Version of 3.9.13

Appendix to Volume 4

Useful facts

As is to be expected, we are coming in this volume to depend on a wide variety of more or less recondite information, and only an exceptionally broad mathematical education will have covered it all. While all the principal ideas are fully expressed in standard textbooks, there are many minor points where I need to develop variations on the familiar formulations. A little under half the material, by word-count, is in general topology (§4A2), where I begin with some pages of definitions. I follow this with a section on Borel and Baire σ-algebras, Baire-property algebras and cylindrical algebras (§4A3), worked out a little more thoroughly than the rest of the material. The other sections are on set theory (§4A1), linear analysis (§4A4), topological groups (§4A5) and Banach algebras (§4A6).

Version of 27.1.13

4A1 Set theory

For this volume, we need fragments from four topics in set theory and one in Boolean algebra. The most important are the theory of closed cofinal sets and stationary sets (4A1B-4A1C) and infinitary combinatorics (4A1D-4A1H). Rather more specialized, we have the theory of normal (ultra)filters (4A1J-4A1L) and a mention of Ostaszewski's ♣ (4A1M-4A1N), used for an example in §439. I conclude with a simple result on the cardinality of σ-algebras (4A1O).

4A1A Cardinals again (a) An infinite cardinal κ is **regular** if it is not the supremum of fewer than κ smaller cardinals, that is, if $\mathrm{cf}\,\kappa = \kappa$. Any infinite successor cardinal is regular. (Kunen 80, I.10.37; Just & Weese 96, 11.18; Jech 78, p. 40; Levy 79, IV.3.11.) In particular, $\omega_1 = \omega^+$ is regular.

(b) If ζ is an ordinal and X is a set then I say that a family $\langle x_\xi\rangle_{\xi<\zeta}$ in X **runs over X with cofinal repetitions** if $\{\xi : \xi < \zeta,\, x_\xi = x\}$ is cofinal with ζ for every $x \in X$. Now if X is any non-empty set and κ is a cardinal greater than or equal to $\max(\omega, \#(X))$, there is a family $\langle x_\xi\rangle_{\xi<\kappa}$ running over X with cofinal repetitions. **P** By 3A1Ca, there is a surjection $\xi \mapsto (x_\xi, \alpha_\xi) : \kappa \to X \times \kappa$. **Q**

(c) The cardinal $\mathfrak{c}$ (i) Every non-trivial interval in $\mathbb{R}$ has cardinal $\mathfrak{c}$. (Enderton 77, p. 131.)

(ii) If $\#(A) \le \mathfrak{c}$ and D is countable, then $\#(A^D) \le \mathfrak{c}$. ($\#(A^D) \le \#(\mathcal{P}\mathbb{N})^{\mathbb{N}}) = \#(\mathcal{P}(\mathbb{N}\times\mathbb{N})) = \#(\mathcal{P}\mathbb{N})$.)

(iii) $\mathrm{cf}(2^\kappa) > \kappa$ for every infinite cardinal κ; in particular, $\mathrm{cf}\,\mathfrak{c} > \omega$. (Kunen 80, I.10.40; Just & Weese 96, 11.24; Jech 78, p. 46; Jech 03, 5.11; Levy 79, V.5.2.)

(d) The Continuum Hypothesis This is the statement '$\mathfrak{c} = \omega_1$'; it is neither provable nor disprovable from the ordinary axioms of mathematics, including the Axiom of Choice. As such, it belongs to Volume 5 rather than to the present volume. But I do at one point refer to one of its immediate consequences. If the continuum hypothesis is true, then there is a well-ordering $\preccurlyeq$ of $[0,1]$ such that $([0,1], \preccurlyeq)$ has order type ω_1 (because there is a bijection $f : [0,1] \to \omega_1$, and we can set $s \preccurlyeq t$ if $f(s) \le f(t)$).

4A1B Closed cofinal sets Let α be an ordinal.

(a) Note that a subset F of α is closed in the order topology iff $\sup A \in F$ whenever $A \subseteq F$ is non-empty and $\sup A < \alpha$. (4A2S(a-ii).)

(b) If α has uncountable cofinality, and $A \subseteq \alpha$ has supremum α, then $A' = \{\xi : 0 < \xi < \alpha, \xi = \sup(A \cap \xi)\}$ is a closed cofinal set in α. **P** (α) For any $\eta < \alpha$ we can choose inductively a strictly increasing sequence $\langle \xi_n \rangle_{n \in \mathbb{N}}$ in A starting from $\xi_0 \geq \eta$; now $\xi = \sup_{n \in \mathbb{N}} \xi_n \in A'$ and $\xi \geq \eta$; this shows that A' is cofinal with α. (β) If $\emptyset \neq B \subseteq A'$ and $\sup B = \xi < \alpha$, then $A \cap \xi \supseteq A \cap \eta$ for every $\eta \in B$, so

$$\xi = \sup B = \sup_{\eta \in B} \sup(A \cap \eta) = \sup(\bigcup_{\eta \in B} A \cap \eta) \leq \sup A \cap \xi \leq \xi$$

and $\xi \in A'$. **Q**

In particular, taking $A = \alpha = \omega_1$, the set of non-zero countable limit ordinals is a closed cofinal set in ω_1.

(c)(i) If $\langle F_\xi \rangle_{\xi < \alpha}$ is a family of subsets of α, the **diagonal intersection** of $\langle F_\xi \rangle_{\xi < \alpha}$ is $\{\xi : \xi < \alpha, \xi \in F_\eta$ for every $\eta < \xi\}$.

(ii) If κ is a regular uncountable cardinal and $\langle F_\xi \rangle_{\xi < \kappa}$ is any family of closed cofinal sets in κ, its diagonal intersection F is again a closed cofinal set in κ. **P** $F = \bigcap_{\xi < \kappa}(F_\xi \cup [0, \xi])$ is certainly closed. To see that it is cofinal, argue as follows. Start from any $\zeta_0 < \kappa$. Given $\zeta_n < \kappa$, set

$$\zeta_{n+1} = \sup_{\xi < \zeta_n}(\min(F_\xi \setminus \zeta_n) + 1);$$

this is defined because every F_ξ is cofinal with κ, and is less than κ because $\operatorname{cf}\kappa = \kappa$. At the end of the induction, set $\zeta^* = \sup_{n \in \mathbb{N}} \zeta_n$; then $\zeta_0 \leq \zeta^*$ and $\zeta^* < \kappa$ because $\operatorname{cf}\kappa > \omega$. If ξ, $\eta < \zeta^*$, there is an $n \in \mathbb{N}$ such that $\max(\xi, \eta) < \zeta_n$, in which case $F_\xi \cap (\zeta^* \setminus \eta) \supseteq F_\xi \cap \zeta_{n+1} \setminus \zeta_n$ is non-empty. As η is arbitrary and F_ξ is closed, $\zeta^* \in F_\xi$; as ξ is arbitrary, $\zeta^* \in F$; as ζ_0 is arbitrary, F is cofinal. **Q**

(iii) In particular, if $f : \kappa \to \kappa$ is any function, then $\{\xi : \xi < \kappa, f(\eta) < \xi$ for every $\eta < \xi\}$ is a closed cofinal set in κ, being the diagonal intersection of $\langle \kappa \setminus (f(\xi) + 1) \rangle_{\xi < \kappa}$.

(d) If α has uncountable cofinality, $\mathcal{F}$ is a non-empty family of closed cofinal sets in α and $\#(\mathcal{F}) < \operatorname{cf}\alpha$, then $\bigcap \mathcal{F}$ is a closed cofinal set in α. **P** Being the intersection of closed sets it is surely closed. Set $\lambda = \max(\omega, \#(\mathcal{F}))$ and let $\langle F_\xi \rangle_{\xi < \lambda}$ run over $\mathcal{F}$ with cofinal repetitions. Starting from any $\zeta_0 < \alpha$, we can choose $\langle \zeta_\xi \rangle_{1 \leq \xi \leq \lambda}$ such that

– if $\xi < \lambda$ then $\zeta_\xi \leq \zeta_{\xi+1} \in F_\xi$;

– if $\xi \leq \lambda$ is a non-zero limit ordinal, $\zeta_\xi = \sup_{\eta < \xi} \zeta_\eta$.

(Because $\lambda < \operatorname{cf}\alpha$, $\zeta_\xi < \alpha$ for every ξ.) Now $\zeta_0 \leq \zeta_\lambda < \alpha$, and if $F \in \mathcal{F}$, $\zeta < \zeta_\lambda$ there is a $\xi < \lambda$ such that $F = F_\xi$ and $\zeta \leq \zeta_\xi$, in which case $\zeta \leq \zeta_{\xi+1} \leq \zeta_\lambda$ and $\zeta_{\xi+1} \in F$. This shows that either $\zeta_\lambda \in F$ or $\zeta_\lambda = \sup(F \cap \zeta_\lambda)$, in which case again $\zeta_\lambda \in F$. As F is arbitrary, $\zeta_\lambda \in \bigcap \mathcal{F}$; as ζ_0 is arbitrary, $\bigcap \mathcal{F}$ is cofinal. **Q**

In particular, the intersection of any sequence of closed cofinal sets in ω_1 is again a closed cofinal set in ω_1.

4A1C Stationary sets (a) Let κ be a cardinal. A subset of κ is **stationary** in κ if it meets every closed cofinal set in κ; otherwise it is **non-stationary**.

(b) If κ is a cardinal of uncountable cofinality, the intersection of any stationary subset of κ with a closed cofinal set in κ is again a stationary set (because the intersection of two closed cofinal sets is a closed cofinal set); the family of non-stationary subsets of κ is a σ-ideal, the **non-stationary ideal** of κ. (Kunen 80, II.6.9; Just & Weese 97, Lemma 21.11; Jech 03, p. 93; Levy 79, IV.4.35.)

(c) Pressing-Down Lemma (Fodor's theorem) If κ is a regular uncountable cardinal, $A \subseteq \kappa$ is stationary and $f : A \to \kappa$ is such that $f(\xi) < \xi$ for every $\xi \in A$, then there is a stationary set $B \subseteq A$ such that f is constant on B. (Kunen 80, II.6.15; Just & Weese 97, Theorem 21.2; Jech 78, Theorem 22; Jech 03, 8.7; Levy 79, IV.4.40.)

(d) There are disjoint stationary sets A, $B \subseteq \omega_1$. (This is easily deduced from 419G or 438Cd, and is also a special case of very much stronger results. See 541Ya in Volume 5, or Kunen 80, II.6.12; Just & Weese 97, Corollary 23.4; Jech 78, p. 59; Jech 03, 8.8; Levy 79, IV.4.48.)

4A1D Δ-systems (a) A family $\langle I_\xi \rangle_{\xi \in A}$ of sets is a **Δ-system** with **root** I if $I_\xi \cap I_\eta = I$ for all distinct ξ, $\eta \in A$.

(b) Δ-system Lemma If $\#(A)$ is a regular uncountable cardinal and $\langle I_\xi\rangle_{\xi\in A}$ is any family of finite sets, there is a set $D\subseteq A$ such that $\#(D)=\#(A)$ and $\langle I_\xi\rangle_{\xi\in D}$ is a Δ-system. (Kunen 80, II.1.6; Just & Weese 97, Theorem 16.3. For the present volume we need only the case $\#(A)=\omega_1$, which is treated in Jech 78, p. 225 and Jech 03, 9.18.)

4A1E Free sets (a) Let A be a set with cardinal at least ω_2, and $\langle J_\xi\rangle_{\xi\in A}$ a family of countable sets. Then there are distinct ξ, $\eta\in A$ such that $\xi\notin J_\eta$ and $\eta\notin J_\xi$. **P** Let $K\subseteq A$ be a set with cardinal ω_1, and set $L=K\cup\bigcup_{i\in K}J_i$. Then L has cardinal ω_1, so there is a $\xi\in A\setminus L$. Now there is an $\eta\in K\setminus J_\xi$, and this pair (ξ,η) serves. **Q**

(b) If $\langle K_\xi\rangle_{\xi\in A}$ is a disjoint family of sets indexed by an uncountable subset A of ω_1, and $\langle J_\eta\rangle_{\eta<\omega_1}$ is a family of countable sets, there is an uncountable $B\subseteq A$ such that $K_\xi\cap J_\eta=\emptyset$ whenever η, $\xi\in B$ and $\eta<\xi$. **P** Choose $\langle\zeta_\xi\rangle_{\xi<\omega_1}$ inductively in such a way that $\zeta_\xi\in A$ and $K_{\zeta_\xi}\cap J_{\zeta_\eta}=\emptyset$, $\zeta_\xi>\zeta_\eta$ for every $\eta<\xi$. Set $B=\{\zeta_\xi:\xi<\omega_1\}$. **Q**

4A1F Selecting subsequences (a) Let $\langle K_i\rangle_{i\in I}$ be a countable family of sets such that $\bigcap_{i\in J}K_i$ is infinite for every finite subset J of I. Then there is an infinite set K such that $K\setminus K_i$ is finite and $K_i\setminus K$ is infinite for every $i\in I$. **P** We can suppose that $I\subseteq\mathbb{N}$. Choose $\langle k_n\rangle_{n\in\mathbb{N}}$ inductively such that $k_n\in\bigcap_{i\in I,i\le n}K_i\setminus\{k_i:i<n\}$ for every $n\in\mathbb{N}$, and set $K=\{k_{2n}:n\in\mathbb{N}\}$. **Q**

Consequently there is a family $\langle K_\xi\rangle_{\xi<\omega_1}$ of infinite subsets of $\mathbb{N}$ such that $K_\xi\setminus K_\eta$ is finite if $\eta\le\xi$, infinite if $\xi<\eta$. (Choose the K_ξ inductively.)

(b) Let $\langle\mathcal{J}_i\rangle_{i\in I}$ be a countable family of subsets of $[\mathbb{N}]^\omega$ such that $\mathcal{J}_i\cap\mathcal{P}K\ne\emptyset$ for every $K\in[\mathbb{N}]^\omega$ and $i\in I$. Then there is an infinite $K\subseteq\mathbb{N}$ such that for every $i\in I$ there is a $J\in\mathcal{J}_i$ such that $K\setminus J$ is finite. **P** The case $I=\emptyset$ is trivial; suppose that $\langle i_n\rangle_{n\in\mathbb{N}}$ runs over I. Choose K_n, k_n inductively, for $n\in\mathbb{N}$, by taking

$$K_0=\mathbb{N},\quad k_n\in K_n,\quad K_{n+1}\subseteq K_n\setminus\{k_n\},\quad K_{n+1}\in\mathcal{J}_{i_n}$$

for every n; set $K=\{k_n:n\in\mathbb{N}\}$. **Q**

4A1G Ramsey's theorem If $n\in\mathbb{N}$, K is finite and $h:[\mathbb{N}]^n\to K$ is any function, there is an infinite $I\subseteq\mathbb{N}$ such that h is constant on $[I]^n$. (Bollobás 79, p. 105, Theorem 3; Just & Weese 97, 15.3; Jech 78, 29.1; Jech 03, 9.1; Levy 79, IX.3.7. For the present volume we need only the case $n=\#(K)=2$.)

4A1H The Marriage Lemma again In 449L it will be useful to have an infinitary version of the Marriage Lemma available.

Proposition Let X and Y be sets, and $R\subseteq X\times Y$ a set such that $R[\{x\}]$ is finite for every $x\in X$ and $\#(R[I])\ge\#(I)$ for every finite set $I\subseteq X$. Then there is an injective function $f:X\to Y$ such that $(x,f(x))\in R$ for every $x\in X$.

proof For each finite $J\subseteq X$ there is an injective function $f_J:J\to Y$ such that $f_J\subseteq R$ (identifying f_J with its graph), by the ordinary Marriage Lemma (3A1K) applied to $R\cap(J\times R[J])$. Let $\mathcal{F}$ be any ultrafilter on $[X]^{<\omega}$ containing $\{J:I\subseteq J\in[X]^{<\omega}\}$ for every $I\in[X]^{<\omega}$; then for each $x\in X$ there must be an $f(x)\in R[\{x\}]$ such that $\{J:f_J(x)=f(x)\}\in\mathcal{F}$, because $R[\{x\}]$ is finite. Now $f\subseteq R$ is a function from X to Y and must be injective, because for any x, $x'\in X$ there is a $J\in[X]^{<\omega}$ such that f and f_J agree on $\{x,x'\}$.

4A1I Filters (a) Let X be a non-empty set. If $\mathcal{E}\subseteq\mathcal{P}X$ is non-empty and has the finite intersection property,

$$\mathcal{F}=\{A:A\subseteq X,\,A\supseteq\bigcap\mathcal{E}'\text{ for some non-empty finite }\mathcal{E}'\subseteq\mathcal{E}\}$$

is the smallest filter on X including $\mathcal{E}$, the filter **generated** by $\mathcal{E}$.

If $\mathcal{E}\subseteq\mathcal{P}X$ is non-empty and downwards-directed, then it has the finite intersection property iff it does not contain $\emptyset$; in this case we say that $\mathcal{E}$ is a **filter base**; $\mathcal{F}=\{A:A\subseteq X,\,A\supseteq E\text{ for some }E\in\mathcal{E}\}$, and $\mathcal{E}$ is a base for the filter $\mathcal{F}$.

In general, if $\mathcal{E}$ is a family of subsets of X, then there is a filter on X including $\mathcal{E}$ iff $\mathcal{E}$ has the finite intersection property; in this case, there is an ultrafilter on X including $\mathcal{E}$ (2A1O).

(b) If κ is a cardinal and $\mathcal{F}$ is a filter then $\mathcal{F}$ is κ-**complete** if $\bigcap \mathcal{E} \in \mathcal{F}$ whenever $\mathcal{E} \subseteq \mathcal{F}$ and $0 < \#(\mathcal{E}) < \kappa$. Every filter is ω-complete.

(c) A filter $\mathcal{F}$ on a regular uncountable cardinal κ is **normal** if (α) $\kappa \setminus \xi \in \mathcal{F}$ for every $\xi < \kappa$ (β) whenever $\langle F_\xi \rangle_{\xi<\kappa}$ is a family in $\mathcal{F}$, its diagonal intersection belongs to $\mathcal{F}$.

4A1J Lemma A normal filter $\mathcal{F}$ on a regular uncountable cardinal κ is κ-complete.

proof If $\lambda < \kappa$ and $\langle F_\xi \rangle_{\xi<\lambda}$ is a family in $\mathcal{F}$, set $F_\xi = \kappa$ for $\lambda \le \xi < \kappa$, and let F be the diagonal intersection of $\langle F_\xi \rangle_{\xi<\kappa}$; then $\bigcap_{\xi<\lambda} F_\xi \supseteq F \setminus \lambda$ belongs to $\mathcal{F}$.

4A1K Theorem Let X be a set and $\mathcal{F}$ a non-principal ω_1-complete ultrafilter on X. Let κ be the least cardinal of any non-empty set $\mathcal{E} \subseteq \mathcal{F}$ such that $\bigcap \mathcal{E} \notin \mathcal{F}$. Then κ is a regular uncountable cardinal, $\mathcal{F}$ is κ-complete, and there is a function $g : X \to \kappa$ such that $g[[\mathcal{F}]]$ is a normal ultrafilter on κ.

proof (a) By the definition of κ, $\mathcal{F}$ is κ-complete. Because $\mathcal{F}$ is ω_1-complete, $\kappa > \omega$. Let H be the set of all functions $h : X \to \kappa$ such that $h^{-1}[\kappa \setminus \xi] \in \mathcal{F}$ for every $\xi < \kappa$. Then H is not empty. **P** Let $\langle E_\xi \rangle_{\xi<\kappa}$ be a family in $\mathcal{F}$ such that $E = \bigcap_{\xi<\kappa} E_\xi \notin \mathcal{F}$. Because $\mathcal{F}$ is an ultrafilter, $X \setminus E \in \mathcal{F}$. Set $h(x) = 0$ if $x \in E$, $h(x) = \min\{\xi : x \notin E_\xi\}$ if $x \in X \setminus E$; then

$$h^{-1}[\kappa \setminus \xi] \supseteq (X \setminus E) \cap \bigcap\nolimits_{\eta<\xi} E_\eta \in \mathcal{F}$$

for every $\xi < \kappa$, because $\mathcal{F}$ is κ-complete, so $h \in H$. **Q**

(b) For h, $h' \in H$, say that $h \prec h'$ if $\{x : h(x) < h'(x)\} \in \mathcal{F}$. Then there is a $g \in H$ such that $h \not\prec g$ for any $h \in H$. **P?** Otherwise, there is a sequence $\langle h_n \rangle_{n\in\mathbb{N}}$ in H such that $h_{n+1} \prec h_n$ for every $n \in \mathbb{N}$. In this case $E_n = \{x : h_{n+1}(x) < h_n(x)\} \in \mathcal{F}$ for every n. Because $\mathcal{F}$ is ω_1-complete, there is an $x \in \bigcap_{n\in\mathbb{N}} E_n$; but now $\langle h_n(x) \rangle_{n\in\mathbb{N}}$ is a strictly decreasing sequence of ordinals, which is impossible. **XQ**

(c) I should check that κ is regular. **P** If $\langle \alpha_\xi \rangle_{\xi<\lambda}$ is any family in κ with $\lambda < \kappa$, then $g^{-1}[\kappa \setminus \alpha_\xi] \in \mathcal{F}$ for every ξ, so (because $\mathcal{F}$ is κ-complete)

$$g^{-1}[\kappa \setminus \sup\nolimits_{\xi<\lambda} \alpha_\xi] = \bigcap\nolimits_{\xi<\lambda} g^{-1}[\kappa \setminus \alpha_\xi] \in \mathcal{F},$$

and $\sup_{\xi<\lambda} \alpha_\xi \ne \kappa$. **Q**

(d) The image filter $g[[\mathcal{F}]]$ is an ultrafilter on κ, by 2A1N. Because $g \in H$, $g^{-1}[\kappa \setminus \xi] \in \mathcal{F}$ and $\kappa \setminus \xi \in g[[\mathcal{F}]]$ for any $\xi < \kappa$. **?** Suppose, if possible, that $g[[\mathcal{F}]]$ is not normal. Then there is a family $\langle A_\xi \rangle_{\xi<\kappa}$ in $g[[\mathcal{F}]]$ such that its diagonal intersection A does not belong to $g[[\mathcal{F}]]$, that is, $g^{-1}[A] \notin \mathcal{F}$ and $X \setminus g^{-1}[A] \in \mathcal{F}$. Define $h : X \to \kappa$ by setting

$$\begin{aligned} h(x) &= 0 \text{ if } g(x) \in A, \\ &= \min\{\eta : \eta < g(x),\ g(x) \notin A_\eta\} \text{ if } g(x) \notin A. \end{aligned}$$

Then

$$h^{-1}[\kappa \setminus \xi] \supseteq (X \setminus g^{-1}[A]) \cap \bigcap\nolimits_{\eta<\xi} g^{-1}[A_\eta] \in \mathcal{F}$$

for every $\xi < \kappa$. Thus $h \in H$. But also $h(x) < g(x)$ for every $x \in X \setminus g^{-1}[A]$, so $h \prec g$, contrary to the choice of g. **X**

Thus $g[[\mathcal{F}]]$ is a normal filter, and the theorem is proved.

4A1L Theorem Let κ be a regular uncountable cardinal, and $\mathcal{F}$ a normal ultrafilter on κ. If $S \subseteq [\kappa]^{<\omega}$, there is a set $F \in \mathcal{F}$ such that, for each $n \in \mathbb{N}$, $[F]^n$ is either a subset of S or disjoint from S.

proof (a) For each $n \in \mathbb{N}$ there is an $F_n \in \mathcal{F}$ such that either $[F_n]^n \subseteq S$ or $[F_n]^n \cap S = \emptyset$. **P** Induce on n. If $n = 0$ we can take $F_n = \kappa$, because $[\kappa]^0 = \{\emptyset\}$. For the inductive step to $n+1$, set $S_\xi = \{I : I \in [\kappa]^{<\omega}, I \cup \{\xi\} \in S\}$ for each $\xi < \kappa$. By the inductive hypothesis, there is for each $\xi < \kappa$ a set $E_\xi \in \mathcal{F}$ such that either $[E_\xi]^n \subseteq S_\xi$ or $[E_\xi]^n \cap S_\xi = \emptyset$. Let E be the diagonal intersection of $\langle E_\xi \rangle_{\xi<\kappa}$, so that $E \in \mathcal{F}$.

Suppose that $A = \{\xi : [E_\xi]^n \subseteq S_\xi\}$ belongs to $\mathcal{F}$. Then $E \cap A \in \mathcal{F}$. If $I \in [E \cap A]^{n+1}$, set $\xi = \min I$. Then $I \setminus \{\xi\} \subseteq E_\xi$, so that $I \setminus \{\xi\} \in S_\xi$ and $I \in S$. Thus $[E \cap A]^{n+1} \subseteq S$. Similarly, if $A \notin \mathcal{F}$, then $E \setminus A \in \mathcal{F}$ and $[E \setminus A]^{n+1} \cap S = \emptyset$. Thus we can take one of $E \cap A$, $E \setminus A$ for F_{n+1}, and the induction continues. **Q**

(b) At the end of the induction, take $F = \bigcap_{n\in\mathbb{N}} F_n$; this serves.

4A1M Ostaszewski's ♣ This is the statement

> Let Ω be the family of non-zero countable limit ordinals. Then there is a family $\langle\theta_\xi(n)\rangle_{\xi\in\Omega,n\in\mathbb{N}}$ such that (α) for each $\xi\in\Omega$, $\langle\theta_\xi(n)\rangle_{n\in\mathbb{N}}$ is a strictly increasing sequence with supremum ξ (β) for any uncountable $A\subseteq\omega_1$ there is a $\xi\in\Omega$ such that $\theta_\xi(n)\in A$ for every $n\in\mathbb{N}$.

This is an immediate consequence of Jensen's $\diamondsuit$ (Just & Weese 97, Exercise 22.9), which is itself a consequence of Gödel's Axiom of Constructibility (Kunen 80, §II.7; Just & Weese 97, §22; Jech 78, §22; Jech 03, 13.21).

4A1N Lemma Assume ♣. Then there is a family $\langle C_\xi\rangle_{\xi<\omega_1}$ of sets such that (i) $C_\xi\subseteq\xi$ for every $\xi<\omega_1$ (ii) $C_\xi\cap\eta$ is finite whenever $\eta<\xi<\omega_1$ (iii) for any uncountable sets A, $B\subseteq\omega_1$ there is a $\xi<\omega_1$ such that $A\cap C_\xi$ and $B\cap C_\xi$ are both infinite.

proof (a) Let $\langle\theta_\xi(n)\rangle_{\xi\in\Omega,n\in\mathbb{N}}$ be a family as in 4A1M. Let $f:\omega_1\to[\omega_1]^2$ be a surjection (3A1Cd). For $\xi\in\Omega$, set

$$C_\xi=\bigcup_{i\in\mathbb{N}}f(\theta_\xi(i+1))\cap\xi\setminus\theta_\xi(i).$$

Then $C_\xi\subseteq\xi$, and if $\eta<\xi$ there is some $n\in\mathbb{N}$ such that $\theta_\xi(n)\ge\eta$, so that

$$C_\xi\cap\eta\subseteq\bigcup_{i\le n}f(\theta_\xi(i))$$

is finite. For $\xi\in\omega_1\setminus\Omega$ set $C_\xi=\emptyset$. Then $\langle C_\xi\rangle_{\xi<\omega_1}$ satisfies (i) and (ii) above.

(b) Now suppose that A, $B\subseteq\omega_1$ are uncountable. Choose $\langle\alpha_\xi\rangle_{\xi<\omega_1}$, $\langle\beta_\xi\rangle_{\xi<\omega_1}$, $\langle I_\xi\rangle_{\xi<\omega_1}$ inductively, as follows. β_ξ is to be the smallest ordinal such that $\{\alpha_\eta:\eta<\xi\}\cup\bigcup_{\eta<\xi}I_\eta\subseteq\beta_\xi$; I_ξ is to be a doubleton subset of $\omega_1\setminus(\beta_\xi\cup\bigcup_{\eta\le\beta_\xi}f(\eta))$ meeting both A and B; and $\alpha_\xi<\omega_1$ is to be such that $f(\alpha_\xi)=I_\xi$. Set $D=\{\alpha_\xi:\xi<\omega_1\}$. This construction ensures that $\langle\alpha_\xi\rangle_{\xi<\omega_1}$ and $\langle\beta_\xi\rangle_{\xi<\omega_1}$ are strictly increasing, with $\beta_\xi<\alpha_\xi<\beta_{\xi+1}$ for every ξ, so that $f(\delta)$ meets both A and B for every $\delta\in D$, while $f(\delta)\subseteq\delta'$ and $f(\delta')\cap(\delta\cup f(\delta))=\emptyset$ whenever $\delta<\delta'$ in D.

By the choice of $\langle\theta_\xi(n)\rangle_{\xi\in\Omega,n\in\mathbb{N}}$, there is a $\xi\in\Omega$ such that $\theta_\xi(n)\in D$ for every $n\in\mathbb{N}$. But this means that

$$f(\theta_\xi(i))\subseteq\theta_\xi(i+1)\subseteq\xi,\quad f(\theta_\xi(i+1))\cap(\theta_\xi(i)\cup f(\theta_\xi(i)))=\emptyset$$

for every $i\in\mathbb{N}$, so $C_\xi=\bigcup_{i\ge1}f(\theta_\xi(i))$ meets both A and B in infinite sets.

4A1O The size of σ-algebras: Proposition Let $\mathfrak{A}$ be a Boolean algebra, B a subset of $\mathfrak{A}$, and $\mathfrak{B}$ the σ-subalgebra of $\mathfrak{A}$ generated by B (331E). Then $\#(\mathfrak{B})\le\max(4,\#(B^{\mathbb{N}}))$. In particular, if $\#(B)\le\mathfrak{c}$ then $\#(\mathfrak{B})\le\mathfrak{c}$.

proof (a) If $\#(B)\le1$, this is trivial, since then $\#(\mathfrak{B})\le4$. So we need consider only the case $\#(B)\ge2$.

(b) Set $\kappa=\#(B^{\mathbb{N}})$; then whenever $\#(A)\le\kappa$, that is, there is an injection from A to $B^{\mathbb{N}}$, then

$$\#(A^{\mathbb{N}})\le\#((B^{\mathbb{N}})^{\mathbb{N}})=\#(B^{\mathbb{N}\times\mathbb{N}})=\#(B^{\mathbb{N}})=\kappa.$$

As we are supposing that B has more than one element, $\kappa\ge\#(\{0,1\}^{\mathbb{N}})=\mathfrak{c}\ge\omega_1$.

(c) Define $\langle B_\xi\rangle_{\xi<\omega_1}$ inductively, as follows. $B_0=B\cup\{0\}$. Given $\langle B_\eta\rangle_{\eta<\xi}$, where $0<\xi<\omega_1$, set $B'_\xi=\bigcup_{\eta<\xi}B_\eta$ and

$$\begin{aligned}B_\xi=\{1\setminus b:b\in B'_\xi\}&\\ \cup\{\sup_{n\in\mathbb{N}}b_n:\langle b_n\rangle_{n\in\mathbb{N}}&\text{ is a sequence in }B'_\xi\text{ with a supremum in }\mathfrak{A}\};\end{aligned}$$

continue.

An easy induction on ξ (relying on 3A1Cc and (b) above) shows that every B_ξ has cardinal at most κ. So $C=\bigcup_{\xi<\omega_1}B_\xi$ has cardinal at most κ.

(d) Now $\langle B_\xi\rangle_{\xi<\omega_1}$ is a non-decreasing family, so if $\langle c_n\rangle_{n\in\mathbb{N}}$ is any sequence in C there is some $\xi<\omega_1$ such that every c_n belongs to $B_\xi\subseteq B'_{\xi+1}$. But this means that if $\sup_{n\in\mathbb{N}}c_n$ is defined in $\mathfrak{A}$, it belongs to $B_{\xi+1}\subseteq C$. At the same time,

$$1 \setminus c_0 \in B_{\xi+1} \subseteq C.$$

This shows that C is closed under complementation and countable suprema; since it contains 0, it is a σ-subalgebra of $\mathfrak{A}$; since it includes B, it includes $\mathfrak{B}$, and $\#(\mathfrak{B}) \le \#(C) \le \kappa$, as claimed.

(d) Finally, if $\#(B) \le \mathfrak{c}$, $\#(\mathfrak{B}) \le \max(4, \#(B^{\mathbb{N}})) \le \mathfrak{c}$ by 4A1A(c-ii).

4A1P An incidental fact If I is a countable set and $\epsilon > 0$, there is a family $\langle \epsilon_i \rangle_{i \in I}$ of strictly positive real numbers such that $\sum_{i \in I} \epsilon_i \le \epsilon$. **P** Let $f : I \to \mathbb{N}$ be an injection and set $\epsilon_i = 2^{-f(i)-1}\epsilon$. **Q**

Version of 21.4.13/28.12.18

4A2 General topology

Even more than in previous volumes, naturally enough, the work of this volume depends on results from general topology. We have now reached the point where some of the facts I rely on are becoming hard to find as explicitly stated theorems in standard textbooks. I find myself therefore writing out rather a lot of proofs. You should not suppose that the results to which I attach proofs, rather than references, are particularly deep; on the contrary, in many cases I am merely spelling out solutions to classic exercises.

The style of 'general' topology, as it has evolved over the last hundred years, is to develop a language capable of squeezing the utmost from every step of argument. While this does sometimes lead to absurdly obscure formulations, it remains a natural, and often profitable, response to the remarkably dense network of related ideas in this area. I therefore follow the spirit of the subject in giving the results I need in the full generality achievable within the terminology I use. For the convenience of anyone coming to the theory for the first time, I repeat some of them in the forms in which they are actually applied. I should remark, however, that in some cases materially stronger results can be proved with little extra effort; as always, this appendix is to be thought of not as a substitute for a thorough study of the subject, but as a guide connecting standard approaches to the general theory with the special needs of this volume.

4A2A Definitions I begin the section with a glossary of terms not defined elsewhere.

Baire space A topological space X is a **Baire space** if $\bigcap_{n \in \mathbb{N}} G_n$ is dense in X whenever $\langle G_n \rangle_{n \in \mathbb{N}}$ is a sequence of dense open subsets of X.

Base of neighbourhoods If X is a topological space and $x \in X$, a **base of neighbourhoods** of x is a family $\mathcal{V}$ of neighbourhoods of x such that every neighbourhood of x includes some member of $\mathcal{V}$.

boundary If X is a topological space and $A \subseteq X$, the **boundary** of A is $\partial A = \overline{A} \setminus \operatorname{int} A = \overline{A} \cap \overline{X \setminus A}$.

càdlàg If X is a Hausdorff space, a function $f : [0, \infty[\to X$ is **càdlàg** ('continue à droit, limite à gauche') (or **RCLL** ('right continuous, left limits'), an ***R*-function**,) if $\lim_{s \downarrow t} f(s) = f(t)$ for every $t \ge 0$ and $\lim_{s \uparrow t} f(s)$ is defined in X for every $t > 0$.

càllàl If X is a Hausdorff space, a function $f : [0, \infty[\to X$ is **càllàl** ('continue à l'une, limite à l'autre') if $f(0) = \lim_{s \downarrow 0} f(s)$ and, for every $t > 0$, $\lim_{s \downarrow t} f(s)$ and $\lim_{s \uparrow t} f(s)$ are defined in X, and at least one of them is equal to $f(t)$.

Čech-complete A completely regular Hausdorff topological space X is **Čech-complete** if it is homeomorphic to a G_δ subset of a compact Hausdorff space.

closed interval Let X be a totally ordered set. A **closed interval** in X is an interval of one of the forms $\emptyset$, $[x, y]$, $]-\infty, y]$, $[x, \infty[$ or $X =]-\infty, \infty[$ where x, $y \in X$ (see the definition of 'interval' below).

coarser topology If $\mathfrak{S}$ and $\mathfrak{T}$ are two topologies on a set X, we say that $\mathfrak{S}$ is **coarser** than $\mathfrak{T}$ if $\mathfrak{S} \subseteq \mathfrak{T}$. (Equality allowed.)

compact support Let X be a topological space and $f : X \to \mathbb{R}$ a function. I say that f has **compact support** if $\overline{\{x : x \in X, f(x) \ne 0\}}$ is compact in X.

countably compact A topological space X is **countably compact** if every countable open cover of X has a finite subcover. (**Warning!** some authors reserve the term for Hausdorff spaces.) A subset of a topological space is countably compact if it is countably compact in its subspace topology.

countably paracompact A topological space X is **countably paracompact** if given any countable open cover $\mathcal{G}$ of X there is a locally finite family $\mathcal{H}$ of open sets which refines $\mathcal{G}$ and covers X. (**Warning!** some authors reserve the term for Hausdorff spaces.)

countably tight A topological space X is **countably tight** (or has **countable tightness**) if whenever $A \subseteq X$ and $x \in \overline{A}$ there is a countable set $B \subseteq A$ such that $x \in \overline{B}$.

direct sum, disjoint union Let $\langle (X_i, \mathfrak{T}_i)\rangle_{i\in I}$ be a family of topological spaces, and set $X = \{(x,i) : i \in I, x \in X_i\}$. The **disjoint union topology** on X is $\mathfrak{T} = \{G : G \subseteq X, \{x : (x,i) \in G\} \in \mathfrak{T}_i$ for every $i \in I\}$; $(X, \mathfrak{T})$ is the **(direct) sum** of $\langle (X_i, \mathfrak{T}_i)\rangle_{i\in I}$.

If X is a set, $\langle X_i\rangle_{i\in I}$ a partition of X, and $\mathfrak{T}_i$ a topology on X_i for every $i \in I$, then the **disjoint union topology** on X is $\{G : G \subseteq X, G \cap X_i \in \mathfrak{T}_i$ for every $i \in I\}$.

dyadic A Hausdorff space is **dyadic** if it is a continuous image of $\{0,1\}^I$ for some set I.

equicontinuous If X is a topological space, $(Y, \mathcal{W})$ a uniform space, and F a set of functions from X to Y, then F is **equicontinuous** if for every $x \in X$ and $W \in \mathcal{W}$ the set $\{y : (f(x), f(y)) \in W$ for every $f \in F\}$ is a neighbourhood of x.

finer topology If $\mathfrak{S}$ and $\mathfrak{T}$ are two topologies on a set X, we say that $\mathfrak{S}$ is **finer** than $\mathfrak{T}$ if $\mathfrak{S} \supseteq \mathfrak{T}$. (Equality allowed.)

first-countable A topological space X is **first-countable** if every point has a countable base of neighbourhoods.

half-open Let X be a totally ordered set. A **half-open interval** in X is a set of one of the forms $[x, y[$, $]x, y]$ where x, $y \in X$ and $x < y$ (see the definition of 'interval' below).

hereditarily Lindelöf A topological space is **hereditarily Lindelöf** if every subspace is Lindelöf.

hereditarily metacompact A topological space is **hereditarily metacompact** if every subspace is metacompact.

hereditarily separable A topological space is **hereditarily separable** if every subspace is separable.

indiscrete If X is any set, the **indiscrete** topology on X is the topology $\{\emptyset, X\}$.

interval Let $(P, \le)$ be a partially ordered set. An **interval** in P is a set of one of the forms $[p, q] = \{r : p \le r \le q\}$, $[p, q[= \{r : p \le r < q\}$, $]p, q] = \{r : p < r \le q\}$, $]p, q[= \{r : p < r < q\}$, $[p, \infty[= \{r : p \le r\}$, $]-\infty, q] = \{r : r \le q\}$, $]p, \infty[= \{r : p < r\}$, $]-\infty, q[= \{r : r < q\}$, $]-\infty, \infty[= P$, where p, $q \in P$. Note that every interval is order-convex, but even in a totally ordered set not every order-convex set need be an interval in this sense; an interval always has end-points, if we allow $\pm\infty$.

irreducible If X and Y are topological spaces, a continuous surjection $f : X \to Y$ is **irreducible** if $f[F] \ne Y$ for any closed proper subset F of X.

isolated If X is a topological space, a family $\mathcal{A}$ of subsets of X is **isolated** if $A \cap \overline{\bigcup(\mathcal{A} \setminus \{A\})}$ is empty for every $A \in \mathcal{A}$; that is, if $\mathcal{A}$ is disjoint and every member of $\mathcal{A}$ is a relatively open set in $\bigcup \mathcal{A}$.

Lindelöf A topological space is **Lindelöf** if every open cover has a countable subcover. (**Warning!** some authors reserve the term for regular spaces.)

Lipschitz If (X, ρ) and (Y, σ) are metric spaces, a function $f : X \to Y$ is **γ-Lipschitz**, or **(γ, ρ, σ)-Lipschitz**, where $\gamma \ge 0$, if $\sigma(f(x), f(y)) \le \gamma\rho(x, y)$ for all x, $y \in X$. $f : X \to Y$ is **Lipschitz** or **(ρ, σ)-Lipschitz** if it is (γ, ρ, σ)-Lipschitz for some $\gamma \ge 0$.

locally finite If X is a topological space, a family $\mathcal{A}$ of subsets of X is **locally finite** if for every $x \in X$ there is an open set which contains x and meets only finitely many members of $\mathcal{A}$.

lower semi-continuous If X is a topological space and T a totally ordered set, a function $f : X \to T$ is **lower semi-continuous** if $\{x : f(x) > t\}$ is open for every $t \in T$. (Cf. 225H, 3A3Cf.)

metacompact A topological space is **metacompact** if every open cover has a point-finite refinement which is an open cover. (**Warning!** some authors reserve the term for Hausdorff spaces.)

neighbourhood If X is a topological space and $x \in X$, a **neighbourhood** of x is any subset of X including an open set which contains x.

network Let $(X, \mathfrak{T})$ be a topological space. A **network** for $\mathfrak{T}$ is a family $\mathcal{E} \subseteq \mathcal{P}X$ such that whenever $x \in G \in \mathfrak{T}$ there is an $E \in \mathcal{E}$ such that $x \in E \subseteq G$.

normal A topological space X is **normal** if for any disjoint closed sets E, $F \subseteq X$ there are disjoint open sets G, H such that $E \subseteq G$ and $F \subseteq H$. (**Warning!** some authors reserve the term for Hausdorff spaces.)

open interval Let X be a totally ordered set. An **open interval** in X is a set of one of the the forms $]x, y[$, $]x, \infty[$, $]-\infty, x[$ or $]-\infty, \infty[= X$ where x, $y \in X$ (see the definition of 'interval' above).

open map If $(X,\mathfrak{T})$ and $(Y,\mathfrak{S})$ are topological spaces, a function $f : X \to Y$ is **open** if $f[G] \in \mathfrak{S}$ for every $G \in \mathfrak{T}$.

order-convex Let $(P,\le)$ be a partially ordered set. A subset C of P is **order-convex** if $[p,q] = \{r : p \le r \le q\}$ is included in C whenever p, $q \in C$.

order topology Let $(X,\le)$ be a totally ordered set. Its **order topology** is that generated by intervals of the form $]x,\infty[\ = \{y : y > x\}$, $]-\infty,x[\ = \{y : y < x\}$ as x runs over X.

paracompact A topological space is **paracompact** if every open cover has a locally finite refinement which is an open cover. (**Warning!** some authors reserve the term for Hausdorff spaces.)

perfect A topological space is **perfect** if it is compact and has no isolated points.

perfectly normal A topological space is **perfectly normal** if it is normal and every closed set is a G_δ set. (**Warning!** remember that some authors reserve the term 'normal' for Hausdorff spaces.)

point-countable, point-finite A family $\mathcal{A}$ of sets is **point-countable** if no point belongs to more than countably many members of $\mathcal{A}$. Similarly, an indexed family $\langle A_i\rangle_{i\in I}$ of sets is **point-finite** if $\{i : x \in A_i\}$ is finite for every x.

Polish A topological space X is **Polish** if it is separable and its topology can be defined from a metric under which X is complete.

pseudometrizable A topological space $(X,\mathfrak{T})$ is **pseudometrizable** if $\mathfrak{T}$ is defined by a single pseudometric (2A3F).

refine(ment) If $\mathcal{A}$ is a family of sets, a **refinement** of $\mathcal{A}$ is a family $\mathcal{B}$ of sets such that every member of $\mathcal{B}$ is included in some member of $\mathcal{A}$; in this case I say that $\mathcal{B}$ **refines** $\mathcal{A}$. (**Warning!** I do not suppose that $\bigcup\mathcal{B} = \bigcup\mathcal{A}$.)

relatively countably compact If X is a topological space, a subset A of X is **relatively countably compact** if every sequence in A has a cluster point in X. (**Warning!** This is *not* the same as supposing that A is included in a countably compact subset of X.)

scattered A topological space X is **scattered** if every non-empty subset of X has an isolated point (in its subspace topology).

second-countable A topological space is **second-countable** if the topology has a countable base, that is, if its weight is at most ω.

semi-continuous *see* lower semi-continuous, upper semi-continuous.

sequential A topological space is **sequential** if every sequentially closed set in X is closed.

sequentially closed If X is a topological space, a subset A of X is **sequentially closed** if $x \in A$ whenever $\langle x_n\rangle_{n\in\mathbb{N}}$ is a sequence in A converging to $x \in X$.

sequentially compact A topological space is **sequentially compact** if every sequence has a convergent sequence. A subset of a topological space is sequentially compact if it is sequentially compact in its subspace topology. (**Warning!** some authors reserve the term for Hausdorff spaces.)

sequentially continuous If X and Y are topological spaces, a function $f : X \to Y$ is **sequentially continuous** if $\langle f(x_n)\rangle_{n\in\mathbb{N}} \to f(x)$ in Y whenever $\langle x_n\rangle_{n\in\mathbb{N}} \to x$ in X.

subbase If $(X,\mathfrak{T})$ is a topological space, a **subbase** for $\mathfrak{T}$ is a family $\mathcal{U} \subseteq \mathfrak{T}$ which generates $\mathfrak{T}$, in the sense that $\mathfrak{T}$ is the coarsest topology on X including $\mathcal{U}$. (**Warning!** most authors reserve the term for families $\mathcal{U}$ with union X.)

totally bounded If $(X,\mathcal{W})$ is a uniform space, a subset A of X is **totally bounded** if for every $W \in \mathcal{W}$ there is a finite set $I \subseteq X$ such that $A \subseteq W[I]$. If (X,ρ) is a metric space, a subset of X is totally bounded if it is totally bounded for the associated uniformity (3A4B).

uniform convergence If X is a set, (Y,σ) is a metric space and $\mathcal{A}$ is a family of subsets of X then the **topology of uniform convergence** on members of $\mathcal{A}$ is the topology on Y^X generated by the pseudometrics $(f,g) \mapsto \min(1, \sup_{x\in A}\sigma(f(x),g(x)))$ as A runs over $\mathcal{A}\setminus\{\emptyset\}$. (It is elementary to verify that the formula here defines a pseudometric.)

upper semi-continuous If X is a topological space and T is a totally ordered set, a function $f : X \to T$ is **upper semi-continuous** if $\{x : f(x) < t\}$ is open for every $t \in T$.

weakly α-favourable A topological space $(X,\mathfrak{T})$ is **weakly α-favourable** if there is a function $\sigma : \bigcup_{n\in\mathbb{N}}(\mathfrak{T}\setminus\{\emptyset\})^{n+1} \to \mathfrak{T}\setminus\{\emptyset\}$ such that (i) $\sigma(G_0,\dots,G_n) \subseteq G_n$ whenever $G_0,\dots,G_n$ are non-empty open sets (ii) whenever $\langle G_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{T}\setminus\{\emptyset\}$ such that $G_{n+1} \subseteq \sigma(G_0,\dots,G_n)$ for every n, then $\bigcap_{n\in\mathbb{N}} G_n$ is non-empty.

weight If X is a topological space, its **weight** $w(X)$ is the smallest cardinal of any base for the topology.

