Version of 9.1.15

Introduction to Volume 5

For the final volume of this treatise, I have collected results which demand more sophisticated set theory than elsewhere. The line is not sharp, but typically we are much closer to questions which are undecidable in ZFC. Only in Chapter 55 are these brought to the forefront of the discussion, but elsewhere much of the work depends on formulations carefully chosen to express, as arguments in ZFC, ideas which arose in contexts in which some special axiom – Martin's axiom, for instance – was being assumed. This has forced the development of concepts – e.g., cardinal functions of structures – which have taken on vigorous lives of their own, and which stand outside the territory marked by the techniques of earlier volumes.

In terms of the classification I have used elsewhere, this volume has one preparatory chapter and five working chapters. There is practically no measure theory in Chapter 51, which is an introduction to some of the methods which have been devised to make sense of abstract analysis in the vast range of alternative mathematical worlds which have become open to us in the last fifty years. It is centered on a study of partially ordered sets, which provide a language in which many of the most important principles can be expressed. Chapter 52 looks at manifestations of these ideas in measure theory. In Chapter 53 I continue the work of Volumes 3 and 4, examining questions which arise more or less naturally if we approach the topics of those volumes with the new techniques.

The Banach-Ulam problem got a mention in Volume 2, a paragraph in Volume 3 and a section in Volume 4; at last, in Chapter 54 of the present volume, I try to give a proper account of the extraordinary ideas to which it has led. I have regretfully abandoned the idea of describing even a representative sample of the forcing models which have been devised to show that measure-theoretic propositions are consistent, but in Chapter 55 I set out some of the basic properties of random real forcing. Finally, in Chapter 56, I look at what measure theory becomes in ZF alone, with countable or dependent choice, and with the axiom of determinacy.

While I should like to believe that most of the material of this volume will be accessible to those who have learnt measure theory from other sources, it has obviously been written with earlier volumes constantly in mind, and I have to advise you to make sure that Volumes 3 and 4, at least, will be available in case of need. Apart from these, I do of course assume that readers will be at ease with modern set theory. It is not so much that I demand a vast amount of knowledge – §§5A1-5A2 have a good many proofs to help cover any gaps – as that I present arguments without much consideration for the inexperienced, and some of them may be indigestible at first if you have not cut your teeth on Just & Weese 96 or Jech 78. What you may not need is any prior knowledge of forcing. But of course for Chapter 55 you will have to take a proper introduction to forcing, e.g., Kunen 80, in parallel with §5A3, since nothing here will make sense without an acquaintance with forcing languages and the fundamental theorem of forcing.

Note on second printing

There has been the usual crop of errors (most, but not all, minor) to be corrected, and I have added a few new results. The most important is P.Larson's proof that it is relatively consistent with ZFC to suppose that there is no medial limit. In the process of preparing new editions of Volumes 1-4, I have I hope covered all the items listed in the old §5A6 ('Later editions only'), which I have therefore dropped, even though there are one or two further entries under this heading. As before, these can be found on the Web edition at `http://www1.essex.ac.uk/maths/people/fremlin/mtcont.htm`.

Version of 3.1.15

Chapter 51

Cardinal functions

The primary object of this volume is to explore those topics in measure theory in which questions arise which are undecided by the ordinary axioms of set theory. We immediately face a new kind of interaction between the propositions we consider. If two statements are undecidable, we can ask whether either implies the other. Almost at once we find ourselves trying to make sense of a bewildering tangle of uncoordinated patterns. The most successful method so far found of listing the multiple connexions present is to reduce as many arguments as possible to investigations of the relationships between specially defined cardinal numbers. In any particular model of set theory (so long as we are using the axiom of choice) these numbers must be in a linear order, so we can at least estimate the number of potential configurations, and focus our attention on the possibilities which seem most accessible or most interesting. At the very beginning of the theory, for instance, we can ask whether $\mathfrak{c} = 2^\omega$ is equal to ω_1, or ω_2, or ω_{ω_1}, or 2^{ω_1}. For Lebesgue measure, perhaps the first question to ask is: if $\langle E_\xi\rangle_{\xi<\omega_1}$ is a family of measurable sets, is $\bigcup_{\xi<\omega_1} E_\xi$ necessarily measurable? If the continuum hypothesis is true, certainly not; but if $\mathfrak{c} > \omega_1$, either 'yes' or 'no' becomes possible. The way in which it is now customary to express this is to say that '$\omega_1 \le \operatorname{add}\mathcal{N} \le \mathfrak{c}$, and $\omega_1 \le \operatorname{add}\mathcal{N} < \mathfrak{c}$, $\omega_1 < \operatorname{add}\mathcal{N} \le \mathfrak{c}$ and $\omega_1 < \operatorname{add}\mathcal{N} < \mathfrak{c}$ are all possible', where $\operatorname{add}\mathcal{N}$ is defined as the least cardinal of any family $\mathcal{E}$ of Lebesgue measurable sets such that $\bigcup\mathcal{E}$ is not measurable. (Actually it is not usually defined in quite this way, but that is what it comes to.)

At this point I suggest that you turn to 522B, where you will find a classic picture ('Cichoń's diagram') of the relationships between ten cardinals intermediate between ω_1 and $\mathfrak{c}$, with $\operatorname{add}\mathcal{N}$ immediately above ω_1. As this diagram already makes clear, one can define rather a lot of cardinal numbers. Furthermore, the relationships between them are not entirely expressible in terms of the partial order in which we say that $\kappa_\mathfrak{a} \preccurlyeq \kappa_\mathfrak{b}$ if we can prove in ZFC that $\kappa_\mathfrak{a} \le \kappa_\mathfrak{b}$. Even in Cichoń's diagram we have results of the type $\operatorname{add}\mathcal{M} = \min(\mathfrak{b}, \operatorname{cov}\mathcal{M})$ in which three cardinals are involved. It is clear that the framework which has been developed over the last thirty-five years is only a beginning. Nevertheless, I am confident that it will maintain a leading role as the theory evolves. The point is that at least some of the cardinals ($\operatorname{add}\mathcal{N}$, $\mathfrak{b}$ and $\operatorname{cov}\mathcal{M} = \mathfrak{m}_{\text{countable}}$, for instance) describe such important features of such important structures that they appear repeatedly in arguments relating to diverse topics, and give us a chance to notice unexpected connexions.

The first step is to list and classify the relevant cardinals. This is the purpose of the present chapter. In fact the definitions here are mostly of a general type. Associated with any ideal of sets, for instance, we have four cardinals ('additivity', 'cofinality', 'unformity' and 'covering number'; see 511F). Most of the cardinals examined in this volume can be defined by one of a limited number of processes from some more or less naturally arising structure; thus $\operatorname{add}\mathcal{N}$, already mentioned, is normally defined as the additivity of the ideal of Lebesgue negligible subsets of $\mathbb{R}$, and $\operatorname{cov}\mathcal{M}$ is the covering number of the ideal of meager subsets of $\mathbb{R}$. Another important type of definition is in terms of whole classes of structure: thus Martin's cardinal $\mathfrak{m}$ can be regarded as the least Martin number (definition: 511Dg) of any ccc Boolean algebra.

§511 lists some of the cardinals associated with partially ordered sets, Boolean algebras, topological spaces and ideals of sets. Which structures count as 'naturally arising' is a matter of taste and experience, but it turns out that many important ideas can be expressed in terms of cardinals associated with relations, and some of these are investigated in §512. The core ideas of the chapter are most clearly manifest in their application to partially ordered sets, which I look at in §513. In §514 I run through the elementary results connecting the cardinal functions of topological spaces and associated Boolean algebras and partially ordered sets. §515 is a brief excursion into abstract Boolean algebra. §516 is a discussion of 'precalibers'. §517 is an introduction to the theory of 'Martin numbers', which (following the principles I have just tried to explain) I will use as vehicles for the arguments which have been used to make deductions from Martin's axiom. §518 gives results on Freese-Nation numbers and tight filtrations of Boolean algebras which can be expressed in general terms and are relevant to questions in measure theory.

Version of 10.10.13

511 Definitions

A large proportion of the ideas of this volume will be expressed in terms of cardinal numbers associated with the structures of measure theory. For any measure space (X, Σ, μ) we have, at least, the structures (X, Σ), $(X, \Sigma, \mathcal{N}(\mu))$ (where $\mathcal{N}(\mu)$ is the null ideal of μ) and the measure algebra $\mathfrak{A} = \Sigma/\Sigma \cap \mathcal{N}(\mu)$; each of these types of structure has a family of cardinal functions associated with it, starting from the obvious ones $\#(X)$, $\#(\Sigma)$ and $\#(\mathfrak{A})$. For the measure algebra $\mathfrak{A}$, we quickly find that we have cardinals naturally associated with its Boolean structure and others naturally associated with the topological structure of its

Stone space; of course the most important ones are those which can be described in both languages. The actual measure $\mu : \Sigma \to [0, \infty]$, and its daughter $\bar{\mu} : \mathfrak{A} \to [0, \infty]$, will be less conspicuous here; for most of the questions addressed in this volume, replacing a measure by another with the same measurable sets and the same negligible sets will make no difference.

In this section I list the definitions on which the rest of the chapter depends, with a handful of elementary results to give you practice with the definitions.

511A Pre-ordered sets When we come to the theory of forcing in Chapter 55, there will be technical advantages in using a generalization of the concept of 'partial order'. A **pre-ordered set** is a set P together with a relation $\le$ on P such that

if $p \le q$ and $q \le r$ then $p \le r$,

$p \le p$ for every $p \in P$;

that is, $\le$ is transitive and reflexive but need not be antisymmetric. As with partial orders, I will write $p \ge q$ to mean $q \le p$; $[p, q] = \{r : p \le r \text{ and } r \le q\}$; $[p, \infty[= \{q : p \le q\}$, $]-\infty, p] = \{q : q \le p\}$. An **upper** (resp. **lower**) **bound** for a set $A \subseteq P$ will be a $p \in P$ such that $q \le p$ (resp. $p \le q$) for every $q \in A$. If $(Q, \le)$ is another pre-ordered set, I will say that $f : P \to Q$ is **order-preserving** if $f(p) \le f(p')$ whenever $p \le p'$ in P. If $\langle (P_i, \le_i) \rangle_{i \in I}$ is a family of pre-ordered sets, their **product** is the pre-ordered set $(P, \le)$ where $P = \prod_{i \in I} P_i$ and, for $p, q \in P$, $p \le q$ iff $p(i) \le_i q(i)$ for every $i \in I$ (cf. 315C).

If $(P, \le)$ is a pre-ordered set, we have an equivalence relation $\sim$ on P defined by saying that $p \sim q$ if $p \le q$ and $q \le p$. Now we have a canonical partial order on the set $\tilde{P}$ of equivalence classes defined by saying that $p^\bullet \le q^\bullet$ iff $p \le q$. For all ordinary purposes, $(P, \le)$ and $(\tilde{P}, \le)$ carry the same structural information, and the move to the true partial order is natural and convenient. It occasionally happens (see 512Ee below, for instance, and also the theory of iterated forcing in Kunen 80, chap. VIII) that it is helpful to have a language which enables us to dispense with this step, thereby simplifying some basic definitions. However the extra generality leads to no new ideas, and I expect that most readers will prefer to do nearly all their thinking in the context of partially ordered sets.

511B Definitions Let $(P, \le)$ be any pre-ordered set.

(a) A subset Q of P is **cofinal** with P if for every $p \in P$ there is a $q \in Q$ such that $p \le q$. The **cofinality** of P, $\operatorname{cf} P$, is the least cardinal of any cofinal subset of P.

(b) The **additivity** of P, $\operatorname{add} P$, is the least cardinal of any subset of P with no upper bound in P. If there is no such set, write $\operatorname{add} P = \infty$.

(c) A subset Q of P is **coinitial** with P if for every $p \in P$ there is a $q \in Q$ such that $q \le p$. The **coinitiality** of P, $\operatorname{ci} P$, is the least cardinal of any coinitial subset of P.

(d) Two elements p, p' of P are **compatible upwards** if $[p, \infty[\cap [p', \infty[\ne \emptyset$, that is, if $\{p, p'\}$ has an upper bound in P; otherwise they are **incompatible upwards**. A subset A of P is an **up-antichain** if no two distinct elements of A are compatible upwards. The **upwards cellularity** of P is $c^\uparrow(P) = \sup\{\#(A) : A \subseteq P$ is an up-antichain in $P\}$; the **upwards saturation** of P, $\operatorname{sat}^\uparrow(P)$, is the least cardinal κ such that there is no up-antichain in P with cardinal κ. P is called **upwards-ccc** if it has no uncountable up-antichain, that is, $c^\uparrow(P) \le \omega$, that is, $\operatorname{sat}^\uparrow(P) \le \omega_1$.

(e) Two elements p, p' of P are **compatible downwards** if $]-\infty, p] \cap]-\infty, p'] \ne \emptyset$, that is, if $\{p, p'\}$ has a lower bound in P; otherwise they are **incompatible downwards**. A subset A of P is a **down-antichain** if no two distinct elements of A are compatible downwards. The **downwards cellularity** of P is $c^\downarrow(P) = \sup\{\#(A) : A \subseteq P$ is a down-antichain in $P\}$; the **downwards saturation** of P, $\operatorname{sat}^\downarrow(P)$, is the least κ such that there is no down-antichain in P with cardinal κ. P is called **downwards-ccc** if it has no uncountable down-antichain, that is, $c^\downarrow(P) \le \omega$, that is, $\operatorname{sat}^\downarrow(P) \le \omega_1$.

(f) If κ is a cardinal, a subset A of P is **upwards-$<\kappa$-linked** in P if every subset of A of cardinal less than κ is bounded above in P. The **upwards $<\kappa$-linking number** of P, $\operatorname{link}^\uparrow_{<\kappa}(P)$, is the smallest cardinal of any cover of P by upwards-$<\kappa$-linked sets.

A subset A of P is **upwards-κ-linked** in P if it is upwards-$<\kappa^+$-linked, that is, every member of $[A]^{\le\kappa}$ is bounded above in P. The **upwards κ-linking number** of P, $\mathrm{link}^{\uparrow}_{\kappa}(P) = \mathrm{link}^{\uparrow}_{<\kappa^+}(P)$, is the smallest cardinal of any cover of P by upwards-κ-linked sets.

Similarly, a subset A of P is **downwards-$<\kappa$-linked** if every member of $[A]^{<\kappa}$ has a lower bound in P, and **downwards-κ-linked** if it is downwards-$<\kappa^+$-linked; the **downwards $<\kappa$-linking number** of P, $\mathrm{link}^{\downarrow}_{<\kappa}(P)$, is the smallest cardinal of any cover of P by downwards-$<\kappa$-linked sets, and $\mathrm{link}^{\downarrow}_{\kappa}(P) = \mathrm{link}^{\downarrow}_{<\kappa^+}(P)$.

(g) The most important cases of (f) above are $\kappa = 2$ and $\kappa = \omega$. A subset A of P is **upwards-linked** if any two members of A are compatible upwards in P, and **upwards-centered** if it is upwards-$<\omega$-linked, that is, any finite subset of A has an upper bound in P. The **upwards linking number** of P, $\mathrm{link}^{\uparrow}(P) = \mathrm{link}^{\uparrow}_{2}(P)$, is the least cardinal of any cover of P by upwards-linked sets, and the **upwards centering number** of P, $d^{\uparrow}(P) = \mathrm{link}^{\uparrow}_{<\omega}(P)$, is the least cardinal of any cover of P by upwards-centered sets.

Similarly, $A \subseteq P$ is **downwards-linked** if any two members of A are compatible downwards in P, and **downwards-centered** if any finite subset of A has a lower bound in P; the **downwards linking number** of P is $\mathrm{link}^{\downarrow}(P) = \mathrm{link}^{\downarrow}_{2}(P)$, and the **downwards centering number** of P is $d^{\downarrow}(P) = \mathrm{link}^{\downarrow}_{<\omega}(P)$.

If $\mathrm{link}^{\uparrow}(P) \le \omega$ (resp. $\mathrm{link}^{\downarrow}(P) \le \omega$) we say that P is **σ-linked upwards** (resp. **downwards**). If $d^{\uparrow}(P) \le \omega$ (resp. $d^{\downarrow}(P) \le \omega$) we say that P is **σ-centered upwards** (resp. **downwards**).

(h) The **upwards Martin number** $\mathfrak{m}^{\uparrow}(P)$ of P is the smallest cardinal of any family $\mathcal{Q}$ of cofinal subsets of P such that there is some $p \in P$ such that no upwards-linked subset of P containing p meets every member of $\mathcal{Q}$; if there is no such family $\mathcal{Q}$, write $\mathfrak{m}^{\uparrow}(P) = \infty$.

Similarly, the **downwards Martin number** $\mathfrak{m}^{\downarrow}(P)$ of P is the smallest cardinal of any family $\mathcal{Q}$ of coinitial subsets of P such that there is some $p \in P$ such that no downwards-linked subset of P containing p meets every member of $\mathcal{Q}$, or ∞ if there is no such $\mathcal{Q}$.

(i) A **Freese-Nation function** on P is a function $f : P \to \mathcal{P}P$ such that whenever $p \le q$ in P then $[p, q] \cap f(p) \cap f(q)$ is non-empty. The **Freese-Nation number** of P, $\mathrm{FN}(P)$, is the least κ such that there is a Freese-Nation function $f : P \to [P]^{<\kappa}$. The **regular Freese-Nation number** of P, $\mathrm{FN}^*(P)$, is the least regular infinite κ such that there is a Freese-Nation function $f : P \to [P]^{<\kappa}$. If Q is a subset of P, the **Freese-Nation index** of Q in P is the least cardinal κ such that $\mathrm{cf}(Q \cap \mathopen]-\infty, p]) < \kappa$ and $\mathrm{ci}(Q \cap [p, \infty\mathclose[) < \kappa$ for every $p \in P$.

(j) The **(principal) bursting number** $\mathrm{bu}\,P$ of P is the least cardinal κ such that there is a cofinal subset Q of P such that

$$\#(\{q : q \in Q,\ q \le p,\ p \not\le q\}) < \kappa$$

for every $p \in P$.

(k) It will be convenient to have a phrase for the following phenomenon. I will say that P is **separative upwards** if whenever $p, q \in P$ and $p \not\le q$ there is a $q' \ge q$ which is incompatible upwards with p. Similarly, of course, P is **separative downwards** if whenever $p, q \in P$ and $p \not\ge q$ there is a $q' \le q$ which is incompatible downwards with p.

511C On the symbol ∞ I note that in the definitions above I have introduced expressions of the form '$\mathrm{add}\,P = \infty$'. The '∞' here must be rigorously distinguished from the '∞' of ordinary measure theory, which can be regarded as a top point added to the set of real numbers. The '∞' of 511B is rather a top point added to the class of ordinals. But it is convenient, and fairly safe, to use formulae like '$\mathrm{add}\,P \le \mathrm{add}\,Q$' on the understanding that $\mathrm{add}\,P \le \infty$ for every pre-ordered set P, while $\infty \le \mathrm{add}\,Q$ only when $\mathrm{add}\,Q = \infty$. Of course we have to be careful to distinguish between '$\mathrm{add}\,P < \infty$' (meaning that there is a subset of P with no upper bound in P) and '$\mathrm{add}\,P$ is finite' (meaning that $\mathrm{add}\,P < \omega$).

511D Definitions Let $\mathfrak{A}$ be a Boolean algebra. I write $\mathfrak{A}^+$ for the set $\mathfrak{A} \setminus \{0\}$ of non-zero elements of $\mathfrak{A}$ and $\mathfrak{A}^-$ for $\mathfrak{A} \setminus \{1\}$, so that the partially ordered sets $(\mathfrak{A}^-, \subseteq)$ and $(\mathfrak{A}^+, \supseteq)$ are isomorphic.

(a) The **Maharam type** $\tau(\mathfrak{A})$ of $\mathfrak{A}$ is the smallest cardinal of any subset B of $\mathfrak{A}$ which τ-generates $\mathfrak{A}$ in the sense that the order-closed subalgebra of $\mathfrak{A}$ including B is $\mathfrak{A}$ itself. (See Chapter 33.)

(b) The **cellularity** of $\mathfrak{A}$ is

$$c(\mathfrak{A})=c^{\uparrow}(\mathfrak{A}^-)=c^{\downarrow}(\mathfrak{A}^+)=\sup\{\#(C):C\subseteq\mathfrak{A}^+\text{ is disjoint}\}.$$

The **saturation** of $\mathfrak{A}$ is

$$\operatorname{sat}(\mathfrak{A})=\operatorname{sat}^{\uparrow}(\mathfrak{A}^-)=\operatorname{sat}^{\downarrow}(\mathfrak{A}^+)=\sup\{\#(C)^+:C\subseteq\mathfrak{A}^+\text{ is disjoint}\},$$

that is, the smallest cardinal κ such that there is no disjoint family in $\mathfrak{A}^+$ with cardinal κ.

(c) The **π-weight** or **density** $\pi(\mathfrak{A})$ of $\mathfrak{A}$ is $\operatorname{cf}\mathfrak{A}^-=\operatorname{ci}\mathfrak{A}^+$, that is, the smallest cardinal of any order-dense subset of $\mathfrak{A}$.

(d) Let κ be a cardinal. A subset A of $\mathfrak{A}^+$ is **$<\kappa$-linked** if it is downwards-$<\kappa$-linked in $\mathfrak{A}^+$, that is, no $B\in[A]^{<\kappa}$ has infimum 0, and **κ-linked** if it is $<\kappa^+$-linked, that is, every $B\in[A]^{\le\kappa}$ has a non-zero lower bound. The **$<\kappa$-linking number** $\operatorname{link}_{<\kappa}(\mathfrak{A})$ of $\mathfrak{A}$ is $\operatorname{link}^{\downarrow}_{<\kappa}(\mathfrak{A}^+)$, the least cardinal of any family of $<\kappa$-linked sets covering $\mathfrak{A}^+$; and the **κ-linking number** $\operatorname{link}_{\kappa}(\mathfrak{A})$ of $\mathfrak{A}$ is $\operatorname{link}_{<\kappa^+}(\mathfrak{A})$, that is, the least cardinal of any cover of $\mathfrak{A}^+$ by κ-linked sets.

(e) As in 511Bg, I say that $A\subseteq\mathfrak{A}^+$ is **linked** if no two members of A are disjoint; the **linking number** of $\mathfrak{A}$ is $\operatorname{link}(\mathfrak{A})=\operatorname{link}_2(\mathfrak{A})$, the least cardinal of any cover of $\mathfrak{A}^+$ by linked sets. Similarly, $A\subseteq\mathfrak{A}^+$ is **centered** if $\inf I\ne0$ for any finite $I\subseteq A$; that is, if A is downwards-centered in $\mathfrak{A}^+$. The **centering number** $d(\mathfrak{A})$ of $\mathfrak{A}$ is $d^{\uparrow}(\mathfrak{A}^-)=d^{\downarrow}(\mathfrak{A}^+)$, that is, the smallest cardinal of any cover of $\mathfrak{A}^+$ by centered sets. $\mathfrak{A}$ is **σ-m-linked** if $\operatorname{link}_m(\mathfrak{A})\le\omega$; in particular, it is **σ-linked** iff $\operatorname{link}(\mathfrak{A})\le\omega$. $\mathfrak{A}$ is **σ-centered** if $d(\mathfrak{A})\le\omega$.

(f) If κ is any cardinal, $\mathfrak{A}$ is **weakly (κ,∞)-distributive** if whenever $\langle A_\xi\rangle_{\xi<\kappa}$ is a family of partitions of unity in $\mathfrak{A}$, there is a partition B of unity such that $\{a:a\in A_\xi,\ a\cap b\ne0\}$ is finite for every $b\in B$ and $\xi<\kappa$. Now the **weak distributivity** $\operatorname{wdistr}(\mathfrak{A})$ of $\mathfrak{A}$ is the least cardinal κ such that $\mathfrak{A}$ is not weakly (κ,∞)-distributive. (If there is no such cardinal, write $\operatorname{wdistr}(\mathfrak{A})=\infty$.)

(g) The **Martin number** $\mathfrak{m}(\mathfrak{A})$ of $\mathfrak{A}$ is the downwards Martin number of $\mathfrak{A}^+$, that is, the smallest cardinal of any family $\mathcal{B}$ of coinitial subsets of $\mathfrak{A}^+$ for which there is some $a\in\mathfrak{A}^+$ such that no linked subset of $\mathfrak{A}$ containing a meets every member of $\mathcal{B}$; or ∞ if there is no such $\mathcal{B}$.

(h) The **Freese-Nation number** of $\mathfrak{A}$, $\operatorname{FN}(\mathfrak{A})$, is the Freese-Nation number of the partially ordered set $(\mathfrak{A},\subseteq)$. The **regular Freese-Nation number** $\operatorname{FN}^*(\mathfrak{A})$ of $\mathfrak{A}$ is the regular Freese-Nation number of $(\mathfrak{A},\subseteq)$, that is, the smallest regular infinite cardinal greater than or equal to $\operatorname{FN}(\mathfrak{A})$.

(i) If κ is a cardinal, a **tight κ-filtration** of $\mathfrak{A}$ is a family $\langle a_\xi\rangle_{\xi<\zeta}$ in $\mathfrak{A}$, where ζ is an ordinal, such that, writing $\mathfrak{A}_\alpha$ for the subalgebra of $\mathfrak{A}$ generated by $\{a_\xi:\xi<\alpha\}$, (α) $\mathfrak{A}_\zeta=\mathfrak{A}$ (β) for every $\alpha<\zeta$, the Freese-Nation index of $\mathfrak{A}_\alpha$ in $\mathfrak{A}$ is at most κ. If $\mathfrak{A}$ has a tight κ-filtration, I will say that it is **tightly κ-filtered**.

511E Precalibers (a) Let $(P,\le)$ be a pre-ordered set.

(i) I will say that $(\kappa,\lambda,<\theta)$ is an **upwards precaliber triple** of P if κ, λ and θ are cardinals, and whenever $\langle p_\xi\rangle_{\xi<\kappa}$ is a family in P then there is a set $\Gamma\in[\kappa]^\lambda$ such that $\{p_\xi:\xi\in I\}$ has an upper bound in P for every $I\in[\Gamma]^{<\theta}$.

Similarly, $(\kappa,\lambda,<\theta)$ is a **downwards precaliber triple** of P if κ, λ and θ are cardinals and whenever $\langle p_\xi\rangle_{\xi<\kappa}$ is a family in P then there is a set $\Gamma\in[\kappa]^\lambda$ such that $\{p_\xi:\xi\in I\}$ has a lower bound in P for every $I\in[\Gamma]^{<\theta}$.

(ii) An **upwards precaliber pair** of P is a pair (κ,λ) of cardinals such that $(\kappa,\lambda,<\omega)$ is an upwards precaliber triple of P, that is, whenever $\langle p_\xi\rangle_{\xi<\kappa}$ is a family in P there is a $\Gamma\in[\kappa]^\lambda$ such that $\{p_\xi:\xi\in\Gamma\}$ is upwards-centered in P.

A **downwards precaliber pair** of P is a pair (κ, λ) of cardinals such that $(\kappa, \lambda, <\omega)$ is a downwards precaliber triple of P.

(iii) An **up-** (resp. **down-**) **precaliber** of P is a cardinal κ such that (κ, κ) is an upwards (resp. downwards) precaliber pair of P.

(b) Let $(X, \mathfrak{T})$ be a topological space. Then $(\kappa, \lambda, <\theta)$ is a **precaliber triple** of X if it is a downwards precaliber triple of $\mathfrak{T} \setminus \{\emptyset\}$; (κ, λ) is a **precaliber pair** of X if it is a downwards precaliber pair of $\mathfrak{T} \setminus \{\emptyset\}$; and κ is a **precaliber** of X if it is a down-precaliber of $\mathfrak{T} \setminus \{\emptyset\}$.

(c) Let $\mathfrak{A}$ be a Boolean algebra. Then $(\kappa, \lambda, <\theta)$ is a **precaliber triple** of $\mathfrak{A}$ if it is a downwards precaliber triple of $\mathfrak{A}^+$; (κ, λ) is a **precaliber pair** of $\mathfrak{A}$ if it is a downwards precaliber pair of $\mathfrak{A}^+$; and κ is a **precaliber** of $\mathfrak{A}$ if it is a down-precaliber of $\mathfrak{A}^+$.

(d) If $(\mathfrak{A}, \bar{\mu})$ is a measure algebra, then $(\kappa, \lambda, <\theta)$ is a **measure-precaliber triple** of $(\mathfrak{A}, \bar{\mu})$ if whenever $\langle a_\xi \rangle_{\xi<\kappa}$ is a family in $\mathfrak{A}$ such that $\inf_{\xi<\kappa} \bar{\mu} a_\xi > 0$, then there is a $\Gamma \in [\kappa]^\lambda$ such that $\{a_\xi : \xi \in I\}$ has a non-zero lower bound for every $I \in [\Gamma]^{<\theta}$. Now (κ, λ) is a **measure-precaliber pair** of $(\mathfrak{A}, \bar{\mu})$ if $(\kappa, \lambda, <\omega)$ is a measure-precaliber triple, and κ is a **measure-precaliber** of $(\mathfrak{A}, \bar{\mu})$ if (κ, κ) is a measure-precaliber pair.

(e) In this context, I will say that $(\kappa, \lambda, \theta)$ is a precaliber triple (in any sense) if $(\kappa, \lambda, <\theta^+)$ is a precaliber triple as defined above; and similarly for measure-precaliber triples.

(f) I will say that one of the structures here satisfies **Knaster's condition** if it has $(\omega_1, \omega_1, 2)$ as a precaliber triple, that is, if every uncountable set has an uncountable linked subset. (For pre-ordered sets I will speak of 'Knaster's condition upwards' or 'Knaster's condition downwards'.) A structure satisfying Knaster's condition must be ccc, because an uncountable set of mutually incompatible elements surely cannot have an uncountable linked subset.

511F Definitions Let X be a set and $\mathcal{I}$ an ideal of subsets of X.

(a) Taking $\mathcal{I}$ to be partially ordered by $\subseteq$, we can speak of $\operatorname{add} \mathcal{I}$ and $\operatorname{cf} \mathcal{I}$ in the sense of 511B. $\mathcal{I}$ is called **κ-additive** or **κ-complete** if $\kappa \le \operatorname{add} \mathcal{I}$, that is, if $\bigcup \mathcal{E} \in \mathcal{I}$ for every $\mathcal{E} \in [\mathcal{I}]^{<\kappa}$.

In addition we have three other cardinals which will be important to us.

(b) The **uniformity** of $\mathcal{I}$ is

$$\operatorname{non} \mathcal{I} = \min\{\#(A) : A \subseteq X,\ A \notin \mathcal{I}\},$$

or ∞ if there is no such set A. (Note the hidden variable X in this notation; if any confusion seems possible, I will write $\operatorname{non}(X, \mathcal{I})$. Many authors prefer $\operatorname{unif} \mathcal{I}$.)

(c) The **shrinking number** of $\mathcal{I}$, $\operatorname{shr} \mathcal{I}$, is the smallest cardinal κ such that whenever $A \in \mathcal{P}X \setminus \mathcal{I}$ there is a $B \in [A]^{\le\kappa} \setminus \mathcal{I}$. (Again, we need to know X as well as $\mathcal{I}$ to determine $\operatorname{shr} \mathcal{I}$, and if necessary I will write $\operatorname{shr}(X, \mathcal{I})$.) The **augmented shrinking number** $\operatorname{shr}^+(\mathcal{I})$ is the smallest κ such that whenever $A \in \mathcal{P}X \setminus \mathcal{I}$ there is a $B \in [A]^{<\kappa} \setminus \mathcal{I}$.

(d) The **covering number** of $\mathcal{I}$ is

$$\operatorname{cov} \mathcal{I} = \min\{\#(\mathcal{E}) : \mathcal{E} \subseteq \mathcal{I},\ \bigcup \mathcal{E} = X\},$$

or ∞ if there is no such set $\mathcal{E}$. (Once more, X is a hidden variable here, and I may write $\operatorname{cov}(X, \mathcal{I})$.)

511G Definition Let (X, Σ, μ) be a measure space.

(a) If κ is a cardinal, μ is **κ-additive** if $\bigcup \mathcal{E} \in \Sigma$ and $\mu(\bigcup \mathcal{E}) = \sum_{E \in \mathcal{E}} \mu E$ for every disjoint family $\mathcal{E} \in [\Sigma]^{<\kappa}$. The **additivity** $\operatorname{add} \mu$ of μ is the largest cardinal κ such that μ is κ-additive, or ∞ if μ is κ-additive for every κ.

(b) The **π-weight** $\pi(\mu)$ of μ is the coinitiality of $\Sigma \setminus \mathcal{N}(\mu)$, where $\mathcal{N}(\mu)$ is the null ideal of μ.

(c) Recall that the Maharam type $\tau(\mu)$ of μ is the Maharam type of the measure algebra of μ (331Fc).

511H Elementary facts: pre-ordered sets Let P be a pre-ordered set.

(a) If $\tilde{P}$ is the partially ordered set of equivalence classes in P, as described in 511A, all the cardinal functions defined in 511B have the same values for P and $\tilde{P}$. (The point is that p is an upper bound for $A \subseteq P$ iff $p^\bullet$ is an upper bound for $\{q^\bullet : q \in A\} \subseteq \tilde{P}$.) Similarly, P and $\tilde{P}$ will have the same triple precalibers, precaliber pairs and precalibers.

(b) Obviously, $c^\uparrow(P) \le \mathrm{sat}^\uparrow(P)$. (In fact $c^\uparrow(P)$ is determined by $\mathrm{sat}^\uparrow(P)$; see 513Bc below.) If $\kappa \le \lambda$ are cardinals then

$$\mathrm{link}^\uparrow_{<\kappa}(P) \le \mathrm{link}^\uparrow_{<\lambda}(P) \le \mathrm{cf}\, P,$$

because every upwards-$<\lambda$-linked set is upwards-$<\kappa$-linked and every set $]-\infty, p]$ is upwards-$<\lambda$-linked. $c^\uparrow(P) \le \mathrm{link}^\uparrow(P)$, because if $A \subseteq P$ is an up-antichain then no upwards-linked set can contain more than one point of A. It follows that

$$\mathrm{link}^\uparrow(P) = \mathrm{link}^\uparrow_{<3}(P) \le \mathrm{link}^\uparrow_{<\omega}(P) = d^\uparrow(P) \le \mathrm{cf}\, P.$$

Of course $\mathrm{cf}\, P \le \#(P)$. Similarly,

$$\mathrm{link}^\downarrow_{<\kappa}(P) \le \mathrm{link}^\downarrow_{<\lambda}(P) \le \mathrm{ci}\, P$$

whenever $\kappa \le \lambda$, and

$$c^\downarrow(P) \le \mathrm{link}^\downarrow(P) \le d^\downarrow(P) \le \mathrm{ci}\, P \le \#(P), \quad c^\downarrow(P) \le \mathrm{sat}^\downarrow P.$$

(c) P is empty iff $\mathrm{cf}\, P = 0$ iff $\mathrm{ci}\, P = 0$ iff $\mathrm{add}\, P = 0$ iff $d^\uparrow(P) = 0$ iff $d^\downarrow(P) = 0$ iff $\mathrm{link}^\uparrow(P) = 0$ iff $\mathrm{link}^\downarrow(P) = 0$ iff $c^\uparrow(P) = 0$ iff $c^\downarrow(P) = 0$ iff $\mathrm{sat}^\uparrow(P) = 1$ iff $\mathrm{sat}^\downarrow(P) = 1$ iff $\mathrm{FN}(P) = 0$.

(d) P is upwards-directed iff $c^\uparrow(P) \le 1$ iff $\mathrm{sat}^\uparrow(P) \le 2$ iff $\mathrm{link}^\uparrow(P) \le 1$ iff $d^\uparrow(P) \le 1$. Similarly, P is downwards-directed iff $c^\downarrow(P) \le 1$ iff $\mathrm{sat}^\downarrow(P) \le 2$ iff $\mathrm{link}^\downarrow(P) \le 1$ iff $d^\downarrow(P) \le 1$.

If P is not empty, it is upwards-directed iff $\mathrm{add}\, P > 2$ iff $\mathrm{add}\, P \ge \omega$.

(e) If P is partially ordered, it has a greatest element iff $\mathrm{cf}\, P = 1$ iff $\mathrm{add}\, P = \infty$. Otherwise, $\mathrm{add}\, P \le \mathrm{cf}\, P$, since no cofinal subset of P can have an upper bound in P.

(f) If P is totally ordered, then $\mathrm{cf}\, P \le \mathrm{add}\, P$. **P** If $A \subseteq P$ has no upper bound in P it must be cofinal with P. **Q**

(g) If $\langle P_i \rangle_{i \in I}$ is a non-empty family of non-empty pre-ordered sets with product P, then $\mathrm{add}\, P = \min_{i \in I} \mathrm{add}\, P_i$. **P** A set $A \subseteq P$ lacks an upper bound in P iff there is an $i \in I$ such that $\{p(i) : p \in A\}$ is unbounded above in P_i. **Q**

(h) If P is partially ordered and $f : P \to \mathcal{P}P$ is a Freese-Nation function then $p \in f(p)$ for every $p \in P$, because $[p, p]$ meets $f(p) \cap f(p)$.

511I Elementary facts: Boolean algebras Let $\mathfrak{A}$ be a Boolean algebra.

(a)

$$\mathrm{link}_{<\kappa}(\mathfrak{A}) \le \mathrm{link}_{<\lambda}(\mathfrak{A}) \le \pi(\mathfrak{A})$$

whenever $\kappa \le \lambda$,

$$c(\mathfrak{A}) \le \mathrm{link}(\mathfrak{A}) \le d(\mathfrak{A}) \le \pi(\mathfrak{A}) \le \#(\mathfrak{A}), \quad c(\mathfrak{A}) \le \mathrm{sat}(\mathfrak{A}).$$

In addition, $\tau(\mathfrak{A}) \le \pi(\mathfrak{A})$ because any order-dense subset of $\mathfrak{A}$ τ-generates $\mathfrak{A}$.

(b)(i) $\mathfrak{A} = \{0\}$ iff $\pi(\mathfrak{A}) = 0$ iff $\operatorname{link}(\mathfrak{A}) = 0$ iff $d(\mathfrak{A}) = 0$ iff $c(\mathfrak{A}) = 0$ iff $\operatorname{sat}(\mathfrak{A}) = 1$.

(ii) $\tau(\mathfrak{A}) = 0$ iff $\mathfrak{A}$ is either $\{0\}$ or $\{0,1\}$.

(c) If $\mathfrak{A}$ is finite, then $c(\mathfrak{A}) = \operatorname{link}(\mathfrak{A}) = d(\mathfrak{A}) = \pi(\mathfrak{A})$ is the number of atoms of $\mathfrak{A}$, $\operatorname{sat}(\mathfrak{A}) = c(\mathfrak{A}) + 1$ and $\#(\mathfrak{A}) = 2^{c(\mathfrak{A})}$, while $\tau(\mathfrak{A}) = \lceil \log_2 c(\mathfrak{A}) \rceil$, unless $\mathfrak{A} = \{0\}$, in which case $\tau(\mathfrak{A}) = 0$. If $\mathfrak{A}$ is infinite then $c(\mathfrak{A})$, $\operatorname{link}(\mathfrak{A})$, $d(\mathfrak{A})$, $\pi(\mathfrak{A})$, $\operatorname{sat}(\mathfrak{A})$ and $\tau(\mathfrak{A})$ are all infinite.

(d) Note that $\mathfrak{A}$ is 'ccc' just when $c(\mathfrak{A}) \le \omega$, that is, $\operatorname{sat}(\mathfrak{A}) \le \omega_1$. $\mathfrak{A}$ is weakly (σ,∞)-distributive, in the sense of 316G, iff $\operatorname{wdistr}(\mathfrak{A}) \ge \omega_1$.

(e)(i) If $\mathfrak{A}$ is purely atomic, $\operatorname{wdistr}(\mathfrak{A}) = \infty$. **P** Suppose that $\langle A_\xi\rangle_{\xi<\kappa}$ is any family of partitions of unity in $\mathfrak{A}$. Then the set B of atoms of $\mathfrak{A}$ is a partition of unity, and $\{a : a \in A_\xi,\ a \cap b \ne 0\}$ has just one member for every $b \in B$ and $\xi < \kappa$. As $\langle A_\xi\rangle_{\xi<\kappa}$ is arbitrary, $\operatorname{wdistr}(\mathfrak{A}) = \infty$. **Q**

(ii) If $\mathfrak{A}$ is not purely atomic, $\operatorname{wdistr}(\mathfrak{A}) \le \pi(\mathfrak{A})$. **P** Let $c \in \mathfrak{A}^+$ be disjoint from every atom of $\mathfrak{A}$, and $D \subseteq \mathfrak{A}$ an order-dense set of size $\pi(\mathfrak{A})$; let D' be $\{d : d \in D,\ d \subseteq c\}$. For $d \in D'$, there is a disjoint sequence of non-zero elements included in d; let A_d be a partition of unity in $\mathfrak{A}$ including such a sequence. If B is any partition of unity in $\mathfrak{A}$, there are a $b \in B$ such that $b \cap c \ne 0$, and a $d \in D'$ such that $d \subseteq b \cap c$; now $\{a : a \in A_d,\ b \cap a \ne 0\}$ is infinite. So $\langle A_d\rangle_{d\in D'}$ witnesses that $\operatorname{wdistr}(\mathfrak{A}) \le \#(D') \le \pi(\mathfrak{A})$. **Q**

(f) $\mathfrak{m}(\mathfrak{A}) = \infty$ iff $\mathfrak{A}$ is purely atomic. **P** Write $\mathcal{B}$ for the family of all coinitial subsets of $\mathfrak{A}^+$. (i) If $\mathfrak{A}$ is purely atomic and $a \in \mathfrak{A}^+$, then there is an atom $d \subseteq a$; now $d \in B$ for every $B \in \mathcal{B}$, so $\{d, a\}$ is a linked subset of $\mathfrak{A}$ meeting every member of $\mathcal{B}$. Accordingly $\mathfrak{m}(\mathfrak{A}) = \infty$. (ii) If $\mathfrak{A}$ is not purely atomic, let $a \in \mathfrak{A}^+$ be such that no atom of $\mathfrak{A}$ is included in a. **?** If A is a linked subset of $\mathfrak{A}$ containing a and meeting every member of $\mathcal{B}$, set $B = \mathfrak{A}^+ \setminus A$. If $b \in \mathfrak{A}^+$, then either $b \cap a = 0$ and $b \in B$, or there are non-zero disjoint b', $b'' \subseteq b \cap a$ and one of b', b'' must belong to B. So $B \in \mathcal{B}$, which is impossible. **X** So $\mathfrak{m}(\mathfrak{A}) \le \#(\mathcal{B}) < \infty$. **Q**

511J Elementary facts: ideals of sets Let X be a set and $\mathcal{I}$ an ideal of subsets of X.

(a) $\operatorname{add}\mathcal{I} \ge \omega$, by the definition of 'ideal of sets'.

(b) $\operatorname{shr}\mathcal{I} = \sup\{\operatorname{non}(A, \mathcal{I} \cap \mathcal{P}A) : A \in \mathcal{P}X \setminus \mathcal{I}\}$, counting $\sup\emptyset$ as 0; $\operatorname{shr}\mathcal{I} \le \#(X)$; $\operatorname{shr}\mathcal{I} \le \operatorname{shr}^+\mathcal{I} \le (\operatorname{shr}\mathcal{I})^+$; if $\operatorname{shr}\mathcal{I}$ is a successor cardinal, $\operatorname{shr}^+\mathcal{I} = (\operatorname{shr}\mathcal{I})^+$.

(c) Suppose that $\mathcal{I}$ covers X but does not contain X. Then $\operatorname{add}\mathcal{I} \le \operatorname{cov}\mathcal{I} \le \operatorname{cf}\mathcal{I}$ and $\operatorname{add}\mathcal{I} \le \operatorname{non}\mathcal{I} \le \operatorname{shr}\mathcal{I} \le \operatorname{cf}\mathcal{I}$. **P** Let $\mathcal{J}$ be a subset of $\mathcal{I}$ with cardinal $\operatorname{cov}\mathcal{I}$ covering X; let $\mathcal{K}$ be a cofinal subset of $\mathcal{I}$ with cardinal $\operatorname{cf}\mathcal{I}$; let $A \in \mathcal{P}X \setminus \mathcal{I}$ be such that $\#(A) = \operatorname{non}\mathcal{I}$. (i) $\mathcal{J}$ cannot have an upper bound in $\mathcal{I}$, so $\operatorname{add}\mathcal{I} \le \#(\mathcal{J}) = \operatorname{cov}\mathcal{I}$. (ii) $\bigcup\mathcal{K} = \bigcup\mathcal{I} = X$, so $\operatorname{cov}\mathcal{I} \le \#(\mathcal{K}) = \operatorname{cf}\mathcal{I}$. (iii) For each $x \in A$ we can find an $I_x \in \mathcal{I}$ containing x; now $\{I_x : x \in A\}$ cannot have an upper bound in $\mathcal{I}$, so $\operatorname{add}\mathcal{I} \le \#(A) = \operatorname{non}\mathcal{I}$. (iv) By (b), $\operatorname{shr}\mathcal{I} \ge \operatorname{non}\mathcal{I}$. (v) Take any $B \subseteq X$ such that $B \notin \mathcal{I}$. Then for each $K \in \mathcal{K}$ we can find an $x_K \in B \setminus K$; now $B' = \{x_K : K \in \mathcal{K}\}$ is not included in any member of $\mathcal{K}$, so cannot belong to $\mathcal{I}$, while $B' \subseteq B$ and $\#(B') \le \#(\mathcal{K}) = \operatorname{cf}\mathcal{I}$. As B is arbitrary, $\operatorname{shr}\mathcal{I} \le \operatorname{cf}\mathcal{I}$. **Q**

(d) Suppose that $X \in \mathcal{I}$. Then $\operatorname{add}\mathcal{I} = \operatorname{non}\mathcal{I} = \infty$, $\operatorname{cov}\mathcal{I} \le 1$ (with $\operatorname{cov}\mathcal{I} = 0$ iff $X = \emptyset$) and $\operatorname{shr}\mathcal{I} = 0$.

(e) Suppose that $\mathcal{I}$ has a greatest member which is not X. Then $\operatorname{add}\mathcal{I} = \operatorname{cov}\mathcal{I} = \infty$ and $\operatorname{non}\mathcal{I} = \operatorname{shr}\mathcal{I} = \operatorname{cf}\mathcal{I} = 1$.

(f) Suppose that $\mathcal{I}$ has no greatest member and does not cover X. Then $\operatorname{add}\mathcal{I} \le \operatorname{cf}\mathcal{I}$ (511He), $\operatorname{non}\mathcal{I} = \operatorname{shr}\mathcal{I} = 1$ and $\operatorname{cov}\mathcal{I} = \infty$.

(g) Suppose that $Y \subseteq X$, and set $\mathcal{I}_Y = \mathcal{I} \cap \mathcal{P}Y$, regarded as an ideal of subsets of Y. Then $\operatorname{add}\mathcal{I}_Y \ge \operatorname{add}\mathcal{I}$, $\operatorname{non}\mathcal{I}_Y \ge \operatorname{non}\mathcal{I}$, $\operatorname{shr}\mathcal{I}_Y \le \operatorname{shr}\mathcal{I}$, $\operatorname{shr}^+\mathcal{I}_Y \le \operatorname{shr}^+\mathcal{I}$, $\operatorname{cov}\mathcal{I}_Y \le \operatorname{cov}\mathcal{I}$ and $\operatorname{cf}\mathcal{I}_Y \le \operatorname{cf}\mathcal{I}$.

511X Basic exercises (a) Let P be a partially ordered set and $\kappa \ge 3$ a cardinal. Show that $\operatorname{add}P \ge \kappa$ iff $(\lambda, \lambda, \lambda)$ is an upwards precaliber triple of P for every $\lambda < \kappa$.

>(b) Let X be a compact Hausdorff space. Show that a pair (κ,λ) of cardinals is a precaliber pair of X iff whenever $\langle G_\xi\rangle_{\xi<\kappa}$ is a family of non-empty open subsets of X there is an $x\in X$ such that $\{\xi:x\in G_\xi\}$ has cardinal at least λ.

(c) Let (X,Σ,μ) be a measure space. For $A\subseteq X$ write μ_A for the subspace measure on A, and $\mathcal{N}(\mu)$, $\mathcal{N}(\mu_A)$ for the corresponding null ideals. Show that $\operatorname{shr}(X,\mathcal{N}(\mu))=\sup\{\operatorname{non}(A,\mathcal{N}(\mu_A)):A\in\mathcal{P}X\setminus\mathcal{N}(\mu)\}$.

(d) Let (X,Σ,μ) be a measure space, and let $\hat\mu$, $\tilde\mu$ be the completion and c.l.d. version of μ. (i) Let $\mathcal{N}(\mu)=\mathcal{N}(\hat\mu)$ and $\mathcal{N}(\tilde\mu)$ be the corresponding null ideals. Show that $\operatorname{add}\mathcal{N}(\mu)\le\operatorname{add}\mathcal{N}(\tilde\mu)$, $\operatorname{cov}\mathcal{N}(\mu)\ge\operatorname{cov}\mathcal{N}(\tilde\mu)$, $\operatorname{non}\mathcal{N}(\mu)\le\operatorname{non}\mathcal{N}(\tilde\mu)$, $\operatorname{shr}\mathcal{N}(\mu)\ge\operatorname{shr}\mathcal{N}(\tilde\mu)$ and $\operatorname{shr}^+\mathcal{N}(\mu)\ge\operatorname{shr}^+\mathcal{N}(\tilde\mu)$. (ii) Show that $\operatorname{add}\mu\le\operatorname{add}\hat\mu\le\operatorname{add}\tilde\mu$, $\pi(\mu)=\pi(\hat\mu)\le\pi(\tilde\mu)$ and $\tau(\mu)=\tau(\hat\mu)\ge\tau(\tilde\mu)$.

(e) Show that if P is a partially ordered set and $c^\uparrow(P)<\omega$ then $c^\uparrow(P)=\operatorname{link}^\uparrow(P)=d^\uparrow(P)$ and $\mathfrak{m}^\uparrow(P)=\infty$.

(f) Let P be a partially ordered set. Show that ω is an up-precaliber of P iff $c^\uparrow(P)<\omega$.

>(g)(i) Show that if P is a partially ordered set and κ is an up-precaliber of P, then $\operatorname{cf}\kappa$ is also an up-precaliber of P. (ii) Show that if κ is a cardinal and $\operatorname{cf}\kappa>\operatorname{cf}P$ then κ is an up-precaliber of P.

>(h) Give $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{Q}$ their usual total orders. Show that $\mathrm{FN}(\mathbb{N})=\mathrm{FN}(\mathbb{Q})=\omega$ and that $\mathrm{FN}(\mathbb{R})=\omega_1$.

(i) Show that if P is a partially ordered set and $\#(P)\ge3$ then $\mathrm{FN}(P)\le\#(P)$. (*Hint*: consider separately the cases P infinite, P finite with no greatest member, and P finite with greatest and least members.)

(j) Let P be a partially ordered set and $\mathcal{Q}$ a family of subsets of P with $\#(\mathcal{Q})<\operatorname{add}P$. Show that if $\bigcup\mathcal{Q}$ is cofinal with P then one of the members of $\mathcal{Q}$ is cofinal with P.

(k) Let U be a Riesz space and κ a cardinal. Then U is **weakly** (κ,∞)**-distributive** if whenever $\langle A_\xi\rangle_{\xi<\kappa}$ is a family of non-empty downwards-directed subsets of U^+, each with infimum 0, and $\bigcup_{\xi<\kappa}A_\xi$ has an upper bound in U, then

$$\{u:u\in U,\text{ for every }\xi<\kappa\text{ there is a }v\in A_\xi\text{ such that }v\le u\}$$

has infimum 0 in U. Show that an Archimedean Riesz space is weakly (κ,∞)-distributive iff its band algebra is. (*Hint*: 368R.)

(l) Let X be a set and $\mathcal{I}$ an ideal of subsets of X. Show that the coinitiality $\operatorname{ci}(\mathcal{P}X\setminus\mathcal{I})$ is at most $\#(X)^{\operatorname{shr}\mathcal{I}}$.

511Y Further exercises (a)(i) Show that $d^\uparrow(P)\le2^{\operatorname{link}^\uparrow(P)}$ for every partially ordered set P. (ii) Show that there is a partially ordered set P such that $d^\uparrow(P)=\omega$ but P cannot be covered by countably many upwards-directed sets.

(b) Let κ be an infinite cardinal, with its usual well-ordering. Show that $\mathrm{FN}(\kappa)=\kappa$.

(c)(i) Find a semi-finite measure space (X,Σ,μ) such that $\operatorname{cf}\mathcal{N}(\mu)<\operatorname{cf}\mathcal{N}(\tilde\mu)$, where $\mathcal{N}(\mu)$ and $\mathcal{N}(\tilde\mu)$ are the null ideals of μ and its c.l.d. version. (ii) Find a semi-finite measure space (X,Σ,μ) such that $\operatorname{add}\mathcal{N}(\tilde\mu)>\operatorname{add}\mathcal{N}(\mu)$ and $\operatorname{cf}\mathcal{N}(\tilde\mu)<\operatorname{cf}\mathcal{N}(\mu)$.

(d) Show that, for a set I, (ω_1,ω,ω) is a precaliber triple of $\mathbb{N}^I$ iff I is countable.

511 Notes and comments Because $(P,\ge)$ is a pre-ordered set whenever $(P,\le)$ is, any cardinal function on pre-ordered sets is bound to appear in two mirror-image forms. It does not quite follow that we have to set up a language with a complete set of mirror pairs of definitions, and indeed I have omitted the reflections of 'additivity' and 'bursting number'; but the naturally arising pre-ordered sets to which we shall want to apply these ideas may appear in either orientation. The most natural conversions to topological spaces and Boolean algebras use the families of non-empty open sets and non-zero elements, which are 'active downwards', so that we have such formulae as $\pi(\mathfrak{A}) = \operatorname{ci}\mathfrak{A}^+$ and $c(X) = c^{\downarrow}(\mathfrak{T} \setminus \{\emptyset\})$; but we could equally well say that $\pi(\mathfrak{A}) = \operatorname{cf}\mathfrak{A}^-$ or that $c(X)$ is the upwards-cellularity of the partially ordered set of proper closed subsets of X.

Most readers, especially those acquainted with Volumes 3 and 4 of this treatise, will be more familiar with topological spaces and Boolean algebras than with general pre-ordered sets, and will prefer to approach the concepts here through the formulations in 5A4A and 511D. But even in the present chapter we shall be looking at questions which demand substantial fragments of the theory of general partially ordered sets, and I think it is useful to grapple with these immediately. The list of definitions above is a long one, and the functions here vary widely in importance; but I hope you will come to agree that all are associated with interesting questions.

I apologise for introducing two cardinal functions to represent the 'breadth' of a pre-ordered set (or topological space or Boolean algebra), its 'cellularity' and 'saturation'. It turns out that the saturation of a space determines its cellularity (513Bc), which seems to render the concept of 'cellularity' unnecessary; but it is well-established and makes some formulae simpler. This is an example of a standard problem: whenever we give a name to a supremum, we find ourselves asking whether the supremum is attained. The question of whether cellularity is attained turns out to be rather interesting (513B again). In the case of shrinking numbers, the ordinary shrinking number $\operatorname{shr}\mathcal{I}$ is the one which has been most studied, but I shall have some results which are more elegantly expressed in terms of the augmented shrinking number $\operatorname{shr}^+\mathcal{I}$.

I give very little space here to the functions $\mathfrak{m}()$ and wdistr() and to precalibers; these are bound to be a bit mysterious. Later in the chapter I will explore their relations with each other and with other cardinal functions. You may recognise them as belonging to the general area associated with Martin's axiom (Fremlin 84a, or §517 below). 'Precaliber pairs' have a slightly more direct description in the context of compact Hausdorff spaces (511Xb). 'Freese-Nation numbers' relate to quite different aspects of the structure of ordered sets. As will be made clear in the next two sections, all the other cardinal functions defined in 511B refer to the cofinal (or coinitial) structure of a partially ordered set; the Freese-Nation number, by contrast, tells us something about the nature of intervals inside it. We see a difference already in the formula for the Freese-Nation number of a Boolean algebra, which refers to the whole algebra $\mathfrak{A}$ rather than to $\mathfrak{A}^+$. Another signal is the fact that it is not a trivial matter to calculate the Freese-Nation number of a finite partially ordered set.

The only cardinal functions I have explicitly defined for measure spaces are the additivity and π-weight of a measure (511G), and even these are, in the most important cases, reducible to the additivity of the null ideal (521A) and the π-weight of the measure algebra (521Da). I give a pair of warming-up exercises (511Xc-511Xd), but we shall hardly see 'measure' again until Chapter 52. For the questions studied in this volume, the important cardinals associated with a measure μ are those defined from its measure algebra together with the four cardinals $\operatorname{add}\mathcal{N}(\mu)$, $\operatorname{cov}\mathcal{N}(\mu)$, $\operatorname{non}\mathcal{N}(\mu)$ and $\operatorname{cf}\mathcal{N}(\mu)$. In particular, the additivity of Lebesgue measure will have a special position. In the case of a topological measure space, of course, we can investigate relationships between the cardinal functions of the topology and the cardinal functions of the measure. I will come to such questions in Chapter 53.

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512 Galois-Tukey connections

One of the most powerful methods of relating the cardinals associated with two partially ordered sets P and Q is to identify a 'Tukey function' from one to the other (513D). It turns out that the idea can be usefully generalized to other relational structures through the concept of 'Galois-Tukey connection' (512A). In this section I give the elementary theory of these connections and their effect on simple cardinal functions.

512A Definitions (a) A **supported relation** is a triple (A, R, B) where A and B are sets and R is a subset of $A \times B$.

It will be convenient, and I think not dangerous, to abuse notation by writing $(A, \in, B)$ or $(A, \subseteq, B)$ to mean (A, R, B) where R is $\{(a, b) : a \in A,\ b \in B,\ a \in b\}$ or $\{(a, b) : a \in A,\ b \in B,\ a \subseteq b\}$.

(b) If (A,R,B) is a supported relation, its **dual** is the supported relation $(A,R,B)^\perp=(B,S,A)$ where

$$S=(B\times A)\setminus R^{-1}=\{(b,a):a\in A,\ b\in B,\ (a,b)\notin R\}.$$

(c) If (A,R,B) and (C,S,D) are supported relations, a **Galois-Tukey connection** from (A,R,B) to (C,S,D) is a pair (ϕ,ψ) such that $\phi:A\to C$ and $\psi:D\to B$ are functions and $(a,\psi(d))\in R$ whenever $(\phi(a),d)\in S$.

(d) (Vojtáš 93) If (A,R,B) and (C,S,D) are supported relations, I write $(A,R,B)\preccurlyeq_{\mathrm{GT}}(C,S,D)$ if there is a Galois-Tukey connection from (A,R,B) to (C,S,D), and $(A,R,B)\equiv_{\mathrm{GT}}(C,S,D)$ if $(A,R,B)\preccurlyeq_{\mathrm{GT}}(C,S,D)$ and $(C,S,D)\preccurlyeq_{\mathrm{GT}}(A,R,B)$.

512B Definitions (a) If (A,R,B) is a supported relation, its **covering number** $\operatorname{cov}(A,R,B)$ (sometimes called **norm** $\|(A,R,B)\|$) is the least cardinal of any set $C\subseteq B$ such that $A\subseteq R^{-1}[C]$; or ∞ if $A\not\subseteq R^{-1}[B]$. Its **additivity** is $\operatorname{add}(A,R,B)=\operatorname{cov}(A,R,B)^\perp$, that is, the smallest cardinal of any subset $C\subseteq A$ such that $C\not\subseteq R^{-1}[\{b\}]$ for any $b\in B$; or ∞ if there is no such C.

Note that $\operatorname{add}(A,R,B)=0$ iff $B=\emptyset$, and that $\operatorname{add}(A,R,B)=1$ iff $B\ne\emptyset$ and $\operatorname{cov}(A,R,B)=\infty$.

(b) If (A,R,B) is a supported relation, its **saturation** $\operatorname{sat}(A,R,B)$ is the least cardinal κ such that whenever $\langle a_\xi\rangle_{\xi<\kappa}$ is a family in A then there are distinct ξ, $\eta<\kappa$ and a $b\in B$ such that (a_ξ,b) and (a_η,b) both belong to R; if there is no such κ (that is, if $\operatorname{cov}(A,R,B)=\infty$) I write $\operatorname{sat}(A,R,B)=\infty$.

(c) If (A,R,B) is a supported relation and κ is a cardinal, say that a subset A' of A is **$<\kappa$-linked** if for every $I\in[A']^{<\kappa}$ there is a $b\in B$ such that $I\subseteq R^{-1}[\{b\}]$, and **κ-linked** if it is $<\kappa^+$-linked, that is, for every $I\in[A']^{\le\kappa}$ there is a $b\in B$ such that $I\subseteq R^{-1}[\{b\}]$. Now the **$<\kappa$-linking number** $\operatorname{link}_{<\kappa}(A,R,B)$ of (A,R,B) is the least cardinal of any cover of A by $<\kappa$-linked sets, if there is such a cover, and otherwise is ∞; and the **κ-linking number** $\operatorname{link}_\kappa(A,R,B)$ of (A,R,B) is $\operatorname{link}_{<\kappa^+}(A,R,B)$, that is, the least cardinal of any cover of A by κ-linked sets.

If $\kappa\le\lambda$, then every $<\lambda$-linked set is $<\kappa$-linked, so $\operatorname{link}_{<\kappa}(A,R,B)\le\operatorname{link}_{<\lambda}(A,R,B)$. Note also that $\operatorname{link}_\kappa(A,R,B)$ is equal to $\operatorname{cov}(A,R,B)$ for every $\kappa\ge\#(A)$, so that $\operatorname{link}_{<\theta}(A,R,B)\le\operatorname{cov}(A,R,B)$ for every θ.

512C There are two things which should be done at once: to plainly state enough of the elementary theory to show at least that the definitions here lead to a coherent structure; and to give examples. I begin with the theory, which really is elementary.

Theorem Let (A,R,B), (C,S,D) and (E,T,F) be supported relations.

(a) $(A,R,B)^{\perp\perp}=(A,R,B)$.

(b) If (ϕ,ψ) is a Galois-Tukey connection from (A,R,B) to (C,S,D) and (ϕ',ψ') is a Galois-Tukey connection from (C,S,D) to (E,T,F), then $(\phi'\phi,\psi\psi')$ is a Galois-Tukey connection from (A,R,B) to (E,T,F).

(c) If (ϕ,ψ) is a Galois-Tukey connection from (A,R,B) to (C,S,D), then (ψ,ϕ) is a Galois-Tukey connection from $(C,S,D)^\perp$ to $(A,R,B)^\perp$.

(d) If $R'\subseteq R$ then $(A,R,B)\preccurlyeq_{\mathrm{GT}}(A,R',B)$.

(e) If $(A,R,B)\preccurlyeq_{\mathrm{GT}}(C,S,D)$ and $(C,S,D)\preccurlyeq_{\mathrm{GT}}(E,T,F)$ then $(A,R,B)\preccurlyeq_{\mathrm{GT}}(E,T,F)$.

(f) $\equiv_{\mathrm{GT}}$ is an equivalence relation on the class of supported relations.

(g) If $(A,R,B)\preccurlyeq_{\mathrm{GT}}(C,S,D)$ then $(C,S,D)^\perp\preccurlyeq_{\mathrm{GT}}(A,R,B)^\perp$. So if $(A,R,B)\equiv_{\mathrm{GT}}(C,S,D)$ then $(A,R,B)^\perp\equiv_{\mathrm{GT}}(C,S,D)^\perp$.

proof (a)-(c) are immediate from the definitions. (d) is trivial because the identity functions from A and B to themselves form a Galois-Tukey connection from (A,R,B) to (A,R',B). (e) follows from (b), and (g) from (c). (f) is immediate from (d) and (e) and the symmetry of the definition of $\equiv_{\mathrm{GT}}$.

512D Theorem Let (A,R,B) and (C,S,D) be supported relations such that $(A,R,B)\preccurlyeq_{\mathrm{GT}}(C,S,D)$. Then

(a) $\operatorname{cov}(A,R,B)\le\operatorname{cov}(C,S,D)$;

(b) $\operatorname{add}(C,S,D)\le\operatorname{add}(A,R,B)$;
(c) $\operatorname{sat}(A,R,B)\le\operatorname{sat}(C,S,D)$;
(d) $\operatorname{link}_{<\kappa}(A,R,B)\le\operatorname{link}_{<\kappa}(C,S,D)$ for every cardinal κ.

proof Let (ϕ,ψ) be a Galois-Tukey connection from (A,R,B) to (C,S,D). If $D_0\subseteq D$ is such that $C=S^{-1}[D_0]$, then $A=R^{-1}[\psi[D_0]]$; this shows that $\operatorname{cov}(A,R,B)\le\operatorname{cov}(C,S,D)$. If $\kappa=\operatorname{sat}(C,S,D)$ and $\langle a_\xi\rangle_{\xi<\kappa}$ is any family in A, then there are a $d\in D$ and distinct ξ, $\eta<\kappa$ such that $(\phi(a_\xi),d)\in S$ and $(\phi(a_\eta),d)\in S$, in which case $(a_\xi,\psi(d))$ and $(a_\eta,\psi(d))$ both belong to R; so $\operatorname{sat}(A,R,B)\le\kappa$. If $\mathcal{C}$ is a cover of C by $<\kappa$-linked sets, then $\{\phi^{-1}[C']:C'\in\mathcal{C}\}$ is a cover of A by $<\kappa$-linked sets; this shows that $\operatorname{link}_{<\kappa}(A,R,B)\le\operatorname{link}_{<\kappa}(C,S,D)$.

Finally, $(C,S,D)^\perp\preccurlyeq_{\mathrm{GT}}(A,R,B)^\perp$, by 512Cc, so

$$\operatorname{add}(C,S,D)=\operatorname{cov}(C,S,D)^\perp\le\operatorname{cov}(A,R,B)^\perp=\operatorname{add}(A,R,B).$$

512E Examples Of course 'supported relations' appear everywhere in mathematics. They are important to us here because covering numbers, saturation and linking numbers, as defined above, correspond to important cardinal functions as defined in §511, and because surprising Galois-Tukey connections exist, as we shall see in Chapter 52. The simplest examples are the following.

(a) Let $(P,\le)$ be a pre-ordered set. Then $(P,\le,P)$ and $(P,\ge,P)$ are supported relations, with duals $(P,\not\ge,P)$ and $(P,\not\le,P)$. $\operatorname{cov}(P,\le,P)=\operatorname{cf}P$, $\operatorname{cov}(P,\ge,P)=\operatorname{ci}P$, $\operatorname{add}(P,\le,P)=\operatorname{add}P$ and $\operatorname{sat}(P,\le,P)=\operatorname{sat}^\uparrow(P)$. For any cardinal κ, a subset of P is upwards-$<\kappa$-linked in the sense of 511Bf iff it is $<\kappa$-linked in $(P,\le,P)$ in the sense of 512Bc. So $\operatorname{link}^\uparrow_{<\kappa}(P)=\operatorname{link}_{<\kappa}(P,\le,P)$. In particular, $d^\uparrow(P)=\operatorname{link}_{<\omega}(P,\le,P)$ (511Bg).

(b) Let $(X,\mathfrak{T})$ be a topological space. Then

$$\pi(X)=\operatorname{cov}(\mathfrak{T}\setminus\{\emptyset\},\supseteq,\mathfrak{T}\setminus\{\emptyset\}),$$

$$d(X)=\operatorname{cov}(\mathfrak{T}\setminus\{\emptyset\},\ni,X)=\operatorname{add}(X,\notin,\mathfrak{T}\setminus\{\emptyset\}),$$

$$\operatorname{sat}(X)=\operatorname{sat}(\mathfrak{T}\setminus\{\emptyset\},\supseteq,\mathfrak{T}\setminus\{\emptyset\})=\operatorname{sat}(\mathfrak{T}\setminus\{\emptyset\},\ni,X),$$

$$n(X)=\operatorname{cov}(X,\in,\mathcal{N}\mathrm{wd}(X))=\operatorname{cov}(X,\mathcal{N}\mathrm{wd}(X))$$

where $\mathcal{N}\mathrm{wd}(X)$ is the ideal of nowhere dense subsets of X. Note that if $\mathcal{M}(X)$ is the ideal of meager subsets of X, then $\operatorname{cov}(X,\mathcal{M}(X))=n(X)$ unless $n(X)=\omega$, in which case $\operatorname{cov}(X,\mathcal{M}(X))=1$.

(c) Let $\mathfrak{A}$ be a Boolean algebra. Write $\mathfrak{A}^+$ for $\mathfrak{A}\setminus\{0\}$ and $\mathfrak{A}^-$ for $\mathfrak{A}\setminus\{1\}$. Then

$$\pi(\mathfrak{A})=\operatorname{cov}(\mathfrak{A}^+,\supseteq,\mathfrak{A}^+)=\operatorname{cov}(\mathfrak{A}^-,\subseteq,\mathfrak{A}^-),$$

$$\operatorname{sat}(\mathfrak{A})=\operatorname{sat}(\mathfrak{A}^+,\supseteq,\mathfrak{A}^+)=\operatorname{sat}(\mathfrak{A}^-,\subseteq,\mathfrak{A}^-),$$

$$d(\mathfrak{A})=\operatorname{link}_{<\omega}(\mathfrak{A}^+,\supseteq,\mathfrak{A}^+)=\operatorname{link}_{<\omega}(\mathfrak{A}^-,\subseteq,\mathfrak{A}^-),$$

$$\operatorname{link}(\mathfrak{A})=\operatorname{link}_2(\mathfrak{A}^+,\supseteq,\mathfrak{A}^+)=\operatorname{link}_2(\mathfrak{A}^-,\subseteq,\mathfrak{A}^-)$$

and generally

$$\operatorname{link}_{<\kappa}(\mathfrak{A})=\operatorname{link}_{<\kappa}(\mathfrak{A}^+,\supseteq,\mathfrak{A}^+)=\operatorname{link}_{<\kappa}(\mathfrak{A}^-,\subseteq,\mathfrak{A}^-),$$

$$\operatorname{link}_{\kappa}(\mathfrak{A})=\operatorname{link}_{\kappa}(\mathfrak{A}^+,\supseteq,\mathfrak{A}^+)=\operatorname{link}_{\kappa}(\mathfrak{A}^-,\subseteq,\mathfrak{A}^-)$$

for every cardinal κ.

(d) Let X be a set and $\mathcal{I}$ an ideal of subsets of X. Then the dual of $(X,\in,\mathcal{I})$ is $(\mathcal{I},\not\ni,X)$; $\operatorname{cov}(X,\in,\mathcal{I})=\operatorname{cov}\mathcal{I}$ and $\operatorname{add}(X,\in,\mathcal{I})=\operatorname{non}\mathcal{I}$.

(e) For a Boolean algebra $\mathfrak{A}$, write $\operatorname{Pou}(\mathfrak{A})$ for the set of partitions of unity in $\mathfrak{A}$. For C, $D\in\operatorname{Pou}(\mathfrak{A})$, say that $C\sqsubseteq^* D$ if every element of D meets only finitely many members of C. Then $\sqsubseteq^*$ is a pre-order on $\operatorname{Pou}(\mathfrak{A})$. Translating the definition 511Df into this language, we see that $\operatorname{wdistr}(\mathfrak{A})=\operatorname{add}\operatorname{Pou}(\mathfrak{A})$.

512F I now turn to some constructions involving supported relations and Galois-Tukey connections which will be useful later.

Dominating sets For any supported relation (A,R,B) and any cardinal κ, we can form a corresponding supported relation $(A,R',[B]^{<\kappa})$, where

$$R' = \{(a,I) : I \in [B]^{<\kappa},\ a \in R^{-1}[I]\}.$$

The most important cases to us will be $\kappa = \omega$ and $\kappa = \omega_1$. When κ is a successor cardinal I will normally write $(A,R',[B]^{\le\lambda})$ rather than $(A,R',[B]^{<\lambda^+})$.

512G Proposition Let (A,R,B) and (C,S,D) be supported relations and κ, λ cardinals.

(a) (A,R,B) is isomorphic to $(A,R',[B]^1)$.

(b) If $(A,R,B) \preccurlyeq_{\mathrm{GT}} (C,S,D)$ and $\lambda \le \kappa$ then $(A,R',[B]^{<\kappa}) \preccurlyeq_{\mathrm{GT}} (C,S',[D]^{<\lambda})$.

(c) In particular, $(A,R',[B]^{<\kappa}) \preccurlyeq_{\mathrm{GT}} (A,R,B)$ if $\kappa \ge 2$.

(d) If $\operatorname{cf}\kappa \ge \lambda$ and $(A,R',[B]^{<\kappa}) \preccurlyeq_{\mathrm{GT}} (C,S,D)$ then $(A,R',[B]^{<\kappa}) \preccurlyeq_{\mathrm{GT}} (C,S',[D]^{<\lambda})$.

(e)(i) If $\operatorname{cov}(A,R,B) = \infty$ then $\operatorname{add}(A,R',[B]^{<\kappa}) \le 1$.

(ii) If $\operatorname{cov}(A,R,B) < \infty$ then $\operatorname{add}(A,R',[B]^{<\kappa}) \ge \kappa$.

