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Appendix to Volume 4

Useful facts

As is to be expected, we are coming in this volume to depend on a wide variety of more or less recondite information, and only an exceptionally broad mathematical education will have covered it all. While all the principal ideas are fully expressed in standard textbooks, there are many minor points where I need to develop variations on the familiar formulations. A little under half the material, by word-count, is in general topology (§4A2), where I begin with some pages of definitions. I follow this with a section on Borel and Baire σ-algebras, Baire-property algebras and cylindrical algebras (§4A3), worked out a little more thoroughly than the rest of the material. The other sections are on set theory (§4A1), linear analysis (§4A4), topological groups (§4A5) and Banach algebras (§4A6).

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4A1 Set theory

For this volume, we need fragments from four topics in set theory and one in Boolean algebra. The most important are the theory of closed cofinal sets and stationary sets (4A1B-4A1C) and infinitary combinatorics (4A1D-4A1H). Rather more specialized, we have the theory of normal (ultra)filters (4A1J-4A1L) and a mention of Ostaszewski's ♣ (4A1M-4A1N), used for an example in §439. I conclude with a simple result on the cardinality of σ-algebras (4A1O).

4A1A Cardinals again (a) An infinite cardinal κ is **regular** if $\operatorname{cf}\kappa = \kappa$. Any infinite successor cardinal is regular. $\omega_1 = \omega^+$ is regular.

(b) If ζ is an ordinal and X is a set then I say that a family $\langle x_\xi \rangle_{\xi<\zeta}$ in X **runs over X with cofinal repetitions** if $\{\xi : \xi < \zeta,\, x_\xi = x\}$ is cofinal with ζ for every $x \in X$. Now if X is any non-empty set and κ is a cardinal greater than or equal to $\max(\omega, \#(X))$, there is a family $\langle x_\xi \rangle_{\xi<\kappa}$ running over X with cofinal repetitions.

(c) The cardinal $\mathfrak{c}$ (i) Every non-trivial interval in $\mathbb{R}$ has cardinal $\mathfrak{c}$.

(ii) If $\#(A) \le \mathfrak{c}$ and D is countable, then $\#(A^D) \le \mathfrak{c}$.

(iii) $\operatorname{cf}(2^\kappa) > \kappa$ for every infinite cardinal κ; in particular, $\operatorname{cf}\mathfrak{c} > \omega$.

(d) The Continuum Hypothesis This is the statement '$\mathfrak{c} = \omega_1$'; it is neither provable nor disprovable from the ordinary axioms of mathematics. If the continuum hypothesis is true, then there is a well-ordering $\preccurlyeq$ of $[0,1]$ such that $([0,1], \preccurlyeq)$ has order type ω_1.

4A1B Closed cofinal sets Let α be an ordinal.

(a) Note that a subset F of α is closed in the order topology iff $\sup A \in F$ whenever $A \subseteq F$ is non-empty and $\sup A < \alpha$.

(b) If α has uncountable cofinality, and $A \subseteq \alpha$ has supremum α, then $A' = \{\xi : 0 < \xi < \alpha,\, \xi = \sup(A \cap \xi)\}$ is a closed cofinal set in α. In particular, the set of non-zero countable limit ordinals is a closed cofinal set in ω_1.

(c)(i) If $\langle F_\xi \rangle_{\xi<\alpha}$ is a family of subsets of α, the **diagonal intersection** of $\langle F_\xi \rangle_{\xi<\alpha}$ is $\{\xi : \xi < \alpha,\, \xi \in F_\eta$ for every $\eta < \xi\}$.

(ii) If κ is a regular uncountable cardinal and $\langle F_\xi\rangle_{\xi<\kappa}$ is any family of closed cofinal sets in κ, its diagonal intersection F is again a closed cofinal set in κ.

(iii) In particular, if $f : \kappa \to \kappa$ is any function, then $\{\xi : \xi < \kappa,\ f(\eta) < \xi$ for every $\eta < \xi\}$ is a closed cofinal set in κ.

(d) If α has uncountable cofinality, $\mathcal{F}$ is a non-empty family of closed cofinal sets in α and $\#(\mathcal{F}) < \operatorname{cf}\alpha$, then $\bigcap\mathcal{F}$ is a closed cofinal set in α. In particular, the intersection of any sequence of closed cofinal sets in ω_1 is again a closed cofinal set in ω_1.

4A1C Stationary sets (a) Let κ be a cardinal. A subset of κ is **stationary** in κ if it meets every closed cofinal set in κ; otherwise it is **non-stationary**.

(b) If κ is a cardinal of uncountable cofinality, the intersection of any stationary subset of κ with a closed cofinal set in κ is again a stationary set; the family of non-stationary subsets of κ is a σ-ideal, the **non-stationary ideal** of κ.

(c) Pressing-Down Lemma If κ is a regular uncountable cardinal, $A \subseteq \kappa$ is stationary and $f : A \to \kappa$ is such that $f(\xi) < \xi$ for every $\xi \in A$, then there is a stationary set $B \subseteq A$ such that f is constant on B.

(d) There are disjoint stationary sets A, $B \subseteq \omega_1$.

4A1D Δ-systems (a) A family $\langle I_\xi\rangle_{\xi\in A}$ of sets is a **Δ-system** with **root** I if $I_\xi \cap I_\eta = I$ for all distinct ξ, $\eta \in A$.

(b) Δ-system Lemma If $\#(A)$ is a regular uncountable cardinal and $\langle I_\xi\rangle_{\xi\in A}$ is any family of finite sets, there is a set $D \subseteq A$ such that $\#(D) = \#(A)$ and $\langle I_\xi\rangle_{\xi\in D}$ is a Δ-system.

4A1E Free sets (a) Let A be a set with cardinal at least ω_2, and $\langle J_\xi\rangle_{\xi\in A}$ a family of countable sets. Then there are distinct ξ, $\eta \in A$ such that $\xi \notin J_\eta$ and $\eta \notin J_\xi$.

(b) If $\langle K_\xi\rangle_{\xi\in A}$ is a disjoint family of sets indexed by an uncountable subset A of ω_1, and $\langle J_\eta\rangle_{\eta<\omega_1}$ is a family of countable sets, there is an uncountable $B \subseteq A$ such that $K_\xi \cap J_\eta = \emptyset$ whenever η, $\xi \in B$ and $\eta < \xi$.

4A1F Selecting subsequences (a) Let $\langle K_i\rangle_{i\in I}$ be a countable family of sets such that $\bigcap_{i\in J} K_i$ is infinite for every finite subset J of I. Then there is an infinite set K such that $K \setminus K_i$ is finite and $K_i \setminus K$ is infinite for every $i \in I$. Consequently there is a family $\langle K_\xi\rangle_{\xi<\omega_1}$ of infinite subsets of $\mathbb{N}$ such that $K_\xi \setminus K_\eta$ is finite if $\eta \le \xi$, infinite if $\xi < \eta$.

(b) Let $\langle \mathcal{J}_i\rangle_{i\in I}$ be a countable family of subsets of $[\mathbb{N}]^\omega$ such that $\mathcal{J}_i \cap \mathcal{P}K \ne \emptyset$ for every $K \in [\mathbb{N}]^\omega$ and $i \in I$. Then there is an infinite $K \subseteq \mathbb{N}$ such that for every $i \in I$ there is a $J \in \mathcal{J}_i$ such that $K \setminus J$ is finite.

4A1G Ramsey's theorem If $n \in \mathbb{N}$, K is finite and $h : [\mathbb{N}]^n \to K$ is any function, there is an infinite $I \subseteq \mathbb{N}$ such that h is constant on $[I]^n$.

4A1H Proposition Let X and Y be sets, and $R \subseteq X \times Y$ a set such that $R[\{x\}]$ is finite for every $x \in X$ and $\#(R[I]) \ge \#(I)$ for every finite set $I \subseteq X$. Then there is an injective function $f : X \to Y$ such that $(x, f(x)) \in R$ for every $x \in X$.

4A1I Filters (a) Let X be a non-empty set. If $\mathcal{E} \subseteq \mathcal{P}X$ is non-empty and has the finite intersection property,

$$\mathcal{F} = \{A : A \subseteq X,\ A \supseteq \bigcap\mathcal{E}' \text{ for some non-empty finite } \mathcal{E}' \subseteq \mathcal{E}\}$$

is the smallest filter on X including $\mathcal{E}$, the filter **generated** by $\mathcal{E}$.

If $\mathcal{E} \subseteq \mathcal{P}X$ is non-empty and downwards-directed, then it has the finite intersection property iff it does not contain $\emptyset$; in this case we say that $\mathcal{E}$ is a **filter base**; $\mathcal{F} = \{A : A \subseteq X,\ A \supseteq E$ for some $E \in \mathcal{E}\}$, and $\mathcal{E}$ is a base for the filter $\mathcal{F}$.

In general, if $\mathcal{E}$ is a family of subsets of X, then there is a filter on X including $\mathcal{E}$ iff $\mathcal{E}$ has the finite intersection property; in this case, there is an ultrafilter on X including $\mathcal{E}$.

(b) If κ is a cardinal and $\mathcal{F}$ is a filter then $\mathcal{F}$ is κ**-complete** if $\bigcap\mathcal{E} \in \mathcal{F}$ whenever $\mathcal{E} \subseteq \mathcal{F}$ and $0 < \#(\mathcal{E}) < \kappa$. Every filter is ω-complete.

(c) A filter $\mathcal{F}$ on a regular uncountable cardinal κ is **normal** if (α) $\kappa\setminus\xi \in \mathcal{F}$ for every $\xi < \kappa$ (β) whenever $\langle F_\xi\rangle_{\xi<\kappa}$ is a family in $\mathcal{F}$, its diagonal intersection belongs to $\mathcal{F}$.

4A1J Lemma A normal filter $\mathcal{F}$ on a regular uncountable cardinal κ is κ-complete.

4A1K Theorem Let X be a set and $\mathcal{F}$ a non-principal ω_1-complete ultrafilter on X. Let κ be the least cardinal of any non-empty set $\mathcal{E} \subseteq \mathcal{F}$ such that $\bigcap\mathcal{E} \notin \mathcal{F}$. Then κ is a regular uncountable cardinal, $\mathcal{F}$ is κ-complete, and there is a function $g : X \to \kappa$ such that $g[[\mathcal{F}]]$ is a normal ultrafilter on κ.

4A1L Theorem Let κ be a regular uncountable cardinal, and $\mathcal{F}$ a normal ultrafilter on κ. If $S \subseteq [\kappa]^{<\omega}$, there is a set $F \in \mathcal{F}$ such that, for each $n \in \mathbb{N}$, $[F]^n$ is either a subset of S or disjoint from S.

4A1M Ostaszewski's ♣ This is the statement

> Let Ω be the family of non-zero countable limit ordinals. Then there is a family $\langle\theta_\xi(n)\rangle_{\xi\in\Omega,n\in\mathbb{N}}$ such that (α) for each $\xi \in \Omega$, $\langle\theta_\xi(n)\rangle_{n\in\mathbb{N}}$ is a strictly increasing sequence with supremum ξ (β) for any uncountable $A \subseteq \omega_1$ there is a $\xi \in \Omega$ such that $\theta_\xi(n) \in A$ for every $n \in \mathbb{N}$.

4A1N Lemma Assume ♣. Then there is a family $\langle C_\xi\rangle_{\xi<\omega_1}$ of sets such that (i) $C_\xi \subseteq \xi$ for every $\xi < \omega_1$ (ii) $C_\xi \cap \eta$ is finite whenever $\eta < \xi < \omega_1$ (iii) for any uncountable sets A, $B \subseteq \omega_1$ there is a $\xi < \omega_1$ such that $A \cap C_\xi$ and $B \cap C_\xi$ are both infinite.

4A1O The size of σ-algebras: Proposition Let $\mathfrak{A}$ be a Boolean algebra, B a subset of $\mathfrak{A}$, and $\mathfrak{B}$ the σ-subalgebra of $\mathfrak{A}$ generated by B. Then $\#(\mathfrak{B}) \le \max(4, \#(B^{\mathbb{N}}))$. In particular, if $\#(B) \le \mathfrak{c}$ then $\#(\mathfrak{B}) \le \mathfrak{c}$.

4A1P An incidental fact If I is a countable set and $\epsilon > 0$, there is a family $\langle\epsilon_i\rangle_{i\in I}$ of strictly positive real numbers such that $\sum_{i\in I}\epsilon_i \le \epsilon$.

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4A2 General topology

Even more than in previous volumes, naturally enough, the work of this volume depends on results from general topology. We have now reached the point where some of the facts I rely on are becoming hard to find as explicitly stated theorems in standard textbooks. I find myself therefore writing out rather a lot of proofs. You should not suppose that the results to which I attach proofs, rather than references, are particularly deep; on the contrary, in many cases I am merely spelling out solutions to classic exercises.

The style of 'general' topology, as it has evolved over the last hundred years, is to develop a language capable of squeezing the utmost from every step of argument. While this does sometimes lead to absurdly obscure formulations, it remains a natural, and often profitable, response to the remarkably dense network of related ideas in this area. I therefore follow the spirit of the subject in giving the results I need in the full generality achievable within the terminology I use. For the convenience of anyone coming to the theory for the first time, I repeat some of them in the forms in which they are actually applied. I should remark, however, that in some cases materially stronger results can be proved with little extra effort; as always, this appendix is to be thought of not as a substitute for a thorough study of the subject, but as a guide connecting standard approaches to the general theory with the special needs of this volume.

4A2A Definitions

Baire space A topological space X is a **Baire space** if $\bigcap_{n\in\mathbb{N}} G_n$ is dense in X whenever $\langle G_n\rangle_{n\in\mathbb{N}}$ is a sequence of dense open subsets of X.

Base of neighbourhoods If X is a topological space and $x \in X$, a **base of neighbourhoods** of x is a family $\mathcal{V}$ of neighbourhoods of x such that every neighbourhood of x includes some member of $\mathcal{V}$.

boundary If X is a topological space and $A \subseteq X$, the **boundary** of A is $\partial A = \overline{A} \setminus \operatorname{int} A = \overline{A} \cap \overline{X \setminus A}$.

càdlàg If X is a Hausdorff space, a function $f : [0,\infty[\to X$ is **càdlàg** ('continue à droit, limite à gauche') (or **RCLL** ('right continuous, left limits'), an ***R*-function**,) if $\lim_{s\downarrow t} f(s) = f(t)$ for every $t \geq 0$ and $\lim_{s\uparrow t} f(s)$ is defined in X for every $t > 0$.

càllàl If X is a Hausdorff space, a function $f : [0,\infty[\to X$ is **càllàl** ('continue à l'une, limite à l'autre') if $f(0) = \lim_{s\downarrow 0} f(s)$ and, for every $t > 0$, $\lim_{s\downarrow t} f(s)$ and $\lim_{s\uparrow t} f(s)$ are defined in X, and at least one of them is equal to $f(t)$.

Čech-complete A completely regular Hausdorff topological space X is **Čech-complete** if it is homeomorphic to a G_δ subset of a compact Hausdorff space.

closed interval Let X be a totally ordered set. A **closed interval** in X is an interval of one of the forms $\emptyset$, $[x,y]$, $]-\infty,y]$, $[x,\infty[$ or $X =]-\infty,\infty[$ where x, $y \in X$.

coarser topology If $\mathfrak{S}$ and $\mathfrak{T}$ are two topologies on a set X, we say that $\mathfrak{S}$ is **coarser** than $\mathfrak{T}$ if $\mathfrak{S} \subseteq \mathfrak{T}$. (Equality allowed.)

compact support Let X be a topological space and $f : X \to \mathbb{R}$ a function. I say that f has **compact support** if $\overline{\{x : x \in X,\, f(x) \neq 0\}}$ is compact in X.

countably compact A topological space X is **countably compact** if every countable open cover of X has a finite subcover. A subset of a topological space is countably compact if it is countably compact in its subspace topology.

countably paracompact A topological space X is **countably paracompact** if given any countable open cover $\mathcal{G}$ of X there is a locally finite family $\mathcal{H}$ of open sets which refines $\mathcal{G}$ and covers X.

countably tight A topological space X is **countably tight** (or has **countable tightness**) if whenever $A \subseteq X$ and $x \in \overline{A}$ there is a countable set $B \subseteq A$ such that $x \in \overline{B}$.

direct sum, disjoint union Let $\langle (X_i,\mathfrak{T}_i)\rangle_{i\in I}$ be a family of topological spaces, and set $X = \{(x,i) : i \in I,\, x \in X_i\}$. The **disjoint union topology** on X is $\mathfrak{T} = \{G : G \subseteq X,\, \{x : (x,i) \in G\} \in \mathfrak{T}_i$ for every $i \in I\}$; $(X,\mathfrak{T})$ is the **(direct) sum** of $\langle (X_i,\mathfrak{T}_i)\rangle_{i\in I}$.