C_b If X is a topological space, $C_b(X)$ is the space of bounded continuous real-valued functions defined on X.

F_σ If X is a topological space, an $\mathbf{F}_\sigma$ set in X is one expressible as the union of a sequence of closed sets.

G_δ If X is a topological space, a $\mathbf{G}_\delta$ set in X is one expressible as the intersection of a sequence of open sets.

K_σ If X is a topological space, a $\mathbf{K}_\sigma$ set in X is one expressible as the union of a sequence of compact sets.

$\mathcal{P}X$ If X is any set, the **usual topology** on $\mathcal{P}X$ is that generated by the sets $\{a : a \subseteq X,\ a \cap J = K\}$ where $J \subseteq X$ is finite and $K \subseteq J$.

T_0 If $(X, \mathfrak{T})$ is a topological space, we say that it is $\mathbf{T}_0$ if for any two distinct points of X there is an open set containing one but not the other.

T_1 If $(X, \mathfrak{T})$ is a topological space, we say that it is $\mathbf{T}_1$ if singleton sets are closed.

π-base If $(X, \mathfrak{T})$ is a topological space, a **π-base** for $\mathfrak{T}$ is a set $\mathcal{U} \subseteq \mathfrak{T}$ such that every non-empty open set includes a non-empty member of $\mathcal{U}$.

σ-compact A topological space X is **σ-compact** if there is a sequence of compact subsets of X covering X.

σ-disjoint A family of sets is **σ-disjoint** if it is expressible as $\bigcup_{n\in\mathbb{N}} \mathcal{A}_n$ where every $\mathcal{A}_n$ is disjoint.

σ-isolated If X is a topological space, a family of subsets of X is **σ-isolated** if it is expressible as $\bigcup_{n\in\mathbb{N}} \mathcal{A}_n$ where every $\mathcal{A}_n$ is an isolated family.

σ-metrically-discrete If (X, ρ) is a metric space, a family of subsets of X is **σ-metrically-discrete** if it is expressible as $\bigcup_{n\in\mathbb{N}} \mathcal{A}_n$ where $\rho(x, y) \ge 2^{-n}$ whenever $n \in \mathbb{N}$, A and B are distinct members of $\mathcal{A}_n$, $x \in A$ and $y \in B$.

4A2B Elementary facts about general topological spaces (a) Bases and networks (i) Let $(X, \mathfrak{T})$ be a topological space and $\mathcal{U}$ a subbase for $\mathfrak{T}$. Then $\{X\} \cup \{U_0 \cap U_1 \cap \ldots \cap U_n : U_0, \ldots, U_n \in \mathcal{U}\}$ is a base for $\mathfrak{T}$. (For this is a base for a topology, by 3A3Mc.)

(ii) Let X and Y be topological spaces, and $\mathcal{U}$ a subbase for the topology of Y. Then a function $f : X \to Y$ is continuous iff $f^{-1}[U]$ is open for every $U \in \mathcal{U}$. (Engelking 89, 1.4.1(ii)).

(iii) If X and Y are topological spaces, $\mathcal{E}$ is a network for the topology of Y, and $f : X \to Y$ is a function such that $f^{-1}[E]$ is open for every $E \in \mathcal{E}$, then f is continuous. (The topology generated by $\mathcal{E}$ includes the given topology on Y.)

(iv) If X is a topological space and $\mathcal{U}$ is a subbase for the topology of X, then a filter $\mathcal{F}$ on X converges to $x \in X$ iff $\{U : x \in U \in \mathcal{U}\} \subseteq \mathcal{F}$. (If the condition is satisfied, $\mathcal{F} \cup \{A : A \subseteq X,\ x \notin A\}$ is a topology on X including $\mathcal{U}$.)

(v) If X and Y are topological spaces with subbases $\mathcal{U}$, $\mathcal{V}$ respectively, then $\{U \times Y : U \in \mathcal{U}\} \cup \{X \times V : V \in \mathcal{V}\}$ is a subbase for the product topology of $X \times Y$. (Kuratowski 66, §15.I.)

(vi) If $\mathcal{U}$ is a (sub-)base for a topology on X, and $Y \subseteq X$, then $\{Y \cap U : U \in \mathcal{U}\}$ is a (sub-)base for the subspace topology of Y. (Császár 78, 2.3.13(e)-(f).)

(vii) If X is a topological space, $\mathcal{E}$ is a network for the topology of X, and Y is a subset of X, then $\{E \cap Y : E \in \mathcal{E}\}$ is a network for the topology of Y.

(viii) If X is a topological space and $\mathcal{A}$ is a (σ-)isolated family of subsets of X, then $\{A \cap Y : A \in \mathcal{A}'\}$ is (σ-)isolated whenever $Y \subseteq X$ and $\mathcal{A}' \subseteq \mathcal{A}$.

(ix) If a topological space X has a σ-isolated network, so has every subspace of X.

(b) If $\langle H_i \rangle_{i\in I}$ is a partition of a topological space X into open sets and $F_i \subseteq H_i$ is closed (either in X or in H_i) for each $i \in I$, then $F = \bigcup_{i\in I} F_i$ is closed in X. ($X \setminus F = \bigcup_{i\in I}(H_i \setminus F_i)$.)

(c) If X is a topological space, $A \subseteq X$ and $x \in X$, then $x \in \overline{A}$ iff there is an ultrafilter on X, containing A, which converges to x. ($\{A\} \cup \{G : x \in G \subseteq X,\ G \text{ is open}\}$ has the finite intersection property; use 4A1Ia.)

(d) Semi-continuity Let X be a topological space.

(i) A function $f : X \to \mathbb{R}$ is lower semi-continuous iff $-f$ is upper semi-continuous. (Čech 66, 18D.8.) A function $f : X \to \mathbb{R}$ is lower semi-continuous iff $\Omega = \{(x, \alpha) : x \in X,\ \alpha \ge f(x)\}$ is closed in $X \times \mathbb{R}$. (If f

is lower semi-continuous and $\alpha < \beta < f(x)$ then $\{y : f(y) > \beta\} \times]-\infty, \beta[$ is a neighbourhood of (x, α); so Ω is closed. If Ω is closed then for any $\gamma \in \mathbb{R}$ the set $\{x : f(x) > \gamma\} = \{x : (x, \gamma) \notin \Omega\}$ is open; so f is lower semi-continuous.)

(ii) If T is a totally ordered set, $f : X \to T$ is lower semi-continuous, Y is another topological space, and $g : Y \to X$ is continuous, then $fg : Y \to T$ is lower semi-continuous. ($\{y : (fg)(y) > t\} = g^{-1}[\{x : f(x) > t\}]$.) In particular, if $f : X \to T$ is lower semi-continuous and $Y \subseteq X$, then $f \restriction Y$ is lower semi-continuous. Similarly, if $f : X \to T$ is upper semi-continuous and $g : Y \to X$ is continuous, then $fg : Y \to T$ is upper semi-continuous.

(iii) If f, $g : X \to]-\infty, \infty]$ are lower semi-continuous so is $f + g : X \to]-\infty, \infty]$. (Čech 66, 18D.8.)

(iv) If f, $g : X \to [0, \infty]$ are lower semi-continuous so is $f \times g : X \to [0, \infty]$. (Čech 66, 18D.8.)

(v) If Φ is any non-empty set of lower semi-continuous functions from X to $[-\infty, \infty]$, then $x \mapsto \sup_{f \in \Phi} f(x) : X \to [-\infty, \infty]$ is lower semi-continuous.

(vi) $f : X \to \mathbb{R}$ is continuous iff f is both upper semi-continuous and lower semi-continuous iff f and $-f$ are both lower semi-continuous.

(vii) If $f : X \to [-\infty, \infty]$ is lower semi-continuous, and $\mathcal{F}$ is a filter on X converging to $y \in X$, then $f(y) \le \liminf_{x \to \mathcal{F}} f(x)$.

(viii) If X is compact and not empty, and $f : X \to [-\infty, \infty]$ is lower semi-continuous then $K = \{x : f(x) = \inf_{y \in X} f(y)\}$ is non-empty and compact. **P** Setting $\gamma = \inf_{y \in X} f(y) \in [-\infty, \infty]$, $\{\{x : f(x) \le \alpha\} : \alpha > \gamma\}$ is a downwards-directed family of non-empty closed sets, so its intersection K is a non-empty closed set. **Q**

(ix) If f, $g : X \to [0, \infty]$ are lower semi-continuous and $f + g$ is continuous at $x \in X$ and finite there, then f and g are continuous at x. **P** If $\epsilon > 0$ there is a neighbourhood G of x such that $(f + g)(y) \le (f + g)(x) + \epsilon$ for every $y \in G$ and $g(y) \ge g(x) - \epsilon$ for every $y \in G$, so that $f(y) \le f(x) + 2\epsilon$ for every $y \in G$. **Q**

(e) Separable spaces (i) If $\langle A_i \rangle_{i \in I}$ is a countable family of separable subsets of a topological space X then $\bigcup_{i \in I} A_i$ and $\overline{\bigcup_{i \in I} A_i}$ are separable. (If $D_i \subseteq A_i$ is countable and dense for each i, $\bigcup_{i \in I} D_i$ is countable and dense in both $\bigcup_{i \in I} A_i$ and its closure.)

(ii) If $\langle X_i \rangle_{i \in I}$ is a family of separable topological spaces and $\#(I) \le \mathfrak{c}$, then $\prod_{i \in I} X_i$ is separable. (Engelking 89, 2.3.16.)

(iii) A continuous image of a separable topological space is separable. (Engelking 89, 1.4.11.)

(f) Open maps (i) Let $\langle X_i \rangle_{i \in I}$ be any family of topological spaces, with product X. If $J \subseteq I$ is any set, and we write X_J for $\prod_{i \in J} X_i$, then the canonical map $x \mapsto x \restriction J : X \to X_J$ is open. (Engelking 89, p. 79.)

(ii) Let X and Y be topological spaces and $f : X \to Y$ a continuous open map. Then $\operatorname{int} f^{-1}[B] = f^{-1}[\operatorname{int} B]$ and $\overline{f^{-1}[B]} = f^{-1}[\overline{B}]$ for every $B \subseteq Y$. **P** Because f is continuous, $f^{-1}[\operatorname{int} B]$ is an open set included in $f^{-1}[B]$, so is included in $\operatorname{int} f^{-1}[B]$. Because f is open, $f[\operatorname{int} f^{-1}[B]]$ is an open set included in $f[f^{-1}[B]] \subseteq B$, so $f[\operatorname{int} f^{-1}[B]] \subseteq \operatorname{int} B$, that is, $\operatorname{int} f^{-1}[B] \subseteq f^{-1}[\operatorname{int} B]$. Now apply this to $Y \setminus B$ and take complements. **Q**

It follows that $f^{-1}[B]$ is nowhere dense in X whenever $B \subseteq Y$ is nowhere dense in Y. ($\operatorname{int} \overline{f^{-1}[B]} = \operatorname{int} f^{-1}[\overline{B}] = f^{-1}[\operatorname{int} \overline{B}] = \emptyset$.) If f is surjective and $B \subseteq Y$, then B is nowhere dense in Y iff $f^{-1}[B]$ is nowhere dense in X. (For $\operatorname{int} \overline{f^{-1}[B]} = f^{-1}[\operatorname{int} \overline{B}]$ is empty iff $\operatorname{int} \overline{B}$ is empty.)

(iii) Let X and Y be topological spaces and $f : X \to Y$ a continuous open map. Then $H \mapsto f^{-1}[H]$ is an order-continuous Boolean homomorphism from the regular open algebra of Y to the regular open algebra of X. **P** If $H \subseteq Y$ is a regular open set,

$$\operatorname{int} \overline{f^{-1}[H]} = \operatorname{int} f^{-1}[\overline{H}] = f^{-1}[\operatorname{int} \overline{H}] = f^{-1}[H]$$

by (ii), so $f^{-1}[H]$ is a regular open set in X. If $F \subseteq Y$ is nowhere dense, then $f^{-1}[F]$ is nowhere dense in X, as noted in (ii) above. By 314Ra, $H \mapsto f^{-1}[H] = \operatorname{int} \overline{f^{-1}[H]}$ is an order-continuous Boolean homomorphism from $\mathrm{RO}(Y)$ to $\mathrm{RO}(X)$. **Q** If f is surjective, then the homomorphism is injective (because $f^{-1}[H] \ne \emptyset$ whenever $H \ne \emptyset$), and for $H \subseteq Y$, H is a regular open set in Y iff $f^{-1}[H]$ is a regular open set in X (because in this case $f^{-1}[H] = f^{-1}[\operatorname{int} \overline{H}]$).

(iv) If X_0, Y_0, X_1, Y_1 are topological spaces, and $f_i : X_i \to Y_i$ is an open map for each i, then $(x_0, x_1) \mapsto (f_0(x_0), f_1(x_1)) : X_0 \times X_1 \to Y_0 \times Y_1$ is open. (Engelking 89, 2.3.29.)

(g) Let $\langle X_i\rangle_{i\in I}$ be a family of topological spaces with product X.

(i) If $A \subseteq X$ is determined by coordinates in $J \subseteq I$ in the sense of 254M, then $\overline{A}$ and $\operatorname{int} A$ are also determined by coordinates in J. **P** Let $\pi : X \to \prod_{i\in J} X_i$ be the canonical map. Then $A = \pi^{-1}[\pi[A]]$, so (f) tells us that $\operatorname{int} A = \pi^{-1}[\operatorname{int}\pi[A]]$ and $\overline{A} = \pi^{-1}[\overline{\pi[A]}]$; but these are both determined by coordinates in J. **Q**

(ii) If $F \subseteq X$ is closed, there is a smallest set $J^* \subseteq I$ such that F is determined by coordinates in J^*.

P Let $\mathcal{J}$ be the family of all those sets $J \subseteq I$ such that F is determined by coordinates in $\mathcal{J}$. If J_1, $J_2 \in \mathcal{J}$, then $J_1 \cap J_2 \in \mathcal{J}$ (254Ta). Set $J^* = \bigcap \mathcal{J}$. **?** Suppose, if possible, that F is not determined by coordinates in J^*. Then there are $x \in F$, $y \in X \setminus F$ such that $x\restriction J^* = y\restriction J^*$. Because $X \setminus F$ is open, there is a finite set $K \subseteq I$ such that $z \notin F$ whenever $z \in X$ and $z\restriction K = y\restriction K$. Because $\mathcal{J}$ is closed under finite intersections, there is a $J \in \mathcal{J}$ such that $K \cap J = K \cap J^*$. Define $z \in X$ by setting $z(i) = x(i)$ for $i \in J$, $z(i) = y(i)$ for $i \in I \setminus J$. Then $z\restriction J = x\restriction J$, so $z \in F$, but $z\restriction K = y\restriction K$, so $z \notin F$. **X**

Thus $J^* \in \mathcal{J}$ and is the required smallest member of $\mathcal{J}$. **Q**

(h) Let X be a topological space.

(i) If $\mathcal{E}$ is a locally finite family of closed subsets of X, then $\bigcup \mathcal{E}'$ is closed for every $\mathcal{E}' \subseteq \mathcal{E}$. (Engelking 89, 1.1.11.)

(ii) If $\langle f_i\rangle_{i\in I}$ is a family in $C(X)$ such that $\langle\{x : f_i(x) \neq 0\}\rangle_{i\in I}$ is locally finite, then we have a continuous function $f : X \to \mathbb{R}$ defined by setting $f(x) = \sum_{i\in I} f_i(x)$ for every $x \in X$. **P** For any x, $\{i : f_i(x) \neq 0\}$ is finite, so f is well-defined. If $x_0 \in X$ and $\epsilon > 0$, there is a neighbourhood V of x_0 such that $J = \{i : i \in I,\ f_i(x) \neq 0 \text{ for some } x \in V\}$ is finite; now there is a neighbourhood W of x_0, included in V, such that $\sum_{i\in J} |f_i(x) - \sum_{i\in J} f_i(x_0)| < \epsilon$ for every $x \in W$, so that $|f(x) - f(x_0)| < \epsilon$ for every $x \in W$. As x_0 and ϵ are arbitrary, f is continuous. **Q**

(i) Let X be a topological space and A, B two subsets of X. Then the boundary $\partial(A * B)$ is included in $\partial A \cup \partial B$, where $*$ is any of $\cup$, $\cap$, $\setminus$, $\triangle$. (Generally, if $F \subseteq X$, $\{A : \partial A \subseteq F\} = \{A : \overline{A} \setminus F \subseteq \operatorname{int} A\}$ is a subalgebra of $\mathcal{P}X$.)

(j) Let X be a topological space and D a dense subset of X, endowed with its subspace topology.

(i) A set $A \subseteq D$ is nowhere dense in D iff it is nowhere dense in X. **P**

$$A \text{ is nowhere dense in } X \iff X \setminus \overline{A} \text{ is dense in } X$$

(writing $\overline{A}$ for the closure of A in X)

$$\iff D \setminus \overline{A} \text{ is dense in } X$$

(3A3Ea)

$$\iff D \setminus \overline{A} \text{ is dense in } D$$
$$\iff D \setminus (D \cap \overline{A}) \text{ is dense in } D$$
$$\iff A \text{ is nowhere dense in } D$$

because $D \cap \overline{A} = \overline{A}^{(D)}$ is the closure of A in D. **Q**

(ii) A set $G \subseteq D$ is a regular open set in D iff it is expressible as $D \cap H$ for some regular open set $H \subseteq X$. **P** (α) If G is a regular open subset of D, set $H = \operatorname{int}\overline{G}$, taking both the closure and the interior in X. Then H is a regular open set in X. Now $D \cap H$ is a relatively open subset of D included in $D \cap \overline{G} = \overline{G}^{(D)}$, so $D \cap H \subseteq \operatorname{int}_D \overline{G}^{(D)} = G$. In the other direction, $\overline{G} \cup \overline{D \setminus G} = \overline{D} = X$, so $\overline{G} \supseteq X \setminus \overline{D \setminus G}$ and $H \supseteq X \setminus \overline{D \setminus G} \supseteq G$. So $G = H \cap D$ is of the required form. (β) If $H \subseteq X$ is a regular open set such that $G = D \cap H$, set $V = X \setminus \overline{H}$; then $H = X \setminus \overline{V}$. Now

$$\overline{V \cap D}^{(D)} = D \cap \overline{V \cap D} = D \cap \overline{V} = D \setminus H = D \setminus G,$$

so $G = D \setminus \overline{V \cap D}^{(D)}$ is the complement of the closure of an open set in D, and is a regular open set in D. **Q**

4A2C G_δ, F_σ, zero and cozero sets Let X be a topological space.

(a)(i) The union of two G_δ sets in X is a G_δ set. (Engelking 89, p. 26; Kuratowski 66, §5.V.)

(ii) The intersection of countably many G_δ sets is a G_δ set. (Engelking 89, p. 26; Kuratowski 66, §5.V.)

(iii) If Y is another topological space, $f : X \to Y$ is continuous and $E \subseteq Y$ is G_δ in Y, then $f^{-1}[E]$ is G_δ in X. ($f^{-1}[\bigcap_{n\in\mathbb{N}} H_n] = \bigcap_{n\in\mathbb{N}} f^{-1}[H_n]$.)

(iv) If Y is a G_δ set in X and $Z \subseteq Y$ is a G_δ set for the subspace topology of Y, then Z is a G_δ set in X. (Kuratowski 66, §5.V.)

(v) A set $E \subseteq X$ is an F_σ set iff $X \setminus E$ is a G_δ set. (Kuratowski 66, §5.V.)

(b)(i) A zero set is closed. A cozero set is open.

(ii) The union of two zero sets is a zero set. (Császár 78, 4.2.36.) The intersection of two cozero sets is a cozero set.

(iii) The intersection of a sequence of zero sets is a zero set. (If $f_n : X \to \mathbb{R}$ is continuous for each n, $x \mapsto \sum_{n=0}^{\infty} \min(2^{-n}, |f_n(x)|)$ is continuous.) The union of a sequence of cozero sets is a cozero set.

(iv) If Y is another topological space, $f : X \to Y$ is continuous and $L \subseteq Y$ is a zero set, then $f^{-1}[L]$ is a zero set. If $f : X \to Y$ is continuous and $H \subseteq Y$ is a cozero set, then $f^{-1}[H]$ is a cozero set. (Čech 66, 28B.3.) If $K \subseteq X$ and $L \subseteq Y$ are zero sets then $K \times L$ is a zero set in $X \times Y$. ($K \times L = \pi_1^{-1}[K] \cap \pi_2^{-1}[L]$.)

(v) If $H \subseteq X$ is a (co-)zero set and $Y \subseteq X$, then $H \cap Y$ is a (co-)zero set in Y. (Use (iv).)

(vi) A cozero set is the union of a non-decreasing sequence of zero sets. (If $f : X \to \mathbb{R}$ is continuous, $X \setminus f^{-1}[\{0\}]) = \bigcup_{n\in\mathbb{N}} g_n^{-1}[\{0\}]$, where $g_n(x) = \max(0, 2^{-n} - |f(x)|)$.) In particular, a cozero set is an F_σ set; taking complements, a zero set is a G_δ set.

(vii) If $\mathcal{G}$ is a partition of X into open sets, and $H \subseteq X$ is such that $H \cap G$ is a cozero set in G for every $G \in \mathcal{G}$, then H is a cozero set in X. (If $f_G : G \to \mathbb{R}$ is continuous for every $G \in \mathcal{G}$, then $f : X \to \mathbb{R}$ is continuous, where $f(x) = f_G(x)$ for $x \in G \in \mathcal{G}$.) Similarly, if $F \subseteq X$ is such that $F \cap G$ is a zero set in G for every $G \in \mathcal{G}$, then F is a zero set in X.

4A2D Weight Let X be a topological space.

(a)(i) $w(Y) \le w(X)$ for every subspace Y of X (4A2B(a-vi)).

(ii) If $X = \prod_{i\in I} X_i$ then $w(X) \le \max(\omega, \#(I), \sup_{i\in I} w(X_i))$. (Engelking 89, 2.3.13.)

(b) A disjoint family $\mathcal{G}$ of non-empty open sets in X has cardinal at most $w(X)$. (If $\mathcal{U}$ is a base for the topology of X, then every non-empty member of $\mathcal{G}$ includes a non-empty member of $\mathcal{U}$, so we have an injective function from $\mathcal{G}$ to $\mathcal{U}$.)

(c) A point-countable family $\mathcal{G}$ of open sets in X has cardinal at most $\max(\omega, w(X))$. **P** If $X = \emptyset$, this is trivial. Otherwise, let $\mathcal{U}$ be a base for the topology of X with $\#(\mathcal{U}) = w(X) > 0$. Choose a function $f : \mathcal{G} \to \mathcal{U}$ such that $\emptyset \ne f(G) \subseteq G$ whenever $G \in \mathcal{G} \setminus \{\emptyset\}$. Then $\mathcal{G}_U = \{G : f(G) = U\}$ is countable for every $U \in \mathcal{U}$, so there is an injection $h_U : \mathcal{G}_U \to \mathbb{N}$; now $G \mapsto (f(G), h_{f(G)}(G)) : \mathcal{G} \to \mathcal{U} \times \mathbb{N}$ is injective, so $\#(\mathcal{G}) \le \#(\mathcal{U} \times \mathbb{N}) = \max(\omega, w(X))$. **Q**

(d) If X is a dyadic Hausdorff space then X is a continuous image of $\{0,1\}^{w(X)}$. **P** There are a set I and a continuous surjection $f : \{0,1\}^I \to X$; because any power of $\{0,1\}$ is compact, so is X. If $w(X)$ is finite, $\#(X) = w(X) \le \#(\{0,1\}^{w(X)})$ and the result is trivial; so we may suppose that $w(X)$ is infinite. Let $\mathcal{U}$ be a base for the topology of X with cardinality $w(X)$. Set $Z = \{0,1\}^I$ and let $\mathcal{E}$ be the algebra of subsets of Z determined by coordinates in finite sets, so that $\mathcal{E}$ is an algebra of subsets of Z and is a base for the topology of Z. For each pair U, V of members of $\mathcal{U}$ such that $\overline{U} \subseteq V$, $f^{-1}[V] \subseteq Z$ is open; the set $\{E : E \in \mathcal{E},\ E \subseteq f^{-1}[V]\}$ is upwards-directed and covers the compact set $f^{-1}[\overline{U}]$, so there is an $E_{UV} \in \mathcal{E}$ such that $f^{-1}[\overline{U}] \subseteq E_{UV} \subseteq f^{-1}[V]$. Let $J \subseteq I$ be a set with cardinal at most $\max(\omega, w(X))$ such that every E_{UV} is determined by coordinates in J. Fix any $w \in \{0,1\}^{I\setminus J}$ and define $g : \{0,1\}^J \to X$ by setting $g(z) = f(z, w)$ for every $z \in \{0,1\}^J$, identifying Z with $\{0,1\}^J \times \{0,1\}^{I\setminus J}$. Then g is continuous. **?** If g is not surjective, set $H = X \setminus g[\{0,1\}^J]$. Take $x \in H$; take $V \in \mathcal{U}$ such that $x \in V \subseteq H$; take an open set G such that $x \in G \subseteq \overline{G} \subseteq V$ (this must be possible because X, being compact and Hausdorff, is regular – see 3A3Bb); take $U \in \mathcal{U}$ such that $x \in U \subseteq G$, so that $x \in U \subseteq \overline{U} \subseteq V$. Because f is surjective, there is a $(u,v) \in \{0,1\}^J \times \{0,1\}^{I\setminus J}$ such that $f(u,v) = x$. Now $(u,v) \in f^{-1}[U] \subseteq E_{UV}$; as E_{UV} is determined by coordinates in J, $(u,w) \in E_{UV} \subseteq f^{-1}[V]$ and $g(u) = f(u,w) \in V$; but V is supposed to be disjoint from $g[\{0,1\}^J]$. **X** So g is surjective, and X is a continuous image of $\{0,1\}^J$. Since $\#(J) \le \max(\omega, w(X)) = w(X)$, $\{0,1\}^J$ and X are continuous images of $\{0,1\}^{w(X)}$. **Q**

(e) If X is a dyadic Hausdorff space then X is separable iff it is a continuous image of $\{0,1\}^{\mathfrak{c}}$. **P** $\{0,1\}^{\mathfrak{c}}$ is separable (4A2B(e-ii)), so any continuous image of it is separable. If X is a separable dyadic Hausdorff space, let $A \subseteq X$ be a countable dense set. If G, $G' \subseteq X$ are distinct regular open sets, then $G \cap A \ne G \cap A'$. Thus X has at most $\mathfrak{c}$ regular open sets; since X is compact and Hausdorff, therefore regular, its regular open sets form a base (4A2F(b-ii)), and $w(X) \le \mathfrak{c}$. By (d), X is a continuous image of $\{0,1\}^{\max(\omega,w(X))}$ which is in turn a continuous image of $\{0,1\}^{\mathfrak{c}}$. **Q**

4A2E The countable chain condition (a)(i) Let $\langle X_i\rangle_{i\in I}$ be a family of topological spaces. If $\prod_{i\in J} X_i$ is ccc for every finite $J \subseteq I$, then $\prod_{i\in I} X_i$ is ccc. (Kunen 80, II.1.9; Fremlin 84, 12I.)

(ii) A separable topological space is ccc. (If D is a countable dense set and $\mathcal{G}$ is a disjoint family of non-empty open sets, we have a surjection from a subset of D onto $\mathcal{G}$.)

(iii) The product of any family of separable topological spaces is ccc. **P** By 4A2B(e-ii) and (ii) here, the product of finitely many separable spaces is separable, therefore ccc; so we can apply (i). **Q**

(iv) Any continuous image of a ccc topological space is ccc. (If $f : X \to Y$ is a continuous surjection and $\mathcal{H}$ is an uncountable disjoint family of open subsets of Y, then $\{f^{-1}[H] : H \in \mathcal{H}\}$ is an uncountable disjoint family of open subsets of X.)

(b) Let $\langle X_i\rangle_{i\in I}$ be a family of topological spaces, and suppose that $X = \prod_{i\in I} X_i$ is ccc. For $J \subseteq I$ and $x \in X$ set $X_J = \prod_{i\in J} X_i$, $\pi_J(x) = x\restriction J$.

(i) If $G \subseteq X$ is open, there is an open set $W \subseteq G$ determined by coordinates in a countable subset of I such that $G \subseteq \overline{W}$. **P** Let $\mathcal{W}$ be the family of subsets of X determined by coordinates in countable sets. Then $\mathcal{W}$ is a σ-algebra (254Mb) including the standard base $\mathcal{U}$ for the topology of X. Let $\mathcal{U}_0$ be a maximal disjoint family in $\{U : U \in \mathcal{U}, U \subseteq G\}$. Then $\mathcal{U}_0$ is countable, so $W = \bigcup\mathcal{U}_0$ belongs to $\mathcal{W}$. No member of $\mathcal{U}$ can be included in $G \setminus W$, so $G \setminus \overline{W}$ must be empty, and we have a suitable set. **Q** So $\overline{G} = \overline{W}$ and $\operatorname{int}\overline{G}$ are determined by coordinates in a countable set (4A2B(g-i)); in particular, if G is a regular open set, then it is determined by coordinates in a countable set.

(ii) If $f : X \to \mathbb{R}$ is continuous, there are a countable set $J \subseteq I$ and a continuous function $g : X_J \to \mathbb{R}$ such that $f = g\pi_J$. **P** For each $q \in \mathbb{Q}$, set $F_q = \overline{\{x : f(x) < q\}}$. By (i), F_q is determined by coordinates in a countable set. Because $\mathbb{Q}$ is countable, there is a countable $J \subseteq I$ such that every F_q is determined by coordinates in J. Also $\{x : f(x) < \alpha\} = \bigcup_{q\in\mathbb{Q},q<\alpha} F_q$ is determined by coordinates in J for every $\alpha \in \mathbb{R}$, so $f(x) = f(y)$ whenever $x\restriction J = y\restriction J$, and there is a $g : X_J \to \mathbb{R}$ such that $f = g\pi_J$. Now if $H \subseteq \mathbb{R}$ is open, $g^{-1}[H] = \pi_J[f^{-1}[H]]$ is open (4A2B(f-i)), so g is continuous. **Q**

(iii) If $A \subseteq X$ is nowhere dense there is a countable set $J \subseteq I$ such that $\pi_J^{-1}[\pi_J[A]]$ is nowhere dense. **P** By (ii), there are a countable set J and an open set $W \subseteq X \setminus \overline{A}$ such that W is determined by coordinates in J and $X \setminus \overline{A} \subseteq \overline{W}$; now W is dense in X and $\pi_J^{-1}[\pi_J[A]] \subseteq X \setminus W$ is nowhere dense. **Q**

4A2F Separation axioms (a) Hausdorff spaces (i) A Hausdorff space is T_1. (Čech 66, 27A.1.)

(ii) If X is a Hausdorff space and $\langle x_n\rangle_{n\in\mathbb{N}}$ is a sequence in X, then a point x of X is a cluster point of $\langle x_n\rangle_{n\in\mathbb{N}}$ iff there is a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$ such that $x = \lim_{n\to\mathcal{F}} x_n$. (If x is a cluster point of $\langle x_n\rangle_{n\in\mathbb{N}}$, apply 4A1Ia to $\{\{n : n \ge n_0,\, x_n \in G\} : n_0 \in \mathbb{N},\, G \subseteq X \text{ is open}, x \in G\}$.)

(iii) A topological space X is Hausdorff iff $\{(x,x) : x \in X\}$ is closed in $X \times X$. (Čech 66 27A.7; Kuratowski 66, I.15.IV.)

(b) Regular spaces (i) A regular T_1 space is Hausdorff. (Čech 66, 27B.7; Gaal 64, p. 81.) Any subspace of a regular space is regular. (Engelking 89, 2.1.6; Kuratowski 66, §14.I.)

(ii) If X is a regular topological space, the regular open subsets of X form a base for the topology. **P** If G is open and $x \in G$, there is an open set H such that $x \in H \subseteq \overline{H} \subseteq G$; now $\operatorname{int}\overline{H}$ is a regular open set containing x and included in G. **Q**

(c) Completely regular spaces In a completely regular space, the cozero sets form a base for the topology. (Čech 66, 28B.5.)

(d) Normal spaces (i) **Urysohn's Lemma** If X is normal and E, F are disjoint closed subsets of X, then there is a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ for $x \in E$ and $f(x) = 1$ for $x \in F$. (Engelking 89, 1.5.11; Kuratowski 66, §14.IV.)

(ii) A regular normal space is completely regular.

(iii) A normal T_1 space is Hausdorff (Gaal 64, p. 86) and completely regular (Császár 78, 4.2.5; Gaal 64, p. 110).

(iv) If X is normal and E, F are disjoint closed sets in X there is a zero set including E and disjoint from F. (Take a continuous function f which is zero on E and 1 on F, and set $Z = \{x : f(x) = 0\}$.)

(v) In a normal space a closed G_δ set is a zero set. (Engelking 89, 1.5.12.)

(vi) If X is a normal space and $\langle G_i\rangle_{i\in I}$ is a point-finite cover of X by open sets, there is a family $\langle H_i\rangle_{i\in I}$ of open sets, still covering X, such that $\overline{H}_i \subseteq G_i$ for every i. (Engelking 89, 1.5.18; Čech 66, 29C.1; Gaal 64, p. 89.)

(vii) If X is a normal space and $\langle G_i\rangle_{i\in I}$ is a point-finite cover of X by open sets, there is a family $\langle H'_i\rangle_{i\in I}$ of cozero sets, still covering X, such that $H'_i \subseteq G_i$ for every i. (Take $\langle H_i\rangle_{i\in I}$ from (vi), and apply (iv) to the disjoint closed sets $X \setminus G_i$, $\overline{H}_i$ to find a suitable cozero set H'_i for each i.)

(viii) If X is a normal space and $\langle G_i\rangle_{i\in I}$ is a locally finite cover of X by open sets, there is a family $\langle g_i\rangle_{i\in I}$ of continuous functions from X to $[0,1]$ such that $g_i \le \chi G_i$ for every $i \in I$ and $\sum_{i\in I} g_i(x) = 1$ for every $x \in X$. (Engelking 89, proof of 5.1.9.)

(ix) **Tietze's theorem** Let X be a normal space, F a closed subset of X and $f : F \to \mathbb{R}$ a continuous function. Then there is a continuous function $g : X \to \mathbb{R}$ extending f. (Engelking 89, 2.1.8; Kuratowski 66, §14.IV; Gaal 64, p. 203.) It follows that if $F \subseteq X$ is closed and $f : F \to [0,1]^I$ is a continuous function from F to any power of the unit interval, there is a continuous function from X to $[0,1]^I$ extending f. (Extend each of the functionals $x \mapsto f(x)(i)$ for $i \in I$.)

(e) Paracompact spaces A Hausdorff paracompact space is regular. (Engelking 89, 5.1.5.) A regular paracompact space is normal. (Engelking 89, 5.1.5; Gaal 64, p. 160.)

(f) Countably paracompact spaces A normal space X is countably paracompact iff whenever $\langle F_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of closed subsets of X with empty intersection, there is a sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ of open sets, also with empty intersection, such that $F_n \subseteq G_n$ for every $n \in \mathbb{N}$. (Engelking 89, 5.2.2; Császár 78, 8.3.56(f).)

(g) Metacompact spaces (i) A paracompact space is metacompact.

(ii) A closed subspace of a metacompact space is metacompact.

(iii) A normal metacompact space is countably paracompact. (Engelking 89, 5.2.6; Császár 78, 8.3.56(c).)

(h) Separating compact sets (i) If X is a Hausdorff space and K and L are disjoint compact subsets of X, there are disjoint open sets G, $H \subseteq X$ such that $K \subseteq G$ and $L \subseteq H$. (Császár 78, 5.3.18.) If T is an algebra of subsets of X including a subbase for the topology of X, there is an open $V \in \mathrm{T}$ such that $K \subseteq V \subseteq X \setminus L$. **P** By 4A2B(a-i), T includes a base for the topology of X. So $\mathcal{E} = \{U : U \in \mathrm{T}$ is open, $U \subseteq G\}$ has union G and there must be a finite $\mathcal{E}_0 \subseteq \mathcal{E}$ covering K; set $V = \bigcup \mathcal{E}_0$. **Q**

(ii) If X is a regular space, $F \subseteq X$ is closed, and $K \subseteq X \setminus F$ is compact, there are disjoint open sets G, $H \subseteq X$ such that $K \subseteq G$ and $F \subseteq H$. (Engelking 89, 3.1.6.)

(iii) If X is a completely regular space, $G \subseteq X$ is open and $K \subseteq G$ is compact, there is a continuous function $f : X \to [0,1]$ such that $f(x) = 1$ for $x \in K$ and $f(x) = 0$ for $x \in X \setminus G$. **P** For each $x \in K$ there is a continuous function $f_x : X \to [0,1]$ such that $f_x(x) = 1$ and $f_x(y) = 0$ for $y \in X \setminus G$. Set $H_x = \{y : f_x(y) > \frac{1}{2}\}$. Then $\bigcup_{x\in K} H_x \supseteq K$, so there is a finite set $I \subseteq K$ such that $K \subseteq \bigcup_{x\in I} H_x$. Set $f(y) = \min(1, 2\sum_{x\in I} f_x(y))$ for $y \in X$. **Q**

(iv) If X is a completely regular Hausdorff space and K and L are disjoint compact subsets of X, there are disjoint cozero sets G, $H \subseteq X$ such that $K \subseteq G$ and $L \subseteq H$. **P** By (i), there are disjoint open sets G', H' such that $K \subseteq G'$ and $L \subseteq H'$. By (iii), there is a continuous function $f : X \to [0,1]$ such that $f(x) = 1$

for $x \in K$ and $f(x) = 0$ for $x \in X \setminus G'$; set $G = \{x : f(x) \neq 0\}$, so that G is a cozero set and $K \subseteq G \subseteq G'$. Similarly there is a cozero set H including L and included in H'. **Q**

(v) If X is a completely regular space and $K \subseteq X$ is a compact G_δ set, then K is a zero set. **P** Let $\langle G_n \rangle_{n\in\mathbb{N}}$ be a sequence of open sets with intersection K. For each $n \in \mathbb{N}$ there is a continuous function $f_n : X \to [0,1]$ such that $f_n(x) = 1$ for $x \in K$ and $f_n(x) = 0$ for $x \in X \setminus G_n$, by (iii). Now $K = \bigcap_{n\in\mathbb{N}}\{x : 1 - f_n(x) = 0\}$ is a zero set, by 4A2C(b-iii). **Q**

(vi) If $\langle X_n \rangle_{n\in\mathbb{N}}$ is a sequence of topological spaces with product X, $K \subseteq X$ is compact, $F \subseteq X$ is closed and $K \cap F = \emptyset$, there is some $n \in \mathbb{N}$ such that $\underline{x{\restriction}n \neq y{\restriction}n}$ for any $x \in F$ and $y \in K$.

P For $n \in \mathbb{N}$ and $x \in X$ set $\pi_n(x) = x{\restriction}n$; set $F_n = \pi_n^{-1}[\pi_n[F]]$. Since $\langle \pi_n^{-1}[\pi_n[F]] \rangle_{n\in\mathbb{N}}$ is non-increasing, so is $\langle F_n \rangle_{n\in\mathbb{N}}$. If $x \in K$, there is an open set $G \subseteq X$, determined by coordinates in a finite set, such that $x \in G \subseteq X \setminus F$; in this case there is an $n \in \mathbb{N}$ such that $\pi_n^{-1}[\pi_n[G]] = G$ is disjoint from F, so that $\pi_n[G] \cap \pi_n[F] = \emptyset$, G does not meet $\pi_n^{-1}[\pi_n[F]]$ and $x \notin F_n$. As x is arbitrary, $\langle K \cap F_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of relatively closed subsets of K with empty intersection; as K is compact, there is an n such that $K \cap F_n = \emptyset$, so that $K \cap \pi_n^{-1}[\pi_n[F]] = \emptyset$ and $x{\restriction}n \neq y{\restriction}n$ whenever $x \in F$ and $y \in K$. **Q**

(vii) If X is a compact Hausdorff space, $f : X \to \mathbb{R}$ is continuous, and $\mathcal{U}$ is a subbase for $\mathfrak{T}$, then there is a countable set $\mathcal{U}_0 \subseteq \mathcal{U}$ such that $f(x) = f(y)$ whenever $\{U : x \in U \in \mathcal{U}_0\} = \{U : y \in U \in \mathcal{U}_0\}$. (Apply (i) to sets of the form $K = \{x : f(x) \leq \alpha\}$, $L = \{x : f(x) \geq \beta\}$.)

(i) Perfectly normal spaces A topological space X is perfectly normal iff every closed set is a zero set. (Engelking 89, 1.4.9.)

Consequently, every open set in a perfectly normal space is a cozero set (and, of course, an F_σ set).

(j) Covers of compact sets Let X be a Hausdorff space, K a compact subset of X, and $\langle G_i \rangle_{i\in I}$ a family of open subsets of X covering K. Then there are a finite set $J \subseteq I$ and a family $\langle K_i \rangle_{i\in J}$ of compact sets such that $K = \bigcup_{i\in J} K_i$ and $K_i \subseteq G_i$ for every $i \in J$. **P** (i) Suppose first that $I = \{i, j\}$ has just two members. Then $K \setminus G_j$ and $K \setminus G_i$ are disjoint compact sets. By (h-i), there are disjoint open sets H_i, H_j such that $K \setminus G_j \subseteq H_i$ and $K \setminus G_i \subseteq H_j$; setting $K_i = K \setminus H_j$ and $K_j = K \setminus H_i$ we have a suitable pair K_i, K_j. (ii) Inducing on $\#(I)$ we get the result for finite I. (iii) In general, there is certainly a finite $J \subseteq I$ such that $K \subseteq \bigcup_{i\in J} G_i$, and we can apply the result to $\langle G_i \rangle_{i\in J}$. **Q**

4A2G Compact and locally compact spaces (a) In any topological space, the union of two compact subsets is compact.

(b) A compact Hausdorff space is normal. (Engelking 89, 3.1.9; Császár 78, 5.3.23; Gaal 64, p. 139.)

(c)(i) If X is a compact Hausdorff space, $Y \subseteq X$ is a zero set and $Z \subseteq Y$ is a zero set in Y, then Z is a zero set in X. (By 4A2C(b-vi) and 4A2C(a-iv), Z is a G_δ set in X; now use 4A2F(d-v).)

(ii) Let X and Y be compact Hausdorff spaces, $f : X \to Y$ a continuous open map and $Z \subseteq X$ a zero set in X. Then $f[Z]$ is a zero set in Y. **P** Let $g : X \to \mathbb{R}$ be a continuous function such that $Z = g^{-1}[\{0\}]$. Set $G_n = \{x : x \in X, |g(x)| < 2^{-n}\}$ for each $n \in \mathbb{N}$. If $y \in \bigcap_{n\in\mathbb{N}} f[G_n]$, then $f^{-1}[\{y\}]$ is a compact set meeting all the closed sets $\overline{G}_n$, so meets their intersection, which is Z. Thus $f[Z] = \bigcap_{n\in\mathbb{N}} f[G_n]$ is a G_δ set. By 4A2F(d-v), it is a zero set. **Q**

(d) If X is a Hausdorff space, $\mathcal{V}$ is a downwards-directed family of compact neighbourhoods of a point x of X and $\bigcap \mathcal{V} = \{x\}$, then $\mathcal{V}$ is a base of neighbourhoods of x. **P** Let G be any open set containing x. Fix any $V_0 \in \mathcal{V}$. Note that because X is Hausdorff, every member of $\mathcal{V}$ is closed (3A3Dc). So $\{V_0 \cap V \setminus G : V \in \mathcal{V}\}$ is a family of (relatively) closed subsets of V_0 with empty intersection, cannot have the finite intersection property (3A3Da), and there is a $V \in \mathcal{V}$ such that $V_0 \cap V \setminus G = \emptyset$. Now there is a $V' \in \mathcal{V}$ such that $V' \subseteq V_0 \cap V$ and $V' \subseteq G$. As G is arbitrary, $\mathcal{V}$ is a base of neighbourhoods of x. **Q**

(e) Let $(X, \mathfrak{T})$ be a locally compact Hausdorff space.

(i) If $K \subseteq X$ is a compact set and $G \supseteq K$ is open, then there is a continuous $f : X \to [0,1]$ with compact support such that $\chi K \leq f \leq \chi G$. (Let $\mathcal{V}$ be the family of relatively compact open subsets of X. Then $\mathcal{V}$ is upwards-directed and covers X, so there is a $V \in \mathcal{V}$ including K. By 3A3Bb, $\mathfrak{T}$ is completely regular; now 4A2F(h-iii) tells us that there is a continuous $f : X \to [0,1]$ such that $\chi K \le f \le \chi(G \cap V)$, so that f has compact support.)