(f) $\operatorname{cov}(A,R,B) \le \max(\omega,\kappa,\operatorname{cov}(A,R',[B]^{\le\kappa}))$; if $\kappa \ge 1$ and $\operatorname{cov}(A,R,B) > \max(\kappa,\omega)$ then $\operatorname{cov}(A,R,B) = \operatorname{cov}(A,R',[B]^{\le\kappa})$.

proof (a) is trivial.

(b) If (ϕ,ψ) is a Galois-Tukey connection from (A,R,B) to (C,S,D), then (ϕ,ψ') is a Galois-Tukey connection from $(A,R',[B]^{<\kappa})$ to $(C,S',[D]^{<\lambda})$, where $\psi'(J) = \psi[J]$ for every $J \in [D]^{<\lambda}$.

(c) Setting $\phi(a) = a$ for $a \in A$ and $\psi(b) = \{b\}$ for $b \in B$, (ψ,ϕ) is a Galois-Tukey connection from $(A,R',[B]^{<\kappa})$ to (A,R,B).

(d) Let (ϕ,ψ) be a Galois-Tukey connection from $(A,R',[B]^{<\kappa})$ to (C,S,D). Set $\psi'(I) = \bigcup_{d\in I}\psi(d)$ for $I \in [D]^{<\lambda}$; then (ϕ,ψ') is a Galois-Tukey connection from $(A,R',[B]^{<\kappa})$ to $(C,S',[D]^{<\lambda})$.

(e)(i) There is an $a \in A\setminus R^{-1}[B]$; now $(a,I) \notin R'$ for any $I \in [B]^{<\kappa}$, so $\operatorname{add}(A,R',[B]^{<\kappa}) \le 1$.

(ii) For every $a \in A$ there is a $b_a \in B$ such that $(a,b_a) \in R$. If $A' \subseteq A$ and $\#(A') < \kappa$, then $I = \{b_a : a \in A'\}$ belongs to $[B]^{<\kappa}$, and $(a,I) \in R'$ for every $a \in A'$; as A' is arbitrary, $\operatorname{add}(A,R',[B]^{<\kappa}) \ge \kappa$.

(f) If $\lambda = \operatorname{cov}(A,R',[B]^{\le\kappa})$ is not ∞, let $\mathcal{D} \subseteq [B]^{\le\kappa}$ be a set with cardinal λ such that $A = (R')^{-1}[\mathcal{D}]$, and set $D = \bigcup\mathcal{D}$; then $A \subseteq R^{-1}[D]$, so $\operatorname{cov}(A,R,B) \le \#(D) \le \max(\omega,\kappa,\lambda)$.

If $\kappa \ge 1$, then $\operatorname{cov}(A,R',[B]^{\le\kappa}) \le \operatorname{cov}(A,R,B)$, by (c) and 512Da, so if the latter is greater than $\max(\kappa,\omega)$ they are equal.

512H Simple products (a) If $\langle(A_i,R_i,B_i)\rangle_{i\in I}$ is any family of supported relations, its **simple product** is $(\prod_{i\in I}A_i, T, \prod_{i\in I}B_i)$ where $T = \{(a,b) : (a(i),b(i)) \in R_i$ for every $i \in I\}$.

(b) Let $\langle(A_i,R_i,B_i)\rangle_{i\in I}$ and $\langle(C_i,S_i,D_i)\rangle_{i\in I}$ be two families of supported relations, with simple products (A,R,B) and (C,S,D). If $(A_i,R_i,B_i) \preccurlyeq_{\mathrm{GT}} (C_i,S_i,D_i)$ for every i, then $(A,R,B) \preccurlyeq_{\mathrm{GT}} (C,S,D)$. **P** For each i, let (ϕ_i,ψ_i) be a Galois-Tukey connection from (A_i,R_i,B_i) to (C_i,S_i,D_i). Define $\phi : A \to C$ and $\psi : D \to B$ by setting $\phi(\langle a_i\rangle_{i\in I}) = \langle\phi_i(a_i)\rangle_{i\in I}$, $\psi(\langle d_i\rangle_{i\in I}) = \langle\psi_i(d_i)\rangle_{i\in I}$ for $\langle a_i\rangle_{i\in I} \in A$, $\langle d_i\rangle_{i\in I} \in D$; then

$$\begin{aligned}(\phi(\langle a_i\rangle_{i\in I}),\langle d_i\rangle_{i\in I}) \in S &\Longrightarrow (\phi_i(a_i),d_i) \in S_i \text{ for every } i \in I\\ &\Longrightarrow (a_i,\psi_i(d_i)) \in R_i \text{ for every } i \in I\\ &\Longrightarrow (\langle a_i\rangle_{i\in I},\psi(\langle d_i\rangle_{i\in I})) \in R.\end{aligned}$$

So (ϕ,ψ) is a Galois-Tukey connection and $(A,R,B) \preccurlyeq_{\mathrm{GT}} (C,S,D)$. **Q**

(c) Let $\langle(A_i,R_i,B_i)\rangle_{i\in I}$ be a family of supported relations with simple product (A,R,B). Suppose that no A_i is empty. Then $\operatorname{add}(A,R,B) = \min_{i\in I}\operatorname{add}(A_i,R_i,B_i)$, interpreting $\min\emptyset$ as ∞ if $I = \emptyset$. **P** Set $\kappa = \operatorname{add}(A,R,B)$ and $\kappa' = \min_{i\in I}\operatorname{add}(A_i,R_i,B_i)$. If $I = \emptyset$ then $A = B = \{\emptyset\}$ and $R = \{(\emptyset,\emptyset)\}$

so $\operatorname{add}(A,R,B)=\infty$. Otherwise, if $C\subseteq A$ and $\#(C)<\kappa'$, then, for each i, $\#(\{c(i):c\in C\})<\operatorname{add}(A_i,R_i,B_i)$, so there is a $b_i\in B_i$ such that $(c(i),b_i)\in R_i$ for every $c\in C$; now $(c,\langle b_i\rangle_{i\in I})\in R$ for every $c\in C$; as C is arbitrary, $\kappa\ge\kappa'$. In the other direction, if $i\in I$ and $C'\in[A_i]^{<\kappa}$, then (because no A_j is empty) there is a $C\in[A]^{<\kappa}$ such that $C'=\{c(i):c\in C\}$. Now there is a $b\in B$ such that $(c,b)\in R$ for every $c\in C$, so that $(c',b(i))\in R_i$ for every $c'\in C'$. As i and C' are arbitrary, $\kappa'\le\kappa$. **Q**

(d) Suppose that (A,R,B) and (C,S,D) are supported relations with simple product $(A\times C,T,B\times D)$. Let κ be an infinite cardinal and define $(A,R',[B]^{<\kappa})$, $(C,S',[D]^{<\kappa})$ and $(A\times C,T',[B\times D]^{<\kappa})$ as in 512F. Then

$$(A,R',[B]^{<\kappa})\times(C,S',[D]^{<\kappa})\equiv_{\mathrm{GT}}(A\times C,T',[B\times D]^{<\kappa}).$$

P Express $(A,R',[B]^{<\kappa})\times(C,S',[D]^{<\kappa})$ as $(A\times C,\tilde T,[B]^{<\kappa}\times[D]^{<\kappa})$.

(i) Set $\phi(a,c)=(a,c)$ for all $a\in A$, $c\in C$, and for $I\in[B\times D]^{<\kappa}$ set

$$\psi(I)=(\pi_1[I],\pi_2[I])\in[B]^{<\kappa}\times[D]^{<\kappa},$$

where $\pi_1(b,d)=b$ and $\pi_2(b,d)=d$ for $b\in B$, $d\in D$. If $a\in A$, $c\in C$ and $I\in[B\times D]^{<\kappa}$ are such that $(\phi(a,c),I)\in T'$, then there must be a $(b,d)\in I$ such that $((a,c),(b,d))\in T$, that is, $(a,b)\in R$ and $(c,d)\in S$; now $b\in\pi_1[I]$ and $d\in\pi_2[I]$, so $(a,\pi_1[I])\in R'$ and $(c,\pi_2[I])\in S'$ and

$$((a,c),\psi(I))=((a,c),(\pi_1[I],\pi_2[I]))\in\tilde T.$$

As a, c and I are arbitrary, (ϕ,ψ) is a Galois-Tukey connection and

$$(A,R',[B]^{<\kappa})\times(C,S',[D]^{<\kappa})\preccurlyeq_{\mathrm{GT}}(A\times C,T',[B\times D]^{<\kappa}).$$

(ii) In the other direction, given $(J,K)\in[B]^{<\kappa}\times[D]^{<\kappa}$ set $\psi'(J,K)=J\times K\in[B\times D]^{<\kappa}$. (This is where we need to suppose that κ is infinite.) If now $(\phi(a,c),(J,K))\in\tilde T$, that is, $(a,J)\in R'$ and $(c,K)\in S'$, there are $b\in J$ and $d\in K$ such that $(a,b)\in R$ and $(c,d)\in S$, so that $((a,c),(b,d))\in T$ and $((a,c),\psi'(J,K))\in T'$. As a, c, J and K are arbitrary, (ϕ,ψ') is a Galois-Tukey connection and

$$(A\times C,T',[B\times D]^{<\kappa})\preccurlyeq_{\mathrm{GT}}(A,R',[B]^{<\kappa})\times(C,S',[D]^{<\kappa}).\ \mathbf{Q}$$

(e) If $\langle(P_i,\le_i)\rangle_{i\in I}$ is a family of pre-ordered sets, with product $(P,\le)$ (511A), then $(P,\le,P)$ is just $\prod_{i\in I}(P_i,\le_i,P_i)$ in the sense here.

512I Sequential compositions Let (A,R,B) and (C,S,D) be supported relations. Their **sequential composition** $(A,R,B)\ltimes(C,S,D)$ is $(A\times C^B,T,B\times D)$, where

$$T=\{((a,f),(b,d)):(a,b)\in R,\ f\in C^B,\ (f(b),d)\in S\}.$$

Their **dual sequential composition** $(A,R,B)\rtimes(C,S,D)$ is $(A\times C,\tilde T,B\times D^A)$ where

$$\begin{aligned}\tilde T=\{((a,c),(b,g)):a\in A,\ b\in B,\ c\in C,\ g\in D^A\\ \text{and either }(a,b)\in R\text{ or }(c,g(a))\in S\}.\end{aligned}$$

512J Proposition Let (A,R,B) and (C,S,D) be supported relations.

(a) $(A,R,B)\rtimes(C,S,D)=((A,R,B)^\perp\ltimes(C,S,D)^\perp)^\perp$.

(b) $\operatorname{cov}((A,R,B)\ltimes(C,S,D))$ is the cardinal product $\operatorname{cov}(A,R,B)\cdot\operatorname{cov}(C,S,D)$ unless $B=C=\emptyset\ne A$, if we use the interpretations

$$0\cdot\infty=\infty\cdot0=0,\quad\kappa\cdot\infty=\infty\cdot\kappa=\infty\cdot\infty=\infty\text{ for every cardinal }\kappa\ge1.$$

(c) $\operatorname{add}((A,R,B)\ltimes(C,S,D))=\min(\operatorname{add}(A,R,B),\operatorname{add}(C,S,D))$ unless $A\times C=\emptyset\ne B\times D$.

proof (a) is just a matter of disentangling the definitions.

(b) Define $T\subseteq(A\times C^B)\times(B\times D)$ as in 512I.

(i) Suppose first that neither A nor C is empty, that $A \subseteq R^{-1}[B]$ and that $C \subseteq S^{-1}[D]$. If $B_0 \subseteq B$ and $D_0 \subseteq D$ are such that $A \subseteq R^{-1}[B_0]$ and $C \subseteq S^{-1}[D_0]$, then for any $a \in A$ and $f \in C^B$ there are $b \in B_0$ and $d \in D_0$ such that $(a,b) \in R$ and $(f(b),d) \in S$, so that $(a,f) \in T^{-1}[B_0 \times D_0]$. So $\operatorname{cov}((A,R,B) \ltimes (C,S,D)) \le \operatorname{cov}(A,R,B) \cdot \operatorname{cov}(C,S,D)$.

On the other hand, if $H \subseteq B \times D$ is such that $A \times C^B \subseteq T^{-1}[H]$, set $B_0 = \{b : C \subseteq S^{-1}[H[\{b\}]]\}$. Then $\#(H[\{b\}]) \ge \operatorname{cov}(C,S,D)$ for $b \in B_0$. Also $A \subseteq R^{-1}[B_0]$. **P?** Otherwise, take $a \in A \setminus R^{-1}[B_0]$. For $b \in B \setminus B_0$, choose $f(b) \in C \setminus S^{-1}[H[\{b\}]]$; for $b \in B_0$, take $f(b)$ to be any member of C. There is supposed to be a member (b,d) of H such that $((a,f),(b,d)) \in T$, that is, $(a,b) \in R$ and $(f(b),d) \in S$. But now $b \notin B_0$, by the choice of a, and $(f(b),d) \notin S$, by the choice of f; so we have a contradiction. **XQ**

So $\#(B_0) \ge \operatorname{cov}(A,R,B)$ and $\#(H) \ge \operatorname{cov}(A,R,B) \cdot \operatorname{cov}(C,S,D)$; as H is arbitrary, $\operatorname{cov}((A,R,B) \ltimes (C,S,D)) \ge \operatorname{cov}(A,R,B) \cdot \operatorname{cov}(C,S,D)$.

(ii) If $A = \emptyset$ then $A \times C^B = \emptyset$ so $\operatorname{cov}(A,R,B)$ and $\operatorname{cov}((A,R,B) \ltimes (C,S,D))$ are both zero. If $C = \emptyset$ and $B \ne \emptyset$ then $\operatorname{cov}(C,S,D) = \operatorname{cov}((A,R,B) \ltimes (C,S,D)) = 0$. If A and C are non-empty and $A \not\subseteq R^{-1}[B]$, then $A \times C^B \not\subseteq T^{-1}[B \times D]$, so $\operatorname{cov}(A,R,B) = \operatorname{cov}((A,R,B) \ltimes \operatorname{cov}(C,S,D)) = \infty$, while $\operatorname{cov}(C,S,D) \ge 1$. If A and C are non-empty and $A \subseteq R^{-1}[B]$ and $C \not\subseteq S^{-1}[D]$, then $B \ne \emptyset$; if we take $c \in C \setminus S^{-1}[D]$ and any member a of A, and set $f(b) = c$ for every $b \in B$, then $(a,f) \notin T^{-1}[B \times D]$, so $\operatorname{cov}((A,R,B) \ltimes (C,S,D)) = \operatorname{cov}(C,S,D) = \infty$, while $\operatorname{cov}(A,R,B) \ge 1$. So with the single exception of $B = C = \emptyset \ne A$ (in which case the empty function belongs to C^B, so that $\operatorname{cov}((A,R,B) \ltimes (C,S,D)) = \infty$, while $\operatorname{cov}(C,S,D) = 0$) we have $\operatorname{cov}((A,R,B) \ltimes (C,S,D)) = \operatorname{cov}(A,R,B) \cdot \operatorname{cov}(C,S,D)$.

(c) Assume throughout that either $A \times C \ne \emptyset$ (so that $A \times C^B \ne \emptyset$) or that $B \times D = \emptyset$.

(i) $\operatorname{add}((A,R,B) \ltimes (C,S,D)) \le \operatorname{add}(A,R,B)$. **P** If $\operatorname{add}(A,R,B) = \infty$ the result is trivial. If $B \times D = \emptyset$ then $\operatorname{add}((A,R,B) \ltimes (C,S,D)) = 0 \le \operatorname{add}(A,R,B)$. Otherwise, our hypothesis ensures that C is not empty; take $A_0 \subseteq A$ such that $\#(A_0) = \operatorname{add}(A,R,B)$ and $A_0 \not\subseteq R^{-1}[\{b\}]$ for any $b \in B$, take any $b_0 \in B$ and any $f_0 \in C^B$; then there is no $(b,d) \in B \times D$ such that $((a,f_0(b_0)),(b,d)) \in T$ for every $a \in A_0$, so $\operatorname{add}((A,R,B) \ltimes (C,S,D)) \le \#(A_0) = \operatorname{add}(A,R,B)$. **Q**

(ii) $\operatorname{add}((A,R,B) \ltimes (C,S,D)) \le \operatorname{add}(C,S,D)$. **P** Again, if $\operatorname{add}(C,S,D) = \infty$ or $B \times D = \emptyset$ the result is immediate. Otherwise, $A \ne \emptyset$. Take $C_0 \subseteq C$ such that $\#(C_0) = \operatorname{add}(C,S,D)$ and there is no $d \in D$ such that $C_0 \subseteq S^{-1}[\{d\}]$, for $c \in C_0$ set $f_c(b) = c$ for every $b \in B$, and fix any $a_0 \in A$; then there is no $(b,d) \in B \times D$ such that $((a_0,f_c),(b,d)) \in T$ for every $c \in C_0$, so $\operatorname{add}((A,R,B) \ltimes (C,S,D)) \le \#(C_0) = \operatorname{add}(C,S,D)$. **Q**

(iii) $\operatorname{add}((A,R,B) \ltimes (C,S,D)) \ge \min(\operatorname{add}(A,R,B), \operatorname{add}(C,S,D))$. **P** If $H \subseteq A \times C^B$ and $\#(H)$ is less than $\min(\operatorname{add}(A,R,B), \operatorname{add}(C,S,D))$, set $A_0 = \{a : (a,f) \in H\}$ and $F = \{f : (a,f) \in H\}$. Then there are a $b \in B$ such that $(a,b) \in R$ for any $a \in A_0$, and a $d \in D$ such that $(f(b),d) \in S$ for any $f \in F$, so that $((a,f),(b,d)) \in T$ for any $(a,f) \in H$. As H is arbitrary, $\operatorname{add}((A,R,B) \ltimes (C,S,D)) \ge \min(\operatorname{add}(A,R,B), \operatorname{add}(C,S,D))$. **Q**

512K The following fact will be used in §526.

Lemma Suppose that (A,R,B) and (C,S,D) are supported relations, and P is a partially ordered set. Suppose that $\langle A_p \rangle_{p \in P}$ is a family of subsets of A such that

$$(A_p,R,B) \preccurlyeq_{\mathrm{GT}} (C,S,D) \text{ for every } p \in P,$$

$$A_p \subseteq A_q \text{ whenever } p \le q \text{ in } P, \quad \bigcup_{p \in P} A_p = A.$$

Then $(A,R,B) \preccurlyeq_{\mathrm{GT}} (P,\le,P) \ltimes (C,S,D)$.

proof If $C = \emptyset$ the result is trivial, since every A_p is empty and B can be empty only if D is. So we may suppose that $C \ne \emptyset$. For each $p \in P$, let (ϕ_p,ψ_p) be a Galois-Tukey connection from (A_p,R,B) to (C,S,D). For $a \in A$, let $r(a) \in P$ be such that $a \in A_{r(a)}$, and set $f_a(p) = \phi_p(a)$ whenever $p \in P$ and $a \in A_p$; for other $p \in P$ take $f_a(p)$ to be any member of C. Set $\phi(a) = (r(a),f_a)$ for $a \in A$. For $q \in P$, $d \in D$ set $\psi(q,d) = \psi_q(d) \in B$. Now (ϕ,ψ) is a Galois-Tukey connection from (A,R,B) to $(P,\le,P) \ltimes (C,S,D)$. **P** Suppose that $a \in A$ and $(q,d) \in P \times D$ are such that $r(a) \le q$ and $(f_a(q),d) \in S$. Then $a \in A_{r(a)} \subseteq A_q$ so $f_a(q) = \phi_q(a)$. Because (ϕ_q,ψ_q) is a Galois-Tukey connection, $(a,\psi(q,d)) = (a,\psi_q(d)) \in R$. **Q**

So we have the result.

512X Basic exercises (a)(i) Suppose that $A \subseteq A'$, $B' \subseteq B$ and that R is any relation. Show that $(A, R, B) \preccurlyeq_{\mathrm{GT}} (A', R, B')$. (ii) Show that $(\emptyset, \emptyset, \{\emptyset\}) \preccurlyeq_{\mathrm{GT}} (A, R, B) \preccurlyeq_{\mathrm{GT}} (\{\emptyset\}, \emptyset, \emptyset)$ for every supported relation (A, R, B).

(b) Let (A, R, B) be any supported relation. Show that $\mathrm{sat}(A, R, B) \le (\mathrm{link}_2(A, R, B))^+$.

(c) Let $(X, \mathfrak{T})$ be a topological space and $(Y, \mathfrak{T}_Y)$ an open subspace. Show that $(\mathfrak{T}_Y \setminus \{\emptyset\}, \supseteq, \mathfrak{T}_Y \setminus \{\emptyset\}) \preccurlyeq_{\mathrm{GT}} (\mathfrak{T} \setminus \{\emptyset\}, \supseteq, \mathfrak{T} \setminus \{\emptyset\})$.

(d) Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{S})$ be topological spaces. (i) Show that if Y is a continuous image of X, $(\mathfrak{S} \setminus \{\emptyset\}, \ni, Y) \preccurlyeq_{\mathrm{GT}} (\mathfrak{T} \setminus \{\emptyset\}, \ni, X)$. (ii) Show that if X and Y are compact and Hausdorff and there is an irreducible continuous surjection from X onto Y, then $(\mathfrak{T} \setminus \{\emptyset\}, \ni, X) \equiv_{\mathrm{GT}} (\mathfrak{S} \setminus \{\emptyset\}, \ni, Y)$ and $(\mathfrak{T} \setminus \{\emptyset\}, \supseteq, \mathfrak{T} \setminus \{\emptyset\}) \equiv_{\mathrm{GT}} (\mathfrak{S} \setminus \{\emptyset\}, \supseteq, \mathfrak{S} \setminus \{\emptyset\})$, so $d(X) = d(Y)$ and $\pi(X) = \pi(Y)$.

(e) Let $\langle (A_i, R_i, B_i) \rangle_{i \in I}$ be a family of supported relations with simple product (A, R, B). Show that $(A, R, B)^\perp$ can be naturally identified with the simple product of $\langle (A_i, R_i, B_i)^\perp \rangle_{i \in I}$.

(f) Let (A, R, B) be a supported relation and $\kappa > 0$ a cardinal. Show that $(A, R', [B]^{\le \kappa}) \preccurlyeq_{\mathrm{GT}} (A, R, B)^\kappa$, where $(A, R, B)^\kappa$ is the simple product of κ copies of (A, R, B) and $R' = \{(a, J) : a \in R^{-1}[J]\}$ as usual.

(g) Let (A, R, B) and (C, S, D) be supported relations, and $(A \times C, T, B \times D)$ their simple product. (i) Show that if $C \ne \emptyset$, then $(A, R, B) \preccurlyeq_{\mathrm{GT}} (A \times C, T, B \times D)$. (ii) Show that $(A \times C, T, B \times D) \preccurlyeq_{\mathrm{GT}} (A, R, B) \ltimes (C, S, D)$. (iii) Show that (using the conventions of 512Jb) $\mathrm{cov}(A \times C, T, B \times D) = \mathrm{cov}(A, R, B) \cdot \mathrm{cov}(C, S, D)$.

(h) Let (A_0, R_0, B_0), (A_1, R_1, B_1), (C_0, S_0, D_0) and (C_1, S_1, D_1) be supported relations such that $(A_0, R_0, B_0) \preccurlyeq_{\mathrm{GT}} (A_1, R_1, B_1)$ and $(C_0, S_0, D_0) \preccurlyeq_{\mathrm{GT}} (C_1, S_1, D_1)$. Show that

$$(A_0, R_0, B_0) \ltimes (C_0, S_0, D_0) \preccurlyeq_{\mathrm{GT}} (A_1, R_1, B_1) \ltimes (C_1, S_1, D_1),$$

$$(A_0, R_0, B_0) \rtimes (C_0, S_0, D_0) \preccurlyeq_{\mathrm{GT}} (A_1, R_1, B_1) \rtimes (C_1, S_1, D_1).$$

512 Notes and comments Much of this section is cluttered by the repeated names (A, R, B) of 'supported relations'. In fact these could probably be dispensed with. While I am reluctant to alter the general convention I use in this book, that a 'relation' is neither more nor less than a class of ordered pairs, it is clear that in all significant cases our supported relation (A, R, B) will be such that $A = \{a : (a, b) \in R\}$ and $B = \{b : (a, b) \in R\}$, so that A and B can be recovered from the set R. But this would make impossible the very useful convention that '$(X, \in, \mathcal{A})$' is to be interpreted as '$(X, \{(x, A) : A \in \mathcal{A}, x \in X \cap A\}, \mathcal{A})$', and since nearly every mathematical argument in this context demands names for the domains and codomains of the relations, it seems easier to write these in each time.

An important feature of the theory here is that while it is very common for our relations to be reasonably well-behaved by some criterion (for instance, we may have Polish spaces A and B and a coanalytic set $R \subseteq A \times B$), the functions in a Galois-Tukey connection are not required to have any properties beyond those declared in the definition. Of course the most important Galois-Tukey connections are those which are 'natural' in some sense, and are constructed in a way which does not involve totally unscrupulous use of the axiom of choice. I will return to this question in the next section.

Version of 23.2.14

513 Partially ordered sets

In §§511-512 I have given long lists of definitions. It is time I filled in details of the most elementary relationships between the various concepts introduced. Here I treat some of those which can be expressed in the language of partially ordered sets. I begin with notes on cofinality and saturation, with the Erdős-Tarski theorem (513B). In this context, Galois-Tukey connections take on particularly direct forms (513D-513E);

for directed sets, we have an alternative definition of Tukey equivalence (513F). The majority of the cardinal functions defined so far on partially ordered sets are determined by their cofinal structure (513G, 513If; see also 516Ga below).

In the last third of the section (513K-513O), I discuss Tukey functions between directed sets with a special kind of topological structure, which I call 'metrizably compactly based'; the point is that for Polish metrizably compactly based directed sets, if there is any Tukey function between them, there must be one which is measurable in an appropriate sense (513O).

513A It will help to have an elementary lemma on maximal antichains.

Lemma Let P be a partially ordered set.

(a) If $Q \subseteq P$ is cofinal and $A \subseteq Q$ is an up-antichain, there is a maximal up-antichain A' in P such that $A \subseteq A' \subseteq Q$. In particular, Q includes a maximal up-antichain.

(b) If $A \subseteq P$ is a maximal up-antichain, $Q = \bigcup_{q\in A} [q,\infty[$ is cofinal with P.

proof (a) Let $A' \subseteq Q$ be maximal subject to being an up-antichain in P including A. Then for any $p \in P \setminus A'$, there are a $q \in Q$ such that $p \le q$, and an $r \in A'$ such that

$$\emptyset \ne [r,\infty[\cap [q,\infty[\subseteq [r,\infty[\cap [p,\infty[,$$

so $A' \cup \{p\}$ is not an up-antichain. But this means that A' is a maximal up-antichain in P.

Starting from $A = \emptyset$, we see that Q includes a maximal up-antichain.

(b) If $p \in P$, then either $p \in A \subseteq Q$, or $A \cup \{p\}$ is not an up-antichain, so that there is some $q \in A$ such that

$$\emptyset \ne [p,\infty[\cap [q,\infty[\subseteq Q \cap [p,\infty[.$$

513B Theorem Let P be a partially ordered set.

(a) $\operatorname{bu} P \le \operatorname{cf} P \le \#(P)$.

(b) $\operatorname{sat}^\uparrow(P)$ is either finite or a regular uncountable cardinal.

(c) $c^\uparrow(P)$ is the predecessor of $\operatorname{sat}^\uparrow(P)$ if $\operatorname{sat}^\uparrow(P)$ is a successor cardinal, and otherwise is equal to $\operatorname{sat}^\uparrow(P)$.

proof (a) To see that $\operatorname{cf} P \le \#(P)$ all we have to note is that P is a cofinal subset of itself. To see that $\operatorname{bu} P \le \operatorname{cf} P$, set $\kappa = \operatorname{cf} P$ and let $\langle p_\xi\rangle_{\xi<\kappa}$ enumerate a cofinal subset of P. Set

$$A = \{\xi : \xi < \kappa,\ p_\xi \not\le p_\eta \text{ for any } \eta < \xi\}, \quad Q = \{p_\xi : \xi \in A\}.$$

If $p \in P$ there is a least $\xi < \kappa$ such that $p \le p_\xi$, and now $\xi \in A$; so Q is cofinal with P. If $p \in P$, there is some $\xi \in A$ such that $p \le p_\xi$, and now $\{q : q \in Q,\ q \le p,\ p \not\le q\} \subseteq \{p_\eta : \eta < \xi\}$ has cardinal less than κ, so that Q witnesses that $\operatorname{bu} P \le \kappa$.

(b)(i) Set $\kappa = \operatorname{sat}^\uparrow(P)$. For $p \in P$, set $\theta(p) = \operatorname{sat}^\uparrow([p,\infty[)$. Note that if $p \in P$ and $B \subseteq P$ is any up-antichain, then $B_p = \{q : q \in B,\ [q,\infty[\cap [p,\infty[\ne \emptyset\}$ has less than $\theta(p)$ members. **P** For $q \in B_p$, choose $q' \in [q,\infty[\cap [p,\infty[$. Because B is an up-antichain, $\{q' : q \in B_p\}$ is an up-antichain and $q \mapsto q'$ is injective; so $\#(B_p) = \#(\{q' : q \in B_p\}) < \theta(p)$. **Q**

If $p \le q$ in P, any up-antichain in $[q,\infty[$ is also an up-antichain in $[p,\infty[$, so $\theta(q) \le \theta(p)$. It follows that $Q = \{p : p \in P,\ \theta(q) = \theta(p) \text{ for every } q \ge p\}$ is cofinal with P. Let $A \subseteq Q$ be a maximal up-antichain (513Aa); then $\#(A) < \kappa$.

(ii) ? Suppose, if possible, that $\kappa = \omega$.

case 1 Suppose there is a $p \in A$ such that $\theta(p) = \omega$. Then we can choose $\langle p_n\rangle_{n\in\mathbb{N}}$, $\langle q_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. $p_0 = p$. Given that $p_n \in Q$ and $\theta(p_n) = \omega$, there must be p_{n+1}, $q_n \in [p_n,\infty[$ such that $[p_{n+1},\infty[\cap [q_n,\infty[= \emptyset$; now $p_{n+1} \in Q$ and $\theta(p_{n+1}) = \omega$. Continue. At the end of the induction, $\{q_n : n \in \mathbb{N}\}$ is an infinite up-antichain in P, which is impossible.

case 2 Suppose that $\theta(p) < \omega$ for every $p \in A$. Then $n = \sum_{p\in A} \theta(p)$ is finite. Let $B \subseteq P$ be any up-antichain. For each $p \in A$, set $B_p = \{q : q \in B,\ [q,\infty[\cap [p,\infty[\ne \emptyset\}$; as noted in (i), $\#(B_p) < \theta(p)$ for every $p \in A$, so $\#(\bigcup_{p\in A} B_p) \le n$. But $B = \bigcup_{p\in A} B_p$, because A is a maximal up-antichain, so $\#(B) \le n$. As B is arbitrary, $\operatorname{sat}^{\uparrow}(P) \le n+1 < \omega$. **X**

Thus $\kappa \ne \omega$.

(iii) ? Suppose, if possible, that κ is a singular infinite cardinal. Set $\lambda = \operatorname{cf}\kappa$ and let $\langle\kappa_\xi\rangle_{\xi<\lambda}$ be a strictly increasing family of cardinals with supremum κ.

case 1 Suppose there is a $p \in Q$ such that $\theta(p) = \kappa$. Then, because $\lambda < \kappa$, there is an up-antichain $B \subseteq [p,\infty[$ with cardinal λ; enumerate B as $\langle p_\xi\rangle_{\xi<\lambda}$. For each $\xi < \lambda$, $\theta(p_\xi) > \kappa_\xi$, so there is an up-antichain $C_\xi \subseteq [p_\xi,\infty[$ with cardinal κ_ξ. Now $C = \bigcup_{\xi<\lambda} C_\xi$ is an up-antichain in P with cardinal κ, which is supposed to be impossible.

case 2 Suppose that $\theta(p) < \kappa$ for every $p \in Q$.

case 2a Suppose that $\sup_{p\in A}\theta(p) < \kappa$. Then there is an up-antichain $C \subseteq P$ such that $\#(C)$ is greater than $\max(\omega, \#(A), \sup_{p\in A}\theta(p))$. For each $p \in A$ set $C_p = \{q : q \in C,\ [q,\infty[\, \cap\, [p,\infty[\, \ne \emptyset\}$, so that $\#(C_p) < \theta(p)$. It follows that $C \ne \bigcup_{p\in A} C_p$. But if $q \in C \setminus \bigcup_{p\in A} C_p$, there is a $q' \in Q$ such that $q' \ge q$, and now $A \cup \{q'\}$ is an up-antichain in Q strictly including A, which is impossible.

case 2b Suppose that $\sup_{p\in A}\theta(p) = \kappa$. Then we can choose inductively a family $\langle p_\xi\rangle_{\xi<\lambda}$ in A such that $\theta(p_\xi) > \max(\kappa_\xi, \sup_{\eta<\xi}\theta(p_\eta))$ for each ξ; the point being that when we come to choose p_ξ, $\langle\theta(p_\eta)\rangle_{\eta<\xi}$ is a family of fewer than $\operatorname{cf}\kappa$ cardinals less than κ, so has supremum less than κ. Now all the p_ξ must be distinct. For each ξ, let $B_\xi \subseteq [p_\xi,\infty[$ be an up-antichain with cardinal κ_ξ; then $\bigcup_{\xi<\lambda} B_\xi$ is an up-antichain in P with cardinal κ, which is impossible. **X**

Thus κ cannot be a singular infinite cardinal.

(c) All we need to know is that $c^{\uparrow}(P) = \sup\{\kappa : \kappa < \operatorname{sat}^{\uparrow}(P)\}$.

Remark (b) is sometimes called the **Erdős-Tarski theorem**.

513C Cofinalities of cardinal functions We can say a little about the possible cofinalities of the cardinals which have appeared so far.

Proposition (a) Let P be a partially ordered set with no greatest member.

(i) If $\operatorname{add} P$ is greater than 2 (that is, P is upwards-directed), it is a regular infinite cardinal, and there is a family $\langle p_\xi\rangle_{\xi<\operatorname{add} P}$ in P such that $p_\eta < p_\xi$ whenever $\eta < \xi < \operatorname{add} P$, but $\{p_\xi : \xi < \operatorname{add} P\}$ has no upper bound in P.

(ii) If $\operatorname{cf} P$ is infinite, its cofinality is at least $\operatorname{add} P$.

(b) Let $\mathcal{I}$ be an ideal of subsets of a set X such that $\bigcup\mathcal{I} = X \notin \mathcal{I}$.

(i) $\operatorname{cf}(\operatorname{add}\mathcal{I}) = \operatorname{add}\mathcal{I} \le \operatorname{cf}(\operatorname{cf}\mathcal{I})$.

(ii) $\operatorname{cf}(\operatorname{non}\mathcal{I}) \ge \operatorname{add}\mathcal{I}$.

(iii) If $\operatorname{cov}\mathcal{I} = \operatorname{cf}\mathcal{I}$ then $\operatorname{cf}(\operatorname{cf}\mathcal{I}) \ge \operatorname{non}\mathcal{I}$.

proof (a)(i) By 511Hd, $\operatorname{add} P \ge \omega$; by 511He, $\operatorname{add} P < \infty$, so $\operatorname{add} P$ is an infinite cardinal. Let $\langle q_\xi\rangle_{\xi<\operatorname{add} P}$ be a family in P with no upper bound in P. Choose $\langle p_\xi\rangle_{\xi<\operatorname{add} P}$ inductively, as follows. Given p_ξ, where $\xi < \operatorname{add} P$, there is a $p'_\xi \in P$ such that $p'_\xi \not\le p_\xi$; let $p_{\xi+1}$ be an upper bound of $\{p_\xi, p'_\xi, q_\xi\}$. For a limit ordinal $\xi < \operatorname{add} P$, let p_ξ be an upper bound of $\{p_\eta : \eta < \xi\}$. This will ensure that $p_\eta < p_\xi$ whenever $\xi < \eta < \operatorname{add} P$ and that $\{p_\xi : \xi < \operatorname{add} P\}$ has no upper bound, since such a bound would have to be a bound for $\{q_\xi : \xi < \operatorname{add} P\}$.

? If $\operatorname{add} P$ is singular, express it as $\sup_{\xi<\lambda}\kappa_\xi$, where $\lambda < \operatorname{add} P$ and $\kappa_\xi < \operatorname{add} P$ for each $\xi < \lambda$. Then for each $\xi < \lambda$, $\{p_\eta : \eta < \kappa_\xi\}$ has an upper bound r_ξ in P; but now $\{r_\xi : \xi < \lambda\}$ has an upper bound in P, which is also an upper bound of $\{p_\eta : \eta < \operatorname{add} P\}$. **X** Thus $\operatorname{add} P$ is regular.

(ii) ? If $\operatorname{cf}(\operatorname{cf} P) < \operatorname{add} P$, express $\operatorname{cf} P$ as $\sup_{\xi<\lambda}\kappa_\xi$ where $\lambda < \operatorname{add} P$ and $\kappa_\xi < \operatorname{cf} P$ for each $\xi < \lambda$. Let $\langle p_\eta\rangle_{\eta<\operatorname{cf} P}$ enumerate a cofinal subset of P. Then $\{p_\eta : \eta < \kappa_\xi\}$ is never cofinal with P, so there is for each $\xi < \lambda$ a $q_\xi \in P$ such that $q_\xi \not\le p_\eta$ for every $\eta < \kappa_\xi$. But now there is a $q \in P$ which is an upper bound for $\{q_\xi : \xi < \lambda\}$, and $q \not\le p_\eta$ for any $\eta < \operatorname{cf} P$. **X**

(b)(i) Because $\bigcup\mathcal{I} = X \notin \mathcal{I}$, $\operatorname{cf}\mathcal{I}$ and $\operatorname{add}\mathcal{I}$ are both infinite, so this is just a special case of (a).

(ii) ? If $\mathrm{cf}(\mathrm{non}\,\mathcal{I}) < \mathrm{add}\,\mathcal{I}$, express $\mathrm{non}\,\mathcal{I}$ as $\sup_{\xi<\lambda}\kappa_\xi$ where $\lambda < \mathrm{add}\,\mathcal{I}$ and $\kappa_\xi < \mathrm{non}\,\mathcal{I}$ for each $\xi < \lambda$. Let $\langle x_\eta\rangle_{\eta<\mathrm{non}\,\mathcal{I}}$ enumerate a subset of X not belonging to $\mathcal{I}$. Then $I_\xi = \{x_\eta : \eta < \kappa_\xi\}$ belongs to $\mathcal{I}$ for each $\xi < \lambda$; but this means that $\bigcup_{\xi<\lambda} I_\xi = \{x_\eta : \eta < \mathrm{non}\,\mathcal{I}\}$ belongs to $\mathcal{I}$. **X**

(iii) Set $\mathrm{cf}(\mathrm{cf}\,\mathcal{I}) = \kappa$. Let $\langle\mathcal{A}_\xi\rangle_{\xi<\kappa}$ be a family of subsets of $\mathcal{I}$, all with cardinal less than $\mathrm{cf}\,\mathcal{I} = \mathrm{cov}\,\mathcal{I}$, such that $\bigcup_{\xi<\kappa}\mathcal{A}_\xi$ is cofinal with $\mathcal{I}$. Because $\#(\mathcal{A}_\xi) < \mathrm{cov}\,\mathcal{I}$, there is an $x_\xi \in X\setminus\bigcup\mathcal{A}_\xi$ for each $\xi < \kappa$. Now $\{x_\xi : \xi < \kappa\}$ is not included in any member of $\mathcal{A}_\xi$ for any ξ, so cannot belong to $\mathcal{I}$ and witnesses that $\mathrm{non}\,\mathcal{I} \le \kappa$.

513D Galois-Tukey connections between partial orders have some distinctive features which make a special language appropriate.

Definition Let P and Q be pre-ordered sets. A function $\phi : P \to Q$ is a **Tukey function** if $\phi^{-1}[B]$ is bounded above in P whenever $B \subseteq Q$ is bounded above in Q. A function $\psi : Q \to P$ is a **dual Tukey function** (also called 'cofinal function', 'convergent function') if $\psi[B]$ is cofinal with P whenever $B \subseteq Q$ is cofinal with Q.

If P and Q are pre-ordered sets, I will write '$P \preccurlyeq_{\mathrm{T}} Q$' if $(P,\le,P) \preccurlyeq_{\mathrm{GT}} (Q,\le,Q)$, and '$P \equiv_{\mathrm{T}} Q$' if $(P,\le,P) \equiv_{\mathrm{GT}} (Q,\le,Q)$; in the latter case I say that P and Q are **Tukey equivalent**. It follows immediately from 512C that $\preccurlyeq_{\mathrm{T}}$ is reflexive and transitive, and of course $P \equiv_{\mathrm{T}} Q$ iff $P \preccurlyeq_{\mathrm{T}} Q$ and $Q \preccurlyeq_{\mathrm{T}} P$.

513E Theorem Let P and Q be pre-ordered sets.

(a) If (ϕ,ψ) is a Galois-Tukey connection from $(P,\le,P)$ to $(Q,\le,Q)$ then $\phi : P \to Q$ is a Tukey function and $\psi : Q \to P$ is a dual Tukey function.

(b)(i) A function $\phi : P \to Q$ is a Tukey function iff there is a function $\psi : Q \to P$ such that (ϕ,ψ) is a Galois-Tukey connection from $(P,\le,P)$ to $(Q,\le,Q)$.

(ii) A function $\psi : Q \to P$ is a dual Tukey function iff there is a function $\phi : P \to Q$ such that (ϕ,ψ) is a Galois-Tukey connection from $(P,\le,P)$ to $(Q,\le,Q)$.

(iii) If $\psi : Q \to P$ is order-preserving and $\psi[Q]$ is cofinal with P, then ψ is a dual Tukey function.

(c) The following are equiveridical, that is, if one is true so are the others:

(i) $P \preccurlyeq_{\mathrm{T}} Q$;

(ii) there is a Tukey function $\phi : P \to Q$;

(iii) there is a dual Tukey function $\psi : Q \to P$.

(d)(i) Let $f : P \to Q$ be such that $f[P]$ is cofinal with Q and, for p, $p' \in P$, $f(p) \le f(p')$ iff $p \le p'$. Then $P \equiv_{\mathrm{T}} Q$.

(ii) Suppose that $A \subseteq P$ is cofinal with P. Then $A \equiv_{\mathrm{T}} P$

(iii) For p, $q \in P$ say that $p \equiv q$ if $p \le q$ and $q \le p$; let $\tilde{P}$ be the partially ordered set of equivalence classes in P under the equivalence relation $\equiv$ (511A, 511Ha). Then $P \equiv_{\mathrm{T}} \tilde{P}$.

(e) Suppose now that $P \preccurlyeq_{\mathrm{T}} Q$. Then

(i) $\mathrm{cf}\,P \le \mathrm{cf}\,Q$;

(ii) $\mathrm{add}\,P \ge \mathrm{add}\,Q$;

(iii) $\mathrm{sat}^\uparrow(P) \le \mathrm{sat}^\uparrow(Q)$, $c^\uparrow(P) \le c^\uparrow(Q)$;

(iv) $\mathrm{link}^\uparrow_{<\kappa}(P) \le \mathrm{link}^\uparrow_{<\kappa}(Q)$ for any cardinal κ;

(v) $\mathrm{link}^\uparrow(P) \le \mathrm{link}^\uparrow(Q)$, $d^\uparrow(P) \le d^\uparrow(Q)$.

(f) If P and Q are Tukey equivalent, then $\mathrm{cf}\,P = \mathrm{cf}\,Q$ and $\mathrm{add}\,P = \mathrm{add}\,Q$.

(g) If $\langle P_i\rangle_{i\in I}$ and $\langle Q_i\rangle_{i\in I}$ are families of pre-ordered ordered sets such that $P_i \preccurlyeq_{\mathrm{T}} Q_i$ for every i, then $\prod_{i\in I} P_i \preccurlyeq_{\mathrm{T}} \prod_{i\in I} Q_i$.

(h) If $0 < \kappa < \mathrm{add}\,P$ then $P \equiv_{\mathrm{T}} P^\kappa$. In particular, if P is upwards-directed then $P \equiv_{\mathrm{T}} P \times P$.

proof (a) To say that (ϕ,ψ) is a Galois-Tukey connection from $(P,\le,P)$ to $(Q,\le,Q)$ means just that $p \le \psi(q)$ whenever $\phi(p) \le q$. Now if $B \subseteq Q$ has an upper bound q, $\psi(q)$ is an upper bound for $\phi^{-1}[B]$; as B is arbitrary, ϕ is a Tukey function. Similarly, if $B \subseteq Q$ is cofinal, then for any $p \in P$ there is a $q \in B$ such that $\phi(p) \le q$ and $p \le \psi(q)$, so $\psi[B]$ is cofinal with P. As B is arbitrary, ψ is a dual Tukey function.

(b)(i) If $\phi : P \to Q$ is a Tukey function, then for each $q \in Q$ set $A_q = \{p : p \in P,\ \phi(p) \le q\}$. A_q must be bounded above in P; take $\psi(q)$ to be any upper bound for A_q in P. Then we see that $p \le \psi(q)$ whenever

$\phi(p) \le q$, so that (ϕ, ψ) is a Galois-Tukey connection from $(P, \le, P)$ to $(Q, \le, Q)$. Together with (a), this proves (i).

(ii) If $\psi : Q \to P$ is a dual Tukey function, then for each $p \in P$ set $B_p = \{q : q \in Q, \psi(q) \not\ge p\}$. Then $\psi[B_p]$ is not cofinal with P, so B_p cannot be cofinal with Q, and there must be a $\phi(p) \in P$ such that $\phi(p) \not\le q$ for any $q \in B_p$. Turning this round, if $\phi(p) \le q$ then $q \notin B_p$ and $p \le \psi(q)$; so (ϕ, ψ) is a Galois-Tukey connection from $(P, \le, P)$ to $(Q, \le, Q)$. Together with (a), this proves (ii).

(iii) Because $\psi[Q]$ is cofinal with P, we have a function $\phi : P \to Q$ such that $p \le \psi\phi(p)$ for every $p \in P$. Now, for any $p \in P$ and $q \in Q$,

$$\phi(p) \le q \Longrightarrow p \le \psi\phi(p) \le \psi(q),$$

so (ϕ, ψ) is a Galois-Tukey connection and ψ is a dual Tukey function.

(c) This follows immediately from (a) and (b).

(d)(i) f is a Tukey function. **P** If $A \subseteq P$ and $f[A]$ is bounded above in Q, let q be an upper bound for $f[A]$. Because $f[P]$ is cofinal with Q, there is a $p_0 \in P$ such that $q \le f(p_0)$. If now $p \in A$, we have $f(p) \le q \le f(p_0)$ so $p \le p_0$; thus A is bounded above in P. As A is arbitrary, f is a Tukey function. **Q** So $P \preccurlyeq_{\mathrm{T}} Q$.

f is a dual Tukey function. **P** If $A \subseteq P$ is cofinal with P, and $q \in Q$, there are a $p \in P$ such that $q \le f(p)$, and a $p' \in A$ such that $p \le p'$; in which case

$$q \le f(p) \le f(p') \in f[A].$$

Thus $f[A]$ is cofinal with Q; as A is arbitrary, f is a dual Tukey function. **Q** So $Q \preccurlyeq_{\mathrm{T}} P$ and $P \equiv_{\mathrm{T}} Q$.

(ii) Apply (i) to the identity map from A to P.

(iii) Apply (i) to the canonical map from P to $\tilde{P}$.

(e) This is just a restatement of the results in 512D, using the identifications listed in 512Ea. The only omission concerns cellularities, for which I have not set out a formal definition in the context of supported relations; but if $A \subseteq P$ is an up-antichain and $\phi : P \to Q$ is a Tukey function, then $\{\phi(a), \phi(a')\}$ can have no upper bound in Q for any distinct a, $a' \in A$, so $\phi[A]$ is an up-antichain in Q with the same cardinality as A, and $\#(A) \le c^{\uparrow}(Q)$. As A is arbitrary, $c^{\uparrow}(P) \le c^{\uparrow}(Q)$ and $\mathrm{sat}^{\uparrow}(P) \le \mathrm{sat}^{\uparrow}(Q)$.

(f) follows at once from (e).

(g) This is a special case of 512H.

(h) Let $Q \subseteq P^{\kappa}$ be the set of constant functions. Because $\kappa \ge 1$, Q is isomorphic to P; because $\kappa < \operatorname{add} P$, Q is cofinal with P^{κ}; so $P \cong Q \equiv_{\mathrm{GT}} P^{\kappa}$.

513F Theorem (Tukey 1940) Suppose that P and Q are upwards-directed partially ordered sets. Then P and Q are Tukey equivalent iff there is a partially ordered set R such that P and Q are both isomorphic, as partially ordered sets, to cofinal subsets of R.

proof (a) Suppose that P and Q are Tukey equivalent. Then there are Tukey functions $\phi : P \to Q$ and $\psi : Q \to P$. Set $S = (P \times \{0\}) \cup (Q \times \{1\})$, with a relation $\le$ defined by saying that

$(p, 0) \le (q, 1)$ iff (α) there is a $p' \ge p$ in P such that $\phi(p') \le q$ in Q (β) $q' \le q$ in Q whenever $q' \in Q$ and $\psi(q') \le p$ in P,

$(q, 1) \le (p, 0)$ iff (α) there is a $q' \ge q$ in Q such that $\psi(q') \le p$ in P (β) $p' \le p$ in P whenever $p' \in P$ and $\phi(p') \le q$ in Q,

$(p', 0) \le (p, 0)$ iff $p' \le p$ in P,

$(q', 1) \le (q, 1)$ iff $q' \le q$ in Q.

Of course $\le$ is reflexive. To see that it is transitive, observe that if $(p, 0) \le (q, 1) \le (\tilde{p}, 0)$ then there is a $p' \ge p$ such that $\phi(p') \le q$, and now $p' \le \tilde{p}$, so $(p, 0) \le (\tilde{p}, 0)$. Similarly, $(q, 1) \le (\tilde{q}, 1)$ whenever $(q, 1) \le (p, 0) \le (\tilde{q}, 1)$. The other cases to check are equally easy. It is *not* necessarily the case that $\le$ is antisymmetric, since it is possible to have $(p, 0) \le (q, 1) \le (p, 0)$; but we have an equivalence relation $\cong$ on

S defined by saying that $s \cong t$ if $s \le t$ and $t \le s$, and a natural partial order on the set R of equivalence classes defined by saying that $s^\bullet \le t^\bullet$ iff $s \le t$.

The map $p \mapsto (p,0)^\bullet : P \to R$ is an order-isomorphism between P and its image $\tilde{P} \subseteq R$. Now for any $q \in Q$ there is a $p \in P$ such that $(q,1) \le (p,0)$. **P** Since ϕ is a Tukey function, $A = \{p' : \phi(p') \le q\}$ must be bounded above in P; let p_0 be an upper bound for A. Now because P is upwards-directed, there is a $p \in P$ such that $p_0 \le p$ and $\psi(q) \le p$, and in this case $(q,1) \le (p,0)$. **Q** This is what we need to see that $\tilde{P}$ is cofinal with R. Similarly, Q is order-isomorphic to its canonical image in R, and this too is cofinal with R. So both P and Q are isomorphic to cofinal subsets of R.

(b) Conversely, if P and Q are both isomorphic to cofinal subsets of a partially ordered set R, then P, R and Q are all Tukey equivalent, by 513Ed.

513G We shall repeatedly want to use some elementary facts about cofinal subsets.

Proposition Let P be a pre-ordered set and Q a cofinal subset of P. Then

(a) $\operatorname{add} Q = \operatorname{add} P$;

(b) $\operatorname{cf} Q = \operatorname{cf} P$;

(c) $\operatorname{sat}^\uparrow(Q) = \operatorname{sat}^\uparrow(P)$, $c^\uparrow(Q) = c^\uparrow(P)$;

(d) $\operatorname{link}^\uparrow_{<\kappa}(Q) = \operatorname{link}^\uparrow_{<\kappa}(P)$ for any cardinal κ; in particular, $\operatorname{link}^\uparrow(Q) = \operatorname{link}^\uparrow(P)$ and $d^\uparrow(Q) = d^\uparrow(P)$;

(e) $\operatorname{bu} Q = \operatorname{bu} P$.

proof All except (e) are consequences of 513Ed and 513Ee. As for bursting numbers, every cofinal subset of Q is also cofinal with P, so $\operatorname{bu} P \le \operatorname{bu} Q$. For the reverse inequality, let Q_1 be a cofinal subset of P such that $\#(\{q : q \in Q_1,\ q \le p,\ p \not\le q\}) < \operatorname{bu} P$ for every $p \in p$. Let $\phi : Q_1 \to Q$ be any function such that $\phi(q) \ge q$ for every $q \in Q_1$, so that $\phi[Q_1]$ is cofinal with Q. If $q \in Q$, then

$$\{q' : q' \in \phi[Q_1],\ q' \le q,\ q \not\le q'\} \subseteq \{\phi(q'') : q'' \in Q_1,\ q'' \le q,\ q \not\le q''\}$$

has cardinal less than $\operatorname{bu} P$, and $\phi[Q_1]$ witnesses that $\operatorname{bu} Q \le \operatorname{bu} P$.

513H Definition Let P be a partially ordered set. Its **σ-additivity** $\operatorname{add}_\omega P$ is the smallest cardinal of any subset A of P such that $A \not\subseteq \bigcup_{q\in D} \left]-\infty,q\right]$ for any countable set $D \subseteq P$. If there is no such set, that is, if $\operatorname{cf} P \le \omega$, I write $\operatorname{add}_\omega P = \infty$.

513I Proposition Let P be a partially ordered set. As in 512F, write $p \le' A$, for $p \in P$ and $A \subseteq P$, if there is a $q \in A$ such that $p \le q$.

(a) $\operatorname{add}_\omega P = \operatorname{add}(P, \le', [P]^{\le\omega})$.

(b) $\max(\omega_1, \operatorname{add} P) \le \operatorname{add}_\omega(P)$.

(c) If $\operatorname{add}_\omega P$ is an infinite cardinal, it is regular.

(d) If $2 \le \kappa \le \operatorname{add} P$, then $(P, \le', [P]^{<\kappa}) \equiv_{\mathrm{GT}} (P, \le, P)$. So if $\operatorname{add} P > \omega$, $\operatorname{add}_\omega(P) = \operatorname{add} P$.

(e) If Q is another partially ordered set and $(P, \le', [P]^{\le\omega}) \preccurlyeq_{\mathrm{GT}} (Q, \le', [Q]^{\le\omega})$ (in particular, if $P \preccurlyeq_{\mathrm{T}} Q$) then $\operatorname{add}_\omega P \ge \operatorname{add}_\omega Q$.

(f) If $Q \subseteq P$ is cofinal with P, then $\operatorname{add}_\omega Q = \operatorname{add}_\omega P$.

(g) If $\kappa \le \operatorname{cf} P$ then $\operatorname{add}(P, \le', [P]^{<\kappa}) \le \operatorname{cf} P$. So if $\operatorname{cf} P > \omega$ then $\operatorname{add}_\omega P \le \operatorname{cf} P$.

(h) If $\operatorname{cf}(\operatorname{cf} P) > \omega$ then $\operatorname{cf}(\operatorname{cf} P) \ge \operatorname{add}_\omega P$.

proof (a) All we have to do is to disentangle the definitions in 512Ba, 512F and 513H.

(b) is immediate from the definition of $\operatorname{add}_\omega$.

(c) ? Suppose, if possible, that $\operatorname{add}_\omega P = \kappa$ where $\kappa > \max(\omega, \operatorname{cf}\kappa)$. Express κ as $\sup_{\xi<\lambda} \kappa_\xi$ where $\kappa_\xi < \kappa$ for every $\xi < \lambda = \operatorname{cf}\kappa$. Let $A \subseteq P$ be a set with cardinal κ such that $A \not\subseteq \bigcup_{q\in D} \left]-\infty,q\right]$ for any countable set $D \subseteq P$. Express A as $\bigcup_{\xi<\lambda} A_\xi$ where $\#(A_\xi) = \kappa_\xi$ for each $\xi < \lambda$. For each $\xi < \lambda$, there is a countable set $D_\xi \subseteq P$ such that $A_\xi \subseteq \bigcup_{q\in D_\xi} \left]-\infty,q\right]$. Set $B = \bigcup_{\xi<\lambda} D_\xi$; then $\#(B) \le \lambda < \kappa$, so there is a countable set $D \subseteq P$ such that $B \subseteq \bigcup_{q\in D} \left]-\infty,q\right]$. But now $A \subseteq \bigcup_{q\in D} \left]-\infty,q\right]$. **X**

(d) By 512Gc, $(P, \le', [P]^{<\kappa}) \preccurlyeq_{\mathrm{GT}} (P, \le, P)$. In the other direction, because $\kappa \le \operatorname{add} P$, we have a function $\psi : [P]^{<\kappa} \to P$ such that $I \subseteq \left]-\infty, \psi(I)\right]$ for every $I \in [P]^{<\kappa}$; so if we set $\phi(p) = p$ for $p \in P$, (ϕ, ψ) will be a Galois-Tukey connection from $(P, \le, P)$ to $(P, \le', [P]^{<\kappa})$, and $(P, \le, P) \preccurlyeq_{\mathrm{GT}} (P, \le', [P]^{<\kappa})$.

Now if $\operatorname{add} P > \omega$,

$$\operatorname{add}_\omega P = \operatorname{add}(P, \leq', [P]^{\leq\omega}) = \operatorname{add}(P, \leq, P) = \operatorname{add} P.$$

(e) Use (a) with 512Db and 512Gb.

(f) Use (e) and 513Ed.

(g) Let $Q \subseteq P$ be a cofinal subset of P with cardinal $\operatorname{cf} P$. If $A \subseteq P$ is such that every member of Q is dominated by a member of A, then A also is cofinal, so $\#(A) \geq \kappa$; thus Q witnesses that $\operatorname{add}(P, \leq', [P]^{<\kappa}) \leq \operatorname{cf} P$. Putting $\kappa = \omega_1$ we see that if $\operatorname{cf} P > \omega$ then $\operatorname{add}_\omega P \leq \operatorname{cf} P$.

(h) ? If $\omega < \operatorname{cf}(\operatorname{cf} P) = \lambda < \operatorname{add}_\omega P$ let $Q \subseteq P$ be a cofinal set with cardinal $\operatorname{cf} P$ and express Q as $\bigcup_{\xi<\lambda} Q_\xi$ where $\#(Q_\xi) < \operatorname{cf} P$ and $Q_\xi \subseteq Q_\eta$ whenever $\xi \leq \eta < \lambda$. For each $\xi < \lambda$, Q_ξ cannot be cofinal with P, so there is a $p_\xi \in P$ such that $p_\xi \not\leq q$ for any $q \in Q_\xi$. Now $A = \{p_\xi : \xi < \lambda\}$ has cardinal less than $\operatorname{add}_\omega P$, so there is a countable set $D \subseteq P$ such that $A \subseteq \bigcup_{r\in D}]-\infty, r]$. For each $r \in D$ there is a $q_r \in Q$ such that $r \leq q_r$; let $\xi_r < \lambda$ be such that $q_r \in Q_{\xi_r}$. Because λ is uncountable and regular (being the cofinality of a cardinal), $\zeta = \sup_{r\in D} \xi_r$ is less than λ, and $q_r \in Q_\zeta$ for every $r \in D$. But now there is an $r \in D$ such that $p_\zeta \leq r \leq q_r \in Q_\zeta$, contrary to the choice of p_ζ. **X**

Remark The point of (b) and (d) here is that there are significant cases in which $\operatorname{add} P < \omega_1 < \operatorname{add}_\omega P$.

***513J Cofinalities of products** It is easy to find the additivity of a product of partially ordered sets (511Hg). Calculating the cofinality of a product of partially ordered sets is surprisingly difficult, and there are some extraordinary results in this area. (See Burke & Magidor 90; there is a taster in 542J below.) Here I will give just one special fact which will be useful.

Proposition Suppose that the generalized continuum hypothesis is true. Let $\langle P_i \rangle_{i\in I}$ be a family of non-empty partially ordered sets with product P. Set

$$\kappa = \#(\{i : i \in I, \operatorname{cf} P_i > 1\}), \quad \lambda = \sup_{i\in I} \operatorname{cf} P_i.$$

Then

(i) if κ and λ are both finite, $\operatorname{cf} P$ is the cardinal product $\prod_{i\in I} \operatorname{cf} P_i$;

(ii) if $\lambda > \kappa$ and there is some $\gamma < \lambda$ such that $\operatorname{cf} \lambda > \#(\{i : i \in I, \operatorname{cf} P_i > \gamma\})$, then $\operatorname{cf} P = \lambda$;

(iii) otherwise, $\operatorname{cf} P = \max(\kappa^+, \lambda^+)$.

proof (a) For each $i \in I$, let $Q_i \subseteq P_i$ be a cofinal set with cardinal $\operatorname{cf} P_i$. Then $Q = \prod_{i\in I} Q_i$ is cofinal with $P = \prod_{i\in I} P_i$, so $\operatorname{cf} P \leq \#(Q)$. If $\lambda < \omega$, then every Q_i must be just the set of maximal elements of P_i, so Q is the set of maximal elements of P, and $\operatorname{cf} P = \#(Q)$. This deals with case (i).

(b) $\operatorname{cf} P > \kappa$. **P** Set $J = \{i : i \in I, \operatorname{cf} P_i > 1\}$. If $\langle p_i \rangle_{i\in J}$ is any family in P, then we can choose $q \in P$ such that $q(i) \not\leq p_i(i)$ for every $i \in J$; accordingly $\{p_i : i \in J\}$ cannot be cofinal with P. **Q** So if $\max(\omega, \lambda) \leq \kappa$,

$$\operatorname{cf} P \geq \kappa^+ = 2^\kappa$$

(because we are assuming the generalized continuum hypothesis)

$$= 2^{\max(\kappa,\lambda)} \geq \#(\mathcal{P}\{(i, q) : i \in J, q \in Q_i\}) \geq \#(\prod_{i\in J} Q_i) = \#(Q)$$

(because $\#(Q_i) = 1$ for $i \in I \setminus J$)

$$\geq \operatorname{cf} P,$$

and $\operatorname{cf} P = \kappa^+ = \max(\kappa^+, \lambda^+)$, as required by (iii).

(c) Note that $\operatorname{cf} P \geq \lambda$, because if $R \subseteq P$ is cofinal with P then $\{p(i) : p \in R\}$ is cofinal with P_i for each i. So if κ is finite and λ is infinite,

$$\lambda \leq \operatorname{cf} P \leq \#(Q) \leq \max(\omega, \sup_{i\in J} \#(Q_i)) = \lambda$$

and $\operatorname{cf} P = \lambda$, as required by (ii).

(d) If λ is infinite and $\kappa < \operatorname{cf}\lambda$ then every function from κ to λ is bounded above in λ. So

$$\lambda \le \operatorname{cf} P \le \#(Q) \le \#(\lambda^\kappa)$$

(where λ^κ, for once, denotes the set of functions from κ to λ)

$$= \#(\bigcup_{\zeta<\lambda} \zeta^\kappa) \le \max(\omega, \lambda, \sup_{\zeta<\lambda} \#(\zeta^\kappa)) \le \max(\omega, \lambda, \sup_{\zeta<\lambda} 2^{\max(\zeta,\kappa)}) = \lambda,$$

again using GCH. Thus in this case also we have $\operatorname{cf} P = \lambda$, as required by (ii).

(e) So we are left with the case in which $\operatorname{cf}\lambda = \theta \le \kappa < \lambda$. Let $\langle\lambda_\eta\rangle_{\eta<\theta}$ be a family of cardinals less than λ with supremum λ.

(α) Suppose that we are in case (iii), that is, $\#(\{i : i \in I,\ \operatorname{cf} P_i > \gamma\}) \ge \theta$ for every $\gamma < \lambda$. Then $\operatorname{cf} P > \lambda$. **P** We can choose $\langle i(\eta)\rangle_{\eta<\theta}$ inductively in I so that $\operatorname{cf} P_{i(\eta)} > \lambda_\eta$ and $i(\eta) \ne i(\xi)$ when $\xi < \eta < \theta$. If $\langle p_\xi\rangle_{\xi<\lambda}$ is any family in P, we can find $q \in P$ such that $q(i(\eta)) \not\le p_\xi(i(\eta))$ for any $\eta < \theta$ and $\xi < \lambda_\eta$, so that $q \not\le p_\xi$ for any $\xi < \lambda$. As $\langle p_\xi\rangle_{\xi<\lambda}$ is arbitrary, $\operatorname{cf} P > \lambda$. **Q** Now

$$\operatorname{cf} P \le \#(Q) \le \#(\lambda^\kappa) \le 2^{\max(\kappa,\lambda)} = \lambda^+ \le \operatorname{cf} P,$$

so $\operatorname{cf} P = \lambda^+ = \max(\kappa^+, \lambda^+)$, as required.

(β) Otherwise, we are in case (ii), and there is a cardinal $\gamma < \lambda$ such that $\#(K) < \theta$, where $K = \{i : i \in I,\ \operatorname{cf} P_i > \gamma\}$. Then $\sup_{i\in K} \operatorname{cf} P_i = \lambda$, so (d) tells us that $\operatorname{cf}(\prod_{i\in K} P_i) = \lambda$. On the other hand,

$$\#(\prod_{i\in I\setminus K} \operatorname{cf} P_i) \le 2^{\max(\gamma,\kappa)} \le \lambda.$$

Since we can identify P with the product of $\prod_{i\in K} P_i$ and $\prod_{i\in I\setminus K} P_i$, $\operatorname{cf} P \le \#(\lambda \times \lambda) = \lambda$. But we noted in (c) that $\operatorname{cf} P \ge \lambda$, so $\operatorname{cf} P = \lambda$, as required. This completes the proof.

***513K** I remarked in the notes to §512 that Galois-Tukey correspondences are not required to have any special properties, and of course the same is true of Tukey functions. But it is also the case that the 'natural' Tukey functions arising in Chapter 52 can in many cases be derived from Borel measurable functions between Polish spaces. I now present some ideas taken from SOLECKI & TODORČEVIĆ 04 which may be regarded as a partial explanation of the phenomenon.

Definition I will say that a **metrizably compactly based directed set** is a partially ordered set P endowed with a metrizable topology such that

(i) $p \vee q = \sup\{p, q\}$ is defined for all $p, q \in P$, and $\vee : P \times P \to P$ is continuous;

(ii) $\{p : p \le q\}$ is compact for every $q \in P$;

(iii) every convergent sequence in P has a subsequence which is bounded above.

In this context, I will say that P is 'separable' or 'analytic' if it is separable, or analytic, in the topological sense.

I leave it to you to check that many significant partially ordered sets are compactly based in the sense defined here (513Xj-513Xn, 513Yg).

***513L Proposition** Let P be a metrizably compactly based directed set.

(a) The ordering of P is a closed subset of $P \times P$.

(b) P is Dedekind complete.

(c)(i) A non-decreasing sequence in P has an upper bound iff it is topologically convergent, and in this case its supremum is its limit.

(ii) A non-increasing sequence in P converges topologically to its infimum.

(d) If $p \in P$ and $\langle p_i\rangle_{i\in\mathbb{N}}$ is a sequence in P, then $\langle p_i\rangle_{i\in\mathbb{N}}$ is topologically convergent to p iff for every $I \in [\mathbb{N}]^\omega$ there is a $J \in [I]^\omega$ such that $p = \inf_{n\in\mathbb{N}} \sup_{i\in J\setminus n} p_i$.

(e) Suppose that $p \in P$ and a double sequence $\langle p_{ni}\rangle_{n,i\in\mathbb{N}}$ in P are such that $\lim_{i\to\infty} p_{ni} = p_n$ is defined in P and less than or equal to p for each n. Then there is a $q \in P$ such that $\{i : p_{ni} \le q\}$ is infinite for every $n \in \mathbb{N}$.

proof Let ρ be a metric on P inducing its topology.

(a) We have only to observe that $\{(p,q) : p \le q\} = \{(p,q) : p \vee q = q\}$.

(b) Suppose that $A \subseteq P$ is non-empty and bounded above. Let B be the set of upper bounds of A. Then $\mathcal{E} = \{[p,q] : p \in A,\ q \in B\}$ is a non-empty family of compact sets with the finite intersection property, because any non-empty finite subset of A has a least upper bound. So there is a $q_0 \in \bigcap \mathcal{E}$ and now q_0 must be the supremum of A.

(c)(i) Suppose that $\langle p_i \rangle_{i \in \mathbb{N}}$ is a non-decreasing sequence in P. (α) If it has a topological limit p, then

$$p \vee p_j = \lim_{i \to \infty} p_i \vee p_j = \lim_{i \to \infty} p_i = p$$

for each j, so p is an upper bound for $\{p_i : i \in \mathbb{N}\}$; while if q is an upper bound for $\{p_i : i \in \mathbb{N}\}$ then $p \le q$ by (a). Thus $p = \sup_{i \in \mathbb{N}} p_i$. (β) If $\{p_i : i \in \mathbb{N}\}$ is bounded above, then it has a least upper bound p, by (b). Now $]-\infty, p]$ is compact, therefore sequentially compact, and every subsequence of $\langle p_i \rangle_{i \in \mathbb{N}}$ has a convergent sub-subsequence; by (α), the limit of this sub-subsequence is always its supremum, which must be p; so $\langle p_i \rangle_{i \in \mathbb{N}}$ itself converges to p.

(ii) Suppose that $\langle p_i \rangle_{i \in \mathbb{N}}$ is a non-increasing sequence in P. Then it lies in the compact set $]-\infty, p_0]$ so has a convergent subsequence $\langle p'_i \rangle_{i \in \mathbb{N}}$ with limit p say. As in (i) just above,

$$p \vee p'_j = \lim_{i \to \infty} p'_i \vee p'_j = \lim_{i \to \infty} p'_j = p'_j$$

for each j, so p is a lower bound for $\{p'_i : i \in \mathbb{N}\}$; while if q is a lower bound for $\{p'_i : i \in \mathbb{N}\}$ then $q \le p$ by (a). Thus $p = \inf_{i \in \mathbb{N}} p'_i = \inf_{i \in \mathbb{N}} p_i$. What this shows is that $\inf_{i \in \mathbb{N}} p_i$ is the only cluster point of $\langle p_i \rangle_{i \in \mathbb{N}}$ and is therefore its topological limit.

(d)(i) Suppose that $p = \lim_{i \to \infty} p_i$. Note first that if $q \in P$ then

$$\limsup_{i \to \infty} \rho(q \vee p_i, p) \le \lim_{i \to \infty} \rho(q \vee p_i, q \vee p) + \rho(q \vee p, p) = \rho(q \vee p, p),$$

$$\limsup_{i \to \infty} \rho(p \vee q \vee p_i, p) \le \rho((p \vee q) \vee p, p) = \rho(q \vee p, p)$$

because $\vee$ is continuous. Now let $I \subseteq \mathbb{N}$ be infinite. By 513K(iii), there is an infinite $I' \subseteq I$ such that $\{p_i : i \in I'\}$ is bounded above. We can choose inductively a strictly increasing sequence $\langle i_n \rangle_{n \in \mathbb{N}}$ in I' such that

$$\rho(\sup_{j \le n \le k} p_{i_n}, p) < 2^{-j}, \qquad \rho(p \vee \sup_{j \le n \le k} p_{i_n}, p) < 2^{-j}$$

whenever $j \le k$ in $\mathbb{N}$. Set $J = \{i_n : n \in \mathbb{N}\}$; then $\rho(\sup_{j \in J, m \le j \le k} p_j, p) < 2^{-m}$ whenever $m \le k \in \mathbb{N}$ and $[m,k]$ meets J. For each m, $q_m = \sup_{j \in J \setminus m} p_j$ is defined in P, by (b) above; moreover, (c-i) tells us that $q_m = \lim_{k \to \infty} \sup_{j \in J, m \le j \le k} p_j$ so

$$\rho(q_m, p) = \lim_{k \to \infty} \rho(\sup_{j \in J, m \le j \le k} p_j, p) \le 2^{-m}.$$

But this means that p is the topological limit of the non-increasing sequence $\langle q_m \rangle_{m \in \mathbb{N}}$ and must be $\inf_{m \in \mathbb{N}} q_m$. Thus $\langle p_i \rangle_{i \in \mathbb{N}}$ satisfies the condition proposed.

(ii) Now suppose that for every $I \in [\mathbb{N}]^\omega$ there is a $J \in [I]^\omega$ such that $p = \inf_{n \in \mathbb{N}} \sup_{i \in J \setminus n} p_i$. Then any convergent subsequence of $\langle p_i \rangle_{i \in \mathbb{N}}$ has limit p. **P** Suppose the subsequence is $\langle p_{i_n} \rangle_{n \in \mathbb{N}}$ where $\langle i_n \rangle_{n \in \mathbb{N}}$ is strictly increasing. Set $I = \{i_n : n \in \mathbb{N}\}$. Then we must have an infinite $J \subseteq \mathbb{N}$ such that $p = \inf_{m \in \mathbb{N}} \sup_{k \in J \setminus m} p_{i_k}$. Now (i) tells us that we also have an infinite $K \subseteq J$ such that the limit p' of $\langle p_{i_n} \rangle_{n \in \mathbb{N}}$ is $\inf_{m \in \mathbb{N}} \sup_{k \in K \setminus m} p_{i_k}$. Since $\sup_{k \in K \setminus m} p_{i_k} \le \sup_{k \in J \setminus m} p_{i_k}$ for every m, $p' \le p$. On the other hand, we also have an infinite $L \subseteq K$ such that $p = \inf_{m \in \mathbb{N}} \sup_{k \in L \setminus m} p_{i_k}$; so that $p \le p'$ and $p = p'$. **Q**

Since the condition tells us also that every subsequence of $\langle p_i \rangle_{i \in \mathbb{N}}$ has a sub-subsequence which is bounded above, and therefore has a convergent sub-sub-subsequence, p is actually the limit of $\langle p_i \rangle_{i \in \mathbb{N}}$.

