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Chapter 49

Further topics

I conclude the volume with notes on six almost unconnected special topics. In §491 I look at equidistributed sequences and the ideal $\mathcal{Z}$ of sets with asymptotic density zero. I give the principal theorems on the existence of equidistributed sequences in abstract topological measure spaces, and examine the way in which an equidistributed sequence can induce an embedding of a measure algebra in the quotient algebra $\mathcal{P}\mathbb{N}/\mathcal{Z}$. The next three sections are linked. In §492 I present some forms of 'concentration of measure' which echo ideas from §476 in combinatorial, rather than geometric, contexts, with theorems of Talagrand and Maurey on product measures and the Haar measure of a permutation group. In §493 I show how the ideas of §§449, 476 and 492 can be put together in the theory of 'extremely amenable' topological groups. Some of the important examples of extremely amenable groups are full groups of measure-preserving automorphisms of measure algebras, previously treated in §383; these are the subject of §494, where I look also at some striking algebraic properties of these groups. In §495, I move on to Poisson point processes, with notes on disintegrations and some special cases in which they can be represented by Radon measures. In §496, I revisit the Maharam submeasures of Chapter 39, showing that various ideas from the present volume can be applied in this more general context. In §497, I give a version of Tao's proof of Szemerédi's theorem on arithmetic progressions, based on a deep analysis of relative independence, as introduced in §458. Finally, in §498 I give a pair of simple, but perhaps surprising, results on subsets of sets of positive measure in product spaces.

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491 Equidistributed sequences

In many of the most important topological probability spaces, starting with Lebesgue measure, there are sequences which are equidistributed in the sense that, in the limit, they spend the right proportion of their time in each part of the space. I give the basic results on existence of equidistributed sequences in 491E-491H, 491Q and 491R. Investigating such sequences, we are led to some interesting properties of the asymptotic density ideal $\mathcal{Z}$ and the quotient algebra $\mathfrak{Z} = \mathcal{P}\mathbb{N}/\mathcal{Z}$ (491A, 491I-491K, 491P). For 'effectively regular' measures (491L-491M), equidistributed sequences lead to embeddings of measure algebras in $\mathfrak{Z}$ (491N).

491A The asymptotic density ideal (a) If I is a subset of $\mathbb{N}$, its **upper asymptotic density** is $d^*(I) = \limsup_{n\to\infty} \frac{1}{n}(I \cap n)$, and its **asymptotic density** is $d(I) = \lim_{n\to\infty} \frac{1}{n}\#(I \cap n)$ if this is defined. d^* is a submeasure on $\mathcal{P}\mathbb{N}$, so

$$\mathcal{Z} = \{I : I \subseteq \mathbb{N},\, d^*(I) = 0\} = \{I : I \subseteq \mathbb{N},\, d(I) = 0\}$$

is an ideal, the **asymptotic density ideal**.

(b)

$$\mathcal{Z} = \{I : I \subseteq \mathbb{N},\, \lim_{n\to\infty} 2^{-n}\#(I \cap 2^{n+1} \setminus 2^n) = 0\}.$$

(c) Writing $\mathcal{D}$ for the domain of d,

$$\begin{aligned}\mathcal{D} &= \{I : I \subseteq \mathbb{N},\, \limsup_{n\to\infty} \frac{1}{n}\#(I\cap n) = \liminf_{n\to\infty} \frac{1}{n}\#(I\cap n)\}\\ &= \{I : I \subseteq \mathbb{N},\, d^*(I) = 1 - d^*(\mathbb{N}\setminus I)\},\end{aligned}$$

$$\mathbb{N}\in\mathcal{D}, \quad \text{if } I,\, J\in\mathcal{D} \text{ and } I\subseteq J \text{ then } J\setminus I\in\mathcal{D},$$

$$\text{if } I,\, J\in\mathcal{D} \text{ and } I\cap J=\emptyset \text{ then } I\cup J\in\mathcal{D} \text{ and } d(I\cup J)=d(I)+d(J).$$

It follows that if $\mathcal{I}\subseteq\mathcal{D}$ and $I\cap J\in\mathcal{I}$ for all $I,\, J\in\mathcal{I}$, then the subalgebra of $\mathcal{P}\mathbb{N}$ generated by $\mathcal{I}$ is included in $\mathcal{D}$.

(d) If $\langle l_n\rangle_{n\in\mathbb{N}}$ is a strictly increasing sequence in $\mathbb{N}$ such that $\lim_{n\to\infty} l_{n+1}/l_n = 1$, and $I\subseteq\mathbb{R}$, then

$$d^*(I) \le \limsup_{n\to\infty} \frac{1}{l_{n+1}-l_n}\#(I\cap l_{n+1}\setminus l_n).$$

***(e)** If $\langle I_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\mathcal{Z}$, there is an $I\in\mathcal{Z}$ such that $I_n\setminus I$ is finite for every n.

491B Equidistributed sequences Let X be a topological space and μ a probability measure on X. I say that a sequence $\langle x_i\rangle_{i\in\mathbb{N}}$ in X is **(asymptotically) equidistributed** if $d^*(\{i : x_i\in F\})\le\mu F$ for every measurable closed set $F\subseteq X$; equivalently, if $\liminf_{n\to\infty}\frac{1}{n}\#(\{i : i<n,\, x_i\in G\})\ge\mu G$ for every measurable open set $G\subseteq X$.

491C Proposition Let X be a topological space, μ a probability measure on X and $\langle x_i\rangle_{i\in\mathbb{N}}$ a sequence in X.

(a) $\langle x_i\rangle_{i\in\mathbb{N}}$ is equidistributed iff $\liminf_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n f(x_i)\ge\int f d\mu$ for every measurable bounded lower semi-continuous function $f : X\to\mathbb{R}$.

(b) If μ measures every zero set and $\langle x_i\rangle_{i\in\mathbb{N}}$ is equidistributed, then $\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n f(x_i)=\int f d\mu$ for every $f\in C_b(X)$.

(c) Suppose that μ measures every zero set in X. If $\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n f(x_i)=\int f d\mu$ for every $f\in C_b(X)$, then $d^*(\{i : x_i\in F\})\le\mu F$ for every zero set $F\subseteq X$.

(d) Suppose that X is normal and that μ measures every zero set and is inner regular with respect to the closed sets. If $\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n f(x_i)=\int f d\mu$ for every $f\in C_b(X)$, then $\langle x_i\rangle_{i\in\mathbb{N}}$ is equidistributed.

(e) Suppose that μ is τ-additive and there is a base $\mathcal{G}$ for the topology of X, consisting of measurable sets and closed under finite unions, such that $\liminf_{n\to\infty}\frac{1}{n+1}\#(\{i : i\le n,\, x_i\in G\})\ge\mu G$ for every $G\in\mathcal{G}$. Then $\langle x_i\rangle_{i\in\mathbb{N}}$ is equidistributed.

(f) Suppose that X is completely regular and that μ measures every zero set and is τ-additive. Then $\langle x_i\rangle_{i\in\mathbb{N}}$ is equidistributed iff the limit $\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n f(x_i)$ is defined and equal to $\int f d\mu$ for every $f\in C_b(X)$.

(g) Suppose that X is metrizable and that μ is a topological measure. Then $\langle x_i\rangle_{i\in\mathbb{N}}$ is equidistributed iff the limit $\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n f(x_i)$ is defined and equal to $\int f d\mu$ for every $f\in C_b(X)$.

(h) Suppose that X is compact, Hausdorff and zero-dimensional, and that μ is a Radon measure on X. Then $\langle x_i\rangle_{i\in\mathbb{N}}$ is equidistributed iff $d(\{i : x_i\in G\})=\mu G$ for every open-and-closed subset G of X.

491D Lemma Let X be a topological space and μ a probability measure on X. Suppose that there is a sequence $\langle\nu_n\rangle_{n\in\mathbb{N}}$ of point-supported probability measures on X such that $\limsup_{n\to\infty}\nu_n F\le\mu F$ for every measurable closed set $F\subseteq X$. Then μ has an equidistributed sequence.

491E Proposition (a)(i) Suppose that X and Y are topological spaces, μ is a probability measure on X and $f : X\to Y$ is a continuous function. If $\langle x_i\rangle_{i\in\mathbb{N}}$ is a sequence in X which is equidistributed with respect to μ, then $\langle f(x_i)\rangle_{i\in\mathbb{N}}$ is equidistributed with respect to the image measure μf^{-1}.

