Version of 9.1.15

Introduction to Volume 5

For the final volume of this treatise, I have collected results which demand more sophisticated set theory than elsewhere. The line is not sharp, but typically we are much closer to questions which are undecidable in ZFC. Only in Chapter 55 are these brought to the forefront of the discussion, but elsewhere much of the work depends on formulations carefully chosen to express, as arguments in ZFC, ideas which arose in contexts in which some special axiom – Martin's axiom, for instance – was being assumed. This has forced the development of concepts – e.g., cardinal functions of structures – which have taken on vigorous lives of their own, and which stand outside the territory marked by the techniques of earlier volumes.

In terms of the classification I have used elsewhere, this volume has one preparatory chapter and five working chapters. There is practically no measure theory in Chapter 51, which is an introduction to some of the methods which have been devised to make sense of abstract analysis in the vast range of alternative mathematical worlds which have become open to us in the last fifty years. It is centered on a study of partially ordered sets, which provide a language in which many of the most important principles can be expressed. Chapter 52 looks at manifestations of these ideas in measure theory. In Chapter 53 I continue the work of Volumes 3 and 4, examining questions which arise more or less naturally if we approach the topics of those volumes with the new techniques.

The Banach-Ulam problem got a mention in Volume 2, a paragraph in Volume 3 and a section in Volume 4; at last, in Chapter 54 of the present volume, I try to give a proper account of the extraordinary ideas to which it has led. I have regretfully abandoned the idea of describing even a representative sample of the forcing models which have been devised to show that measure-theoretic propositions are consistent, but in Chapter 55 I set out some of the basic properties of random real forcing. Finally, in Chapter 56, I look at what measure theory becomes in ZF alone, with countable or dependent choice, and with the axiom of determinacy.

While I should like to believe that most of the material of this volume will be accessible to those who have learnt measure theory from other sources, it has obviously been written with earlier volumes constantly in mind, and I have to advise you to make sure that Volumes 3 and 4, at least, will be available in case of need. Apart from these, I do of course assume that readers will be at ease with modern set theory. It is not so much that I demand a vast amount of knowledge – §§5A1-5A2 have a good many proofs to help cover any gaps – as that I present arguments without much consideration for the inexperienced, and some of them may be indigestible at first if you have not cut your teeth on Just & Weese 96 or Jech 78. What you may not need is any prior knowledge of forcing. But of course for Chapter 55 you will have to take a proper introduction to forcing, e.g., Kunen 80, in parallel with §5A3, since nothing here will make sense without an acquaintance with forcing languages and the fundamental theorem of forcing.

Note on second printing

There has been the usual crop of errors (most, but not all, minor) to be corrected, and I have added a few new results. The most important is P.Larson's proof that it is relatively consistent with ZFC to suppose that there is no medial limit. In the process of preparing new editions of Volumes 1-4, I have I hope covered all the items listed in the old §5A6 ('Later editions only'), which I have therefore dropped, even though there are one or two further entries under this heading. As before, these can be found on the Web edition at `http://www1.essex.ac.uk/maths/people/fremlin/mtcont.htm`.

Version of 3.1.15

Chapter 51

Cardinal functions

The primary object of this volume is to explore those topics in measure theory in which questions arise which are undecided by the ordinary axioms of set theory. We immediately face a new kind of interaction between the propositions we consider. If two statements are undecidable, we can ask whether either implies the other. Almost at once we find ourselves trying to make sense of a bewildering tangle of uncoordinated patterns. The most successful method so far found of listing the multiple connexions present is to reduce as many arguments as possible to investigations of the relationships between specially defined cardinal numbers. In any particular model of set theory (so long as we are using the axiom of choice) these numbers must be in a linear order, so we can at least estimate the number of potential configurations, and focus our attention on the possibilities which seem most accessible or most interesting. At the very beginning of the theory, for instance, we can ask whether $\mathfrak{c} = 2^\omega$ is equal to ω_1, or ω_2, or ω_{ω_1}, or 2^{ω_1}. For Lebesgue measure, perhaps the first question to ask is: if $\langle E_\xi\rangle_{\xi<\omega_1}$ is a family of measurable sets, is $\bigcup_{\xi<\omega_1} E_\xi$ necessarily measurable? If the continuum hypothesis is true, certainly not; but if $\mathfrak{c} > \omega_1$, either 'yes' or 'no' becomes possible. The way in which it is now customary to express this is to say that '$\omega_1 \le \operatorname{add}\mathcal{N} \le \mathfrak{c}$, and $\omega_1 \le \operatorname{add}\mathcal{N} < \mathfrak{c}$, $\omega_1 < \operatorname{add}\mathcal{N} \le \mathfrak{c}$ and $\omega_1 < \operatorname{add}\mathcal{N} < \mathfrak{c}$ are all possible', where $\operatorname{add}\mathcal{N}$ is defined as the least cardinal of any family $\mathcal{E}$ of Lebesgue measurable sets such that $\bigcup\mathcal{E}$ is not measurable. (Actually it is not usually defined in quite this way, but that is what it comes to.)

At this point I suggest that you turn to 522B, where you will find a classic picture ('Cichoń's diagram') of the relationships between ten cardinals intermediate between ω_1 and $\mathfrak{c}$, with $\operatorname{add}\mathcal{N}$ immediately above ω_1. As this diagram already makes clear, one can define rather a lot of cardinal numbers. Furthermore, the relationships between them are not entirely expressible in terms of the partial order in which we say that $\kappa_{\mathfrak{a}} \preccurlyeq \kappa_{\mathfrak{b}}$ if we can prove in ZFC that $\kappa_{\mathfrak{a}} \le \kappa_{\mathfrak{b}}$. Even in Cichoń's diagram we have results of the type $\operatorname{add}\mathcal{M} = \min(\mathfrak{b}, \operatorname{cov}\mathcal{M})$ in which three cardinals are involved. It is clear that the framework which has been developed over the last thirty-five years is only a beginning. Nevertheless, I am confident that it will maintain a leading role as the theory evolves. The point is that at least some of the cardinals ($\operatorname{add}\mathcal{N}$, $\mathfrak{b}$ and $\operatorname{cov}\mathcal{M} = \mathfrak{m}_{\text{countable}}$, for instance) describe such important features of such important structures that they appear repeatedly in arguments relating to diverse topics, and give us a chance to notice unexpected connexions.

The first step is to list and classify the relevant cardinals. This is the purpose of the present chapter. In fact the definitions here are mostly of a general type. Associated with any ideal of sets, for instance, we have four cardinals ('additivity', 'cofinality', 'unformity' and 'covering number'; see 511F). Most of the cardinals examined in this volume can be defined by one of a limited number of processes from some more or less naturally arising structure; thus $\operatorname{add}\mathcal{N}$, already mentioned, is normally defined as the additivity of the ideal of Lebesgue negligible subsets of $\mathbb{R}$, and $\operatorname{cov}\mathcal{M}$ is the covering number of the ideal of meager subsets of $\mathbb{R}$. Another important type of definition is in terms of whole classes of structure: thus Martin's cardinal $\mathfrak{m}$ can be regarded as the least Martin number (definition: 511Dg) of any ccc Boolean algebra.

§511 lists some of the cardinals associated with partially ordered sets, Boolean algebras, topological spaces and ideals of sets. Which structures count as 'naturally arising' is a matter of taste and experience, but it turns out that many important ideas can be expressed in terms of cardinals associated with relations, and some of these are investigated in §512. The core ideas of the chapter are most clearly manifest in their application to partially ordered sets, which I look at in §513. In §514 I run through the elementary results connecting the cardinal functions of topological spaces and associated Boolean algebras and partially ordered sets. §515 is a brief excursion into abstract Boolean algebra. §516 is a discussion of 'precalibers'. §517 is an introduction to the theory of 'Martin numbers', which (following the principles I have just tried to explain) I will use as vehicles for the arguments which have been used to make deductions from Martin's axiom. §518 gives results on Freese-Nation numbers and tight filtrations of Boolean algebras which can be expressed in general terms and are relevant to questions in measure theory.

Version of 10.10.13

511 Definitions

A large proportion of the ideas of this volume will be expressed in terms of cardinal numbers associated with the structures of measure theory. For any measure space (X,Σ,μ) we have, at least, the structures (X,Σ), $(X,\Sigma,\mathcal{N}(\mu))$ (where $\mathcal{N}(\mu)$ is the null ideal of μ) and the measure algebra $\mathfrak{A} = \Sigma/\Sigma\cap\mathcal{N}(\mu)$; each of these types of structure has a family of cardinal functions associated with it, starting from the obvious ones $\#(X)$, $\#(\Sigma)$ and $\#(\mathfrak{A})$. For the measure algebra $\mathfrak{A}$, we quickly find that we have cardinals naturally associated with its Boolean structure and others naturally associated with the topological structure of its

Stone space; of course the most important ones are those which can be described in both languages. The actual measure $\mu : \Sigma \to [0,\infty]$, and its daughter $\bar\mu : \mathfrak{A} \to [0,\infty]$, will be less conspicuous here; for most of the questions addressed in this volume, replacing a measure by another with the same measurable sets and the same negligible sets will make no difference.

In this section I list the definitions on which the rest of the chapter depends, with a handful of elementary results to give you practice with the definitions.

511A Pre-ordered sets A **pre-ordered set** is a set P together with a relation $\le$ on P such that

if $p \le q$ and $q \le r$ then $p \le r$,

$p \le p$ for every $p \in P$.

As with partial orders, I will write $p \ge q$ to mean $q \le p$; $[p,q] = \{r : p \le r \text{ and } r \le q\}$; $[p,\infty[= \{q : p \le q\}$, $]{-\infty},p] = \{q : q \le p\}$. An **upper** (resp. **lower**) **bound** for a set $A \subseteq P$ will be a $p \in P$ such that $q \le p$ (resp. $p \le q$) for every $q \in A$. If $(Q,\le)$ is another pre-ordered set, I will say that $f : P \to Q$ is **order-preserving** if $f(p) \le f(p')$ whenever $p \le p'$ in P. If $\langle (P_i,\le_i)\rangle_{i\in I}$ is a family of pre-ordered sets, their **product** is the pre-ordered set $(P,\le)$ where $P = \prod_{i\in I} P_i$ and, for p, $q \in P$, $p \le q$ iff $p(i) \le_i q(i)$ for every $i \in I$.

If $(P,\le)$ is a pre-ordered set, we have an equivalence relation $\sim$ on P defined by saying that $p \sim q$ if $p \le q$ and $q \le p$. Now we have a canonical partial order on the set $\tilde P$ of equivalence classes defined by saying that $p^\bullet \le q^\bullet$ iff $p \le q$.

511B Definitions Let $(P,\le)$ be any pre-ordered set.

(a) A subset Q of P is **cofinal** with P if for every $p \in P$ there is a $q \in Q$ such that $p \le q$. The **cofinality** of P, $\operatorname{cf} P$, is the least cardinal of any cofinal subset of P.

(b) The **additivity** of P, $\operatorname{add} P$, is the least cardinal of any subset of P with no upper bound in P. If there is no such set, write $\operatorname{add} P = \infty$.

(c) A subset Q of P is **coinitial** with P if for every $p \in P$ there is a $q \in Q$ such that $q \le p$. The **coinitiality** of P, $\operatorname{ci} P$, is the least cardinal of any coinitial subset of P.

(d) Two elements p, p' of P are **compatible upwards** if $[p,\infty[\cap [p',\infty[\ne \emptyset$; otherwise they are **incompatible upwards**. A subset A of P is an **up-antichain** if no two distinct elements of A are compatible upwards. The **upwards cellularity** of P is $c^\uparrow(P) = \sup\{\#(A) : A \subseteq P$ is an up-antichain in $P\}$; the **upwards saturation** of P, $\operatorname{sat}^\uparrow(P)$, is the least cardinal κ such that there is no up-antichain in P with cardinal κ. P is called **upwards-ccc** if it has no uncountable up-antichain.

(e) Two elements p, p' of P are **compatible downwards** if $]{-\infty},p] \cap]{-\infty},p'] \ne \emptyset$; otherwise they are **incompatible downwards**. A subset A of P is a **down-antichain** if no two distinct elements of A are compatible downwards. The **downwards cellularity** of P is $c^\downarrow(P) = \sup\{\#(A) : A \subseteq P$ is a down-antichain in $P\}$; the **downwards saturation** of P, $\operatorname{sat}^\downarrow(P)$, is the least κ such that there is no down-antichain in P with cardinal κ. P is called **downwards-ccc** if it has no uncountable down-antichain.

(f) If κ is a cardinal, a subset A of P is **upwards-$<\kappa$-linked** in P if every subset of A of cardinal less than κ is bounded above in P. The **upwards $<\kappa$-linking number** of P, $\operatorname{link}^\uparrow_{<\kappa}(P)$, is the smallest cardinal of any cover of P by upwards-$<\kappa$-linked sets.

A subset A of P is **upwards-κ-linked** in P if it is upwards-$<\kappa^+$-linked. The **upwards κ-linking number** of P, $\operatorname{link}^\uparrow_\kappa(P) = \operatorname{link}^\uparrow_{<\kappa^+}(P)$, is the smallest cardinal of any cover of P by upwards-κ-linked sets.

Similarly, a subset A of P is **downwards-$<\kappa$-linked** if every member of $[A]^{<\kappa}$ has a lower bound in P, and **downwards-κ-linked** if it is downwards-$<\kappa^+$-linked; the **downwards $<\kappa$-linking number** of P, $\operatorname{link}^\downarrow_{<\kappa}(P)$, is the smallest cardinal of any cover of P by downwards-$<\kappa$-linked sets, and $\operatorname{link}^\downarrow_\kappa(P) = \operatorname{link}^\downarrow_{<\kappa^+}(P)$.

(g) The most important cases of (f) above are $\kappa = 2$ and $\kappa = \omega$. A subset A of P is **upwards-linked** if any two members of A are compatible upwards in P, and **upwards-centered** if it is upwards-$<\omega$-linked. The **upwards linking number** of P, $\operatorname{link}^\uparrow(P) = \operatorname{link}^\uparrow_2(P)$, is the least cardinal of any cover of P by upwards-linked sets, and the **upwards centering number** of P, $d^\uparrow(P) = \mathrm{link}^\uparrow_{<\omega}(P)$, is the least cardinal of any cover of P by upwards-centered sets.

$A \subseteq P$ is **downwards-linked** if any two members of A are compatible downwards in P, and **downwards-centered** if any finite subset of A has a lower bound in P; the **downwards linking number** of P is $\mathrm{link}^\downarrow(P) = \mathrm{link}^\downarrow_2(P)$, and the **downwards centering number** of P is $d^\downarrow(P) = \mathrm{link}^\downarrow_{<\omega}(P)$.

If $\mathrm{link}^\uparrow(P) \le \omega$ (resp. $\mathrm{link}^\downarrow(P) \le \omega$) we say that P is **σ-linked upwards** (resp. **downwards**). If $d^\uparrow(P) \le \omega$ (resp. $d^\downarrow(P) \le \omega$) we say that P is **σ-centered upwards** (resp. **downwards**).

(h) The **upwards Martin number** $\mathfrak{m}^\uparrow(P)$ of P is the smallest cardinal of any family $\mathcal{Q}$ of cofinal subsets of P such that there is some $p \in P$ such that no upwards-linked subset of P containing p meets every member of $\mathcal{Q}$; if there is no such family $\mathcal{Q}$, write $\mathfrak{m}^\uparrow(P) = \infty$.

Similarly, the **downwards Martin number** $\mathfrak{m}^\downarrow(P)$ of P is the smallest cardinal of any family $\mathcal{Q}$ of coinitial subsets of P such that there is some $p \in P$ such that no downwards-linked subset of P containing p meets every member of $\mathcal{Q}$, or ∞ if there is no such $\mathcal{Q}$.

(i) A **Freese-Nation function** on P is a function $f : P \to \mathcal{P}P$ such that whenever $p \le q$ in P then $[p,q] \cap f(p) \cap f(q)$ is non-empty. The **Freese-Nation number** of P, $\mathrm{FN}(P)$, is the least κ such that there is a Freese-Nation function $f : P \to [P]^{<\kappa}$. The **regular Freese-Nation number** of P, $\mathrm{FN}^*(P)$, is the least regular infinite κ such that there is a Freese-Nation function $f : P \to [P]^{<\kappa}$. If Q is a subset of P, the **Freese-Nation index** of Q in P is the least cardinal κ such that $\mathrm{cf}(Q \cap]-\infty, p]) < \kappa$ and $\mathrm{ci}(Q \cap [p, \infty[) < \kappa$ for every $p \in P$.

(j) The **(principal) bursting number** $\mathrm{bu}\, P$ of P is the least cardinal κ such that there is a cofinal subset Q of P such that

$$\#(\{q : q \in Q,\ q \le p,\ p \not\le q\}) < \kappa$$

for every $p \in P$.

(k) I will say that P is **separative upwards** if whenever $p, q \in P$ and $p \not\le q$ there is a $q' \ge q$ which is incompatible upwards with p. P is **separative downwards** if whenever $p, q \in P$ and $p \not\ge q$ there is a $q' \le q$ which is incompatible downwards with p.

511D Definitions Let $\mathfrak{A}$ be a Boolean algebra. I write $\mathfrak{A}^+$ for the set $\mathfrak{A} \setminus \{0\}$ of non-zero elements of $\mathfrak{A}$ and $\mathfrak{A}^-$ for $\mathfrak{A} \setminus \{1\}$, so that the partially ordered sets $(\mathfrak{A}^-, \subseteq)$ and $(\mathfrak{A}^+, \supseteq)$ are isomorphic.

(a) The **Maharam type** $\tau(\mathfrak{A})$ of $\mathfrak{A}$ is the smallest cardinal of any subset of $\mathfrak{A}$ which τ-generates $\mathfrak{A}$.

(b) The **cellularity** of $\mathfrak{A}$ is

$$c(\mathfrak{A}) = c^\uparrow(\mathfrak{A}^-) = c^\downarrow(\mathfrak{A}^+) = \sup\{\#(C) : C \subseteq \mathfrak{A}^+ \text{ is disjoint}\}.$$

The **saturation** of $\mathfrak{A}$ is

$$\mathrm{sat}(\mathfrak{A}) = \mathrm{sat}^\uparrow(\mathfrak{A}^-) = \mathrm{sat}^\downarrow(\mathfrak{A}^+) = \sup\{\#(C)^+ : C \subseteq \mathfrak{A}^+ \text{ is disjoint}\}.$$

(c) The **π-weight** or **density** $\pi(\mathfrak{A})$ of $\mathfrak{A}$ is $\mathrm{cf}\, \mathfrak{A}^- = \mathrm{ci}\, \mathfrak{A}^+$.

(d) Let κ be a cardinal. A subset A of $\mathfrak{A}^+$ is **$<\kappa$-linked** if it is downwards-$<\kappa$-linked in $\mathfrak{A}^+$, and **κ-linked** if it is $<\kappa^+$-linked. The **$<\kappa$-linking number** $\mathrm{link}_{<\kappa}(\mathfrak{A})$ of $\mathfrak{A}$ is $\mathrm{link}^\downarrow_{<\kappa}(\mathfrak{A}^+)$, the least cardinal of any family of $<\kappa$-linked sets covering $\mathfrak{A}^+$; and the **κ-linking number** $\mathrm{link}_\kappa(\mathfrak{A})$ of $\mathfrak{A}$ is $\mathrm{link}_{<\kappa^+}(\mathfrak{A})$.