If X is a set, $\langle X_i\rangle_{i\in I}$ a partition of X, and $\mathfrak{T}_i$ a topology on X_i for every $i \in I$, then the **disjoint union topology** on X is $\{G : G \subseteq X,\, G \cap X_i \in \mathfrak{T}_i$ for every $i \in I\}$.

dyadic A Hausdorff space is **dyadic** if it is a continuous image of $\{0,1\}^I$ for some set I.

equicontinuous If X is a topological space, $(Y,\mathcal{W})$ a uniform space, and F a set of functions from X to Y, then F is **equicontinuous** if for every $x \in X$ and $W \in \mathcal{W}$ the set $\{y : (f(x),f(y)) \in W$ for every $f \in F\}$ is a neighbourhood of x.

finer topology If $\mathfrak{S}$ and $\mathfrak{T}$ are two topologies on a set X, we say that $\mathfrak{S}$ is **finer** than $\mathfrak{T}$ if $\mathfrak{S} \supseteq \mathfrak{T}$. (Equality allowed.)

first-countable A topological space X is **first-countable** if every point has a countable base of neighbourhoods.

half-open Let X be a totally ordered set. A **half-open interval** in X is a set of one of the forms $[x,y[$, $]x,y]$ where x, $y \in X$ and $x < y$.

hereditarily Lindelöf A topological space is **hereditarily Lindelöf** if every subspace is Lindelöf.

hereditarily metacompact A topological space is **hereditarily metacompact** if every subspace is metacompact.

hereditarily separable A topological space is **hereditarily separable** if every subspace is separable.

indiscrete If X is any set, the **indiscrete** topology on X is the topology $\{\emptyset, X\}$.

interval Let $(P,\leq)$ be a partially ordered set. An **interval** in P is a set of one of the forms $[p,q] = \{r : p \leq r \leq q\}$, $[p,q[= \{r : p \leq r < q\}$, $]p,q] = \{r : p < r \leq q\}$, $]p,q[= \{r : p < r < q\}$, $[p,\infty[= \{r : p \leq r\}$, $]-\infty,q] = \{r : r \leq q\}$, $]p,\infty[= \{r : p < r\}$, $]-\infty,q[= \{r : r < q\}$, $]-\infty,\infty[= P$, where p, $q \in P$.

irreducible If X and Y are topological spaces, a continuous surjection $f : X \to Y$ is **irreducible** if $f[F] \neq Y$ for any closed proper subset F of X.

isolated If X is a topological space, a family $\mathcal{A}$ of subsets of X is **isolated** if $A \cap \overline{\bigcup(\mathcal{A} \setminus \{A\})}$ is empty for every $A \in \mathcal{A}$.

Lindelöf A topological space is **Lindelöf** if every open cover has a countable subcover.

Lipschitz If (X,ρ) and (Y,σ) are metric spaces, a function $f:X\to Y$ is **γ-Lipschitz**, or **(γ,ρ,σ)-Lipschitz**, where $\gamma\ge 0$, if $\sigma(f(x),f(y))\le\gamma\rho(x,y)$ for all x, $y\in X$. $f:X\to Y$ is **Lipschitz** or **(ρ,σ)-Lipschitz** if it is (γ,ρ,σ)-Lipschitz for some $\gamma\ge 0$.

locally finite If X is a topological space, a family $\mathcal{A}$ of subsets of X is **locally finite** if for every $x\in X$ there is an open set which contains x and meets only finitely many members of $\mathcal{A}$.

lower semi-continuous If X is a topological space and T a totally ordered set, a function $f:X\to T$ is **lower semi-continuous** if $\{x:f(x)>t\}$ is open for every $t\in T$.

metacompact A topological space is **metacompact** if every open cover has a point-finite refinement which is an open cover.

neighbourhood If X is a topological space and $x\in X$, a **neighbourhood** of x is any subset of X including an open set which contains x.

network Let $(X,\mathfrak{T})$ be a topological space. A **network** for $\mathfrak{T}$ is a family $\mathcal{E}\subseteq\mathcal{P}X$ such that whenever $x\in G\in\mathfrak{T}$ there is an $E\in\mathcal{E}$ such that $x\in E\subseteq G$.

normal A topological space X is **normal** if for any disjoint closed sets E, $F\subseteq X$ there are disjoint open sets G, H such that $E\subseteq G$ and $F\subseteq H$.

open interval Let X be a totally ordered set. An **open interval** in X is a set of one of the the forms $]x,y[$, $]x,\infty[$, $]-\infty,x[$ or $]-\infty,\infty[\,=X$ where x, $y\in X$.

open map If $(X,\mathfrak{T})$ and $(Y,\mathfrak{S})$ are topological spaces, a function $f:X\to Y$ is **open** if $f[G]\in\mathfrak{S}$ for every $G\in\mathfrak{T}$.

order-convex Let $(P,\le)$ be a partially ordered set. A subset C of P is **order-convex** if $[p,q]=\{r:p\le r\le q\}$ is included in C whenever p, $q\in C$.

order topology Let $(X,\le)$ be a totally ordered set. Its **order topology** is that generated by intervals of the form $]x,\infty[$, $]-\infty,x[$ as x runs over X.

paracompact A topological space is **paracompact** if every open cover has a locally finite refinement which is an open cover.

perfect A topological space is **perfect** if it is compact and has no isolated points.

perfectly normal A topological space is **perfectly normal** if it is normal and every closed set is a G_δ set.

point-countable, point-finite A family $\mathcal{A}$ of sets is **point-countable** if no point belongs to more than countably many members of $\mathcal{A}$. Similarly, an indexed family $\langle A_i\rangle_{i\in I}$ of sets is **point-finite** if $\{i:x\in A_i\}$ is finite for every x.

Polish A topological space X is **Polish** if it is separable and its topology can be defined from a metric under which X is complete.

pseudometrizable A topological space $(X,\mathfrak{T})$ is **pseudometrizable** if $\mathfrak{T}$ is defined by a single pseudometric.

refine(ment) If $\mathcal{A}$ is a family of sets, a **refinement** of $\mathcal{A}$ is a family $\mathcal{B}$ of sets such that every member of $\mathcal{B}$ is included in some member of $\mathcal{A}$; in this case I say that $\mathcal{B}$ **refines** $\mathcal{A}$.

relatively countably compact If X is a topological space, a subset A of X is **relatively countably compact** if every sequence in A has a cluster point in X.

scattered A topological space X is **scattered** if every non-empty subset of X has an isolated point (in its subspace topology).

second-countable A topological space is **second-countable** if the topology has a countable base.

semi-continuous *see* lower semi-continuous, upper semi-continuous.

sequential A topological space is **sequential** if every sequentially closed set in X is closed.

sequentially closed If X is a topological space, a subset A of X is **sequentially closed** if $x\in A$ whenever $\langle x_n\rangle_{n\in\mathbb{N}}$ is a sequence in A converging to $x\in X$.

sequentially compact A topological space is **sequentially compact** if every sequence has a convergent sequence. A subset of a topological space is sequentially compact if it is sequentially compact in its subspace topology.

sequentially continuous If X and Y are topological spaces, a function $f:X\to Y$ is **sequentially continuous** if $\langle f(x_n)\rangle_{n\in\mathbb{N}}\to f(x)$ in Y whenever $\langle x_n\rangle_{n\in\mathbb{N}}\to x$ in X.

subbase If $(X,\mathfrak{T})$ is a topological space, a **subbase** for $\mathfrak{T}$ is a family $\mathcal{U}\subseteq\mathfrak{T}$ which generates $\mathfrak{T}$, in the sense that $\mathfrak{T}$ is the coarsest topology on X including $\mathcal{U}$.

totally bounded If $(X, \mathcal{W})$ is a uniform space, a subset A of X is **totally bounded** if for every $W \in \mathcal{W}$ there is a finite set $I \subseteq X$ such that $A \subseteq W[I]$. If (X, ρ) is a metric space, a subset of X is totally bounded if it is totally bounded for the associated uniformity.

uniform convergence If X is a set, (Y, σ) is a metric space and $\mathcal{A}$ is a family of subsets of X then the **topology of uniform convergence** on members of $\mathcal{A}$ is the topology on Y^X generated by the pseudometrics $(f, g) \mapsto \min(1, \sup_{x \in A} \sigma(f(x), g(x)))$ as A runs over $\mathcal{A} \setminus \{\emptyset\}$.

upper semi-continuous If X is a topological space and T is a totally ordered set, a function $f : X \to T$ is **upper semi-continuous** if $\{x : f(x) < t\}$ is open for every $t \in T$.

weakly α-favourable A topological space $(X, \mathfrak{T})$ is **weakly α-favourable** if there is a function $\sigma : \bigcup_{n \in \mathbb{N}} (\mathfrak{T} \setminus \{\emptyset\})^{n+1} \to \mathfrak{T} \setminus \{\emptyset\}$ such that (i) $\sigma(G_0, \dots, G_n) \subseteq G_n$ whenever $G_0, \dots, G_n$ are non-empty open sets (ii) whenever $\langle G_n \rangle_{n \in \mathbb{N}}$ is a sequence in $\mathfrak{T} \setminus \{\emptyset\}$ such that $G_{n+1} \subseteq \sigma(G_0, \dots, G_n)$ for every n, then $\bigcap_{n \in \mathbb{N}} G_n$ is non-empty.

weight If X is a topological space, its **weight** $w(X)$ is the smallest cardinal of any base for the topology.

C_b If X is a topological space, $C_b(X)$ is the space of bounded continuous real-valued functions defined on X.

F_σ If X is a topological space, an $\mathbf{F}_\sigma$ set in X is one expressible as the union of a sequence of closed sets.

G_δ If X is a topological space, a $\mathbf{G}_\delta$ set in X is one expressible as the intersection of a sequence of open sets.

K_σ If X is a topological space, a $\mathbf{K}_\sigma$ set in X is one expressible as the union of a sequence of compact sets.

$\mathcal{P}X$ If X is any set, the **usual topology** on $\mathcal{P}X$ is that generated by the sets $\{a : a \subseteq X,\ a \cap J = K\}$ where $J \subseteq X$ is finite and $K \subseteq J$.

T_0 If $(X, \mathfrak{T})$ is a topological space, we say that it is $\mathbf{T}_0$ if for any two distinct points of X there is an open set containing one but not the other.

T_1 If $(X, \mathfrak{T})$ is a topological space, we say that it is $\mathbf{T}_1$ if singleton sets are closed.

π-base If $(X, \mathfrak{T})$ is a topological space, a **π-base** for $\mathfrak{T}$ is a set $\mathcal{U} \subseteq \mathfrak{T}$ such that every non-empty open set includes a non-empty member of $\mathcal{U}$.

σ-compact A topological space X is **σ-compact** if there is a sequence of compact subsets of X covering X.

σ-disjoint A family of sets is **σ-disjoint** if it is expressible as $\bigcup_{n \in \mathbb{N}} \mathcal{A}_n$ where every $\mathcal{A}_n$ is disjoint.

σ-isolated If X is a topological space, a family of subsets of X is **σ-isolated** if it is expressible as $\bigcup_{n \in \mathbb{N}} \mathcal{A}_n$ where every $\mathcal{A}_n$ is an isolated family.

σ-metrically-discrete If (X, ρ) is a metric space, a family of subsets of X is **σ-metrically-discrete** if it is expressible as $\bigcup_{n \in \mathbb{N}} \mathcal{A}_n$ where $\rho(x, y) \ge 2^{-n}$ whenever $n \in \mathbb{N}$, A and B are distinct members of $\mathcal{A}_n$, $x \in A$ and $y \in B$.

4A2B Elementary facts about general topological spaces (a) Bases and networks (i) Let $(X, \mathfrak{T})$ be a topological space and $\mathcal{U}$ a subbase for $\mathfrak{T}$. Then $\{X\} \cup \{U_0 \cap U_1 \cap \ldots \cap U_n : U_0, \dots, U_n \in \mathcal{U}\}$ is a base for $\mathfrak{T}$.

(ii) Let X and Y be topological spaces, and $\mathcal{U}$ a subbase for the topology of Y. Then a function $f : X \to Y$ is continuous iff $f^{-1}[U]$ is open for every $U \in \mathcal{U}$.

(iii) If X and Y are topological spaces, $\mathcal{E}$ is a network for the topology of Y, and $f : X \to Y$ is a function such that $f^{-1}[E]$ is open for every $E \in \mathcal{E}$, then f is continuous.

(iv) If X is a topological space and $\mathcal{U}$ is a subbase for the topology of X, then a filter $\mathcal{F}$ on X converges to $x \in X$ iff $\{U : x \in U \in \mathcal{U}\} \subseteq \mathcal{F}$.

(v) If X and Y are topological spaces with subbases $\mathcal{U}$, $\mathcal{V}$ respectively, then $\{U \times Y : U \in \mathcal{U}\} \cup \{X \times V : V \in \mathcal{V}\}$ is a subbase for the product topology of $X \times Y$.

(vi) If $\mathcal{U}$ is a (sub-)base for a topology on X, and $Y \subseteq X$, then $\{Y \cap U : U \in \mathcal{U}\}$ is a (sub-)base for the subspace topology of Y.

(vii) If X is a topological space, $\mathcal{E}$ is a network for the topology of X, and Y is a subset of X, then $\{E \cap Y : E \in \mathcal{E}\}$ is a network for the topology of Y.

(viii) If X is a topological space and $\mathcal{A}$ is a (σ-)isolated family of subsets of X, then $\{A \cap Y : A \in \mathcal{A}'\}$ is (σ-)isolated whenever $Y \subseteq X$ and $\mathcal{A}' \subseteq \mathcal{A}$.

(ix) If a topological space X has a σ-isolated network, so has every subspace of X.

(b) If $\langle H_i\rangle_{i\in I}$ is a partition of a topological space X into open sets and $F_i\subseteq H_i$ is closed for each $i\in I$, then $F=\bigcup_{i\in I}F_i$ is closed in X.

(c) If X is a topological space, $A\subseteq X$ and $x\in X$, then $x\in\overline{A}$ iff there is an ultrafilter on X, containing A, which converges to x.

(d) Semi-continuity Let X be a topological space.

(i) A function $f:X\to\mathbb{R}$ is lower semi-continuous iff $-f$ is upper semi-continuous. A function $f:X\to\mathbb{R}$ is lower semi-continuous iff $\Omega=\{(x,\alpha):x\in X,\ \alpha\ge f(x)\}$ is closed in $X\times\mathbb{R}$.

(ii) If T is a totally ordered set, $f:X\to T$ is lower semi-continuous, Y is another topological space, and $g:Y\to X$ is continuous, then $fg:Y\to T$ is lower semi-continuous. In particular, if $f:X\to T$ is lower semi-continuous and $Y\subseteq X$, then $f\restriction Y$ is lower semi-continuous. Similarly, if $f:X\to T$ is upper semi-continuous and $g:Y\to X$ is continuous, then $fg:Y\to T$ is upper semi-continuous.

(iii) If f, $g:X\to\,]-\infty,\infty]$ are lower semi-continuous so is $f+g:X\to\,]-\infty,\infty]$.

(iv) If f, $g:X\to[0,\infty]$ are lower semi-continuous so is $f\times g:X\to[0,\infty]$.

(v) If Φ is any non-empty set of lower semi-continuous functions from X to $[-\infty,\infty]$, then $x\mapsto\sup_{f\in\Phi}f(x):X\to[-\infty,\infty]$ is lower semi-continuous.

(vi) $f:X\to\mathbb{R}$ is continuous iff f is both upper semi-continuous and lower semi-continuous iff f and $-f$ are both lower semi-continuous.

(vii) If $f:X\to[-\infty,\infty]$ is lower semi-continuous, and $\mathcal{F}$ is a filter on X converging to $y\in X$, then $f(y)\le\liminf_{x\to\mathcal{F}}f(x)$.

(viii) If X is compact and not empty, and $f:X\to[-\infty,\infty]$ is lower semi-continuous then $K=\{x:f(x)=\inf_{y\in X}f(y)\}$ is non-empty and compact.

(ix) If f, $g:X\to[0,\infty]$ are lower semi-continuous and $f+g$ is continuous at $x\in X$ and finite there, then f and g are continuous at x.

(e) Separable spaces (i) If $\langle A_i\rangle_{i\in I}$ is a countable family of separable subsets of a topological space X then $\bigcup_{i\in I}A_i$ and $\overline{\bigcup_{i\in I}A_i}$ are separable.

(ii) If $\langle X_i\rangle_{i\in I}$ is a family of separable topological spaces and $\#(I)\le\mathfrak{c}$, then $\prod_{i\in I}X_i$ is separable.

(iii) A continuous image of a separable topological space is separable.

(f) Open maps (i) Let $\langle X_i\rangle_{i\in I}$ be any family of topological spaces, with product X. If $J\subseteq I$ is any set, and we write X_J for $\prod_{i\in J}X_i$, then the canonical map $x\mapsto x\restriction J:X\to X_J$ is open.