(ii) Suppose that (X,μ) and (Y,ν) are topological probability spaces and $f : X\to Y$ is a continuous inverse-measure-preserving function. If $\langle x_i\rangle_{i\in\mathbb{N}}$ is a sequence in X which is equidistributed with respect to μ, then $\langle f(x_i)\rangle_{i\in\mathbb{N}}$ is equidistributed with respect to ν.

(b) Let X be a topological space and μ a probability measure on X, and suppose that X has a countable network consisting of sets measured by μ. Let λ be the ordinary product measure on $X^{\mathbb{N}}$. Then λ-almost every sequence in X is μ-equidistributed.

491F Theorem Let $\langle(X_\alpha, \mathfrak{T}_\alpha, \Sigma_\alpha, \mu_\alpha)\rangle_{\alpha\in A}$ be a family of τ-additive topological probability spaces, each of which has an equidistributed sequence. If $\#(A) \le \mathfrak{c}$, then the τ-additive product measure λ on $X = \prod_{\alpha\in A} X_\alpha$ has an equidistributed sequence.

491G Corollary The usual measure of $\{0,1\}^{\mathfrak{c}}$ has an equidistributed sequence.

491H Theorem Any separable compact Hausdorff topological group has an equidistributed sequence for its Haar probability measure.

491I The quotient $\mathcal{P}\mathbb{N}/\mathcal{Z}$ Since $\mathcal{Z} \triangleleft \mathcal{P}\mathbb{N}$, we can form the quotient **asymptotic density algebra** $\mathfrak{Z} = \mathcal{P}\mathbb{N}/\mathcal{Z}$. The functional d^* descends to $\mathfrak{Z}$ if we set

$$\bar{d}^*(I^\bullet) = d^*(I) \text{ for every } I \subseteq \mathbb{N}.$$

(a) $\bar{d}^*$ is a strictly positive submeasure on $\mathfrak{Z}$.

(b) Let $\bar\rho$ be the metric on $\mathfrak{Z}$ defined by saying that $\bar\rho(a,b) = \bar{d}^*(a \triangle b)$ for all a, $b \in \mathfrak{Z}$. Under $\bar\rho$, the Boolean operations $\cup$, $\cap$, $\triangle$ and $\setminus$ and the function $\bar{d}^* : \mathfrak{Z} \to [0,1]$ are uniformly continuous, and $\mathfrak{Z}$ is complete.

***(c)** If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{Z}$, there is an $a \in \mathfrak{Z}$ such that $a \subseteq a_n$ for every n and $\bar{d}^*(a) = \inf_{n\in\mathbb{N}} \bar{d}^*(a_n)$.

***(d)** $\bar{d}^*$ is a Maharam submeasure on $\mathfrak{Z}$.

491J Lemma Let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $\mathfrak{Z} = \mathcal{P}\mathbb{N}/\mathcal{Z}$ such that $\lim_{n\to\infty} \bar{d}^*(a_n) + \bar{d}^*(1 \setminus a_n) = 1$. Then $\langle a_n\rangle_{n\in\mathbb{N}}$ is topologically convergent to a member a of $\mathfrak{Z}$; $a = \sup_{n\in\mathbb{N}} a_n$ in $\mathfrak{Z}$ and $d^*(a) + d^*(1 \setminus a) = 1$.

491K Corollary Set $D = \{a : a \in \mathfrak{Z},\ \bar{d}^*(a) + \bar{d}^*(1 \setminus a) = 1\}$, and write $\bar{d}$ for $\bar{d}^*\restriction D$.

(a) If $I \subseteq \mathbb{N}$ then its asymptotic density $d(I)$ is defined iff $I^\bullet \in D$, and in this case $d(I) = \bar{d}(I^\bullet)$.

(b) If $a \in D$ then its complement $1 \setminus a$ in $\mathfrak{Z}$ belongs to D; if a, $b \in D$ and $a \cap b = 0$, then $a \cup b \in D$ and $\bar{d}(a \cup b) = \bar{d}(a) + \bar{d}(b)$; if a, $b \in D$ and $a \subseteq b$ then $b \setminus a \in D$ and $\bar{d}(b \setminus a) = \bar{d}(b) - \bar{d}(a)$.

(c) D is a topologically closed subset of $\mathfrak{Z}$.

(d) If $A \subseteq D$ is upwards-directed, then $\sup A$ is defined in $\mathfrak{Z}$ and belongs to D; moreover there is a sequence in A with the same supremum as A, and $\sup A$ belongs to the topological closure of A.

(e) Let $\mathfrak{B} \subseteq D$ be a subalgebra of $\mathfrak{Z}$. Then the following are equiveridical:

(i) $\mathfrak{B}$ is topologically closed in $\mathfrak{Z}$;

(ii) $\mathfrak{B}$ is order-closed in $\mathfrak{Z}$;

(iii) setting $\bar\nu = \bar{d}^*\restriction\mathfrak{B} = \bar{d}\restriction\mathfrak{B}$, $(\mathfrak{B}, \bar\nu)$ is a probability algebra.

In this case, $\mathfrak{B}$ is regularly embedded in $\mathfrak{Z}$.

(f) If $I \subseteq D$ is closed under either $\cap$ or $\cup$, then the topologically closed subalgebra of $\mathfrak{Z}$ generated by I, which is also the order-closed subalgebra of $\mathfrak{Z}$ generated by I, is included in D.

491L Effectively regular measures Let (X, Σ, μ) be a measure space, and $\mathfrak{T}$ a topology on X.

(a) I will say that a measurable subset K of X of finite measure is **regularly enveloped** if for every $\epsilon > 0$ there are an open measurable set G and a closed measurable set F such that $K \subseteq G \subseteq F$ and $\mu(F \setminus K) \le \epsilon$.

(b) Note that the family of regularly enveloped measurable sets of finite measure is closed under finite unions and countable intersections.

(c) μ is **effectively regular** if it is inner regular with respect to the regularly enveloped sets of finite measure.

491M Examples (a) Any totally finite Radon measure is effectively regular.

(b) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space such that $\mathfrak{T}$ is a regular topology. Then μ is effectively regular.

(c) Any totally finite Baire measure is effectively regular.

(d) A totally finite completion regular topological measure is effectively regular.

491N Theorem Let X be a topological space and μ an effectively regular probability measure on X, with measure algebra $(\mathfrak{A}, \bar{\mu})$. Suppose that $\langle x_i \rangle_{i \in \mathbb{N}}$ is an equidistributed sequence in X. Then we have a unique order-continuous Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{Z} = \mathcal{P}\mathbb{N}/\mathcal{Z}$ such that $\pi G^\bullet \subseteq \{i : x_i \in G\}^\bullet$ for every measurable open set $G \subseteq X$, and $\bar{d}^*(\pi a) = \bar{\mu} a$ for every $a \in \mathfrak{A}$.

491O Proposition Let X be a topological space and μ an effectively regular probability measure on X which measures every zero set, and suppose that $\langle x_i \rangle_{i \in \mathbb{N}}$ is an equidistributed sequence in X. Let $\mathfrak{A}$ be the measure algebra of μ and $\pi : \mathfrak{A} \to \mathfrak{Z} = \mathcal{P}\mathbb{N}/\mathcal{Z}$ the regular embedding described in 491N; let $T_\pi : L^\infty(\mathfrak{A}) \to L^\infty(\mathfrak{Z})$ be the corresponding order-continuous Banach algebra embedding. Let $S : \ell^\infty(X) \to \ell^\infty$ be the Riesz homomorphism defined by setting $(Sf)(i) = f(x_i)$ for $f \in \ell^\infty(X)$ and $i \in \mathbb{N}$, and $R : \ell^\infty \to L^\infty(\mathfrak{Z})$ the Riesz homomorphism corresponding to the Boolean homomorphism $I \mapsto I^\bullet : \mathcal{P}\mathbb{N} \to \mathfrak{Z}$. For $f \in \mathcal{L}^\infty(\mu)$ let $f^\bullet$ be the corresponding member of $L^\infty(\mu) \cong L^\infty(\mathfrak{A})$. Then $T_\pi(f^\bullet) = RSf$ for every $f \in C_b(X)$.