(e) Note first that if $\langle q_i \rangle_{i \in \mathbb{N}}$ is a sequence in P converging to $q^* \in P$, and $\epsilon > 0$, there is a $q' \in P$ such that $\rho(q', q^*) \le \epsilon$ and $\{i : q_i \le q'\}$ is infinite. **P** By (d), there is an infinite $J \subseteq \mathbb{N}$ such that $q^* = \inf_{n \in \mathbb{N}} \sup_{i \in J \setminus n} q_i$; by (c-ii), we can take $q' = \sup_{i \in J \setminus n} q_i$ for some n. **Q**

For m, $i \in \mathbb{N}$, set $q_{mi} = p \vee \sup_{n \le m} p_{ni}$. Then $\lim_{i \to \infty} q_{mi} = p \vee \sup_{n \le m} p_n = p$ for each m. We can therefore find, for each $m \in \mathbb{N}$, a $q_m \in P$ such that $\rho(q_m, p) \le 2^{-m}$ and $\{i : q_{mi} \le q_m\}$ is infinite. As

$\langle q_m\rangle_{m\in\mathbb{N}} \to p$, there is a $q \in P$ such that $\{m : q_m \le q\}$ is infinite. Now, for any $n \in \mathbb{N}$, there is an $m \ge n$ such that $q_m \le q$, so that

$$\{i : p_{ni} \le q\} \supseteq \{i : q_{mi} \le q_m\}$$

is infinite.

***513M Proposition** Let P be a separable metrizably compactly based directed set, and give the set $\mathcal{C}$ of closed subsets of P its Vietoris topology. Let $\mathcal{K}_b \subseteq \mathcal{C}$ be the family of non-empty compact subsets of P which are bounded above in P. Then $K \mapsto \sup K : \mathcal{K}_b \to P$ is Borel measurable.

proof Writing $\mathcal{K}$ for the family of non-empty compact subsets of P, we have a sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ of Borel measurable functions from $\mathcal{K}$ to P such that $K = \overline{\{f_n(K) : n \in \mathbb{N}\}}$ for every $K \in \mathcal{K}$ (5A4Dc). Set $g_n(K) = \sup_{i\le n} f_i(K)$ for each $K \in \mathcal{K}$ and $n \in \mathbb{N}$; because P is separable, every g_n is Borel measurable (put 418Bd and 418Ac together). For $K \in \mathcal{K}_b$, $\langle g_n(K)\rangle_{n\in\mathbb{N}}$ is a non-decreasing bounded sequence, so converges to $g(K) \in P$, by 513L(c-i); now $g : \mathcal{K}_b \to P$ is Borel measurable (418Ba). Since $\{q : q \le g(K)\}$ is a closed set including $\{f_i(K) : i \in \mathbb{N}\}$, it includes K, and $g(K)$ is an upper bound for K; because $g(K) = \sup_{i\in\mathbb{N}} f_i(K)$, $g(K) = \sup K$. So we have the result.

***513N Lemma** Let P and Q be non-empty metrizably compactly based directed sets of which P is separable, and $\phi : P \to Q$ a Tukey function. Set

$$R = \overline{\{(q,p) : p \in P,\, q \in Q,\, \phi(p) \le q\}}.$$

Then

(a) $R[\,]-\infty,q]\,]$ is bounded above in P for every $q \in Q$;

(b) $R \subseteq Q \times P$ is usco-compact.

proof (a) Because P is non-empty, we need consider only the case in which $R[\,]-\infty,q]\,]$ is non-empty. Let $\langle p_n\rangle_{n\in\mathbb{N}}$ be a sequence running over a dense subset of $R[\,]-\infty,q]\,]$. For each $n \in \mathbb{N}$ we have sequences $\langle p_{ni}\rangle_{i\in\mathbb{N}}$ in P and $\langle q_{ni}\rangle_{n\in\mathbb{N}}$ in Q such that $\phi(p_{ni}) \le q_{ni}$, $\lim_{i\to\infty} p_{ni} = p_n$ and $\lim_{i\to\infty} q_{ni} = q_n \le q$. By 513Le, there is a $q' \in Q$ such that $I_n = \{i : q_{ni} \le q'\}$ is infinite for every $n \in \mathbb{N}$. Because ϕ is a Tukey function, there is a $p' \in P$ such that $p_{ni} \le p'$ whenever $n \in \mathbb{N}$ and $i \in I_n$. But now $\{p : p \le p'\}$ is closed, so it contains every p_n and p' is an upper bound for $R[\,]-\infty,q]\,]$.

(b) In particular, for any $q \in Q$, $R[\{q\}]$ is bounded above in P, therefore relatively compact; since R is closed, every $R[\{q\}]$ is closed and therefore compact. Now suppose that $F \subseteq P$ is closed and that $\langle q_n\rangle_{n\in\mathbb{N}}$ is a sequence in $R^{-1}[F]$ converging to $q \in Q$. Then there is a $q^* \in Q$ such that $J = \{n : q_n \le q^*\}$ is infinite. For $n \in J$, let $p_n \in F$ be such that $(q_n,p_n) \in R$. Then $\{p_n : n \in J\}$ is included in the order-bounded set $R^{-1}[\,]-\infty,q^*]\,]$, so is relatively compact, and $\langle p_n\rangle_{n\in\mathbb{N}}$ has a cluster point p say. Of course $p \in F$; also (q,p) is a cluster point of $\langle(q_n,p_n)\rangle_{n\in\mathbb{N}}$, so belongs to $\overline{R} = R$, and $q \in R^{-1}[F]$. As q is arbitrary, $R^{-1}[F]$ is closed; as F is arbitrary, R is usco-compact.

***513O Theorem** (Solecki & Todorčević 04) Let P and Q be metrizably compactly based directed sets such that $P \preccurlyeq_{\mathrm{T}} Q$. Let Σ be the σ-algebra of subsets of P generated by the Souslin-F sets.

(a) If P is separable, there is a Borel measurable dual Tukey function $\psi : Q \to P$.

(b) If P is separable and Q is analytic, there is a Σ-measurable Tukey function $\phi : P \to Q$.

proof If either P or Q is empty, so is the other, and the result is trivial; suppose that they are non-empty.

(a) Let $\phi_0 : P \to Q$ be a Tukey function, and set $R = \overline{\{(q,p) : p \in P,\, q \in Q,\, \phi_0(p) \le q\}}$, so that R is usco-compact (513N). Let $\mathcal{C}$ be the set of closed subsets of P with its Vietoris topology; then $q \mapsto R[\{q\}]$ is Borel measurable (5A4Db). Since $\emptyset$ is an isolated point of $\mathcal{C}$, $Q_0 = \{y : R[\{y\}] \ne \emptyset\}$ is a Borel set in Q. If $q \in Q_0$, then $R[\{q\}]$ is a non-empty compact subset of P which is bounded above (513Na), so 513M tells us that $q \mapsto \sup R[\{q\}] : Q_0 \to P$ is Borel measurable. Fix any $p_0 \in P$ and set $\psi(q) = \sup R[\{q\}]$ if $q \in Q_0$, p_0 if $q \in Q \setminus Q_0$. Then ψ is Borel measurable. If $p \in P$, $q \in Q$ and $\phi_0(p) \le q$, then $(q,p) \in R$, $p \in R[\{q\}]$ and $p \le \psi(q)$; thus (ϕ_0,ψ) is a Galois-Tukey connection and ψ is a dual Tukey function.

(b) $R \subseteq Q \times P$ is a closed set (422Da) and $R[Q] = P$. Because P and Q are separable and metrizable, R can be obtained by Souslin's operation from products of closed sets. By the von Neumann-Jankow selection theorem (423N), there is a Σ-measurable $\phi : P \to Q$ such that $(\phi(p), p) \in R$ for every $p \in P$. If $q \in Q$, then $\{p : \phi(p) \le q\} \subseteq R[\,]-\infty, q]\,]$ is bounded above in P, so ϕ is a Tukey function.

513P The last result in this section is entirely unconnected with the rest, and is a standard trick; but it will be useful later and contains an implicit challenge (513Yj).

Lemma Let P and Q be non-empty partially ordered sets, and suppose that (i) every non-decreasing sequence in P has an upper bound in P (ii) there is no strictly increasing family $\langle q_\xi \rangle_{\xi<\omega_1}$ in Q. Let $f : P \to Q$ be an order-preserving function. Then there is a $p \in P$ such that $f(p') = f(p)$ whenever $p' \in P$ and $p' \ge p$.

proof ? Otherwise, we can choose $\langle p_\xi \rangle_{\xi<\omega_1}$ inductively so that

$p_0 \in P$,

$p_{\xi+1} \ge p_\xi$ and $f(p_{\xi+1}) > f(p_\xi)$ for every $\xi < \omega_1$,

p_ξ is an upper bound for $\{p_\eta : \eta < \xi\}$ for every non-zero limit ordinal $\xi < \omega_1$.

But now $\langle f(p_\xi) \rangle_{\xi<\omega_1}$ is strictly increasing, which is impossible. **X**

513X Basic exercises (a) Let P be a partially ordered set, and $\mathcal{A}$ the family of subsets of P which are not cofinal with P. Show that $(\mathcal{A}, \not\ni, P) \preccurlyeq_{\mathrm{GT}} (P, \le, P)$. Explain the relation of this fact to 511Xj, 513C(a-ii) and 513Xb.

(b) Let P be a partially ordered set such that $\operatorname{bu} P \ge \omega$. Show that $\operatorname{cf}(\operatorname{bu} P) \ge \operatorname{add} P$.

(c) Let P, Q and R be partially ordered sets. (i) Show that if $\phi_1 : P \to Q$ and $\phi_2 : Q \to R$ are Tukey functions, then $\phi_2\phi_1 : P \to R$ is a Tukey function. (ii) Show that if $\psi_1 : P \to Q$ and $\psi_2 : Q \to R$ are dual Tukey functions, then $\psi_2\psi_1 : P \to R$ is a dual Tukey function.

(d) Let P and Q be partially ordered sets, and $g : Q \to P$ a function. Show that g is a dual Tukey function iff for every $p_0 \in P$ there is a $q_0 \in Q$ such that $g(q) \ge p_0$ for every $q \ge q_0$.

(e)(i) Show that if P, Q are partially ordered sets, P is Dedekind complete and $P \preccurlyeq_{\mathrm{T}} Q$, there is an order-preserving dual Tukey function from Q to P. (ii) Set $P = [\{0,1,2\}]^{\le 2}$ and $Q = [\{0,1,2\}]^2$. Show that there is no order-preserving Tukey function from P to Q.

(r) Let P be a partially ordered set. Show that if $\kappa \ge \operatorname{cf} P$ and $\lambda \le \operatorname{add} P$ then $P \preccurlyeq_{\mathrm{T}} [\kappa]^{<\lambda}$.

(f) Suppose that P is a partially ordered set and $\operatorname{add} P = \operatorname{cf} P = \kappa \ge \omega$. Show that $P \equiv_{\mathrm{T}} \kappa$.

(g) Prove (a)-(d) of 513G directly, without mentioning Tukey functions or Galois-Tukey connections.

>(h)(i) Show that if P and Q are two partially ordered sets such that $\operatorname{sat}^{\uparrow}(P) = \#(P)^+ = \#(Q)^+ = \operatorname{sat}^{\uparrow}(Q)$ then P and Q are Tukey equivalent. (*Hint*: if $B \subseteq Q$ is an up-antichain, any injective function $\phi : P \to B$ is a Tukey function from P to Q.) (ii) Give an example of such a pair P, Q such that $\mathfrak{m}(P) \ne \mathfrak{m}(Q)$ and $\operatorname{bu} P \ne \operatorname{bu} Q$.

(i) Let P, Q_1, Q_2 be partially ordered sets such that $(P, \le, P) \preccurlyeq_{\mathrm{GT}} (Q_1, \le, Q_1) \ltimes (Q_2, \le, Q_2)$ (definition: 512I). Show that $\operatorname{add}_\omega P \ge \min(\operatorname{add}_\omega Q_1, \operatorname{add}_\omega Q_2)$.

(j) Show that $\mathbb{N}^{\mathbb{N}}$, with its usual ordering and topology, is a metrizably compactly based directed set.

(k) Let X be a set, $1 \le p < \infty$ and P the positive cone $(\ell^p(X))^+$ of the Banach lattice $\ell^p(X)$, with the topology induced by the norm of $\ell^p(X)$. Show that P is a metrizably compactly based directed set.

(l) Let $\mathcal{Z}$ be the ideal of subsets of $\mathbb{N}$ with asymptotic density zero, with its natural ordering and the topology induced by the metric $(a, b) \mapsto \sup_{n\ge1} \frac{1}{n}\#((a \triangle b) \cap n)$. Show that $\mathcal{Z}$ is a metrizably compactly based directed set.

(m) Let X be a metrizable space, and $\mathcal{F}$ the set of nowhere dense compact subsets of X. Show that $\mathcal{F}$, with its natural ordering and its Vietoris topology, is a metrizably compactly based directed set. (*Hint*: use a Hausdorff metric.)

(n) Let X be a metrizable space, $\mathcal{K}$ the family of compact subsets of X, and $\mathcal{I}$ a σ-ideal of subsets of X. Show that $\mathcal{K}\cap\mathcal{I}$, with the natural partial order and the Vietoris topology, is a metrizably compactly based directed set.

(o) Let $\langle P_i\rangle_{i\in I}$ be a countable family of metrizably compactly based directed sets, with product P. Show that P is metrizably compactly based.

(p) Let P be a metrizably compactly based directed set. (i) Show that P is a lattice iff it has a least element. (ii) Show that if we adjoin a least element $-\infty$ to P as an isolated point, $P\cup\{-\infty\}$ is a metrizably compactly based directed set.

(q) Let $\langle P_i\rangle_{i\in I}$ be a family of partially ordered sets, and P their product. (i) Show that $\operatorname{cf} P$ is at most the cardinal product $\prod_{i\in I}\operatorname{cf} P_i$, with equality if I is finite. (ii) Show that if $P\ne\emptyset$ then $\sup_{i\in I}\operatorname{cf} P_i\le\operatorname{cf} P$. (iii) Show that if $P\ne\emptyset$ and for every $i\in I$ there is a $j\in I$ such that $\operatorname{cf} P_i<\operatorname{cf} P_j$, then $\sup_{i\in I}\operatorname{cf} P_i<\operatorname{cf} P$.

513Y Further exercises (a) Show that for a cardinal κ, there is a partially ordered set P such that $c^{\uparrow}(P)=\operatorname{sat}^{\uparrow}(P)=\kappa$ iff κ is weakly inaccessible. (*Hint*: for such a κ, take X to be a product of discrete spaces, one of each cardinality less than κ, and P the family of proper closed subsets of X.)

(b) Show that for any cardinal $\kappa>0$ there is a supported relation (A,R,B) such that $\operatorname{sat}(A,R,B)=\kappa$.

(c) For a non-empty upwards-directed set P, a topological space X and $A\subseteq X$, write $\operatorname{cl}_P(A)$ for the set of points $x\in X$ for which there is a function $f:P\to A$ such that $x\in\overline{f[C]}$ for every cofinal set $C\subseteq P$; equivalently, $f[[\mathcal{F}^{\uparrow}(P)]]\to x$, where $\mathcal{F}^{\uparrow}(P)$ is the filter on P generated by sets of the form $[p,\infty[$ as p runs over P. Now let P and Q be upwards-directed sets. Show that $P\preccurlyeq_{\mathrm{T}} Q$ iff $\operatorname{cl}_P(A)\subseteq\operatorname{cl}_Q(A)$ for any subset A of any topological space.

(d) For partially ordered sets P and Q, say that $P\approx Q$ if there is a partially ordered set R into which both P and Q can be embedded as cofinal subsets. (i) Show that $P\approx Q$ iff there is a Galois-Tukey connection (ϕ,ψ) from $(P,\le,P)$ to $(Q,\le,Q)$ such that (ψ,ϕ) is a Galois-Tukey connection from $(Q,\le,Q)$ to $(P,\le,P)$. (ii) Show that if P, R and R' are partially ordered sets such that P can be embedded as a cofinal subset into both R and R', then $R\approx R'$. (iii) Show that $\approx$ is an equivalence relation on the class of partially ordered sets. (iv) Show that if $\mathcal{P}$ is a set of partially ordered sets, and $P\approx P'$ for all P, $P'\in\mathcal{P}$, then there is a partially ordered set R such that every member of $\mathcal{P}$ can be embedded into R as a cofinal set. (v) Give an example of partially ordered sets P and Q such that $P\equiv_{\mathrm{T}} Q$ but $P\not\approx Q$.

(e) For a cardinal κ and a supported relation (A,R,B) set $\operatorname{add}_{<\kappa}(A,R,B)=\operatorname{add}(A,R',[B]^{<\kappa})$. Which of the ideas of 513I can be extended to the general context?

(f) Show that there are two families $\langle(A_i,R_i,B_i)\rangle_{i\in I}$ and $\langle(C_i,S_i,D_i)\rangle_{i\in I}$ of supported relations, with simple products (A,R,B) and (C,S,D) respectively, such that $\operatorname{cov}(A_i,R_i,B_i)=\operatorname{cov}(C_i,S_i,D_i)$ for each i, but $\operatorname{cov}(A,R,B)\ne\operatorname{cov}(C,S,D)$. (*Hint*: examine the proof of 513J.)

(g) Let X be a set and U a solid linear subspace of $\mathbb{R}^X$ with an order-continuous norm under which it is a Banach lattice. Show that its positive cone, with its norm topology, is a metrizably compactly based directed set.

(h) Explore possible definitions of 'compactly based' partially ordered set which do not require the topology to be metrizable.

(i) Let P be an analytic metrizably compactly based directed set. Show that P is Polish. (*Hint*: Solecki & Todorčević 04.)

(j) For partially ordered sets P and Q, say that $Q\maltese P$ if for every order-preserving $f:P\to Q$ there is a $p\in P$ such that $f(p')=f(p)$ for every $p'\ge p$. Explore the properties of the relation $\maltese$.

513 Notes and comments Most of the first part of this section consists of elementary verifications; an exception is the Erdős-Tarski theorem on the cellularity and saturation of a partially ordered set (513Bb-513Bc), which can equally well be regarded as a theorem about topological spaces or Boolean algebras (see 514Da and 514Nc). As usual, I have presented the ideas of the last two sections in an ahistorical manner; the original objective of Tukey 1940 was to classify directed sets from the point of view of net-convergence (513Yc).

I have starred 513K-513O because I do not expect to rely on them in the rest of this work. Nevertheless I think that they give a useful support to the ideas here, particularly in the context of §526, where these 'compactly based' partial orders appear naturally. Note that 513Ld tells us that if a directed set P is metrizably compactly based, there is a unique witnessing topology; every topological property of P must be a reflection of a property of the ordering.

Version of 16.5.14

514 Boolean algebras

The cardinal functions of Boolean algebras and topological spaces are intimately entwined; necessarily so, because we have a functorial connexion between Boolean algebras and zero-dimensional compact Hausdorff spaces (312Q). In this section I run through the elementary ideas. In 514D-514E I list properties of cardinal functions of Boolean algebras, corresponding to the relatively familiar results in 5A4B for topological spaces; Stone spaces (514B), regular open algebras (514H) and category algebras (514I) provide links of different kinds between the two theories. It turns out that some of the most important features of the cofinal structure of a pre-ordered set can also be described in terms of its 'up-topology' (514L-514M) and the associated regular open algebra (514N-514S). I conclude with a brief note on finite-support products (514T-514U).

514A I put a special property of locally compact spaces into the language of this chapter.

Lemma Let $(X,\mathfrak{T})$ be a topological space. Then $d(X)\ge d^{\downarrow}(\mathfrak{T}\setminus\{\emptyset\})$. If X is locally compact and Hausdorff, then $d(X)=d^{\downarrow}(\mathfrak{T}\setminus\{\emptyset\})$.

proof If $x\in X$, then $\{G:x\in G\in\mathfrak{T}\}$ is downwards-centered in $\mathfrak{T}\setminus\{\emptyset\}$. So

$$d^{\downarrow}(\mathfrak{T}\setminus\{\emptyset\})\le\operatorname{cov}(\mathfrak{T}\setminus\{\emptyset\},\ni,X)=d(X).$$

Now suppose that X is locally compact and Hausdorff. Set $\kappa=d^{\downarrow}(\mathfrak{T}\setminus\{\emptyset\})$, and let $\langle\mathcal{H}_\xi\rangle_{\xi<\kappa}$ be a cover of $\mathfrak{T}\setminus\{\emptyset\}$ by downwards-centered sets. For $\xi<\kappa$ set $F_\xi=\bigcap\{\overline{H}:H\in\mathcal{H}_\xi\}$, and let $D\subseteq X$ be a set with cardinal at most κ such that $D\cap F_\xi\ne\emptyset$ whenever $\xi<\kappa$ and $F_\xi\ne\emptyset$. If $G\subseteq X$ is a non-empty open set, then there is a non-empty relatively compact open set H_0 such that $\overline{H}_0\subseteq G$ (recall that X, being locally compact and Hausdorff, is certainly regular). There is some $\xi<\kappa$ such that $H_0\in\mathcal{H}_\xi$; because $\{\overline{H}:H\in\mathcal{H}_\xi\}$ is a family of closed sets with the finite intersection property containing the compact set $\overline{H}_0$, its intersection F_ξ is not empty. Also $F_\xi\subseteq\overline{H}_0\subseteq G$, so $D\cap G\supseteq D\cap F_\xi$ is non-empty. As G is arbitrary, D is dense, and $d(X)\le\#(D)\le\kappa$. We know already that $\kappa\le d(X)$, so they are equal.

514B Stone spaces Necessarily, any cardinal function ζ of topological spaces corresponds to a cardinal function $\tilde\zeta$ of Boolean algebras, taking $\tilde\zeta(\mathfrak{A})=\zeta(Z)$ where Z is the Stone space of $\mathfrak{A}$. Working through the functions described in 5A4A and 511D, we have the following results.

Theorem Let $\mathfrak{A}$ be any Boolean algebra and Z its Stone space. For $a\in\mathfrak{A}$ let $\widehat{a}$ be the corresponding open-and-closed subset of Z.

(a) $\#(\mathfrak{A})$ is $2^{w(Z)}=2^{\#(Z)}$ if $\mathfrak{A}$ is finite, $w(Z)$ otherwise.

(b) $\operatorname{sat}(\mathfrak{A})=\operatorname{sat}(Z)$, $c(\mathfrak{A})=c(Z)$.

(c) $\pi(\mathfrak{A})=\pi(Z)$.

(d) $d(\mathfrak{A})=d(Z)$.

(e) Let $\mathcal{N}\mathrm{wd}(Z)$ be the ideal of nowhere dense subsets of Z. Then $\mathrm{wdistr}(\mathfrak{A})=\mathrm{add}\,\mathcal{N}\mathrm{wd}(Z)$.

proof Let $\mathcal{E}$ be the algebra of open-and-closed subsets of Z, so that $a\mapsto\widehat{a}$ is an isomorphism from $\mathfrak{A}$ to $\mathcal{E}$. The essential fact here is that $\mathcal{E}\setminus\{\emptyset\}$ is coinitial with $\mathfrak{T}\setminus\{\emptyset\}$, where $\mathfrak{T}$ is the topology of Z, so that (writing $\mathfrak{A}^+$ for $\mathfrak{A}\setminus\{0\}$, as usual)

$$(\mathfrak{A}^+,\supseteq,\mathfrak{A}^+)\cong(\mathcal{E}\setminus\{\emptyset\},\supseteq,\mathcal{E}\setminus\{\emptyset\})\equiv_{\mathrm{GT}}(\mathfrak{T}\setminus\{\emptyset\},\supseteq,\mathfrak{T}\setminus\{\emptyset\})$$

by 513E(d-ii), inverted.

(a) If $\mathfrak{A}$ is finite, so is Z, and in this case $\mathfrak{A}\cong\mathcal{P}Z$ has cardinal $2^{\#(Z)}$. If $\mathfrak{A}$ is infinite, so are Z and $w(Z)$. Because $\mathcal{E}$ is a base for the topology of Z, $w(Z)\le\#(\mathcal{E})=\#(\mathfrak{A})$. On the other hand, let $\mathcal{U}$ be a base for the topology of Z with $\#(\mathcal{U})=w(Z)$. Then every member of $\mathcal{E}$ is expressible as the union of a finite subset of $\mathcal{U}$, so

$$\#(\mathfrak{A})=\#(\mathcal{E})\le\#([\mathcal{U}]^{<\omega})\le\max(\omega,\#(\mathcal{U}))=w(Z).$$

(b)-(c)

$$c(\mathfrak{A})=c(\mathcal{E})=c^{\downarrow}(\mathcal{E}\setminus\{\emptyset\})=c^{\downarrow}(\mathfrak{T}\setminus\{\emptyset\})=c(Z),$$

$$\mathrm{sat}(\mathfrak{A})=\mathrm{sat}(\mathcal{E})=\mathrm{sat}^{\downarrow}(\mathcal{E}\setminus\{\emptyset\})=\mathrm{sat}^{\downarrow}(\mathfrak{T}\setminus\{\emptyset\})=\mathrm{sat}(Z),$$

$$\pi(\mathfrak{A})=\pi(\mathcal{E})=\mathrm{ci}(\mathcal{E}\setminus\{\emptyset\})=\mathrm{ci}(\mathfrak{T}\setminus\{\emptyset\})=\pi(Z)$$

using 513Gb, inverted, to move between $\mathcal{E}\setminus\{\emptyset\}$ and $\mathfrak{T}\setminus\{\emptyset\}$.

(d)

$$d(\mathfrak{A})=d^{\downarrow}(\mathfrak{A}^+)=d^{\downarrow}(\mathcal{E}\setminus\{\emptyset\})=d^{\downarrow}(\mathfrak{T}\setminus\{\emptyset\})$$

(513Gd, inverted)

$$=d(Z)$$

because Z is compact and Hausdorff (514A).

(e) Let $(\mathrm{Pou}(\mathfrak{A}),\sqsubseteq^*)$ be the pre-ordered set of partitions of unity in $\mathfrak{A}$ as described in 512Ee. For $C\in\mathrm{Pou}(\mathfrak{A})$, set

$$f(C)=Z\setminus\bigcup_{c\in C}\widehat{c}.$$

Then $f(C)\in\mathcal{N}\mathrm{wd}(Z)$. **P?** Otherwise, since $f(C)$ is certainly closed, its interior is non-empty, and there is a non-zero $a\in\mathfrak{A}$ such that $\widehat{a}\subseteq f(C)$; but in this case $a\cap c=0$ for every $c\in C$ and C is not a partition of unity. **XQ**

If C, $D\in\mathrm{Pou}(\mathfrak{A})$ and $C\sqsubseteq^* D$ then $f(C)\subseteq f(D)$. **P** If $d\in D$, $C_0=\{c:c\in C,\,c\cap d\ne0\}$ is finite and $d\subseteq\sup C_0$; so $\widehat{d}\subseteq\bigcup_{c\in C_0}\widehat{c}$ is disjoint from $f(C)$. Thus $Z\setminus f(D)\subseteq Z\setminus f(C)$ and $f(C)\subseteq f(D)$. **Q**

If C, $D\in\mathrm{Pou}(\mathfrak{A})$ and $f(C)\subseteq f(D)$ then $C\sqsubseteq^* D$. **P** If $d\in D$ then the compact set $\widehat{d}$ is included in the open set $\bigcup_{c\in C}\widehat{c}$. So there is a finite set $C_0\subseteq C$ such that $\widehat{d}\subseteq\bigcup_{c\in C_0}\widehat{c}$ and $\{c:c\in C,\,d\cap c\ne0\}\subseteq C_0$ is finite. **Q**

$f[\mathrm{Pou}(\mathfrak{A})]$ is cofinal with $\mathcal{N}\mathrm{wd}(Z)$. **P** If $F\in\mathcal{N}\mathrm{wd}(Z)$, let $C\subseteq\mathfrak{A}$ be a maximal disjoint set such that $F\cap\widehat{c}=\emptyset$ for every $c\in C$. **?** If C is not a partition of unity in $\mathfrak{A}$, let $a\in\mathfrak{A}^+$ be such that $a\cap c=0$ for every $c\in C$. Then $\widehat{a}\setminus F$ is a non-empty open set, so there is a non-zero $b\in\mathfrak{A}$ such that $\widehat{b}\subseteq\widehat{a}\setminus F$; in which case we ought to have added b to C. **X** So $C\in\mathrm{Pou}(\mathfrak{A})$ and $F\subseteq f(C)$. **Q**

By 513E(d-i), $\mathrm{Pou}(\mathfrak{A})$ and $\mathcal{N}\mathrm{wd}(Z)$ are Tukey equivalent, and

$$\mathrm{add}\,\mathcal{N}\mathrm{wd}(Z)=\mathrm{add}\,\mathrm{Pou}(\mathfrak{A})=\mathrm{wdistr}(\mathfrak{A})$$

as remarked in 512Ee.

514C I begin the detailed study of cardinal functions of Boolean algebras with two elementary remarks.

Lemma Let $\mathfrak{A}$ be a Boolean algebra.

(a) $d(\mathfrak{A})$ is the smallest cardinal κ such that $\mathfrak{A}$ is isomorphic, as Boolean algebra, to a subalgebra of $\mathcal{P}\kappa$.

(b) $\operatorname{link}(\mathfrak{A})$ is the smallest cardinal κ such that $\mathfrak{A}$ is isomorphic, as partially ordered set, to a subset of $\mathcal{P}\kappa$.

proof (a)(i) If we have an isomorphism π from $\mathfrak{A}$ to a subalgebra of $\mathcal{P}\kappa$, then $A_\xi = \{a : \xi \in \pi a\}$ is centered for each $\xi < \kappa$, and $\bigcup_{\xi<\kappa} A_\xi = \mathfrak{A}^+$; so $d(\mathfrak{A}) \le \kappa$.

(ii) Let Z be the Stone space of $\mathfrak{A}$, and for $a \in \mathfrak{A}$ let $\widehat{a} \subseteq Z$ be the corresponding open-and-closed set. There is a dense set $D \subseteq Z$ with cardinal $d(\mathfrak{A})$ (514Bd), and $a \mapsto D \cap \widehat{a} : \mathfrak{A} \to \mathcal{P}D$ is an injective Boolean homomorphism; so $\mathfrak{A}$ is isomorphic to a subalgebra of $\mathcal{P}D \cong \mathcal{P}(d(\mathfrak{A}))$.

(b)(i) $\kappa \le \operatorname{link}(\mathfrak{A})$. **P** Let $\langle A_\xi\rangle_{\xi<\operatorname{link}(\mathfrak{A})}$ be a family of linked subsets of $\mathfrak{A}^+$ covering $\mathfrak{A}^+$. Set $A'_\xi = \{b : \exists\ a \in A_\xi,\ b \supseteq a\}$; then each A'_ξ is still linked in $\mathfrak{A}$. Define $h : \mathfrak{A} \to \mathcal{P}\kappa$ by setting $h(a) = \{\xi : a \in A'_\xi\}$. Then h is order-preserving. Now if a, $b \in \mathfrak{A}$ and $a \not\subseteq b$, there is a $\xi < \kappa$ such that $a \setminus b \in A_\xi$, in which case $\xi \in h(a) \setminus h(b)$. Thus h is an embedding and $\kappa \le \operatorname{link}(\mathfrak{A})$. **Q**

(ii) $\operatorname{link}(\mathfrak{A}) \le \kappa$. **P** Let $h : \mathfrak{A} \to \mathcal{P}\kappa$ be an order-isomorphism between $\mathfrak{A}$ and a subset of $\mathcal{P}\kappa$. For each ξ, set

$$A_\xi = \{a : a \in \mathfrak{A},\ \xi \in h(a) \setminus h(1 \setminus a)\}.$$

If a, $b \in A_\xi$ then $\xi \in h(b) \setminus h(1 \setminus a)$ so $b \not\subseteq 1 \setminus a$ and $a \cap b \ne 0$; thus A_ξ is linked. If $a \in \mathfrak{A}^+$ then $a \not\subseteq 1 \setminus a$ so $h(a) \not\subseteq h(1 \setminus a)$ and there is a $\xi < \kappa$ such that $\xi \in h(a) \setminus h(1 \setminus a)$; thus $\mathfrak{A}^+ = \bigcup_{\xi<\kappa} A_\xi$ and $\operatorname{link}(\mathfrak{A}) \le \kappa$. **Q**

514D Theorem Let $\mathfrak{A}$ be a Boolean algebra.

(a)

$$c(\mathfrak{A}) \le \operatorname{link}(\mathfrak{A}) \le d(\mathfrak{A}) \le \pi(\mathfrak{A}) \le \#(\mathfrak{A}) \le 2^{\operatorname{link}(\mathfrak{A})}, \quad \tau(\mathfrak{A}) \le \pi(\mathfrak{A}),$$

$\operatorname{sat}(\mathfrak{A}) = c(\mathfrak{A})^+$ unless $\operatorname{sat}(\mathfrak{A})$ is weakly inaccessible, in which case $\operatorname{sat}(\mathfrak{A}) = c(\mathfrak{A})$.

(b) If $A \subseteq \mathfrak{A}$, there is a $B \in [A]^{<\operatorname{sat}(\mathfrak{A})}$ with the same upper bounds as A; similarly, there is a $B \in [A]^{<\operatorname{sat}(\mathfrak{A})}$ with the same lower bounds as A.

(c) $\operatorname{link}_{c(\mathfrak{A})}(\mathfrak{A}) = \operatorname{link}_{<\operatorname{sat}(\mathfrak{A})}(\mathfrak{A}) = \pi(\mathfrak{A})$.

(d) If $\mathfrak{A}$ is not purely atomic, $\operatorname{wdistr}(\mathfrak{A}) \le \min(d(\mathfrak{A}), 2^{\tau(\mathfrak{A})})$ is a regular infinite cardinal.

(e) $\#(\mathfrak{A}) \le \max(4, \sup_{\lambda<\operatorname{sat}(\mathfrak{A})} \tau(\mathfrak{A})^\lambda)$, where $\tau(\mathfrak{A})^\lambda$ is the cardinal power.

proof Let Z be the Stone space of $\mathfrak{A}$; for $a \in \mathfrak{A}$, let $\widehat{a} \subseteq Z$ be the corresponding open-and-closed set.

(a) This is mostly a repetition of 511Ia. By 514Cb, $\#(\mathfrak{A}) \le 2^{\operatorname{link}(\mathfrak{A})}$. By 513Bc, inverted, and the definitions in 511Db,

$$\operatorname{sat}(\mathfrak{A}) = \operatorname{sat}^\downarrow(\mathfrak{A}^+) = c^\downarrow(\mathfrak{A}^+)^+ = c(\mathfrak{A})^+$$

unless $\operatorname{sat}(\mathfrak{A})$ is a regular uncountable limit cardinal, that is, is weakly inaccessible, and otherwise $\operatorname{sat}(\mathfrak{A}) = c(\mathfrak{A})$. (See also 5A4Ba.)

(b) By 5A4Bd, applied to $\{\widehat{a} : a \in A\}$, there is a $B \in [A]^{<\operatorname{sat}(\mathfrak{A})}$ such that $\overline{\bigcup_{b\in B} \widehat{b}} = \overline{\bigcup_{a\in A} \widehat{a}}$. Now if c is an upper bound of B, then $\widehat{c}$ is a closed set including $\widehat{b}$ for every $b \in B$, so it also includes $\widehat{a}$ for every $a \in A$, and c is an upper bound of A.

Applying this to $\{1 \setminus a : a \in A\}$ we see that there is a set $B' \in [A]^{<\operatorname{sat}(\mathfrak{A})}$ with the same lower bounds as A.

(c) Set $\kappa = \operatorname{link}_{<\operatorname{sat}(\mathfrak{A})}(\mathfrak{A})$. By 511Ia, $\kappa \le \operatorname{link}_{c(\mathfrak{A})}(\mathfrak{A}) \le \pi(\mathfrak{A})$. On the other hand, if $A \subseteq \mathfrak{A}^+$ is $<\operatorname{sat}(\mathfrak{A})$-linked, it has a lower bound in $\mathfrak{A}^+$. **P** By (b), there is a set $B \subseteq A$, with the same lower bounds as A, such that $\#(B) < \operatorname{sat}(\mathfrak{A})$. Now B has a non-zero lower bound because A is $<\operatorname{sat}(\mathfrak{A})$-linked, so A also has a non-zero lower bound. **Q** We have a cover $\langle A_\xi\rangle_{\xi<\kappa}$ of $\mathfrak{A}^+$ by $<\operatorname{sat}(\mathfrak{A})$-linked sets; each A_ξ has a non-zero lower bound a_ξ say; and $\{a_\xi : \xi < \kappa\}$ is a π-base for $\mathfrak{A}$, so $\pi(\mathfrak{A}) \le \kappa$.

(d)(i) Because $\mathrm{wdistr}(\mathfrak{A}) = \mathrm{add}\,\mathcal{N}\mathrm{wd}(Z)$, where $\mathcal{N}\mathrm{wd}(Z)$ is the ideal of nowhere dense subsets of Z (514Be), and is not ∞ (511Ie), it must be a regular infinite cardinal (513C(a-i)). (Or argue directly from 511Df.)

(ii) As for the upper bound for $\mathrm{wdistr}(\mathfrak{A})$, suppose that $a \in \mathfrak{A}^+$ includes no atom and that $D \in [\mathfrak{A}]^{\tau(\mathfrak{A})}$ τ-generates $\mathfrak{A}$. Since $\mathfrak{A}$ and $\tau(\mathfrak{A})$ are surely infinite, the subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ generated by $D \cup \{a\}$ is still with cardinal $\tau(\mathfrak{A})$ (331Gc). For $B \subseteq \mathfrak{B}$ set $E_B = Z \cap \bigcap_{b \in B} \widehat{b}$, and set $\mathcal{C} = \{B : B \subseteq \mathfrak{B},\ E_B$ is nowhere dense$\}$. Then $\bigcup_{B \in \mathcal{C}} E_B \supseteq \widehat{a}$. **P** Take any $z \in \widehat{a}$. Set $B = \{b : b \in \mathfrak{B},\ z \in \widehat{b}\}$. **?** If E_B has non-empty interior, it includes $\widehat{c}$ for some non-zero $c \subseteq a$. But now, for any $d \in D$, either $d \in B$ and $c \subseteq d$, or $1 \setminus d \in B$ and $c \cap d = 0$. So the order-closed subalgebra $\{d :$ either $c \subseteq d$ or $c \cap d = 0\}$ includes D and must be the whole of $\mathfrak{A}$, and $c \subseteq a$ is an atom. **X** So $\mathrm{int}\,E_B = \emptyset$, $B \in \mathcal{C}$ and $z \in E_B$. As z is arbitrary, $\widehat{a} \subseteq \bigcup_{B \in \mathcal{C}} E_B$. **Q**

By 514Be, with 514Bd,

$$\mathrm{wdistr}(\mathfrak{A}) \le \#(\mathcal{C}) \le 2^{\#(\mathfrak{B})} = 2^{\tau(\mathfrak{A})}.$$

At the same time, if $Y \subseteq Z$ is any dense set with cardinal $d(Z)$, then $\{\{y\} : y \in Y \cap \widehat{a}\}$ is a family of nowhere dense sets with no upper bound in the ideal of nowhere dense subsets of Z; so 514Be also tells us that

$$\mathrm{wdistr}(\mathfrak{A}) \le \#(Y \cap \widehat{a}) \le d(Z) = d(\mathfrak{A}).$$

(e) (Compare 4A1O.) Set $\kappa = \sup_{\lambda < \mathrm{sat}(\mathfrak{A})} \tau(\mathfrak{A})^\lambda$. If $\#(\mathfrak{A}) > 4$ then $\tau(\mathfrak{A}) \ge 2$ so $\kappa \ge \sup_{\lambda < \mathrm{sat}(\mathfrak{A})} 2^\lambda$, and the result is immediate from 511Ic if $\mathfrak{A}$ is finite. If $\mathfrak{A}$ is infinite, so is $\mathrm{sat}(\mathfrak{A})$, while $\lambda < \kappa$ for every $\lambda < \mathrm{sat}(\mathfrak{A})$, so $\mathrm{sat}(\mathfrak{A}) \le \kappa$. Let $D \subseteq \mathfrak{A}$ be a set with cardinal $\tau(\mathfrak{A})$ which τ-generates $\mathfrak{A}$. Define $\langle D_\xi \rangle_{\xi < \kappa}$ inductively by setting

$$D_0 = D, \quad D_\xi = \{1 \setminus a : a \in \mathfrak{A},\ a = \sup C \text{ for some } C \subseteq \textstyle\bigcup_{\eta < \xi} D_\eta\}$$

for $\xi < \kappa$. Then $\#(D_\xi) \le \kappa$ for every $\xi < \kappa$. **P** The point is that, by (b), every member of D_ξ is expressible in the form $1 \setminus \sup C$ for some $C \in [\bigcup_{\eta < \xi} D_\eta]^{<\mathrm{sat}(\mathfrak{A})}$. But the inductive hypothesis tells us that $\bigcup_{\eta < \xi} D_\eta$ has cardinal at most κ, so the number of its subsets with cardinal less than $\mathrm{sat}(\mathfrak{A})$ is also κ (5A1Ff, because $\mathrm{sat}(\mathfrak{A})$ is regular), and $\#(D_\xi) \le \kappa$. **Q**

At the end of the induction, set $\mathfrak{B} = \bigcup_{\xi < \mathrm{sat}(\mathfrak{A})} D_\xi$. Then $1 \setminus (b \cup b') \in \mathfrak{B}$ for every b, $b' \in \mathfrak{B}$, so $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$. Also it is order-closed. **P** If $B \subseteq \mathfrak{B}$ has a supremum $a \in \mathfrak{A}$, there is a $C \subseteq B$ such that $\#(C) < \mathrm{sat}(\mathfrak{A})$ and $a = \sup C$. Now there must be some set $J \subseteq \mathrm{sat}(\mathfrak{A})$ such that $\#(J) < \mathrm{sat}(\mathfrak{A})$ and $C \subseteq \bigcup_{\eta \in J} D_\eta$. Since $\mathrm{sat}(\mathfrak{A})$ is regular (513Bb), $\zeta = \sup J$ is less than $\mathrm{sat}(\mathfrak{A})$. Now $1 \setminus a \in D_{\zeta+1}$ and $a \in \mathfrak{B}$. **Q**

By the choice of D, $\mathfrak{B} = \mathfrak{A}$, so $\#(\mathfrak{A}) = \#(\mathfrak{B}) \le \kappa$.

514E Subalgebras, homomorphic images, products: Theorem Let $\mathfrak{A}$ be a Boolean algebra.

(a) If $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$, then

$$\mathrm{sat}(\mathfrak{B}) \le \mathrm{sat}(\mathfrak{A}), \quad c(\mathfrak{B}) \le c(\mathfrak{A}),$$

$$\mathrm{link}_{<\kappa}(\mathfrak{B}) \le \mathrm{link}_{<\kappa}(\mathfrak{A})$$

for every $\kappa \le \omega$, in particular,

$$d(\mathfrak{B}) \le d(\mathfrak{A}), \quad \mathrm{link}(\mathfrak{B}) \le \mathrm{link}(\mathfrak{A}).$$

(b) If $\mathfrak{B}$ is a regularly embedded subalgebra of $\mathfrak{A}$ then, in addition, $\mathrm{link}_{<\kappa}(\mathfrak{B}) \le \mathrm{link}_{<\kappa}(\mathfrak{A})$ for $\kappa > \omega$, $\pi(\mathfrak{B}) \le \pi(\mathfrak{A})$ and $\mathrm{wdistr}(\mathfrak{A}) \le \mathrm{wdistr}(\mathfrak{B})$.

(c) If $\mathfrak{B}$ is a Boolean algebra and $\phi : \mathfrak{A} \to \mathfrak{B}$ is a surjective order-continuous Boolean homomorphism, then

$$\mathrm{sat}(\mathfrak{B}) \le \mathrm{sat}(\mathfrak{A}), \quad c(\mathfrak{B}) \le c(\mathfrak{A}), \quad \pi(\mathfrak{B}) \le \pi(\mathfrak{A}),$$

$$\mathrm{link}_{<\kappa}(\mathfrak{B}) \le \mathrm{link}_{<\kappa}(\mathfrak{A}) \text{ for every cardinal } \kappa,$$

$$d(\mathfrak{B}) \le d(\mathfrak{A}), \quad \mathrm{link}(\mathfrak{B}) \le \mathrm{link}(\mathfrak{A}),$$

and also

$$\mathrm{wdistr}(\mathfrak{A}) \le \mathrm{wdistr}(\mathfrak{B}), \quad \tau(\mathfrak{B}) \le \tau(\mathfrak{A}).$$

(d) If $\mathfrak{B}$ is a principal ideal of $\mathfrak{A}$, then

$$\mathrm{sat}(\mathfrak{B}) \le \mathrm{sat}(\mathfrak{A}), \quad c(\mathfrak{B}) \le c(\mathfrak{A}), \quad \pi(\mathfrak{B}) \le \pi(\mathfrak{A}),$$

$$\mathrm{link}_{<\kappa}(\mathfrak{B}) \le \mathrm{link}_{<\kappa}(\mathfrak{A}) \text{ for every } \kappa,$$

$$d(\mathfrak{B}) \le d(\mathfrak{A}), \quad \mathrm{link}(\mathfrak{B}) \le \mathrm{link}(\mathfrak{A});$$

moreover,

$$\mathrm{wdistr}(\mathfrak{A}) \le \mathrm{wdistr}(\mathfrak{B}), \quad \tau(\mathfrak{B}) \le \tau(\mathfrak{A}).$$

(e) If $\mathfrak{B}$ is an order-dense subalgebra of $\mathfrak{A}$ then

$$\mathrm{sat}(\mathfrak{B}) = \mathrm{sat}(\mathfrak{A}), \quad c(\mathfrak{B}) = c(\mathfrak{A}), \quad \pi(\mathfrak{B}) = \pi(\mathfrak{A}),$$

$$\mathrm{link}_{<\kappa}(\mathfrak{B}) = \mathrm{link}_{<\kappa}(\mathfrak{A}) \text{ for every } \kappa,$$

$$d(\mathfrak{B}) = d(\mathfrak{A}), \quad \mathrm{link}(\mathfrak{B}) = \mathrm{link}(\mathfrak{A}),$$

and finally

$$\mathrm{wdistr}(\mathfrak{B}) = \mathrm{wdistr}(\mathfrak{A}), \quad \tau(\mathfrak{A}) \le \tau(\mathfrak{B}).$$

(f) If $\mathfrak{A}$ is the simple product of a family $\langle\mathfrak{A}_i\rangle_{i\in I}$ of Boolean algebras, then

$$\tau(\mathfrak{A}) \le \max(\omega, \sup_{i\in I}\tau(\mathfrak{A}_i), \min\{\lambda : \#(I) \le 2^\lambda\}),$$

$$\mathrm{sat}(\mathfrak{A}) \le \max(\omega, \#(I)^+, \sup_{i\in I}\mathrm{sat}(\mathfrak{A}_i)),$$

$$c(\mathfrak{A}) \le \max(\omega, \#(I), \sup_{i\in I} c(\mathfrak{A}_i)),$$

$$\pi(\mathfrak{A}) \le \max(\omega, \#(I), \sup_{i\in I} \pi(\mathfrak{A}_i)),$$

$$\mathrm{link}_{<\kappa}(\mathfrak{A}) \le \max(\omega, \#(I), \sup_{i\in I}\mathrm{link}_{<\kappa}(\mathfrak{A}_i)) \text{ for every } \kappa,$$

$$\mathrm{link}(\mathfrak{A}) \le \max(\omega, \#(I), \sup_{i\in I}\mathrm{link}(\mathfrak{A}_i)),$$

$$d(\mathfrak{A}) \le \max(\omega, \#(I), \sup_{i\in I} d(\mathfrak{A}_i)),$$

and

$$\mathrm{wdistr}(\mathfrak{A}) = \min_{i\in I}\mathrm{wdistr}(\mathfrak{A}_i).$$

proof Write Z for the Stone space of $\mathfrak{A}$.

(a) Any disjoint subset of $\mathfrak{B}^+$ is a disjoint subset of $\mathfrak{A}^+$, so $\mathrm{sat}(\mathfrak{B}) \le \mathrm{sat}(\mathfrak{A})$ and $c(\mathfrak{B}) \le c(\mathfrak{A})$. If $\kappa \le \omega$ and $\mathcal{A}$ is a cover of $\mathfrak{A}^+$ by sets which are downwards $<\kappa$-linked in $\mathfrak{A}^+$, then $A\cap\mathfrak{B}$ is downwards $<\kappa$-linked in $\mathfrak{B}^+$ for each $A \in \mathcal{A}$, so $\mathrm{link}_{<\kappa}(\mathfrak{B}) \le \mathrm{link}_{<\kappa}(\mathfrak{A})$.

(b) For each non-zero $a \in \mathfrak{A}$, the set $B_a = \{b : b \in \mathfrak{B},\ a \subseteq b\}$ does not have infimum 0 in $\mathfrak{A}$ so cannot have infimum 0 in $\mathfrak{B}$; let $\psi(a) \in \mathfrak{B}^+$ be a lower bound for B_a. If now we set $\phi(b) = b$ for $b \in \mathfrak{B}$, (ϕ,ψ) is a Galois-Tukey connection from $(\mathfrak{B}^+,\supseteq,\mathfrak{B}^+)$ to $(\mathfrak{A}^+,\supseteq,\mathfrak{A}^+)$. It follows at once that

$$\mathrm{link}_{<\kappa}(\mathfrak{B}) = \mathrm{link}_{<\kappa}(\mathfrak{B}^+,\supseteq,\mathfrak{B}^+) \le \mathrm{link}_{<\kappa}(\mathfrak{A}^+,\supseteq,\mathfrak{A}^+) = \mathrm{link}_{<\kappa}(\mathfrak{A})$$

for arbitrary κ (512Dd), and that

$$\pi(\mathfrak{B}) = \mathrm{cov}(\mathfrak{B}^+,\supseteq,\mathfrak{B}^+) \le \mathrm{cov}(\mathfrak{A}^+,\supseteq,\mathfrak{A}^+) = \pi(\mathfrak{A})$$

(512Da).

Now suppose that $\kappa < \mathrm{wdistr}(\mathfrak{A})$ and that $\langle B_\xi\rangle_{\xi<\kappa}$ is a family of partitions of unity in $\mathfrak{B}$. Then

$$D = \{d : d \in \mathfrak{B},\ \{b : b \in B_\xi,\ b \cap d \neq 0\} \text{ is finite for every } \xi < \kappa\}$$

is order-dense in $\mathfrak{B}$. **P** Take any non-zero $d \in \mathfrak{B}$. $\sup B_\xi = 1$ in $\mathfrak{A}$, that is, B_ξ is still a partition of unity in $\mathfrak{A}$, for each ξ. So there is a partition A of unity in $\mathfrak{A}$ such that $\{b : b \in B_\xi,\ b \cap a \neq 0\}$ is finite for every $\xi < \kappa$ and $a \in A$. Let $a \in A$ be such that $d \cap a \neq 0$, and set $e_\xi = \sup\{b : b \in B_\xi,\ b \cap a \neq 0\}$ for each $\xi < \kappa$. Then $a \subseteq e_\xi \in \mathfrak{B}$ for each ξ. This means that $\{d\} \cup \{e_\xi : \xi < \kappa\}$ has a non-zero lower bound $d \cap a$ in $\mathfrak{A}$; as $\mathfrak{B}$ is regularly embedded in $\mathfrak{A}$, there is a non-zero $d' \subseteq d$ which is also a lower bound for $\{e_\xi : \xi < \kappa\}$. But this means that $d' \in D$. As d is arbitrary, D is order-dense in $\mathfrak{B}$. **Q**

There is therefore a partition of unity included in D. As $\langle B_\xi\rangle_{\xi<\kappa}$ is arbitrary, $\mathrm{wdistr}(\mathfrak{B}) \geq \mathrm{wdistr}(\mathfrak{A})$.

(c)(i) For any $b \in \mathfrak{B}^+$ there is a $\psi(b) \in \mathfrak{A}^+$ such that $\phi\psi(b) \subseteq b$ and $a = 0$ whenever $a \subseteq \psi(b)$ and $\phi a = 0$. **P** Consider $D = \{d : d \in \mathfrak{A},\ \phi d \supseteq b\}$. This is a non-empty downwards-directed subset of $\mathfrak{A}$ and b is a non-zero lower bound of $\phi[D]$. As ϕ is supposed to be order-continuous, D must have a non-zero lower bound in $\mathfrak{A}$; let $\psi(b)$ be such a lower bound. Since there is a $d \in \mathfrak{A}$ such that $\phi d = b$, and now $d \in D$, we must have $\phi\psi(b) \subseteq \phi d = b$. If $a \subseteq \psi(b)$ and $\phi a = 0$, then $\phi(1 \subseteq a) = 1 \supseteq b$, $1 \setminus a \in D$ and $a \subseteq \psi(b) \subseteq 1 \setminus a$, so $a = 0$. **Q**

(ii) If $\kappa = \mathrm{sat}(\mathfrak{A})$ and $\langle b_\xi\rangle_{\xi<\kappa}$ is a family in $\mathfrak{B}^+$, then $\langle \psi(b_\xi)\rangle_{\xi<\kappa}$ is a family in $\mathfrak{A}^+$ and there are distinct ξ, $\eta < \kappa$ such that $a = \psi(b_\xi) \cap \psi(b_\eta)$ is non-zero. Now

$$0 \neq \phi a \subseteq \phi\psi(b_{xi}) \cap \phi\psi(b_\eta) \subseteq b_\xi \cap b_\eta.$$

As $\langle b_\xi\rangle_{\xi<\kappa}$ is arbitrary, $\mathrm{sat}(\mathfrak{B}) \leq \mathrm{sat}(\mathfrak{A})$. By a similar argument, or using 514Da, we see that $c(\mathfrak{B}) \leq c(\mathfrak{A})$.

(iii) Let A be a coinitial subset of $\mathfrak{A}^+$ of cardinal $\pi(\mathfrak{A})$. Set $B = \phi[A] \setminus \{0\}$. If $b \in \mathfrak{B}^+$, there is an $a \in A$ such that $a \subseteq \psi(b)$, and now $\phi a \in B$ and $\phi a \subseteq \phi\psi(b) \subseteq b$. So B is cofinal with $\mathfrak{B}^+$ and

$$\pi(\mathfrak{B}) \leq \#(B) \leq \#(A) = \pi(\mathfrak{A}).$$

(iv) Let W be the Stone space of $\mathfrak{B}$. Write $\mathcal{N}\mathrm{wd}(Z)$, $\mathcal{N}\mathrm{wd}(W)$ for the ideals of nowhere dense subsets of Z and W, so that $\mathrm{add}\,\mathcal{N}\mathrm{wd}(Z) = \mathrm{wdistr}(\mathfrak{A})$ and $\mathrm{add}\,\mathcal{N}\mathrm{wd}(W) = \mathrm{wdistr}(\mathfrak{B})$ (514Be). Corresponding to $\phi : \mathfrak{A} \to \mathfrak{B}$ we have an injective continuous function $\theta : W \to Z$ such that $\theta^{-1}[E] \in \mathcal{N}\mathrm{wd}(W)$ for every $E \in \mathcal{N}\mathrm{wd}(Z)$ (312Sb, 313R). Also $\theta[F] \in \mathcal{N}\mathrm{wd}(Z)$ for every $F \in \mathcal{N}\mathrm{wd}(W)$. **P?** Otherwise, because $\theta[\overline{F}]$ is compact, therefore closed, there is a non-empty open set $G \subseteq \theta[\overline{F}]$. Now $\theta^{-1}[G]$ is a non-empty open set, and is included in $\overline{F}$, because θ is injective; but this is impossible. **XQ** So if $\mathcal{J}_0 \subseteq \mathcal{N}\mathrm{wd}(W)$ and $\#(\mathcal{J}_0) < \mathrm{wdistr}(\mathfrak{A})$, $E = \bigcup\{\theta[F] : F \in \mathcal{J}_0\}$ belongs to $\mathcal{N}\mathrm{wd}(Z)$ and $\bigcup \mathcal{J}_0 \subseteq \theta^{-1}[E]$ belongs to $\mathcal{N}\mathrm{wd}(W)$. This shows that $\mathrm{add}\,\mathcal{N}\mathrm{wd}(W) \geq \mathrm{add}\,\mathcal{N}\mathrm{wd}(Z)$, so that $\mathrm{wdistr}(\mathfrak{B}) \geq \mathrm{wdistr}(\mathfrak{A})$.

(v) As for $\tau(\mathfrak{B})$, we have only to recall that if $D \subseteq \mathfrak{A}$ is a τ-generating set with cardinal $\tau(\mathfrak{A})$, the order-closed subalgebra of $\mathfrak{B}$ generated by $\phi[D]$ includes $\phi[\mathfrak{A}] = \mathfrak{B}$ (313Mb), and

$$\tau(\mathfrak{B}) \leq \#(\phi[D]) \leq \tau(\mathfrak{A}).$$

(d) If $\mathfrak{B}$ is the principal ideal generated by b, then $a \mapsto a \cap b : \mathfrak{A} \to \mathfrak{B}$ is an order-continuous surjection, so we can repeat the list in (c).

(e)(i) Because $\mathfrak{B}^+$ is coinitial with $\mathfrak{A}^+$ we can use 513Gc, inverted, to see that

$$\mathrm{sat}(\mathfrak{B}) = \mathrm{sat}^{\downarrow}(\mathfrak{B}^+) = \mathrm{sat}^{\downarrow}(\mathfrak{A}^+) = \mathrm{sat}(\mathfrak{A}), \quad c(\mathfrak{B}) = c(\mathfrak{A}),$$

$$\pi(\mathfrak{B}) = \mathrm{ci}(\mathfrak{B}^+) = \mathrm{ci}(\mathfrak{A}^+) = \pi(\mathfrak{A}),$$

$$\mathrm{link}_{<\kappa}(\mathfrak{B}) = \mathrm{link}^{\downarrow}_{<\kappa}(\mathfrak{B}^+) = \mathrm{link}^{\downarrow}_{<\kappa}(\mathfrak{A}^+) = \mathrm{link}_{<\kappa}(\mathfrak{A}).$$

(ii) From (b) we know that $\mathrm{wdistr}(\mathfrak{B}) \geq \mathrm{wdistr}(\mathfrak{A})$. For the reverse inequality, suppose that $\kappa < \mathrm{wdistr}(\mathfrak{B})$ and that $\langle A_\xi\rangle_{\xi<\kappa}$ is any family of partitions of unity in $\mathfrak{A}$. For each $\xi < \kappa$ set $B_\xi = \{b : b \in \mathfrak{B},\ \exists\ a \in A_\xi,\ b \subseteq a\}$. Then B_ξ is order-dense in $\mathfrak{B}$ and includes a partition of unity B'_ξ (313K). Now there is a partition C of unity in $\mathfrak{B}$ such that $D'_{\xi c} = \{b : b \in B'_\xi,\ b \cap c \neq 0\}$ is finite for any $\xi < \kappa$ and $c \in C$. C is still a partition of unity in $\mathfrak{A}$, and $D_{\xi c} = \{a : a \in A_\xi,\ a \cap c \neq 0\}$ is finite for every $c \in C$ and $\xi < \kappa$. (For if a, a' are distinct elements of $D_{\xi c}$, then $\{b : b \in D'_{\xi c},\ b \subseteq a\}$ and $\{b : b \in D'_{\xi c},\ b \subseteq a'\}$ are disjoint and not empty.) As $\langle A_\xi\rangle_{\xi<\kappa}$ is arbitrary, $\mathrm{wdistr}(\mathfrak{B}) \leq \mathrm{wdistr}(\mathfrak{A})$.

(iii) If $D \subseteq \mathfrak{B}$ τ-generates $\mathfrak{B}$, then D also τ-generates $\mathfrak{A}$. **P** Applying 313Mb to the identity map from $\mathfrak{B}$ to $\mathfrak{A}$, we see that the order-closed subalgebra $\mathfrak{D}$ of $\mathfrak{A}$ generated by D includes $\mathfrak{B}$; but as any member of $\mathfrak{A}$ is the supremum of a subset of $\mathfrak{B}$, $\mathfrak{D} = \mathfrak{A}$. **Q** So $\tau(\mathfrak{A}) \le \tau(\mathfrak{B})$.

(f) We can identify each $\mathfrak{A}_i$ with the principal ideal of $\mathfrak{A}$ generated by an element a_i, where $\langle a_i \rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$ (315E). If $\kappa = \max(\omega, \sup_{i\in I} \tau(\mathfrak{A}_i), \min\{\lambda : \#(I) \le 2^\lambda\})$, then for each $i \in I$ choose $\langle a_{i\xi}\rangle_{\xi<\kappa}$ in $\mathfrak{A}_i$ such that $\{a_{i\xi} : \xi < \kappa\}$ τ-generates $\mathfrak{A}_i$, and let $\phi : I \to \mathcal{P}\kappa$ be injective. For $\xi < \kappa$, set

$$b_\xi = \sup_{i\in I} a_{i\xi}, \quad c_\xi = \sup_{i\in\phi(\xi)} a_i.$$

Let $\mathfrak{B}$ be the order-closed subalgebra of $\mathfrak{A}$ generated by $\{b_\xi : \xi < \kappa\} \cup \{c_\xi : \xi < \kappa\}$. Then

$$a_i = \inf\{c_\xi : \xi \in \phi(i)\} \setminus \sup\{c_\xi : \xi \in \kappa \setminus \phi(i)\} \in \mathfrak{B}$$

for each i. Because $\{b : b \in \mathfrak{B}, b \subseteq a_i\}$ is an order-closed subalgebra of $\mathfrak{A}_i$ containing $b_\xi \cap a_i = a_{i\xi}$ for every $\xi < \kappa$, it is the whole of $\mathfrak{A}_i$, so $\mathfrak{A}_i \subseteq \mathfrak{B}$ for every $i \in I$. It follows at once that $\mathfrak{B} = \mathfrak{A}$, so that $\tau(\mathfrak{A}) \le \kappa$.

The other parts are all elementary.

514F For measure algebras, Maharam type is not only the cardinal function which gives most information, but is also, as a rule, easy to calculate. For other Boolean algebras, it may not be obvious what the Maharam type is. The following result sometimes helps.

Proposition Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, and $\langle a_{ij}\rangle_{i\in I, j\in J}$ a τ-generating family in $\mathfrak{A}$ such that

$$\langle a_{ij}\rangle_{j\in J} \text{ is disjoint for every } i \in I, \quad \sup_{i\in I} a_{ij} = 1 \text{ for every } j \in J.$$

Then $\tau(\mathfrak{A}) \le \max(\omega, \#(I))$.

proof (a) We may suppose that $J = \kappa$ is a cardinal. For i, $j \in I$ set

$$a_i^* = \sup_{\xi<\kappa} a_{i\xi}, \quad b_{ij} = \sup_{\xi<\eta<\kappa} a_{i\eta} \cap a_{j\xi}.$$

Then

$$\sup_{\eta\le\zeta} a_{i\eta} = a_i^* \setminus \sup_{j\in I}(b_{ij} \setminus \sup_{\xi<\zeta} a_{j\xi})$$

whenever $i \in I$ and $\zeta < \kappa$. **P** (i) If $\eta \le \zeta$ and $j \in I$, then $a_{i\eta} \subseteq a_i^*$ and

$$a_{i\eta} \cap b_{ij} = \sup_{\xi<\theta<\kappa} a_{i\eta} \cap a_{i\theta} \cap a_{j\xi} = \sup_{\xi<\eta} a_{i\eta} \cap a_{j\xi}$$

(because $\langle a_{i\theta}\rangle_{\theta<\kappa}$ is disjoint)

$$\subseteq \sup_{\xi<\zeta} a_{j\xi},$$

so

$$\sup_{\eta\le\zeta} a_{i\eta} \subseteq a_i^* \setminus \sup_{j\in I}(b_{ij} \setminus \sup_{\xi<\zeta} a_{j\xi}).$$

(ii) If $0 \ne c \subseteq a_i^*$ and $c \cap a_{i\eta} = 0$ for every $\eta \le \zeta$, there are an $\eta > \zeta$ such that $c' = c \cap a_{i\eta}$ is non-zero, and a $j \in I$ such that $c'' = c' \cap a_{j\zeta}$ is non-zero. In this case $c'' \subseteq b_{ij} \setminus \sup_{\xi<\zeta} a_{j\xi}$, so $c \cap b_{ij} \setminus \sup_{\xi<\zeta} a_{j\xi} \ne 0$. Accordingly

$$\sup_{\eta\le\zeta} a_{i\eta} \supseteq a_i^* \setminus \sup_{j\in I}(b_{ij} \setminus \sup_{\xi<\zeta} a_{j\xi}). \text{ } \mathbf{Q}$$

(b) Let $\mathfrak{B}$ be the order-closed subalgebra of $\mathfrak{A}$ generated by $\{a_i^* : i \in I\} \cup \{b_{ij} : i, j \in I\}$. Using (a) for the inductive step, we see that $\sup_{\xi\le\zeta} a_{i\xi} \in \mathfrak{B}$ for every $i \in I$ and $\zeta < \kappa$. Consequently $a_{i\zeta} = \sup_{\xi\le\zeta} a_{i\xi} \setminus \sup_{\xi<\zeta} a_{i\xi}$ belongs to $\mathfrak{B}$ whenever $i \in I$ and $\zeta < \kappa$, and $\mathfrak{A} = \mathfrak{B}$ is τ-generated by $\{a_i^* : i \in I\} \cup \{b_{ij} : i, j \in I\}$, so has Maharam type at most $\max(\omega, \#(I))$.

514G Order-preserving functions of Boolean algebras (a) Let F be an ordinal function of Boolean algebras, that is, a function defined on the class of Boolean algebras, taking ordinal values, and such that $F(\mathfrak{A}) = F(\mathfrak{B})$ whenever $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic. We say that F is **order-preserving** if $F(\mathfrak{B}) \le F(\mathfrak{A})$

whenever $\mathfrak{B}$ is a principal ideal of $\mathfrak{A}$. It is easy to check that all the cardinal functions defined in 511D are order-preserving; see 514Ed. Now a Boolean algebra $\mathfrak{A}$ is **F-homogeneous** if $F(\mathfrak{B}) = F(\mathfrak{A})$ for every non-zero principal ideal $\mathfrak{B}$ of $\mathfrak{A}$. Of course any principal ideal of an F-homogeneous Boolean algebra is again F-homogeneous.

We have already seen 'Maharam-type-homogeneous' algebras in Chapter 33. I mention **cellularity-homogeneous** algebras as a class which will be used later. The proof of the Erdős-Tarski theorem in 513Bb is based on the idea of upwards-saturation-homogeneous partially ordered set. Of course all the most important ordinal functions of Boolean algebras actually take cardinal values.

(b) If F is any order-preserving ordinal function of Boolean algebras, and $\mathfrak{A}$ is a Boolean algebra, then (writing $\mathfrak{A}_a$ for the principal ideal generated by a) $\{a : a \in \mathfrak{A}, \mathfrak{A}_a$ is F-homogeneous$\}$ is order-dense in $\mathfrak{A}$. **P** If $a \in \mathfrak{A}^+$, set $\xi = \min\{F(\mathfrak{A}_b) : 0 \ne b \subseteq a\}$, and let b be such that $0 \ne b \subseteq a$ and $F(\mathfrak{A}_b) = \xi$; then $\mathfrak{A}_b$ is F-homogeneous. **Q** So if $\mathfrak{A}$ is a Dedekind complete Boolean algebra, it is isomorphic to a simple product of F-homogeneous Boolean algebras. (Argue as in the proof of 332B.)

(c) Similarly, if $F_0, \dots, F_n$ are order-preserving ordinal functions of Boolean algebras, and $\mathfrak{A}$ is any Boolean algebra, then $\{a : \mathfrak{A}_a$ is F_i-homogeneous for every $i \le n\}$ is order-dense in $\mathfrak{A}$; and if $\mathfrak{A}$ is Dedekind complete, it is isomorphic to a simple product of Boolean algebras all of which are F_i-homogeneous for every $i \le n$.

(d) Of course any Boolean algebra which is homogeneous in the full sense (316N) is F-homogeneous for every function F of Boolean algebras. Maharam's theorem tells us that a Maharam-type-homogeneous *measurable* algebra is homogeneous (331N).

514H Regular open algebras: Proposition Let $(X, \mathfrak{T})$ be a topological space and $\mathrm{RO}(X)$ its regular open algebra (314O *et seq.*).

(a)(i) $(\mathrm{RO}(X)^+, \supseteq, \mathrm{RO}(X)^+) \preccurlyeq_{\mathrm{GT}} (\mathfrak{T} \setminus \{\emptyset\}, \supseteq, \mathfrak{T} \setminus \{\emptyset\})$.

(ii) If X is regular, $(\mathrm{RO}(X)^+, \supseteq, \mathrm{RO}(X)^+) \equiv_{\mathrm{GT}} (\mathfrak{T} \setminus \{\emptyset\}, \supseteq, \mathfrak{T} \setminus \{\emptyset\})$.

(b)(i) $\mathrm{sat}(\mathrm{RO}(X)) = \mathrm{sat}(X)$, $c(\mathrm{RO}(X)) = c(X)$, $\pi(\mathrm{RO}(X)) \le \pi(X)$ and $d(\mathrm{RO}(X)) \le d(X)$.

(ii) If X is regular, $\pi(\mathrm{RO}(X)) = \pi(X)$.

(iii) If X is locally compact and Hausdorff, $d(\mathrm{RO}(X)) = d(X)$.

(c) Let $\mathcal{N}\mathrm{wd}(X)$ be the ideal of nowhere dense subsets of X.

(i) If X is regular, $\mathrm{wdistr}(\mathrm{RO}(X)) \le \mathrm{add}\,\mathcal{N}\mathrm{wd}(X)$.

(ii) If X is locally compact and Hausdorff, $\mathrm{wdistr}(\mathrm{RO}(X)) = \mathrm{add}\,\mathcal{N}\mathrm{wd}(X)$.

(d) If $Y \subseteq X$ is dense, then $G \mapsto G \cap Y$ is a Boolean isomorphism from $\mathrm{RO}(X)$ to $\mathrm{RO}(Y)$.

proof (a) For $G \in \mathfrak{T} \setminus \{\emptyset\}$, set $\psi(G) = \mathrm{int}\,\overline{G}$. If we set $\phi(G) = G$ for $G \in \mathrm{RO}(X)^+$, then (ϕ, ψ) is a Galois-Tukey connection from $(\mathrm{RO}(X)^+, \supseteq, \mathrm{RO}(X)^+)$ to $(\mathfrak{T} \setminus \{\emptyset\}, \supseteq, \mathfrak{T} \setminus \{\emptyset\})$.

If X is regular, then $\mathrm{RO}(X)^+$ is coinitial with $\mathfrak{T} \setminus \{\emptyset\}$, so 513Ed, inverted, shows that they are equivalent.

(b)(i) Any disjoint family in $\mathrm{RO}(X)^+$ is a disjoint family of non-empty open subsets of X, so $c(\mathrm{RO}(X)) \le c(X)$ and $\mathrm{sat}(\mathrm{RO}(X)) \le \mathrm{sat}(X)$. On the other hand, if $\mathcal{G}$ is a disjoint family of non-empty open subsets of X, then $\langle \mathrm{int}\,\overline{G} \rangle_{G \in \mathcal{G}}$ is a disjoint family in $\mathrm{RO}(X)^+$, so $c(X) \le c(\mathrm{RO}(X))$ and $\mathrm{sat}(\mathrm{RO}(X)) \le \mathrm{sat}(X)$.

By (a) and 513Ee, inverted,

$$\pi(\mathrm{RO}(X)) = \mathrm{ci}(\mathrm{RO}(X)^+) \le \mathrm{ci}(\mathfrak{T} \setminus \{\emptyset\}) = \pi(X),$$

$$d(\mathrm{RO}(X)) = d^{\downarrow}(\mathrm{RO}(X)^+) \le d^{\downarrow}(\mathfrak{T} \setminus \{\emptyset\} \le d(X)$$

by 514A.

(ii) If X is regular, $\mathrm{RO}(X)^+$ is coinitial with $\mathfrak{T} \setminus \{\emptyset\}$, so $\mathrm{ci}(\mathrm{RO}(X)^+) = \mathrm{ci}(\mathfrak{T} \setminus \{\emptyset\})$ and $\pi(\mathrm{RO}(X)) = \pi(X)$.