(ii) Let X and Y be topological spaces and $f:X\to Y$ a continuous open map. Then $\operatorname{int}f^{-1}[B]=f^{-1}[\operatorname{int}B]$ and $\overline{f^{-1}[B]}=f^{-1}[\overline{B}]$ for every $B\subseteq Y$.

It follows that $f^{-1}[B]$ is nowhere dense in X whenever $B\subseteq Y$ is nowhere dense in Y. If f is surjective and $B\subseteq Y$, then B is nowhere dense in Y iff $f^{-1}[B]$ is nowhere dense in X.

(iii) Let X and Y be topological spaces and $f:X\to Y$ a continuous open map. Then $H\mapsto f^{-1}[H]$ is an order-continuous Boolean homomorphism from the regular open algebra of Y to the regular open algebra of X. If f is surjective, then the homomorphism is injective, and for $H\subseteq Y$, H is a regular open set in Y iff $f^{-1}[H]$ is a regular open set in X.

(iv) If X_0, Y_0, X_1, Y_1 are topological spaces, and $f_i:X_i\to Y_i$ is an open map for each i, then $(x_0,x_1)\mapsto(f_0(x_0),f_1(x_1)):X_0\times X_1\to Y_0\times Y_1$ is open.

(g) Let $\langle X_i\rangle_{i\in I}$ be a family of topological spaces with product X.

(i) If $A\subseteq X$ is determined by coordinates in $J\subseteq I$, then $\overline{A}$ and $\operatorname{int}A$ are also determined by coordinates in J.

(ii) If $F\subseteq X$ is closed, there is a smallest set $J^*\subseteq I$ such that F is determined by coordinates in J^*.

(h) Let X be a topological space.

(i) If $\mathcal{E}$ is a locally finite family of closed subsets of X, then $\bigcup\mathcal{E}'$ is closed for every $\mathcal{E}'\subseteq\mathcal{E}$.

(ii) If $\langle f_i\rangle_{i\in I}$ is a family in $C(X)$ such that $\langle\{x:f_i(x)\ne0\}\rangle_{i\in I}$ is locally finite, then we have a continuous function $f:X\to\mathbb{R}$ defined by setting $f(x)=\sum_{i\in I}f_i(x)$ for every $x\in X$.

(i) Let X be a topological space and A, B two subsets of X. Then the boundary $\partial(A*B)$ is included in $\partial A\cup\partial B$, where $*$ is any of $\cup$, $\cap$, $\setminus$, $\triangle$.

(j) Let X be a topological space and D a dense subset of X, endowed with its subspace topology.

(i) A set $A\subseteq D$ is nowhere dense in D iff it is nowhere dense in X.

(ii) A set $G\subseteq D$ is a regular open set in D iff it is expressible as $D\cap H$ for some regular open set $H\subseteq X$.

4A2C G_δ, F_σ, zero and cozero sets Let X be a topological space.

(a)(i) The union of two G_δ sets in X is a G_δ set.

(ii) The intersection of countably many G_δ sets is a G_δ set.

(iii) If Y is another topological space, $f:X\to Y$ is continuous and $E\subseteq Y$ is G_δ in Y, then $f^{-1}[E]$ is G_δ in X.

(iv) If Y is a G_δ set in X and $Z\subseteq Y$ is a G_δ set for the subspace topology of Y, then Z is a G_δ set in X.

(v) A set $E\subseteq X$ is an F_σ set iff $X\setminus E$ is a G_δ set.

(b)(i) A zero set is closed. A cozero set is open.

(ii) The union of two zero sets is a zero set. The intersection of two cozero sets is a cozero set.

(iii) The intersection of a sequence of zero sets is a zero set. The union of a sequence of cozero sets is a cozero set.

(iv) If Y is another topological space, $f:X\to Y$ is continuous and $L\subseteq Y$ is a zero set, then $f^{-1}[L]$ is a zero set. If $f:X\to Y$ is continuous and $H\subseteq Y$ is a cozero set, then $f^{-1}[H]$ is a cozero set. If $K\subseteq X$ and $L\subseteq Y$ are zero sets then $K\times L$ is a zero set in $X\times Y$.

(v) If $H\subseteq X$ is a (co-)zero set and $Y\subseteq X$, then $H\cap Y$ is a (co-)zero set in Y.

(vi) A cozero set is the union of a non-decreasing sequence of zero sets. In particular, a cozero set is an F_σ set; a zero set is a G_δ set.

(vii) If $\mathcal{G}$ is a partition of X into open sets, and $H\subseteq X$ is such that $H\cap G$ is a cozero set in G for every $G\in\mathcal{G}$, then H is a cozero set in X. Similarly, if $F\subseteq X$ is such that $F\cap G$ is a zero set in G for every $G\in\mathcal{G}$, then F is a zero set in X.

4A2D Weight Let X be a topological space.

(a)(i) $w(Y)\le w(X)$ for every subspace Y of X.

(ii) If $X=\prod_{i\in I}X_i$ then $w(X)\le\max(\omega,\#(I),\sup_{i\in I}w(X_i))$.

(b) A disjoint family of non-empty open sets in X has cardinal at most $w(X)$.

(c) A point-countable family of open sets in X has cardinal at most $\max(\omega,w(X))$.

(d) If X is a dyadic Hausdorff space then X is a continuous image of $\{0,1\}^{w(X)}$.

(e) If X is a dyadic Hausdorff space then X is separable iff it is a continuous image of $\{0,1\}^{\mathfrak{c}}$.

4A2E The countable chain condition (a)(i) Let $\langle X_i\rangle_{i\in I}$ be a family of topological spaces. If $\prod_{i\in J}X_i$ is ccc for every finite $J\subseteq I$, then $\prod_{i\in I}X_i$ is ccc.

(ii) A separable topological space is ccc.

(iii) The product of any family of separable topological spaces is ccc.

(iv) Any continuous image of a ccc topological space is ccc.

(b) Let $\langle X_i\rangle_{i\in I}$ be a family of topological spaces, and suppose that $X=\prod_{i\in I}X_i$ is ccc. For $J\subseteq I$ and $x\in X$ set $X_J=\prod_{i\in J}X_i$, $\pi_J(x)=x{\restriction}J$.

(i) If $G\subseteq X$ is open, there is an open set $W\subseteq G$ determined by coordinates in a countable subset of I such that $G\subseteq\overline{W}$. So $\overline{G}=\overline{W}$ and $\operatorname{int}\overline{G}$ are determined by coordinates in a countable set; in particular, if G is a regular open set, then it is determined by coordinates in a countable set.

(ii) If $f:X\to\mathbb{R}$ is continuous, there are a countable set $J\subseteq I$ and a continuous function $g:X_J\to\mathbb{R}$ such that $f=g\pi_J$.

(iii) If $A\subseteq X$ is nowhere dense there is a countable set $J\subseteq I$ such that $\pi_J^{-1}[\pi_J[A]]$ is nowhere dense.

4A2F Separation axioms (a) Hausdorff spaces (i) A Hausdorff space is T_1.

(ii) If X is a Hausdorff space and $\langle x_n\rangle_{n\in\mathbb{N}}$ is a sequence in X, then a point x of X is a cluster point of $\langle x_n\rangle_{n\in\mathbb{N}}$ iff there is a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$ such that $x=\lim_{n\to\mathcal{F}}x_n$.

(iii) A topological space X is Hausdorff iff $\{(x,x):x\in X\}$ is closed in $X\times X$.

(b) Regular spaces (i) A regular T_1 space is Hausdorff. Any subspace of a regular space is regular.

(ii) If X is a regular topological space, the regular open subsets of X form a base for the topology.

(c) Completely regular spaces In a completely regular space, the cozero sets form a base for the topology.

(d) Normal spaces (i) **Urysohn's Lemma** If X is normal and E, F are disjoint closed subsets of X, then there is a continuous function $f:X\to[0,1]$ such that $f(x)=0$ for $x\in E$ and $f(x)=1$ for $x\in F$.

(ii) A regular normal space is completely regular.

(iii) A normal T_1 space is Hausdorff and completely regular .

(iv) If X is normal and E, F are disjoint closed sets in X there is a zero set including E and disjoint from F.

(v) In a normal space a closed G_δ set is a zero set.

(vi) If X is a normal space and $\langle G_i\rangle_{i\in I}$ is a point-finite cover of X by open sets, there is a family $\langle H_i\rangle_{i\in I}$ of open sets, still covering X, such that $\overline{H}_i\subseteq G_i$ for every i.

(vii) If X is a normal space and $\langle G_i\rangle_{i\in I}$ is a point-finite cover of X by open sets, there is a family $\langle H'_i\rangle_{i\in I}$ of cozero sets, still covering X, such that $H'_i\subseteq G_i$ for every i.

(viii) If X is a normal space and $\langle G_i\rangle_{i\in I}$ is a locally finite cover of X by open sets, there is a family $\langle g_i\rangle_{i\in I}$ of continuous functions from X to $[0,1]$ such that $g_i\le\chi G_i$ for every $i\in I$ and $\sum_{i\in I}g_i(x)=1$ for every $x\in X$.

(ix) **Tietze's theorem** Let X be a normal space, F a closed subset of X and $f:F\to\mathbb{R}$ a continuous function. Then there is a continuous function $g:X\to\mathbb{R}$ extending f. It follows that if $F\subseteq X$ is closed and $f:F\to[0,1]^I$ is a continuous function from F to any power of the unit interval, there is a continuous function from X to $[0,1]^I$ extending f.

(e) Paracompact spaces A Hausdorff paracompact space is regular. A regular paracompact space is normal.

(f) Countably paracompact spaces A normal space X is countably paracompact iff whenever $\langle F_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of closed subsets of X with empty intersection, there is a sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ of open sets, also with empty intersection, such that $F_n\subseteq G_n$ for every $n\in\mathbb{N}$.

(g) Metacompact spaces (i) A paracompact space is metacompact.

(ii) A closed subspace of a metacompact space is metacompact.

(iii) A normal metacompact space is countably paracompact.

(h) Separating compact sets (i) If X is a Hausdorff space and K and L are disjoint compact subsets of X, there are disjoint open sets G, $H\subseteq X$ such that $K\subseteq G$ and $L\subseteq H$. If T is an algebra of subsets of X including a subbase for the topology of X, there is an open $V\in\mathrm{T}$ such that $K\subseteq V\subseteq X\setminus L$.

(ii) If X is a regular space, $F\subseteq X$ is closed, and $K\subseteq X\setminus F$ is compact, there are disjoint open sets G, $H\subseteq X$ such that $K\subseteq G$ and $F\subseteq H$.

(iii) If X is a completely regular space, $G\subseteq X$ is open and $K\subseteq G$ is compact, there is a continuous function $f:X\to[0,1]$ such that $f(x)=1$ for $x\in K$ and $f(x)=0$ for $x\in X\setminus G$.

(iv) If X is a completely regular Hausdorff space and K and L are disjoint compact subsets of X, there are disjoint cozero sets G, $H\subseteq X$ such that $K\subseteq G$ and $L\subseteq H$.

(v) If X is a completely regular space and $K\subseteq X$ is a compact G_δ set, then K is a zero set.

(vi) If $\langle X_n\rangle_{n\in\mathbb{N}}$ is a sequence of topological spaces with product X, $K\subseteq X$ is compact, $F\subseteq X$ is closed and $K\cap F=\emptyset$, there is some $n\in\mathbb{N}$ such that $x\restriction n\ne y\restriction n$ for any $x\in F$ and $y\in K$.

(vii) If X is a compact Hausdorff space, $f:X\to\mathbb{R}$ is continuous, and $\mathcal{U}$ is a subbase for $\mathfrak{T}$, then there is a countable set $\mathcal{U}_0\subseteq\mathcal{U}$ such that $f(x)=f(y)$ whenever $\{U:x\in U\in\mathcal{U}_0\}=\{U:y\in U\in\mathcal{U}_0\}$.

(i) Perfectly normal spaces A topological space X is perfectly normal iff every closed set is a zero set.

Consequently, every open set in a perfectly normal space is a cozero set (and an F_σ set).

(j) Covers of compact sets Let X be a Hausdorff space, K a compact subset of X, and $\langle G_i\rangle_{i\in I}$ a family of open subsets of X covering K. Then there are a finite set $J\subseteq I$ and a family $\langle K_i\rangle_{i\in J}$ of compact sets such that $K=\bigcup_{i\in J}K_i$ and $K_i\subseteq G_i$ for every $i\in J$.

4A2G Compact and locally compact spaces (a) In any topological space, the union of two compact subsets is compact.

(b) A compact Hausdorff space is normal.

(c)(i) If X is a compact Hausdorff space, $Y\subseteq X$ is a zero set and $Z\subseteq Y$ is a zero set in Y, then Z is a zero set in X.

(ii) Let X and Y be compact Hausdorff spaces, $f:X\to Y$ a continuous open map and $Z\subseteq X$ a zero set in X. Then $f[Z]$ is a zero set in Y.

(d) If X is a Hausdorff space, $\mathcal{V}$ is a downwards-directed family of compact neighbourhoods of a point x of X and $\bigcap\mathcal{V}=\{x\}$, then $\mathcal{V}$ is a base of neighbourhoods of x.

(e) Let $(X,\mathfrak{T})$ be a locally compact Hausdorff space.

(i) If $K\subseteq X$ is a compact set and $G\supseteq K$ is open, then there is a continuous $f:X\to[0,1]$ with compact support such that $\chi K\le f\le\chi G$.

(ii) $\mathfrak{T}$ is the coarsest topology on X such that every $\mathfrak{T}$-continuous real-valued function with compact support is continuous.

(f)(i) A topological space X is countably compact iff every sequence in X has a cluster point in X

(ii) If X is a countably compact topological space and $\langle F_n\rangle_{n\in\mathbb{N}}$ is a sequence of closed sets such that $\bigcap_{i\le n}F_i\ne\emptyset$ for every $n\in\mathbb{N}$, then $\bigcap_{n\in\mathbb{N}}F_n\ne\emptyset$.

(iii) In any topological space, a relatively compact set is relatively countably compact.

(iv) Let X and Y be topological spaces and $f:X\to Y$ a continuous function. If $A\subseteq X$ is relatively countably compact in X, then $f[A]$ is relatively countably compact in Y.

(v) A relatively countably compact set in $\mathbb{R}$ must be bounded. So if X is a topological space, $A\subseteq X$ is relatively countably compact and $f:X\to\mathbb{R}$ is continuous, then $f[A]$ is bounded.

(vi) If X and Y are topological spaces and $f:X\to Y$ is continuous, then $f[A]$ is countably compact whenever $A\subseteq X$ is countably compact.

(g)(i) Let X and Y be topological spaces and $\phi:X\times Y\to\mathbb{R}$ a continuous function. Define $\theta:X\to C(Y)$ by setting $\theta(x)(y)=\phi(x,y)$ for $x\in X$, $y\in Y$. Then θ is continuous if we give $C(Y)$ the topology of uniform convergence on compact subsets of Y.

(ii) In particular, if Y is compact then θ is continuous if we give $C(Y)$ its usual norm topology.

(iii) Let X be a locally compact topological space, and give $C(X)$ the topology of uniform convergence on compact subsets of X. Then the function $(f,x)\mapsto f(x):C(X)\times X\to\mathbb{R}$ is continuous.

(h)(i) Suppose that X is a compact space such that there are no non-trivial convergent sequences in X. If $\langle F_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of infinite closed subsets of X, then $F=\bigcap_{n\in\mathbb{N}}F_n$ is infinite.

(ii) If X is an infinite scattered compact Hausdorff space it has a non-trivial convergent sequence.

(iii) If X is an extremally disconnected Hausdorff space, it has no non-trivial convergent sequence.

(i)(i) If X and Y are compact Hausdorff spaces and $f:X\to Y$ is a continuous surjection then there is a closed set $K\subseteq X$ such that $f[K]=Y$ and $f\restriction K$ is irreducible.

(ii) If X and Y are compact Hausdorff spaces and $f:X\to Y$ is an irreducible continuous surjection, then (α) if $\mathcal{U}$ is a π-base for the topology of Y then $\{f^{-1}[U]:U\in\mathcal{U}\}$ is a π-base for the topology of X (β) if Y has a countable π-base so does X (γ) if x is an isolated point in X then $f(x)$ is an isolated point in Y (δ) if Y has no isolated points, nor does X.

(j)(i) Let X be a non-empty compact Hausdorff space without isolated points. Then there are a closed set $F \subseteq X$ and a continuous surjection $f : F \to \{0,1\}^{\mathbb{N}}$.

(ii) If X is a non-empty compact Hausdorff space without isolated points, then $\#(X) \geq \mathfrak{c}$.

(iii) If X is a compact Hausdorff space which is not scattered, it has an infinite closed subset with a countable π-base and no isolated points.