(e) $A \subseteq \mathfrak{A}^+$ is **linked** if no two members of A are disjoint; the **linking number** of $\mathfrak{A}$ is $\mathrm{link}(\mathfrak{A}) = \mathrm{link}_2(\mathfrak{A})$, the least cardinal of any cover of $\mathfrak{A}^+$ by linked sets. $A \subseteq \mathfrak{A}^+$ is **centered** if $\inf I \ne 0$ for any finite $I \subseteq A$. The **centering number** $d(\mathfrak{A})$ of $\mathfrak{A}$ is $d^\uparrow(\mathfrak{A}^-) = d^\downarrow(\mathfrak{A}^+)$. $\mathfrak{A}$ is **σ-m-linked** if $\mathrm{link}_m(\mathfrak{A}) \le \omega$; it is **$\sigma$-linked** iff $\mathrm{link}(\mathfrak{A}) \le \omega$. $\mathfrak{A}$ is **σ-centered** if $d(\mathfrak{A}) \le \omega$.

(f) If κ is any cardinal, $\mathfrak{A}$ is **weakly** (κ,∞)**-distributive** if whenever $\langle A_\xi\rangle_{\xi<\kappa}$ is a family of partitions of unity in $\mathfrak{A}$, there is a partition B of unity such that $\{a : a\in A_\xi,\ a\cap b\ne 0\}$ is finite for every $b\in B$ and $\xi<\kappa$. Now the **weak distributivity** $\operatorname{wdistr}(\mathfrak{A})$ of $\mathfrak{A}$ is the least cardinal κ such that $\mathfrak{A}$ is not weakly (κ,∞)-distributive. (If there is no such cardinal, write $\operatorname{wdistr}(\mathfrak{A})=\infty$.)

(g) The **Martin number** $\mathfrak{m}(\mathfrak{A})$ of $\mathfrak{A}$ is the downwards Martin number of $\mathfrak{A}^+$.

(h) The **Freese-Nation number** of $\mathfrak{A}$, $\mathrm{FN}(\mathfrak{A})$, is the Freese-Nation number of the partially ordered set $(\mathfrak{A},\subseteq)$. The **regular Freese-Nation number** $\mathrm{FN}^*(\mathfrak{A})$ of $\mathfrak{A}$ is the regular Freese-Nation number of $(\mathfrak{A},\subseteq)$.

(i) If κ is a cardinal, a **tight** κ**-filtration** of $\mathfrak{A}$ is a family $\langle a_\xi\rangle_{\xi<\zeta}$ in $\mathfrak{A}$, where ζ is an ordinal, such that, writing $\mathfrak{A}_\alpha$ for the subalgebra of $\mathfrak{A}$ generated by $\{a_\xi : \xi<\alpha\}$, (α) $\mathfrak{A}_\zeta=\mathfrak{A}$ (β) for every $\alpha<\zeta$, the Freese-Nation index of $\mathfrak{A}_\alpha$ in $\mathfrak{A}$ is at most κ. If $\mathfrak{A}$ has a tight κ-filtration, I will say that it is **tightly** κ**-filtered**.

511E Precalibers (a) Let $(P,\le)$ be a pre-ordered set.

(i) I will say that $(\kappa,\lambda,{<}\theta)$ is an **upwards precaliber triple** of P if κ, λ and θ are cardinals, and whenever $\langle p_\xi\rangle_{\xi<\kappa}$ is a family in P then there is a set $\Gamma\in[\kappa]^\lambda$ such that $\{p_\xi : \xi\in I\}$ has an upper bound in P for every $I\in[\Gamma]^{<\theta}$.

$(\kappa,\lambda,{<}\theta)$ is a **downwards precaliber triple** of P if κ, λ and θ are cardinals and whenever $\langle p_\xi\rangle_{\xi<\kappa}$ is a family in P then there is a set $\Gamma\in[\kappa]^\lambda$ such that $\{p_\xi : \xi\in I\}$ has a lower bound in P for every $I\in[\Gamma]^{<\theta}$.

(ii) An **upwards precaliber pair** of P is a pair (κ,λ) of cardinals such that $(\kappa,\lambda,{<}\omega)$ is an upwards precaliber triple of P.

A **downwards precaliber pair** of P is a pair (κ,λ) of cardinals such that $(\kappa,\lambda,{<}\omega)$ is a downwards precaliber triple of P.

(iii) An **up-** (resp. **down-**) **precaliber** of P is a cardinal κ such that (κ,κ) is an upwards (resp. downwards) precaliber pair of P.

(b) Let $(X,\mathfrak{T})$ be a topological space. Then $(\kappa,\lambda,{<}\theta)$ is a **precaliber triple** of X if it is a downwards precaliber triple of $\mathfrak{T}\setminus\{\emptyset\}$; (κ,λ) is a **precaliber pair** of X if it is a downwards precaliber pair of $\mathfrak{T}\setminus\{\emptyset\}$; and κ is a **precaliber** of X if it is a down-precaliber of $\mathfrak{T}\setminus\{\emptyset\}$.

(c) Let $\mathfrak{A}$ be a Boolean algebra. Then $(\kappa,\lambda,{<}\theta)$ is a **precaliber triple** of $\mathfrak{A}$ if it is a downwards precaliber triple of $\mathfrak{A}^+$; (κ,λ) is a **precaliber pair** of $\mathfrak{A}$ if it is a downwards precaliber pair of $\mathfrak{A}^+$; and κ is a **precaliber** of $\mathfrak{A}$ if it is a down-precaliber of $\mathfrak{A}^+$.

(d) If $(\mathfrak{A},\bar\mu)$ is a measure algebra, then $(\kappa,\lambda,{<}\theta)$ is a **measure-precaliber triple** of $(\mathfrak{A},\bar\mu)$ if whenever $\langle a_\xi\rangle_{\xi<\kappa}$ is a family in $\mathfrak{A}$ such that $\inf_{\xi<\kappa}\bar\mu a_\xi>0$, then there is a $\Gamma\in[\kappa]^\lambda$ such that $\{a_\xi : \xi\in I\}$ has a non-zero lower bound for every $I\in[\Gamma]^{<\theta}$. Now (κ,λ) is a **measure-precaliber pair** of $(\mathfrak{A},\bar\mu)$ if $(\kappa,\lambda,{<}\omega)$ is a measure-precaliber triple, and κ is a **measure-precaliber** of $(\mathfrak{A},\bar\mu)$ if (κ,κ) is a measure-precaliber pair.

(e) In this context, I will say that (κ,λ,θ) is a precaliber triple (in any sense) if $(\kappa,\lambda,{<}\theta^+)$ is a precaliber triple as defined above; and similarly for measure-precaliber triples.

(f) I will say that one of the structures here satisfies **Knaster's condition** if it has $(\omega_1,\omega_1,2)$ as a precaliber triple. (For pre-ordered sets I will speak of 'Knaster's condition upwards' or 'Knaster's condition downwards'.) A structure satisfying Knaster's condition must be ccc.

511F Definitions Let X be a set and $\mathcal{I}$ an ideal of subsets of X.

(a) Taking $\mathcal{I}$ to be partially ordered by $\subseteq$, we can speak of $\operatorname{add}\mathcal{I}$ and $\operatorname{cf}\mathcal{I}$. $\mathcal{I}$ is called κ**-additive** or κ**-complete** if $\kappa\le\operatorname{add}\mathcal{I}$.

(b) The **uniformity** of $\mathcal{I}$ is

$$\operatorname{non}\mathcal{I} = \min\{\#(A) : A \subseteq X,\ A \notin \mathcal{I}\},$$

or ∞ if there is no such set A.

(c) The **shrinking number** of $\mathcal{I}$, $\operatorname{shr}\mathcal{I}$, is the smallest cardinal κ such that whenever $A \in \mathcal{P}X \setminus \mathcal{I}$ there is a $B \in [A]^{\le\kappa} \setminus \mathcal{I}$. The **augmented shrinking number** $\operatorname{shr}^+(\mathcal{I})$ is the smallest κ such that whenever $A \in \mathcal{P}X \setminus \mathcal{I}$ there is a $B \in [A]^{<\kappa} \setminus \mathcal{I}$.

(d) The **covering number** of $\mathcal{I}$ is

$$\operatorname{cov}\mathcal{I} = \min\{\#(\mathcal{E}) : \mathcal{E} \subseteq \mathcal{I},\ \bigcup\mathcal{E} = X\},$$

or ∞ if there is no such set $\mathcal{E}$.

511G Definition Let (X, Σ, μ) be a measure space.

(a) If κ is a cardinal, μ is **κ-additive** if $\bigcup\mathcal{E} \in \Sigma$ and $\mu(\bigcup\mathcal{E}) = \sum_{E\in\mathcal{E}} \mu E$ for every disjoint family $\mathcal{E} \in [\Sigma]^{<\kappa}$. The **additivity** $\operatorname{add}\mu$ of μ is the largest cardinal κ such that μ is κ-additive, or ∞ if μ is κ-additive for every κ.

(b) The **π-weight** $\pi(\mu)$ of μ is the coinitiality of $\Sigma \setminus \mathcal{N}(\mu)$, where $\mathcal{N}(\mu)$ is the null ideal of μ.

511H Elementary facts: pre-ordered sets Let P be a pre-ordered set.

(a) If $\tilde{P}$ is the partially ordered set of equivalence classes in P, as described in 511A, all the cardinal functions defined in 511B have the same values for P and $\tilde{P}$. P and $\tilde{P}$ will have the same triple precalibers, precaliber pairs and precalibers.

(b) $c^\uparrow(P) \le \operatorname{sat}^\uparrow(P)$. If $\kappa \le \lambda$ are cardinals then

$$\operatorname{link}^\uparrow_{<\kappa}(P) \le \operatorname{link}^\uparrow_{<\lambda}(P) \le \operatorname{cf} P.$$

$c^\uparrow(P) \le \operatorname{link}^\uparrow(P)$.

$$\operatorname{link}^\uparrow(P) = \operatorname{link}^\uparrow_{<3}(P) \le \operatorname{link}^\uparrow_{<\omega}(P) = d^\uparrow(P) \le \operatorname{cf} P.$$

$\operatorname{cf} P \le \#(P)$.

$$\operatorname{link}^\downarrow_{<\kappa}(P) \le \operatorname{link}^\downarrow_{<\lambda}(P) \le \operatorname{ci} P$$

whenever $\kappa \le \lambda$, and

$$c^\downarrow(P) \le \operatorname{link}^\downarrow(P) \le d^\downarrow(P) \le \operatorname{ci} P \le \#(P), \quad c^\downarrow(P) \le \operatorname{sat}^\downarrow P.$$

(c) P is empty iff $\operatorname{cf} P = 0$ iff $\operatorname{ci} P = 0$ iff $\operatorname{add} P = 0$ iff $d^\uparrow(P) = 0$ iff $d^\downarrow(P) = 0$ iff $\operatorname{link}^\uparrow(P) = 0$ iff $\operatorname{link}^\downarrow(P) = 0$ iff $c^\uparrow(P) = 0$ iff $c^\downarrow(P) = 0$ iff $\operatorname{sat}^\uparrow(P) = 1$ iff $\operatorname{sat}^\downarrow(P) = 1$ iff $\operatorname{FN}(P) = 0$.

(d) P is upwards-directed iff $c^\uparrow(P) \le 1$ iff $\operatorname{sat}^\uparrow(P) \le 2$ iff $\operatorname{link}^\uparrow(P) \le 1$ iff $d^\uparrow(P) \le 1$. P is downwards-directed iff $c^\downarrow(P) \le 1$ iff $\operatorname{sat}^\downarrow(P) \le 2$ iff $\operatorname{link}^\downarrow(P) \le 1$ iff $d^\downarrow(P) \le 1$.

If P is not empty, it is upwards-directed iff $\operatorname{add} P > 2$ iff $\operatorname{add} P \ge \omega$.

(e) If P is partially ordered, it has a greatest element iff $\operatorname{cf} P = 1$ iff $\operatorname{add} P = \infty$. Otherwise, $\operatorname{add} P \le \operatorname{cf} P$.

(f) If P is totally ordered, then $\operatorname{cf} P \le \operatorname{add} P$.

(g) If $\langle P_i\rangle_{i\in I}$ is a non-empty family of non-empty pre-ordered sets with product P, then $\operatorname{add} P = \min_{i\in I} \operatorname{add} P_i$.

(h) If P is partially ordered and $f : P \to \mathcal{P}P$ is a Freese-Nation function then $p \in f(p)$ for every $p \in P$

511I Elementary facts: Boolean algebras Let $\mathfrak{A}$ be a Boolean algebra.

(a)

$$\mathrm{link}_{<\kappa}(\mathfrak{A}) \le \mathrm{link}_{<\lambda}(\mathfrak{A}) \le \pi(\mathfrak{A})$$

whenever $\kappa \le \lambda$,

$$c(\mathfrak{A}) \le \mathrm{link}(\mathfrak{A}) \le d(\mathfrak{A}) \le \pi(\mathfrak{A}) \le \#(\mathfrak{A}), \quad c(\mathfrak{A}) \le \mathrm{sat}(\mathfrak{A}).$$

$\tau(\mathfrak{A}) \le \pi(\mathfrak{A})$.

(b)(i) $\mathfrak{A} = \{0\}$ iff $\pi(\mathfrak{A}) = 0$ iff $\mathrm{link}(\mathfrak{A}) = 0$ iff $d(\mathfrak{A}) = 0$ iff $c(\mathfrak{A}) = 0$ iff $\mathrm{sat}(\mathfrak{A}) = 1$.

(ii) $\tau(\mathfrak{A}) = 0$ iff $\mathfrak{A}$ is either $\{0\}$ or $\{0, 1\}$.

(c) If $\mathfrak{A}$ is finite, then $c(\mathfrak{A}) = \mathrm{link}(\mathfrak{A}) = d(\mathfrak{A}) = \pi(\mathfrak{A})$ is the number of atoms of $\mathfrak{A}$, $\mathrm{sat}(\mathfrak{A}) = c(\mathfrak{A}) + 1$ and $\#(\mathfrak{A}) = 2^{c(\mathfrak{A})}$, while $\tau(\mathfrak{A}) = \lceil \log_2 c(\mathfrak{A}) \rceil$, unless $\mathfrak{A} = \{0\}$, in which case $\tau(\mathfrak{A}) = 0$. If $\mathfrak{A}$ is infinite then $c(\mathfrak{A})$, $\mathrm{link}(\mathfrak{A})$, $d(\mathfrak{A})$, $\pi(\mathfrak{A})$, $\mathrm{sat}(\mathfrak{A})$ and $\tau(\mathfrak{A})$ are all infinite.

(d) $\mathfrak{A}$ is 'ccc' just when $c(\mathfrak{A}) \le \omega$, that is, $\mathrm{sat}(\mathfrak{A}) \le \omega_1$. $\mathfrak{A}$ is weakly (σ, ∞)-distributive iff $\mathrm{wdistr}(\mathfrak{A}) \ge \omega_1$.

(e)(i) If $\mathfrak{A}$ is purely atomic, $\mathrm{wdistr}(\mathfrak{A}) = \infty$.

(ii) If $\mathfrak{A}$ is not purely atomic, $\mathrm{wdistr}(\mathfrak{A}) \le \pi(\mathfrak{A})$.

(f) $\mathfrak{m}(\mathfrak{A}) = \infty$ iff $\mathfrak{A}$ is purely atomic.

511J Elementary facts: ideals of sets Let X be a set and $\mathcal{I}$ an ideal of subsets of X.

(a) $\mathrm{add}\,\mathcal{I} \ge \omega$.

(b) $\mathrm{shr}\,\mathcal{I} = \sup\{\mathrm{non}(A, \mathcal{I} \cap \mathcal{P}A) : A \in \mathcal{P}X \setminus \mathcal{I}\}$, counting $\sup\emptyset$ as 0; $\mathrm{shr}\,\mathcal{I} \le \#(X)$; $\mathrm{shr}\,\mathcal{I} \le \mathrm{shr}^+\,\mathcal{I} \le (\mathrm{shr}\,\mathcal{I})^+$; if $\mathrm{shr}\,\mathcal{I}$ is a successor cardinal, $\mathrm{shr}^+\,\mathcal{I} = (\mathrm{shr}\,\mathcal{I})^+$.

(c) Suppose that $\mathcal{I}$ covers X but does not contain X. Then $\mathrm{add}\,\mathcal{I} \le \mathrm{cov}\,\mathcal{I} \le \mathrm{cf}\,\mathcal{I}$ and $\mathrm{add}\,\mathcal{I} \le \mathrm{non}\,\mathcal{I} \le \mathrm{shr}\,\mathcal{I} \le \mathrm{cf}\,\mathcal{I}$.

(d) Suppose that $X \in \mathcal{I}$. Then $\mathrm{add}\,\mathcal{I} = \mathrm{non}\,\mathcal{I} = \infty$, $\mathrm{cov}\,\mathcal{I} \le 1$ (with $\mathrm{cov}\,\mathcal{I} = 0$ iff $X = \emptyset$) and $\mathrm{shr}\,\mathcal{I} = 0$.

(e) Suppose that $\mathcal{I}$ has a greatest member which is not X. Then $\mathrm{add}\,\mathcal{I} = \mathrm{cov}\,\mathcal{I} = \infty$ and $\mathrm{non}\,\mathcal{I} = \mathrm{shr}\,\mathcal{I} = \mathrm{cf}\,\mathcal{I} = 1$.

(f) Suppose that $\mathcal{I}$ has no greatest member and does not cover X. Then $\mathrm{add}\,\mathcal{I} \le \mathrm{cf}\,\mathcal{I}$, $\mathrm{non}\,\mathcal{I} = \mathrm{shr}\,\mathcal{I} = 1$ and $\mathrm{cov}\,\mathcal{I} = \infty$.

(g) Suppose that $Y \subseteq X$, and set $\mathcal{I}_Y = \mathcal{I} \cap \mathcal{P}Y$, regarded as an ideal of subsets of Y. Then $\mathrm{add}\,\mathcal{I}_Y \ge \mathrm{add}\,\mathcal{I}$, $\mathrm{non}\,\mathcal{I}_Y \ge \mathrm{non}\,\mathcal{I}$, $\mathrm{shr}\,\mathcal{I}_Y \le \mathrm{shr}\,\mathcal{I}$, $\mathrm{shr}^+\,\mathcal{I}_Y \le \mathrm{shr}^+\,\mathcal{I}$, $\mathrm{cov}\,\mathcal{I}_Y \le \mathrm{cov}\,\mathcal{I}$ and $\mathrm{cf}\,\mathcal{I}_Y \le \mathrm{cf}\,\mathcal{I}$.

Version of 27.11.13

512 Galois-Tukey connections

One of the most powerful methods of relating the cardinals associated with two partially ordered sets P and Q is to identify a 'Tukey function' from one to the other (513D). It turns out that the idea can be usefully generalized to other relational structures through the concept of 'Galois-Tukey connection' (512A). In this section I give the elementary theory of these connections and their effect on simple cardinal functions.

512A Definitions (a) A **supported relation** is a triple (A,R,B) where A and B are sets and R is a subset of $A\times B$.