(ii) $\mathfrak{T}$ is the coarsest topology on X such that every $\mathfrak{T}$-continuous real-valued function with compact support is continuous. **P** Let Φ be the set of continuous functions of compact support for $\mathfrak{T}$. If $\mathfrak{S}$ is a topology on X such that every member of Φ is continuous, and $x \in G \in \mathfrak{T}$, then there is an $f \in \Phi$ such that $f(x) = 1$ and $f(y) = 0$ for $y \in X \setminus G$, by (i). Now f is $\mathfrak{S}$-continuous, by hypothesis, so $H = \{y : f(y) > \frac{1}{2}\}$ belongs to $\mathfrak{S}$ and $x \in H \subseteq G$. As x is arbitrary, $G = \operatorname{int}_{\mathfrak{S}} G$ belongs to $\mathfrak{S}$; as G is arbitrary, $\mathfrak{T} \subseteq \mathfrak{S}$. **Q**

(f)(i) A topological space X is countably compact iff every sequence in X has a cluster point in X, that is, X is relatively countably compact in itself. (Engelking 89, 3.10.3; Császár 78, 5.3.31(e); Gaal 64, p. 129.)

(ii) If X is a countably compact topological space and $\langle F_n\rangle_{n\in\mathbb{N}}$ is a sequence of closed sets such that $\bigcap_{i\le n} F_i \ne \emptyset$ for every $n \in \mathbb{N}$, then $\bigcap_{n\in\mathbb{N}} F_n \ne \emptyset$. (Engelking 89, 3.10.3; Császár 78, 5.3.31(c).)

(iii) In any topological space, a relatively compact set is relatively countably compact (2A3Ob).

(iv) Let X and Y be topological spaces and $f : X \to Y$ a continuous function. If $A \subseteq X$ is relatively countably compact in X, then $f[A]$ is relatively countably compact in Y. **P** Let $\langle y_n\rangle_{n\in\mathbb{N}}$ be a sequence in $f[A]$. Then there is a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in A such that $f(x_n) = y_n$ for every $n \in \mathbb{N}$. Because A is relatively countably compact, $\langle x_n\rangle_{n\in\mathbb{N}}$ has a cluster point $x \in X$. If $n_0 \in \mathbb{N}$ and H is an open set containing $f(x)$, there is an $n \ge n_0$ such that $x_n \in f^{-1}[H]$, so that $y_n \in H$. Thus $f(x)$ is a cluster point of $\langle y_n\rangle_{n\in\mathbb{N}}$; as $\langle y_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $f[A]$ is relatively countably compact. **Q**

(v) A relatively countably compact set in $\mathbb{R}$ must be bounded. (If $A \subseteq \mathbb{R}$ is unbounded there is a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in A such that $|x_n| \ge n$ for every n.) So if X is a topological space, $A \subseteq X$ is relatively countably compact and $f : X \to \mathbb{R}$ is continuous, then $f[A]$ is bounded.

(vi) If X and Y are topological spaces and $f : X \to Y$ is continuous, then $f[A]$ is countably compact whenever $A \subseteq X$ is countably compact. (Engelking 89, 3.10.5.)

(g)(i) Let X and Y be topological spaces and $\phi : X \times Y \to \mathbb{R}$ a continuous function. Define $\theta : X \to C(Y)$ by setting $\theta(x)(y) = \phi(x, y)$ for $x \in X$, $y \in Y$. Then θ is continuous if we give $C(Y)$ the topology of uniform convergence on compact subsets of Y. (As noted in Engelking 89, pp. 157-158, the topology of uniform convergence on compact sets is the 'compact-open' topology of $C(Y)$, as defined in 441Yi, so the result here is covered by Engelking 89, 3.4.1.)

(ii) In particular, if Y is compact then θ is continuous if we give $C(Y)$ its usual norm topology.

(iii) Let X be a locally compact topological space, and give $C(X)$ the topology of uniform convergence on compact subsets of X. Then the function $(f, x) \mapsto f(x) : C(X) \times X \to \mathbb{R}$ is continuous. **P** Take $g \in C(X)$, $y \in X$ and $\epsilon > 0$. Let $K \subseteq X$ be a compact set such that $y \in \operatorname{int} K$. Then $V = \{f : f \in C(X), |f(x) - g(x)| \le \frac{1}{2}\epsilon$ for every $x \in K\}$ is a neighbourhood of g, and $U = \{x : x \in K, |g(x) - g(y)| \le \frac{1}{2}\epsilon\}$ is a neighbourhood of y. If $f \in V$ and $x \in U$, then

$$|f(x) - g(y)| \le |f(x) - g(x)| + |g(x) - g(y)| \le \epsilon. \quad \mathbf{Q}$$

(h)(i) Suppose that X is a compact space such that there are no non-trivial convergent sequences in X, that is, no convergent sequences which are not eventually constant. If $\langle F_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of infinite closed subsets of X, then $F = \bigcap_{n\in\mathbb{N}} F_n$ is infinite. **P** Because X is compact, F cannot be empty (3A3Da). Choose a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ such that $x_n \in F_n \setminus \{x_i : i < n\}$ for each $n \in \mathbb{N}$. If $G \supseteq F$ is an open set, then $\bigcap_{n\in\mathbb{N}} F_n \setminus G = \emptyset$, so there must be some $n \in \mathbb{N}$ such that $F_n \subseteq G$, and $x_i \in G$ for $i \ge n$. **?** If $F = \{y_0, \dots, y_k\}$, let $l \le k$ be the first point such that whenever $G \supseteq \{y_0, \dots, y_l\}$ is open, then $\{i : x_i \notin G\}$ is finite. Then there is an open set $G' \supseteq \{y_j : j < l\}$ such that $I = \{i : x_i \notin G'\}$ is infinite. But if H is any open set containing y_l, then $\{i : x_i \notin G' \cup H\}$ is finite, so $\{i : i \in I, x_i \notin H\}$ is finite. Thus if we re-enumerate $\langle x_i\rangle_{i\in I}$ as $\langle x'_n\rangle_{n\in\mathbb{N}}$, $\langle x'_n\rangle_{n\in\mathbb{N}}$ converges to y_l and is a non-trivial convergent sequence. **X** Thus F is infinite, as claimed. **Q**

(ii) If X is an infinite scattered compact Hausdorff space it has a non-trivial convergent sequence. **P** Let $\langle x_n\rangle_{n\in\mathbb{N}}$ be any sequence of distinct points in X. Set $F_n = \overline{\{x_i : i \ge n\}}$ for each n, so that $F = \bigcap_{n\in\mathbb{N}} F_n$ is a non-empty set. Because X is scattered, F has an isolated point z say; let G be an open set such that $F \cap G = \{z\}$, and H an open set such that $z \in H \subseteq \overline{H} \subseteq G$ (3A3Bb). In this case, $I = \{i : x_i \in H\}$ must be infinite; re-enumerate $\langle x_i\rangle_{i\in I}$ as $\langle x'_n\rangle_{n\in\mathbb{N}}$. **?** If $\langle x'_n\rangle_{n\in\mathbb{N}}$ does not converge to z, there is an open set H' containing z such that $\{n : x'_n \notin H'\}$ is infinite, that is, $\{i : i \in I, x_i \notin H'\}$ is infinite. In this case, $F_n \cap \overline{H} \setminus H'$ is non-empty for every $n \in \mathbb{N}$, but $F \cap \overline{H} \setminus H' = \emptyset$, which is impossible. **X** Thus $\langle x'_n\rangle_{n\in\mathbb{N}}$ is a non-trivial convergent sequence in X. **Q**

(iii) If X is an extremally disconnected Hausdorff space (definition: 3A3Af), it has no non-trivial convergent sequence. **P?** Suppose, if possible, that there is a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ converging to $x \in X$ such that $\{n : x_n \ne x\}$ is infinite. Choose $\langle n_i\rangle_{i\in\mathbb{N}}$ and $\langle G_i\rangle_{i\in\mathbb{N}}$ inductively, as follows. Given that $x \notin \overline{G}_j$ for $j < i$, there is an n_i such that $x_{n_i} \ne x$ and $x_{n_i} \notin \overline{G}_j$ for every $j < i$; now let G_i be an open set such that $x_{n_i} \in G_i$ and $x \notin \overline{G}_i$, and continue.

Since all the n_i must be distinct, $\langle x_{n_i}\rangle_{i\in\mathbb{N}} \to x$. But consider

$$G = \bigcup_{i\in\mathbb{N}} G_{2i} \setminus \bigcup_{j<2i} \overline{G}_j, \quad H = \bigcup_{i\in\mathbb{N}} G_{2i+1} \setminus \bigcup_{j\le 2i} \overline{G}_j.$$

Then G and H are disjoint open sets and $x_{n_{2i}} \in G$, $x_{n_{2i+1}} \in H$ for every i. So $x \in \overline{G} \cap \overline{H}$. But $\overline{G}$ is open (because X is extremally disconnected), and is disjoint from H, and now $\overline{H}$ is disjoint from $\overline{G}$; so they cannot both contain x. **XQ**

(i)(i) If X and Y are compact Hausdorff spaces and $f : X \to Y$ is a continuous surjection then there is a closed set $K \subseteq X$ such that $f[K] = Y$ and $f{\restriction}K$ is irreducible. **P** Let $\mathcal{E}$ be the family of closed sets $F \subseteq X$ such that $f[F] = Y$. If $\mathcal{F} \subseteq \mathcal{E}$ is non-empty and downwards-directed, then for any $y \in Y$ the family $\{F \cap f^{-1}[\{y\}] : F \in \mathcal{F}\}$ is a downwards-directed family of non-empty closed sets, so (because X is compact) has non-empty intersection; this shows that $\bigcap\mathcal{F} \in \mathcal{E}$. By Zorn's Lemma, $\mathcal{E}$ has a minimal element K say. Now $f[K] = Y$ but $f[F] \ne Y$ for any closed proper subset of K, so $f{\restriction}K$ is irreducible. **Q**

(ii) If X and Y are compact Hausdorff spaces and $f : X \to Y$ is an irreducible continuous surjection, then (α) if $\mathcal{U}$ is a π-base for the topology of Y then $\{f^{-1}[U] : U \in \mathcal{U}\}$ is a π-base for the topology of X (β) if Y has a countable π-base so does X (γ) if x is an isolated point in X then $f(x)$ is an isolated point in Y (δ) if Y has no isolated points, nor does X. **P** (α) If $G \subseteq X$ is a non-empty open set then $f[X \setminus G] \ne Y$. As $f[X \setminus G]$ is closed, there is a non-empty $U \in \mathcal{U}$ disjoint from $f[X \setminus G]$. Now $f^{-1}[U]$ is a non-empty subset of G. (β) Follows at once from (α). (γ) By (α), with $\mathcal{U}$ the family of all open subsets of Y, there is a non-empty open set $U \subseteq Y$ such that $f^{-1}[U] \subseteq \{x\}$, that is, $U = \{f(x)\}$. (δ) Follows at once from (γ). **Q**

(j)(i) Let X be a non-empty compact Hausdorff space without isolated points. Then there are a closed set $F \subseteq X$ and a continuous surjection $f : F \to \{0,1\}^{\mathbb{N}}$. **P** For $\sigma \in \bigcup_{n\in\mathbb{N}}\{0,1\}^n$ choose closed sets $V_\sigma \subseteq X$ inductively, as follows. $V_\emptyset = X$. Given that V_σ is a closed set with non-empty interior, there are distinct points x, $y \in \operatorname{int} V_\sigma$ (because X has no isolated points); let G, H be disjoint open subsets of X such that $x \in G$ and $y \in H$; and let $V_{\sigma^\frown<0>}$ and $V_{\sigma^\frown<1>}$ be closed sets such that

$$x \in \operatorname{int} V_{\sigma^\frown<0>} \subseteq G \cap \operatorname{int} V_\sigma, \quad y \in \operatorname{int} V_{\sigma^\frown<1>} \subseteq H \cap \operatorname{int} V_\sigma.$$

(This is possible because X is regular.) The construction ensures that $V_\tau \subseteq V_\sigma$ whenever $\tau \in \{0,1\}^n$ extends $\sigma \in \{0,1\}^m$, and that $V_\tau \cap V_\sigma = \emptyset$ whenever τ, $\sigma \in \{0,1\}^n$ are different. Set $F = \bigcap_{n\in\mathbb{N}} \bigcup_{\sigma\in\{0,1\}^n} V_\sigma$; then F is a closed subset of X and we have a continuous function $f : F \to \{0,1\}^{\mathbb{N}}$ defined by saying that $f(x)(i) = \sigma(i)$ whenever $n \in \mathbb{N}$, $\sigma \in \{0,1\}^n$, $i < n$ and $x \in V_\sigma$. Finally, f is surjective, because if $z \in \{0,1\}^{\mathbb{N}}$ then $\langle V_{z\restriction n}\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of closed sets in the compact space X, so has non-empty intersection V say, and $f(x) = z$ for any $x \in V$. **Q**

(ii) If X is a non-empty compact Hausdorff space without isolated points, then $\#(X) \ge \mathfrak{c}$. (Use (i).)

(iii) If X is a compact Hausdorff space which is not scattered, it has an infinite closed subset with a countable π-base and no isolated points. **P** Because X is not scattered, it has a non-empty subset A without isolated points. Then $\overline{A}$ is compact and has no isolated points; by (i), there are a closed set $F_0 \subseteq \overline{A}$ and a continuous surjection $f : F_0 \to \{0,1\}^{\mathbb{N}}$. By (i-i) above, there is a closed $F \subseteq F_0$ such that $f[F] = \{0,1\}^{\mathbb{N}}$ and $f{\restriction}F$ is irreducible. Of course F is infinite; by (i-ii), it has a countable π-base and no isolated points. **Q**

(iv) Let X be a compact Hausdorff space. Then there is a continuous surjection from X onto $[0,1]$ iff X is not scattered. **P** (α) Suppose that $f : X \to [0,1]$ is a continuous surjection. By (i-i) again, there is a closed set $F \subseteq X$ such that $f[F] = [0,1]$ and $f{\restriction}F$ is irreducible; by (i-ii) F has no isolated points. So X is not scattered. (β) If X is not scattered, let $A \subseteq X$ be a non-empty set with no isolated points. Then $\overline{A}$ is a non-empty compact subset of X with no isolated points, so there is a continuous surjection $g : \overline{A} \to \{0,1\}^{\mathbb{N}}$ ((i) of this subparagraph). Now there is a continuous surjection $h : \{0,1\}^{\mathbb{N}} \to [0,1]$ (e.g., set $h(y)=\sum_{n=0}^{\infty}2^{-n-1}y(n)$ for $y\in\{0,1\}^{\mathbb{N}}$), so we have a continuous surjection $hg:\overline{A}\to[0,1]$. By Tietze's theorem (4A2F(d-ix)), there is a continuous function $f_0:X\to\mathbb{R}$ extending hg; setting $f(x)=\operatorname{med}(0,f_0(x),1)$ for $x\in X$, we have a continuous surjection $f:X\to[0,1]$. **Q**

(v) A Hausdorff continuous image of a scattered compact Hausdorff space is scattered. (Immediate from (iv).)

(vi) If X is an uncountable first-countable compact Hausdorff space, it is not scattered. **P** Let $\mathcal{G}$ be the family of countable open subsets of X, and G^* its union. No finite subset of $\mathcal{G}$ can cover X, so $X\setminus G^*$ is non-empty. **?** If x is an isolated point of $X\setminus G^*$, then $\{x\}\cup G^*$ is a neighbourhood of x; let $\langle U_n\rangle_{n\in\mathbb{N}}$ run over a base of open neighbourhoods of x with $\overline{U}_0\subseteq\{x\}\cup G^*$. For each $n\in\mathbb{N}$, $F_n=\overline{U}_0\setminus U_n$ is a compact set included in G^*, so is covered by finitely many members of $\mathcal{G}$, and is countable. But this means that $U_0=\{x\}\cup\bigcup_{n\in\mathbb{N}}U_0\setminus U_n$ is countable, and $x\in G^*$. **X** Thus $X\setminus G^*$ is a non-empty set with no isolated points, and X is not scattered. **Q**

It follows that there is a continuous surjection from X onto $[0,1]$, by (iv).

(k) A locally compact Hausdorff space is Čech-complete. (Engelking 89, p. 196.)

(l) If X is a topological space, $f:X\to\mathbb{R}$ is lower semi-continuous, and $K\subseteq X$ is compact and not empty, then there is an $x_0\in K$ such that $f(x_0)=\inf_{x\in K}f(x)$. (Gaal 64, p. 209 Theorem 3.) Similarly, if $g:X\to\mathbb{R}$ is upper semi-continuous, there is an $x_1\in K$ such that $g(x_1)=\sup_{x\in K}g(x)$.

(m) If X is a Hausdorff space, Y is a compact space and $F\subseteq X\times Y$ is closed, then its projection $\{x:(x,y)\in F\}$ is a closed subset of X. (Engelking 89, 3.1.16.)

(n) If X is a locally compact topological space, Y is a topological space and $f:X\to Y$ is a continuous open surjection, then Y is locally compact. (Engelking 89, 3.3.15.)

4A2H Lindelöf spaces (a) If X is a topological space, then a subset Y of X is Lindelöf (in its subspace topology) iff for every family $\mathcal{G}$ of open subsets of X covering Y there is a countable subfamily of $\mathcal{G}$ still covering Y.

(b)(i) A regular Lindelöf space X is normal (therefore completely regular) and paracompact. (Engelking 89, 3.8.11 & 5.1.2.)

(ii) If X is a Lindelöf space and $\mathcal{A}$ is a locally finite family of subsets of X then $\mathcal{A}$ is countable. **P** The family $\mathcal{G}$ of open sets meeting only finitely many members of $\mathcal{A}$ is an open cover of X. If $\mathcal{G}_0\subseteq\mathcal{G}$ is a countable cover of X then $\{A:A\in\mathcal{A},\ A$ meets some member of $\mathcal{G}_0\}=\mathcal{A}\setminus\{\emptyset\}$ is countable. **Q**

(c)(i) A topological space X is hereditarily Lindelöf iff for any family $\mathcal{G}$ of open subsets of X there is a countable family $\mathcal{G}_0\subseteq\mathcal{G}$ such that $\bigcup\mathcal{G}_0=\bigcup\mathcal{G}$. **P** ($\alpha$) If X is hereditarily Lindelöf and $\mathcal{G}$ is a family of open subsets of X, then $\mathcal{G}$ is an open cover of $\bigcup\mathcal{G}$, so has a countable subcover. (β) If X is not hereditarily Lindelöf, let $Y\subseteq X$ be a non-Lindelöf subspace, and $\mathcal{H}$ a cover of Y by relatively open sets which has no countable subcover; setting $\mathcal{G}=\{G:G\subseteq X$ is open, $G\cap Y\in\mathcal{H}\}$, there can be no countable $\mathcal{G}_0\subseteq\mathcal{G}$ with union $\bigcup\mathcal{G}$. **Q**

(ii) Let X be a regular hereditarily Lindelöf space. Then X is perfectly normal. **P** Let $F\subseteq X$ be closed. Let $\mathcal{G}$ be the family of open sets $G\subseteq X$ such that $\overline{G}\cap F=\emptyset$; because X is regular, $\bigcup\mathcal{G}=X\setminus F$; because X is hereditarily Lindelöf, there is a sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ in G such that $X\setminus F=\bigcup_{n\in\mathbb{N}}G_n$. This means that $F=\bigcap_{n\in\mathbb{N}}X\setminus\overline{G_n}$ is a G_δ set. But X is normal ((b) above), so is perfectly normal. **Q**

(d) Any σ-compact topological space is Lindelöf. (Engelking 89, 3.8.5.)

4A2I Stone-Čech compactifications (a) Let X be a completely regular Hausdorff space. Then there is a compact Hausdorff space βX, the **Stone-Čech compactification** of X, in which X can be embedded as a dense subspace. If Y is another compact Hausdorff space, then every continuous function from X to Y has a unique continuous extension to a continuous function from βX to Y. (Engelking 89, 3.6.1; Császár 78, 6.4d; Čech 66, 41D.5.)

(b) Let I be any set, and write βI for its Stone-Čech compactification when I is given its discrete topology. Let Z be the Stone space of the Boolean algebra $\mathcal{P}I$.

(i) There is a canonical homeomorphism $\phi : \beta I \to Z$ defined by saying that $\phi(i)(a) = \chi a(i)$ for every $i \in I$ and $a \subseteq I$. **P** Recall that Z is the set of ring homomorphisms from $\mathcal{P}I$ onto $\mathbb{Z}_2$ (311E). If $i \in I$, let $\hat{\imath}$ be the corresponding member of Z defined by setting $\hat{\imath}(a) = \chi a(i)$ for every $a \subseteq I$. Then Z is compact and Hausdorff (311I), and $i \mapsto \hat{\imath} : I \to Z$ is continuous, so has a unique extension to a continuous function $\phi : \beta I \to Z$.

If $G \subseteq Z$ is open and not empty, it includes a set of the form $\widehat{a} = \{\theta : \theta \in Z, \theta(a) = 1\}$ where $a \subseteq I$ is not empty; if i is any member of a, $\hat{\imath} \in \widehat{a} \subseteq G$ so $G \cap \phi[\beta I] \neq \emptyset$. This shows that $\phi[\beta I]$ is dense in Z; as βI is compact, $\phi[\beta I]$ is compact, therefore closed, and is equal to Z. Thus ϕ is surjective.

If t, u are distinct points of βI, there is an open subset H of βI such that $t \in H$ and $u \notin \overline{H}$. Set $a = H \cap I$. Then $t \in \overline{a}$, the closure of a regarded as a subset of βI, so $\phi(t) \in \overline{\phi[a]}$ (3A3Cd). But $\phi[a] = \{\hat{\imath} : i \in a\} \subseteq \widehat{a}$, which is open-and-closed, so $\phi(t) \in \widehat{a}$. Similarly, setting $b = I \setminus \overline{H}$, $\phi(u) \in \widehat{b}$; since $\widehat{a} \cap \widehat{b} = \widehat{a \cap b}$ is empty, $\phi(t) \neq \phi(u)$. This shows that ϕ is injective, therefore a homeomorphism between βI and Z (3A3Dd). **Q**

Note that if $z : \mathcal{P}I \to \mathbb{Z}_2$ is a Boolean homomorphism, then $\{J : z(J) = 1\}$ is an ultrafilter on I; and conversely, if $\mathcal{F}$ is an ultrafilter on I, we have a Boolean homomorphism $z : \mathcal{P}I \to \mathbb{Z}_2$ such that $\mathcal{F} = z^{-1}[\{1\}]$. So we can identify βI with the set of ultrafilters on I. Under this identification, the canonical embedding of I in βI corresponds to matching each member of I with the corresponding principal ultrafilter on I.

(ii) $C(\beta I)$ is isomorphic, as Banach lattice, to $\ell^\infty(I)$. **P** By 363Ha, we can identify $\ell^\infty(I)$, as Banach lattice, with $L^\infty(\mathcal{P}I) = C(Z)$. But (i) tells us that we have a canonical identification between $C(Z)$ and $C(\beta I)$. **Q**

(iii) We have a one-to-one correspondence between filters $\mathcal{F}$ on I and non-empty closed sets $F \subseteq \beta I$, got by matching $\mathcal{F}$ with $\bigcap\{\widehat{a} : a \in \mathcal{F}\}$, or F with $\{a : a \subseteq I, F \subseteq \widehat{a}\}$, where $\widehat{a} \subseteq \beta I$ is the open-and-closed set corresponding to $a \subseteq I$. **P** The identification of βI with Z means that we can regard the map $a \mapsto \widehat{a}$ as a Boolean isomorphism between $\mathcal{P}I$ and the algebra of open-and-closed subsets of βI (311I). For any filter $\mathcal{F}$ on I, set $H(\mathcal{F}) = \bigcap\{\widehat{a} : a \in \mathcal{F}\}$; because $\{\widehat{a} : a \in \mathcal{F}\}$ is a downwards-directed family of non-empty closed sets in the compact Hausdorff space βI, $H(\mathcal{F})$ is a non-empty closed set. If $F \subseteq \beta I$ is a non-empty closed set, then it is elementary to check that $\mathcal{H}(F) = \{a : F \subseteq \widehat{a}\}$ is a filter on I, and evidently $H(\mathcal{H}(F)) \supseteq F$. But if $t \in \beta I \setminus F$, then (because $\{\widehat{a} : a \subseteq I\}$ is a base for the topology of βI, see 311I again) there is an $a \subseteq I$ such that $t \in \widehat{a}$ and $F \cap \widehat{a} = \emptyset$, that is, $F \subseteq \widehat{I \setminus a}$; so $I \setminus a \in \mathcal{H}(F)$ and $H(\mathcal{H}(F)) \subseteq \widehat{I \setminus a}$ and $t \notin H(\mathcal{H}(F))$. Thus $H(\mathcal{H}(F)) = F$ for every non-empty closed set $F \subseteq \beta I$.

If $\mathcal{F}_1$ and $\mathcal{F}_2$ are filters on I and $a \in \mathcal{F}_1 \setminus \mathcal{F}_2$, then $\{\widehat{b \setminus a} : b \in \mathcal{F}_2\}$ is a downwards-directed family of non-empty closed sets in βI, so has non-empty intersection; if $t \in \widehat{b \setminus a} = \widehat{b} \setminus \widehat{a}$ for every $b \in \mathcal{F}_2$, then $t \in H(\mathcal{F}_2) \setminus H(\mathcal{F}_1)$. This shows that $\mathcal{F} \mapsto H(\mathcal{F})$ is injective. It follows that $\mathcal{F} \mapsto H(\mathcal{F})$, $F \mapsto \mathcal{H}(F)$ are the two halves of a bijection, as claimed. **Q**

(iv) βI is extremally disconnected. (Because $\mathcal{P}I$ is Dedekind complete, Z is extremally disconnected (314S).)

(v) There are no non-trivial convergent sequences in βI. (4A2G(h-iii). Compare Engelking 89, 3.6.15.)

4A2J Uniform spaces (See §3A4.) Let $(X, \mathcal{W})$ be a uniform space; give X the induced topology $\mathfrak{T}$ (3A4Ab).

(a) $\mathcal{W}$ is generated by a family of pseudometrics. (Engelking 89, 8.1.10; Bourbaki 66, IX.1.4; Császár 78, 4.2.32.) More precisely: if $\langle W_n \rangle_{n \in \mathbb{N}}$ is any sequence in $\mathcal{W}$, there is a pseudometric ρ on X such that (α) $\{(x, y) : \rho(x, y) \leq \epsilon\} \in \mathcal{W}$ for every $\epsilon > 0$ (β) whenever $n \in \mathbb{N}$ and $\rho(x, y) < 2^{-n}$ then $(x, y) \in W_n$ (Engelking 89, 8.1.10).

It follows that $\mathfrak{T}$ is completely regular, therefore regular (3A3Be). $\mathfrak{T}$ is defined by the bounded uniformly continuous real-valued functions on X, in the sense that it is the coarsest topology $\mathfrak{S}$ on X such that these are all continuous. **P** Let P be the family of pseudometrics compatible with $\mathcal{W}$ in the sense of (α) just above. If $x \in G \in \mathfrak{T}$, there is a $\rho \in \mathrm{P}$ such that $\{y : \rho(x, y) < 1\} \subseteq G$; setting $f(y) = \rho(x, y)$, we see that f is uniformly continuous, therefore $\mathfrak{S}$-continuous, and that $x \in \mathrm{int}_{\mathfrak{S}} G$. As x is arbitrary, $G \in \mathfrak{S}$; as G is arbitrary, $\mathfrak{T} \subseteq \mathfrak{S}$; but of course $\mathfrak{S} \subseteq \mathfrak{T}$ just because uniformly continuous functions are continuous. **Q**

(b) If $\mathcal{W}$ is countably generated and $\mathfrak{T}$ is Hausdorff, there is a metric ρ on X defining $\mathcal{W}$ and $\mathfrak{T}$. (Engelking 89, 8.1.21.)

(c) If $W \in \mathcal{W}$ and $x \in X$ then $x \in \operatorname{int} W[\{x\}]$. (Engelking 89, 8.1.3.) If $A \subseteq X$ then $\overline{A} = \bigcap_{W\in\mathcal{W}} W[A]$. (Engelking 89, 8.1.4.)

(d) Any subset of a totally bounded set in X is totally bounded. (Engelking 89, 8.3.2; Császár 78, 3.2.70.) The closure of a totally bounded set is totally bounded. **P** If A is totally bounded and $W \in \mathcal{W}$, take $W' \in \mathcal{W}$ such that $W' \circ W' \subseteq W$. Then there is a finite set $I \subseteq X$ such that $A \subseteq W'[I]$. In this case

$$\overline{A} \subseteq W'[A] \subseteq W'[W'[I]] = (W' \circ W')[I] \subseteq W[I]$$

by (b). As W is arbitrary, $\overline{A}$ is totally bounded. **Q**

(e) A subset of X is compact iff it is complete (definition: 3A4F) (for its subspace uniformity) and totally bounded. (Engelking 89, 8.3.16; Čech 66, 41A.8; Császár 78, 5.2.22; Gaal 64, pp. 278-279.) So if X is complete, every closed totally bounded subset of X is compact, and the totally bounded sets are just the relatively compact sets. (A closed subspace of a complete space is complete.)

(f) If $f : X \to \mathbb{R}$ is a continuous function with compact support, it is uniformly continuous. **P** Set $K = \overline{\{x : f(x) \neq 0\}}$. Let $\epsilon > 0$. For each $x \in X$, there is a $W_x \in \mathcal{W}$ such that $|f(y) - f(x)| \le \frac{1}{2}\epsilon$ whenever $y \in W_x[\{x\}]$. Let $W'_x \in \mathcal{W}$ be such that $W'_x \circ W'_x \subseteq W_x$. Set $G_x = \operatorname{int} W'_x[\{x\}]$; then $x \in G_x$, by (b). Because K is compact, there is a finite set $I \subseteq K$ such that $K \subseteq \bigcup_{x\in I} G_x$. Set $W = (X \times X) \cap \bigcap_{x\in I} W'_x \in \mathcal{W}$. Take any $(y, z) \in W \cap W^{-1}$. If neither y nor z belongs to K, then of course $|f(y) - f(z)| \le \epsilon$. If $y \in K$, let $x \in I$ be such that $y \in G_x$. Then

$$y \in W'_x[\{x\}] \subseteq W_x[\{x\}], \quad z \in W[W'_x[\{x\}]] \subseteq W'_x[W'_x[\{x\}]] \subseteq W_x[\{x\}],$$

so

$$|f(y) - f(z)| \le |f(y) - f(x)| + |f(z) - f(x)| \le \epsilon.$$

The same idea works if $z \in K$. So $|f(y) - f(z)| \le \epsilon$ for all y, $z \in W \cap W^{-1}$; as ϵ is arbitrary, f is uniformly continuous. **Q**

(g)(i) If $(Y, \mathfrak{S})$ is a completely regular space, there is a uniformity on Y which induces $\mathfrak{S}$. (Engelking 89, 8.1.20.)

(ii) If $(Y, \mathfrak{S})$ is a compact completely regular topological space, there is exactly one uniformity on Y which induces $\mathfrak{S}$; it is defined by the set of all those pseudometrics on Y which are continuous as functions from $Y \times Y$ to $\mathbb{R}$. (Engelking 89, 8.3.13; Gaal 64, p. 304.)

(iii) If $(Y, \mathfrak{S})$ is a compact completely regular space and $\mathcal{V}$ is the uniformity on Y inducing $\mathfrak{S}$, then any continuous function from Y to X is uniformly continuous. (Gaal 64, p. 305 Theorem 8.)

(h) The set U of uniformly continuous real-valued functions on X is a Riesz subspace of $\mathbb{R}^X$ containing the constant functions. If a sequence in U converges uniformly, the limit function again belongs to U. (Császár 78, 3.2.64; Gaal 64, p. 237 Lemma 4.)

(i) Let $(Y, \mathcal{V})$ be another uniform space. If $\mathcal{F}$ is a Cauchy filter on X and $f : X \to Y$ is a uniformly continuous function, then $f[[\mathcal{F}]]$ is a Cauchy filter on Y. (Császár 78, 5.1.2.)

4A2K First-countable, sequential and countably tight spaces **(a)** Let X be a countably tight topological space. If $\langle F_\xi \rangle_{\xi<\zeta}$ is a non-decreasing family of closed subsets of X indexed by an ordinal ζ, then $E = \bigcup_{\xi<\zeta} F_\xi$ is an F_σ set, and is closed unless $\operatorname{cf}\zeta = \omega$. **P** If $\operatorname{cf}\zeta = 0$, that is, $\zeta = 0$, then $E = \emptyset$ is closed. If $\operatorname{cf}\zeta = 1$, that is, $\zeta = \xi + 1$ for some ordinal ξ, then $E = F_\xi$ is closed. If $\operatorname{cf}\zeta = \omega$, there is a sequence $\langle \xi_n \rangle_{n\in\mathbb{N}}$ in ζ with supremum ζ, so that $E = \bigcup_{n\in\mathbb{N}} F_{\xi_n}$ is F_σ. If $\operatorname{cf}\zeta > \omega$, take $x \in \overline{E}$. Then there is a sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ in E such that $x \in \overline{\{x_n : n \in \mathbb{N}\}}$. For each n there is a $\xi_n < \zeta$ such that $x_n \in F_{\xi_n}$, and now $\xi = \sup_{n\in\mathbb{N}} \xi_n < \zeta$ and $x \in \overline{F}_\xi = F_\xi \subseteq E$. As x is arbitrary, E is closed. **Q**

(b) If X is countably tight, any subspace of X is countably tight, just because if $A \subseteq Y \subseteq X$ then the closure of A in Y is the intersection of Y with the closure of A in X. If X is compact and countably tight, then any Hausdorff continuous image of X is countably tight. **P** Let $f : X \to Y$ be a continuous surjection, where Y is Hausdorff, B a subset of Y and $y \in \overline{B}$. Set $A = f^{-1}[B]$. Then $\overline{A}$ is compact, so $f[\overline{A}]$ is compact, therefore closed; because f is surjective, $y \in \overline{f[A]} \subseteq f[\overline{A}]$, and there is an $x \in \overline{A}$ such that $f(x) = y$. Now there is a countable set $A_0 \subseteq A$ such that $x \in \overline{A}_0$, in which case

$$y = f(x) \in f[\overline{A}_0] \subseteq \overline{f[A_0]},$$

while $f[A_0]$ is a countable subset of B. **Q**

(c) If X is a sequential space, it is countably tight. **P** Suppose that $A \subseteq X$ and $x \in \overline{A}$. Set $B = \bigcup\{\overline{C} : C \subseteq A \text{ is countable}\}$. If $\langle y_n \rangle_{n\in\mathbb{N}}$ is a sequence in B converging to $y \in X$, then for each $n \in \mathbb{N}$ we can find a countable set $C_n \subseteq A$ such that $y_n \in \overline{C}_n$, and now $C = \bigcup_{n\in\mathbb{N}} C_n$ is a countable subset of A such that $y \in \overline{C} \subseteq B$. So B is sequentially closed, therefore closed, and $x \in B$. As A and x are arbitrary, X is countably tight. **Q**

(d) If X is a sequential space, Y is a topological space and $f : X \to Y$ is sequentially continuous, then f is continuous. (Engelking 89, 1.6.15.)

(e) First-countable spaces are sequential. (Engelking 89, 1.6.14.)

(f) Let X be a locally compact Hausdorff space in which every singleton set is G_δ. Then X is first-countable. **P** If $\{x\} = \bigcap_{n\in\mathbb{N}} G_n$ where each G_n is open, then for each $n \in \mathbb{N}$ we can find a compact set F_n such that $x \in \operatorname{int} F_n \subseteq G_n$. By 4A2Gd, $\{\bigcap_{i\le n} F_i : n \in \mathbb{N}\}$ is a base of neighbourhoods of x. **Q**

4A2L (Pseudo-)metrizable spaces 'Pseudometrizable' spaces, as such, hardly appear in this volume, for the usual reasons; they surface briefly in §463. It is perhaps worth noting, however, that all the ideas, and very nearly all the results, in this paragraph apply equally well to pseudometrics and pseudometrizable topologies. If X is a set and ρ is a pseudometric on X, set $U(x, \delta) = \{y : \rho(x, y) < \delta\}$ for $x \in X$ and $\delta > 0$.

(a) Any subspace of a (pseudo-)metrizable space is (pseudo-)metrizable (2A3J). A topological space is metrizable iff it is pseudometrizable and Hausdorff (2A3L).

(b) Metrizable spaces are paracompact (Engelking 89, 5.1.3; Császár 78, 8.3.16; Čech 66, 30C.2; Gaal 64, p. 155), therefore hereditarily metacompact ((a) above and 4A2F(g-i)).

(c) A metrizable space is perfectly normal (Engelking 89, 4.1.13; Császár 78, 8.4.5.), so every closed set is a zero set and every open set is a cozero set (in particular, is F_σ).

(d) If X is a pseudometrizable space, it is first-countable. (If ρ is a pseudometric defining the topology of X, and x is any point of X, then $\{\{y : \rho(y, x) < 2^{-n}\} : n \in \mathbb{N}\}$ is a base of neighbourhoods of x.) So X is sequential and countably tight (4A2Ke, 4A2Kc), and if Y is another topological space and $f : X \to Y$ is sequentially continuous, then f is continuous (4A2Kd).

(e) Relative compactness Let X be a pseudometrizable space and A a subset of X. Then the following are equiveridical: (α) A is relatively compact; (β) A is relatively countably compact; (γ) every sequence in A has a subsequence with a limit in X. **P** Fix a pseudometric ρ defining the topology of X. (α)⇒(β) by 4A2G(f-iii). If $\langle x_n \rangle_{n\in\mathbb{N}}$ is a sequence in A with a cluster point $x \in X$, then we can choose $\langle n_i \rangle_{i\in\mathbb{N}}$ inductively such that $\rho(x_{n_i}, x) \le 2^{-i}$ and $n_{i+1} > n_i$ for every i; now $\langle x_{n_i} \rangle_{i\in\mathbb{N}} \to x$; it follows that ($\beta$)⇒($\gamma$). Now assume that ($\alpha$) is false. Then there is an ultrafilter $\mathcal{F}$ on X containing A which has no limit in X (3A3Be, 3A3De). If $\mathcal{F}$ is a Cauchy filter, choose $F_n \in \mathcal{F}$ such that $\rho(x, y) \le 2^{-n}$ whenever $x, y \in F_n$, and $x_n \in A \cap \bigcap_{i\le n} F_i$ for each n; then it is easy to see that $\langle x_n \rangle_{n\in\mathbb{N}}$ is a sequence in A with no convergent subsequence. If $\mathcal{F}$ is not a Cauchy filter, let $\epsilon > 0$ be such that there is no $F \in \mathcal{F}$ such that $\rho(x, y) \le \epsilon$ for every $x, y \in F$. Then $X \setminus U(x, \frac{1}{2}\epsilon) \in \mathcal{F}$ for every $x \in X$, so we can choose $\langle x_n \rangle_{n\in\mathbb{N}}$ inductively such that $x_n \in A \setminus \bigcup_{i<n} U(x_i, \frac{1}{2}\epsilon)$ for every $n \in \mathbb{N}$, and again we have a sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ in A with no convergent subsequence in X. Thus not-(α)⇒ not-(γ) and the proof is complete. **Q**

(f) Compactness If X is a pseudometrizable space, it is compact iff it is countably compact iff it is sequentially compact. ((e) above, using 4A2G(f-i). Compare Engelking 89, 4.1.17, and Császár 78, 5.3.33 & 5.3.47.)

(g)(i) If (X,ρ) is a metric space, its topology has a base which is σ-metrically-discrete. **P** Enumerate X as $\langle x_\xi\rangle_{\xi<\kappa}$ where κ is a cardinal. Let $\langle(q_n,q'_n)\rangle_{n\in\mathbb{N}}$ be a sequence running over $\{(q,q'):q,\,q'\in\mathbb{Q},\,0<q<q'\}$ in such a way that $q'_n-q_n\ge 2^{-n}$ for every $n\in\mathbb{N}$. For $n\in\mathbb{N}$, $\xi<\kappa$ set $G_{n\xi}=\{x:\rho(x,x_\xi)<q_n,\,\inf_{\eta<\xi}\rho(x,x_\eta)>q'_n\}$ (interpreting $\inf\emptyset$ as ∞). Then $\mathcal{U}=\langle G_{n\xi}\rangle_{\xi<\kappa,n\in\mathbb{N}}$ is a σ-metrically-discrete family of open sets. If $G\subseteq X$ is open and $x\in G$, let $\epsilon>0$ be such that $U(x,2\epsilon)\subseteq G$. Let $\xi<\kappa$ be minimal such that $\rho(x,x_\xi)<\epsilon$, and let $n\in\mathbb{N}$ be such that $\rho(x,x_\xi)<q_n<q'_n<\epsilon$; then $x\in G_{n\xi}\subseteq G$. As x and G are arbitrary, $\mathcal{U}$ is a base for the topology of X. **Q**

(ii) Consequently, any metrizable space has a σ-disjoint base. (Compare Engelking 89, 4.4.3; Császár 78, 8.4.5; Kuratowski 66, §21.XVII.)

(h) The product of a countable family of metrizable spaces is metrizable. (Engelking 89, 4.2.2; Császár 78, 7.3.27.)

(i) Let X be a metrizable space and $\kappa\ge\omega$ a cardinal. Then $w(X)\le\kappa$ iff X has a dense subset with cardinal at most κ. (Engelking 89, 4.1.15.)

(j) If (X,ρ) is any metric space, then the balls $B(x,\delta)=\{y:\rho(y,x)\le\delta\}$ are all closed sets (cf. 1A2G). In particular, in a normed space $(X,\|\,\|)$, the balls $B(x,\delta)=\{y:\|y-x\|\le\delta\}$ are closed.

4A2M Complete metric spaces (a) Baire's theorem for complete metric spaces Every complete metric space is a Baire space. (Engelking 89, 4.3.36 & 3.9.4; Kechris 95, 8.4; Császár 78, 9.2.1 & 9.2.8; Gaal 64, p. 287.) So a non-empty complete metric space is not meager (cf. 3A3Ha).

(b) Let $\langle(X_i,\rho_i)\rangle_{i\in I}$ be a countable family of complete metric spaces. Then there is a complete metric on $X=\prod_{i\in I}X_i$ which defines the product topology on X. (Engelking 89, 4.3.12; Kuratowski 66, §33.III.)

(c) Let (X,ρ) be a complete metric space, and $E\subseteq X$ a G_δ set. Then there is a complete metric on E which defines the subspace topology of E. (Engelking 89, 4.3.23; Kuratowski 66, §33.VI; Kechris 95, 3.11.)

(d) Let (X,ρ) be a complete metric space. Then it is Čech-complete. (Engelking 89, 4.3.26.)

(e) A non-empty complete metric space without isolated points is uncountable. (If x is not isolated, $\{x\}$ is nowhere dense.)

4A2N Countable networks: Proposition (a) If X is a topological space with a countable network, any subspace of X has a countable network.

(b) Let X be a space with a countable network. Then X is hereditarily Lindelöf. If it is regular, it is perfectly normal.

(c) If X is a topological space, and $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of subsets of X each of which has a countable network (for its subspace topology), then $A=\bigcup_{n\in\mathbb{N}}A_n$ has a countable network.

(d) A continuous image of a space with a countable network has a countable network.

(e) Let $\langle X_i\rangle_{i\in I}$ be a countable family of topological spaces with countable networks, with product X. Then X has a countable network.

(f) If X is a Hausdorff space with a countable network, there is a countable family $\mathcal{G}$ of open sets such that whenever x, y are distinct points in X there are disjoint G, $H\in\mathcal{G}$ such that $x\in G$ and $y\in H$.

(g) If X is a regular topological space with a countable network, it has a countable network consisting of closed sets.

(h) A compact Hausdorff space with a countable network is second-countable.

(i) If a topological space X has a countable network, then any dense set in X includes a countable dense set; in particular, X is separable.

(j) If a topological space X has a countable network, then $C(X)$, with the topology of pointwise convergence inherited from the product topology of $\mathbb{R}^X$, has a countable network.

proof (a) If $\mathcal{E}$ is a countable network for the topology of X, and $Y \subseteq X$, then $\{Y \cap E : E \in \mathcal{E}\}$ is a countable network for the topology of Y.

(b) By Engelking 89, 3.8.12 X is Lindelöf. Since any subspace of X has a countable network ((a) above), it also is Lindelöf, and X is hereditarily Lindelöf. By 4A2H(c-ii), if X is regular, it is perfectly normal.

(c) If $\mathcal{E}_n$ is a countable network for the topology of A_n for each n, then $\bigcup_{n\in\mathbb{N}} \mathcal{E}_n$ is a countable network for the topology of A.

(d) Let X be a topological space with a countable network $\mathcal{E}$, and Y a continuous image of X. Let $f : X \to Y$ be a continuous surjection. Then $\{f[E] : E \in \mathcal{E}\}$ is a network for the topology of Y. **P** If $H \subseteq Y$ is open and $y \in H$, then $f^{-1}[H]$ is an open subset of X and there is an $x \in X$ such that $f(x) = y$. Now there must be an $E \in \mathcal{E}$ such that $x \in E \subseteq f^{-1}[H]$, so that $y \in f[E] \subseteq H$. **Q** But $\{f[E] : E \in \mathcal{E}\}$ is countable, so Y has a countable network.

(e) For each $i \in I$ let $\mathcal{E}_i$ be a countable network for the topology of X_i. For each finite $J \subseteq I$, let $\mathcal{C}_J$ be the family of sets expressible as $\prod_{i\in I} E_i$ where $E_i \in \mathcal{E}_i$ for each $i \in J$ and $E_i = X_i$ for $i \in I \setminus J$; then $\mathcal{C}_J$ is countable because $\mathcal{E}_i$ is countable for each $i \in J$. Because the family $[I]^{<\omega}$ of finite subsets of I is countable (3A1Cd), $\mathcal{E} = \bigcup\{\mathcal{C}_J : J \in [I]^{<\omega}\}$ is countable. But $\mathcal{E}$ is a network for the topology of X. **P** If $G \subseteq X$ is open and $x \in G$, then there is a family $\langle G_i\rangle_{i\in I}$ such that every $G_i \subseteq X_i$ is open, $J = \{i : G_i \ne X_i\}$ is finite, and $x \in \prod_{i\in I} G_i$. For $i \in J$, there is an $E_i \in \mathcal{E}_i$ such that $x(i) \in E_i \subseteq G_i$; set $E_i = X_i$ for $i \in I \setminus J$. Then

$$E = \prod_{i\in I} E_i \in \mathcal{C}_J \subseteq \mathcal{E}$$

and $x \in E \subseteq G$. **Q**

So $\mathcal{E}$ is a countable network for the topology of X.