(iii) If X is locally compact and Hausdorff it is also regular, so $\mathrm{RO}(X)^+$ is coinitial with $\mathfrak{T} \setminus \{\emptyset\}$, and

$$d(X) = d^{\downarrow}(\mathfrak{T} \setminus \{\emptyset\})$$

(514A)

$$= d^{\downarrow}(\mathrm{RO}(X)^+) = d(\mathrm{RO}(X)).$$

(c)(i) Let $\langle E_\xi\rangle_{\xi<\kappa}$ be a family of nowhere dense sets in X, where $\kappa < \mathrm{wdistr}(\mathrm{RO}(X))$. For each $\xi < \kappa$, set $\mathcal{G}_\xi = \{G : G \in \mathrm{RO}(X),\ \overline{G} \cap E_\xi = \emptyset\}$. Then $\mathcal{G}_\xi$ is upwards-directed, and $\bigcup \mathcal{G}_\xi = X \setminus \overline{E}_\xi$, because any point of $X \setminus \overline{E}_\xi$ belongs to a regular open set with closure disjoint from E_ξ. But this means that $\sup \mathcal{G}_\xi = X$ in $\mathrm{RO}(X)$ (314P), and there is a partition $\mathcal{G}'_\xi$ of unity included in $\mathcal{G}_\xi$. Because $\kappa < \mathrm{wdistr}(\mathrm{RO}(X))$, there is a partition $\mathcal{H}$ of unity in $\mathrm{RO}(X)$ such that $\{G : G \in \mathcal{G}'_\xi,\ G \cap H \neq \emptyset\}$ is finite for each ξ and $H \in \mathcal{H}$. It follows that $H \subseteq \bigcup\{\overline{G} : G \in \mathcal{G}'_\xi\}$ is disjoint from E_ξ whenever $\xi < \kappa$ and $H \in \mathcal{H}$. Accordingly $\bigcup_{\xi<\kappa} E_\xi$ is disjoint from $\bigcup \mathcal{H}$ and is nowhere dense. As $\langle E_\xi\rangle_{\xi<\kappa}$ is arbitrary, $\mathrm{add}\,\mathcal{N}\mathrm{wd}(X) \geq \mathrm{wdistr}(\mathrm{RO}(X))$.

(ii) If X is locally compact and Hausdorff, suppose that $\kappa < \mathrm{add}\,\mathcal{N}\mathrm{wd}(X)$ and that $\langle \mathcal{G}_\xi\rangle_{\xi<\kappa}$ is a family of partitions of unity in $\mathrm{RO}(X)$. Then $E_\xi = X \setminus \bigcup \mathcal{G}_\xi$ is a nowhere dense closed set for each ξ (314P again). So $E = \bigcup_{\xi<\kappa} E_\xi$ is nowhere dense. Set

$$\mathcal{U} = \{U : U \subseteq X \text{ is open, } \overline{U} \subseteq X \setminus E \text{ is compact}\};$$

then $\mathcal{U}$ is an upwards-directed family with union $X \setminus \overline{E}$, so includes a partition $\mathcal{G}$ of unity. But if $H \in \mathcal{G}$ and $\xi < \kappa$, $\overline{H}$ is a compact set disjoint from E_ξ, so must be included in the union of some finite subfamily from $\mathcal{G}_\xi$, and $\{G : G \in \mathcal{G}_\xi,\ G \cap H \neq \emptyset\}$ is finite. As $\langle \mathcal{G}_\xi\rangle_{\xi<\kappa}$ is arbitrary, $\mathrm{wdistr}(\mathrm{RO}(X)) \geq \mathrm{add}\,\mathcal{N}\mathrm{wd}(X)$ and we have equality.

(d) If $Y \subseteq X$ is dense, and we write int_Y, $\overline{}^{(Y)}$ for interior and closure in the subspace topology of Y, we have

$$\mathrm{int}_Y\,\overline{G \cap Y}^{(Y)} = \mathrm{int}_Y(Y \cap \overline{G \cap Y}) = \mathrm{int}_Y(Y \cap \overline{G}) = Y \cap \mathrm{int}\,\overline{G}$$

for every open set $G \subseteq X$. Let $f : Y \to X$ be the identity map. Then f is continuous and $f^{-1}[M] = Y \cap M$ is nowhere dense in Y whenever $M \subseteq X$ is nowhere dense in X, so we have a corresponding Boolean homomorphism $\pi : \mathrm{RO}(X) \to \mathrm{RO}(Y)$ defined by setting

$$\pi G = \mathrm{int}_Y\,\overline{f^{-1}[G]}^{(Y)} = \mathrm{int}_Y\,\overline{G \cap Y}^{(Y)} = Y \cap \mathrm{int}\,\overline{G} = G \cap Y$$

for every $G \in \mathrm{RO}(X)$ (314Ra). Because Y is dense, $\pi G \neq \emptyset$ for every non-empty G, and π is injective. If $H \in \mathrm{RO}(Y) \setminus \{\emptyset\}$, then there is an open set $G \subseteq X$ such that $H = G \cap Y$, so that

$$\pi(\mathrm{int}\,\overline{G}) = Y \cap \mathrm{int}\,\overline{G} = \mathrm{int}_Y\,\overline{G \cap Y}^{(Y)} = H;$$

thus π is surjective and is an isomorphism.

514I Category algebras For many topological spaces, their regular open algebras can be understood better through their expressions as quotients of Baire-property algebras. It is time I brought this approach into the main line of the argument.

(a) Let X be a topological space, and $\mathcal{M}$ the σ-ideal of meager subsets of X. Recall that the Baire-property algebra of X is the σ-algebra $\widehat{\mathcal{B}} = \{G \triangle A : G \subseteq X \text{ is open, } A \in \mathcal{M}\}$, and that the category algebra of X is the quotient Boolean algebra $\mathfrak{G} = \widehat{\mathcal{B}}/\mathcal{M}$ (4A3R[1]). Note that if $G \subseteq X$ is any open set, then $\overline{G} \setminus G$ and $\overline{G} \setminus \mathrm{int}\,\overline{G}$ are nowhere dense, so

$$G^\bullet = \overline{G}^\bullet = (\mathrm{int}\,\overline{G})^\bullet$$

in $\mathfrak{G}$.

(b) For $G \in \mathrm{RO}(X)$, set $\pi G = G^\bullet \in \mathfrak{G}$. Then $\pi : \mathrm{RO}(X) \to \mathfrak{G}$ is an order-continuous surjective Boolean homomorphism. **P** (i) If G, $H \in \mathrm{RO}(X)$, then

$$G \cap_{\mathrm{RO}(X)} H = G \cap H, \quad X \setminus_{\mathrm{RO}(X)} G = X \setminus \overline{G},$$

(314P), so

[1] Formerly 4A3Q.

$$(G \cap_{\mathrm{RO}(X)} H)^\bullet = (G \cap H)^\bullet = G^\bullet \cap H^\bullet,$$

$$(X \setminus_{\mathrm{RO}(X)} G)^\bullet = (X \setminus \overline{G})^\bullet = 1 \setminus \overline{G}^\bullet = 1 \setminus G^\bullet.$$

By 312H(ii), this is enough to show that π is a Boolean homomorphism. (ii) If $E \in \widehat{\mathcal{B}}$, let $G_0 \subseteq X$ be an open set such that $G_0 \triangle E \in \mathcal{M}$; then $G = \operatorname{int} \overline{G}_0$ belongs to $\mathrm{RO}(X)$ and

$$\pi G = G^\bullet = G_0^\bullet = E^\bullet.$$

Thus π is surjective. (iii) There is a regular open set W such that $X \setminus W$ is meager and every non-empty open subset of W is non-meager (4A3Sa[2]); now the kernel of π is just $\{G : G \in \mathrm{RO}(X), G \cap W = \emptyset\}$ which has a largest member $\mathrm{int}(X \setminus W)$. This shows that the kernel of π is order-closed, so that π is order-continuous (313P(a-ii)). **Q**

(c) From the last part of the proof of (b), we see that the kernel of π is the principal ideal of $\mathrm{RO}(X)$ generated by $X \setminus \overline{W}$, so that in fact π includes an isomorphism between the complementary principal ideal generated by W and $\mathfrak{G}$.

In particular, being isomorphic to a principal ideal in the Dedekind complete Boolean algebra $\mathrm{RO}(X)$, $\mathfrak{G}$ is Dedekind complete (314Xd, 314Ea).

(d) It is useful to know that if $G \subseteq X$ is open, then the category algebra of G can be identified with the principal ideal of $\mathfrak{G}$ generated by $G^\bullet$; this is because a subset of G is nowhere dense regarded as a subset of G iff it is nowhere dense regarded as a subset of X, so that $\mathcal{M} \cap \mathcal{P}G$ is exactly the ideal of meager subsets of G for the subspace topology, while the Borel σ-algebra of G is $\{G \cap E : E \subseteq X \text{ is Borel}\}$ (4A3Ca).

(e) Recall from 431Fa that every $A \subseteq X$ has a Baire-property envelope, that is, a set $E \in \widehat{\mathcal{B}}$ such that $A \subseteq E$ and $E \setminus F$ is meager whenever $F \in \widehat{\mathcal{B}}$ and $A \subseteq F$. If $\langle A_n \rangle_{n \in \mathbb{N}}$ is any sequence of subsets of X, and E_n is a Baire-property envelope of A_n for each n, then $E = \bigcup_{n \in \mathbb{N}} E_n$ is a Baire-property envelope of $A = \bigcup_{n \in \mathbb{N}} A_n$. **P** Of course $A \subseteq E \in \widehat{\mathcal{B}}$. If $A \subseteq F \in \widehat{\mathcal{B}}$, then $A_n \subseteq F$ for every n, so $E_n \setminus F$ is meager for every n and $E \setminus F$ is meager. **Q**

If $A \subseteq X$, we can define $\psi(A) \in \mathfrak{G}$ by setting $\psi(A) = \inf\{F^\bullet : A \subseteq F \in \widehat{\mathcal{B}}\}$, because $\mathfrak{G}$ is Dedekind complete. Note that $\psi(A) = E^\bullet$ for any Baire-property envelope E of A. It follows that $\psi(\bigcup_{n \in \mathbb{N}} A_n) = \sup_{n \in \mathbb{N}} \psi(A_n)$ for any sequence $\langle A_n \rangle_{n \in \mathbb{N}}$ of subsets of X; also $\psi(A) = 0$ in $\mathfrak{G}$ iff A is meager.

(f) The construction here is most useful when X is a Baire space, so that no non-empty open set is meager, π is injective and is an isomorphism between $\mathrm{RO}(X)$ and $\mathfrak{G}$.

(g) If X is a zero-dimensional space, then the algebra $\mathcal{E}$ of open-and-closed sets in X is an order-dense subalgebra of $\mathrm{RO}(X)$, so that $\mathrm{RO}(X)$ can be identified with the Dedekind completion of $\mathcal{E}$; and if X is a zero-dimensional compact Hausdorff space, then the category algebra of X can equally be identified with the Dedekind completion of $\mathcal{E}$.

(h) Finally, I note that if X is an extremally disconnected compact Hausdorff space, so that its algebra $\mathcal{E}$ of open-and-closed sets is already Dedekind complete (314S), then $\mathcal{E} = \mathrm{RO}(X)$. So if X is the Stone space of a Dedekind complete Boolean algebra $\mathfrak{A}$, we have a Boolean isomorphism $a \mapsto \widehat{a}^\bullet$ from $\mathfrak{A}$ to $\mathfrak{G}$, writing $\widehat{a}$ for the open-and-closed subset of X corresponding to $a \in \mathfrak{A}$.

514J Now we have the following.

Proposition Let X be a topological space and $\mathfrak{C}$ its category algebra.

(a) $\mathrm{sat}(\mathfrak{C}) \le \mathrm{sat}(X)$, $c(\mathfrak{C}) \le c(X)$, $\pi(\mathfrak{C}) \le \pi(X)$ and $d(\mathfrak{C}) \le d(X)$.

(b) If X is a Baire space, $\mathrm{sat}(\mathfrak{C}) = \mathrm{sat}(X)$ and $c(\mathfrak{C}) = c(X)$.

(c) If X is regular, $\mathrm{wdistr}(\mathfrak{C}) \le \mathrm{add}\,\mathcal{N}\mathit{wd}(X)$, where $\mathcal{N}\mathit{wd}(X)$ is the ideal of nowhere dense subsets of X.

proof All we need to know is that $\mathfrak{C}$ is isomorphic to a principal ideal of $\mathrm{RO}(X)$, which is the whole of $\mathrm{RO}(X)$ if X is a Baire space (514Ic, 514If), and apply 514H and 514Ed.

[2] Formerly 4A3Ra.

514K Later in this volume, we shall see that the Lebesgue measure algebra, in particular, can have weak distributivity large compared with its cellularity and its Maharam type. For such algebras the following result gives us significant information.

Proposition Let $\mathfrak{A}$ be a Boolean algebra such that $\mathrm{sat}(\mathfrak{A}) \le \mathrm{wdistr}(\mathfrak{A})$. Then whenever $A \subseteq \mathfrak{A}$ and $\#(A) < \mathrm{wdistr}(\mathfrak{A})$ there is a set $C \subseteq \mathfrak{A}$ such that $\#(C) \le \max(c(\mathfrak{A}), \tau(\mathfrak{A}))$ and $a = \sup\{c : c \in C,\, c \subseteq a\}$ for every $a \in A$.

proof (a) If $\mathfrak{A}$ is finite we can take C to be the set of atoms of $\mathfrak{A}$; so let us henceforth suppose that $\mathfrak{A}$ is infinite. Let $D \subseteq \mathfrak{A}$ be a τ-generating set of cardinal $\tau(\mathfrak{A})$, and $\mathfrak{D}$ the subalgebra of $\mathfrak{A}$ generated by D, so that (because $\mathfrak{A}$ is infinite) $\#(\mathfrak{D}) = \tau(\mathfrak{A})$. For any $a \in \mathfrak{A}$, write

$$Q(a) = \{b : b \in \mathfrak{A},\, \exists\ d \in \mathfrak{D},\, (a \triangle d) \cap b = 0\},$$

$$\mathcal{E}(a) = \{B : B \text{ is a maximal antichain, } \sup B' \in Q(a) \text{ for every finite } B' \subseteq B\}.$$

Now the first fact to establish is that $\mathcal{E}(a) \ne \emptyset$ for any $a \in \mathfrak{A}$.

P Set $E = \{a : \mathcal{E}(a) \ne \emptyset\}$. Then $\mathfrak{D} \subseteq E$, because $1 \in Q(d)$ and $\{1\} \in \mathcal{E}(d)$ for every $d \in \mathfrak{D}$. If $a \in E$, then $Q(1 \setminus a) = Q(a)$ (because $1 \setminus d \in \mathfrak{D}$ for every $d \in \mathfrak{D}$), so $\mathcal{E}(1 \setminus a) = \mathcal{E}(a)$ is non-empty, and $1 \setminus a \in E$. If $F \subseteq E$ is non-empty and has supremum $a \in \mathfrak{A}$, then there is a non-empty set $F_0 \subseteq F$, still with supremum a, such that $\#(F_0) < \mathrm{sat}(\mathfrak{A})$ (514Db). For each $c \in F_0$ choose $B_c \in \mathcal{E}(c)$. Because $\#(F_0) < \mathrm{wdistr}(\mathfrak{A})$, there is a maximal antichain $B \subseteq \mathfrak{A}$ such that $\{e : e \in B_c,\, e \cap b \ne 0\}$ is finite for every $c \in F_0$. If $B' \subseteq B$ is finite and $c \in F_0$, then $\sup B' \subseteq \sup B'_c$ where $B'_c = \{e : e \in B_c,\, e \cap \sup B' \ne 0\}$, so $\sup B' \in Q(c)$. Set

$$\tilde{D} = \{b : \text{there are } b' \in B \text{ and } c \in F_0 \text{ such that } b \subseteq b' \setminus (a \setminus c)\}.$$

Because $\sup F_0 = a$ and $\sup B = 1$, $\sup \tilde{D} = 1$ and there is a maximal antichain $\tilde{B} \subseteq \tilde{D}$. If $B' \subseteq \tilde{B}$ is finite, with supremum b^*, there are $c_0, \dots, c_n \in F_0$ such that b^* is disjoint from $a \setminus \sup_{i \le n} c_i$; also $b^* \in Q(c_i)$ for each i. So we can find $d_i \in \mathfrak{D}$ such that $c_i \triangle d_i$ is disjoint from b^* for each $i \le n$; accordingly $c \triangle d$ is disjoint from b^*, where $c = \sup_{i \le n} c_i$ and $d = \sup_{i \le n} d_i$, and

$$a \triangle d \subseteq (a \triangle c) \cup (c \triangle d) \subseteq (a \setminus c) \cup (c \triangle d) \subseteq 1 \setminus b^*,$$

while $d \in \mathfrak{D}$. This shows that $b^* \in Q(a)$; as B' is arbitrary, $\tilde{B} \in \mathcal{E}(a)$ and $a \in E$.

This shows that E is closed under complements and arbitrary suprema. It is therefore an order-closed subalgebra of $\mathfrak{A}$ (312B(iii), 313E(a-i)); since it includes $\mathfrak{D}$, it is the whole of $\mathfrak{A}$, which is what we need to know. **Q**

(b) Now turn to the given set A. For each $a \in A$ choose $B_a \in \mathcal{E}(a)$. Then there is a maximal antichain B such that $\{e : e \in B_a,\, e \cap b \ne 0\}$ is finite for every $b \in B$ and $a \in A$. Of course $\#(B) < \mathrm{sat}(\mathfrak{A})$. Set $C = \{d \cap b : d \in \mathfrak{D},\, b \in B\}$. Then

$$\#(C) \le \max(\omega, \#(B), \#(\mathfrak{D})) \le \max(c(\mathfrak{A}), \tau(\mathfrak{A})).$$

? Suppose that $a \in A$ is not the supremum of $C' = \{c : c \in C,\, c \subseteq a\}$. Let $a' \subseteq a$ be non-zero and disjoint from every member of C'. Then there is a $b \in B$ such that $b \cap a' \ne 0$. As b is covered by finitely many members of B_a it belongs to $Q(a)$, and there is a $d \in D$ such that $(a \triangle d) \cap b = 0$; which means that

$$0 \ne a' \cap b \subseteq a \cap b = d \cap b,$$

while $d \cap b \in C$. Thus $d \cap b \in C'$; but a' is supposed to be disjoint from every member of C'. **X**

Thus C has the properties we need.

514L The regular open algebra of a pre-ordered set Many important features of pre-ordered sets, at least in those aspects which are of concern to us here, can be related to the regular open algebras of suitable topologies.

Definitions (a) For any pre-ordered set P, a subset G of P is **up-open** if $[p, \infty[\ \subseteq G$ whenever $p \in G$. The family of such sets is a topology on P, the **up-topology**. Similarly, the **down-topology** of P is the family of **down-open** sets H such that $p \le q \in H \Rightarrow p \in H$. Note that $G \subseteq P$ is up-open iff it is closed for the down-topology, and vice versa. In particular, the intersection of any non-empty family of up-open sets is again up-open..

(b) I will write $\mathrm{RO}^\uparrow(P)$ for the regular open algebra of P when P is given its up-topology, and $\mathrm{RO}^\downarrow(P)$ for the regular open algebra of P when P is given its down-topology.

514M These up- and down-topologies, entirely unrelated to the usual 'order topology' on a totally ordered set (4A2A) and the ideas of order-convergence considered in Volume 3, take a bit of getting used to. Their characteristic property is that every point p has a smallest neighbourhood $[p,\infty[$; see 514Xj. I begin with an elementary lemma for practice.

Lemma Let P be a pre-ordered set endowed with its up-topology.

(a)(i) For any $A \subseteq P$, $\overline{A} = \{p : A \cap [p,\infty[\ne \emptyset\}$.

(ii) For any $p \in P$, $\overline{[p,\infty[}$ is the set of elements of P which are compatible upwards with p.

(iii) For any p, $q \in P$, the following are equiveridical: (α) $q \in \operatorname{int}\overline{[p,\infty[}$; (β) every member of $[q,\infty[$ is compatible upwards with p; (γ) q is incompatible upwards with every $r \in P$ which is incompatible upwards with p.

(b) A subset of P is dense iff it is cofinal.

(c) If Q is another pre-ordered set with its up-topology, a function $f : P \to Q$ is continuous iff it is order-preserving.

(d)(i) A subset G of P is a regular open set iff

$$G = \{p : G \cap [q,\infty[\ne \emptyset \text{ for every } q \ge p\}.$$

(ii) If $\mathcal{G}$ is a non-empty family of regular open subsets of P, then $\bigcap\mathcal{G}$ is a regular open subset of P, and is $\inf\mathcal{G}$ in the regular open algebra $\mathrm{RO}^\uparrow(P)$.

(e) P is separative upwards iff all the sets $[p,\infty[$ are regular open sets.

(f) If P is separative upwards and $A \subseteq P$ has a supremum p in P, then $[p,\infty[= \inf_{q\in A}[q,\infty[$ in $\mathrm{RO}^\uparrow(P)$.

proof (a) For (i), we need only note that $[p,\infty[$ is the smallest open set containing p. Now (ii) amounts to a restatement of the definition of 'compatible upwards'. As for (iii),

$$\begin{aligned} q \in \operatorname{int}\overline{[p,\infty[} &\iff [q,\infty[\subseteq \overline{[p,\infty[} \\ &\iff [q',\infty[\cap [p,\infty[\ne \emptyset \text{ for every } q' \ge q \end{aligned}$$

(by (i))

$$\iff [q,\infty[\cap [r,\infty[= \emptyset \text{ whenever } [r,\infty[\cap [p,\infty[= \emptyset$$

because

$$P \setminus \overline{[p,\infty[} = \bigcup\{[r,\infty[: [r,\infty[\cap [p,\infty[= \emptyset\}.$$

(b) $\mathcal{U} = \{[p,\infty[: p \in P\}$ is a base for the up-topology, so a subset of P is dense iff it meets every member of $\mathcal{U}$; but this is the same thing as saying that it is cofinal.

(c) If f is order-preserving and $H \subseteq Q$ is up-open, then

$$p' \ge p \in f^{-1}[H] \Longrightarrow f(p') \ge f(p) \in H \Longrightarrow f(p') \in H,$$

so $f^{-1}[H]$ is up-open; as H is arbitrary, f is continuous. If f is continuous and $p \le p'$ in P, then $H = [f(p),\infty[$ is up-open, so $f^{-1}[H]$ is up-open and must contain p', that is, $f(p') \ge f(p)$; as p and p' are arbitrary, f is order-preserving.

(d)(i) For any set $A \subseteq P$,

$$\{p : A \cap [q,\infty[\ne \emptyset \text{ for every } q \ge p\} = \{p : [p,\infty[\subseteq \overline{A}\} = \operatorname{int}\overline{A}$$

(using (a)).

(ii) As noted in 514L, $\bigcap\mathcal{G}$ is open, so is equal to its interior; but 314P tells us that $\operatorname{int}\bigcap\mathcal{G}$ is $\inf\mathcal{G}$ in $\mathrm{RO}(P)$.

(e)

P is separative upwards

$$\iff \forall\ p,\ q \in P,\ \text{ either } p \le q \text{ or } \ \exists\ r,\ r \ge q,\ [r,\infty[\,\cap\,[p,\infty[\, = \emptyset$$

(511Bk)

$$\iff \forall\ p,\ q \in P,\ \text{ either } q \in [p,\infty[\ \text{ or } q \notin \operatorname{int}\overline{[p,\infty[}$$

((a-iii) above)

$$\iff \forall\ p \in P,\ \operatorname{int}\overline{[p,\infty[} \subseteq [p,\infty[$$

$$\iff \forall\ p \in P,\ [p,\infty[\ \text{ is a regular open set.}$$

(f) $[p,\infty[$ is actually the intersection $\bigcap_{q\in A}[q,\infty[$.

514N Proposition Let $(P,\le)$ be a pre-ordered set, and write $\mathfrak{T}^\uparrow$ for the up-topology of P and $\mathrm{RO}^\uparrow(P)$ for the regular open algebra of $(P,\mathfrak{T}^\uparrow)$.

(a) $(\mathrm{RO}^\uparrow(P)^+,\supseteq,\mathrm{RO}^\uparrow(P)^+) \preccurlyeq_{\mathrm{GT}} (\mathfrak{T}^\uparrow\setminus\{\emptyset\},\supseteq,\mathfrak{T}^\uparrow\setminus\{\emptyset\}) \equiv_{\mathrm{GT}} (P,\le,P)$. If P is separative upwards, then $(\mathrm{RO}^\uparrow(P)^+,\supseteq,\mathrm{RO}^\uparrow(P)^+) \equiv_{\mathrm{GT}} (P,\le,P)$.

(b) $\pi(\mathrm{RO}^\uparrow(P)) \le \pi(P,\mathfrak{T}^\uparrow) = d(P,\mathfrak{T}^\uparrow) = \operatorname{cf}P$. If P is separative upwards, then we have equality.

(c) $\mathrm{sat}^\uparrow(P,\le) = \mathrm{sat}(P,\mathfrak{T}^\uparrow) = \mathrm{sat}(\mathrm{RO}^\uparrow(P))$ and $c^\uparrow(P,\le) = c(P,\mathfrak{T}^\uparrow) = c(\mathrm{RO}^\uparrow(P))$.

(d) For any cardinal κ,

$$\mathrm{link}_{<\kappa}(\mathrm{RO}^\uparrow(P)) \le \mathrm{link}^\uparrow_{<\kappa}(P,\le),$$

with equality if *either* P is separative upwards *or* $\kappa \le \omega$. In particular, we always have

$$\mathrm{link}^\uparrow(P,\le) = \mathrm{link}(\mathrm{RO}^\uparrow(P)),\qquad d^\uparrow(P,\le) = d(\mathrm{RO}^\uparrow(P)).$$

(e) If $Q \subseteq P$ is cofinal, then $\mathrm{RO}^\uparrow(Q) \cong \mathrm{RO}^\uparrow(P)$.

(f) If $A \subseteq P$ is a maximal up-antichain, then $\mathrm{RO}^\uparrow(P) \cong \prod_{a\in A}\mathrm{RO}^\uparrow([a,\infty[)$.

(g) If $\tilde P$ is the partially ordered set of equivalence classes associated with P, then $\mathrm{RO}^\uparrow(\tilde P) \cong \mathrm{RO}^\uparrow(P)$.

proof (a) By 514Ha,

$$(\mathrm{RO}^\uparrow(P)^+,\supseteq,\mathrm{RO}^\uparrow(P)^+) \preccurlyeq_{\mathrm{GT}} (\mathfrak{T}^\uparrow\setminus\{\emptyset\},\supseteq,\mathfrak{T}^\uparrow\setminus\{\emptyset\}).$$

Next, observe that $\mathcal{U} = \{[p,\infty[\,: p \in P\}$ is a base for $\mathfrak{T}^\uparrow$, so that

$$(\mathfrak{T}^\uparrow\setminus\{\emptyset\},\supseteq,\mathfrak{T}^\uparrow\setminus\{\emptyset\}) \equiv_{\mathrm{GT}} (\mathcal{U},\supseteq,\mathcal{U})$$

by 513Ed (inverted, as usual). If we set $\phi(p) = [p,\infty[$ for $p \in P$, and choose $\psi(U) \in P$ such that $U = [\psi(U),\infty[$ for $U \in \mathcal{U}$, then (ϕ,ψ) is a Galois-Tukey connection from $(P,\le,P)$ to $(\mathcal{U},\supseteq,\mathcal{U})$, while (ψ,ϕ) is a Galois-Tukey connection in the reverse direction; so $(P,\le) \equiv_{\mathrm{GT}} (\mathcal{U},\supseteq)$.

If P is separative upwards, then $\mathcal{U}$ is included in $\mathrm{RO}^\uparrow(P)$ (514Me) and is coinitial with $\mathrm{RO}^\uparrow(P)^+$, so

$$(\mathrm{RO}^\uparrow(P)^+,\supseteq,\mathrm{RO}^\uparrow(P)^+) \equiv_{\mathrm{GT}} (\mathcal{U},\supseteq,\mathcal{U}) \equiv_{\mathrm{GT}} (P,\le,P).$$

(b) Now

$$\pi(\mathrm{RO}^\uparrow(P)) \le \pi(P,\mathfrak{T}^\uparrow)$$

(514H(b-i))

$$= \operatorname{ci}(\mathfrak{T}^\uparrow\setminus\{\emptyset\}) = \operatorname{ci}\mathcal{U} = \operatorname{cf}P,$$

defining $\mathcal{U}$ as in (a) above. By 514Mb, $\operatorname{cf}P = d(P,\mathfrak{T}^\uparrow)$. If P is separative upwards, then $\pi(\mathrm{RO}^\uparrow(P)) = \operatorname{cf}P$ because $(\mathrm{RO}^\uparrow(P)^+,\supseteq,\mathrm{RO}^\uparrow(P)^+) \equiv_{\mathrm{GT}} (P,\le,P)$.

(c) Similarly, again using 514H(b-i), and with 512Dc at the last step,

$$\mathrm{sat}(\mathrm{RO}^\uparrow(P)) = \mathrm{sat}(P,\mathfrak{T}^\uparrow) = \mathrm{sat}^\downarrow(\mathfrak{T}^\uparrow\setminus\{\emptyset\}) = \mathrm{sat}^\downarrow(\mathcal{U}) = \mathrm{sat}^\uparrow(P).$$

Now we saw in 514Da and 513Bc that cellularity is determined by saturation both for partially ordered sets and for Boolean algebras, so $c(\mathrm{RO}^\uparrow(P)) = c^\uparrow(P)$. (Of course this is easily shown by a direct argument.)

(d) Using (a) and 512Dd, we see that

$$\begin{aligned}\mathrm{link}_{<\kappa}(\mathrm{RO}^{\uparrow}(P)) &= \mathrm{link}_{<\kappa}(\mathrm{RO}^{\uparrow}(P)^+, \supseteq, \mathrm{RO}^{\uparrow}(P)^+)\\ &\le \mathrm{link}_{<\kappa}(P, \le, P) = \mathrm{link}^{\uparrow}_{<\kappa}(P, \le),\end{aligned}$$

with equality if P is separative upwards. For other P, if $\kappa \le \omega$, set $\lambda = \mathrm{link}_{<\kappa}(\mathrm{RO}^{\uparrow}(P))$ and let $\langle \mathcal{H}_\xi \rangle_{\xi<\lambda}$ be a cover of $\mathrm{RO}^{\uparrow}(P)^+$ by $<\kappa$-linked sets. Set $A_\xi = \{p : \mathrm{int}\,\overline{[p,\infty[} \in \mathcal{H}_\xi\}$ for each $\xi < \kappa$. Then any A_ξ is upwards-$<\kappa$-linked in P. **P?** Otherwise, there is an $I \in [A_\xi]^{<\kappa}$ which has no upper bound in P, that is, $\bigcap_{p\in I} [p,\infty[= \emptyset$. Now

$$\bigcap_{p\in I} \mathrm{int}\,\overline{[p,\infty[} \subseteq \bigcup_{i\in I}(\overline{[p,\infty[} \setminus [p,\infty[)$$

is an open set covered by finitely many nowhere dense sets and is therefore empty, so we have a finite subset of $\mathcal{H}_\xi$ with empty intersection. **XQ** So $\langle A_\xi \rangle_{\xi<\lambda}$ witnesses that $\mathrm{link}^{\uparrow}_{<\kappa}(P, \le) \le \lambda$ and again we have equality. In particular,

$$\mathrm{link}(\mathrm{RO}^{\uparrow}(P)) = \mathrm{link}_{<3}(\mathrm{RO}^{\uparrow}(P)) = \mathrm{link}^{\uparrow}_{<3}(P, \le) = \mathrm{link}^{\uparrow}(P, \le),$$

$$d(\mathrm{RO}^{\uparrow}(P)) = \mathrm{link}_{<\omega}(\mathrm{RO}^{\uparrow}(P)) = \mathrm{link}^{\uparrow}_{<\omega}(P, \le) = d^{\uparrow}(P, \le).$$

(e) Put 514Mb and 514Hd together.

(f) Because A is an up-antichain, $\langle [a,\infty[\rangle_{a\in A}$ is a disjoint family of open sets in P; because A is maximal, $\bigcup_{a\in A} [a,\infty[$ is cofinal, therefore dense. So 315H gives the result.

(g) Let $Q \subseteq P$ be a set meeting each equivalence class in just one point, so that $q \mapsto q^\bullet : Q \to \tilde{P}$ is a bijection. Then Q is cofinal with P, while with its subspace ordering Q is isomorphic to $\tilde{P}$. So

$$\mathrm{RO}^{\uparrow}(\tilde{P}) \cong \mathrm{RO}^{\uparrow}(Q) \cong \mathrm{RO}^{\uparrow}(P)$$

by (e).

514O Of course we very much want to be able to recognise cases in which two partially ordered sets have isomorphic regular open algebras; and it is also important to know when one $\mathrm{RO}^{\uparrow}(P)$ can be regularly embedded in another. The next four results give some of the known sufficient conditions for these.

Proposition Suppose that P and Q are pre-ordered sets and $f : P \to Q$ is an order-preserving function such that $f^{-1}[Q_0]$ is cofinal with P for every up-open cofinal $Q_0 \subseteq Q$. Then there is an order-continuous Boolean homomorphism $\pi : \mathrm{RO}^{\uparrow}(Q) \to \mathrm{RO}^{\uparrow}(P)$ defined by setting $\pi H = \mathrm{int}\,\overline{f^{-1}[H]}$ (taking the closure and interior with respect to the up-topology on P) for every $H \in \mathrm{RO}^{\uparrow}(Q)$. If $f[P]$ is cofinal with Q then π is injective, so is a regular embedding of $\mathrm{RO}^{\uparrow}(Q)$ in $\mathrm{RO}^{\uparrow}(P)$.

proof By 514Mc, f is continuous for the up-topologies. Moreover, $f^{-1}[M]$ is nowhere dense in P whenever $M \subseteq Q$ is nowhere dense in Q. **P** $Q_0 = Q \setminus \overline{M}$ is up-open and dense, therefore cofinal (514Mb), so $f^{-1}[Q_0]$ is up-open and dense, and $f^{-1}[M] \subseteq P \setminus f^{-1}[Q_0]$ is nowhere dense. **Q**

By 314Ra again, there is an order-continuous Boolean homomorphism $\pi : \mathrm{RO}^{\uparrow}(Q) \to \mathrm{RO}^{\uparrow}(P)$ defined by setting $\pi H = \mathrm{int}\,\overline{f^{-1}[H]}$ for every $H \in \mathrm{RO}^{\uparrow}(Q)$. Now

$$\begin{aligned} f[P] \text{ is cofinal } &\iff f[P] \text{ is dense}\\ &\implies f[P] \cap H \ne \emptyset \text{ for every } H \in \mathrm{RO}^{\uparrow}(Q) \setminus \{\emptyset\}\\ &\iff f^{-1}[H] \ne \emptyset \text{ for every } H \in \mathrm{RO}^{\uparrow}(Q) \setminus \{\emptyset\}\\ &\iff \pi H \ne \emptyset \text{ for every } H \in \mathrm{RO}^{\uparrow}(Q) \setminus \{\emptyset\} \iff \pi \text{ is injective.}\end{aligned}$$

So in this case π is a regular embedding of $\mathrm{RO}^{\uparrow}(Q)$ in $\mathrm{RO}^{\uparrow}(P)$.

514P Corollary Suppose that P and Q are pre-ordered sets, that $f : P \to Q$ is an order-preserving function and whenever $p \in P$, $q \in Q$ and $f(p) \le q$, there is a $p' \ge p$ such that $f(p') \ge q$. If $f[P]$ is *either* cofinal with Q *or* coinitial with Q, then $\mathrm{RO}^{\uparrow}(Q)$ can be regularly embedded in $\mathrm{RO}^{\uparrow}(P)$.

proof If $Q_0 \subseteq Q$ is up-open and cofinal, then $f^{-1}[Q_0]$ is cofinal with P. **P** Take any $p \in P$. Then there are a $q \in Q_0$ such that $q \geq f(p)$ and a $p' \geq p$ such that $f(p') \geq q$; as Q_0 is up-open, $p' \in f^{-1}[Q_0]$; as p is arbitrary, $f^{-1}[Q_0]$ is cofinal. **Q** So if $f[P]$ is cofinal with Q, we can use 514O. On the other hand, if $f[P]$ is coinitial with Q it is also cofinal with Q. **P** For $q \in Q$ there is a $p \in P$ such that $f(p) \leq q$; now our main hypothesis tells us that there is a $p' \in P$ such that $f(p') \geq q$. **Q** So we have the result in this case also.

514Q Proposition Let P and Q be pre-ordered sets, endowed with their up-topologies, and $f : P \to Q$ a function such that

whenever $A \subseteq P$ is a maximal up-antichain then $f{\restriction}A$ is injective and $f[A]$ is a maximal up-antichain in Q.

Then there is an injective order-continuous Boolean homomorphism $\pi : \mathrm{RO}^{\uparrow}(P) \to \mathrm{RO}^{\uparrow}(Q)$ defined by setting $\pi(\operatorname{int}\overline{[p,\infty[}) = \operatorname{int}\overline{[f(p),\infty[}$ for every $p \in P$. In particular, $\mathrm{RO}^{\uparrow}(P)$ can be regularly embedded in $\mathrm{RO}^{\uparrow}(Q)$. If $f[P]$ is cofinal with Q, then π is an isomorphism.

proof (a) For $p \in P$, set $H_p = \operatorname{int}\overline{[f(p),\infty[} \in \mathrm{RO}^{\uparrow}(Q)$. If $A \subseteq P$ is a maximal up-antichain, $\langle [f(p),\infty[\rangle_{p\in A}$ is a disjoint family of up-open subsets of Q with dense union, so $\langle H_p\rangle_{p\in A}$ is a partition of unity in $\mathrm{RO}^{\uparrow}(Q)$. It follows that $\langle H_p\rangle_{p\in A}$ must be disjoint for every up-antichain $A \subseteq P$. Moreover, if $p_0 \in P$ and $p_1 \in \operatorname{int}\overline{[p_0,\infty[}$ in P, we have a maximal up-antichain A containing p_0, and $A' = (A \setminus \{p_0\}) \cup \{p_1\}$ is an up-antichain; as $H_{p_1} \cap \bigcup_{p\in A, p\neq p_0} H_p = \emptyset$, H_{p_1} must be included in H_{p_0}.

(b) For $G \in \mathrm{RO}^{\uparrow}(P)$, set $\pi G = \sup\{H_p : p \in G\}$, the supremum being taken in $\mathrm{RO}^{\uparrow}(Q)$. If G, $G' \in \mathrm{RO}^{\uparrow}(P)$ are disjoint, then p and p' are incompatible upwards, so H_p and $H_{p'}$ are disjoint, whenever $p \in G$ and $p' \in G'$; accordingly πG and $\pi G'$ must be disjoint.

(c) If $p \in P$, then of course $H_p \subseteq \pi(\operatorname{int}\overline{[p,\infty[})$. On the other hand, if $p' \in \operatorname{int}\overline{[p,\infty[}$, then we saw in (a) that $H_{p'} \subseteq H_p$, so that $\pi(\operatorname{int}\overline{[p,\infty[})$ must be exactly H_p.

(d) If $\mathcal{G} \subseteq \mathrm{RO}^{\uparrow}(P)$ has supremum G_0 in $\mathrm{RO}^{\uparrow}(P)$, $\pi G_0 = \sup_{G\in\mathcal{G}} \pi G$ in $\mathrm{RO}^{\uparrow}(Q)$. **P** Of course $\pi G_0 \supseteq \pi G$ for every $G \in \mathcal{G}$. Let A be maximal among the up-antichains included in $\bigcup\mathcal{G}$, and extend A to a maximal up-antichain $A' \subseteq P$. Then $\langle H_p\rangle_{p\in A'}$ is a partition of unity in $\mathrm{RO}^{\uparrow}(Q)$, so $H = \sup_{p\in A} H_p$ and $H' = \sup_{p\in A'\setminus A} H_p$ are complementary elements of $\mathrm{RO}^{\uparrow}(Q)$. For every $p \in A$ there is a $G \in \mathcal{G}$ with $p \in G$, so that $H_p \subseteq \pi G$; accordingly $H \subseteq \sup_{G\in\mathcal{G}} \pi G$. On the other hand, take any $p \in G_0$. By the maximality of A, $G \cap [p',\infty[= \emptyset$ for every $p' \in A' \setminus A$ and $G \in \mathcal{G}$, so $[p,\infty[\cap [p',\infty[\subseteq G_0 \cap [p',\infty[= \emptyset$ for every $p' \in A' \setminus A$ and $H_p \cap H_{p'} = \emptyset$ for every $p' \in A' \setminus A$, that is, $H_p \cap H' = \emptyset$ and $H_p \subseteq H$. As p is arbitrary,

$$\pi G_0 \subseteq H \subseteq \sup_{G\in\mathcal{G}} \pi G \subseteq \pi G_0$$

and we have equality. **Q**

(e) Now we see that

$$\pi\emptyset = \emptyset,$$

$$\pi P = Q$$

(because if we take any maximal up-antichain $A \subseteq P$, πP includes $\sup_{p\in A} H_p$),

$$\pi G \cap \pi H = \emptyset \text{ whenever } G,\, H \in \mathrm{RO}^{\uparrow}(P) \text{ and } G \cap H = \emptyset,$$

$$\pi(\sup\mathcal{G}) = \sup \pi[\mathcal{G}] \text{ for every } \mathcal{G} \subseteq \mathrm{RO}^{\uparrow}(P).$$

By 312H(iv), π is a Boolean homomorphism, and by 313L(b-iv) it is order-continuous. Finally, $\pi G \neq \emptyset$ whenever $G \in \mathrm{RO}^{\uparrow}(P) \setminus \{\emptyset\}$, so π is injective and is a regular embedding.

(f) If $f[P]$ is cofinal with Q, then $\pi[\mathrm{RO}^{\uparrow}(P)]$ is order-dense in $\mathrm{RO}^{\uparrow}(Q)$. **P** Let $H \in \mathrm{RO}^{\uparrow}(Q)$ be non-empty. As $f[P]$ is dense, there is a $p \in P$ such that $f(p) \in H$. Now

$$\emptyset \neq \pi(\operatorname{int}\overline{[p,\infty[}) = \operatorname{int}\overline{[f(p),\infty[} \subseteq \operatorname{int}\overline{H} = H;$$

as H is arbitrary, we have the result. **Q** By 314Ia, π is an isomorphism. This completes the proof.

514R Corollary Let P and Q be pre-ordered sets. Suppose that there is a function $f : P \to Q$ such that $f[P]$ is cofinal with Q and, for p, $p' \in P$, p and p' are compatible upwards in P iff $f(p)$ and $f(p')$ are compatible upwards in Q. Then $\mathrm{RO}^{\uparrow}(P) \cong \mathrm{RO}^{\uparrow}(Q)$.

proof The point is that f satisfies the condition of 514Q. **P** Suppose that $A \subseteq P$ is a maximal up-antichain. If p, p' are distinct elements of A, then p and p' are incompatible upwards in P, so $f(p)$ and $f(p')$ are incompatible upwards in Q. This shows simultaneously that $f{\restriction}A$ is injective and that $f[A]$ is an up-antichain in Q. If q is any element of Q, there is a $p \in P$ such that $f(p) \ge q$; now there must be a $p' \in A$ such that p' is compatible upwards with p, in which case $f(p')$ is compatible upwards with $f(p)$ and therefore with q. So $f[A]$ is maximal; as A is arbitrary, we have the result. **Q**

So 514Q gives the result.

514S Proposition (a) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and P a pre-ordered set. Suppose that we have a function $f : P \to \mathfrak{A}^+$ such that, for p, $q \in P$,

$$f(p) \subseteq f(q) \text{ whenever } p \le q,$$

$$f(p) \cap f(q) = 0 \text{ whenever } p \text{ and } q \text{ are incompatible downwards in } P,$$

$$f[P] \text{ is order-dense in } \mathfrak{A}.$$

Then $\mathrm{RO}^{\downarrow}(P) \cong \mathfrak{A}$.

(b) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and $D \subseteq \mathfrak{A}$ an order-dense set not containing 0. Give D the ordering $\subseteq$, and write $\mathrm{RO}^{\downarrow}(D)$ for the regular open algebra of D with its down-topology. Then $\mathrm{RO}^{\downarrow}(D) \cong \mathfrak{A}$.

(c) Let $(X, \mathfrak{T})$ be a topological space and P a pre-ordered set. Suppose we have a function $g : P \to \mathfrak{T} \setminus \{\emptyset\}$ such that, for p, $q \in P$,

$$g(p) \subseteq g(q) \text{ whenever } p \le q,$$

$$g(p) \cap g(q) = \emptyset \text{ whenever } p \text{ and } q \text{ are incompatible downwards in } P,$$

$$g[P] \text{ is a } \pi\text{-base for } \mathfrak{T}.$$

Then $\mathrm{RO}^{\downarrow}(P) \cong \mathrm{RO}(X)$.

(d) Let $(X, \mathfrak{T})$ be a topological space and $\mathcal{U}$ a π-base for the topology of X not containing $\{\emptyset\}$. Give $\mathcal{U}$ the ordering $\subseteq$. Then $\mathrm{RO}^{\downarrow}(\mathcal{U}) \cong \mathrm{RO}(X)$.

proof (a)(i) The key is the following fact: if $p \in P$, $a \in \mathfrak{A}$ and $a \cap f(p) \ne 0$, then there is a $q \le p$ such that $f(q) \subseteq a$. **P** There is a $q_0 \in P$ such that $f(q_0) \subseteq a \cap f(p)$. Now q_0 and p cannot be incompatible downwards, so there is a $q \in \left]-\infty, q_0\right] \cap \left]-\infty, p\right]$, and in this case $f(q) \subseteq f(q_0) \subseteq a$. **Q**

(ii) For $G \in \mathrm{RO}^{\downarrow}(P)$, set $\pi G = \sup f[G]$ in $\mathfrak{A}$. Then $\pi : \mathrm{RO}^{\downarrow}(P) \to \mathfrak{A}$ is order-preserving. Of course $\pi(\emptyset) = 0$.

$\pi(G \cap H) = \pi G \cap \pi H$ for all G, $H \in \mathrm{RO}^{\downarrow}(P)$. **P** Because π is order-preserving, $\pi(G \cap H) \subseteq \pi G \cap \pi H$. **?** If $a = \pi G \cap \pi H \setminus \pi(G \cap H) \ne 0$, take $p \in G$ such that $a \cap f(p) \ne 0$; then there is a $p' \le p$ such that $f(p') \subseteq a$. Next, there must be a $q \in H$ such that $f(q) \cap f(p') \ne 0$, and a $q' \le q$ such that $f(q') \subseteq f(p')$. But now $q' \in \left]-\infty, p\right] \cap \left]-\infty, q\right] \subseteq G \cap H$, so $f(q') \subseteq \pi(G \cap H)$; while at the same time $f(q') \subseteq a$. **X** Thus $\pi(G \cap H) = \pi G \cap \pi H$. **Q**

$\pi(P \setminus \overline{G}) = 1 \setminus \pi G$ for every $G \in \mathrm{RO}^{\downarrow}(P)$. **P** (Perhaps I should say that $\overline{G}$ here is the closure of G for the down-topology of P.) Set $H = P \setminus \overline{G}$. Then $\pi G \cap \pi H = \pi(G \cap H) = 0$ by what we have just seen. **?** If $a = 1 \setminus (\pi G \cup \pi H)$ is non-zero, let $p_0 \in P$ be such that $f(p_0) \subseteq a$. Then $\left]-\infty, p_0\right]$ is a non-empty open set so must meet one of G, H. But if $p \in G \cup H$ then $f(p_0) \cap f(p) = 0$ so p_0 and p are incompatible downwards and, in particular, $p \not\le p_0$. **XQ**

So π is a Boolean homomorphism, and it is injective because $\pi G \supseteq f(p) \ne 0$ whenever $p \in G \in \mathrm{RO}^{\downarrow}(P)$. Finally, π is surjective. **P** If $a \in \mathfrak{A}$, set $G = \{p : f(p) \subseteq a\}$. Then G is down-open. If $q \notin G$, $f(q) \setminus a \ne 0$, so there is a $q_1 \le q$ such that $f(q_1) \cap a = 0$ and $\left]-\infty, q_1\right]$ does not meet G; accordingly $\left]-\infty, q\right] \not\subseteq \overline{G}$ and $q \notin \operatorname{int} \overline{G}$. So $G \in \mathrm{RO}^{\downarrow}(P)$. Because $f[P]$ is order-dense, $a = \sup f[G] = \pi G$ belongs to $\pi[\mathrm{RO}^{\downarrow}(P)]$. **Q**

Thus we have an isomorphism between $\mathrm{RO}^{\downarrow}(P)$ and $\mathfrak{A}$.

(b) Apply (a) to the identity map from D to $\mathfrak{A}$.

(c) Apply (a) to the map $p \mapsto \operatorname{int}\overline{g(p)} : P \to \mathrm{RO}(X)$.

(d) Apply (c) to the identity function from $\mathcal{U}$ to $\mathfrak{T}$.

514T Finite-support products At many points in this chapter we find ourselves seeking to relate partially ordered sets to Boolean algebras and topological spaces. In 511D and 512Eb I sought to describe the cardinal functions of topological spaces and Boolean algebras in terms of naturally associated partially ordered sets, and in 514L and 514N of this section I described constructions of topologies and Boolean algebras from partial orders. One of the most important constructions of general topology is that of 'product'. The matching construction in Boolean algebra is that of 'free product' (315I). I now come to the corresponding idea for partial orders.

Definition Let $\langle P_i\rangle_{i\in I}$ be a family of non-empty partially ordered sets. The **upwards finite-support product** $\bigotimes^{\uparrow}_{i\in I} P_i$ of $\langle P_i\rangle_{i\in I}$ is the set $\bigcup\{\prod_{i\in J} P_i : J \in [I]^{<\omega}\}$, ordered by saying that $p \le q$ iff $\operatorname{dom} p \subseteq \operatorname{dom} q$ and $p(i) \le q(i)$ for every $i \in \operatorname{dom} p$. Similarly, the **downwards finite-support product** $\bigotimes^{\downarrow}_{i\in I} P_i$ of $\langle P_i\rangle_{i\in I}$ is the same set $\bigcup\{\prod_{i\in J} P_i : J \in [I]^{<\omega}\}$, but ordered by saying that $p \le q$ iff $\operatorname{dom} q \subseteq \operatorname{dom} p$ and $p(i) \le q(i)$ for every $i \in \operatorname{dom} q$.

514U Proposition Let $\langle P_i\rangle_{i\in I}$ be a family of non-empty partially ordered sets, with upwards finite-support product $P = \bigotimes^{\uparrow}_{i\in I} P_i$.

(a) The regular open algebra $\mathrm{RO}^{\uparrow}(P)$ is isomorphic to the regular open algebra of $P^* = \prod_{i\in I} P_i$ when every P_i is given its up-topology.

(b) If I is finite, P^* is a cofinal subset of P, and the ordering of P^*, regarded as a subset of P, is the usual product partial order on P^*.

(c) If $Q_i \subseteq P_i$ is cofinal for each $i \in I$, then $\bigcup_{J\in[I]^{<\omega}} \prod_{i\in J} Q_i$ is cofinal with P. So $\operatorname{cf} P$ is at most $\max(\omega, \#(I), \sup_{i\in I} \operatorname{cf} P_i)$.

(d) $c^{\uparrow}(P) = \sup_{J\in[I]^{<\omega}} c^{\uparrow}(\prod_{i\in J} P_i)$.

proof (a) For $p \in P$, set

$$G_p = \{q : q \in P^*,\ q(i) \ge p(i) \text{ for every } i \in \operatorname{dom} p\}.$$

Then G_p is a non-empty open set in P^*. If $p \le p'$ in P, then $G_p \supseteq G_{p'}$. If p, $p' \in P$ are incompatible upwards in P, there must be an $i \in \operatorname{dom} p \cap \operatorname{dom} p'$ such that $p(i)$ and $p'(i)$ are incompatible upwards in P_i, in which case $G_p \cap G_{p'}$ is empty. If $V \subseteq P^*$ is a non-empty open set, take any $q \in V$. There is a finite set $J \subseteq I$ such that $V \supseteq \{q' : q' \in P^*,\ q'(i) \ge q(i) \text{ for every } i \in J\}$. Set $p = q{\restriction}J$; then $G_p \subseteq V$. So $p \mapsto G_p$ satisfies the conditions of 514Sc, inverted, and $\mathrm{RO}^{\uparrow}(P)$ is isomorphic to $\mathrm{RO}(P^*)$.

(b)-(c) These are immediate from the definition of the ordering of P. For the estimate of the cofinality of P, just take cofinal sets $Q_i \subseteq P_i$ such that $\#(Q_i) = \operatorname{cf} P_i$ for each i, and estimate $\#(\bigcup_{J\in[I]^{<\omega}} \prod_{i\in J} Q_i)$.

(d) We have

$$c^{\uparrow}(P) = c(\mathrm{RO}^{\uparrow}(P))$$

(514Nc)

$$= c(\mathrm{RO}^{\uparrow}(P^*))$$

((a) above))

$$= c(P^*)$$

(514Hb)

$$= \sup_{J\in[I]^{<\omega}} c(\prod_{i\in J} P_i)$$

(5A4Be, here taking the product topology on $\prod_{i\in I} P_i$)

$$= \sup_{J\in[I]^{<\omega}} c^{\uparrow}(\prod_{i\in J} P_i)$$

because if J is finite then the up-topology $\mathfrak{T}_J^{\uparrow}$ on $\prod_{i\in J} P_i$ is just the product of the up-topologies on the P_i, so we can use the identification of $c(\prod_{i\in J} P_i, \mathfrak{T}_J^{\uparrow})$ with $c^{\uparrow}(\prod_{i\in J} P_i)$.

514X Basic exercises (a) Let $\mathfrak{A}$ be a Boolean algebra. Show that $\operatorname{link}_{<\kappa}(\mathfrak{A}) = \pi(\mathfrak{A})$ for any $\kappa \ge \operatorname{sat}(\mathfrak{A})$.

(b) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras and $\mathfrak{A}$ their free product. Show that

$$d(\mathfrak{A}) \le \max(\omega, \#(I), \sup_{i\in I} d(\mathfrak{A}_i)), \quad \pi(\mathfrak{A}) \le \max(\omega, \#(I), \sup_{i\in I} \pi(\mathfrak{A}_i)),$$

$$c(\mathfrak{A}) \le \max(\omega, \sup_{i\in I} 2^{c(\mathfrak{A}_i)}).$$

(*Hint*: 4A1Db, 5A1Ha.)

(c) Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{B}$ a chargeable Boolean algebra (definition: 391Bb). Suppose that $\mathfrak{A}\setminus\{1\} \preccurlyeq_{\mathrm{T}} \mathfrak{B}\setminus\{1\}$. Show that $\mathfrak{A}$ is chargeable. (*Hint*: 391J.)

(d) Let κ be a cardinal and $\mathfrak{A}$ a Boolean algebra with cardinal at most 2^{κ}. (i) Show that $\mathfrak{A}$ is a homomorphic image of a κ-centered Boolean algebra. (*Hint*: if κ is infinite, $\{0,1\}^{2^{\kappa}}$ has density κ.) (ii) Show that if $\mathfrak{A}$ is Dedekind complete it is a homomorphic image of $\mathcal{P}\kappa$. (*Hint*: 514Ca, 314K.)

(e) Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{B}$ *either* a regularly embedded subalgebra of $\mathfrak{A}$ *or* a quotient $\mathfrak{A}/I$ where I is an order-closed ideal in $\mathfrak{A}$. Let $\operatorname{Pou}(\mathfrak{A})$, $\operatorname{Pou}(\mathfrak{B})$ be the pre-ordered sets of partitions of unity in $\mathfrak{A}$, $\mathfrak{B}$ respectively (512Ee). Show that $\operatorname{Pou}(\mathfrak{B}) \preccurlyeq_{\mathrm{T}} \operatorname{Pou}(\mathfrak{A})$, and hence that $\operatorname{wdistr}(\mathfrak{B}) \ge \operatorname{wdistr}(\mathfrak{A})$.

(f) Let X be a set. Show that $\tau(\mathcal{P}X)$ is the least cardinal λ such that $\#(X) \le 2^{\lambda}$.

(g) Let Σ be the countable-cocountable algebra of ω_1. Show that Σ is an order-dense subalgebra of $\mathcal{P}\omega_1$, that $\tau(\Sigma) = \omega_1$, and that $\tau(\mathcal{P}\omega_1) = \omega$.

(h) For a Boolean algebra $\mathfrak{A}$, write $\operatorname{hc}(\mathfrak{A}) = \min\{c(\mathfrak{B}) : \mathfrak{B}$ is a non-zero principal ideal of $\mathfrak{A}\}$, counting $\min\emptyset$ as ∞. (i) Show that if $\mathfrak{B}$ is a regularly embedded subalgebra of $\mathfrak{A}$, then $\operatorname{hc}(\mathfrak{B}) \le \operatorname{hc}(\mathfrak{A})$. (ii) Show that if $\mathfrak{B}$ is a Boolean algebra and there is a surjective order-continuous Boolean homomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$, then $\operatorname{hc}(\mathfrak{B}) \le \operatorname{hc}(\mathfrak{A})$. (iii) Show that if $\mathfrak{B}$ is a principal ideal of $\mathfrak{A}$ then $\operatorname{hc}(\mathfrak{B}) \ge \operatorname{hc}(\mathfrak{A})$. (iv) Show that if $\mathfrak{B}$ is an order-dense subalgebra of $\mathfrak{A}$ then $\operatorname{hc}(\mathfrak{B}) = \operatorname{hc}(\mathfrak{A})$. (v) Show that if $\mathfrak{A}$ is the simple product of a family $\langle\mathfrak{A}_i\rangle_{i\in I}$ of Boolean algebras, then $\operatorname{hc}(\mathfrak{A}) = \min_{i\in I} \operatorname{hc}(\mathfrak{A}_i)$.

>(i)(i) (Solovay 66) Let I be any set, with its discrete topology, and $X = I^{\mathbb{N}}$ with the product topology. Show that $\tau(\mathrm{RO}(X)) = \omega$. (*Hint*: take I to be well-ordered; set $G_{ij} = \{x : x(i) < x(j)\}$ for i, $j \in \mathbb{N}$; show that the closed subalgebra of $\mathrm{RO}(X)$ generated by the G_{ij} contains $\{x : x(n) \ge \xi\}$ for every $n \in \mathbb{N}$ and $\xi \in I$.) (ii) Show that the subalgebra $\mathfrak{B}$ of $\mathrm{RO}(X)$ generated by $\{\{x : x(n) = i\} : n \in \mathbb{N}, i \in I\}$ is an order-dense subalgebra of $\mathrm{RO}(X)$ and that $\tau(\mathfrak{B}) \ge \#(I)$ if $\#(I) > 1$.

(j) Let $(X,\mathfrak{T})$ be a T_0 topological space. Show that we have a partial order on X defined by saying that $x \le y$ iff $x \in \overline{\{y\}}$. Show that $\mathfrak{T}$ is the up-topology on X iff the family of $\mathfrak{T}$-closed sets is a topology.

(k) Let P be a partially ordered set. Show that a subset of P is a regular open set for the up-topology iff it is of the form $\bigcap_{q\in A}\{p : p \in P,\ [p,\infty[\cap [q,\infty[= \emptyset\}$ for some set $A \subseteq P$.

>(l) Rewrite the statement and proof of the Erdős-Tarski theorem (513Bb) (i) in terms of topological spaces (ii) in terms of Boolean algebras.

(m) Find partially ordered sets P and Q such that the regular open algebras of P and Q for their up-topologies are isomorphic, but $\operatorname{add} P \ne \operatorname{add} Q$ and $\operatorname{cf} P \ne \operatorname{cf} Q$.

(n) Let P be a non-empty partially ordered set such that its regular open algebra $\mathrm{RO}^{\uparrow}(P)$ for the up-topology is atomless, and let Q be a set of the same size as P with the trivial partial order in which $q \le q'$ iff $q = q'$. Show that Q and the product partially ordered set $P \times Q$ are Tukey equivalent but $\mathrm{RO}^{\uparrow}(P \times Q)$ is atomless, while $\mathrm{RO}^{\uparrow}(Q)$ is purely atomic.

(o) Let P be a partially ordered set and κ an infinite cardinal. Show that $\kappa < \operatorname{wdistr}(\mathrm{RO}^{\uparrow}(P))$ iff for every family $\langle Q_\xi\rangle_{\xi<\kappa}$ of cofinal subsets of P there is a cofinal $Q \subseteq P$ such that for every $q \in Q$ and $\xi < \kappa$ there is an $I \in [Q_\xi]^{<\omega}$ such that for every $p \ge q$ there is an $r \in I$ which is compatible upwards with p.

(p) Suppose that P is a partially ordered set and that $A \subseteq P$ is such that

$$Q = \{q : q \in P,\, q = \sup\{a : a \in A,\, a \le q\}\}$$

is cofinal with P. Show that if P is separative upwards, then $\tau(\mathrm{RO}^{\uparrow}(P)) \le \#(A)$.

(q) Let $\mathfrak{A}$ be the measure algebra of Lebesgue measure. Show that the simple products $\{0,1\} \times \mathfrak{A}$ and $\mathcal{P}\mathbb{N} \times \mathfrak{A}$ are not isomorphic, but that each can be regularly embedded in the other.

(r) Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{S})$ be topological spaces. Suppose we have a π-base $\mathcal{U}$ for $\mathfrak{T}$ and a function $f : \mathcal{U} \to \mathfrak{S}$ such that $f[\mathcal{U}]$ is a π-base for $\mathfrak{S}$ and, for $U, U' \in \mathcal{U}$, $U \cap U' = \emptyset$ iff $f(U) \cap f(U') = \emptyset$. Show that $\mathrm{RO}(X) \cong \mathrm{RO}(Y)$.

(s) Let $\langle P_i\rangle_{i\in I}$ be a family of non-empty partially ordered sets and $\langle I_j\rangle_{j\in J}$ a partition (that is, disjoint cover) of I. Show that the upwards finite-support product $\bigotimes^{\uparrow}_{i\in I} P_i$ can be naturally identified with $\bigotimes^{\uparrow}_{j\in J} \bigotimes^{\uparrow}_{i\in I_j} P_i$.

514Y Further exercises (a) For a partially ordered set P, its **order-dimension** is the smallest cardinal κ such that P is isomorphic, as partially ordered set, to a subset of a product $\prod_{\xi<\kappa} X_\xi$ where every X_ξ is a totally ordered set (and the product is given its product partial order, as in 315C). Show that the order-dimension of a Boolean algebra $\mathfrak{A}$ is $\operatorname{link}(\mathfrak{A})$.

(b) Show that $\mathcal{P}\mathbb{N}$ has a subalgebra with uncountable π-weight. (*Hint*: 515H.)

(c) Let $\mathfrak{A}$ be a Boolean algebra such that $c(\mathfrak{A}) \ne 1$, and A a subset of $\mathfrak{A}$. Show that there is a $B \in [A]^{\le c(\mathfrak{A})}$ with the same upper and lower bounds as A.

(d) Show that for any cardinal κ there are a ccc Boolean algebra $\mathfrak{A}$ and an ideal $\mathcal{I}$ of $\mathfrak{A}$ such that $c(\mathfrak{A}/\mathcal{I}) = \kappa$.

(e) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, not $\{0\}$, $\mathbb{P}$ the forcing notion $(\mathfrak{A}^+, \subseteq, 1, \downarrow)$ (5A3M), and κ a cardinal. Show that the following are equiveridical: (i) there is an atomless order-closed subalgebra of $\mathfrak{A}$ with Maharam type at most κ; (ii) $\Vdash_{\mathbb{P}} \mathcal{P}\check{\kappa} \ne (\mathcal{P}\kappa)\check{}$.

(f) Let $\mathfrak{A}$ be a Boolean algebra and κ a cardinal. I will say that $\mathfrak{A}$ has the **$<\kappa$-interpolation property** if whenever A, $B \subseteq \mathfrak{A}$, $a \subseteq b$ whenever $a \in A$ and $b \in B$, and $\#(A \cup B) < \kappa$, then there is a $c \in \mathfrak{A}$ such that $a \subseteq c \subseteq b$ for every $a \in A$, $b \in B$. (Thus the σ-interpolation property of 466G is the $<\omega_1$-interpolation property.) (i) Suppose that $\mathfrak{A}$ has the $<\kappa$-interpolation property and I is an ideal of $\mathfrak{A}$ such that $\kappa \le (\operatorname{add} I)^+$. Show that the quotient $\mathfrak{A}/I$ has the $<\kappa$-interpolation property. (ii) Suppose that $\mathfrak{A}$ has the $<\kappa$-interpolation property, $\mathfrak{B}$ is a Boolean algebra with cardinal at most κ, $\mathfrak{C}$ is a subalgebra of $\mathfrak{B}$ and $\phi : \mathfrak{C} \to \mathfrak{A}$ is a Boolean homomorphism. Show that ϕ has an extension to a Boolean homomorphism from $\mathfrak{B}$ to $\mathfrak{A}$. (Compare 314K.) (iii) Show that if $\mathfrak{A}$ has the $<\operatorname{sat}(\mathfrak{A})$-interpolation property it is Dedekind complete.

(g) Let $\mathfrak{A}$ be a ccc Dedekind complete Boolean algebra with Maharam type κ. Show that there is a σ-ideal $\mathcal{J}$ of the Baire σ-algebra $\mathcal{B}\mathfrak{a}(\{0,1\}^\kappa)$ such that $\mathfrak{A} \cong \mathcal{B}\mathfrak{a}(\{0,1\}^\kappa)/\mathcal{J}$.

514 Notes and comments With any mathematical object, the set-theorist's first concern is simply to establish its cardinality. There is therefore a natural distinction to make between cardinal invariants which control the cardinality of a space, as linking number, centering number and π-weight do for Boolean algebras (514Da), and others, like weak distributivity, which are measures of some kind of complexity not directly linked with cardinality. Observe that for general Boolean algebras $\mathfrak{A}$ not even the cellularity is controlled by the Maharam type (514Xi); in fact, of the cardinals here, only wdistr($\mathfrak{A}$) is controlled by $\tau(\mathfrak{A})$ alone (514Dd). Maharam type and cellularity together control the size of the algebra (514De), and for measurable algebras, of course, Maharam type almost completely determines the algebra and even the measure (see Chapter 33).

I use the language of Galois-Tukey connections in many of the proofs of this section. This is not because there is any real need for it (there is no depth to any of the results I quote) but because I think that it shows some common strands running through a rather long list of facts. Also it points up the proofs which are *not* reducible to simple applications of ideas in §512; for instance, those relating to weak distributivity. And, finally, it will provide useful practice for the ideas of Chapter 52.

I have deliberately arranged the lists of cardinal functions of topological spaces and Boolean algebras in such a way that the cardinals of Boolean algebras and their Stone spaces will naturally correspond. There are of course important exceptions. The Maharam type of a Boolean algebra, and the tightness of a topological space, do not seem to have significant natural analogues in the other category. Note that the correspondences depend to a significant degree on the compactness of Stone spaces. This is perhaps more important than their zero-dimensionality. The point about the open-and-closed algebra of a zero-dimensional space is that it is order-dense in the regular open algebra, and that our cardinal functions of Boolean algebras are nearly all unchanged by Dedekind completion (514Ee). For arbitrary topological spaces, we can still investigate their regular open algebras, and we find that the cardinal functions of a regular open algebra are much more closely related to those of the topological space if the space is locally compact (514A, 514H(b)-(c)).

You will not be surprised to recognise some of the results and arguments of this section as direct generalizations of special cases already treated; thus 316B becomes 514Bb, 316E (or 215B(iv)) becomes 514Db, 316I becomes 514Be and 4A1O becomes 514De.

I have to admit that there are rather more pages than ideas in this section. What it is really here for is to provide a compendium of useful facts in the language which I wish to use in the rest of the volume. Perhaps I should say 'languages', because much of the space is taken up by repeating results in three forms, as they apply to partially ordered sets, to Boolean algebras and to topological spaces. The point is of course that we frequently find that a fact which is obvious in one of its three manifestations is a surprise in another. And some care is needed in the translations. The theory of finite-support products of partially ordered sets (514T-514U), for instance, is supposed to mimic the theory of products of topological spaces. But actually it reflects the theory of π-bases of topologies rather than the theory of spaces-with-points. And while we have straightforward functors between the *categories* of Boolean algebras and topological spaces, with Boolean homomorphisms corresponding to continuous functions (312Q-312S), such results as we have concerning functions between partially ordered sets and their actions on the corresponding regular open algebras are partial and delicate (514O-514R).

The Tukey classification (513D) and the regular open algebras of 514N are both attempts to reduce the multitudinous variety of partially ordered sets to relatively coherent schemes. They carry rather different information; the Tukey classification tells us about additivity and cofinality (513E) and precalibers (516C below), while the regular open algebra determines linking numbers (514N). It is easy to find partially ordered sets with the same regular open algebras but different additivity and cofinality (514Xm), or with the same Tukey classification but different regular open algebras (514Xn). The regular open algebras studied here are primarily of interest in relation to the use of partially ordered sets in the theory of forcing; I hope to return to such questions later in this volume.

Of the cardinal functions of Boolean algebras defined in §511, I have not mentioned Martin numbers or Freese-Nation numbers. These will be dealt with at length in §§517-518.

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515 The Balcar-Franěk theorem

I interpolate a section to give two basic results on Dedekind complete Boolean algebras: the Balcar-Franěk theorem (515H) on independent sets and the Pierce-Koppelberg theorem (515L) on cardinalities. The concept of 'Boolean-independence' (515A) provides a tool for some useful results on regular open algebras (515N-515Q).

515A Definition Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$.

(a) I say that a family $\langle\mathfrak{B}_i\rangle_{i\in I}$ of subalgebras of $\mathfrak{A}$ is **Boolean-independent** if $\inf_{i\in J} b_i \ne 0$ whenever $J \subseteq I$ is finite and $b_i \in \mathfrak{B}_i^+ = \mathfrak{B}_i \setminus \{0\}$ for every $i \in J$.

(b) I say that a family $\langle a_i\rangle_{i\in I}$ in $\mathfrak{A}$ is **Boolean-independent** if $\inf_{j\in J} a_j \setminus \sup_{k\in K} a_k$ is non-zero whenever J, $K \subseteq I$ are disjoint finite sets. Similarly, a set $B \subseteq \mathfrak{A}$ is **Boolean-independent** if $\inf J \setminus \sup K \ne 0$ for any disjoint finite sets J, $K \subseteq B$.

(c) I say that a family $\langle D_i\rangle_{i\in I}$ of partitions of unity in $\mathfrak{A}$ is **Boolean-independent** if $\inf_{i\in J} d_i \ne 0$ whenever $J \subseteq I$ is finite and $d_i \in D_i$ for every $i \in J$.

(Many authors write 'independent' rather than 'Boolean-independent', and in the proofs of this section I may do the same. But in this book as a whole it is more often natural to read 'independent' as 'stochastically independent', as in 458L and 525H.)

515B Lemma (Compare 272D.) Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$.

(a) A family $\langle a_i\rangle_{i\in I}$ in $\mathfrak{A}$ is Boolean-independent iff no a_i is 0 or 1 and $\langle\{0, a_i, 1\setminus a_i, 1\}\rangle_{i\in I}$ is a Boolean-independent family of subalgebras of $\mathfrak{A}$.

(b) Let $\langle\mathfrak{B}_i\rangle_{i\in I}$ be a family of subalgebras of $\mathfrak{A}$. Let $\mathfrak{B}$ be the free product of $\langle\mathfrak{B}_i\rangle_{i\in I}$, and $\varepsilon_i : \mathfrak{B}_i \to \mathfrak{B}$ the canonical homomorphism for each $i \in I$ (315I). Then we have a unique Boolean homomorphism $\phi : \mathfrak{B} \to \mathfrak{A}$ such that $\phi\varepsilon_i(b) = b$ whenever $i \in I$ and $b \in \mathfrak{B}_i$, and $\langle\mathfrak{B}_i\rangle_{i\in I}$ is Boolean-independent iff ϕ is injective; in which case $\mathfrak{B}$ is isomorphic to the subalgebra of $\mathfrak{A}$ generated by $\bigcup_{i\in I}\mathfrak{B}_i$.

(c) If $\langle\mathfrak{B}_i\rangle_{i\in I}$ is a Boolean-independent family of subalgebras of $\mathfrak{A}$, $\langle I_j\rangle_{j\in J}$ is a disjoint family of subsets of I, and $\mathfrak{C}_j$ is the subalgebra of $\mathfrak{A}$ generated by $\bigcup_{i\in I_j}\mathfrak{B}_i$ for each j, then $\langle\mathfrak{C}_j\rangle_{j\in J}$ is Boolean-independent.

(d) Suppose that $B \subseteq \mathfrak{A}$ is a Boolean-independent set and that $\langle C_j\rangle_{j\in J}$ is a disjoint family of subsets of B. For $j \in J$ write $\mathfrak{C}_j$ for the subalgebra of $\mathfrak{A}$ generated by C_j. Then $\langle\mathfrak{C}_j\rangle_{j\in J}$ is Boolean-independent.

(e) Suppose that $\langle\mathfrak{B}_i\rangle_{i\in I}$ is a Boolean-independent family of subalgebras of $\mathfrak{A}$, and that for each $i \in I$ we have a Boolean-independent subset B_i of $\mathfrak{B}_i$. Then $\langle B_i\rangle_{i\in I}$ is disjoint and $\bigcup_{i\in I} B_i$ is Boolean-independent.