(b) If (A,R,B) is a supported relation, its **dual** is the supported relation $(A,R,B)^\perp=(B,S,A)$ where

$$S=(B\times A)\setminus R^{-1}=\{(b,a):a\in A,\ b\in B,\ (a,b)\notin R\}.$$

(c) If (A,R,B) and (C,S,D) are supported relations, a **Galois-Tukey connection** from (A,R,B) to (C,S,D) is a pair (ϕ,ψ) such that $\phi:A\to C$ and $\psi:D\to B$ are functions and $(a,\psi(d))\in R$ whenever $(\phi(a),d)\in S$.

(d) If (A,R,B) and (C,S,D) are supported relations, I write $(A,R,B)\preccurlyeq_{\mathrm{GT}}(C,S,D)$ if there is a Galois-Tukey connection from (A,R,B) to (C,S,D), and $(A,R,B)\equiv_{\mathrm{GT}}(C,S,D)$ if $(A,R,B)\preccurlyeq_{\mathrm{GT}}(C,S,D)$ and $(C,S,D)\preccurlyeq_{\mathrm{GT}}(A,R,B)$.

512B Definitions (a) If (A,R,B) is a supported relation, its **covering number** $\mathrm{cov}(A,R,B)$ is the least cardinal of any set $C\subseteq B$ such that $A\subseteq R^{-1}[C]$; or ∞ if $A\not\subseteq R^{-1}[B]$. Its **additivity** is $\mathrm{add}(A,R,B)=\mathrm{cov}(A,R,B)^\perp$, that is, the smallest cardinal of any subset $C\subseteq A$ such that $C\not\subseteq R^{-1}[\{b\}]$ for any $b\in B$; or ∞ if there is no such C.

(b) If (A,R,B) is a supported relation, its **saturation** $\mathrm{sat}(A,R,B)$ is the least cardinal κ such that whenever $\langle a_\xi\rangle_{\xi<\kappa}$ is a family in A then there are distinct ξ, $\eta<\kappa$ and a $b\in B$ such that (a_ξ,b) and (a_η,b) both belong to R; if there is no such κ I write $\mathrm{sat}(A,R,B)=\infty$.

(c) If (A,R,B) is a supported relation and κ is a cardinal, say that a subset A' of A is **$<\kappa$-linked** if for every $I\in[A']^{<\kappa}$ there is a $b\in B$ such that $I\subseteq R^{-1}[\{b\}]$, and **κ-linked** if it is $<\kappa^+$-linked. Now the **$<\kappa$-linking number** $\mathrm{link}_{<\kappa}(A,R,B)$ of (A,R,B) is the least cardinal of any cover of A by $<\kappa$-linked sets, if there is such a cover, and otherwise is ∞; and the **κ-linking number** $\mathrm{link}_\kappa(A,R,B)$ of (A,R,B) is $\mathrm{link}_{<\kappa^+}(A,R,B)$.

If $\kappa\le\lambda$, then $\mathrm{link}_{<\kappa}(A,R,B)\le\mathrm{link}_{<\lambda}(A,R,B)$. $\mathrm{link}_\kappa(A,R,B)$ is equal to $\mathrm{cov}(A,R,B)$ for every $\kappa\ge\#(A)$, so that $\mathrm{link}_{<\theta}(A,R,B)\le\mathrm{cov}(A,R,B)$ for every θ.

512C Theorem Let (A,R,B), (C,S,D) and (E,T,F) be supported relations.

(a) $(A,R,B)^{\perp\perp}=(A,R,B)$.

(b) If (ϕ,ψ) is a Galois-Tukey connection from (A,R,B) to (C,S,D) and (ϕ',ψ') is a Galois-Tukey connection from (C,S,D) to (E,T,F), then $(\phi'\phi,\psi\psi')$ is a Galois-Tukey connection from (A,R,B) to (E,T,F).

(c) If (ϕ,ψ) is a Galois-Tukey connection from (A,R,B) to (C,S,D), then (ψ,ϕ) is a Galois-Tukey connection from $(C,S,D)^\perp$ to $(A,R,B)^\perp$.

(d) If $R'\subseteq R$ then $(A,R,B)\preccurlyeq_{\mathrm{GT}}(A,R',B)$.

(e) If $(A,R,B)\preccurlyeq_{\mathrm{GT}}(C,S,D)$ and $(C,S,D)\preccurlyeq_{\mathrm{GT}}(E,T,F)$ then $(A,R,B)\preccurlyeq_{\mathrm{GT}}(E,T,F)$.

(f) $\equiv_{\mathrm{GT}}$ is an equivalence relation on the class of supported relations.

(g) If $(A,R,B)\preccurlyeq_{\mathrm{GT}}(C,S,D)$ then $(C,S,D)^\perp\preccurlyeq_{\mathrm{GT}}(A,R,B)^\perp$. So if $(A,R,B)\equiv_{\mathrm{GT}}(C,S,D)$ then $(A,R,B)^\perp\equiv_{\mathrm{GT}}(C,S,D)^\perp$.

512D Theorem Let (A,R,B) and (C,S,D) be supported relations such that $(A,R,B)\preccurlyeq_{\mathrm{GT}}(C,S,D)$. Then

(a) $\mathrm{cov}(A,R,B)\le\mathrm{cov}(C,S,D)$;

(b) $\mathrm{add}(C,S,D)\le\mathrm{add}(A,R,B)$;

(c) $\mathrm{sat}(A,R,B)\le\mathrm{sat}(C,S,D)$;

(d) $\mathrm{link}_{<\kappa}(A,R,B)\le\mathrm{link}_{<\kappa}(C,S,D)$ for every cardinal κ.

512E Examples (a) Let $(P,\le)$ be a pre-ordered set. Then $(P,\le,P)$ and $(P,\ge,P)$ are supported relations, with duals $(P,\not\ge,P)$ and $(P,\not\le,P)$. $\mathrm{cov}(P,\le,P)=\mathrm{cf}\,P$, $\mathrm{cov}(P,\ge,P)=\mathrm{ci}\,P$, $\mathrm{add}(P,\le,P)=\mathrm{add}\,P$ and $\mathrm{sat}(P,\le,P)=\mathrm{sat}^\uparrow(P)$. $\mathrm{link}^\uparrow_{<\kappa}(P)=\mathrm{link}_{<\kappa}(P,\le,P)$. In particular, $d^\uparrow(P)=\mathrm{link}_{<\omega}(P,\le,P)$ (511Bg).

(b) Let $(X, \mathfrak{T})$ be a topological space. Then

$$\pi(X) = \operatorname{cov}(\mathfrak{T} \setminus \{\emptyset\}, \supseteq, \mathfrak{T} \setminus \{\emptyset\}),$$

$$d(X) = \operatorname{cov}(\mathfrak{T} \setminus \{\emptyset\}, \ni, X) = \operatorname{add}(X, \notin, \mathfrak{T} \setminus \{\emptyset\}),$$

$$\operatorname{sat}(X) = \operatorname{sat}(\mathfrak{T} \setminus \{\emptyset\}, \supseteq, \mathfrak{T} \setminus \{\emptyset\}) = \operatorname{sat}(\mathfrak{T} \setminus \{\emptyset\}, \ni, X),$$

$$n(X) = \operatorname{cov}(X, \in, \mathcal{N}\mathrm{wd}(X)) = \operatorname{cov}(X, \mathcal{N}\mathrm{wd}(X))$$

where $\mathcal{N}\mathrm{wd}(X)$ is the ideal of nowhere dense subsets of X. Note that if $\mathcal{M}(X)$ is the ideal of meager subsets of X, then $\operatorname{cov}(X, \mathcal{M}(X)) = n(X)$ unless $n(X) = \omega$, in which case $\operatorname{cov}(X, \mathcal{M}(X)) = 1$.

(c) Let $\mathfrak{A}$ be a Boolean algebra. Write $\mathfrak{A}^+$ for $\mathfrak{A} \setminus \{0\}$ and $\mathfrak{A}^-$ for $\mathfrak{A} \setminus \{1\}$. Then

$$\pi(\mathfrak{A}) = \operatorname{cov}(\mathfrak{A}^+, \supseteq, \mathfrak{A}^+) = \operatorname{cov}(\mathfrak{A}^-, \subseteq, \mathfrak{A}^-),$$

$$\operatorname{sat}(\mathfrak{A}) = \operatorname{sat}(\mathfrak{A}^+, \supseteq, \mathfrak{A}^+) = \operatorname{sat}(\mathfrak{A}^-, \subseteq, \mathfrak{A}^-),$$

$$d(\mathfrak{A}) = \operatorname{link}_{<\omega}(\mathfrak{A}^+, \supseteq, \mathfrak{A}^+) = \operatorname{link}_{<\omega}(\mathfrak{A}^-, \subseteq, \mathfrak{A}^-),$$

$$\operatorname{link}(\mathfrak{A}) = \operatorname{link}_2(\mathfrak{A}^+, \supseteq, \mathfrak{A}^+) = \operatorname{link}_2(\mathfrak{A}^-, \subseteq, \mathfrak{A}^-)$$

and generally

$$\operatorname{link}_{<\kappa}(\mathfrak{A}) = \operatorname{link}_{<\kappa}(\mathfrak{A}^+, \supseteq, \mathfrak{A}^+) = \operatorname{link}_{<\kappa}(\mathfrak{A}^-, \subseteq, \mathfrak{A}^-),$$

$$\operatorname{link}_{\kappa}(\mathfrak{A}) = \operatorname{link}_{\kappa}(\mathfrak{A}^+, \supseteq, \mathfrak{A}^+) = \operatorname{link}_{\kappa}(\mathfrak{A}^-, \subseteq, \mathfrak{A}^-)$$

for every cardinal κ.

(d) Let X be a set and $\mathcal{I}$ an ideal of subsets of X. Then the dual of $(X, \in, \mathcal{I})$ is $(\mathcal{I}, \not\ni, X)$; $\operatorname{cov}(X, \in, \mathcal{I}) = \operatorname{cov} \mathcal{I}$ and $\operatorname{add}(X, \in, \mathcal{I}) = \operatorname{non} \mathcal{I}$.

(e) For a Boolean algebra $\mathfrak{A}$, write $\operatorname{Pou}(\mathfrak{A})$ for the set of partitions of unity in $\mathfrak{A}$. For C, $D \in \operatorname{Pou}(\mathfrak{A})$, say that $C \sqsubseteq^* D$ if every element of D meets only finitely many members of C. $\sqsubseteq^*$ is a pre-order on $\operatorname{Pou}(\mathfrak{A})$. $\operatorname{wdistr}(\mathfrak{A}) = \operatorname{add} \operatorname{Pou}(\mathfrak{A})$.

512F Dominating sets For any supported relation (A, R, B) and any cardinal κ, we can form a corresponding supported relation $(A, R', [B]^{<\kappa})$, where

$$R' = \{(a, I) : I \in [B]^{<\kappa},\ a \in R^{-1}[I]\}.$$

When κ is a successor cardinal I will normally write $(A, R', [B]^{\le\lambda})$ rather than $(A, R', [B]^{<\lambda^+})$.

512G Proposition Let (A, R, B) and (C, S, D) be supported relations and κ, λ cardinals.

(a) (A, R, B) is isomorphic to $(A, R', [B]^1)$.

(b) If $(A, R, B) \preccurlyeq_{\mathrm{GT}} (C, S, D)$ and $\lambda \le \kappa$ then $(A, R', [B]^{<\kappa}) \preccurlyeq_{\mathrm{GT}} (C, S', [D]^{<\lambda})$.

(c) In particular, $(A, R', [B]^{<\kappa}) \preccurlyeq_{\mathrm{GT}} (A, R, B)$ if $\kappa \ge 2$.

(d) If $\operatorname{cf} \kappa \ge \lambda$ and $(A, R', [B]^{<\kappa}) \preccurlyeq_{\mathrm{GT}} (C, S, D)$ then $(A, R', [B]^{<\kappa}) \preccurlyeq_{\mathrm{GT}} (C, S', [D]^{<\lambda})$.

(e)(i) If $\operatorname{cov}(A, R, B) = \infty$ then $\operatorname{add}(A, R', [B]^{<\kappa}) \le 1$.

(ii) If $\operatorname{cov}(A, R, B) < \infty$ then $\operatorname{add}(A, R', [B]^{<\kappa}) \ge \kappa$.

(f) $\operatorname{cov}(A, R, B) \le \max(\omega, \kappa, \operatorname{cov}(A, R', [B]^{\le\kappa}))$; if $\kappa \ge 1$ and $\operatorname{cov}(A, R, B) > \max(\kappa, \omega)$ then $\operatorname{cov}(A, R, B) = \operatorname{cov}(A, R', [B]^{\le\kappa})$.

512H Simple products (a) If $\langle (A_i, R_i, B_i) \rangle_{i \in I}$ is any family of supported relations, its **simple product** is $(\prod_{i \in I} A_i, T, \prod_{i \in I} B_i)$ where $T = \{(a, b) : (a(i), b(i)) \in R_i \text{ for every } i \in I\}$.

(b) Let $\langle (A_i, R_i, B_i) \rangle_{i \in I}$ and $\langle (C_i, S_i, D_i) \rangle_{i \in I}$ be two families of supported relations, with simple products (A, R, B) and (C, S, D). If $(A_i, R_i, B_i) \preccurlyeq_{\mathrm{GT}} (C_i, S_i, D_i)$ for every i, then $(A, R, B) \preccurlyeq_{\mathrm{GT}} (C, S, D)$.

(c) Let $\langle(A_i,R_i,B_i)\rangle_{i\in I}$ be a family of supported relations with simple product (A,R,B). Suppose that no A_i is empty. Then $\operatorname{add}(A,R,B)=\min_{i\in I}\operatorname{add}(A_i,R_i,B_i)$, interpreting $\min\emptyset$ as ∞ if $I=\emptyset$.

(d) Suppose that (A,R,B) and (C,S,D) are supported relations with simple product $(A\times C,T,B\times D)$. Let κ be an infinite cardinal and define $(A,R',[B]^{<\kappa})$, $(C,S',[D]^{<\kappa})$ and $(A\times C,T',[B\times D]^{<\kappa})$ as in 512F. Then

$$(A,R',[B]^{<\kappa})\times(C,S',[D]^{<\kappa})\equiv_{\mathrm{GT}}(A\times C,T',[B\times D]^{<\kappa}).$$

(e) If $\langle(P_i,\le_i)\rangle_{i\in I}$ is a family of pre-ordered sets, with product $(P,\le)$, then $(P,\le,P)$ is just $\prod_{i\in I}(P_i,\le_i,P_i)$ in the sense here.

512I Sequential compositions Let (A,R,B) and (C,S,D) be supported relations. Their **sequential composition** $(A,R,B)\ltimes(C,S,D)$ is $(A\times C^B,T,B\times D)$, where

$$T=\{((a,f),(b,d)):(a,b)\in R,\ f\in C^B,\ (f(b),d)\in S\}.$$

Their **dual sequential composition** $(A,R,B)\rtimes(C,S,D)$ is $(A\times C,\tilde T,B\times D^A)$ where

$$\begin{aligned}\tilde T=\{((a,c),(b,g)):a\in A,\ b\in B,\ c\in C,\ g\in D^A&\\ \text{and either }(a,b)\in R\text{ or }(c,g(a))\in S\}.&\end{aligned}$$

512J Proposition Let (A,R,B) and (C,S,D) be supported relations.

(a) $(A,R,B)\rtimes(C,S,D)=((A,R,B)^\perp\ltimes(C,S,D)^\perp)^\perp$.

(b) $\operatorname{cov}((A,R,B)\ltimes(C,S,D))$ is the cardinal product $\operatorname{cov}(A,R,B)\cdot\operatorname{cov}(C,S,D)$ unless $B=C=\emptyset\ne A$, if we use the interpretations

$$0\cdot\infty=\infty\cdot 0=0,\quad \kappa\cdot\infty=\infty\cdot\kappa=\infty\cdot\infty=\infty\text{ for every cardinal }\kappa\ge 1.$$

(c) $\operatorname{add}((A,R,B)\ltimes(C,S,D))=\min(\operatorname{add}(A,R,B),\operatorname{add}(C,S,D))$ unless $A\times C=\emptyset\ne B\times D$.

512K Lemma Suppose that (A,R,B) and (C,S,D) are supported relations, and P is a partially ordered set. Suppose that $\langle A_p\rangle_{p\in P}$ is a family of subsets of A such that

$$(A_p,R,B)\preccurlyeq_{\mathrm{GT}}(C,S,D)\text{ for every }p\in P,$$

$$A_p\subseteq A_q\text{ whenever }p\le q\text{ in }P,\quad \bigcup_{p\in P}A_p=A.$$

Then $(A,R,B)\preccurlyeq_{\mathrm{GT}}(P,\le,P)\ltimes(C,S,D)$.

Version of 23.2.14

513 Partially ordered sets

In §§511-512 I have given long lists of definitions. It is time I filled in details of the most elementary relationships between the various concepts introduced. Here I treat some of those which can be expressed in the language of partially ordered sets. I begin with notes on cofinality and saturation, with the Erdős-Tarski theorem (513B). In this context, Galois-Tukey connections take on particularly direct forms (513D-513E); for directed sets, we have an alternative definition of Tukey equivalence (513F). The majority of the cardinal functions defined so far on partially ordered sets are determined by their cofinal structure (513G, 513If).

In the last third of the section (513K-513O), I discuss Tukey functions between directed sets with a special kind of topological structure, which I call ‘metrizably compactly based’; the point is that for Polish metrizably compactly based directed sets, if there is any Tukey function between them, there must be one which is measurable in an appropriate sense (513O).

513A Lemma Let P be a partially ordered set.

(a) If $Q \subseteq P$ is cofinal and $A \subseteq Q$ is an up-antichain, there is a maximal up-antichain A' in P such that $A \subseteq A' \subseteq Q$. In particular, Q includes a maximal up-antichain.

(b) If $A \subseteq P$ is a maximal up-antichain, $Q = \bigcup_{q \in A} [q, \infty[$ is cofinal with P.

513B Theorem Let P be a partially ordered set.

(a) $\operatorname{bu} P \le \operatorname{cf} P \le \#(P)$.

(b) $\operatorname{sat}^\uparrow(P)$ is either finite or a regular uncountable cardinal.

(c) $c^\uparrow(P)$ is the predecessor of $\operatorname{sat}^\uparrow(P)$ if $\operatorname{sat}^\uparrow(P)$ is a successor cardinal, and otherwise is equal to $\operatorname{sat}^\uparrow(P)$.

513C Cofinalities of cardinal functions: Proposition (a) Let P be a partially ordered set with no greatest member.

(i) If $\operatorname{add} P$ is greater than 2, it is a regular infinite cardinal, and there is a family $\langle p_\xi \rangle_{\xi < \operatorname{add} P}$ in P such that $p_\eta < p_\xi$ whenever $\eta < \xi < \operatorname{add} P$, but $\{p_\xi : \xi < \operatorname{add} P\}$ has no upper bound in P.

(ii) If $\operatorname{cf} P$ is infinite, its cofinality is at least $\operatorname{add} P$.

(b) Let $\mathcal{I}$ be an ideal of subsets of a set X such that $\bigcup \mathcal{I} = X \notin \mathcal{I}$.

(i) $\operatorname{cf}(\operatorname{add} \mathcal{I}) = \operatorname{add} \mathcal{I} \le \operatorname{cf}(\operatorname{cf} \mathcal{I})$.

