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Chapter 52

Cardinal functions of measure theory

From the point of view of this book, the most important cardinals are those associated with measures and measure algebras, especially, of course, Lebesgue measure and the usual measure ν_I of $\{0,1\}^I$. In this chapter I try to cover the principal known facts about these which are theorems of ZFC. I start with a review of the theory for general measure spaces in §521, including some material which returns to the classification scheme of Chapter 21, exploring relationships between (strict) localizability, magnitude and Maharam type. §522 examines Lebesgue measure and the surprising connexions found by BARTOSZYŃSKI 84 and RAISONNIER & STERN 85 between the cardinals associated with the Lebesgue null ideal and the corresponding ones based on the ideal of meager subsets of $\mathbb{R}$. §523 looks at the measures ν_I for uncountable sets I, giving formulae for the additivities and cofinalities of their null ideals, and bounds for their covering numbers, uniformities and shrinking numbers. Remarkably, these cardinals are enough to tell us most of what we want to know concerning the cardinal functions of general Radon measures and semi-finite measure algebras (§524). These three sections are heavily dependent on the Galois-Tukey connections and Tukey functions of §§512-513. Precalibers do not seem to fit into this scheme, and the relatively partial information I have is in §525. The second half of the chapter deals with special topics which can be approached with the methods so far developed. In §526 I return to the ideal of subsets of $\mathbb{N}$ with asymptotic density zero, seeking to locate it in the Tukey classification. Further σ-ideals which are of interest in measure theory are the 'skew products' of §527. In §528 I examine some interesting Boolean algebras, the 'amoeba algebras' first introduced by MARTIN & SOLOVAY 70, giving the results of TRUSS 88 on the connexions between different amoeba algebras and localization posets. Finally, in §529, I look at a handful of other structures, concentrating on results involving cardinals already described.

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521 Basic theory

In the first half of this section (down to 521L) I collect facts about the cardinal functions add, cf, non, cov, shr and shr$^+$ when applied to the null ideal $\mathcal{N}(\mu)$ of a measure μ, and also the π-weight of a measure. In particular I look at their relations with the constructions introduced earlier in this treatise: measure algebras and function spaces (521B), subspace measures (521F), direct sums (521G), inverse-measure-preserving functions and image measures (521H), products (521J), perfect measures (521K) and compact measures (521L). The list is long just because I have four volumes' worth of miscellaneous concepts to examine; nearly all the individual arguments are elementary.

In the second half of the section, I give a handful of easy results which may clarify some patterns from earlier volumes. In 521M-521P I look again at 'strict localizability' as considered in Chapter 21, importing the concept of 'magnitude' of a measure space from §332, hoping to throw light on the examples of §216. In 521E I consider the topological densities of measure algebras. In 521R-521S I explore possibilities for the 'countably separated' measure spaces of §§343-344, examining in particular their Maharam types. Finally, in 521T, I review some measures which arose in §464 while analyzing the L-space $\ell^\infty(I)^*$.

521A Proposition Let (X,Σ,μ) be a measure space.

(a) If $\mathcal{E}\subseteq\Sigma$ and $\#(\mathcal{E})<\operatorname{add}\mu$ then $\bigcup\mathcal{E}\in\Sigma$ and

$$\mu(\bigcup\mathcal{E})=\sup\{\mu(\bigcup\mathcal{E}_0):\mathcal{E}_0\subseteq\mathcal{E}\text{ is finite}\}.$$

(b) $\omega_1\le\operatorname{add}\mu\le\operatorname{add}\mathcal{N}(\mu)$.

(c) If μ is the measure defined by Carathéodory's method from an outer measure θ on X, then $\operatorname{add}\mu = \operatorname{add}\mathcal{N}(\mu)$.

(d) If μ is complete and locally determined, $\operatorname{add}\mu = \operatorname{add}\mathcal{N}(\mu)$.

521B Proposition Let (X,Σ,μ) be a measure space and $(\mathfrak{A},\bar\mu)$ its measure algebra.

(a) If $\mathcal{E}\subseteq\Sigma$ and $\#(\mathcal{E})<\operatorname{add}\mu$, then $(\bigcup\mathcal{E})^\bullet=\sup_{E\in\mathcal{E}}E^\bullet$ and $(X\cap\bigcap\mathcal{E})^\bullet=\inf_{E\in\mathcal{E}}E^\bullet$ in $\mathfrak{A}$.

(b) Suppose that $A\subseteq[-\infty,\infty]^X$ is a non-empty family of Σ-measurable functions with $\#(A)<\operatorname{add}\mu$, and that $g(x)=\sup_{f\in A}f(x)$ in $[-\infty,\infty]$ for every $f\in A$. Then g is Σ-measurable.

(c) Write $\mathcal{L}^0$ for the family of μ-virtually measurable real-valued functions defined almost everywhere in X, and L^0 for the corresponding space of equivalence classes, as in §241. Suppose that $A\subseteq\mathcal{L}^0$ is such that $0<\#(A)<\operatorname{add}\mu$ and $\{f^\bullet:f\in A\}$ is bounded above in L^0. Set $g(x)=\sup_{f\in A}f(x)$ whenever this is defined in $\mathbb{R}$; then $g\in\mathcal{L}^0$ and $g^\bullet=\sup_{f\in A}f^\bullet$ in L^0.

(d)(i) If, in (b), A consists of non-negative integrable functions and is upwards-directed, then $\int g\,d\mu=\sup_{f\in A}\int f d\mu$.

(ii) If, in (b), $f_1\wedge f_2=0$ a.e. for all distinct f_1, $f_2\in A$, then $\int g\,d\mu=\sum_{f\in A}\int f d\mu$.

521C Proposition Let (X,Σ,μ) be a measure space, and $A\subseteq X$.

(a) If $\gamma<\mu^*A$ there is a $B\subseteq A$ such that $\#(B)<\operatorname{shr}^+\mathcal{N}(\mu)$ and $\mu^*B>\gamma$.

(b) There is a $B\subseteq A$ such that $\#(B)\le\max(\omega,\operatorname{shr}\mathcal{N}(\mu))$ and $\mu^*B=\mu^*A$.

521D Proposition Let (X,Σ,μ) be a measure space and $(\mathfrak{A},\bar\mu)$ its measure algebra.

(a) $\pi(\mathfrak{A})\le\pi(\mu)\le\max(\pi(\mathfrak{A}),\operatorname{cf}\mathcal{N}(\mu))$.

(b) If $\mu X>0$, then $\operatorname{non}\mathcal{N}(\mu)\le\pi(\mu)$.

(c) If (X,Σ,μ) has locally determined negligible sets, then $\operatorname{shr}\mathcal{N}(\mu)\le\pi(\mu)$.

(d) Suppose that there is a topology $\mathfrak{T}$ on X such that $(X,\mathfrak{T},\Sigma,\mu)$ is a quasi-Radon measure space. Then, writing $\mathfrak{A}^+$ for $\mathfrak{A}\setminus\{0\}$, the partially ordered sets $(\Sigma\setminus\mathcal{N}(\mu),\supseteq)$ and $(\mathfrak{A}^+,\supseteq)$ are Tukey equivalent and $\pi(\mu)=\pi(\mathfrak{A})$.

521E Proposition Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra.

(a) Give $\mathfrak{A}$ its measure-algebra topology.

(i) If $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$, it is topologically dense iff it τ-generates $\mathfrak{A}$.

(ii) If $\mathfrak{A}$ is finite, then its topological density is $\#(\mathfrak{A})$; if $\mathfrak{A}$ is infinite, its topological density is equal to its Maharam type $\tau(\mathfrak{A})$.

(b) Let $\mathfrak{A}^f$ be the set of elements of $\mathfrak{A}$ with finite measure, with its strong measure-algebra topology. Then the topological density of $\mathfrak{A}^f$ is $\#(\mathfrak{A}^f)=\#(\mathfrak{A})$ if $\mathfrak{A}$ is finite, and $\max(c(\mathfrak{A}),\tau(\mathfrak{A}))$ if $\mathfrak{A}$ is infinite.

521F Proposition Let (X,Σ,μ) be a measure space, A a subset of X and μ_A the subspace measure on A.

(a) $\mathcal{N}(\mu_A)\preccurlyeq_{\mathrm{T}}\mathcal{N}(\mu)$, so $\operatorname{add}\mathcal{N}(\mu_A)\ge\operatorname{add}\mathcal{N}(\mu)$ and $\operatorname{cf}\mathcal{N}(\mu_A)\le\operatorname{cf}\mathcal{N}(\mu)$.

(b) $(A,\in,\mathcal{N}(\mu_A))\preccurlyeq_{\mathrm{GT}}(X,\in,\mathcal{N}(\mu))$, so $\operatorname{non}\mathcal{N}(\mu_A)\ge\operatorname{non}\mathcal{N}(\mu)$ and $\operatorname{cov}\mathcal{N}(\mu_A)\le\operatorname{cov}\mathcal{N}(\mu)$.

(c) $\operatorname{add}\mu_A\ge\operatorname{add}\mu$.

(d) $\operatorname{shr}\mathcal{N}(\mu_A)\le\operatorname{shr}\mathcal{N}(\mu)$ and $\operatorname{shr}^+\mathcal{N}(\mu_A)\le\operatorname{shr}^+\mathcal{N}(\mu)$.

(e) If either $A\in\Sigma$ or (X,Σ,μ) has locally determined negligible sets, $\pi(\mu_A)\le\pi(\mu)$.

(f) If μ_A is semi-finite, then $\tau(\mu_A)\le\tau(\mu)$.

521G Proposition Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a non-empty family of measure spaces with direct sum (X,Σ,μ). Then

$$\operatorname{add}\mathcal{N}(\mu)=\min_{i\in I}\operatorname{add}\mathcal{N}(\mu_i),\quad \operatorname{add}\mu=\min_{i\in I}\operatorname{add}\mu_i,$$

$$\operatorname{cov}\mathcal{N}(\mu)=\sup_{i\in I}\operatorname{cov}\mathcal{N}(\mu_i),\quad \operatorname{non}\mathcal{N}(\mu)=\min_{i\in I}\operatorname{non}\mathcal{N}(\mu_i),$$

$$\operatorname{shr}\mathcal{N}(\mu)=\sup_{i\in I}\operatorname{shr}\mathcal{N}(\mu_i),\quad \operatorname{shr}^+\mathcal{N}(\mu)=\sup_{i\in I}\operatorname{shr}^+\mathcal{N}(\mu_i),$$

$$\tau(\mu) \le \max(\omega, \sup_{i\in I} \tau(\mu_i), \min\{\lambda : \#(I) \le 2^\lambda\})$$

and $\pi(\mu)$ is the cardinal sum $\sum_{i\in I} \pi(\mu_i)$. If I is finite, then

$$\mathrm{cf}\,\mathcal{N}(\mu) = \max_{i\in I} \mathrm{cf}\,\mathcal{N}(\mu_i).$$

521H Proposition Let (X, Σ, μ) and (Y, T, ν) be measure spaces, and $f : X \to Y$ an inverse-measure-preserving function.

(a)(i) $(X, \in, \mathcal{N}(\mu)) \preccurlyeq_{\mathrm{GT}} (Y, \in, \mathcal{N}(\nu))$, so $\mathrm{non}\,\mathcal{N}(\mu) \ge \mathrm{non}\,\mathcal{N}(\nu)$ and $\mathrm{cov}\,\mathcal{N}(\mu) \le \mathrm{cov}\,\mathcal{N}(\nu)$.

(ii) If there is a topology on Y such that ν is a topological measure inner regular with respect to the closed sets, then $\pi(\nu) \le \pi(\mu)$.

(iii) If ν is σ-finite, then $\tau(\nu) \le \tau(\mu)$.

(b) If ν is the image measure μf^{-1}, then $\mathrm{add}\,\nu \ge \mathrm{add}\,\mu$. If, moreover, μ is complete, $\mathcal{N}(\nu) \preccurlyeq_{\mathrm{T}} \mathcal{N}(\mu)$, so $\mathrm{add}\,\mathcal{N}(\mu) \le \mathrm{add}\,\mathcal{N}(\nu)$ and $\mathrm{cf}\,\mathcal{N}(\mu) \ge \mathrm{cf}\,\mathcal{N}(\nu)$; also $\mathrm{shr}\,\mathcal{N}(\mu) \ge \mathrm{shr}\,\mathcal{N}(\nu)$ and $\mathrm{shr}^+\,\mathcal{N}(\mu) \ge \mathrm{shr}^+\,\mathcal{N}(\nu)$.

521I Corollary Let (X, Σ, μ) be an atomless strictly localizable measure space. Then $\mathrm{non}\,\mathcal{N}(\mu) \ge \mathrm{non}\,\mathcal{N}$ and $\mathrm{cov}\,\mathcal{N}(\mu) \le \mathrm{cov}\,\mathcal{N}$, where $\mathcal{N}$ is the null ideal of Lebesgue measure on $\mathbb{R}$.

521J Proposition Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a non-empty family of probability spaces with product (X, Σ, μ).

(a)

$$\mathrm{non}\,\mathcal{N}(\mu) \ge \sup_{i\in I} \mathrm{non}\,\mathcal{N}(\mu_i), \quad \mathrm{cov}\,\mathcal{N}(\mu) \le \min_{i\in I} \mathrm{cov}\,\mathcal{N}(\mu_i),$$

$$\mathrm{add}\,\mu = \mathrm{add}\,\mathcal{N}(\mu) \le \min_{i\in I} \mathrm{add}\,\mathcal{N}(\mu_i), \quad \mathrm{cf}\,\mathcal{N}(\mu) \ge \sup_{i\in I} \mathrm{cf}\,\mathcal{N}(\mu_i),$$

$$\mathrm{shr}\,\mathcal{N}(\mu) \ge \sup_{i\in I} \mathrm{shr}\,\mathcal{N}(\mu_i), \quad \mathrm{shr}^+\,\mathcal{N}(\mu) \ge \sup_{i\in I} \mathrm{shr}^+\,\mathcal{N}(\mu_i),$$

$$\pi(\mu) \ge \sup_{i\in I} \pi(\mu_i).$$

(b) Set $\kappa = \#(\{i : i \in I, \Sigma_i \ne \{\emptyset, X_i\}\})$. Then $[\kappa]^{\le\omega} \preccurlyeq_{\mathrm{T}} \mathcal{N}(\mu)$; consequently $\mathrm{add}\,\mu = \mathrm{add}\,\mathcal{N}(\mu)$ is ω_1 if κ is uncountable, while $\mathrm{cf}\,\mathcal{N}(\mu)$ is at least $\mathrm{cf}[\kappa]^{\le\omega}$.