(f) Let $\langle D_i\rangle_{i\in I}$ be a family of partitions of unity in $\mathfrak{A}$, none containing 0. For each $i \in I$ let $\mathfrak{B}_i$ be the order-closed subalgebra of $\mathfrak{A}$ generated by D_i. Then $\langle\mathfrak{B}_i\rangle_{i\in I}$ is Boolean-independent iff $\langle D_i\rangle_{i\in I}$ is Boolean-independent.

proof (a) The point is just that in 515Aa we need consider only $b_i \in \mathfrak{B}_i \setminus \{0, 1\}$, while in 515Ab we have

$$\inf_{j\in J} a_j \setminus \sup_{k\in K} a_k = \inf_{j\in J} a_j \cap \inf_{i\in K} 1 \setminus a_k.$$

(b) 315Jb, applied to the identity maps from the $\mathfrak{B}_i$ to $\mathfrak{A}$, assures us that there is a unique Boolean homomorphism $\phi : \mathfrak{B} \to \mathfrak{A}$ such that $\phi\varepsilon_i(b) = b$ for every $i \in I$ and $b \in \mathfrak{B}_i$.

(i) By 315K(e-ii), $\langle\varepsilon_i[\mathfrak{B}_i]\rangle_{i\in I}$ is a Boolean-independent family of subalgebras of $\mathfrak{B}$. So if ϕ is injective, $\langle\mathfrak{B}_i\rangle_{i\in I} = \langle\pi_i[\varepsilon_i[\mathfrak{B}_i]]\rangle_{i\in I}$ is Boolean-independent in $\phi[\mathfrak{B}]$ and therefore in $\mathfrak{A}$. In this case, because $\mathfrak{B}$ is the subalgebra of itself generated by $\bigcup_{i\in I}\varepsilon_i[\mathfrak{B}_i]$ (315Ka), the subalgebra of $\mathfrak{A}$ generated by $\bigcup_{i\in I}\mathfrak{B}_i$ is $\phi[\mathfrak{B}]$ and is isomorphic to $\mathfrak{B}$.

(ii) If $\langle\mathfrak{B}_i\rangle_{i\in I}$ is Boolean-independent and $b \in \mathfrak{B}^+$, there are a finite $J \subseteq I$ and a family $\langle b_j\rangle_{j\in J} \in \prod_{j\in J}\mathfrak{B}_j^+$ such that $b \supseteq \inf_{j\in J}\varepsilon_j(b_j)$ (315Kb). Now $\phi(b) \supseteq \inf_{j\in J} b_j$ is non-zero; as b is arbitrary, ϕ is injective.

(c) Let $L \subseteq J$ be a finite set and suppose that $c_j \in \mathfrak{C}_j^+$ for each $j \in L$. As observed in (b), the embeddings $\mathfrak{B}_i \subsetneqq \mathfrak{C}_j$ identify $\mathfrak{C}_j$ with the free product of $\langle\mathfrak{B}_i\rangle_{i\in I_j}$, so 315Kb tells us that there must be a finite set

$K_j \subseteq I_j$ and elements $b_i \in \mathfrak{B}_i^+$, for $i \in K_j$, such that $\inf_{i\in K_j} b_i \subseteq c_j$. Now $\inf_{j\in L} c_j \supseteq \inf\{b_i : i \in \bigcup_{j\in L} K_j\}$ is non-zero. As $\langle c_j\rangle_{j\in L}$ is arbitrary, $\langle\mathfrak{C}_j\rangle_{j\in J}$ is Boolean-independent. (Compare 315L.)

(d) Set $\mathfrak{B}_b = \{0, b, 1\setminus b, 1\}$ for $b \in B$, so that $\langle\mathfrak{B}_b\rangle_{b\in B}$ is Boolean-independent, by (a); now apply (c).

(e)(i) If i, $j \in I$ are distinct, $b \in B_i$ and $b' \in B_j$, then $b \in \mathfrak{B}_i^+$ and $1\setminus b' \in \mathfrak{B}_j^+$, so $b\setminus b' \ne 0$ and $b \ne b'$.

(ii) If J, K are disjoint finite subsets of $\bigcup_{i\in I} B_i$, then $J\cap B_i$ and $K\cap B_i$ are disjoint finite subsets of B_i, so that

$$b_i = \inf(J\cap B_i)\setminus\sup(K\cap B_i) \in \mathfrak{B}_i^+$$

for each $i \in I$. Let $L \subseteq I$ be a finite set such that $J\cup K \subseteq \bigcup_{i\in L} B_i$; then

$$\inf J\setminus\sup K = \inf_{i\in L} b_i \ne 0.$$

As J and K are arbitrary, $\bigcup_{i\in I} B_i$ is Boolean-independent.

(f) Since $D_i \subseteq \mathfrak{B}_i$, $\langle D_i\rangle_{i\in I}$ must be Boolean-independent if $\langle\mathfrak{B}_i\rangle_{i\in I}$ is.

On the other hand, each D_i is order-dense in $\mathfrak{B}_i$. **P** For $d \in D_i$, the set $\{b : d \subseteq b \text{ or } d\cap b = 0\}$ is an order-closed subalgebra of $\mathfrak{A}$ including D_i, so includes $\mathfrak{B}_i$. If $b \in \mathfrak{B}_i^+$, then (because $\sup D_i = 1$ in $\mathfrak{A}$) there must be a $d \in D_i$ such that $b\cap d \ne 0$, in which case $0 \ne d \subseteq b$. As b is arbitrary, D_i is order-dense in $\mathfrak{B}_i$. **Q**

Now suppose that $\langle D_i\rangle_{i\in I}$ is Boolean-independent, $J \subseteq I$ is finite and $b_i \in \mathfrak{B}_i^+$ for each $i \in J$. Then we have non-zero $d_i \in D_i$ such that $d_i \subseteq b_i$ for each i. So $\inf_{i\in J} b_i \supseteq \inf_{i\in J} d_i$ is non-zero. As $\langle b_i\rangle_{i\in J}$ is arbitrary, $\langle\mathfrak{B}_i\rangle_{i\in I}$ is Boolean-independent.

515C Proposition Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$, and κ a cardinal.

(a) There is a Boolean-independent subset of $\mathfrak{A}$ with cardinal κ iff there is a subalgebra of $\mathfrak{A}$ which is isomorphic to the algebra of open-and-closed subsets of $\{0,1\}^\kappa$.

(b) If $\mathfrak{A}$ is Dedekind complete, there is a Boolean-independent subset of $\mathfrak{A}$ with cardinal κ iff there is a subalgebra of $\mathfrak{A}$ which is isomorphic to the regular open algebra of $\{0,1\}^\kappa$.

proof Set $Z = \{0,1\}^\kappa$; write $\mathcal{E}$ for the algebra of open-and-closed subsets of Z and $\mathfrak{G}$ for the regular open algebra of Z.

(a)(i) Suppose that $\mathfrak{A}$ has a Boolean-independent subset of cardinal κ, enumerated as $\langle a_\xi\rangle_{\xi<\kappa}$. Setting $\mathfrak{A}_\xi = \{0, a_\xi, 1\setminus a_\xi, 1\}$ for each ξ, $\langle\mathfrak{A}_\xi\rangle_{\xi<\kappa}$ is a Boolean-independent family of subalgebras of $\mathfrak{A}$, and the subalgebra $\mathfrak{C}$ of $\mathfrak{A}$ generated by $\bigcup_{\xi<\kappa}\mathfrak{A}_\xi$ can be identified with the free product of $\langle\mathfrak{A}_\xi\rangle_{\xi<\kappa}$ (515Bb). But since the Stone space of each $\mathfrak{A}_\xi$ has just two points, the construction of 315I makes it plain that the Stone space of $\mathfrak{C}$ is homeomorphic to Z, so that $\mathfrak{C}$ is isomorphic to $\mathcal{E}$.

(ii) In the other direction, the sets $E_\xi = \{z : z \in Z,\ z(\xi) = 1\}$ are Boolean-independent in $\mathcal{E}$, so if $\mathcal{E}$ can be embedded in $\mathfrak{A}$ there must be a Boolean-independent subset of $\mathfrak{A}$ with cardinal κ.

(b) Now suppose that $\mathfrak{A}$ is Dedekind complete. $\mathcal{E}$ is an order-dense subalgebra of $\mathfrak{G}$ (314T). So if $\mathfrak{A}$ has a subalgebra isomorphic to $\mathfrak{G}$ it certainly has one isomorphic to $\mathcal{E}$. On the other hand, if $\mathfrak{A}$ has a subalgebra isomorphic to $\mathcal{E}$, so that there is an injective Boolean homomorphism $\pi : \mathcal{E} \to \mathfrak{A}$, then (because $\mathfrak{A}$ is Dedekind complete) π has an extension to a Boolean homomorphism $\pi_1 : \mathfrak{G} \to \mathfrak{A}$ (314K); because $\mathcal{E}$ is order-dense in $\mathfrak{G}$ and π is injective, π_1 is injective, so that $\pi_1[\mathfrak{G}]$ is a subalgebra of $\mathfrak{A}$ isomorphic to $\mathfrak{G}$.

Putting this together with (a), we see that $\mathfrak{A}$ has a Boolean-independent subset with cardinal κ iff it has a subalgebra isomorphic to $\mathfrak{G}$.

515D Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, not $\{0\}$, and $\mathfrak{B}$ an order-closed subalgebra of $\mathfrak{A}$ such that $\mathfrak{A}$ is relatively atomless over $\mathfrak{B}$. Then there is an $a^* \in \mathfrak{A}\setminus\{0,1\}$ such that $\mathfrak{B}$ and $\{0, a^*, 1\setminus a^*, 1\}$ are Boolean-independent subalgebras of $\mathfrak{A}$.

Remark Recall from 331A that a Boolean algebra $\mathfrak{A}$ is 'relatively atomless' over an order-closed subalgebra $\mathfrak{B}$ if for every $a \in \mathfrak{A}^+$ there is a $c \subseteq a$ which is not of the form $a\cap b$ for any $b \in \mathfrak{B}$.

proof Set

$$C = \{c : c \in \mathfrak{A},\ c \ne 0,\ \mathfrak{B}\cap\mathfrak{A}_c = \{0\}\},$$

where $\mathfrak{A}_c$ is the principal ideal of $\mathfrak{A}$ generated by c. Then C is order-dense in $\mathfrak{A}$. **P** If $a \in \mathfrak{A}^+$, there is a $c \in \mathfrak{A}_a \setminus \{a \cap b : b \in \mathfrak{B}\}$. Set $b_0 = \sup\{b : b \in \mathfrak{B}, b \subseteq c\}$; then $c \setminus b_0 \subseteq a$ and $c \setminus b_0 \in C$. **Q**

For $a \in \mathfrak{A}$ set $\operatorname{upr}(a, \mathfrak{B}) = \inf\{b : a \subseteq b \in \mathfrak{B}\}$, as in 313S. Set $E = \{\operatorname{upr}(c, \mathfrak{B}) : c \in C\}$. Then E is order-dense in $\mathfrak{B}$, because if $b \in \mathfrak{B}^+$ there is a $c \in C$ such that $c \subseteq b$, and now $\operatorname{upr}(c, \mathfrak{B})$ belongs to E and is included in b. So there is a partition D of unity in $\mathfrak{B}$ included in E (313K). For each $d \in D$ choose $c_d \in C$ such that $d = \operatorname{upr}(c_d, \mathfrak{B})$, and set $a^* = \sup\{c_d : d \in D\}$. If $b \in \mathfrak{B}^+$, there is a $d \in D$ such that $b \cap d \neq 0$, that is, $d \not\subseteq 1 \setminus b$, so $c_d \not\subseteq 1 \setminus b$ and $b \cap c_d \neq 0$; so $b \cap a^* \neq 0$. Also, because $c_d \in C$, $b \cap d \not\subseteq c_d = a^* \cap d$, so $b \not\subseteq a^*$ and $b \cap (1 \setminus a^*) \neq 0$. As b is arbitrary, $\mathfrak{B}$ and $\{0, a^*, 1 \setminus a^*, 1\}$ are Boolean-independent.

515E Lemma (Balcar & Vojtáš 77) Let $\mathfrak{A}$ be a Boolean algebra. Suppose that $C \subseteq \mathfrak{A}^+$ and that $\#(C) < c(\mathfrak{A}_c)$ for every $c \in C$, where $\mathfrak{A}_c$ is the principal ideal of $\mathfrak{A}$ generated by c. Then there is a partition D of unity in $\mathfrak{A}$ such that every member of C includes a non-zero member of D.

proof Enumerate C as $\langle c_\xi \rangle_{\xi<\kappa}$. For each $\xi < \kappa$, let B_ξ be a disjoint set in $\mathfrak{A}^+_{c_\xi}$ with cardinal κ^+, and set

$$A_\xi = \{\eta : \eta < \kappa, \#(\{b : b \in B_\xi, b \cap c_\eta \neq 0\}) \leq \kappa\},$$

$$B'_\xi = B_\xi \setminus \bigcup_{\eta \in A_\xi} \{b : b \in B_\xi, b \cap c_\eta \neq 0\}.$$

Then B'_ξ is a disjoint set in $\mathfrak{A}_{c_\xi}$, $\#(B'_\xi) = \kappa^+$, and $\{b : b \in B'_\xi, b \cap c_\eta \neq 0\}$ is empty if $\eta \in A_\xi$ and has cardinal κ^+ otherwise. Now define $A \subseteq \kappa$ inductively by saying that $\xi \in A$ iff $\xi \in A_\eta$ whenever $\eta \in A \cap \xi$, and set $B = \bigcup_{\xi \in A} B'_\xi$.

B is disjoint. **P** If $\eta, \xi \in A$, $\eta \leq \xi$, $b \in B'_\eta$, $b' \in B'_\xi$ and $b \neq b'$, then either $\eta = \xi$ and $b \cap b' = 0$ because B'_ξ is disjoint, or $\eta < \xi$ and $\xi \in A_\eta$ and $b \cap b' \subseteq b \cap c_\xi = 0$. **Q** Also $D_\xi = \{b : b \in B, b \cap c_\xi \neq 0\}$ has cardinal κ^+ for every $\xi < \kappa$. **P** If $\xi \in A$, then $D_\xi \supseteq B'_\xi$ has cardinal κ^+. If $\xi \notin A$ there is some $\eta \in A \cap \xi$ such that $\xi \notin A_\eta$ and $D_\xi \supseteq \{b : b \in B'_\eta, b \cap c_\xi \neq 0\}$ has cardinal κ^+. **Q**

We can therefore find an injection $\xi \mapsto b_\xi : \kappa \to B$ such that $c_\xi \cap b_\xi \neq 0$ for every ξ. Let D be any partition of unity including $\{c_\xi \cap b_\xi : \xi < \kappa\}$; this works.

515F Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra such that $c(\mathfrak{A}) = \operatorname{sat}(\mathfrak{A})$ and $\mathfrak{A}$ is cellularity-homogeneous. Then there is a Boolean-independent family $\langle D_i \rangle_{i \in I}$ of partitions of unity in $\mathfrak{A}$ such that $\#(I) = \sup_{i \in I} \#(D_i) = c(\mathfrak{A})$.

proof Write κ for $c(\mathfrak{A}) = \operatorname{sat}(\mathfrak{A})$. Choose $\langle D_\xi \rangle_{\xi<\kappa}$ inductively, as follows. Given D_η for $\eta < \xi$, let $\mathfrak{C}_\xi$ be the subalgebra of $\mathfrak{A}$ generated by $\bigcup_{\eta<\xi} D_\eta$; then $\#(\mathfrak{C}_\xi) < \kappa$. (Recall from 513Bb that $\kappa = \operatorname{sat}^\downarrow(\mathfrak{A}^+)$ must be a regular uncountable cardinal, while of course $\#(D_\eta) < \kappa$ for every η.) By 515E we have a partition D of unity in $\mathfrak{A}$, not containing $\{0\}$, such that every non-zero element of $\mathfrak{C}_\xi$ includes an element of D. For each $d \in D$ the principal ideal of $\mathfrak{A}$ generated by d has cellularity $\kappa > \#(\xi)$ so there is a disjoint family $\langle b_{d\eta} \rangle_{\eta \leq \xi}$ of non-zero elements with supremum d. Set $b_\eta = \sup_{d \in D} b_{d\eta}$ for $\eta \leq \xi$, and $D_\xi = \{b_\eta : \eta \leq \xi\}$; then D_ξ is a partition of unity in $\mathfrak{A}$.

The construction ensures that whenever $d \in D_\xi$ and $c \in \mathfrak{C}^+_\xi$ then $d \cap c \neq 0$. It follows that $\langle D_\xi \rangle_{\xi<\kappa}$ is Boolean-independent. **P** I show by induction on $\#(J)$ that if $J \subseteq \kappa$ is finite and $d_\xi \in D_\xi$ for each $\xi \in J$, then $\inf_{\xi \in J} d_\xi \neq 0$. If J is empty this is trivial. For the inductive step to $\#(J) = n+1$, set $\xi = \max J$ and $J' = \xi \cap J$. By the inductive hypothesis, $c = \inf_{\eta \in J'} d_\eta$ is non-zero; but $c \in \mathfrak{C}_\xi$, so, by the construction of D_ξ, $c \cap d_\xi = \inf_{\eta \in J} d_\eta$ is non-empty. So the induction proceeds. **Q**

515G Lemma Let $\langle \mathfrak{A}_i \rangle_{i \in I}$ be a non-empty family of Boolean algebras with simple product $\mathfrak{A}$. Suppose that for each $i \in I$ the algebra $\mathfrak{A}_i$ has a Boolean-independent set with cardinal $\kappa_i \geq \omega$. Then $\mathfrak{A}$ has a Boolean-independent set with cardinal $\kappa = \#(\prod_{i \in I} \kappa_i)$.

proof For each $i \in I$ let B_i be a Boolean-independent set in $\mathfrak{A}_i$ with cardinal κ_i. Let $\langle a_\xi \rangle_{\xi<\kappa}$ be a family in $\prod_{i \in I} B_i \subseteq \mathfrak{A}$ such that for every finite $J \subseteq \kappa$ there is an $i \in I$ such that $a_\xi(i) \neq a_\eta(i)$ whenever $\xi, \eta \in J$ are distinct (5A1L). Now $\langle a_\xi \rangle_{\xi<\kappa}$ is Boolean-independent in $\mathfrak{A}$. **P** Suppose that J, $K \subseteq \kappa$ are finite and disjoint. Then there is an $i \in I$ such that $a_\xi(i) \neq a_\eta(i)$ whenever $\xi, \eta \in J \cup K$ are distinct. But this means that $\langle a_\xi(i) \rangle_{\xi \in J \cup K}$ is Boolean-independent in $\mathfrak{A}_i$, so that, setting $a = \inf_{\xi \in J} a_\xi \setminus \sup_{\xi \in K} a_\xi$,

$$a(i) = \inf_{\xi \in J} a_\xi(i) \setminus \sup_{\xi \in K} a_\xi(i) \neq 0,$$

and $a \neq 0$. As J and K are arbitrary, $\langle a_\xi \rangle_{\xi<\kappa}$ is a Boolean-independent family. **Q**

Accordingly $\{a_\xi : \xi < \kappa\}$ is a Boolean-independent set of size κ.

515H The Balcar-Franěk theorem (Balcar & Franěk 82) Let $\mathfrak{A}$ be an infinite Dedekind complete Boolean algebra. Then there is a Boolean-independent set $A \subseteq \mathfrak{A}$ such that $\#(A) = \#(\mathfrak{A})$.

proof Set $\kappa = \#(\mathfrak{A})$. For $a \in \mathfrak{A}$ write $\mathfrak{A}_a$ for the principal ideal of $\mathfrak{A}$ generated by a.

(a) Suppose that $\mathfrak{A}$ is purely atomic. Then $\mathfrak{A}$ has an independent set with cardinal κ. **P** Let B be the set of its atoms; because $\mathfrak{A}$ is infinite, so is B; set $\lambda = \#(B)$, so that

$$\mathfrak{A} \cong \prod_{b \in B} \mathfrak{A}_b \cong \mathcal{P}\lambda$$

(315F(iii)), and $\kappa = 2^\lambda$. There is a dense subset D of $\{0,1\}^\kappa$ with $\#(D) \le \lambda$ (5A4Be); let $f : B \to D$ be a surjection. For $\xi < \kappa$ set

$$a_\xi = \sup\{b : b \in B,\ f(b)(\xi) = 1\}.$$

If J, $K \subseteq \kappa$ are disjoint finite sets, the set

$$G = \{x : x \in \{0,1\}^\kappa,\ x(\xi) = 1\ \forall\ \xi \in J,\ x(\eta) = 0\ \forall\ \eta \in K\}$$

is a non-empty open set, so there is a $b \in B$ such that $f(b) \in G$; but this means that $\inf_{\xi \in J} a_\xi \setminus \sup_{\eta \in K} a_\eta \supseteq b$ is non-zero. As J and K are arbitrary, $\{a_\xi : \xi < \kappa\}$ is an independent subset of $\mathfrak{A}$ with cardinal κ. **Q**

(b) Suppose that $\mathfrak{A}$ is Maharam-type-homogeneous, and that $\mathfrak{B}$ is an order-closed subalgebra of $\mathfrak{A}$ with Maharam type less than $\tau(\mathfrak{A})$. Then there is a subalgebra $\mathfrak{C}$ of $\mathfrak{A}$, Boolean-independent of $\mathfrak{B}$, such that $\mathfrak{C}$ has a Boolean-independent subset with cardinal $\tau(\mathfrak{A})$. **P** Let $B \subseteq \mathfrak{B}$ be a set with cardinal less than $\tau(\mathfrak{A})$ which τ-generates $\mathfrak{B}$. Choose $\langle c_\xi \rangle_{\xi<\tau(\mathfrak{A})}$ inductively, as follows. Given $\langle c_\eta \rangle_{\eta<\xi}$, where $\xi < \tau(\mathfrak{A})$, let $\mathfrak{B}_\xi$ be the order-closed subalgebra of $\mathfrak{A}$ generated by $B \cup \{c_\eta : \eta < \xi\}$. If $a \in \mathfrak{A}^+$, the order-closed subalgebra $\mathfrak{D} = \{a \cap b : b \in \mathfrak{B}_\xi\}$ of $\mathfrak{A}_a$ is τ-generated by $\{a \cap c_\eta : \eta < \xi\} \cup \{a \cap b : b \in B\}$ (314Hb), so $\tau(\mathfrak{D}) < \tau(\mathfrak{A}) = \tau(\mathfrak{A}_a)$ and $\mathfrak{D} \neq \mathfrak{A}_a$. Thus $\mathfrak{A}$ is relatively atomless over $\mathfrak{B}_\xi$; by 515D, there is a $c_\xi \in \mathfrak{A} \setminus \{0,1\}$ such that $\mathfrak{B}_\xi$ and $\{0, c_\xi, 1 \setminus c_\xi, 1\}$ are Boolean-independent. Continue. Now an easy induction on $\#(J \cup K)$ (as in the last part of the proof of 515F) shows that if J, K are disjoint finite subsets of $\tau(\mathfrak{A})$, and $b \in \mathfrak{B}$ is non-zero, $b \cap \inf_{\xi \in J} c_\xi \setminus \sup_{\eta \in K} c_\eta \neq 0$. So if we take $\mathfrak{C}$ to be the subalgebra of $\mathfrak{A}$ generated by $C = \{c_\xi : \xi < \tau(\mathfrak{A})\}$, $\mathfrak{C}$ and $\mathfrak{B}$ are Boolean-independent and $C \subseteq \mathfrak{C}$ is a Boolean-independent set with cardinal $\tau(\mathfrak{A})$. **Q**

(c) Suppose that $\mathfrak{A}$ is Maharam-type-homogeneous and that $c(\mathfrak{A}) < \mathrm{sat}(\mathfrak{A})$. Then $\mathfrak{A}$ has a Boolean-independent subset with cardinal κ. **P** Because $\mathfrak{A}$ is infinite, $c(\mathfrak{A})$ is infinite. Let $D \subseteq \mathfrak{A}^+$ be a disjoint set with cardinal $c(\mathfrak{A})$; adding $1 \setminus \sup D$ if necessary, we may suppose that D is a partition of unity. For each $d \in D$, $\mathfrak{A}_d$ has a Boolean-independent set with cardinal $\tau(\mathfrak{A}_d) = \tau(\mathfrak{A})$ (apply (b) above to $\mathfrak{A}_d$, with $\mathfrak{D} = \{0, d\}$). By 315F(iii) again, $\mathfrak{A} \cong \prod_{d \in D} \mathfrak{A}_d$; by 515G, $\prod_{d \in D} \mathfrak{A}_d$ has a Boolean-independent subset with cardinal the cardinal power $\tau(\mathfrak{A})^{\#(D)} = \tau(\mathfrak{A})^{c(\mathfrak{A})}$, so $\mathfrak{A}$ also has. But

$$\kappa \le \sup_{\lambda < \mathrm{sat}(\mathfrak{A})} \tau(\mathfrak{A})^\lambda = \tau(\mathfrak{A})^{c(\mathfrak{A})}$$

by 514De, so $\mathfrak{A}$ has a Boolean-independent set of cardinal κ. **Q**

(d) Suppose that $\mathfrak{A}$ is cellularity-homogeneous and Maharam-type-homogeneous and $c(\mathfrak{A}) = \mathrm{sat}(\mathfrak{A})$. Then $\mathfrak{A}$ has a Boolean-independent subset with cardinal κ.

P (i) By 515F, we can find a Boolean-independent family $\langle D_i \rangle_{i \in I}$ of partitions of unity in $\mathfrak{A}$ such that $\#(I) = \sup_{i \in I} \#(D_i) = \mathrm{sat}(\mathfrak{A})$. We know that $\mathrm{sat}(\mathfrak{A}) \ge \omega_1$, so we can suppose that all the D_i are infinite. For each $i \in I$, let $\mathfrak{D}_i$ be the order-closed subalgebra of $\mathfrak{A}$ generated by D_i. By (a) above, $\mathfrak{D}_i$ has a Boolean-independent subset B_i with cardinal $2^{\#(D_i)}$, so that $B = \bigcup_{i \in I} B_i$ has cardinal $\sup_{\lambda < \mathrm{sat}(\mathfrak{A})} 2^\lambda$. By 515Df, $\langle \mathfrak{D}_i \rangle_{i \in I}$ is Boolean-independent. By 515De, B is Boolean-independent.

(ii) If $\#(B) = \kappa$, we can stop. Otherwise, let $\mathfrak{D}$ be the order-closed subalgebra of $\mathfrak{A}$ generated by $D = \bigcup_{i \in I} D_i$. Because

$$\sup_{\lambda<\mathrm{sat}(\mathfrak{A})} \#(B)^\lambda = \#(B)$$

(5A1Ff)

$$< \kappa \le \sup_{\lambda<\mathrm{sat}(\mathfrak{A})} \tau(\mathfrak{A})^\lambda$$

(514Be), we must have

$$\tau(\mathfrak{A}) > \#(B) = \sup_{\lambda<\mathrm{sat}(\mathfrak{A})} 2^\lambda \ge \mathrm{sat}(\mathfrak{A}) = \#(D) \ge \tau(\mathfrak{D}).$$

By (b), we have a subalgebra $\mathfrak{C}$ of $\mathfrak{A}$, Boolean-independent of $\mathfrak{D}$, such that $\mathfrak{C}$ has a Boolean-independent subset C with cardinal $\tau(\mathfrak{A})$. Let $\langle C_i\rangle_{i\in I}$ be a disjoint family of subsets of C all with cardinal $\tau(\mathfrak{A})$.

For $i \in I$, let $\mathfrak{C}_{i0}$ be the subalgebra of $\mathfrak{A}$ generated by D_i and $\mathfrak{C}_{i1}$ the subalgebra generated by C_i. Let $\mathfrak{E}_i$ be the subalgebra generated by $\mathfrak{C}_{i0} \cup \mathfrak{C}_{i1}$ and $\widehat{\mathfrak{E}}_i$ its Dedekind completion (314T-314U). In $\widehat{\mathfrak{E}}_i$ we have the partition of unity D_i and the Boolean-independent set C_i with cardinal $\tau(\mathfrak{A})$. For each $b \in D_i$, the principal ideal $(\widehat{\mathfrak{E}}_i)_b$ of $\widehat{\mathfrak{E}}_i$ generated by b has a Boolean-independent set $\{b \cap c : c \in C_i\}$ with cardinal $\tau(\mathfrak{A})$. Because $\widehat{\mathfrak{E}}_i$ is Dedekind complete, it is isomorphic to $\prod_{b\in D_i}(\widehat{\mathfrak{E}}_i)_b$, and has a Boolean-independent subset with cardinal $\tau(\mathfrak{A})^{\#(D_i)}$ (515G again).

Because $\mathfrak{A}$ is Dedekind complete, the embedding $\mathfrak{E}_i \subsetneqq \mathfrak{A}$ extends to a Boolean homomorphism $\pi_i : \widehat{\mathfrak{E}}_i \to \mathfrak{A}$ (314K). Because $\mathfrak{E}_i$ is order-dense in $\widehat{\mathfrak{E}}_i$, π_i is injective. So $\mathfrak{E}_i^* = \pi_i[\widehat{\mathfrak{E}}_i]$ is a subalgebra of $\mathfrak{A}$ isomorphic to $\widehat{\mathfrak{E}}_i$, and has a Boolean-independent subset E_i with cardinal $\tau(\mathfrak{A})^{\#(D_i)}$.

(iii) By 515Df and 515Dd, $\langle\mathfrak{C}_{i0}\rangle_{i\in I}$ and $\langle\mathfrak{C}_{i1}\rangle_{i\in I}$ are both Boolean-independent families; because $\mathfrak{C}_{i0} \subseteq \mathfrak{D}$ and $\mathfrak{C}_{j1} \subseteq \mathfrak{C}$ whenever i, $j \in I$, and $\mathfrak{D}$ and $\mathfrak{C}$ are Boolean-independent, $\langle\mathfrak{C}_{ij}\rangle_{i\in I,j\in\{0,1\}}$ is Boolean-independent, so $\langle\mathfrak{E}_i\rangle_{i\in I}$ is Boolean-independent (515Dc). If $J \subseteq I$ is finite, and $e_i \in (\mathfrak{E}_i^*)^+$ for each $i \in J$, then there are $e_i' \in \mathfrak{E}_i$ such that $0 \ne e_i' \subseteq e_i$ for each i. Now $\inf_{i\in J} e_i \supseteq \inf_{i\in J} e_i' \ne 0$. As $\langle e_i\rangle_{i\in J}$ is arbitrary, $\langle\mathfrak{E}_i^*\rangle_{i\in I}$ is Boolean-independent. But this means that $E = \bigcup_{i\in I} E_i$ is Boolean-independent (515De), while

$$\#(E) \ge \sup_{i\in I} \tau(\mathfrak{A})^{\#(D_i)} = \sup_{\lambda<\mathrm{sat}(\mathfrak{A})} \tau(\mathfrak{A})^\lambda \ge \kappa.$$

Of course $\#(E) \le \#(\mathfrak{A}) = \kappa$, so we have a Boolean-independent set with cardinal κ in this case also. **Q**

(e) If $\mathfrak{A}$ is atomless it has a Boolean-independent subset with cardinal κ. **P** Because Maharam type and cellularity are both order-preserving cardinal functions (514Ed), $\mathfrak{A}$ is isomorphic to the product of a family $\langle\mathfrak{A}_i\rangle_{i\in I}$ of Maharam-type-homogeneous cellularity-homogeneous algebras, none of them $\{0\}$ (514Gc). Now, for each i, $\mathfrak{A}_i$ is an atomless (therefore infinite) Maharam-type-homogeneous cellularity-homogeneous Dedekind complete Boolean algebra, so by (c)-(d) above has a Boolean-independent set with cardinal $\#(\mathfrak{A}_i)$. By 515G once more, $\mathfrak{A}$ has a Boolean-independent set with cardinal $\#(\prod_{i\in I}\mathfrak{A}_i) = \kappa$. **Q**

(f) Finally, for the general case, let A be the set of atoms of A and set $c = \sup A$, so that the principal ideal $\mathfrak{A}_c$ is purely atomic and the principal ideal $\mathfrak{A}_{1\setminus c}$ is atomless. Because $\mathfrak{A} \cong \mathfrak{A}_c \times \mathfrak{A}_{1\setminus c}$ is infinite, one of $\mathfrak{A}_c$, $\mathfrak{A}_{1\setminus c}$ has cardinal κ, and therefore (by (a) or (f)) has a Boolean-independent subset with cardinal κ; which is now a Boolean independent subset of $\mathfrak{A}$ with cardinal κ.

This completes the proof.

515I Corollary If $\mathfrak{A}$ is an infinite Dedekind complete Boolean algebra and $\kappa \le \#(\mathfrak{A})$, $\mathfrak{A}$ has a subalgebra isomorphic to the regular open algebra of $\{0,1\}^\kappa$.

proof By 515H, $\mathfrak{A}$ has a Boolean-independent family $\langle a_\xi\rangle_{\xi<\kappa}$. By 515Cb, $\mathfrak{A}$ has a subalgebra isomorphic to the regular open algebra of $\{0,1\}^\kappa$.

515J Corollary If $\mathfrak{A}$ is an infinite Dedekind complete Boolean algebra with Stone space Z, then $\#(Z) = 2^{\#(\mathfrak{A})}$.

proof Since Z may be identified with the set of uniferent ring homomorphisms from $\mathfrak{A}$ to $\mathbb{Z}_2$ (311E), $\#(Z) \le 2^{\#(\mathfrak{A})}$. On the other hand, writing $W = \{0,1\}^{\#(\mathfrak{A})}$, we have a subalgebra of $\mathfrak{A}$ isomorphic to the algebra $\mathcal{E}$ of open-and-closed subsets of W (515I). If $\pi : \mathcal{E} \to \mathfrak{A}$ is an injective Boolean homomorphism, it corresponds to a surjective continuous function $\psi : Z \to W$ (312Sa), so that $\#(Z) \ge \#(W) = 2^{\#(\mathfrak{A})}$.

515K I extract part of the proof of the next theorem as a lemma.

Lemma Let $\mathfrak{A}$ be an infinite Boolean algebra with the σ-interpolation property.

(a) Let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a disjoint sequence in $\mathfrak{A}$. Then $\#(\mathfrak{A}) \ge \#(\prod_{n\in\mathbb{N}}\mathfrak{A}_{a_n})$, writing $\mathfrak{A}_d$ for the principal ideal of $\mathfrak{A}$ generated by d, as usual.

(b) Set $\kappa = \#(\mathfrak{A})$, and let I be the set of those $a \in \mathfrak{A}$ such that $\#(\mathfrak{A}_a) < \kappa$. Then I is an ideal of $\mathfrak{A}$, and

either $\mathfrak{A}/I$ is infinite,

or there is a set $J \subseteq I$ with cardinal κ such that every sequence in J has an upper bound in J,

or $\#(\prod_{n\in\mathbb{N}}\mathfrak{A}_{a_n}) = \kappa$ for some sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in I.

Remark Recall from 466G that $\mathfrak{A}$ has the 'σ-interpolation property' if whenever A, $B \subseteq \mathfrak{A}$ are countable and $a \subseteq b$ for every $a \in A$ and $b \in B$, then there is a $c \in \mathfrak{A}$ such that $a \subseteq c \subseteq b$ for every $a \in A$ and $b \in B$. See also 514Yf above.

proof (a) The point is that the map $a \mapsto \langle a \cap a_n\rangle_{n\in\mathbb{N}} : \mathfrak{A} \to \prod_{n\in\mathbb{N}}\mathfrak{A}_{a_n}$ is surjective. **P** If $\langle b_n\rangle_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}}\mathfrak{A}_{a_n}$, there must be an $a \in \mathfrak{A}$ such that $b_n \subseteq a \subseteq 1 \setminus (a_n \setminus b_n)$ for every n, so that $a \cap a_n = b_n$ for every n. **Q** The result follows at once.

(b) If a, $b \in I$ then $(c,d) \mapsto c \cup d$ is a surjection from $\mathfrak{A}_a \times \mathfrak{A}_b$ onto $\mathfrak{A}_{a\cup b}$, so $a \cup b \in I$; of course $b \in I$ whenever $b \subseteq a \in I$, so I is an ideal of $\mathfrak{A}$.

? Suppose, if possible, that all three alternatives are false. Then $\mathfrak{A}/I$ is finite; let $v_0, \dots, v_m$ be its atoms. Let $c_0, \dots, c_m \in \mathfrak{A}$ be such that $c_i^\bullet = v_i$ for every i. Observe that $\mathfrak{A}$ is the union of finitely many sets with cardinal $\#(I)$, so I itself must have cardinal κ, and there is a sequence $\langle b'_n\rangle_{n\in\mathbb{N}}$ in I with no upper bound in I; setting $b_n = b'_n \setminus \sup_{m<n} b'_m$ for each n, we get a disjoint sequence $\langle b_n\rangle_{n\in\mathbb{N}}$ in I with no upper bound in I. Now there is some $k \le m$ such that $\langle b_n \cap c_k\rangle_{n\in\mathbb{N}}$ has no upper bound in I. Set $K = \{d : d \subseteq c_k,\ d \cap b_n = 0 \text{ for every } n \in \mathbb{N}\}$. Then $K \vartriangleleft \mathfrak{A}_{c_k}$. If $d \in K$, $c_k \setminus d$ is an upper bound for $\{b_n \cap c_k : n \in \mathbb{N}\}$, so does not belong to I; as $c_k^\bullet$ is an atom in $\mathfrak{A}/I$, d must belong to I. Thus $K \subseteq I$. The function $d \mapsto \langle d \cap b_n\rangle_{n\in\mathbb{N}} : \mathfrak{A}_{c_k} \to \prod_{n\in\mathbb{N}}\mathfrak{A}_{c_k\cap b_n}$ is a Boolean homomomorphism with kernel K, so

$$\#(\mathfrak{A}_{c_k}/K) \le \#(\textstyle\prod_{n\in\mathbb{N}}\mathfrak{A}_{c_k\cap b_n}) < \kappa$$

(since the third alternative is false, and $\#(\prod_{n\in\mathbb{N}}\mathfrak{A}_{c_k\cap b_n}) \le \kappa$ by (a)); as $\#(\mathfrak{A}_{c_k}) = \kappa$, $\#(K) = \kappa$. There is therefore a sequence $\langle d_n\rangle_{n\in\mathbb{N}}$ in K with no upper bound in K. But there is a $d \in \mathfrak{A}$ such that $d_n \subseteq d \subseteq 1 \setminus b_n$ for every $n \in \mathbb{N}$, because $\mathfrak{A}$ has the σ-interpolation property; so that $d \cap c_k \in K$ is an upper bound for $\{d_n : n \in \mathbb{N}\}$. **X**

515L Theorem (Koppelberg 75) If $\mathfrak{A}$ is an infinite Boolean algebra with the σ-interpolation property, then $\#(\mathfrak{A})$ is equal to the cardinal power $\#(\mathfrak{A})^\omega$.

proof Induce on $\kappa = \#(\mathfrak{A})$.

(a) If $\kappa \le \mathfrak{c}$, then (because $\mathfrak{A}$ is infinite) there is a disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}^+$, so that

$$\mathfrak{c} \le \#(\textstyle\prod_{n\in\mathbb{N}}\mathfrak{A}_{a_n}) \le \#(\mathfrak{A})$$

by 515Ka, and $\kappa = \mathfrak{c}$. So $\kappa^\omega = (2^\omega)^\omega = \kappa$.

(b) For the inductive step to $\kappa > \mathfrak{c}$, set $I = \{a : a \in \mathfrak{A},\ \#(\mathfrak{A}_a) < \kappa\}$, as in 515Kb. It is easy to see that every principal ideal of $\mathfrak{A}$ has the σ-interpolation property, so that $\#(\mathfrak{A}_a)^\omega \le \max(\mathfrak{c}, \#(\mathfrak{A}_a))$ for every $a \in I$. Now consider the three possibilities of 515Kb.

case 1 If the quotient algebra $\mathfrak{A}/I$ is infinite, then $\kappa^\omega = \kappa$. **P** There is a disjoint sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}/I$. For each $n \in \mathbb{N}$ take $a_n \in \mathfrak{A}$ such that $a_n^\bullet = u_n$; now setting $a'_n = a_n \setminus \sup_{i<n} a_i$ for each n, $\langle a'_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A} \setminus I$. So

$$\kappa \le \kappa^\omega = \#(\textstyle\prod_{n\in\mathbb{N}}\mathfrak{A}_{a'_n}) \le \kappa$$

by 515Ka again. **Q**

case 2 Suppose that there is a set $J \subseteq I$ such that $\#(J) = \kappa$ and every sequence in J has an upper bound in J. Then $\kappa^\omega = \kappa$. **P**

$$\kappa^\omega = \#(J^{\mathbb{N}}) \le \#(\bigcup_{a\in J} \mathfrak{A}_a^{\mathbb{N}})$$

(because every sequence in J is included in $\mathfrak{A}_a$ for some $a \in J$)

$$\le \max(\omega, \#(J), \sup_{a\in I} \#(\mathfrak{A}_a^{\mathbb{N}})) \le \max(\kappa, \sup_{a\in J} \#(\mathfrak{A}_a)) = \kappa \le \kappa^\omega. \text{ } \mathbf{Q}$$

case 3 Suppose there is a sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in I such that $\#(\prod_{n\in\mathbb{N}} \mathfrak{A}_{a_n}) = \kappa$. Then $\kappa^\omega = \kappa$. **P** Set $L = \{n : n \in \mathbb{N}, \mathfrak{A}_{a_n}$ is infinite$\}$. Then

$$\kappa^\omega = \#((\prod_{n\in\mathbb{N}} \mathfrak{A}_{a_n})^{\mathbb{N}}) = \#(\prod_{n\in\mathbb{N}\setminus L} \mathfrak{A}_{a_n}^{\mathbb{N}} \times \prod_{n\in L} \mathfrak{A}_{a_n}^{\mathbb{N}})$$

$$\le \#(\mathfrak{c} \times \prod_{n\in L} \mathfrak{A}_{a_n}) \le \max(\mathfrak{c}, \kappa) = \kappa. \text{ } \mathbf{Q}$$

Thus in all three cases we have $\kappa^\omega = \kappa$, and the induction proceeds.

515M Corollary (a) If $\mathfrak{A}$ is an infinite ccc Dedekind σ-complete Boolean algebra then $\#(\mathfrak{A}) = \tau(\mathfrak{A})^\omega$.

(b) If $\mathfrak{A}$ is any infinite Dedekind σ-complete Boolean algebra, then $\#(L^0(\mathfrak{A})) = \#(L^\infty(\mathfrak{A})) = \#(\mathfrak{A})$.

proof (a) Of course $\mathfrak{A}$, being Dedekind σ-complete, has the σ-interpolation property, as noted in 466G. So by 515L and 514De,

$$\tau(\mathfrak{A})^\omega \le \#(\mathfrak{A})^\omega = \#(\mathfrak{A}) \le \tau(\mathfrak{A})^\omega.$$

(b) $a \mapsto \chi a : \mathfrak{A} \to L^\infty(\mathfrak{A})$ and $u \mapsto \langle [\![u > q]\!]\rangle_{q\in\mathbb{Q}} : L^0(\mathfrak{A}) \to \mathfrak{A}^{\mathbb{Q}}$ are injective, so

$$\#(\mathfrak{A}) \le \#(L^\infty(\mathfrak{A})) \le \#(L^0(\mathfrak{A})) \le \#(\mathfrak{A})^\omega = \#(\mathfrak{A}).$$

515N It will be convenient later to know a little more about the regular open algebras of powers of $\{0, 1\}$.

Proposition Let I be a set. Write $\mathfrak{G}$ for the regular open algebra $\mathrm{RO}(\{0, 1\}^I)$.

(a) $\mathfrak{G}$ is ccc and Dedekind complete and isomorphic to the category algebra of $\{0, 1\}^I$. The algebra of open-and-closed subsets of $\{0, 1\}^I$ is an order-dense subalgebra of $\mathfrak{G}$.

(b) Let $\mathfrak{A}$ be a Boolean algebra. Then $\mathfrak{A}$ is isomorphic to $\mathfrak{G}$ iff it is Dedekind complete and there is a Boolean-independent family $\langle a_i\rangle_{i\in I}$ in $\mathfrak{A}$ such that the subalgebra generated by $\{a_i : i \in I\}$ is order-dense in $\mathfrak{A}$.

(c) If I is infinite, $\mathfrak{G}$ is homogeneous.

proof Write $\mathcal{E}$ for the algebra of open-and-closed subsets of $\{0, 1\}^I$ and $e_i = \{x : x \in \{0, 1\}^I, x(i) = 1\}$ for $i \in I$. Note that the set

$$\{e : e \subseteq \{0, 1\}^I \text{ is determined by coordinates in a finite set}\}$$

is an algebra of subsets of $\{0, 1\}^I$ (254Ma) which contains e_i for every i so includes $\mathcal{E}$. We also know that every member of $\mathfrak{G}$ is determined by coordinates in a countable subset of I (4A2E(b-i)).

(a) $\mathfrak{G}$, being a regular open algebra, is Dedekind complete (314P). By 4A2E(a-iii), $\{0, 1\}^I$ is ccc; by 514H(b-i), $\mathfrak{G}$ is ccc. By 514Kg, $\mathcal{E}$ is an order-dense subalgebra of $\mathfrak{G}$ and $\mathfrak{G}$ is isomorphic to the category algebra of $\{0, 1\}^I$.

(b)(i) Because $\{0, 1\}^I$ is zero-dimensional, $\mathcal{E}$ is order-dense in $\mathfrak{G}$. Now $\langle e_i\rangle_{i\in I}$ is independent and generates $\mathcal{E}$ (cf. 315Ka). So $\mathfrak{G}$ has the declared properties.

(ii) If $\mathfrak{A}$ satisfies the conditions, then, as in (a-i) of the proof of 515C, the subalgebra of $\mathfrak{A}$ generated by $\{e_i : i \in I\}$ is isomorphic to $\mathcal{E}$. Now both $\mathfrak{A}$ and $\mathfrak{G}$ are Dedekind completions of $\mathcal{E}$, so they are isomorphic (314U).

(c)(i) Suppose first that I is countable, and that $a \in \mathfrak{G}^+$. Then the principal ideal $\mathfrak{G}_a$ is atomless and Dedekind complete and has countable π-weight (514Eb), so has a countable order-dense subalgebra $\mathfrak{B}$. $\mathfrak{B}$ is countable, atomless and not $\{0\}$, so is isomorphic to the algebra of open-and-closed subsets of $\{0,1\}^{\mathbb{N}}$ and to $\mathcal{E}$ (316M). Now any isomorphism between $\mathfrak{B}$ and $\mathcal{E}$ extends to an isomorphism between their completions, which by 314Ub can be identified with $\mathfrak{G}_a$ and $\mathfrak{G}$.

(ii) Now suppose that I is uncountable. Again take $a \in \mathfrak{G}^+$. Then a is determined by coordinates in some countable subset J of I. We can of course suppose that J is infinite. Now we can express a as $b \times \{0,1\}^{I\setminus J}$ where b is a non-empty open subset of $\{0,1\}^J$, and in fact is a regular open subset (use the formulae in 4A2B(g-i) to see that $\operatorname{int}\overline{b} \times \{0,1\}^{I\setminus J} = \operatorname{int}\overline{a}$). Let $\mathfrak{B}$ be the principal ideal of $\mathrm{RO}(\{0,1\}^J)$ generated by b. By (i) here, $\mathfrak{B}$ is isomorphic to $\mathrm{RO}(\{0,1\}^J)$ and there is an independent family $\langle b_j\rangle_{j\in J}$ in $\mathfrak{B}$ generating an order-dense subalgebra $\mathcal{E}_1$ of $\mathfrak{B}$.

Next, we have a independent family $\langle c_k\rangle_{k\in I\setminus J}$ in $\mathrm{RO}(\{0,1\}^{I\setminus J})$ generating an order-dense subalgebra $\mathcal{E}_2$ of $\mathrm{RO}(\{0,1\}^{I\setminus J})$. Note that, by 4A2B(g-i) again,

$$b' \mapsto b' \times \{0,1\}^{I\setminus J} : \mathrm{RO}(\{0,1\}^I) \to \mathfrak{G}, \quad c' \mapsto \{0,1\}^J \times c' : \mathrm{RO}(\{0,1\}^{I\setminus J} \to \mathfrak{G}$$

are injective Boolean homomorphisms. So if we set

$$a_j = b_j \times \{0,1\}^{I\setminus J} \text{ for } j \in J, \quad a_j = \{0,1\}^J \times c_k \text{ for } k \in I \setminus J,$$

$\langle a_i\rangle_{i\in I}$ will be an independent family in $\mathfrak{G}$ and the subalgebra $\mathcal{E}$ it generates will contain $b'\times c'$ for every $b \in \mathcal{E}_1$ and $c \in \mathcal{E}_2$; as every member of $\mathfrak{G}^+$, being a non-empty open set in $\{0,1\}^I \cong \{0,1\}^J \times \{0,1\}^{I\setminus J}$, includes a product $b' \times c'$ where $b' \in \mathrm{RO}(\{0,1\}^I)^+$ and $c' \in \mathrm{RO}(\{0,1\}^{I\setminus J})^+$, $\mathcal{E}$ is order-dense in $\mathfrak{G}_a$. Accordingly (b) tells us that $\mathfrak{G}_a \cong \mathfrak{G}$, as required.

Remark Algebras of this kind will appear regularly as the volume proceeds; they are among the basic building blocks from which Dedekind complete Boolean algebras are constructed. I will occasionally use the phrase '**Cohen algebra**' indifferently to mean either the category algebra, or the regular open algebra, of a set $\{0,1\}^I$ where I is infinite.

515O We need to know some elementary facts about the algebra $\mathrm{RO}(\{0,1\}^{\mathbb{N}})$ which I have not yet spelt out.

Proposition (a) A Boolean algebra $\mathfrak{A}$ is isomorphic to $\mathfrak{G} = \mathrm{RO}(\{0,1\}^{\mathbb{N}})$ iff it is Dedekind complete, atomless, has countable π-weight and is not $\{0\}$. In particular, the regular open algebra $\mathrm{RO}(\mathbb{R})$ is isomorphic to $\mathfrak{G}$.

(b) Every atomless order-closed subalgebra of $\mathfrak{G}$ is isomorphic to $\mathfrak{G}$.

proof (a)(i) By 515N, $\mathfrak{G}$ is Dedekind complete (514Ic) and has π-weight ω. As in the proof of 515N, the algebra $\mathcal{E}$ of open-and-closed subsets of $\{0,1\}^{\mathbb{N}}$ is order-dense in $\mathfrak{G}$; as $\mathcal{E}$ is atomless, so is $\mathfrak{G}$.

(ii) If $\mathfrak{A}$ satisfies the conditions, let B be a countable order-dense subset of $\mathfrak{A}$ and $\mathfrak{B}$ the subalgebra of $\mathfrak{A}$ generated by B. Then $\mathfrak{B}$ is countable, atomless and not $\{0\}$, so is isomorphic to $\mathcal{E}$ (316M). Now any isomorphism between $\mathfrak{B}$ and $\mathcal{E}$ extends to an isomorphism between their completions, which by 314Ub can be identified with $\mathfrak{A}$ and $\mathfrak{G}$ respectively.

(iii) All regular open algebras are Dedekind complete. Because $\mathbb{R}$ is not empty, $\mathrm{RO}(\mathbb{R}) \neq \{0\}$. Because $\mathbb{R}$ is Hausdorff and has no isolated points, $\mathrm{RO}(\mathbb{R})$ is atomless. Because $\mathbb{R}$ has countable π-weight, so has $\mathrm{RO}(\mathbb{R})$ (514H(b-ii)). So $\mathrm{RO}(\mathbb{R}) \cong \mathfrak{G}$ by (ii) here.

(b) All we have to observe is that any atomless order-closed subalgebra of $\mathfrak{G}$ satisfies the conditions of (a) (see 514E).

515P There is a more complicated, but still manageable, characterization of $\mathrm{RO}(\{0,1\}^{\omega_1})$.

Proposition A Boolean algebra $\mathfrak{A}$ is isomorphic to $\mathrm{RO}(\{0,1\}^{\omega_1})$ iff

(α) it is non-zero, ccc and Dedekind complete,

(β) every non-zero principal ideal of $\mathfrak{A}$ has π-weight ω_1,

(γ) there is a non-decreasing family $\langle A_\xi\rangle_{\xi<\omega_1}$ of countable subsets of $\mathfrak{A}$ such that

each A_ξ is order-dense in the order-closed subalgebra of $\mathfrak{A}$ which it generates,

$A_\zeta = \bigcup_{\xi<\zeta} A_\xi$ for every non-zero countable limit ordinal ζ,
$\bigcup_{\xi<\omega_1} A_\xi$ is order-dense in $\mathfrak{A}$.

proof Write $\mathfrak{G}$ for $\mathrm{RO}(\{0,1\}^{\omega_1})$.

(a)(i) We know from 515N that $\mathfrak{G}$ satisfies the conditions (α) and (β) above.

(ii) For $\xi < \omega_1$, let $\mathcal{E}_\xi$ be the set of open-and-closed subsets of $\{0,1\}^{\omega_1}$ which are determined by coordinates in ξ, and $\mathfrak{G}_\xi$ the set of regular open subsets of $\{0,1\}^{\omega_1}$ which are determined by coordinates in ξ. Then $\mathfrak{G}_\xi$ is an order-closed subalgebra of $\mathfrak{G}$. **P** Reviewing the formulae of 314O, we see that the Boolean operations of $\mathfrak{G}$ can all be expressed in terms of (arbitrary) unions and intersections, set difference, closure and interior. Since all of these can be done within the family of subsets of $\{0,1\}^{\omega_1}$ determined by coordinates in ξ (254Ma, 4A2B(g-i)), $\mathfrak{G}_\xi$ is an order-closed subalgebra of $\mathfrak{G}$. **Q** Next, $\mathcal{E}_\xi$ is a subalgebra of the algebra $\mathcal{E}$ of open-and-closed subsets of $\{0,1\}^{\omega_1}$, and is order-dense in $\mathfrak{G}_\xi$, so $\mathfrak{G}_\xi$ is the order-closed subalgebra of $\mathfrak{G}$ generated by $\mathcal{E}_\xi$.

Of course $\langle\mathcal{E}_\xi\rangle_{\xi<\omega_1}$ is a non-decreasing family of countable subsets of $\mathfrak{G}$. If ζ is a non-zero countable limit ordinal, then every member of $\mathcal{E}_\zeta$ is determined by coordinates in a finite subset of ζ, so $\mathcal{E}_\zeta = \bigcup_{\xi<\zeta} \mathcal{E}_\xi$. And $\bigcup_{\xi<\omega_1} \mathcal{E}_\xi = \mathcal{E}$ is order-dense in $\mathfrak{G}$. So $\langle\mathcal{E}_\xi\rangle_{\xi<\omega_1}$ witnesses that $\mathfrak{G}$ satisfies condition (γ).

(b) Now suppose that $\mathfrak{A}$ and $\langle A_\xi\rangle_{\xi<\omega_1}$ satisfy the conditions (α)-(γ). For each $\xi < \omega_1$ let $\mathfrak{A}_\xi$ be the order-closed subalgebra generated by A_ξ.

(i) We need to know that $\mathfrak{A} = \bigcup_{\xi<\omega_1} \mathfrak{A}_\xi$. **P** Because $\mathfrak{A}$ is ccc and $\langle\mathfrak{A}_\xi\rangle_{\xi<\omega_1}$ is a non-decreasing family of order-closed subalgebras of $\mathfrak{A}$, $\bigcup_{\xi<\omega_1} \mathfrak{A}_\xi$ is an order-closed subalgebra (use 316Fb). But $\bigcup_{\xi<\omega_1} \mathfrak{A}_\xi$ includes the order-dense set $\bigcup_{\xi<\omega_1} A_\xi$ so is the whole of $\mathfrak{A}$. **Q**

(ii) Suppose that $\mathfrak{C}$ is an order-closed subalgebra of $\mathfrak{C}$ with countable π-weight. Then there is an $a \in \mathfrak{A}$ which is independent of $\mathfrak{C}$ in the sense that $c \cap a$ and $c \setminus a$ are non-zero for every non-zero $c \in \mathfrak{C}$. **P** Let $\langle(a_i, c_i)\rangle_{i\in I}$ be a maximal family such that $\langle c_i\rangle_{i\in I}$ is a disjoint family in $\mathfrak{C}$ and, for each $i \in I$, $a_i \subseteq c_i$ is such that $c \cap a_i$ and $c \setminus a_i$ are non-zero for every $c \in \mathfrak{C}$ such that $c \cap c_i$ is non-zero. Set $c^* = 1 \setminus \sup_{i\in I} c_i$. **?** If $c^* \neq 0$, consider the principal ideal $\mathfrak{A}_{c^*}$ of $\mathfrak{A}$ generated by c^*. By hypothesis, this must have uncountable π-weight, so the countable set $\mathfrak{C}_{c^*}$ cannot be order-dense in $\mathfrak{A}_{c^*}$, and there is a non-zero $a' \in \mathfrak{A}_{c^*}$ not including any non-zero member of $\mathfrak{C}_{c^*}$, therefore not including any non-zero member of $\mathfrak{C}$. Set $c' = \mathrm{upr}(a', \mathfrak{C})$; then c' is non-zero and $c' \subseteq c^*$, so $c' \cap c_i = 0$ for every $i \in I$. If $c \in \mathfrak{C}$ is non-zero and meets c', then $c \cap c' \not\subseteq a$ so $c \setminus a \neq 0$, while $a \not\subseteq c' \setminus c$ so $c \cap a \neq 0$. But this means that we ought to have added (a', c') to the family $\langle(a_i, c_i)\rangle_{i\in I}$. **X**

Thus $\sup_{i\in I} c_i = 1$. Set $a = \sup_{i\in I} a_i$. If $c \in \mathfrak{C}^+$, there is an $i \in I$ such that $c \cap c_i \neq 0$, in which case $c \cap a \supseteq c \cap a_i$ and $c \setminus a \supseteq (c \cap c_i) \setminus a_i$ are both non-zero. Accordingly a is independent of $\mathfrak{C}$, as required. **Q**

(iii) Suppose that $\mathfrak{C}$ is an order-closed subalgebra of $\mathfrak{A}$ with countable π-weight, and $b \in \mathfrak{A}$. Then there are an an $a \in \mathfrak{A}$ and a $d \in \mathfrak{C}^+$ such that a is independent of $\mathfrak{C}$ and $a \cap d \subseteq b$. **P** By (ii), we have an $a' \in \mathfrak{A}$ which is independent of $\mathfrak{C}$. If there is a non-zero $d \in \mathfrak{C}$ such that $d \subseteq b$ we can set $a = a'$. Otherwise, set $d = \mathrm{upr}(b, \mathfrak{C})$ and $a = b \cup (a' \setminus d)$. If $c \in \mathfrak{C}$ is non-zero, either $c \cap d \neq 0$ and

$$c \cap a \supseteq c \cap d \cap b \neq 0, \quad c \setminus a \supseteq c \cap d \setminus b \neq 0,$$

or $c \setminus d \neq 0$ and

$$c \cap a \supseteq (c \setminus d) \cap a' \neq 0, \quad c \setminus a \supseteq (c \setminus d) \setminus a' \neq 0.$$

So a is independent of $\mathfrak{C}$, while of course $0 \neq a \cap d \subseteq b$. **Q**

(iv) Suppose that $\zeta < \omega_1$. Then there are a $\zeta' \geq \zeta$ and an independent sequence $\langle a_i\rangle_{i\in\mathbb{N}}$ in $\mathfrak{A}_{\zeta'}$ such that $\mathfrak{A}_\zeta$ and the subalgebra $\mathfrak{B}$ generated by $\{a_i : i \in \mathbb{N}\}$ are independent and the subalgebra generated by $\mathfrak{A}_\zeta \cup \mathfrak{B}$ is order-dense in $\mathfrak{G}_{\zeta'}$. **P** We need a couple of bookkeeping devices. For each $\xi < \omega_1$ le $\langle b(\xi, n)\rangle_{n\in\mathbb{N}}$ be a sequence running over $A_\xi \setminus \{0\}$, and let $\langle(j_i, k_i)\rangle_{i\in\mathbb{N}}$ be an enumeration of $\mathbb{N} \times \mathbb{N}$ with $j_i \leq i$ for every $i \in \mathbb{N}$. Now choose sequences $\langle\zeta_i\rangle_{i\in\mathbb{N}}$ in ω_1, $\langle a_i\rangle_{i\in\mathbb{N}}$ in $\mathfrak{A}$ and $\langle\mathfrak{D}_i\rangle_{i\in\mathbb{N}}$ inductively so that

$$\zeta_i = \min\{\xi : \zeta \leq \xi,\, a_j \in \mathfrak{A}_\xi \text{ for every } j < i\}$$

(by (i) above, ζ_i will always be countable, and $\zeta = \zeta_0 \leq \zeta_1 \leq \dots$)

$$\mathfrak{D}_i \text{ is the subalgebra of } \mathfrak{A} \text{ generated by } \mathfrak{A}_\zeta \cup \{a_j : j < i\}$$

(so that $\mathfrak{D}_i$ is always an order-closed subalgebra of $\mathfrak{A}$, by 314Ja),

$$a_i \text{ is independent of } \mathfrak{D}_i \text{ and there is a } d \in \mathfrak{D}_i \text{ such that } 0 \ne a_i \cap d \subseteq b(\zeta_{j_i}, k_i)$$

(using (iii)). At the end of the induction, set $\zeta' = \sup_{i\in\mathbb{N}} \zeta_i$.

Because a_i is always independent of $\mathfrak{D}_i$, $\langle a_i\rangle_{i\in\mathbb{N}}$ is independent and $\mathfrak{B}$, the subalgebra generated by $\{a_i : i \in \mathbb{N}\}$, is independent of $\mathfrak{A}_\zeta$. Next, if $a \in \mathfrak{A}_{\zeta'}$ is non-zero, there is an $a' \in A_{\zeta'}$ such that $0 \ne a' \subseteq a$; there are a $\xi < \zeta'$ such that $a' \in A_\xi$, a $j \in \mathbb{N}$ such that $\xi \le \zeta_j$ and $a' \in A_{\zeta_j}$, and an $i \in \mathbb{N}$ such that $j_i = j$ and $a' = b(\zeta_{j_i}, k_i)$. There is now a $d \in \mathfrak{D}_i^+$ such that $a_i \cap d \subseteq a'$. Because $\mathfrak{D}_i$ is generated by $\mathfrak{A}_\zeta \cup \{a_l : l < i\}$, there are $c \in \mathfrak{A}_\zeta^+$ and $J \subseteq i$ such that $a'' = c \cap \inf_{l\in J} a_l \setminus \sup_{l\in i\setminus J} a_l$ is included in a'. But now a'' belongs to the subalgebra generated by $\mathfrak{A}_\zeta \cup \mathfrak{B}$ and is included in a. As a is arbitrary, the subalgebra generated by $\mathfrak{A}_\zeta \cup \mathfrak{B}$ is order-dense in $\mathfrak{A}_{\zeta'}$, as required. **Q**

(v) We can therefore build inductively families $\langle\zeta_\xi\rangle_{\xi<\omega_1}$ and $\langle a_{\xi i}\rangle_{\xi<\omega_1, i\in\mathbb{N}}$ such that, for each $\xi < \omega_1$,

$$\langle a_{\eta i}\rangle_{\eta<\xi, i\in\mathbb{N}} \text{ is independent,}$$

$$\text{the subalgebra generated by } \{a_{\xi' i} : \xi' < \xi,\ i \in \mathbb{N}\} \text{ is an order-dense subset of } \mathfrak{A}_{\zeta_i}.$$

P Start with $\zeta_0 = 0$. For the inductive step to $\xi + 1$, where $\xi < \omega_1$, apply (iv) with $\zeta = \zeta_\xi$ to choose $\langle a_{\xi i}\rangle_{i\in\mathbb{N}}$ and ζ', and set $\zeta_{\xi+1} = \zeta'$. If $a \in \mathfrak{A}^+_{\zeta_{\xi+1}}$ then a includes $c \cap b$ for some non-zero $c \in \mathfrak{A}_\zeta$ and b in the algebra generated by $\{a_{\xi i} : i \in \mathbb{N}\}$; now c includes c' for some non-zero c' in the subalgebra generated by $\{a_{\xi' i} : \xi' < \xi,\ i \in \mathbb{N}\}$, and we see that $c' \cap b$ is a non-zero element of the subalgebra generated by $\{a_{\xi' i} : \xi' \le \xi + 1,\ i \in \mathbb{N}\}$ and is included in a.

For the inductive step to a non-zero limit ordinal $\xi < \omega_1$ set $\zeta_\xi = \sup_{\eta<\xi} \zeta_\eta$. In this case, if $a \in \mathfrak{A}^+_{\zeta_\xi}$, there is a non-zero $a' \in A_{\zeta_\xi}$ included in a, there is an $\eta < \xi$ such that $a' \in A_{\zeta_\eta}$, and there is a non-zero a'' in the subalgebra generated by $\{a_{\eta' i} : \eta' < \eta,\ i \in \mathbb{N}\}$ such that $a'' \subseteq a'$. So again the subalgebra generated by $\{a_{\xi' i} : \xi' < \xi,\ i \in \mathbb{N}\}$ is order-dense in $\mathfrak{A}_{\zeta_\xi}$ and the induction proceeds. **Q**

(vi) At the end of the induction, $\langle a_{\xi i}\rangle_{\xi<\omega_1, i\in\mathbb{N}}$ is independent, because all its finite subfamilies are independent. Because $a_{\xi 0} \in \mathfrak{A}_{\zeta_{\xi+1}} \setminus \mathfrak{A}_{\zeta_\xi}$ for every ξ, $\langle\zeta_\xi\rangle_{\xi<\omega_1}$ is strictly increasing and $\sup_{\xi<\omega_1} \zeta_\xi = \omega_1$, so that $\mathfrak{A} = \bigcup_{\xi<\omega_1} \mathfrak{A}_{\zeta_\xi}$. Accordingly the subalgebra generated by $\{a_{\xi i} : \xi < \omega_1,\ i \in \mathbb{N}\}$ will be order-dense in $\mathfrak{A}$. But 515Nc now tells us that $\mathfrak{A} \cong \mathfrak{G}$. So the conditions listed are sufficient as well as necessary.

515Q Concerning closed subalgebras, the position with $\mathrm{RO}(\{0,1\}^{\omega_1})$ is nearly as straightforward as with $\mathrm{RO}(\{0,1\}^{\omega})$, though the proof is a good deal deeper.

Proposition Let $\mathfrak{A}$ be an atomless order-closed subalgebra of $\mathfrak{G} = \mathrm{RO}(\{0,1\}^{\omega_1})$. Then $\mathfrak{A}$ is isomorphic either to $\mathrm{RO}(\{0,1\}^{\omega})$ or to $\mathfrak{G}$ or to the simple product $\mathrm{RO}(\{0,1\}^{\omega}) \times \mathfrak{G}$.

proof (a) We know that there is a non-decreasing family $\langle B_\xi\rangle_{\xi<\omega_1}$ of countable subsets of $\mathfrak{G}$ such that $\bigcup_{\xi<\omega_1} B_\xi$ is order-dense in $\mathfrak{G}$, $B_\xi = \bigcup_{\eta<\xi} B_\eta$ for every non-zero countable limit ordinal ξ, every B_ξ is order-dense in the order-closed subalgebra $\mathfrak{G}_\xi$ of $\mathfrak{G}$ which it generates, and $\mathfrak{G} = \bigcup_{\xi<\omega_1} \mathfrak{G}_\xi$.

Consider the set

$$C = \{\zeta : \zeta < \omega_1 \text{ is a non-zero limit ordinal, } \mathrm{upr}(b, \mathfrak{A}) \in \mathfrak{G}_\zeta \text{ for every } b \in \bigcup_{\xi<\zeta} B_\xi\}.$$

Then C is a closed cofinal subset of ω_1, and $\mathrm{upr}(b, \mathfrak{A}) \in \mathfrak{G}_\zeta$ whenever $\zeta \in C$ and $b \in \bigcup_{\xi<\zeta} B_\xi = B_\zeta$. Set $A_\zeta = \{\mathrm{upr}(b, \mathfrak{A}) : b \in B_\zeta\}$ for each $\zeta \in C$. Then $\langle A_\zeta\rangle_{\zeta\in C}$ is a non-decreasing family of countable subsets of $\mathfrak{A}$ and $A_\zeta = \bigcup_{\xi\in C\cap\zeta} A_\xi$ whenever $\zeta \in C$ is $\sup(C \cap \zeta)$. Moreover, if $\zeta \in \mathfrak{C}$, A_ζ is an order-dense subset of $\mathfrak{A} \cap \mathfrak{G}_\zeta$. **P** Surely $A_\zeta \subseteq \mathfrak{A}$, and $A_\zeta \subseteq \mathfrak{G}_\zeta$ by the definition of C. If $a \in (\mathfrak{A} \cap \mathfrak{G}_\zeta)^+$, there is a non-zero $b \in B_\zeta$ such that $b \subseteq a$, and now $\mathrm{upr}(b, \mathfrak{A}) \in A_\zeta$ and $0 \ne \mathrm{upr}(b, \mathfrak{A}) \subseteq a$. **Q** Since $\mathfrak{A} \cap \mathfrak{G}_\zeta$ is an order-closed subalgebra of $\mathfrak{G}$ it is also an order-closed subalgebra of $\mathfrak{A}$ and must be the order-closed subalgebra of $\mathfrak{A}$ generated by A_ζ.

What this means is that $\langle A_\zeta\rangle_{\zeta\in C}$, suitably re-indexed, witnesses that $\mathfrak{A}$ satisfies condition (γ) of 515P.

(b) So if $\mathfrak{A}$ is π-weight-homogeneous with π-weight ω_1, that is, all its non-zero principal ideals have π-weight ω_1, then $\mathfrak{A} \cong \mathfrak{G}$. **P** As surely it is non-zero, ccc and Dedekind complete (because $\mathfrak{G}$ is), it satisfies all three conditions of 515P. **Q**

On the other hand, if $\pi(\mathfrak{A})$ is countable, then 515O tells us that $\mathfrak{A} \cong \mathrm{RO}(\{0,1\}^\omega)$. So let us suppose from now on that $\mathfrak{A}$ has uncountable π-weight and has a principal ideal which has π-weight different from ω_1.

(c) Because π-weight is an order-preserving ordinal function of Boolean algebras (514Ed), and $\mathfrak{A}$ is Dedekind complete, it is isomorphic to a simple product of π-weight-homogeneous principal ideals (514Gc), all of which are Dedekind complete and have π-weight at most $\pi(\mathfrak{A}) \le \pi(\mathfrak{G}) = \omega_1$ (514Eb). Because $c(\mathfrak{A}) \le c(\mathfrak{G})$ is countable (514Ea), we are dealing with a countable product. As $\pi(\mathfrak{A}) > \omega$, not all the terms have countable π-weight, and we are supposing also that not all the terms have π-weight ω_1.

The terms with π-weight ω can be joined together as a single principal ideal $\mathfrak{B}$ with π-weight ω (514Ef), which must now be isomorphic to $\mathrm{RO}(\{0,1\}^\omega)$. Similarly the terms with π-weight ω_1 can be joined together as a single π-weight-homogeneous principal ideal $\mathfrak{C}$ of π-weight ω_1. So $\mathfrak{A} \cong \mathfrak{B} \times \mathfrak{C}$.

Express $\mathfrak{C}$ as $\mathfrak{A}_c$ where $c \in \mathfrak{A}$. Consider the corresponding principal ideal $\mathfrak{G}_c$ of $\mathfrak{G}$. This is isomorphic to $\mathfrak{G}$ (515Nc) and $\mathfrak{C}$ is an order-closed subalgebra of $\mathfrak{G}_c$. So (b) tells us that $\mathfrak{C} \cong \mathfrak{G}$, and our decomposition of $\mathfrak{A}$ is of the required form.

515X Basic exercises (a) Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$, and $\langle D_i\rangle_{i\in I}$ a family of partitions of unity in $\mathfrak{A}$, none containing 0. Show that the following are equiveridical: (i) $\langle D_i\rangle_{i\in I}$ is Boolean-independent; (ii) $\langle \mathfrak{B}_i\rangle_{i\in I}$ is Boolean-independent, where $\mathfrak{B}_i$ is the subalgebra of $\mathfrak{A}$ generated by D_i for each $i \in I$.

(b) Give an example of a Boolean algebra $\mathfrak{A}$ with Boolean-independent subalgebras $\mathfrak{B}$, $\mathfrak{C}$ such that the order-closed subalgebras generated by $\mathfrak{B}$ and $\mathfrak{C}$ are not Boolean-independent.

(c) For a Boolean algebra $\mathfrak{A}$, not $\{0\}$, write $\mathrm{ind}(\mathfrak{A})$ for $\sup\{\#(A) : A \subseteq \mathfrak{A}$ is Boolean-independent$\}$. (If $\mathfrak{A} = \{0\}$, say $\mathrm{ind}(\mathfrak{A}) = 0$.) (i) Show that if $\mathfrak{B}$ is either a subalgebra or a principal ideal or a homomorphic image of $\mathfrak{A}$ then $\mathrm{ind}(\mathfrak{B}) \le \mathrm{ind}(\mathfrak{A})$. (ii) Show that $\mathfrak{A}$ is infinite iff $\mathrm{ind}(\mathfrak{A})$ is infinite. (iii) Show that if $\mathfrak{A}$ is finite and not $\{0\}$ then $\mathrm{ind}(\mathfrak{A})$ is the largest n such that $2^{2^n} \le \#(\mathfrak{A})$. (iv) Show that if $\mathfrak{A}$ is the finite-cofinite algebra of subsets of an infinite set X, then $\mathrm{ind}(\mathfrak{A}) = \omega$ but $\mathfrak{A}$ has no infinite Boolean-independent set. (v) Show that if $\mathfrak{A}$ and $\mathfrak{B}$ are Boolean algebras then $\mathrm{ind}(\mathfrak{A} \times \mathfrak{B})$ is at most the cardinal sum $\mathrm{ind}(\mathfrak{A}) + \mathrm{ind}(\mathfrak{B})$. (vi) Show that if $\mathfrak{A}$ is infinite and has the σ-interpolation property then $\mathrm{ind}(\mathfrak{A}) \ge \mathfrak{c}$.