(ii) $\operatorname{cf}(\operatorname{non} \mathcal{I}) \ge \operatorname{add} \mathcal{I}$.

(iii) If $\operatorname{cov} \mathcal{I} = \operatorname{cf} \mathcal{I}$ then $\operatorname{cf}(\operatorname{cf} \mathcal{I}) \ge \operatorname{non} \mathcal{I}$.

513D Definition Let P and Q be pre-ordered sets. A function $\phi : P \to Q$ is a **Tukey function** if $\phi^{-1}[B]$ is bounded above in P whenever $B \subseteq Q$ is bounded above in Q. A function $\psi : Q \to P$ is a **dual Tukey function** (also called 'cofinal function', 'convergent function') if $\psi[B]$ is cofinal with P whenever $B \subseteq Q$ is cofinal with Q.

If P and Q are pre-ordered sets, I will write '$P \preccurlyeq_{\mathrm{T}} Q$' if $(P, \le, P) \preccurlyeq_{\mathrm{GT}} (Q, \le, Q)$, and '$P \equiv_{\mathrm{T}} Q$' if $(P, \le, P) \equiv_{\mathrm{GT}} (Q, \le, Q)$; in the latter case I say that P and Q are **Tukey equivalent**. It follows immediately from 512C that $\preccurlyeq_{\mathrm{T}}$ is reflexive and transitive, and of course $P \equiv_{\mathrm{T}} Q$ iff $P \preccurlyeq_{\mathrm{T}} Q$ and $Q \preccurlyeq_{\mathrm{T}} P$.

513E Theorem Let P and Q be pre-ordered sets.

(a) If (ϕ, ψ) is a Galois-Tukey connection from $(P, \le, P)$ to $(Q, \le, Q)$ then $\phi : P \to Q$ is a Tukey function and $\psi : Q \to P$ is a dual Tukey function.

(b)(i) A function $\phi : P \to Q$ is a Tukey function iff there is a function $\psi : Q \to P$ such that (ϕ, ψ) is a Galois-Tukey connection from $(P, \le, P)$ to $(Q, \le, Q)$.

(ii) A function $\psi : Q \to P$ is a dual Tukey function iff there is a function $\phi : P \to Q$ such that (ϕ, ψ) is a Galois-Tukey connection from $(P, \le, P)$ to $(Q, \le, Q)$.

(iii) If $\psi : Q \to P$ is order-preserving and $\psi[Q]$ is cofinal with P, then ψ is a dual Tukey function.

(c) The following are equiveridical, that is, if one is true so are the others:

(i) $P \preccurlyeq_{\mathrm{T}} Q$;

(ii) there is a Tukey function $\phi : P \to Q$;

(iii) there is a dual Tukey function $\psi : Q \to P$.

(d)(i) Let $f : P \to Q$ be such that $f[P]$ is cofinal with Q and, for p, $p' \in P$, $f(p) \le f(p')$ iff $p \le p'$. Then $P \equiv_{\mathrm{T}} Q$.

(ii) Suppose that $A \subseteq P$ is cofinal with P. Then $A \equiv_{\mathrm{T}} P$

(iii) For p, $q \in P$ say that $p \equiv q$ if $p \le q$ and $q \le p$; let $\tilde{P}$ be the partially ordered set of equivalence classes in P under the equivalence relation $\equiv$. Then $P \equiv_{\mathrm{T}} \tilde{P}$.

(e) Suppose now that $P \preccurlyeq_{\mathrm{T}} Q$. Then

(i) $\operatorname{cf} P \le \operatorname{cf} Q$;

(ii) $\operatorname{add} P \ge \operatorname{add} Q$;

(iii) $\operatorname{sat}^\uparrow(P) \le \operatorname{sat}^\uparrow(Q)$, $c^\uparrow(P) \le c^\uparrow(Q)$;

(iv) $\operatorname{link}^\uparrow_{<\kappa}(P) \le \operatorname{link}^\uparrow_{<\kappa}(Q)$ for any cardinal κ;

(v) $\operatorname{link}^\uparrow(P) \le \operatorname{link}^\uparrow(Q)$, $d^\uparrow(P) \le d^\uparrow(Q)$.

(f) If P and Q are Tukey equivalent, then $\operatorname{cf} P = \operatorname{cf} Q$ and $\operatorname{add} P = \operatorname{add} Q$.

(g) If $\langle P_i \rangle_{i \in I}$ and $\langle Q_i \rangle_{i \in I}$ are families of pre-ordered ordered sets such that $P_i \preccurlyeq_{\mathrm{T}} Q_i$ for every i, then $\prod_{i \in I} P_i \preccurlyeq_{\mathrm{T}} \prod_{i \in I} Q_i$.

(h) If $0 < \kappa < \operatorname{add} P$ then $P \equiv_{\mathrm{T}} P^\kappa$. In particular, if P is upwards-directed then $P \equiv_{\mathrm{T}} P \times P$.

513F Theorem Suppose that P and Q are upwards-directed partially ordered sets. Then P and Q are Tukey equivalent iff there is a partially ordered set R such that P and Q are both isomorphic, as partially ordered sets, to cofinal subsets of R.

513G Proposition Let P be a pre-ordered set and Q a cofinal subset of P. Then
(a) $\operatorname{add} Q = \operatorname{add} P$;
(b) $\operatorname{cf} Q = \operatorname{cf} P$;
(c) $\operatorname{sat}^{\uparrow}(Q) = \operatorname{sat}^{\uparrow}(P)$, $c^{\uparrow}(Q) = c^{\uparrow}(P)$;
(d) $\operatorname{link}^{\uparrow}_{<\kappa}(Q) = \operatorname{link}^{\uparrow}_{<\kappa}(P)$ for any cardinal κ; in particular, $\operatorname{link}^{\uparrow}(Q) = \operatorname{link}^{\uparrow}(P)$ and $d^{\uparrow}(Q) = d^{\uparrow}(P)$;
(e) $\operatorname{bu} Q = \operatorname{bu} P$.

513H Definition Let P be a partially ordered set. Its **σ-additivity** $\operatorname{add}_\omega P$ is the smallest cardinal of any subset A of P such that $A \not\subseteq \bigcup_{q\in D} \left]-\infty, q\right]$ for any countable set $D \subseteq P$. If there is no such set, I write $\operatorname{add}_\omega P = \infty$.

513I Proposition Let P be a partially ordered set. As in 512F, write $p \le' A$, for $p \in P$ and $A \subseteq P$, if there is a $q \in A$ such that $p \le q$.
(a) $\operatorname{add}_\omega P = \operatorname{add}(P, \le', [P]^{\le\omega})$.
(b) $\max(\omega_1, \operatorname{add} P) \le \operatorname{add}_\omega(P)$.
(c) If $\operatorname{add}_\omega P$ is an infinite cardinal, it is regular.
(d) If $2 \le \kappa \le \operatorname{add} P$, then $(P, \le', [P]^{<\kappa}) \equiv_{\mathrm{GT}} (P, \le, P)$. So if $\operatorname{add} P > \omega$, $\operatorname{add}_\omega(P) = \operatorname{add} P$.
(e) If Q is another partially ordered set and $(P, \le', [P]^{\le\omega}) \preccurlyeq_{\mathrm{GT}} (Q, \le', [Q]^{\le\omega})$ then $\operatorname{add}_\omega P \ge \operatorname{add}_\omega Q$.
(f) If $Q \subseteq P$ is cofinal with P, then $\operatorname{add}_\omega Q = \operatorname{add}_\omega P$.
(g) If $\kappa \le \operatorname{cf} P$ then $\operatorname{add}(P, \le', [P]^{<\kappa}) \le \operatorname{cf} P$. So if $\operatorname{cf} P > \omega$ then $\operatorname{add}_\omega P \le \operatorname{cf} P$.
(h) If $\operatorname{cf}(\operatorname{cf} P) > \omega$ then $\operatorname{cf}(\operatorname{cf} P) \ge \operatorname{add}_\omega P$.

***513J Cofinalities of products: Proposition** Suppose that the generalized continuum hypothesis is true. Let $\langle P_i\rangle_{i\in I}$ be a family of non-empty partially ordered sets with product P. Set

$$\kappa = \#(\{i : i \in I,\ \operatorname{cf} P_i > 1\}), \quad \lambda = \sup_{i\in I} \operatorname{cf} P_i.$$

Then
(i) if κ and λ are both finite, $\operatorname{cf} P$ is the cardinal product $\prod_{i\in I} \operatorname{cf} P_i$;
(ii) if $\lambda > \kappa$ and there is some $\gamma < \lambda$ such that $\operatorname{cf} \lambda > \#(\{i : i \in I,\ \operatorname{cf} P_i > \gamma\})$, then $\operatorname{cf} P = \lambda$;
(iii) otherwise, $\operatorname{cf} P = \max(\kappa^+, \lambda^+)$.

***513K Definition** I will say that a **metrizably compactly based directed set** is a partially ordered set P endowed with a metrizable topology such that
(i) $p \vee q = \sup\{p, q\}$ is defined for all p, $q \in P$, and $\vee : P \times P \to P$ is continuous;
(ii) $\{p : p \le q\}$ is compact for every $q \in P$;
(iii) every convergent sequence in P has a subsequence which is bounded above.
In this context, I will say that P is 'separable' or 'analytic' if it is separable, or analytic, in the topological sense.

***513L Proposition** Let P be a metrizably compactly based directed set.
(a) The ordering of P is a closed subset of $P \times P$.
(b) P is Dedekind complete.
(c)(i) A non-decreasing sequence in P has an upper bound iff it is topologically convergent, and in this case its supremum is its limit.
(ii) A non-increasing sequence in P converges topologically to its infimum.
(d) If $p \in P$ and $\langle p_i\rangle_{i\in\mathbb{N}}$ is a sequence in P, then $\langle p_i\rangle_{i\in\mathbb{N}}$ is topologically convergent to p iff for every $I \in [\mathbb{N}]^\omega$ there is a $J \in [I]^\omega$ such that $p = \inf_{n\in\mathbb{N}} \sup_{i\in J\setminus n} p_i$.
(e) Suppose that $p \in P$ and a double sequence $\langle p_{ni}\rangle_{n,i\in\mathbb{N}}$ in P are such that $\lim_{i\to\infty} p_{ni} = p_n$ is defined in P and less than or equal to p for each n. Then there is a $q \in P$ such that $\{i : p_{ni} \le q\}$ is infinite for every $n \in \mathbb{N}$.

***513M Proposition** Let P be a separable metrizably compactly based directed set, and give the set $\mathcal{C}$ of closed subsets of P its Vietoris topology. Let $\mathcal{K}_b \subseteq \mathcal{C}$ be the family of non-empty compact subsets of P which are bounded above in P. Then $K \mapsto \sup K : \mathcal{K}_b \to P$ is Borel measurable.

***513N Lemma** Let P and Q be non-empty metrizably compactly based directed sets of which P is separable, and $\phi : P \to Q$ a Tukey function. Set

$$R = \overline{\{(q,p) : p \in P,\, q \in Q,\, \phi(p) \le q\}}.$$

Then

(a) $R[\,]-\infty, q]\,]$ is bounded above in P for every $q \in Q$;

(b) $R \subseteq Q \times P$ is usco-compact.

***513O Theorem** Let P and Q be metrizably compactly based directed sets such that $P \preccurlyeq_{\mathrm{T}} Q$. Let Σ be the σ-algebra of subsets of P generated by the Souslin-F sets.

(a) If P is separable, there is a Borel measurable dual Tukey function $\psi : Q \to P$.

(b) If P is separable and Q is analytic, there is a Σ-measurable Tukey function $\phi : P \to Q$.

513P Lemma Let P and Q be non-empty partially ordered sets, and suppose that (i) every non-decreasing sequence in P has an upper bound in P (ii) there is no strictly increasing family $\langle q_\xi \rangle_{\xi<\omega_1}$ in Q. Let $f : P \to Q$ be an order-preserving function. Then there is a $p \in P$ such that $f(p') = f(p)$ whenever $p' \in P$ and $p' \ge p$.

Version of 16.5.14

514 Boolean algebras

The cardinal functions of Boolean algebras and topological spaces are intimately entwined; necessarily so, because we have a functorial connexion between Boolean algebras and zero-dimensional compact Hausdorff spaces. In this section I run through the elementary ideas. In 514D-514E I list properties of cardinal functions of Boolean algebras, corresponding to the relatively familiar results in 5A4B for topological spaces; Stone spaces (514B), regular open algebras (514H) and category algebras (514I) provide links of different kinds between the two theories. It turns out that some of the most important features of the cofinal structure of a pre-ordered set can also be described in terms of its 'up-topology' (514L-514M) and the associated regular open algebra (514N-514S). I conclude with a brief note on finite-support products (514T-514U).

514A Lemma Let $(X, \mathfrak{T})$ be a topological space. Then $d(X) \ge d^{\downarrow}(\mathfrak{T} \setminus \{\emptyset\})$. If X is locally compact and Hausdorff, then $d(X) = d^{\downarrow}(\mathfrak{T} \setminus \{\emptyset\})$.

514B Stone spaces: Theorem Let $\mathfrak{A}$ be any Boolean algebra and Z its Stone space. For $a \in \mathfrak{A}$ let $\widehat{a}$ be the corresponding open-and-closed subset of Z.

(a) $\#(\mathfrak{A})$ is $2^{w(Z)} = 2^{\#(Z)}$ if $\mathfrak{A}$ is finite, $w(Z)$ otherwise.

(b) $\mathrm{sat}(\mathfrak{A}) = \mathrm{sat}(Z)$, $c(\mathfrak{A}) = c(Z)$.

(c) $\pi(\mathfrak{A}) = \pi(Z)$.

(d) $d(\mathfrak{A}) = d(Z)$.

(e) Let $\mathcal{N}\mathrm{wd}(Z)$ be the ideal of nowhere dense subsets of Z. Then $\mathrm{wdistr}(\mathfrak{A}) = \mathrm{add}\,\mathcal{N}\mathrm{wd}(Z)$.

514C Lemma Let $\mathfrak{A}$ be a Boolean algebra.

(a) $d(\mathfrak{A})$ is the smallest cardinal κ such that $\mathfrak{A}$ is isomorphic, as Boolean algebra, to a subalgebra of $\mathcal{P}\kappa$.

(b) $\mathrm{link}(\mathfrak{A})$ is the smallest cardinal κ such that $\mathfrak{A}$ is isomorphic, as partially ordered set, to a subset of $\mathcal{P}\kappa$.

514D Theorem Let $\mathfrak{A}$ be a Boolean algebra.

(a)

$$c(\mathfrak{A}) \le \mathrm{link}(\mathfrak{A}) \le d(\mathfrak{A}) \le \pi(\mathfrak{A}) \le \#(\mathfrak{A}) \le 2^{\mathrm{link}(\mathfrak{A})}, \quad \tau(\mathfrak{A}) \le \pi(\mathfrak{A}),$$

$\mathrm{sat}(\mathfrak{A}) = c(\mathfrak{A})^+$ unless $\mathrm{sat}(\mathfrak{A})$ is weakly inaccessible, in which case $\mathrm{sat}(\mathfrak{A}) = c(\mathfrak{A})$.

(b) If $A \subseteq \mathfrak{A}$, there is a $B \in [A]^{<\mathrm{sat}(\mathfrak{A})}$ with the same upper bounds as A; there is a $B \in [A]^{<\mathrm{sat}(\mathfrak{A})}$ with the same lower bounds as A.

(c) $\mathrm{link}_{c(\mathfrak{A})}(\mathfrak{A}) = \mathrm{link}_{<\mathrm{sat}(\mathfrak{A})}(\mathfrak{A}) = \pi(\mathfrak{A})$.

(d) If $\mathfrak{A}$ is not purely atomic, $\mathrm{wdistr}(\mathfrak{A}) \le \min(d(\mathfrak{A}), 2^{\tau(\mathfrak{A})})$ is a regular infinite cardinal.

(e) $\#(\mathfrak{A}) \le \max(4, \sup_{\lambda<\mathrm{sat}(\mathfrak{A})} \tau(\mathfrak{A})^\lambda)$.

514E Subalgebras, homomorphic images, products: Theorem Let $\mathfrak{A}$ be a Boolean algebra.

(a) If $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$, then

$$\mathrm{sat}(\mathfrak{B}) \le \mathrm{sat}(\mathfrak{A}), \quad c(\mathfrak{B}) \le c(\mathfrak{A}),$$

$$\mathrm{link}_{<\kappa}(\mathfrak{B}) \le \mathrm{link}_{<\kappa}(\mathfrak{A})$$

for every $\kappa \le \omega$,

$$d(\mathfrak{B}) \le d(\mathfrak{A}), \quad \mathrm{link}(\mathfrak{B}) \le \mathrm{link}(\mathfrak{A}).$$

(b) If $\mathfrak{B}$ is a regularly embedded subalgebra of $\mathfrak{A}$ then $\mathrm{link}_{<\kappa}(\mathfrak{B}) \le \mathrm{link}_{<\kappa}(\mathfrak{A})$ for $\kappa > \omega$, $\pi(\mathfrak{B}) \le \pi(\mathfrak{A})$ and $\mathrm{wdistr}(\mathfrak{A}) \le \mathrm{wdistr}(\mathfrak{B})$.