(c) Now suppose that I is countable and that we have for each $i \in I$ a probability space $(Y_i, \mathrm{T}_i, \nu_i)$ and an inverse-measure-preserving function $f_i : X_i \to Y_i$ which represents an isomorphism of the measure algebras of μ_i and ν_i. Let (Y, T, ν) be the product of $\langle(Y_i, \mathrm{T}_i, \nu_i)\rangle_{i\in I}$. Then

$$\mathcal{N}(\mu) \preccurlyeq_{\mathrm{T}} \mathcal{N}(\nu) \times \prod_{i\in I} \mathcal{N}(\mu_i).$$

Consequently

$$\mathrm{add}\,\mathcal{N}(\mu) \ge \min(\mathrm{add}\,\mathcal{N}(\nu), \min_{i\in I} \mathrm{add}\,\mathcal{N}(\mu_i)),$$

and if I is finite

$$\mathrm{cf}\,\mathcal{N}(\mu) \le \max(\mathrm{cf}\,\mathcal{N}(\nu), \max_{i\in I} \mathrm{cf}\,\mathcal{N}(\mu_i)).$$

(d) If I is finite, then

$$\mathrm{non}\,\mathcal{N}(\mu) = \max_{i\in I} \mathrm{non}\,\mathcal{N}(\mu_i), \quad \mathrm{cov}\,\mathcal{N}(\mu) = \min_{i\in I} \mathrm{cov}\,\mathcal{N}(\mu_i).$$

521K Proposition Let (X, Σ, μ) be a perfect semi-finite measure space which is not purely atomic. Then

$$\mathrm{add}\,\mathcal{N}(\mu) \le \mathrm{add}\,\mathcal{N}, \quad \mathrm{cf}\,\mathcal{N}(\mu) \ge \mathrm{cf}\,\mathcal{N},$$

$$\mathrm{shr}\,\mathcal{N}(\mu) \ge \mathrm{shr}\,\mathcal{N}, \quad \mathrm{shr}^+\,\mathcal{N}(\mu) \ge \mathrm{shr}^+\,\mathcal{N}, \quad \pi(\mu) \ge \pi(\mu_L)$$

where $\mathcal{N}$ is the null ideal of Lebesgue measure on $\mathbb{R}$ and μ_L is Lebesgue measure on $\mathbb{R}$.

521L Proposition (a) Let (X, Σ, μ) be a strictly localizable measure space and (Y, T, ν) a locally compact semi-finite measure space, and suppose that they have isomorphic measure algebras. Then $(X, \in, \mathcal{N}(\mu)) \preccurlyeq_{\mathrm{GT}} (Y, \in, \mathcal{N}(\nu))$; consequently $\mathrm{cov}\,\mathcal{N}(\mu) \le \mathrm{cov}\,\mathcal{N}(\nu)$ and $\mathrm{non}\,\mathcal{N}(\nu) \le \mathrm{non}\,\mathcal{N}(\mu)$.

(b) Let (X,Σ,μ) be a Maharam-type-homogeneous compact probability space with Maharam type κ. Then $\operatorname{cov}\mathcal{N}(\mu)=\operatorname{cov}\mathcal{N}_\kappa$ and $\operatorname{non}\mathcal{N}(\mu)=\operatorname{non}\mathcal{N}_\kappa$, where $\mathcal{N}_\kappa$ is the null ideal of the usual measure on $\{0,1\}^\kappa$.

(c) Let (X,Σ,μ) be a compact strictly localizable measure space with measure algebra $\mathfrak{A}$. Then

$$d(\mathfrak{A})=\min\{\#(A):A\subseteq X\text{ has full outer measure}\}.$$

521M Proposition Let (X,Σ,μ) be a complete locally determined measure space of magnitude at most $\operatorname{add}\mu$. Then it is strictly localizable.

521N Proposition Let (X,Σ,μ) be a complete locally determined localizable measure space of magnitude at most $\mathfrak{c}$. Then it is strictly localizable.

521O Proposition (a) If (X,Σ,μ) is a semi-finite measure space, its magnitude is at most $\max(\omega,2^{\#(X)})$.

(b) If (X,Σ,μ) is a strictly localizable measure space, its magnitude is at most $\max(\omega,\#(X))$.

(c) There is an infinite semi-finite measure space (X,Σ,μ) with magnitude $2^{\#(X)}$.

(d) If $\langle A_i\rangle_{i\in I}$ is a disjoint family of subsets of X and $\#(I)>\max(\omega,\operatorname{mag}(\mu))$ then there is an $i\in I$ such that $X\setminus A_i$ has full outer measure.

521P Proposition (a) If $2^\lambda<2^\kappa$ whenever $\mathfrak{c}\le\lambda<\kappa$ and $\operatorname{cf}\lambda>\omega$, then the magnitude of μ is at most $\max(\omega,\#(X))$ for every localizable measure space (X,Σ,μ).

(b) Suppose that $2^{\mathfrak{c}}=2^{\mathfrak{c}^+}$. Then there is a localizable measure space (Y,T,ν) with $\#(Y)=\mathfrak{c}$ and $\operatorname{mag}\nu=\mathfrak{c}^+$.

Remark $2^{\mathfrak{c}^+}$ here is $\#(\mathcal{P}(\mathfrak{c}^+))$.

521Q Free products: Proposition (a) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be semi-finite measure algebras and $(\mathfrak{C},\bar\lambda)$ their localizable measure algebra free product. Then

$$c(\mathfrak{C})\le\max(\omega,c(\mathfrak{A}),c(\mathfrak{B})),$$

$$\tau(\mathfrak{C})\le\max(\omega,\tau(\mathfrak{A}),\tau(\mathfrak{B})).$$

(b) Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a family of probability algebras, and $(\mathfrak{C},\bar\lambda)$ their probability algebra free product. Then

$$\tau(\mathfrak{C})\le\max(\omega,\#(I),\sup_{i\in I}\tau(\mathfrak{A}_i)).$$

521R Proposition If (X,Σ,μ) is any measure space, its Maharam type is at most $2^{\#(X)}$.

521S Proposition (a) A countably separated measure space has Maharam type at most $2^{\mathfrak{c}}$.

(b) There is a countably separated quasi-Radon probability space with Maharam type $2^{\mathfrak{c}}$.

(c) A countably separated semi-finite measure space has magnitude at most $2^{\mathfrak{c}}$.

(d) There is a countably separated semi-finite measure space with magnitude $2^{\mathfrak{c}}$.

521T Proposition Let I be a set, and suppose that a non-zero $\theta\in(M_{\mathrm{m}}\cap M_\tau^\perp)^+$, as defined in §464, corresponds to the Radon measure μ_θ on βI. Let ν be the usual measure on $\mathcal{P}I$. Then the Maharam type of μ_θ is at least $\operatorname{cov}\mathcal{N}(\nu)$.

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522 Cichoń's diagram

In this section I describe some extraordinary relationships between the cardinals associated with the ideals of meager and negligible sets in the real line. I concentrate on the strikingly symmetric pattern of Cichoń's diagram (522B); the first half of the section is taken up with proofs of the facts encapsulated in this diagram. I include a handful of results characterizing some of the most important cardinals here (522C, 522S), notes on Martin cardinals associated with the diagram (522T) and the Freese-Nation number of $\mathcal{P}\mathbb{N}$ (522U), and a brief discussion of cofinalities (522V).

522A Notation In this section, I will use the symbols $\mathcal{M}$ and $\mathcal{N}$ for the ideals of meager and negligible subsets of $\mathbb{R}$ respectively. Associated with these we have the eight cardinals $\operatorname{add}\mathcal{M}$, $\operatorname{cov}\mathcal{M}$, $\operatorname{non}\mathcal{M}$, $\operatorname{cf}\mathcal{M}$, $\operatorname{add}\mathcal{N}$, $\operatorname{cov}\mathcal{N}$, $\operatorname{non}\mathcal{N}$ and $\operatorname{cf}\mathcal{N}$. In addition we have two cardinals associated with the partially ordered set $\mathbb{N}^{\mathbb{N}}$: the **bounding number** $\mathfrak{b} = \operatorname{add}_\omega \mathbb{N}^{\mathbb{N}}$ and the **dominating number** $\mathfrak{d} = \operatorname{cf}\mathbb{N}^{\mathbb{N}}$. I use the notions of Galois-Tukey connection and Tukey function, and the associated relations $\preccurlyeq_{\mathrm{GT}}$, $\equiv_{\mathrm{GT}}$ and $\preccurlyeq_{\mathrm{T}}$, as described in §§512-513.

522B Cichoń's diagram The diagram itself is the following:

$$\begin{tikzcd}
& \operatorname{cov}\mathcal{N} \arrow[r, dash] \arrow[dd, dash] & \operatorname{non}\mathcal{M} \arrow[r, dash] \arrow[d, dash] & \operatorname{cf}\mathcal{M} \arrow[r, dash] \arrow[d, dash] & \operatorname{cf}\mathcal{N} \arrow[r, dash] \arrow[dd, dash] & \mathfrak{c} \\
& & \mathfrak{b} \arrow[r, dash] \arrow[d, dash] & \mathfrak{d} \arrow[d, dash] & & \\
\omega_1 \arrow[r, dash] & \operatorname{add}\mathcal{N} \arrow[r, dash] & \operatorname{add}\mathcal{M} \arrow[r, dash] & \operatorname{cov}\mathcal{M} \arrow[r, dash] & \operatorname{non}\mathcal{N} &
\end{tikzcd}$$

The cardinals here increase from bottom left to top right; that is,

$$\omega_1 \le \operatorname{add}\mathcal{N} \le \operatorname{add}\mathcal{M} \le \mathfrak{b} \le \mathfrak{d} \le \operatorname{cf}\mathcal{M} \le \operatorname{cf}\mathcal{N} \le \mathfrak{c},$$

etc. In addition, we have two equalities:

$$\operatorname{add}\mathcal{M} = \min(\mathfrak{b}, \operatorname{cov}\mathcal{M}), \quad \operatorname{cf}\mathcal{M} = \max(\mathfrak{d}, \operatorname{non}\mathcal{M}).$$

522C Lemma (i) On $\mathbb{N}^{\mathbb{N}}$ define a relation $\le^*$ by saying that $f \le^* g$ if if $\{n : f(n) > g(n)\}$ is finite. Then $\le^*$ is a pre-order on $\mathbb{N}^{\mathbb{N}}$; $\operatorname{add}(\mathbb{N}^{\mathbb{N}}, \le^*) = \mathfrak{b}$ and $\operatorname{cf}(\mathbb{N}^{\mathbb{N}}, \le^*) = \mathfrak{d}$.

(ii) On $\mathbb{N}^{\mathbb{N}}$ define a relation $\preceq$ by saying that $f \preceq g$ if either $f \le g$ or $\{n : g(n) \le f(n)\}$ is finite. Then $\preceq$ is a partial order on $\mathbb{N}^{\mathbb{N}}$, $\operatorname{add}(\mathbb{N}^{\mathbb{N}}, \preceq) = \mathfrak{b}$ and $\operatorname{cf}(\mathbb{N}^{\mathbb{N}}, \preceq) = \mathfrak{d}$.

(iii) $(\mathbb{N}^{\mathbb{N}}, \le^*) \equiv_{\mathrm{T}} (\mathbb{N}^{\mathbb{N}}, \preceq)$.

522D Proposition $\mathfrak{b} \le \mathfrak{d}$.

522E Proposition $\operatorname{add}\mathcal{N} \le \operatorname{cov}\mathcal{N}$, $\operatorname{add}\mathcal{M} \le \operatorname{cov}\mathcal{M}$, $\operatorname{non}\mathcal{M} \le \operatorname{cf}\mathcal{M}$ and $\operatorname{non}\mathcal{N} \le \operatorname{cf}\mathcal{N}$.

522F Proposition $\omega_1 \le \operatorname{add}\mathcal{N}$ and $\operatorname{cf}\mathcal{N} \le \mathfrak{c}$.

522G Proposition $\operatorname{cov}\mathcal{N} \le \operatorname{non}\mathcal{M}$ and $\operatorname{cov}\mathcal{M} \le \operatorname{non}\mathcal{N}$.

522H Proposition $\operatorname{add}\mathcal{M} \le \mathfrak{b}$ and $\mathfrak{d} \le \operatorname{cf}\mathcal{M}$.

522I Proposition $\mathfrak{b} \le \operatorname{non}\mathcal{M}$ and $\operatorname{cov}\mathcal{M} \le \mathfrak{d}$.

522J Theorem $\operatorname{add}\mathcal{M} = \min(\mathfrak{b}, \operatorname{cov}\mathcal{M})$ and $\operatorname{cf}\mathcal{M} = \max(\mathfrak{d}, \operatorname{non}\mathcal{M})$.

522K Localization Let I be any set. Write $\mathcal{S}_I$ for the family of sets $S \subseteq \mathbb{N} \times I$ such that each vertical section $S[\{n\}]$ has at most 2^n members. For $f \in I^{\mathbb{N}}$ and $S \subseteq \mathbb{N} \times I$ say that $f \subseteq^* S$ if $\{n : n \in \mathbb{N}, (n, f(n)) \notin S\}$ is finite. I will say that the supported relation $(I^{\mathbb{N}}, \subseteq^*, \mathcal{S}_I)$ is the ***I*-localization relation**.

***522L Lemma** Let I be an infinite set. For any $\alpha \in \mathbb{N}^{\mathbb{N}}$ write

$$\mathcal{S}_I^{(\alpha)} = \{S : S \subseteq \mathbb{N} \times I,\ \#(S[\{n\}]) \le \alpha(n) \text{ for every } n \in \mathbb{N}\}.$$

Then $(I^{\mathbb{N}}, \subseteq^*, \mathcal{S}_I^{(\alpha)}) \equiv_{\mathrm{GT}} (I^{\mathbb{N}}, \subseteq^*, \mathcal{S}_I^{(\beta)})$ whenever α, $\beta \in \mathbb{N}^{\mathbb{N}}$ and $\lim_{n\to\infty} \alpha(n) = \lim_{n\to\infty} \beta(n) = \infty$.

522M Proposition Let $(\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S})$ be the $\mathbb{N}$-localization relation. Then $(\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S}) \equiv_{\mathrm{GT}} (\mathcal{N}, \subseteq, \mathcal{N})$.

522N Lemma Let X be a topological space with a countable π-base. Then there is for each $n \in \mathbb{N}$ a countable family $\mathcal{U}_n$ of open subsets of X such that every dense open subset of X includes some member of $\mathcal{U}_n$ and $\bigcap \mathcal{V} \ne \emptyset$ for every $\mathcal{V} \in [\mathcal{U}_n]^{\le n}$.

522O Proposition Let $(\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S})$ be the $\mathbb{N}$-localization relation. Then $(\mathcal{M}, \subseteq, \mathcal{M}) \preccurlyeq_{\mathrm{GT}} (\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S})$.

522P Corollary $\mathcal{M} \preccurlyeq_{\mathrm{T}} \mathcal{N}$.

522Q Theorem $\mathrm{add}\,\mathcal{N} \le \mathrm{add}\,\mathcal{M}$ and $\mathrm{cf}\,\mathcal{M} \le \mathrm{cf}\,\mathcal{N}$.

522R The exactness of Cichoń's diagram The list of inequalities displayed in Cichoń's diagram is complete in the following sense: it is known that all assignments of the values ω_1, ω_2 to the eleven cardinals of the diagram which are allowed by the diagram together with the two equalities $\mathrm{add}\,\mathcal{M} = \min(\mathfrak{b}, \mathrm{cov}\,\mathcal{M})$, $\mathrm{cf}\,\mathcal{M} = \max(\mathfrak{d}, \mathrm{non}\,\mathcal{M})$ are relatively consistent with the axioms of ZFC.

522S The cardinals non $\mathcal{M}$, cov $\mathcal{M}$: Theorem (a) $n(\mathbb{R}) = \mathrm{cov}\,\mathcal{M} = \mathfrak{m}_{\mathrm{countable}}$.

(b) $\mathrm{cov}\,\mathcal{M}$ is the least cardinal of any set $A \subseteq \mathbb{N}^{\mathbb{N}}$ such that for every $g \in \mathbb{N}^{\mathbb{N}}$ there is an $f \in A$ such that $f(n) \ne g(n)$ for every $n \in \mathbb{N}$.

(c) (Bartoszyński 87) $\mathrm{non}\,\mathcal{M}$ is the least cardinal of any set $A \subseteq \mathbb{N}^{\mathbb{N}}$ such that for every $g \in \mathbb{N}^{\mathbb{N}}$ there is an $f \in A$ such that $\{n : f(n) = g(n)\}$ is infinite.