(f) By (b) and (e), $X \times X$ is hereditarily Lindelöf. In particular, $W = \{(x, y) : x \ne y\}$ is Lindelöf. Set

$$\mathcal{V} = \{G \times H : G,\, H \subseteq X \text{ are open, } G \cap H = \emptyset\}.$$

Because X is Hausdorff, $\mathcal{V}$ is a cover of W. So there is a countable $\mathcal{V}_0 \subseteq \mathcal{V}$ covering W. Set

$$\mathcal{G} = \{G : G \times H \in \mathcal{V}_0\} \cup \{H : G \times H \in \mathcal{V}_0\}.$$

Then $\mathcal{G}$ is a countable family of open sets separating the points of X.

(g) Let $\mathcal{E}$ be a countable network for the topology of X. Set $\mathcal{E}' = \{\overline{E} : E \in \mathcal{E}\}$. If $G \subseteq X$ is open and $x \in G$, then (because the topology is regular) there is an open set H such that $x \in H \subseteq \overline{H} \subseteq G$. Now there is an $E \in \mathcal{E}$ such that $x \in E \subseteq H$, in which case $\overline{E} \in \mathcal{E}'$ and $x \in \overline{E} \subseteq G$. So $\mathcal{E}'$ is a countable network for X consisting of closed sets.

(h) Engelking 89, 3.1.19.

(i) Let $D \subseteq X$ be dense, and $\mathcal{E}$ a countable network for the topology of X. Let $D' \subseteq D$ be a countable set such that $D' \cap E \ne \emptyset$ whenever $E \in \mathcal{E}$ and $D \cap E \ne \emptyset$. If $G \subseteq X$ is open and not empty, there is an $x \in D \cap G$; now there is an $E \in \mathcal{E}$ such that $x \in E \subseteq G$, and as $x \in D \cap E$ there must be an $x' \in D' \cap E$, so that $x' \in D' \cap G$. As G is arbitrary, D' is dense in X.

Taking $D = X$, we see that X has a countable dense subset.

(j) Let $\mathcal{E}$ be a countable network for the topology of X and $\mathcal{U}$ a countable base for the topology of $\mathbb{R}$ (4A2Ua). For $E \in \mathcal{E}$ and $U \in \mathcal{U}$ set $H(E, U) = \{f : f \in C(X),\, E \subseteq f^{-1}[U]\}$. Then the set of finite intersections of sets of the form $H(E, U)$ is a countable network for the topology of pointwise convergence on $C(X)$. (Compare 4A2Oe.)

4A2O Second-countable spaces (a) Let $(X, \mathfrak{T})$ be a topological space and $\mathcal{U}$ a countable subbase for $\mathfrak{T}$. Then $\mathfrak{T}$ is second-countable. ($\{X\} \cup \{U_0 \cap U_1 \cap \ldots \cap U_n : U_0, \ldots, U_n \in \mathcal{U}\}$ is countable and is a base for $\mathfrak{T}$, by 4A2B(a-i).)

(b) Any base of a second-countable space includes a countable base. (Császár 78, 2.4.17.)

(c) A second-countable space has a countable network (because a base is also a network), so is separable and hereditarily Lindelöf (Engelking 89, 1.3.8 & 3.8.1, 4A2Nb, 4A2Ni).

(d) The product of a countable family of second-countable spaces is second-countable. (Engelking 89, 2.3.14.)

(e) If X is a second-countable space then $C(X)$, with the topology of uniform convergence on compact sets, has a countable network. **P** (See Engelking 89, Ex. 3.4H.) Let $\mathcal{U}$ be a countable base for the topology of X and $\mathcal{V}$ a countable base for the topology of $\mathbb{R}$ (4A2Ua). For $U \in \mathcal{U}$, $V \in \mathcal{V}$ set $H(U,V) = \{f : f \in C(X), U \subseteq f^{-1}[V]\}$. Then the set of finite intersections of sets of the form $H(U,V)$ is a countable network for the topology of uniform convergence on compact subsets of X. **Q**

4A2P Separable metrizable spaces (a)(i) A metrizable space is second-countable iff it is separable. (Engelking 89, 4.1.16; Császár 78, 2.4.16; Gaal 64 p. 120.)

(ii) A compact metrizable space is separable (Engelking 89, 4.1.18; Császár 78, 5.3.35; Kuratowski 66, §21.IX), so is second-countable and has a countable network.

(iii) Any base of a separable metrizable space includes a countable base (4A2Ob), which is also a countable network, so the space is hereditarily Lindelöf (4A2Nb).

(iv) Any subspace of a separable metrizable space is separable and metrizable (4A2La, 4A2Na, 4A2Ni).

(v) A countable product of separable metrizable spaces is separable and metrizable (4A2B(e-ii), 4A2Lh).

(b) A topological space is separable and metrizable iff it is second-countable, regular and Hausdorff. (Engelking 89, 4.2.9; Császár 78, 7.1.57; Kuratowski 66, §22.II.)

(c) A Hausdorff continuous image of a compact metrizable space is metrizable. (It is a compact Hausdorff space, by 2A3N(b-ii), with a countable network, by 4A2Nd, so is metrizable, by 4A2Nh.)

(d) A metrizable space is separable iff it is ccc iff it is Lindelöf. (Engelking 89, 4.1.16.)

(e) If X is a compact metrizable space, then $C(X)$ is separable under its usual norm topology defined from the norm $\| \|_\infty$. (4A2Oe, or Engelking 89, 3.4.16.)

4A2Q Polish spaces: Proposition (a) A countable discrete space is Polish.

(b) A compact metrizable space is Polish.

(c) The product of a countable family of Polish spaces is Polish.

(d) A G_δ subset of a Polish space is Polish in its subspace topology; in particular, a set which is either open or closed is Polish.

(e) The disjoint union of countably many Polish spaces is Polish.

(f) If X is any set and $\langle \mathfrak{T}_n \rangle_{n\in\mathbb{N}}$ is a sequence of Polish topologies on X such that $\mathfrak{T}_m \cap \mathfrak{T}_n$ is Hausdorff for all m, $n \in \mathbb{N}$, then the topology $\mathfrak{T}_\infty$ generated by $\bigcup_{n\in\mathbb{N}} \mathfrak{T}_n$ is Polish.

(g) If X is a Polish space, it is homeomorphic to a G_δ set in a compact metrizable space.

(h) If X is a locally compact Hausdorff space, it is Polish iff it has a countable network iff it is metrizable and σ-compact.

proof (a) Any set X is complete under the discrete metric ρ defined by setting $\rho(x,y) = 1$ whenever x, $y \in X$ are distinct. This defines the discrete topology, and if X is countable it is separable, therefore Polish.

(b) By 4A2P(a-ii), it is separable; by 4A2Je, any metric defining the topology is complete.

(c) If $\langle X_i \rangle_{i\in I}$ is a countable family of Polish spaces with product X, then surely X is separable (4A2B(e-ii)); and 4A2Mb tells us that its topology is defined by a complete metric.

(d) If X is Polish and E is a G_δ set in X, then E is separable, by 4A2P(a-iv), and its topology is defined by a complete metric, by 4A2Mc. So E is Polish. Any open set is of course a G_δ set, and any closed set is a G_δ set by 4A2Lc.

(e) Let $\langle X_i\rangle_{i\in I}$ be a countable disjoint family of Polish spaces, and $X = \bigcup_{i\in I} X_i$. For each $i \in I$ let ρ_i be a complete metric on X_i defining the topology of X_i. Define $\rho : X \times X \to [0,\infty[$ by setting $\rho(x,y) = \min(1,\rho_i(x,y))$ if $i \in I$ and x, $y \in X_i$, $\rho(x,y) = 1$ otherwise. It is easy to check that ρ is a complete metric on X defining the disjoint union topology on X. X is separable, by 4A2B(e-i), therefore Polish.

(f) This result is in Kechris 95, 13.3; but I spell out the proof because it is an essential element of some measure-theoretic arguments. On $X^{\mathbb{N}}$ take the product topology $\mathfrak{T}$ of the topologies $\mathfrak{T}_n$. This is Polish, by (c). Consider the diagonal $\Delta = \{x : x \in X^{\mathbb{N}},\, x(m) = x(n) \text{ for all } m,\, n \in \mathbb{N}\}$. This is closed in $X^{\mathbb{N}}$. **P** If $x \in X^{\mathbb{N}} \setminus \Delta$, let m, $n \in \mathbb{N}$ be such that $x(m) \neq x(n)$. Because $\mathfrak{T}_m \cap \mathfrak{T}_n$ is Hausdorff, there are disjoint G, $H \in \mathfrak{T}_m \cap \mathfrak{T}_n$ such that $x(m) \in G$ and $x(n) \in H$. Now $\{y : y \in X^{\mathbb{N}},\, y(m) \in G,\, y(n) \in H\}$ is an open set in $X^{\mathbb{N}}$ containing x and disjoint from Δ. As x is arbitrary, Δ is closed. **Q**

By (d), Δ, with its subspace topology, is a Polish space. Let $f : X \to \Delta$ be the natural bijection, setting $f(t) = x$ if $x(n) = t$ for every n, and let $\mathfrak{S}$ be the topology on X which makes f a homeomorphism. The topology on Δ is generated by $\{\{x : x \in \Delta,\, x(n) \in G\} : n \in \mathbb{N},\, G \in \mathfrak{T}_n\}$, so $\mathfrak{S}$ is generated by $\{\{t : t \in X,$ $t \in G\} : n \in \mathbb{N},\, G \in \mathfrak{T}_n\} = \bigcup_{n\in\mathbb{N}} \mathfrak{T}_n$. Thus $\mathfrak{S} = \mathfrak{T}_\infty$ and $\mathfrak{T}_\infty$ is Polish.

(g) Kechris 95, 4.14.

(h) If X is Polish, then it is separable, therefore Lindelöf (4A2P(a-iii)). Since the family $\mathcal{G}$ of relatively compact open subsets of X covers X, there is a countable $\mathcal{G}_0 \subseteq \mathcal{G}$ covering X, and $\{\overline{G} : G \in \mathcal{G}_0\}$ witnesses that X is σ-compact. Also, of course, X is metrizable.

If X is metrizable and σ-compact, let $\langle K_n\rangle_{n\in\mathbb{N}}$ be a sequence of compact sets covering X; each K_n has a countable network (4A2P(a-ii)), so $X = \bigcup_{n\in\mathbb{N}} K_n$ has a countable network (4A2Nc).

If X has a countable network, let $Z = X \cup \{\infty\}$ be its one-point compactification (3A3O). This is compact and Hausdorff and has a countable network, by 4A2Nc again, so is second-countable (4A2Nh) and metrizable (4A2Pb) and Polish ((b) above). So X also, being an open set in Z, is Polish ((d) above).

4A2R Order topologies Let $(X,\le)$ be a totally ordered set and $\mathfrak{T}$ its order topology.

(a) The set $\mathcal{U}$ of open intervals in X (definition: 4A2A) is a base for $\mathfrak{T}$.

(b) $[x,y]$, $[x,\infty[$ and $]-\infty,x]$ are closed sets for all x, $y \in X$.

(c) $\mathfrak{T}$ is Hausdorff, normal and countably paracompact.

(d) If $A \subseteq X$ then $\overline{A}$ is the set of elements of X expressible as either suprema or infima of non-empty subsets of A.

(e) A subset of X is closed iff it is order-closed.

(f) If $\langle x_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in X with supremum x, then $x = \lim_{n\to\infty} x_n$.

(g) A set $K \subseteq X$ is compact iff $\sup A$ and $\inf A$ are defined in X and belong to K for every non-empty $A \subseteq K$.

(h) X is Dedekind complete iff $[x,y]$ is compact for all x, $y \in X$.

(i) X is compact iff it is either empty or Dedekind complete with greatest and least elements.

(j) Any open set $G \subseteq X$ is expressible as a union of disjoint open order-convex sets; if X is Dedekind complete, these will be open intervals.

(k) If X is well-ordered it is locally compact.

(l) In $X \times X$, $\{(x,y) : x < y\}$ is open and $\{(x,y) : x \le y\}$ is closed.

(m) If $F \subseteq X$ and *either* F is order-convex *or* F is compact *or* X is Dedekind complete and F is closed, then the subspace topology on F is induced by the inherited order of F.

(n) If X is ccc it is hereditarily Lindelöf, therefore perfectly normal.

(o) If Y is another totally ordered set with its order topology, an order-preserving function from X to Y is continuous iff it is order-continuous.

proof (a) Put the definition of 'order topology' (4A2A) together with 4A2B(a-i).

(b) Their complements are either X, or members of $\mathcal{U}$, or unions of two members of $\mathcal{U}$.

(c) Fix a well-ordering $\preccurlyeq$ of X.

(i) If $x < y \in X$, define U_{xy}, U_{yx} as follows: if $]x,y[$ is empty, $U_{xy} =]-\infty,x] =]-\infty,y[$ and $U_{yx} = [y,\infty[=]x,\infty[$; otherwise, let z be the $\preccurlyeq$-least member of $]x,y[$ and set $U_{xy} =]-\infty,z[$, $U_{yx} =]z,\infty[$.

This construction ensures that if x, y are any distinct points of X, U_{xy} and U_{yx} are disjoint open sets containing x, y respectively, so $\mathfrak{T}$ is Hausdorff.

(ii) Now suppose that $F \subseteq X$ is closed and that $x \in X \setminus F$. Then $V_{xF} = \operatorname{int}(X \cap \bigcap_{y \in F} U_{xy})$ contains x. **P** There are u, $v \in X \cup \{-\infty, \infty\}$ such that $x \in \,]u, v[\, \subseteq X \setminus F$. If $]u, x[\, = \emptyset$, set $u' = u$; otherwise, let u' be the $\preccurlyeq$-least member of $]u, x[$. Similarly, if $]x, v[\, = \emptyset$, set $v' = v$; otherwise, let v' be the $\preccurlyeq$-least member of $]x, v[$. Then $u' < x < v'$. Now suppose that $y \in F$ and $y > x$. If $]x, v[\, = \emptyset$, then $U_{xy} \supseteq \,]-\infty, x] = \,]-\infty, v'[$. Otherwise, $v' \in \,]x, v[\, \subseteq \,]x, y[$, so $U_{xy} = \,]-\infty, z[$ where z is the $\preccurlyeq$-least member of $]x, y[$. But this means that $z \preccurlyeq v'$ and either $z = v'$ or $z \notin \,]x, v[$; in either case, $v' \le z$ and $]-\infty, v'[\, \subseteq U_{xy}$.

Similarly, $]u', \infty[\, \subseteq U_{xy}$ whenever $y \in F$ and $y < x$. So $x \in \,]u', v'[\, \subseteq V_{xF}$. **Q**

(iii) Let E and F be disjoint closed sets. Set $G = \bigcup_{x \in E} V_{xF}$, $H = \bigcup_{y \in F} V_{yE}$. Then G and H are open sets including E, F respectively. If $x \in E$ and $y \in F$, then $V_{xF} \cap V_{yE} \subseteq U_{xy} \cap U_{yx} = \emptyset$, so $G \cap H = \emptyset$. As E and F are arbitrary, $\mathfrak{T}$ is normal.

(iv) Let $\langle F_n \rangle_{n \in \mathbb{N}}$ be a non-increasing sequence of closed sets with empty intersection. Let $\mathcal{I}$ be the family of open intervals $I \subseteq X$ such that $I \cap F_n = \emptyset$ for some n. Because the F_n are closed and have empty intersection, $\mathcal{I}$ covers X. If I, $I' \in \mathcal{I}$ are not disjoint, $I \cup I' \in \mathcal{I}$; so we have an equivalence relation $\sim$ on X defined by saying that $x \sim y$ if there is some $I \in \mathcal{I}$ containing both x and y. The corresponding equivalence classes are open and therefore closed, and are order-convex. Let $\mathcal{G}$ be the set of equivalence classes for $\sim$.

For each $G \in \mathcal{G}$, fix $x_G \in G$. Set $G^+ = G \cap [x_G, \infty[$. Then we have a non-decreasing sequence $\langle G_n^+ \rangle_{n \in \mathbb{N}}$ of closed sets, with union G^+, such that $G_n^+ \cap F_n = \emptyset$ for each n. **P** If there is some $m \in \mathbb{N}$ such that $G^+ \cap F_m = \emptyset$, set $G_n^+ = \emptyset$ if $G^+ \cap F_n \ne \emptyset$, G^+ if $G^+ \cap F_n = \emptyset$. Otherwise, given $x \in G^+$ and $n \in \mathbb{N}$, there is some m such that $[x_G, x]$ does not meet F_m, and an $x' \in G^+ \cap F_{\max(m,n)}$, so that $x' \in F_n$ and $x' > x$. We can therefore choose a strictly increasing sequence $\langle x_k \rangle_{k \in \mathbb{N}}$ such that $x_0 = x_G$ and $x_{k+1} \in G^+ \cap F_k$ for each k. If x is any upper bound of $\{x_k : k \in \mathbb{N}\}$ then $x \not\sim x_G$, so $G^+ = \bigcup_{k \in \mathbb{N}} [x_G, x_k]$. Now, for each n, there is a least $k(n)$ such that $[x_G, x_{k(n)}] \cap F_n \ne \emptyset$; set $G_n^+ = \emptyset$ if $k(n) = 0$, $[x_G, x_{k(n-1)}]$ otherwise. As $F_{n+1} \subseteq F_n$, $k(n+1) \ge k(n)$ for each n. Since each $[x_G, x_k]$ is disjoint from some F_n, and therefore from all but finitely many F_n, $\lim_{n \to \infty} k(n) = \infty$ and $G^+ = \bigcup_{n \in \mathbb{N}} G_n^+$. **Q**

Similarly, $G^- = G \cap \,]-\infty, x_G]$ can be expressed as the union of a non-decreasing sequence $\langle G_n^- \rangle_{n \in \mathbb{N}}$ of closed sets such that $G_n^- \cap F_n = \emptyset$ for every n. Now set $F'_n = \bigcup_{G \in \mathcal{G}} G_n^+ \cup G_n^-$ for each n. Because every G_n^+ and G_n^- is closed, and every G is open-and-closed, F'_n is closed. So $\langle F'_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence of closed sets with union X, and F'_n is disjoint from F_n for each n. Accordingly $\langle X \setminus F'_n \rangle_{n \in \mathbb{N}}$ is a non-increasing sequence of open sets with empty intersection enveloping the F_n. As $\langle F_n \rangle_{n \in \mathbb{N}}$ is arbitrary, $\mathfrak{T}$ is countably paracompact (4A2Ff).

(d) Let B be the set of such suprema and infima. For $x \in X$ set $A_x = A \cap \,]-\infty, x]$, $A'_x = A \cap [x, \infty[$. Then $x \in B$ iff either $x = \sup A_x$ or $x = \inf A'_x$, so

$$\begin{aligned} x \notin B &\iff x \ne \sup A_x \text{ and } x \ne \inf A'_x \\ &\iff \text{there are } u < x,\ v > x \text{ such that } A_x \subseteq \,]-\infty, u] \text{ and } A'_x \subseteq [v, \infty[\\ &\iff \text{there are } u,\ v \text{ such that } x \in \,]u, v[\, \subseteq X \setminus A \\ &\iff x \notin \overline{A}. \end{aligned}$$

Thus $B = \overline{A}$, as claimed.

(e) Because X is totally ordered, all its subsets are both upwards-directed and downwards-directed; so we have only to join the definition in 313Da to (d) above.

(f) If $x \in \,]u, v[$ then there is some $n \in \mathbb{N}$ such that $x_n \ge u$, and now $x_i \in \,]u, v[$ for every $i \ge n$.

(g)(i) If K is compact and $A \subseteq K$ is non-empty, let B be the set of upper bounds for A in $X \cup \{-\infty, \infty\}$, and set $\mathcal{G} = \{]-\infty, a[\, : a \in A\} \cup \{]b, \infty[\, : b \in B\}$. Then no finite subfamily of $\mathcal{G}$ can cover K; and if $c \in K \setminus \bigcup \mathcal{G}$ then $c = \sup A$. Similarly, any non-empty subset of K has an infimum in X which belongs to K.

(ii) Now suppose that K satisfies the condition. By (d) above, it is closed. If it is empty it is certainly compact. Otherwise, $a_0 = \inf K$ and $b_0 = \sup K$ are defined in X and belong to K. Let $\mathcal{G}$ be an open cover of K. Set

$$A = \{x : x \in X,\ K \cap [a_0, x] \text{ is not covered by any finite } \mathcal{G}_0 \subseteq \mathcal{G}\}.$$

Note that A is bounded below by a_0. **?** If $b_0 \in A$, then $c = \inf A$ is defined and belongs to $[a_0, b_0]$, because X is Dedekind complete. If $c \notin K$ then there are u, v such that $c \in \,]u, v[\, \subseteq X \setminus K$; if $c \in K$ then there are u, v such that $c \in \,]u, v[\, \subseteq G$ for some $G \in \mathcal{G}$. In either case, $u \notin A$, so that $K \cap [a_0, v[\, \subseteq (K \cap [a_0, u]) \cup \,]u, v[$ is covered by a finite subset of $\mathcal{G}$, and A does not meet $[a_0, v[$, that is, $A \subseteq [v, \infty[$ and v is a lower bound of A. **X** Thus $b_0 \notin A$, and $K = K \cap [a_0, b_0]$ is covered by a finite subset of $\mathcal{G}$. As $\mathcal{G}$ is arbitrary, K is compact.

(h) Use (g).

(i) Use (h).

(j) (Compare 2A2I.) For x, $y \in G$ write $x \sim y$ if either $x \le y$ and $[x, y] \subseteq G$ or $y \le x$ and $[y, x] \subseteq G$. It is easy to check that $\sim$ is an equivalence relation on G. Let $\mathcal{C}$ be the set of equivalence classes under $\sim$. Then $\mathcal{C}$ is a partition of G into order-convex sets. Now every $C \in \mathcal{C}$ is open. **P** If $x \in C \in \mathcal{C}$ then there are u, $v \in X \cup \{-\infty, \infty\}$ such that $x \in \,]u, v[\, \subseteq G$; now $]u, v[\, \subseteq C$. **Q** So we have our partition of G into disjoint open order-convex sets.

If X is Dedekind complete, then every member of $\mathcal{C}$ is an open interval. **P** Take $C \in \mathcal{C}$. Set

$$A = \{u : u \in X \cup \{-\infty\},\ u < x \text{ for every } x \in C\},$$

$$B = \{v : v \in X \cup \{\infty\},\ x < v \text{ for every } x \in C\},$$

$$a = \sup A, \quad b = \inf B;$$

these are defined because X is Dedekind complete. If $a < x < b$, there are y, $z \in C$ such that $y \le x \le z$, so that $[y, x] \subseteq [y, z] \subseteq G$ and $y \sim x$ and $x \in C$; thus $]a, b[\, \subseteq C$. If $x \in C$, there is an open interval $]u, v[$ containing x and included in G; now $]u, v[\, \subseteq C$, so $a \le u < x < v \le b$ and $x \in \,]a, b[$. Thus $C = \,]a, b[$ is an open interval. **Q**

(k) Use (h).

(l) Write W for $\{(x, y) : x < y\}$. If $x < y$, then either there is a z such that $x < z < y$, in which case $]-\infty, z[\, \times \,]z, \infty[$ is an open set containing (x, y) and included in W, or $]x, y[\, = \emptyset$, so $]-\infty, y[\, \times \,]x, \infty[$ is an open set containing (x, y) and included in W. Thus W is open.

Now $\{(x, y) : x \le y\} = (X \times X) \setminus \{(x, y) : y < x\}$ is closed.

(m) The subspace topology $\mathfrak{T}_F$ on F is generated by sets of the form $F \cap]-\infty, x[$, $F \cap]x, \infty[$ where $x \in X$ (4A2B(a-vi)), while the order topology $\mathfrak{S}$ on F is generated by sets of the form $F \cap \,]-\infty, x[$, $F \cap \,]x, \infty[$ where $x \in F$. So $\mathfrak{S} \subseteq \mathfrak{T}_F$.

Now suppose that one of the three conditions is satisfied, and that $x \in X$. If $x \in F$, or $F \cap \,]-\infty, x[$ is either F or $\emptyset$, then of course $F \cap \,]-\infty, x[\, \in \mathfrak{S}$. Otherwise, F meets both $]-\infty, x[$ and $[x, \infty[$ and does not contain x, so is not order-convex. In this case $x' = \inf(F \cap [x, \infty[)$ is defined and belongs to F. **P** If F is compact, this is covered by (g). If X is Dedekind complete and F is closed, then x' is defined, and belongs to F by (e). **Q** Now $F \cap \,]-\infty, x[\, = F \cap \,]-\infty, x'[\, \in \mathfrak{S}$.

Similarly, $F \cap]x, \infty[\, \in \mathfrak{S}$ for every $x \in X$. But this means that $\mathfrak{T}_F \subseteq \mathfrak{S}$, so the two topologies are equal, as stated.

(n)(i) Let $\mathcal{G}$ be a family of open subsets of X with union H. Set

$$\mathcal{A} = \{A : A \subseteq \bigcup \mathcal{G}_0 \text{ for some countable } \mathcal{G}_0 \subseteq \mathcal{G}\}.$$

(I seek to show that $H \in \mathcal{A}$.) Of course $\bigcup \mathcal{A}_0 \in \mathcal{A}$ for every countable subset $\mathcal{A}_0$ of $\mathcal{A}$.

(ii) Let $\mathcal{C}$ be the family of order-convex members of $\mathcal{A}$, and $\mathcal{C}^*$ the family of maximal members of $\mathcal{C}$. If $C \in \mathcal{C}$ is not included in any member of $\mathcal{C}^*$, then there is a $C' \in \mathcal{C}$ such that $C \subseteq C'$ and $\operatorname{int}(C' \setminus C) \ne \emptyset$. **P** Since no member of $\mathcal{C}$ including C can be maximal, we can find $C' \in \mathcal{C}$ such that $C \subseteq C'$ and $\#(C' \setminus C) \ge 5$.

Because C is order-convex, every point of $X \setminus C$ is either a lower bound or an upper bound of C, and there must be three points $x < y < z$ of $C' \setminus C$ on the same side of C. In this case,

$$y \in]x, z[\subseteq \operatorname{int}(C' \setminus C),$$

so we have an appropriate C'. **Q**

(iii) In fact, every member of $\mathcal{C}$ is included in a member of $\mathcal{C}^*$. **P?** Suppose, if possible, otherwise. Then we can choose a strictly increasing family $\langle C_\xi \rangle_{\xi<\omega_1}$ in $\mathcal{C}$ inductively, as follows. Start from any non-empty $C_0 \in \mathcal{C}$ not included in any member of $\mathcal{C}^*$. Given that $C_0 \subseteq C_\xi \in \mathcal{C}$, then C_ξ cannot be included in any member of $\mathcal{C}^*$, so by (β) above there is a $C_{\xi+1} \in \mathcal{C}$ such that $\operatorname{int}(C_{\xi+1} \setminus C_\xi)$ is non-empty. Given $\langle C_\eta \rangle_{\eta<\xi}$ where $\xi < \omega_1$ is a non-zero countable limit ordinal, set $C_\xi = \bigcup_{\eta<\xi} C_\eta$; then C_ξ is order-convex, because $\{C_\eta : \eta < \xi\}$ is upwards-directed, and belongs to $\mathcal{A}$, because $\mathcal{A}$ is closed under countable unions, so $C_\xi \in \mathcal{C}$ and the induction proceeds.

Now, however, $\langle \operatorname{int}(C_{\xi+1} \setminus C_\xi) \rangle_{\xi<\omega_1}$ is an uncountable disjoint family of non-empty open sets, and X is not ccc. **XQ**

(iv) Since $C \cup C'$ is order-convex whenever $C, C' \in \mathcal{C}$ and $C \cap C' \neq \emptyset$, $\mathcal{C}^*$ is a disjoint family. Moreover, if $x \in H$, there is some open interval containing x and belonging to $\mathcal{C}$, so $x \in \operatorname{int} C$ for some $C \in \mathcal{C}^*$; this shows that $\mathcal{C}^*$ is an open cover of H. Because X is ccc, $\mathcal{C}^*$ is countable, so $H = \bigcup \mathcal{C}^* \in \mathcal{A}$. Thus there is some countable $\mathcal{G}_0 \subseteq \mathcal{G}$ with union H; as $\mathcal{G}$ is arbitrary, X is hereditarily Lindelöf, by 4A2H(c-i).

(v) By 4A2H(c-ii), X is perfectly normal.

(o)(i) Suppose that f is continuous. If $A \subseteq X$ is a non-empty set with supremum x in X, then $x \in \overline{A}$, by (d), so $f(x) \in \overline{f[A]}$ (3A3Cc) and $f(x)$ is less than or equal to any upper bound of $f[A]$; but $f(x)$ is an upper bound of $f[A]$, because f is order-preserving, so $f(x) = \sup f[A]$. Similarly, $f(\inf A) = \inf f[A]$ whenever A is non-empty and has an infimum, so f is order-continuous.

(ii) Now suppose that f is order-continuous. Take any $y \in Y$ and consider $A = f^{-1}[]-\infty, y[]$, $B = X \setminus A$. If $x \in A$ then $f(x)$ cannot be $\inf f[B]$ so x cannot be $\inf B$ and there is an $x' \in X$ such that $x < x' \leq z$ for every $z \in B$; in which case $x \in]-\infty, x'[\subseteq A$. So A is open. Similarly, $f^{-1}[]y, \infty[]$ is open. By 4A2B(a-ii), f is continuous.

4A2S Order topologies on ordinals (a) Let ζ be an ordinal with its order topology.

(i) ζ is locally compact (4A2Rk); all the sets $[0, \eta] =]-\infty, \eta+1[$, for $\eta < \zeta$, are open and compact (4A2Rh). If ζ is a successor ordinal, it is compact, being of the form $[0, \eta]$ where $\zeta = \eta + 1$.

(ii) For any $A \subseteq \zeta$, $\overline{A} = \{\sup B : \emptyset \neq B \subseteq A, \sup B < \zeta\}$. (4A2Rd, because $\inf B \in B \subseteq A$ for every non-empty $B \subseteq A$.)

(iii) If $\xi \leq \zeta$, then the subspace topology on ξ induced by the order topology of ζ is the order topology of ξ. (4A2Rm.)

(b) Give ω_1 its order topology.

(i) ω_1 is first-countable. **P** If $\xi < \omega_1$ is either zero or a successor ordinal, then $\{\xi\}$ is open so $\{\{\xi\}\}$ is a base of neighbourhoods of ξ. If ξ is a non-zero limit ordinal, there is a sequence $\langle \xi_n \rangle_{n\in\mathbb{N}}$ in ξ with supremum ξ, and $\{]\xi_n, \xi] : n \in \mathbb{N}\}$ is a base of neighbourhoods of ξ. **Q**

(ii) Singleton subsets of ω_1 are zero sets. (Assemble 4A2F(d-v), 4A2Rc and (i) above.)

(iii) If $f : \omega_1 \to \mathbb{R}$ is continuous, there is a $\xi < \omega_1$ such that $f(\eta) = f(\xi)$ for every $\eta \geq \xi$. **P?** Otherwise, we may define a strictly increasing family $\langle \zeta_\xi \rangle_{\xi<\omega_1}$ in ω_1 by saying that $\zeta_0 = 0$,

$$\zeta_{\xi+1} = \min\{\eta : \eta > \zeta_\xi,\ f(\eta) \neq f(\zeta_\xi)\}$$

for every $\xi < \omega_1$,

$$\zeta_\xi = \sup\{\zeta_\eta : \eta < \xi\}$$

for non-zero countable limit ordinals ξ. Now

$$\omega_1 = \bigcup_{k\in\mathbb{N}} \{\xi : |f(\zeta_{\xi+1}) - f(\zeta_\xi)| \geq 2^{-k}\},$$

so there is a $k \in \mathbb{N}$ such that $A = \{\xi : |f(\zeta_{\xi+1}) - f(\zeta_\xi)| \ge 2^{-k}\}$ is infinite. Take a strictly increasing sequence $\langle \xi_n \rangle_{n\in\mathbb{N}}$ in A and set $\xi = \sup_{n\in\mathbb{N}} \xi_n = \sup_{n\in\mathbb{N}}(\xi_n + 1)$. Then $\langle \zeta_{\xi_n} \rangle_{n\in\mathbb{N}}$ and $\langle \zeta_{\xi_n+1} \rangle_{n\in\mathbb{N}}$ are strictly increasing sequences with supremum ζ_ξ, so both converge to ζ_ξ in the order topology of ω_1 (4A2Rf), and

$$f(\zeta_\xi) = \lim_{n\to\infty} f(\zeta_{\xi_n}) = \lim_{n\to\infty} f(\zeta_{\xi_n+1}).$$

But this means that

$$\lim_{n\to\infty} f(\zeta_{\xi_n}) - f(\zeta_{\xi_n+1}) = 0,$$

which is impossible, because $|f(\zeta_{\xi_n}) - f(\zeta_{\xi_n+1})| \ge 2^{-k}$ for every n. **XQ**

4A2T Topologies on spaces of subsets In §§446, 476 and 479 it will be useful to be able to discuss topologies on spaces of closed sets. In fact everything we really need can be expressed in terms of Fell topologies ((a-ii) here), but it may help if I put these in the context of two other constructions, Vietoris topologies and Hausdorff metrics (see (a) and (g) below), which may be more familiar to some readers. Let X be a topological space, and $\mathcal{C} = \mathcal{C}_X$ the family of closed subsets of X.

(a)(i) The **Vietoris topology** on $\mathcal{C}$ is the topology generated by sets of the forms

$$\{F : F \in \mathcal{C},\, F \cap G \ne \emptyset\}, \quad \{F : F \in \mathcal{C},\, F \subseteq G\}$$

where $G \subseteq X$ is open.

(ii) The **Fell topology** on $\mathcal{C}$ is the topology generated by sets of the forms

$$\{F : F \in \mathcal{C},\, F \cap G \ne \emptyset\}, \quad \{F : F \in \mathcal{C},\, F \cap K = \emptyset\}$$

where $G \subseteq X$ is open and $K \subseteq X$ is compact. If X is Hausdorff then the Fell topology is coarser than the Vietoris topology. If X is compact and Hausdorff the two topologies agree.

(iii) Suppose X is metrizable, and that ρ is a metric on X inducing its topology. For a non-empty subset A of X, write $\rho(x, A) = \inf_{y\in A} \rho(x, y)$ for every $x \in X$. Note that $\rho(x, A) \le \rho(x, y) + \rho(y, A)$ for all x, $y \in X$, so that $x \mapsto \rho(x, A) : X \to \mathbb{R}$ is 1-Lipschitz.

For E, $F \in \mathcal{C} \setminus \{\emptyset\}$, set

$$\tilde\rho(E, F) = \min(1, \max(\sup_{x\in E} \rho(x, F), \sup_{y\in F} \rho(y, E))).$$

If E, $F \in \mathcal{C} \setminus \{\emptyset\}$ and $z \in X$, then $\rho(z, F) \le \rho(z, E) + \sup_{x\in E} \rho(x, F)$; from this it is easy to see that $\tilde\rho$ is a metric on $\mathcal{C} \setminus \{\emptyset\}$, the **Hausdorff metric**. Observe that $\tilde\rho(\{x\}, \{y\}) = \min(1, \rho(x, y))$ for all x, $y \in X$.

Remarks The formula I give for $\tilde\rho$ has a somewhat arbitrary feature 'min(1, . . .)'. Any number strictly greater than 0 can be used in place of '1' here. Many authors prefer to limit themselves to the family of non-empty closed sets of finite diameter, rather than the whole of $\mathcal{C} \setminus \{\emptyset\}$; this makes it more more natural to omit the truncation, and work with $(E, F) \mapsto \max(\sup_{x\in E} \rho(x, F), \sup_{y\in F} \rho(y, E))$. All such variations produce uniformly equivalent metrics. A more radical approach redefines 'metric' to allow functions which take the value ∞; but this seems a step too far.

Given that I am truncating my Hausdorff metrics by the value 1, there would be no extra problems if I defined $\tilde\rho(\emptyset, E) = 1$ for every non-empty closed set E; but I think I am following the majority in regarding Hausdorff distance as defined only for non-empty sets.

(b)(i) The Fell topology is T_1. **P** If $F \subseteq X$ is closed and $x \in X$, then $\{E : E \in \mathcal{C},\, E \cap (X \setminus F) = \emptyset\}$ and $\{E : E \in \mathcal{C},\, E \cap \{x\} \ne \emptyset\}$ are complements of open sets, so are closed. Now if $F \in \mathcal{C}$ then

$$\{F\} = \{E : E \subseteq F\} \cap \bigcap_{x\in F} \{E : x \in E\}$$

is closed. **Q**

(ii) The map $(E, F) \mapsto E \cup F : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is continuous for the Fell topology. **P** If $G \subseteq X$ is open and $K \subseteq X$ is compact, then

$$\{(E, F) : (E \cup F) \cap G \ne \emptyset\} = \{(E, F) : E \cap G \ne \emptyset \text{ or } F \cap G \ne \emptyset\},$$

$$\{(E, F) : (E \cup F) \cap K = \emptyset\} = \{E : E \cap K = \emptyset\} \times \{F : F \cap K = \emptyset\}$$

are open in the product topology. So the result follows by 4A2B(a-ii). **Q**

(iii) $\mathcal{C}$ is compact in the Fell topology. **P** Let $\mathfrak{F}$ be an ultrafilter on $\mathcal{C}$. Set $F_0 = \bigcap_{\mathcal{E}\in\mathfrak{F}} \overline{\bigcup\mathcal{E}}$. ($\alpha$) If $G \subseteq X$ is open and $G \cap F_0 \ne \emptyset$, set $\mathcal{E} = \{F : F \in \mathcal{C},\, F \cap G = \emptyset\}$. Then $\overline{\bigcup\mathcal{E}}$ does not meet G, so does not include F_0, and $\mathcal{E} \notin \mathfrak{F}$. Accordingly $\{F : F \cap G \ne \emptyset\} = \mathcal{C} \setminus \mathcal{E}$ belongs to $\mathfrak{F}$. (β) If $K \subseteq X$ is compact and $F_0 \cap K = \emptyset$, then $\{K \cap \overline{\bigcup\mathcal{E}} : \mathcal{E} \in \mathfrak{F}\}$ is a downwards-directed family of relatively closed subsets of K with empty intersection so must contain the empty set, and there is an $\mathcal{E} \in \mathfrak{F}$ such that $K \cap F = \emptyset$ for every $F \in \mathcal{E}$, that is, $\{F : F \cap K = \emptyset\}$ belongs to $\mathfrak{F}$. (γ) By 4A2B(a-iv), $\mathfrak{F} \to F_0$. As $\mathfrak{F}$ is arbitrary, $\mathcal{C}$ is compact. **Q**

(c) If X is Hausdorff, $x \mapsto \{x\}$ is continuous for the Fell topology on $\mathcal{C}$. **P** If $G \subseteq X$ is open, then $\{x : \{x\} \cap G \ne \emptyset\} = G$ is open. If $K \subseteq X$ is compact, then $\{x : \{x\} \cap K = \emptyset\} = X \setminus K$ is open (3A3Dc). **Q**

(d) If X and another topological space Y are regular, and $\mathcal{C}_Y$, $\mathcal{C}_{X\times Y}$ are the families of closed subsets of Y and $X \times Y$ respectively, then $(E,F) \mapsto E \times F : \mathcal{C}_X \times \mathcal{C}_Y \to \mathcal{C}_{X\times Y}$ is continuous when each space is given its Fell topology. **P** (i) Suppose that $W \subseteq X \times Y$ is open, and consider $\mathcal{V}_W = \{(E,F) : E \in \mathcal{C}_X,\, F \in \mathcal{C}_Y,\, (E \times F) \cap W \ne \emptyset\}$. If $(E_0,F_0) \in \mathcal{V}_W$, take $(x_0,y_0) \in (E_0 \times F_0) \cap W$. Let $G \subseteq X$ and $H \subseteq Y$ be open sets such that $(x_0,y_0) \in G \times H \subseteq W$. Then $\{(E,F) : E \in \mathcal{C}_X,\, F \in \mathcal{C}_Y,\, E \cap G \ne \emptyset,\, F \cap H \ne \emptyset\}$ is an open set in $\mathcal{C}_X \times \mathcal{C}_Y$ containing (E_0,F_0) and included in $\mathcal{V}_W$. As (E_0,F_0) is arbitrary, $\mathcal{V}_W$ is open in $\mathcal{C}_X \times \mathcal{C}_Y$. (ii) Suppose that $K \subseteq X \times Y$ is compact, and consider $\mathcal{W}_K = \{(E,F) : E \in \mathcal{C}_X,\, F \in \mathcal{C}_Y,\, (E \times F) \cap K = \emptyset\}$. If $(E_0,F_0) \in \mathcal{W}_K$, then set $K_1 = K \cap (E_0 \times Y)$ and $K_2 = K \cap (X \times F_0)$. These are disjoint closed subsets of K, so there are disjoint open subsets U_1, U_2 of $X \times Y$ including K_1, K_2 respectively (4A2F(h-ii)). Now $K_1' = K \setminus U_1$ and $K_2' = K \setminus U_2$ are compact subsets of K with union K.

Let π_1, π_2 be the projections from $X \times Y$ onto X, Y respectively; then $\{(E,F) : E \in \mathcal{C}_X,\, E \cap \pi_1[K_1'] = F \cap \pi_2[K_2'] = \emptyset\}$ is an open set in $\mathcal{C}_X \times \mathcal{C}_Y$ containing (E_0,F_0) and included in $\mathcal{W}_K$. As (E_0,F_0) is arbitrary, $\mathcal{W}_K$ is open. (iii) Putting these together with 4A2B(a-ii), we see that $(E,F) \mapsto E \times F$ is continuous. **Q**

(e) Suppose that X is locally compact and Hausdorff.

(i) The set $\{(E,F) : E,\, F \in \mathcal{C},\, E \subseteq F\}$ is closed in $\mathcal{C} \times \mathcal{C}$ for the product topology defined from the Fell topology on $\mathcal{C}$. **P** Suppose that E_0, $F_0 \in \mathcal{C}$ and $E_0 \not\subseteq F_0$. Take $x \in E_0 \setminus F_0$. Because X is locally compact and Hausdorff, there is a relatively compact open set G such that $x \in G$ and $\overline{G} \cap F_0 = \emptyset$. Now $\mathcal{V} = \{E : E \cap G \ne \emptyset\}$ and $\mathcal{W} = \{F : F \cap \overline{G} = \emptyset\}$ are open sets in $\mathcal{C}$ containing E_0, F_0 respectively, and $E \not\subseteq F$ for every $E \in \mathcal{V}$ and $F \in \mathcal{W}$. This shows that $\{(E,F) : E \not\subseteq F\}$ is open, so that its complement is closed. **Q**

It follows that $\{(x,F) : x \in F\} = \{(x,F) : \{x\} \subseteq F\}$ is closed in $X \times \mathcal{C}$ when $\mathcal{C}$ is given its Fell topology, since $x \mapsto \{x\}$ is continuous, by (c) above.

(ii) The Fell topology on $\mathcal{C}$ is Hausdorff. **P** The set

$$\{(E,E) : E \in \mathcal{C}\} = \{(E,F) : E \subseteq F \text{ and } F \subseteq E\}$$

is closed in $\mathcal{C} \times \mathcal{C}$, by (i). So 4A2F(a-iii) applies. **Q**

It follows that if $\langle F_i\rangle_{i\in I}$ is a family in $\mathcal{C}$, and $\mathcal{F}$ is an ultrafilter on I, then we have a well-defined limit $\lim_{i\to\mathcal{F}} F_i$ defined in $\mathcal{C}$ for the Fell topology, because $\mathcal{C}$ is compact ((b-iii) above).

(iii) If $\mathcal{L} \subseteq \mathcal{C}$ is compact, then $\bigcup\mathcal{L}$ is a closed subset of X. **P** Take $x \in X \setminus \bigcup\mathcal{L}$. For every $C \in \mathcal{L}$, there is a relatively compact open set G containing x such that $C \cap \overline{G}$ is empty; now finitely many such open sets G must suffice for every $C \in \mathcal{L}$, and the intersection of these G is a neighbourhood of x not meeting $\bigcup\mathcal{L}$. **Q** (Compare 4A2Gm.)

(f) Suppose that X is metrizable, locally compact and separable. Then the Fell topology on $\mathcal{C}$ is metrizable. **P** X is second-countable (4A2P(a-i)); let $\mathcal{U}$ be a countable base for the topology of X consisting of relatively compact open sets (4A2Ob) and closed under finite unions. Let $\mathbb{V}$ be the set of open sets in $\mathcal{C}$ expressible in the form

$$\{F : F \cap U \ne \emptyset \text{ for every } U \in \mathcal{U}_0,\, F \cap \overline{V} = \emptyset\}$$

where $\mathcal{U}_0 \subseteq \mathcal{U}$ is finite and $V \in \mathcal{U}$. Then $\mathbb{V}$ is countable. If $\mathcal{V} \subseteq \mathcal{C}$ is open and $F_0 \in \mathcal{V}$, then there are a finite family $\mathcal{G}$ of open sets in X and a compact $K \subseteq X$ such that

$$F_0 \in \{F : F \cap G \ne \emptyset \text{ for every } G \in \mathcal{G},\, F \cap K = \emptyset\} \subseteq \mathcal{V}.$$

For each $G \in \mathcal{G}$ there is a $U_G \in \mathcal{U}$ such that $U_G \subseteq G$ and $F \cap U_G \neq \emptyset$. Next, each point of K belongs to a member of $\mathcal{U}$ with closure disjoint from F_0, so (because K is compact and $\mathcal{U}$ is closed under finite unions) there is a $V \in \mathcal{U}$ such that $K \subseteq V$ and $F_0 \cap \overline{V} = \emptyset$. Now

$$\mathcal{V}' = \{F : F \cap U_G \neq \emptyset \text{ for every } G \in \mathcal{G},\ F \cap \overline{V} = \emptyset\}$$

belongs to $\mathbb{V}$, contains F_0 and is included in $\mathcal{V}$. This shows that $\mathbb{V}$ is a base for the Fell topology, and the Fell topology is second-countable. Since we already know that it is compact and Hausdorff, therefore regular, it is metrizable (4A2Pb). **Q**

(g) Suppose that X is metrizable, and that ρ is a metric inducing the topology of X; let $\tilde{\rho}$ be the corresponding Hausdorff metric on $\mathcal{C} \setminus \{\emptyset\}$.