(c) If $\mathfrak{B}$ is a Boolean algebra and $\phi : \mathfrak{A} \to \mathfrak{B}$ is a surjective order-continuous Boolean homomorphism, then

$$\mathrm{sat}(\mathfrak{B}) \le \mathrm{sat}(\mathfrak{A}), \quad c(\mathfrak{B}) \le c(\mathfrak{A}), \quad \pi(\mathfrak{B}) \le \pi(\mathfrak{A}),$$

$$\mathrm{link}_{<\kappa}(\mathfrak{B}) \le \mathrm{link}_{<\kappa}(\mathfrak{A}) \text{ for every cardinal } \kappa,$$

$$d(\mathfrak{B}) \le d(\mathfrak{A}), \quad \mathrm{link}(\mathfrak{B}) \le \mathrm{link}(\mathfrak{A}),$$

$$\mathrm{wdistr}(\mathfrak{A}) \le \mathrm{wdistr}(\mathfrak{B}), \quad \tau(\mathfrak{B}) \le \tau(\mathfrak{A}).$$

(d) If $\mathfrak{B}$ is a principal ideal of $\mathfrak{A}$, then

$$\mathrm{sat}(\mathfrak{B}) \le \mathrm{sat}(\mathfrak{A}), \quad c(\mathfrak{B}) \le c(\mathfrak{A}), \quad \pi(\mathfrak{B}) \le \pi(\mathfrak{A}),$$

$$\mathrm{link}_{<\kappa}(\mathfrak{B}) \le \mathrm{link}_{<\kappa}(\mathfrak{A}) \text{ for every } \kappa,$$

$$d(\mathfrak{B}) \le d(\mathfrak{A}), \quad \mathrm{link}(\mathfrak{B}) \le \mathrm{link}(\mathfrak{A});$$

$$\mathrm{wdistr}(\mathfrak{A}) \le \mathrm{wdistr}(\mathfrak{B}), \quad \tau(\mathfrak{B}) \le \tau(\mathfrak{A}).$$

(e) If $\mathfrak{B}$ is an order-dense subalgebra of $\mathfrak{A}$ then

$$\mathrm{sat}(\mathfrak{B}) = \mathrm{sat}(\mathfrak{A}), \quad c(\mathfrak{B}) = c(\mathfrak{A}), \quad \pi(\mathfrak{B}) = \pi(\mathfrak{A}),$$

$$\mathrm{link}_{<\kappa}(\mathfrak{B}) = \mathrm{link}_{<\kappa}(\mathfrak{A}) \text{ for every } \kappa,$$

$$d(\mathfrak{B}) = d(\mathfrak{A}), \quad \mathrm{link}(\mathfrak{B}) = \mathrm{link}(\mathfrak{A}),$$

$$\mathrm{wdistr}(\mathfrak{B}) = \mathrm{wdistr}(\mathfrak{A}), \quad \tau(\mathfrak{A}) \le \tau(\mathfrak{B}).$$

(f) If $\mathfrak{A}$ is the simple product of a family $\langle \mathfrak{A}_i \rangle_{i\in I}$ of Boolean algebras, then

$$\tau(\mathfrak{A}) \le \max(\omega, \sup_{i\in I} \tau(\mathfrak{A}_i), \min\{\lambda : \#(I) \le 2^\lambda\}),$$

$$\mathrm{sat}(\mathfrak{A}) \le \max(\omega, \#(I)^+, \sup_{i\in I} \mathrm{sat}(\mathfrak{A}_i)),$$

$$c(\mathfrak{A}) \le \max(\omega, \#(I), \sup_{i\in I} c(\mathfrak{A}_i)),$$

$$\pi(\mathfrak{A}) \le \max(\omega, \#(I), \sup_{i\in I} \pi(\mathfrak{A}_i)),$$

$$\mathrm{link}_{<\kappa}(\mathfrak{A}) \le \max(\omega, \#(I), \sup_{i\in I} \mathrm{link}_{<\kappa}(\mathfrak{A}_i)) \text{ for every } \kappa,$$

$$\mathrm{link}(\mathfrak{A}) \le \max(\omega, \#(I), \sup_{i\in I} \mathrm{link}(\mathfrak{A}_i)),$$

$$d(\mathfrak{A}) \le \max(\omega, \#(I), \sup_{i\in I} d(\mathfrak{A}_i)),$$

$$\mathrm{wdistr}(\mathfrak{A}) = \min_{i\in I} \mathrm{wdistr}(\mathfrak{A}_i).$$

514F Proposition Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, and $\langle a_{ij}\rangle_{i\in I,j\in J}$ a τ-generating family in $\mathfrak{A}$ such that

$$\langle a_{ij}\rangle_{j\in J} \text{ is disjoint for every } i \in I, \quad \sup_{i\in I} a_{ij} = 1 \text{ for every } j \in J.$$

Then $\tau(\mathfrak{A}) \le \max(\omega, \#(I))$.

514G Order-preserving functions of Boolean algebras (a) Let F be an ordinal function of Boolean algebras, that is, a function defined on the class of Boolean algebras, taking ordinal values, and such that $F(\mathfrak{A}) = F(\mathfrak{B})$ whenever $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic. We say that F is **order-preserving** if $F(\mathfrak{B}) \le F(\mathfrak{A})$ whenever $\mathfrak{B}$ is a principal ideal of $\mathfrak{A}$. Now a Boolean algebra $\mathfrak{A}$ is **F-homogeneous** if $F(\mathfrak{B}) = F(\mathfrak{A})$ for every non-zero principal ideal $\mathfrak{B}$ of $\mathfrak{A}$. Of course any principal ideal of an F-homogeneous Boolean algebra is again F-homogeneous.

(b) If F is any order-preserving ordinal function of Boolean algebras, and $\mathfrak{A}$ is a Boolean algebra, then $\{a : a \in \mathfrak{A},\ \mathfrak{A}_a \text{ is } F\text{-homogeneous}\}$ is order-dense in $\mathfrak{A}$. So if $\mathfrak{A}$ is a Dedekind complete Boolean algebra, it is isomorphic to a simple product of F-homogeneous Boolean algebras.

(c) Similarly, if $F_0, \dots, F_n$ are order-preserving ordinal functions of Boolean algebras, and $\mathfrak{A}$ is any Boolean algebra, then $\{a : \mathfrak{A}_a \text{ is } F_i\text{-homogeneous for every } i \le n\}$ is order-dense in $\mathfrak{A}$; and if $\mathfrak{A}$ is Dedekind complete, it is isomorphic to a simple product of Boolean algebras all of which are F_i-homogeneous for every $i \le n$.

514H Regular open algebras: Proposition Let $(X, \mathfrak{T})$ be a topological space and $\mathrm{RO}(X)$ its regular open algebra.

(a)(i) $(\mathrm{RO}(X)^+, \supseteq, \mathrm{RO}(X)^+) \preccurlyeq_{\mathrm{GT}} (\mathfrak{T}\setminus\{\emptyset\}, \supseteq, \mathfrak{T}\setminus\{\emptyset\})$.

(ii) If X is regular, $(\mathrm{RO}(X)^+, \supseteq, \mathrm{RO}(X)^+) \equiv_{\mathrm{GT}} (\mathfrak{T}\setminus\{\emptyset\}, \supseteq, \mathfrak{T}\setminus\{\emptyset\})$.

(b)(i) $\mathrm{sat}(\mathrm{RO}(X)) = \mathrm{sat}(X)$, $c(\mathrm{RO}(X)) = c(X)$, $\pi(\mathrm{RO}(X)) \le \pi(X)$ and $d(\mathrm{RO}(X)) \le d(X)$.

(ii) If X is regular, $\pi(\mathrm{RO}(X)) = \pi(X)$.

(iii) If X is locally compact and Hausdorff, $d(\mathrm{RO}(X)) = d(X)$.

(c) Let $\mathcal{N}\mathcal{w}\mathsf{d}(X)$ be the ideal of nowhere dense subsets of X.

(i) If X is regular, $\mathrm{wdistr}(\mathrm{RO}(X)) \le \mathrm{add}\,\mathcal{N}\mathcal{w}\mathsf{d}(X)$.

(ii) If X is locally compact and Hausdorff, $\mathrm{wdistr}(\mathrm{RO}(X)) = \mathrm{add}\,\mathcal{N}\mathcal{w}\mathsf{d}(X)$.

(d) If $Y \subseteq X$ is dense, then $G \mapsto G \cap Y$ is a Boolean isomorphism from $\mathrm{RO}(X)$ to $\mathrm{RO}(Y)$.

514I Category algebras (a) Let X be a topological space, and $\mathcal{M}$ the σ-ideal of meager subsets of X. Recall that the Baire-property algebra of X is the σ-algebra $\widehat{\mathcal{B}} = \{G\triangle A : G \subseteq X \text{ is open}, A \in \mathcal{M}\}$, and that the category algebra of X is the quotient Boolean algebra $\mathfrak{G} = \widehat{\mathcal{B}}/\mathcal{M}$ (4A3R[1]). Note that if $G \subseteq X$ is any open set, then

$$G^\bullet = \overline{G}^\bullet = (\mathrm{int}\,\overline{G})^\bullet$$

in $\mathfrak{G}$.

(b) For $G \in \mathrm{RO}(X)$, set $\pi G = G^\bullet \in \mathfrak{G}$. Then $\pi : \mathrm{RO}(X) \to \mathfrak{G}$ is an order-continuous surjective Boolean homomorphism.

[1]Formerly 4A3Q.

(c) π includes an isomorphism between a principal ideal of $\mathrm{RO}(X)$ and $\mathfrak{G}$. $\mathfrak{G}$ is Dedekind complete.

(d) It is useful to know that if $G \subseteq X$ is open, then the category algebra of G can be identified with the principal ideal of $\mathfrak{G}$ generated by $G^\bullet$.

(e) If $\langle A_n\rangle_{n\in\mathbb{N}}$ is any sequence of subsets of X, and E_n is a Baire-property envelope of A_n for each n, then $E = \bigcup_{n\in\mathbb{N}} E_n$ is a Baire-property envelope of $A = \bigcup_{n\in\mathbb{N}} A_n$.

If $A \subseteq X$, we can define $\psi(A) \in \mathfrak{G}$ by setting $\psi(A) = \inf\{F^\bullet : A \subseteq F \in \widehat{\mathcal{B}}\}$. $\psi(A) = E^\bullet$ for any Baire-property envelope E of A. It follows that $\psi(\bigcup_{n\in\mathbb{N}} A_n) = \sup_{n\in\mathbb{N}} \psi(A_n)$ for any sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of subsets of X; also $\psi(A) = 0$ in $\mathfrak{G}$ iff A is meager.

(f) When X is a Baire space, π is an isomorphism between $\mathrm{RO}(X)$ and $\mathfrak{G}$.

(g) If X is a zero-dimensional space, then the algebra $\mathcal{E}$ of open-and-closed sets in X is an order-dense subalgebra of $\mathrm{RO}(X)$, so that $\mathrm{RO}(X)$ can be identified with the Dedekind completion of $\mathcal{E}$; and if X is a zero-dimensional compact Hausdorff space, then the category algebra of X can be identified with the Dedekind completion of $\mathcal{E}$.

(h) I note that if X is an extremally disconnected compact Hausdorff space, so that its algebra $\mathcal{E}$ of open-and-closed sets is already Dedekind complete, then $\mathcal{E} = \mathrm{RO}(X)$. So if X is the Stone space of a Dedekind complete Boolean algebra $\mathfrak{A}$, we have a Boolean isomorphism $a \mapsto \widehat{a}^\bullet$ from $\mathfrak{A}$ to $\mathfrak{G}$, writing $\widehat{a}$ for the open-and-closed subset of X corresponding to $a \in \mathfrak{A}$.

514J Proposition Let X be a topological space and $\mathfrak{C}$ its category algebra.
(a) $\mathrm{sat}(\mathfrak{C}) \le \mathrm{sat}(X)$, $c(\mathfrak{C}) \le c(X)$, $\pi(\mathfrak{C}) \le \pi(X)$ and $d(\mathfrak{C}) \le d(X)$.
(b) If X is a Baire space, $\mathrm{sat}(\mathfrak{C}) = \mathrm{sat}(X)$ and $c(\mathfrak{C}) = c(X)$.
(c) If X is regular, $\mathrm{wdistr}(\mathfrak{C}) \le \mathrm{add}\,\mathcal{N}\mathit{wd}(X)$, where $\mathcal{N}\mathit{wd}(X)$ is the ideal of nowhere dense subsets of X.

514K Proposition Let $\mathfrak{A}$ be a Boolean algebra such that $\mathrm{sat}(\mathfrak{A}) \le \mathrm{wdistr}(\mathfrak{A})$. Then whenever $A \subseteq \mathfrak{A}$ and $\#(A) < \mathrm{wdistr}(\mathfrak{A})$ there is a set $C \subseteq \mathfrak{A}$ such that $\#(C) \le \max(c(\mathfrak{A}), \tau(\mathfrak{A}))$ and $a = \sup\{c : c \in C,\, c \subseteq a\}$ for every $a \in A$.

514L The regular open algebra of a pre-ordered set: Definitions (a) For any pre-ordered set P, a subset G of P is **up-open** if $[p,\infty[\subseteq G$ whenever $p \in G$. The family of such sets is a topology on P, the **up-topology**. Similarly, the **down-topology** of P is the family of **down-open** sets H such that $p \le q \in H \Rightarrow p \in H$. $G \subseteq P$ is up-open iff it is closed for the down-topology.

(b) I will write $\mathrm{RO}^\uparrow(P)$ for the regular open algebra of P when P is given its up-topology, and $\mathrm{RO}^\downarrow(P)$ for the regular open algebra of P when P is given its down-topology.

514M Lemma Let P be a pre-ordered set endowed with its up-topology.
(a)(i) For any $A \subseteq P$, $\overline{A} = \{p : A \cap [p,\infty[\ne \emptyset\}$.
(ii) For any $p \in P$, $\overline{[p,\infty[}$ is the set of elements of P which are compatible upwards with p.
(iii) For any p, $q \in P$, the following are equiveridical: (α) $q \in \mathrm{int}\,\overline{[p,\infty[}$; ($\beta$) every member of $[q,\infty[$ is compatible upwards with p; (γ) q is incompatible upwards with every $r \in P$ which is incompatible upwards with p.
(b) A subset of P is dense iff it is cofinal.
(c) If Q is another pre-ordered set with its up-topology, a function $f : P \to Q$ is continuous iff it is order-preserving.
(d)(i) A subset G of P is a regular open set iff

$$G = \{p : G \cap [q,\infty[\ne \emptyset \text{ for every } q \ge p\}.$$

(ii) If $\mathcal{G}$ is a non-empty family of regular open subsets of P, then $\bigcap\mathcal{G}$ is a regular open subset of P, and is $\inf\mathcal{G}$ in the regular open algebra $\mathrm{RO}^\uparrow(P)$.
(e) P is separative upwards iff all the sets $[p,\infty[$ are regular open sets.
(f) If P is separative upwards and $A \subseteq P$ has a supremum p in P, then $[p,\infty[= \inf_{q\in A} [q,\infty[$ in $\mathrm{RO}^\uparrow(P)$.

514N Proposition Let $(P,\le)$ be a pre-ordered set, and write $\mathfrak{T}^\uparrow$ for the up-topology of P and $\mathrm{RO}^\uparrow(P)$ for the regular open algebra of $(P,\mathfrak{T}^\uparrow)$.

(a) $(\mathrm{RO}^\uparrow(P)^+,\supseteq,\mathrm{RO}^\uparrow(P)^+)\preccurlyeq_{\mathrm{GT}}(\mathfrak{T}^\uparrow\setminus\{\emptyset\},\supseteq,\mathfrak{T}^\uparrow\setminus\{\emptyset\})\equiv_{\mathrm{GT}}(P,\le,P)$. If P is separative upwards, then $(\mathrm{RO}^\uparrow(P)^+,\supseteq,\mathrm{RO}^\uparrow(P)^+)\equiv_{\mathrm{GT}}(P,\le,P)$.

(b) $\pi(\mathrm{RO}^\uparrow(P))\le\pi(P,\mathfrak{T}^\uparrow)=d(P,\mathfrak{T}^\uparrow)=\operatorname{cf}P$. If P is separative upwards, then we have equality.

(c) $\operatorname{sat}^\uparrow(P,\le)=\operatorname{sat}(P,\mathfrak{T}^\uparrow)=\operatorname{sat}(\mathrm{RO}^\uparrow(P))$ and $c^\uparrow(P,\le)=c(P,\mathfrak{T}^\uparrow)=c(\mathrm{RO}^\uparrow(P))$.

(d) For any cardinal κ,

$$\operatorname{link}_{<\kappa}(\mathrm{RO}^\uparrow(P))\le\operatorname{link}^\uparrow_{<\kappa}(P,\le),$$

with equality if *either* P is separative upwards *or* $\kappa\le\omega$. In particular, we always have

$$\operatorname{link}^\uparrow(P,\le)=\operatorname{link}(\mathrm{RO}^\uparrow(P)),\quad d^\uparrow(P,\le)=d(\mathrm{RO}^\uparrow(P)).$$

(e) If $Q\subseteq P$ is cofinal, then $\mathrm{RO}^\uparrow(Q)\cong\mathrm{RO}^\uparrow(P)$.

(f) If $A\subseteq P$ is a maximal up-antichain, then $\mathrm{RO}^\uparrow(P)\cong\prod_{a\in A}\mathrm{RO}^\uparrow([a,\infty[)$.

(g) If $\tilde P$ is the partially ordered set of equivalence classes associated with P, then $\mathrm{RO}^\uparrow(\tilde P)\cong\mathrm{RO}^\uparrow(P)$.

514O Proposition Suppose that P and Q are pre-ordered sets and $f:P\to Q$ is an order-preserving function such that $f^{-1}[Q_0]$ is cofinal with P for every up-open cofinal $Q_0\subseteq Q$. Then there is an order-continuous Boolean homomorphism $\pi:\mathrm{RO}^\uparrow(Q)\to\mathrm{RO}^\uparrow(P)$ defined by setting $\pi H=\operatorname{int}\overline{f^{-1}[H]}$ (taking the closure and interior with respect to the up-topology on P) for every $H\in\mathrm{RO}^\uparrow(Q)$. If $f[P]$ is cofinal with Q then π is injective, so is a regular embedding of $\mathrm{RO}^\uparrow(Q)$ in $\mathrm{RO}^\uparrow(P)$.

514P Corollary Suppose that P and Q are pre-ordered sets, that $f:P\to Q$ is an order-preserving function and whenever $p\in P$, $q\in Q$ and $f(p)\le q$, there is a $p'\ge p$ such that $f(p')\ge q$. If $f[P]$ is *either* cofinal with Q *or* coinitial with Q, then $\mathrm{RO}^\uparrow(Q)$ can be regularly embedded in $\mathrm{RO}^\uparrow(P)$.

514Q Proposition Let P and Q be pre-ordered sets, endowed with their up-topologies, and $f:P\to Q$ a function such that

whenever $A\subseteq P$ is a maximal up-antichain then $f\restriction A$ is injective and $f[A]$ is a maximal up-antichain in Q.

Then there is an injective order-continuous Boolean homomorphism $\pi:\mathrm{RO}^\uparrow(P)\to\mathrm{RO}^\uparrow(Q)$ defined by setting $\pi(\operatorname{int}\overline{[p,\infty[})=\operatorname{int}\overline{[f(p),\infty[}$ for every $p\in P$. In particular, $\mathrm{RO}^\uparrow(P)$ can be regularly embedded in $\mathrm{RO}^\uparrow(Q)$. If $f[P]$ is cofinal with Q, then π is an isomorphism.

514R Corollary Let P and Q be pre-ordered sets. Suppose that there is a function $f:P\to Q$ such that $f[P]$ is cofinal with Q and, for p, $p'\in P$, p and p' are compatible upwards in P iff $f(p)$ and $f(p')$ are compatible upwards in Q. Then $\mathrm{RO}^\uparrow(P)\cong\mathrm{RO}^\uparrow(Q)$.

514S Proposition (a) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and P a pre-ordered set. Suppose that we have a function $f:P\to\mathfrak{A}^+$ such that, for p, $q\in P$,

$$f(p)\subseteq f(q)\text{ whenever }p\le q,$$

$$f(p)\cap f(q)=0\text{ whenever }p\text{ and }q\text{ are incompatible downwards in }P,$$

$$f[P]\text{ is order-dense in }\mathfrak{A}.$$

Then $\mathrm{RO}^\downarrow(P)\cong\mathfrak{A}$.

(b) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and $D\subseteq\mathfrak{A}$ an order-dense set not containing 0. Give D the ordering $\subseteq$, and write $\mathrm{RO}^\downarrow(D)$ for the regular open algebra of D with its down-topology. Then $\mathrm{RO}^\downarrow(D)\cong\mathfrak{A}$.

(c) Let $(X,\mathfrak{T})$ be a topological space and P a pre-ordered set. Suppose we have a function $g:P\to\mathfrak{T}\setminus\{\emptyset\}$ such that, for p, $q\in P$,

$$g(p)\subseteq g(q)\text{ whenever }p\le q,$$

$$g(p) \cap g(q) = \emptyset \text{ whenever } p \text{ and } q \text{ are incompatible downwards in } P,$$

$$g[P] \text{ is a } \pi\text{-base for } \mathfrak{T}.$$

Then $\mathrm{RO}^{\downarrow}(P) \cong \mathrm{RO}(X)$.