522T Martin numbers Following the identification of $\mathrm{cov}\,\mathcal{M}$ with $\mathfrak{m}_{\mathrm{countable}}$, we can amalgamate the diagrams in 522B and 517Ob, as follows:

$$\begin{tikzcd}
 & & \mathrm{cov}\,\mathcal{N} \arrow[r, dash] \arrow[dd, dash] & \mathrm{non}\,\mathcal{M} \arrow[r, dash] \arrow[d, dash] & \mathrm{cf}\,\mathcal{M} \arrow[r, dash] \arrow[d, dash] & \mathrm{cf}\,\mathcal{N} \arrow[r, dash] \arrow[dd, dash] & \mathfrak{c} \\
 & & & \mathfrak{b} \arrow[r, dash] \arrow[d, dash] & \mathfrak{d} \arrow[d, dash] & & \\
 & & \mathrm{add}\,\mathcal{N} \arrow[r, dash] \arrow[d, dash] & \mathrm{add}\,\mathcal{M} \arrow[r, dash] \arrow[d, dash] & \mathrm{cov}\,\mathcal{M} \arrow[r, dash] & \mathrm{non}\,\mathcal{N} & \\
 & & \mathfrak{m}_{\sigma\text{-linked}} \arrow[r, dash] \arrow[d, dash] & \mathfrak{p} \arrow[d, dash] & & & \\
\omega_1 \arrow[r, dash] & \mathfrak{m} \arrow[r, dash] & \mathfrak{m}_{\mathrm{K}} \arrow[r, dash] & \mathfrak{m}_{\mathrm{pc}\omega_1} & & &
\end{tikzcd}$$

***522U FN($\mathcal{P}\mathbb{N}$): Proposition** (a) $\mathrm{FN}(\mathcal{P}\mathbb{N}) \ge \mathfrak{b}$.

(b) $\mathrm{FN}(\mathcal{P}\mathbb{N}) \ge \mathrm{cov}\,\mathcal{N}$.

(c) If $\mathrm{FN}(\mathcal{P}\mathbb{N}) = \omega_1$ then $\mathrm{shr}\,\mathcal{M} = \omega_1$, so

$$\mathfrak{m} = \mathfrak{m}_{\mathrm{K}} = \mathfrak{m}_{\mathrm{pc}\omega_1} = \mathfrak{m}_{\sigma\text{-linked}} = \mathfrak{p} = \mathrm{add}\,\mathcal{N} = \mathrm{add}\,\mathcal{M} = \mathfrak{b} = \mathrm{cov}\,\mathcal{N} = \mathrm{non}\,\mathcal{M} = \omega_1.$$

(d) If $\mathrm{FN}(\mathcal{P}\mathbb{N}) = \omega_1$ and $\kappa \ge \mathfrak{m}_{\mathrm{countable}}$ is such that $\mathrm{cf}[\kappa]^{\le\omega} \le \kappa \le \mathfrak{c}$, then $\kappa = \mathfrak{c}$. So if $\mathrm{FN}(\mathcal{P}\mathbb{N}) = \omega_1$ and $\mathfrak{m}_{\mathrm{countable}} < \omega_\omega$, then

$$\mathfrak{m}_{\mathrm{countable}} = \mathrm{non}\,\mathcal{N} = \mathfrak{d} = \mathrm{cf}\,\mathcal{M} = \mathrm{cf}\,\mathcal{N} = \mathfrak{c}.$$

(e) There is a set $A \subseteq \mathbb{R}$ with cardinal $\mathfrak{m}_{\mathrm{countable}}$ such that every meager set meets A in a set with cardinal less than $\mathrm{FN}^*(\mathcal{P}\mathbb{N})$.

522V Cofinalities: Proposition (a) $\mathrm{cf}\,\mathfrak{c} \ge \mathfrak{p}$.
(b) $\mathrm{add}\,\mathcal{N}$, $\mathrm{add}\,\mathcal{M}$ and $\mathfrak{b}$ are regular.
(c) $\mathrm{cf}(\mathrm{cf}\,\mathcal{N}) \ge \mathrm{add}\,\mathcal{N}$, $\mathrm{cf}(\mathrm{cf}\,\mathcal{M}) \ge \mathrm{add}\,\mathcal{M}$ and $\mathrm{cf}\,\mathfrak{d} \ge \mathfrak{b}$.
(d) $\mathrm{cf}(\mathrm{non}\,\mathcal{N}) \ge \mathrm{add}\,\mathcal{N}$, $\mathrm{cf}(\mathrm{non}\,\mathcal{M}) \ge \mathrm{add}\,\mathcal{M}$.
(e) If $\mathrm{cf}\,\mathcal{M} = \mathfrak{m}_{\mathrm{countable}}$ then $\mathrm{cf}(\mathrm{cf}\,\mathcal{M}) \ge \mathrm{non}\,\mathcal{M}$; if $\mathrm{cf}\,\mathcal{N} = \mathrm{cov}\,\mathcal{N}$, then $\mathrm{cf}(\mathrm{cf}\,\mathcal{N}) \ge \mathrm{non}\,\mathcal{N}$.
(f) $\mathrm{cf}(\mathfrak{m}_{\mathrm{countable}}) \ge \mathrm{add}\,\mathcal{N}$.

522W Other spaces (a)(i) Let (X,Σ,μ) be an atomless countably separated σ-finite perfect measure space of non-zero measure, and $\mathcal{N}(\mu)$ the null ideal of μ. Then $(X,\mathcal{N}(\mu))$ is isomorphic to $(\mathbb{R},\mathcal{N})$; in particular, $\mathrm{add}\,\mathcal{N}(\mu) = \mathrm{add}\,\mathcal{N}$, $\mathrm{cov}\,\mathcal{N}(\mu) = \mathrm{cov}\,\mathcal{N}$, $\mathrm{non}\,\mathcal{N}(\mu) = \mathrm{non}\,\mathcal{N}$ and $\mathrm{cf}\,\mathcal{N}(\mu) = \mathrm{cf}\,\mathcal{N}$.

(ii) On a Hausdorff space with a countable network any non-zero atomless Radon measure will have a null ideal isomorphic to $\mathcal{N}$.

(b)(i) $(\mathbb{R},\mathcal{M})$ is duplicated in any non-empty Polish space X without isolated points, in the sense that $(X,\mathcal{B}(X),\mathcal{M}(X)) \cong (\mathbb{R},\mathcal{B},\mathcal{M})$, where $\mathcal{B}$ and $\mathcal{B}(X)$ are the Borel σ-algebras of $\mathbb{R}$ and X respectively, and $\mathcal{M}(X)$ is the ideal of meager subsets of X.

(ii) Again, the most important special cases here are $X = [0,1]$, $X = \{0,1\}^{\mathbb{N}}$ and $X = \mathbb{N}^{\mathbb{N}}$.

522Z Problem Is it the case that $(\mathbb{R},\in,\mathcal{M}) \equiv_{\mathrm{GT}} (\mathbb{N}^{\mathbb{N}},\mathtt{finint},\mathbb{N}^{\mathbb{N}})$?

Version of 24.8.24

523 The measure of $\{0,1\}^I$

In §522 I tried to give an account of current knowledge concerning the most important cardinals associated with Lebesgue measure. The next step is to investigate the usual measure ν_I on $\{0,1\}^I$ for an arbitrary set I. Here I discuss the cardinals associated with these measures. Obviously they depend only on $\#(I)$, and are trivial if I is finite. I start with the basic diagram relating the cardinal functions of ν_κ and ν_λ for different cardinals κ and λ (523B). I take the opportunity to mention some simple facts about the measures ν_I (523C-523D). Then I look at additivities (523E), covering numbers (523F-523G), uniformities (523H-523L), shrinking numbers (523M) and cofinalities (523N). I end with a description of these cardinals under the generalized continuum hypothesis (523P).

523A Notation For any set I, I will write ν_I for the usual measure on $\{0,1\}^I$ and $\mathcal{N}_I$ for its null ideal.

523B The basic diagram Suppose that κ and λ are infinite cardinals, with $\kappa \le \lambda$. Then we have the following diagram:

$$\begin{tikzcd}
 & \mathrm{cov}\,\mathcal{N}_\lambda \arrow[r, dash] \arrow[dd, dash] & \mathrm{cov}\,\mathcal{N}_\kappa \arrow[r, dash] \arrow[dd, dash] & \mathrm{cf}\,\mathcal{N}_\kappa \arrow[r, dash] \arrow[d, dash] & \mathrm{cf}\,\mathcal{N}_\lambda \arrow[r, dash] \arrow[d, dash] & \lambda^\omega \\
 & & & \mathrm{shr}\,\mathcal{N}_\kappa \arrow[r, dash] \arrow[d, dash] & \mathrm{shr}\,\mathcal{N}_\lambda \arrow[d, dash] & \\
\omega_1 \arrow[r, dash] & \mathrm{add}\,\mathcal{N}_\lambda \arrow[r, dash] & \mathrm{add}\,\mathcal{N}_\kappa \arrow[r, dash] & \mathrm{non}\,\mathcal{N}_\kappa \arrow[r, dash] & \mathrm{non}\,\mathcal{N}_\lambda &
\end{tikzcd}$$

(As in 522B, the cardinals here increase from bottom left to top right.)

523C Lemma Let I be any set, and $\mathcal{J}$ a family of subsets of I such that every countable subset of I is included in some member of $\mathcal{J}$. Then a subset A of $\{0,1\}^I$ belongs to $\mathcal{N}_I$ iff there is some $J \in \mathcal{J}$ such that $\{x \restriction J : x \in A\} \in \mathcal{N}_J$.

523D Proposition Let κ be an infinite cardinal, and T the domain of ν_κ. For $A \subseteq \{0,1\}^\kappa$ write T_A for the subspace σ-algebra on A.

(a) If $E \subseteq \{0,1\}^\kappa$ is measurable and not negligible, then $(E, \mathrm{T}_E, \mathcal{N}_\kappa \cap \mathcal{P}E)$ is isomorphic to $(\{0,1\}^\kappa, \mathrm{T}, \mathcal{N}_\kappa)$.

(b) If $\mathcal{E} \subseteq \mathcal{N}_\kappa$ and $\#(\mathcal{E}) < \operatorname{cov} \mathcal{N}_\kappa$, then $(\nu_\kappa)_*(\bigcup \mathcal{E}) = 0$.

(c) If $A \subseteq \{0,1\}^\kappa$ is non-negligible, then there is a set $B \subseteq \{0,1\}^\kappa$, of full outer measure, such that $(A, \mathrm{T}_A, \mathcal{N}_\kappa \cap \mathcal{P}A)$ is isomorphic to $(B, \mathrm{T}_B, \mathcal{N}_\kappa \cap \mathcal{P}B)$.

(d) There is a set $A \subseteq \{0,1\}^\kappa$ with cardinal $\operatorname{non} \mathcal{N}_\kappa$ which has full outer measure.

523E Additivities If κ is any uncountable cardinal, then $\operatorname{add} \mathcal{N}_\kappa = \operatorname{add} \nu_\kappa = \omega_1$.

523F Covering numbers Still on the left-hand side of the diagram, we have a non-increasing function $\kappa \mapsto \operatorname{cov} \mathcal{N}_\kappa$, and a critical value κ_c after which it is constant.

$$\omega \le \kappa_c \le \operatorname{cov} \mathcal{N}_{\kappa_c} \le \operatorname{cov} \mathcal{N}_\omega = \operatorname{cov} \mathcal{N} \le \mathfrak{c}.$$

Another way of putting the same idea is to say that

$$\text{if } \lambda \text{ is a cardinal such that } \operatorname{cov} \mathcal{N}_\lambda \ge \lambda \text{ then } \operatorname{cov} \mathcal{N}_\kappa \ge \lambda \text{ for every } \kappa.$$

523G Proposition If κ is a cardinal and $\operatorname{cov} \mathcal{N}_\kappa < \operatorname{add} \mathcal{N}$, then $\operatorname{cov} \mathcal{N}_\kappa \le \operatorname{cf}[\kappa]^{\le\omega}$.

523H Uniformities: Lemma Suppose that I is a set and F a family of functions with domain I such that for every countable $J \subseteq I$ there is an $f \in F$ such that $f \restriction J$ is injective. Then

$$\operatorname{non} \mathcal{N}_I \le \max(\#(F), \sup_{f \in F} \operatorname{non} \mathcal{N}_{f[I]}).$$

523I Theorem (a) For any cardinal κ,

(i) $\operatorname{non} \mathcal{N}_\kappa \le \max(\operatorname{non} \mathcal{N}, \operatorname{cf}[\kappa]^{\le\omega})$,

(ii) $\operatorname{non} \mathcal{N}_{\kappa^+} \le \max(\kappa^+, \operatorname{non} \mathcal{N}_\kappa)$,

(iii) $\operatorname{non} \mathcal{N}_{2^\kappa} \le \max(\mathfrak{c}, \operatorname{cf}[\kappa]^{\le\omega})$,

(iv) $\operatorname{non} \mathcal{N}_{2^{\kappa^+}} \le \max(\kappa^+, \operatorname{non} \mathcal{N}_{2^\kappa})$.

(b) If $\operatorname{cf} \kappa > \omega$, then $\operatorname{non} \mathcal{N}_{\kappa^+} \le \max(\operatorname{cf} \kappa, \sup_{\lambda<\kappa} \operatorname{non} \mathcal{N}_\lambda)$.

523J Corollary (a) $\operatorname{non} \mathcal{N}_{\omega_2} = \operatorname{non} \mathcal{N}_{\omega_1} = \operatorname{non} \mathcal{N}$.

(b) For any $n \in \mathbb{N}$, $\operatorname{non} \mathcal{N}_{\omega_{n+1}} \le \max(\omega_n, \operatorname{non} \mathcal{N})$.

(c) $\operatorname{non} \mathcal{N}_{2^{\omega_1}} = \operatorname{non} \mathcal{N}_\mathfrak{c}$.

(d) If $n \in \mathbb{N}$ then $\operatorname{non} \mathcal{N}_{2^{\omega_n}} \le \max(\omega_n, \operatorname{non} \mathcal{N}_\mathfrak{c}$).

523K Corollary For any sets I, K let $\Upsilon_\omega(I,K)$ be the least cardinal of any family F of functions from I to K such that for every countable $J \subseteq I$ there is an $f \in F$ which is injective on J. (If $\#(K) < \min(\omega, \#(I))$ take $\Upsilon_\omega(I,K) = \infty$.) Then

(a) $\operatorname{non} \mathcal{N}_I \le \max(\Upsilon_\omega(I,K), \operatorname{non} \mathcal{N}_K)$ for all sets I and K;

(b) if $\kappa \ge \mathfrak{c}$ is a cardinal, then $\operatorname{non} \mathcal{N}_\kappa = \max(\Upsilon_\omega(\kappa,\mathfrak{c}), \operatorname{non} \mathcal{N}_\mathfrak{c})$.

523L Proposition (a) If λ and κ are infinite cardinals with $\kappa > 2^\lambda$, then $\operatorname{non} \mathcal{N}_\kappa > \lambda$.

(b) If κ is a strong limit cardinal of countable cofinality then $\operatorname{non} \mathcal{N}_\kappa > \kappa$.

523M Shrinking numbers: Proposition (a)(i) For any non-zero cardinals κ and λ,

$$\operatorname{shr} \mathcal{N}_\kappa \le \max(\operatorname{cov}_{\mathrm{Sh}}(\kappa, \lambda, \omega_1, 2), \sup_{\theta<\lambda} \operatorname{shr} \mathcal{N}_\theta).$$

(ii) For any infinite cardinal κ, $\operatorname{shr} \mathcal{N}_\kappa \le \max(\operatorname{shr} \mathcal{N}, \operatorname{cf}[\kappa]^{\le\omega})$.