(iv) Let X be a compact Hausdorff space. Then there is a continuous surjection from X onto $[0,1]$ iff X is not scattered.

(v) A Hausdorff continuous image of a scattered compact Hausdorff space is scattered.

(vi) If X is an uncountable first-countable compact Hausdorff space, it is not scattered. It follows that there is a continuous surjection from X onto $[0,1]$.

(k) A locally compact Hausdorff space is Čech-complete.

(l) If X is a topological space, $f : X \to \mathbb{R}$ is lower semi-continuous, and $K \subseteq X$ is compact and not empty, then there is an $x_0 \in K$ such that $f(x_0) = \inf_{x \in K} f(x)$. Similarly, if $g : X \to \mathbb{R}$ is upper semi-continuous, there is an $x_1 \in K$ such that $g(x_1) = \sup_{x \in K} g(x)$.

(m) If X is a Hausdorff space, Y is a compact space and $F \subseteq X \times Y$ is closed, then its projection $\{x : (x,y) \in F\}$ is a closed subset of X.

(n) If X is a locally compact topological space, Y is a topological space and $f : X \to Y$ is a continuous open surjection, then Y is locally compact.

4A2H Lindelöf spaces (a) If X is a topological space, then a subset Y of X is Lindelöf iff for every family $\mathcal{G}$ of open subsets of X covering Y there is a countable subfamily of $\mathcal{G}$ still covering Y.

(b)(i) A regular Lindelöf space X is normal (therefore completely regular) and paracompact.

(ii) If X is a Lindelöf space and $\mathcal{A}$ is a locally finite family of subsets of X then $\mathcal{A}$ is countable.

(c)(i) A topological space X is hereditarily Lindelöf iff for any family $\mathcal{G}$ of open subsets of X there is a countable family $\mathcal{G}_0 \subseteq \mathcal{G}$ such that $\bigcup \mathcal{G}_0 = \bigcup \mathcal{G}$.

(ii) Let X be a regular hereditarily Lindelöf space. Then X is perfectly normal.

(d) Any σ-compact topological space is Lindelöf.

4A2I Stone-Čech compactifications (a) Let X be a completely regular Hausdorff space. Then there is a compact Hausdorff space βX, the **Stone-Čech compactification** of X, in which X can be embedded as a dense subspace. If Y is another compact Hausdorff space, then every continuous function from X to Y has a unique continuous extension to a continuous function from βX to Y.

(b) Let I be any set, and write βI for its Stone-Čech compactification when I is given its discrete topology. Let Z be the Stone space of the Boolean algebra $\mathcal{P}I$.

(i) There is a canonical homeomorphism $\phi : \beta I \to Z$ defined by saying that $\phi(i)(a) = \chi a(i)$ for every $i \in I$ and $a \subseteq I$. We can identify βI with the set of ultrafilters on I. Under this identification, the canonical embedding of I in βI corresponds to matching each member of I with the corresponding principal ultrafilter on I.

(ii) $C(\beta I)$ is isomorphic, as Banach lattice, to $\ell^\infty(I)$.

(iii) We have a one-to-one correspondence between filters $\mathcal{F}$ on I and non-empty closed sets $F \subseteq \beta I$, got by matching $\mathcal{F}$ with $\bigcap\{\widehat{a} : a \in \mathcal{F}\}$, or F with $\{a : a \subseteq I,\ F \subseteq \widehat{a}\}$, where $\widehat{a} \subseteq \beta I$ is the open-and-closed set corresponding to $a \subseteq I$.

(iv) βI is extremally disconnected.

(v) There are no non-trivial convergent sequences in βI.

4A2J Uniform spaces Let $(X, \mathcal{W})$ be a uniform space; give X the induced topology $\mathfrak{T}$.

(a) $\mathcal{W}$ is generated by a family of pseudometrics. More precisely: if $\langle W_n \rangle_{n\in\mathbb{N}}$ is any sequence in $\mathcal{W}$, there is a pseudometric ρ on X such that (α) $\{(x,y) : \rho(x,y) \le \epsilon\} \in \mathcal{W}$ for every $\epsilon > 0$ (β) whenever $n \in \mathbb{N}$ and $\rho(x,y) < 2^{-n}$ then $(x,y) \in W_n$.

It follows that $\mathfrak{T}$ is completely regular, therefore regular. $\mathfrak{T}$ is defined by the bounded uniformly continuous real-valued functions on X, in the sense that it is the coarsest topology on X such that these are all continuous.

(b) If $\mathcal{W}$ is countably generated and $\mathfrak{T}$ is Hausdorff, there is a metric ρ on X defining $\mathcal{W}$ and $\mathfrak{T}$.

(c) If $W \in \mathcal{W}$ and $x \in X$ then $x \in \operatorname{int} W[\{x\}]$. If $A \subseteq X$ then $\overline{A} = \bigcap_{W\in\mathcal{W}} W[A]$.

(d) Any subset of a totally bounded set in X is totally bounded. The closure of a totally bounded set is totally bounded.

(e) A subset of X is compact iff it is complete (for its subspace uniformity) and totally bounded. So if X is complete, every closed totally bounded subset of X is compact, and the totally bounded sets are just the relatively compact sets.

(f) If $f : X \to \mathbb{R}$ is a continuous function with compact support, it is uniformly continuous.

(g)(i) If $(Y, \mathfrak{S})$ is a completely regular space, there is a uniformity on Y which induces $\mathfrak{S}$.

(ii) If $(Y, \mathfrak{S})$ is a compact completely regular topological space, there is exactly one uniformity on Y which induces $\mathfrak{S}$; it is defined by the set of all those pseudometrics on Y which are continuous as functions from $Y \times Y$ to $\mathbb{R}$.

(iii) If $(Y, \mathfrak{S})$ is a compact completely regular space and $\mathcal{V}$ is the uniformity on Y inducing $\mathfrak{S}$, then any continuous function from Y to X is uniformly continuous.

(h) The set U of uniformly continuous real-valued functions on X is a Riesz subspace of $\mathbb{R}^X$ containing the constant functions. If a sequence in U converges uniformly, the limit function again belongs to U.

(i) Let $(Y, \mathcal{V})$ be another uniform space. If $\mathcal{F}$ is a Cauchy filter on X and $f : X \to Y$ is a uniformly continuous function, then $f[[\mathcal{F}]]$ is a Cauchy filter on Y.

4A2K First-countable, sequential and countably tight spaces (a) Let X be a countably tight topological space. If $\langle F_\xi \rangle_{\xi<\zeta}$ is a non-decreasing family of closed subsets of X indexed by an ordinal ζ, then $E = \bigcup_{\xi<\zeta} F_\xi$ is an F_σ set, and is closed unless $\operatorname{cf}\zeta = \omega$.

(b) If X is countably tight, any subspace of X is countably tight. If X is compact and countably tight, then any Hausdorff continuous image of X is countably tight.

(c) If X is a sequential space, it is countably tight.

(d) If X is a sequential space, Y is a topological space and $f : X \to Y$ is sequentially continuous, then f is continuous.

(e) First-countable spaces are sequential.

(f) Let X be a locally compact Hausdorff space in which every singleton set is G_δ. Then X is first-countable.

4A2L (Pseudo-)metrizable spaces

(a) Any subspace of a (pseudo-)metrizable space is (pseudo-)metrizable. A topological space is metrizable iff it is pseudometrizable and Hausdorff.

(b) Metrizable spaces are paracompact, therefore hereditarily metacompact.

(c) A metrizable space is perfectly normal, so every closed set is a zero set and every open set is a cozero set (in particular, is F_σ).

(d) If X is a pseudometrizable space, it is first-countable. So X is sequential and countably tight, and if Y is another topological space and $f : X \to Y$ is sequentially continuous, then f is continuous.

(e) Relative compactness Let X be a pseudometrizable space and A a subset of X. Then the following are equiveridical: (α) A is relatively compact; (β) A is relatively countably compact; (γ) every sequence in A has a subsequence with a limit in X.

(f) Compactness If X is a pseudometrizable space, it is compact iff it is countably compact iff it is sequentially compact.

(g)(i) If (X, ρ) is a metric space, its topology has a base which is σ-metrically-discrete.
(ii) Consequently, any metrizable space has a σ-disjoint base.

(h) The product of a countable family of metrizable spaces is metrizable.

(i) Let X be a metrizable space and $\kappa \geq \omega$ a cardinal. Then $w(X) \leq \kappa$ iff X has a dense subset with cardinal at most κ.

(j) If (X, ρ) is any metric space, then the balls $B(x, \delta) = \{y : \rho(y, x) \leq \delta\}$ are all closed sets. In particular, in a normed space $(X, \| \|)$, the balls $B(x, \delta) = \{y : \|y - x\| \leq \delta\}$ are closed.

4A2M Complete metric spaces (a) Baire's theorem for complete metric spaces Every complete metric space is a Baire space. So a non-empty complete metric space is not meager.

(b) Let $\langle (X_i, \rho_i) \rangle_{i \in I}$ be a countable family of complete metric spaces. Then there is a complete metric on $X = \prod_{i \in I} X_i$ which defines the product topology on X.

(c) Let (X, ρ) be a complete metric space, and $E \subseteq X$ a G_δ set. Then there is a complete metric on E which defines the subspace topology of E.

(d) Let (X, ρ) be a complete metric space. Then it is Čech-complete.

(e) A non-empty complete metric space without isolated points is uncountable.

4A2N Countable networks: Proposition (a) If X is a topological space with a countable network, any subspace of X has a countable network.

(b) Let X be a space with a countable network. Then X is hereditarily Lindelöf. If it is regular, it is perfectly normal.

(c) If X is a topological space, and $\langle A_n \rangle_{n \in \mathbb{N}}$ is a sequence of subsets of X each of which has a countable network (for its subspace topology), then $A = \bigcup_{n \in \mathbb{N}} A_n$ has a countable network.

(d) A continuous image of a space with a countable network has a countable network.

(e) Let $\langle X_i \rangle_{i \in I}$ be a countable family of topological spaces with countable networks, with product X. Then X has a countable network.

(f) If X is a Hausdorff space with a countable network, there is a countable family $\mathcal{G}$ of open sets such that whenever x, y are distinct points in X there are disjoint G, $H \in \mathcal{G}$ such that $x \in G$ and $y \in H$.

(g) If X is a regular topological space with a countable network, it has a countable network consisting of closed sets.

(h) A compact Hausdorff space with a countable network is second-countable.

(i) If a topological space X has a countable network, then any dense set in X includes a countable dense set; in particular, X is separable.

(j) If a topological space X has a countable network, then $C(X)$, with the topology of pointwise convergence inherited from the product topology of $\mathbb{R}^X$, has a countable network.

4A2O Second-countable spaces (a) Let $(X, \mathfrak{T})$ be a topological space and $\mathcal{U}$ a countable subbase for $\mathfrak{T}$. Then $\mathfrak{T}$ is second-countable.

(b) Any base of a second-countable space includes a countable base.

(c) A second-countable space has a countable network, so is separable and hereditarily Lindelöf.

(d) The product of a countable family of second-countable spaces is second-countable.

(e) If X is a second-countable space then $C(X)$, with the topology of uniform convergence on compact sets, has a countable network.

4A2P Separable metrizable spaces (a)(i) A metrizable space is second-countable iff it is separable.

(ii) A compact metrizable space is separable, so is second-countable and has a countable network.

(iii) Any base of a separable metrizable space includes a countable base, which is also a countable network, so the space is hereditarily Lindelöf.

(iv) Any subspace of a separable metrizable space is separable and metrizable.

(v) A countable product of separable metrizable spaces is separable and metrizable.

(b) A topological space is separable and metrizable iff it is second-countable, regular and Hausdorff.

(c) A Hausdorff continuous image of a compact metrizable space is metrizable.

(d) A metrizable space is separable iff it is ccc iff it is Lindelöf.

(e) If X is a compact metrizable space, then $C(X)$ is separable under its usual norm topology defined from the norm $\| \ \|_\infty$.

4A2Q Polish spaces: Proposition (a) A countable discrete space is Polish.

(b) A compact metrizable space is Polish.

(c) The product of a countable family of Polish spaces is Polish.

(d) A G_δ subset of a Polish space is Polish; in particular, a set which is either open or closed is Polish.

(e) The disjoint union of countably many Polish spaces is Polish.

(f) If X is any set and $\langle \mathfrak{T}_n \rangle_{n \in \mathbb{N}}$ is a sequence of Polish topologies on X such that $\mathfrak{T}_m \cap \mathfrak{T}_n$ is Hausdorff for all m, $n \in \mathbb{N}$, then the topology generated by $\bigcup_{n \in \mathbb{N}} \mathfrak{T}_n$ is Polish.

(g) If X is a Polish space, it is homeomorphic to a G_δ set in a compact metrizable space.

(h) If X is a locally compact Hausdorff space, it is Polish iff it has a countable network iff it is metrizable and σ-compact.

4A2R Order topologies Let $(X, \le)$ be a totally ordered set and $\mathfrak{T}$ its order topology.

(a) The set $\mathcal{U}$ of open intervals in X is a base for $\mathfrak{T}$.

(b) $[x, y]$, $[x, \infty[$ and $]{-\infty}, x]$ are closed sets for all x, $y \in X$.

(c) $\mathfrak{T}$ is Hausdorff, normal and countably paracompact.

(d) If $A \subseteq X$ then $\overline{A}$ is the set of elements of X expressible as either suprema or infima of non-empty subsets of A.

(e) A subset of X is closed iff it is order-closed.

(f) If $\langle x_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence in X with supremum x, then $x = \lim_{n \to \infty} x_n$.

(g) A set $K \subseteq X$ is compact iff $\sup A$ and $\inf A$ are defined in X and belong to K for every non-empty $A \subseteq K$.

(h) X is Dedekind complete iff $[x, y]$ is compact for all x, $y \in X$.

(i) X is compact iff it is either empty or Dedekind complete with greatest and least elements.

(j) Any open set $G \subseteq X$ is expressible as a union of disjoint open order-convex sets; if X is Dedekind complete, these will be open intervals.

(k) If X is well-ordered it is locally compact.

(l) In $X \times X$, $\{(x, y) : x < y\}$ is open and $\{(x, y) : x \le y\}$ is closed.

(m) If $F \subseteq X$ and *either* F is order-convex *or* F is compact *or* X is Dedekind complete and F is closed, then the subspace topology on F is induced by the inherited order of F.

(n) If X is ccc it is hereditarily Lindelöf, therefore perfectly normal.

(o) If Y is another totally ordered set with its order topology, an order-preserving function from X to Y is continuous iff it is order-continuous.

4A2S Order topologies on ordinals (a) Let ζ be an ordinal with its order topology.

(i) ζ is locally compact; all the sets $[0,\eta] =]{-\infty},\eta+1[$, for $\eta < \zeta$, are open and compact. If ζ is a successor ordinal, it is compact.

(ii) For any $A \subseteq \zeta$, $\overline{A} = \{\sup B : \emptyset \ne B \subseteq A, \sup B < \zeta\}$.

(iii) If $\xi \le \zeta$, then the subspace topology on ξ induced by the order topology of ζ is the order topology of ξ.

(b) Give ω_1 its order topology.

(i) ω_1 is first-countable.

(ii) Singleton subsets of ω_1 are zero sets.

(iii) If $f : \omega_1 \to \mathbb{R}$ is continuous, there is a $\xi < \omega_1$ such that $f(\eta) = f(\xi)$ for every $\eta \ge \xi$.

4A2T Topologies on spaces of subsets Let X be a topological space, and $\mathcal{C} = \mathcal{C}_X$ the family of closed subsets of X.

(a)(i) The **Vietoris topology** on $\mathcal{C}$ is the topology generated by sets of the forms

$$\{F : F \in \mathcal{C},\ F \cap G \ne \emptyset\}, \quad \{F : F \in \mathcal{C},\ F \subseteq G\}$$

where $G \subseteq X$ is open.

(ii) The **Fell topology** on $\mathcal{C}$ is the topology generated by sets of the forms

$$\{F : F \in \mathcal{C},\ F \cap G \ne \emptyset\}, \quad \{F : F \in \mathcal{C},\ F \cap K = \emptyset\}$$

where $G \subseteq X$ is open and $K \subseteq X$ is compact. If X is Hausdorff then the Fell topology is coarser than the Vietoris topology. If X is compact and Hausdorff the two topologies agree.

(iii) Suppose X is metrizable, and that ρ is a metric on X inducing its topology. For a non-empty subset A of X, write $\rho(x,A) = \inf_{y \in A} \rho(x,y)$ for every $x \in X$. Note that $x \mapsto \rho(x,A) : X \to \mathbb{R}$ is 1-Lipschitz.

