $\text{link}_n(\mathfrak{A}) \le \max(\omega, \tau(\mathfrak{A}))$ for every $n \ge 2$.

(b) Show that, in the language of §522, $\mathfrak{p} \le \mathfrak{s} \le \min(\text{non}\,\mathcal{N}, \text{non}\,\mathcal{M}, \mathfrak{d})$.

(c) Let $\mathfrak{A}$ be a Maharam algebra. (i) Show that if

(α) $\text{cf}[\lambda]^{\le\omega} \le \lambda^+$ for every cardinal $\lambda \le \tau(\mathfrak{A})$,

(β) $\square_\lambda$ is true for every uncountable cardinal $\lambda \le \tau(\mathfrak{A})$ of countable cofinality,

then $\text{FN}(\mathfrak{A}) \le \text{FN}(\mathcal{P}\mathbb{N})$, with equality unless $\mathfrak{A}$ is finite. (*Hint*: 518D, 518I.) (ii) Show that if $\#(\mathfrak{A}) \le \omega_2$ and $\text{FN}(\mathcal{P}\mathbb{N}) = \omega_1$, then $\mathfrak{A}$ is tightly ω_1-filtered. (*Hint*: 518M.)

(d) Let X be a set, Σ a σ-algebra of subsets of X, and $\nu : \Sigma \to [0, \infty[$ a non-zero Maharam submeasure; set $\mathcal{I} = \{E : E \in \Sigma, \nu E = 0\}$ and $\mathfrak{A} = \Sigma/\mathcal{I}$. Suppose that $\#(\mathfrak{A}) \le \omega_2$ and $\text{FN}(\mathcal{P}\mathbb{N}) = \omega_1$. Show that there is a lifting for ν, that is, a Boolean homomorphism $\theta : \mathfrak{A} \to \Sigma$ such that $(\theta a)^\bullet = a$ for every $a \in \mathfrak{A}$. (*Hint*: 518L.)

(e) Let $\mathfrak{A}$ be a Boolean algebra, ν an exhaustive submeasure on $\mathfrak{A}$, and $\langle a_i\rangle_{i\in\mathbb{N}}$ a sequence in $\mathfrak{A}$ such that $\inf_{i\in\mathbb{N}}\nu a_i>0$. Let $\mathcal{F}$ be a Ramsey ultrafilter on $\mathbb{N}$. (i) Show that there is an $I\in\mathcal{F}$ such $\inf_{i,j\in I}\nu(a_i\cap a_j)>0$. (ii) Show that for every $k\in\mathbb{N}$ there is an $I\in\mathcal{F}$ such that $\inf\{\nu(\inf_{i\in K}a_i):K\in[I]^k\}>0$. (iii) Show that there is an $I\in\mathcal{F}$ such that $\{a_i:i\in I\}$ is centered. (*Hint*: 538Hc.)

(f) Let X be a set, Σ a σ-algebra of subsets of X, and $\mathcal{I}\lhd\mathcal{P}X$ a σ-ideal; suppose that $\Sigma/\Sigma\cap\mathcal{I}$ is ccc. Let Y be a set, T a σ-algebra of subsets of Y, and $\nu:\mathrm{T}\to[0,\infty[$ a Maharam submeasure; let $\mathcal{I}\ltimes\mathcal{N}(\nu)$ be the skew product. Show that $(\Sigma\widehat{\otimes}\mathrm{T})/(\Sigma\widehat{\otimes}\mathrm{T})\cap(\mathcal{I}\ltimes\mathcal{N}(\nu))$ is ccc. (*Hint*: 527L.)

539Y Further exercises (a) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra with a countable σ-generating set (331E), and ν a Maharam submeasure on $\mathfrak{A}$. Set $\mathcal{I}=\{a:\nu a=0\}$. Show that $\mathcal{I}\preccurlyeq_{\mathrm{T}}\mathcal{N}$.

(b) Let X be a set, Σ a σ-algebra of subsets of X, and $\mathcal{I}$ a proper σ-ideal of subsets of X generated by $\Sigma\cap\mathcal{I}$; let Σ_L be the algebra of Lebesgue measurable subsets of $\mathbb{R}$. Write $\mathfrak{A}$ for $\Sigma/\Sigma\cap\mathcal{I}$, $\mathcal{L}$ for $(\Sigma\widehat{\otimes}\Sigma_L)\cap(\mathcal{I}\ltimes\mathcal{N})$ and $\mathfrak{C}$ for $\Sigma\widehat{\otimes}\Sigma_L/\mathcal{L}$. (i) Show that $c(\mathfrak{C})=\max(\omega,c(\mathfrak{A}))$ and $\tau(\mathfrak{C})=\max(\omega,\tau(\mathfrak{A}))$. (ii) Show that $\mathfrak{C}$ is weakly (σ,∞)-distributive iff $\mathfrak{A}$ is. (iii) Show that $\mathfrak{C}$ is measurable iff $\mathfrak{A}$ is. (iv) Show that $\mathfrak{C}$ is a Maharam algebra iff $\mathfrak{A}$ is.