(iii) If $\operatorname{cf} \kappa > \omega$, then $\operatorname{shr} \mathcal{N}_\kappa \le \max(\kappa, \sup_{\theta<\kappa} \operatorname{shr} \mathcal{N}_\theta)$.

(b) For any infinite cardinal κ,

(i) $\operatorname{shr} \mathcal{N}_\kappa \ge \kappa$;

(ii) $\operatorname{cf}(\operatorname{shr} \mathcal{N}_\kappa) > \omega$;

(iii) $\operatorname{cf}(\operatorname{shr}^+ \mathcal{N}_\kappa) > \kappa$.

523N Cofinalities: Theorem For any infinite cardinal κ,

$$\kappa \le \operatorname{cf}\mathcal{N}_\kappa = \max(\operatorname{cf}\mathcal{N}, \operatorname{cf}[\kappa]^{\le\omega}) \le \kappa^\omega.$$

523P The generalized continuum hypothesis: Proposition Suppose that the generalized continuum hypothesis is true. Then, for any infinite cardinal κ,

$$\operatorname{add}\mathcal{N}_\kappa = \operatorname{add}\nu_\kappa = \operatorname{cov}\mathcal{N}_\kappa = \omega_1;$$

$$\begin{aligned}\operatorname{non}\mathcal{N}_\kappa &= \lambda \text{ if } \kappa = \lambda^+ \text{ where } \operatorname{cf}\lambda > \omega,\\ &= \kappa^+ \text{ if } \operatorname{cf}\kappa = \omega,\\ &= \kappa \text{ otherwise;}\end{aligned}$$

$$\begin{aligned}\operatorname{shr}\mathcal{N}_\kappa = \operatorname{cf}\mathcal{N}_\kappa &= \kappa^+ \text{ if } \operatorname{cf}\kappa = \omega,\\ &= \kappa \text{ otherwise;}\end{aligned}$$

$$\begin{aligned}\operatorname{shr}^+\mathcal{N}_\kappa = (\operatorname{shr}\mathcal{N}_\kappa)^+ &= \kappa^{++} \text{ if } \operatorname{cf}\kappa = \omega,\\ &= \kappa^+ \text{ otherwise.}\end{aligned}$$

523Z Problem Is there a proof in ZFC that $\operatorname{shr}\mathcal{N}_\kappa \ge \operatorname{cf}[\kappa]^{\le\omega}$ for every cardinal κ?

Version of 29.9.10/10.9.13

524 Radon measures

It is a remarkable fact that for a Radon measure the principal cardinal functions are determined by its measure algebra (524J), so can in most cases be calculated in terms of the cardinals of the last section (524P-524Q). The proof of this seems to require a substantial excursion involving not only measure algebras but also the Banach lattices $\ell^1(\kappa)$ and/or the κ-localization relation (524D, 524E). The same machinery gives us formulae for the cardinal functions of measurable algebras (524M). The results of §518 can be translated directly to give partial information on the Freese-Nation numbers of measurable algebras (524O). For covering number and uniformity, we can see from 521L that strictly localizable compact measures follow Radon measures. I know of no such general results for any other class of measure, but there are some bounds for cardinal functions of countably compact and quasi-Radon measures, which I give in 524R-524T.

524A Notation If (X,Σ,μ) is a measure space, $\mathcal{N}(\mu)$ will be the null ideal of μ. For any cardinal κ, ν_κ will be the usual measure on $\{0,1\}^\kappa$, T_κ its domain and $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$ its measure algebra. As in §§522-523, I will write $\mathcal{N}_\kappa$ for $\mathcal{N}(\nu_\kappa)$ and $\mathcal{N}$ for the null ideal of Lebesgue measure on $\mathbb{R}$, so that $(\mathbb{R},\mathcal{N})$ and $(\{0,1\}^\omega,\mathcal{N}_\omega)$ are isomorphic. If $\mathfrak{A}$ is any Boolean algebra, I write $\mathfrak{A}^+$ for $\mathfrak{A}\setminus\{0\}$ and $\mathfrak{A}^-$ for $\mathfrak{A}\setminus\{1\}$. If (A,R,B) is a supported relation, R' is the relation $\{(a,I) : a \in R^{-1}[I]\}$. For any cardinal κ, $(\kappa^{\mathbb{N}},\subseteq^*,\mathcal{S}_\kappa)$ will be the κ-localization relation.

524B Proposition Let $(X,\mathfrak{T},\Sigma,\mu)$ be a σ-finite Radon measure space with Maharam type κ. Then $\mathcal{N}(\mu) \preccurlyeq_{\mathrm{T}} \mathfrak{B}_\kappa^-$.

524C Lemma Let P be a partially ordered set such that $p \vee q = \sup\{p,q\}$ is defined for all p, $q \in P$. Suppose that ρ is a metric on P such that P is complete and $\vee : P\times P \to P$ is uniformly continuous with respect to ρ. Let $Q \subseteq P$ be an open set, and $\kappa \ge d(Q)$ a cardinal. Then $(Q,\le', [Q]^{<\omega}) \preccurlyeq_{\mathrm{GT}} (\ell^1(\kappa),\le,\ell^1(\kappa))$. If Q is upwards-directed, then $Q \preccurlyeq_{\mathrm{T}} \ell^1(\kappa)$.

524D Proposition If κ is any cardinal,

$$(\mathfrak{B}_\kappa^-, \subseteq', [\mathfrak{B}_\kappa^-]^{<\omega}) \preccurlyeq_{\mathrm{GT}} (\ell^1(\kappa), \le, \ell^1(\kappa)).$$

524E Proposition Let κ be an infinite cardinal. Then

$$(\ell^1(\kappa), \le', [\ell^1(\kappa)]^{\le\omega}) \preccurlyeq_{\mathrm{GT}} (\kappa^{\mathbb{N}}, \subseteq^*, \mathcal{S}_\kappa).$$

524F Lemma Let (X,Σ,μ) be a countably compact measure space with Maharam type κ.

(a) If μ is a Maharam-type-homogeneous probability measure, there is a family $\langle E_\xi\rangle_{\xi<\kappa}$ in $\mathcal{N}(\mu)$ such that $\bigcup_{\xi\in A} E_\xi$ has full outer measure for every uncountable $A \subseteq \kappa$.

(b) If μ is σ-finite, there is a family $\langle E_\xi\rangle_{\xi<\kappa}$ in $\mathcal{N}(\mu)$ such that $\bigcup_{\xi\in A} E_\xi$ is non-negligible for every uncountable $A \subseteq \kappa$.

524G Proposition Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Maharam-type-homogeneous Radon probability space with Maharam type $\kappa \ge \omega$. Then $(\kappa^{\mathbb{N}}, \subseteq^*, \mathcal{S}_\kappa) \preccurlyeq_{\mathrm{GT}} (\mathcal{N}(\mu), \subseteq, \mathcal{N}(\mu))$.

524H Corollary Let κ be an infinite cardinal, and μ a Maharam-type-homogeneous Radon probability measure with Maharam type κ. Then $(\mathfrak{B}_\kappa^+, \supseteq', [\mathfrak{B}_\kappa^+]^{\le\omega})$, $(\ell^1(\kappa), \le', [\ell^1(\kappa)]^{\le\omega})$, $(\kappa^{\mathbb{N}}, \subseteq^*, \mathcal{S}_\kappa)$ and $(\mathcal{N}(\mu), \subseteq, \mathcal{N}(\mu))$ are Galois-Tukey equivalent.

524I Corollary Let μ be a Maharam-type-homogeneous Radon probability measure with infinite Maharam type κ. Then

$$\operatorname{add}\mathcal{N}(\mu) = \operatorname{add}\mathcal{N}_\kappa = \operatorname{add}_\omega \ell^1(\kappa),$$

$$\operatorname{cf}\mathcal{N}(\mu) = \operatorname{cf}\mathcal{N}_\kappa = \operatorname{cf}\ell^1(\kappa).$$

524J Theorem Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be Radon measure spaces with non-zero measure and isomorphic measure algebras.

(a) $\mathcal{N}(\mu)$ and $\mathcal{N}(\nu)$ are Tukey equivalent, so $\operatorname{add}\mu = \operatorname{add}\mathcal{N}(\mu) = \operatorname{add}\mathcal{N}(\nu) = \operatorname{add}\nu$ and $\operatorname{cf}\mathcal{N}(\mu) = \operatorname{cf}\mathcal{N}(\nu)$.

(b) $(X, \in, \mathcal{N}(\mu))$ and $(Y, \in, \mathcal{N}(\nu))$ are Galois-Tukey equivalent, so $\operatorname{cov}\mathcal{N}(\mu) = \operatorname{cov}\mathcal{N}(\nu)$ and $\operatorname{non}\mathcal{N}(\mu) = \operatorname{non}\mathcal{N}(\nu)$.

524K Corollary Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be Radon measure spaces with measure algebras $\mathfrak{A}$, $\mathfrak{B}$ respectively. If $\mathfrak{A}$ can be regularly embedded in $\mathfrak{B}$, then $\mathcal{N}(\mu) \preccurlyeq_{\mathrm{T}} \mathcal{N}(\nu)$.

524L Proposition Let κ be an infinite cardinal. Then for any $n \ge 2$ the n-linking number $\operatorname{link}_n(\mathfrak{B}_\kappa)$ is the least λ such that $\kappa \le 2^\lambda$.

524M Theorem Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra. Let K be the set of infinite cardinals κ such that $\mathfrak{A}$ has a homogeneous principal ideal with Maharam type κ.

$$\begin{aligned}
\text{(a)} \qquad \#(\mathfrak{A}) &= 2^{c(\mathfrak{A})} \text{ if } \mathfrak{A} \text{ is finite,}\\
&= \tau(\mathfrak{A})^{\omega} \text{ if } \mathfrak{A} \text{ is ccc and infinite.}\\
\text{(b)} \qquad \operatorname{wdistr}(\mathfrak{A}) &= \infty \text{ if } \mathfrak{A} \text{ is purely atomic,}\\
&= \operatorname{add}\mathcal{N} \text{ if } K = \{\omega\},\\
&= \omega_1 \text{ otherwise.}\\
\text{(c)} \qquad \pi(\mathfrak{A}) &= c(\mathfrak{A}) \text{ if } \mathfrak{A} \text{ is purely atomic,}\\
&= \max(c(\mathfrak{A}), \operatorname{cf}\mathcal{N}, \sup_{\kappa\in K} \operatorname{cf}[\kappa]^{\le\omega}) \text{ otherwise.}\\
\text{(d)} \qquad \mathfrak{m}(\mathfrak{A}) &= \infty \text{ if } \mathfrak{A} \text{ is purely atomic,}\\
&= \min_{\kappa\in K} \operatorname{cov}\mathcal{N}_{\kappa} \text{ otherwise.}\\
\text{(e)} \qquad d(\mathfrak{A}) &= c(\mathfrak{A}) \text{ if } \mathfrak{A} \text{ is purely atomic,}\\
&= \max(c(\mathfrak{A}), \sup_{\kappa\in K} \operatorname{non}\mathcal{N}_{\kappa}) \text{ otherwise.}
\end{aligned}$$

(f) For $2 \le n < \omega$,

$$\begin{aligned}
\operatorname{link}_n(\mathfrak{A}) &= c(\mathfrak{A}) \text{ if } \mathfrak{A} \text{ is purely atomic,}\\
&= \max(c(\mathfrak{A}), \min\{\lambda : \tau(\mathfrak{A}) \le 2^{\lambda}\}) \text{ otherwise.}
\end{aligned}$$

524N Corollary (a) If (X, Σ, μ) is a semi-finite locally compact measure space, with $\mu X > 0$, then $\operatorname{cov}\mathcal{N}(\mu) \ge \mathfrak{m}_{\sigma\text{-linked}}$.

(b) If $\mathfrak{A}$ is any measurable algebra, then $\mathfrak{m}(\mathfrak{A}) \ge \mathfrak{m}_{\sigma\text{-linked}}$.

524O Freese-Nation numbers: Proposition (a) Let $(\mathfrak{A}, \bar{\mu})$ be an infinite measure algebra. Then $\operatorname{FN}(\mathfrak{A}) \ge \operatorname{FN}(\mathcal{P}\mathbb{N})$.

(b) Let $\mathfrak{A}$ be a measurable algebra.

(i) $\operatorname{FN}(\mathfrak{A}) \le \mathfrak{c}^+$.

(ii) If $\tau(\mathfrak{A}) \le \mathfrak{c}$ then $\operatorname{FN}(\mathfrak{A}) \le \operatorname{FN}(\mathcal{P}\mathbb{N})$.

(iii) If

(α) $\operatorname{cf}([\lambda]^{\le\omega}) \le \lambda^+$ for every cardinal $\lambda \le \tau(\mathfrak{A})$,

(β) $\square_{\lambda}$ is true for every uncountable cardinal $\lambda \le \tau(\mathfrak{A})$ of countable cofinality,

then $\operatorname{FN}(\mathfrak{A}) \le \operatorname{FN}^*(\mathcal{P}\mathbb{N})$.

(c) Suppose that the continuum hypothesis and $\operatorname{CTP}(\omega_{\omega+1}, \omega_{\omega})$ are both true. If $\mathfrak{A}$ is a measurable algebra, then

$$\begin{aligned}
\operatorname{FN}(\mathfrak{A}) &= \mathfrak{c} = \omega_1 \text{ if } \omega \le \tau(\mathfrak{A}) < \omega_{\omega},\\
&= \mathfrak{c}^+ = \omega_2 \text{ otherwise .}
\end{aligned}$$

524P The Maharam classification: Theorem Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space, and $\mathfrak{A}$ its measure algebra. Let K be the set of infinite cardinals κ such that the Maharam-type-κ component of $\mathfrak{A}$ is non-zero.

$$\begin{aligned}
\text{(a)} \qquad \operatorname{add}\mu = \operatorname{add}\mathcal{N}(\mu) &= \infty \text{ if } K = \emptyset,\\
&= \operatorname{add}\mathcal{N} \text{ if } K = \{\omega\},\\
&= \omega_1 \text{ otherwise.}
\end{aligned}$$

(b)
$$\begin{aligned}\pi(\mu) = \pi(\mathfrak{A}) &= c(\mathfrak{A}) \text{ if } K = \emptyset,\\ &= \max(c(\mathfrak{A}), \operatorname{cf}\mathcal{N}, \sup_{\kappa\in K} \operatorname{cf}[\kappa]^{\le\omega}) \text{ otherwise.}\end{aligned}$$

(c)
$$\begin{aligned}\operatorname{cov}\mathcal{N}(\mu) &= 1 \text{ if } \mathfrak{A} = \{0\},\\ &= \infty \text{ if } \mathfrak{A} \text{ has an atom,}\\ &= \operatorname{cov}\mathcal{N}_{\min K} \text{ otherwise.}\end{aligned}$$

(d)
$$\begin{aligned}\operatorname{non}\mathcal{N}(\mu) &= \infty \text{ if } \mathfrak{A} = \{0\},\\ &= 1 \text{ if } \mathfrak{A} \text{ has an atom,}\\ &= \operatorname{non}\mathcal{N}_{\min K} \text{ otherwise.}\end{aligned}$$

(e)
$$\begin{aligned}\operatorname{shr}\mathcal{N}(\mu) &= 0 \text{ if } \mathfrak{A} = \{0\},\\ &= 1 \text{ if } \mathfrak{A} \text{ has an atom,}\\ &\ge \operatorname{shr}\mathcal{N} \text{ otherwise.}\end{aligned}$$

(f) If μ is σ-finite,
$$\begin{aligned}\operatorname{cf}\mathcal{N}(\mu) &= 1 \text{ if } K = \emptyset,\\ &= \max(\operatorname{cf}\mathcal{N}, \operatorname{cf}[\tau(\mathfrak{A})]^{\le\omega}) \text{ otherwise.}\end{aligned}$$

***524Q Proposition** Suppose that the generalized continuum hypothesis is true. Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon measure space and $(\mathfrak{A},\bar\mu)$ its measure algebra. For each cardinal κ, write e_κ for the Maharam-type-κ component of $\mathfrak{A}$, and $\mathfrak{C}_\kappa$ for the principal ideal of $\mathfrak{A}$ generated by $\sup_{\kappa'>\kappa} e_{\kappa'}$; set $\lambda = \sup\{\kappa : e_\kappa \ne 0\}$. Then $\operatorname{cf}\mathcal{N}(\mu) = \max(c(\mathfrak{C}_0)^+, \lambda^+)$ unless $\lambda > c(\mathfrak{C}_0)$ and there is some $\gamma < \lambda$ such that $\operatorname{cf}\lambda > c(\mathfrak{C}_\gamma)$, in which case $\operatorname{cf}\mathcal{N}(\mu) = \lambda$.