(i) The topology $\mathfrak{S}_{\tilde{\rho}}$ defined by $\tilde{\rho}$ is finer than the Fell topology $\mathfrak{S}_F$ on $\mathcal{C} \setminus \{\emptyset\}$. **P** Let $G \subseteq X$ be open, and consider the set $\mathcal{V}_G = \{F : F \in \mathcal{C},\ F \cap G \neq \emptyset\}$. If $E \in \mathcal{V}_G$, take $x \in E \cap G$ and $\epsilon > 0$ such that $U(x, \epsilon) \subseteq G$; then $\{F : \tilde{\rho}(F, E) < \epsilon\} \subseteq \mathcal{V}_G$. As E is arbitrary, $\mathcal{V}_G$ is $\mathfrak{S}_{\tilde{\rho}}$-open. Next, suppose that $K \subseteq X$ is compact, and consider the set $\mathcal{W}_K = \{F : F \in \mathcal{C} \setminus \{\emptyset\},\ F \cap K = \emptyset\}$. If $E \in \mathcal{W}_K$, the function $x \mapsto \rho(x, E) : K \to]0, \infty[$ is continuous, so has a non-zero lower bound ϵ say; now $\{F : \tilde{\rho}(F, E) < \epsilon\} \subseteq \mathcal{W}_K$. As E is arbitrary, $\mathcal{W}_K$ is $\mathfrak{S}_{\tilde{\rho}}$-open. So $\mathfrak{S}_{\tilde{\rho}}$ is finer than the topology $\mathfrak{S}_F$ generated by the sets $\mathcal{V}_G$ and $\mathcal{W}_K$. **Q**

(ii) If X is compact, then $\mathfrak{S}_{\tilde{\rho}}$ and $\mathfrak{S}_F$ are the same, and both are compact. **P** Suppose that $E \in \mathcal{V} \in \mathfrak{S}_{\tilde{\rho}}$. Let $\epsilon \in]0, 1[$ be such that $F \in \mathcal{V}$ whenever $F \in \mathcal{C} \setminus \{\emptyset\}$ and $\tilde{\rho}(E, F) < 2\epsilon$. Because X is compact, E is ρ-totally bounded (4A2Je) and there is a finite set $I \subseteq E$ such that $E \subseteq \bigcup_{x \in I} U(x, \epsilon)$. Because $x \mapsto \rho(x, E)$ is continuous, $K = \{x : \rho(x, E) \geq \epsilon\}$ is closed, therefore compact; now

$$\mathcal{W} = \{F : F \in \mathcal{C},\ F \cap K = \emptyset,\ F \cap U(x, \epsilon) \neq \emptyset \text{ for every } x \in I\}$$

is a neighbourhood of E for $\mathfrak{S}_F$ included in $\mathcal{V}$. Thus $\mathcal{V}$ is a neighbourhood of E for $\mathfrak{S}_F$; as E and $\mathcal{V}$ are arbitrary, $\mathfrak{S}_F$ is finer than $\mathfrak{S}_{\tilde{\rho}}$. So the two topologies are equal.

Observe finally that $\{\emptyset\} = \{F : F \in \mathcal{C},\ F \cap X = \emptyset\}$ is open for the Fell topology on $\mathcal{C}$, so $\mathcal{C} \setminus \{\emptyset\}$ is closed, therefore compact, by (b-iii). So $\mathfrak{S}_{\tilde{\rho}} = \mathfrak{S}_F$ is compact. **Q**

4A2U Old friends (a) $\mathbb{R}$, with its usual topology, is metrizable (2A3Ff) and separable (the countable set $\mathbb{Q}$ is dense), so is second-countable (4A2P(a-i)). Every subset of $\mathbb{R}$ is separable (4A2P(a-iv)); in particular, every dense subset of $\mathbb{R}$ has a countable subset which is still dense.

(b) $\mathbb{N}^{\mathbb{N}}$ is Polish in its usual topology (4A2Qc), so has a countable network (4A2P(a-iii) or 4A2Ne), and is hereditarily Lindelöf (4A2Nb or 4A2Pd). Moreover, it is homeomorphic to $[0,1] \setminus \mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ (Kuratowski 66, §36.II; Kechris 95, 7.7).

(c) The map $x \mapsto \frac{2}{3} \sum_{j=0}^{\infty} 3^{-j} x(j)$ (cf. 134Gb) is a homeomorphism between $\{0,1\}^{\mathbb{N}}$ and the Cantor set $C \subseteq [0,1]$. (It is a continuous bijection.)

(d) If I is any set, then the map $A \mapsto \chi A : \mathcal{P}I \to \{0,1\}^I$ is a homeomorphism (for the usual topologies on $\mathcal{P}I$ and $\{0,1\}^I$, as described in 4A2A and 3A3K). So $\mathcal{P}I$ is zero-dimensional, compact (3A3K) and Hausdorff. If I is countable, then $\mathcal{P}I$ is metrizable, therefore Polish (4A2Qb).

(e) Give the space $C([0, \infty[)$ the topology $\mathfrak{T}_c$ of uniform convergence on compact sets.

(i) $C([0, \infty[)$ is a Polish locally convex linear topological space. **P** $\mathfrak{T}_c$ is determined by the seminorms $f \mapsto \sup_{t \leq n} |f(t)|$ for $n \in \mathbb{N}$, so it is a metrizable linear space topology. By 4A2Oe, it has a countable network, so is separable. Any function which is continuous on every set $[0, n]$ is continuous on $[0, \infty[$, so $C([0, \infty[)$ is complete under the metric $(f, g) \mapsto \sum_{n=0}^{\infty} \min(2^{-n}, \sup_{t \leq n} |f(t) - g(t)|)$; as this metric defines $\mathfrak{T}_c$, $\mathfrak{T}_c$ is Polish. **Q**

(ii) Suppose that $A \subseteq C([0, \infty[)$ is such that $\{f(0) : f \in A\}$ is bounded and for every $a \geq 0$ and $\epsilon > 0$ there is a $\delta > 0$ such that $|f(s) - f(t)| \leq \epsilon$ whenever $f \in A$, $s, t \in [0, a]$ and $|s - t| \leq \delta$. Then A is relatively compact for $\mathfrak{T}_c$. **P** Note first that $\{f(a) : f \in A\}$ is bounded for every $a \geq 0$, since if $\delta > 0$ is such that $|f(s) - f(t)| \leq 1$ whenever $f \in A$, $s, t \in [0, a]$ and $|s - t| \leq \delta$, then $|f(a)| \leq |f(0)| + \lceil \frac{a}{\delta} \rceil$ for every $f \in A$.

So if $\mathcal{F}$ is an ultrafilter on $C([0,\infty[)$ containing A, $g(a)=\lim_{f\to\mathcal{F}} f(a)$ is defined for every $a\ge 0$. If $a\ge 0$ and $\epsilon>0$, let $\delta\in\,]0,1]$ be such that $|f(s)-f(t)|\le\epsilon$ whenever $f\in A$, s, $t\in[0,a+1]$ and $|s-t|\le\delta$; then $|g(s)-g(a)|\le\epsilon$ whenever $|s-a|\le\delta$; as a and ϵ are arbitrary, $g\in C([0,\infty[)$. If $a\ge 0$ and $\epsilon>0$, let $\delta>0$ be such that $|f(s)-f(t)|\le\frac{1}{3}\epsilon$ whenever $f\in A$, s, $t\in[0,a]$ and $|s-t|\le\delta$. Then

$$A'=\{f:f\in A,\,|f(i\delta)-g(i\delta)|\le\frac{1}{3}\epsilon\text{ for every }i\le\lceil\frac{a}{\delta}\rceil\}$$

belongs to $\mathcal{F}$. Now $|f(t)-g(t)|\le\epsilon$ for every $f\in A'$ and $t\in[0,a]$. As a and ϵ are arbitrary, $\mathcal{F}\to g$ for $\mathfrak{T}_c$. As $\mathcal{F}$ is arbitrary, A is relatively compact (3A3De). **Q**

Version of 7.1.17

4A3 Topological σ-algebras

I devote a section to some σ-algebras which can be defined on topological spaces. While 'measures' will not be mentioned here, the manipulation of these σ-algebras is an essential part of the technique of measure theory, and I will give proofs and exercises as if this were part of the main work. I look at Borel σ-algebras (4A3A-4A3J), Baire σ-algebras (4A3K-4A3P), spaces of càdlàg functions (4A3Q), Baire-property algebras (4A3R, 4A3S) and cylindrical σ-algebras on linear spaces (4A3U-4A3W).

4A3A Borel sets If $(X,\mathfrak{T})$ is a topological space, the **Borel σ-algebra** of X is the σ-algebra $\mathcal{B}(X)$ of subsets of X generated by $\mathfrak{T}$. Its elements are the **Borel sets** of X. If $(Y,\mathfrak{S})$ is another topological space with Borel σ-algebra $\mathcal{B}(Y)$, a function $f:X\to Y$ is **Borel measurable** if $f^{-1}[H]\in\mathcal{B}(X)$ for every $H\in\mathfrak{S}$, and is a **Borel isomorphism** if it is a bijection and $\mathcal{B}(Y)=\{F:F\subseteq Y,\,f^{-1}[F]\in\mathcal{B}(X)\}$, that is, f is an isomorphism between the structures $(X,\mathcal{B}(X))$ and $(Y,\mathcal{B}(Y))$.

4A3B (Σ,T)-measurable functions It is time I put the following idea into bold type.

(a) Let X and Y be sets, with σ-algebras $\Sigma\subseteq\mathcal{P}X$ and $\mathrm{T}\subseteq\mathcal{P}Y$. A function $f:X\to Y$ is **(Σ,T)-measurable** if $f^{-1}[F]\in\Sigma$ for every $F\in\mathrm{T}$.

(b) If Σ, T and Υ are σ-algebras of subsets of X, Y and Z respectively, and $f:X\to Y$ is (Σ,T)-measurable while $g:Y\to Z$ is (T,Υ)-measurable, then $gf:X\to Z$ is (Σ,Υ)-measurable. (If $H\in\Upsilon$, $g^{-1}[H]\in\mathrm{T}$ so $(gf)^{-1}[H]=f^{-1}[g^{-1}[H]]\in\Sigma$.)

(c) Let $\langle X_i\rangle_{i\in I}$ be a family of sets with product X, Y another set, and $f:X\to Y$ a function. If $\mathrm{T}\subseteq\mathcal{P}Y$, $\Sigma_i\subseteq\mathcal{P}X_i$ are σ-algebras for each i, then f is $(\mathrm{T},\widehat{\bigotimes}_{i\in I}\Sigma_i)$-measurable iff $\pi_i f:Y\to X_i$ is (T,Σ_i)-measurable for every i, where $\pi_i:X\to X_i$ is the coordinate map. **P** π_i is $(\widehat{\bigotimes}_{j\in I}\Sigma_j,\Sigma_i)$-measurable, so if f is $(\mathrm{T},\widehat{\bigotimes}_{j\in I}\Sigma_j)$-measurable then $\pi_i f$ must be (T,Σ_i)-measurable. In the other direction, if every $\pi_i f$ is measurable, then $\{H:H\subseteq X,\,f^{-1}[E]\in\mathrm{T}\}$ is a σ-algebra of subsets of X containing $\pi_i^{-1}[E]$ whenever $i\in I$ and $E\in\Sigma_i$, so includes $\widehat{\bigotimes}_{i\in I}\Sigma_i$, and f is measurable. **Q**

4A3C Elementary facts (a) If X is a topological space and Y is a subspace of X, then $\mathcal{B}(Y)$ is just the subspace σ-algebra $\{E\cap Y:E\in\mathcal{B}(X)\}$. **P** $\{E:E\subseteq X,\,E\cap Y\in\mathcal{B}(Y)\}$ and $\{E\cap Y:E\in\mathcal{B}(X)\}$ are σ-algebras containing all open sets, so include $\mathcal{B}(X)$, $\mathcal{B}(Y)$ respectively. **Q**

(b) If X is a set, Σ is a σ-algebra of subsets of X, $(Y,\mathfrak{S})$ is a topological space and $f:X\to Y$ is a function, then f is $(\Sigma,\mathcal{B}(Y))$-measurable iff $f^{-1}[H]\in\Sigma$ for every $H\in\mathfrak{S}$. **P** If f is $(\Sigma,\mathcal{B}(Y))$-measurable then $f^{-1}[H]\in\Sigma$ for every $H\in\mathfrak{S}$ just because $\mathfrak{S}\subseteq\mathcal{B}(Y)$. If $f^{-1}[H]\in\Sigma$ for every $H\in\mathfrak{S}$, then $\{F:F\subseteq Y,\,f^{-1}[F]\in\Sigma\}$ is a σ-algebra of subsets of Y (111Xc; cf. 234C) including $\mathfrak{S}$, so includes $\mathcal{B}(Y)$, and f is $(\Sigma,\mathcal{B}(Y))$-measurable. **Q**

(c) If X and Y are topological spaces, and $f : X \to Y$ is a function, then f is Borel measurable iff it is $(\mathcal{B}(X), \mathcal{B}(Y))$-measurable. (Apply (b) with $\Sigma = \mathcal{B}(X)$.) So if X, Y and Z are topological spaces and $f : X \to Y$, $g : Y \to Z$ are Borel measurable functions, then $gf : X \to Z$ is Borel measurable. (Use 4A3Bb.)

(d) If X and Y are topological spaces and $f : X \to Y$ is continuous, it is Borel measurable. (Immediate from the definitions in 4A3A.)

(e) If X is a topological space and $f : X \to [-\infty, \infty]$ is lower semi-continuous, then it is Borel measurable. (The inverse image of a half-open interval $]\alpha, \beta]$ is a difference of open sets, so is a Borel set, and every open subset of $[-\infty, \infty]$ is a countable union of such half-open intervals.)

(f) If $\langle X_i \rangle_{i \in I}$ is a family of topological spaces with product X, then $\mathcal{B}(X) \supseteq \widehat{\bigotimes}_{i \in I} \mathcal{B}(X_i)$. (Put (d) and 4A3Bc together.)

(g) Let X be a topological space.

(i) The algebra $\mathfrak{A}$ of subsets generated by the open sets is precisely the family of sets expressible as a disjoint union $\bigcup_{i \le n} G_i \cap F_i$ where every G_i is open and every F_i is closed. **P** Write $\mathcal{A}$ for the family of sets expressible in this form. Of course $\mathcal{A} \subseteq \mathfrak{A}$. In the other direction, observe that

$X \in \mathcal{A}$,

if E, $E' \in \mathcal{A}$ then $E \cap E' \in \mathcal{A}$,

if $E \in \mathcal{A}$ then $X \setminus E \in \mathcal{A}$

because if G_i is open and F_i is closed for $i \le n$, then (identifying $\{0, \dots, n\}$ with $n + 1$)

$$X \setminus \bigcup_{i \le n} (G_i \cap F_i) = \bigcap_{i \le n} (X \setminus G_i) \cup (G_i \setminus F_i)$$
$$= \bigcup_{I \subseteq n+1} \bigl(\bigcap_{i \in I} (G_i \setminus F_i) \cap \bigcap_{i \in (n+1) \setminus I} (X \setminus G_i)\bigr)$$

belongs to $\mathcal{A}$. So $\mathcal{A}$ is an algebra of sets and must be equal to $\mathfrak{A}$. **Q**

(ii) $\mathcal{B}(X)$ is the smallest family $\mathcal{E} \supseteq \mathfrak{A}$ such that $\bigcup_{n \in \mathbb{N}} E_n \in \mathcal{E}$ for every non-decreasing sequence $\langle E_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{E}$ and $\bigcap_{n \in \mathbb{N}} E_n \in \mathcal{E}$ for every non-increasing sequence $\langle E_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{E}$. (136G.)

4A3D Hereditarily Lindelöf spaces (a) Suppose that X is a hereditarily Lindelöf space and $\mathcal{U}$ is a subbase for the topology of X. Then $\mathcal{B}(X)$ is the σ-algebra of subsets of X generated by $\mathcal{U}$. **P** Write Σ for the σ-algebra generated by $\mathcal{U}$. Of course $\Sigma \subseteq \mathcal{B}(X)$ just because every member of $\mathcal{U}$ is open. In the other direction, set

$$\mathcal{V} = \{X\} \cup \{U_0 \cap U_1 \cap \ldots \cap U_n : U_0, \dots, U_n \in \mathcal{U}\};$$

then $\mathcal{V} \subseteq \Sigma$ and $\mathcal{V}$ is a base for the topology of X (4A2B(a-i)). If $G \subseteq X$ is open, set $\mathcal{V}_1 = \{V : V \in \mathcal{V}, V \subseteq G\}$; then $G = \bigcup \mathcal{V}_1$. Because X is hereditarily Lindelöf, there is a countable set $\mathcal{V}_0 \subseteq \mathcal{V}_1$ such that $G = \bigcup \mathcal{V}_0$ (4A2H(c-i)), so that $G \in \Sigma$. Thus every open set belongs to Σ and $\mathcal{B}(X) \subseteq \Sigma$. **Q**

(b) Let X be a set, Σ a σ-algebra of subsets of X, Y a hereditarily Lindelöf space, $\mathcal{U}$ a subbase for the topology of Y, and $f : X \to Y$ a function. If $f^{-1}[U] \in \Sigma$ for every $U \in \mathcal{U}$, then f is $(\Sigma, \mathcal{B}(Y))$-measurable. **P** $\{F : F \subseteq Y, f^{-1}[F] \in \Sigma\}$ is a σ-algebra of subsets of Y including $\mathcal{U}$, so contains every open set, by (a), and therefore every Borel set, as in 4A3Cb. **Q**

(c) Let $\langle X_i \rangle_{i \in I}$ be a family of topological spaces with product X. Suppose that X is hereditarily Lindelöf.

(i) $\mathcal{B}(X) = \widehat{\bigotimes}_{i \in I} \mathcal{B}(X_i)$. **P** By 4A3Cf, $\mathcal{B}(X) \supseteq \widehat{\bigotimes}_{i \in I} \mathcal{B}(X_i)$. On the other hand, $\widehat{\bigotimes}_{i \in I} \mathcal{B}(X_i)$ is a σ-algebra including

$$\mathcal{U} = \{\pi_i^{-1}[G] : i \in I, G \subseteq X_i \text{ is open}\},$$

where $\pi_i(x) = x(i)$ for $i \in I$ and $x \in X$; since $\mathcal{U}$ is a subbase for the topology of X, (a) tells us that $\widehat{\bigotimes}_{i \in I} \mathcal{B}(X_i)$ includes $\mathcal{B}(X)$. **Q**

(ii) If Y is another topological space, then a function $f : Y \to X$ is Borel measurable iff $\pi_i f : Y \to X_i$ is Borel measurable for every $i \in I$, where $\pi_i : X \to X_i$ is the canonical map. (Use 4A3Bc.)

4A3E Applications Recall that any topological space with a countable network (in particular, any second-countable space, any separable metrizable space) is hereditarily Lindelöf (4A2Nb, 4A2Oc, 4A2P(a-iii)); and so is any ccc totally ordered space (4A2Rn). So 4A3Da-4A3Db will be applicable to these. As for product spaces, the product of a countable family of spaces with countable networks again has a countable network (4A2Ne), so for such spaces we shall be able to use 4A3Dc. For instance, $\mathcal{B}(\{0,1\}^{\mathbb{N}})$ is the σ-algebra of subsets of $\{0,1\}^{\mathbb{N}}$ generated by the sets $\{x : x(n) = 1\}$ for $n \in \mathbb{N}$.

4A3F Spaces with countable networks (a) Let X be a topological space with a countable network. Then $\#(\mathcal{B}(X)) \le \mathfrak{c}$. **P** Let $\mathcal{E}$ be a countable network for the topology of X and Σ the σ-algebra of subsets of X generated by $\mathcal{E}$. Then $\#(\Sigma) \le \mathfrak{c}$ (4A1O). If $G \subseteq X$ is open, there is a subset $\mathcal{E}'$ of $\mathcal{E}$ such that $\bigcup \mathcal{E}' = G$; but $\mathcal{E}'$ is necessarily countable, so $G \in \Sigma$. It follows that $\mathcal{B}(X) \subseteq \Sigma$ and $\#(\mathcal{B}(X)) \le \mathfrak{c}$. **Q**

(b) $\#(\mathcal{B}(\mathbb{N}^{\mathbb{N}})) = \mathfrak{c}$. **P** $\mathbb{N}^{\mathbb{N}}$ has a countable network (4A2Ub), so $\#(\mathcal{B}(\mathbb{N}^{\mathbb{N}})) \le \mathfrak{c}$. On the other hand, $\mathcal{B}(\mathbb{N}^{\mathbb{N}})$ contains all singletons, so

$$\#(\mathcal{B}(\mathbb{N}^{\mathbb{N}})) \ge \#(\mathbb{N}^{\mathbb{N}}) \ge \#(\{0,1\}^{\mathbb{N}}) = \mathfrak{c}. \text{ } \mathbf{Q}$$

4A3G Second-countable spaces (a) Suppose that X is a second-countable space and Y is any topological space. Then $\mathcal{B}(X \times Y) = \mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$. **P** By 4A3Cf, $\mathcal{B}(X \times Y) \supseteq \mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$. On the other hand, let $\mathcal{U}$ be a countable base for the topology of X. If $W \subseteq X \times Y$ is open, set

$$V_U = \bigcup\{H : H \subseteq Y \text{ is open, } U \times H \subseteq W\}$$

for $U \in \mathcal{U}$. Then $W = \bigcup_{U \in \mathcal{U}} U \times V_U$ belongs to $\mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$. As W is arbitrary, $\mathcal{B}(X \times Y) \subseteq \mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$. **Q**

(b) If X is any topological space, Y is a T_0 second-countable space, and $f : X \to Y$ is Borel measurable, then (the graph of) f is a Borel set in $X \times Y$. **P** Let $\mathcal{U}$ be a countable base for the topology of Y. Because Y is T_0,

$$f = \bigcap_{U \in \mathcal{U}}(\{(x,y) : x \in f^{-1}[U],\, y \in U\} \cup \{(x,y) : x \in X \setminus f^{-1}[U],\, y \in Y \setminus U\})$$

which is a Borel subset of $X \times Y$ by 4A3Cc and 4A3Cf. **Q**

4A3H Borel sets in Polish spaces: Proposition Let $(X,\mathfrak{T})$ be a Polish space and $E \subseteq X$ a Borel set. Then there is a Polish topology $\mathfrak{S}$ on X, including $\mathfrak{T}$, for which E is open.

proof Let $\mathcal{E}$ be the union of all the Polish topologies on X including $\mathfrak{T}$. Of course $X \in \mathcal{E}$. If $E \in \mathcal{E}$ then $X \setminus E \in \mathcal{E}$. **P** There is a Polish topology $\mathfrak{S} \supseteq \mathfrak{T}$ such that $E \in \mathfrak{S}$. As both E and $X \setminus E$ are Polish in the subspace topologies $\mathfrak{S}_E$, $\mathfrak{S}_{X\setminus E}$ induced by $\mathfrak{S}$ (4A2Qd), the disjoint union topology $\mathfrak{S}'$ of $\mathfrak{S}_E$ and $\mathfrak{S}_{X\setminus E}$ is also Polish (4A2Qe). Now $\mathfrak{S}' \supseteq \mathfrak{S} \supseteq \mathfrak{T}$ and $X \setminus E \in \mathfrak{S}'$, so $X \setminus E \in \mathcal{E}$. **Q** Moreover, the union of any sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$ belongs to $\mathcal{E}$. **P** For each $n \in \mathbb{N}$ let $\mathfrak{S}_n \supseteq \mathfrak{T}$ be a Polish topology containing E_n. If m, $n \in \mathbb{N}$ then $\mathfrak{S}_m \cap \mathfrak{S}_n$ includes $\mathfrak{T}$, so is Hausdorff. By 4A2Qf, the topology $\mathfrak{S}$ generated by $\bigcup_{n\in\mathbb{N}} \mathfrak{S}_n$ is Polish. Of course $\mathfrak{S} \supseteq \mathfrak{T}$, and $\bigcup_{n\in\mathbb{N}} E_n \in \mathfrak{S}$, so $\bigcup_{n\in\mathbb{N}} E_n \in \mathcal{E}$. **Q**

Thus $\mathcal{E}$ is a σ-algebra. Since it surely includes $\mathfrak{T}$, it includes $\mathcal{B}(X,\mathfrak{T})$, as claimed.

4A3I Corollary If $(X,\mathfrak{T})$ is a Polish space and $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence of Borel subsets of X, then there is a zero-dimensional Polish topology $\mathfrak{S}$ on X, including $\mathfrak{T}$, for which every E_n is open-and-closed.

proof (a) By 4A3H, we can find for each $n \in \mathbb{N}$ a Polish topology $\mathfrak{T}_n \supseteq \mathfrak{T}$ containing E_k (if $n = 2k$ is even) or $X \setminus E_k$ (if $n = 2k+1$ is odd); now the topology $\mathfrak{T}'$ generated by $\bigcup_{n\in\mathbb{N}} \mathfrak{T}_n$ is Polish, by 4A2Qf, and every E_n is open-and-closed for $\mathfrak{T}'$.

(b) Now choose $\langle \mathcal{V}_n\rangle_{n\in\mathbb{N}}$ and $\langle \mathfrak{T}'_n\rangle_{n\in\mathbb{N}}$ inductively such that

$\mathfrak{T}'_0 = \mathfrak{T}'$,

given that $\mathfrak{T}'_n$ is a Polish topology on X, then $\mathcal{V}_n$ is a countable base for $\mathfrak{T}'_n$ and $\mathfrak{T}'_{n+1}$ is a Polish topology on X including $\mathfrak{T}'_n \cup \{X \setminus V : V \in \mathcal{V}_n\}$

(using (a) for the inductive step). Now the topology $\mathfrak{S}$ generated by $\bigcup_{n\in\mathbb{N}} \mathfrak{T}'_n$ is Polish, includes $\mathfrak{T}$, contains every E_n and its complement and has a base $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ consisting of open-and-closed sets.

4A3J Borel sets in ω_1: Proposition A set $E \subseteq \omega_1$ is a Borel set iff either E or its complement includes a closed cofinal set.

proof (a) Let Σ be the family of all those sets $E \subseteq \omega_1$ such that either E or $\omega_1 \setminus E$ includes a closed cofinal set. Of course Σ is closed under complements. Because the intersection of a sequence of closed cofinal sets is a closed cofinal set (4A1Bd), the union of any sequence in Σ belongs to Σ; so Σ is a σ-algebra. If E is closed, then either it is cofinal with ω_1, and is a closed cofinal set, or there is a $\xi < \omega_1$ such that $E \subseteq \xi$, in which case $\omega_1 \setminus E$ includes the closed cofinal set $\omega_1 \setminus \xi$; in either case, $E \in \Sigma$. Thus every open set belongs to Σ, and Σ includes $\mathcal{B}(\omega_1)$.

(b) Now suppose that $E \subseteq \omega_1$ is such that there is a closed cofinal set $F \subseteq \omega_1 \setminus E$. For each $\xi < \omega_1$ let $f_\xi : \xi \to \mathbb{N}$ be an injective function. Define $g : E \to \mathbb{N}$ by setting $g(\eta) = f_{\alpha(\eta)}(\eta)$, where $\alpha(\eta) = \min(F \setminus \eta)$ for $\eta \in E$. Set $A_n = g^{-1}[\{n\}]$ for $n \in \mathbb{N}$, so that $E = \bigcup_{n\in\mathbb{N}} A_n$. If $n \in \mathbb{N}$, $\xi \in \overline{A}_n \setminus A_n$ and $\xi' < \xi$, there must be η, $\eta' \in A_n$ such that $\xi' \le \eta < \eta' < \xi$. But now, because $f_{\alpha(\eta')}$ is injective, while $g(\eta) = g(\eta') = n$, $\alpha(\eta) \ne \alpha(\eta')$, so $\alpha(\eta) \in F \cap]\xi', \xi[$. As ξ' is arbitrary, $\xi \in \overline{F} = F$. This shows that $\overline{A}_n \subseteq A_n \cup F$ and $A_n = \overline{A}_n \setminus F$ is a Borel set. This is true for every $n \in \mathbb{N}$, so $E = \bigcup_{n\in\mathbb{N}} A_n$ is a Borel set.

(c) If $E \subseteq \omega_1$ includes a closed cofinal set, then (b) tells us that $\omega_1 \setminus E$ and E are Borel sets. Thus $\Sigma \subseteq \mathcal{B}(\omega_1)$ and $\Sigma = \mathcal{B}(\omega_1)$, as claimed.

4A3K Baire sets When we come to study measures in terms of the integrals of continuous functions (§436), we find that it is sometimes inconvenient or even impossible to apply them to arbitrary Borel sets, and we need to use a smaller σ-algebra, as follows.

(a) Definition Let X be a topological space. The **Baire σ-algebra** $\mathcal{B}\mathfrak{a}(X)$ of X is the σ-algebra generated by the zero sets. Members of $\mathcal{B}\mathfrak{a}(X)$ are called **Baire** sets. (**Warning!** Do not confuse 'Baire sets' in this sense with 'sets with the Baire property' in the sense of 4A3R, nor with 'sets which are Baire spaces in their subspace topologies'.)

(b) For any topological space X, $\mathcal{B}\mathfrak{a}(X) \subseteq \mathcal{B}(X)$ (because every zero set is closed, therefore Borel). If $\mathfrak{T}$ is perfectly normal – for instance, if it is metrizable (4A2Lc), or is regular and hereditarily Lindelöf (4A2H(c-ii)) – then $\mathcal{B}\mathfrak{a}(X) = \mathcal{B}(X)$ (because every closed set is a zero set, by 4A2Fi, so every open set belongs to $\mathcal{B}\mathfrak{a}(X)$).

(c) Let X and Y be topological spaces, with Baire σ-algebras $\mathcal{B}\mathfrak{a}(X)$, $\mathcal{B}\mathfrak{a}(Y)$ respectively. If $f : X \to Y$ is continuous, it is $(\mathcal{B}\mathfrak{a}(X), \mathcal{B}\mathfrak{a}(Y))$-measurable. **P** Let T be the σ-algebra $\{F : F \subseteq Y,\ f^{-1}[F] \in \mathcal{B}\mathfrak{a}(X)\}$. If $g : Y \to \mathbb{R}$ is continuous, then $gf : X \to \mathbb{R}$ is continuous, so

$$f^{-1}[\{y : g(y) = 0\}] = \{x : gf(x) = 0\} \in \mathcal{B}\mathfrak{a}(X),$$

and $\{y : g(y) = 0\} \in \mathrm{T}$. Thus every zero set belongs to T, and $\mathrm{T} \supseteq \mathcal{B}\mathfrak{a}(Y)$. **Q**

(d) In particular, if X is a subspace of Y, then $E \cap X \in \mathcal{B}\mathfrak{a}(X)$ whenever $E \in \mathcal{B}\mathfrak{a}(Y)$. More fundamentally, $F \cap X$ is a zero set in X for every zero set $F \subseteq Y$, just because $g \restriction X$ is continuous for any continuous $g : Y \to \mathbb{R}$.

(e) If X is a topological space and Y is a separable metrizable space, a function $f : X \to Y$ is **Baire measurable** if $f^{-1}[H] \in \mathcal{B}\mathfrak{a}(X)$ for every open $H \subseteq Y$. Observe that f is Baire measurable in this sense iff it is $(\mathcal{B}\mathfrak{a}(X), \mathcal{B}(Y))$-measurable iff it is $(\mathcal{B}\mathfrak{a}(X), \mathcal{B}\mathfrak{a}(Y))$-measurable.

4A3L Lemma Let $(X, \mathfrak{T})$ be a topological space. Then $\mathcal{B}\mathfrak{a}(X)$ is just the smallest σ-algebra of subsets of X with respect to which every continuous real-valued function on X is measurable.

proof (a) Let $f : X \to \mathbb{R}$ be a continuous function and $\alpha \in \mathbb{R}$. Set $g(x) = \max(0, f(x) - \alpha)$ for $x \in X$; then g is continuous, so

$$\{x : f(x) \le \alpha\} = \{x : g(x) = 0\}$$

is a zero set and belongs to $\mathcal{B}\mathfrak{a}(X)$. As α is arbitrary, f is $\mathcal{B}\mathfrak{a}(X)$-measurable.

(b) On the other hand, if Σ is any σ-algebra of subsets of X such that every continuous real-valued function on X is Σ-measurable, and $F \subseteq X$ is a zero set, then there is a continuous g such that $F = g^{-1}[\{0\}]$, so that $F \in \Sigma$; as F is arbitrary, $\Sigma \supseteq \mathcal{B}\mathfrak{a}(X)$.

4A3M Product spaces Let $\langle X_i \rangle_{i \in I}$ be a family of topological spaces with product X.

(a) $\mathcal{B}\mathfrak{a}(X) \supseteq \widehat{\bigotimes}_{i \in I} \mathcal{B}\mathfrak{a}(X_i)$. (Apply 4A3Bc to the identity map from X to itself; compare 4A3Cf.)

(b) Suppose that X is ccc. Then every Baire subset of X is determined by coordinates in a countable set. **P** By 254Mb, the family $\mathcal{W}$ of sets determined by coordinates in countable sets is a σ-algebra of subsets of X. By 4A2E(b-ii), every continuous real-valued function is $\mathcal{W}$-measurable, so $\mathcal{W}$ contains every zero set and every Baire set. **Q**

4A3N Products of separable metrizable spaces: Proposition Let $\langle X_i \rangle_{i \in I}$ be a family of separable metrizable spaces, with product X.

(a) $\mathcal{B}\mathfrak{a}(X) = \widehat{\bigotimes}_{i \in I} \mathcal{B}\mathfrak{a}(X_i) = \widehat{\bigotimes}_{i \in I} \mathcal{B}(X_i)$.

(b) $\mathcal{B}\mathfrak{a}(X)$ is the family of those Borel subsets of X which are determined by coordinates in countable sets.

(c) A set $Z \subseteq X$ is a zero set iff it is closed and determined by coordinates in a countable set.

(d) If Y is a dense subset of X, then the Baire σ-algebra $\mathcal{B}\mathfrak{a}(Y)$ of Y is just the subspace σ-algebra $\mathcal{B}\mathfrak{a}(X)_Y$ induced by $\mathcal{B}\mathfrak{a}(X)$.

(e) If Y is a set, T is a σ-algebra of subsets of Y, and $f : Y \to X$ is a function, then f is $(\mathrm{T}, \mathcal{B}\mathfrak{a}(X))$-measurable iff $\pi_i f : Y \to X_i$ is $(\mathrm{T}, \mathcal{B}(X_i))$-measurable for every $i \in I$, where $\pi_i(x) = x(i)$ for $x \in X$ and $i \in I$.

proof (a) X is ccc (4A2E(a-iii)), so if $f : X \to \mathbb{R}$ is continuous, there are a countable set $J \subseteq I$ and a continuous function $g : X_J \to \mathbb{R}$ such that $f = g\tilde{\pi}_J$, where $X_J = \prod_{i \in J} X_i$ and $\tilde{\pi}_J : X \to X_J$ is the canonical map (4A2E(b-ii)). Now X_J is separable and metrizable (4A2P(a-v)), therefore hereditarily Lindelöf (4A2P(a-iii)), so $\mathcal{B}(X_J) = \widehat{\bigotimes}_{i \in J} \mathcal{B}(X_i)$, by 4A3D(c-i). By 4A3Bc, $\tilde{\pi}_J$ is $(\widehat{\bigotimes}_{i \in I} \mathcal{B}(X_i), \widehat{\bigotimes}_{j \in J} \mathcal{B}(X_j))$-measurable, so f is $\widehat{\bigotimes}_{i \in I} \mathcal{B}(X_i)$-measurable. As f is arbitrary, $\mathcal{B}\mathfrak{a}(X) \subseteq \widehat{\bigotimes}_{i \in I} \mathcal{B}(X_i)$ (4A3L). Also $\mathcal{B}(X_i) = \mathcal{B}\mathfrak{a}(X_i)$ for every i (4A3Kb), so $\widehat{\bigotimes}_{i \in I} \mathcal{B}(X_i) = \widehat{\bigotimes}_{i \in I} \mathcal{B}\mathfrak{a}(X_i)$. With 4A3Ma, this proves the result.

(b)(i) If $W \in \mathcal{B}\mathfrak{a}(X)$ it is certainly a Borel set (4A3Kb), and by 4A3Mb it is determined by coordinates in a countable set.

(ii) If W is a Borel subset of X determined by coordinates in a countable subset J of X, then write $X_J = \prod_{i \in J} X_i$ and $X_{I \setminus J} = \prod_{i \in I \setminus J} X_i$; let $\tilde{\pi}_J : X \to X_J$ be the canonical map. We can identify W with $W' \times X_{I \setminus J}$, where W' is some subset of X_J. Now if $z \in X_{I \setminus J}$, $W' = \{w : w \in X_J, (w, z) \in W\}$ is a Borel subset of X_J, because $w \mapsto (w, z) : X_J \to X$ is continuous. (I am passing over the trivial case $X = \emptyset$.) Since X_J is metrizable (4A2P(a-v) again), $W' \in \mathcal{B}\mathfrak{a}(X_J)$ (4A3Kb) and $W = \tilde{\pi}_J^{-1}[W']$ is a Baire set (4A3Kc).

(c)(i) If Z is a zero set, it is surely closed; and it is determined by coordinates in a countable set by (b) above, or directly from 4A2E(b-ii) again.

(ii) If Z is closed and determined by coordinates in a countable set J, then (in the language of (b)) it can be identified with $Z' \times X_{I \setminus J}$ for some $Z' \subseteq X_J$. As in the proof of (b), Z' is closed (at least, if $X_{I \setminus J} \neq \emptyset$), so is a zero set (4A2Lc), and $Z = \tilde{\pi}_J^{-1}[Z']$ is a zero set (4A2C(b-iv)).

(d)(i) $\mathcal{B}\mathfrak{a}(Y) \supseteq \mathcal{B}\mathfrak{a}(X)_Y$ by 4A3Kd.

(ii) Let $f : Y \to \mathbb{R}$ be any continuous function. For each $n \in \mathbb{N}$, there is an open set $G_n \subseteq X$ such that $G_n \cap Y = \{y : f(y) > 2^{-n}\}$. Now $\overline{G}_n$ is determined by coordinates in a countable set (4A2E(b-i)), so is a zero set, by (c) here. Because Y is dense in X, $\overline{G}_n = \overline{G_n \cap Y}$ does not meet $\{y : f(y) = 0\}$, and $\{y : f(y) > 0\} = Y \cap \bigcup_{n \in \mathbb{N}} \overline{G}_n$ belongs to $\mathcal{B}\mathfrak{a}(X)_Y$. Thus $\mathcal{B}\mathfrak{a}(X)_Y$ contains every cozero subset of Y and includes $\mathcal{B}\mathfrak{a}(Y)$.

(e) Put (a) and 4A3Bc together.

4A3O Compact spaces (a) Let $(X,\mathfrak{T})$ be a topological space, $\mathcal{U}$ a subbase for $\mathfrak{T}$, and $\mathfrak{A}$ the algebra of subsets of X generated by $\mathcal{U}$. If $H \subseteq X$ is open and $K \subseteq H$ is compact, there is an open $E \in \mathfrak{A}$ such that $K \subseteq E \subseteq H$. **P** Set $\mathcal{V} = \{X\} \cup \{U_0 \cap U_1 \cap \ldots \cap U_n : U_0, \ldots, U_n \in \mathcal{U}\}$, so that $\mathcal{V}$ is a base for $\mathfrak{T}$ and $\mathcal{V} \subseteq \mathfrak{A}$. $\{U : U \in \mathcal{V}, U \subseteq H\}$ is an open cover of the compact set K, so there is a finite set $\mathcal{U}_0 \subseteq \mathcal{V}$ such that $E = \bigcup \mathcal{U}_0$ includes K and is included in H; now $E \in \mathfrak{A}$. **Q**

(b) Let $(X,\mathfrak{T})$ be a compact space and $\mathcal{U}$ a subbase for $\mathfrak{T}$. Then every open-and-closed subset of X belongs to the algebra of subsets of X generated by $\mathcal{U}$. (If $F \subseteq X$ is open-and-closed, it is also compact; apply (a) here with $K = H = F$.)

(c) Let $(X,\mathfrak{T})$ be a compact space and $\mathcal{U}$ a subbase for $\mathfrak{T}$. Then $\mathcal{B}\mathfrak{a}(X)$ is included in the σ-algebra of subsets of X generated by $\mathcal{U}$. **P** Let Σ be the σ-algebra generated by $\mathcal{U}$. If $Z \subseteq X$ is a zero set, there is a sequence $\langle H_n\rangle_{n\in\mathbb{N}}$ of open sets with intersection Z (4A2C(b-vi)); now we can find a sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in Σ such that $Z \subseteq E_n \subseteq H_n$ for every n, by (a), so that $Z = \bigcap_{n\in\mathbb{N}} E_n \in \Sigma$. This shows that every zero set belongs to Σ, so Σ must include $\mathcal{B}\mathfrak{a}(X)$. **Q**

(d) In a compact Hausdorff zero-dimensional space the Baire σ-algebra is the σ-algebra generated by the open-and-closed sets. (Apply (c) with $\mathcal{U}$ the family of open-and-closed sets.)

(e) Let $\langle X_i\rangle_{i\in I}$ be a family of compact Hausdorff spaces with product X. Then $\mathcal{B}\mathfrak{a}(X) = \widehat{\bigotimes}_{i\in I}\mathcal{B}\mathfrak{a}(X_i)$. **P** By 4A3Ma, $\mathcal{B}\mathfrak{a}(X) \supseteq \widehat{\bigotimes}_{i\in I}\mathcal{B}\mathfrak{a}(X_i)$. On the other hand, let $\mathcal{U}_i$ be the family of cozero sets in X_i for each i. Because X_i is completely regular (3A3Bb), $\mathcal{U}_i$ is a base for its topology (4A2Fc). Set

$$\mathcal{W} = \{\textstyle\prod_{i\in I} U_i : U_i \in \mathcal{U}_i \text{ for every } i \in I,\ \{i : U_i \neq X_i\} \text{ is finite}\},$$

so that $\mathcal{W} \subseteq \widehat{\bigotimes}_{i\in I}\mathcal{B}\mathfrak{a}(X_i)$ is a base for the topology of X. By (c) above, $\mathcal{B}\mathfrak{a}(X)$ is the σ-algebra generated by $\mathcal{W}$, and $\mathcal{B}\mathfrak{a}(X) \subseteq \widehat{\bigotimes}_{i\in I}\mathcal{B}\mathfrak{a}(X_i)$. **Q**

(f) In particular, for any set I, $\mathcal{B}\mathfrak{a}(\{0,1\}^I)$ is the σ-algebra generated by sets of the form $\{x : x(i) = 1\}$ as i runs over I.

4A3P Proposition The Baire σ-algebra $\mathcal{B}\mathfrak{a}(\omega_1)$ of ω_1 is just the countable-cocountable algebra (211R).

proof We see from 4A2S(b-iii) that every continuous function is measurable with respect to the countable-cocountable algebra, so $\mathcal{B}\mathfrak{a}(\omega_1)$ is included in the countable-cocountable algebra. On the other hand,

$$[0,\xi] = \{\eta : \eta \le \xi\} = [0,\xi+1[\ = \omega_1 \setminus\ ]\xi,\omega_1[$$

is an open-and-closed set (4A2S(a-i)), therefore a zero set, therefore belongs to $\mathcal{B}\mathfrak{a}(\omega_1)$, for every $\xi < \omega_1$. Now if $\xi < \omega_1$, it is itself a countable set, so

$$[0,\xi[\ = \textstyle\bigcup_{\eta<\xi}[0,\eta] \in \mathcal{B}\mathfrak{a}(\omega_1), \quad \{\xi\} = [0,\xi] \setminus [0,\xi[\ \in \mathcal{B}\mathfrak{a}(\omega_1).$$

It follows that every countable set belongs to $\mathcal{B}\mathfrak{a}(\omega_1)$ and the countable-cocountable algebra is included in $\mathcal{B}\mathfrak{a}(\omega_1)$.

4A3Q Càdlàg functions Let X be a Polish space, and C_{dlg} the set of càdlàg functions (definition: 4A2A) from $[0,\infty[$ to X, with its topology of pointwise convergence inherited from $X^{[0,\infty[}$.

(a) $\mathcal{B}\mathfrak{a}(C_{\text{dlg}})$ is the subspace σ-algebra induced by $\mathcal{B}\mathfrak{a}(X^{[0,\infty[})$.

(b) $(C_{\text{dlg}}, \mathcal{B}\mathfrak{a}(C_{\text{dlg}}))$ is a standard Borel space.

(c)(i) For any $t \ge 0$, let $\mathcal{B}\mathfrak{a}_t(C_{\text{dlg}})$ be the σ-algebra of subsets of C_{dlg} generated by the functions $\omega \mapsto \omega(s)$ for $s \le t$. Then $(\omega,s) \mapsto \omega(s) : C_{\text{dlg}} \times [0,t] \to X$ is $\mathcal{B}\mathfrak{a}_t(C_{\text{dlg}})\widehat{\otimes}\mathcal{B}([0,t])$-measurable.

(ii) $(\omega,t) \mapsto \omega(t) : C_{\text{dlg}} \times [0,\infty[\ \to X$ is $\mathcal{B}\mathfrak{a}(C_{\text{dlg}})\widehat{\otimes}\mathcal{B}([0,\infty[)$-measurable.

(d) The set $C([0,\infty[\,;X)$ of continuous functions from $[0,\infty[$ to X belongs to $\mathcal{B}\mathfrak{a}(C_{\text{dlg}})$.

proof (a) Use 4A3Nd.

(b)(i) Fix a complete metric ρ on X defining its topology. For $A \subseteq B \subseteq \mathbb{R}$, $f \in X^B$ and $\epsilon > 0$, set

$$\mathrm{jump}_A(f,\epsilon) = \sup\{n : \text{ there is an } I \in [A]^n \text{ such that } \rho(f(s),f(t)) > \epsilon \text{ whenever } s < t \text{ are successive elements of } I\}$$

(see 438P). Set $D = \mathbb{Q} \cap [0,\infty[$. Then any set of the form $\{f : \mathrm{jump}_A(f,\epsilon) > m\}$ is open, so

$$E = \bigcap_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \{f : f \in X^D,\ \mathrm{jump}_{D\cap[0,n]}(f, 2^{-n}) \le m\} \cap \bigcap_{\substack{q\in D\\ n\in\mathbb{N}}} \bigcup_{m\ge n} \{f : f \in X^D,\ \rho(f(q+2^{-m}), f(q)) \le 2^{-n}\}$$

is Borel in the Polish space X^D.