(d) Let Z be an infinite extremally disconnected compact Hausdorff space. Show that there is a continuous surjection from Z onto $\{0,1\}^{w(Z)}$.

(e) Let $\mathfrak{A}$ be a Boolean algebra with the σ-interpolation property. Show that any homomorphic image of $\mathfrak{A}$ has the σ-interpolation property.

(f) Let κ be an infinite cardinal. Show that the following are equiveridical: (i) there is a measure algebra with cardinal κ; (ii) there is a measurable algebra with cardinal κ; (iii) $\kappa^\omega = \kappa$.

515Y Further exercises (a)(i) Show that if $\mathfrak{A}$ is any Boolean algebra, other than $\{0\}$, with cardinal at most ω_1, it is isomorphic to a subalgebra of $\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}$. (ii) Show that an atomless Boolean algebra with cardinal ω_1 and the σ-interpolation property is isomorphic to $\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}$. (This is a version of **Parovičenko's theorem**.)

(b) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, and $\kappa \le \#(\mathfrak{A})$ a regular uncountable cardinal. Show that there is a strictly increasing family $\langle\mathfrak{A}_\xi\rangle_{\xi<\kappa}$ of subalgebras of $\mathfrak{A}$ with union $\mathfrak{A}$. (Compare 494Yk.)

515 Notes and comments The material of this section is taken from Koppelberg 89, where you can find a good deal more. I have picked out the results which are essential to a proper understanding of measure algebras. Of course there are short cuts, using Maharam's theorem (332B), if we know that we are dealing with a localizable measure algebra; but I should not like to leave you with the impression that the theorems here are restricted to measure algebras.

Any theorem about Boolean algebras is also a theorem about zero-dimensional compact Hausdorff spaces; thus 515H and 515Xd have an equal right to be called the Balcar-Franěk theorem. 515D and part (b) of the proof of 515H may be regarded as a simple form of some of the ideas of §331.

Clearly some of the ideas of this section can be expressed in terms of the independence number $\operatorname{ind}(\mathfrak{A})$ (515Xc). But the expression is complicated by the fact that (like cellularity) the independence number may not be attained (see 515Xc(iv)), while the theorems here mostly need actual Boolean-independent families. Since $\operatorname{ind}(\mathfrak{A}) = \#(\mathfrak{A})$ for infinite Dedekind complete Boolean algebras (515H), we shall not have to grapple with these difficulties.

The results of 515O-515Q give us a good grip on the regular open algebras of $\{0,1\}^\omega$ and $\{0,1\}^{\omega_1}$ and their order-closed subalgebras. There are serious obstacles in the way of extending these ideas; order-closed subalgebras of the category algebra of $\{0,1\}^{\omega_2}$, for instance, can be very different in character. For examples see Koppelberg & Shelah 96 and Balcar Jech & Zapletal 97.

Version of 9.10.14

516 Precalibers

In this section I will try to display the elementary connexions between 'precalibers', as defined in 511E, and the cardinal functions we have looked at so far. The first step is to generalize the idea of precaliber from partially ordered sets to supported relations (516A); the point is that Galois-Tukey connections give us information on precalibers (516C), and in particular give quick proofs that partially ordered sets, topological spaces and Boolean algebras related in the canonical ways explored in §514 have many of the same precalibers (516G, 516H, 516M). Much of the section is taken up with lists of expected facts, but for some results the hypotheses need to be chosen with care. I end with a fundamental theorem on the saturation of product spaces (516T).

516A Definition If (A,R,B) is a supported relation, a **precaliber triple** of (A,R,B) is a triple $(\kappa,\lambda,{<}\theta)$ where κ, λ and θ are cardinals and whenever $\langle a_\xi\rangle_{\xi<\kappa}$ is a family in A then there is a set $\Gamma\in[\kappa]^\lambda$ such that $\langle a_\xi\rangle_{\xi\in\Gamma}$ is ${<}\theta$-linked in the sense of 512Bc, that is, for every $I\in[\Gamma]^{<\theta}$ there is a $b\in B$ such that $(a_\xi,b)\in R$ for every $\xi\in I$. Similarly, (κ,λ,θ) is a precaliber triple of (A,R,B) if whenever $\langle a_\xi\rangle_{\xi<\kappa}$ is a family in A then there is a set $\Gamma\in[\kappa]^\lambda$ such that $\langle a_\xi\rangle_{\xi\in\Gamma}$ is θ-linked; that is, if $(\kappa,\lambda,{<}\theta^+)$ is a precaliber triple.

Now (κ,λ) is a **precaliber pair** of (A,R,B) if $(\kappa,\lambda,{<}\omega)$ is a precaliber triple of (A,R,B), and κ is a **precaliber** of (A,R,B) if (κ,κ) is a precaliber pair.

516B Elementary remarks I ought perhaps to spell out the following immediate consequences of the definitions. Let (A,R,B) be a supported relation.

(a) If $\kappa'\ge\kappa$, $\lambda'\le\lambda$, $\theta'\le\theta$ and $(\kappa,\lambda,{<}\theta)$ is a precaliber triple of (A,R,B), then $(\kappa',\lambda',{<}\theta')$ is a precaliber triple of (A,R,B). So if $\kappa'\ge\kappa$, $\lambda'\le\lambda$ and (κ,λ) is a precaliber pair of (A,R,B), then (κ',λ') is a precaliber pair of (A,R,B).

(b) If $\theta>0$, then $(0,0,{<}\theta)$ is a precaliber triple of (A,R,B) iff $B\ne\emptyset$. If $A=\emptyset$ then $(\kappa,\lambda,{<}\theta)$ is a precaliber triple of (A,R,B) whenever $\kappa\ge1$. If $A\ne\emptyset$ and $A\ne R^{-1}[B]$, that is, $\operatorname{cov}(A,R,B)=\infty$, then the only precaliber triples of (A,R,B) are of the form $(\kappa,0,{<}\theta)$. If $A\ne\emptyset$ and $(\kappa,\lambda,{<}\theta)$ is a precaliber triple of (A,R,B), then $\lambda\le\kappa$. $\operatorname{cov}(A,R,B)=\infty$ iff 1 is not a precaliber of (A,R,B).

(c) If (κ,λ,λ) is a precaliber triple of (A,R,B) then $(\kappa,\lambda,{<}\theta)$ is a precaliber triple of (A,R,B) for every θ; in particular, (κ,λ) is a precaliber pair of (A,R,B).

(d) If $(\kappa,\kappa,{<}\theta)$ is a precaliber triple of (A,R,B), so is $(\operatorname{cf}\kappa,\operatorname{cf}\kappa,{<}\theta)$. **P** If $\operatorname{cf}\kappa=\kappa$ there is nothing to prove. If $2\le\kappa<\omega$ and A is empty the result is trivial. If $2\le\kappa<\omega$ and A is not empty, then B is not empty, so if $\theta\le1$ the result is trivial. If $2\le\kappa<\omega$ and A is not empty and $\theta>1$, then $R^{-1}[B]=A$ so $(\operatorname{cf}\kappa,\operatorname{cf}\kappa,{<}\theta)=(1,1,{<}\theta)$ is a precaliber triple of (A,R,B).

If $\kappa>\operatorname{cf}\kappa$ is infinite, let $\langle\gamma_\xi\rangle_{\xi<\operatorname{cf}\kappa}$ be a strictly increasing family with supremum κ. For $\eta<\kappa$, set $f(\eta)=\min\{\xi:\eta\le\gamma_\xi\}$. If $\langle a_\xi\rangle_{\xi<\operatorname{cf}\kappa}$ is a family in A, set $a'_\eta=a_{f(\eta)}$ for each $\eta<\kappa$. Then there is a $\Gamma\in[\kappa]^\kappa$

such that $\langle a'_\eta\rangle_{\eta\in\Gamma}$ is $<\theta$-linked. Set $\Gamma'=\{f(\eta):\eta\in\Gamma\}$; then $\langle a_\xi\rangle_{\xi\in\Gamma'}$ is $<\theta$-linked. Also Γ must be cofinal with κ, so Γ' is cofinal with $\operatorname{cf}\kappa$ and $\#(\Gamma')=\operatorname{cf}\kappa$. As $\langle a_\xi\rangle_{\xi<\operatorname{cf}\kappa}$ is arbitrary, $(\operatorname{cf}\kappa,\operatorname{cf}\kappa,<\theta)$ is a precaliber triple of (A,R,B). **Q**

In particular, if κ is a precaliber of (A,R,B), so is $\operatorname{cf}\kappa$.

516C Theorem Suppose that (A,R,B) and (C,S,D) are supported relations, and that $(A,R,B)\preccurlyeq_{\mathrm{GT}}(C,S,D)$. Then $(\kappa,\lambda,<\theta)$ or (κ,λ,θ) is a precaliber triple of (A,R,B) whenever it is a precaliber triple of (C,S,D), so (κ,λ) is a precaliber pair of (A,R,B) whenever it is a precaliber pair of (C,S,D), and κ is a precaliber of (A,R,B) whenever it is a precaliber of (C,S,D).

proof Let (ϕ,ψ) be a Galois-Tukey connection from (A,R,B) to (C,R,S). If $(\kappa,\lambda,<\theta)$ is a precaliber triple of (C,S,D), and $\langle a_\xi\rangle_{\xi<\kappa}$ is a family in A, then there is a set $\Gamma\in[\kappa]^\lambda$ such that whenever $I\in[\Gamma]^{<\theta}$ there is a $d\in D$ such that $(f(a_\xi),d)\in S$ for every $\xi\in I$, and now $(a_\xi,g(d))\in R$ for every $\xi\in I$. Thus $(\kappa,\lambda,<\theta)$ is a precaliber triple of (A,R,B). The results for precaliber pairs and precalibers follow at once.

516D Corollary If $(A,R,B)\equiv_{\mathrm{GT}}(C,S,D)$ then (A,R,B) and (C,S,D) have the same precaliber triples, the same precaliber pairs and the same precalibers.

516E Remark Because all the definitions in 516A start from precaliber triples $(\kappa,\lambda,<\theta)$, any theorem about such precaliber triples is likely to lead at once to corresponding results concerning precaliber triples (κ,λ,θ), precaliber pairs and precalibers. In the rest of this section I shall not always take the space to spell these out systematically, and when later I wish to use a fact about precalibers I may direct you, without comment, to a fact about precaliber triples or pairs from which it may be deduced.

516F The next step is to check the connexion between the definition in 516A and those of §511. But this is elementary.

Proposition (a) If P is a partially ordered set, $(\kappa,\lambda,<\theta)$ or (κ,λ,θ) is a precaliber triple of $(P,\le,P)$ iff it is an upwards precaliber triple of P.

(b) If $\mathfrak{A}$ is a Boolean algebra, then $\mathfrak{A}$ and $(\mathfrak{A}^+,\supseteq,\mathfrak{A}^+)$ have the same precaliber triples, where $\mathfrak{A}^+=\mathfrak{A}\setminus\{0\}$.

(c) If $(X,\mathfrak{T})$ is a topological space, then X and $(\mathfrak{T}\setminus\{\emptyset\},\supseteq,\mathfrak{T}\setminus\{\emptyset\})$ have the same precaliber triples.

proof Read the definitions in 511E and 516A.

516G Corollary Let $(P,\le)$ be a partially ordered set.

(a) If Q is a cofinal subset of P, then P and Q have the same upwards precaliber triples.

(b) Let $\mathfrak{T}^\uparrow$ be the up-topology of P (definition: 514L). Then $(\kappa,\lambda,<\theta)$ is an upwards precaliber triple for $(P,\le)$ iff it is a precaliber triple for $(P,\mathfrak{T}^\uparrow)$.

proof (a) By 513E(d-ii), $(P,\le,P)\equiv_{\mathrm{GT}}(Q,\le,Q)$.

(b) By 514Na, $(P,\le,P)\equiv_{\mathrm{GT}}(\mathfrak{T}^\uparrow\setminus\{\emptyset\},\supseteq,\mathfrak{T}^\uparrow\setminus\{\emptyset\})$.

516H Corollary Let $\mathfrak{A}$ be a Boolean algebra.

(a) If Z is the Stone space of $\mathfrak{A}$, then $\mathfrak{A}$ and Z have the same precaliber triples.

(b) If $\mathfrak{B}$ is an order-dense subalgebra of $\mathfrak{A}$, then $\mathfrak{A}$ and $\mathfrak{B}$ have the same precaliber triples.

proof (a) Write $\mathfrak{T}$ for the topology of Z and $\mathcal{E}$ for the algebra of open-and-closed sets. Because Z is zero-dimensional, $\mathcal{E}^+$ is coinitial with $\mathfrak{T}\setminus\{\emptyset\}$, so $(\mathfrak{A}^+,\supseteq,\mathfrak{A}^+)\cong(\mathcal{E}^+,\supseteq,\mathcal{E}^+)$ and $(\mathfrak{T}\setminus\{\emptyset\},\supseteq,\mathfrak{T}\setminus\{\emptyset\})$ have the same precaliber triples, by 516Ga, inverted.

(b) $\mathfrak{B}^+$ is coinitial with $\mathfrak{A}^+$, so we can use the same idea.

516I Corollary Let $(X,\mathfrak{T})$ be a topological space.

(a) If Y is an open subspace of X, then every precaliber triple of X is a precaliber triple of Y.

(b) If Y is a dense subspace of X, then every precaliber triple of X is a precaliber triple of Y.

(c) If X is regular and Y is a dense subspace of X, then X and Y have the same precaliber triples.

(d) Suppose that Y is a topological space, and that there is a continuous surjection $f : X \to Y$ such that $\operatorname{int} f[G] \neq \emptyset$ whenever $G \subseteq X$ is a non-empty open set. Then every precaliber triple of X is a precaliber triple of Y.

proof (a) Write $\mathfrak{S}$ for the topology of Y. For $H \in \mathfrak{S}\setminus\{\emptyset\}$, set $\phi(H) = H$; for $G \in \mathfrak{T}\setminus\{\emptyset\}$, set $\psi(G) = G\cap Y$ if this is non-empty, G otherwise. Then (ϕ, ψ) is a Galois-Tukey connection from $(\mathfrak{S} \setminus \{\emptyset\}, \supseteq, \mathfrak{S} \setminus \{\emptyset\})$ to $(\mathfrak{T} \setminus \{\emptyset\}, \supseteq, \mathfrak{T} \setminus \{\emptyset\})$, so 516C and 516Fc give the result.

(b) Again write $\mathfrak{S}$ for the topology of Y. For $H \in \mathfrak{S}\setminus\{\emptyset\}$, set $\phi(H) = X\setminus\overline{Y \setminus H}$, where the closure is taken in X; for $G \in \mathfrak{T}\setminus\{\emptyset\}$, set $\psi(G) = G\cap Y$. Then (ϕ, ψ) is a Galois-Tukey connection from $(\mathfrak{S}\setminus\{\emptyset\}, \supseteq, \mathfrak{S}\setminus\{\emptyset\})$ to $(\mathfrak{T} \setminus \{\emptyset\}, \supseteq, \mathfrak{T} \setminus \{\emptyset\})$, so again we have the result.

(c) If now X is regular, then for each $G \in \mathfrak{T} \setminus \{\emptyset\}$ choose $V_G \in \mathfrak{T} \setminus \{\emptyset\}$ such that $\overline{V}_G \subseteq G$ and set $\psi'(G) = V_G\cap Y$. Then (ψ', ϕ) is a Galois-Tukey connection from $(\mathfrak{T}\setminus\{\emptyset\}, \supseteq, \mathfrak{T}\setminus\{\emptyset\})$ to $(\mathfrak{S}\setminus\{\emptyset\}, \supseteq, \mathfrak{S}\setminus\{\emptyset\})$, so every precaliber triple of Y is a precaliber triple of X.

(d) Once more writing $\mathfrak{S}$ for the topology of Y, set $\phi(H) = f^{-1}[H]$ for every $H \in \mathfrak{S} \setminus \{\emptyset\}$ and $\psi(G) = \operatorname{int} f[G]$ for every $G \in \mathfrak{T} \setminus \{\emptyset\}$; then again (ϕ, ψ) is a Galois-Tukey connection from $(\mathfrak{S} \setminus \{\emptyset\}, \supseteq, \mathfrak{S} \setminus \{\emptyset\})$ to $(\mathfrak{T} \setminus \{\emptyset\}, \supseteq, \mathfrak{T} \setminus \{\emptyset\})$.

Remark For variations on (b) and (d) here, see 516Xh and 516Oa.

516J Straightforward counting arguments give us some connexions between precalibers and other cardinal functions, as follows.

Proposition Let (A, R, B) be a supported relation.

(a) $\operatorname{sat}(A, R, B)$ is the least cardinal κ, if there is one, such that $(\kappa, 2)$ is a precaliber pair of (A, R, B); if there is no such κ, $\operatorname{sat}(A, R, B) = \infty$. In particular, if $\kappa \geq 2$ is a precaliber of (A, R, B), then $\kappa \geq \operatorname{sat}(A, R, B)$.

(b) If $\kappa > \max(\omega, \lambda, \operatorname{link}_{<\theta}(A, R, B))$ then $(\kappa, \lambda^+, {<\theta})$ is a precaliber triple of (A, R, B). In particular, if $\kappa > \max(\omega, \lambda, \operatorname{cov}(A, R, B))$ then $(\kappa, \lambda^+, {<\theta})$ is a precaliber triple of (A, R, B) for every θ.

(c) If $\operatorname{cf}\kappa > \operatorname{link}_{<\theta}(A, R, B)$ then $(\kappa, \kappa, {<\theta})$ is a precaliber triple of (A, R, B).

proof (a) If $(\kappa, 2)$ is a precaliber pair of (A, R, B), and $\langle a_\xi\rangle_{\xi<\kappa}$ is any family in A, then there must be a $\Gamma \in [\kappa]^2$ such that for every finite $I \subseteq \Gamma$ there is a $b \in B$ such that $(a_\xi, b) \in R$ for every $\xi \in I$. But this means that if $\Gamma = \{\xi, \eta\}$ then ξ, η are distinct members of κ such that, for some $b \in B$, both (a_ξ, b) and (a_η, b) belong to R. As $\langle a_\xi\rangle_{\xi<\kappa}$ is arbitrary, $\operatorname{sat}(A, R, B) \leq \kappa$.

Conversely, any witness that $(\kappa, 2)$ is not a precaliber pair of (A, R, B) will provide a witness that $\operatorname{sat}(A, R, B) > \kappa$.

Now if $\kappa \geq 2$ is a precaliber of (A, R, B), that is, (κ, κ) is a precaliber pair, then $(\kappa, 2)$ is a precaliber pair of (A, R, B), by 516Ba, so $\kappa \geq \operatorname{sat}(A, R, B)$.

(b) Write δ for $\operatorname{link}_{<\theta}(A, R, B)$, and let $\langle A_\eta\rangle_{\eta<\delta}$ be a cover of A by ${<\theta}$-linked sets. Let $\langle a_\xi\rangle_{\xi<\kappa}$ be any family in A. For $\eta < \delta$ set $C_\eta = \{\xi : a_\xi \in A_\eta\}$; then $\kappa = \bigcup_{\eta<\delta} C_\eta$ so there must be some $\eta < \delta$ such that $\#(C_\eta) > \lambda$. Now if $\Gamma \subseteq C_\eta$ is a set with cardinal λ^+, $\{a_\xi : \xi \in \Gamma\}$ is ${<\theta}$-linked in (A, R, B). As $\langle a_\xi\rangle_{\xi<\kappa}$ is arbitrary, $(\kappa, \lambda^+, {<\theta})$ is a precaliber triple of (A, R, B).

The special case is now elementary, if we remember that $\operatorname{link}_{<\theta}(A, R, B) \leq \operatorname{cov}(A, R, B)$ for every θ (512Bc).

(c) If $\operatorname{link}_{<\theta}(A, R, B) = 0$ then $A = \emptyset$ and the result is trivial. Otherwise, $\operatorname{cf}\kappa \geq \omega$. Choose δ and $\langle C_\eta\rangle_{\eta<\delta}$ as in (b) above. Let $\langle a_\xi\rangle_{\xi<\kappa}$ be any family in A. Then there must be some $\eta < \delta$ such that $\#(C_\eta) = \kappa$, and $\{a_\xi : \xi \in C_\eta\}$ is ${<\theta}$-linked in (A, R, B). As $\langle a_\xi\rangle_{\xi<\kappa}$ is arbitrary, $(\kappa, \kappa, {<\theta})$ is a precaliber triple of (A, R, B).

516K For partially ordered sets, we have translations of the results above, and a further useful fact.

Proposition Let P be a partially ordered set.

(a) $\operatorname{sat}^\uparrow(P)$ is the least cardinal κ such that $(\kappa, 2)$ is an upwards precaliber pair of P.

(b) If $\kappa > \max(\omega, \lambda, \mathrm{link}^{\uparrow}_{<\theta}(P))$ then $(\kappa, \lambda^+, {<}\theta)$ is an upwards precaliber triple of P. In particular, if $\kappa > \max(\omega, \lambda, \mathrm{cf}\, P)$ then $(\kappa, \lambda^+, {<}\theta)$ is an upwards precaliber triple of P for every θ, and if $\kappa > \max(\omega, \lambda, d^{\uparrow}(P))$ then (κ, λ^+) is an upwards precaliber pair of P.

(c) If $\mathrm{cf}\, \kappa > \mathrm{cf}\, P$ then $(\kappa, \kappa, {<}\theta)$ is an upwards precaliber triple of P for every θ. If $\mathrm{cf}\, \kappa > d^{\uparrow}(P)$ then κ is an up-precaliber of P.

(d) If $\mathrm{sat}^{\uparrow}(P) \geq \omega$, $(\mathrm{sat}^{\uparrow}(P), \omega)$ is an upwards precaliber pair of P.

proof (a)-(c) We need only identify $\mathrm{cf}\, P$ with $\mathrm{cov}(P, \leq, P) \geq \sup_\theta \mathrm{link}_{<\theta}(P, \leq, P)$ (512Bc) and $d^{\uparrow}(P)$ with the centering number $\mathrm{link}_{<\omega}(P, \leq, P)$, as in 512Ea.

(d)(i) Set $\kappa = \mathrm{sat}^{\uparrow}(P)$. By 513Bb, κ is a regular uncountable cardinal. The first thing to note is that if $\langle p_\xi \rangle_{\xi<\kappa}$ is any family in P, then there is a $\zeta < \kappa$ such that $\{\xi : \xi < \kappa,\ p_\xi$ and p_ζ are compatible upwards in $P\}$ has cardinal κ. **P?** Otherwise, for each $\zeta < \kappa$ there is an $\alpha_\zeta < \kappa$ such that p_ζ and p_ξ are upwards-incompatible for every $\xi \geq \alpha_\zeta$. Set $C = \{\xi : \xi < \kappa,\ \alpha_\eta \leq \xi$ for every $\eta < \xi\}$. Then $\#(C) = \kappa$ and $\langle p_\xi \rangle_{\xi \in C}$ is an up-antichain in P, which is impossible. **XQ**

(ii) Now let $\langle p_\xi \rangle_{\xi<\kappa}$ be a family in P. Choose inductively sets $A_n \in [\kappa]^\kappa$, ordinals $\zeta_n \in A_n$ and families $\langle p_{n\xi} \rangle_{\xi \in A_n}$ in P, as follows. $A_0 = \kappa$, $p_{0\xi} = p_\xi$ for each $\xi < \kappa$. Given $\langle p_{n\xi} \rangle_{\xi \in A_n}$, then by (i) there is a $\zeta_n \in A_n$ such that

$$A_{n+1} = \{\xi : \xi \in A_n,\ \xi \neq \zeta_n,\ p_{n\xi} \text{ is compatible upwards with } p_{n,\zeta_n}\}$$

has cardinal κ. Now, for $\xi \in A_{n+1}$, let $p_{n+1,\xi}$ be an upper bound of $\{p_{n,\zeta_n}, p_{n\xi}\}$; continue.

At the end of the induction, observe that $\langle p_{n,\zeta_n} \rangle_{n\in\mathbb{N}}$ is non-decreasing. At the same time, we see that $p_\xi \leq p_{n\xi}$ whenever $n \in \mathbb{N}$ and $\xi \in A_n$. So $\{p_{\zeta_n} : n \in \mathbb{N}\}$ is upwards-centered. Also the ζ_n are all different, so $\Gamma = \{\zeta_n : n \in \mathbb{N}\}$ is infinite. As $\langle p_\xi \rangle_{\xi<\kappa}$ is arbitrary, $(\mathrm{sat}^{\uparrow}(P), \omega)$ is an upwards precaliber pair of P.

Remark There will be a stronger form of (d) in 517Fa below.

516L Corollary Let $\mathfrak{A}$ be a Boolean algebra.

(a) $\mathrm{sat}(\mathfrak{A})$ is the least cardinal κ such that $(\kappa, 2)$ is a precaliber pair of $\mathfrak{A}$.

(b) If $\kappa > \max(\omega, \lambda, \mathrm{link}_{<\theta}(\mathfrak{A}))$ then $(\kappa, \lambda^+, {<}\theta)$ is a precaliber triple of $\mathfrak{A}$. In particular, if $\kappa > \max(\omega, \lambda, \pi(\mathfrak{A}))$ then $(\kappa, \lambda^+, {<}\theta)$ is a precaliber triple of $\mathfrak{A}$ for every θ, and if $\kappa > \max(\omega, \lambda, d(\mathfrak{A}))$ then (κ, λ^+) is a precaliber pair of $\mathfrak{A}$.

(c) If $\mathrm{cf}\, \kappa > d(\mathfrak{A})$ then κ is a precaliber of $\mathfrak{A}$.

(d) If $\mathfrak{A}$ is infinite, $(\mathrm{sat}(\mathfrak{A}), \omega)$ is a precaliber pair of $\mathfrak{A}$.

proof Apply 516K, inverted, to $\mathfrak{A}^+$, recalling that $\pi(\mathfrak{A}) = \mathrm{ci}(\mathfrak{A}^+)$.

516M When we turn to topological spaces, we can refine the results slightly, using the following elementary facts.

Lemma Let $(X, \mathfrak{T})$ be a topological space and $\mathrm{RO}(X)$ its regular open algebra. If κ, λ and θ are cardinals, and $\theta \leq \omega$, then the following are equiveridical:

(i) $(\kappa, \lambda, {<}\theta)$ is a precaliber triple of $(X, \mathfrak{T})$;

(ii) $(\kappa, \lambda, {<}\theta)$ is a precaliber triple of $(\mathfrak{T} \setminus \{\emptyset\}, \ni, X)$;

(iii) $(\kappa, \lambda, {<}\theta)$ is a precaliber triple of $\mathrm{RO}(X)$.

proof (a)(i)⇒(ii) If we set $\phi(G) = G$ and choose a point $\psi(G) \in G$ for every non-empty open set $G \subseteq X$, then (ϕ, ψ) is a Galois-Tukey connection from $(\mathfrak{T} \setminus \{\emptyset\}, \ni, X)$ to $(\mathfrak{T} \setminus \{\emptyset\}, \supseteq, \mathfrak{T} \setminus \{\emptyset\})$, so any precaliber triple of the latter is a precaliber triple of the former.

(b)(ii)⇒(iii) Assume (ii), and let $\langle G_\xi \rangle_{\xi<\kappa}$ be a family in $\mathrm{RO}(X)^+$. Then there is a $\Gamma \in [\kappa]^\lambda$ such that $\bigcap_{\xi \in I} G_\xi \neq \emptyset$ for every $I \in [\Gamma]^{<\theta}$. But in this case, because I is finite, $\bigcap_{\xi\in I} G_\xi$ is a lower bound for $\{G_\xi : \xi \in I\}$ in $\mathrm{RO}(X)^+$. As $\langle G_\xi \rangle_{\xi<\kappa}$ is arbitrary, $(\kappa, \lambda, {<}\theta)$ is a precaliber triple of $\mathrm{RO}(X)$.

(c)(iii)⇒(i) Assume (iii), and let $\langle G_\xi \rangle_{\xi<\kappa}$ be a family in $\mathfrak{T} \setminus \{\emptyset\}$. Then there is a $\Gamma \in [\kappa]^\lambda$ such that $\bigcap_{\xi\in I} \mathrm{int}\, \overline{G}_\xi \neq \emptyset$ for every $I \in [\Gamma]^{<\theta}$. But in this case, because I is finite, $\bigcap_{\xi \in I} G_\xi$ is not empty, and is a lower bound for $\{G_\xi : \xi \in I\}$ in $\mathfrak{T} \setminus \{\emptyset\}$. As $\langle G_\xi \rangle_{\xi<\kappa}$ is arbitrary, $(\kappa, \lambda, {<}\theta)$ is a precaliber triple of $(X, \mathfrak{T})$.

516N Corollary Let X be a topological space.

(a) $\operatorname{sat}(X)$ is the least cardinal κ such that $(\kappa,2)$ is a precaliber pair of X.

(b) If $\kappa>\max(\omega,\lambda,d(X))$ then (κ,λ^+) is a precaliber pair of X.

(c) If $\operatorname{cf}\kappa>d(X)$ then κ is a precaliber of X.

(d) If $\operatorname{sat}(X)$ is infinite, then $(\operatorname{sat}(X),\omega)$ is a precaliber pair of X.

proof Here we need to know that $\operatorname{sat}(X)=\operatorname{sat}(\mathrm{RO}(X))$ and $d(X)\ge d(\mathrm{RO}(X))$ (514H(b-i)).

516O The idea of 516M leads to further results about precalibers of topological spaces.

Proposition Let $(X,\mathfrak{T})$ be a topological space.

(a) If Y is a continuous image of X and $\theta\le\omega$, then $(\kappa,\lambda,{<\theta})$ is a precaliber triple of Y whenever it is a precaliber triple of X.

(b) Suppose that X is the product of a family $\langle X_i\rangle_{i\in I}$ of topological spaces. If $(\kappa,\kappa,{<\theta})$ is a precaliber triple of every X_i and *either* I is finite *or* $\theta\le\omega$ and κ is a regular uncountable cardinal, then $(\kappa,\kappa,{<\theta})$ is a precaliber triple of X.

proof (a) Let $f:X\to Y$ be a continuous surjection. Writing $\mathfrak{S}$ for the topology of Y, we have a Galois-Tukey connection (ϕ,f) from $(\mathfrak{S}\setminus\{\emptyset\},\ni,Y)$ to $(\mathfrak{T}\setminus\{\emptyset\},\ni,X)$, if we set $\phi(H)=f^{-1}[H]$ for $H\in\mathfrak{S}\setminus\{\emptyset\}$. Now if $\theta\le\omega$ and $(\kappa,\lambda,{<\theta})$ is a precaliber triple of $(X,\mathfrak{T})$, it is a precaliber triple of $(\mathfrak{T}\setminus\{\emptyset\},\ni,X)$, $(\mathfrak{S}\setminus\{\emptyset\},\ni,Y)$ and $(Y,\mathfrak{S})$, using 516M and 516C.

(b) If $X=\emptyset$ then $(\kappa,\kappa,{<\theta})$ is a precaliber triple of X just because $X=X_i$ for some i; so let us suppose that $X\ne\emptyset$.

(i) If $I=\{0,1\}$ then $(\kappa,\kappa,{<\theta})$ is a precaliber triple of X. **P** Let $\langle W_\xi\rangle_{\xi<\kappa}$ be a family of non-empty open sets in X. For each $\xi<\kappa$, let $G_{\xi0}\subseteq X_0$ and $G_{\xi1}\subseteq X_1$ be non-empty open sets such that $G_{\xi0}\times G_{\xi1}\subseteq W_\xi$. Because $(\kappa,\kappa,{<\theta})$ is a precaliber triple of X_0, there is a $\Gamma\in[\kappa]^\kappa$ such that $H^{(0)}_K=\operatorname{int}(\bigcap_{\xi\in K}G_{\xi0})$ is non-empty for every $K\in[\Gamma]^{<\theta}$. Because $(\kappa,\kappa,{<\theta})$ is a precaliber triple of X_1, there is a $\Delta\in[\Gamma]^\kappa$ such that $H^{(1)}_K=\operatorname{int}(\bigcap_{\xi\in K}G_{\xi1})$ is non-empty for every $K\in[\Delta]^{<\theta}$. Now $\bigcap_{\xi\in K}W_\xi\supseteq H^{(0)}_K\times H^{(1)}_K$ has non-empty interior for every $K\in[\Delta]^{<\theta}$. As $\langle W_\xi\rangle_{\xi<\kappa}$ is arbitrary, $(\kappa,\kappa,{<\theta})$ is a precaliber triple of X. **Q**

(ii) If I is finite, then $(\kappa,\kappa,{<\theta})$ is a precaliber triple of X. **P** Induce on $\#(I)$, using (i) for the inductive step. **Q**

(iii) Now suppose that I is infinite, κ is regular and uncountable and $\theta\le\omega$. Then $(\kappa,\kappa,{<\theta})$ is a precaliber triple of X. **P** Let $\langle W_\xi\rangle_{\xi<\kappa}$ be a family of non-empty open sets in X. Let $\mathcal{V}$ be the standard base for the topology of X consisting of sets of the form $\prod_{\xi<\kappa}U_\xi$ where $U_\xi\subseteq X_\xi$ is open for every ξ and $\{\xi:U_\xi\ne X_\xi\}$ is finite. For each $\xi<\kappa$ let $W'_\xi\subseteq W_\xi$ be a non-empty member of $\mathcal{V}$, so that W'_ξ is determined by a coordinates in a finite subset I_ξ of I. By the Δ-system Lemma (4A1Db) there is a set $A\subseteq\kappa$, with cardinal κ, such that $\langle I_\xi\rangle_{\xi\in A}$ is a Δ-system with root J say. For $\xi\in A$ express W'_ξ as $U_\xi\cap V_\xi$ where U_ξ is determined by coordinates in J and V_ξ is determined by coordinates in $I_\xi\setminus J$. Now U_ξ is of the form $\pi_J^{-1}[H_\xi]$ where $H_\xi\subseteq\prod_{i\in J}X_i$ is a non-empty open set and $\pi_J:X\to\prod_{i\in J}X_i$ is the canonical map. By (ii), $(\kappa,\kappa,{<\theta})$ is a precaliber triple of $\prod_{i\in J}X_i$, so there is a $\Gamma\in[A]^\kappa$ such that $\bigcap_{\xi\in K}H_\xi$ is non-empty whenever $K\in[\Gamma]^{<\theta}$. Now take any $K\in[\Gamma]^{<\theta}$. Then $U=\pi_J^{-1}[\bigcap_{\xi\in K}H_\xi]$ is a non-empty set determined by coordinates in J, while V_ξ is a non-empty open set determined by coordinates in $I_\xi\setminus J$ for each $\xi\in K$; because the $I_\xi\setminus J$ are disjoint and K is finite, $U\cap\bigcap_{\xi\in J}V_\xi$ is non-empty, and $\bigcap_{\xi\in K}W_\xi$ is a non-empty set, necessarily open because K is finite. As $\langle W_\xi\rangle_{\xi<\kappa}$ is arbitrary, $(\kappa,\kappa,{<\theta})$ is a precaliber triple of X. **Q**

516P Corollary Let $\langle P_i\rangle_{i\in I}$ be a family of non-empty partially ordered sets, with upwards finite-support product $P=\bigotimes^{\uparrow}_{i\in I}P_i$ (definition: 514T). If $(\kappa,\kappa,{<\theta})$ is an upwards precaliber triple of every P_i and either I is finite or $\theta\le\omega$ and κ is a regular uncountable cardinal, then $(\kappa,\kappa,{<\theta})$ is an upwards precaliber triple of P.

proof Suppose first that θ is countable. By 516Gb and 516M we can identify the relevant upwards precaliber triples of each P_i and P with the precaliber triples of their regular open algebras. But $\mathrm{RO}^{\uparrow}(P)\cong\mathrm{RO}(\prod_{i\in I}P_i)$ (514Ua), so 516Ob gives the result at once.

For finite I, $P^* = \prod_{i\in I} P_i$ is a cofinal subset of P (514Ub), so that it has the same upwards precaliber triples (516Ga); at the same time, it is easy to see that the up-topology of P^* is just the product of the up-topologies on the P_i. So this time we do not need to look at regular open algebras and can use 516Gb and 516Ob directly.

516Q For locally compact spaces, as usual, we have further results.

Proposition Let X be a locally compact Hausdorff topological space.

(a) (κ,λ) is a precaliber pair of X iff whenever $\langle G_\xi\rangle_{\xi<\kappa}$ is a family of non-empty open subsets of X, then there is an $x\in X$ such that $\#(\{\xi : x\in G_\xi\})\ge\lambda$.

(b) Suppose that κ is a regular infinite cardinal. Then κ is a precaliber of X iff $\operatorname{sat}(X)\le\kappa$ and whenever $\langle E_\xi\rangle_{\xi<\kappa}$ is a non-decreasing family of nowhere dense subsets of X then $\bigcup_{\xi<\kappa} E_\xi$ has empty interior.

proof (a)(i) The condition asserts that (κ,λ,λ) is a precaliber triple of $(\mathfrak{T}\setminus\{\emptyset\},\ni,X)$. It follows at once that (κ,λ) is a precaliber pair of $(\mathfrak{T}\setminus\{\emptyset\},\ni,X)$ and therefore of $(X,\mathfrak{T})$, by 516M.

(ii) Now suppose that (κ,λ) is a precaliber pair of X, and that $\langle G_\xi\rangle_{\xi<\kappa}$ is a family of non-empty open subsets of X. For each $\xi<\kappa$ choose a non-empty relatively compact open set H_ξ such that $\overline{H}_\xi\subseteq G_\xi$. Then there is a $\Gamma\in[\kappa]^\lambda$ such that $\{H_\xi : \xi\in\Gamma\}$ is centered. In this case, $\{\overline{H}_\xi : \xi\in\Gamma\}$ has the finite intersection property, so has non-empty intersection. If x is any point of this intersection, then $\{\xi : x\in G_\xi\}\supseteq\Gamma$ has cardinal at least λ.

(b)(i) Suppose that κ is a precaliber of X. Then surely $\operatorname{sat}(X)\le\kappa$ (516Ja). If $\langle E_\xi\rangle_{\xi<\kappa}$ is a non-decreasing family of nowhere dense subsets of X, take any non-empty open set $G\subseteq X$. For each $\xi<\kappa$, $G_\xi = G\setminus\overline{E}_\xi$ is a non-empty open set, so by (a) there is an $x\in X$ such that $\Gamma=\{\xi : x\in G_\xi\}$ has cardinal κ. But as $\langle G_\xi\rangle_{\xi<\kappa}$ is non-increasing, this means that $\Gamma=\kappa$ and $x\in G\setminus\bigcup_{\xi<\kappa}E_\xi$. As G is arbitrary, $\bigcup_{\xi<\kappa}E_\xi$ has empty interior.

(ii) Now suppose that the condition is satisfied. Let $\langle G_\xi\rangle_{\xi<\kappa}$ be a family of non-empty open subsets of X. For $\xi<\kappa$ set $H_\xi=\bigcup_{\eta\ge\xi}G_\eta$, $W_\xi = X\setminus\overline{H}_\xi$. By 5A4Bd, there is a set $I\subseteq\kappa$ such that $\#(I)<\operatorname{sat}(X)$ and

$$\overline{\bigcup_{\xi\in I}W_\xi}=\overline{\bigcup_{\xi<\kappa}W_\xi}.$$

Because $\#(I)<\operatorname{cf}\kappa$, $\zeta=\sup I$ is less than κ, and $H_\zeta\cap W_\xi=\emptyset$ for every $\xi\in I$, so $H_\zeta\cap W_\xi=\emptyset$ for every $\xi<\kappa$, that is, $H_\zeta\subseteq\overline{H}_\xi$ for every $\xi<\kappa$.

Setting

$$E_\xi = H_\zeta\setminus H_\xi\subseteq\overline{H}_\xi\setminus H_\xi$$

for each ξ, $\langle E_\xi\rangle_{\xi<\kappa}$ is a non-decreasing family of nowhere dense sets, and cannot cover H_ζ. If $x\in H_\zeta\setminus\bigcup_{\xi<\kappa}E_\xi$, then $x\in H_\xi$ for every $\xi<\kappa$, so $\Gamma=\{\eta : x\in G_\eta\}$ is cofinal with κ. Because κ is regular, $\Gamma\in[\kappa]^\kappa$, and $\bigcap_{\xi\in I}G_\xi$ is non-empty for every $I\in[\Gamma]^{<\omega}$. As $\langle G_\xi\rangle_{\xi<\kappa}$ is arbitrary, κ is a precaliber of X, by (a).

516R We can use the last proposition to give corresponding characterizations of precaliber pairs of Boolean algebras in terms of their Stone spaces.

Corollary Let $\mathfrak{A}$ be a Boolean algebra and Z its Stone space.

(a) A pair (κ,λ) of cardinals is a precaliber pair of $\mathfrak{A}$ iff whenever $\langle G_\xi\rangle_{\xi<\kappa}$ is a family of non-empty open sets in Z there is a $z\in Z$ such that $\#(\{\xi : z\in G_\xi\})\ge\lambda$.

(b) Suppose that $\kappa\ge\operatorname{sat}(\mathfrak{A})$ is a regular infinite cardinal. Then κ is a precaliber of $\mathfrak{A}$ iff whenever $\langle E_\xi\rangle_{\xi<\kappa}$ is a non-decreasing family of nowhere dense subsets of Z then $\bigcup_{\xi<\kappa}E_\xi$ has empty interior.

proof Put 516Ha and 516Q together.

516S I collect some further results relating precalibers to the standard constructions involving Boolean algebras as considered in 514E.

Proposition Let $\mathfrak{A}$ be a Boolean algebra.

(a) If $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$ and $(\kappa,\lambda,<\theta)$ is a precaliber triple of $\mathfrak{A}$ such that $\theta\le\omega$, then $(\kappa,\lambda,<\theta)$ is a precaliber triple of $\mathfrak{B}$. In particular, every precaliber pair of $\mathfrak{A}$ is a precaliber pair of $\mathfrak{B}$ and $\mathfrak{B}$ will satisfy Knaster's condition if $\mathfrak{A}$ does.

(b) If $\mathfrak{B}$ is a regularly embedded subalgebra of $\mathfrak{A}$, then every precaliber triple of $\mathfrak{A}$ is a precaliber triple of $\mathfrak{B}$.

(c) If $\mathfrak{B}$ is a Boolean algebra and $\phi:\mathfrak{A}\to\mathfrak{B}$ is a surjective order-continuous Boolean homomorphism, then every precaliber triple of $\mathfrak{A}$ is a precaliber triple of $\mathfrak{B}$.

(d) If $\mathfrak{B}$ is a principal ideal of $\mathfrak{A}$ then every precaliber triple of $\mathfrak{A}$ is a precaliber triple of $\mathfrak{B}$.

(e) If $\mathfrak{A}$ is the simple product of a family $\langle\mathfrak{A}_i\rangle_{i\in I}$ of Boolean algebras, $(\kappa,\lambda,<\theta)$ is a precaliber triple of $\mathfrak{A}_i$ for every $i\in I$ and $\operatorname{cf}\kappa>\#(I)$, then $(\kappa,\lambda,<\theta)$ is a precaliber triple of $\mathfrak{A}$.

proof (a) The Stone space of $\mathfrak{B}$ is a continuous image of the Stone space of $\mathfrak{A}$ (312Sa). So all we have to do is to put 516Ha and 516Oa together.

Taking $\theta=\omega$ we see that a precaliber pair of $\mathfrak{A}$ will be a precaliber pair of $\mathfrak{B}$. Taking $\lambda=\kappa=\omega_1$ and $\theta=3$, we see that if $\mathfrak{A}$ satisfies Knaster's condition so does $\mathfrak{B}$.

(b) Any subset of $\mathfrak{B}^+$ with a lower bound in $\mathfrak{A}^+$ has a lower bound in $\mathfrak{B}^+$, so the identity map from $\mathfrak{B}^+$ to $\mathfrak{A}^+$ is a Tukey function from $(\mathfrak{B}^+,\supseteq)$ to $(\mathfrak{A}^+,\supseteq)$ and we can put 516C together with 516Fb.

(c) If $A\subseteq\mathfrak{A}^+$ has no lower bound in $\mathfrak{A}^+$, that is, $\inf A=0$, then $\inf\phi[A]=0$ and $\phi[A]$ has no lower bound in $\mathfrak{B}^+$. Accordingly $\phi\restriction\mathfrak{A}^+$ is a dual Tukey function from $(\mathfrak{A}^+,\supseteq)$ to $(\mathfrak{B}^+,\supseteq)$, $(\mathfrak{B}^+,\supseteq)\preccurlyeq_{\mathrm{T}}(\mathfrak{A}^+,\supseteq)$ and we can proceed as in (c).

(d) As in (b), the identity map from $\mathfrak{B}^+$ to $\mathfrak{A}^+$ is a Tukey function from $(\mathfrak{B}^+,\supseteq)$ to $(\mathfrak{A}^+,\supseteq)$. (Or put 312T and 516Ia together.)

(e) Let $\langle a_\xi\rangle_{\xi<\kappa}$ be a family in $\mathfrak{A}^+$. Then for each $\xi<\kappa$ there is an $i\in I$ such that $a_\xi(i)$ is non-zero in $\mathfrak{A}_i$. As $\operatorname{cf}\kappa>\#(I)$ there is an $i\in I$ such that $\Delta=\{\xi:\xi<\kappa,\ a_\xi(i)\ne0\}$ has cardinal κ. As $(\#(\Delta),\lambda,<\theta)$ is a precaliber triple of $\mathfrak{A}_i$, there is a $\Gamma\in[\Delta]^\lambda$ such that $\{a_\xi(i):\xi\in I\}$ has a non-zero lower bound in $\mathfrak{A}_i$ for every $I\in[\Gamma]^{<\theta}$. But now $\{a_\xi:\xi\in I\}$ has a non-zero lower bound in $\mathfrak{A}$ for every $I\in[\Gamma]^{<\theta}$, while $\Gamma\in[\kappa]^\lambda$. As $\langle a_\xi\rangle_{\xi<\kappa}$ is arbitrary, $(\kappa,\lambda,<\theta)$ is a precaliber triple of $\mathfrak{A}$.

516T A central problem from the very beginning of set-theoretic topology concerns the saturation of product spaces. Here I describe one of the principal methods of showing that product spaces have small saturation, in a form adapted to partially ordered sets.

Theorem (a) Let P and Q be partially ordered sets, and κ a cardinal such that $(\kappa,\operatorname{sat}^\uparrow(Q),2)$ is an upwards precaliber triple of P. Then $\operatorname{sat}^\uparrow(P\times Q)\le\kappa$.

(b) Let $\langle P_i\rangle_{i\in I}$ be a family of non-empty partially ordered sets with upwards finite-support product P. Suppose that κ is a regular uncountable cardinal such that $(\kappa,\kappa,2)$ is an upwards precaliber triple of every P_i. Then $\operatorname{sat}^\uparrow(P)\le\kappa$.

proof (a) ? Otherwise, there is an up-antichain $\langle(p_\xi,q_\xi)\rangle_{\xi<\kappa}$ in $P\times Q$. Let $\Gamma\subseteq\kappa$ be a set with cardinal $\operatorname{sat}^\uparrow(Q)$ such that $\{p_\xi:\xi\in\Gamma\}$ is upwards-linked. Then $\langle q_\xi\rangle_{\xi\in\Gamma}$ must be an up-antichain in Q; but this is impossible. **X**

(b) By 516P, $(\kappa,\kappa,2)$ is an upwards precaliber triple of P. So $(\kappa,2,2)$ and $(\kappa,2,<\omega)$ also are (516Ba, 516Bc), and $\operatorname{sat}^\uparrow(P)\le\kappa$ (516Ka).

516U It will be useful to be able to quote what amounts to a simple special case of the above result.

Corollary Let $\mathfrak{A}$ be a Boolean algebra satisfying Knaster's condition (511Ef) and $\mathfrak{B}$ a ccc Boolean algebra. Then their free product $\mathfrak{A}\otimes\mathfrak{B}$ is ccc.

proof By 516Ta, inverted, $(\mathfrak{A}\setminus\{0\})\times(\mathfrak{B}\setminus\{0\})$ is downwards-ccc. But $(a,b)\mapsto a\otimes b$ is an order-preserving bijection between $(\mathfrak{A}\setminus\{0\})\times(\mathfrak{B}\setminus\{0\})$ and an order-dense (that is, coinitial) subset of $(\mathfrak{A}\otimes\mathfrak{B})\setminus\{0\}$ (315Kb); so $(\mathfrak{A}\otimes\mathfrak{B})\setminus\{0\}$ is downwards-ccc (513Gc, inverted), that is, $\mathfrak{A}\otimes\mathfrak{B}$ is ccc.

516V An elementary, but not quite obvious, fact will turn out to be useful.

Proposition Let $\mathfrak{A}$ be an atomless Boolean algebra which satisfies Knaster's condition. Then $\mathfrak{A}$ has an atomless order-closed subalgebra with countable Maharam type.

proof For each $a \in \mathfrak{A} \setminus \{0\}$ choose a' such that $a' \subseteq a$ and $a' \notin \{0, a\}$. Define $\langle \mathfrak{B}_\xi \rangle_{\xi<\omega_1}$ and $\langle B_\xi \rangle_{\xi<\omega_1}$ inductively, as follows. $\mathfrak{B}_\xi$ will be the order-closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{\eta<\xi} \mathfrak{B}_\eta$; B_ξ will be $\{b : b$ is an atom of $\mathfrak{B}_\xi\} \cup \{b' : b$ is an atom of $\mathfrak{B}_\xi\}$. Because $\mathfrak{A}$ is ccc (511Ef), every B_ξ will be countable and every $\mathfrak{B}_\xi$ will be countably τ-generated. Of course $\langle \mathfrak{B}_\xi \rangle_{\xi<\omega_1}$ is non-decreasing.

? If no $\mathfrak{B}_\xi$ is atomless, then choose an atom b_ξ of $\mathfrak{B}_\xi$ for each $\xi < \omega_1$. Because $\mathfrak{A}$ satisfies Knaster's condition, there is an uncountable set $\Gamma \subseteq \omega_1$ such that $b_\xi \cap b_\eta \neq 0$ for all ξ, $\eta \in \Gamma$. If $\xi < \eta$ in ω_1, then $b_\xi \in \mathfrak{B}_\eta$ and b_η is an atom of $\mathfrak{B}_\eta$ so either $b_\eta \subseteq b_\xi$ or $b_\eta \cap b_\xi = 0$. So we see that if $\xi < \eta$ in Γ, then $b_\eta \subseteq b_\xi$. But we know also that $b'_\xi \in \mathfrak{B}_\eta$, so b_ξ is not an atom of $\mathfrak{B}_\eta$ and $b_\xi \setminus b_\eta \neq 0$.

This means that if we define $f : \Gamma \to \Gamma$ by taking $f(\xi) = \min(\Gamma \setminus (\xi + 1))$ for $\xi \in \Gamma$, $\langle b_\xi \setminus b_{f(\xi)} \rangle_{\xi\in\Gamma}$ will be a disjoint family in $\mathfrak{A} \setminus \{0\}$, which is impossible. **X**

So one of the $\mathfrak{B}_\xi$ will serve for $\mathfrak{B}$.

516X Basic exercises (a) Let (A, R, B) be a supported relation, and $n \geq 1$ an integer. Show that n is a precaliber of (A, R, B) iff $\operatorname{add}(A, R, B) > n$.

(b) Let P and Q be partially ordered sets, and $f : P \to Q$ a surjection such that, for any finite set $I \subseteq P$, I is bounded above in P iff $f[I]$ is bounded above in Q. Show that P and Q have the same upwards precaliber pairs.

(c)(i) Show that if P is a partially ordered set and $\kappa > \operatorname{cf} P$ is an infinite cardinal such that $\operatorname{cf} \kappa$ is an up-precaliber of P, then κ is an up-precaliber of P. (ii) Show that if $\mathfrak{A}$ is a Boolean algebra and $\kappa > \pi(\mathfrak{A})$ is an infinite cardinal such that $\operatorname{cf} \kappa$ is a precaliber of $\mathfrak{A}$, then κ is a precaliber of $\mathfrak{A}$.

(d) Let P be a partially ordered set and κ an infinite cardinal. Show that $\operatorname{sat}^\uparrow(P) \leq \kappa$ iff (κ, ω) is an upwards precaliber pair of P. (*Hint*: if $\kappa = \operatorname{sat}^\uparrow(P)$ and $\langle p_\xi \rangle_{\xi<\kappa}$ is a family in P, choose ξ_n, q_n such that $p_{\xi_i} \leq q_n$ for $i \leq n$ and $\{\xi : q_n$ is compatible upwards with $p_\xi\}$ is always cofinal with κ.)

(e) Let $(X, \mathfrak{T})$ be a topological space, $\mathrm{RO}(X)$ its regular open algebra and $\mathfrak{G}$ its category algebra (definition: 514I). (i) Show that any precaliber triple of $(X, \mathfrak{T})$ is also a precaliber triple of $\mathrm{RO}(X)$, $(\mathfrak{T} \setminus \{\emptyset\}, \ni, X)$ and $\mathfrak{G}$. (ii) Show that if $(X, \mathfrak{T})$ is regular, then $(X, \mathfrak{T})$ and $\mathrm{RO}(X)$ have the same precaliber triples. (iii) Show that if $(X, \mathfrak{T})$ is locally compact and Hausdorff, then $(X, \mathfrak{T})$ and $\mathfrak{G}$ have the same precaliber triples.

(f) Let $(P, \leq)$ be the totally ordered set ω_1, $\mathfrak{T}^\uparrow$ its up-topology and $\mathrm{RO}^\uparrow(P)$ the regular open algebra of $(P, \mathfrak{T}^\uparrow)$. Show that $(\omega_1, \omega_1, \omega_1)$ is a precaliber triple of $\mathrm{RO}^\uparrow(P)$ but not of $(P, \leq)$ or $(P, \mathfrak{T}^\uparrow)$.

(g) Let $\langle \mathfrak{A}_i \rangle_{i\in I}$ be a family of Boolean algebras and $\mathfrak{A}$ their free product. Show that if $(\kappa, \kappa, {<}\theta)$ is a precaliber triple of every $\mathfrak{A}_i$ and either I is finite or $\theta \leq \omega$ and κ is a regular infinite cardinal, then $(\kappa, \kappa, {<}\theta)$ is a precaliber triple of $\mathfrak{A}$.

(h) Suppose that X is a topological space and Y is a dense subset of X and $\theta \leq \omega$. Show that $(\kappa, \lambda, {<}\theta)$ is a precaliber triple of Y iff it is a precaliber triple of X.

(i) Let X be a locally compact Hausdorff space, and κ a precaliber of X. Show that whenever $\langle E_\xi \rangle_{\xi<\kappa}$ is a non-decreasing family of nowhere dense subsets of X then $\bigcup_{\xi<\kappa} E_\xi$ has empty interior.

(j) Prove 516Sa-516Sc without mentioning Stone spaces.

(k)(i) Let X and Y be topological spaces, and κ a cardinal such that $(\kappa, \operatorname{sat}(Y), 2)$ is a precaliber triple of X. Show that $\operatorname{sat}(X \times Y) \leq \kappa$. (ii) Let $\langle X_i \rangle_{i\in I}$ be a family of topological spaces with product X. Suppose that κ is a regular uncountable cardinal such that $(\kappa, \kappa, 2)$ is a precaliber triple of every X_i. Show that $\operatorname{sat}(X) \leq \kappa$.

(l)(i) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, and $\mathfrak{A}\otimes\mathfrak{B}$ their free product. Suppose that κ is a cardinal such that $(\kappa,\operatorname{sat}(\mathfrak{B}),2)$ is a precaliber triple of $\mathfrak{A}$. Show that $\operatorname{sat}(\mathfrak{A}\otimes\mathfrak{B})\le\kappa$. (ii) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras with free product $\mathfrak{A}$. Suppose that κ is a regular uncountable cardinal such that $(\kappa,\kappa,2)$ is a precaliber triple of every $\mathfrak{A}_i$. Show that $\operatorname{sat}(\mathfrak{A})\le\kappa$.

(m) Let X and Y be topological spaces. Show that if $(\kappa,\kappa',{<}\theta)$ is a precaliber triple of X and $(\kappa',\lambda,{<}\theta)$ is a precaliber triple of Y, then $(\kappa,\lambda,{<}\theta)$ is a precaliber triple of $X\times Y$.

(n) Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are Boolean algebras and that there is a surjective Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. Show that if $(\kappa,\lambda,{<}\theta)$ is a precaliber triple of $\mathfrak{A}$ and θ is countable, then $(\kappa,\lambda,{<}\theta)$ is a precaliber triple of $\mathfrak{B}$.

516Y Further exercises (a) Let $\mathfrak{A}$ be an atomless Boolean algebra such that $(\kappa,\kappa,2)$ is a precaliber triple of $\mathfrak{A}$ for every regular uncountable cardinal κ. Show that there is a countable $B\subseteq\mathfrak{A}$ such that for every non-zero $a\in\mathfrak{A}$ there is a $b\in\mathfrak{B}$ such that a meets both b and $1\setminus b$.

516 Notes and comments 'Precaliber triples' are visibly complex. With three cardinals in action, there is a promise of a powerful method of describing special features of a partially ordered set or Boolean algebra, but at the same time a threat of alarming demands on our memory. In fact none of the arguments in this section are deep, and they are here mainly for reference. Some of the results depend in not-quite-obvious ways on the exact hypotheses, and it will be useful later to have clear statements to hand. In the proofs I have emphasized Galois-Tukey connections whenever possible; at the cost of possibly tedious repetititions of such formulae as $(\mathfrak{T}\setminus\{\emptyset\},\supseteq,\mathfrak{T}\setminus\{\emptyset\})$ naming the supported relations involved, they can save us the trouble of negotiating the quantifiers in the definition

$$\forall\ \langle a_\xi\rangle_{\xi<\kappa}\in A^\kappa\ \exists\ \Gamma\in[\kappa]^\lambda\ \forall\ I\in[\Gamma]^{<\theta}\ \exists\ b\in B\ldots.$$

But of course it is a useful exercise to find proofs from first principles, not mentioning supported relations and not (for instance) using Stone spaces to deal with Boolean algebras.

'Supported relations' form a materially more various class of structures than partially ordered sets, topological spaces or Boolean algebras. But the constructions already developed in this book (Stone spaces, regular open algebras, up-topologies) give us functorial relations between the last three categories which mean that from the point of view of this section they are nearly the same. So such results as 516T can be expected to apply to topological spaces and Boolean algebras as well (516Xk, 516Xl). (But note 516Xf.)

Precaliber triples belong with saturation and linking numbers as parameters describing the 'breadth' of a topological space or Boolean algebra; see Comfort & Negrepontis 82. In the first place, they address a classic problem: when is the product of ccc topological spaces ccc? (This is the case $\kappa=\omega_1$ of 516Xk.) But with the exception of saturation, there do not appear to be simple connexions between precalibers and the cardinal functions we have looked at so far. Precalibers seem to correspond to new features of the structures considered here. When we come to look at the most important objects of measure theory (in particular, measure algebras), we shall find that their precalibers are relatively fluid; I mean that while cellularity, Maharam types and many linking numbers, for instance, are determined by simple formulae in ZFC, precalibers are not.

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517 Martin numbers

I devote a section to the study of 'Martin numbers' of partially ordered sets and Boolean algebras. Like additivity and cofinality they enable us to frame as theorems of ZFC some important arguments which were first used in special models of set theory, and to pose challenging questions on the relationships between classical structures in analysis. I begin with some general remarks on the Martin numbers of partially ordered sets (517A-517E); most of these are perfectly elementary but the equivalent conditions of 517B, in particular, are useful and not all obvious. Much of the importance of Martin numbers comes from their effect on precalibers (517F, 517H) and hence on saturation of products (517G). The same ideas can be expressed in terms of Boolean algebras, with no surprises (517I). I have not set out a definition of 'Martin number' for a topological space, but the Novák number of a locally compact Hausdorff space is closely related to the Martin numbers of its regular open algebra and its algebra of open-and-closed sets (517J-517K). Consequently we have connexions between the Martin number and the weak distributivity of a Boolean algebra (517L). A striking fact, which will have a prominent role in the next chapter, is that non-trivial countable partially ordered sets all have the same Martin number $\mathfrak{m}_{\text{countable}}$ (517P).

517A Proposition For any partially ordered set P, $\mathfrak{m}^{\uparrow}(P) \ge \omega_1$.

proof If $\mathcal{Q}$ is a countable family of cofinal subsets of P and $p_0 \in P$, let $\langle Q_n\rangle_{n\in\mathbb{N}}$ be a sequence running over $\mathcal{Q} \cup \{P\}$, and choose $\langle p_n\rangle_{n\ge1}$ inductively such that $p_{n+1} \ge p_n$ and $p_{n+1} \in Q_n$ for every $n \in \mathbb{N}$. Then $\{p_n : n \in \mathbb{N}\}$ is an upwards-linked subset of P meeting every member of $\mathcal{Q}$.

517B Lemma Let P be a partially ordered set, and κ a cardinal. Then the following are equiveridical:

(i) $\kappa < \mathfrak{m}^{\uparrow}(P)$;

(ii) whenever $p_0 \in P$ and $\mathcal{Q}$ is a family of up-open cofinal subsets of P with $\#(\mathcal{Q}) \le \kappa$, there is an upwards-linked subset of P which contains p_0 and meets every member of $\mathcal{Q}$;

(iii) whenever $p_0 \in P$ and $\mathcal{A}$ is a family of maximal up-antichains in P with $\#(\mathcal{A}) \le \kappa$, there is an upwards-linked subset of P which contains p_0 and meets every member of $\mathcal{A}$;

(iv) whenever $p_0 \in P$ and $\mathcal{Q}$ is a family of cofinal subsets of P with $\#(\mathcal{Q}) \le \kappa$, there is an upwards-directed subset of P which contains p_0 and meets every member of $\mathcal{Q}$;

(v) whenever $p_0 \in P$ and $\mathcal{Q}$ is a family of up-open cofinal subsets of P with $\#(\mathcal{Q}) \le \kappa$, there is an upwards-directed subset of P which contains p_0 and meets every member of $\mathcal{Q}$;

(vi) whenever $p_0 \in P$ and $\mathcal{A}$ is a family of maximal up-antichains in P with $\#(\mathcal{A}) \le \kappa$, there is an upwards-directed subset of P which contains p_0 and meets every member of $\mathcal{A}$.

proof (a) Most of the circuit is elementary.

(vi)⇒(iv) because every cofinal subset of P includes a maximal up-antichain (513Aa).

(iv)⇒(v)⇒(ii) are trivial.

(ii)⇒(i) Assuming (ii), let $\mathcal{Q}$ be a family of cofinal subsets of P with $\#(\mathcal{Q}) \le \kappa$. For each $Q \in \mathcal{Q}$, $U_Q = \bigcup_{q\in Q} [q,\infty[$ is up-open and cofinal with P (513Ab). If $p_0 \in P$, (ii) tells us that there is an upwards-linked subset R_0 of P containing p_0 and meeting U_Q for every $Q \in \mathcal{Q}$. Set $R = \bigcup_{p\in R_0}]-\infty,p]$; then R is an upwards-linked subset of P containing p_0 and meeting every member of $\mathcal{Q}$. As $\mathcal{Q}$ and p_0 are arbitrary, (i) is true.

(i)⇒(iii) Assuming (i), let $\mathcal{A}$ be a family of maximal up-antichains in P with $\#(\mathcal{A}) \le \kappa$. For each $A \in \mathcal{A}$, $U_A = \bigcup_{p\in A} [p,\infty[$ is cofinal with P. So (i) tells us that if $p_0 \in P$ there is an upwards-linked subset R_0 of P containing p_0 and meeting U_A for every $A \in \mathcal{A}$. As just above, set $R = \bigcup_{p\in R_0}]-\infty,p]$; then R is upwards-linked, contains p_0 and meets every member of $\mathcal{A}$. As $\mathcal{A}$ and p_0 are arbitrary, (iii) is true.

(b) So we are left with (iii)⇒(vi). Assume (iii), and take $p_0 \in P$ and a family $\mathcal{A}$ of maximal up-antichains in P with $\#(\mathcal{A}) \le \kappa$. Let $\mathcal{C}$ be the set of all maximal up-antichains in P. For $A \in \mathcal{C}$, set $U_A = \bigcup_{q\in A} [q,\infty[$. Then U_A is cofinal with P. Consequently $U_A \cap U_B$ is cofinal with P for any A, $B \in \mathcal{C}$, because if $p \in P$ there are $q \in U_A$ and $r \in U_B$ such that $p \le q \le r$, and now $r \in U_A \cap U_B$. It follows that $U_A \cap U_B$ includes a maximal up-antichain $D(A,B)$.

Take any $A_0 \in \mathcal{C}$ such that $p_0 \in A_0$. Let $\mathcal{A}^* \subseteq \mathcal{C}$ be such that $\{A_0\} \cup \mathcal{A} \subseteq \mathcal{A}^*$, $D(A,B) \in \mathcal{A}^*$ for all A, $B \in \mathcal{A}^*$, and $\#(\mathcal{A}^*) \le \max(\omega,\kappa)$ (5A1Gb). Then there is an upwards-linked subset R_0 of P containing p_0 and meeting every member of $\mathcal{A}^*$. **P** If $\#(\mathcal{A}^*) \le \kappa$, this is immediate from (iii); if $\#(\mathcal{A}^*) \le \omega$, it is because $\omega < \mathfrak{m}^{\uparrow}(P)$, by 517A, and (i)⇒(iii). **Q**

Set $R = R_0 \cap \bigcup \mathcal{A}^*$. Then R contains p_0 (because $p_0 \in R_0 \cap A_0$) and R meets every member of $\mathcal{A}$. Also R is upwards-directed. **P** If p, $q \in R$, take A, $B \in \mathcal{A}^*$ such that $p \in A$ and $q \in B$. Then $D(A,B) \in \mathcal{A}^*$, so there is an $r \in R_0 \cap D(A,B)$, and $r \in R$. As $r \in U_A \cap U_B$, there must be $p' \in A$ and $q' \in B$ such that $p' \le r$ and $q' \le r$. But R_0 is upwards-linked, so

$$\emptyset \neq [p,\infty[\cap [r,\infty[\subseteq [p,\infty[\cap [p',\infty[;$$

as A is an up-antichain, $p = p'$. Similarly, $q = q'$ and $r \in R$ is an upper bound of $\{p, q\}$. As p and q are arbitrary, R is upwards-directed. **Q**

So we have a set R of the kind required by (vi).

517C Lemma Let P_0 and P_1 be partially ordered sets, and suppose that there is a relation $S \subseteq P_0 \times P_1$ such that $S[P_0]$ is cofinal with P_1, $S^{-1}[Q]$ is cofinal with P_0 for every cofinal $Q \subseteq P_1$, and $S[R]$ is upwards-linked in P_1 for every upwards-linked $R \subseteq P_0$. Then $\mathfrak{m}^\uparrow(P_1) \ge \mathfrak{m}^\uparrow(P_0)$.

proof Suppose that $p_1 \in P_1$ and that $\mathcal{Q}$ is a family of cofinal subsets of P_1 with $\#(\mathcal{Q}) < \mathfrak{m}^\uparrow(P_0)$. Then there is a pair $(p_0, p_1') \in S$ such that $p_1' \ge p_1$. Now $S^{-1}[Q]$ is cofinal with P_0 for every $Q \in \mathcal{Q}$, so there is an upwards-linked $R \subseteq P_0$ containing p_0 and meeting $S^{-1}[Q]$ for every $Q \in \mathcal{Q}$. In this case $p_1' \in S[R]$ and $S[R]$ is upwards-linked, so $\{p_1\} \cup S[R]$ is an upwards-linked subset of P_1 containing p_1 and meeting every member of $\mathcal{Q}$. As p_1 and $\mathcal{Q}$ are arbitrary, $\mathfrak{m}^\uparrow(P_1) \ge \mathfrak{m}^\uparrow(P_0)$.

517D Proposition (a) If P is a partially ordered set and Q is a cofinal subset of P, then $\mathfrak{m}^\uparrow(P) = \mathfrak{m}^\uparrow(Q)$.

(b) If P is any partially ordered set and $\mathrm{RO}^\uparrow(P)$ is its regular open algebra when it is given its up-topology, then $\mathfrak{m}^\uparrow(P) = \mathfrak{m}(\mathrm{RO}^\uparrow(P))$.

(c) If P is a partially ordered set and $p_0 \in P$, then $\mathfrak{m}^\uparrow([p_0,\infty[) \ge \mathfrak{m}^\uparrow(P)$.

proof (a) Let P_0, P_1 be cofinal subsets of P, and set $S = \{(p_0, p_1) : p_0 \in P_0,\ p_1 \in P_1,\ p_0 \ge p_1\}$. Then S satisfies the conditions of 517C, so $\mathfrak{m}^\uparrow(P_1) \ge \mathfrak{m}^\uparrow(P_0)$. It follows at once that all cofinal subsets of P, including P itself, have the same Martin number.

(b)(i) Setting $S = \{(p, G) : p \in G \in \mathrm{RO}^\uparrow(P)\}$, S satisfies the conditions of 517C with $P_0 = (P, \le)$ and $P_1 = (\mathrm{RO}^\uparrow(P)^+, \supseteq)$, so $\mathfrak{m}(\mathrm{RO}^\uparrow(P)) \ge \mathfrak{m}^\uparrow(P)$.

(ii) Setting $S' = \{(G, p) : G \in \mathrm{RO}^\uparrow(P)^+,\ p \in P,\ G \subseteq \overline{[p,\infty[}\}$, S' satisfies the conditions of 517C with $P_0 = (\mathrm{RO}^\uparrow(P)^+, \supseteq)$ and $P_1 = (P, \le)$, so $\mathfrak{m}^\uparrow(P) \ge \mathfrak{m}(\mathrm{RO}^\uparrow(P))$.

(c) Let $\mathcal{Q}$ be a family of upwards-cofinal subsets of $[p_0,\infty[$ with $\#(\mathcal{Q}) < \mathfrak{m}^\uparrow(P)$, and $p_1 \in [p_0,\infty[$. For each $Q \in \mathcal{Q}$, set $Q' = Q \cup \{p : p \in P,\ [p,\infty[\cap [p_0,\infty[= \emptyset\}$. Then every Q' is cofinal with P. So there is an upwards-linked set $R \subseteq P$ containing p_1 and meeting Q' for every $Q \in \mathcal{Q}$. If $Q \in \mathcal{Q}$ and $r \in R \cap Q'$, then $[r,\infty[\cap [p_0,\infty[\supseteq [r,\infty[\cap [p_1,\infty[$ is non-empty, so $r \in Q$. Thus $R \cap [p_0,\infty[$ is an upwards-linked subset of $[p_0,\infty[$ containing p_1 and meeting every member of $\mathcal{Q}$. As $\mathcal{Q}$ and p_1 are arbitrary, $\mathfrak{m}^\uparrow([p_0,\infty[) \ge \mathfrak{m}^\uparrow(P)$.

517E Corollary Let P be a partially ordered set such that $\mathfrak{m}^\uparrow(P)$ is not ∞. Then $\mathfrak{m}^\uparrow(P) \le 2^{\mathrm{cf}\,P}$.

proof Let Q_0 be a cofinal subset of P with $\#(Q_0) = \mathrm{cf}\,P$. Then $\mathfrak{m}^\uparrow(Q_0) = \mathfrak{m}^\uparrow(P) < \infty$. So there are $q_0 \in Q_0$ and a family $\mathcal{Q}$ of cofinal subsets of Q_0 such that no upwards-linked subset of Q_0 containing p_0 can meet every member of $\mathcal{Q}$. Now

$$\mathfrak{m}^\uparrow(P) = \mathfrak{m}^\uparrow(Q_0) \le \#(\mathcal{Q}) \le 2^{\#(Q_0)} = 2^{\mathrm{cf}\,P}.$$

517F Proposition Let P be a non-empty partially ordered set.

(a) Suppose that κ and λ are cardinals such that $\mathrm{sat}^\uparrow(P) \le \mathrm{cf}\,\kappa$, $\lambda \le \kappa$ and $\lambda < \mathfrak{m}^\uparrow(P)$. Then (κ, λ) is an upwards precaliber pair of P.