524R Proposition Let (X,Σ,μ) be a countably compact σ-finite measure space with Maharam type κ. Then $[\kappa]^{\le\omega} \preccurlyeq_{\mathrm{T}} \mathcal{N}(\mu)$. Consequently $\operatorname{cf}[\kappa]^{\le\omega} \le \operatorname{cf}\mathcal{N}(\mu)$, and if κ is uncountable then $\operatorname{add}\mathcal{N}(\mu) = \omega_1$ and $\operatorname{cf}\mathcal{N}(\mu) \ge \operatorname{cf}\mathcal{N}_\kappa$.

524S Proposition Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon measure space, with $\mu X > 0$, and $(Y,\mathfrak{S},\mathrm{T},\nu)$ a quasi-Radon measure space such that the measure algebras of μ and ν are isomorphic. Then

(a) $\mathcal{N}(\nu) \preccurlyeq_{\mathrm{T}} \mathcal{N}(\mu)$, so $\operatorname{add}\nu = \operatorname{add}\mathcal{N}(\nu) \ge \operatorname{add}\mathcal{N}(\mu) = \operatorname{add}\mu$ and $\operatorname{cf}\mathcal{N}(\nu) \le \operatorname{cf}\mathcal{N}(\mu)$;

(b) $(Y,\in,\mathcal{N}(\nu)) \preccurlyeq_{\mathrm{GT}} (X,\in,\mathcal{N}(\mu))$, so $\operatorname{cov}\mathcal{N}(\nu) \le \operatorname{cov}\mathcal{N}(\mu)$ and $\operatorname{non}\mathcal{N}(\nu) \ge \operatorname{non}\mathcal{N}(\mu)$.

524T Corollary Let $(Y,\mathfrak{S},\mathrm{T},\nu)$ be a quasi-Radon measure space, and $\mathfrak{B}$ its measure algebra. Let K be the set of infinite cardinals κ such that the Maharam-type-κ component of $\mathfrak{B}$ is non-zero.

(a)
$$\begin{aligned}\operatorname{add}\nu = \operatorname{add}\mathcal{N}(\nu) &= \infty \text{ if } K = \emptyset,\\ &\ge \operatorname{add}\mathcal{N} \text{ if } K = \{\omega\}.\end{aligned}$$

(b)
$$\begin{aligned}\pi(\nu) = \pi(\mathfrak{B}) &= c(\mathfrak{B}) \text{ if } K = \emptyset,\\ &= \max(c(\mathfrak{B}), \operatorname{cf}\mathcal{N}, \sup_{\kappa\in K} \operatorname{cf}[\kappa]^{\le\omega}) \text{ otherwise.}\end{aligned}$$

(c)
$$\begin{aligned}\operatorname{cov}\mathcal{N}(\nu) &= 1 \text{ if } \mathfrak{B} = \{0\},\\ &= \infty \text{ if } \mathfrak{B} \text{ has an atom,}\\ &\le \operatorname{cov}\mathcal{N}_{\min K} \text{ otherwise.}\end{aligned}$$

(d)
$$\begin{aligned}\operatorname{non}\mathcal{N}(\nu) &= \infty \text{ if } \mathfrak{B} = \{0\},\\ &= 1 \text{ if } \mathfrak{B} \text{ has an atom,}\\ &\ge \operatorname{non}\mathcal{N}_{\min K} \text{ otherwise.}\end{aligned}$$

(e) If ν is σ-finite,

$$\begin{aligned}\operatorname{cf}\mathcal{N}(\nu) &= 1 \text{ if } K = \emptyset,\\ &\le \max(\operatorname{cf}\mathcal{N}, \operatorname{cf}[\tau(\mathfrak{B})]^{\le\omega}) \text{ otherwise.}\end{aligned}$$

524U Lemma Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra. Then there is a Radon probability measure on $\{0,1\}^{\tau(\mathfrak{A})}$ with measure algebra isomorphic to $(\mathfrak{A}, \bar\mu)$.

524Z Problems (a) Let (Z, μ) be the Stone space of $(\mathfrak{B}_\omega, \bar\nu_\omega)$. Is $\operatorname{shr}\mathcal{N}(\mu)$ necessarily equal to $\operatorname{shr}\mathcal{N}$?

(b) Can there be a quasi-Radon probability measure μ with Maharam type greater than $\mathfrak{c}$ such that $\operatorname{add}\mathcal{N}(\mu) > \omega_1$?

Version of 11.9.13

525 Precalibers

I continue the discussion of precalibers in §516 with results applying to measure algebras. I start with connexions between measure spaces and precalibers of their measure algebras (525B-525C). The next step is to look at measure-precalibers. Elementary facts are in 525D-525G. When we come to ask which cardinals are precalibers of which measure algebras, there seem to be real difficulties; partial answers, largely based on infinitary combinatorics, are in 525I-525O. 525P is a note on a particular pair of cardinals. Finally, 525T deals with precaliber triples (κ, κ, k) where k is finite; I approach it through a general result on correlations in uniformly bounded families of random variables (525S).

525A Notation If (X, Σ, μ) is a measure space, $\mathcal{N}(\mu)$ will be the null ideal of μ. For any set I, ν_I will be the usual measure on $\{0,1\}^I$, T_I its domain, $\mathcal{N}_I = \mathcal{N}(\nu_I)$ its null ideal and $(\mathfrak{B}_I, \bar\nu_I)$ its measure algebra. In this context, set $e_i = \{x : x \in \{0,1\}^I, x(i) = 1\}^\bullet$ in $\mathfrak{B}_I$ for $i \in I$. Then $\langle e_i\rangle_{i\in I}$ is a stochastically independent family of elements of measure $\frac{1}{2}$ in $\mathfrak{B}_I$, and $\{e_i : i \in I\}$ τ-generates $\mathfrak{B}_I$; I will say that $\langle e_i\rangle_{i\in I}$ is the **standard generating family** in $\mathfrak{B}_I$.

525B Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space, and $\mathfrak{A}$ its measure algebra. Then the downwards precaliber triples of the partially ordered set $(\Sigma \setminus \mathcal{N}(\mu), \subseteq)$ are just the precaliber triples of the Boolean algebra $\mathfrak{A}$.

525C Theorem Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space and $(\mathfrak{A}, \bar\mu)$ its measure algebra.

(a) A pair (κ, λ) of cardinals is a precaliber pair of $\mathfrak{A}$ iff whenever $\langle E_\xi\rangle_{\xi<\kappa}$ is a family in $\Sigma \setminus \mathcal{N}(\mu)$ there is an $x \in X$ such that $\#(\{\xi : x \in E_\xi\}) \ge \lambda$.

(b) A pair (κ, λ) of cardinals is a measure-precaliber pair of $(\mathfrak{A}, \bar\mu)$ iff whenever $\langle E_\xi\rangle_{\xi<\kappa}$ is a family in $\Sigma \setminus \mathcal{N}(\mu)$ such that $\inf_{\xi<\kappa} \mu E_\xi > 0$ then there is an $x \in X$ such that $\#(\{\xi : x \in E_\xi\}) \ge \lambda$.

(c) Suppose that $\kappa \ge \operatorname{sat}(\mathfrak{A})$ is an infinite regular cardinal. Then the following are equiveridical:

(i) κ is a precaliber of $\mathfrak{A}$;

(ii) $\mu_*(\bigcup_{\xi<\kappa} E_\xi) = 0$ whenever $\langle E_\xi\rangle_{\xi<\kappa}$ is a non-decreasing family in $\mathcal{N}(\mu)$;

(iii) whenever $\langle A_\xi\rangle_{\xi<\kappa}$ is a non-decreasing family of sets such that $\bigcup_{\xi<\kappa} A_\xi = X$, then there is some $\xi < \kappa$ such that A_ξ has full outer measure in X.

525D Proposition Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra.

(a) Any precaliber triple of $\mathfrak{A}$ is a measure-precaliber triple of $(\mathfrak{A}, \bar\mu)$.

(b) If $(\kappa, \lambda, {<}\theta)$ is a measure-precaliber triple of $(\mathfrak{A}, \bar\mu)$ and κ has uncountable cofinality, then $(\kappa, \lambda, {<}\theta)$ is a precaliber triple of $\mathfrak{A}$.

(c) If κ is a measure-precaliber of $(\mathfrak{A}, \bar\mu)$, so is $\operatorname{cf}\kappa$.

525E Proposition (a) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra and κ an infinite cardinal. Then κ is a precaliber of $\mathfrak{A}$ iff either $\mathfrak{A}$ is finite or κ is a measure-precaliber of $(\mathfrak{A},\bar{\mu})$ and $\operatorname{cf}\kappa > \omega$.

(b) An infinite cardinal κ is a precaliber of every measurable algebra iff it is a measure-precaliber of every probability algebra and has uncountable cofinality.

525F Proposition Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra.

(a) ω is a measure-precaliber of $(\mathfrak{A},\bar{\mu})$.

(b) If $\omega \le \kappa < \mathfrak{m}(\mathfrak{A})$, then κ is a measure-precaliber of $(\mathfrak{A},\bar{\mu})$.

525G Proposition (a) Let $(\mathfrak{A},\bar{\mu})$ be a totally finite measure algebra. Let K be the set of infinite cardinals κ' such that the Maharam-type-κ' component of $\mathfrak{A}$ is non-zero. If κ, λ and θ are cardinals, of which κ is infinite, then $(\kappa,\lambda,{<}\theta)$ is a measure-precaliber triple of $(\mathfrak{A},\bar{\mu})$ iff it is a measure-precaliber triple of $(\mathfrak{B}_{\kappa'},\bar{\nu}_{\kappa'})$ for every $\kappa' \in K$.

(b) Suppose that $\omega \le \kappa < \operatorname{cov}\mathcal{N}_{\kappa'}$. Then κ is a measure-precaliber of $\mathfrak{B}_{\kappa'}$.

(c) For any cardinal κ', ω_1 is a precaliber of $\mathfrak{B}_{\kappa'}$ iff $\operatorname{cov}\mathcal{N}_{\kappa'} > \omega_1$.

(d) If κ, κ' are cardinals such that $\operatorname{non}\mathcal{N}_{\kappa'} < \operatorname{cf}\kappa$, then κ is a precaliber of $\mathfrak{B}_{\kappa'}$.

525H The structure of $\mathfrak{B}_I$ Let I be any set and $\langle e_i\rangle_{i\in I}$ the standard generating family in $\mathfrak{B}_I$. If $a \in \mathfrak{B}_I$, there is a smallest countable set $J \subseteq I$ such that a belongs to the closed subalgebra $\mathfrak{C}_J$ of $\mathfrak{B}_I$ generated by $\{e_i : i \in J\}$.

Now suppose that $\langle a_\xi\rangle_{\xi\in\Gamma}$ is a family in $\mathfrak{B}_I$, that for each $\xi \in \Gamma$ we are given a set $I_\xi \subseteq I$ such that $a_\xi \in \mathfrak{C}_{I_\xi}$, and that $J \subseteq I$ is such that $I_\xi \cap I_\eta \subseteq J$ for all distinct ξ, $\eta \in \Gamma$. Then $\langle a_\xi\rangle_{\xi\in\Gamma}$ is relatively stochastically independent over $\mathfrak{C}_J$. It follows that if $\Delta \subseteq \Gamma$ is finite and $\inf_{\xi\in\Delta} \operatorname{upr}(a_\xi,\mathfrak{C}_J) \ne 0$, then $\inf_{\xi\in\Delta} a_\xi \ne 0$; if $\langle \operatorname{upr}(a_\xi,\mathfrak{C}_J)\rangle_{\xi\in\Gamma}$ is centered, so is $\langle a_\xi\rangle_{\xi\in\Gamma}$.

525I Theorem (a)(i) If $\kappa > 0$ and $(\kappa,\lambda,{<}\theta)$ is a measure-precaliber triple of $(\mathfrak{B}_\kappa,\bar{\nu}_\kappa)$, then it is a measure-precaliber triple of every probability algebra.

(ii) If $\kappa > 0$ and $(\kappa,\lambda,{<}\theta)$ is a precaliber triple of $\mathfrak{B}_\kappa$, then it is a precaliber triple of every measurable algebra.

(b) Suppose that $\operatorname{cf}\kappa \ge \omega_2$. If (κ,λ) is a precaliber pair of $\mathfrak{B}_{\kappa'}$ for every $\kappa' < \kappa$, then it is a precaliber pair of every measurable algebra.

(c) Suppose that $(\kappa,\lambda,{<}\theta)$ is a measure-precaliber triple of $(\mathfrak{B}_\omega,\bar{\nu}_\omega)$ and that κ' is such that $\operatorname{cf}[\kappa']^{\le\omega} < \operatorname{cf}\kappa$. Then $(\kappa,\lambda,{<}\theta)$ is a measure-precaliber triple of $(\mathfrak{B}_{\kappa'},\bar{\nu}_{\kappa'})$.

525J Corollary Suppose that κ is an infinite cardinal and $\kappa < \operatorname{cov}\mathcal{N}_\kappa$. Then κ is a measure-precaliber of every probability algebra.

525K Proposition Let $\kappa > \operatorname{non}\mathcal{N}_\omega$ be a regular cardinal such that $\operatorname{cf}[\lambda]^{\le\omega} < \kappa$ for every $\lambda < \kappa$. Then κ is a precaliber of every measurable algebra.

525L Proposition Suppose that λ and κ are infinite cardinals such that $\lambda^\omega < \operatorname{cf}\kappa \le \kappa \le 2^\lambda$, where λ^ω is the cardinal power. Then κ is a precaliber of every measurable algebra.

525M Proposition Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra and κ an infinite cardinal such that $\operatorname{cf}\kappa$ is a measure-precaliber of $(\mathfrak{A},\bar{\mu})$ and $\lambda^\omega < \kappa$ for every $\lambda < \kappa$. Then κ is a measure-precaliber of $(\mathfrak{A},\bar{\mu})$.

525N Proposition Let κ be either ω or a strong limit cardinal of countable cofinality, and suppose that $2^\kappa = \kappa^+$. Then κ^+ is not a precaliber of $\mathfrak{B}_\kappa$.

525O Proposition Suppose that the generalized continuum hypothesis is true.