(ii) If $\omega \in C_{\mathrm{dlg}}$, then $\mathrm{jump}_{[0,n]}(\omega, 2^{-n})$ is finite for every $n \in \mathbb{N}$. (Apply 438Pa to an extension of ω to a member of $X^{\mathbb{R}}$ which is constant on $]-\infty,0[$; or make the trifling required changes to the argument of 438Pa.) Since $\lim_{n\to\infty} \omega(q+2^{-n}) = \omega(q)$ for every $q \in D$, $\omega\restriction D \in E$.

(iii) Conversely, given $f \in E$, $\mathrm{jump}_{[0,n]}(f,\epsilon)$ is finite for every $n \in \mathbb{N}$ and $\epsilon > 0$, so $\lim_{q\in D, q\downarrow t} f(q)$ is defined in X for every $t \ge 0$, and $\lim_{q\in D, q\uparrow t} f(q)$ is defined in X for every $t > 0$. (Apply the argument of (a-ii) of the proof of 438P.) Set $\omega_f(t) = \lim_{q\in D, q\downarrow t} f(q)$ for $t \ge 0$. Because $f(q)$ is a cluster point of $\langle f(q+2^{-n})\rangle_{n\in\mathbb{N}}$ for every $q \in D$, ω_f extends f. It is easy to see that $\mathrm{jump}_{[0,n[}(\omega_f,\epsilon) = \mathrm{jump}_{[0,n[}(f,\epsilon)$ is finite for every $n \in \mathbb{N}$ and $\epsilon > 0$, so $\lim_{s\downarrow t} \omega_f(s)$ is defined for every $t \ge 0$ and $\lim_{s\uparrow t} \omega_f(s)$ is defined for every $t > 0$. Also, for $t \ge 0$,

$$\lim_{s\downarrow t} \omega_f(s) = \lim_{q\in D, q\downarrow t} \omega_f(q) = \lim_{q\in D, q\downarrow t} f(q) = \omega_f(t).$$

Thus $\omega_f \in C_{\mathrm{dlg}}$. Clearly a member of C_{dlg} is uniquely determined by its values on D, so $f \mapsto \omega_f$ and $\omega \mapsto \omega\restriction D$ are the two halves of a bijection between E and C_{dlg}.

(iv) By 424G, E, with its Borel (or Baire) σ-algebra $\mathcal{B}(E)$, is a standard Borel space. Of course the map $\omega \mapsto \omega\restriction D$ is $(\mathcal{B}\mathfrak{a}(C_{\mathrm{dlg}}), \mathcal{B}(E))$-measurable. But also the map $f \mapsto \omega_f(t) : E \to X$ is $\mathcal{B}(E)$-measurable for every $t \ge 0$. **P** If $\langle q_n\rangle_{n\in\mathbb{N}}$ is a sequence in D decreasing to t, $\omega_f(t) = \lim_{n\to\infty} f(q_n)$ for every $f \in E$, and we can use 418Ba. **Q** Since $\mathcal{B}\mathfrak{a}(C_{\mathrm{dlg}})$ is the σ-algebra induced by $\mathcal{B}\mathfrak{a}(X^{[0,\infty[})$ (4A3Nd), and $\mathcal{B}\mathfrak{a}(X^{[0,\infty[}) = \widehat{\bigotimes}_{[0,\infty[}\mathcal{B}(X)$ (4A3Na), this is enough to show that $f \mapsto \omega_f$ is $(\mathcal{B}(E), \mathcal{B}\mathfrak{a}(C_{\mathrm{dlg}}))$-measurable.

Thus $(C_{\mathrm{dlg}}, \mathcal{B}\mathfrak{a}(C_{\mathrm{dlg}})) \cong (E, \mathcal{B}(E))$ is a standard Borel space.

(c)(i) For $n \in \mathbb{N}$, $\omega \in C_{\mathrm{dlg}}$ and $s \in [0,t]$, set $h_n(\omega,s) = \omega(\min(t, 2^{-n}i))$ if $i \in \mathbb{N}$ and $2^{-n}(i-1) < s \le 2^{-n}i$. If $G \subseteq X$ is open then

$$\{(\omega,s) : s \le t,\ h_n(\omega,s) \in G\} = \bigcup_{i\in\mathbb{N}} \{\omega : \omega(\min(t,2^{-n}i)) \in G\} \times ([0,t] \cap \,]2^{-n}(i-1), 2^{-n}i]) \in \mathcal{B}\mathfrak{a}_t(C_{\mathrm{dlg}})\widehat{\otimes}\mathcal{B}([0,t]),$$

so h_n is $\mathcal{B}\mathfrak{a}_t(C_{\mathrm{dlg}})\widehat{\otimes}\mathcal{B}([0,t])$-measurable. Now $\omega(s) = \lim_{n\to\infty} h_n(\omega,s)$ for every $\omega \in C_{\mathrm{dlg}}$ and $s \in [0,t]$, so $(\omega,s) \mapsto \omega(s)$ is measurable in the same sense, by 418Ba again.

(ii) If $G \subseteq X$ is open then

$$\{(\omega,t) : t \ge 0,\ \omega(t) \in G\} = \bigcup_{n\in\mathbb{N}} \{(\omega,t) : t \in [0,n],\ \omega(t) \in G\} \in \mathcal{B}\mathfrak{a}(C_{\mathrm{dlg}})\widehat{\otimes}\mathcal{B}([0,\infty[).$$

(d)

$$C([0,\infty[\,;X) = \bigcap_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \{\omega : \omega \in C_{\mathrm{dlg}},\ \rho(\omega(q),\omega(q')) \le 2^{-n} \text{ whenever } q,\ q' \in \mathbb{Q}\cap[0,n] \text{ and } |q-q'| \le 2^{-m}\}.$$

4A3R Baire property Let X be a topological space, and $\mathcal{M}$ the ideal of meager subsets of X. A subset X has the **Baire property** if it is expressible in the form $G\triangle M$ where $G \subseteq X$ is open and $M \in \mathcal{M}$; that is, $A \subseteq X$ has the Baire property if there is an open set $G \subseteq X$ such that $G\triangle A$ is meager. (For $A = G\triangle M$ iff $M = G\triangle A$.) The family $\widehat{\mathcal{B}}(X)$ of all such sets is the **Baire-property algebra** of X. (See 4A3S.) (**Warning!** do not confuse the 'Baire-property algebra' $\widehat{\mathcal{B}}$ with the 'Baire σ-algebra' $\mathcal{B}\mathfrak{a}$ as defined in 4A3K.) The quotient algebra $\widehat{\mathcal{B}}(X)/\mathcal{M}$ is the **category algebra** of X.

4A3S Proposition Let X be a topological space.

(a) Let $A \subseteq X$ be any set.

(i) There is a largest open set $G \subseteq X$ such that $A \cap G$ is meager.

(ii) $H = X \setminus \overline{G}$ is the smallest regular open subset of X such that $A \setminus H$ is meager; $H \subseteq \overline{A}$.

(iii) H is in itself a Baire space.

(iv) If $A \in \widehat{\mathcal{B}}(X)$, $H\triangle A$ is meager.

(v) If X is a Baire space and $A \in \widehat{\mathcal{B}}(X)$, then H is the largest open subset of X such that $H \setminus A$ is meager.

(b)(i) $\widehat{\mathcal{B}}(X)$ is a σ-algebra of subsets of X including $\mathcal{B}(X)$.

(ii) $\widehat{\mathcal{B}}(X) = \{G\triangle M : G \subseteq X \text{ is a regular open set}, M \in \mathcal{M}\}$.

(c) If X has a countable network, its category algebra has a countable order-dense set (definition: 313J).

proof (a) (See Kechris 95, 8.29.)

(i) Set $\mathcal{U} = \{U : U \subseteq X \text{ is open}, A \cap U \text{ is meager}\}$. Let $\mathcal{U}_0 \subseteq \bigcup\mathcal{U}$ be a maximal disjoint set, and $G_0 = \bigcup\mathcal{U}_0$. Then $A \cap G_0$ is meager. **P** For each $U \in \mathcal{U}_0$, let $\langle F_{Un}\rangle_{n\in\mathbb{N}}$ be a sequence of nowhere dense closed sets covering $A \cap U$. Set $A_n = \bigcup_{U\in\mathcal{U}_0} F_{Un}$. If $V \subseteq X$ is any non-empty open set, either $V \cap A_n$ is empty or there is a $U \in \mathcal{U}_0$ such that $V \cap U \neq \emptyset$, in which case $V \cap U \setminus F_{Un}$ is a non-empty open subset of U not meeting A_n. Thus A_n is nowhere dense. This is true for every n, so $A \cap G_0 \subseteq \bigcup_{n\in\mathbb{N}} A_n$ is meager. **Q**

If $U \in \mathcal{U}$, then $V = U \setminus \overline{G}_0$ belongs to $\mathcal{U}$ and cannot meet any member of $\mathcal{U}_0$, so must be empty. Thus $U \subseteq \overline{G}_0$; as U is arbitrary, $G = \bigcup\mathcal{U}$ is included in $\overline{G}_0$ and $G \setminus G_0$ is nowhere dense. It follows that $A \cap G$ is meager, and G is the largest open set for which this is true.

(ii) $H = X \setminus \overline{G} = \operatorname{int}(X \setminus G)$ is now a regular open set (314O), and $A \setminus H = (A \cap G) \cup (\overline{G} \setminus G)$ is meager. If H' is another regular open set such that $A \setminus H'$ is meager, then $H \setminus \overline{H'}$ is open and meets A in a meager set, so is included in G and must be empty. So $H \subseteq \operatorname{int}\overline{H'} = H'$. Thus H is the smallest regular open set such that $A \setminus H$ is meager. Of course $X \setminus \overline{A}$ belongs to $\mathcal{U}$ so is included in G and does not meet H, that is, $H \subseteq \overline{A}$.

(iii) If now $\langle H_n\rangle_{n\in\mathbb{N}}$ is a sequence of open subsets of H which are dense in H, and $H' \subseteq H$ is any non-empty open set, $H' \setminus H_n$ is nowhere dense for every n, so $H' \setminus \bigcap_{n\in\mathbb{N}} H_n$ is meager. On the other hand, $A \cap H'$ is non-meager so H' also is, and H' must meet $\bigcap_{n\in\mathbb{N}} H_n$. As H' is arbitrary, $\bigcap_{n\in\mathbb{N}} H_n$ is dense in H; as $\langle H_n\rangle_{n\in\mathbb{N}}$ is arbitrary, H is a Baire space in its subspace topology.

(iv) If A has the Baire property, there is an open set U such that $A\triangle U$ is meager. In this case, $A \cap H \setminus \overline{U}$ must be meager, so $H \subseteq \overline{U}$ and $H \setminus A \subseteq (U \setminus A) \cup (\overline{U} \setminus G)$ is meager and $H\triangle A = (A \setminus H) \cup (H \setminus A)$ is meager.

(v) By (iv), $H \setminus A$ is meager. If $U \subseteq X$ is open and $U \setminus A$ is meager, set $V = U \setminus \overline{H}$. Then $V \setminus A$ and $V \cap A$ are both meager, so V is meager, and must be empty, since X is a Baire space. Thus $U \subseteq \overline{H}$ and $G \subseteq H$, because H is a regular open set.

(b)(i) (See Čech 66, §22C; Kuratowski 66, §11.III; Kechris 95, 8.22.) Of course $X = X\triangle\emptyset$ belongs to $\widehat{\mathcal{B}}(X)$. If $E \in \widehat{\mathcal{B}}(X)$, let $G \subseteq X$ be an open set such that $E\triangle G$ is meager. Then

$$(X \setminus \overline{G})\triangle(X \setminus E) \subseteq (\overline{G} \setminus G) \cup (G\triangle E)$$

is meager, so $X \setminus E \in \widehat{\mathcal{B}}(X)$. If $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\widehat{\mathcal{B}}(X)$, then for each $n \in \mathbb{N}$ we can find an open set G_n such that $G_n\triangle E_n$ is meager, and now

$$(\bigcup_{n\in\mathbb{N}} G_n)\triangle(\bigcup_{n\in\mathbb{N}} E_n) \subseteq \bigcup_{n\in\mathbb{N}} G_n\triangle E_n$$

is meager, so $\bigcup_{n\in\mathbb{N}} E_n \in \widehat{\mathcal{B}}(X)$.

This shows that $\widehat{\mathcal{B}}(X)$ is a σ-algebra of subsets of X. Since it contains every open set, it must include the Borel σ-algebra.

(ii) Of course $G\triangle M \in \widehat{B}(X)$ whenever G is a regular open set and M is meager. In the other direction, given a set A with the Baire property, let G_0 be an open set such that $G_0\triangle A$ is meager. Then $G = \operatorname{int}\overline{G}_0$ is a regular open set and $G\triangle G_0$ is nowhere dense, so $M = G\triangle A$ is meager, while $A = G\triangle M$.

(c) Suppose that X has a countable network $\mathcal{A}$. For $A \in \mathcal{A}$, set $d_A = \overline{A}^\bullet$, the equivalence class of $\overline{A} \in \widehat{\mathcal{B}}$ in $\widehat{\mathcal{B}}/\mathcal{M}$. If $b \in \widehat{\mathcal{B}}/\mathcal{M}$ is non-zero, take $E \in \widehat{\mathcal{B}}$ such that $E^\bullet = b$, and an open set $G \subseteq X$ such that $E\triangle G \in \mathcal{M}$. Then G is not meager. Set $\mathcal{A}_1 = \{A : A \in \mathcal{A},\ A \subseteq G\}$; then $\mathcal{A}_1$ is countable and $G = \bigcup \mathcal{A}_1$, so there is a non-meager $A \in \mathcal{A}_1$. In this case $\overline{A}$ is not meager, so $d_A \neq 0$, while $\overline{A} \setminus G \subseteq \overline{G} \setminus G$ is nowhere dense, so $d_A \subseteq b$. As b is arbitrary, $\{d_A : A \in \mathcal{A}\}$ is order-dense in $\widehat{\mathcal{B}}/\mathcal{M}$, and is countable because $\mathcal{A}$ is.

***4A3T** The following result does not mention any topology, but its principal applications are with $\mathcal{J}$ an ideal of meager sets, so I slip it in here. It will be useful in §424 and in Volume 5.

Lemma Let X and Y be sets, Σ a σ-algebra of subsets of X, T a σ-algebra of subsets of Y and $\mathcal{J}$ a σ-ideal of T. Suppose that the quotient Boolean algebra $\mathrm{T}/\mathcal{J}$ has a countable order-dense set.

(a) $\{x : x \in X,\ W[\{x\}] \cap A \in \mathcal{J}\}$ belongs to Σ for any $W \in \Sigma\widehat{\otimes}\mathrm{T}$ and $A \subseteq Y$.

(b) For every $W \in \Sigma\widehat{\otimes}\mathrm{T}$ there are sequences $\langle E_n\rangle_{n\in\mathbb{N}}$ in Σ, $\langle V_n\rangle_{n\in\mathbb{N}}$ in T such that $(W\triangle W_1)[\{x\}] \in \mathcal{J}$ for every $x \in X$, where $W_1 = \bigcup_{n\in\mathbb{N}} E_n \times V_n$.

proof (Compare Kechris 95, 16.1.)

(a) Let $D \subseteq \mathrm{T}/\mathcal{J}$ be a countable order-dense set. Let $\mathcal{V} \subseteq \mathrm{T}$ be a countable set such that $D = \{V^\bullet : V \in \mathcal{V}\}$. Let Λ be the family of subsets W of $X \times Y$ such that

$$W[\{x\}] \in \mathrm{T} \text{ for every } x \in X,$$

$$\{x : x \in X,\ W[\{x\}] \cap A \in \mathcal{J}\} \in \Sigma \text{ for every } A \subseteq Y.$$

(i) Of course $\emptyset \in \Lambda$, because if $W = \emptyset$ then $W[\{x\}] = \emptyset$ for every $x \in X$, and $\{x : W[\{x\}]\cap A \in \mathcal{J}\} = X$ for every $A \subseteq Y$.

(ii) Suppose that $W \in \Lambda$, and set $W' = (X \times Y) \setminus W$. Then

$$W'[\{x\}] = Y \setminus W[\{x\}] \in \mathrm{T}$$

for every $x \in X$. Now suppose that $A \subseteq Y$. Set $\mathcal{V}^* = \{V : V \in \mathcal{V},\ A \cap V \notin \mathcal{J}\}$,

$$E = \{x : W'[\{x\}] \cap A \in \mathcal{J}\},$$

$$E' = \{x : V \cap W[\{x\}] \notin \mathcal{J} \text{ for every } V \in \mathcal{V}^*\}.$$

Then $E = E'$. **P** (α) If $x \in E$ and $V \in \mathcal{V}^*$, then $A\cap W'[\{x\}] = A\setminus W[\{x\}]$ belongs to $\mathcal{J}$, so $V\cap A\setminus W[\{x\}] \in \mathcal{J}$ and $V \cap W[\{x\}] \supseteq V \cap A \cap W[\{x\}]$ is not in $\mathcal{J}$. As V is arbitrary, $x \in E'$; as x is arbitrary, $E \subseteq E'$. (β) If $x \notin E$, then $W'[\{x\}] \cap A \notin \mathcal{J}$. Set $\mathcal{V}_1 = \{V : V \in \mathcal{V},\ V \setminus W'[\{x\}] \in \mathcal{J}\}$, $D_1 = \{V^\bullet : V \in \mathcal{V}_1\}$. Then $D_1 = \{d : d \in D,\ d \subseteq W'[\{x\}]^\bullet\}$, so $W'[\{x\}]^\bullet = \sup D_1$ in $\mathrm{T}/\mathcal{J}$ (313K). Because $\mathcal{J}$ is a σ-ideal, the map $F \mapsto F^\bullet : \mathrm{T} \to \mathrm{T}/\mathcal{J}$ is sequentially order-continuous (313Qb), and $(\bigcup \mathcal{V}_1)^\bullet = \sup D_1$ (313Lc), that is, $W'[\{x\}]\triangle \bigcup \mathcal{V}_1 \in \mathcal{J}$. There must therefore be a $V \in \mathcal{V}_1$ such that $V \cap A \notin \mathcal{J}$, that is, $V \in \mathcal{V}^*$. At the same time, $V \cap W[\{x\}] = V \setminus W'[\{x\}]$ belongs to $\mathcal{J}$, so V witnesses that $x \notin E'$. As x is arbitrary, $E' \subseteq E$. **Q**

Since $W \in \Lambda$,

$$E = E' = X \setminus \bigcup_{V\in\mathcal{V}^*} \{x : V \cap W[\{x\}] \in \mathcal{J}\}$$

belongs to T. As A is arbitrary, $W' \in \Lambda$. Thus the complement of any member of Λ belongs to Λ.

(iii) If $\langle W_n\rangle_{n\in\mathbb{N}}$ is any sequence in Λ with union W, then

$$W[\{x\}] = \bigcup_{n\in\mathbb{N}} W_n[\{x\}] \in \mathrm{T}$$

for every x, while

$$\{x : W[\{x\}] \cap A \in \mathcal{J}\} = \bigcap_{n\in\mathbb{N}}\{x : W_n[\{x\}] \cap A \in \mathcal{J}\} \in \Sigma$$

for every $A \subseteq Y$. So $W \in \Lambda$.

(iv) What this shows is that Λ is a σ-algebra of subsets of $X \times Y$. But if $W = E \times F$, where $E \in \Sigma$ and $F \in \mathrm{T}$, then

$$W[\{x\}] \in \{\emptyset, F\} \subseteq \mathrm{T} \text{ for every } x \in X,$$

$$\{x : W[\{x\}] \cap A \in \mathcal{J}\} \in \{X \setminus E, X\} \subseteq \Sigma$$

for every $A \subseteq Y$; so $W \in \Lambda$. Accordingly Λ must include $\Sigma\widehat{\otimes}\mathrm{T}$, as claimed.

(b) Let $\langle V_n\rangle_{n\in\mathbb{N}}$ be a sequence running over $\mathcal{V} \cup \{\emptyset\}$. For each $n \in \mathbb{N}$, set

$$E_n = \{x : V_n \setminus W[\{x\}] \in \mathcal{J}\};$$

by (a), $E_n \in \Sigma$. Set $W_1 = \bigcup_{n\in\mathbb{N}} E_n \times V_n$. Take any $x \in X$. Then $W_1[\{x\}] = \bigcup_{n\in I} V_n$ where $I = \{n : x \in E_n\}$. Since $V_n \setminus W[\{x\}] \in \mathcal{J}$ for every $n \in I$, $W_1[\{x\}] \setminus W[\{x\}] \in \mathcal{J}$. **?** If $W[\{x\}] \setminus W_1[\{x\}] \notin \mathcal{J}$, there is a $d \in D$ such that $0 \ne d \subseteq W[\{x\}]^\bullet \setminus W_1[\{x\}]^\bullet$. Let $n \in \mathbb{N}$ be such that $V_n^\bullet = d$; then $V_n \setminus W[\{x\}] \in \mathcal{J}$, so $n \in I$ and $V_n \subseteq W_1[\{x\}]$ and $d \subseteq W_1[\{x\}]^\bullet$, which is impossible. **X**

As x is arbitrary, W_1 has the required properties.

4A3U Cylindrical σ-algebras I offer a note on a particular type of Baire σ-algebra.

Definition Let X be a linear topological space. Then the **cylindrical σ-algebra** of X is the smallest σ-algebra Σ of subsets of X such that every continuous linear functional on X is Σ-measurable.

4A3V Proposition Let X be a linear topological space and $\mathfrak{T}_s = \mathfrak{T}_s(X, X^*)$ its weak topology. Then the cylindrical σ-algebra of X is just the Baire σ-algebra of $(X, \mathfrak{T}_s)$.

proof (a) Let $\langle f_i\rangle_{i\in I}$ be a Hamel basis for X^* (4A4Ab). For $x \in X$, set $Tx = \langle f_i(x)\rangle_{i\in I}$; then $T : X \to \mathbb{R}^I$ is a linear operator. Now $Y = T[X]$ is dense in $\mathbb{R}^I$. **P** Y is a linear subspace of $\mathbb{R}^I$, so its closure $\overline{Y}$ also is (2A5Ec). If $\phi \in (\mathbb{R}^I)^*$ is such that $\phi(Tx) = 0$ for every $x \in X$, there are a finite set $J \subseteq I$ and a family $\langle \alpha_i\rangle_{i\in J} \in \mathbb{R}^J$ such that $\phi(y) = \sum_{i\in J} \alpha_i y(i)$ for every $y \in \mathbb{R}^I$ (4A4Be). In this case, $\sum_{i\in J} \alpha_i f_i(x) = \phi(Tx) = 0$ for every $x \in X$; but $\langle f_i\rangle_{i\in I}$ is linearly independent, so $\alpha_i = 0$ for every $i \in J$ and $\phi = 0$. By 4A4Eb, $\overline{Y}$ must be the whole of $\mathbb{R}^I$. **Q**

(b)(i) Set $\mathfrak{T}'_s = \{T^{-1}[H] : H \subseteq Y \text{ is open}\}$. Then $\mathfrak{T}'_s = \mathfrak{T}_s$. **P** Because every f_i is continuous, T is continuous, so $\mathfrak{T}'_s \subseteq \mathfrak{T}_s$. On the other hand, any $f \in X^*$ is a linear combination of the f_i, so is $\mathfrak{T}'_s$-continuous, and $\mathfrak{T}_s \subseteq \mathfrak{T}'_s$. **Q**

(ii) If $g : X \to \mathbb{R}$ is $\mathfrak{T}_s$-continuous, there is a continuous $g_1 : Y \to \mathbb{R}$ such that $g = g_1T$. **P** If x, $x' \in X$ are such that $Tx = Tx'$, then every $\mathfrak{T}_s$-open set containing one must contain the other, so $g(x) = g(x')$. This means that there is a function $g_1 : Y \to \mathbb{R}$ such that $g = g_1T$. Next, if $U \subseteq \mathbb{R}$ is open, $T^{-1}[g_1^{-1}[U]] = g^{-1}[U]$ belongs to $\mathfrak{T}_s = \mathfrak{T}'_s$, so $g_1^{-1}[U]$ must be open in Y; as U is arbitrary, g_1 is continuous. **Q**

(c) Now let Σ be the cylindrical σ-algebra of X. Then every $f_i : X \to \mathbb{R}$ is Σ-measurable, so $T : X \to \mathbb{R}^I$ is $(\Sigma, \mathcal{B}\mathfrak{a}(\mathbb{R}^I))$-measurable, by 4A3Ne. Because Y is dense in $\mathbb{R}^I$, $\mathcal{B}\mathfrak{a}(Y)$ is the subspace σ-algebra induced by $\mathcal{B}\mathfrak{a}(\mathbb{R}^I)$ (4A3Nd). So $T : X \to Y$ is $(\Sigma, \mathcal{B}\mathfrak{a}(Y))$-measurable. Now if $g : X \to \mathbb{R}$ is a continuous function, there is a continuous function $g_1 : Y \to \mathbb{R}$ such that $g = g_1T$; g_1 is $\mathcal{B}\mathfrak{a}(Y)$-measurable, so g is Σ-measurable. As this is true for every $\mathfrak{T}_s$-continuous $g : X \to \mathbb{R}$, $\mathcal{B}\mathfrak{a}(X) \subseteq \Sigma$.

(d) On the other hand, $\Sigma \subseteq \mathcal{B}\mathfrak{a}(X)$ just because every member of X^* is $\mathfrak{T}_s$-continuous. So $\Sigma = \mathcal{B}\mathfrak{a}(X)$, as claimed.

4A3W Proposition Let $(X, \mathfrak{T})$ be a separable metrizable locally convex linear topological space, and $\mathfrak{T}_s = \mathfrak{T}_s(X, X^*)$ its weak topology. Then the cylindrical σ-algebra of X is also both the Baire σ-algebra and the Borel σ-algebra for both $\mathfrak{T}$ and $\mathfrak{T}_s$.

proof (a) For any linear topological space, we have

$$\Sigma \subseteq \mathcal{B}\mathfrak{a}(X,\mathfrak{T}_s) \subseteq \mathcal{B}\mathfrak{a}(X,\mathfrak{T}) \subseteq \mathcal{B}(X,\mathfrak{T}), \quad \mathcal{B}\mathfrak{a}(X,\mathfrak{T}_s) \subseteq \mathcal{B}(X,\mathfrak{T}_s) \subseteq \mathcal{B}(X,\mathfrak{T}),$$

writing Σ for the cylindrical σ-algebra and $\mathcal{B}\mathfrak{a}(X,\mathfrak{T}_s)$, $\mathcal{B}\mathfrak{a}(X,\mathfrak{T})$, $\mathcal{B}(X,\mathfrak{T}_s)$ and $\mathcal{B}(X,\mathfrak{T})$ for the Baire and Borel σ-algebras of the two topologies. So all I have to do is to show that $\mathcal{B}(X,\mathfrak{T}) \subseteq \Sigma$.

(b) Let $\langle\tau_n\rangle_{n\in\mathbb{N}}$ be a sequence of seminorms defining the topology of X (4A4Cf), and $D \subseteq X$ a countable dense set; for $n \in \mathbb{N}$, $\delta > 0$ and $x \in X$, set $U_n(x,\delta) = \{y : \tau_i(y-x) < \delta$ for every $i \le n\}$. Then every $U_n(x,\delta)$ is a convex open set. Set

$$\mathcal{U} = \{U_n(z,2^{-m}) : z \in D,\ m,\ n \in \mathbb{N}\}.$$

(c) If $G \subseteq X$ is any convex open set, $\overline{G} \in \Sigma$. **P** Set $\mathcal{U}_G = \{U : U \in \mathcal{U},\ U \cap G = \emptyset\}$. For each $U \in \mathcal{U}_G$, there are $f_U \in X^*$, $\alpha_U \in \mathbb{R}$ such that $f_U(x) < \alpha_U$ for every $x \in G$ and $f_U(x) > \alpha_U$ for every $x \in U$ (4A4Db). So $F = \{x : f_U(x) \le \alpha_U$ for every $U \in \mathcal{U}_G\}$ belongs to Σ. Of course $F \supseteq \overline{G}$. On the other hand, if $x \notin \overline{G}$, there are m, $n \in \mathbb{N}$ such that $G \cap U_n(x,2^{-m}) = \emptyset$; if we take $z \in D \cap U_n(x,2^{-m-1})$, $U = U_n(z,2^{-m-1})$ then $x \in U \in \mathcal{U}_G$, so $f_U(x) > \alpha_U$ and $x \notin F$. Thus $\overline{G} = F$ belongs to Σ. **Q**

(d) In particular, $\mathcal{V} = \{\overline{U} : U \in \mathcal{U}\}$ is included in Σ. But $\mathcal{V}$ is a countable network for $\mathfrak{T}$. So every member of $\mathfrak{T}$ is a union of countably many members of Σ and belongs to Σ. It follows at once that $\mathcal{B}(X,\mathfrak{T}) \subseteq \Sigma$, as required.

4A3X Basic exercises (a) Let X be a regular space with a countable network and Y any topological space. Show that $\mathcal{B}(X \times Y) = \mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$. (*Hint*: 4A2Ng.)

(b) Let X be a topological space, and $E \subseteq X$. Show that the following are equiveridical: (α) $E \in \mathcal{B}\mathfrak{a}(X)$; (β) there are a continuous function $f : X \to [0,1]^{\mathbb{N}}$ and $F \in \mathcal{B}([0,1]^{\mathbb{N}})$ such that $E = f^{-1}[F]$.

>(c) Let X be a topological space, and $K \subseteq X$ a compact set such that $K \in \mathcal{B}\mathfrak{a}(X)$. Show that K is a zero set. (*Hint*: 4A3Xb.)

(d) Let X be a compact Hausdorff space such that $\mathcal{B}\mathfrak{a}(X) = \mathcal{B}(X)$. Show that X is perfectly normal.

(e) Let $\mathfrak{S}$ be the topology on $\mathbb{R}$ generated by the usual topology and $\{\{x\} : x \in \mathbb{R}\setminus\mathbb{Q}\}$. Show that $\mathfrak{S}$ is completely regular and Hausdorff and that $\mathbb{Q}$ is a closed Baire set which is not a zero set.

>(f) Let X be a ccc completely regular topological space. Show that any nowhere dense set is included in a nowhere dense zero set.

(g) Let $\langle X_i\rangle_{i\in I}$ be a family of spaces with countable networks, and X their product. Show that $\mathcal{B}\mathfrak{a}(X) = \widehat{\bigotimes}_{i\in I}\mathcal{B}\mathfrak{a}(X_i)$.

>(h)(i) Let I be an uncountable set with its discrete topology, and X the one-point compactification of I. Show that $\mathcal{B}\mathfrak{a}(I)$ is not the subspace σ-algebra generated by $\mathcal{B}\mathfrak{a}(X)$. (ii) Show that ω_1, with its order topology, has a subset I such that $\mathcal{B}\mathfrak{a}(I)$ is not the subspace σ-algebra induced by $\mathcal{B}\mathfrak{a}(\omega_1)$.

4A3Y Further exercises (a) Give an example of a Hausdorff space X with a countable network and a metrizable space Y such that $\mathcal{B}(X \times Y) \ne \mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$.

(b) Let X be a Čech-complete space and $\langle E_n\rangle_{n\in\mathbb{N}}$ a sequence of Borel sets in X. Show that there is a Čech-complete topology on X, finer than the given topology, with the same weight, and containing every E_n.

(c) Give an example of compact Hausdorff spaces X, Y and a function $f : X \to Y$ which is $(\mathcal{B}\mathfrak{a}(X),\mathcal{B}\mathfrak{a}(Y))$-measurable but not Borel measurable.

(d) Let $\langle X_i\rangle_{i\in I}$ be a family of regular spaces with countable networks, with product X. Show that (i) $\mathcal{B}\mathfrak{a}(X)$ is the family of Borel subsets of X which are determined by coordinates in countable sets; (ii) a set $Z \subseteq X$ is a zero set iff it is closed and determined by coordinates in a countable set; (iii) if $Y \subseteq X$ is dense, then $\mathcal{B}\mathfrak{a}(Y)$ is just the subspace σ-algebra $\mathcal{B}\mathfrak{a}(X)_Y$ induced by $\mathcal{B}\mathfrak{a}(X)$.

(e) Let X be a normal topological space and Y a closed subset of X. Show that every Baire subset of Y is the intersection of Y with a Baire subset of X.

4A3 Notes and comments Much of this section consists of easy technicalities. It is however not always easy to guess at the exact results obtainable by these methods. It is important to notice that Baire σ-algebras on subspaces can give difficulties which do not arise with Borel σ-algebras (4A3Xh, 4A3Ca). I have expressed 4A3D in terms of 'hereditarily Lindelöf' spaces. Of course the separable metrizable spaces form by far the most important class of these, but there are others (the split interval, for instance) which are of great interest in measure theory. Similarly, there are important products of non-metrizable spaces which are ccc (e.g., 417Xt(vii)), so that 4A3Mb has something to say.

4A3H-4A3I are a most useful tool in studying Borel subsets of Polish spaces, especially in conjunction with the First Separation Theorem (422I); see 424G and 424H. I include 4A3Yb and 4A3Yd to show that some more of the arguments here can be adapted to non-separable or non-metrizable spaces.

You will note my caution in the definition of 'Baire measurable' function (4A3Ke). This is supposed to cover the case of functions taking values in $[-\infty,\infty]$ without taking a position on functions between general topological spaces (4A3Yc).

It is relatively easy to show that spaces of càdlàg functions have standard Borel structures (4A3Qb). To exhibit usable complete metrics generating these is another matter; see chap. 3 of Billingsley 99.

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4A4 Locally convex spaces

As in §3A5, all the ideas, and nearly all the results as stated below, are applicable to complex linear spaces; but for the purposes of this volume the real case will almost always be sufficient, and for definiteness you may take it that the scalar field is $\mathbb{R}$, except in 4A4J-4A4K. (Complex Hilbert spaces arise naturally in §445.)

4A4A Linear spaces (a) If U is a linear space, a **Hamel basis** for U is a maximal linearly independent family $\langle u_i\rangle_{i\in I}$ in U, so that every member of U is uniquely expressible as $\sum_{i\in J}\alpha_i u_i$ for some finite $J\subseteq I$ and $\langle\alpha_i\rangle_{i\in J}\in(\mathbb{R}\setminus\{0\})^J$.

(b) Every linear space has a Hamel basis. (Schaefer 71, p. 10; Köthe 69, §7.3.)

(c) If U is a linear space, I write U' for the **algebraic dual** of U, the linear space of all linear functionals from U to $\mathbb{R}$.

4A4B Linear topological spaces (see §2A5) **(a)** If U is a linear topological space, and V is a linear subspace of U, then V, with the linear structure and topology induced by those of U, is again a linear topological space. (Bourbaki 87, I.1.3; Schaefer 71, §I.2; Köthe 69, §15.2.)

(b) If $\langle U_i\rangle_{i\in I}$ is any family of linear topological spaces, then $U=\prod_{i\in I}U_i$, with the product linear space structure and topology, is again a linear topological space. (Bourbaki 87, I.1.3; Schaefer 71, §I.2; Köthe 69, §15.4.) In particular, $\mathbb{R}^X$, with its usual linear and topological structures, is a linear topological space, for any set X.

(c) If U and V are linear topological spaces, the set of continuous linear operators from U to V is a linear subspace of the space $\mathsf{L}(U;V)$ of all linear operators from U to V. If U, V and W are linear topological spaces, and $T:U\to V$ and $S:V\to W$ are continuous linear operators, then $ST:U\to W$ is a continuous linear operator.

(d) If U is a linear topological space, I will write U^* for the **dual** of U, the space of all continuous linear functionals from U to $\mathbb{R}$ (compare 2A4H). U^* is a linear subspace of U' as defined in 4A4Ac. The **weak**

topology on U, $\mathfrak{T}_s(U,U^*)$, is that defined by the method of 2A5B from the seminorms $u \mapsto |f(u)|$ as f runs over U^* (compare 2A5Ia). The **weak* topology** on U^*, $\mathfrak{T}_s(U^*,U)$, is that defined from the seminorms $f \mapsto |f(u)|$ as u runs over U (compare 2A5Ig). By 2A5B, both are linear space topologies. If U and V are linear topological spaces, $T : U \to V$ is a continuous linear operator, and $g \in V^*$, then $gT \in U^*$ ((c) above); consequently T is $(\mathfrak{T}_s(U,U^*), \mathfrak{T}_s(V,V^*))$-continuous.

(e) If $U = \prod_{i\in I} U_i$ is a product of linear topological spaces, then every element of U^* is of the form $u \mapsto \sum_{i\in J} f_i(u(i))$ where $J \subseteq I$ is finite and $f_i \in U_i^*$ for every $i \in J$. (Bourbaki 87, II.6.6; Schaefer 71, IV.4.3; Köthe 69, §22.5.) Consequently the weak topology on U is the product of the weak topologies on the U_i.

(f) Let U be a linear topological space. For $A \subseteq U$ write A° for its **polar** set $\{f : f \in U^*,\ f(x) \le 1$ for every $x \in A\}$ in U^*. If G is a neighbourhood of 0 in U, then G° is a $\mathfrak{T}_s(U^*,U)$-compact subset of U^* (compare 3A5F). (Schaefer 71, III.4.3; Köthe 69, §20.9; Rudin 91, 3.15.)

(g) Let U be a linear topological space. If $D \subseteq U$ is non-empty and closed under addition and multiplication by rationals, $\overline{D}$ is a linear subspace of U. **P** The linear span

$$V = \{\sum_{i=0}^n \alpha_i u_i : u_0, \dots , u_n \in D,\ \alpha_0, \dots , \alpha_n \in \mathbb{R}\}$$

of

$$D = \{\sum_{i=0}^n \alpha_i u_i : u_0, \dots , u_n \in D,\ \alpha_0, \dots , \alpha_n \in \mathbb{Q}\}$$

is included in $\overline{D}$, because addition and scalar multiplication are continuous; so $\overline{D} = \overline{V}$ is a linear subspace. **Q** If $A \subseteq U$ is separable, then the closed linear subspace generated by A is separable. **P** Let $D_0 \subseteq A$ be a countable dense subset; then

$$D = \{\sum_{i=0}^n \alpha_i u_i : u_0, \dots , u_n \in D_0,\ \alpha_0, \dots , \alpha_n \in \mathbb{Q}\}$$

is countable, and $\overline{D}$ is separable; but $\overline{D}$ is the closed linear subspace generated by A. **Q**

(h) If $\langle u_i\rangle_{i\in I}$ is an indexed family in a Hausdorff linear topological space U and $u \in U$, we say that $u = \sum_{i\in I} u_i$ if for every neighbourhood G of u there is a finite set $J \subseteq I$ such that $\sum_{i\in K} u_i \in G$ whenever $K \subseteq I$ is finite and $J \subseteq K$ (compare 226Ad).

If $\langle v_i\rangle_{i\in I}$ is another family with the same index set, and $v = \sum_{i\in I} v_i$ is defined, then $\sum_{i\in I}(u_i + v_i)$ is defined and equal to $u + v$. **P** If G is a neighbourhood of $u + v$, there are neighbourhoods H, H' of u, v respectively such that $H + H' \subseteq G$; there are finite sets J, $J' \subseteq I$ such that $\sum_{i\in K} u_i \in H$ whenever $J \subseteq K \in [I]^{<\omega}$ and $\sum_{i\in K} v_i \in H'$ whenever $J' \subseteq K \in [I]^{\omega}$; now $\sum_{i\in K} u_i + v_i \in G$ whenever $J \cup J' \subseteq K \in [I]^{<\omega}$. **Q**

If now V is another Hausdorff linear topological space and $T : U \to V$ is a continuous linear operator, $\sum_{i\in I} Tu_i = T(\sum_{i\in I} u_i)$ if the right-hand-side is defined. **P** Set $u = \sum_{i\in I} u_i$. If H is an open set containing Tu, then $T^{-1}[H]$ is an open set containing u, so there is a $J \in [I]^{<\omega}$ (notation: 3A1J) such that $\sum_{i\in K} u_i \in T^{-1}[H]$ and $\sum_{i\in K} Tu_i \in H$ whenever $J \subseteq K \in [I]^{<\omega}$. **Q**

(i) If U is a Hausdorff linear topological space, then any finite-dimensional linear subspace of U is closed. (Schaefer 71I.3.3; Taylor 64, 3.12-C; Rudin 91, 1.2.1.)

(j) If U is a first-countable Hausdorff linear topological space which (regarded as a linear topological space) is complete, then there is a metric ρ on U, defining its topology, under which U is complete. **P** Let $\mathcal{W}$ be the uniformity of U (3A4Ad). We know there is a sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ running over a base of neighbourhoods of 0 in U; setting $W_n = \{(u,v) : u - v \in G_n\}$ for each n, $\{W_n : n \in \mathbb{N}\}$ generates $\mathcal{W}$. So there is a metric ρ on U defining $\mathcal{W}$ (4A2Jb). Because X is $\mathcal{W}$-complete, it is ρ-complete (Engelking 89, 8.3.5). **Q**

4A4C Locally convex spaces (a) A linear topological space is **locally convex** if the convex open sets form a base for the topology.

(b) A linear topological space is locally convex iff its topology can be defined by a family of seminorms (2A5B, 2A5D). (Bourbaki 87, II.4.1; Schaefer 71, §II.4; Köthe 69, §18.1.)

(c) Let U be a linear space and τ a seminorm on U. Then $N_\tau = \{u : \tau(u) = 0\}$ is a linear subspace of X. On the quotient space U/N_τ we have a norm defined by setting $\|u^\bullet\| = \tau(u)$ for every $u \in U$. (Bourbaki 87, II.1.3; Schaefer 71, II.5.4; Rudin 91, 1.43.)

(d) Let U be a locally convex linear topological space, and T the family of continuous seminorms on U. For each $\tau \in \mathrm{T}$, write $N_\tau = \{u : \tau(u) = 0\}$, as in (c) above, and π_τ for the canonical map from U to $U_\tau = U/N_\tau$. Give each U_τ its norm, and set $\mathcal{G}_\tau = \{\pi_\tau^{-1}[H] : H \subseteq U_\tau \text{ is open}\}$. Then $\bigcup_{\tau\in\mathrm{T}} \mathcal{G}_\tau$ is a base for the topology of X closed under finite unions. (Schaefer 71, II.5.4.)

(e) A linear subspace of a locally convex linear topological space is locally convex. (Bourbaki 87, II.4.3; Köthe 69, §18.3.) The product of any family of locally convex linear topological spaces (4A4Bb) is locally convex. (Bourbaki 87, II.4.3; Köthe 69, §18.3.)

(f) If U is a metrizable locally convex linear topological space, its topology can be defined by a sequence of seminorms. (Bourbaki 87, II.4.1; Köthe 69, §18.2.)

(g) Let U be a linear space and V a linear subspace of the space U' of all linear functionals on U. Let $\mathfrak{T}_s(V, U)$ be the topology on V generated by the seminorms $f \mapsto |f(u)|$ as u runs over U (compare 4A4Bd), and let $\phi : V \to \mathbb{R}$ be a $\mathfrak{T}_s(V, U)$-continuous linear functional. Then there is a $u \in U$ such that $\phi(f) = f(u)$ for every $f \in V$. (Bourbaki 87, IV.1.1; Schaefer 71, IV.1.2; Köthe 69, §20.2; Rudin 91, 3.10; Dunford & Schwartz 57, II.3.9; Taylor 64, 3.81-A.)

(h) Grothendieck's theorem If U is a complete locally convex Hausdorff linear topological space, and ϕ is a linear functional on the dual U^* such that $\phi\restriction G^\circ$ is $\mathfrak{T}_s(U^*, U)$-continuous for every neighbourhood G of 0 in U, then ϕ is of the form $f \mapsto f(u)$ for some $u \in U$. (Bourbaki 87, III.3.6; Schaefer 71, IV.6.2; Köthe 69, §21.9.)

4A4D Hahn-Banach theorem (a) Let U be a linear space and $\theta : U \to [0, \infty[$ a seminorm.

(i) If $V \subseteq U$ is a linear subspace and $g : V \to \mathbb{R}$ is a linear functional such that $|g(v)| \le \theta(v)$ for every $v \in V$, then there is a linear functional $f : U \to \mathbb{R}$, extending g, such that $|f(u)| \le \theta(u)$ for every $u \in U$.

(ii) If $u_0 \in U$ then there is a linear functional $f : U \to \mathbb{R}$ such that $f(u_0) = \theta(u_0)$ and $|f(u)| \le \theta(u)$ for every $u \in U$. (Bourbaki 87, II.3.2; Rudin 91, 3.3; Dunford & Schwartz 57, II.3.11; Taylor 64, 3.7-C; or use 3A5Aa.)

(b) Let U be a linear topological space and G, H two disjoint convex sets in U, of which one has non-empty interior. Then there are a non-zero $f \in U^*$ and an $\alpha \in \mathbb{R}$ such that $f(u) \le \alpha \le f(v)$ for every $u \in G$, $v \in H$, so that $f(u) < \alpha$ for every $u \in \operatorname{int} G$ and $\alpha < f(v)$ for every $u \in \operatorname{int} H$. (Bourbaki 87, II.5.2; Schaefer 71, II.9.1; Köthe 69, §17.1.)

4A4E The Hahn-Banach theorem in locally convex spaces Let U be a locally convex linear topological space.

(a) If $V \subseteq U$ is a linear subspace, then every member of V^* extends to a member of U^* (compare 3A5Ab). (Bourbaki 87, II.4.1; Schaefer 71, II.4.2; Köthe 69, §20.1; Rudin 91, 3.6; Taylor 64, 3.8-D.)

Consequently $\mathfrak{T}_s(V, V^*)$ is just the subspace topology on V induced by $\mathfrak{T}_s(U, U^*)$.

(b) Let $C \subseteq U$ be a non-empty closed convex set. If $u \in U$ then $u \in C$ iff $f(u) \le \sup_{v\in C} f(v)$ for every $f \in U^*$ iff $f(u) \ge \inf_{v\in C} f(v)$ for every $f \in U^*$. (Bourbaki 87, II.5.3; Schaefer 71, II.9.2; Köthe 69, §20.7; Dunford & Schwartz 57, V.2.12.)

If $V \subseteq U$ is a closed linear subspace and $u \in U \setminus V$ there is an $f \in U^*$ such that $f(u) \ne 0$ and $f(v) = 0$ for every $v \in V$. (Bourbaki 87, II.5.3; Köthe 69, §20.1; Rudin 91, 3.5; Taylor 64, 3.8-E.)

(c) If U is Hausdorff, U^* separates its points (compare 3A5Ae). (Bourbaki 87, II.4.1; Rudin 91, 3.4.)