(b) In particular, if $\mathrm{sat}^\uparrow(P) \le \mathrm{cf}\,\kappa \le \kappa < \mathfrak{m}^\uparrow(P)$ then κ is an up-precaliber of P.

proof (a) Since P is not empty, $\mathrm{sat}^\uparrow(P) \ge 2$ and κ is infinite. Write θ for $\mathrm{sat}^\uparrow(P)$. Let $\langle p_\xi \rangle_{\xi<\kappa}$ be a family in P. For $I \subseteq \kappa$, set

$$U_I = \bigcup_{\xi \in I} [p_\xi,\infty[, \quad V_I = \{q : q \in P,\ [q,\infty[\cap [p_\xi,\infty[= \emptyset \text{ for every } \xi \in I\}.$$

Then for every $J \subseteq \kappa$ there is an $I \in [J]^{<\theta}$ such that $V_J \cup U_I$ is cofinal with P. **P** $V_J \cup U_J$ is cofinal with P, so there is a maximal up-antichain $A \subseteq V_J \cup U_J$. Now $\#(A \cap U_J) < \mathrm{sat}^\uparrow(P) = \theta$, so there is a set $I \in [J]^{<\theta}$ such that $A \cap U_J \subseteq U_I$, and $A \subseteq V_J \cup U_I$. Now $V_J \cup U_I \supseteq \bigcup_{q \in A} [q,\infty[$, so is cofinal with P. **Q**

Next, $Q = \bigcup\{V_{\kappa\setminus I} : I \in [\kappa]^{<\kappa}\}$ is not cofinal with P. **P?** If it were, there would be a maximal up-antichain $A \subseteq Q$. For each $q \in A$, let $I_q \in [\kappa]^{<\kappa}$ be such that $q \in V_{\kappa\setminus I_q}$. Because $\#(A) < \theta \le \operatorname{cf}\kappa$, $\bigcup_{q\in A} I_q \ne \kappa$, and there is a $\xi \in \kappa \setminus \bigcup_{q\in A} I_q$. But now $[q,\infty[\, \cap\, [p_\xi,\infty[\, = \emptyset$ for every $q \in A$, and A is not a maximal antichain. **XQ**

Let $q_0 \in P$ be such that $Q \cap [q_0,\infty[\, = \emptyset$. Choose $\langle I_\xi\rangle_{\xi<\lambda}$ inductively in such a way that, writing $J_\xi = \kappa\setminus\bigcup_{\eta<\xi} I_\eta$, $I_\xi \in [J_\xi]^{<\theta}$ and $Q_\xi = V_{J_\xi} \cup U_{I_\xi}$ is cofinal with P for every $\xi < \lambda$. Because $\lambda < \mathfrak{m}^\uparrow(P)$, there is an upwards-directed set $R \subseteq P$ containing q_0 and meeting every Q_ξ. Set $\Gamma = \{\eta : \eta < \kappa,\ R\cap[p_\eta,\infty[\, \ne \emptyset\}$; then $\{p_\eta : \eta \in \Gamma\}$ is upwards-centered. Next, $\kappa \setminus J_\xi = \bigcup_{\eta<\xi} I_\eta$ has cardinal less than κ for every $\xi < \lambda$. **P** If $\theta = \kappa$ or $\kappa = \omega$, this is because $\#(\xi) < \kappa$ and $\#(I_\eta) < \kappa$ for every $\eta < \xi$ and κ is regular (use 513Bb). Otherwise it's because $\max(\omega,\theta,\#(\xi)) < \kappa$. **Q**

This means that $V_{J_\xi} \cap [q_0,\infty[\, = \emptyset$ and $R \cap V_{J_\xi}$ must be empty, for every $\xi < \lambda$. We must therefore have $R \cap U_{I_\xi} \ne \emptyset$ for each $\xi < \lambda$, so that $\Gamma \cap I_\xi \ne \emptyset$; as $\langle I_\xi\rangle_{\xi<\lambda}$ is disjoint, $\#(\Gamma) \ge \lambda$.

As $\langle p_\xi\rangle_{\xi<\kappa}$ is arbitrary, (κ,λ) is an upwards precaliber pair of P.

(b) This follows at once, setting $\lambda = \kappa$.

517G Corollary (a) If P and Q are partially ordered sets and $\operatorname{sat}^\uparrow(Q) < \mathfrak{m}^\uparrow(P)$, then $\operatorname{sat}^\uparrow(P\times Q)$ is at most $\max(\omega,\operatorname{sat}^\uparrow(P),\operatorname{sat}^\uparrow(Q))$.

(b) Let $\langle P_i\rangle_{i\in I}$ be a family of non-empty partially ordered sets with upwards finite-support product P. Let κ be a regular uncountable cardinal such that $\operatorname{sat}^\uparrow(P_i) \le \kappa < \mathfrak{m}^\uparrow(P_i)$ for every $i \in I$. Then $\operatorname{sat}^\uparrow(P) \le \kappa$.

proof (a) Set $\lambda = \operatorname{sat}^\uparrow(Q)$, $\kappa = \max(\omega,\operatorname{sat}^\uparrow(P),\operatorname{sat}^\uparrow(Q))$. Then κ is regular (513Bb again), $\lambda \le \kappa$ and $\lambda < \mathfrak{m}^\uparrow(P)$, so (κ,λ) is an upwards precaliber pair of P and $(\kappa,\lambda,2)$ is an upwards precaliber triple of P. By 516Ta, $\operatorname{sat}^\uparrow(P\times Q) \le \kappa$.

(b) By 517Fb, κ is an up-precaliber of P_i for every i, so $(\kappa,\kappa,2)$ is an upwards precaliber triple of every P_i, and we can use 516Tb.

517H Proposition Let P be a non-empty partially ordered set, and let P^* be the upwards finite-support product of the family $\langle P_n\rangle_{n\in\mathbb{N}}$ where $P_n = P$ for every n. Suppose that $\kappa < \mathfrak{m}^\uparrow(P^*)$.

(a) Every subset of P with κ or fewer members can be covered by a sequence of upwards-directed sets.

(b) In particular, if κ is uncountable then (κ,λ) is an upwards precaliber pair of P for every $\lambda < \kappa$, and if κ has uncountable cofinality then κ is an up-precaliber of P.

proof (a) Let $A \in [P]^{<\kappa}$. For each $p \in A$, set $Q_p = \{q : q \in P^*, \exists\ n \in \operatorname{dom} q,\ q(n) = p\}$; then Q_p is cofinal with P^*. So there is an upwards-directed set $R \subseteq P^*$ such that $R\cap Q_p \ne \emptyset$ for every $p \in A$. For each $n \in \mathbb{N}$, set $R_n = \{q(n) : q \in R,\ n \in \operatorname{dom} q\}$. Then $A \subseteq \bigcup_{n\in\mathbb{N}} R_n$. If $n \in \mathbb{N}$ and r, $r' \in R_n$, there are q, $q' \in R$ such that $q(n) = r$ and $q'(n) = r'$. Now there is a $q'' \in R$ such that $q'' \ge q$ and $q'' \ge q'$, in which case $q''(n)$ belongs to $R_n \cap [r,\infty[\, \cap\, [r',\infty[$. As r and r' are arbitrary, R_n is upwards-directed. Thus $\langle R_n\rangle_{n\in\mathbb{N}}$ is an appropriate sequence.

(b) Suppose that κ is uncountable and that either $\lambda < \kappa$ or $\operatorname{cf}\kappa > \omega$ and $\lambda = \kappa$. Let $\langle p_\xi\rangle_{\xi<\kappa}$ be any family in P. Let $\langle R_n\rangle_{n\in\mathbb{N}}$ be a sequence of upwards-directed sets covering $\{p_\xi : \xi < \kappa\}$, and for each $n \in \mathbb{N}$ set $\Gamma_n = \{\xi : p_\xi \in R_n\}$. There must be some n such that $\#(\Gamma_n) \ge \lambda$, and $\{p_\xi : \xi \in \Gamma_n\}$ is upwards-centered.

517I Proposition Let $\mathfrak{A}$ be a Boolean algebra.

(a) If $\mathfrak{B}$ is a regularly embedded subalgebra of $\mathfrak{A}$, then $\mathfrak{m}(\mathfrak{B}) \ge \mathfrak{m}(\mathfrak{A})$.

(b) If $\mathfrak{B}$ is a principal ideal of $\mathfrak{A}$, then $\mathfrak{m}(\mathfrak{B}) \ge \mathfrak{m}(\mathfrak{A})$.

(c) If $\mathfrak{B}$ is an order-dense subalgebra of $\mathfrak{A}$, then $\mathfrak{m}(\mathfrak{B}) = \mathfrak{m}(\mathfrak{A})$.

(d) If $\widehat{\mathfrak{A}}$ is the Dedekind completion of $\mathfrak{A}$, then $\mathfrak{m}(\widehat{\mathfrak{A}}) = \mathfrak{m}(\mathfrak{A})$.

(e) If $D \subseteq \mathfrak{A}$ is non-empty and $\sup D = 1$, then $\mathfrak{m}(\mathfrak{A}) = \min_{d\in D}\mathfrak{m}(\mathfrak{A}_d)$, where $\mathfrak{A}_d$ is the principal ideal generated by d.

(f) If $\mathfrak{A}$ is the simple product of a non-empty family $\langle\mathfrak{A}_i\rangle_{i\in I}$ of Boolean algebras, then $\mathfrak{m}(\mathfrak{A}) = \min_{i\in I}\mathfrak{m}(\mathfrak{A}_i)$.

(g) Suppose that κ and λ are infinite cardinals such that $\operatorname{sat}(\mathfrak{A}) \le \operatorname{cf}\kappa$, $\lambda \le \kappa$ and $\lambda < \mathfrak{m}(\mathfrak{A})$. Then (κ,λ) is a precaliber pair of $\mathfrak{A}$.

proof (a) Setting $S = \{(a,b) : a \in \mathfrak{A}^+,\ a \subseteq b \in \mathfrak{B}\}$, S satisfies the conditions of 517C for $P_0 = (\mathfrak{A}^+,\supseteq)$ and $P_1 = (\mathfrak{B}^+,\supseteq)$. **P** The only non-trivial part is the check that if Q is coinitial with $\mathfrak{B}^+$ then $S^{-1}[Q]$ is coinitial with $\mathfrak{A}^+$. But $\sup Q = 1$ in $\mathfrak{B}$; as $\mathfrak{B}$ is regularly embedded in $\mathfrak{A}$, $\sup Q = 1$ in $\mathfrak{A}$. So if $a \in \mathfrak{A}^+$, there is a $b \in Q$ such that $a \cap b \ne 0$, and now $a \cap b \in S^{-1}[Q]$ and $a \cap b \subseteq a$. As a is arbitrary, $S^{-1}[Q]$ is coinitial with $\mathfrak{A}^+$. **Q** So $\mathfrak{m}(\mathfrak{B}) \ge \mathfrak{m}(\mathfrak{A})$.

(b) If $\mathfrak{A}_a$ is the principal ideal generated by $a \in \mathfrak{A}^+ = \mathfrak{A} \setminus \{0\}$, we have

$$\mathfrak{m}(\mathfrak{A}) = \mathfrak{m}^{\downarrow}(\mathfrak{A}^+) \le \mathfrak{m}^{\downarrow}(]0,a])$$

(by 517Dc, inverted)

$$= \mathfrak{m}(\mathfrak{A}_a).$$

On my definitions the trivial ideal $\{0\}$ also is a principal ideal, but of course $\mathfrak{m}(\{0\}) = \infty \ge \mathfrak{m}(\mathfrak{A})$.

(c) Apply 517Da (inverted) to $\mathfrak{A}^+$ and $\mathfrak{B}^+$.

(d) This follows from (c), because $\mathfrak{A}$ is order-dense in $\widehat{\mathfrak{A}}$.

(e) By (b), $\mathfrak{m}(\mathfrak{A}) \le \mathfrak{m}(\mathfrak{A}_d)$ for every d. In the other direction, let $\mathcal{Q}$ be a family of coinitial subsets of $\mathfrak{A}^+$ such that $\#(\mathcal{Q}) < \min_{d\in D} \mathfrak{m}(\mathfrak{A}_d)$, and take any $c \in \mathfrak{A}^+$. Then there is a $d \in D$ such that $c \cap d \ne 0$. For $Q \in \mathcal{Q}$ set $Q' = \{a \cap d : a \in Q\} \setminus \{0\}$; then Q' is coinitial with $\mathfrak{A}_d^+$. Since $\#(\{Q' : Q \in \mathcal{Q}\}) < \mathfrak{m}(\mathfrak{A}_d)$, there is a downwards-linked set $R' \subseteq \mathfrak{A}_d^+$ meeting every Q' and containing $c \cap d$. Set $R = \{a : a \in \mathfrak{A},\ a \cap d \in R'\}$; then R is a downwards-linked subset of $\mathfrak{A}^+$ meeting every member of $\mathcal{Q}$ and containing c. As c and $\mathcal{Q}$ are arbitrary, $\mathfrak{m}(\mathfrak{A}) \ge \min_{d\in D} \mathfrak{m}(\mathfrak{A}_d)$.

(f) This is, in effect, a special case of (e), since we can identify the $\mathfrak{A}_i$ with principal ideals of $\mathfrak{A}$ (315E).

(g) Apply 517Fa (inverted) to $\mathfrak{A}^+$.

517J Proposition Let X be a locally compact Hausdorff space, and κ a cardinal. Then the following are equiveridical:

(i) $\kappa < \mathfrak{m}(\mathrm{RO}(X))$, where $\mathrm{RO}(X)$ is the regular open algebra of X;
(ii) $X \cap \bigcap \mathcal{G}$ is dense in X whenever $\mathcal{G}$ is a family of dense open subsets of X and $\#(\mathcal{G}) \le \kappa$;
(iii) $\kappa < n(H)$ for every non-empty open set $H \subseteq X$.

proof (i)⇒(iii) Suppose that $\kappa < \mathfrak{m}(\mathrm{RO}(X))$. Let $H \subseteq X$ be a non-empty open set and $\mathcal{E}$ a family of nowhere dense subsets of H with $\#(\mathcal{E}) \le \kappa$. Note that every member of $\mathcal{E}$ is nowhere dense in X. Because X is locally compact and regular, we have a non-empty regular open set H_0 such that $K = \overline{H}_0$ is compact and included in H. For each $E \in \mathcal{E}$, set $\mathcal{G}_E = \{G : G \in \mathrm{RO}(X)^+,\ \overline{G} \cap E = \emptyset\}$; then $\mathcal{G}_E$ is coinitial with $\mathrm{RO}(X)^+$. Because $\kappa < \mathfrak{m}^{\downarrow}(\mathrm{RO}(X)^+)$, there is a centered $\mathcal{G} \subseteq \mathrm{RO}(X)^+$ containing H_0 and meeting every $\mathcal{G}_E$. But in this case $\{K\} \cup \{\overline{G} : G \in \mathcal{G}\}$ is a family of closed sets in X containing the compact set K and with the finite intersection property, so has non-empty intersection F, which is included in $H \setminus \bigcup \mathcal{E}$. As H and $\mathcal{E}$ are arbitrary, (iii) is true.

(iii)⇒(ii) This is easy. If (iii) is true, $\mathcal{G}$ is a family of dense open subsets of X with $\#(\mathcal{G}) \le \kappa$, and $H \subseteq X$ is a non-empty open set, then $\mathcal{E} = \{H \setminus G : G \in \mathcal{G}\}$ is a family of nowhere dense subsets of H, so cannot cover H, and $H \cap \bigcap \mathcal{G} \ne \emptyset$. As $\mathcal{G}$ and H are arbitrary, (ii) is true.

(ii)⇒(i) Suppose that (ii) is true. Take $H \in \mathrm{RO}(X)^+$ and a family $\mathfrak{G}$ of coinitial subsets of $\mathrm{RO}(X)^+$ with $\#(\mathfrak{G}) \le \kappa$. For each $\mathcal{G} \in \mathfrak{G}$, $\bigcup \mathcal{G}$ is a dense open subset of X. Accordingly there is a point $x \in H \cap \bigcap_{\mathcal{G}\in\mathfrak{G}} \bigcup \mathcal{G}$. Set $R = \{G : G \in \mathrm{RO}(X),\ x \in G\}$. Then R is a downwards-linked subset of $\mathrm{RO}(X)^+$ containing H and meeting every member of $\mathfrak{G}$. As H and $\mathfrak{G}$ are arbitrary, $\kappa < \mathfrak{m}(\mathrm{RO}(X))$.

517K Corollary Let $\mathfrak{A}$ be a Boolean algebra with Stone space Z.

(a) $\mathfrak{m}(\mathfrak{A}) = \mathfrak{m}(\mathrm{RO}(Z))$.
(b) For any cardinal κ, the following are equiveridical:

  (i) $\kappa < \mathfrak{m}(\mathfrak{A})$;

(ii) $Z \cap \bigcap \mathcal{G}$ is dense in Z whenever $\mathcal{G}$ is a family of dense open subsets of Z and $\#(\mathcal{G}) \le \kappa$;
(iii) $\kappa < n(H)$ for every non-empty open set $H \subseteq Z$.

proof (a) $\mathfrak{A}$ is isomorphic to the algebra of open-and-closed subsets of Z, which is an order-dense subalgebra of $\mathrm{RO}(Z)$ (314Ta). So $\mathfrak{m}(\mathfrak{A}) = \mathfrak{m}(\mathrm{RO}(Z))$ by 517Ic.

(b) now follows from 517J.

517L These identifications make the following results easy.

Proposition Let $\mathfrak{A}$ be a Boolean algebra.
(a) $\mathrm{wdistr}(\mathfrak{A}) \le \mathfrak{m}(\mathfrak{A})$.
(b) If $\mathrm{wdistr}(\mathfrak{A})$ is a precaliber of $\mathfrak{A}$ then $\mathrm{wdistr}(\mathfrak{A}) < \mathfrak{m}(\mathfrak{A})$.

proof (a) Let Z be the Stone space of $\mathfrak{A}$ and $\mathcal{N}\mathit{wd}$ the ideal of nowhere dense subsets of Z. Then $\mathrm{wdistr}(\mathfrak{A}) = \mathrm{add}\,\mathcal{N}\mathit{wd}$ (514Be), while $\mathfrak{m}(\mathfrak{A})$ is the least cardinal of any subset of $\mathcal{N}\mathit{wd}$ covering a non-empty open subset of Z, if there is one (517Kb). Since no non-empty open subset of Z can belong to $\mathcal{N}\mathit{wd}$, $\mathrm{wdistr}(\mathfrak{A}) \le \mathfrak{m}(\mathfrak{A})$.

(b) Because $\mathrm{wdistr}(\mathfrak{A}) = \mathrm{add}\,\mathcal{N}\mathit{wd} \ge \omega$, it is a regular infinite cardinal (513C(a-i)). If $\langle G_\xi \rangle_{\xi < \mathrm{wdistr}(\mathfrak{A})}$ is a family of dense open subsets of Z, and $H \subseteq Z$ is open and not empty, then $H_\xi = H \cap \mathrm{int}(\bigcap_{\eta < \xi} G_\eta)$ is non-empty for every $\xi < \mathrm{wdistr}(\mathfrak{A})$. So if also $\mathrm{wdistr}(\mathfrak{A})$ is a precaliber of $\mathfrak{A}$ and therefore of Z (516Ha), there is a point z of Z such that $\{\xi : z \in H_\xi\}$ has cardinal $\mathrm{wdistr}(\mathfrak{A})$ (516Qb) and is therefore cofinal with $\mathrm{wdistr}(\mathfrak{A})$; which means that $z \in H \cap \bigcap_{\xi < \mathrm{wdistr}(\mathfrak{A})} G_\xi$. Thus $n(H) > \mathrm{wdistr}(\mathfrak{A})$; as H is arbitrary, $\mathfrak{m}(\mathfrak{A}) > \mathrm{wdistr}(\mathfrak{A})$, by 517Kb again.

517M It is worth extracting an idea from the proofs just above as a general result.

Proposition Let X be any topological space. Then the Novák number $n(X)$ of X (5A4Af) is at most $\sup\{\mathfrak{m}(\mathrm{RO}(G)) : G \subseteq X \text{ is open and not empty}\}$, where $\mathrm{RO}(G)$ is the regular open algebra of G.

proof (a) If there is a non-empty open subset G of X such that $\mathfrak{m}(\mathrm{RO}(G)) = \infty$, the result is trivial; suppose otherwise. Set $\kappa = \sup\{\mathfrak{m}(\mathrm{RO}(G)) : G \subseteq X \text{ is open and not empty}\}$. Then for any non-empty open set $G \subseteq X$ there is a family $\langle E_\xi \rangle_{\xi < \kappa}$ of nowhere dense sets such that $\#(\mathcal{E}) \le \kappa$ and $G \cap \mathrm{int}(\bigcup_{\xi < \kappa} E_\xi) \ne \emptyset$. **P** We have a family $\langle \mathcal{Q}_\xi \rangle_{\xi < \kappa}$ of order-dense subsets of $\mathrm{RO}(G)^+$ and an $H \in \mathrm{RO}(G)^+$ such that there is no downwards-directed family in $\mathrm{RO}(G)^+$ containing H and meeting every $\mathcal{Q}_\xi$. Set $E_\xi = G \setminus \bigcup \mathcal{Q}_\xi$ for each ξ; then E_ξ must be nowhere dense in the topological sense because any open set meeting G at all must meet some member of $\mathcal{Q}_\xi$. If $x \in H$, then $\mathcal{R} = \{U : U \in \mathrm{RO}(G),\ x \in U\}$ is a downwards-directed family in $\mathrm{RO}(G)^+$ containing H, so does not meet every $\mathcal{Q}_\xi$, and there must be a $\xi < \kappa$ such that $x \notin \bigcup \mathcal{Q}_\xi$, that is, $x \in E_\xi$. As x is arbitrary, $G \cap \mathrm{int}(\bigcup_{\xi < \kappa} E_\xi) \supseteq H$ is not empty. **Q**

(b) Let $\langle H_i \rangle_{i \in I}$ be a maximal disjoint family of non-empty open sets in X such that every H_i can be covered by a family of at most κ nowhere dense sets. By (a), $\bigcup_{i \in I} H_i$ is dense. For each $i \in I$, let $\langle E_{i\xi} \rangle_{\xi < \kappa}$ be a family of nowhere dense sets covering H_i. Set $E_\xi = \bigcup_{i \in I} H_i \cap E_{i\xi}$ for each $\xi < \kappa$; then E_ξ is nowhere dense (5A4Ea). Also $\bigcup_{\xi < \kappa} E_\xi = \bigcup_{i \in I} H_i$ is a dense open set, so that $\{E_\xi : \xi < \kappa\} \cup (X \setminus \bigcup_{i \in I} H_i)$ is a cover of X by nowhere dense sets, and $n(X) \le \kappa$. (Of course κ is infinite, by 517A, except in the trivial case $X = \emptyset$.)

517N Corollary If $\mathfrak{A}$ is a Martin-number-homogeneous Boolean algebra with Stone space Z, then $\mathfrak{m}(\mathfrak{A}) = n(Z)$.

proof By 517Kb(i)$\Rightarrow$(iii), $\mathfrak{m}(\mathfrak{A}) \le n(Z)$. In the other direction, given $a \in \mathfrak{A}$, write $\widehat{a}$ for the open-and-closed subset of Z corresponding to a, and $\mathfrak{A}_a$ for the principal ideal generated by a. If $G \subseteq Z$ is a non-empty regular open set, let $a \in \mathfrak{A} \setminus \{0\}$ be such that $\widehat{a} \subseteq G$. Then

$$\mathfrak{m}(\mathrm{RO}(G)) \le \mathfrak{m}(\mathrm{RO}(\widehat{a}))$$

(by 517Ib, because $\mathrm{RO}(\widehat{a})$ can be regarded as a principal ideal of $\mathrm{RO}(G)$)

$$= \mathfrak{m}(\mathfrak{A}_a)$$

(because we can identify $\widehat{a}$ with the Stone space of $\mathfrak{A}_a$, by 312T, and use 517Ka)

$$= \mathfrak{m}(\mathfrak{A}).$$

By 517M, $n(Z) \le \mathfrak{m}(\mathfrak{A})$.

517O Martin cardinals (a) For any class $\mathcal{P}$ of partially ordered sets, we have an associated cardinal

$$\mathfrak{m}^{\uparrow}_{\mathcal{P}} = \min\{\mathfrak{m}^{\uparrow}(P) : P \in \mathcal{P}\}.$$

Much the most important of these is the cardinal

$$\mathfrak{m} = \min\{\mathfrak{m}^{\uparrow}(P) : P \text{ is upwards-ccc}\}.$$

Others of great interest are

$$\mathfrak{p} = \min\{\mathfrak{m}^{\uparrow}(P) : P \text{ is } \sigma\text{-centered upwards}\},$$

$$\mathfrak{m}_{\mathrm{K}} = \min\{\mathfrak{m}^{\uparrow}(P) : P \text{ satisfies Knaster's condition upwards}\},$$

$$\mathfrak{m}_{\text{countable}} = \min\{\mathfrak{m}^{\uparrow}(P) : P \text{ is a countable partially ordered set}\}.$$

Two more which are worth examining are

$$\mathfrak{m}_{\sigma\text{-linked}} = \min\{\mathfrak{m}^{\uparrow}(P) : P \text{ is } \sigma\text{-linked upwards}\},$$

$$\mathfrak{m}_{\mathrm{pc}\omega_1} = \min\{\mathfrak{m}^{\uparrow}(P) : \omega_1 \text{ is an up-precaliber of } P\}.$$

(b) These cardinals are related as follows:

$$\begin{tikzcd}[arrows=-]
 & & \mathfrak{m}_{\sigma\text{-linked}} \arrow[r] & \mathfrak{p} \arrow[r] & \mathfrak{m}_{\text{countable}} \arrow[r] & \mathfrak{c} \\
\omega_1 \arrow[r] & \mathfrak{m} \arrow[r] & \mathfrak{m}_{\mathrm{K}} \arrow[u] \arrow[r] & \mathfrak{m}_{\mathrm{pc}\omega_1} \arrow[u] & &
\end{tikzcd}$$

The numbers here increase from bottom left to top right; that is,

$$\omega_1 \le \mathfrak{m} \le \mathfrak{m}_{\mathrm{K}} \le \mathfrak{m}_{\mathrm{pc}\omega_1} \le \mathfrak{p} \le \mathfrak{m}_{\text{countable}} \le \mathfrak{c},$$

$$\mathfrak{m}_{\mathrm{K}} \le \mathfrak{m}_{\sigma\text{-linked}} \le \mathfrak{p}.$$

From 517A we see that $\omega_1 \le \mathfrak{m}$. For the proof that $\mathfrak{m}_{\text{countable}} \le \mathfrak{c}$, see 517P below. As for the intermediate inequalities involving Martin cardinals, they follow directly from inclusions between the corresponding classes of partially ordered set. These are all immediate from the definitions; I give references to the general results of this chapter which cover the relevant facts, as follows.

(i) Every partially ordered set satisfying Knaster's condition upwards is ccc. (If $(\omega_1, 2)$ is an upwards precaliber pair of P, then $\mathrm{sat}^{\uparrow}(P) \le \omega_1$ (516Ka).)

(ii) If ω_1 is an up-precaliber of P, then P satisfies Knaster's condition upwards. (If $(\omega_1, \omega_1, {<}\omega)$ is a triple precaliber of P, so is $(\omega_1, 2, {<}\omega)$, by 516Ba.)

(iii) If P is σ-linked upwards, it satisfies Knaster's condition upwards. (As $\omega_1 > \max(\omega, \omega, \mathrm{link}^{\uparrow}(P))$, $(\omega_1, \omega_1, {<}3)$ is an upwards precaliber triple of P (516Kb), so $(\omega_1, 2, {<}3)$ and $(\omega_1, 2, {<}\omega)$ also are, by 516Ba again.)

(iv) If P is σ-centered upwards, it is σ-linked upwards. ($\mathrm{link}(P) \le \mathrm{link}_{<\omega}(P)$, by 511Hb.)

(v) If P is σ-centered upwards, ω_1 is an up-precaliber of P. (As $\omega_1 > \max(\omega, \omega, \mathrm{link}^{\uparrow}_{<\omega}(P))$, (ω_1, ω_1) is an upwards precaliber pair of P, by 516Kb again.)

(vi) If P is countable, it is σ-centered upwards. (Singleton subsets are centered.)

(c) I should note a special feature of the bottom row of this diagram. In the chain $\omega_1 \le \mathfrak{m} \le \mathfrak{m}_{\mathrm{K}} \le \mathfrak{m}_{\mathrm{pc}\omega_1}$, at most one of the inequalities can be strict. **P** Suppose that P is upwards-ccc and $\mathfrak{m}^{\uparrow}(P) > \omega_1$. Then ω_1 is an up-precaliber of P (517Fb), so $\mathfrak{m}^{\uparrow}(P) \ge \mathfrak{m}_{\mathrm{pc}\omega_1}$. So if, for instance, $\mathfrak{m}_{\mathrm{K}} > \omega_1$ and P satisfies Knaster's condition upwards, $\mathfrak{m}^{\uparrow}(P) > \omega_1$ and $\mathfrak{m}^{\uparrow}(P) \ge \mathfrak{m}_{\mathrm{pc}\omega_1}$; as P is arbitrary, $\mathfrak{m}_{\mathrm{K}} \ge \mathfrak{m}_{\mathrm{pc}\omega_1}$. Similarly, if $\mathfrak{m} > \omega_1$ then $\mathfrak{m} = \mathfrak{m}_{\mathrm{pc}\omega_1}$. **Q**

(d) Now **Martin's Axiom** is the assertion

$$‘\mathfrak{m} = \mathfrak{c}’.$$

From the diagram above, we see that this is a consequence of the continuum hypothesis (‘$\omega_1 = \mathfrak{c}$’), and fixes all the intermediate cardinals.

(e) All the partially ordered sets considered in (b) are ccc, which is why $\mathfrak{m}$ appears at bottom left. The same idea can be applied to larger classes, e.g. ‘proper’ or ‘stationary-set-preserving’ partial orders. For the moment I will not even define these classes; I mention them only for the sake of readers who are already familiar with them and may be expecting a reference here. There is an important difference, however, in that if the cardinal which we might call

$$\mathfrak{m}_{\mathrm{proper}} = \min\{\mathfrak{m}^{\uparrow}(P) : P \text{ is upwards-proper}\}$$

is greater than ω_1, then $\mathfrak{c} = \mathfrak{m}_{\mathrm{proper}} = \omega_2$ (Veličković 92, or Moore 05); so that we have only to say whether the Proper Forcing Axiom (‘$\mathfrak{m}_{\mathrm{proper}} > \omega_1$’) is true or false to determine the value of $\mathfrak{m}_{\mathrm{proper}}$.

517P All the cardinals here have special features, but the ones I will concentrate on just now are the two largest, $\mathfrak{m}_{\mathrm{countable}}$ and $\mathfrak{p}$.

Proposition (a) $\omega_1 \le \mathfrak{m}_{\mathrm{countable}} \le \mathfrak{c}$.

(b) Let $\mathfrak{A}$ be a Boolean algebra with countable π-weight. If $\mathfrak{A}$ is purely atomic, then $\mathfrak{m}(\mathfrak{A}) = \infty$; otherwise, $\mathfrak{m}(\mathfrak{A}) = \mathfrak{m}_{\mathrm{countable}}$.

(c) If P is a partially ordered set of countable cofinality and $\mathfrak{m}^{\uparrow}(P)$ is not ∞, then $\mathfrak{m}^{\uparrow}(P) = \mathfrak{m}_{\mathrm{countable}}$.

(d)(i) Let X be a topological space such that its category algebra is atomless and has countable π-weight. Then $n(X) \le \mathfrak{m}_{\mathrm{countable}}$.

(ii) If X is a non-empty locally compact Hausdorff space with countable π-weight and no isolated points, then $n(X) = \mathfrak{m}_{\mathrm{countable}}$.

(iii) If X is a non-empty Polish space with no isolated points, then $n(X) = \mathfrak{m}_{\mathrm{countable}}$.

proof Let $\mathfrak{B}$ be the algebra of open-and-closed subsets of $\{0,1\}^{\mathbb{N}}$. The argument will go more smoothly if I prove (a)-(c) with $\mathfrak{m}(\mathfrak{B})$ in place of $\mathfrak{m}_{\mathrm{countable}}$, and at an appropriate moment point out that I have shown that the two are equal.

(a) $\mathfrak{m}(\mathfrak{B}) = \mathfrak{m}^{\downarrow}(\mathfrak{B}^+)$ is uncountable, by 517A. To see that $\mathfrak{m}(\mathfrak{B}) \le \mathfrak{c}$, let $\mathcal{Q}$ be the set of all coinitial subsets of $\mathfrak{B}^+$; then $\#(\mathcal{Q}) \le \mathfrak{c}$ because $\mathfrak{B}$ is countable. **?** If $\mathfrak{m}(\mathfrak{B}) > \mathfrak{c}$, there must be a linked set $R \subseteq \mathfrak{B}^+$ meeting every member of $\mathcal{Q}$. But now consider $Q = \mathfrak{B}^+ \setminus R$. If $a \in \mathfrak{B}^+$, there are disjoint non-zero a', $a'' \subseteq a$ which cannot both belong to R, so at least one belongs to Q. But this means that Q is order-dense in $\mathfrak{B}$ and ought to meet R. **X** (Compare 517E.)

(b)(i) If $\mathfrak{A}$ is purely atomic, $\mathfrak{m}(\mathfrak{A}) = \infty$, by 511If.

(ii) Suppose that $\mathfrak{A}$ is not purely atomic. Because $\pi(\mathfrak{A})$ is countable, there is a countable order-dense set $C \subseteq \mathfrak{A}$. Let $\mathfrak{C}$ be the subalgebra of $\mathfrak{A}$ generated by C, so that $\mathfrak{C}$ is a countable order-dense subalgebra of $\mathfrak{A}$, and is not purely atomic. Consider the free product $\mathfrak{C} \otimes \mathfrak{B}$ (315N). This is a countable atomless Boolean algebra (use 315O), so is isomorphic to $\mathfrak{B}$ (316M). Also we have an injective order-continuous Boolean homomorphism from $\mathfrak{C}$ to $\mathfrak{C} \otimes \mathfrak{B}$ (315K), so that $\mathfrak{C}$ is isomorphic to a regularly embedded subalgebra of $\mathfrak{B}$ and $\mathfrak{m}(\mathfrak{C}) \ge \mathfrak{m}(\mathfrak{B})$ (517Ia).

Next, $\mathfrak{C}$ has a non-trivial atomless principal ideal $\mathfrak{C}_a$ say. Because $\mathfrak{C}_a$ is still countable, it is itself isomorphic to $\mathfrak{B}$. So 517Ib tells us that $\mathfrak{m}(\mathfrak{C}) \le \mathfrak{m}(\mathfrak{C}_a) = \mathfrak{m}(\mathfrak{B})$, and $\mathfrak{m}(\mathfrak{C}) = \mathfrak{m}(\mathfrak{B})$.

Finally, $\mathfrak{m}(\mathfrak{A}) = \mathfrak{m}(\mathfrak{C})$ by 517Ic.

(c) We know that $\mathfrak{m}^{\uparrow}(P) = \mathfrak{m}(\mathrm{RO}^{\uparrow}(P))$ (517Db) and that $\pi(\mathrm{RO}^{\uparrow}(P)) \le \mathrm{cf}\, P$ (514Nb) is countable. Let $D \subseteq \mathrm{RO}^{\uparrow}(P)^+$ be a countable order-dense set, and $\mathfrak{A}$ the subalgebra of $\mathrm{RO}^{\uparrow}(P)$ generated by D. Then $\mathfrak{m}(\mathrm{RO}^{\uparrow}(P)) = \mathfrak{m}(\mathfrak{A})$ by 517Ic, and $\mathfrak{A}$ is countable. By (b), $\mathfrak{m}^{\uparrow}(P) = \mathfrak{m}(\mathfrak{A})$ is either ∞ or $\mathfrak{m}(\mathfrak{B})$; since the former is ruled out by hypothesis, we are left with the latter.

What this shows, however, is that

$$\begin{aligned}\mathfrak{m}(\mathfrak{B}) &\le \min\{\mathfrak{m}^{\uparrow}(P) : \mathrm{cf}\, P \le \omega\} \le \min\{\mathfrak{m}^{\uparrow}(P) : \#(P) \le \omega\}\\ &= \mathfrak{m}_{\mathrm{countable}} \le \mathfrak{m}^{\downarrow}(\mathfrak{B}^+) = \mathfrak{m}(\mathfrak{B})\end{aligned}$$

so that $\mathfrak{m}_{\mathrm{countable}} = \mathfrak{m}(\mathfrak{B})$ and we can rewrite the results so far in the forms given in the statement of the proposition.

(d)(i) Consider first the case in which X is a non-empty Baire space, so that its category algebra is isomorphic to $\mathrm{RO}(X)$ (514If). Since $\mathrm{RO}(X)$ is atomless and not $\{\emptyset\}$, and in particular is not purely atomic, but has countable π-weight, $\mathfrak{m}(\mathrm{RO}(X)) = \mathfrak{m}_{\mathrm{countable}}$, by (b). The same applies to any non-empty open subset G of X, recalling that the category algebra of G can be identified with a principal ideal of the category algebra of X (514Id). So $n(X) \le \mathfrak{m}_{\mathrm{countable}}$ by 517M.

If X is not a Baire space, then it has a smallest comeager regular open set H, which is itself a Baire space (4A3Sa[3]), and X and H have isomorphic category algebras (514Ic), so we see from the argument just above that $n(H) \le \mathfrak{m}_{\mathrm{countable}}$. But $X \setminus H$ is a countable union of nowhere dense subsets of X, and every subset of H which is nowhere dense in H is also nowhere dense in X, so $n(X) \le \max(\omega, n(H)) \le \mathfrak{m}_{\mathrm{countable}}$.

(ii) Because X is Hausdorff and has no isolated points, $\mathrm{RO}(X)$ is atomless. Next, $\pi(\mathrm{RO}(X)) \le \pi(X)$ is countable (514H(b-i)), and $\mathrm{RO}(X)$ is isomorphic to the category algebra of X, by Baire's theorem. So the first part of the proof of (i) tells us that $n(X) \le \mathfrak{m}_{\mathrm{countable}} = \mathfrak{m}(\mathrm{RO}(X))$. From 517J we now see that

$$\mathfrak{m}(\mathrm{RO}(X)) = \min\{n(H) : H \subseteq X \text{ is a non-empty open set}\} \le n(X),$$

so $n(X) = \mathfrak{m}_{\mathrm{countable}}$ exactly.

(iii) Now suppose that X is a non-empty Polish space without isolated points. As in (ii), the category algebra of X is atomless and has countable π-weight, so $n(X) \le \mathfrak{m}_{\mathrm{countable}}$. In the other direction, suppose that $\kappa < \mathfrak{m}_{\mathrm{countable}}$ and that $\langle E_\xi \rangle_{\xi<\kappa}$ is a family of nowhere dense subsets of X. Let ρ be a metric defining the topology of X under which X is complete, and $\mathcal{U}$ a countable base for the topology of X, not containing $\emptyset$. For $\xi < \kappa$, set $\mathcal{Q}_\xi = \{U : U \in \mathcal{U},\, \overline{U} \cap E_\xi = \emptyset\}$; for $n \in \mathbb{N}$ set $\mathcal{Q}'_n = \{U : U \in \mathcal{U},\, \mathrm{diam}\, U \le 2^{-n}\}$. Then every $\mathcal{Q}_\xi$ and every $\mathcal{Q}'_n$ is coinitial with $\mathcal{U}$. By (c) above,

$$\mathfrak{m}^{\downarrow}(\mathcal{U}) \ge \mathfrak{m}_{\mathrm{countable}} > \max(\kappa, \omega),$$

so there is a downwards-directed $\mathcal{V} \subseteq \mathcal{U}$ meeting every $\mathcal{Q}_\xi$ and every $\mathcal{Q}_n$. Now $\{\overline{V} : V \in \mathcal{V}\}$ is a downwards-directed set containing sets of arbitrarily small diameter, so generates a Cauchy filter and (because (X, ρ) is complete) has non-empty intersection. Take any $x \in \bigcap_{V \in \mathcal{V}} \overline{V}$. Because $\mathcal{V}$ meets every $\mathcal{Q}_\xi$, $x \notin \bigcup_{\xi<\kappa} E_\xi$ and $\langle E_\xi \rangle_{\xi<\kappa}$ does not cover X. As $\langle E_\xi \rangle_{\xi<\kappa}$ is arbitrary, $n(X) \ge \mathfrak{m}_{\mathrm{countable}}$ and the two are equal.

517Q Lemma If P is any partially ordered set, $\mathfrak{m}^{\uparrow}(P) \ge \min(\mathrm{add}_\omega P, \mathfrak{m}_{\mathrm{countable}})$.

proof (For the definition of $\mathrm{add}_\omega P$, see 513H.) Take $\kappa < \min(\mathrm{add}_\omega P, \mathfrak{m}_{\mathrm{countable}})$, $p_0 \in P$ and a family $\langle Q_\xi \rangle_{\xi<\kappa}$ of cofinal subsets of P. Choose $\langle R_n \rangle_{n\in\mathbb{N}}$ and $\langle Q_{n\xi} \rangle_{n\in\mathbb{N},\xi<\kappa}$ as follows. $R_0 = \{p_0\}$. Given that $R_n \subseteq P$ is countable, then for each $\xi < \kappa$ choose a countable set $Q_{n\xi} \subseteq Q_\xi$ such that for every $p \in R_n$ there is a $q \in Q_{n\xi}$ such that $p \le q$. Now, because $\mathrm{add}_\omega P > \kappa$ (and, of course, $\mathrm{add}_\omega P > \omega$, as noted in 513Ib), we can find a countable set $R_{n+1} \subseteq P$ such that whenever $q \in \bigcup_{\xi<\kappa} Q_{n\xi}$ there is an $r \in R_{n+1}$ such that $q \le r$. This will ensure that whenever $p \in R_n$ and $\xi < \kappa$ there are $q \in Q_\xi$ and $p' \in R_{n+1}$ such that $p \le q \le p'$.

At the end of the induction, consider the countable partially ordered set $R = \bigcup_{n\in\mathbb{N}} R_n$. For $\xi < \kappa$ set

$$Q'_\xi = \{r : r \in R, \exists\, q \in Q_\xi,\, q \le r\};$$

[3]Formerly 4A3Ra.

then Q'_ξ is cofinal with R. Because $\kappa < \mathfrak{m}_{\text{countable}}$, there is an upwards-linked subset S of R meeting every Q'_ξ and containing p_0. But now $\{p : p \in P, \exists\, s \in S, p \le s\}$ is an upwards-linked subset of p containing p_0 and meeting every Q_ξ. As p_0 and $\langle Q_\xi\rangle_{\xi<\kappa}$ are arbitrary, $\mathfrak{m}^\uparrow(P) \ge \min(\operatorname{add}_\omega P, \mathfrak{m}_{\text{countable}})$.

517R Proposition (a) ('Booth's Lemma'; see Booth 70) Suppose that $\mathcal{A}$ is a family of subsets of $\mathbb{N}$ such that $\#(\mathcal{A}) < \mathfrak{p}$ and $\bigcap\mathcal{J}$ is infinite for every finite $\mathcal{J} \subseteq \mathcal{A}$. Then there is an infinite $I \subseteq \mathbb{N}$ such that $I \setminus A$ is finite for every $A \in \mathcal{A}$.

(b) $2^\kappa \le \mathfrak{c}$ for every $\kappa < \mathfrak{p}$.

(c) Suppose that X is a set and $\#(X) < \mathfrak{p}$. Then there is a countable set $\mathcal{A} \subseteq \mathcal{P}X$ such that $\mathcal{P}X$ is the σ-algebra generated by $\mathcal{A}$.

proof (a) Let P be $[\mathbb{N}]^{<\omega} \times [\mathcal{A}]^{<\omega}$, ordered by saying that $(K,\mathcal{J}) \le (K',\mathcal{J}')$ if $K \subseteq K' \subseteq K \cup \bigcap\mathcal{J}$ and $\mathcal{J} \subseteq \mathcal{J}'$. If $(K,\mathcal{J}) \le (K',\mathcal{J}') \le (K'',\mathcal{J}'')$ then of course $K \subseteq K''$ and $\mathcal{J} \subseteq \mathcal{J}''$; also

$$K'' \subseteq K' \cup \bigcap\mathcal{J}' \subseteq K \cup \bigcap\mathcal{J} \cup \bigcap\mathcal{J}' \subseteq K \cup \bigcap\mathcal{J}.$$

So $\le$ is a partial ordering of P. For any $K \in [\mathbb{N}]^{<\omega}$, $\{(K,\mathcal{J}) : J \in [\mathcal{A}]^{<\omega}\}$ is upwards-centered; so P is σ-centered upwards.

For each $A \in \mathcal{A}$, set $Q_A = \{(K,\mathcal{J}) : (K,\mathcal{J}) \in P,\ A \in \mathcal{J}\}$; since $(K,\mathcal{J}) \le (K,\mathcal{J} \cup \{A\})$ whenever $(K,\mathcal{J}) \in P$, Q_A is cofinal with P. For $n \in \mathbb{N}$, set $Q'_n = \{(K,\mathcal{J}) : (K,\mathcal{J}) \in P,\ K \not\subseteq n\}$. If $(K,\mathcal{J}) \in P$, $\bigcap\mathcal{J}$ must be infinite, and there is an $m \in \mathbb{N} \cap \bigcap\mathcal{J} \setminus n$; now $(K,\mathcal{J}) \le (K \cup \{m\},\mathcal{J}) \in Q'_n$. So Q'_n is cofinal with P.

Because $\max(\omega,\#(\mathcal{A})) < \mathfrak{p}$ (517Ob), there is an upwards-linked $R \subseteq P$ meeting every Q_A and every Q'_n. Set $I = \bigcup\{K : (K,\mathcal{J}) \in R\}$. If $n \in \mathbb{N}$, there is a $(K,\mathcal{J}) \in R \cap Q'_n$; now $K \not\subseteq n$ and $K \subseteq I$, so $I \not\subseteq n$; as n is arbitrary, I is infinite. If $A \in \mathcal{A}$, there is $(K_0,\mathcal{J}_0) \in R \cap Q_A$. **?** If $I \not\subseteq K_0 \cup A$, there is a $(K,\mathcal{J}) \in R$ such that $K \not\subseteq K_0 \cup A$. Now there is a $(K',\mathcal{J}') \in P$ such that $(K,\mathcal{J}) \le (K',\mathcal{J}')$ and $(K_0,\mathcal{J}_0) \le (K',\mathcal{J}')$. But in this case

$$K \subseteq K' \subseteq K_0 \cup \bigcap\mathcal{J}_0 \subseteq K_0 \cup A. \text{ **X**}$$

So $I \setminus A \subseteq K_0$ is finite. As A is arbitrary, we have a suitable I.

(b) We may suppose that κ is infinite. By 515H, or otherwise, there is a Boolean-independent family $\langle J_\xi\rangle_{\xi<\kappa}$ in $\mathcal{P}\mathbb{N}$. Note that $I = \bigcap_{\xi\in K} J_\xi \setminus \bigcup_{\xi\in L} J_\xi$ must be infinite whenever K, $L \subseteq \kappa$ are disjoint finite sets, because $\langle I \cap J_\xi\rangle_{\xi\in\kappa\setminus(K\cup L)}$ is Boolean-independent. For $C \subseteq \kappa$ set

$$\mathcal{A}_C = \{J_\xi : \xi \in C\} \cup \{\mathbb{N} \setminus J_\xi : \xi \in \kappa \setminus C\}.$$

By (a), there is an infinite $I_C \subseteq \mathbb{N}$ such that $I_C \setminus A$ is finite for every $A \in \mathcal{A}_C$. If C, $D \subseteq \kappa$ and $\xi \in C \setminus D$, then $I_C \setminus J_\xi$ and $I_D \cap J_\xi$ are finite, so $I_C \cap I_D$ is finite and $I_C \ne I_D$. Thus $C \mapsto I_C$ is injective and $2^\kappa \le \mathfrak{c}$.

(c) Let $\langle I_x\rangle_{x\in X}$ be a family of infinite subsets of $\mathbb{N}$ such that $I_x \cap I_y$ is finite for all distinct x, $y \in X$ (5A1Ga). Set $A_n = \{x : n \in I_x\}$ for $n \in \mathbb{N}$.

Take any $A \subseteq X$ and set $P_A = \mathrm{Fn}_{<\omega}(\mathbb{N};\{0,1\}) \times [X \setminus A]^{<\omega}$, partially ordered by saying that

$(f,J) \le (f',J')$ if f' extends f, $J' \supseteq J$ and whenever $x \in J$ and $i \in I_x \cap \operatorname{dom} f' \setminus \operatorname{dom} f$, then $f'(i) = 0$.

Then P_A is σ-centered upwards because $\{(f,J) : J \in [X \setminus A]^{<\omega}\}$ is upwards-centered for every $f \in \mathrm{Fn}_{<\omega}(\mathbb{N};\{0,1\})$. For $x \in A$ and $m \in \mathbb{N}$ set

$$Q_{xm} = \{(f,J) : (f,J) \in P_A,\ f(i) = 1 \text{ for some } i \in I_x \setminus m\};$$

for $x \in X \setminus A$ set

$$Q'_x = \{(f,J) : (f,J) \in P_A,\ x \in J\}.$$

Then every Q_{xm} and every Q'_x is cofinal with P_A. Because $\#(X) < \mathfrak{p}$, there is an upwards-directed $R \subseteq P_A$ meeting every Q_{xm} and every Q'_x. Set $L = \bigcup_{(f,J)\in R}\{i : f(i) = 1\}$. Now

— if $x \in A$ and $m \in \mathbb{N}$ then $L \cap I_x \setminus m$ is non-empty, so $L \cap I_x$ is infinite,

— if $x \in X \setminus A$, there is a pair $(f_0,J_0) \in R$ such that $x \in J_0$; now $f(i) = 0$ whenever $(f,J) \in R$ and $i \in \operatorname{dom} f \setminus \operatorname{dom} f_0$, so $L \cap I_x \subseteq \operatorname{dom} f_0$.

Accordingly

$$A=\{x:x\in X,\,I_x\cap L\text{ is infinite}\}=\bigcap_{n\in\mathbb{N}}\bigcup_{m\in L\setminus n}A_m$$

belongs to the σ-algebra generated by $\mathcal{A}=\{A_n:n\in\mathbb{N}\}$, and we have a suitable family.

517S Proposition Let P be a partially ordered set which satisfies Knaster's condition upwards. If $A\subseteq P$ and $\#(A)<\mathfrak{m}_{\text{K}}$, then A can be covered by a sequence of upwards-directed subsets of P.

proof By 516P, the upwards finite-support product P^* of countably many copies of P also satisfies Knaster's condition upwards. So we can use 517Ha.

517X Basic exercises (a) Let P be a partially ordered set and κ a cardinal. Show that the following are equiveridical: (i) $\kappa<\mathfrak{m}^\uparrow(P)$; (ii) whenever $p_0\in P$ and $\mathcal{Q}$ is a family of cofinal subsets of P with $\#(\mathcal{Q})\le\kappa$, there is an upwards-centered subset of P which contains p_0 and meets every member of $\mathcal{Q}$; (iii) whenever $p_0\in P$ and $\mathcal{Q}$ is a family of up-open cofinal subsets of P with $\#(\mathcal{Q})\le\kappa$, there is an upwards-centered subset of P which contains p_0 and meets every member of $\mathcal{Q}$; (iv) whenever $p_0\in P$ and $\mathcal{A}$ is a family of maximal up-antichains in P with $\#(\mathcal{A})\le\kappa$, there is an upwards-centered subset of P which contains p_0 and meets every member of $\mathcal{A}$.

(b) Let P be a partially ordered set and A a maximal up-antichain in P. Show that

$$\mathfrak{m}^\uparrow(P)=\min_{p\in A}\mathfrak{m}^\uparrow([p,\infty[).$$

(c)(i) Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$. For $a\in\mathfrak{A}$ let $\mathfrak{A}_a$ be the corresponding principal ideal. Show that there is an $a\in\mathfrak{A}^+$ such that $\mathfrak{m}(\mathfrak{A}_b)=\mathfrak{m}(\mathfrak{A}_a)$ whenever $0\ne b\subseteq a$. (ii) Show that any Dedekind complete Boolean algebra is isomorphic to a simple product of Martin-number-homogeneous Boolean algebras.

>(d) Let P be a partially ordered set. Show that $\mathfrak{m}^\uparrow(P)=\infty$ iff $\{p:[p,\infty[\text{ is upwards-linked}\}$ is cofinal with P.

(e) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras and $\mathfrak{A}$ its free product; suppose that κ is a regular uncountable cardinal such that $\operatorname{sat}(\mathfrak{A}_i)\le\kappa<\mathfrak{m}(\mathfrak{A}_i)$ for every $i\in I$. Show that $\operatorname{sat}(\mathfrak{A})\le\kappa$.

(f) Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{C}$ the free product of a sequence of copies of $\mathfrak{A}$. Suppose that $\kappa<\mathfrak{m}(\mathfrak{C})$. (i) Show if $A\in[\mathfrak{A}^+]^{\le\kappa}$ then A can be covered by a sequence of centered subsets of $\mathfrak{A}^+$. (ii) Show that if $\operatorname{cf}\kappa\ge\omega_1$ then κ is a precaliber of $\mathfrak{A}$.

(g) Let $\mathfrak{A}$ be a Boolean algebra and Z its Stone space. Show that $\mathfrak{m}(\mathfrak{A})=\min\{n(Y):Y\subseteq Z$ is a non-meager set with the Baire property$\}$.

>(h) Let P be a non-empty partially ordered set and P^* the upwards finite-support product of a sequence of copies of P. Show that if $\mathfrak{m}^\uparrow(P^*)>\omega_1$ then P must be upwards-ccc.

(i) Let X be any topological space. Show that $\mathfrak{m}(\mathrm{RO}(X))\ge\min\{n(G):G\subseteq X$ is a non-empty open set$\}$.

(j) Let X be a locally compact Hausdorff space such that $\mathrm{RO}(X)$ is Martin-number-homogeneous. Show that $\mathfrak{m}(\mathrm{RO}(X))=n(X)$.

(k)(i) Let P be a partially ordered set which is σ-linked upwards. Show that if $A\subseteq P$ and $\#(A)<\mathfrak{m}_{\sigma\text{-linked}}$, then A can be covered by a sequence of upwards-directed subsets of P. (ii) Let P be a partially ordered set such that ω_1 is an up-precaliber of P. Show that if $A\subseteq P$ and $\#(A)<\mathfrak{m}_{\text{pc}\omega_1}$, then A can be covered by a sequence of upwards-directed subsets of P. (iii) Let P be a partially ordered set which is σ-centered upwards. Show that if $A\subseteq P$ and $\#(A)<\mathfrak{p}$, then A can be covered by a sequence of upwards-directed subsets of P.

517Y Further exercises (a) For a partially ordered set P, write A_P for the family of upwards-linked subsets of P, B_P for the family of cofinal subsets of P, and T_P for $\{(R,Q) : R \in A_P,\ Q \in B_P,\ R \cap Q = \emptyset\}$. (i) Show that $\mathfrak{m}^{\uparrow}(P) = \min_{p\in P} \operatorname{cov}(A_{[p,\infty[}, T_{[p,\infty[}, B_{[p,\infty[})$. (ii) Show that if Q is another partially ordered set and there is a relation $S \subseteq P \times Q$ with the properties described in 517C, then for every $q \in Q$ there is a $p \in P$ such that $(A_{[p,\infty[}, T_{[p,\infty[}, B_{[p,\infty[}) \preccurlyeq_{\mathrm{GT}} (A_{[q,\infty[}, T_{[q,\infty[}, B_{[q,\infty[})$.

(b) Show that for every infinite regular cardinal κ there is a partially ordered set with Martin number κ^+.

(c) Show that $\mathfrak{m}(\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}) \ge \mathfrak{p}$.

(d) (A.Szymański, 1981) (i) Suppose that $\mathcal{A}$, $\mathcal{B} \in [\mathcal{P}\mathbb{N}]^{<\mathfrak{p}}$, $\mathcal{A}$ is downwards-directed and $A \cap B$ is infinite for all $A \in \mathcal{A}$, $B \in \mathcal{B}$. Show that there is a set $D \subseteq \mathbb{N}$ such that $D \setminus A$ is finite for every $A \in \mathcal{A}$ and $D \cap B$ is infinite for every $B \in \mathcal{B}$. (ii) Show that $\mathfrak{p}$ is regular.

517 Notes and comments The study of 'Martin numbers' is a natural extension of investigations into consequences of Martin's axiom. Most of the results here are straightforward expressions of techniques developed for deducing consequences from $\mathfrak{m} = \mathfrak{c}$ or $\mathfrak{m} > \omega_1$. In particular, 517F, 517G and 517H correspond to the now-classical theorems that if $\mathfrak{m} > \omega_1$ then ω_1 is a precaliber of every ccc partially ordered set, the product of any family of ccc topological spaces is ccc, and a ccc partially ordered set with cardinal ω_1 is a countable union of directed sets (see Fremlin 84a, §41). For those familiar with the use of Martin's axiom there are no surprises here, though some refinements in the arguments are necessary. The cardinal $\mathfrak{m}_{\text{countable}}$ is probably most commonly known as the Novák number of $\mathbb{R}$ (517Pd), the covering number of the ideal of meager subsets of $\mathbb{R}$. In countable partially ordered sets, most of the arguments above short-circuit to some degree; precalibers become trivial, finite-support products are automatically ccc, and directed sets have cofinal totally ordered subsets, so that the ideas take on new colours.

In Fremlin 84a I found that focusing on the cardinals $\mathfrak{p}$, $\mathfrak{m}_{\mathrm{K}}$ and $\mathfrak{m}$ broke the arguments up into reasonably balanced chapters. Within the chapter on $\mathfrak{m}_{\mathrm{K}}$, however, there is a natural division between arguments applying to $\mathfrak{m}_{\mathrm{pc}\omega_1}$ and those applying to $\mathfrak{m}_{\sigma\text{-linked}}$, which in the present book I intend to make explicit. The notation $\mathfrak{p}$ is the standard name for the cardinal $\mathfrak{m}_{\sigma\text{-centered}}$; its special position comes in part from the fact that it had been studied under a different, combinatorial, definition for a decade before M.G.Bell showed that it could also be described by the definition here (Bell 81, or Fremlin 84a, 14C).

Version of 24.12.14

518 Freese-Nation numbers

I run through those elements of the theory of Freese-Nation numbers, as developed by S.Fuchino, S.Geschke, S.Koppelberg, S.Shelah and L.Soukup, which seem relevant to questions concerning measure spaces and measure algebras. The first part of the section (518A-518K) examines the calculation of Freese-Nation numbers of familiar partially ordered sets and Boolean algebras. In 518L-518S I look at 'tight filtrations', which are of interest to us because of their use in lifting theorems (518L, §535).

For the definitions of 'Freese-Nation number' and 'Freese-Nation index' see 511Bi and 511Dh.

518A Proposition (Fuchino Koppelberg & Shelah 96) Let P be a partially ordered set.

(a) $\mathrm{FN}(P) \le \max(3, \#(P))$.

(b) $\mathrm{FN}(P, \ge) = \mathrm{FN}(P, \le)$.

(c) If P has no maximal element, then $\operatorname{add} P \le \mathrm{FN}(P)$.

proof (a)(i) Suppose first that P is finite and totally ordered. If $\#(P) \le 2$, set $f(p) = P$ for every $p \in P$. Otherwise, take $p_0 \in P$ such that $]-\infty, p_0[$ and $]p_0, \infty[$ are both non-empty, and set $f(p) = [p_0, \infty[$ if $p \ge p_0$, $]-\infty, p_0]$ if $p < p_0$; then f is a Freese-Nation function witnessing that $\mathrm{FN}(P) \le \#(P)$.

(ii) Next suppose that P is finite and not totally ordered. For $p \in P$ set $A_p =]-\infty, p] \cup [p, \infty[$, and take $B = \{p : A_p = P\}$; then $B \ne P$. Set $f(p) = A_p$ for $p \in P \setminus B$, B for $p \in B$; then f is a Freese-Nation function so again $\mathrm{FN}(P) \le \#(P)$.

(iii) If P is infinite, enumerate it as $\langle p_\xi\rangle_{\xi<\#(P)}$ and set $f(p_\xi)=\{p_\eta:\eta\le\xi\}$ for each ξ; once more we have a Freese-Nation function witnessing that $\mathrm{FN}(P)\le\#(P)$.

(b) A function $f:P\to\mathcal{P}P$ is a Freese-Nation function for $\le$ iff it is a Freese-Nation function for the reverse ordering $\ge$.

(c) Set $\kappa=\mathrm{FN}(P)$. Then we have a Freese-Nation function $f:P\to[P]^{<\kappa}$.

(i) I had better sort out the trivial cases. If P is empty, then $\kappa=\operatorname{add}P=0$. Otherwise, $p\in f(p)$ for every $p\in P$, so $\kappa\ge 2$; if $\operatorname{add}P\le 2$ we can stop. So we may suppose that $\operatorname{add}P>2$, that is, that P is upwards-directed.

(ii) ? If $\kappa<\operatorname{add}P$, choose $\langle p_\xi\rangle_{\xi\le\kappa}$ inductively, as follows. Given $\langle p_\eta\rangle_{\eta<\xi}$, where $\xi\le\kappa$, then $\bigcup_{\eta<\xi}f(p_\eta)$ has an upper bound p'_ξ in P. **P** If κ is infinite, this is because $\#(\bigcup_{\eta<\xi}f(p_\eta))\le\kappa<\operatorname{add}P$. If κ is finite, it is because $\#(\bigcup_{\eta<\xi}f(p_\eta))<\omega\le\operatorname{add}P$. **Q**

As P has no maximal element, we can find $p_\xi>p'_\xi$, and continue. At the end of the induction, we have $p_\xi<p_\kappa$, so there is a $q_\xi\in f(p_\xi)\cap f(p_\kappa)\cap[p_\xi,p_\kappa]$, for each $\xi<\kappa$. If $\eta<\xi<\kappa$, then

$$q_\eta\le p'_\xi<p_\xi\le q_\xi$$

and $q_\eta\ne q_\xi$. But this means that $f(p_\kappa)\supseteq\{q_\xi:\xi<\kappa\}$ has at least κ elements. **X**

518B Proposition Let P be a partially ordered set and Q a subset of P.

(a) If Q is order-convex (that is, $[q,q']\subseteq Q$ whenever $q,\ q'\in Q$), then $\mathrm{FN}(Q)\le\mathrm{FN}(P)$.

(b) If Q is a retract of P (that is, there is an order-preserving $h:P\to Q$ such that $h(q)=q$ for every $q\in Q$), then $\mathrm{FN}(Q)\le\mathrm{FN}(P)$.

(c) If Q is, in itself, Dedekind complete (that is, every non-empty subset of Q with an upper bound in Q has a supremum in Q for the induced ordering), then $\mathrm{FN}(Q)\le\mathrm{FN}(P)$.

proof (a) If $f:P\to\mathcal{P}P$ is a Freese-Nation function on P, then $q\mapsto Q\cap f(q):Q\to\mathcal{P}Q$ is a Freese-Nation function on Q.

(b) If f is a Freese-Nation function on P, then $q\mapsto h[f(q)]$ is a Freese-Nation function on Q.

(c) Set $Q_1=\bigcup_{q,q'\in Q}[q,q']$, so that Q_1 is an order-convex subset of P and $\mathrm{FN}(Q_1)\le\mathrm{FN}(P)$. For $p\in Q_1$, set $h(p)=\sup(Q\cap\,]-\infty,p])$, the supremum being taken in Q; then $h:Q_1\to Q$ is a retraction, so $\mathrm{FN}(Q)\le\mathrm{FN}(Q_1)$.

518C Corollary (a) If $\mathfrak{A}$ is an infinite Dedekind σ-complete Boolean algebra then $\mathrm{FN}(\mathfrak{A})\ge\mathrm{FN}(\mathcal{P}\mathbb{N})$.

(b) Let $\mathfrak{A}$ be an infinite Dedekind complete Boolean algebra. Then

$$\mathrm{FN}(\mathrm{RO}(\{0,1\}^{\#(\mathfrak{A})}))\le\mathrm{FN}(\mathfrak{A})\le\mathrm{FN}(\mathcal{P}(\mathrm{link}(\mathfrak{A})))\le\max(3,2^{\mathrm{link}(\mathfrak{A})}).$$

(c) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra. If $\mathfrak{B}$ is either an order-closed subalgebra or a principal ideal of $\mathfrak{A}$, then $\mathrm{FN}(\mathfrak{B})\le\mathrm{FN}(\mathfrak{A})$.

proof (a) Take any disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$; then $I\mapsto\sup_{n\in I}a_n$ is an embedding of the partially ordered set $\mathcal{P}\mathbb{N}$ into $\mathfrak{A}$. As $\mathcal{P}\mathbb{N}$ is Dedekind complete, 518Bc tells us that $\mathrm{FN}(\mathcal{P}\mathbb{N})\le\mathrm{FN}(\mathfrak{A})$.

(b)(i) By 515I, $\mathfrak{A}$ has a subalgebra $\mathfrak{B}$ isomorphic to the regular open algebra $\mathrm{RO}(\{0,1\}^{\#(\mathfrak{A})})$; by 518Bc, $\mathrm{FN}(\mathfrak{B})\le\mathrm{FN}(\mathfrak{A})$.

(ii) By 514Cb, we have a subset Q of $\mathcal{P}(\mathrm{link}(\mathfrak{A}))$ which is order-isomorphic to $\mathfrak{A}$, and 518Bc tells us that $\mathrm{FN}(Q)\le\mathrm{FN}(\mathcal{P}(\mathrm{link}(\mathfrak{A})))$.

(iii) By 518Aa, $\mathrm{FN}(\mathcal{P}(\mathrm{link}(\mathfrak{A})))\le 2^{\mathrm{link}(\mathfrak{A})}$ except in the trivial case $\mathfrak{A}=\{0,1\}$.

(c) Immediate from 518Bc.

518D Corollary The following sets all have the same Freese-Nation number:
(i) $\mathcal{P}\mathbb{N}$;
(ii) $\mathbb{N}^{\mathbb{N}}$, with its usual ordering $\le$;
(iii) any infinite σ-linked Dedekind complete Boolean algebra;
(iv) the family of open subsets of any infinite Hausdorff second-countable topological space.

proof (a) The map $I \mapsto \chi I : \mathcal{P}\mathbb{N} \to \mathbb{N}^{\mathbb{N}}$ is an order-preserving embedding; because $\mathcal{P}\mathbb{N}$ is Dedekind complete, $\mathrm{FN}(\mathcal{P}\mathbb{N}) \le \mathrm{FN}(\mathbb{N}^{\mathbb{N}})$ (518Bc).

(b) The map $f \mapsto \{(i,j) : j \le f(i)\} : \mathbb{N}^{\mathbb{N}} \to \mathcal{P}(\mathbb{N}\times\mathbb{N})$ is an order-preserving embedding; because $\mathbb{N}^{\mathbb{N}}$ is Dedekind complete,

$$\mathrm{FN}(\mathbb{N}^{\mathbb{N}}) \le \mathrm{FN}(\mathcal{P}(\mathbb{N}\times\mathbb{N})) = \mathrm{FN}(\mathcal{P}\mathbb{N}).$$

(c) Now let $\mathfrak{A}$ be an infinite σ-linked Dedekind complete Boolean algebra. By 518Ca, $\mathrm{FN}(\mathcal{P}\mathbb{N}) \le \mathrm{FN}(\mathfrak{A})$; by 518Cb, $\mathrm{FN}(\mathfrak{A}) \le \mathrm{FN}(\mathcal{P}\mathbb{N})$.

(d) Let $(X,\mathfrak{T})$ be an infinite Hausdorff second-countable space. (α) Because X is Hausdorff and infinite, it has a disjoint sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ of non-empty open sets; now $I \mapsto \bigcup_{n\in I} G_n$ is an embedding of $\mathcal{P}\mathbb{N}$ in $\mathfrak{T}$, so $\mathrm{FN}(\mathcal{P}\mathbb{N}) \le \mathrm{FN}(\mathfrak{T})$. (β) Let $\mathcal{U}$ be a countable base for $\mathfrak{T}$. Then $G \mapsto \{U : U \in \mathcal{U}, U \subseteq G\}$ is an embedding of $\mathfrak{T}$ in $\mathcal{P}\mathcal{U}$; as $\mathfrak{T}$, regarded as a partially ordered set, is Dedekind complete, $\mathrm{FN}(\mathfrak{T}) \le \mathrm{FN}(\mathcal{P}\mathcal{U}) = \mathrm{FN}(\mathcal{P}\mathbb{N})$.

518E There is a simple result in general topology which will be used a couple of times in the next chapter.

Lemma Let $(X,\mathfrak{T})$ be a T_1 topological space without isolated points, and $\mathcal{N}\mathit{wd}(X)$ the ideal of nowhere dense sets. Then there is a set $A \subseteq X$, with cardinal $\operatorname{cov}\mathcal{N}\mathit{wd}(X)$, such that $\#(A\cap F) < \mathrm{FN}^*(\mathfrak{T})$ for every $F \in \mathcal{N}\mathit{wd}(X)$.

Remark Perhaps I should say here that $\mathrm{FN}^*(\mathfrak{T})$ is the regular Freese-Nation number of the partially ordered set $(\mathfrak{T},\subseteq)$.

proof As X has no isolated points, $\operatorname{cov}\mathcal{N}\mathit{wd}(X) \le \#(X)$. Set $\kappa = \operatorname{cov}\mathcal{N}\mathit{wd}(X)$ and $\lambda = \mathrm{FN}^*(\mathfrak{T})$. If $\kappa < \lambda$ the result is trivial and we can stop. Otherwise, let $f : \mathfrak{T} \to [\mathfrak{T}]^{<\lambda}$ be a Freese-Nation function. Then we can choose $\langle x_\xi\rangle_{\xi<\kappa}$ inductively so that whenever $\eta < \xi$ and $G \in f(X\setminus\{x_\eta\})$ is dense, then $x_\xi \in G$. **P** When we come to choose x_ξ, set $\theta = \#(\bigcup_{\eta<\xi}\{G : G \in f(X\setminus\{x_\eta\}) \text{ is dense}\})$. If $\lambda < \kappa$ then $\theta \le \max(\#(\xi),\omega,\lambda) < \kappa$. If $\lambda = \kappa$ then κ is regular and infinite and $\#(f(X\setminus\{x_\eta\}) < \kappa$ for every $\eta < \xi$ so again $\theta < \kappa$. So we have fewer than $\operatorname{cov}\mathcal{N}\mathit{wd}(X)$ dense open sets and can find a point x_ξ in all of them. **Q**

Note that as $X\setminus\{x_\eta\}$ is itself dense for every $\eta < \xi$, and $H \in f(H)$ for every $H \in \mathfrak{T}$, all the x_ξ must be distinct, and $A = \{x_\xi : \xi < \kappa\}$ has cardinal κ. Now suppose that $F \in \mathcal{N}\mathit{wd}(X)$ and set $B = \{\xi : \xi < \kappa, x_\xi \in F\}$. For each $\xi \in B$, $X\setminus\overline{F} \subseteq X\setminus\{x_\xi\}$, so there is a $G_\xi \in f(X\setminus\overline{F}) \cap f(X\setminus\{x_\xi\})$ such that $X\setminus\overline{F} \subseteq G_\xi$ and $x_\xi \notin G_\xi$. If η, $\xi \in B$ and $\eta < \xi$, then $G_\eta \in f(X\setminus\{x_\eta\})$ is dense, so contains x_ξ, and cannot be equal to G_ξ. Thus $\xi \mapsto G_\xi$ is an injective function from B to $f(X\setminus\overline{F})$, and $\#(B) < \lambda$. Thus we have an appropriate set A.

518F Lemma Let $\mathfrak{A}$ be a Boolean algebra, $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$ and κ an infinite cardinal.

(a) If $\mathrm{cf}(\mathfrak{B}\cap[0,a]) < \kappa$ for every $a \in \mathfrak{A}$, then the Freese-Nation index of $\mathfrak{B}$ in $\mathfrak{A}$ is at most κ.

(b) Suppose that $I \in [\mathfrak{A}]^{<\mathrm{cf}\,\kappa}$ and $\mathfrak{B}_I$ is the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{B}\cup I$. If the Freese-Nation index of $\mathfrak{B}$ in $\mathfrak{A}$ is less than or equal to κ, so is the Freese-Nation index of $\mathfrak{B}_I$.

(c) If $\mathfrak{B}$ is expressible as the union of fewer than κ order-closed subalgebras of $\mathfrak{A}$, each of them Dedekind complete in itself, then the Freese-Nation index of $\mathfrak{B}$ in $\mathfrak{A}$ is at most κ.

proof (a) For any $a \in \mathfrak{A}$,

$$\mathrm{ci}(\mathfrak{B}\cap[a,1]) = \mathrm{cf}(\mathfrak{B}\cap[0,1\setminus a]) < \kappa.$$

(b)(i) Suppose first that I is a singleton $\{d\}$. In this case

$$\mathfrak{B}_I = \{(b\cap d)\cup(c\setminus d) : b, c \in \mathfrak{B}\}.$$

Take $a \in \mathfrak{A}$. Then there are sets B, $C \subseteq \mathfrak{B}$, with cardinal less than κ, which are cofinal in

$$\mathfrak{B} \cap [0, a \cup (1 \setminus d)], \quad \mathfrak{B} \cap [0, a \cup d]$$

respectively. Set $D = \{b \cap d : b \in B\} \cup \{c \setminus d : c \in C\}$, so that $D \subseteq \mathfrak{B}_I \cap [0, a]$ and $\#(D) < \kappa$. If b, $c \in \mathfrak{B}$ and $(b \cap d) \cup (c \setminus d) \subseteq a$, then $b \subseteq a \cup (1 \setminus d)$ and $c \subseteq a \cup d$, so there are $b' \in B$, $c' \in C$ such that $b \subseteq b'$ and $c \subseteq c'$; now

$$(b \cap d) \cup (c \setminus d) \subseteq (b' \cap d) \cup (c' \setminus d) \in D.$$

Thus D witnesses that $\mathrm{cf}(\mathfrak{C} \cap [0, a]) < \kappa$. By (a), this is enough to show that the Freese-Nation index of $\mathfrak{B}_I$ in $\mathfrak{A}$ is at most κ.