(a) An infinite cardinal κ is a measure-precaliber of every probability algebra iff $\operatorname{cf}\kappa$ is not the successor of a cardinal of countable cofinality.

(b) An infinite cardinal κ is a precaliber of every measurable algebra iff $\operatorname{cf}\kappa$ is neither ω nor the successor of a cardinal of countable cofinality.

***525P Proposition** $(\mathfrak{m}_{\text{countable}}, \mathrm{FN}^*(\mathcal{P}\mathbb{N}))$ is not a precaliber pair of $\mathfrak{B}_\omega$.

525Q Lemma Let $(\mathfrak{A}, \bar\mu)$ be a semi-finite measure algebra, $\langle u_n\rangle_{n\in\mathbb{N}}$ a $\| \|_2$-bounded sequence in $L^2 = L^2(\mathfrak{A}, \bar\mu)^+$, and $\mathcal{F}$ a non-principal ultrafilter on $\mathbb{N}$. Suppose that $p \in [0, \infty[$ is such that $\sup_{n\in\mathbb{N}} \|u_n^p\|_2$ is finite, and set $v = \lim_{n\to\mathcal{F}} u_n$, $w = \lim_{n\to\mathcal{F}} u_n^p$, the limits being taken for the weak topology in L^2. Then $v^p \le w$.

525R Lemma Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra and $\langle u_n\rangle_{n\in\mathbb{N}}$ a $\| \|_\infty$-bounded sequence in $L^\infty(\mathfrak{A}, \bar\mu)^+$ such that $\delta = \inf_{n\in\mathbb{N}} \int u_n > 0$. Let $k_0, \dots, k_m$ be strictly positive integers with sum k. Suppose that $\gamma < \delta^k$.

(a) There are integers $n_0 < n_1 < \dots < n_m$ such that $\int \prod_{j=0}^m u_{n_j}^{k_j} \ge \gamma$.

(b) In fact, there is an infinite set $I \subseteq \mathbb{N}$ such that $\int \prod_{j=0}^m u_{n_j}^{k_j} \ge \gamma$ whenever $n_0, \dots, n_m$ belong to I and $n_0 < n_1 < \dots < n_m$.

525S Theorem Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra and κ an infinite cardinal. Let $\langle u_\xi\rangle_{\xi<\kappa}$ be a $\| \|_\infty$-bounded family in $L^\infty(\mathfrak{A})^+$. Set $\delta = \inf_{\xi<\kappa} \int u_\xi$. Then for any $k \in \mathbb{N}$ and $\gamma < \delta^{k+1}$ there is a $\Gamma \in [\kappa]^\kappa$ such that $\int \prod_{i=0}^k u_{\xi_i} \ge \gamma$ for all $\xi_0, \dots, \xi_k \in \Gamma$.

525T Corollary (a) If κ is an infinite cardinal and $k \in \mathbb{N}$, (κ, κ, k) is a measure-precaliber triple of every probability algebra.

(b) If κ is a cardinal of uncountable cofinality and $k \in \mathbb{N}$, (κ, κ, k) is a precaliber triple of every measurable algebra. In particular, every measurable algebra satisfies Knaster's condition.

(c) If κ is a cardinal of uncountable cofinality, $(\mathfrak{A}, \bar\mu)$ is a probability algebra, $k \ge 1$ and $\langle a_\xi\rangle_{\xi<\kappa}$ is a family in $\mathfrak{A} \setminus \{0\}$, then there are a $\delta > 0$ and a $\Gamma \in [\kappa]^\kappa$ such that $\bar\mu(\inf_{\xi\in I} a_\xi) \ge \delta$ for every $I \in [\Gamma]^k$.

(d) For any measurable algebra $\mathfrak{A}$, $\mathfrak{m}(\mathfrak{A}) \ge \mathfrak{m}_{\mathrm{K}}$; and if $\mathfrak{m}(\mathfrak{A}) > \omega_1$, then $\mathfrak{m}(\mathfrak{A}) \ge \mathfrak{m}_{\mathrm{pc}\omega_1}$. So if $\omega \le \kappa < \mathfrak{m}_K$, κ is a measure-precaliber of every probability algebra.

525Z Problem Can we, in ZFC, find an infinite cardinal κ which is not a measure-precaliber of all probability algebras?

Version of 24.1.14

526 Asymptotic density zero

In §491, I devoted some paragraphs to the ideal $\mathcal{Z}$ of subsets of $\mathbb{N}$ with asymptotic density zero, as part of an investigation into equidistributed sequences in topological measure spaces. Here I return to $\mathcal{Z}$ to examine its place in the Tukey ordering of partially ordered sets. We find that it lies strictly between $\mathbb{N}^\mathbb{N}$ and ℓ^1 (526B, 526J, 526L) but in some sense is closer to ℓ^1 (526Ga). On the way, I mention the ideal $\mathcal{N}wd$ of nowhere dense subsets of $\mathbb{N}^\mathbb{N}$ (526H-526L) and ideals of sets with negligible closures (526I-526M).

526A Proposition For $I \subseteq \mathbb{N}$, set $\nu I = \sup_{n\ge1} \frac{1}{n}\#(I \cap n)$.

(a) ν is a strictly positive submeasure on $\mathcal{P}\mathbb{N}$. We have a metric ρ on $\mathcal{P}\mathbb{N}$ defined by setting $\rho(I, J) = \nu(I \triangle J)$ for all I, $J \subseteq \mathbb{N}$, under which the Boolean operations $\cup$, $\cap$, $\triangle$ and $\setminus$ and upper asymptotic density $d^* : \mathcal{P}\mathbb{N} \to [0, 1]$ are uniformly continuous and $\mathcal{P}\mathbb{N}$ is complete.

(b) $\mathcal{Z}$ is a separable closed subset of $\mathcal{P}\mathbb{N}$.

(c) If $\mathcal{I} \subseteq \mathcal{Z}$ is such that $\sum_{I\in\mathcal{I}} \nu I$ is finite, then $\bigcup \mathcal{I} \in \mathcal{Z}$.

(d) With the subspace topology, $(\mathcal{Z}, \subseteq)$ is a metrizably compactly based directed set.

526B Proposition $\mathbb{N}^\mathbb{N} \preccurlyeq_{\mathrm{T}} \mathcal{Z} \preccurlyeq_{\mathrm{T}} \ell^1$.

526C Lemma Let $\langle(\mathfrak{A}_n, \bar\mu_n)\rangle_{n\in\mathbb{N}}$ be a sequence of purely atomic probability algebras, and $\mathfrak{A} = \prod_{n\in\mathbb{N}} \mathfrak{A}_n$ the simple product algebra. Then there is an order-continuous Boolean homomorphism $\pi : \mathfrak{A} \to \mathcal{P}\mathbb{N}$ such that $\limsup_{n\to\infty} \bar\mu_n a(n)$ is the upper asymptotic density $d^*(\pi a)$ for every $a \in \mathfrak{A}$; consequently, $\lim_{n\to\infty} \bar\mu_n a(n)$ is the asymptotic density $d(\pi a)$ of πa if either is defined.

526D Lemma Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra, and $\kappa \ge \max(\omega, c(\mathfrak{A}), \tau(\mathfrak{A}))$ a cardinal. Let $(\mathfrak{B}_\kappa, \bar{\nu}_\kappa)$ be the measure algebra of the usual measure on $\{0,1\}^\kappa$, and $\gamma > 0$. Then there is a function $\theta : \mathfrak{A} \to \mathfrak{B}_\kappa$ such that

(i) $\theta(\sup A) = \sup \theta[A]$ for every non-empty $A \subseteq \mathfrak{A}$ such that $\sup A$ is defined in $\mathfrak{A}$;

(ii) $\bar{\nu}_\kappa \theta(a) = 1 - e^{-\gamma\bar{\mu}a}$ for every $a \in \mathfrak{A}$, interpreting $e^{-\infty}$ as 0;

(iii) if $\langle a_i \rangle_{i\in I}$ is a disjoint family in $\mathfrak{A}$, and $\mathfrak{C}_i$ is the closed subalgebra of $\mathfrak{B}_\kappa$ generated by $\{\theta(a) : a \subseteq a_i\}$ for each i, then $\langle \mathfrak{C}_i \rangle_{i\in I}$ is stochastically independent.

526E Lemma Let $\langle(\mathfrak{A}_n, \bar{\mu}_n)\rangle_{n\in\mathbb{N}}$ be a sequence of finite probability algebras and $\langle \gamma_n \rangle_{n\in\mathbb{N}}$ a sequence in $]0,\infty[$. Write P for the set

$$\{p : p \in \prod_{n\in\mathbb{N}} \mathfrak{A}_n,\ \lim_{n\to\infty} \gamma_n \bar{\mu}_n p(n) = 0\},$$

with the ordering inherited from the product partial order on $\prod_{n\in\mathbb{N}} \mathfrak{A}_n$. Then $P \preccurlyeq_{\mathrm{T}} \mathcal{Z}$.

526F Theorem $(\ell^1, \le, \ell^1) \preccurlyeq_{\mathrm{GT}} (\mathbb{N}^{\mathbb{N}}, \le, \mathbb{N}^{\mathbb{N}}) \ltimes (\mathcal{Z}, \subseteq, \mathcal{Z})$.

526G Corollary Let $\mathcal{N}$ be the ideal of Lebesgue negligible subsets of $\mathbb{R}$.

(a) $\mathrm{add}_\omega \mathcal{Z} = \mathrm{add}\,\mathcal{N} = \mathrm{add}_\omega \ell^1$ and $\mathrm{cf}\,\mathcal{Z} = \mathrm{cf}\,\mathcal{N} = \mathrm{cf}\,\ell^1$.

(b) If $\mathcal{A} \subseteq \mathcal{Z}$ and $\#(\mathcal{A}) < \mathrm{add}\,\mathcal{N}$, there is a $J \in \mathcal{Z}$ such that $I \setminus J$ is finite for every $I \in \mathcal{A}$.

526H Proposition Let $\mathcal{N}\mathrm{wd}$ be the ideal of nowhere dense subsets of $\mathbb{N}^{\mathbb{N}}$ and $\mathcal{M}$ the ideal of meager subsets of $\mathbb{N}^{\mathbb{N}}$.

(a) $\mathcal{N}\mathrm{wd}$ is isomorphic, as partially ordered set, to $(\mathcal{N}\mathrm{wd})^{\mathbb{N}}$.

(b) $(\mathcal{N}\mathrm{wd}, \subseteq', [\mathcal{N}\mathrm{wd}]^{\le\omega}) \equiv_{\mathrm{GT}} (\mathcal{M}, \subseteq, \mathcal{M})$.

(c) $\mathcal{N}\mathrm{wd} \preccurlyeq_{\mathrm{T}} \ell^1$.

(d) Let X be a set and $\mathcal{V}$ a countable family of subsets of X. Set

$$\mathcal{D} = \{D : D \subseteq X, \text{ for every } V \in \mathcal{V} \text{ there is a } V' \in \mathcal{V} \text{ such that } V' \subseteq V \setminus D\}.$$

Then $\mathcal{D} \preccurlyeq_{\mathrm{T}} \mathcal{N}\mathrm{wd}$.

(e) If X is any non-empty Polish space without isolated points, and $\mathcal{N}\mathrm{wd}(X)$ is the ideal of nowhere dense subsets of X, then $\mathcal{N}\mathrm{wd} \equiv_{\mathrm{T}} \mathcal{N}\mathrm{wd}(X)$.

(f) If X is a compact metrizable space and $\mathcal{C}_{\mathrm{nwd}}$ is the family of closed nowhere dense subsets of X with the Fell topology, then $(\mathcal{C}_{\mathrm{nwd}}, \subseteq)$ is a metrizably compactly based directed set.

526I Proposition Let X be a second-countable topological space and μ a σ-finite topological measure on X. Let $\mathcal{E}$ be the ideal of subsets of X with negligible closures. Then, writing $\mathcal{N}\mathrm{wd}$ for the ideal of nowhere dense subsets of $\mathbb{N}^{\mathbb{N}}$, $\mathcal{E} \preccurlyeq_{\mathrm{T}} \mathcal{N}\mathrm{wd}$ and $\mathcal{E} \preccurlyeq_{\mathrm{T}} \mathcal{Z}$.

526J Proposition Let $\mathcal{E}_{\mathrm{Leb}}$ be the ideal of subsets of $\mathbb{R}$ whose closures are Lebesgue negligible. Then $\mathbb{N}^{\mathbb{N}} \preccurlyeq_{\mathrm{T}} \mathcal{E}_{\mathrm{Leb}}$ but $\mathcal{E}_{\mathrm{Leb}} \not\preccurlyeq_{\mathrm{T}} \mathbb{N}^{\mathbb{N}}$; consequently $\mathcal{Z} \not\preccurlyeq_{\mathrm{T}} \mathbb{N}^{\mathbb{N}}$, $\mathcal{N}\mathrm{wd} \not\preccurlyeq_{\mathrm{T}} \mathbb{N}^{\mathbb{N}}$ and $\ell^1 \not\preccurlyeq_{\mathrm{T}} \mathbb{N}^{\mathbb{N}}$.

526K Proposition Let $\mathcal{N}\mathrm{wd}$ be the ideal of nowhere dense subsets of $\mathbb{N}^{\mathbb{N}}$. Then $\mathcal{Z} \not\preccurlyeq_{\mathrm{T}} \mathcal{N}\mathrm{wd}$, so $\mathcal{Z} \not\preccurlyeq_{\mathrm{T}} \mathcal{E}_{\mathrm{Leb}}$ and $\ell^1 \not\preccurlyeq_{\mathrm{T}} \mathcal{N}\mathrm{wd}$.

526L Proposition $\mathcal{N}\mathrm{wd} \not\preccurlyeq_{\mathrm{T}} \mathcal{Z}$, so $\mathcal{N}\mathrm{wd} \not\preccurlyeq_{\mathrm{T}} \mathcal{E}_{\mathrm{Leb}}$ and $\ell^1 \not\preccurlyeq_{\mathrm{T}} \mathcal{Z}$.

526M Proposition Let X be a second-countable topological space and μ a σ-finite topological measure on X. Let $\mathcal{E}$ be the ideal of subsets of X with negligible closures, $\mathcal{N}(\mu)$ the null ideal of μ, and $\mathcal{M}$ the ideal of meager subsets of $\mathbb{N}^{\mathbb{N}}$. Then

$$(\mathcal{E}, \subseteq, \mathcal{N}(\mu)) \preccurlyeq_{\mathrm{GT}} (\mathcal{M}, \not\ni, \mathbb{N}^{\mathbb{N}});$$

consequently $\mathrm{add}(\mathcal{E}, \subseteq, \mathcal{N}(\mu)) \ge \mathfrak{m}_{\mathrm{countable}}$.

Version of 22.9.21

527 Skew products of ideals

The methods of this chapter can be applied to a large proportion of the partially ordered sets which arise in analysis. In this section I look at skew products of ideals, constructed by a method suggested by Fubini's theorem and the Kuratowski-Ulam theorem (527E). At the end of the section I introduce 'harmless' algebras (527M-527O).

527A Notation If (X,Σ,μ) is a measure space, $\mathcal{N}(\mu)$ will be the null ideal of μ; $\mathcal{N}$ will be the null ideal of Lebesgue measure on $\mathbb{R}$. If X is a topological space, $\mathcal{B}(X)$ will be the Borel σ-algebra of X and $\mathcal{M}(X)$ the σ-ideal of meager subsets of X; $\mathcal{M}$ will be the ideal $\mathcal{M}(\mathbb{R})$ of meager subsets of $\mathbb{R}$.