(d) If $u \in U$ belongs to the $\mathfrak{T}_s(U, U^*)$-closure of a convex set $C \subseteq U$, it belongs to the closure of C (compare 3A5Ee). (Schaefer 71, II.9.2; Köthe 69, §20.7; Rudin 91, 3.12.) In particular, if C is closed, it is $\mathfrak{T}_s(U, U^*)$-closed. (Dunford & Schwartz 57, V.2.13.)

(e) If C, $C' \subseteq U$ are disjoint non-empty closed convex sets, of which one is compact, there is an $f \in U^*$ such that $\sup_{u\in C} f(u) < \inf_{u\in C'} f(u)$. (Apply (b) to $C - C'$. See Bourbaki 87, II.5.3; Schaefer 71, II.9.2; Köthe 69, §20.7; Rudin 91, 3.4; Dunford & Schwartz 57, V.3.13.)

(f) Let V be a linear subspace of U'. Let $K \subseteq U$ be a non-empty $\mathfrak{T}_s(U, V)$-compact convex set, and $\phi_0 : V \to \mathbb{R}$ a linear functional such that $\phi_0(f) \le \sup_{u\in K} f(u)$ for every $f \in V$. Then there is a $u_0 \in K$ such that $\phi_0(f) = f(u_0)$ for every $f \in V$. **P** Give U the topology $\mathfrak{T}_s(U, V)$ and V' the topology $\mathfrak{T}_s(V', V)$ (4A4Cg). For $u \in U$, $f \in V$ set $\hat{u}(f) = f(u)$; then $u \mapsto \hat{u}$ is a continuous linear operator from U to V' (use 2A3H), so $\hat{K} = \{\hat{u} : u \in K\}$ is a compact convex subset of V'.

? Suppose, if possible, that $\phi_0 \notin \hat{K}$. By (b), there is a continuous linear functional $\boldsymbol{\theta} : V' \to \mathbb{R}$ such that $\boldsymbol{\theta}(\phi_0) > \sup_{u\in K} \boldsymbol{\theta}(\hat{u})$. But there is an $f \in V$ such that $\boldsymbol{\theta}(\phi) = \phi(f)$ for every $\phi \in V'$ (4A4Cg), so that $\phi_0(f) > \sup_{u\in K} f(u)$, contrary to hypothesis. **X** So there is a $u_0 \in K$ such that $\phi_0 = \hat{u}_0$, as claimed. **Q**

(g) The Bipolar Theorem Let $A \subseteq U'$ be a non-empty set. Set $A^\circ = \{u : u \in U,\ f(u) \le 1$ for every $f \in A\}$ (compare 4A4Bf). If $g \in U'$ is such that $g(u) \le 1$ for every $u \in A^\circ$, then g belongs to the $\mathfrak{T}_s(U', U)$-closed convex hull of $A \cup \{0\}$. **P** Put 4A4Cg and (b) above together, as in (f). See Bourbaki 87, II.6.3; Schaefer 71, IV.1.5; Köthe 69, 20.8. **Q**

(h) Let W be a linear subspace of U' separating the points of U. Then W is $\mathfrak{T}_s(U', U)$-dense in U'. (For $W^0 = \{0\}$.)

4A4F The Mackey topology Let U be a linear space and V a linear subspace of U'. The **Mackey topology** $\mathfrak{T}_k(V, U)$ on V is the topology of uniform convergence on convex $\mathfrak{T}_s(U, V)$-compact subsets of U. Every $\mathfrak{T}_k(V, U)$-continuous linear functional on V is of the form $f \mapsto f(u)$ for some $u \in U$ (use 4A4Ef). So every $\mathfrak{T}_k(V, U)$-closed convex set is $\mathfrak{T}_s(V, U)$-closed, by 4A4Ed. (See Bourbaki 87, IV.1.1; Schaefer 71, IV.3.2; Köthe 69, 21.4.)

4A4G Extreme points (a) Let X be a real linear space, and $C \subseteq X$ a convex set. An element of C is an **extreme** point of C if it is not expressible as a convex combination of two other members of C; equivalently, if it is not expressible as $\frac{1}{2}(x + y)$ where x, $y \in C$ are distinct.

(b) The Kreĭn-Mil'man theorem Let U be a Hausdorff locally convex linear topological space and $K \subseteq U$ a compact convex set. Then K is the closed convex hull of the set of its extreme points. (Bourbaki 87, II.7.1; Schaefer 71, II.10.4; Köthe 69, §25.1; Rudin 91, 3.22.)

(c) Let U and V be Hausdorff locally convex linear topological spaces, $T : U \to V$ a continuous linear operator, $K \subseteq X$ a compact convex set and v any extreme point of $T[K] \subseteq V$. Then there is an extreme point u of K such that $Tu = v$. **P** Set $K_1 = \{u' : u' \in K, Tu' = v\}$. Then K_1 is a compact convex set so has an extreme point u. **?** If u is not an extreme point of K, it is expressible as $\alpha u_1 + (1 - \alpha)u_2$ where u_1, u_2 are distinct points of K and $\alpha \in \,]0, 1[$. So $v = \alpha Tu_1 + (1 - \alpha)Tu_2$, and we must have $Tu_1 = Tv = Tu_2$, because v is an extreme point of $T[K]$; but this means that u_1, $u_2 \in K_1$ and u is not an extreme point of K_1. **X** So u has the required properties. **Q**

4A4H Proposition Let I be a set, W a closed linear subspace of $\mathbb{R}^I$, U a linear topological space and V a Hausdorff linear topological space. Let $K \subseteq U$ be a compact set and $T : U \times \mathbb{R}^I \to V$ a continuous linear operator. Then $T[K \times W]$ is closed.

proof Take $v_0 \in \overline{T[K \times W]}$. Let $J \subseteq I$ be a maximal set such that

> whenever $L \subseteq J$ is finite and $H \subseteq V$ is an open set containing v_0, there are a $u \in K$ and an $x \in W$ such that $x(i) = 0$ for every $i \in L$ and $T(u, x) \in H$.

Let $\mathcal{F}$ be the filter on $U \times \mathbb{R}^I$ generated by the closed set $K \times W$, the sets $\{(u,x) : x(i) = 0\}$ for $i \in J$, and the sets $T^{-1}[H]$ for open sets H containing v_0. Then for any $j \in I$ there is an $F \in \mathcal{F}$ such that $\{x(j) : (u,x) \in F\}$ is bounded. **P?** Suppose, if possible, otherwise. Then, in particular, $j \notin J$. So there must be a finite set $L \subseteq J$ and an open set H containing v_0 such that $x(j) \neq 0$ whenever $u \in K$, $x \in W$, $x(i) = 0$ for every $i \in L$ and $T(u,x) \in H$. By 2A5C, or otherwise, there is a neighbourhood G_0 of 0 in V such that $v_0 + G_0 - G_0 \subseteq H$ and $\alpha v \in G_0$ whenever $v \in G_0$ and $|\alpha| \le 1$. Fix $u^* \in K$, $x^* \in W$ such that $x^*(i) = 0$ for every $i \in L$ and $T(u^*,x^*) \in v_0 + G_0$. If $x \in W$ and $x(i) = 0$ for every $i \in L$ and $x(j) = x^*(j)$, then $T(u^*, x^* - x) \notin v_0 + G_0 - G_0$ so $T(0,x) \notin G_0$. It follows that $T(0,x) \notin G_0$ whenever $x \in W$ and $x(i) = 0$ for every $i \in L$ and $|x(j)| \ge |x^*(j)|$.

Let G be a neighbourhood of 0 in V such that $G + G - G - G \subseteq G_0$. We are supposing that $\{x(j) : (u,x) \in F\}$ is unbounded for every $F \in \mathcal{F}$. So for every $n \in \mathbb{N}$ there are $u_n \in K$ and $x_n \in W$ such that $x_n(i) = 0$ for $i \in L$, $T(u_n,x_n) \in v_0 + G$ and $|x_n(j)| \ge n$. Let $u \in K$ be a cluster point of $\langle u_n\rangle_{n\in\mathbb{N}}$. Then $T(u,0)$ is a cluster point of $\langle T(u_n,0)\rangle_{n\in\mathbb{N}}$, so $M = \{n : T(u_n,0) \in T(u,0) + G\}$ is infinite. For $n \in M$, $T(0,x_n) = T(u_n,x_n) - T(u_n,0) \in v_0 - T(u,0) + G - G$; so if $m, n \in M$, $T(0, x_m - x_n) \in G - G - (G - G) \subseteq G_0$. But note now that $(x_m - x_n)(i) = 0$ for all $m, n \in \mathbb{N}$ and $i \in L$, and that because M is infinite there are certainly $m, n \in M$ such that $|x_m(j) - x_n(j)| \ge |x^*(j)|$; which contradicts the last paragraph. **XQ**

Now let $\mathcal{G}$ be any ultrafilter on $U \times \mathbb{R}^I$ including $\mathcal{F}$. Then for every $i \in I$ there is a $\gamma_i < \infty$ such that $\mathcal{G}$ contains $\{(u,x) : |x(i)| \le \gamma_i\}$. It follows that $\mathcal{G}$ has a limit $(\hat u, \hat x)$ in $K \times W$. Now the image filter $T[[\mathcal{G}]]$ (2A1Ib) converges to $T(\hat u, \hat x)$; since $T[[\mathcal{F}]] \to v_0$, and the topology of V is Hausdorff, $v_0 = T(\hat u, \hat x) \in T[K \times W]$. As v_0 is arbitrary, $T[K \times W]$ is closed.

4A4I Normed spaces (a) Two norms $\| \,\|$, $\| \,\|'$ on a linear space U give rise to the same topology iff they are **equivalent** in the sense that, for some $M \ge 0$,

$$\|x\| \le M\|x\|', \quad \|x\|' \le M\|x\|$$

for every $x \in U$. (Köthe 69, §14.2; Taylor 64, 3.1-D; Jameson 74, 2.8.)

(b) If U and V are normed spaces, $T : U \to V$ is a linear operator and $gT : U \to \mathbb{R}$ is continuous for every $g \in V^*$, then T is a bounded operator. (Jameson 74, 27.6.)

(c) If U is any normed space, its dual U^*, under its usual norm (2A4H), is a Banach space. (Rudin 91, 4.1; Dunford & Schwartz 57, II.3.9; Köthe 69, §14.5.)

(d) If U is a separable normed space, its dual U^* (regarded as a normed space) is isometrically isomorphic to a closed linear subspace of ℓ^∞. **P** Let $\langle x_n\rangle_{n\in\mathbb{N}}$ run over a dense subset of the unit ball of U (4A2P(a-iv)); define $T : U^* \to \ell^\infty$ by setting $(Tf)(n) = f(x_n)$ for every n. T is a linear isometry between U^* and $T[U^*]$, which is closed because U^* is complete (4A4Ic, 3A4Fd). **Q**

(e) Let U be a Banach space. Suppose that $\langle u_i\rangle_{i\in I}$ is a family in U such that $\gamma = \sum_{i\in I} \|u_i\| < \infty$.

(i) $\sum_{i\in I} u_i$ is defined in the sense of 4A4Bh. **P** For $J \in [I]^{<\omega}$, set $v_J = \sum_{i\in J} u_i$. For each $n \in \mathbb{N}$, there is a $J_n \in [I]^{<\omega}$ such that $\gamma - \sum_{i\in K} \|u_i\| \le 2^{-n}$ whenever $J_n \subseteq K \in [I]^{<\omega}$. Now

$$\|v_{J_m} - v_{J_n}\| \le \textstyle\sum_{i\in J_m \triangle J_n} \|u_i\| \le 2^{-m} + 2^{-n}$$

for all $m, n \in \mathbb{N}$, so $\langle v_{J_n}\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence and has a limit v say. If now $n \in \mathbb{N}$ and $J_n \subseteq K \in [I]^{<\omega}$,

$$\begin{aligned}\|v - \sum_{i\in K} u_i\| &= \lim_{m\to\infty} \|v_{J_m} - \sum_{i\in K} u_i\| \le \limsup_{m\to\infty} \sum_{i\in J_m \triangle K} \|u_i\| \\ &\le \lim_{m\to\infty} 2^{-m} + 2^{-n} = 2^{-n},\end{aligned}$$

so $v = \sum_{i\in I} u_i$. **Q**

(ii) Now if $\langle I_j\rangle_{j\in J}$ is any partition of I, $w_j = \sum_{i\in I_j} u_i$ is defined for every j, and $\sum_{j\in J} w_j$ is defined and equal to $\sum_{i\in I} u_i$.

(f) Let U be a normed space. For $u \in U$, define $\hat u \in U^{**} = (U^*)^*$ by setting $\hat u(f) = f(u)$ for every $f \in U^*$. Then $\{\hat u : u \in U, \|u\| \le 1\}$ is weak*-dense in $\{\phi : \phi \in U^{**}, \|\phi\| = 1\}$. (Apply 4A4Eg with $A = \{\hat u : \|u\| \le 1\}$.)

4A4J Inner product spaces (a) Let U be an inner product space over $\begin{smallmatrix}\mathbb{R}\\ \mathbb{C}\end{smallmatrix}$ (3A5M). An **orthonormal family** in U is a family $\langle e_i\rangle_{i\in I}$ in U such that $(e_i|e_j)=0$ if $i\ne j$, 1 if $i=j$. An **orthonormal basis** in U is an orthonormal family $\langle e_i\rangle_{i\in I}$ in U such that the closed linear subspace of U generated by $\{e_i:i\in I\}$ is U itself.

(b) If U, V are inner product spaces over $\mathbb{R}$ and $T:U\to V$ is an isometry such that $T(0)=0$, then $(Tu|Tv)=(u|v)$ for all u, $v\in U$ and T is linear. **P** (α)

$$(Tu|Tv)=\frac{1}{2}(\|Tu\|^2+\|Tv\|^2-\|Tu-Tv\|^2)=\frac{1}{2}(\|u\|^2+\|v\|^2-\|u-v\|^2)=(u|v).$$

(β) For any u, $v\in U$,

$$\begin{aligned}\|T(u+v)-Tu-Tv\|^2&=\|T(u+v)\|^2+\|Tu\|^2+\|Tv\|^2\\&\qquad-2(T(u+v)|Tu)-2(T(u+v)|Tv)+2(Tu|Tv)\\&=\|u+v\|^2+\|u\|^2+\|v\|^2\\&\qquad-2(u+v|u)-2(u+v|v)+2(u|v)\\&=0.\end{aligned}$$

So T is additive. (γ) Consequently $T(qu)=qTu$ for every $u\in U$ and $q\in\mathbb{Q}$; as T is continuous, it is linear. **Q**

(c) If U, V are inner product spaces over $\mathbb{C}$ and $T:U\to V$ is a linear operator such that $\|Tu\|=\|u\|$ for every $u\in U$, then $(Tu|Tv)=(u|v)$ for all u, $v\in U$. **P**

$$\mathcal{R}\mathrm{e}(Tu|Tv)=\frac{1}{2}(\|Tu\|^2+\|Tv\|^2-\|Tu-Tv\|^2)=\mathcal{R}\mathrm{e}(u|v),$$

$$\mathcal{I}\mathrm{m}(Tu|Tv)=-\mathcal{R}\mathrm{e}(i(Tu|Tv))=-\mathcal{R}\mathrm{e}(T(iu)|Tv)=-\mathcal{R}\mathrm{e}(iu|v)=\mathcal{I}\mathrm{m}(u|v).\ \mathbf{Q}$$

(d) If U is an inner product space over $\begin{smallmatrix}\mathbb{R}\\ \mathbb{C}\end{smallmatrix}$, a linear operator $T:U\to U$ is **self-adjoint** if $(Tu|v)=(u|Tv)$ for all u, $v\in U$.

(e) If U is a finite-dimensional inner product space over $\mathbb{R}$, it is isomorphic to Euclidean space $\mathbb{R}^r$, where $r=\dim U$. (Taylor 64, 3.21-A.) In particular, any finite-dimensional inner product space is a Hilbert space.

(f) If U is an inner product space over $\begin{smallmatrix}\mathbb{R}\\ \mathbb{C}\end{smallmatrix}$ and $V\subseteq U$ is a linear subspace of U, then $V^\perp=\{x:x\in U,\ (x|y)=0$ for every $y\in V\}$ is a linear subspace of U, and $\|x+y\|^2=\|x\|^2+\|y\|^2$ for $x\in V$, $y\in V^\perp$. If V is complete (in particular, if V is finite-dimensional), then $U=V\oplus V^\perp$. (Rudin 91, 12.4; Bourbaki 87, V.1.6; Taylor 64, 4.82-A.)

(g) If U is an inner product space over $\mathbb{R}$ and v_1, $v_2\in U$ are such that $\|v_1\|=\|v_2\|=1$, there is a linear operator $T:U\to U$ such that $Tv_1=v_2$ and $\|Tu\|=\|u\|$ and $\|Tu-u\|\le\|v_1-v_2\|\|u\|$ for every $u\in U$. **P** If v_2 is a multiple of v_1, say $v_2=\alpha v_1$, take $Tu=\alpha u$ for every u. Otherwise, set $w=v_2-(v_2|v_1)v_1$ and $w_1=\frac{1}{\|w\|}w$, so that $v_2=\cos\theta v_1+\sin\theta w_1$, where $\theta=\arccos(v_2|v_1)$. Let V be the two-dimensional linear subspace of U generated by v_1 and w_1, so that $U=V\oplus V^\perp$. Define a linear operator $T:U\to U$ by saying that $Tv_1=v_2$, $Tw_1=\cos\theta w_1-\sin\theta v_1$ and $Tu=u$ for $u\in V^\perp$. Then T acts on V as a simple rotation through an angle θ, so $\|Tv\|=1$ and $\|Tv-v\|=\|v_2-v_1\|$ whenever $v\in V$ and $\|v\|=1$; generally, if $u\in U$, then $\|Tu\|=\|u\|$ and

$$\|Tu-u\|=\|T(Pu)-Pu\|=\|v_2-v_1\|\|Pu\|\le\|v_2-v_1\|\|u\|,$$

where P is the orthogonal projection of U onto V. **Q**

(h) Let U be an inner product space over $\begin{smallmatrix}\mathbb{R}\\ \mathbb{C}\end{smallmatrix}$, and $\langle u_i\rangle_{i\in I}$ a countable family in U. Then there is a countable orthonormal family $\langle v_j\rangle_{j\in J}$ in U such that $\{v_j:j\in J\}$ and $\{u_i:i\in I\}$ span the same linear subspace of U. **P** We can suppose that $I \subseteq \mathbb{N}$; set $u_i = 0$ for $i \in \mathbb{N} \setminus I$. Define $\langle v_n \rangle_{n\in\mathbb{N}}$ inductively by setting $v'_n = u_n - \sum_{i<n}(u_n|v_i)v_i$, $v_n = 0$ if $v'_n = 0$, $\frac{1}{\|v'_n\|}v'_n$ otherwise. Set $J = \{n : v_n \ne 0\}$. **Q**

(i) Let U be an inner product space over $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$, and $\langle e_i \rangle_{i\in I}$ an orthonormal family in U. Then $\sum_{i\in I} |(u|e_i)|^2 \le \|u\|^2$ for every $u \in U$. (DUNFORD & SCHWARTZ 57, p. 252; TAYLOR 64, 3.2-D.)

(j) Let U be an inner product space over $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$, and $C \subseteq U$ a convex set. Then there is at most one point $u \in C$ such that $\|u\| \le \|v\|$ for every $v \in C$. **P** If u, u' both have this property, then $v = \frac{1}{2}(u + u') \in C$, and $\|u\| = \|u'\| \le \|v\|$; but $4\|v\|^2 + \|u - u'\|^2 = 2(\|u\|^2 + \|u'\|^2)$, so $\|u - u'\| = 0$ and $u = u'$. **Q**

For such a u, $\|u\|^2 \le \mathcal{R}\mathrm{e}(u|v)$ for every $v \in C$. **P** For $\alpha \in]0, 1]$,

$$\|u\|^2 \le \|\alpha v + (1-\alpha)u\|^2 = \|u\|^2 + 2\alpha(\mathcal{R}\mathrm{e}(u|v) - \|u\|^2) + \alpha^2\|v - u\|^2,$$

so $\mathcal{R}\mathrm{e}(u|v) - \|u\|^2 \ge -\lim_{\alpha\downarrow 0} \frac{1}{2}\alpha\|v - u\|^2$. **Q**

4A4K Hilbert spaces (a) If U is a real or complex Hilbert space, its unit ball is compact in the weak topology $\mathfrak{T}_s(U, U^*)$; any bounded set is relatively compact for $\mathfrak{T}_s(U, U^*)$. (BOURBAKI 87, V.1.7; DUNFORD & SCHWARTZ 57, IV.4.6.)

(b) If U is a real or complex Hilbert space, any norm-bounded sequence in U has a weakly convergent subsequence. (462D; DUNFORD & SCHWARTZ 57, IV.4.7.)

(c) If U is a real or complex Hilbert space and $\langle u_i \rangle_{i\in I}$ is any orthonormal family in U, then it can be extended to an orthonormal basis. (DUNFORD & SCHWARTZ 57, IV.4.10; TAYLOR 64, 3.2-I.) In particular, U has an orthonormal basis.

4A4L Compact operators (see 3A5La) **(a)** Let U, V and W be Banach spaces. If $T \in \mathsf{B}(U; V)$ and $S \in \mathsf{B}(V; W)$ and either S or T is a compact operator, then ST is compact. (DUNFORD & SCHWARTZ 57, VI.5.4; JAMESON 74, 34.2.)

(b) If U is a Banach space, $T \in \mathsf{B}(U; U)$ is a compact linear operator and $\gamma \ne 0$ then $\{u : Tu = \gamma u\}$ is finite-dimensional. (RUDIN 91, 4.18; TAYLOR 64, 5.5-C; DUNFORD & SCHWARTZ 57, VII.4.5; JAMESON 74, 34.8.)

4A4M Self-adjoint compact operators If U is a Hilbert space and $T : U \to U$ is a self-adjoint compact linear operator, then $T[U]$ is included in the closed linear span of $\{Tv : v$ is an eigenvector of $T\}$. (TAYLOR 64, 6.4-B.)

4A4N Max-flow Min-cut Theorem (FORD & FULKERSON 56) Let (V, E, γ) be a (finite) transportation network, that is,

V is a finite set of 'vertices',

$E \subseteq \{(v, v') : v, v' \in V, v \ne v'\}$ is a set of (directed) 'edges',

$\gamma : E \to [0, \infty[$ is a function;

we regard a member $e = (v, v')$ of E as 'starting' at v and 'ending' at v', and $\gamma(e)$ is the 'capacity' of the edge e. Suppose that v_0, $v_1 \in V$ are distinct vertices such that no edge ends at v_0 and no edge starts at v_1. Then we have a 'flow' $\phi : E \to [0, \infty[$ and a 'cut' $X \subseteq E$ such that

(i) for every $v \in V \setminus \{v_0, v_1\}$,

$$\sum_{e\in E, e \text{ starts at } v} \phi(e) = \sum_{e\in E, e \text{ ends at } v} \phi(e),$$

(ii) $\phi(e) \le \gamma(e)$ for every $e \in E$,

(iii) there is no path from v_0 to v_1 using only edges in $E \setminus X$,

(iv) $\sum_{e\in E, e \text{ starts at } v_0} \phi(e) = \sum_{e\in E, e \text{ ends at } v_1} \phi(e) = \sum_{e\in X} \gamma(e)$.

proof BOLLOBÁS 79, §III.1; ANDERSON 87, 12.3.1.

Version of 4.8.13

4A5 Topological groups

For Chapter 44 we need a variety of facts about topological groups. Most are essentially elementary, and all the non-trivial ideas are covered by at least one of Császár 78 and Hewitt & Ross 63. In 4A5A-4A5C and 4A5I I give some simple definitions concerning groups and group actions. Topological groups, properly speaking, appear in 4A5D. Their simplest properties are in 4A5E-4A5G. I introduce 'right' and 'bilateral' uniformities in 4A5H; the latter are the more interesting (4A5M-4A5O), but the former are also important (see the proof of 4A5P). 4A5J-4A5L deal with quotient spaces, including spaces of cosets of non-normal subgroups. I conclude with notes on metrizable groups (4A5Q-4A5S).

4A5A Notation If X is a group, $x_0 \in X$, and A, $B \subseteq X$ I write

$$x_0A = \{x_0x : x \in A\}, \quad Ax_0 = \{xx_0 : x \in A\},$$

$$AB = \{xy : x \in A,\, y \in B\}, \quad A^{-1} = \{x^{-1} : x \in A\}.$$

A is **symmetric** if $A = A^{-1}$. Observe that $(AB)C = A(BC)$, $(AB)^{-1} = B^{-1}A^{-1}$ for any A, B, $C \subseteq X$.

4A5B Group actions (a) If X is a group and Z is a set, an **action** of X on Z is a function $(x, z) \mapsto x{\bullet}z : X \times Z \to Z$ such that

$$(xy){\bullet}z = x{\bullet}(y{\bullet}z) \text{ for all } x,\, y \in X \text{ and } z \in Z,$$

$$e{\bullet}z = z \text{ for every } z \in Z$$

where e is the identity of X.

In this context I may say that 'X acts on Z', taking the operation $\bullet$ for granted.

(b) An action $\bullet$ of a group X on a set Z is **transitive** if for every w, $z \in Z$ there is an $x \in X$ such that $x{\bullet}w = z$.

(c) If $\bullet$ is an action of a group X on a set Z, I write $x{\bullet}A = \{x{\bullet}z : z \in A\}$ whenever $x \in X$ and $A \subseteq Z$.

(d) If $\bullet$ is an action of a group X on a set Z, then $z \mapsto x{\bullet}z : Z \to Z$ is a permutation for every $x \in X$. (For it has an inverse $z \mapsto x^{-1}{\bullet}z$.) So if Z is a topological space and $z \mapsto x{\bullet}z$ is continuous for every x, it is a homeomorphism for every x.

(e) An action $\bullet$ of a group X on a set Z is **faithful** if whenever x, $y \in X$ are distinct there is a $z \in Z$ such that $x{\bullet}z \neq y{\bullet}z$; that is, the natural homomorphism from X to the group of permutations of Z is injective. An action of X on Z is faithful iff for any $x \in X$ which is not the identity there is a $z \in Z$ such that $x{\bullet}z \neq z$.

(f) If $\bullet$ is an action of a group X on a set Z, then $Y_z = \{x : x \in X,\, x{\bullet}z = z\}$ is a subgroup of X (the **stabilizer** of z) for every $z \in Z$. If $\bullet$ is transitive, then Y_w and Y_z are conjugate subgroups for all w, $z \in Z$. (If $x{\bullet}w = z$, then $Y_z = xY_wx^{-1}$.)

(g) If $\bullet$ is an action of a group X on a set Z, then sets of the form $\{a{\bullet}z : a \in X\}$ are called **orbits** of the action; they are the equivalence classes under the equivalence relation $\sim$, where $z \sim z'$ if there is an $a \in X$ such that $z' = a{\bullet}z$.

4A5C Examples Let X be any group.

(a) Write

$$x{\bullet}_l y = xy, \quad x{\bullet}_r y = yx^{-1}, \quad x{\bullet}_c y = xyx^{-1}$$

for x, $y \in X$. These are all actions of X on itself, the **left**, **right** and **conjugacy** actions.

(b) If $A \subseteq X$, we have an action of X on the set $\{yA : y \in X\}$ of left cosets of A defined by setting $x\bullet(yA) = xyA$ for x, $y \in X$.

(c)(i) Let $\bullet$ be an action of a group X on a set Z. If f is any function defined on a subset of Z, and $x \in X$, write $x\bullet f$ for the function defined by saying that $(x\bullet f)(z) = f(x^{-1}\bullet z)$ whenever $z \in Z$ and $x^{-1}\bullet z \in \operatorname{dom} f$. It is easy to check that this defines an action of X on the class of all functions with domains included in Z. Observe that

$$x\bullet(f+g) = (x\bullet f) + (x\bullet g), \quad x\bullet(f \times g) = (x\bullet f) \times (x\bullet g), \quad x\bullet(f/g) = (x\bullet f)/(x\bullet g)$$

whenever $x \in X$ and f, g are real-valued functions with domains included in Z.

(ii) In (i), if $X = Z$, we have corresponding actions $\bullet_l$, $\bullet_r$ and $\bullet_c$ of X on the class of functions with domains included in X:

$$(x\bullet_l f)(y) = f(x^{-1}y), \quad (x\bullet_r f)(y) = f(yx), \quad (x\bullet_c f)(y) = f(x^{-1}yx)$$

whenever these are defined. These are the left, right and conjugacy **shift actions**.

Note that

$$x\bullet_l \chi A = \chi(xA), \quad x\bullet_r \chi A = \chi(Ax^{-1}), \quad x\bullet_c \chi A = \chi(xAx^{-1})$$

whenever $A \subseteq X$ and $x \in X$. In this context, the following idea is sometimes useful. If f is a function with domain included in X, set $\overset{\leftrightarrow}{f}(y) = f(y^{-1})$ when $y \in X$ and $y^{-1} \in \operatorname{dom} f$. Then

$$(\overset{\leftrightarrow}{f})^{\leftrightarrow} = f, \quad x\bullet_l \overset{\leftrightarrow}{f} = (x\bullet_r f)^{\leftrightarrow}, \quad x\bullet_r \overset{\leftrightarrow}{f} = (x\bullet_l f)^{\leftrightarrow}, \quad x\bullet_c \overset{\leftrightarrow}{f} = (x\bullet_c f)^{\leftrightarrow}$$

for any such f and any $x \in X$.

(d) If $\bullet$ is an action of a group X on a set Z, $Y \subseteq X$ is a subgroup of X, and $W \subseteq Z$ is Y-invariant in the sense that $y\bullet w \in W$ whenever $y \in Y$ and $w \in W$, then $\bullet\restriction Y \times W$ is an action of Y on W. In the context of (c-i) above, this means that if V is any set of functions with domains included in W such that $y\bullet f \in V$ whenever $y \in Y$ and $f \in V$, then we have an action of Y on V.

4A5D Definitions (a) A **topological group** is a group X endowed with a topology such that the operations $(x, y) \mapsto xy : X \times X \to X$ and $x \mapsto x^{-1} : X \to X$ are continuous.

(b) A **Polish group** is a topological group in which the topology is Polish.

4A5E Elementary facts Let X be any topological group.

(a) For any $x \in X$, the functions $y \mapsto xy$, $y \mapsto yx$ and $y \mapsto y^{-1}$ are all homeomorphisms from X to itself. (Hewitt & Ross 63, 4.2; Folland 95, 2.1.)

(b) The maps $(x, y) \mapsto x^{-1}y$, $(x, y) \mapsto xy^{-1}$ and $(x, y) \mapsto xyx^{-1}$ from $X \times X$ to X are continuous.

(c) $\{G : G$ is open, $e \in G$, $G^{-1} = G\}$ is a base of neighbourhoods of the identity e of X. (Hewitt & Ross 63, 4.6; Folland 95, 2.1.)

(d) If $G \subseteq X$ is an open set, then AG and GA are open for any set $A \subseteq X$. (Hewitt & Ross 63, 4.4.)

(e) If $F \subseteq X$ is closed and $x \in X \setminus F$, there is a neighbourhood U of e such that $UxUU \cap FUU = \emptyset$. **P** Set $U_1 = X \setminus x^{-1}F$. Let U_2 be a neighbourhood of e such that $U_2U_2U_2U_2^{-1}U_2^{-1} \subseteq U_1$. Let U be a neighbourhood of e such that $U \subseteq U_2 \cap xU_2x^{-1}$; this works. **Q**

(f) If $K \subseteq X$ is compact and $F \subseteq X$ is closed then KF and FK are closed. If K, $L \subseteq X$ are compact so is KL. (Hewitt & Ross 63, 4.4.)

(g) If there is any compact set $K \subseteq X$ such that $\operatorname{int} K$ is non-empty, then X is locally compact.

(h) If $K \subseteq X$ is compact and $\mathcal{F}$ is a downwards-directed family of closed subsets of X with intersection F_0, then $KF_0 = \bigcap_{F\in\mathcal{F}} KF$ and $F_0K = \bigcap_{F\in\mathcal{F}} FK$. **P** Of course $KF_0 \subseteq \bigcap_{F\in\mathcal{F}} KF$. If $x \in X \setminus KF_0$, then $K^{-1}x \cap F_0$ is empty; because $K^{-1}x$ is compact, there is some $F \in \mathcal{F}$ such that $K^{-1}x \cap F = \emptyset$ (3A3Db), so that $x \notin KF$. Accordingly $KF_0 = \bigcap_{F\in\mathcal{F}} KF$. Similarly, $F_0K = \bigcap_{F\in\mathcal{F}} FK$. **Q**

(i) If $K \subseteq X$ is compact and $G \subseteq X$ is open, then $W = \{(x,y) : xKy \subseteq G\}$ is open in $X \times X$. **P** It is enough to deal with the case $K \neq \emptyset$. Take $(x_0, y_0) \in W$. For each $z \in K$, there is an open neighbourhood U_z of e such that $U_z z U_z U_z \subseteq x_0^{-1}Gy_0^{-1}$ (apply (e) with $F = X \setminus x_0^{-1}Gy_0^{-1}$). Now $\{zU_z : z \in K\}$ is an open cover of K so there are $z_0, \dots, z_n \in K$ such that $K \subseteq \bigcup_{i\le n} z_iU_{z_i}$. Set $U = \bigcap_{i\le n} U_{z_i}$; then $UKU \subseteq x_0^{-1}Gy_0^{-1}$ and $(x,y) \in W$ whenever $x \in x_0U$ and $y \in Uy_0$. Accordingly $(x_0,y_0) \in \operatorname{int} W$; as (x_0,y_0) is arbitrary, W is open. **Q**

It follows that $\{x : xK \subseteq G\}$, $\{x : Kx \subseteq G\}$ and $\{x : xKx^{-1} \subseteq G\}$ are open in X.

(j) If X is Hausdorff, $K \subseteq X$ is compact and U is a neighbourhood of e, there is a neighbourhood V of e such that $xy \in U$ whenever $x, y \in K$ and $yx \in V$; that is, $y^{-1}zy \in U$ whenever $z \in V$ and $y \in K$. **P** If U is open, then $\{yx : x, y \in K, xy \notin U\}$ is a closed set not containing e. Compare 4A5Oc below. **Q**

(k) Any open subgroup of X is also closed. (Császár 78, 11.2.12; Hewitt & Ross 63, 5.5; Folland 95, 2.1.)

(l) If X is locally compact, it has an open subgroup which is σ-compact. (Hewitt & Ross 63, 5.14; Folland 95, 2.3.)

(m) If Y is a subgroup of X, its closure $\overline{Y}$ is a subgroup of X. (Hewitt & Ross 63, 5.3; Folland 95, 2.1.)

(n) Let X be a group and $\mathcal{V}$ a family of subsets of X. Then there is a topology of X under which X is a topological group and $\mathcal{V}$ is a base of neighbourhoods of the identity iff

(α) $\mathcal{V}$ is a filter base;

(β) for every $V \in \mathcal{V}$ there is a $W \in \mathcal{V}$ such that $W^2 \subseteq V$;

(γ) for every $V \in \mathcal{V}$ there is a $W \in \mathcal{V}$ such that $W^{-1} \subseteq V$;

(δ) for every $V \in \mathcal{V}$ and $z \in X$ there is a $W \in \mathcal{V}$ such that $zWz^{-1} \subseteq V$.

In this case, there is exactly one such topology, and it is Hausdorff iff $\bigcap \mathcal{V} = \{e\}$.

Császár 78, 11.2.4; Hewitt & Ross 63, II.4.5.

4A5F Proposition (a) Let $(X,\mathfrak{T})$ and $(Y,\mathfrak{S})$ be topological groups. If $\phi : X \to Y$ is a group homomorphism which is continuous at the identity of X, it is continuous. (Császár 78, 11.2.17; Hewitt & Ross 63, 5.40.)

(b) Let X be a group and $\mathfrak{S}$, $\mathfrak{T}$ two topologies on X both making X a topological group. If every $\mathfrak{S}$-neighbourhood of the identity is a $\mathfrak{T}$-neighbourhood of the identity, then $\mathfrak{S} \subseteq \mathfrak{T}$. (Apply (a) to the identity map from $(X,\mathfrak{T})$ to $(X,\mathfrak{S})$.)

4A5G Proposition If $\langle X_i\rangle_{i\in I}$ is any family of topological groups, then $\prod_{i\in I} X_i$, with the product topology and the product group structure, is again a topological group. (Hewitt & Ross 63, 6.2.)

4A5H The uniformities of a topological group Let $(X,\mathfrak{T})$ be a topological group. Write $\mathcal{U}$ for the set of open neighbourhoods of the identity e of X.

(a) For $U \in \mathcal{U}$, set $W_U = \{(x,y) : xy^{-1} \in U\} \subseteq X \times X$. The family $\{W_U : U \in \mathcal{U}\}$ is a filter base, and the filter on $X \times X$ which it generates is a uniformity on X, the **right uniformity** of X. (**Warning!!** Some authors call this the 'left uniformity'.) This uniformity induces the topology $\mathfrak{T}$ (Császár 78, 11.2.7). It follows that $\mathfrak{T}$ is completely regular, therefore regular (4A2Ja, or Hewitt & Ross 63, 8.4).

(b) For $U \in \mathcal{U}$, set $\tilde{W}_U = \{(x,y) : xy^{-1} \in U,\, x^{-1}y \in U\} \subseteq X \times X$. The family $\{\tilde{W}_U : U \in \mathcal{U}\}$ is a filter base, and the filter on $X \times X$ which it generates is a uniformity on X, the **bilateral uniformity** of X. This uniformity induces the topology $\mathfrak{T}$. (Császár 78, 11.3.c.)

(c) $x \mapsto x^{-1}$ is uniformly continuous for the bilateral uniformity. (The check is elementary.)

(d) If X and Y are topological groups and $\phi : X \to Y$ is a continuous homomorphism, then ϕ is uniformly continuous for the bilateral uniformities. **P** If V is a neighbourhood of the identity in Y and $W_V = \{(y,z) : yz^{-1},\ y^{-1}z \text{ both belong to } V\}$ is the corresponding member of the bilateral uniformity on Y, then $U = \phi^{-1}[V]$ is a neighbourhood of the identity in X and $(\phi(x), \phi(w)) \in W_V$ whenever $(x,w) \in W_U$. **Q**

(e) If X is an abelian topological group, then the right and bilateral uniformities on X coincide, and may be called 'the' topological group uniformity of X; cf. 3A4Ad.

4A5I Definitions If X is a topological group and Z a topological space, an action of X on Z is 'continuous' or 'Borel measurable' if it is continuous, or Borel measurable, when regarded as a function from $X \times Z$ to Z.

Of course the left, right and conjugacy actions of a topological group on itself are all continuous.

4A5J Quotients under group actions, and quotient groups: Theorem (a) Let X be a topological space, Y a topological group, and $\bullet$ a continuous action of Y on X. Let Z be the set of orbits of the action, and for $x \in X$ write $\pi(x) \in Z$ for the orbit containing x.

(i) We have a topology on Z defined by saying that $V \subseteq Z$ is open iff $\pi^{-1}[V]$ is open in X. The canonical map $\pi : X \to Z$ is continuous and open.

(ii)(α) If Y is compact and X is Hausdorff, then Z is Hausdorff.

(β) If X is locally compact then Z is locally compact.

(b) Let X be a topological group, Y a subgroup of X, and Z the set of left cosets of Y in X. Set $\pi(x) = xY$ for $x \in X$.

(i) We have a topology on Z defined by saying that $V \subseteq Z$ is open iff $\pi^{-1}[V]$ is open in X. The canonical map $\pi : X \to Z$ is continuous and open.

(ii)(α) Z is Hausdorff iff Y is closed.

(β) If X is locally compact, so is Z.

(γ) If X is locally compact and Polish and Y is closed, then Z is Polish.

(δ) If X is locally compact and σ-compact and Y is closed and Z is metrizable, then Z is Polish.

(iii) We have a continuous action of X on Z defined by saying that $x\bullet\pi(x') = \pi(xx')$ for any x, $x' \in X$.

(iv) If Y is a normal subgroup of X, then the group operation on Z renders it a topological group.

proof (a)(i) It is elementary to check that $\{V : \pi^{-1}[V] \text{ is open}\}$ is a topology such that $\pi : X \to Z$ is continuous. To see that π is open, take an open set $U \subseteq X$ and consider

$$\pi^{-1}[\pi[U]] = \bigcup_{x' \in U}\{x : \pi(x) = \pi(x')\} = \bigcup_{x' \in U, y \in Y}\{x : x = y\bullet x'\} = \bigcup_{y \in Y} y\bullet U.$$

But as $x \mapsto y\bullet x$ is a homeomorphism for every $y \in Y$ (4A5Ea), every $y\bullet U$ is open, and the union $\pi^{-1}[\pi[U]]$ is open. So $\pi[U]$ is open in Z; as U is arbitrary, π is an open map.

(ii)($\boldsymbol{\alpha}$) Set $F = \{((x,x'),y) : x \in X,\ y \in Y,\ y\bullet x = x'\}$. Because the function $((x,x'),y) \mapsto (y\bullet x, x') : (X \times X) \times Y \to X \times X$ is continuous and $\{(x,x) : x \in X\}$ is closed in $X \times X$ (4A2F(a-iii)), F is closed. By 4A2Gm, the projection $\{(x,x') : \exists\ y \in Y,\ y\bullet x = x'\} = \{(x,x') : \pi(x) = \pi(x')\}$ is closed in $X \times X'$ and $\{(x,x') : \pi(x) \ne \pi(x')\}$ is open. Since $(x,x') \mapsto (\pi(x), \pi(x')) : X \times X \to Z \times Z$ is an open mapping (4A2B(f-iv)), $\{(z,z') : z \ne z'\}$ is open in $Z \times Z$, and Z is Hausdorff by 4A2F(a-iii) in the other direction.

($\boldsymbol{\beta}$) Use 4A2Gn.

(b)(i) Apply (a-i) to the right action $(y,x) \mapsto xy^{-1}$ of Y on X, or see Hewitt & Ross 63, 5.15-5.16.

(ii)($\boldsymbol{\alpha}$) By Hewitt & Ross 63, 5.21, Z is Hausdorff iff Y is closed.

($\boldsymbol{\beta}$) Use (a-ii-β), or see Hewitt & Ross 63, 5.22 or Folland 95, 2.2.

($\boldsymbol{\gamma}$) X has a countable network (4A2P(a-ii)), so Z also has (4A2Nd); since we have just seen that Z is locally compact and Hausdorff, it must be Polish (4A2Qh).

($\boldsymbol{\delta}$) Because X is σ-compact, so is its continuous image Z; we know from (α)-(β) that Z is locally compact and Hausdorff; we are supposing that it is metrizable; so it is Polish, by the other half of 4A2Qh.

(iii) I have noted in 4A5Cb that the formula given defines an action. If $V \subseteq Z$ is open and $x_0 \in X$, $z_0 \in Z$ are such that $x_0{\bullet}z_0 \in V$, take $x_0' \in X$ such that $\pi(x_0') = z_0$, and observe that $x_0x_0' \in \pi^{-1}[V]$, which is open. So there are open neighbourhoods V_0, V_0' of x_0, x_0' respectively such that $V_0V_0' \subseteq \pi^{-1}[V]$, and $x{\bullet}z \in V$ whenever $x \in V_0$ and $z \in \pi[V_0']$. Since $\pi[V_0']$ is an open neighbourhood of z_0, this is enough to show that $\bullet$ is continuous at (x_0, z_0).

(iv) Császár 78, 11.2.15; Hewitt & Ross 63, 5.26; Folland 95, 2.2.

4A5K Proposition Let X be a topological group with identity e.

(a) $Y = \overline{\{e\}}$ is a closed normal subgroup of X.

(b) Writing $\pi : X \to X/Y$ for the canonical map,

(i) a subset of X is open iff it is the inverse image of an open subset of X/Y,

(ii) a subset of X is closed iff it is the inverse image of a closed subset of X/Y,

(iii) $\pi[G]$ is a regular open set in X/Y for every regular open set $G \subseteq X$,

(iv) $\pi[F]$ is nowhere dense in X/Y for every nowhere dense set $F \subseteq X$,

(v) $\pi^{-1}[V]$ is nowhere dense in X for every nowhere dense $V \subseteq X/Y$.

proof (a) Császár 78, 11.2.13; Hewitt & Ross 63, 5.4; Folland 95, 2.3.

(b)(i)-(ii) Because π is continuous, the inverse image of an open or closed set is open or closed. In the other direction, if $G \subseteq X$ is open and $x \in G$, then $x\overline{\{e\}} = \overline{\{x\}} \subseteq G$, because X is regular (4A5Ha). So $G = GY = \pi^{-1}[\pi[G]]$. Since π is an open map (4A5J(a-i)), $\pi[G]$ is open and G is the inverse image of an open set. If $F \subseteq X$ is closed, $\pi[F] = (X/Y) \setminus \pi[X \setminus F]$ is closed and

$$F = X \setminus \pi^{-1}[\pi[X \setminus F]] = \pi^{-1}[(X/Y) \setminus \pi[X \setminus F]]$$

is the inverse image of a closed set.

(iii) If $A \subseteq X$, then $\pi[\overline{A}]$ is a closed set included in $\overline{\pi[A]}$ (because π is continuous), so $\overline{\pi[A]} = \pi[\overline{A}]$. If $G \subseteq X$ is a regular open set, then $\pi^{-1}[\operatorname{int}\pi[\overline{G}]]$ is an open subset of $\pi^{-1}[\pi[\overline{G}]] = \overline{G}$, so is included in $\operatorname{int}\overline{G} = G$. But this means that the open set $\pi[G]$ includes $\operatorname{int}\pi[\overline{G}] = \operatorname{int}\overline{\pi[G]}$, and $\pi[G] = \operatorname{int}\overline{\pi[G]}$ is a regular open set.

(iv) If $F \subseteq X$ is nowhere dense, then its closure is of the form $\pi^{-1}[V]$ for some closed set $V \subseteq X/Y$. Now if $H \subseteq X/Y$ is a non-empty open set, $\pi^{-1}[H]$ is a non-empty open subset of X, so is not included in F, and H cannot be included in V. Thus V is nowhere dense; but $V \supseteq \pi[F]$, so $\pi[F]$ is nowhere dense.

(v) If $V \subseteq X/Y$ is nowhere dense, and $G \subseteq X$ is open and not empty, then $G = \pi^{-1}[H]$ for some non-empty open $H \subseteq X/Y$. In this case, $H \setminus \overline{V}$ is non-empty, so $\pi^{-1}[H \setminus \overline{V}]$ is a non-empty open subset of G disjoint from $\pi^{-1}[V]$. As G is arbitrary, $\pi^{-1}[V]$ is nowhere dense.