(ii) An elementary induction now shows that the Freese-Nation index of $\mathfrak{B}_I$ in $\mathfrak{A}$ is at most κ for every finite subset I of $\mathfrak{A}$. If $\omega \leq \#(I) < \mathrm{cf}\,\kappa$ and $a \in \mathfrak{A}$, then $\mathfrak{B}_I = \bigcup\{\mathfrak{B}_J : J \in [I]^{<\omega}\}$. For each $J \in [I]^{<\omega}$, let B_J be a cofinal subset of $\mathfrak{B}_J \cap [0, a]$ with cardinal less than κ. Then $B = \bigcup\{B_J : J \in [I]^{<\omega}\}$ is cofinal in $\mathfrak{B}_I \cap [0, a]$; and as $\#([I]^{<\omega}) < \mathrm{cf}\,\kappa$, $\#(B) < \kappa$. So again we have $\mathrm{cf}(\mathfrak{B}_I \cap [0, a]) < \kappa$ for every $a \in \mathfrak{A}$, and the Freese-Nation index of $\mathfrak{B}_I$ in $\mathfrak{A}$ is at most κ.

(c) Suppose that $\langle \mathfrak{B}_\xi \rangle_{\xi<\lambda}$ is a family of order-closed subalgebras with union $\mathfrak{B}$, where $\lambda < \kappa$. If $a \in \mathfrak{A}$, then $b_\xi = \sup(\mathfrak{B}_\xi \cap [0, a])$ is defined in $\mathfrak{B}_\xi$, and belongs to $[0, a]$, for each $\xi < \lambda$, and $\{b_\xi : \xi < \lambda\}$ is cofinal with $\mathfrak{B} \cap [0, a]$; so we can apply (a).

518G Lemma (Fuchino Koppelberg & Shelah 96) Let P be a partially ordered set, ζ an ordinal, and $\langle A_\xi \rangle_{\xi<\zeta}$ a family with union P; set $P_\alpha = \bigcup_{\xi<\alpha} A_\xi$ for each $\alpha \leq \zeta$. Let κ be a regular infinite cardinal such that, for each $\alpha < \zeta$, $\mathrm{FN}(P_{\alpha+1}) \leq \kappa$ and the Freese-Nation index of P_α in $P_{\alpha+1}$ is at most κ. Then $\mathrm{FN}(P) \leq \kappa$.

proof For each $\alpha < \zeta$ set $A'_\alpha = A_\alpha \setminus P_\alpha$ and choose a Freese-Nation function $f_\alpha : P_{\alpha+1} \to [P_{\alpha+1}]^{<\kappa}$. For $p \in P$, let $\gamma(p)$ be that $\alpha < \zeta$ such that $p \in A'_\alpha$, and let $D_p \subseteq P_{\gamma(p)}$ be a set with cardinal less than κ such that $D_p \cap]-\infty, p]$ is cofinal with $P_{\gamma(p)} \cap]-\infty, p]$ and $D_p \cap [p, \infty[$ is cofinal with $P_{\gamma(p)} \cap [p, \infty[$. Define g inductively, on each A'_α in turn, by setting $g(p) = f_{\gamma(p)}(p) \cup \bigcup_{q \in D_p} g(q)$ for every $p \in P$. Because κ is regular, g is a function from P to $[P]^{<\kappa}$.

Now g is a Freese-Nation function on P. **P** I induce on α to show that if p, $q \in P$ and $\max(\gamma(p), \gamma(q)) = \alpha$ then $g(p) \cap g(q) \cap [p, q] \neq \emptyset$. For the inductive step to $\alpha < \zeta$, if $\gamma(p) = \gamma(q) = \alpha$ then

$$g(p) \cap g(q) \cap [p, q] \supseteq f_\alpha(p) \cap f_\alpha(q) \cap [p, q] \neq \emptyset.$$

If $\gamma(p) < \gamma(q) = \alpha$, then there is an $r \in D_q$ such that $p \leq r \leq q$; now $\max(\gamma(p), \gamma(r)) < \alpha$, so

$$g(p) \cap g(q) \cap [p, q] \supseteq g(p) \cap g(r) \cap [p, r] \neq \emptyset$$

by the inductive hypothesis. The same argument works if $\gamma(q) < \gamma(p)$. **Q**

518H Lemma Suppose that κ is an uncountable cardinal of countable cofinality such that $\square_\kappa$ is true and $\mathrm{cf}[\lambda]^{\leq\omega} \leq \lambda^+$ for every $\lambda \leq \kappa$. Then there are families $\langle M_{\alpha n} \rangle_{\alpha<\kappa^+, n\in\mathbb{N}}$, $\langle M_\alpha \rangle_{\alpha<\kappa^+}$ of sets and a function sk such that

(i) $\#(M_{\alpha n}) < \kappa$ whenever $\alpha < \kappa^+$ and $n \in \mathbb{N}$;

(ii) $\langle M_{\alpha n} \rangle_{n\in\mathbb{N}}$ is non-decreasing for each $\alpha < \kappa^+$;

(iii) $\langle M_\alpha \rangle_{\alpha<\kappa^+}$ is a non-decreasing family, $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for every non-zero limit ordinal $\alpha < \kappa^+$, and $\kappa^+ \subseteq \bigcup_{\alpha<\kappa^+} M_\alpha$;

(iv) if $\alpha < \kappa^+$ has uncountable cofinality, $M_\alpha = \bigcup_{n\in\mathbb{N}} M_{\alpha n}$;

(v) $X \subseteq \mathrm{sk}(X)$ for every set X;

(vi) $\mathrm{sk}(X)$ is countable whenever X is countable;

(vii) $A \subseteq \mathrm{sk}(X)$ whenever $A \in \mathrm{sk}(X)$ is countable;

(viii) $\mathrm{sk}(X) \subseteq \mathrm{sk}(Y)$ whenever $X \subseteq \mathrm{sk}(Y)$;

(ix) for every $\alpha < \kappa^+$ of uncountable cofinality there is an $m \in \mathbb{N}$ such that whenever $n \geq m$ and $A \subseteq M_{\alpha n}$ is countable there is a countable set $D \in M_{\alpha n}$ such that $A \subseteq \mathrm{sk}(D)$;

(x) $\bigcup_{\alpha<\kappa^+} M_\alpha \cap [\kappa]^{\leq\omega}$ is cofinal with $[\kappa]^{\leq\omega}$.

proof (a) There is a strictly increasing sequence $\langle\kappa_n\rangle_{n\in\mathbb{N}}$ of infinite cardinals with supremum κ; since

$$\kappa_n^+ \le \mathrm{cf}[\kappa_n^+]^{\le\omega} \le \max(\kappa_n^+, \mathrm{cf}[\kappa_n]^{\le\omega}) = \kappa_n^+$$

for each n (5A1F(e-iv)), we can suppose that in fact $\mathrm{cf}[\kappa_n]^{\le\omega} = \kappa_n$ for every n. Take $\langle C_\alpha\rangle_{\alpha<\kappa^+}$ witnessing $\square_\kappa$, so that

for every $\alpha<\kappa^+$, $C_\alpha \subseteq \alpha$ is a closed cofinal set in α of order type at most κ,

whenever $\delta<\alpha<\kappa^+$ and $\delta = \sup(\delta\cap C_\alpha)$, then $C_\delta = \delta\cap C_\alpha$

(5A6D(a-ii)). For $\alpha<\kappa^+$ set

$$C'_\alpha = \{\delta : \delta<\alpha,\, \delta=\sup(\delta\cap C_\alpha)\} \subseteq C_\alpha$$

and

$$C'_{\alpha n} = \{\delta : \delta\in C'_\alpha,\, \mathrm{otp}(\delta\cap C_\alpha)<\kappa_n\}$$

for each n. Because $\mathrm{otp}(C_\alpha)\le\kappa$, $C'_\alpha = \bigcup_{n\in\mathbb{N}} C'_{\alpha n}$, while $\#(C'_{\alpha n})\le\kappa_n$ for each n. Note that if α has uncountable cofinality, C'_α will be cofinal with α.

(b) Let $g:\kappa^+\to[\kappa]^{\le\omega}$ be such that $g[\kappa^+]$ is cofinal with $[\kappa]^{\le\omega}$. For each non-zero $\alpha<\kappa^+$, fix on a surjective function $f_\alpha:\kappa\to\alpha$. For each $n\in\mathbb{N}$, let $g_n:\kappa_n\to[\kappa_n]^{\le\omega}$ be such that $g_n[\kappa_n]$ is cofinal with $[\kappa_n]^{\le\omega}$. For each $\alpha<\kappa^+$, let $h_\alpha:\#(C_\alpha)\to C_\alpha$ be a bijection. Now, for any set X, write $\mathrm{sk}(X)$ for the smallest set including X and such that

$g(\alpha)\in\mathrm{sk}(X)$ whenever $\alpha\in\mathrm{sk}(X)\cap\kappa^+$,

$f_\alpha(\xi)\in\mathrm{sk}(X)$ whenever $0<\alpha<\kappa^+$, $\xi<\kappa$ and α, $\xi\in\mathrm{sk}(X)$,

$g_n(\xi)\in\mathrm{sk}(X)$ whenever $n\in\mathbb{N}$ and $\xi\in\kappa_n\cap\mathrm{sk}(X)$,

$h_\alpha[A]\in\mathrm{sk}(X)$ whenever $\alpha\in\mathrm{sk}(X)\cap\kappa^+$ and $A\in\mathrm{sk}(X)$,

$A\cup B\in\mathrm{sk}(X)$ whenever A, $B\in\mathrm{sk}(X)$,

$A\subseteq\mathrm{sk}(X)$ whenever $A\in\mathrm{sk}(X)$ is countable.

Of course we always have

$$\mathrm{sk}(\mathrm{sk}(X)) = \mathrm{sk}(X) = \bigcup\{\mathrm{sk}(I) : I\in[X]^{<\omega}\}$$

and $\#(\mathrm{sk}(X))\le\max(\omega,\#(X))$, so (v)-(viii) are all true.

(c) For each $\alpha<\kappa^+$ and $n\in\mathbb{N}$, set $M_{\alpha n} = \mathrm{sk}(\kappa_n\cup C'_{\alpha n})$ and $M_\alpha = \mathrm{sk}(\kappa\cup\alpha)$. Then

$$\#(M_{\alpha n})\le\max(\omega,\kappa_n,\#(C'_{\alpha n}))<\kappa.$$

Also $\langle M_\alpha\rangle_{\alpha<\kappa^+}$ is non-decreasing and $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ whenever $\alpha<\kappa$ is a non-zero limit ordinal, while $\kappa^+\subseteq\bigcup_{\alpha<\kappa^+} M_\alpha$. This deals with (i)-(iii).

(d) Now for (iv): $M_\alpha = \bigcup_{n\in\mathbb{N}} M_{\alpha n}$ whenever $\alpha<\kappa^+$ has uncountable cofinality. **P** Of course $M_{\alpha n}\subseteq M_\alpha$ for every n just because $\mathrm{sk}(X)\subseteq\mathrm{sk}(Y)$ whenever $X\subseteq Y$. On the other hand, if $\beta<\alpha$, take $\delta\in C'_\alpha$ such that $\beta<\delta$, and $\xi<\kappa$ such that $f_\delta(\xi)=\beta$; then if $n\in\mathbb{N}$ is such that $\xi<\kappa_n$ and $\mathrm{otp}(\delta\cap C_\alpha)<\kappa_n$, β will be in $M_{\alpha n}$. So $\bigcup_{n\in\mathbb{N}} M_{\alpha_n}\supseteq\alpha$. Moreover,

$$\kappa = \bigcup_{n\in\mathbb{N}}\kappa_n \subseteq \bigcup_{n\in\mathbb{N}} M_{\alpha n}.$$

So $\alpha\cup\kappa\subseteq\bigcup_{n\in\mathbb{N}} M_{\alpha n}$ and M_α must be exactly $\bigcup_{n\in\mathbb{N}} M_{\alpha n}$. **Q**

(e) Again suppose that $\alpha<\kappa^+$ has uncountable cofinality. Then there must be an $m\in\mathbb{N}$ such that $C'_{\alpha m}$ is cofinal with α. Suppose that $n\ge m$ and $A\subseteq M_{\alpha n}$ is countable. Then there must be a countable set $C\subseteq\kappa_n\cup C'_{\alpha n}$ such that $A\subseteq\mathrm{sk}(C)$. Let $\delta\in C'_{\alpha m}$ be such that $C\cap\alpha\subseteq\delta$. Then $C_\delta=\delta\cap C_\alpha$ has cardinal less than $\kappa_m\le\kappa_n$, so $(C\cap\kappa_n)\cup h_\delta^{-1}[C]$ is a countable subset of κ_n and is included in $g_n(\xi)$ for some $\xi<\kappa_n$. Now ξ and δ belong to $M_{\alpha n}$, so $g_n(\xi)$ and $h_\delta[g_n(\xi)]$ and $D = g_n(\xi)\cup h_\delta[g_n(\xi)]$ all belong to $M_{\alpha n}$, and are countable. But $C\subseteq D$, so $A\subseteq\mathrm{sk}(D)$, as required by (ix).

(f) Finally, if $A\subseteq\kappa$ is countable, there is a $\beta<\kappa^+$ such that $g(\beta)\supseteq A$, and now $g(\beta)\in M_{\beta+1}$. So (x) is true.

518I Theorem (Fuchino & Soukup 97) Let $\mathfrak{A}$ be a ccc Dedekind complete Boolean algebra. Suppose that

(α) $\mathrm{cf}[\lambda]^{\le\omega} \le \lambda^+$ for every cardinal $\lambda \le \tau(\mathfrak{A})$,

(β) $\square_\lambda$ is true for every uncountable cardinal $\lambda \le \tau(\mathfrak{A})$ of countable cofinality.

Let $\mathfrak{A}$ be a ccc Dedekind complete Boolean algebra, and κ a regular uncountable cardinal such that $\mathrm{FN}(\mathfrak{B}) \le \kappa$ for every countably generated order-closed subalgebra $\mathfrak{B}$ of $\mathfrak{A}$. Then $\mathrm{FN}(\mathfrak{A}) \le \kappa$.

proof Induce on the Maharam type $\tau(\mathfrak{A})$ of $\mathfrak{A}$.

(a) If $\tau(\mathfrak{A}) \le \omega$ the result is trivial.

(b) For the inductive step to $\tau(\mathfrak{A}) = \lambda$, where λ is an infinite cardinal of uncountable cofinality, let $\langle a_\xi \rangle_{\xi<\lambda}$ enumerate a τ-generating subset of $\mathfrak{A}$. For each $\beta < \lambda$, let $\mathfrak{B}_\beta$ be the order-closed subalgebra of $\mathfrak{A}$ generated by $\{a_\xi : \xi < \beta\}$, and for $\alpha \le \lambda$ set $\mathfrak{A}_\alpha = \bigcup_{\beta<\alpha} \mathfrak{B}_\beta$. By the inductive hypothesis, $\mathrm{FN}(\mathfrak{B}_\beta) \le \kappa$ for every $\beta < \lambda$. Also, for $\alpha < \kappa$, either α has uncountable cofinality, in which case (because $\mathfrak{A}$ is ccc) $\mathfrak{A}_\alpha = \mathfrak{B}_\alpha$ is order-closed, or α has countable cofinality, in which case $\mathfrak{A}_\alpha$ is a countable union of order-closed subalgebras. In either case, the Freese-Nation index of $\mathfrak{A}_\alpha$ in $\mathfrak{A}_{\alpha+1}$ is countable (518Fc). Because $\mathrm{cf}\,\lambda > \omega$, $\mathfrak{A} = \mathfrak{A}_\lambda$. By 518G, $\mathrm{FN}(\mathfrak{A}) \le \kappa$.

(c)(i) For the inductive step to $\tau(\mathfrak{A}) = \lambda$, where λ is an uncountable cardinal of countable cofinality, we may use the method of Lemma 518H to construct $\langle M_{\alpha n} \rangle_{\alpha<\lambda^+, n\in\mathbb{N}}$, $\langle M_\alpha \rangle_{\alpha<\lambda^+}$ and sk as described there. Enumerate a τ-generating set in $\mathfrak{A}$ as $\langle a_\xi \rangle_{\xi<\lambda}$, and for any set X write $\mathfrak{B}_X$ for the order-closed subalgebra of $\mathfrak{A}$ generated by $\{a_\xi : \xi \in X \cap \lambda\}$. For each $\alpha < \lambda^+$ set $\mathfrak{C}_\alpha = \bigcup\{\mathfrak{B}_{\mathrm{sk}(D)} : D \in M_\alpha \text{ is countable}\}$.

(ii) If $\alpha < \lambda^+$ has uncountable cofinality, then $\mathfrak{C}_\alpha$ is the union of a non-decreasing sequence of order-closed subalgebras of $\mathfrak{A}$ with Maharam type less than λ. **P** By (ix) of 518H, there is an $m \in \mathbb{N}$ such that whenever $n \ge m$ and $\mathcal{D} \subseteq M_{\alpha n}$ is countable there is a countable set $F \in M_{\alpha n}$ such that $\mathcal{D} \subseteq \mathrm{sk}(F)$. For each $n \ge m$, set $\mathfrak{C}_n = \bigcup\{\mathfrak{B}_{\mathrm{sk}(D)} : D \in M_{\alpha n} \text{ is countable}\}$. Then for any countable set $C \subseteq \mathfrak{C}_n$, there is a countable set $\mathcal{D}$ of countable sets belonging to $M_{\alpha n}$ such that $C \subseteq \bigcup_{D\in\mathcal{D}} \mathfrak{B}_{\mathrm{sk}(D)}$. So there is a countable set $F \in M_{\alpha n}$ such that $\mathcal{D} \subseteq \mathrm{sk}(F)$; by 518H(vii), $D \subseteq \mathrm{sk}(F)$ and $\mathfrak{B}_{\mathrm{sk}(D)} \subseteq \mathfrak{B}_{\mathrm{sk}(F)}$ (518H(viii)) for every $D \in \mathcal{D}$. But this means that $C \subseteq \mathfrak{B}_{\mathrm{sk}}(F) \subseteq \mathfrak{C}_n$, while $\mathfrak{B}_{\mathrm{sk}}(F)$ is an order-closed subalgebra of $\mathfrak{A}$. Because $\mathfrak{A}$ is ccc, this is enough to show that $\mathfrak{C}_n$ is an order-closed subalgebra of $\mathfrak{A}$; by 518H(ii) and 518H(iv), $\langle \mathfrak{C}_n \rangle_{n\ge m}$ is non-decreasing and has union $\mathfrak{C}_\alpha$. Each $\mathfrak{C}_n$ is τ-generated by

$$\{a_\eta : \text{there is a countable } D \in M_{\alpha n} \text{ such that } \eta \in \mathrm{sk}(D) \cap \lambda^+\},$$

so (using 518H(vi))

$$\tau(\mathfrak{C}_n) \le \max(\omega, \#(M_{\alpha n})) < \lambda. \textbf{ Q}$$

It follows from 518Fc again that the Freese-Nation index of $\mathfrak{C}_\alpha$ in $\mathfrak{A}$ is countable, and from the inductive hypothesis we see also that $\mathrm{FN}(\mathfrak{C}_n) \le \kappa$ for every $n \ge m$, so that (using 518G, as usual) $\mathrm{FN}(\mathfrak{C}_\alpha) \le \kappa$.

(iii) If $\alpha < \lambda^+$ is the union of a sequence $\langle \alpha_n \rangle_{n\in\mathbb{N}}$ of ordinals of uncountable cofinality, then $M_\alpha = \bigcup_{n\in\mathbb{N}} M_{\alpha_n}$ (518H(iii)), so $\mathfrak{C}_\alpha = \bigcup_{n\in\mathbb{N}} \mathfrak{C}_{\alpha_n}$ is again a countable union of order-closed subalgebras of $\mathfrak{A}$, and the Freese-Nation index of $\mathfrak{C}_\alpha$ in $\mathfrak{A}$ is at most ω. Moreover, because $\mathrm{FN}(\mathfrak{C}_{\alpha_n}) \le \kappa$ for each n, $\mathrm{FN}(\mathfrak{C}_\alpha) \le \kappa$.

(iv) If $a \in \mathfrak{A}$, there is a countable set $D \subseteq \lambda$ such that $a \in \mathfrak{B}_D$. But now there is an $\alpha < \lambda^+$, of uncountable cofinality, such that $D \subseteq D'$ for some countable $D' \in M_\alpha$ (518H(x)), and

$$a \in \mathfrak{B}_D \subseteq \mathfrak{B}_{D'} \subseteq \mathfrak{B}_{\mathrm{sk}(D')} \subseteq \mathfrak{C}_\alpha,$$

by 518H(v).

(v) Let $F \subseteq \lambda^+$ be the set of ordinals which are either of uncountable cofinality, or the union of a sequence of such ordinals; so that F is a closed cofinal set in λ^+, and $\mathfrak{C}_\alpha$ has countable Freese-Nation index in $\mathfrak{A}$ for every $\alpha \in F$. By (iv), $\bigcup_{\alpha\in F} \mathfrak{C}_\alpha = \mathfrak{A}$. So if we enumerate F in ascending order as $\langle \alpha_\xi \rangle_{\xi<\lambda^+}$ and set $P_\xi = \mathfrak{C}_{\alpha_\xi}$ for each ξ, $P_{\lambda^+} = \mathfrak{A}$ then $\langle P_\xi \rangle_{\xi\le\lambda^+}$ satisfies the conditions of 518G, so $\mathrm{FN}(\mathfrak{A}) \le \kappa$, and the induction proceeds.

518J Lemma Let λ be an infinite cardinal and $\mathfrak{G}$ the regular open algebra of $\{0,1\}^\lambda$. Suppose that κ is the least cardinal of uncountable cofinality greater than or equal to $\mathrm{FN}(\mathfrak{G})$. Then $\kappa \le \mathfrak{c}^+$ and we have a family $\mathcal{V} \subseteq [\lambda]^{\le\mathfrak{c}}$, cofinal with $[\lambda]^{\le\mathfrak{c}}$, such that $\#(\{A \cap V : V \in \mathcal{V}\}) < \kappa$ for every countable set $A \subseteq \lambda$.

proof Actually it is more convenient to work with $\mathfrak{G}=\mathrm{RO}(\{0,1\}^{\lambda\times\mathbb{N}})$; of course this makes no difference.

(a) I will use the phrase 'cylinder set' to mean a subset of $X=\{0,1\}^{\lambda\times\mathbb{N}}$ of the form $\{x:x\restriction J=z\}$, where $J\subseteq\lambda\times\mathbb{N}$ is finite. For $I\subseteq\lambda$, let $\mathfrak{G}_I$ be the order-closed subalgebra of $\mathfrak{G}$ consisting of those regular open sets determined by coordinates in $I\times\mathbb{N}$. For $G\in\mathfrak{G}$, there is a smallest subset $J(G)$ of λ such that $G\in\mathfrak{G}_{J(G)}$ (use 4A2B(g-ii)). Recall that $J(G)$ is always countable (use 4A2E(b-i)), so that $\#(\mathfrak{G}_I)\le\mathfrak{c}$ whenever $\#(I)\le\mathfrak{c}$.

(b) The function $G\mapsto\mathfrak{G}_{J(G)}$ is a Freese-Nation function. **P** Suppose that $G_1\subseteq G_2$ in $\mathfrak{G}$. Set $K=J(G_1)$ and $L=J(G_2)$, and let $\phi:X\to\{0,1\}^{L\times\mathbb{N}}$ be the canonical map, so that $\phi^{-1}[\overline{\phi[G_2]}]=G_2$. Set $H=\phi^{-1}[\operatorname{int}\overline{\phi[G_1]}]$; because ϕ is continuous and open (4A2B(f-i)), $H=\operatorname{int}\overline{\phi^{-1}[\phi[G_1]]}$ (4A2B(f-ii)). In particular, H is a regular open set; at the same time, $H\supseteq G_1$ and $H\subseteq\operatorname{int}\overline{\phi^{-1}[\phi[G_2]]}=G_2$ and H is determined by coordinates in $L\times\mathbb{N}$, so $H\in\mathfrak{G}_L$. Next, $\phi[G_1]\subseteq\{0,1\}^{L\times\mathbb{N}}$ is determined by coordinates in $(K\cap L)\times\mathbb{N}$, so $\operatorname{int}\overline{\phi[G_1]}$ also is (4A2B(g-i)) and H is determined by coordinates in $K\times\mathbb{N}$. Thus $H\in\mathfrak{G}_{J(G_1)}\cap\mathfrak{G}_{J(G_2)}$, which is what we need. **Q**

Since $\#(\mathfrak{G}_{J(G)})\le\mathfrak{c}$ for every G, $\mathrm{FN}(\mathfrak{G})\le\mathfrak{c}^+$; as $\operatorname{cf}\mathfrak{c}^+$ is surely uncountable, $\kappa\le\mathfrak{c}^+$.

(c) Now let $f:\mathfrak{G}\to[\mathfrak{G}]^{<\kappa}$ be a Freese-Nation function. Let $\mathcal{V}$ be the family of those sets $V\in[\lambda]^{\le\mathfrak{c}}$ such that $f(G)\subseteq\mathfrak{G}_V$ for every $G\in\mathfrak{G}_V$; because $\#(f(G))\le\mathfrak{c}$ for every G, and $\#(\mathfrak{G}_V)\le\mathfrak{c}$ whenever $V\in[\lambda]^{\le\mathfrak{c}}$, $\mathcal{V}$ is cofinal with $[\lambda]^{\le\mathfrak{c}}$.

(d) Fix a countable set $A\subseteq\lambda$ and $\zeta\in A$ for the moment. Let $\langle C_\xi\rangle_{\xi\in A}$ be a disjoint family of non-empty cylinder sets determined by coordinates in $\{\zeta\}\times\mathbb{N}$; for each $\xi\in A$, set $C'_\xi=\{x:x\in X,\,x(\xi,0)=1\}$. Set

$$G^*=\sup_{\xi\in A}C_\xi\cap C'_\xi\in\mathfrak{G}_A.$$

Next, for $V\in\mathcal{V}$, set

$$G_V=\sup_{\xi\in A\cap V}C_\xi\cap C'_\xi,\quad G'_V=\sup\{H:H\in\mathfrak{G}_V,\,H\subseteq G^*\}$$

so that $G_V\subseteq G^*$ and $G'_V\in\mathfrak{G}_V$. Now if $\zeta\in V\in\mathcal{V}$, $G_V=G'_V$. **P** Since $C_\xi\cap C'_\xi\in\mathfrak{G}_V$ for every $\xi\in V\cap A$, $G_V\in\mathfrak{G}_V$ and $G_V\subseteq G'_V$. **?** Suppose, if possible, that $G_V\ne G'_V$. Then $G'_V\setminus\overline{G}_V$ is a non-empty set belonging to $\mathfrak{G}_V$, so includes a non-empty cylinder set D determined by coordinates in $V\times\mathbb{N}$. Express D as $D'\cap D''$, where D' is determined by coordinates in $(V\cap A)\times\mathbb{N}$ and D'' by coordinates in $(V\setminus A)\times\mathbb{N}$. As $D'\cap D''\subseteq G^*\in\mathfrak{G}_A$, $D'\subseteq G^*$, so $D'\cap C_\xi\subseteq C'_\xi$ for $\xi\in A$.

If $\xi\in A\setminus V$, $D\cap C'_\xi$ is determined by coordinates in a set not containing $\{(\xi,0)\}$, but is included in C'_ξ, so must be empty. Thus

$$D\subseteq D'=\sup_{\xi\in A\cap V}D'\cap C_\xi\cap C'_\xi\subseteq G_V,$$

which is impossible. **X** Accordingly $G_V=G'_V$, as claimed. **Q**

Note next that if V, $V'\in\mathcal{V}$ and $V\cap A\ne V'\cap A$, then $G_V\ne G_{V'}$, because if $\xi\in A\cap(V\triangle V')$ then $C_\xi\cap C'_\xi\subseteq G_V\triangle G_{V'}$.

At this point, consider $f(G^*)$. For each $V\in\mathcal{V}$ such that $\zeta\in V$, there must be some $H_V\in f(G^*)\cap f(G_V)$ such that $G_V\subseteq H_V\subseteq G^*$. By the definition of $\mathcal{V}$, $H_V\in\mathfrak{G}_V$ so $H_V\subseteq G'_V=G_V$ and $H_V=G_V$. But this shows that

$$\#(\{V\cap A:\zeta\in V\in\mathcal{V}\})\le\#(\{G_V:\zeta\in V\in\mathcal{V}\})\le\#(f(G^*))<\kappa.$$

(e) Now take any countable $A\subseteq\lambda$. By (d), we see that $\#(\{A\cap V:\zeta\in V\in\mathcal{V}\})<\kappa$ for every $\zeta\in A$. But now

$$\{A\cap V:V\in\mathcal{V}\}\subseteq\{\emptyset\}\cup\bigcup_{\zeta\in A}\{A\cap V:\zeta\in V\in\mathcal{V}\}$$

has size less than κ, because $\operatorname{cf}\kappa>\omega$. This completes the proof.

518K Theorem (Fuchino Geschke Shelah & Soukup 01) Suppose that $\lambda>\mathfrak{c}$ is a cardinal of countable cofinality such that $\mathrm{CTP}(\lambda^+,\lambda)$ is true (definition: 5A6F). Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra with cardinal at least λ. Then $\mathrm{FN}(\mathfrak{A})\ge\omega_2$.

proof (a) By 518Cb and 518Cc, it is enough to show that $\mathrm{FN}(\mathfrak{G}) \geq \omega_2$, where $\mathfrak{G}$ is the regular open algebra of $\{0,1\}^\lambda$.

(b) ? Suppose, if possible, that $\mathrm{FN}(\mathfrak{G}) \leq \omega_1$. Let $\mathcal{V} \subseteq [\lambda]^{\leq \mathfrak{c}}$ be as in 518J, with $\kappa = \omega_1$. Note first that if $\mathcal{V}' \subseteq \mathcal{V}$ and $\#(\mathcal{V}') \leq \lambda$ then there is an $A \in [\lambda]^{\leq \omega}$ such that $A \not\subseteq V$ for every $V \in \mathcal{V}'$. **P** Let $\langle \lambda_n \rangle_{n \in \mathbb{N}}$ be a sequence of cardinals less than λ with supremum λ. Express $\mathcal{V}'$ as $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ where $\#(\mathcal{V}_n) \leq \lambda_n$ for each n. For each $n \in \mathbb{N}$, $\#(\bigcup \mathcal{V}_n) < \lambda$, so we can find an $\alpha_n \in \lambda \setminus \bigcup \mathcal{V}_n$; now $A = \{\alpha_n : n \in \mathbb{N}\}$ is not included in any member of $\mathcal{V}'$. **Q**

Choose $\langle A_\xi \rangle_{\xi<\lambda^+}$ and $\langle V_\xi \rangle_{\xi<\lambda^+}$ inductively, as follows. Given $V_\eta \in \mathcal{V}$ for $\eta < \xi$, choose $A_\xi \in [\lambda]^{\leq \omega}$ such that $A_\xi \not\subseteq V_\eta$ for every $\eta < \xi$; now take $V_\xi \in \mathcal{V}$ such that $A_\xi \subseteq V_\xi$, and continue.

Because $\mathrm{CTP}(\lambda^+, \lambda)$ is true, there is an uncountable set $B \subseteq \lambda^+$ such that $C = \bigcup_{\xi\in B} A_\xi$ is countable (5A6F(b-ii)). If η, $\xi \in B$ and $\eta < \xi$, then $A_\xi = A_\xi \cap C \cap V_\xi \not\subseteq V_\eta$, so $C \cap V_\xi \neq C \cap V_\eta$. But this means that $\{C \cap V : V \in \mathcal{V}\}$ is uncountable, contrary to the choice of $\mathcal{V}$. **X**

Thus $\mathrm{FN}(\mathfrak{G}) \geq \omega_2$, and the proof is complete.

Remark Compare Fuchino & Soukup 97, Theorem 12, where it is shown that if the generalized continuum hypothesis and $\mathrm{CTP}(\omega_{\omega+1}, \omega_\omega)$ are both true the Freese-Nation number of $[\omega_\omega]^{\leq \omega}$ is greater than ω_1, and also Fuchino Geschke Shelah & Soukup 01, Theorem 4.2, where a different special axiom is used to find a ccc Dedekind complete Boolean algebra with cardinal $\omega_{\omega+1}$ with Freese-Nation number greater than ω_1.

518L I turn now to the associated idea of 'tight filtration' (511Di). Before discussing conditions ensuring the existence of such filtrations, I give the application of the idea which is most important for this book.

Theorem Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, $\mathfrak{B}$ a tightly ω_1-filtered Boolean algebra, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a surjective sequentially order-continuous Boolean homomorphism; suppose that $\mathfrak{B} \neq \{0\}$. Then there is a Boolean homomorphism $\theta : \mathfrak{B} \to \mathfrak{A}$ such that $\pi\theta b = b$ for every $b \in \mathfrak{B}$.

proof Let $\langle b_\xi \rangle_{\xi<\zeta}$ be a tight ω_1-filtration in $\mathfrak{B}$; for $\alpha \leq \zeta$, write $\mathfrak{C}_\alpha$ for the subalgebra of $\mathfrak{B}$ generated by $\{b_\xi : \xi < \alpha\}$. Define Boolean homomorphisms $\theta_\alpha : \mathfrak{C}_\alpha \to \mathfrak{A}$ inductively, as follows. Start with $\mathfrak{C}_0 = \{0,1\}$, $\theta_0 0 = \emptyset$, $\theta_0 1 = 1$. Given θ_α, let B, $B' \subseteq \mathfrak{C}_\alpha$ be countable sets such that B is a cofinal subset of $\{b : b \in \mathfrak{C}_\alpha, b \subseteq b_\alpha\}$ and B' is a cofinal subset of $\{b : b \in \mathfrak{C}_\alpha, b \subseteq 1 \setminus b_\alpha\}$. Choose any $a \in \mathfrak{A}$ such that $\pi a = b_\alpha$ and set

$$a_\alpha = (a \cup \sup_{b\in B} \theta_\alpha b) \setminus \sup_{b\in B'} \theta_\alpha b.$$

Because B and B' are both countable and $\mathfrak{A}$ is Dedekind σ-complete, a_α is defined in $\mathfrak{A}$. Because B and B' are cofinal with $\{b : b \in \mathfrak{C}_\alpha, b \subseteq b_\alpha\}$ and $\{b : b \in \mathfrak{C}_\alpha, b \subseteq 1 \setminus b_\alpha\}$ respectively, $\theta b \subseteq a_\alpha$ whenever $b \in \mathfrak{C}_\alpha$ and $b \subseteq b_\alpha$, while $\theta b \cap a_\alpha = \emptyset$ whenever $b \in \mathfrak{C}_\alpha$ and $b \subseteq 1 \setminus b_\alpha$. This means that we can define a Boolean homomorphism $\theta_{\alpha+1} : \mathfrak{C}_{\alpha+1} \to \mathfrak{A}$ by setting

$$\theta_{\alpha+1}((b \cap b_\alpha) \cup (c \setminus b_\alpha)) = (\theta_\alpha b \cap a_\alpha) \cup (\theta_\alpha c \setminus a_\alpha)$$

for all b, $c \in \mathfrak{C}_\alpha$ (312O).

This is the inductive step to a successor ordinal. For the inductive step to a non-zero limit ordinal $\alpha \leq \zeta$, $\mathfrak{C}_\alpha = \bigcup_{\xi<\alpha} \mathfrak{C}_\xi$ and we can define θ_α by setting $\theta_\alpha a = \theta_\xi a$ whenever $\xi < \alpha$ and $a \in \mathfrak{C}_\xi$.

An easy induction (using the hypothesis that π is sequentially order-continuous) now shows that $c = \pi\theta_\alpha c$ whenever $\alpha \leq \zeta$ and $c \in \mathfrak{C}_\alpha$, so that $\pi\theta_\zeta$ is the identity homomorphism on $\mathfrak{C}_\zeta = \mathfrak{B}$.

518M Theorem Let $\mathfrak{A}$ be a Boolean algebra and κ a regular infinite cardinal such that $\mathrm{FN}(\mathfrak{A}) \leq \kappa$ and $\#(\mathfrak{A}) \leq \kappa^+$. Then $\mathfrak{A}$ is tightly κ-filtered.

proof (a) Let $\langle a_\xi \rangle_{\xi<\kappa^+}$ run over $\mathfrak{A}$, and let $f : \mathfrak{A} \to [\mathfrak{A}]^{<\kappa}$ be a Freese-Nation function. For each $\alpha < \kappa^+$, let $\mathfrak{A}_\alpha$ be the smallest subalgebra of $\mathfrak{A}$ containing a_ξ for every $\xi < \alpha$ and such that $f(a) \subseteq \mathfrak{A}_\alpha$ for every $a \in \mathfrak{A}_\alpha$. Then $\langle \mathfrak{A}_\alpha \rangle_{\alpha<\kappa^+}$ is a non-decreasing family with union $\mathfrak{A}$, and $\#(\mathfrak{A}_\alpha) \leq \kappa$ for every $\alpha < \kappa^+$.

(b)(i) If $\alpha < \kappa^+$, the Freese-Nation index of $\mathfrak{A}_\alpha$ in $\mathfrak{A}$ is at most κ. **P** If $a \in \mathfrak{A}$, then whenever $b \in \mathfrak{A}_\alpha$ and $b \subseteq a$, there is a $c \in f(a) \cap f(b) \cap [b, a]$. Now $c \in f(a) \cap \mathfrak{A}_\alpha$. This shows that $f(a) \cap \mathfrak{A}_\alpha \cap [0, a]$ is cofinal with $\mathfrak{A}_\alpha \cap [0, a]$, so that $\mathrm{cf}(\mathfrak{A}_\alpha \cap [0, a]) < \kappa$. By 518Fa, this is what we need to know. **Q**

(ii) If $\alpha < \kappa^+$, $I \in [\mathfrak{A}]^{<\kappa}$ and $\mathfrak{B}$ is the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_\alpha \cup I$, then the Freese-Nation index of $\mathfrak{B}$ in $\mathfrak{A}$ is at most κ, by 518Fb.

(c) For each $\alpha < \kappa^+$ enumerate $\mathfrak{A}_{\alpha+1} \setminus \mathfrak{A}_\alpha$ as $\langle a_{\alpha\xi}\rangle_{\xi<\kappa_\alpha}$, where $\kappa_\alpha \le \kappa$. Well-order $\mathfrak{A}$ by setting $a \preccurlyeq a'$ if *either* there is some $\alpha < \kappa^+$ such that $a \in \mathfrak{A}_\alpha$ and $a' \notin \mathfrak{A}_\alpha$ *or* there are $\alpha < \kappa^+$ and $\xi \le \eta < \kappa_\alpha$ such that $a = a_{\alpha\xi}$ and $a' = a_{\alpha\eta}$. Let $\zeta \in \mathrm{On}$ be the order type of this well-ordering and $\langle b_\xi\rangle_{\xi<\zeta}$ the corresponding enumeration of $\mathfrak{A}$. For each $\beta \le \zeta$, let $\mathfrak{B}_\beta$ be the subalgebra of $\mathfrak{A}$ generated by $\{b_\xi : \xi < \beta\}$. Then the Freese-Nation index of $\mathfrak{B}_\beta$ in $\mathfrak{A}$ is at most κ. **P** If $\beta < \zeta$, there is a largest $\alpha < \kappa^+$ such that $\mathfrak{A}_\alpha \subseteq B_\beta$, and in this case $\mathfrak{A}_\alpha = \mathfrak{B}_\gamma$ for some $\gamma \le \beta$, while $\mathfrak{A}_{\alpha+1} = \mathfrak{B}_{\gamma'}$ for some $\gamma' > \beta$; moreover, $\#(\beta\setminus\gamma) < \kappa_\alpha \le \kappa$, because $\mathrm{otp}(\gamma' \setminus \gamma) = \kappa_\alpha$. But this means that $\mathfrak{B}$, which is the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_\alpha \cup \{b_\xi : \gamma \le \xi < \beta\}$, has Freese-Nation index at most κ, by (b-ii) above. **Q**

Thus $\langle b_\xi\rangle_{\xi<\zeta}$ is a tight κ-filtration of $\mathfrak{A}$.

518N Definition Let $\mathfrak{A}$ be a Boolean algebra and κ a cardinal. Then a **κ-Geschke system** for $\mathfrak{A}$ is a family $\mathbb{G}$ of subalgebras of $\mathfrak{A}$ such that

(α) every element of $\mathfrak{A}$ belongs to an element of $\mathbb{G}$ with cardinal less than κ;

(β) for any $\mathbb{G}_0 \subseteq \mathbb{G}$, the subalgebra of $\mathfrak{A}$ generated by $\bigcup\mathbb{G}_0$ belongs to $\mathbb{G}$;

(γ) whenever $\mathfrak{B}_1$, $\mathfrak{B}_2 \in \mathbb{G}$, $a \in \mathfrak{B}_1$, $b \in \mathfrak{B}_2$ and $a \subseteq b$, then there is a $c \in \mathfrak{B}_1 \cap \mathfrak{B}_2$ such that $a \subseteq c \subseteq b$.

(Of course (γ) can be rephrased as '$\mathfrak{B}_1 \cap \mathfrak{B}_2 \cap [0,b]$ is cofinal with $\mathfrak{B}_1 \cap [0,b]$ whenever $\mathfrak{B}_1$, $\mathfrak{B}_2 \in \mathbb{G}$ and $b \in \mathfrak{B}_2$'.)

518O Lemma Let $\mathfrak{A}$ be a Boolean algebra, κ a cardinal and $\mathbb{G}$ a κ-Geschke system for $\mathfrak{A}$. Suppose that $\lambda \ge \kappa$ is a regular uncountable cardinal and that $f : [\mathfrak{A}]^{<\omega} \to [\mathfrak{A}]^{<\lambda}$ is a function. Then there is a $\mathfrak{B} \in \mathbb{G}$ such that $\#(\mathfrak{B}) < \lambda$ and $f(I) \subseteq \mathfrak{B}$ whenever $I \in [\mathfrak{B}]^{<\omega}$.

proof Enlarging f if necessary, we may suppose that $f(I)$ always includes the subalgebra of $\mathfrak{A}$ generated by I, and that $f(\{a\})$ includes a member of $\mathbb{G}$, with cardinal less than κ and containing a, for every $a \in \mathfrak{A}$. If now we take $A_0 = \emptyset$ and $A_{n+1} = \bigcup\{f(I) : I \in [A_n]^{<\omega}\}$ for each $n \in \mathbb{N}$, $\mathfrak{B} = \bigcup_{n\in\mathbb{N}} A_n$ will be a subalgebra of $\mathfrak{A}$, of size less than λ, and a union of members of $\mathbb{G}$, so belongs to $\mathbb{G}$; while $f(I) \subseteq \mathfrak{B}$ for every $I \in [\mathfrak{B}]^{<\omega}$.

518P Lemma (Geschke 02) Let κ be a regular uncountable cardinal and $\mathfrak{A}$ a Boolean algebra. Then $\mathfrak{A}$ is tightly κ-filtered iff there is a κ-Geschke system for $\mathfrak{A}$.

proof (a) Suppose that $\mathfrak{A}$ is tightly κ-filtered.

(i) Let $\langle a_\xi\rangle_{\xi<\zeta}$ be a tight κ-filtration of $\mathfrak{A}$. For $I \subseteq \zeta$ let $\mathfrak{A}_I$ be the subalgebra of $\mathfrak{A}$ generated by $\{a_\xi : \xi \in I\}$. For $\alpha < \zeta$, there must be subsets U_α, V_α of $\mathfrak{A}_\alpha$, with cardinal less than κ, such that U_α is cofinal with $\mathfrak{A}_\alpha \cap [0, a_\alpha]$ and V_α is cofinal with $\mathfrak{A}_\alpha \cap [0, 1 \setminus a_\alpha]$. Let $K_\alpha \in [\alpha]^{<\kappa}$ be such that $U_\alpha \cup V_\alpha \subseteq \mathfrak{A}_{K_\alpha}$.

Write $\mathcal{M}$ for the family of those subsets M of ζ such that $K_\alpha \subseteq M$ for every $\alpha \in M$.

(ii) If M, $N \in \mathcal{M}$, $\gamma \le \zeta$, $a \in \mathfrak{A}_{M\cap\gamma}$, $b \in \mathfrak{A}_{N\cap\gamma}$ and $a \subseteq b$, then there is a $c \in \mathfrak{A}_{M\cap N\cap\gamma}$ such that $a \subseteq c \subseteq b$. **P** Induce on γ.

(α) If $\gamma = 0$ then

$$\mathfrak{A}_{M\cap\gamma} = \mathfrak{A}_{N\cap\gamma} = \mathfrak{A}_{M\cap N\cap\gamma} = \{0, 1\}$$

and the result is trivial.

(β) For the inductive step to $\gamma = \alpha + 1$, consider the following cases.

case 1 If $\alpha \notin M \cup N$ then $a \in \mathfrak{A}_{M\cap\alpha}$ and $b \in \mathfrak{A}_{N\cap\alpha}$, so the inductive hypothesis gives us a $c \in \mathfrak{A}_{M\cap N\cap\alpha}$ such that $a \subseteq c \subseteq b$.

case 2 If $\alpha \in N \setminus M$, then $a \in \mathfrak{A}_{M\cap\alpha}$ and b is of the form $(b' \cap a_\alpha) \cup (b'' \setminus a_\alpha)$ where b', $b'' \in \mathfrak{A}_{N\cap\alpha}$. Now $a \setminus b' \in \mathfrak{A}_\alpha$ and $a \setminus b' \subseteq 1 \setminus a_\alpha$, so there is a $v \in V_\alpha$ such that $a \setminus b' \subseteq v$. Since $K_\alpha \subseteq N \cap \alpha$, $v \in \mathfrak{A}_{N\cap\alpha}$. Similarly, there is a $u \in U_\alpha \subseteq \mathfrak{A}_{N\cap\alpha}$ such that $a \setminus b'' \subseteq u$. We have

$$a \subseteq (u \cap b') \cup (v \cap b'') \cup (b' \cap b'') \in \mathfrak{A}_{N\cap\alpha},$$

so the inductive hypothesis tells us that there is a $c \in \mathfrak{A}_{M\cap N\cap\alpha}$ such that

$$a \subseteq c \subseteq (u \cap b') \cup (v \cap b'') \cup (b' \cap b'') \subseteq b.$$

case 3 Similarly, if $\alpha \in M \setminus N$, then we express a as $(a' \cap a_\alpha) \cup (a'' \setminus a_\alpha)$ where a', $a'' \in \mathfrak{A}_{M\cap\alpha}$, and find $v \in V_\alpha \subseteq \mathfrak{A}_{M\cap\alpha}$, $u \in U_\alpha \subseteq \mathfrak{A}_{M\cap\alpha}$, $c \in \mathfrak{A}_{M\cap N\cap\alpha}$ such that

$$a' \setminus b \subseteq v, \quad a'' \setminus b \subseteq u,$$

$$a \subseteq c \subseteq (b \cup u) \cap (b \cup v) \subseteq b.$$

case 4 Finally, if $\alpha \in M \cap N$, express a as $(a' \cap a_\alpha) \cup (a'' \setminus a_\alpha)$ and b as $(b' \cap a_\alpha) \cup (b'' \setminus a_\alpha)$ where a', a'' belong to $\mathfrak{A}_{M\cap\alpha}$ and b', b'' belong to $\mathfrak{A}_{N\cap\alpha}$. As $a' \setminus b'$ belongs to $\mathfrak{A}_\alpha$ and is included in $1 \setminus a_\alpha$, there is a $v \in V_\alpha$ such that $a' \setminus b' \subseteq v$; as $K_\alpha \subseteq M \cap N \cap \alpha$, $v \in \mathfrak{A}_{M\cap N\cap\alpha}$. Now $a' \setminus v \in \mathfrak{A}_{M\cap\alpha}$, $b' \setminus v \in \mathfrak{A}_{N\cap\alpha}$ and $a' \setminus v \subseteq b' \setminus v$, so the inductive hypothesis tells us that there is a $c' \in \mathfrak{A}_{M\cap N\cap\alpha}$ such that $a' \setminus v \subseteq c' \subseteq b' \setminus v$; in which case $c' \cap a_\alpha \in \mathfrak{A}_{M\cap N\cap\gamma}$ and

$$a' \cap a_\alpha = a' \cap a_\alpha \setminus v \subseteq c' \cap a_\alpha \subseteq b' \cap a_\alpha \setminus v = b' \cap a_\alpha.$$

Similarly, there are $u \in \mathfrak{A}_{M\cap N\cap\alpha}$, $c'' \in \mathfrak{A}_{M\cap N\cap\gamma}$ such that

$$a'' \setminus b'' \subseteq v, \quad a'' \setminus u \subseteq c'' \subseteq b'' \setminus u, \quad a'' \setminus a_\alpha \subseteq c'' \setminus a_\alpha \subseteq b'' \setminus a_\alpha.$$

Putting these together, $c = (c' \cap a_\alpha) \cup (c'' \setminus a_\alpha)$ belongs to $\mathfrak{A}_{M\cap N\cap\gamma}$ and $a \subseteq c \subseteq b$.

Thus the induction proceeds to a successor ordinal γ.

(γ) If $\gamma > 0$ is a limit ordinal, $a \in \mathfrak{A}_{M\cap\gamma}$ and $b \in \mathfrak{A}_{N\cap\gamma}$ and $a \subseteq b$, then there is some $\alpha < \gamma$ such that $a \in \mathfrak{A}_{M\cap\alpha}$ and $b \in \mathfrak{A}_{N\cap\alpha}$, so the inductive hypothesis gives us a $c \in \mathfrak{A}_{M\cap N\cap\alpha} \subseteq \mathfrak{A}_{M\cap N\cap\gamma}$ with $a \subseteq c \subseteq b$, and again the induction proceeds. **Q**

(ii) Now set $\mathbb{G} = \{\mathfrak{A}_M : M \in \mathcal{M}\}$, and consider the conditions (α)-(γ) of 518N.

(α) For any $a \in \mathfrak{A}$, there is a finite set $I \subseteq \zeta$ such that $a \in \mathfrak{A}_I$. Let M be the smallest element of $\mathcal{M}$ including I; then (because κ is regular and uncountable) $\#(M) < \kappa$, so $\#(\mathfrak{A}_M) < \kappa$, while $a \in \mathfrak{A}_M \in \mathbb{G}$.

(β) If $\mathbb{G}_0 \subseteq \mathbb{G}$, consider $\mathcal{M}^* = \{M : M \in \mathcal{M}, \mathfrak{A}_M \in \mathbb{G}_0\}$. Then $M^* = \bigcup \mathcal{M}^*$ belongs to $\mathcal{M}$, and $\mathfrak{A}_{M^*} \in \mathbb{G}$ must be the subalgebra of $\mathfrak{A}$ generated by $\bigcup \mathbb{G}_0$.

(γ) Finally, condition (γ) is just (ii) above with $\gamma = \zeta$.

So $\mathbb{G}$ is a κ-Geschke system for $\mathfrak{A}$.

(b) Suppose that $\mathfrak{A}$ has a κ-Geschke system $\mathbb{G}$. I seek to use the ideas of the proof of 518M.

(i) Enumerate $\mathfrak{A}$ as $\langle a_\xi \rangle_{\xi<\lambda}$, and for each $\xi < \lambda$ let $\mathfrak{C}_\xi \in \mathbb{G}$ be such that $a_\xi \in \mathfrak{C}_\xi$ and $\#(\mathfrak{C}_\xi) < \kappa$. For $\alpha \le \lambda$ let $\mathfrak{A}_\alpha$ be the subalgebra of $\mathfrak{A}$ generated by $\bigcup_{\xi<\alpha} \mathfrak{C}_\xi$, so that $\mathfrak{A}_\alpha \in \mathbb{G}$. Set $C_\alpha = \mathfrak{C}_\alpha \setminus \mathfrak{A}_\alpha$ for each $\alpha < \lambda$. An easy induction shows that, for any $\alpha \le \lambda$, $\mathfrak{A}_\alpha$ is the subalgebra of $\mathfrak{A}$ generated by $\bigcup_{\xi<\alpha} C_\xi$.

(ii) If $\alpha \le \lambda$, the Freese-Nation index of $\mathfrak{A}_\alpha$ in $\mathfrak{A}$ is at most κ. **P** For any $\xi < \lambda$ and $b \in \mathfrak{A}_\alpha \cap [0, a_\xi]$ there must be a $c \in \mathfrak{A}_\alpha \cap \mathfrak{C}_\xi$ such that $b \subseteq c \subseteq a_\xi$, because both $\mathfrak{A}_\alpha$ and $\mathfrak{C}_\xi$ belong to $\mathbb{G}$; so $\mathfrak{C}_\xi \cap \mathfrak{A}_\alpha \cap [0, a_\xi]$ is cofinal with $\mathfrak{A}_\alpha \cap [0, a_\xi]$ and $\mathrm{cf}(\mathfrak{A}_\alpha \cap [0, a_\xi]) \le \#(\mathfrak{C}_\xi) < \kappa$. Similarly, $\mathfrak{C}_\xi \cap \mathfrak{A}_\alpha \cap [a_\xi, 1]$ is coinitial with $\mathfrak{A}_\alpha \cap [a_\xi, 1]$ and $\mathrm{ci}(\mathfrak{A}_\alpha \cap [a_\xi, 1]) < \kappa$. **Q**

(iii) List $\bigcup_{\alpha<\lambda} C_\alpha$ as $\langle b_\xi \rangle_{\xi<\zeta}$, where ζ is an ordinal, in such a way that whenever $\xi \le \eta < \zeta$, $b_\xi \in C_\alpha$ and $b_\eta \in C_\beta$, then $\alpha \le \beta$. Then $\{b_\xi : \xi < \zeta\}$ generates $\mathfrak{A}$. If $\beta < \zeta$ and $\mathfrak{B}_\beta$ is the subalgebra of $\mathfrak{A}$ generated by $\{b_\xi : \xi < \beta\}$, let α be such that $b_\xi \in C_\alpha$; then $\mathfrak{A}_\alpha = \mathfrak{B}_\gamma$ for some $\gamma \le \beta$, $\#(\beta \setminus \gamma) < \#(C_\alpha) < \kappa$ and $\mathfrak{B}_\beta$ is the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_\alpha \cup \{b_\xi : \gamma \le \xi < \beta\}$, so has Freese-Nation index at most κ in $\mathfrak{A}$, by 518Fb. This shows that $\langle b_\xi \rangle_{\xi<\zeta}$ is a tight κ-filtration of $\mathfrak{A}$, and $\mathfrak{A}$ is tightly κ-filtered.

518Q Corollary Let κ be a regular uncountable cardinal and $\mathfrak{A}$ a tightly κ-filtered Boolean algebra.

(a) If $\mathfrak{C}$ is a retract of $\mathfrak{A}$ (that is, $\mathfrak{C}$ is a subalgebra of $\mathfrak{A}$ and there is a Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{C}$ such that $\pi c = c$ for every $c \in \mathfrak{C}$), then $\mathfrak{C}$ is tightly κ-filtered.

(b) If $\mathfrak{C}$ is a subalgebra of $\mathfrak{A}$ which is (in itself) Dedekind complete, then $\mathfrak{C}$ is tightly κ-filtered.

proof (a) By 518P there is a κ-Geschke system $\mathbb{G}$ for $\mathfrak{A}$. Let $\mathbb{G}_1$ be the set of those $\mathfrak{B} \in \mathbb{G}$ such that $\pi[\mathfrak{B}] \subseteq \mathfrak{B}$. Then $\mathbb{G}_1$ is a κ-Geschke system. **P** Of course $\mathbb{G}_1$ satisfies (γ) of 518N, just because $\mathbb{G}_1 \subseteq \mathbb{G}$.

As for (β), if $\mathbb{G}_0 \subseteq \mathbb{G}_1$ and $\mathfrak{B}^*$ is the subalgebra generated by $\bigcup \mathbb{G}_0$, then $\mathfrak{B}^* \in \mathbb{G}$ and $\pi[\mathfrak{B}^*]$ must be the subalgebra generated by $\bigcup_{\mathfrak{B} \in \mathbb{G}_0} \pi[\mathfrak{B}] \subseteq \mathfrak{B}^*$, so $\pi[\mathfrak{B}^*] \subseteq \mathfrak{B}^*$ and $\mathfrak{B}^* \in \mathbb{G}_1$. Finally, if $a \in \mathfrak{A}$, 518O (taking $\lambda = \kappa$ and $f(I) = \{a\} \cup \pi[I]$) tells us that there is a $\mathfrak{B} \in \mathbb{G}_1$ containing a and with cardinal less than κ. **Q**

Observe next that because $\pi c = c$ for every $c \in \mathfrak{C}$, $\pi[\mathfrak{B}] = \mathfrak{B} \cap \mathfrak{C}$ for every $\mathfrak{B} \in \mathbb{G}_1$. Set $\mathbb{H} = \{\mathfrak{B} \cap \mathfrak{C} : \mathfrak{B} \in \mathbb{G}_1\}$. Then $\mathbb{H}$ is a κ-Geschke system for $\mathfrak{C}$. **P** (α) If $c \in \mathfrak{C}$ there is a $\mathfrak{B} \in \mathbb{G}_1$ such that $c \in \mathfrak{B}$ and $\#(\mathfrak{B}) < \kappa$; now $c \in \mathfrak{B} \cap \mathfrak{C} \in \mathbb{H}$ and $\#(\mathfrak{B} \cap \mathfrak{C}) < \kappa$. ($\beta$) If $\mathbb{H}' \subseteq \mathbb{H}$, set $\mathbb{G}_1' = \{\mathfrak{B} : \mathfrak{B} \in \mathbb{G}_1, \mathfrak{B} \cap \mathfrak{C} \in \mathbb{H}'\}$. Then the subalgebra $\mathfrak{B}^*$ generated by $\bigcup \mathbb{G}_1'$ belongs to $\mathbb{G}_1$, and $\pi[\mathfrak{B}^*] \in \mathbb{H}$ is the subalgebra generated by $\bigcup\{\pi[\mathfrak{B}] : \mathfrak{B} \in \mathbb{G}_1'\} = \bigcup \mathbb{H}$. ($\gamma$) If $b_1 \in \mathfrak{D}_1 \in \mathbb{H}$, $b_2 \in \mathfrak{D}_2 \in \mathbb{H}$ and $b_1 \subseteq b_2$, express $\mathfrak{D}_1$, $\mathfrak{D}_2$ as $\mathfrak{B}_1 \cap \mathfrak{C}$, $\mathfrak{B}_2 \cap \mathfrak{C}$ where $\mathfrak{B}_1$ and $\mathfrak{B}_2$ belong to $\mathbb{G}_1$. Then there is a $b \in \mathfrak{B}_1 \cap \mathfrak{B}_2$ such that $b_1 \subseteq b \subseteq b_2$; in which case $\pi b \in \mathfrak{D}_1 \cap \mathfrak{D}_2$ and

$$b_1 = \pi b_1 \subseteq \pi b \subseteq \pi b_2 = b_2.$$

Thus $\mathbb{H}$ satisfies (γ) of 518N and is a κ-Geschke system for $\mathfrak{C}$. By 518P in the other direction, $\mathfrak{C}$ is tightly κ-filtered. **Q**

(b) In this case, the identity map from $\mathfrak{C}$ to itself extends to a Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{C}$ (314K), so we can use (a).

518R Lemma (a) Let I be a set and $\mathfrak{G}$ the regular open algebra of $\{0,1\}^I$. For $J \subseteq I$ let $\mathfrak{G}_J$ be the order-closed subalgebra of $\mathfrak{G}$ consisting of regular open sets determined by coordinates in J. Suppose that J and K are disjoint subsets of I, and $\langle a_q \rangle_{q \in \mathbb{Q}}$, $\langle b_q \rangle_{q \in \mathbb{Q}}$ disjoint families in $\mathfrak{G}_J \setminus \{\emptyset\}$ and $\mathfrak{G}_K \setminus \{\emptyset\}$ respectively. For $t \in \mathbb{R}$ set $w_t = \sup_{p,q \in \mathbb{Q}, p \le t \le q} a_q \cap b_p$, the supremum being taken in $\mathfrak{G}$; set $w = \sup_{p,q \in \mathbb{Q}, p \le q} a_q \cap b_p$. If $w' \subseteq w$ belongs to the subalgebra of $\mathfrak{G}$ generated by $\mathfrak{G}_{I \setminus K} \cup \mathfrak{G}_{I \setminus J}$, then $\{t : w_t \subseteq w'\}$ is finite.

(b) If $I = \omega_3$ then $\mathfrak{G}$ is not tightly ω_1-filtered.

proof (a)(i) I had better explain why each $\mathfrak{G}_J$ is an order-closed subalgebra; the point is just that if $A \subseteq \{0,1\}^I$ is determined by coordinates in $J \subseteq I$ then so are its closure and interior (4A2B(g-i) again), so that the operations $\mathcal{H} \mapsto \operatorname{int}(\bigcap \mathcal{H})$, $\mathcal{H} \mapsto \operatorname{int} \overline{\bigcup \mathcal{H}}$ take subsets of $\mathfrak{G}_J$ to members of $\mathfrak{G}_J$.

(ii) w' must be expressible in the form $\sup_{i<n} u_i \cap v_i$ where $u_i \in \mathfrak{G}_{I \setminus K}$ and $v_i \in \mathfrak{G}_{I \setminus J}$ for each i. **?** Suppose, if possible, that there are $t_0 < t_1 < \ldots < t_n$ in $\mathbb{R}$ such that $w_{t_j} \subseteq \sup_{i<n} u_i \cap v_i$ for every j. Take rational numbers q_j and q_j', for $j \le n$, such that $q_0 \le t_0 \le q_0' < q_1 \le t_1 \le q_1' < \ldots < q_n \le t_n \le q_n'$. Set $e_{-1} = \{0,1\}^I$. Choose i_j, e_j, c_j, c_j' and c_j'' inductively, for $j \le n$, as follows. Given that $e_{j-1} \in \mathfrak{G}_{I \setminus (J \cup K)}$ is non-empty, where $j \le n$, then $a_{q_j'}$, b_{q_j} and e_{j-1} are non-empty sets determined by coordinates in J, K and $I \setminus (J \cup K)$ respectively, so have non-empty intersection; also $a_{q_j'} \cap b_{q_j} \subseteq w_{t_j} \subseteq \sup_{i<n} u_i \cap v_i$. There is therefore an $i_j < n$ such that $a_{q_j'} \cap b_{q_j} \cap e_{j-1} \cap u_{i_j} \cap v_{i_j}$ is non-empty, and includes a basic cylinder set c_j say. Now we can express c_j as $c_j' \cap c_j'' \cap e_j$ where c_j' is determined by coordinates in J, c_j'' by coordinates in K and e_j by coordinates in $I \setminus (J \cup K)$; note that $e_j \subseteq e_{j-1}$, and continue.

At the end of this process, there must be $j < k \le n$ such that $i_j = i_k = i$ say. Now $q_j' < q_k$, so

$$a_{q_j'} \cap b_{q_k} \cap u_i \cap v_i \subseteq a_{q_j'} \cap b_{q_k} \cap w = \emptyset.$$

(Recall that $\langle a_q \rangle_{q \in \mathbb{Q}}$ and $\langle b_q \rangle_{q \in \mathbb{Q}}$ are disjoint.) On the other hand, $c_j' \cap c_j'' \cap e_j \subseteq u_i$, which is determined by coordinates in $I \setminus K$, so $c_j' \cap e_j \subseteq u_i$; similarly, $c_k'' \cap e_k \subseteq v_i$; so

$$capc_j' \cap e_j \cap c_k'' \cap e_k \subseteq a_{q_j'} \cap b_{q_k} \cap u_i \cap v_i = \emptyset.$$

But c_j', c_k'' and $e_j \cap e_k$ are all non-empty and determined by coordinates in J, K and $I \setminus (J \cup K)$ respectively, so this is impossible. **X**

Thus $\{t : w_t \subseteq w'\}$ has at most n members, and is finite.

(b) As in 518J, I will work with $I = \omega_3 \times \mathbb{N}$.

(i) Note that every member of $\mathfrak{G}$ belongs to $\mathfrak{G}_J$ for some countable J (4A2E(b-i) again), so we can choose for each $c \in \mathfrak{G}$ a countable $J(c) \subseteq \omega_3$ such that $c \in \mathfrak{G}_{J(c) \times \mathbb{N}}$ for some countable I. For each $\xi < \omega_3$, let $\langle a_{\xi q} \rangle_{q \in \mathbb{Q}}$ be a disjoint family of non-zero elements of $\mathfrak{G}_{\{\xi\} \times \mathbb{N}}$. For $t \in \mathbb{R}$, $\xi < \omega_3$ set $c_{\xi t}' = \sup_{q \in \mathbb{Q}, q \le t} a_{\xi q}$, $c_{\xi t}'' = \sup_{q \in \mathbb{Q}, q \ge t} a_{\xi q}$. Let $T \subseteq \mathbb{R}$ be a set with cardinal ω_1. For $D \subseteq \mathfrak{G}$ set $\tilde{J}(D) = \bigcup_{c \in D} J(c)$.

(ii) ? Suppose, if possible, that $\mathfrak{G}$ is tightly ω_1-filtered. Then it has an ω_1-Geschke system $\mathbb{B}$ say (518P). By 518O, with $\lambda = \omega_3$ and

$$f(\emptyset) = \{c''_{\xi t} : \xi < \omega_2,\, t \in T\} = C$$

say, there is a $\mathfrak{B}_1 \in \mathbb{B}$ such that $C \subseteq \mathfrak{B}_1$ and $\#(\mathfrak{B}_1) \le \omega_2$; take $\xi \in \omega_3 \setminus \tilde{J}(\mathfrak{B}_1)$, and let $\mathfrak{B}_2 \in \mathbb{B}$ be such that $\mathfrak{B}_2$ is countable and $a_{\xi p} \in \mathfrak{B}_2$ for every $p \in \mathbb{Q}$. Then $\tilde{J}(\mathfrak{B}_2)$ is countable, so there is an $\eta \in \omega_2 \setminus \tilde{J}(\mathfrak{B}_2)$.

Set $w = \sup_{p,q\in\mathbb{Q},p\le q} a_{\xi p} \cap a_{\eta q}$, and for $t \in T$ set $w_t = c'_{\xi t} \cap c''_{\eta t}$. Then w belongs to a countable $\mathfrak{B}_0 \in \mathbb{B}$, while the subalgebra $\mathfrak{B}^*$ of $\mathfrak{G}$ generated by $\mathfrak{B}_1 \cup \mathfrak{B}_2$ belongs to $\mathbb{B}$. But if we set $J = \{\xi\} \times \mathbb{N}$, $K = \{\eta\} \times \mathbb{N}$ then we see that $\mathfrak{B}_1 \subseteq \mathfrak{G}_{(\omega_3\times\mathbb{N})\setminus J}$ and $\mathfrak{B}_2 \subseteq \mathfrak{G}_{(\omega_3\times\mathbb{N})\setminus K}$. So (a) tells us that any member of $\mathfrak{B}^*$ included in w can include only finitely many w_t, while $w_t \in \mathfrak{B}^* \cap [0, w]$. Thus $\mathrm{cf}(\mathfrak{B}^* \cap [0, w]) \ge \omega_1$. On the other hand, by (γ) of 518N, the countable set $\mathfrak{B}_0 \cap \mathfrak{B}^* \cap [0, w]$ is cofinal with $\mathfrak{B}^* \cap [0, w]$. **X**

This contradiction proves the result.

518S Theorem (Geschke 02) If $\mathfrak{A}$ is a tightly ω_1-filtered Dedekind complete Boolean algebra then $\#(\mathfrak{A}) \le \omega_2$.

proof ? Otherwise, by 515I, $\mathfrak{A}$ has a subalgebra $\mathfrak{C}$ isomorphic to the regular open algebra of $\{0,1\}^{\omega_3}$. By 518Rb, $\mathfrak{C}$ is not tightly ω_1-filtered; by 518Qb, nor is $\mathfrak{A}$. **X**

518X Basic exercises (a) Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{B}$ a principal ideal of $\mathfrak{A}$. Show that $\mathrm{FN}(\mathfrak{B}) \le \mathrm{FN}(\mathfrak{A})$.

(b) Show that $\mathrm{FN}(\alpha) = \#(\alpha)$ for every infinite ordinal α.

(c) Show that if P and Q are partially ordered sets, then $\mathrm{FN}(P \times Q)$ is at most the cardinal product $\mathrm{FN}(P) \cdot \mathrm{FN}(Q)$.

>(d) Show that $\mathrm{FN}(\mathcal{P}\mathbb{N}) \ge \omega_1$.

(e) Show that $\mathrm{FN}(\mathbb{Q}) = \omega$ and $\mathrm{FN}(\mathbb{R}) = \omega_1$.

(f) Show that $\mathrm{FN}^*(\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}) = \mathrm{FN}^*(\mathcal{P}\mathbb{N})$.

(g) Let P be a partially ordered set and Q a subset of P with Freese-Nation index κ in P. Show that if $\lambda \ge \max(\kappa, \mathrm{FN}(P))$ is a regular infinite cardinal then $\mathrm{FN}(Q) \le \kappa$.

(h) Let P be a partially ordered set and $\langle P_\xi\rangle_{\xi<\zeta}$ a non-decreasing family of subsets of P such that $P_\xi = \bigcup_{\eta<\xi} P_\eta$ for every non-zero limit ordinal $\xi \le \zeta$. Suppose that κ is a regular infinite cardinal such that the Freese-Nation index of P_ξ in $P_{\xi+1}$ is at most κ for every $\xi < \zeta$. Show that the Freese-Nation index of P_0 in P_ζ is at most κ.

(i) Let $\mathfrak{A}$ be a Boolean algebra, κ a regular infinite cardinal and $\langle a_\xi\rangle_{\xi<\zeta}$ a family in $\mathfrak{A}$. For each $\alpha \le \zeta$ let $\mathfrak{A}_\alpha$ be the subalgebra of $\mathfrak{A}$ generated by $\{a_\xi : \xi < \alpha\}$. Suppose that $\mathfrak{A}_\zeta = \mathfrak{A}$ and that the Freese-Nation index of $\mathfrak{A}_\alpha$ in $\mathfrak{A}_{\alpha+1}$ is at most κ for every $\alpha < \zeta$. Show that $\langle a_\xi\rangle_{\xi<\zeta}$ is a tight κ-filtration of $\mathfrak{A}$.

(j) Suppose that $\mathfrak{c} = \omega_1$. Show that any Dedekind complete ccc Boolean algebra with cardinal at most $\mathfrak{c}^+ = \omega_2$ is tightly ω_1-filtered.

(k) Let $\mathfrak{A}$ be a Boolean algebra, $\kappa \le \lambda$ cardinals and $\mathbb{G}$ a κ-Geschke system for $\mathfrak{A}$. Show that $\mathbb{G}$ is a λ-Geschke system for $\mathfrak{A}$.

(l) Let $\kappa \le \mathfrak{c}$ be a regular uncountable cardinal. Show that if $\mathfrak{A}$ is a tightly κ-filtered Dedekind complete Boolean algebra then $\#(\mathfrak{A}) \le \kappa^+$.

518Y Further exercises (a) Show that if P is a finite partially ordered set then $\mathrm{FN}(P) \le 2 + \frac{1}{2}\#(P)$.

(b) Show that $\mathrm{FN}(\mathcal{P}I) > \#(I)$ for every infinite set I. (*Hint*: Fuchino Koppelberg & Shelah 96.)

(c)(i) Let $\mathfrak{A}$ be an infinite Boolean algebra. Show that $\mathrm{FN}(S(\mathfrak{A})) = \mathrm{FN}(\mathfrak{A})$. (ii) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra. Show that $\mathrm{FN}(\mathfrak{A}) \le \mathrm{FN}(L^0(\mathfrak{A})) \le \mathrm{FN}(\mathfrak{A}^{\mathbb{N}})$.

518 Notes and comments 'Freese-Nation numbers' are a relatively recent topic, beginning with the investigation of partially ordered sets with Freese-Nation numbers at most ω (the 'Freese-Nation property') in Freese & Nation 78 and those with Freese-Nation numbers at most ω_1 (the 'weak Freese-Nation property') in Fuchino Koppelberg & Shelah 96. There are interesting puzzles concerning the Freese-Nation numbers of finite and countable partially ordered sets which I pass over here. Unlike most of the cardinals discussed in this chapter, Freese-Nation numbers refer to the internal, rather than cofinal, structure of a partially ordered set.

The Freese-Nation number $\mathrm{FN}(\mathcal{P}\mathbb{N})$ appears in many contexts besides the identifications of 518D. I will mention it again in 522U. I do not know whether it is consistent to suppose that its cofinality is countable.

Of the special axioms used in 518I, (α) has a more familiar aspect; for instance, it is a consequence of GCH, regardless of the value of $\tau(\mathfrak{A})$ (5A6Ab). (β) is believed not to be a consequence of GCH (see 555Yf), but is true in 'ordinary' models of set theory (5A6Db, 5A6Bc). In 518K I call on a form of Chang's transfer principle; this is *false* in ordinary models of set theory (5A6Fc), but is believed to be relatively consistent with ZFC + GCH (5A6Fa). Freese-Nation numbers are therefore a little exceptional among those appearing in measure theory, in that they are not fixed by the generalized continuum hypothesis.

Version of 28.4.23

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