For E, $F \in \mathcal{C} \setminus \{\emptyset\}$, set

$$\tilde\rho(E,F) = \min(1, \max(\sup_{x\in E} \rho(x,F), \sup_{y \in F} \rho(y,E))).$$

$\tilde\rho$ is a metric on $\mathcal{C} \setminus \{\emptyset\}$, the **Hausdorff metric**. $\tilde\rho(\{x\},\{y\}) = \min(1,\rho(x,y))$ for all x, $y \in X$.

(b)(i) The Fell topology is T_1.

(ii) The map $(E,F) \mapsto E \cup F : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is continuous for the Fell topology.

(iii) $\mathcal{C}$ is compact in the Fell topology.

(c) If X is Hausdorff, $x \mapsto \{x\}$ is continuous for the Fell topology on $\mathcal{C}$.

(d) If X and another topological space Y are regular, and $\mathcal{C}_Y$, $\mathcal{C}_{X\times Y}$ are the families of closed subsets of Y and $X \times Y$ respectively, then $(E,F) \mapsto E \times F : \mathcal{C}_X \times \mathcal{C}_Y \to \mathcal{C}_{X\times Y}$ is continuous when each space is given its Fell topology.

(e) Suppose that X is locally compact and Hausdorff.

(i) The set $\{(E,F) : E, F \in \mathcal{C}, E \subseteq F\}$ is closed in $\mathcal{C} \times \mathcal{C}$ for the product topology defined from the Fell topology on $\mathcal{C}$. $\{(x,F) : x \in F\}$ is closed in $X \times \mathcal{C}$ when $\mathcal{C}$ is given its Fell topology.

(ii) The Fell topology on $\mathcal{C}$ is Hausdorff. It follows that if $\langle F_i \rangle_{i\in I}$ is a family in $\mathcal{C}$, and $\mathcal{F}$ is an ultrafilter on I, then we have a well-defined limit $\lim_{i\to\mathcal{F}} F_i$ defined in $\mathcal{C}$ for the Fell topology.

(iii) If $\mathcal{L} \subseteq \mathcal{C}$ is compact, then $\bigcup \mathcal{L}$ is a closed subset of X.

(f) Suppose that X is metrizable, locally compact and separable. Then the Fell topology on $\mathcal{C}$ is metrizable.

(g) Suppose that X is metrizable, and that ρ is a metric inducing the topology of X; let $\tilde{\rho}$ be the corresponding Hausdorff metric on $\mathcal{C} \setminus \{\emptyset\}$.

(i) The topology $\mathfrak{S}_{\tilde{\rho}}$ defined by $\tilde{\rho}$ is finer than the Fell topology $\mathfrak{S}_F$ on $\mathcal{C} \setminus \{\emptyset\}$.

(ii) If X is compact, then $\mathfrak{S}_{\tilde{\rho}}$ and $\mathfrak{S}_F$ are the same, and both are compact.

4A2U Old friends (a) $\mathbb{R}$, with its usual topology, is metrizable and separable, so is second-countable. Every subset of $\mathbb{R}$ is separable; every dense subset of $\mathbb{R}$ has a countable subset which is still dense.

(b) $\mathbb{N}^{\mathbb{N}}$ is Polish in its usual topology, so has a countable network, and is hereditarily Lindelöf. Moreover, it is homeomorphic to $[0,1] \setminus \mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$.

(c) The map $x \mapsto \frac{2}{3}\sum_{j=0}^{\infty} 3^{-j}x(j)$ is a homeomorphism between $\{0,1\}^{\mathbb{N}}$ and the Cantor set $C \subseteq [0,1]$.

(d) If I is any set, then the map $A \mapsto \chi A : \mathcal{P}I \to \{0,1\}^I$ is a homeomorphism. So $\mathcal{P}I$ is zero-dimensional, compact and Hausdorff. If I is countable, then $\mathcal{P}I$ is metrizable, therefore Polish.

(e) Give the space $C([0,\infty[)$ the topology $\mathfrak{T}_c$ of uniform convergence on compact sets.

(i) $C([0,\infty[)$ is a Polish locally convex linear topological space.

(ii) Suppose that $A \subseteq C([0,\infty[)$ is such that $\{f(0) : f \in A\}$ is bounded and for every $a \ge 0$ and $\epsilon > 0$ there is a $\delta > 0$ such that $|f(s) - f(t)| \le \epsilon$ whenever $f \in A$, s, $t \in [0,a]$ and $|s-t| \le \delta$. Then A is relatively compact for $\mathfrak{T}_c$.

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4A3 Topological σ-algebras

I devote a section to some σ-algebras which can be defined on topological spaces. While 'measures' will not be mentioned here, the manipulation of these σ-algebras is an essential part of the technique of measure theory, and I will give proofs and exercises as if this were part of the main work. I look at Borel σ-algebras (4A3A-4A3J), Baire σ-algebras (4A3K-4A3P), spaces of càdlàg functions (4A3Q), Baire-property algebras (4A3R, 4A3S) and cylindrical σ-algebras on linear spaces (4A3U-4A3W).

4A3A Borel sets If $(X,\mathfrak{T})$ is a topological space, the **Borel σ-algebra** of X is the σ-algebra $\mathcal{B}(X)$ of subsets of X generated by $\mathfrak{T}$. Its elements are the **Borel sets** of X. If $(Y,\mathfrak{S})$ is another topological space with Borel σ-algebra $\mathcal{B}(Y)$, a function $f : X \to Y$ is **Borel measurable** if $f^{-1}[H] \in \mathcal{B}(X)$ for every $H \in \mathfrak{S}$, and is a **Borel isomorphism** if it is a bijection and $\mathcal{B}(Y) = \{F : F \subseteq Y,\ f^{-1}[F] \in \mathcal{B}(X)\}$.

4A3B (Σ,T)-measurable functions (a) Let X and Y be sets, with σ-algebras $\Sigma \subseteq \mathcal{P}X$ and $\mathrm{T} \subseteq \mathcal{P}Y$. A function $f : X \to Y$ is (Σ,T)-**measurable** if $f^{-1}[F] \in \Sigma$ for every $F \in \mathrm{T}$.

(b) If Σ, T and Υ are σ-algebras of subsets of X, Y and Z respectively, and $f : X \to Y$ is (Σ,T)-measurable while $g : Y \to Z$ is (T,Υ)-measurable, then $gf : X \to Z$ is (Σ,Υ)-measurable.

(c) Let $\langle X_i \rangle_{i \in I}$ be a family of sets with product X, Y another set, and $f : X \to Y$ a function. If $\mathrm{T} \subseteq \mathcal{P}Y$, $\Sigma_i \subseteq \mathcal{P}X_i$ are σ-algebras for each i, then f is $(\mathrm{T},\widehat{\bigotimes}_{i \in I}\Sigma_i)$-measurable iff $\pi_i f : Y \to X_i$ is (T,Σ_i)-measurable for every i, where $\pi_i : X \to X_i$ is the coordinate map.

4A3C Elementary facts (a) If X is a topological space and Y is a subspace of X, then $\mathcal{B}(Y)$ is just the subspace σ-algebra $\{E \cap Y : E \in \mathcal{B}(X)\}$.

(b) If X is a set, Σ is a σ-algebra of subsets of X, $(Y,\mathfrak{S})$ is a topological space and $f : X \to Y$ is a function, then f is $(\Sigma,\mathcal{B}(Y))$-measurable iff $f^{-1}[H] \in \Sigma$ for every $H \in \mathfrak{S}$.

(c) If X and Y are topological spaces, and $f : X \to Y$ is a function, then f is Borel measurable iff it is $(\mathcal{B}(X), \mathcal{B}(Y))$-measurable. So if X, Y and Z are topological spaces and $f : X \to Y$, $g : Y \to Z$ are Borel measurable functions, then $gf : X \to Z$ is Borel measurable.

(d) If X and Y are topological spaces and $f : X \to Y$ is continuous, it is Borel measurable.

(e) If X is a topological space and $f : X \to [-\infty, \infty]$ is lower semi-continuous, then it is Borel measurable.

(f) If $\langle X_i \rangle_{i \in I}$ is a family of topological spaces with product X, then $\mathcal{B}(X) \supseteq \widehat{\bigotimes}_{i \in I} \mathcal{B}(X_i)$.

(g) Let X be a topological space.

(i) The algebra $\mathfrak{A}$ of subsets generated by the open sets is precisely the family of sets expressible as a disjoint union $\bigcup_{i \le n} G_i \cap F_i$ where every G_i is open and every F_i is closed.

(ii) $\mathcal{B}(X)$ is the smallest family $\mathcal{E} \supseteq \mathfrak{A}$ such that $\bigcup_{n \in \mathbb{N}} E_n \in \mathcal{E}$ for every non-decreasing sequence $\langle E_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{E}$ and $\bigcap_{n \in \mathbb{N}} E_n \in \mathcal{E}$ for every non-increasing sequence $\langle E_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{E}$.

4A3D Hereditarily Lindelöf spaces (a) Suppose that X is a hereditarily Lindelöf space and $\mathcal{U}$ is a subbase for the topology of X. Then $\mathcal{B}(X)$ is the σ-algebra of subsets of X generated by $\mathcal{U}$.

(b) Let X be a set, Σ a σ-algebra of subsets of X, Y a hereditarily Lindelöf space, $\mathcal{U}$ a subbase for the topology of Y, and $f : X \to Y$ a function. If $f^{-1}[U] \in \Sigma$ for every $U \in \mathcal{U}$, then f is $(\Sigma, \mathcal{B}(Y))$-measurable.

(c) Let $\langle X_i \rangle_{i \in I}$ be a family of topological spaces with product X. Suppose that X is hereditarily Lindelöf.

(i) $\mathcal{B}(X) = \widehat{\bigotimes}_{i \in I} \mathcal{B}(X_i)$.

(ii) If Y is another topological space, then a function $f : Y \to X$ is Borel measurable iff $\pi_i f : Y \to X_i$ is Borel measurable for every $i \in I$, where $\pi_i : X \to X_i$ is the canonical map.

4A3F Spaces with countable networks (a) Let X be a topological space with a countable network. Then $\#(\mathcal{B}(X)) \le \mathfrak{c}$.

(b) $\#(\mathcal{B}(\mathbb{N}^{\mathbb{N}})) = \mathfrak{c}$.

4A3G Second-countable spaces (a) Suppose that X is a second-countable space and Y is any topological space. Then $\mathcal{B}(X \times Y) = \mathcal{B}(X) \widehat{\otimes} \mathcal{B}(Y)$.

(b) If X is any topological space, Y is a T_0 second-countable space, and $f : X \to Y$ is Borel measurable, then (the graph of) f is a Borel set in $X \times Y$.

4A3H Borel sets in Polish spaces: Proposition Let $(X, \mathfrak{T})$ be a Polish space and $E \subseteq X$ a Borel set. Then there is a Polish topology $\mathfrak{S}$ on X, including $\mathfrak{T}$, for which E is open.

4A3I Corollary If $(X, \mathfrak{T})$ is a Polish space and $\langle E_n \rangle_{n \in \mathbb{N}}$ is a sequence of Borel subsets of X, then there is a zero-dimensional Polish topology $\mathfrak{S}$ on X, including $\mathfrak{T}$, for which every E_n is open-and-closed.

4A3J Borel sets in ω_1: Proposition A set $E \subseteq \omega_1$ is a Borel set iff either E or its complement includes a closed cofinal set.

4A3K Baire sets (a) Definition Let X be a topological space. The **Baire σ-algebra** $\mathcal{B}\mathfrak{a}(X)$ of X is the σ-algebra generated by the zero sets. Members of $\mathcal{B}\mathfrak{a}(X)$ are called **Baire** sets.

(b) For any topological space X, $\mathcal{B}\mathfrak{a}(X) \subseteq \mathcal{B}(X)$. If $\mathfrak{T}$ is perfectly normal then $\mathcal{B}\mathfrak{a}(X) = \mathcal{B}(X)$.

(c) Let X and Y be topological spaces, with Baire σ-algebras $\mathcal{B}\mathfrak{a}(X)$, $\mathcal{B}\mathfrak{a}(Y)$ respectively. If $f : X \to Y$ is continuous, it is $(\mathcal{B}\mathfrak{a}(X), \mathcal{B}\mathfrak{a}(Y))$-measurable.

(d) In particular, if X is a subspace of Y, then $E \cap X \in \mathcal{B}\mathfrak{a}(X)$ whenever $E \in \mathcal{B}\mathfrak{a}(Y)$. $F \cap X$ is a zero set in X for every zero set $F \subseteq Y$.

(e) If X is a topological space and Y is a separable metrizable space, a function $f : X \to Y$ is **Baire measurable** if $f^{-1}[H] \in \mathcal{B}\mathfrak{a}(X)$ for every open $H \subseteq Y$. f is Baire measurable iff it is $(\mathcal{B}\mathfrak{a}(X), \mathcal{B}(Y))$-measurable iff it is $(\mathcal{B}\mathfrak{a}(X), \mathcal{B}\mathfrak{a}(Y))$-measurable.

4A3L Lemma Let $(X, \mathfrak{T})$ be a topological space. Then $\mathcal{B}\mathfrak{a}(X)$ is just the smallest σ-algebra of subsets of X with respect to which every continuous real-valued function on X is measurable.

4A3M Product spaces Let $\langle X_i \rangle_{i \in I}$ be a family of topological spaces with product X.

(a) $\mathcal{B}\mathfrak{a}(X) \supseteq \widehat{\bigotimes}_{i \in I} \mathcal{B}\mathfrak{a}(X_i)$.

(b) Suppose that X is ccc. Then every Baire subset of X is determined by coordinates in a countable set.

4A3N Products of separable metrizable spaces: Proposition Let $\langle X_i \rangle_{i \in I}$ be a family of separable metrizable spaces, with product X.

(a) $\mathcal{B}\mathfrak{a}(X) = \widehat{\bigotimes}_{i \in I} \mathcal{B}\mathfrak{a}(X_i) = \widehat{\bigotimes}_{i \in I} \mathcal{B}(X_i)$.

(b) $\mathcal{B}\mathfrak{a}(X)$ is the family of those Borel subsets of X which are determined by coordinates in countable sets.

(c) A set $Z \subseteq X$ is a zero set iff it is closed and determined by coordinates in a countable set.

(d) If Y is a dense subset of X, then the Baire σ-algebra $\mathcal{B}\mathfrak{a}(Y)$ of Y is just the subspace σ-algebra $\mathcal{B}\mathfrak{a}(X)_Y$ induced by $\mathcal{B}\mathfrak{a}(X)$.

(e) If Y is a set, T is a σ-algebra of subsets of Y, and $f : Y \to X$ is a function, then f is $(\mathrm{T}, \mathcal{B}\mathfrak{a}(X))$-measurable iff $\pi_i f : Y \to X_i$ is $(\mathrm{T}, \mathcal{B}(X_i))$-measurable for every $i \in I$, where $\pi_i(x) = x(i)$ for $x \in X$ and $i \in I$.

4A3O Compact spaces (a) Let $(X, \mathfrak{T})$ be a topological space, $\mathcal{U}$ a subbase for $\mathfrak{T}$, and $\mathfrak{A}$ the algebra of subsets of X generated by $\mathcal{U}$. If $H \subseteq X$ is open and $K \subseteq H$ is compact, there is an open $E \in \mathfrak{A}$ such that $K \subseteq E \subseteq H$.

(b) Let $(X, \mathfrak{T})$ be a compact space and $\mathcal{U}$ a subbase for $\mathfrak{T}$. Then every open-and-closed subset of X belongs to the algebra of subsets of X generated by $\mathcal{U}$.

(c) Let $(X, \mathfrak{T})$ be a compact space and $\mathcal{U}$ a subbase for $\mathfrak{T}$. Then $\mathcal{B}\mathfrak{a}(X)$ is included in the σ-algebra of subsets of X generated by $\mathcal{U}$.

(d) In a compact Hausdorff zero-dimensional space the Baire σ-algebra is the σ-algebra generated by the open-and-closed sets.

(e) Let $\langle X_i \rangle_{i \in I}$ be a family of compact Hausdorff spaces with product X. Then $\mathcal{B}\mathfrak{a}(X) = \widehat{\bigotimes}_{i \in I} \mathcal{B}\mathfrak{a}(X_i)$.

(f) In particular, for any set I, $\mathcal{B}\mathfrak{a}(\{0,1\}^I)$ is the σ-algebra generated by sets of the form $\{x : x(i) = 1\}$ as i runs over I.

4A3P Proposition The Baire σ-algebra $\mathcal{B}\mathfrak{a}(\omega_1)$ of ω_1 is the countable-cocountable algebra.

4A3Q Càdlàg functions Let X be a Polish space, and C_{dlg} the set of càdlàg functions from $[0, \infty[$ to X, with its topology of pointwise convergence inherited from $X^{[0,\infty[}$.