491P Proposition Any probability algebra $(\mathfrak{A}, \bar{\mu})$ with cardinal at most $\mathfrak{c}$ can be regularly embedded as a subalgebra of $\mathfrak{Z} = \mathcal{P}\mathbb{N}/\mathcal{Z}$ in such a way that $\bar{\mu}$ is identified with the restriction of the submeasure $\bar{d}^*$ to the image of $\mathfrak{A}$.

491Q Corollary Every Radon probability measure on $\{0,1\}^{\mathfrak{c}}$ has an equidistributed sequence.

491R Proposition Let X be a topological space, μ an effectively regular topological probability measure on X which has an equidistributed sequence, and ν a probability measure on X which is an indefinite-integral measure over μ. Then ν has an equidistributed sequence.

491S The asymptotic density filter(a) Set

$$\mathcal{F}_d = \{\mathbb{N} \setminus I : I \in \mathcal{Z}\} = \{I : I \subseteq \mathbb{N}, \lim_{n\to\infty} \frac{1}{n}\#(I \cap n) = 1\}.$$

$\mathcal{F}_d$ is a filter on $\mathbb{N}$, the **(asymptotic) density filter**.

(b) For a bounded sequence $\langle \alpha_n \rangle_{n \in \mathbb{N}}$ in $\mathbb{C}$, $\lim_{n\to\mathcal{F}_d} \alpha_n = 0$ iff $\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^n |\alpha_k| = 0$.

(c) For any $m \in \mathbb{N}$ and $A \subseteq \mathbb{N}$, $A + m \in \mathcal{F}_d$ iff $A \in \mathcal{F}_d$. Hence, or otherwise, for any (real or complex) sequence $\langle \alpha_n \rangle_{n \in \mathbb{N}}$, $\lim_{n\to\mathcal{F}_d} \alpha_n = \lim_{n\to\mathcal{F}_d} \alpha_{m+n}$ if either is defined.

491Z Problem It is known that for almost every $x > 1$ the sequence $\langle <x^i> \rangle_{i \in \mathbb{N}}$ of fractional parts of powers of x is equidistributed for Lebesgue measure on $[0,1]$. But is $\langle <(\frac{3}{2})^n> \rangle_{n \in \mathbb{N}}$ equidistributed?

Version of 30.5.16

492 Combinatorial concentration of measure

'Concentration of measure' takes its most dramatic forms in the geometrically defined notions of concentration explored in §476. But the phenomenon is observable in many other contexts, if we can devise the right abstract geometries to capture it. In this section I present one of Talagrand's theorems on the concentration of measure in product spaces, using the Hamming metric (492D), and Maurey's theorem on concentration of measure in permutation groups (492H).

492A Lemma Let (X,Σ,μ) be a totally finite measure space, $\alpha<\beta$ in $\mathbb{R}$, $\phi:[\alpha,\beta]\to\mathbb{R}$ a convex function, and $f:X\to[\alpha,\beta]$ a Σ-measurable function. Then

$$\int\phi(f(x))\mu(dx)\le\frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\int fd\mu+\frac{\beta\phi(\alpha)-\alpha\phi(\beta)}{\beta-\alpha}\mu X.$$

492B Corollary Let (X,Σ,μ) be a probability space and $f:X\to[\alpha,1]$ a measurable function, where $0<\alpha\le1$. Then $\int\frac{1}{f}d\mu\cdot\int fd\mu\le\frac{(1+\alpha)^2}{4\alpha}$.

492C Lemma $\frac{1}{2}(1+\cosh t)\le e^{t^2/4}$ for every $t\in\mathbb{R}$.

492D Theorem Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i<n}$ be a non-empty finite family of probability spaces with product (X,Λ,λ). Let ρ be the **normalized Hamming metric** on X defined by setting $\rho(x,y)=\frac{1}{n}\#(\{i:i<n,\,x(i)\ne y(i)\})$ for x, $y\in X$. If $W\in\Lambda$ and $\lambda W>0$, then

$$\overline{\int}e^{t\rho(x,W)}\lambda(dx)\le\frac{1}{\lambda W}e^{t^2/4n}$$

for every $t\ge0$.

492E Corollary Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i<n}$ be a non-empty finite family of probability spaces with product (X,Λ,λ).

(a) Let ρ be the normalized Hamming metric on X. If $W\in\Lambda$ and $\lambda W>0$, then

$$\lambda^*\{x:\rho(x,W)\ge\gamma\}\le\frac{1}{\lambda W}e^{-n\gamma^2}$$

for every $\gamma\ge0$.

(b) If W, $W'\in\Lambda$ and $\gamma>0$ are such that $e^{-n\gamma^2}<\lambda W\cdot\lambda W'$ then there are $x\in W$, $x'\in W'$ such that $\#(\{i:i<n,\,x(i)\ne x'(i)\})<n\gamma$.

492F Lemma $e^t\le t+e^{t^2}$ for every $t\in\mathbb{R}$.

492G Lemma Let (X,Σ,μ) be a probability space, and $\langle f_n\rangle_{n\in\mathbb{N}}$ a martingale on X. Suppose that $f_n\in\mathcal{L}^\infty(\mu)$ for every n, and that $\alpha_n\ge\operatorname{ess\,sup}|f_n-f_{n-1}|$ for $n\ge1$. Then for any $n\ge1$ and $\gamma\ge0$,

$$\Pr(f_n-f_0\ge\gamma)\le\exp(-\gamma^2/4\sum_{i=1}^n\alpha_i^2),$$

at least if $\sum_{i=1}^n\alpha_i^2>0$.

492H Theorem Let X be a non-empty finite set and G the group of all permutations of X with its discrete topology. For π, $\phi\in G$ set

$$\rho(\pi,\phi)=\frac{\#(\{x:x\in X,\,\pi(x)\ne\phi(x)\})}{\#(X)}.$$

Then ρ is a metric on G. Give G its Haar probability measure, and let $f : G \to \mathbb{R}$ be a 1-Lipschitz function. Then

$$\Pr(f - \mathbb{E}(f) \geq \gamma) \leq \exp(-\frac{\gamma^2 \#(X)}{16})$$

for any $\gamma \geq 0$.

492I Corollary Let X be a non-empty finite set, with $\#(X) = n$, and G the group of all permutations of X. Let μ be the Haar probability measure of G when given its discrete topology. Suppose that $A \subseteq G$ and $\mu A \geq \frac{1}{2}$. Then

$$\mu\{\pi : \pi \in G, \exists\, \phi \in A, \#(\{x : x \in X, \pi(x) \neq \phi(x)\}) \leq k\} \geq 1 - \exp(-\frac{k^2}{64n})$$

for every $k \leq n$.

Version of 4.1.13

493 Extremely amenable groups

A natural variation on the idea of 'amenable group' is the concept of 'extremely amenable' group (493A). Expectedly, most of the ideas of 449C-449E can be applied to extremely amenable groups (493B); unexpectedly, we find not only that there are interesting extremely amenable groups, but that we need some of the central ideas of measure theory to study them. I give a criterion for extreme amenability of a group in terms of the existence of suitably concentrated measures (493C) before turning to three examples: measure algebras under symmetric difference (493D), L^0 spaces (493E) and isometry groups of spheres in infinite-dimensional Hilbert spaces (493G).

493A Definition Let G be a topological group. Then G is **extremely amenable** or has the **fixed point on compacta property** if every continuous action of G on a compact Hausdorff space has a fixed point.

493B Proposition (a) Let G and H be topological groups such that there is a continuous surjective homomorphism from G onto H. If G is extremely amenable, so is H.

(b) Let G be a topological group and suppose that there is a dense subset A of G such that every finite subset of A is included in an extremely amenable subgroup of G. Then G is extremely amenable.

(c) Let G be a topological group with an extremely amenable normal subgroup H such that G/H is extremely amenable. Then G is extremely amenable.

(d) The product of any family of extremely amenable topological groups is extremely amenable.

(e) Let G be a topological group. Then G is extremely amenable iff there is a point in the greatest ambit Z of G which is fixed by the action of G on Z.

(f) Let G be an extremely amenable topological group. Then every dense subgroup of G is extremely amenable.