4A5L Theorem Let X be a topological group and Y a normal subgroup of X. Let $\pi : X \to X/Y$ be the canonical homomorphism.

(a) If X' is another topological group and $\phi : X \to X'$ a continuous homomorphism with kernel including Y, then we have a continuous homomorphism $\psi : X/Y \to X'$ defined by the formula $\psi\pi = \phi$; ψ is injective iff Y is the kernel of ϕ.

(b) Suppose that K_1, K_2 are two subgroups of X/Y such that $K_2 \lhd K_1$. Set $Y_1 = \pi^{-1}[K_1]$ and $Y_2 = \pi^{-1}[K_2]$. Then $Y_2 \lhd Y_1$ and Y_1/Y_2 and K_1/K_2 are isomorphic as topological groups.

proof (a) This is elementary group theory, except for the claim that ψ is continuous. But if $H \subseteq X'$ is open, then $\psi^{-1}[H] = \pi[\phi^{-1}[H]]$ is open because ϕ is continuous and π is open (4A5J(a-i)); so ψ is continuous.

(b) See Hewitt & Ross 63, 5.35.

4A5M Proposition Let X be a topological group.

(a) Let Y be any subgroup of X. If X is given its bilateral uniformity, then the subspace uniformity on Y is the bilateral uniformity of Y. (Császár 78, 11.3.13.)

(b) If X is locally compact it is complete under its right uniformity. (Császár 78, 11.3.21.) If X is complete under its right uniformity it is complete under its bilateral uniformity. (Császár 78, 11.3.10.)

(c) Suppose that X is Hausdorff and that Y is a subgroup of X which is locally compact in its subspace topology. Then Y is closed in X. **P** Putting (a) and (b) together, we see that Y is complete in its subspace uniformity, therefore closed (3A4Fd). **Q**

4A5N Theorem Let X be a Hausdorff topological group. Then its completion $\widehat{X}$ under its bilateral uniformity can be endowed (in exactly one way) with a group structure rendering it a Hausdorff topological group in which the natural embedding of X in $\widehat{X}$ represents X as a dense subgroup of $\widehat{X}$. (Császár 78, 11.3.15.) If X has a neighbourhood of the identity which is totally bounded for the bilateral uniformity, then $\widehat{X}$ is locally compact. (Császár 78, 11.3.24.)

4A5O Proposition Let X be a topological group.

(a) If $A \subseteq X$, then the following are equiveridical: (i) A is totally bounded for the bilateral uniformity of X; (ii) for every neighbourhood U of the identity there is a finite set $I \subseteq X$ such that $A \subseteq IU \cap UI$.

(b) If A, $B \subseteq X$ are totally bounded for the bilateral uniformity of X, so are $A \cup B$, A^{-1} and AB. In particular, $\bigcup_{i \le n} x_i B$ is totally bounded for any $x_0, \dots , x_n \in X$.

(c) If $A \subseteq X$ is totally bounded for the bilateral uniformity, and U is any neighbourhood of the identity, then $\{y : xyx^{-1} \in U$ for every $x \in A\}$ is a neighbourhood of the identity.

(d) If X is the product of a family $\langle X_i\rangle_{i\in I}$ of topological groups, a subset A of X is totally bounded for the bilateral uniformity of X iff it is included in a product $\prod_{i\in I} A_i$ where $A_i \subseteq X_i$ is totally bounded for the bilateral uniformity of X_i for every $i \in I$.

(e) If X is locally compact, a subset of X is totally bounded for the bilateral uniformity iff it is relatively compact.

proof (a)(i)⇒(ii) Suppose that A is totally bounded, and that U is a neighbourhood of the identity e of X. Set

$$W = \{(x, y) : xy^{-1} \in U^{-1},\, x^{-1}y \in U\} = \{(x, y) : y \in Ux \cap xU\};$$

then W belongs to the uniformity, so there is a finite set $I \subseteq X$ such that $A \subseteq W[I]$. But $W[I] \subseteq UI \cap IU$, so $A \subseteq UI \cap IU$.

(ii)⇒(i) Now suppose that A satisfies the condition, and that W belongs to the uniformity. Then there is a neighbourhood U of e such that $\{(x, y) : xy^{-1} \in U,\, x^{-1}y \in U\} \subseteq W$. Let V be a neighbourhood of e such that $VV^{-1} \subseteq U$ and $V^{-1}V \subseteq U$. Let $I \subseteq X$ be a finite set such that $A \subseteq VI \cap IV$. For w, $z \in I$ set $A_{wz} = A \cap Vw \cap zV$. If x, $y \in A_{wz}$, $xy^{-1} \in Vww^{-1}V^{-1} \subseteq U$ and $x^{-1}y \in V^{-1}z^{-1}zV \subseteq U$. But this means that $A_{wz} \times A_{wz} \subseteq W$. So if we take a finite set J which meets every non-empty A_{wz}, $A \subseteq W[J]$. As W is arbitrary, A is totally bounded.

(b) Of course $A \cup B$ is totally bounded; this is immediate from the definition of 'totally bounded'. If U is a neighbourhood of e, so is U^{-1}, so there is a finite set $I \subseteq X$ such that $A \subseteq IU^{-1} \cap U^{-1}I$ and $A^{-1} \subseteq UI^{-1} \cap I^{-1}U$; as U is arbitrary, A^{-1} is totally bounded.

To see that AB also is totally bounded, let U be a neighbourhood of e, and take a neighbourhood V of e such that $VV \subseteq U$. Then there is a finite set $I \subseteq X$ such that $A \subseteq VI$ and $B \subseteq IV$. Let W be a neighbourhood of e such that $zWz^{-1} \cup z^{-1}Wz \subseteq V$ for every $z \in I$, and J a finite set such that $B \subseteq WJ$ and $A \subseteq JW$. Then

$$zW \subseteq Vz, \quad Wz \subseteq zV$$

for every $z \in I$, so

$$IW \subseteq VI, \quad WI \subseteq IV$$

and

$$AB \subseteq VIWJ \subseteq VVIJ \subseteq UK, \quad AB \subseteq JWIV \subseteq JIVV \subseteq KU$$

where $K = IJ \cup JI$ is finite. As U is arbitrary, AB is totally bounded.

(c) Let V be a neighbourhood of e such that $VVV^{-1} \subseteq U$. Let I be a finite set such that $A \subseteq VI$. Let W be a neighbourhood of e such that $zWz^{-1} \subseteq V$ for every $z \in I$. If now $y \in W$ and $x \in A$, there is a $z \in I$ such that $x \in Vz$, so that

$$xyx^{-1} \in VzWz^{-1}V^{-1} \subseteq VVV^{-1} \subseteq U.$$

Turning this round, $\{y : xyx^{-1} \in U$ for every $x \in A\}$ includes W and is a neighbourhood of e.

(d)(i) Suppose that A is totally bounded. Set $A_i = \pi_i[A]$ for each $i \in I$, where $\pi_i(x) = x(i)$ for $x \in X$. If U is a neighbourhood of the identity in X_i, then $V = \pi_i^{-1}[U]$ is a neighbourhood of the identity in X, so there is a finite set $J \subseteq X$ such that $A \subseteq JV \cap VJ$; now $A_i \subseteq KU \cap UK$, where $K = \pi_i[J]$ is finite. As U is arbitrary, A_i is totally bounded. This is true for every i, while $A \subseteq \prod_{i\in I} A_i$.

(ii) Suppose that $A \subseteq \prod_{i\in I} A_i$ where $A_i \subseteq X_i$ is totally bounded for each $i \in I$. If A is empty, of course it is totally bounded; assume that $A \neq \emptyset$. If $I = \emptyset$, then $X = \{\emptyset\}$ is the trivial group, and again A is totally bounded; so assume that I is non-empty. Let V be a neighbourhood of the identity in X. Then there are a non-empty finite set $L \subseteq I$ and a family $\langle U_i\rangle_{i\in L}$ such that U_i is a neighbourhood of the identity in X_i for each $i \in L$, and $V \supseteq \bigcap_{i\in L} \pi_i^{-1}U_i$. For each $i \in L$, let J_i be a finite subset of X_i such that $A_i \subseteq J_iU_i \cap U_iJ_i$. Set

$$J = \{x : x \in X,\ x(i) \text{ is the identity for } i \in I \setminus L,\ x(i) \in J_i \text{ for } i \in L\}.$$

Then J is finite and $A \subseteq JV \cap VJ$. As V is arbitrary, A is totally bounded.

(e) Use (a).

4A5P Lemma Let X be a locally compact Hausdorff topological group. Take $f \in C_k(X)$, the space of continuous real-valued functions on X with compact supports.

(a) Let $K \subseteq X$ be a compact set. Then for any $\epsilon > 0$ there is a neighbourhood W of the identity e of X such that $|f(xay) - f(xby)| \le \epsilon$ whenever $x \in K$, $y \in X$ and $ab^{-1} \in W$.

(b) For any $x_0 \in X$, there is a non-negative $f^* \in C_k(X)$ such that for every $\epsilon > 0$ there is an open set G containing x_0 such that $|f(xy) - f(x_0y)| \le \epsilon f^*(y)$ for every $x \in G$ and $y \in X$.

proof (a) By 4A2Jf and 4A5Ha, f is uniformly continuous for the right uniformity of X. There is therefore a symmetric neighbourhood U of e such that $|f(y_1) - f(y_2)| \le \epsilon$ whenever y_1, $y_2 \in X$ and $y_1y_2^{-1} \in U$. By 4A5Oc, there is a symmetric neighbourhood W of e such that $xzx^{-1} \in U$ whenever $x \in K$ and $z \in W$.

Now suppose that $x \in K$, $y \in X$ and $ab^{-1} \in W$. Then $(xay)(xby)^{-1} = xab^{-1}x^{-1} \in U$, so $|f(xay) - f(xby)| \le \epsilon$, as required.

(b) We need a trifling refinement of the ideas above.

(i) Suppose for the moment that $x_0 = e$. Set $L = \overline{\{x : f(x) \neq 0\}}$ and let V be a compact symmetric neighbourhood of the identity e, so that L and VL are compact. Let $f^* \in C_k(X)$ be such that $f^* \ge \chi(VL)$ (4A2G(e-i)). Given $\epsilon > 0$, take U as in (a), so that U is a symmetric neighbourhood of e and $|f(y_1)-f(y_2)| \le \epsilon$ whenever $y_1y_2^{-1} \in U$; this time arrange further that $U \subseteq V$. Then if $x \in U$ and $y \in X$,

either y and xy belong to VL, while $(xy)y^{-1} \in U$, so $|f(xy) - f(y)| \le \epsilon \le \epsilon f^*(y)$

or neither y nor xy belongs to L, so $|f(xy) - f(y)| = 0 \le \epsilon f^*(y)$.

(ii) For the general case, set $f_0(x) = f(x_0x)$ for $x \in X$. Because $x \mapsto x_0x$ is a homeomorphism, $f_0 \in C_k(X)$. By (i), we have a non-negative $f^* \in C_k(X)$ such that for every $\epsilon > 0$ there is a neighbourhood G_ϵ of e such that $|f_0(xy) - f_0(y)| \le \epsilon f^*(y)$ whenever $x \in G_\epsilon$ and $y \in X$. Now, given $\epsilon > 0$, $G' = x_0G_\epsilon$ is a neighbourhood of x_0 and $|f(xy) - f(x_0y)| \le \epsilon f^*(y)$ whenever $x \in G'$ and $y \in X$. So f^* witnesses that the result is true.

4A5Q Metrizable groups: Proposition Let $(X, \mathfrak{T})$ be a topological group. Then the following are equiveridical:

(i) X is metrizable;

(ii) the identity e of X has a countable neighbourhood base;

(iii) there is a metric ρ on X, inducing the topology $\mathfrak{T}$, which is **right-translation-invariant**, that is, $\rho(x_1, x_2) = \rho(x_1y, x_2y)$ for all x_1, x_2, $y \in X$;

(iv) there is a right-translation-invariant metric on X which induces the right uniformity of X;

(v) the bilateral uniformity of X is metrizable.

proof Császár 78, 11.2.10 and 11.3.2.

Warning! A Polish group (4A5Db) is of course metrizable, so has a right-translation-invariant metric inducing its topology. At the same time, it has a complete metric inducing its topology. But there is no suggestion that these two metrics should be the same, or even induce the same uniformity (441Xr).

4A5R Corollary If X is a locally compact topological group and $\{e\}$ is a G_δ set in X, then X is metrizable. (Put 4A5Q and 4A2Kf together, or see Hewitt & Ross 63, 8.5.)

4A5S Lemma Let X be a σ-compact locally compact Hausdorff topological group and $\langle U_n\rangle_{n\in\mathbb{N}}$ any sequence of neighbourhoods of the identity in X. Then X has a compact normal subgroup $Y\subseteq\bigcap_{n\in\mathbb{N}}U_n$ such that $Z=X/Y$ is Polish.

proof Let $\langle K_n\rangle_{n\in\mathbb{N}}$ be a sequence of compact sets covering X. Choose inductively a sequence $\langle V_n\rangle_{n\in\mathbb{N}}$ of compact neighbourhoods of e such that, for each $n\in\mathbb{N}$,

(α) $V_{n+1}\subseteq V_n^{-1}$, $\quad V_{n+1}V_{n+1}\subseteq V_n$, $\quad V_n\subseteq U_n$,

(β) $xyx^{-1}\in V_n$ whenever $y\in V_{n+1}$ and $x\in\bigcup_{i\le n}K_i$.

(When we come to choose V_{n+1}, we can achieve (α) because inversion and multiplication are continuous, and (β) by 4A5Oc; and we can then shrink V_{n+1} to a compact neighbourhood of e because X is locally compact.) Set $Y=\bigcap_{n\in\mathbb{N}}V_n$. Then ($\alpha$) is enough to ensure that Y is a compact subgroup of X included in $\bigcap_{n\in\mathbb{N}}U_n$, while ($\beta$) ensures that Y is normal, because for any $x\in X$ there is an $n\in\mathbb{N}$ such that $xV_{m+1}x^{-1}\subseteq V_m$ for every $m\ge n$.

Let $\pi:X\to Z$ be the canonical map. Then $C=\bigcap_{n\in\mathbb{N}}\pi[\operatorname{int}V_n]$ is a G_δ subset of Z, because π is open (4A5J(a-i) again). But

$$\pi^{-1}[C]\subseteq\bigcap_{n\in\mathbb{N}}\pi^{-1}[\pi[V_{n+1}]]=\bigcap_{n\in\mathbb{N}}V_{n+1}Y\subseteq\bigcap_{n\in\mathbb{N}}V_n=Y,$$

so $C=\{e_Z\}$, writing e_Z for the identity of Z. Thus $\{e_Z\}$ is a G_δ set; as Z is locally compact and Hausdorff (4A5J(b-ii)), it is metrizable (4A5R). By 4A5J(b-ii-δ), Z is Polish.

***4A5T** I shall not rely on the following fact, but it will help you to make sense of some of the results of this volume.

Theorem A compact Hausdorff topological group is dyadic.

proof Uspenskiĭ 88.

Version of 8.12.10

4A6 Banach algebras

I give results which are needed for Chapter 44. Those down to 4A6K should be in any introductory text on normed algebras; 4A6L-4A6O, as expressed here, are a little more specialized. As with normed spaces or linear topological spaces, Banach algebras may be defined over either $\mathbb{R}$ or $\mathbb{C}$. In §445 we need complex Banach algebras, but in §446 I think the ideas are clearer in the context of real Banach algebras. Accordingly, as in §2A4, I express as much as possible of the theory in terms applicable equally to either, speaking of 'normed algebras' or 'Banach algebras' without qualification, and using the symbol $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$ to represent the field of scalars. Since (at least, if you keep to the path indicated here) the ideas are independent of which field we work with, you will have no difficulty in applying the arguments given in Folland 95 or Hewitt & Ross 63 for the complex case to the real case. In 4A6B and 4A6I-4A6K, however, we have results which apply only to 'complex' Banach algebras, in which the underlying field is taken to be $\mathbb{C}$.

4A6A Definition (a) I repeat a definition from §2A4. A **normed algebra** is a normed space U together with a multiplication, a binary operator $\times$ on U, such that

$$u\times(v\times w)=(u\times v)\times w,$$

$$u\times(v+w)=(u\times v)+(u\times w),\quad (u+v)\times w=(u\times w)+(v\times w),$$

$$(\alpha u)\times v=u\times(\alpha v)=\alpha(u\times v),$$

$$\|u\times v\|\le\|u\|\|v\|$$

for all u, v, $w\in U$ and $\alpha\in\frac{\mathbb{R}}{\mathbb{C}}$. A normed algebra is **commutative** if its multiplication is commutative.

(b) A **Banach algebra** is a normed algebra which is a Banach space. A **unital Banach algebra** is a Banach algebra with a multiplicative identity e such that $\|e\|=1$. (**Warning:** some authors reserve the term 'Banach algebra' for what I call a 'unital Banach algebra'.)

In a unital Banach algebra I will always use the letter e for the identity.

4A6B Stone-Weierstrass Theorem: fourth form Let X be a locally compact Hausdorff space, and $C_0=C_0(X;\mathbb{C})$ the complex Banach algebra of continuous functions $f:X\to\mathbb{C}$ such that $\{x:|f(x)|\ge\epsilon\}$ is compact for every $\epsilon>0$. Let $A\subseteq C_0$ be such that

A is a linear subspace of C_0,

$f\times g\in A$ for every f, $g\in A$,

the complex conjugate $\bar f$ of f belongs to A for every $f\in A$,

for every $x\in X$ there is an $f\in A$ such that $f(x)\ne0$,

whenever x, y are distinct points of X there is an $f\in A$ such that $f(x)\ne f(y)$.

Then A is $\|\;\|_\infty$-dense in C_0.

proof Let $X_\infty=X\cup\{\infty\}$ be the one-point compactification of X (3A3O). For $f\in C_0$ write $f^\#$ for the extension of f to $X\cup\{\infty\}$ with $f^\#(\infty)=0$, so that $f^\#\in C_b(X_\infty;\mathbb{C})$. Let $B\subseteq C_b(X_\infty;\mathbb{C})$ be the set of all functions of the form $f^\#+\alpha\chi X_\infty$ where $f\in A$ and $\alpha\in\mathbb{C}$. Then B is a subalgebra of $C_b(X\cup\{\infty\})$ which contains complex conjugates of its members and constant functions and separates the points of X_∞.

Take any $h\in C_0$ and $\epsilon>0$. By the 'third form' of the Stone-Weierstrass theorem (281G), there is a $g\in B$ such that $\|g-h^\#\|_\infty\le\frac12\epsilon$. Express g as $f^\#+\alpha\chi X_\infty$ where $f\in A$ and $\alpha\in\mathbb{C}$. Then

$$|\alpha|=|g(\infty)|=|g(\infty)-h^\#(\infty)|\le\tfrac12\epsilon,$$

so

$$\|h-f\|_\infty=\|h^\#-f^\#\|_\infty\le\|h^\#-g\|_\infty+\|g-f^\#\|_\infty\le\tfrac12\epsilon+|\alpha|\le\epsilon.$$

As h and ϵ are arbitrary, A is dense in C_0.

4A6C Proposition If U is any Banach space other than $\{0\}$, then the space $\mathsf{B}(U;U)$ of bounded linear operators from U to itself is a unital Banach algebra. (Köthe 69, 14.6.)

4A6D Proposition Any normed algebra U can be embedded as a subalgebra of a unital Banach algebra V, in such a way that if U is commutative so is V. (Folland 95, §1.3; Hewitt & Ross 63, C.3.)

4A6E Proposition Let U be a unital Banach algebra and $W\subseteq U$ a closed proper ideal. Then U/W, with the quotient linear structure, ring structure and norm, is a unital Banach algebra. (Hewitt & Ross 63, C.2.)

4A6F Proposition If U is a Banach algebra and $\phi:U\to\frac{\mathbb{R}}{\mathbb{C}}$ is a multiplicative linear functional, then $|\phi(u)|\le\|u\|$ for every $u\in U$.

proof ? Otherwise, there is a u such that $|\phi(u)|>\|u\|$; set $v=\frac{1}{\phi(u)}u$, so that $\phi(v)=1$ and $\|v\|<1$. Since $\|v^n\|\le\|v\|^n$ for every $n\ge1$, $w=\sum_{n\in\mathbb{N}\setminus\{0\}}v^n$ is defined in U (4A4Ie), and $w=vw+v$ (because $u\mapsto vu$ is a continuous linear operator, so we can use 4A4Bh to see that $vw=\sum_{n\in\mathbb{N}\setminus\{0\}}v^{n+1}$). But this means that $\phi(w)=\phi(v)\phi(w)+\phi(v)=\phi(w)+1$, which is impossible. **X**

4A6G Definition Let U be a normed algebra and $u \in U$.

(a) For any $u \in U$, $\lim_{n\to\infty} \|u^n\|^{1/n}$ is defined and equal to $\inf_{n\ge1} \|u^n\|^{1/n}$. (Hewitt & Ross 63, C.4.)

(b) This common value is called the **spectral radius** of u.

4A6H Theorem If U is a unital Banach algebra, then the set R of invertible elements is open, and $u \mapsto u^{-1}$ is a continuous function from R to itself. If $v \in U$ and $\|v - e\| < 1$, then $v \in R$ and $\|v^{-1} - e\| \le \frac{\|v-e\|}{1-\|v-e\|}$. (Folland 95, 1.4; Hewitt & Ross 63, C.8 & C.10; Rudin 91, 10.7 & 10.12.)

4A6I Theorem Let U be a complex unital Banach algebra and $u \in U$. Write r for the spectral radius of u.

(a) If $\zeta \in \mathbb{C}$ and $|\zeta| > r$ then $\zeta e - u$ is invertible.

(b) There is a ζ such that $|\zeta| = r$ and $\zeta e - u$ is not invertible.

proof Folland 95, 1.8; Hewitt & Ross 63, C.24; Rudin 91, 1.13.

4A6J Theorem Let U be a commutative complex unital Banach algebra, and $u \in U$. Then for any $\zeta \in \mathbb{C}$ the following are equiveridical:

(i) there is a non-zero multiplicative linear functional $\phi : U \to \mathbb{C}$ such that $\phi(u) = \zeta$;

(ii) $\zeta e - u$ is not invertible.

proof Folland 95, 1.13; Hewitt & Ross 63, C.20; Rudin 91, 11.5.

4A6K Corollary Let U be a commutative complex Banach algebra and $u \in U$. Then its spectral radius $r(u)$ is $\max\{|\phi(u)| : \phi$ is a multiplicative linear functional on $U\}$. (Folland 95, 1.13; Hewitt & Ross 63, C.20; Rudin 91, 11.9.)

4A6L Exponentiation Let U be a unital Banach algebra. For any $u \in U$, $\sum_{k=0}^\infty \|\frac{1}{k!}u^k\| \le \sum_{k=0}^\infty \frac{1}{k!}\|u\|^k$ is finite, so

$$\exp(u) = \sum_{k\in\mathbb{N}} \frac{1}{k!}u^k$$

is defined in U (4A4Ie). (In this formula, interpret u^0 as e for every u.)

4A6M Lemma Let U be a unital Banach algebra.

(a) If u, $v \in U$ and $\max(\|u\|, \|v\|) \le \gamma$ then $\|\exp(u) - \exp(v) - u + v\| \le \|u - v\|(\exp\gamma - 1)$. So if $\max(\|u\|, \|v\|) \le \frac{2}{3}$ and $\exp(u) = \exp(v)$ then $u = v$.

(b) If $\|u - e\| \le \frac{1}{6}$ then there is a v such that $\exp(v) = u$ and $\|v\| \le 2\|u - e\|$.

(c) If u, $v \in U$ and $uv = vu$ then $\exp(u + v) = \exp(u)\exp(v)$.

proof (a) Note first that if $k \ge 1$ then

$$\begin{aligned}\|u^k - v^k\| &= \|\sum_{i=0}^{k-1} u^{k-i}v^i - u^{k-i-1}v^{i+1}\| = \|\sum_{i=0}^{k-1} u^{k-i-1}(u-v)v^i\| \\ &\le \sum_{i=0}^{k-1} \|u\|^{k-i-1}\|u-v\|\|v\|^i \le \sum_{i=0}^{k-1} \gamma^{k-1}\|u-v\| = k\gamma^{k-1}\|u-v\|.\end{aligned}$$

So

$$\|\exp(u)-\exp(v)-u+v\| = \|\sum_{k\in\mathbb{N}\setminus\{0,1\}}\frac{1}{k!}(u^k-v^k)\| \le \sum_{k=2}^{\infty}\frac{1}{k!}\|u^k-v^k\|$$

$$\le \sum_{k=2}^{\infty}\frac{k}{k!}\gamma^{k-1}\|u-v\| = \|u-v\|(\exp\gamma-1).$$

Now if $\exp(u)=\exp(v)$ and $\gamma\le\frac{2}{3}$, $0\le\exp\gamma-1<1$ so $\|u-v\|=0$ and $u=v$.

(b) Set $\gamma=\|u-e\|$. Define $\langle v_n\rangle_{n\in\mathbb{N}}$ in U by setting $v_0=0$, $v_{n+1}=v_n+u-\exp(v_n)$ for $n\in\mathbb{N}$. Then

$$\|v_{n+1}-v_n\| = \|u-\exp(v_n)\|\le 2^{-n}\gamma, \quad \|v_n\|\le 2(1-2^{-n})\gamma$$

for every $n\in\mathbb{N}$. **P** Induce on n. The induction starts with $\|v_0\|=0$ and $\|u-\exp(v_0)\|=\|u-e\|=\gamma$. Given that $\|v_n\|\le 2(1-2^{-n})\gamma$ and $\|u-\exp(v_n)\|\le 2^{-n}\gamma$, then

$$\|v_{n+1}\|\le\|v_n\|+\|u-\exp(v_n)\|\le 2(1-2^{-n})\gamma+2^{-n}\gamma=2(1-2^{-n-1})\gamma.$$

Now $\max(\|v_{n+1}\|,\|v_n\|)\le 2\gamma\le\frac{1}{3}$, so

$$\|u-\exp(v_{n+1})\| = \|v_{n+1}-v_n+\exp(v_n)-\exp(v_{n+1})\| \le \|v_{n+1}-v_n\|(\exp\frac{1}{3}-1)$$

$$\le\frac{1}{2}\|v_{n+1}-v_n\| = \frac{1}{2}\|u-\exp(v_n)\|\le 2^{-n-1}\gamma,$$

and the induction continues. **Q**

Since $\sum_{n=0}^{\infty}\|v_{n+1}-v_n\|$ is finite, $v=\lim_{n\to\infty}v_n$ is defined in U, and $\|v\|=\lim_{n\to\infty}\|v_n\|\le 2\gamma$. Accordingly

$$\|\exp(v)-\exp(v_n)\|\le\|v-v_n\|+\|\exp(v)-\exp(v_n)-v+v_n\|$$

$$\le\|v-v_n\|(1+\exp\frac{1}{3}-1)\to 0$$

as $n\to\infty$, and $\exp(v)=\lim_{n\to\infty}\exp(v_n)=u$.

(c) Because $uv=vu$, $(u+v)^m=\sum_{j=0}^{m}\frac{m!}{j!(m-j)!}u^jv^{m-j}$ for every $m\in\mathbb{N}$ (induce on m; the point is that $uv^j=v^ju$ for every $j\in\mathbb{N}$). Next, $\sum_{j,k\in\mathbb{N}}\frac{1}{j!k!}\|u\|^j\|v\|^k$ is finite. So

$$\exp(u+v)=\sum_{m\in\mathbb{N}}\frac{1}{m!}(u+v)^m$$

$$=\sum_{m\in\mathbb{N}}\Bigl(\sum_{j+k=m}\frac{1}{j!k!}u^jv^k\Bigr)=\sum_{(j,k)\in\mathbb{N}\times\mathbb{N}}\frac{1}{j!k!}u^jv^k$$

(using 4A4I(e-ii))

$$=\sum_{j\in\mathbb{N}}\Bigl(\sum_{k\in\mathbb{N}}\frac{1}{j!k!}u^jv^k\Bigr)$$

(4A4I(e-ii) again)

$$=\sum_{j\in\mathbb{N}}\Bigl(\frac{1}{j!}u^j\sum_{k\in\mathbb{N}}\frac{1}{k!}v^k\Bigr)$$

(by 4A4Bh, because $w\mapsto\frac{1}{j!}u^jw$ is a continuous linear operator for each j)

$$=\sum_{j\in\mathbb{N}}\frac{1}{j!}u^j\exp(v)=\Bigl(\sum_{j\in\mathbb{N}}\frac{1}{j!}u^j\Bigr)\exp(v)$$

(4A4Bh again)

$$=\exp(u)\exp(v),$$

as claimed.

4A6N Lemma If U is a unital Banach algebra, $u \in U$ and $\|u^n - e\| \le \frac{1}{6}$ for every $n \in \mathbb{N}$, then $u = e$.

proof For every $n \in \mathbb{N}$ there is a $v_n \in U$ such that $\exp(v_n) = u^{2^n}$ and $\|v_n\| \le \frac{1}{3}$ (4A6Mb). Then $\exp(v_{n+1}) = \exp(v_n)^2 = \exp(2v_n)$ (4A6Mc), $\|v_{n+1}\| \le \frac{1}{3}$ and $\|2v_n\| \le \frac{1}{3}$ so $v_{n+1} = 2v_n$ for every n (4A6Ma). Inducing on n, $v_n = 2^n v_0$ for every n, so that $\|v_0\| \le 2^{-n}\|v_n\| \to 0$ as $n \to \infty$, and $u = \exp(v_0) = e$.

4A6O Proposition Let U be a normed algebra, and U^*, U^{**} its dual and bidual as a normed space. For a bounded linear operator $T : U \to U$ let $T' : U^* \to U^*$ be the adjoint of T and $T'' : U^{**} \to U^{**}$ the adjoint of T'.

(a) We have bilinear maps, all of norm at most 1,

$$(f, x) \mapsto f \circ x : U^* \times U \to U^*,$$
$$(\phi, f) \mapsto \phi \circ f : U^{**} \times U^* \to U^*,$$
$$(\phi, \psi) \mapsto \phi \circ \psi : U^{**} \times U^{**} \to U^{**}$$

defined by the formulae

$$(f \circ x)(y) = f(xy),$$
$$(\phi \circ f)(x) = \phi(f \circ x),$$
$$(\phi \circ \psi)(f) = \phi(\psi \circ f)$$

for all x, $y \in U$, $f \in U^*$ and ϕ, $\psi \in U^{**}$.

(b)(i) Suppose that $S : U \to U$ is a bounded linear operator such that $S(xy) = (Sx)y$ for all x, $y \in U$. Then $S''(\phi \circ \psi) = (S''\phi) \circ \psi$ for all ϕ, $\psi \in U^{**}$.

(ii) Suppose that $T : U \to U$ is a bounded linear operator such that $T(xy) = x(Ty)$ for all x, $y \in U$. Then $T''(\phi \circ \psi) = \phi \circ (T''\psi)$ for all ϕ, $\psi \in U^{**}$.

proof (a) The calculations are elementary if we take them one at a time.

(b)(i)(α) $(S'f) \circ x = f \circ (Sx)$ for every $f \in U^*$ and $x \in U$. **P**

$$((S'f) \circ x)(y) = (S'f)(xy) = f(S(xy)) = f((Sx)y) = (f \circ (Sx))(y)$$

for every $y \in U$. **Q**

(β) $\psi \circ (S'f) = S'(\psi \circ f)$ for every $f \in U^*$. **P**

$$(\psi \circ (S'f))(x) = \psi((S'f) \circ x) = \psi(f \circ (Sx)) = (\psi \circ f)(Sx) = (S'(\psi \circ f))(x)$$

for every $x \in U$. **Q**

(γ) So

$$\begin{aligned}(S''(\phi \circ \psi))(f) &= (\phi \circ \psi)(S'f) = \phi(\psi \circ (S'f)) \\ &= \phi(S'(\psi \circ f)) = (S''\phi)(\psi \circ f) = ((S''\phi) \circ \psi)(f)\end{aligned}$$

for every $f \in U^*$, and $S''(\phi \circ \psi) = (S''\phi) \circ \psi$.

(ii)(α) $(T'f) \circ x = T'(f \circ x)$ for every $f \in U^*$ and $x \in U$. **P**

$$((T'f) \circ x)(y) = (T'f)(xy) = f(T(xy)) = f(x(Ty)) = (f \circ x)(Ty) = (T'(f \circ x))(y)$$

for every $y \in U$. **Q**

(β) $\psi \circ (T'f) = (T''\psi) \circ f$ for every $f \in U^*$. **P**

$$\begin{aligned}(\psi \circ (T'f))(x) &= \psi((T'f) \circ x) = \psi(T'(f \circ x)) \\ &= (T''\psi)(f \circ x) = ((T''\psi) \circ f)(x)\end{aligned}$$

for every $x \in U$. **Q**

(γ) So

$$(T''(\phi\circ\psi))(f) = (\phi\circ\psi)(T'f) = \phi(\psi\circ(T'f)) = \phi((T''\psi)\circ f) = (\phi\circ(T''\psi))(f)$$

for every $f \in U'$, and $T''(\phi\circ\psi) = \phi\circ(T''\psi)$.

Remark I must not abandon you at this point without telling you that $\circ : U^{**} \times U^{**} \to U^{**}$ is an **Arens multiplication**, and that it is associative, so that that U^{**} is a Banach algebra.

Version of 13.3.22

4A7 Algebraic topology

A fundamental theorem about the topology of Euclidean space is used in §472. (8.7.22) no idea what I was doing here

4A7A Definitions Suppose that X and Y are topological spaces, and $f : X \to Y$, $g : X \to Y$ are continuous functions.

(a) A **homotopy from** f **to** g is a continuous function $F : X \times [0,1] \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for every $x \in X$.

(b) f and g are **homotopic** if there is a homotopy from f to g.

4A7B Theorem If $r \ge 1$ and $S_{r-1} = \partial B(\mathbf{0}.1)$ is the unit sphere $\{x : \|x\| = 1\}$ in $\mathbb{R}^r$, then the identity function from S_{r-1} to itself is not homotopic to a constant function.

proof

4A7C Corollary If $r \ge 1$, $B(0,1)$ is the unit ball in $\mathbb{R}^r$ and $h : B(\mathbf{0},1) \to B(\mathbf{0},1)$ is a continuous function such that $h[S_{r-1}] = S_{r-1}$, then $h[B(\mathbf{0},1)] = B(\mathbf{0},1)$.

proof

472G Theorem (Bagnara Gennaioli Leccese & Luongo p22) Let $r \ge 1$ be an integer, $B \subseteq \mathbb{R}^r$ a closed balls, ρ_E the Euclidean metric on $\mathbb{R}^r$, and ρ a metric on B inducing the usual topology on B. Then there is a (ρ,ρ_E)-Lipschitz surjection from B onto itself.

proof (a) For the time being (down to the end of (c) below), suppose that $B = \{x : x \in \mathbb{R}^r, \|x\| \le 1\}$ is the ordinary unit ball of $\mathbb{R}^r$. Fix i such that $1 \le i \le r$ for the moment, and write π_i for the ith coordinate map from $\mathbb{R}^r$ to $\mathbb{R}$.

(i) For $\alpha > 0$ and $x \in B$, set

$$f_i(\alpha,x) = \inf_{z\in B} \pi_i(z) + \frac{1}{\alpha}\rho(x,z);$$

since B is compact and $z \mapsto \pi_i(z) + \frac{1}{\alpha}\rho(x,z)$ is continuous, the infimum is attained at $z_{x\alpha i}$ say. (We have two metrics in this theorem, but only one topology on B, so 'continuous' and 'compact' in the last sentence are unambiguous.)

(ii) Observe that

$$-1 = \inf_{z\in B} \pi_i(z) \le f_i(\alpha,x) \le \pi_i(x) + \frac{1}{\alpha}\rho(x,x) \le \pi_i(x) \le 1$$

for every $x \in B$.

(iii) If $0 < \alpha \le \beta$ and $x \in B$ then

$$f_i(\beta,x)=\inf_{z\in B}\pi_i(z)+\frac{1}{\beta}\rho(x,z)\le\inf_{z\in B}\pi_i(z)+\frac{1}{\alpha}\rho(x,z)=f_i(\alpha,x)$$

and

$$\begin{aligned}f_i(\alpha,x)-f_i(\beta,x)&\le\sup_{z\in B}\bigl((\pi_i(z)+\frac{1}{\alpha}\rho(x,z))-(\pi_i(z)+\frac{1}{\beta}\rho(x,z))\bigr)\\&=(\frac{1}{\alpha}-\frac{1}{\beta})\sup_{z\in B}\rho(x,z)\le M(\frac{1}{\alpha}-\frac{1}{\beta})\end{aligned}$$

where $M=\sup_{y,z\in B}\rho(y,z)$ is finite, again because B is compact.

(iv) If x, $y\in B$ then

$$\begin{aligned}f_i(\alpha,y)&=\inf_{z\in B}\pi_i(z)+\frac{1}{\alpha}\rho(y,z)\le\inf_{z\in B}\pi_i(z)+\frac{1}{\alpha}(\rho(y,x)+\rho(x,z))\\&=\frac{1}{\alpha}\rho(y,x)+\inf_{z\in B}\pi_i(z)+\frac{1}{\alpha}\rho(x,z)=\frac{1}{\alpha}\rho(y,x)+f_i(\alpha,x)\end{aligned}$$

and similarly $f_i(\alpha,x)\le\frac{1}{\alpha}\rho(x,y)+f_i(\alpha,y)$, so $|f_i(\alpha,x)-f_i(\alpha,y)|\le\frac{1}{\alpha}\rho(x,y)$.

(v) For any $x\in B$, $\alpha\mapsto f_i(\alpha,x)$ is non-increasing and bounded above, so $\lim_{\alpha\downarrow 0}f_i(\alpha,x)=\sup_{\alpha>0}f_i(\alpha,x)$ is defined. But now we have

$$\pi_i(z_{x\alpha i})+\frac{1}{\alpha}\rho(x,z_{x\alpha i})=f_i(\alpha,x)\le\pi_i(x)$$

so $\rho(x,z_{x\alpha i})\le\alpha$ for every $\alpha>0$ and $x=\lim_{\alpha\downarrow 0}z_{x\alpha i}$. It follows that $\pi_i(x)=\lim_{\alpha\downarrow 0}\pi_i(z_{x\alpha i})$ and

$$\begin{aligned}\pi_i(x)&\ge\lim_{\alpha\downarrow 0}f_i(\alpha,x)=\lim_{\alpha\downarrow 0}(\pi_i(z_{x\alpha i})+\frac{1}{\alpha}\rho(x,z_{x\alpha i}))\\&=\pi_i(x)+\lim_{\alpha\downarrow 0}\frac{1}{\alpha}\rho(x,z_{x\alpha i})\ge\pi_i(x);\end{aligned}$$

thus $\lim_{\alpha\downarrow 0}f_i(\alpha,x)=\pi_i(x)$.

As the real-valued functions $x\mapsto\pi_i(x)-f_i(\alpha,x)$ are all continuous and B is compact, Dini's theorem (436Ic) tells us that

$$\lim_{\alpha\downarrow 0}\sup_{x\in B}(\pi_i(x)-f_i(\alpha,x))=\inf_{\alpha>0}\sup_{x\in B}(\pi_i(x)-f_i(\alpha,x))=0.$$

(vi) Extend f_i to the whole of $[0,\infty[\,\times B$ by setting $f_i(0,x)=\pi_i(x)$ for every $x\in B$. Then f_i is continuous. **P** Take $\alpha\ge 0$ and $x\in B$. If $\alpha>0$, then

$$\begin{aligned}|f_i(\beta,y)-f_i(\alpha,x)|&\le|f_i(\beta,y)-f_i(\alpha,y)|+|f_i(\alpha,y)-f_i(\alpha,x)|\\&\le M|\frac{1}{\beta}-\frac{1}{\alpha}|+\frac{1}{\alpha}\rho(y,x)\end{aligned}$$

whenever $\beta>0$ and $y\in B$, by (iii) and (iv) above. So f_i is continuous at (α,x). If $\alpha=0$, then

$$\lim_{\beta\downarrow 0}\sup_{y\in B}|f_i(\beta,y)-\pi_i(y)|=0=\sup_{y\in B}|f_i(0,y)-\pi_i(y)|$$

so

$$\begin{aligned}&\limsup_{(\beta,y)\to(\alpha,x)}|f_i(\beta,y)-f_i(\alpha,x)|\\&\qquad\le\limsup_{(\beta,y)\to(\alpha,x)}|f_i(\beta,y)-\pi_i(y)|+\limsup_{(\beta,y)\to(\alpha,x)}|\pi_i(y)-\pi_i(x)|=0\end{aligned}$$

and again f_i is continuous at (α,x). **Q**

(b) Define $f:[0,\infty[\,\times B\to\mathbb{R}^r$ by setting

$$f(\alpha,x)=(f_1(\alpha,x),\dots,f_r(\alpha,x))$$

for $x \in B$. By (a-vi), f is continuous, and if $\alpha > 0$ then by (a-iv)

$$\|f(\alpha,x) - f(\alpha,y)\| \le \sum_{i=1}^{r} |f_i(\alpha,x) - f_i(\alpha,y)| \le \frac{r}{\alpha}\rho(x,y)$$

for all x, $y \in B$ so $x \mapsto f(\alpha,x)$ is (ρ,ρ_E)-Lipschitz. By (a-v),

$$\limsup_{\alpha\downarrow 0} \sup_{x\in B} \|f(\alpha,x) - x\| \le \sum_{i=1}^{r} \limsup_{\alpha\downarrow 0} \sup_{x\in B} |\pi_i(x) - f_i(\alpha,x)| = 0$$

so there is a $\delta > 0$ such that $\|f(\alpha,x) - x\| \le \frac{1}{2}$ whenever $x \in B$ and $0 < \alpha \le \delta$.

(c) For $x \in \mathbb{R}^r$ let $h(x) \in B$ be such that $\|x - h(x)\| = \min_{z\in B} \|x - z\|$ (3A5Md); then $\|h(x) - h(y)\| \le \|x - y\|$ for all x, $y \in \mathbb{R}^r$ (3A5Me[1]), while $\|h(x)\| = 1$ whenever $\|x\| \ge 1$. So if we set $F(t,x) = h(2f(\delta t,x))$ for $t \in [0,1]$ and $x \in B$, F will be continuous and $x \mapsto F(1,x) : B \to B$ will be (ρ,ρ_E)-Lipschitz.

Moreover, $\|f(\delta t,x) - x\| \le \frac{1}{2}$ for every $x \in B$, by the choice of δ. so if $x \in S_{r-1}$ we shall have $\|f(\delta t,x)\| \ge \frac{1}{2}$ and $\|F(t,x)\| = 1$ for every $t \in [0,1]$. So $F\restriction[0,1] \times S_{r-1}$ is a homotopy between g_0 and $g\restriction S_{r-1}$ where

$$g_0(x) = F(0,x) = h(2f(0,x)) = h(2x) = x$$

for $x \in S_{r-1}$ and $g(x) = F(1,x)$ for $x \in B$. But this means that $g\restriction S_{r-1}$ is homotopic to the identity on S_{r-1}. By 4A7C, $g[B] = B$, while g is (ρ,ρ_E)-Lipschitz.

(d) For a general closed ball $B \subseteq \mathbb{R}^r$, if B is a singleton then the result is trivial. If $B = \{x : x \in \mathbb{R}^r, \|x - z\| \le \delta\}$ where $z \in \mathbb{R}^r$ and $\delta > 0$, then $(\mathbb{R}^r, \rho_E, B)$ and $(\mathbb{R}^r, \rho_E, B(\mathbf{0},1))$ are lipeomorphic in the sense that there is a bijection $\phi : \mathbb{R}^r \to \mathbb{R}^r$ such that ϕ and ϕ^{-1} are (ρ_E,ρ_E)-Lipschitz and $\phi[B(\mathbf{0},1)] = B$. (Take $\phi(x) = z + \delta x$ for $x \in \mathbb{R}^r$.) Now, given a metric ρ on B inducing its topology, set $\tilde\rho(x,y) = \rho(\phi(x),\phi(y))$ for x, $y \in B(\mathbf{0},1)$; then $\tilde\rho$ is a metric on $B(\mathbf{0},1)$ inducing its topology, and $\phi^{-1} : B \to B(\mathbf{0},1)$ is $(1,\rho,\tilde\rho)$-Lipschitz. By (a)-(c), there is a $(\tilde\rho,\rho_E)$-Lipschitz surjection $\tilde g : B(\mathbf{0},1) \to B(\mathbf{0},1)$. But now $g = \phi\tilde g\phi^{-1}$ is a (ρ,ρ_E)-Lipschitz surjection from B onto itself.

472H Corollary Let $r \ge 1$ be an integer, ρ a metric on $\mathbb{R}^r$ inducing the usual topology on $\mathbb{R}^r$, and $\mu_{Hr}^{(\rho)}$ the corresponding r-dimensional Hausdorff measure on $\mathbb{R}^r$. Then $\mu_{Hr}^{(\rho)}$ is strictly positive.

proof If $G \subseteq \mathbb{R}^r$ is a non-empty open set, it includes a non-trivial closed ball B say. By 472G, there is a surjection $g : B \to B$ which is (ρ,ρ_E)-Lipschitz where ρ_E is the usual metric on $\mathbb{R}^r$; let $\gamma > 0$ be such that g is (γ,ρ,ρ_E)-Lipschitz. As the r-dimensional Hausdorff measure $\mu_{Hr}^{(\rho_E)}$ is just a multiple of Lebesgue measure (264I),

$$0 < \mu_{Hr}^{(\rho_E)}B \le \gamma^r \mu_{Hr}^{(\rho)}B$$

(471J) and

$$\gamma^r \mu_{Hr}^{(\rho)}G \ge \gamma^r \mu_{Hr}^{(\rho)}B > 0.$$

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Concordance for Appendix

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this volume, and which have since been changed.

4A2Jf Uniformities on completely regular spaces 4A2Jf, referred to in the 2009 edition of Volume 5, has been moved to 4A2Jg.

4A3Q Baire property and cylindrical σ-algebras 4A3Q-4A3T and 4A3V, referred to in the 2008 and 2015 editions of Volume 5, are now 4A3R-4A3U and 4A3W.

[1] Later editions only.

4A4B Bounded sets in linear topological spaces 4A4Bg, referred to in the 2008 edition of Volume 5, has been moved to 3A5Nb.

Version of 9.2.13

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