(d) Let $(X, \mathfrak{T})$ be a topological space and $\mathcal{U}$ a π-base for the topology of X not containing $\{\emptyset\}$. Give $\mathcal{U}$ the ordering $\subseteq$. Then $\mathrm{RO}^{\downarrow}(\mathcal{U}) \cong \mathrm{RO}(X)$.

514T Finite-support products: Definition Let $\langle P_i \rangle_{i \in I}$ be a family of non-empty partially ordered sets. The **upwards finite-support product** $\bigotimes^{\uparrow}_{i \in I} P_i$ of $\langle P_i \rangle_{i \in I}$ is the set $\bigcup\{\prod_{i \in J} P_i : J \in [I]^{<\omega}\}$, ordered by saying that $p \le q$ iff $\operatorname{dom} p \subseteq \operatorname{dom} q$ and $p(i) \le q(i)$ for every $i \in \operatorname{dom} p$. Similarly, the **downwards finite-support product** $\bigotimes^{\downarrow}_{i \in I} P_i$ of $\langle P_i \rangle_{i \in I}$ is $\bigcup\{\prod_{i \in J} P_i : J \in [I]^{<\omega}\}$ ordered by saying that $p \le q$ iff $\operatorname{dom} q \subseteq \operatorname{dom} p$ and $p(i) \le q(i)$ for every $i \in \operatorname{dom} q$.

514U Proposition Let $\langle P_i \rangle_{i \in I}$ be a family of non-empty partially ordered sets, with upwards finite-support product $P = \bigotimes^{\uparrow}_{i \in I} P_i$.

(a) The regular open algebra $\mathrm{RO}^{\uparrow}(P)$ is isomorphic to the regular open algebra of $P^* = \prod_{i \in I} P_i$ when every P_i is given its up-topology.

(b) If I is finite, P^* is a cofinal subset of P, and the ordering of P^*, regarded as a subset of P, is the usual product partial order on P^*.

(c) If $Q_i \subseteq P_i$ is cofinal for each $i \in I$, then $\bigcup_{J \in [I]^{<\omega}} \prod_{i \in J} Q_i$ is cofinal with P. So $\operatorname{cf} P$ is at most $\max(\omega, \#(I), \sup_{i \in I} \operatorname{cf} P_i)$.

(d) $c^{\uparrow}(P) = \sup_{J \in [I]^{<\omega}} c^{\uparrow}(\prod_{i \in J} P_i)$.

Version of 23.12.15

515 The Balcar-Franěk theorem

I interpolate a section to give two basic results on Dedekind complete Boolean algebras: the Balcar-Franěk theorem (515H) on independent sets and the Pierce-Koppelberg theorem (515L) on cardinalities. The concept of 'Boolean-independence' (515A) provides a tool for some useful results on regular open algebras (515N-515Q).

515A Definition Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$.

(a) I say that a family $\langle \mathfrak{B}_i \rangle_{i \in I}$ of subalgebras of $\mathfrak{A}$ is **Boolean-independent** if $\inf_{i \in J} b_i \ne 0$ whenever $J \subseteq I$ is finite and $b_i \in \mathfrak{B}_i^+ = \mathfrak{B}_i \setminus \{0\}$ for every $i \in J$.

(b) I say that a family $\langle a_i \rangle_{i \in I}$ in $\mathfrak{A}$ is **Boolean-independent** if $\inf_{j \in J} a_j \setminus \sup_{k \in K} a_k$ is non-zero whenever J, $K \subseteq I$ are disjoint finite sets. Similarly, a set $B \subseteq \mathfrak{A}$ is **Boolean-independent** if $\inf J \setminus \sup K \ne 0$ for any disjoint finite sets J, $K \subseteq B$.

(c) I say that a family $\langle D_i \rangle_{i \in I}$ of partitions of unity in $\mathfrak{A}$ is **Boolean-independent** if $\inf_{i \in J} d_i \ne 0$ whenever $J \subseteq I$ is finite and $d_i \in D_i$ for every $i \in J$.

515B Lemma Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$.

(a) A family $\langle a_i \rangle_{i \in I}$ in $\mathfrak{A}$ is Boolean-independent iff no a_i is 0 or 1 and $\langle \{0, a_i, 1 \setminus a_i, 1\} \rangle_{i \in I}$ is a Boolean-independent family of subalgebras of $\mathfrak{A}$.

(b) Let $\langle \mathfrak{B}_i \rangle_{i \in I}$ be a family of subalgebras of $\mathfrak{A}$. Let $\mathfrak{B}$ be the free product of $\langle \mathfrak{B}_i \rangle_{i \in I}$, and $\varepsilon_i : \mathfrak{B}_i \to \mathfrak{B}$ the canonical homomorphism for each $i \in I$. Then we have a unique Boolean homomorphism $\phi : \mathfrak{B} \to \mathfrak{A}$ such that $\phi\varepsilon_i(b) = b$ whenever $i \in I$ and $b \in \mathfrak{B}_i$, and $\langle \mathfrak{B}_i \rangle_{i \in I}$ is Boolean-independent iff ϕ is injective; in which case $\mathfrak{B}$ is isomorphic to the subalgebra of $\mathfrak{A}$ generated by $\bigcup_{i \in I} \mathfrak{B}_i$.

(c) If $\langle \mathfrak{B}_i \rangle_{i \in I}$ is a Boolean-independent family of subalgebras of $\mathfrak{A}$, $\langle I_j \rangle_{j \in J}$ is a disjoint family of subsets of I, and $\mathfrak{C}_j$ is the subalgebra of $\mathfrak{A}$ generated by $\bigcup_{i \in I_j} \mathfrak{B}_i$ for each j, then $\langle \mathfrak{C}_j \rangle_{j \in J}$ is Boolean-independent.

(d) Suppose that $B \subseteq \mathfrak{A}$ is a Boolean-independent set and that $\langle C_j \rangle_{j \in J}$ is a disjoint family of subsets of B. For $j \in J$ write $\mathfrak{C}_j$ for the subalgebra of $\mathfrak{A}$ generated by C_j. Then $\langle \mathfrak{C}_j \rangle_{j \in J}$ is Boolean-independent.

(e) Suppose that $\langle\mathfrak{B}_i\rangle_{i\in I}$ is a Boolean-independent family of subalgebras of $\mathfrak{A}$, and that for each $i\in I$ we have a Boolean-independent subset B_i of $\mathfrak{B}_i$. Then $\langle B_i\rangle_{i\in I}$ is disjoint and $\bigcup_{i\in I}B_i$ is Boolean-independent.

(f) Let $\langle D_i\rangle_{i\in I}$ be a family of partitions of unity in $\mathfrak{A}$, none containing 0. For each $i\in I$ let $\mathfrak{B}_i$ be the order-closed subalgebra of $\mathfrak{A}$ generated by D_i. Then $\langle\mathfrak{B}_i\rangle_{i\in I}$ is Boolean-independent iff $\langle D_i\rangle_{i\in I}$ is Boolean-independent.

515C Proposition Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$, and κ a cardinal.

(a) There is a Boolean-independent subset of $\mathfrak{A}$ with cardinal κ iff there is a subalgebra of $\mathfrak{A}$ which is isomorphic to the algebra of open-and-closed subsets of $\{0,1\}^\kappa$.

(b) If $\mathfrak{A}$ is Dedekind complete, there is a Boolean-independent subset of $\mathfrak{A}$ with cardinal κ iff there is a subalgebra of $\mathfrak{A}$ which is isomorphic to the regular open algebra of $\{0,1\}^\kappa$.

515D Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, not $\{0\}$, and $\mathfrak{B}$ an order-closed subalgebra of $\mathfrak{A}$ such that $\mathfrak{A}$ is relatively atomless over $\mathfrak{B}$. Then there is an $a^*\in\mathfrak{A}\setminus\{0,1\}$ such that $\mathfrak{B}$ and $\{0,a^*,1\setminus a^*,1\}$ are Boolean-independent subalgebras of $\mathfrak{A}$.

515E Lemma Let $\mathfrak{A}$ be a Boolean algebra. Suppose that $C\subseteq\mathfrak{A}^+$ and that $\#(C)<c(\mathfrak{A}_c)$ for every $c\in C$. Then there is a partition D of unity in $\mathfrak{A}$ such that every member of C includes a non-zero member of D.

515F Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra such that $c(\mathfrak{A})=\operatorname{sat}(\mathfrak{A})$ and $\mathfrak{A}$ is cellularity-homogeneous. Then there is a Boolean-independent family $\langle D_i\rangle_{i\in I}$ of partitions of unity in $\mathfrak{A}$ such that $\#(I)=\sup_{i\in I}\#(D_i)=c(\mathfrak{A})$.

515G Lemma Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a non-empty family of Boolean algebras with simple product $\mathfrak{A}$. Suppose that for each $i\in I$ the algebra $\mathfrak{A}_i$ has a Boolean-independent set with cardinal $\kappa_i\ge\omega$. Then $\mathfrak{A}$ has a Boolean-independent set with cardinal $\kappa=\#(\prod_{i\in I}\kappa_i)$.

515H The Balcar-Franěk theorem Let $\mathfrak{A}$ be an infinite Dedekind complete Boolean algebra. Then there is a Boolean-independent set $A\subseteq\mathfrak{A}$ such that $\#(A)=\#(\mathfrak{A})$.

515I Corollary If $\mathfrak{A}$ is an infinite Dedekind complete Boolean algebra and $\kappa\le\#(\mathfrak{A})$, $\mathfrak{A}$ has a subalgebra isomorphic to the regular open algebra of $\{0,1\}^\kappa$.

515J Corollary If $\mathfrak{A}$ is an infinite Dedekind complete Boolean algebra with Stone space Z, then $\#(Z)=2^{\#(\mathfrak{A})}$.

515K Lemma Let $\mathfrak{A}$ be an infinite Boolean algebra with the σ-interpolation property.

(a) Let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a disjoint sequence in $\mathfrak{A}$. Then $\#(\mathfrak{A})\ge\#(\prod_{n\in\mathbb{N}}\mathfrak{A}_{a_n})$.

(b) Set $\kappa=\#(\mathfrak{A})$, and let I be the set of those $a\in\mathfrak{A}$ such that $\#(\mathfrak{A}_a)<\kappa$. Then I is an ideal of $\mathfrak{A}$, and
 either $\mathfrak{A}/I$ is infinite,
 or there is a set $J\subseteq I$ with cardinal κ such that every sequence in J has an upper bound in J,
 or $\#(\prod_{n\in\mathbb{N}}\mathfrak{A}_{a_n})=\kappa$ for some sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in I.

515L Theorem If $\mathfrak{A}$ is an infinite Boolean algebra with the σ-interpolation property, then $\#(\mathfrak{A})$ is equal to the cardinal power $\#(\mathfrak{A})^\omega$.

515M Corollary (a) If $\mathfrak{A}$ is an infinite ccc Dedekind σ-complete Boolean algebra then $\#(\mathfrak{A})=\tau(\mathfrak{A})^\omega$.

(b) If $\mathfrak{A}$ is any infinite Dedekind σ-complete Boolean algebra, then $\#(L^0(\mathfrak{A}))=\#(L^\infty(\mathfrak{A}))=\#(\mathfrak{A})$.

515N Proposition Let I be a set. Write $\mathfrak{G}$ for the regular open algebra $\mathrm{RO}(\{0,1\}^I)$.

(a) $\mathfrak{G}$ is ccc and Dedekind complete and isomorphic to the category algebra of $\{0,1\}^I$. The algebra of open-and-closed subsets of $\{0,1\}^I$ is an order-dense subalgebra of $\mathfrak{G}$.

(b) Let $\mathfrak{A}$ be a Boolean algebra. Then $\mathfrak{A}$ is isomorphic to $\mathfrak{G}$ iff it is Dedekind complete and there is a Boolean-independent family $\langle a_i\rangle_{i\in I}$ in $\mathfrak{A}$ such that the subalgebra generated by $\{a_i:i\in I\}$ is order-dense in $\mathfrak{A}$.

(c) If I is infinite, $\mathfrak{G}$ is homogeneous.

515O Proposition (a) A Boolean algebra is isomorphic to $\mathfrak{G} = \mathrm{RO}(\{0,1\}^{\mathbb{N}})$ iff it is Dedekind complete, atomless, has countable π-weight and is not $\{0\}$. In particular, the regular open algebra $\mathrm{RO}(\mathbb{R})$ is isomorphic to $\mathfrak{G}$.

(b) Every atomless order-closed subalgebra of $\mathfrak{G}$ is isomorphic to $\mathfrak{G}$.

515P Proposition A Boolean algebra $\mathfrak{A}$ is isomorphic to $\mathrm{RO}(\{0,1\}^{\omega_1})$ iff

(α) it is non-zero, ccc and Dedekind complete,

(β) every non-zero principal ideal of $\mathfrak{A}$ has π-weight ω_1,

(γ) there is a non-decreasing family $\langle A_\xi\rangle_{\xi<\omega_1}$ of countable subsets of $\mathfrak{A}$ such that

each A_ξ is order-dense in the order-closed subalgebra of $\mathfrak{A}$ which it generates,

$A_\zeta = \bigcup_{\xi<\zeta} A_\xi$ for every non-zero countable limit ordinal ζ,

$\bigcup_{\xi<\omega_1} A_\xi$ is order-dense in $\mathfrak{A}$.

515Q Proposition Let $\mathfrak{A}$ be an atomless order-closed subalgebra of $\mathfrak{G} = \mathrm{RO}(\{0,1\}^{\omega_1})$. Then $\mathfrak{A}$ is isomorphic either to $\mathrm{RO}(\{0,1\}^\omega)$ or to $\mathfrak{G}$ or to the simple product $\mathrm{RO}(\{0,1\}^\omega) \times \mathfrak{G}$.

Version of 9.10.14

516 Precalibers

In this section I will try to display the elementary connexions between 'precalibers', as defined in 511E, and the cardinal functions we have looked at so far. The first step is to generalize the idea of precaliber from partially ordered sets to supported relations (516A); the point is that Galois-Tukey connections give us information on precalibers (516C), and in particular give quick proofs that partially ordered sets, topological spaces and Boolean algebras related in the canonical ways explored in §514 have many of the same precalibers (516G, 516H, 516M). Much of the section is taken up with lists of expected facts, but for some results the hypotheses need to be chosen with care. I end with a fundamental theorem on the saturation of product spaces (516T).

516A Definition If (A,R,B) is a supported relation, a **precaliber triple** of (A,R,B) is a triple $(\kappa,\lambda,{<}\theta)$ where κ, λ and θ are cardinals and whenever $\langle a_\xi\rangle_{\xi<\kappa}$ is a family in A then there is a set $\Gamma \in [\kappa]^\lambda$ such that $\langle a_\xi\rangle_{\xi\in\Gamma}$ is ${<}\theta$-linked in the sense of 512Bc. Similarly, (κ,λ,θ) is a precaliber triple of (A,R,B) if whenever $\langle a_\xi\rangle_{\xi<\kappa}$ is a family in A then there is a set $\Gamma \in [\kappa]^\lambda$ such that $\langle a_\xi\rangle_{\xi\in\Gamma}$ is θ-linked; that is, if $(\kappa,\lambda,{<}\theta^+)$ is a precaliber triple.

Now (κ,λ) is a **precaliber pair** of (A,R,B) if $(\kappa,\lambda,{<}\omega)$ is a precaliber triple of (A,R,B), and κ is a **precaliber** of (A,R,B) if (κ,κ) is a precaliber pair.

516B Elementary remarks Let (A,R,B) be a supported relation.

(a) If $\kappa' \ge \kappa$, $\lambda' \le \lambda$, $\theta' \le \theta$ and $(\kappa,\lambda,{<}\theta)$ is a precaliber triple of (A,R,B), then $(\kappa',\lambda',{<}\theta')$ is a precaliber triple of (A,R,B). So if $\kappa' \ge \kappa$, $\lambda' \le \lambda$ and (κ,λ) is a precaliber pair of (A,R,B), then (κ',λ') is a precaliber pair of (A,R,B).

(b) If $\theta > 0$, then $(0,0,{<}\theta)$ is a precaliber triple of (A,R,B) iff $B \ne \emptyset$. If $A = \emptyset$ then $(\kappa,\lambda,{<}\theta)$ is a precaliber triple of (A,R,B) whenever $\kappa \ge 1$. If $A \ne \emptyset$ and $A \ne R^{-1}[B]$, then the only precaliber triples of (A,R,B) are of the form $(\kappa,0,{<}\theta)$. If $A \ne \emptyset$ and $(\kappa,\lambda,{<}\theta)$ is a precaliber triple of (A,R,B), then $\lambda \le \kappa$. $\mathrm{cov}(A,R,B) = \infty$ iff 1 is not a precaliber of (A,R,B).

(c) If (κ,λ,λ) is a precaliber triple of (A,R,B) then $(\kappa,\lambda,{<}\theta)$ is a precaliber triple of (A,R,B) for every θ; in particular, (κ,λ) is a precaliber pair of (A,R,B).

(d) If $(\kappa,\kappa,{<}\theta)$ is a precaliber triple of (A,R,B), so is $(\mathrm{cf}\,\kappa,\mathrm{cf}\,\kappa,{<}\theta)$. In particular, if κ is a precaliber of (A,R,B), so is $\mathrm{cf}\,\kappa$.

516C Theorem Suppose that (A,R,B) and (C,S,D) are supported relations, and that $(A,R,B) \preccurlyeq_{\mathrm{GT}} (C,S,D)$. Then $(\kappa,\lambda,{<}\theta)$ or (κ,λ,θ) is a precaliber triple of (A,R,B) whenever it is a precaliber triple of (C,S,D), so (κ,λ) is a precaliber pair of (A,R,B) whenever it is a precaliber pair of (C,S,D), and κ is a precaliber of (A,R,B) whenever it is a precaliber of (C,S,D).

516D Corollary If $(A,R,B) \equiv_{\mathrm{GT}} (C,S,D)$ then (A,R,B) and (C,S,D) have the same precaliber triples, the same precaliber pairs and the same precalibers.

516F Proposition (a) If P is a partially ordered set, $(\kappa,\lambda,{<}\theta)$ or (κ,λ,θ) is a precaliber triple of $(P,\le,P)$ iff it is an upwards precaliber triple of P.

(b) If $\mathfrak{A}$ is a Boolean algebra, then $\mathfrak{A}$ and $(\mathfrak{A}^+,\supseteq,\mathfrak{A}^+)$ have the same precaliber triples.

(c) If $(X,\mathfrak{T})$ is a topological space, then X and $(\mathfrak{T}\setminus\{\emptyset\},\supseteq,\mathfrak{T}\setminus\{\emptyset\})$ have the same precaliber triples.

516G Corollary Let $(P,\le)$ be a partially ordered set.

(a) If Q is a cofinal subset of P, then P and Q have the same upwards precaliber triples.

(b) Let $\mathfrak{T}^\uparrow$ be the up-topology of P. Then $(\kappa,\lambda,{<}\theta)$ is an upwards precaliber triple for $(P,\le)$ iff it is a precaliber triple for $(P,\mathfrak{T}^\uparrow)$.

516H Corollary Let $\mathfrak{A}$ be a Boolean algebra.