493C Theorem Let G be a topological group and $\mathcal{B}$ its Borel σ-algebra. Suppose that for every $\epsilon > 0$, open neighbourhood V of the identity of G, finite set $I \subseteq G$ and finite family $\mathcal{E}$ of zero sets in G there is a finitely additive functional $\nu : \mathcal{B} \to [0, 1]$ such that $\nu G = 1$ and

(i) $\nu(VF) \geq 1 - \epsilon$ whenever $F \in \mathcal{E}$ and $\nu F \geq \frac{1}{2}$,

(ii) for every $a \in I$ there is a $b \in aV$ such that $|\nu(bF) - \nu F| \leq \epsilon$ for every $F \in \mathcal{E}$.

Then G is extremely amenable.

493D Theorem Let $(\mathfrak{A}, \bar{\mu})$ be an atomless measure algebra. Then $\mathfrak{A}$, with the group operation $\triangle$ and the measure-algebra topology, is an extremely amenable group.

493E Theorem Let (X, Σ, μ) be an atomless measure space. Then $L^0(\mu)$, with the group operation $+$ and the topology of convergence in measure, is an extremely amenable group.

493F Lemma For any $m \in \mathbb{N}$ and any $\epsilon > 0$, there is an $r(m,\epsilon) \ge 1$ such that whenever X is a finite-dimensional inner product space over $\mathbb{R}$ of dimension at least $r(m,\epsilon)$, $x_0,\dots,x_{m-1} \in S_X$, Q_1, $Q_2 \subseteq H_X$ are closed sets and $\min(\lambda_X Q_1, \lambda_X Q_2) \ge \epsilon$, then there are $f_1 \in Q_1$, $f_2 \in Q_2$ such that $\|f_1(x_i) - f_2(x_i)\| \le \epsilon$ for every $i < m$.

493G Theorem Let X be an infinite-dimensional inner product space over $\mathbb{R}$. Then the isometry group H_X of its unit sphere S_X, with its topology of pointwise convergence, is extremely amenable.

493H Theorem If G is a locally compact Hausdorff topological group with more than one element, it is not extremely amenable.

Version of 17.5.13

494 Groups of measure-preserving automorphisms

I return to the study of automorphism groups of measure algebras, as in Chapter 38 of Volume 3, but this time with the intention of exploring possible topological group structures. Two topologies in particular have attracted interest, the 'weak' and 'uniform' topologies (494A). After a brief account of their basic properties (494B-494C) I begin work on the four main theorems. The first is the Halmos-Rokhlin theorem that if $(\mathfrak{A},\bar\mu)$ is the Lebesgue probability algebra the set of weakly mixing measure-preserving automorphisms of $\mathfrak{A}$ which are not mixing is comeager for the weak topology on $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ (494E). This depends on a striking characterization of weakly mixing automorphisms of a probability algebra in terms of eigenvectors of the corresponding operators on the complex Hilbert space $L^2_{\mathbb{C}}(\mathfrak{A},\bar\mu)$ (494D). It turns out that there is an elegant example of a weakly mixing automorphism which is not mixing which can be described in terms of a Gaussian distribution of the kind introduced in §456, so I give it here (494F).

We need a couple of preliminary results on fixed-point subalgebras (494G-494H) before approaching the other three theorems. If $(\mathfrak{A},\bar\mu)$ is an atomless probability algebra, then $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is extremely amenable under its weak topology (494L); if $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is given its uniform topology, then every group homomorphism from $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ to a Polish group is continuous (494O); finally, there is no strictly increasing sequence of subgroups with union $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ (494Q). All these results have wide-ranging extensions to full subgroups of $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ subject to certain restrictions on the fixed-point subalgebras.

494A Definitions Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ the group of measure-preserving automorphisms of $\mathfrak{A}$. Write $\mathfrak{A}^f$ for $\{c : c \in \mathfrak{A},\ \bar\mu c < \infty\}$.

(a) I will say that the **weak topology** on $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is that generated by the pseudometrics $(\pi,\phi) \mapsto \bar\mu(\pi c \,\triangle\, \phi c)$ as c runs over $\mathfrak{A}^f$.

(b) I will say that the **uniform topology** on $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is that generated by the pseudometrics

$$(\pi,\phi) \mapsto \sup_{a\in\mathfrak{A}} \bar\mu(c \cap (\pi a \,\triangle\, \phi a))$$

as c runs over $\mathfrak{A}^f$.

494B Proposition Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and give $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ its weak topology.

(a) $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is a topological group.

(b) $(\pi,a) \mapsto \pi a : \mathrm{Aut}_{\bar\mu}\mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ is continuous for the weak topology on $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ and the measure-algebra topology on $\mathfrak{A}$.

(c) If $(\mathfrak{A},\bar\mu)$ is semi-finite, $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is Hausdorff.

(d) If $(\mathfrak{A},\bar\mu)$ is localizable, $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is complete under its bilateral uniformity.

(e) If $(\mathfrak{A},\bar\mu)$ is σ-finite and $\mathfrak{A}$ has countable Maharam type, then $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is a Polish group.

494C Proposition Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and give $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ its uniform topology.

(a) $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is a topological group.

(b) For $c \in \mathfrak{A}^f$ and $\epsilon > 0$, set

$$U(c,\epsilon) = \{\pi : \pi \in \operatorname{Aut}_{\bar\mu}\mathfrak{A},\ \pi \text{ is supported by an } a \in \mathfrak{A} \text{ such that } \bar\mu(c \cap a) \le \epsilon\}.$$

Then $\{U(c,\epsilon) : c \in \mathfrak{A}^f,\ \epsilon > 0\}$ is a base of neighbourhoods of ι.

(c) The set of periodic measure-preserving automorphisms of $\mathfrak{A}$ with supports of finite measure is dense in $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$.

(d) The weak topology on $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is coarser than the uniform topology.

(e) If $(\mathfrak{A},\bar\mu)$ is semi-finite, $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is Hausdorff.

(f) If $(\mathfrak{A},\bar\mu)$ is localizable, $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is complete under its bilateral uniformity.

(g) If $(\mathfrak{A},\bar\mu)$ is semi-finite and G is a full subgroup of $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$, then G is closed.

(h) If $(\mathfrak{A},\bar\mu)$ is σ-finite, then $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is metrizable.

(i) Suppose that $(\mathfrak{A},\bar\mu)$ is σ-finite and $\mathfrak{A}$ has countable Maharam type. If $D \subseteq \operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is countable, then the full subgroup G of $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ generated by D, with its induced topology, is a Polish group.

494D Lemma Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and $\phi \in \operatorname{Aut}_{\bar\mu}\mathfrak{A}$. Let $T = T_\phi : L^2_{\mathbb{C}} \to L^2_{\mathbb{C}}$ be the corresponding operator on the complex Hilbert space $L^2_{\mathbb{C}} = L^2_{\mathbb{C}}(\mathfrak{A},\bar\mu)$. Then the following are equiveridical:

(α) ϕ is weakly mixing;

(β) $\inf_{k\in\mathbb{N}} |(T^k w|w)| < 1$ whenever $w \in L^2_{\mathbb{C}}$, $\|w\|_2 = 1$ and $\int w = 0$;

(γ) $\inf_{k\in\mathbb{N}} |(T^k w|w)| = 0$ whenever $w \in L^2_{\mathbb{C}}$, $\|w\|_2 = 1$ and $\int w = 0$.

494E Theorem Let $(\mathfrak{A},\bar\mu)$ be a probability algebra, and give $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ its weak topology.

(a) If $\mathfrak{A} \ne \{0,1\}$, the set of mixing measure-preserving Boolean automorphisms is meager in $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$.

(b) If $\mathfrak{A}$ is atomless and homogeneous, the set of two-sided Bernouilli shifts on $(\mathfrak{A},\bar\mu)$ is dense in $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$.

(c) If $\mathfrak{A}$ has countable Maharam type, the set of weakly mixing measure-preserving Boolean automorphisms is a G_δ subset of $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$.

(d) If $\mathfrak{A}$ is atomless and has countable Maharam type, the set of weakly mixing measure-preserving Boolean automorphisms which are not mixing is comeager in $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$, and is not empty.

494F Example (a) There is a Radon probability measure ν on $\mathbb{R}$, zero on singletons, such that

$$\int \cos(2\pi \cdot 3^j t)\,\nu(dt) = \int \cos 2\pi t\,\nu(dt) > 0$$

for every $j \in \mathbb{N}$.

(b) Set $\sigma_{jk} = \int \cos 2\pi(k-j)t\,\nu(dt)$ for j, $k \in \mathbb{Z}$. Then there is a centered Gaussian distribution μ on $X = \mathbb{R}^{\mathbb{Z}}$ with covariance matrix $\langle\sigma_{jk}\rangle_{j,k\in\mathbb{Z}}$.