527B Skew products of ideals Suppose that $\mathcal{I} \lhd \mathcal{P}X$ and $\mathcal{J} \lhd \mathcal{P}Y$ are ideals of subsets of sets X, Y respectively.

(a) I will write $\mathcal{I} \ltimes \mathcal{J}$ for their **skew product** $\{W : W \subseteq X \times Y,\ \{x : W[\{x\}] \notin \mathcal{J}\} \in \mathcal{I}\}$. $\mathcal{I} \ltimes \mathcal{J} \lhd \mathcal{P}(X \times Y)$.

$\mathcal{I} \rtimes \mathcal{J}$ will be $\{W : W \subseteq X \times Y,\ \{y : W^{-1}[\{y\}] \notin \mathcal{I}\} \in \mathcal{J}\}$.

(b) Suppose that X and Y are not empty and that $\mathcal{I}$ and $\mathcal{J}$ are proper ideals. Then

$$\operatorname{add}(\mathcal{I} \ltimes \mathcal{J}) = \min(\operatorname{add}\mathcal{I}, \operatorname{add}\mathcal{J}), \quad \operatorname{cf}(\mathcal{I} \ltimes \mathcal{J}) \ge \max(\operatorname{cf}\mathcal{I}, \operatorname{cf}\mathcal{J}),$$

$$\operatorname{non}(\mathcal{I} \ltimes \mathcal{J}) = \max(\operatorname{non}\mathcal{I}, \operatorname{non}\mathcal{J}), \quad \operatorname{cov}(\mathcal{I} \ltimes \mathcal{J}) = \min(\operatorname{cov}\mathcal{I}, \operatorname{cov}\mathcal{J}).$$

(c) If Λ is a family of subsets of $X \times Y$, write $\mathcal{I} \ltimes_\Lambda \mathcal{J} \subseteq \mathcal{I} \ltimes \mathcal{J}$ for the ideal generated by $(\mathcal{I} \ltimes \mathcal{J}) \cap \Lambda$. Note that if $\kappa \le \min(\operatorname{add}\mathcal{I}, \operatorname{add}\mathcal{J})$ and $\bigcup \mathcal{W} \in \Lambda$ for every $\mathcal{W} \in [\Lambda]^{<\kappa}$, then $\operatorname{add}(\mathcal{I} \ltimes_\Lambda \mathcal{J}) \ge \kappa$; in particular, $\mathcal{I} \ltimes_\Lambda \mathcal{J}$ will be a σ-ideal whenever $\mathcal{I}$ and $\mathcal{J}$ are σ-ideals and Λ is a σ-algebra of subsets of $X \times Y$.

527C Theorem Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be σ-finite effectively locally finite τ-additive topological measure spaces, both measures being inner regular with respect to the Borel sets. Let $\tilde\lambda$ be the τ-additive product measure on $X \times Y$. Then $\mathcal{N}(\mu) \ltimes_{\mathcal{B}(X \times Y)} \mathcal{N}(\nu) = \mathcal{N}(\tilde\lambda)$.

527D Theorem Let X and Y be topological spaces, with product $X \times Y$. Write $\mathcal{M}^* = \mathcal{M}(X) \ltimes_{\mathcal{B}(X\times Y)} \mathcal{M}(Y)$ and $\mathcal{M}_1^* = \mathcal{M}(X) \ltimes_{\widehat{\mathcal{B}}(X\times Y)} \mathcal{M}(Y)$, writing $\widehat{\mathcal{B}}(X \times Y)$ for the Baire-property algebra of $X \times Y$.

(a) If $\mathcal{M}(X \times Y) \subseteq \mathcal{M}_1^*$, then $\mathcal{M}^* = \mathcal{M}_1^* = \mathcal{M}(X \times Y)$.

(b) Let $\mathfrak{G}$ be the category algebra of Y. If $\pi(\mathfrak{G}) < \operatorname{add}\mathcal{M}(X)$ then $\mathcal{M}^* = \mathcal{M}(X \times Y)$.

527E Corollary If X and Y are separable metrizable spaces, then $\mathcal{M}(X \times Y) = \mathcal{M}(X) \ltimes_{\mathcal{B}(X\times Y)} \mathcal{M}(Y)$.

527F Lemma Let X be a set, Σ a σ-algebra of subsets of X, and $\mathcal{I}$ a σ-ideal of subsets of X generated by $\Sigma \cap \mathcal{I}$; suppose that the quotient algebra $\Sigma/\Sigma \cap \mathcal{I}$ is non-zero, atomless and has countable π-weight. Let Y be a set, T a σ-algebra of subsets of Y, and $\langle \mathcal{H}_n \rangle_{n\in\mathbb{N}}$ a sequence of finite covers of Y by members of T. Set

$$\mathcal{H}_n^* = \{\bigcup_{m\ge n} H_m : H_m \in \mathcal{H}_m \cup \{\emptyset\} \text{ for every } m \ge n\}$$

for each $n \in \mathbb{N}$. Then there is a sequence $\langle W_n \rangle_{n\in\mathbb{N}}$ of subsets of $\mathbb{N}^{\mathbb{N}} \times X \times Y$ such that

(i) for every $n \in \mathbb{N}$, W_n is expressible as the union of a sequence of sets of the form $I \times E \times F$ where $I \subseteq \mathbb{N}^{\mathbb{N}}$ is open-and-closed, $E \in \Sigma$ and $F \in \mathrm{T}$;

(ii) whenever $n \in \mathbb{N}$, $\alpha \in \mathbb{N}^{\mathbb{N}}$ and $x \in X$ then $\{y : (\alpha,x,y) \in W_n\} \in \mathcal{H}_n^*$;

(iii) setting $W = \bigcap_{n\in\mathbb{N}} W_n$, the set $\{(\alpha,x) : \alpha \in \mathbb{N}^{\mathbb{N}},\ x \in X,\ (\alpha,x,f(x)) \notin W\}$ belongs to $[\mathbb{N}^{\mathbb{N}}]^{\le\omega} \ltimes \mathcal{I}$ for every (Σ,T)-measurable function $f : X \to Y$.

527G Theorem Let X be a set, Σ a σ-algebra of subsets of X, and $\mathcal{I}$ a σ-ideal of subsets of X which is generated by $\Sigma\cap\mathcal{I}$; suppose that the quotient algebra $\Sigma/\Sigma\cap\mathcal{I}$ is non-zero, atomless and has countable π-weight. Let (Y,T,ν) be an atomless perfect semi-finite measure space such that $\nu Y>0$. Set $\mathcal{K}=\mathcal{I}\ltimes_{\Sigma\widehat{\otimes}\mathrm{T}}\mathcal{N}(\nu)$. Then $[\mathfrak{c}]^{\le\omega}\preccurlyeq_{\mathrm{T}}\mathcal{K}$, so $\operatorname{add}\mathcal{K}=\omega_1$ and $\operatorname{cf}\mathcal{K}\ge\mathfrak{c}$.

527H Corollary $\mathcal{M}\ltimes_{\mathcal{B}(\mathbb{R}^2)}\mathcal{N}\equiv_{\mathrm{T}}[\mathfrak{c}]^{\le\omega}$.

527I Lemma Let X be a set, Σ a σ-algebra of subsets of X, and Y a topological space with a countable π-base $\mathcal{H}$. Let $\mathcal{W}$ be the family of subsets of $X\times Y$ of the form $\bigcup_{H\in\mathcal{H}}E_H\times H$, where $E_H\in\Sigma$ for every $H\in\mathcal{H}$, and $\mathcal{D}_0$ the family of sets $D\subseteq X\times Y$ such that $(X\times Y)\setminus D\in\mathcal{W}$ and $D[\{x\}]$ is nowhere dense for every $x\in X$; let $\mathcal{L}_0$ be the σ-ideal of subsets of $X\times Y$ generated by $\mathcal{D}_0$. Then $\Sigma\widehat{\otimes}\mathcal{B}(Y)\subseteq\{W\triangle L:W\in\mathcal{W},\ L\in\mathcal{L}_0\}$.

527J Theorem Let X be a topological space and μ a σ-finite quasi-Radon measure on X with countable Maharam type; let Y be a topological space of countable π-weight. Then $\mathcal{N}(\mu)\ltimes_{\mathcal{B}(X\times Y)}\mathcal{M}(Y)\preccurlyeq_{\mathrm{T}}\mathcal{N}$.

527K Corollary $\mathcal{N}\ltimes_{\mathcal{B}(\mathbb{R}^2)}\mathcal{M}\equiv_{\mathrm{T}}\mathcal{N}$.

527L Theorem Let X be a set, Σ a σ-ideal of subsets of X, and $\mathcal{I}\triangleleft\mathcal{P}X$ a σ-ideal; suppose that $\Sigma/\Sigma\cap\mathcal{I}$ is ccc. Let (Y,T,ν) be a σ-finite measure space. Then $(\Sigma\widehat{\otimes}\mathrm{T})/((\Sigma\widehat{\otimes}\mathrm{T})\cap(\mathcal{I}\ltimes\mathcal{N}(\nu)))$ is ccc.

527M Definition A Boolean algebra $\mathfrak{A}$ is **harmless** if it is ccc and whenever $\mathfrak{B}$ is a countable subalgebra of $\mathfrak{A}$, there is a regularly embedded countable subalgebra of $\mathfrak{A}$ including $\mathfrak{B}$.

527N Lemma (a) A Boolean algebra with a harmless order-dense subalgebra is itself harmless.

(b) If $\mathfrak{A}$ is a Dedekind complete Boolean algebra, then it is harmless iff every order-closed subalgebra of $\mathfrak{A}$ with countable Maharam type has countable π-weight.

(c) For any set I, the regular open algebra $\mathrm{RO}(\{0,1\}^I)$ of $\{0,1\}^I$ is harmless, so the category algebra of $\{0,1\}^I$ is harmless.

(d) If $\mathfrak{A}$ has countable π-weight it is harmless.

(e) If $\mathfrak{A}$ is a harmless Boolean algebra, $\mathfrak{B}$ is a Boolean algebra and $\pi:\mathfrak{A}\to\mathfrak{B}$ is a surjective order-continuous Boolean homomorphism, then $\mathfrak{B}$ is harmless. In particular, any principal ideal of a harmless Boolean algebra is harmless.

527O Theorem Let (X,Σ,μ) be a σ-finite measure space and Y a topological space such that the category algebra $\mathfrak{G}$ of Y is harmless. Write $\mathcal{L}$ for $(\Sigma\widehat{\otimes}\mathcal{B}(Y))\cap(\mathcal{N}(\mu)\ltimes\mathcal{M}(Y))$ and $\mathfrak{A}$ for the measure algebra of μ. Then $\mathfrak{C}=(\Sigma\widehat{\otimes}\mathcal{B}(Y))/\mathcal{L}$ is ccc, and is isomorphic to the Dedekind completion of the free product $\mathfrak{A}\otimes\mathfrak{G}$. If neither $\mathfrak{A}$ nor $\mathfrak{G}$ is trivial, the isomorphism corresponds to embeddings $E^\bullet\mapsto(E\times Y)^\bullet:\mathfrak{A}\to\mathfrak{C}$ and $F^\bullet\mapsto(X\times F)^\bullet:\mathfrak{B}\to\mathfrak{C}$.

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528 Amoeba algebras

In the course of investigating the principal consequences of Martin's axiom, Martin & Solovay 70 introduced the partially ordered set of open subsets of $\mathbb{R}$ with measure strictly less than γ, for $\gamma>0$ (528O). Elementary extensions of this idea lead us to a very interesting class of partially ordered sets, which I study here in terms of their regular open algebras, the 'amoeba algebras' (528A). Of course the most important ones are those associated with Lebesgue measure, and these are closely related to 'localization posets' (528I), themselves intimately connected with the localization relations of 522K. In the second half of the section I look at the cardinal functions of these algebras, of which the most interesting seems to be Maharam type (528V).

As elsewhere in this chapter, I will write $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$ for the measure algebra of the usual measure on $\{0,1\}^\kappa$. In any measure algebra $(\mathfrak{A},\bar\mu)$ I will write $\mathfrak{A}^f=\{a:a\in\mathfrak{A},\ \bar\mu a<\infty\}$.

528A Amoeba algebras Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra.

(a) If $0 < \gamma \le \bar{\mu}1$, the **amoeba algebra** $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$ is the regular open algebra $\mathrm{RO}^{\uparrow}(P)$ where $P = \{a : a \in \mathfrak{A},\ \bar{\mu}a < \gamma\}$, ordered by $\subseteq$.

(b) The **variable-measure amoeba algebra** $\mathrm{AM}^*(\mathfrak{A}, \bar{\mu})$ is the regular open algebra $\mathrm{RO}^{\uparrow}(P')$ where

$$P' = \{(a, \alpha) : a \in \mathfrak{A},\ \alpha \in \,]\bar{\mu}a, \bar{\mu}1]\},$$

ordered by saying that

$$(a, \alpha) \le (b, \beta) \text{ if } a \subseteq b \text{ and } \beta \le \alpha.$$

528B Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra and $0 < \gamma \le \bar{\mu}1$. Set $P = \{a : a \in \mathfrak{A},\ \bar{\mu}a < \gamma\}$.

(a) Two elements a, $b \in P$ are compatible upwards in P iff $\bar{\mu}(a \cup b) < \gamma$.

(b) Suppose that $(\mathfrak{A}, \bar{\mu})$ is semi-finite and atomless.

(i) P is separative upwards, so $[a, \infty[\, \in \mathrm{RO}^{\uparrow}(P)$ for every $a \in P$.

(ii) If $A \subseteq P$ is non-empty, then the infimum $\inf_{a \in A} [a, \infty[$ is empty unless $\sup A$ is defined in $\mathfrak{A}$ and belongs to P, and in this case $\inf_{a \in A} [a, \infty[\, = [\sup A, \infty[$.

528C Proposition Suppose that (X, Σ, μ) is a measure space, $(\mathfrak{A}, \bar{\mu})$ its measure algebra and $0 < \gamma \le \mu X$. If $\mathcal{E} \subseteq \Sigma$ is any family such that μ is outer regular with respect to $\mathcal{E}$, then $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$ is isomorphic to $\mathrm{RO}^{\uparrow}(\{E : E \in \mathcal{E},\ \mu E < \gamma\})$.

528D Proposition (a) Let $(\mathfrak{A}, \bar{\mu})$ be an atomless homogeneous probability algebra. Then the amoeba algebras $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$ and $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma')$ are isomorphic for all γ, $\gamma' \in \,]0, 1[$.

(b) Let $(\mathfrak{A}, \bar{\mu})$ be a non-totally-finite atomless quasi-homogeneous measure algebra. Then all the amoeba algebras $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$, for $\gamma > 0$, are isomorphic.

528E Lemma Let $(\mathfrak{A}, \bar{\mu})$ be an atomless semi-finite measure algebra. Then there is a family $\langle c_\alpha \rangle_{\alpha \in [0, \bar{\mu}1]}$ in $\mathfrak{A}$ such that $c_\alpha \subseteq c_\beta$ and $\bar{\mu}c_\alpha = \alpha$ whenever $0 \le \alpha \le \beta \le \bar{\mu}1$, and $\alpha \mapsto c_\alpha$ is continuous for the measure-algebra topology of $\mathfrak{A}$.

528F Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra, and $\gamma \in \,]0, \infty[$.

(a) Suppose that $e \in \mathfrak{A}$ and $\bar{\mu}e \ge \gamma$. If $\mathfrak{A}_e$ is atomless, then $\mathrm{AM}(\mathfrak{A}_e, \bar{\mu}\restriction\mathfrak{A}_e, \gamma)$ can be regularly embedded in $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$.