(a) If Z is the Stone space of $\mathfrak{A}$, then $\mathfrak{A}$ and Z have the same precaliber triples.

(b) If $\mathfrak{B}$ is an order-dense subalgebra of $\mathfrak{A}$, then $\mathfrak{A}$ and $\mathfrak{B}$ have the same precaliber triples.

516I Corollary Let $(X,\mathfrak{T})$ be a topological space.

(a) If Y is an open subspace of X, then every precaliber triple of X is a precaliber triple of Y.

(b) If Y is a dense subspace of X, then every precaliber triple of X is a precaliber triple of Y.

(c) If X is regular and Y is a dense subspace of X, then X and Y have the same precaliber triples.

(d) Suppose that Y is a topological space, and that there is a continuous surjection $f:X\to Y$ such that $\operatorname{int} f[G]\ne\emptyset$ whenever $G\subseteq X$ is a non-empty open set. Then every precaliber triple of X is a precaliber triple of Y.

516J Proposition Let (A,R,B) be a supported relation.

(a) $\operatorname{sat}(A,R,B)$ is the least cardinal κ, if there is one, such that $(\kappa,2)$ is a precaliber pair of (A,R,B); if there is no such κ, $\operatorname{sat}(A,R,B)=\infty$. In particular, if $\kappa\ge 2$ is a precaliber of (A,R,B), then $\kappa\ge\operatorname{sat}(A,R,B)$.

(b) If $\kappa>\max(\omega,\lambda,\operatorname{link}_{<\theta}(A,R,B))$ then $(\kappa,\lambda^+,{<}\theta)$ is a precaliber triple of (A,R,B). In particular, if $\kappa>\max(\omega,\lambda,\operatorname{cov}(A,R,B))$ then $(\kappa,\lambda^+,{<}\theta)$ is a precaliber triple of (A,R,B) for every θ.

(c) If $\operatorname{cf}\kappa>\operatorname{link}_{<\theta}(A,R,B)$ then $(\kappa,\kappa,{<}\theta)$ is a precaliber triple of (A,R,B).

516K Proposition Let P be a partially ordered set.

(a) $\operatorname{sat}^\uparrow(P)$ is the least cardinal κ such that $(\kappa,2)$ is an upwards precaliber pair of P.

(b) If $\kappa>\max(\omega,\lambda,\operatorname{link}^\uparrow_{<\theta}(P))$ then $(\kappa,\lambda^+,{<}\theta)$ is an upwards precaliber triple of P. In particular, if $\kappa>\max(\omega,\lambda,\operatorname{cf}P)$ then $(\kappa,\lambda^+,{<}\theta)$ is an upwards precaliber triple of P for every θ, and if $\kappa>\max(\omega,\lambda,d^\uparrow(P))$ then (κ,λ^+) is an upwards precaliber pair of P.

(c) If $\operatorname{cf}\kappa>\operatorname{cf}P$ then $(\kappa,\kappa,{<}\theta)$ is an upwards precaliber triple of P for every θ. If $\operatorname{cf}\kappa>d^\uparrow(P)$ then κ is an up-precaliber of P.

(d) If $\operatorname{sat}^\uparrow(P)\ge\omega$, $(\operatorname{sat}^\uparrow(P),\omega)$ is an upwards precaliber pair of P.

516L Corollary Let $\mathfrak{A}$ be a Boolean algebra.

(a) $\operatorname{sat}(\mathfrak{A})$ is the least cardinal κ such that $(\kappa,2)$ is a precaliber pair of $\mathfrak{A}$.

(b) If $\kappa>\max(\omega,\lambda,\operatorname{link}_{<\theta}(\mathfrak{A}))$ then $(\kappa,\lambda^+,{<}\theta)$ is a precaliber triple of $\mathfrak{A}$. In particular, if $\kappa>\max(\omega,\lambda,\pi(\mathfrak{A}))$ then $(\kappa,\lambda^+,{<}\theta)$ is a precaliber triple of $\mathfrak{A}$ for every θ, and if $\kappa>\max(\omega,\lambda,d(\mathfrak{A}))$ then (κ,λ^+) is a precaliber pair of $\mathfrak{A}$.

(c) If $\operatorname{cf}\kappa>d(\mathfrak{A})$ then κ is a precaliber of $\mathfrak{A}$.

(d) If $\mathfrak{A}$ is infinite, $(\operatorname{sat}(\mathfrak{A}),\omega)$ is a precaliber pair of $\mathfrak{A}$.

516M Lemma Let $(X,\mathfrak{T})$ be a topological space and $\mathrm{RO}(X)$ its regular open algebra. If κ, λ and θ are cardinals, and $\theta\le\omega$, then the following are equiveridical:

(i) $(\kappa,\lambda,{<}\theta)$ is a precaliber triple of $(X,\mathfrak{T})$;

(ii) $(\kappa,\lambda,{<}\theta)$ is a precaliber triple of $(\mathfrak{T}\setminus\{\emptyset\},\ni,X)$;

(iii) $(\kappa,\lambda,{<}\theta)$ is a precaliber triple of $\mathrm{RO}(X)$.

516N Corollary Let X be a topological space.

(a) $\mathrm{sat}(X)$ is the least cardinal κ such that $(\kappa,2)$ is a precaliber pair of X.

(b) If $\kappa>\max(\omega,\lambda,d(X))$ then (κ,λ^+) is a precaliber pair of X.

(c) If $\mathrm{cf}\,\kappa>d(X)$ then κ is a precaliber of X.

(d) If $\mathrm{sat}(X)$ is infinite, then $(\mathrm{sat}(X),\omega)$ is a precaliber pair of X.

516O Proposition Let $(X,\mathfrak{T})$ be a topological space.

(a) If Y is a continuous image of X and $\theta\le\omega$, then $(\kappa,\lambda,{<}\theta)$ is a precaliber triple of Y whenever it is a precaliber triple of X.

(b) Suppose that X is the product of a family $\langle X_i\rangle_{i\in I}$ of topological spaces. If $(\kappa,\kappa,{<}\theta)$ is a precaliber triple of every X_i and *either* I is finite *or* $\theta\le\omega$ and κ is a regular uncountable cardinal, then $(\kappa,\kappa,{<}\theta)$ is a precaliber triple of X.

516P Corollary Let $\langle P_i\rangle_{i\in I}$ be a family of non-empty partially ordered sets, with upwards finite-support product $P=\bigotimes^{\uparrow}_{i\in I}P_i$. If $(\kappa,\kappa,{<}\theta)$ is an upwards precaliber triple of every P_i and either I is finite or $\theta\le\omega$ and κ is a regular uncountable cardinal, then $(\kappa,\kappa,{<}\theta)$ is an upwards precaliber triple of P.

516Q Proposition Let X be a locally compact Hausdorff topological space.

(a) (κ,λ) is a precaliber pair of X iff whenever $\langle G_\xi\rangle_{\xi<\kappa}$ is a family of non-empty open subsets of X, then there is an $x\in X$ such that $\#(\{\xi:x\in G_\xi\})\ge\lambda$.

(b) Suppose that κ is a regular infinite cardinal. Then κ is a precaliber of X iff $\mathrm{sat}(X)\le\kappa$ and whenever $\langle E_\xi\rangle_{\xi<\kappa}$ is a non-decreasing family of nowhere dense subsets of X then $\bigcup_{\xi<\kappa}E_\xi$ has empty interior.

516R Corollary Let $\mathfrak{A}$ be a Boolean algebra and Z its Stone space.

(a) A pair (κ,λ) of cardinals is a precaliber pair of $\mathfrak{A}$ iff whenever $\langle G_\xi\rangle_{\xi<\kappa}$ is a family of non-empty open sets in Z there is a $z\in Z$ such that $\#(\{\xi:z\in G_\xi\})\ge\lambda$.

(b) Suppose that $\kappa\ge\mathrm{sat}(\mathfrak{A})$ is a regular infinite cardinal. Then κ is a precaliber of $\mathfrak{A}$ iff whenever $\langle E_\xi\rangle_{\xi<\kappa}$ is a non-decreasing family of nowhere dense subsets of Z then $\bigcup_{\xi<\kappa}E_\xi$ has empty interior.

516S Proposition Let $\mathfrak{A}$ be a Boolean algebra.

(a) If $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$ and $(\kappa,\lambda,{<}\theta)$ is a precaliber triple of $\mathfrak{A}$ such that $\theta\le\omega$, then $(\kappa,\lambda,{<}\theta)$ is a precaliber triple of $\mathfrak{B}$. In particular, every precaliber pair of $\mathfrak{A}$ is a precaliber pair of $\mathfrak{B}$ and $\mathfrak{B}$ will satisfy Knaster's condition if $\mathfrak{A}$ does.

(b) If $\mathfrak{B}$ is a regularly embedded subalgebra of $\mathfrak{A}$, then every precaliber triple of $\mathfrak{A}$ is a precaliber triple of $\mathfrak{B}$.

(c) If $\mathfrak{B}$ is a Boolean algebra and $\phi:\mathfrak{A}\to\mathfrak{B}$ is a surjective order-continuous Boolean homomorphism, then every precaliber triple of $\mathfrak{A}$ is a precaliber triple of $\mathfrak{B}$.

(d) If $\mathfrak{B}$ is a principal ideal of $\mathfrak{A}$ then every precaliber triple of $\mathfrak{A}$ is a precaliber triple of $\mathfrak{B}$.

(e) If $\mathfrak{A}$ is the simple product of a family $\langle\mathfrak{A}_i\rangle_{i\in I}$ of Boolean algebras, $(\kappa,\lambda,{<}\theta)$ is a precaliber triple of $\mathfrak{A}_i$ for every $i\in I$ and $\mathrm{cf}\,\kappa>\#(I)$, then $(\kappa,\lambda,{<}\theta)$ is a precaliber triple of $\mathfrak{A}$.

516T Theorem (a) Let P and Q be partially ordered sets, and κ a cardinal such that $(\kappa,\mathrm{sat}^{\uparrow}(Q),2)$ is an upwards precaliber triple of P. Then $\mathrm{sat}^{\uparrow}(P\times Q)\le\kappa$.

(b) Let $\langle P_i\rangle_{i\in I}$ be a family of non-empty partially ordered sets with upwards finite-support product P. Suppose that κ is a regular uncountable cardinal such that $(\kappa,\kappa,2)$ is an upwards precaliber triple of every P_i. Then $\mathrm{sat}^{\uparrow}(P)\le\kappa$.

516U Corollary Let $\mathfrak{A}$ be a Boolean algebra satisfying Knaster's condition and $\mathfrak{B}$ a ccc Boolean algebra. Then their free product $\mathfrak{A}\otimes\mathfrak{B}$ is ccc.

516V Proposition Let $\mathfrak{A}$ be an atomless Boolean algebra which satisfies Knaster's condition. Then $\mathfrak{A}$ has an atomless order-closed subalgebra with countable Maharam type.

Version of 14.11.14

517 Martin numbers

I devote a section to the study of 'Martin numbers' of partially ordered sets and Boolean algebras. Like additivity and cofinality they enable us to frame as theorems of ZFC some important arguments which were first used in special models of set theory, and to pose challenging questions on the relationships between classical structures in analysis. I begin with some general remarks on the Martin numbers of partially ordered sets (517A-517E); most of these are perfectly elementary but the equivalent conditions of 517B, in particular, are useful and not all obvious. Much of the importance of Martin numbers comes from their effect on precalibers (517F, 517H) and hence on saturation of products (517G). The same ideas can be expressed in terms of Boolean algebras, with no surprises (517I). I have not set out a definition of 'Martin number' for a topological space, but the Novák number of a locally compact Hausdorff space is closely related to the Martin numbers of its regular open algebra and its algebra of open-and-closed sets (517J-517K). Consequently we have connexions between the Martin number and the weak distributivity of a Boolean algebra (517L). A striking fact, which will have a prominent role in the next chapter, is that non-trivial countable partially ordered sets all have the same Martin number $\mathfrak{m}_{\mathrm{countable}}$ (517P).

517A Proposition For any partially ordered set P, $\mathfrak{m}^{\uparrow}(P)\ge\omega_1$.

517B Lemma Let P be a partially ordered set, and κ a cardinal. Then the following are equiveridical:

(i) $\kappa<\mathfrak{m}^{\uparrow}(P)$;

(ii) whenever $p_0\in P$ and $\mathcal{Q}$ is a family of up-open cofinal subsets of P with $\#(\mathcal{Q})\le\kappa$, there is an upwards-linked subset of P which contains p_0 and meets every member of $\mathcal{Q}$;

(iii) whenever $p_0\in P$ and $\mathcal{A}$ is a family of maximal up-antichains in P with $\#(\mathcal{A})\le\kappa$, there is an upwards-linked subset of P which contains p_0 and meets every member of $\mathcal{A}$;

(iv) whenever $p_0\in P$ and $\mathcal{Q}$ is a family of cofinal subsets of P with $\#(\mathcal{Q})\le\kappa$, there is an upwards-directed subset of P which contains p_0 and meets every member of $\mathcal{Q}$;

(v) whenever $p_0\in P$ and $\mathcal{Q}$ is a family of up-open cofinal subsets of P with $\#(\mathcal{Q})\le\kappa$, there is an upwards-directed subset of P which contains p_0 and meets every member of $\mathcal{Q}$;

(vi) whenever $p_0\in P$ and $\mathcal{A}$ is a family of maximal up-antichains in P with $\#(\mathcal{A})\le\kappa$, there is an upwards-directed subset of P which contains p_0 and meets every member of $\mathcal{A}$.

517C Lemma Let P_0 and P_1 be partially ordered sets, and suppose that there is a relation $S\subseteq P_0\times P_1$ such that $S[P_0]$ is cofinal with P_1, $S^{-1}[Q]$ is cofinal with P_0 for every cofinal $Q\subseteq P_1$, and $S[R]$ is upwards-linked in P_1 for every upwards-linked $R\subseteq P_0$. Then $\mathfrak{m}^{\uparrow}(P_1)\ge\mathfrak{m}^{\uparrow}(P_0)$.

517D Proposition (a) If P is a partially ordered set and Q is a cofinal subset of P, then $\mathfrak{m}^{\uparrow}(P)=\mathfrak{m}^{\uparrow}(Q)$.

(b) If P is any partially ordered set and $\mathrm{RO}^{\uparrow}(P)$ is its regular open algebra when it is given its up-topology, then $\mathfrak{m}^{\uparrow}(P)=\mathfrak{m}(\mathrm{RO}^{\uparrow}(P))$.

(c) If P is a partially ordered set and $p_0\in P$, then $\mathfrak{m}^{\uparrow}([p_0,\infty[)\ge\mathfrak{m}^{\uparrow}(P)$.

517E Corollary Let P be a partially ordered set such that $\mathfrak{m}^{\uparrow}(P)$ is not ∞. Then $\mathfrak{m}^{\uparrow}(P)\le 2^{\operatorname{cf}P}$.

517F Proposition Let P be a non-empty partially ordered set.

(a) Suppose that κ and λ are cardinals such that $\operatorname{sat}^{\uparrow}(P)\le\operatorname{cf}\kappa$, $\lambda\le\kappa$ and $\lambda<\mathfrak{m}^{\uparrow}(P)$. Then (κ,λ) is an upwards precaliber pair of P.

(b) In particular, if $\operatorname{sat}^{\uparrow}(P)\le\operatorname{cf}\kappa\le\kappa<\mathfrak{m}^{\uparrow}(P)$ then κ is an up-precaliber of P.

517G Corollary (a) If P and Q are partially ordered sets and $\operatorname{sat}^\uparrow(Q) < \mathfrak{m}^\uparrow(P)$, then $\operatorname{sat}^\uparrow(P \times Q)$ is at most $\max(\omega, \operatorname{sat}^\uparrow(P), \operatorname{sat}^\uparrow(Q))$.

(b) Let $\langle P_i \rangle_{i \in I}$ be a family of non-empty partially ordered sets with upwards finite-support product P. Let κ be a regular uncountable cardinal such that $\operatorname{sat}^\uparrow(P_i) \le \kappa < \mathfrak{m}^\uparrow(P_i)$ for every $i \in I$. Then $\operatorname{sat}^\uparrow(P) \le \kappa$.

517H Proposition Let P be a non-empty partially ordered set, and let P^* be the upwards finite-support product of the family $\langle P_n \rangle_{n \in \mathbb{N}}$ where $P_n = P$ for every n. Suppose that $\kappa < \mathfrak{m}^\uparrow(P^*)$.

(a) Every subset of P with κ or fewer members can be covered by a sequence of upwards-directed sets.

(b) In particular, if κ is uncountable then (κ, λ) is an upwards precaliber pair of P for every $\lambda < \kappa$, and if κ has uncountable cofinality then κ is an up-precaliber of P.

517I Proposition Let $\mathfrak{A}$ be a Boolean algebra.

(a) If $\mathfrak{B}$ is a regularly embedded subalgebra of $\mathfrak{A}$, then $\mathfrak{m}(\mathfrak{B}) \ge \mathfrak{m}(\mathfrak{A})$.

(b) If $\mathfrak{B}$ is a principal ideal of $\mathfrak{A}$, then $\mathfrak{m}(\mathfrak{B}) \ge \mathfrak{m}(\mathfrak{A})$.

(c) If $\mathfrak{B}$ is an order-dense subalgebra of $\mathfrak{A}$, then $\mathfrak{m}(\mathfrak{B}) = \mathfrak{m}(\mathfrak{A})$.

(d) If $\widehat{\mathfrak{A}}$ is the Dedekind completion of $\mathfrak{A}$, then $\mathfrak{m}(\widehat{\mathfrak{A}}) = \mathfrak{m}(\mathfrak{A})$.

(e) If $D \subseteq \mathfrak{A}$ is non-empty and $\sup D = 1$, then $\mathfrak{m}(\mathfrak{A}) = \min_{d \in D} \mathfrak{m}(\mathfrak{A}_d)$, where $\mathfrak{A}_d$ is the principal ideal generated by d.

(f) If $\mathfrak{A}$ is the simple product of a non-empty family $\langle \mathfrak{A}_i \rangle_{i \in I}$ of Boolean algebras, then $\mathfrak{m}(\mathfrak{A}) = \min_{i \in I} \mathfrak{m}(\mathfrak{A}_i)$.

(g) Suppose that κ and λ are infinite cardinals such that $\operatorname{sat}(\mathfrak{A}) \le \operatorname{cf} \kappa$, $\lambda \le \kappa$ and $\lambda < \mathfrak{m}(\mathfrak{A})$. Then (κ, λ) is a precaliber pair of $\mathfrak{A}$.

517J Proposition Let X be a locally compact Hausdorff space, and κ a cardinal. Then the following are equiveridical:

(i) $\kappa < \mathfrak{m}(\mathrm{RO}(X))$, where $\mathrm{RO}(X)$ is the regular open algebra of X;

(ii) $X \cap \bigcap \mathcal{G}$ is dense in X whenever $\mathcal{G}$ is a family of dense open subsets of X and $\#(\mathcal{G}) \le \kappa$;

(iii) $\kappa < n(H)$ for every non-empty open set $H \subseteq X$.

517K Corollary Let $\mathfrak{A}$ be a Boolean algebra with Stone space Z.

(a) $\mathfrak{m}(\mathfrak{A}) = \mathfrak{m}(\mathrm{RO}(Z))$.