(c) Let $S : X \to X$ be the shift operator defined by saying that $(Sx)(j) = x(j+1)$ for $x \in X$ and $j \in \mathbb{Z}$. Then S is an automorphism of (X,μ).

(d) Let $(\mathfrak{A},\bar\mu)$ be the measure algebra of μ and $\phi \in \operatorname{Aut}_{\bar\mu}\mathfrak{A}$ the automorphism represented by S. Then ϕ is not mixing.

(e) ϕ is weakly mixing.

494G Proposition Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and G a full subgroup of $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$, with fixed-point subalgebra $\mathfrak{C}$.

(a) If $a \in \mathfrak{A}^f$ and $\pi \in G$, there is a $\phi \in G$, supported by $a \cup \pi a$, such that $\phi d = \pi d$ for every $d \subseteq a$.

(b) If $(\mathfrak{A},\bar\mu)$ is localizable and a, $b \in \mathfrak{A}^f$, then the following are equiveridical:

(i) there is a $\pi \in G$ such that $\pi a \subseteq b$;

(ii) $\bar\mu(a \cap c) \le \bar\mu(b \cap c)$ for every $c \in \mathfrak{C}$.

(c) If $(\mathfrak{A},\bar\mu)$ is localizable and a, $b \in \mathfrak{A}^f$, then the following are equiveridical:

(i) there is a $\pi \in G$ such that $\pi a = b$;

(ii) $\bar\mu(a \cap c) = \bar\mu(b \cap c)$ for every $c \in \mathfrak{C}$.

(d) If $(\mathfrak{A},\bar\mu)$ is totally finite and $\langle a_i\rangle_{i\in I}$, $\langle b_i\rangle_{i\in I}$ are disjoint families in $\mathfrak{A}$ such that $\bar\mu(a_i \cap c) = \bar\mu(b_i \cap c)$ for every $i \in I$ and $c \in \mathfrak{C}$, there is a $\pi \in G$ such that $\pi a_i = b_i$ for every $i \in I$.

(e) If $(\mathfrak{A},\bar\mu)$ is localizable and $H = \{\pi : \pi \in \operatorname{Aut}_{\bar\mu}\mathfrak{A},\ \pi c = c \text{ for every } c \in \mathfrak{C}\}$, then H is the closure of G for the weak topology of $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$.

494H Proposition Let $\mathfrak{A}$ be a Boolean algebra, G a full subgroup of $\operatorname{Aut}\mathfrak{A}$, and $a \in \mathfrak{A}$. Set $G_a = \{\pi : \pi \in G, \pi \text{ is supported by } a\}$, $H_a = \{\pi{\restriction}\mathfrak{A}_a : \pi \in G_a\}$.

(a) G_a is a full subgroup of $\operatorname{Aut}\mathfrak{A}$ and H_a is a full subgroup of $\operatorname{Aut}\mathfrak{A}_a$, for every $a \in \mathfrak{A}$.

(b) Suppose that $\mathfrak{A}$ is Dedekind complete, and that the fixed-point subalgebra of G is $\mathfrak{C}$. Then the fixed-point subalgebra of H_a is $\{a \cap c : c \in \mathfrak{C}\}$.

494I Lemma Let $(\mathfrak{A}, \bar\mu)$ be an atomless homogeneous probability algebra. Then $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$, with its weak topology, is extremely amenable.

494J Lemma Let $(\mathfrak{C}, \bar\lambda)$ be a totally finite measure algebra, $(\mathfrak{B}, \bar\nu)$ a probability algebra, and $(\mathfrak{A}, \bar\mu)$ the localizable measure algebra free product $(\mathfrak{C}, \bar\lambda)\widehat{\otimes}(\mathfrak{B}, \bar\nu)$. Give $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ its weak topology, and let G be the subgroup $\{\pi : \pi \in \operatorname{Aut}_{\bar\mu}\mathfrak{A}, \pi(c \otimes 1) = c \otimes 1 \text{ for every } c \in \mathfrak{C}\}$. Suppose that $\mathfrak{B}$ is either finite, with all its atoms of the same measure, or homogeneous. Then G is amenable, and if either $\mathfrak{B}$ is homogeneous or $\mathfrak{C}$ is atomless, G is extremely amenable.

494K Lemma Let $(\mathfrak{A}, \bar\mu)$ be a totally finite measure algebra, and give $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ its weak topology. Let G be a subgroup of $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$, and suppose that $G = \{\pi : \pi \in \operatorname{Aut}_{\bar\mu}\mathfrak{A}, \pi c = c \text{ for every } c \in \mathfrak{C}\}$. Then G is amenable, and if every atom of $\mathfrak{A}$ belongs to $\mathfrak{C}$, then G is extremely amenable.

494L Theorem Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra and G a full subgroup of $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$, with the topology induced by the weak topology of $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$. Then G is amenable. If every atom of $\mathfrak{A}$ with finite measure belongs to the fixed-point subalgebra of G, then G is extremely amenable.

494M Lemma Let $\mathfrak{A}$ be a Boolean algebra, G a full subgroup of $\operatorname{Aut}\mathfrak{A}$, and $V \subseteq G$ a symmetric set. Let $\sim_G$ be the orbit equivalence relation on $\mathfrak{A}$ induced by the action of G, so that $a \sim_G b$ iff there is a $\phi \in G$ such that $\phi a = b$. Suppose that $a \in \mathfrak{A}$ and $\pi, \pi' \in G$ are such that

$\pi = (\overleftarrow{b \,{}_{\pi}\, c})$ and $\pi' = (\overleftarrow{b' \,{}_{\pi'}\, c'})$ are exchanging involutions supported by a,

$b \sim_G b'$ and $a \setminus (b \cup c) \sim_G a \setminus (b' \cup c')$,

$\pi \in V$,

whenever $\phi \in G$ is supported by a there is a $\psi \in V$ agreeing with ϕ on $\mathfrak{A}_a$.

Then $\pi' \in V^3$.

494N Lemma Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra and $G \subseteq \operatorname{Aut}_{\bar\mu}\mathfrak{A}$ a full subgroup with fixed-point subalgebra $\mathfrak{C}$. Suppose that $\mathfrak{A}$ is relatively atomless over $\mathfrak{C}$. For $a \in \mathfrak{A}$, let $u_a \in L^\infty(\mathfrak{C})$ be the conditional expectation of χa on $\mathfrak{C}$, and let G_a be $\{\pi : \pi \in G \text{ is supported by } a\}$. Suppose that $a \subseteq e$ in $\mathfrak{A}$ and $V \subseteq G$ are such that

V is symmetric,

for every $\phi \in G_e$ there is a $\psi \in V$ such that ϕ and ψ agree on $\mathfrak{A}_e$,

there is an involution in V with support a,

$u_a \le \frac{2}{3}u_e$.

Then $G_a \subseteq V^{18}$.

494O Theorem Suppose that $(\mathfrak{A}, \bar\mu)$ is an atomless probability algebra and $G \subseteq \operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is a full ergodic subgroup, with the topology induced by the uniform topology of $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$.

(a) If $V \subseteq G$ is symmetric and G can be covered by countably many left translates of V in G, then $V^{38} = \{\pi_1\pi_2 \ldots \pi_{38} : \pi_1, \ldots, \pi_{38} \in V\}$ is a neighbourhood of the identity in G.

(b) If H is a topological group such that for every neighbourhood W of the identity in H there is a countable set $D \subseteq H$ such that $H = DW$, and $\theta : G \to H$ is a group homomorphism, then θ is continuous.

494P Remark Note that if a topological group H is either Lindelöf or ccc, it satisfies the condition of (b) above.

494Q Theorem Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and G a full subgroup of $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ such that $\mathfrak{A}$ is relatively atomless over the fixed-point subalgebra $\mathfrak{C}$ of G. Let $\langle V_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of subsets of G such that $V_n^2\subseteq V_{n+1}$ for every n and $G=\bigcup_{n\in\mathbb{N}}V_n$. Then there is an $n\in\mathbb{N}$ such that $G=V_n$.