(a) $\mathcal{B}\mathfrak{a}(C_{\mathrm{dlg}})$ is the subspace σ-algebra induced by $\mathcal{B}\mathfrak{a}(X^{[0,\infty[})$.

(b) $(C_{\text{dlg}}, \mathcal{B}\mathfrak{a}(C_{\text{dlg}}))$ is a standard Borel space.

(c)(i) For any $t \geq 0$, let $\mathcal{B}\mathfrak{a}_t(C_{\text{dlg}})$ be the σ-algebra of subsets of C_{dlg} generated by the functions $\omega \mapsto \omega(s)$ for $s \leq t$. Then $(\omega, s) \mapsto \omega(s) : C_{\text{dlg}} \times [0, t] \to X$ is $\mathcal{B}\mathfrak{a}_t(C_{\text{dlg}}) \widehat{\otimes} \mathcal{B}([0, t])$-measurable.

(ii) $(\omega, t) \mapsto \omega(t) : C_{\text{dlg}} \times [0, \infty[\to X$ is $\mathcal{B}\mathfrak{a}(C_{\text{dlg}}) \widehat{\otimes} \mathcal{B}([0, \infty[)$-measurable.

(d) The set $C([0, \infty[\,; X)$ of continuous functions from $[0, \infty[$ to X belongs to $\mathcal{B}\mathfrak{a}(C_{\text{dlg}})$.

4A3R Baire property Let X be a topological space, and $\mathcal{M}$ the ideal of meager subsets of X. A subset X has the **Baire property** if it is expressible in the form $G \triangle M$ where $G \subseteq X$ is open and $M \in \mathcal{M}$; $A \subseteq X$ has the Baire property if there is an open set $G \subseteq X$ such that $G \triangle A$ is meager. The family $\widehat{\mathcal{B}}(X)$ of all such sets is the **Baire-property algebra** of X. The quotient algebra $\widehat{\mathcal{B}}(X)/\mathcal{M}$ is the **category algebra** of X.

4A3S Proposition Let X be a topological space.

(a) Let $A \subseteq X$ be any set.

(i) There is a largest open set $G \subseteq X$ such that $A \cap G$ is meager.

(ii) $H = X \setminus \overline{G}$ is the smallest regular open subset of X such that $A \setminus H$ is meager; $H \subseteq \overline{A}$.

(iii) H is in itself a Baire space.

(iv) If $A \in \widehat{\mathcal{B}}(X)$, $H \triangle A$ is meager.

(v) If X is a Baire space and $A \in \widehat{\mathcal{B}}(X)$, then H is the largest open subset of X such that $H \setminus A$ is meager.

(b)(i) $\widehat{\mathcal{B}}(X)$ is a σ-algebra of subsets of X including $\mathcal{B}(X)$.

(ii) $\widehat{\mathcal{B}}(X) = \{G \triangle M : G \subseteq X \text{ is a regular open set}, M \in \mathcal{M}\}$.

(c) If X has a countable network, its category algebra has a countable order-dense set.

***4A3T Lemma** Let X and Y be sets, Σ a σ-algebra of subsets of X, T a σ-algebra of subsets of Y and $\mathcal{J}$ a σ-ideal of T. Suppose that the quotient Boolean algebra $\mathrm{T}/\mathcal{J}$ has a countable order-dense set.

(a) $\{x : x \in X, W[\{x\}] \cap A \in \mathcal{J}\}$ belongs to Σ for any $W \in \Sigma \widehat{\otimes} \mathrm{T}$ and $A \subseteq Y$.

(b) For every $W \in \Sigma \widehat{\otimes} \mathrm{T}$ there are sequences $\langle E_n \rangle_{n \in \mathbb{N}}$ in Σ, $\langle V_n \rangle_{n \in \mathbb{N}}$ in T such that $(W \triangle W_1)[\{x\}] \in \mathcal{J}$ for every $x \in X$, where $W_1 = \bigcup_{n \in \mathbb{N}} E_n \times V_n$.

4A3U Cylindrical σ-algebras: Definition Let X be a linear topological space. Then the **cylindrical σ-algebra** of X is the smallest σ-algebra Σ of subsets of X such that every continuous linear functional on X is Σ-measurable.

4A3V Proposition Let X be a linear topological space and $\mathfrak{T}_s = \mathfrak{T}_s(X, X^*)$ its weak topology. Then the cylindrical σ-algebra of X is just the Baire σ-algebra of $(X, \mathfrak{T}_s)$.

4A3W Proposition Let $(X, \mathfrak{T})$ be a separable metrizable locally convex linear topological space, and $\mathfrak{T}_s = \mathfrak{T}_s(X, X^*)$ its weak topology. Then the cylindrical σ-algebra of X is also both the Baire σ-algebra and the Borel σ-algebra for both $\mathfrak{T}$ and $\mathfrak{T}_s$.

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4A4 Locally convex spaces

As in §3A5, all the ideas, and nearly all the results as stated below, are applicable to complex linear spaces; but for the purposes of this volume the real case will almost always be sufficient, and for definiteness you may take it that the scalar field is $\mathbb{R}$, except in 4A4J-4A4K. (Complex Hilbert spaces arise naturally in §445.)

4A4A Linear spaces (a) If U is a linear space, a **Hamel basis** for U is a maximal linearly independent family $\langle u_i \rangle_{i \in I}$ in U, so that every member of U is uniquely expressible as $\sum_{i \in J} \alpha_i u_i$ for some finite $J \subseteq I$ and $\langle \alpha_i \rangle_{i \in J} \in (\mathbb{R} \setminus \{0\})^J$.

(b) Every linear space has a Hamel basis.

(c) If U is a linear space, I write U' for the **algebraic dual** of U, the linear space of all linear functionals from U to $\mathbb{R}$.

4A4B Linear topological spaces (a) If U is a linear topological space, and V is a linear subspace of U, then V, with the linear structure and topology induced by those of U, is again a linear topological space.

(b) If $\langle U_i\rangle_{i\in I}$ is any family of linear topological spaces, then $U=\prod_{i\in I}U_i$, with the product linear space structure and topology, is again a linear topological space. $\mathbb{R}^X$, with its usual linear and topological structures, is a linear topological space, for any set X.

(c) If U and V are linear topological spaces, the set of continuous linear operators from U to V is a linear subspace of the space $\mathsf{L}(U;V)$ of all linear operators from U to V. If U, V and W are linear topological spaces, and $T:U\to V$ and $S:V\to W$ are continuous linear operators, then $ST:U\to W$ is a continuous linear operator.

(d) If U is a linear topological space, I will write U^* for the **dual** of U, the space of all continuous linear functionals from U to $\mathbb{R}$. U^* is a linear subspace of U'. The **weak topology** on U, $\mathfrak{T}_s(U,U^*)$, is that defined from the seminorms $u\mapsto|f(u)|$ as f runs over U^*. The **weak* topology** on U^*, $\mathfrak{T}_s(U^*,U)$, is that defined from the seminorms $f\mapsto|f(u)|$ as u runs over U. If U and V are linear topological spaces, $T:U\to V$ is a continuous linear operator, and $g\in V^*$, then $gT\in U^*$; T is $(\mathfrak{T}_s(U,U^*),\mathfrak{T}_s(V,V^*))$-continuous.

(e) If $U=\prod_{i\in I}U_i$ is a product of linear topological spaces, then every element of U^* is of the form $u\mapsto\sum_{i\in J}f_i(u(i))$ where $J\subseteq I$ is finite and $f_i\in U_i^*$ for every $i\in J$. Consequently the weak topology on U is the product of the weak topologies on the U_i.

(f) Let U be a linear topological space. For $A\subseteq U$ write A° for its **polar** set $\{f:f\in U^*,\ f(x)\le 1$ for every $x\in A\}$ in U^*. If G is a neighbourhood of 0 in U, then G° is a $\mathfrak{T}_s(U^*,U)$-compact subset of U^*.

(g) Let U be a linear topological space. If $D\subseteq U$ is non-empty and closed under addition and multiplication by rationals, $\overline{D}$ is a linear subspace of U. If $A\subseteq U$ is separable, then the closed linear subspace generated by A is separable.

(h) If $\langle u_i\rangle_{i\in I}$ is an indexed family in a Hausdorff linear topological space U and $u\in U$, we say that $u=\sum_{i\in I}u_i$ if for every neighbourhood G of u there is a finite set $J\subseteq I$ such that $\sum_{i\in K}u_i\in G$ whenever $K\subseteq I$ is finite and $J\subseteq K$.

If $\langle v_i\rangle_{i\in I}$ is another family with the same index set, and $v=\sum_{i\in I}v_i$ is defined, then $\sum_{i\in I}(u_i+v_i)$ is defined and equal to $u+v$.

If now V is another Hausdorff linear topological space and $T:U\to V$ is a continuous linear operator, $\sum_{i\in I}Tu_i=T(\sum_{i\in I}u_i)$ if the right-hand-side is defined.

(i) If U is a Hausdorff linear topological space, then any finite-dimensional linear subspace of U is closed.

(j) If U is a first-countable Hausdorff linear topological space which (regarded as a linear topological space) is complete, then there is a metric ρ on U, defining its topology, under which U is complete.

4A4C Locally convex spaces (a) A linear topological space is **locally convex** if the convex open sets form a base for the topology.

(b) A linear topological space is locally convex iff its topology can be defined by a family of seminorms.

(c) Let U be a linear space and τ a seminorm on U. $N_\tau=\{u:\tau(u)=0\}$ is a linear subspace of X. On the quotient space U/N_τ we have a norm defined by setting $\|u^\bullet\|=\tau(u)$ for every $u\in U$.

(d) Let U be a locally convex linear topological space, and T the family of continuous seminorms on U. For each $\tau \in \mathrm{T}$, write $N_\tau = \{u : \tau(u) = 0\}$ and π_τ for the canonical map from U to $U_\tau = U/N_\tau$. Give each U_τ its norm, and set $\mathcal{G}_\tau = \{\pi_\tau^{-1}[H] : H \subseteq U_\tau \text{ is open}\}$. Then $\bigcup_{\tau \in \mathrm{T}} \mathcal{G}_\tau$ is a base for the topology of X closed under finite unions.

(e) A linear subspace of a locally convex linear topological space is locally convex. The product of any family of locally convex linear topological spaces is locally convex.

(f) If U is a metrizable locally convex linear topological space, its topology can be defined by a sequence of seminorms.

(g) Let U be a linear space and V a linear subspace of U'. Let $\mathfrak{T}_s(V, U)$ be the topology on V generated by the seminorms $f \mapsto |f(u)|$ as u runs over U, and let $\phi : V \to \mathbb{R}$ be a $\mathfrak{T}_s(V, U)$-continuous linear functional. Then there is a $u \in U$ such that $\phi(f) = f(u)$ for every $f \in V$.

(h) Grothendieck's theorem If U is a complete locally convex Hausdorff linear topological space, and ϕ is a linear functional on the dual U^* such that $\phi \restriction G^\circ$ is $\mathfrak{T}_s(U^*, U)$-continuous for every neighbourhood G of 0 in U, then ϕ is of the form $f \mapsto f(u)$ for some $u \in U$.

4A4D Hahn-Banach theorem (a) Let U be a linear space and $\theta : U \to [0, \infty[$ a seminorm.

(i) If $V \subseteq U$ is a linear subspace and $g : V \to \mathbb{R}$ is a linear functional such that $|g(v)| \le \theta(v)$ for every $v \in V$, then there is a linear functional $f : U \to \mathbb{R}$, extending g, such that $|f(u)| \le \theta(u)$ for every $u \in U$.

(ii) If $u_0 \in U$ then there is a linear functional $f : U \to \mathbb{R}$ such that $f(u_0) = \theta(u_0)$ and $|f(u)| \le \theta(u)$ for every $u \in U$.

(b) Let U be a linear topological space and G, H two disjoint convex sets in U, of which one has non-empty interior. Then there are a non-zero $f \in U^*$ and an $\alpha \in \mathbb{R}$ such that $f(u) \le \alpha \le f(v)$ for every $u \in G$, $v \in H$, so that $f(u) < \alpha$ for every $u \in \operatorname{int} G$ and $\alpha < f(v)$ for every $u \in \operatorname{int} H$.

4A4E The Hahn-Banach theorem in locally convex spaces Let U be a locally convex linear topological space.

(a) If $V \subseteq U$ is a linear subspace, then every member of V^* extends to a member of U^*. $\mathfrak{T}_s(V, V^*)$ is the subspace topology on V induced by $\mathfrak{T}_s(U, U^*)$.

(b) Let $C \subseteq U$ be a non-empty closed convex set. If $u \in U$ then $u \in C$ iff $f(u) \le \sup_{v \in C} f(v)$ for every $f \in U^*$ iff $f(u) \ge \inf_{v \in C} f(v)$ for every $f \in U^*$.

If $V \subseteq U$ is a closed linear subspace and $u \in U \setminus V$ there is an $f \in U^*$ such that $f(u) \ne 0$ and $f(v) = 0$ for every $v \in V$.

(c) If U is Hausdorff, U^* separates its points.

(d) If $u \in U$ belongs to the $\mathfrak{T}_s(U, U^*)$-closure of a convex set $C \subseteq U$, it belongs to the closure of C. In particular, if C is closed, it is $\mathfrak{T}_s(U, U^*)$-closed.

(e) If C, $C' \subseteq U$ are disjoint non-empty closed convex sets, of which one is compact, there is an $f \in U^*$ such that $\sup_{u \in C} f(u) < \inf_{u \in C'} f(u)$.

(f) Let V be a linear subspace of U'. Let $K \subseteq U$ be a non-empty $\mathfrak{T}_s(U, V)$-compact convex set, and $\phi_0 : V \to \mathbb{R}$ a linear functional such that $\phi_0(f) \le \sup_{u \in K} f(u)$ for every $f \in V$. Then there is a $u_0 \in K$ such that $\phi_0(f) = f(u_0)$ for every $f \in V$.

(g) The Bipolar Theorem Let $A \subseteq U'$ be a non-empty set. Set $A^\circ = \{u : u \in U,\ f(u) \le 1 \text{ for every } f \in A\}$. If $g \in U'$ is such that $g(u) \le 1$ for every $u \in A^\circ$, then g belongs to the $\mathfrak{T}_s(U', U)$-closed convex hull of $A \cup \{0\}$.

(h) Let W be a linear subspace of U' separating the points of U. Then W is $\mathfrak{T}_s(U',U)$-dense in U'.

4A4F The Mackey topology Let U be a linear space and V a linear subspace of U'. The **Mackey topology** $\mathfrak{T}_k(V,U)$ on V is the topology of uniform convergence on convex $\mathfrak{T}_s(U,V)$-compact subsets of U. Every $\mathfrak{T}_k(V,U)$-continuous linear functional on V is of the form $f \mapsto f(u)$ for some $u \in U$. So every $\mathfrak{T}_k(V,U)$-closed convex set is $\mathfrak{T}_s(V,U)$-closed.

4A4G Extreme points (a) Let X be a real linear space, and $C \subseteq X$ a convex set. An element of C is an **extreme** point of C if it is not expressible as a convex combination of two other members of C.

(b) The Kreĭn-Mil'man theorem Let U be a Hausdorff locally convex linear topological space and $K \subseteq U$ a compact convex set. Then K is the closed convex hull of the set of its extreme points.

(c) Let U and V be Hausdorff locally convex linear topological spaces, $T : U \to V$ a continuous linear operator, $K \subseteq X$ a compact convex set and v any extreme point of $T[K] \subseteq V$. Then there is an extreme point u of K such that $Tu = v$.

4A4H Proposition Let I be a set, W a closed linear subspace of $\mathbb{R}^I$, U a linear topological space and V a Hausdorff linear topological space. Let $K \subseteq U$ be a compact set and $T : U \times \mathbb{R}^I \to V$ a continuous linear operator. Then $T[K \times W]$ is closed.

4A4I Normed spaces (a) Two norms $\| \,\|$, $\| \,\|'$ on a linear space U give rise to the same topology iff they are **equivalent** in the sense that, for some $M \ge 0$,

$$\|x\| \le M\|x\|', \quad \|x\|' \le M\|x\|$$

for every $x \in U$.

(b) If U and V are normed spaces, $T : U \to V$ is a linear operator and $gT : U \to \mathbb{R}$ is continuous for every $g \in V^*$, then T is a bounded operator.

(c) If U is any normed space, its dual U^*, under its usual norm, is a Banach space.

(d) If U is a separable normed space, its dual U^* is isometrically isomorphic to a closed linear subspace of ℓ^∞.

(e) Let U be a Banach space. Suppose that $\langle u_i \rangle_{i \in I}$ is a family in U such that $\gamma = \sum_{i \in I} \|u_i\| < \infty$.

(i) $\sum_{i \in I} u_i$ is defined.

(ii) Now if $\langle I_j \rangle_{j \in J}$ is any partition of I, $w_j = \sum_{i \in I_j} u_i$ is defined for every j, and $\sum_{j \in J} w_j$ is defined and equal to $\sum_{i \in I} u_i$.