(b) For any cardinal κ, the following are equiveridical:

  (i) $\kappa < \mathfrak{m}(\mathfrak{A})$;

  (ii) $Z \cap \bigcap \mathcal{G}$ is dense in Z whenever $\mathcal{G}$ is a family of dense open subsets of Z and $\#(\mathcal{G}) \le \kappa$;

  (iii) $\kappa < n(H)$ for every non-empty open set $H \subseteq Z$.

517L Proposition Let $\mathfrak{A}$ be a Boolean algebra.

(a) $\operatorname{wdistr}(\mathfrak{A}) \le \mathfrak{m}(\mathfrak{A})$.

(b) If $\operatorname{wdistr}(\mathfrak{A})$ is a precaliber of $\mathfrak{A}$ then $\operatorname{wdistr}(\mathfrak{A}) < \mathfrak{m}(\mathfrak{A})$.

517M Proposition Let X be any topological space. Then the Novák number $n(X)$ of X is at most $\sup\{\mathfrak{m}(\mathrm{RO}(G)) : G \subseteq X \text{ is open and not empty}\}$, where $\mathrm{RO}(G)$ is the regular open algebra of G.

517N Corollary If $\mathfrak{A}$ is a Martin-number-homogeneous Boolean algebra with Stone space Z, then $\mathfrak{m}(\mathfrak{A}) = n(Z)$.

517O Martin cardinals (a) For any class $\mathcal{P}$ of partially ordered sets, we have an associated cardinal

$$\mathfrak{m}^\uparrow_{\mathcal{P}} = \min\{\mathfrak{m}^\uparrow(P) : P \in \mathcal{P}\}.$$

Much the most important of these is the cardinal

$$\mathfrak{m} = \min\{\mathfrak{m}^\uparrow(P) : P \text{ is upwards-ccc}\}.$$

Others of great interest are

$$\mathfrak{p} = \min\{\mathfrak{m}^\uparrow(P) : P \text{ is } \sigma\text{-centered upwards}\},$$

$$\mathfrak{m}_{\mathrm{K}} = \min\{\mathfrak{m}^{\uparrow}(P) : P \text{ satisfies Knaster's condition upwards}\},$$

$$\mathfrak{m}_{\mathrm{countable}} = \min\{\mathfrak{m}^{\uparrow}(P) : P \text{ is a countable partially ordered set}\}.$$

Two more which are worth examining are

$$\mathfrak{m}_{\sigma\text{-linked}} = \min\{\mathfrak{m}^{\uparrow}(P) : P \text{ is } \sigma\text{-linked upwards}\},$$

$$\mathfrak{m}_{\mathrm{pc}\omega_1} = \min\{\mathfrak{m}^{\uparrow}(P) : \omega_1 \text{ is an up-precaliber of } P\}.$$

(b) These cardinals are related as follows:

$$\begin{tikzcd}
 & & \mathfrak{m}_{\sigma\text{-linked}} \arrow[r, dash] \arrow[d, dash] & \mathfrak{p} \arrow[r, dash] \arrow[d, dash] & \mathfrak{m}_{\mathrm{countable}} \arrow[r, dash] & \mathfrak{c} \\
\omega_1 \arrow[r, dash] & \mathfrak{m} \arrow[r, dash] & \mathfrak{m}_{\mathrm{K}} \arrow[r, dash] & \mathfrak{m}_{\mathrm{pc}\omega_1} & &
\end{tikzcd}$$

The numbers here increase from bottom left to top right; that is,

$$\omega_1 \le \mathfrak{m} \le \mathfrak{m}_{\mathrm{K}} \le \mathfrak{m}_{\mathrm{pc}\omega_1} \le \mathfrak{p} \le \mathfrak{m}_{\mathrm{countable}} \le \mathfrak{c},$$

$$\mathfrak{m}_{\mathrm{K}} \le \mathfrak{m}_{\sigma\text{-linked}} \le \mathfrak{p}.$$

(c) In the chain $\omega_1 \le \mathfrak{m} \le \mathfrak{m}_{\mathrm{K}} \le \mathfrak{m}_{\mathrm{pc}\omega_1}$, at most one of the inequalities can be strict.

(d) Now **Martin's Axiom** is the assertion

$$\text{‘}\mathfrak{m} = \mathfrak{c}\text{’}.$$

From the diagram above, we see that this is a consequence of the continuum hypothesis (‘$\omega_1 = \mathfrak{c}$’), and fixes all the intermediate cardinals.

517P Proposition (a) $\omega_1 \le \mathfrak{m}_{\mathrm{countable}} \le \mathfrak{c}$.

(b) Let $\mathfrak{A}$ be a Boolean algebra with countable π-weight. If $\mathfrak{A}$ is purely atomic, then $\mathfrak{m}(\mathfrak{A}) = \infty$; otherwise, $\mathfrak{m}(\mathfrak{A}) = \mathfrak{m}_{\mathrm{countable}}$.

(c) If P is a partially ordered set of countable cofinality and $\mathfrak{m}^{\uparrow}(P)$ is not ∞, then $\mathfrak{m}^{\uparrow}(P) = \mathfrak{m}_{\mathrm{countable}}$.

(d)(i) Let X be a topological space such that its category algebra is atomless and has countable π-weight. Then $n(X) \le \mathfrak{m}_{\mathrm{countable}}$.

(ii) If X is a non-empty locally compact Hausdorff space with countable π-weight and no isolated points, then $n(X) = \mathfrak{m}_{\mathrm{countable}}$.

(iii) If X is a non-empty Polish space with no isolated points, then $n(X) = \mathfrak{m}_{\mathrm{countable}}$.

517Q Lemma If P is any partially ordered set, $\mathfrak{m}^{\uparrow}(P) \ge \min(\mathrm{add}_{\omega} P, \mathfrak{m}_{\mathrm{countable}})$.

517R Proposition (a)) Suppose that $\mathcal{A}$ is a family of subsets of $\mathbb{N}$ such that $\#(\mathcal{A}) < \mathfrak{p}$ and $\bigcap \mathcal{J}$ is infinite for every finite $\mathcal{J} \subseteq \mathcal{A}$. Then there is an infinite $I \subseteq \mathbb{N}$ such that $I \setminus A$ is finite for every $A \in \mathcal{A}$.

(b) $2^{\kappa} \le \mathfrak{c}$ for every $\kappa < \mathfrak{p}$.

(c) Suppose that X is a set and $\#(X) < \mathfrak{p}$. Then there is a countable set $\mathcal{A} \subseteq \mathcal{P}X$ such that $\mathcal{P}X$ is the σ-algebra generated by $\mathcal{A}$.

517S Proposition Let P be a partially ordered set which satisfies Knaster's condition upwards. If $A \subseteq P$ and $\#(A) < \mathfrak{m}_{\mathrm{K}}$, then A can be covered by a sequence of upwards-directed subsets of P.

Version of 24.12.14

518 Freese-Nation numbers

I run through those elements of the theory of Freese-Nation numbers, as developed by S.Fuchino, S.Geschke, S.Koppelberg, S.Shelah and L.Soukup, which seem relevant to questions concerning measure spaces and measure algebras. The first part of the section (518A-518K) examines the calculation of Freese-Nation numbers of familiar partially ordered sets and Boolean algebras. In 518L-518S I look at 'tight filtrations', which are of interest to us because of their use in lifting theorems (518L, §535).

518A Proposition Let P be a partially ordered set.
(a) $\mathrm{FN}(P) \le \max(3, \#(P))$.
(b) $\mathrm{FN}(P, \ge) = \mathrm{FN}(P, \le)$.
(c) If P has no maximal element, then $\operatorname{add} P \le \mathrm{FN}(P)$.

518B Proposition Let P be a partially ordered set and Q a subset of P.
(a) If Q is order-convex, then $\mathrm{FN}(Q) \le \mathrm{FN}(P)$.
(b) If Q is a retract of P, then $\mathrm{FN}(Q) \le \mathrm{FN}(P)$.
(c) If Q is Dedekind complete, then $\mathrm{FN}(Q) \le \mathrm{FN}(P)$.

518C Corollary (a) If $\mathfrak{A}$ is an infinite Dedekind σ-complete Boolean algebra then $\mathrm{FN}(\mathfrak{A}) \ge \mathrm{FN}(\mathcal{P}\mathbb{N})$.
(b) Let $\mathfrak{A}$ be an infinite Dedekind complete Boolean algebra. Then

$$\mathrm{FN}(\mathrm{RO}(\{0,1\}^{\#(\mathfrak{A})})) \le \mathrm{FN}(\mathfrak{A}) \le \mathrm{FN}(\mathcal{P}(\mathrm{link}(\mathfrak{A}))) \le \max(3, 2^{\mathrm{link}(\mathfrak{A})}).$$

(c) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra. If $\mathfrak{B}$ is either an order-closed subalgebra or a principal ideal of $\mathfrak{A}$, then $\mathrm{FN}(\mathfrak{B}) \le \mathrm{FN}(\mathfrak{A})$.

518D Corollary The following sets all have the same Freese-Nation number:
(i) $\mathcal{P}\mathbb{N}$;
(ii) $\mathbb{N}^{\mathbb{N}}$, with its usual ordering $\le$;
(iii) any infinite σ-linked Dedekind complete Boolean algebra;
(iv) the family of open subsets of any infinite Hausdorff second-countable topological space.

518E Lemma Let $(X, \mathfrak{T})$ be a T_1 topological space without isolated points, and $\mathcal{N}\mathit{wd}(X)$ the ideal of nowhere dense sets. Then there is a set $A \subseteq X$, with cardinal $\operatorname{cov} \mathcal{N}\mathit{wd}(X)$, such that $\#(A \cap F) < \mathrm{FN}^*(\mathfrak{T})$ for every $F \in \mathcal{N}\mathit{wd}(X)$.

518F Lemma Let $\mathfrak{A}$ be a Boolean algebra, $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$ and κ an infinite cardinal.
(a) If $\mathrm{cf}(\mathfrak{B} \cap [0, a]) < \kappa$ for every $a \in \mathfrak{A}$, then the Freese-Nation index of $\mathfrak{B}$ in $\mathfrak{A}$ is at most κ.
(b) Suppose that $I \in [\mathfrak{A}]^{<\mathrm{cf}\,\kappa}$ and $\mathfrak{B}_I$ is the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{B} \cup I$. If the Freese-Nation index of $\mathfrak{B}$ in $\mathfrak{A}$ is less than or equal to κ, so is the Freese-Nation index of $\mathfrak{B}_I$.
(c) If $\mathfrak{B}$ is expressible as the union of fewer than κ order-closed subalgebras of $\mathfrak{A}$, each of them Dedekind complete in itself, then the Freese-Nation index of $\mathfrak{B}$ in $\mathfrak{A}$ is at most κ.

518G Lemma Let P be a partially ordered set, ζ an ordinal, and $\langle A_\xi \rangle_{\xi<\zeta}$ a family with union P; set $P_\alpha = \bigcup_{\xi<\alpha} A_\xi$ for each $\alpha \le \zeta$. Let κ be a regular infinite cardinal such that, for each $\alpha < \zeta$, $\mathrm{FN}(P_{\alpha+1}) \le \kappa$ and the Freese-Nation index of P_α in $P_{\alpha+1}$ is at most κ. Then $\mathrm{FN}(P) \le \kappa$.

518H Lemma Suppose that κ is an uncountable cardinal of countable cofinality such that $\square_\kappa$ is true and $\mathrm{cf}[\lambda]^{\le\omega} \le \lambda^+$ for every $\lambda \le \kappa$. Then there are families $\langle M_{\alpha n} \rangle_{\alpha<\kappa^+, n\in\mathbb{N}}$, $\langle M_\alpha \rangle_{\alpha<\kappa^+}$ of sets and a function sk such that
(i) $\#(M_{\alpha n}) < \kappa$ whenever $\alpha < \kappa^+$ and $n \in \mathbb{N}$;
(ii) $\langle M_{\alpha n} \rangle_{n\in\mathbb{N}}$ is non-decreasing for each $\alpha < \kappa^+$;
(iii) $\langle M_\alpha \rangle_{\alpha<\kappa^+}$ is a non-decreasing family, $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for every non-zero limit ordinal $\alpha < \kappa^+$, and $\kappa^+ \subseteq \bigcup_{\alpha<\kappa^+} M_\alpha$;

(iv) if $\alpha < \kappa^+$ has uncountable cofinality, $M_\alpha = \bigcup_{n\in\mathbb{N}} M_{\alpha n}$;
(v) $X \subseteq \mathrm{sk}(X)$ for every set X;
(vi) $\mathrm{sk}(X)$ is countable whenever X is countable;
(vii) $A \subseteq \mathrm{sk}(X)$ whenever $A \in \mathrm{sk}(X)$ is countable;
(viii) $\mathrm{sk}(X) \subseteq \mathrm{sk}(Y)$ whenever $X \subseteq \mathrm{sk}(Y)$;
(ix) for every $\alpha < \kappa^+$ of uncountable cofinality there is an $m \in \mathbb{N}$ such that whenever $n \ge m$ and $A \subseteq M_{\alpha n}$ is countable there is a countable set $D \in M_{\alpha n}$ such that $A \subseteq \mathrm{sk}(D)$;
(x) $\bigcup_{\alpha<\kappa^+} M_\alpha \cap [\kappa]^{\le\omega}$ is cofinal with $[\kappa]^{\le\omega}$.

518I Theorem Let $\mathfrak{A}$ be a ccc Dedekind complete Boolean algebra. Suppose that
(α) $\mathrm{cf}[\lambda]^{\le\omega} \le \lambda^+$ for every cardinal $\lambda \le \tau(\mathfrak{A})$,
(β) $\Box_\lambda$ is true for every uncountable cardinal $\lambda \le \tau(\mathfrak{A})$ of countable cofinality.
Let $\mathfrak{A}$ be a ccc Dedekind complete Boolean algebra, and κ a regular uncountable cardinal such that $\mathrm{FN}(\mathfrak{B}) \le \kappa$ for every countably generated order-closed subalgebra $\mathfrak{B}$ of $\mathfrak{A}$. Then $\mathrm{FN}(\mathfrak{A}) \le \kappa$.

518J Lemma Let λ be an infinite cardinal and $\mathfrak{G}$ the regular open algebra of $\{0,1\}^\lambda$. Suppose that κ is the least cardinal of uncountable cofinality greater than or equal to $\mathrm{FN}(\mathfrak{G})$. Then $\kappa \le \mathfrak{c}^+$ and we have a family $\mathcal{V} \subseteq [\lambda]^{\le\mathfrak{c}}$, cofinal with $[\lambda]^{\le\mathfrak{c}}$, such that $\#(\{A \cap V : V \in \mathcal{V}\}) < \kappa$ for every countable set $A \subseteq \lambda$.

518K Theorem Suppose that $\lambda > \mathfrak{c}$ is a cardinal of countable cofinality such that $\mathrm{CTP}(\lambda^+, \lambda)$ is true. Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra with cardinal at least λ. Then $\mathrm{FN}(\mathfrak{A}) \ge \omega_2$.

518L Theorem Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, $\mathfrak{B}$ a tightly ω_1-filtered Boolean algebra, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a surjective sequentially order-continuous Boolean homomorphism; suppose that $\mathfrak{B} \ne \{0\}$. Then there is a Boolean homomorphism $\theta : \mathfrak{B} \to \mathfrak{A}$ such that $\pi\theta b = b$ for every $b \in \mathfrak{B}$.

518M Theorem Let $\mathfrak{A}$ be a Boolean algebra and κ a regular infinite cardinal such that $\mathrm{FN}(\mathfrak{A}) \le \kappa$ and $\#(\mathfrak{A}) \le \kappa^+$. Then $\mathfrak{A}$ is tightly κ-filtered.

518N Definition Let $\mathfrak{A}$ be a Boolean algebra and κ a cardinal. Then a **κ-Geschke system** for $\mathfrak{A}$ is a family $\mathbb{G}$ of subalgebras of $\mathfrak{A}$ such that
(α) every element of $\mathfrak{A}$ belongs to an element of $\mathbb{G}$ with cardinal less than κ;
(β) for any $\mathbb{G}_0 \subseteq \mathbb{G}$, the subalgebra of $\mathfrak{A}$ generated by $\bigcup \mathbb{G}_0$ belongs to $\mathbb{G}$;
(γ) whenever $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathbb{G}$, $a \in \mathfrak{B}_1$, $b \in \mathfrak{B}_2$ and $a \subseteq b$, then there is a $c \in \mathfrak{B}_1 \cap \mathfrak{B}_2$ such that $a \subseteq c \subseteq b$.

518O Lemma Let $\mathfrak{A}$ be a Boolean algebra, κ a cardinal and $\mathbb{G}$ a κ-Geschke system for $\mathfrak{A}$. Suppose that $\lambda \ge \kappa$ is a regular uncountable cardinal and that $f : [\mathfrak{A}]^{<\omega} \to [\mathfrak{A}]^{<\lambda}$ is a function. Then there is a $\mathfrak{B} \in \mathbb{G}$ such that $\#(\mathfrak{B}) < \lambda$ and $f(I) \subseteq \mathfrak{B}$ whenever $I \in [\mathfrak{B}]^{<\omega}$.

518P Lemma Let κ be a regular uncountable cardinal and $\mathfrak{A}$ a Boolean algebra. Then $\mathfrak{A}$ is tightly κ-filtered iff there is a κ-Geschke system for $\mathfrak{A}$.

518Q Corollary Let κ be a regular uncountable cardinal and $\mathfrak{A}$ a tightly κ-filtered Boolean algebra.
(a) If $\mathfrak{C}$ is a retract of $\mathfrak{A}$, then $\mathfrak{C}$ is tightly κ-filtered.
(b) If $\mathfrak{C}$ is a subalgebra of $\mathfrak{A}$ which is Dedekind complete, then $\mathfrak{C}$ is tightly κ-filtered.

518R Lemma (a) Let I be a set and $\mathfrak{G}$ the regular open algebra of $\{0,1\}^I$. For $J \subseteq I$ let $\mathfrak{G}_J$ be the order-closed subalgebra of $\mathfrak{G}$ consisting of regular open sets determined by coordinates in J. Suppose that J and K are disjoint subsets of I, and $\langle a_q\rangle_{q\in\mathbb{Q}}$, $\langle b_q\rangle_{q\in\mathbb{Q}}$ disjoint families in $\mathfrak{G}_J \setminus \{\emptyset\}$ and $\mathfrak{G}_K \setminus \{\emptyset\}$ respectively. For $t \in \mathbb{R}$ set $w_t = \sup_{p,q\in\mathbb{Q},p\le t\le q} a_q \cap b_p$, the supremum being taken in $\mathfrak{G}$; set $w = \sup_{p,q\in\mathbb{Q},p\le q} a_q \cap b_p$. If $w' \subseteq w$ belongs to the subalgebra of $\mathfrak{G}$ generated by $\mathfrak{G}_{I\setminus K} \cup \mathfrak{G}_{I\setminus J}$, then $\{t : w_t \subseteq w'\}$ is finite.
(b) If $I = \omega_3$ then $\mathfrak{G}$ is not tightly ω_1-filtered.

518S Theorem If $\mathfrak{A}$ is a tightly ω_1-filtered Dedekind complete Boolean algebra then $\#(\mathfrak{A}) \le \omega_2$.