494R Theorem Let $(\mathfrak{A},\bar\mu)$ be an atomless localizable measure algebra, and G a full ergodic subgroup of $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$. Let $\langle V_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of subsets of G, covering G, with $V_n^2\subseteq V_{n+1}$ for every n. Then there is an $n\in\mathbb{N}$ such that $G=V_n$.

494Z Problems For $k\in\mathbb{N}$, say that a topological group G is **k-Steinhaus** if whenever $V\subseteq G$ is a symmetric set, containing the identity, such that countably many left translates of V cover G, then V^k is a neighbourhood of the identity. For your favourite groups, determine the smallest k, if any, for which they are k-Steinhaus.

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495 Poisson point processes

A classical challenge in probability theory is to formulate a consistent notion of 'random set'. Simple geometric considerations lead us to a variety of measures which are both interesting and important. All these are manifestly special constructions. Even in the most concrete structures, we have to make choices which come to seem arbitrary as soon as we are conscious of the many alternatives. There is however one construction which has a claim to pre-eminence because it is both robust under the transformations of abstract measure theory and has striking properties when applied to familiar measures (to the point, indeed, that it is relevant to questions in physics and chemistry). This gives the 'Poisson point processes' of 495D-495E. In this section I give a brief introduction to the measure-theoretic aspects of this construction.

495A Poisson distributions(a) The **Poisson distribution** with parameter $\gamma>0$ is the point-supported Radon probability measure ν_γ on $\mathbb{R}$ such that $\nu_\gamma\{n\}=\frac{\gamma^n}{n!}e^{-\gamma}$ for every $n\in\mathbb{N}$. Its expectation is γ. ν_γ can be identified with the corresponding subspace measure on $\mathbb{N}$. It will be convenient to allow $\gamma=0$, so that the Dirac measure on $\mathbb{R}$ or $\mathbb{N}$ concentrated at 0 becomes a 'Poisson distribution with expectation 0'.

(b) The convolution of two Poisson distributions is a Poisson distribution. So if f and g are independent random variables with Poisson distributions then $f+g$ has a Poisson distribution.

(c) If $\langle f_i\rangle_{i\in I}$ is a countable independent family of random variables with Poisson distributions, and $\alpha=\sum_{i\in I}\mathbb{E}(f_i)$ is finite, then $f=\sum_{i\in I}f_i$ is defined a.e. and has a Poisson distribution with expectation α.

(d) $1-e^{-\gamma}(1+\gamma)=\nu_\gamma(\mathbb{N}\setminus\{0,1\})$ is at most $\frac{1}{2}\gamma^2$ for every $\gamma\ge 0$.

495B Theorem Let (X,Σ,μ) be a measure space. Set $\Sigma^f=\{E:E\in\Sigma,\ \mu E<\infty\}$. Then for any $\gamma>0$ there are a probability space (Ω,Λ,λ) and a family $\langle g_E\rangle_{E\in\Sigma^f}$ of random variables on Ω such that

(i) for every $E\in\Sigma^f$, g_E has a Poisson distribution with expectation $\gamma\mu E$;
(ii) whenever $\langle E_i\rangle_{i\in I}$ is a disjoint family in Σ^f, then $\langle g_{E_i}\rangle_{i\in I}$ is stochastically independent;
(iii) whenever $\langle E_i\rangle_{i\in\mathbb{N}}$ is a disjoint sequence in Σ^f with union $E\in\Sigma^f$, then $g_E=_{\text{a.e.}}\sum_{i=0}^\infty g_{E_i}$.

495C Lemma Let X be a set and $\mathcal{E}$ a subring of the Boolean algebra $\mathcal{P}X$. Let $\mathcal{H}$ be the family of sets of the form

$$\{S:S\subseteq X,\ \#(S\cap E_i)=n_i\text{ for every }i\in I\}$$

where $\langle E_i\rangle_{i\in I}$ is a finite disjoint family in $\mathcal{E}$ and $n_i\in\mathbb{N}$ for every $i\in I$. Then the Dynkin class $\mathrm{T}\subseteq\mathcal{P}(\mathcal{P}X)$ generated by $\mathcal{H}$ is the σ-algebra of subsets of $\mathcal{P}X$ generated by $\mathcal{H}$.

495D Theorem Let (X,Σ,μ) be an atomless measure space. Set $\Sigma^f = \{E : E \in \Sigma,\ \mu E < \infty\}$; for $E \in \Sigma^f$, set $f_E(S) = \#(S \cap E)$ when $S \subseteq X$ meets E in a finite set. Let T be the σ-algebra of subsets of $\mathcal{P}X$ generated by sets of the form $\{S : f_E(S) = n\}$ where $E \in \Sigma^f$ and $n \in \mathbb{N}$. Then for any $\gamma > 0$ there is a unique probability measure ν with domain T such that

(i) for every $E \in \Sigma^f$, f_E is measurable and has a Poisson distribution with expectation $\gamma\mu E$;

(ii) whenever $\langle E_i\rangle_{i\in I}$ is a disjoint family in Σ^f, then $\langle f_{E_i}\rangle_{i\in I}$ is stochastically independent.

495E Definition In the context of 495D, I will call the completion of ν the **Poisson point process** on X with **intensity** or **density** γ.

495F Proposition Let (X,Σ,μ) be an atomless measure space, $\langle X_i\rangle_{i\in I}$ a countable partition of X into measurable sets and $\gamma > 0$. Let ν be the Poisson point process of (X,Σ,μ) with intensity γ; for $i \in I$ let ν_i be the Poisson point process of (X_i,Σ_i,μ_i) with intensity γ, where μ_i is the subspace measure on X_i and Σ_i its domain. For $S \subseteq X$ set $\phi(S) = \langle S \cap X_i\rangle_{i\in I} \in \prod_{i\in I}\mathcal{P}X_i$. Then ϕ is an isomorphism between ν and the product measure $\lambda = \prod_{i\in I}\nu_i$ on $Z = \prod_{i\in I}\mathcal{P}X_i$.

495G Proposition Let (X,Σ,μ) be a perfect atomless measure space, and $\gamma > 0$. Then the Poisson point process on X with intensity γ is a perfect probability measure.

495H Lemma Let (X,Σ,μ) be an atomless σ-finite measure space, and $\gamma > 0$; let ν be the Poisson point process on X with intensity γ. Suppose that $f : X \to \mathbb{R}$ is a Σ-measurable function such that $\mu f^{-1}[\{\alpha\}] = 0$ for every $\alpha \in \mathbb{R}$. Then $\nu\{S : S \subseteq X,\ f{\restriction}S \text{ is injective}\} = 1$.

495I Proposition Let (X,Σ,μ) be an atomless countably separated measure space and $\gamma > 0$. Let ν' be a complete probability measure on $\mathcal{P}X$ such that $\nu'\{S : S \subseteq X,\ S \cap E = \emptyset\}$ is defined and equal to $e^{-\gamma\mu E}$ whenever $E \in \Sigma$ has finite measure. Then ν' extends the Poisson point process ν on X with intensity γ.

495J Proposition Let (X_1,Σ_1,μ_1) and (X_2,Σ_2,μ_2) be atomless measure spaces, and $f : X_1 \to X_2$ an inverse-measure-preserving function. Let $\gamma > 0$, and let ν_1, ν_2 be the Poisson point processes on X_1, X_2 respectively with intensity γ. Then $S \mapsto f[S] : \mathcal{P}X_1 \to \mathcal{P}X_2$ is inverse-measure-preserving for ν_1 and ν_2; in particular, $\mathcal{P}A$ has full outer measure for ν_2 whenever $A \subseteq X_2$ has full outer measure for μ_2.

495K Lemma Let $(\tilde X,\tilde\Sigma,\tilde\mu)$ be an atomless σ-finite measure space, and $\gamma > 0$. Write μ_L for Lebesgue measure on $[0,1]$, μ' for the product measure on $X' = \tilde X \times [0,1]$, and λ' for the product measure on $\Omega' = [0,1]^{\tilde X}$. Let $\tilde\nu$, ν' be the Poisson point processes on $\tilde X$, X' respectively with intensity γ. For $T \subseteq \tilde X$ define $\psi_T : \Omega' \to \mathcal{P}X'$ by setting $\psi_T(z) = \{(t,z(t)) : t \in T\}$ for $z \in \Omega'$; let ν'_T be the image measure $\lambda'\psi_T^{-1}$ on $\mathcal{P}X'$. Then $\langle\nu'_T\rangle_{T\subseteq\tilde X}$ is a disintegration of ν' over $\tilde\nu$.