(f) Let U be a normed space. For $u \in U$, define $\hat{u} \in U^{**} = (U^*)^*$ by setting $\hat{u}(f) = f(u)$ for every $f \in U^*$. Then $\{\hat{u} : u \in U,\, \|u\| \le 1\}$ is weak*-dense in $\{\phi : \phi \in U^{**},\, \|\phi\| = 1\}$.

4A4J Inner product spaces (a) Let U be an inner product space over $\begin{smallmatrix}\mathbb{R}\\\mathbb{C}\end{smallmatrix}$. An **orthonormal family** in U is a family $\langle e_i \rangle_{i \in I}$ in U such that $(e_i|e_j) = 0$ if $i \ne j$, 1 if $i = j$. An **orthonormal basis** in U is an orthonormal family $\langle e_i \rangle_{i \in I}$ in U such that the closed linear subspace of U generated by $\{e_i : i \in I\}$ is U itself.

(b) If U, V are inner product spaces over $\mathbb{R}$ and $T : U \to V$ is an isometry such that $T(0) = 0$, then $(Tu|Tv) = (u|v)$ for all u, $v \in U$ and T is linear.

(c) If U, V are inner product spaces over $\mathbb{C}$ and $T : U \to V$ is a linear operator such that $\|Tu\| = \|u\|$ for every $u \in U$, then $(Tu|Tv) = (u|v)$ for all u, $v \in U$.

(d) If U is an inner product space over $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$, a linear operator $T : U \to U$ is **self-adjoint** if $(Tu|v) = (u|Tv)$ for all u, $v \in U$.

(e) If U is a finite-dimensional inner product space over $\mathbb{R}$, it is isomorphic to Euclidean space $\mathbb{R}^r$, where $r = \dim U$. In particular, any finite-dimensional inner product space is a Hilbert space.

(f) If U is an inner product space over $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$ and $V \subseteq U$ is a linear subspace of U, then $V^\perp = \{x : x \in U, \, (x|y) = 0 \text{ for every } y \in V\}$ is a linear subspace of U, and $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ for $x \in V$, $y \in V^\perp$. If V is complete (in particular, if V is finite-dimensional), then $U = V \oplus V^\perp$.

(g) If U is an inner product space over $\mathbb{R}$ and v_1, $v_2 \in U$ are such that $\|v_1\| = \|v_2\| = 1$, there is a linear operator $T : U \to U$ such that $Tv_1 = v_2$ and $\|Tu\| = \|u\|$ and $\|Tu - u\| \le \|v_1 - v_2\|\|u\|$ for every $u \in U$.

(h) Let U be an inner product space over $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$, and $\langle u_i \rangle_{i \in I}$ a countable family in U. Then there is a countable orthonormal family $\langle v_j \rangle_{j \in J}$ in U such that $\{v_j : j \in J\}$ and $\{u_i : i \in I\}$ span the same linear subspace of U.

(i) Let U be an inner product space over $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$, and $\langle e_i \rangle_{i \in I}$ an orthonormal family in U. Then $\sum_{i \in I} |(u|e_i)|^2 \le \|u\|^2$ for every $u \in U$.

(j) Let U be an inner product space over $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$, and $C \subseteq U$ a convex set. Then there is at most one point $u \in C$ such that $\|u\| \le \|v\|$ for every $v \in C$.

For such a u, $\|u\|^2 \le \mathcal{R}\mathrm{e}(u|v)$ for every $v \in C$.

4A4K Hilbert spaces (a) If U is a real or complex Hilbert space, its unit ball is compact in the weak topology $\mathfrak{T}_s(U, U^*)$; any bounded set is relatively compact for $\mathfrak{T}_s(U, U^*)$.

(b) If U is a real or complex Hilbert space, any norm-bounded sequence in U has a weakly convergent subsequence.

(c) If U is a real or complex Hilbert space and $\langle u_i \rangle_{i \in I}$ is any orthonormal family in U, then it can be extended to an orthonormal basis. In particular, U has an orthonormal basis.

4A4L Compact operators (a) Let U, V and W be Banach spaces. If $T \in \mathsf{B}(U;V)$ and $S \in \mathsf{B}(V;W)$ and either S or T is a compact operator, then ST is compact.

(b) If U is a Banach space, $T \in \mathsf{B}(U;U)$ is a compact linear operator and $\gamma \ne 0$ then $\{u : Tu = \gamma u\}$ is finite-dimensional.

4A4M Self-adjoint compact operators If U is a Hilbert space and $T : U \to U$ is a self-adjoint compact linear operator, then $T[U]$ is included in the closed linear span of $\{Tv : v \text{ is an eigenvector of } T\}$.

4A4N Max-flow Min-cut Theorem Let (V, E, γ) be a (finite) transportation network, that is,

V is a finite set of 'vertices',

$E \subseteq \{(v, v') : v, \, v' \in V, \, v \ne v'\}$ is a set of (directed) 'edges',

$\gamma : E \to [0, \infty[$ is a function;

we regard a member $e = (v, v')$ of E as 'starting' at v and 'ending' at v', and $\gamma(e)$ is the 'capacity' of the edge e. Suppose that v_0, $v_1 \in V$ are distinct vertices such that no edge ends at v_0 and no edge starts at v_1. Then we have a 'flow' $\phi : E \to [0, \infty[$ and a 'cut' $X \subseteq E$ such that

(i) for every $v \in V \setminus \{v_0, v_1\}$,

$$\sum_{e \in E, e \text{ starts at } v} \phi(e) = \sum_{e \in E, e \text{ ends at } v} \phi(e),$$

(ii) $\phi(e) \le \gamma(e)$ for every $e \in E$,

(iii) there is no path from v_0 to v_1 using only edges in $E \setminus X$,

(iv) $\sum_{e \in E, e \text{ starts at } v_0} \phi(e) = \sum_{e \in E, e \text{ ends at } v_1} \phi(e) = \sum_{e \in X} \gamma(e)$.

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4A5 Topological groups

For Chapter 44 we need a variety of facts about topological groups. Most are essentially elementary, and all the non-trivial ideas are covered by at least one of Császár 78 and Hewitt & Ross 63. In 4A5A-4A5C and 4A5I I give some simple definitions concerning groups and group actions. Topological groups, properly speaking, appear in 4A5D. Their simplest properties are in 4A5E-4A5G. I introduce 'right' and 'bilateral' uniformities in 4A5H; the latter are the more interesting (4A5M-4A5O), but the former are also important. 4A5J-4A5L deal with quotient spaces, including spaces of cosets of non-normal subgroups. I conclude with notes on metrizable groups (4A5Q-4A5S).

4A5A Notation If X is a group, $x_0 \in X$, and A, $B \subseteq X$ I write

$$x_0A = \{x_0x : x \in A\}, \quad Ax_0 = \{xx_0 : x \in A\},$$

$$AB = \{xy : x \in A,\, y \in B\}, \quad A^{-1} = \{x^{-1} : x \in A\}.$$

A is **symmetric** if $A = A^{-1}$.

4A5B Group actions (a) If X is a group and Z is a set, an **action** of X on Z is a function $(x,z) \mapsto x{\bullet}z : X \times Z \to Z$ such that

$$(xy){\bullet}z = x{\bullet}(y{\bullet}z) \text{ for all } x,\, y \in X \text{ and } z \in Z,$$

$$e{\bullet}z = z \text{ for every } z \in Z$$

where e is the identity of X.

In this context I may say that 'X acts on Z', taking the operation $\bullet$ for granted.

(b) An action $\bullet$ of a group X on a set Z is **transitive** if for every w, $z \in Z$ there is an $x \in X$ such that $x{\bullet}w = z$.

(c) If $\bullet$ is an action of a group X on a set Z, I write $x{\bullet}A = \{x{\bullet}z : z \in A\}$ whenever $x \in X$ and $A \subseteq Z$.

(d) If $\bullet$ is an action of a group X on a set Z, then $z \mapsto x{\bullet}z : Z \to Z$ is a permutation for every $x \in X$. So if Z is a topological space and $z \mapsto x{\bullet}z$ is continuous for every x, it is a homeomorphism for every x.

(e) An action $\bullet$ of a group X on a set Z is **faithful** if whenever x, $y \in X$ are distinct there is a $z \in Z$ such that $x{\bullet}z \neq y{\bullet}z$.

(f) If $\bullet$ is an action of a group X on a set Z, then $Y_z = \{x : x \in X,\, x{\bullet}z = z\}$ is a subgroup of X (the **stabilizer** of z) for every $z \in Z$. If $\bullet$ is transitive, then Y_w and Y_z are conjugate subgroups for all w, $z \in Z$.

(g) If $\bullet$ is an action of a group X on a set Z, then sets of the form $\{a{\bullet}z : a \in X\}$ are called **orbits** of the action; they are the equivalence classes under the equivalence relation $\sim$, where $z \sim z'$ if there is an $a \in X$ such that $z' = a{\bullet}z$.

4A5C Examples Let X be any group.

(a) Write

$$x{\bullet}_l y = xy, \quad x{\bullet}_r y = yx^{-1}, \quad x{\bullet}_c y = xyx^{-1}$$

for x, $y \in X$. These are all actions of X on itself, the **left**, **right** and **conjugacy** actions.

(b) If $A \subseteq X$, we have an action of X on the set $\{yA : y \in X\}$ of left cosets of A defined by setting $x{\bullet}(yA) = xyA$ for x, $y \in X$.

(c)(i) Let $\bullet$ be an action of a group X on a set Z. If f is any function defined on a subset of Z, and $x \in X$, write $x{\bullet}f$ for the function defined by saying that $(x{\bullet}f)(z) = f(x^{-1}{\bullet}z)$ whenever $z \in Z$ and $x^{-1}{\bullet}z \in \operatorname{dom} f$. It is easy to check that this defines an action of X on the class of all functions with domains included in Z.

$$x{\bullet}(f+g) = (x{\bullet}f) + (x{\bullet}g), \quad x{\bullet}(f \times g) = (x{\bullet}f) \times (x{\bullet}g), \quad x{\bullet}(f/g) = (x{\bullet}f)/(x{\bullet}g)$$

whenever $x \in X$ and f, g are real-valued functions with domains included in Z.

(ii) In (i), if $X = Z$, we have corresponding actions $\bullet_l$, $\bullet_r$ and $\bullet_c$ of X on the class of functions with domains included in X. These are the left, right and conjugacy **shift actions**.

Note that

$$x{\bullet_l}\chi A = \chi(xA), \quad x{\bullet_r}\chi A = \chi(Ax^{-1}), \quad x{\bullet_c}\chi A = \chi(xAx^{-1})$$

whenever $A \subseteq X$ and $x \in X$. In this context, the following idea is sometimes useful. If f is a function with domain included in X, set $\overset{\leftrightarrow}{f}(y) = f(y^{-1})$ when $y \in X$ and $y^{-1} \in \operatorname{dom} f$. Then

$$(\overset{\leftrightarrow}{f})^{\leftrightarrow} = f, \quad x{\bullet_l}\overset{\leftrightarrow}{f} = (x{\bullet_r}f)^{\leftrightarrow}, \quad x{\bullet_r}\overset{\leftrightarrow}{f} = (x{\bullet_l}f)^{\leftrightarrow}, \quad x{\bullet_c}\overset{\leftrightarrow}{f} = (x{\bullet_c}f)^{\leftrightarrow}$$

for any such f and any $x \in X$.

(d) If $\bullet$ is an action of a group X on a set Z, $Y \subseteq X$ is a subgroup of X, and $W \subseteq Z$ is Y-invariant in the sense that $y{\bullet}w \in W$ whenever $y \in Y$ and $w \in W$, then $\bullet{\restriction}Y \times W$ is an action of Y on W. In the context of (c-i) above, this means that if V is any set of functions with domains included in W such that $y{\bullet}f \in V$ whenever $y \in Y$ and $f \in V$, then we have an action of Y on V.

4A5D Definitions (a) A **topological group** is a group X endowed with a topology such that the operations $(x, y) \mapsto xy : X \times X \to X$ and $x \mapsto x^{-1} : X \to X$ are continuous.

(b) A **Polish group** is a topological group in which the topology is Polish.

4A5E Elementary facts Let X be any topological group.

(a) For any $x \in X$, the functions $y \mapsto xy$, $y \mapsto yx$ and $y \mapsto y^{-1}$ are all homeomorphisms from X to itself.

(b) The maps $(x, y) \mapsto x^{-1}y$, $(x, y) \mapsto xy^{-1}$ and $(x, y) \mapsto xyx^{-1}$ from $X \times X$ to X are continuous.

(c) $\{G : G$ is open, $e \in G$, $G^{-1} = G\}$ is a base of neighbourhoods of the identity e of X.

(d) If $G \subseteq X$ is an open set, then AG and GA are open for any set $A \subseteq X$.

(e) If $F \subseteq X$ is closed and $x \in X \setminus F$, there is a neighbourhood U of e such that $UxUU \cap FUU = \emptyset$.

(f) If $K \subseteq X$ is compact and $F \subseteq X$ is closed then KF and FK are closed. If K, $L \subseteq X$ are compact so is KL.

(g) If there is any compact set $K \subseteq X$ such that $\operatorname{int} K$ is non-empty, then X is locally compact.

(h) If $K \subseteq X$ is compact and $\mathcal{F}$ is a downwards-directed family of closed subsets of X with intersection F_0, then $KF_0 = \bigcap_{F \in \mathcal{F}} KF$ and $F_0K = \bigcap_{F \in \mathcal{F}} FK$.

(i) If $K \subseteq X$ is compact and $G \subseteq X$ is open, then $W = \{(x, y) : xKy \subseteq G\}$ is open in $X \times X$.
It follows that $\{x : xK \subseteq G\}$, $\{x : Kx \subseteq G\}$ and $\{x : xKx^{-1} \subseteq G\}$ are open in X.

(j) If X is Hausdorff, $K \subseteq X$ is compact and U is a neighbourhood of e, there is a neighbourhood V of e such that $xy \in U$ whenever x, $y \in K$ and $yx \in V$; that is, $y^{-1}zy \in U$ whenever $z \in V$ and $y \in K$.

(k) Any open subgroup of X is also closed.

(l) If X is locally compact, it has an open subgroup which is σ-compact.

(m) If Y is a subgroup of X, its closure $\overline{Y}$ is a subgroup of X.

(n) Let X be a group and $\mathcal{V}$ a family of subsets of X. Then there is a topology of X under which X is a topological group and $\mathcal{V}$ is a base of neighbourhoods of the identity iff

(α) $\mathcal{V}$ is a filter base;
(β) for every $V \in \mathcal{V}$ there is a $W \in \mathcal{V}$ such that $W^2 \subseteq V$;
(γ) for every $V \in \mathcal{V}$ there is a $W \in \mathcal{V}$ such that $W^{-1} \subseteq V$;
(δ) for every $V \in \mathcal{V}$ and $z \in X$ there is a $W \in \mathcal{V}$ such that $zWz^{-1} \subseteq V$.

In this case, there is exactly one such topology, and it is Hausdorff iff $\bigcap \mathcal{V} = \{e\}$.

4A5F Proposition (a) Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{S})$ be topological groups. If $\phi : X \to Y$ is a group homomorphism which is continuous at the identity of X, it is continuous.

(b) Let X be a group and $\mathfrak{S}$, $\mathfrak{T}$ two topologies on X both making X a topological group. If every $\mathfrak{S}$-neighbourhood of the identity is a $\mathfrak{T}$-neighbourhood of the identity, then $\mathfrak{S} \subseteq \mathfrak{T}$.

4A5G Proposition If $\langle X_i \rangle_{i \in I}$ is any family of topological groups, then $\prod_{i \in I} X_i$, with the product topology and the product group structure, is again a topological group.

4A5H The uniformities of a topological group Let $(X, \mathfrak{T})$ be a topological group. Write $\mathcal{U}$ for the set of open neighbourhoods of the identity e of X.

(a) For $U \in \mathcal{U}$, set $W_U = \{(x, y) : xy^{-1} \in U\} \subseteq X \times X$. The family $\{W_U : U \in \mathcal{U}\}$ is a filter base, and the filter on $X \times X$ which it generates is a uniformity on X, the **right uniformity** of X. This uniformity induces the topology $\mathfrak{T}$. It follows that $\mathfrak{T}$ is completely regular, therefore regular.

(b) For $U \in \mathcal{U}$, set $\tilde{W}_U = \{(x, y) : xy^{-1} \in U,\, x^{-1}y \in U\} \subseteq X \times X$. The family $\{\tilde{W}_U : U \in \mathcal{U}\}$ is a filter base, and the filter on $X \times X$ which it generates is a uniformity on X, the **bilateral uniformity** of X. This uniformity induces the topology $\mathfrak{T}$.