(b) Suppose that $\mathfrak{A}$ is atomless, and that $\gamma < \bar{\mu}1$. Let $\langle e_k \rangle_{k \in \mathbb{N}}$ be a non-decreasing sequence in $\mathfrak{A}$ with supremum 1, and suppose that $\bar{\mu}e_k \ge \gamma$ for every $k \in \mathbb{N}$. Then we have a sequence $\langle \pi_k \rangle_{k \in \mathbb{N}}$ such that $\pi_k : \mathrm{AM}(\mathfrak{A}_{e_k}, \bar{\mu}\restriction\mathfrak{A}_{e_k}, \gamma) \to \mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$ is a regular embedding for every $k \in \mathbb{N}$, and $\bigcup_{k \in \mathbb{N}} \pi_k[\mathrm{AM}(\mathfrak{A}_{e_k}, \bar{\mu}\restriction\mathfrak{A}_{e_k}, \gamma)]$ τ-generates $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$.

(c) Now suppose that $(\mathfrak{A}, \bar{\mu})$ is atomless and quasi-homogeneous, and that $\gamma < \bar{\mu}1$. Then $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$ can be regularly embedded in $\mathrm{AM}^*(\mathfrak{A}, \bar{\mu})$.

528G Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, and $\mathfrak{C}$ a σ-subalgebra of $\mathfrak{A}$ such that $\sup(\mathfrak{C} \cap \mathfrak{A}^f) = 1$ in $\mathfrak{A}$. Then $\mathrm{AM}^*(\mathfrak{C}, \bar{\mu}\restriction\mathfrak{C})$ can be regularly embedded in $\mathrm{AM}^*(\mathfrak{A}, \bar{\mu})$.

528H Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra, not $\{0\}$, and let $\kappa \ge \max(\omega, \tau(\mathfrak{A}), c(\mathfrak{A}))$ be a cardinal. Then $\mathrm{AM}^*(\mathfrak{A}, \bar{\mu})$ can be regularly embedded in $\mathrm{AM}(\mathfrak{B}_\kappa, \bar{\nu}_\kappa, \frac{1}{2})$.

528I Definition For any set I, the **(I, ∞)-localization poset** is the set

$$\mathcal{S}_I^\infty = \{p : p \subseteq \mathbb{N} \times I,\ \#(p[\{n\}]) \le 2^n \text{ for every } n,\ \sup_{n \in \mathbb{N}} \#(p[\{n\}]) \text{ is finite}\},$$

ordered by $\subseteq$. For $p \in \mathcal{S}_I^\infty$ set $\|p\| = \max_{n \in \mathbb{N}} \#(p[\{n\}])$. I will write $\mathcal{S}^\infty$ for $\mathcal{S}_{\mathbb{N}}^\infty$.

528J Proposition Let κ be an infinite cardinal, $\mathcal{S}_\kappa^\infty$ the (κ,∞)-localization poset, and $(\mathfrak{A},\bar\mu)$ a semi-finite measure algebra, not $\{0\}$, with $\kappa\ge\max(\omega,c(\mathfrak{A}),\tau(\mathfrak{A}))$. Then the variable-measure amoeba algebra $\mathrm{AM}^*(\mathfrak{A},\bar\mu)$ can be regularly embedded in $\mathrm{RO}^\uparrow(\mathcal{S}_\kappa^\infty)$.

528K Theorem Let $(\mathfrak{A},\bar\mu)$ be an atomless σ-finite measure algebra in which every non-zero principal ideal has Maharam type κ, and $0<\gamma<\bar\mu 1$. Then each of the algebras

$$\mathrm{AM}(\mathfrak{A},\bar\mu,\gamma),\quad \mathrm{AM}^*(\mathfrak{A},\bar\mu),\quad \mathrm{AM}(\mathfrak{B}_\kappa,\bar\nu_\kappa,\tfrac{1}{2})$$

can be regularly embedded in the other two, and all three can be regularly embedded in $\mathrm{RO}^\uparrow(\mathcal{S}_\kappa^\infty)$.

528L Lemma $\mathfrak{m}(\mathrm{AM}(\mathfrak{B}_\omega,\bar\nu_\omega,\frac{1}{2}))\le\operatorname{add}\mathcal{N}$, where $\mathcal{N}$ is the null ideal of Lebesgue measure on $\mathbb{R}$.

528M Lemma $\mathfrak{m}^\uparrow(\mathcal{S}^\infty)\ge\operatorname{add}\mathcal{N}$.

528N Theorem Let $(\mathfrak{A},\bar\mu)$ be an atomless σ-finite measure algebra with countable Maharam type, and $0<\gamma<\bar\mu 1$. Then the algebras $\mathrm{AM}(\mathfrak{A},\bar\mu,\gamma)$ and $\mathrm{AM}^*(\mathfrak{A},\bar\mu)$ and the $(\mathbb{N},\infty)$-localization poset $\mathcal{S}^\infty$ (active upwards) all have Martin numbers equal to $\operatorname{add}\mathcal{N}$.

528O Corollary Let $\gamma>0$. Let $\mathcal{G}$ be the partially ordered set

$$\{G:G\subseteq\mathbb{R}\text{ is open, }\mu_L G<\gamma\},$$

where μ_L is Lebesgue measure. Then $\mathfrak{m}^\uparrow(\mathcal{G})=\operatorname{add}\mathcal{N}$.

528P Proposition Let $(\mathfrak{A},\bar\mu)$ be an atomless semi-finite measure algebra, and $0<\gamma<\bar\mu 1$.
(a) For any integer $m\ge 2$,

$$c(\mathrm{AM}(\mathfrak{A},\bar\mu,\gamma))=\operatorname{link}_m(\mathrm{AM}(\mathfrak{A},\bar\mu,\gamma))=\max(c(\mathfrak{A}),\tau(\mathfrak{A})).$$

(b) $d(\mathrm{AM}(\mathfrak{A},\bar\mu,\gamma))=\pi(\mathrm{AM}(\mathfrak{A},\bar\mu,\gamma))=\max(\operatorname{cf}[c(\mathfrak{A})]^{\le\omega},\pi(\mathfrak{A}))$.

528Q Proposition Let $\mathcal{S}^\infty$ be the $(\mathbb{N},\infty)$-localization poset.
(a) $\pi(\mathrm{RO}^\uparrow(\mathcal{S}^\infty))=\operatorname{cf}\mathcal{S}^\infty=\mathfrak{c}$.
(b) For every $m\ge 2$,

$$c(\mathrm{RO}^\uparrow(\mathcal{S}^\infty))=c^\uparrow(\mathcal{S}^\infty)=\operatorname{link}_m(\mathrm{RO}^\uparrow(\mathcal{S}^\infty))=\operatorname{link}_m^\uparrow(\mathcal{S}^\infty)=\omega.$$

(c) $d(\mathrm{RO}^\uparrow(\mathcal{S}^\infty))=d^\uparrow(\mathcal{S}^\infty)=\operatorname{cf}\mathcal{N}$.

528R Theorem Let κ be any cardinal, and $\mathcal{S}_\kappa^\infty$ the (κ,∞)-localization poset. Then $\mathrm{RO}^\uparrow(\mathcal{S}_\kappa^\infty)$ has countable Maharam type.

528S Definition Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. I will say that a **well-spread basis** for $\mathfrak{A}$ is a non-decreasing sequence $\langle D_n\rangle_{n\in\mathbb{N}}$ of subsets of $\mathfrak{A}$ such that

(i) setting $D=\bigcup_{n\in\mathbb{N}}D_n$, $\#(D)\le\max(\omega,c(\mathfrak{A}),\tau(\mathfrak{A}))$;

(ii) if $a\in\mathfrak{A}$, $\gamma\in\mathbb{R}$ and $\bar\mu a<\gamma$, there is a set $D\subseteq\bigcup_{n\in\mathbb{N}}D_n$ such that $a\subseteq\sup D$ and $\bar\mu(\sup D)<\gamma$;

(iii) if $n\in\mathbb{N}$ and $\langle d_i\rangle_{i\in\mathbb{N}}$ is a sequence in D_n such that $\bar\mu(\sup_{i\in\mathbb{N}}d_i)<\infty$, there is an infinite set $J\subseteq\mathbb{N}$ such that $d=\sup_{i\in J}d_i$ belongs to D_n;

(iv) whenever $n\in\mathbb{N}$, $a\in\mathfrak{A}$ and $\bar\mu a\le\gamma'<\gamma<\bar\mu 1$, there is a $b\in\mathfrak{A}$ such that $a\subseteq b$ and $\gamma'\le\bar\mu b<\gamma$ and $\bar\mu(b\cup d)\ge\gamma$ whenever $d\in D_n$ and $d\not\subseteq a$.

528T Lemma (a) Let κ be an infinite cardinal, and $\langle e_\xi\rangle_{\xi<\kappa}$ the standard generating family in $\mathfrak{B}_\kappa$. For $n\in\mathbb{N}$ let C_n be the set of elements of $\mathfrak{B}_\kappa$ expressible as $\inf_{\xi\in I}e_\xi\cap\inf_{\xi\in J}(1\setminus e_\xi)$ where I, $J\subseteq\kappa$ are disjoint and $\#(I\cup J)\le n$. Then $\langle C_n\rangle_{n\in\mathbb{N}}$ is a well-spread basis for $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$. Moreover,

(*) for each $n \geq 1$, there is a set $C'_n \subseteq C_n$, with cardinal κ, such that $\bar{\nu}_\kappa c = 2^{-n}$ for every $c \in C'_n$, and whenever $a \in \mathfrak{B}_\kappa \setminus \{1\}$ and $I \subseteq C'_n$ is infinite, there is a $c \in I$ such that $c' \not\subseteq a \cup c$ whenever $c \subset c' \in C_n$.

(b) Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra and $e \in \mathfrak{A}$. If $\langle C_n \rangle_{n\in\mathbb{N}}$ is a well-spread basis for $(\mathfrak{A}_e, \bar{\mu}\restriction\mathfrak{A}_e)$ and $\langle D_n \rangle_{n\in\mathbb{N}}$ is a well-spread basis for $(\mathfrak{A}_{1\setminus e}, \bar{\mu}\restriction\mathfrak{A}_{1\setminus e})$, then $\langle C_n \cup D_n \rangle_{n\in\mathbb{N}}$ is a well-spread basis for $(\mathfrak{A}, \bar{\mu})$.

528U Lemma Let $(\mathfrak{A}, \bar{\mu})$ be an atomless semi-finite measure algebra and $0 < \gamma < \bar{\mu}1$. Let E, ϵ, $\preccurlyeq$ and $\mathcal{F}$ be such that

E is a partition of unity in $\mathfrak{A}$ such that $\mathfrak{A}_e$ is homogeneous and $0 < \epsilon \leq \bar{\mu}e < \infty$ for every $e \in E$;

$\preccurlyeq$ is a well-ordering of E such that $\tau(\mathfrak{A}_e) \leq \tau(\mathfrak{A}_{e'})$ whenever $e \preccurlyeq e'$ in E;

$\mathcal{F}$ is a partition of E such that each member of $\mathcal{F}$ is either a singleton or a countable set with no $\preccurlyeq$-greatest member.

Let P_0 be

$$\{a : a \in \mathfrak{A},\ \bar{\mu}a < \gamma,\ \gamma \leq \bar{\mu}(a \cup e) \text{ whenever } \{e\} \in \mathcal{F}\},$$

ordered by $\subseteq$. Then $\mathrm{RO}^{\uparrow}(P_0)$ has countable Maharam type.

528V Theorem Let $(\mathfrak{A}, \bar{\mu})$ be an atomless semi-finite measure algebra and $0 < \gamma < \bar{\mu}1$. Then $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$ has countable Maharam type.

528Z Problems (a) Let $(\mathfrak{A}_L, \bar{\mu}_L)$ be the measure algebra of Lebesgue measure on $\mathbb{R}$. Is the amoeba algebra $\mathrm{AM}(\mathfrak{A}_L, \bar{\mu}_L, 1)$ isomorphic to the amoeba algebra $\mathrm{AM}(\mathfrak{B}_\omega, \bar{\nu}_\omega, \frac{1}{2})$?

(b) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, $\mathfrak{B}$ a closed subalgebra of $\mathfrak{A}$, and $0 < \gamma < 1$. Is it necessarily true that $\mathrm{AM}(\mathfrak{B}, \bar{\mu}\restriction\mathfrak{B}, \gamma)$ can be regularly embedded in $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$?

Version of 26.5.11

529 Further partially ordered sets of measure theory

I end the chapter with notes on some more structures which can be approached by the methods used earlier. The Banach lattices of Chapter 36 are of course partially ordered sets, and many of them can easily be assigned places in the Tukey classification (529C, 529D). More surprising is the fact that the Novák numbers of $\{0,1\}^I$, for large I, are supported by the additivity of Lebesgue measure (529F); this is associated with an interesting property of the localization poset from the last section (529E). There is a similarly unexpected connexion between the covering number of Lebesgue measure and 'reaping numbers' $\mathfrak{r}(\omega_1, \lambda)$ for large λ (529H).

529A Notation I will write $\mathcal{N}(\mu)$ for the null ideal of μ in a measure space (X, Σ, μ), and $\mathcal{N}$ for the null ideal of Lebesgue measure on $\mathbb{R}$.

529B Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra.

(a) For $p \in [1, \infty[$, give $L^p = L^p(\mathfrak{A}, \bar{\mu})$ its norm topology. Then its topological density is

$$\begin{aligned} d(L^p) &= 1 \text{ if } \mathfrak{A} = \{0\}, \\ &= \omega \text{ if } 0 < \#(\mathfrak{A}) < \omega, \\ &= \max(c(\mathfrak{A}), \tau(\mathfrak{A})) \text{ if } \mathfrak{A} \text{ is infinite.} \end{aligned}$$

(b) Give $L^0 = L^0(\mathfrak{A})$ its topology of convergence in measure. Then

$$\begin{aligned} d(L^0) &= 1 \text{ if } \mathfrak{A} = \{0\}, \\ &= \omega \text{ if } 0 < \#(\mathfrak{A}) < \omega, \\ &= \tau(\mathfrak{A}) \text{ if } \mathfrak{A} \text{ is infinite.} \end{aligned}$$

529C Theorem Let U be an L-space. Then $U \equiv_{\mathrm{T}} \ell^1(\kappa)$, where $\kappa = \dim U$ if U is finite-dimensional, and otherwise is the topological density of U.

529D Theorem Let $\mathfrak{A}$ be a homogeneous measurable algebra with Maharam type $\kappa \geq \omega$. Then $L^0(\mathfrak{A}) \equiv_{\mathrm{T}} \ell^1(\kappa)$.

529E Proposition Let $\mathcal{S}^\infty$ be the $(\mathbb{N}, \infty)$-localization poset. Then $\mathrm{RO}(\{0,1\}^{\mathfrak{c}})$ can be regularly embedded in $\mathrm{RO}^{\uparrow}(\mathcal{S}^\infty)$.

529F Corollary $n(\{0,1\}^I) \geq \operatorname{add}\mathcal{N}$ for every set I.

529G Reaping numbers For cardinals $\theta \leq \lambda$ let $\mathfrak{r}(\theta, \lambda)$ be the smallest cardinal of any set $\mathcal{A} \subseteq [\lambda]^\theta$ such that for every $B \subseteq \lambda$ there is an $A \in \mathcal{A}$ such that either $A \subseteq B$ or $A \cap B = \emptyset$.

529H Proposition $\mathfrak{r}(\omega_1, \lambda) \geq \operatorname{cov}\mathcal{N}$ for all uncountable λ.