495L Theorem Let (X,Σ,μ) and $(\tilde X,\tilde\Sigma,\tilde\mu)$ be atomless σ-finite measure spaces and $\gamma > 0$. Let ν, $\tilde\nu$ be the Poisson point processes on X, $\tilde X$ respectively with intensity γ. Suppose that $f : X \to \tilde X$ is inverse-measure-preserving and that $\langle\mu_t\rangle_{t\in\tilde X}$ is a disintegration of μ over $\tilde\mu$ consistent with f such that every μ_t is a probability measure. Write λ for the product measure $\prod_{t\in\tilde X}\mu_t$ on $\Omega = X^{\tilde X}$, and for $T \subseteq \tilde X$ define $\phi_T : \Omega \to \mathcal{P}X$ by setting $\phi_T(z) = z[T]$ for $z \in \Omega$; let ν_T be the image measure $\lambda\phi_T^{-1}$ on $\mathcal{P}X$. Then $\langle\nu_T\rangle_{T\subseteq\tilde X}$ is a disintegration of ν over $\tilde\nu$. Moreover

(i) setting $\tilde f(S) = f[S]$ for $S \subseteq X$, $\langle\nu_T\rangle_{T\subseteq\tilde X}$ is consistent with $\tilde f : \mathcal{P}X \to \mathcal{P}\tilde X$;

(ii) if $\langle\mu_t\rangle_{t\in\tilde X}$ is strongly consistent with f, then $\langle\nu_T\rangle_{T\subseteq\tilde X}$ is strongly consistent with $\tilde f$.

495M Proposition Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra, and $\gamma>0$. Then there are a probability algebra $(\mathfrak{B},\bar{\lambda})$ and a function $\theta:\mathfrak{A}\to\mathfrak{B}$ such that

(i) $\theta(\sup A)=\sup\theta[A]$ for every non-empty $A\subseteq\mathfrak{A}$ such that $\sup A$ is defined in $\mathfrak{A}$;

(ii) $\bar{\lambda}\theta(a)=1-e^{-\gamma\bar{\mu}a}$ for every $a\in\mathfrak{A}$, interpreting $e^{-\infty}$ as 0;

(iii) whenever $\langle a_i\rangle_{i\in I}$ is a disjoint family in $\mathfrak{A}$ and $\mathfrak{C}_i$ is the closed subalgebra of $\mathfrak{B}$ generated by $\{\theta(a):a\subseteq a_i\}$ for each i, then $\langle\mathfrak{C}_i\rangle_{i\in I}$ is stochastically independent.

495N Proposition Let U be any L-space. Then there are a probability space (Ω,Λ,λ) and a positive linear operator $T:U\to L^1(\lambda)$ such that $\|Tu\|_1=\|u\|_1$ whenever $u\in L^1(\mu)^+$ and $\langle Tu_i\rangle_{i\in I}$ is stochastically independent in $L^0(\lambda)$ whenever $\langle u_i\rangle_{i\in I}$ is a disjoint family in $L^1(\mu)$.

Remarks $\langle v_i\rangle_{i\in I}$ in $L^0(\lambda)$ is 'independent' if $\langle g_i\rangle_{i\in I}$ is an independent family of random variables whenever $g_i\in\mathcal{L}^0(\lambda)$ represents v_i for each i.

495O Proposition Let (X,Σ,μ) be an atomless measure space, and ν the Poisson point process on X with intensity $\gamma>0$.

(a) If $f\in\mathcal{L}^1(\mu)$, $Q_f(S)=\sum_{x\in S\cap\operatorname{dom}f}f(x)$ is defined and finite for ν-almost every $S\subseteq X$, and $\int Q_f d\nu=\gamma\int fd\mu$.

(b) If $f\in\mathcal{L}^1(\mu)\cap\mathcal{L}^2(\mu)$, $\int Q_f^2 d\nu$ is defined and equal to $\gamma\int f^2d\mu+(\gamma\int fd\mu)^2$.

(c) We have a positive linear operator $T:L^1(\mu)\to L^1(\nu)$ defined by setting $T(f^\bullet)=Q_f^\bullet$ for every $f\in\mathcal{L}^1(\mu)$.

(d) $\|Tu\|_1=\gamma\|u\|_1$ whenever $u\in L^1(\mu)^+$ and $\langle Tu_i\rangle_{i\in I}$ is stochastically independent in $L^0(\lambda)$ whenever $\langle u_i\rangle_{i\in I}$ is a disjoint family in $L^1(\mu)$.

495P Proposition Let (X,Σ,μ) be an atomless measure space, and ν the Poisson point process on X with intensity $\gamma>0$. For $f\in\mathcal{L}^1(\mu)$ set $Q_f(S)=\sum_{x\in S\cap\operatorname{dom}f}f(x)$ when $S\subseteq X$ and the sum is defined in $\mathbb{R}$. Then

$$\int_{\mathcal{P}X}e^{iyQ_f}d\nu=\exp\Bigl(\gamma\int_X(e^{iyf}-1)d\mu\Bigr)$$

for any $y\in\mathbb{R}$.

495Q Proposition Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon measure space such that μ is outer regular with respect to the open sets, and $\gamma>0$. Give the space $\mathcal{C}$ of closed subsets of X its Fell topology.

(a) There is a unique quasi-Radon probability measure $\tilde{\nu}$ on $\mathcal{C}$ such that

$$\tilde{\nu}\{C:\#(C\cap E)=\emptyset\}=e^{-\gamma\mu E}$$

whenever $E\subseteq X$ is a measurable set of finite measure.

(b) If $E_0,\dots,E_r$ are disjoint sets of finite measure, none including any singleton set of non-zero measure, and $n_i\in\mathbb{N}$ for $i\le r$, then

$$\tilde{\nu}\{C:\#(C\cap E_i)=n_i\text{ for every }i\le r\}=\prod_{i=0}^r\frac{(\gamma\mu E_i)^{n_i}}{n_i!}e^{-\gamma\mu E_i}.$$

(c) Suppose that μ is atomless and ν is the Poisson point process on X with intensity γ.

(i) $\mathcal{C}$ has full outer measure for ν, and $\tilde{\nu}$ extends the subspace measure $\nu_{\mathcal{C}}$.

(ii) If moreover μ is σ-finite, then $\mathcal{C}$ is ν-conegligible.

(d) If X is locally compact then $\tilde{\nu}$ is a Radon measure.

(e) If X is second-countable and μ is atomless then $\tilde{\nu}=\nu_{\mathcal{C}}$.

495R Proposition Let $(X,\mathfrak{T})$ be a σ-compact locally compact Hausdorff space and $M_{\mathrm{R}}^{\infty+}(X)$ the set of Radon measures on X. Give $M_{\mathrm{R}}^{\infty+}(X)$ the topology generated by sets of the form $\{\mu:\mu G>\alpha\}$ and $\{\mu:\mu K<\alpha\}$ where $G\subseteq X$ is open, $K\subseteq X$ is compact and $\alpha\in\mathbb{R}$. Let $\mathcal{C}$ be the space of closed subsets of X with its Fell topology, and $P_{\mathrm{R}}(\mathcal{C})$ the set of Radon probability measures on $\mathcal{C}$ with its narrow topology. For $\mu\in M_{\mathrm{R}}^{\infty+}(X)$ and $\gamma>0$ let $\tilde{\nu}_{\mu,\gamma}$ be the Radon measure on $\mathcal{C}$ defined from μ and γ as in 495Q. Then the function $(\mu,\gamma)\mapsto\tilde{\nu}_{\mu,\gamma}:M_{\mathrm{R}}^{\infty+}(X)\times\,]0,\infty[\,\to P_{\mathrm{R}}(\mathcal{C})$ is continuous.

495S Theorem Let $\gamma > 0$, and let ν be the Poisson point process on $[0,\infty[$, with Lebesgue measure, with intensity γ. Let λ_0 be the exponential distribution with expectation $1/\gamma$, regarded as a Radon probability measure on $]0,\infty[$, and λ the corresponding product measure on $]0,\infty[^{\mathbb{N}}$. Define $\phi :]0,\infty[^{\mathbb{N}} \to \mathcal{P}([0,\infty[)$ by setting $\phi(x) = \{\sum_{i=0}^{n} x(i) : n \in \mathbb{N}\}$ for $x \in]0,\infty[^{\mathbb{N}}$. Then ϕ is a measure space isomorphism between $]0,\infty[^{\mathbb{N}}$ and a ν-conegligible subset of $\mathcal{P}([0,\infty[)$.