(c) $x \mapsto x^{-1}$ is uniformly continuous for the bilateral uniformity.

(d) If X and Y are topological groups and $\phi : X \to Y$ is a continuous homomorphism, then ϕ is uniformly continuous for the bilateral uniformities.

4A5I Definitions If X is a topological group and Z a topological space, an action of X on Z is 'continuous' or 'Borel measurable' if it is continuous, or Borel measurable, when regarded as a function from $X \times Z$ to Z.

4A5J Quotients under group actions, and quotient groups: Theorem (a) Let X be a topological space, Y a topological group, and $\bullet$ a continuous action of Y on X. Let Z be the set of orbits of the action, and for $x \in X$ write $\pi(x) \in Z$ for the orbit containing x.

(i) We have a topology on Z defined by saying that $V \subseteq Z$ is open iff $\pi^{-1}[V]$ is open in X. The canonical map $\pi : X \to Z$ is continuous and open.

(ii)(α) If Y is compact and X is Hausdorff, then Z is Hausdorff.

(β) If X is locally compact then Z is locally compact.

(b) Let X be a topological group, Y a subgroup of X, and Z the set of left cosets of Y in X. Set $\pi(x) = xY$ for $x \in X$.

(i) We have a topology on Z defined by saying that $V \subseteq Z$ is open iff $\pi^{-1}[V]$ is open in X. The canonical map $\pi : X \to Z$ is continuous and open.

(ii)(α) Z is Hausdorff iff Y is closed.

(β) If X is locally compact, so is Z.

(γ) If X is locally compact and Polish and Y is closed, then Z is Polish.

(δ) If X is locally compact and σ-compact and Y is closed and Z is metrizable, then Z is Polish.

(iii) We have a continuous action of X on Z defined by saying that $x\bullet\pi(x') = \pi(xx')$ for any x, $x' \in X$.

(iv) If Y is a normal subgroup of X, then the group operation on Z renders it a topological group.

4A5K Proposition Let X be a topological group with identity e.

(a) $Y = \overline{\{e\}}$ is a closed normal subgroup of X.

(b) Writing $\pi : X \to X/Y$ for the canonical map,

(i) a subset of X is open iff it is the inverse image of an open subset of X/Y,

(ii) a subset of X is closed iff it is the inverse image of a closed subset of X/Y,

(iii) $\pi[G]$ is a regular open set in X/Y for every regular open set $G \subseteq X$,

(iv) $\pi[F]$ is nowhere dense in X/Y for every nowhere dense set $F \subseteq X$,

(v) $\pi^{-1}[V]$ is nowhere dense in X for every nowhere dense $V \subseteq X/Y$.

4A5L Theorem Let X be a topological group and Y a normal subgroup of X. Let $\pi : X \to X/Y$ be the canonical homomorphism.

(a) If X' is another topological group and $\phi : X \to X'$ a continuous homomorphism with kernel including Y, then we have a continuous homomorphism $\psi : X/Y \to X'$ defined by the formula $\psi\pi = \phi$; ψ is injective iff Y is the kernel of ϕ.

(b) Suppose that K_1, K_2 are two subgroups of X/Y such that $K_2 \lhd K_1$. Set $Y_1 = \pi^{-1}[K_1]$ and $Y_2 = \pi^{-1}[K_2]$. Then $Y_2 \lhd Y_1$ and Y_1/Y_2 and K_1/K_2 are isomorphic as topological groups.

4A5M Proposition Let X be a topological group.

(a) Let Y be any subgroup of X. If X is given its bilateral uniformity, then the subspace uniformity on Y is the bilateral uniformity of Y.

(b) If X is locally compact it is complete under its right uniformity. If X is complete under its right uniformity it is complete under its bilateral uniformity.

(c) Suppose that X is Hausdorff and that Y is a subgroup of X which is locally compact in its subspace topology. Then Y is closed in X.

4A5N Theorem Let X be a Hausdorff topological group. Then its completion $\widehat{X}$ under its bilateral uniformity can be endowed (in exactly one way) with a group structure rendering it a Hausdorff topological group in which the natural embedding of X in $\widehat{X}$ represents X as a dense subgroup of $\widehat{X}$. If X has a neighbourhood of the identity which is totally bounded for the bilateral uniformity, then $\widehat{X}$ is locally compact.

4A5O Proposition Let X be a topological group.

(a) If $A \subseteq X$, then the following are equiveridical: (i) A is totally bounded for the bilateral uniformity of X; (ii) for every neighbourhood U of the identity there is a finite set $I \subseteq X$ such that $A \subseteq IU \cap UI$.

(b) If A, $B \subseteq X$ are totally bounded for the bilateral uniformity of X, so are $A \cup B$, A^{-1} and AB. In particular, $\bigcup_{i \le n} x_i B$ is totally bounded for any $x_0, \ldots, x_n \in X$.

(c) If $A \subseteq X$ is totally bounded for the bilateral uniformity, and U is any neighbourhood of the identity, then $\{y : xyx^{-1} \in U \text{ for every } x \in A\}$ is a neighbourhood of the identity.

(d) If X is the product of a family $\langle X_i \rangle_{i \in I}$ of topological groups, a subset A of X is totally bounded for the bilateral uniformity of X iff it is included in a product $\prod_{i \in I} A_i$ where $A_i \subseteq X_i$ is totally bounded for the bilateral uniformity of X_i for every $i \in I$.

(e) If X is locally compact, a subset of X is totally bounded for the bilateral uniformity iff it is relatively compact.

4A5P Lemma Let X be a locally compact Hausdorff topological group. Take $f \in C_k(X)$.

(a) Let $K \subseteq X$ be a compact set. Then for any $\epsilon > 0$ there is a neighbourhood W of the identity e of X such that $|f(xay) - f(xby)| \le \epsilon$ whenever $x \in K$, $y \in X$ and $ab^{-1} \in W$.

(b) For any $x_0 \in X$, there is a non-negative $f^* \in C_k(X)$ such that for every $\epsilon > 0$ there is an open set G containing x_0 such that $|f(xy) - f(x_0 y)| \le \epsilon f^*(y)$ for every $x \in G$ and $y \in X$.

4A5Q Metrizable groups: Proposition Let $(X,\mathfrak{T})$ be a topological group. Then the following are equiveridical:

(i) X is metrizable;

(ii) the identity e of X has a countable neighbourhood base;

(iii) there is a metric ρ on X, inducing the topology $\mathfrak{T}$, which is **right-translation-invariant**, that is, $\rho(x_1,x_2)=\rho(x_1y,x_2y)$ for all x_1, x_2, $y\in X$;

(iv) there is a right-translation-invariant metric on X which induces the right uniformity of X;

(v) the bilateral uniformity of X is metrizable.

4A5R Corollary If X is a locally compact topological group and $\{e\}$ is a G_δ set in X, then X is metrizable.

4A5S Lemma Let X be a σ-compact locally compact Hausdorff topological group and $\langle U_n\rangle_{n\in\mathbb{N}}$ any sequence of neighbourhoods of the identity in X. Then X has a compact normal subgroup $Y\subseteq\bigcap_{n\in\mathbb{N}}U_n$ such that $Z=X/Y$ is Polish.

***4A5T Theorem** A compact Hausdorff topological group is dyadic.

Version of 8.12.10

4A6 Banach algebras

I give results which are needed for Chapter 44. Those down to 4A6K should be in any introductory text on normed algebras; 4A6L-4A6O, as expressed here, are a little more specialized. As with normed spaces or linear topological spaces, Banach algebras may be defined over either $\mathbb{R}$ or $\mathbb{C}$. In §445 we need complex Banach algebras, but in §446 I think the ideas are clearer in the context of real Banach algebras. Accordingly, as in §2A4, I express as much as possible of the theory in terms applicable equally to either, speaking of 'normed algebras' or 'Banach algebras' without qualification, and using the symbol $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$ to represent the field of scalars. Since (at least, if you keep to the path indicated here) the ideas are independent of which field we work with, you will have no difficulty in applying the arguments given in Folland 95 or Hewitt & Ross 63 for the complex case to the real case. In 4A6B and 4A6I-4A6K, however, we have results which apply only to 'complex' Banach algebras, in which the underlying field is taken to be $\mathbb{C}$.

4A6A Definition (a) A **normed algebra** is a normed space U together with a multiplication, a binary operator $\times$ on U, such that

$$u\times(v\times w)=(u\times v)\times w,$$

$$u\times(v+w)=(u\times v)+(u\times w),\quad (u+v)\times w=(u\times w)+(v\times w),$$

$$(\alpha u)\times v=u\times(\alpha v)=\alpha(u\times v),$$

$$\|u\times v\|\le\|u\|\|v\|$$

for all u, v, $w\in U$ and $\alpha\in\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$. A normed algebra is **commutative** if its multiplication is commutative.

(b) A **Banach algebra** is a normed algebra which is a Banach space. A **unital Banach algebra** is a Banach algebra with a multiplicative identity e such that $\|e\|=1$.

In a unital Banach algebra I will always use the letter e for the identity.

4A6B Stone-Weierstrass Theorem: fourth form Let X be a locally compact Hausdorff space, and $C_0=C_0(X;\mathbb{C})$ the complex Banach algebra of continuous functions $f:X\to\mathbb{C}$ such that $\{x:|f(x)|\ge\epsilon\}$ is compact for every $\epsilon>0$. Let $A\subseteq C_0$ be such that

A is a linear subspace of C_0,

$f\times g\in A$ for every f, $g\in A$, the complex conjugate $\bar{f}$ of f belongs to A for every $f \in A$,

for every $x \in X$ there is an $f \in A$ such that $f(x) \neq 0$,

whenever x, y are distinct points of X there is an $f \in A$ such that $f(x) \neq f(y)$.

Then A is $\| \|_\infty$-dense in C_0.

4A6C Proposition If U is any Banach space other than $\{0\}$, then the space $\mathsf{B}(U;U)$ of bounded linear operators from U to itself is a unital Banach algebra.

4A6D Proposition Any normed algebra U can be embedded as a subalgebra of a unital Banach algebra V, in such a way that if U is commutative so is V.

4A6E Proposition Let U be a unital Banach algebra and $W \subseteq U$ a closed proper ideal. Then U/W, with the quotient linear structure, ring structure and norm, is a unital Banach algebra.

4A6F Proposition If U is a Banach algebra and $\phi : U \to {\mathbb{R} \atop \mathbb{C}}$ is a multiplicative linear functional, then $|\phi(u)| \le \|u\|$ for every $u \in U$.

4A6G Definition Let U be a normed algebra and $u \in U$.

(a) For any $u \in U$, $\lim_{n\to\infty} \|u^n\|^{1/n}$ is defined and equal to $\inf_{n\ge 1} \|u^n\|^{1/n}$.

(b) This common value is the **spectral radius** of u.

4A6H Theorem If U is a unital Banach algebra, then the set R of invertible elements is open, and $u \mapsto u^{-1}$ is a continuous function from R to itself. If $v \in U$ and $\|v - e\| < 1$, then $v \in R$ and $\|v^{-1} - e\| \le \frac{\|v-e\|}{1-\|v-e\|}$.

4A6I Theorem Let U be a complex unital Banach algebra and $u \in U$. Write r for the spectral radius of u.

(a) If $\zeta \in \mathbb{C}$ and $|\zeta| > r$ then $\zeta e - u$ is invertible.

(b) There is a ζ such that $|\zeta| = r$ and $\zeta e - u$ is not invertible.

4A6J Theorem Let U be a commutative complex unital Banach algebra, and $u \in U$. Then for any $\zeta \in \mathbb{C}$ the following are equiveridical:

(i) there is a non-zero multiplicative linear functional $\phi : U \to \mathbb{C}$ such that $\phi(u) = \zeta$;

(ii) $\zeta e - u$ is not invertible.

4A6K Corollary Let U be a commutative complex Banach algebra and $u \in U$. Then its spectral radius $r(u)$ is $\max\{|\phi(u)| : \phi$ is a multiplicative linear functional on $U\}$.

4A6L Exponentiation Let U be a unital Banach algebra. For any $u \in U$,

$$\exp(u) = \sum_{k\in\mathbb{N}} \frac{1}{k!} u^k$$

is defined in U.

4A6M Lemma Let U be a unital Banach algebra.

(a) If u, $v \in U$ and $\max(\|u\|, \|v\|) \le \gamma$ then $\|\exp(u) - \exp(v) - u + v\| \le \|u - v\|(\exp\gamma - 1)$. So if $\max(\|u\|, \|v\|) \le \frac{2}{3}$ and $\exp(u) = \exp(v)$ then $u = v$.

(b) If $\|u - e\| \le \frac{1}{6}$ then there is a v such that $\exp(v) = u$ and $\|v\| \le 2\|u - e\|$.

(c) If u, $v \in U$ and $uv = vu$ then $\exp(u + v) = \exp(u)\exp(v)$.

4A6N Lemma If U is a unital Banach algebra, $u \in U$ and $\|u^n - e\| \le \frac{1}{6}$ for every $n \in \mathbb{N}$, then $u = e$.

4A6O Proposition Let U be a normed algebra, and U^*, U^{**} its dual and bidual as a normed space. For a bounded linear operator $T : U \to U$ let $T' : U^* \to U^*$ be the adjoint of T and $T'' : U^{**} \to U^{**}$ the adjoint of T'.

(a) We have bilinear maps, all of norm at most 1,

$$(f, x) \mapsto f \circ x : U^* \times U \to U^*,$$
$$(\phi, f) \mapsto \phi \circ f : U^{**} \times U^* \to U^*,$$
$$(\phi, \psi) \mapsto \phi \circ \psi : U^{**} \times U^{**} \to U^{**}$$

defined by the formulae

$$(f \circ x)(y) = f(xy),$$
$$(\phi \circ f)(x) = \phi(f \circ x),$$
$$(\phi \circ \psi)(f) = \phi(\psi \circ f)$$

for all x, $y \in U$, $f \in U^*$ and ϕ, $\psi \in U^{**}$.

(b)(i) Suppose that $S : U \to U$ is a bounded linear operator such that $S(xy) = (Sx)y$ for all x, $y \in U$. Then $S''(\phi \circ \psi) = (S''\phi) \circ \psi$ for all ϕ, $\psi \in U^{**}$.

(ii) Suppose that $T : U \to U$ is a bounded linear operator such that $T(xy) = x(Ty)$ for all x, $y \in U$. Then $T''(\phi \circ \psi) = \phi \circ (T''\psi)$ for all ϕ, $\psi \in U^{**}$.

Version of 13.3.22

4A7 Algebraic topology

A fundamental theorem about the topology of Euclidean space is used in §472. (8.7.22) no idea what I was doing here

4A7A Definitions Suppose that X and Y are topological spaces, and $f : X \to Y$, $g : X \to Y$ are continuous functions.

(a) A **homotopy from** f **to** g is a continuous function $F : X \times [0, 1] \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for every $x \in X$.

(b) f and g are **homotopic** if there is a homotopy from f to g.

4A7B Theorem If $r \ge 1$ and $S_{r-1} = \partial B(\mathbf{0}.1)$ is the unit sphere $\{x : \|x\| = 1\}$ in $\mathbb{R}^r$, then the identity function from S_{r-1} to itself is not homotopic to a constant function.

4A7C Corollary If $r \ge 1$, $B(0, 1)$ is the unit ball in $\mathbb{R}^r$ and $h : B(\mathbf{0}, 1) \to B(\mathbf{0}, 1)$ is a continuous function such that $h[S_{r-1}] = S_{r-1}$, then $h[B(\mathbf{0}, 1)] = B(\mathbf{0}, 1)$.

472G Theorem (Bagnara Gennaioli Leccese & Luongo p22) Let $r \ge 1$ be an integer, $B \subseteq \mathbb{R}^r$ a closed balls, ρ_E the Euclidean metric on $\mathbb{R}^r$, and ρ a metric on B inducing the usual topology on B. Then there is a (ρ, ρ_E)-Lipschitz surjection from B onto itself.

472H Corollary Let $r \ge 1$ be an integer, ρ a metric on $\mathbb{R}^r$ inducing the usual topology on $\mathbb{R}^r$, and $\mu_{Hr}^{(\rho)}$ the corresponding r-dimensional Hausdorff measure on $\mathbb{R}^r$. Then $\mu_{Hr}^{(\rho)}$ is strictly positive.

Version of 10.1.17

Concordance for Appendix

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this volume, and which have since been changed.

4A2Jf Uniformities on completely regular spaces 4A2Jf, referred to in the 2009 edition of Volume 5, has been moved to 4A2Jg.

4A3Q Baire property and cylindrical σ-algebras 4A3Q-4A3T and 4A3V, referred to in the 2008 and 2015 editions of Volume 5, are now 4A3R-4A3U and 4A3W.

4A4B Bounded sets in linear topological spaces 4A4Bg, referred to in the 2008 edition of Volume 5, has been moved to 3A5Nb.