(c) Let $\mathfrak{A}$ be a Boolean algebra with a strictly positive Maharam submeasure $\hat\nu$, and $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$ which is dense for the associated metric (539Ac); set $\nu=\hat\nu\restriction\mathfrak{B}$, so that ν is an exhaustive submeasure on $\mathfrak{B}$. For $\epsilon>0$ let $r_{\nu\epsilon}:\mathfrak{B}\to\mathrm{On}$ and $r_{\hat\nu\epsilon}:\mathfrak{A}\to\mathrm{On}$ be the rank functions associated with ν and $\hat\nu$ respectively. Show that

$$r_{\nu\delta}(b)\le r_{\hat\nu\delta}(b)\le r_{\nu\epsilon}(b)$$

whenever $b\in\mathfrak{B}$ and $0<\epsilon<\delta$.

(e) (J.Kupka) Let ν be a totally finite submeasure on a Boolean algebra $\mathfrak{A}$, and set

$$\check\nu a=\inf_{n\in\mathbb{N}}\sup\{\min_{i\le n}\nu a_i:a_0,\dots,a_n\subseteq a\text{ are disjoint}\}.$$

for $a\in\mathfrak{A}$, as in the proof of 539U. Show that *either* $\check\nu\ge\frac{1}{3}\nu$ *or* there is a non-zero additive $\mu:\mathfrak{A}\to[0,\infty[$ such that $\mu a\le\nu a$ for every $a\in\mathfrak{A}$. (*Hint*: 392D.)

(f) Show that the exhaustive submeasures constructed by Talagrand's original method, as described in §394 with $\|I\|=\#(I)$ for $I\in[\mathbb{N}]^{<\omega}$, have exhaustivity rank at most the ordinal power ω^{ω^2}.

(g) Suppose that $\mathfrak{A}$ is a non-measurable Maharam algebra. Show that $\mathrm{Mhsr}(\mathfrak{A})=\omega\cdot\mathrm{Mhsr}(\mathfrak{A})$.

539Z Problems (a) Let ν be a non-zero totally finite Radon submeasure on a Hausdorff space X. Must there be a lifting for ν? that is, writing Σ for the domain of ν, must there be a Boolean homomorphism $\phi:\Sigma\to\Sigma$ such that $\nu(E\triangle\phi E)=0$ for every $E\in\Sigma$ and $\phi E=\emptyset$ whenever $\nu E=0$?

(b) Is there a Maharam algebra with uncountable Maharam submeasure rank?

539 Notes and comments During the growth of this treatise, the sections on Maharam submeasures were twice transformed by new discoveries, and I naturally hope that the work I have just presented will be similarly outdated before too long. In the pages above I have tried in the first place to show how the cardinal functions of chapters 51 and 52 can be applied in this more general context. With minor refinements of technique, we can go a fair way. Because we know we have at least two non-trivial atomless Maharam algebras of countable type, we are led to a more detailed analysis, as in 539Ca and 539J.

Equally instructive are the apparent limits to what the methods can achieve, which mostly point to remaining areas of obscurity. I say 'remaining'; but what is most conspicuous about the present situation is our nearly total ignorance concerning the structure of non-measurable Maharam algebras. The Talagrand-Perović-Veličković construction, as described in §394, gives us a family of such algebras, but so far we can answer hardly any of the most elementary questions about them (see 394Z).

The message of Balcar Jech & Pazák 05 is that a Dedekind complete, ccc, weakly (σ,∞)-distributive Boolean algebra is 'nearly' a Maharam algebra. Any further condition (e.g., the σ-finite chain condition, as in 393S) is likely to render it a Maharam algebra; and with a little help from an extra axiom of set theory, it is already necessarily a Maharam algebra (539N). Similarly, much of the work of the last sixty years on submeasures suggests that exhaustive submeasures are 'nearly' uniformly exhaustive, and that an extra condition (e.g., sub- or super-modularity) is enough to tip the balance (413Yh). At both boundaries, there are few examples to limit conjectures about further conditions on which such results might be based. Besides 539P and Talagrand's examples, we have a further important possibility of a not-quite-Maharam algebra in 555K below.

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