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496 Maharam submeasures

The old problem of characterizing measurable algebras led, among other things, to the concepts of 'Maharam submeasure' and 'Maharam algebra' (§393). It is known that these can be very different from measures (§394), but the differences are not well understood. In this section I will continue the work of §393 by showing that some, at least, of the ways in which topologies and measures interact apply equally to Maharam submeasures. The most important of these interactions are associated with the concept of 'Radon measure', so the first step is to find a corresponding notion of 'Radon submeasure' (496C, 496Y). In 496D-496K I run through a handful of theorems which parallel results in §§416 and 431-433. Products of submeasures remain problematic, but something can be done (496L-496M).

496A Definitions If $\mathfrak{A}$ is a Boolean algebra, a **submeasure** on $\mathfrak{A}$ is a functional $\mu : \mathfrak{A} \to [0,\infty]$ such that $\mu 0 = 0$ and $\mu a \le \mu(a \cup b) \le \mu a + \mu b$ for all a, $b \in \mathfrak{A}$. μ is **strictly positive** if $\mu a > 0$ for every $a \in \mathfrak{A} \setminus \{0\}$, **exhaustive** if $\lim_{n\to\infty} \mu a_n = 0$ for every disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$, **totally finite** if $\mu 1 < \infty$, a **Maharam submeasure** if it is totally finite and $\lim_{n\to\infty} \mu a_n = 0$ for every non-increasing sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ with zero infimum. A Maharam submeasure is sequentially order-continuous. If μ and ν are two submeasures on a Boolean algebra $\mathfrak{A}$, then μ is **absolutely continuous** with respect to ν if for every $\epsilon > 0$ there is a $\delta > 0$ such that $\mu a \le \epsilon$ whenever $\nu a \le \delta$. A **Maharam algebra** is a Dedekind σ-complete Boolean algebra which carries a strictly positive Maharam submeasure.

496B Basic facts (a) Let μ be a submeasure on a Boolean algebra $\mathfrak{A}$.

(i) Set $I = \{a : a \in \mathfrak{A}, \mu a = 0\}$. I is an ideal of $\mathfrak{A}$; write $\mathfrak{C}$ for the quotient Boolean algebra $\mathfrak{A}/I$. Then we have a strictly positive submeasure $\bar\mu$ on $\mathfrak{C}$ defined by setting $\bar\mu a^\bullet = \mu a$ for every $a \in \mathfrak{A}$.

(ii) If μ is exhaustive, so is $\bar\mu$.

(iii) If $\mathfrak{A}$ is Dedekind σ-complete and μ is a Maharam submeasure, then $\mathfrak{C}$ is a Maharam algebra.
In this context I will say that $\mathfrak{C}$ is **the Maharam algebra of** μ.

(b) If μ is a strictly positive totally finite submeasure on a Boolean algebra $\mathfrak{A}$, there is an associated metric $(a,b) \mapsto \mu(a \triangle b)$; the corresponding metric completion $\widehat{\mathfrak{A}}$ admits a continuous extension of μ to a strictly positive submeasure $\hat\mu$ on $\widehat{\mathfrak{A}}$. If μ is exhaustive, then $\hat\mu$ is a Maharam submeasure and $\widehat{\mathfrak{A}}$ is a Maharam algebra. A Maharam algebra is ccc, therefore Dedekind complete, and weakly (σ,∞)-distributive.

(c) If μ is a submeasure defined on an algebra Σ of subsets of a set X, I will say that the **null ideal** $\mathcal{N}(\mu)$ of μ is the ideal of subsets of X generated by $\{E : E \in \Sigma, \mu E = 0\}$. If $\mathcal{N}(\mu) \subseteq \Sigma$ I will say that μ is **complete**. Generally, the **completion** of μ is the functional $\hat\mu$ defined by saying that $\hat\mu(E \triangle A) = \mu E$ whenever $E \in \Sigma$ and $A \in \mathcal{N}(\mu)$; $\hat\mu$ is a complete submeasure.

(d) If $\mathfrak{A}$ is a Maharam algebra, and μ, ν are two strictly positive Maharam submeasures on $\mathfrak{A}$, then each is absolutely continuous with respect to the other. Consequently the metrics associated with them induce the same topology, the **Maharam-algebra topology** of $\mathfrak{A}$.

496C Radon submeasures Let X be a Hausdorff space. A **totally finite Radon submeasure** on X is a complete totally finite submeasure μ defined on a σ-algebra Σ of subsets of X such that (i) Σ contains every open set (ii) $\inf\{\mu(E \setminus K) : K \subseteq E \text{ is compact}\} = 0$ for every $E \in \Sigma$.

In this context I will say that a set $E \in \Sigma$ is **self-supporting** if $\mu(E \cap G) > 0$ whenever $G \subseteq X$ is open and $G \cap E \neq \emptyset$.

496D Proposition Let μ be a totally finite Radon submeasure on a Hausdorff space X with domain Σ.
(a) μ is a Maharam submeasure.
(b) $\inf\{\mu(G \setminus E) : G \supseteq E$ is open$\} = 0$ for every $E \in \Sigma$.
(c) If $E \in \Sigma$ there is a relatively closed $F \subseteq E$ such that F is self-supporting and $\mu(E \setminus F) = 0$.
(d) If $E \in \Sigma$ and $\epsilon > 0$ there is a compact self-supporting $K \subseteq E$ such that $\mu(E \setminus K) \le \epsilon$.

496E Theorem Let X be a Hausdorff space and $\mathcal{K}$ the family of compact subsets of X. Let $\phi : \mathcal{K} \to [0,\infty[$ be a bounded functional such that

(α) $\phi\emptyset = 0$ and $\phi K \le \phi(K \cup L) \le \phi K + \phi L$ for all K, $L \in \mathcal{K}$;

(β) whenever $K \in \mathcal{K}$ and $\epsilon > 0$ there is an $L \in \mathcal{K}$ such that $L \subseteq X \setminus K$ and $\phi K' \le \epsilon$ whenever $K' \in \mathcal{K}$ is disjoint from $K \cup L$;

(γ) whenever K, $L \in \mathcal{K}$ and $K \subseteq L$ then $\phi L \le \phi K + \sup\{\phi K' : K' \in \mathcal{K}, K' \subseteq L \setminus K\}$.

Then there is a unique totally finite Radon submeasure on X extending ϕ.

496F Theorem Let X be a zero-dimensional compact Hausdorff space and $\mathcal{E}$ the algebra of open-and-closed subsets of X. Let $\nu : \mathcal{E} \to [0,\infty[$ be an exhaustive submeasure. Then there is a unique totally finite Radon submeasure on X extending ν.

496G Theorem Let $\mathfrak{A}$ be a Maharam algebra, and μ a strictly positive Maharam submeasure on $\mathfrak{A}$. Let Z be the Stone space of $\mathfrak{A}$, and write $\widehat{a}$ for the open-and-closed subset of Z corresponding to each $a \in \mathfrak{A}$. Then there is a unique totally finite Radon submeasure ν on Z such that $\nu\widehat{a} = \mu a$ for every $a \in \mathfrak{A}$. The domain of ν is the Baire-property algebra $\widehat{\mathcal{B}}$ of Z, and the null ideal of ν is the nowhere dense ideal of Z.

496H Theorem Let X be a Hausdorff space, Σ_0 an algebra of subsets of X, and $\mu_0 : \Sigma_0 \to [0,\infty[$ an exhaustive submeasure such that $\inf\{\mu_0(E \setminus K) : K \in \Sigma_0$ is compact, $K \subseteq E\} = 0$ for every $E \in \Sigma_0$. Then μ_0 has an extension to a totally finite Radon submeasure μ_1 on X.

496I Theorem Let X be a set, Σ a σ-algebra of subsets of X, and μ a complete Maharam submeasure on Σ.
(a) Σ is closed under Souslin's operation.
(b) If A is the kernel of a Souslin scheme $\langle E_\sigma\rangle_{\sigma\in S}$ in Σ, and $\epsilon > 0$, there is a $\psi \in \mathbb{N}^{\mathbb{N}}$ such that

$$\mu(A \setminus \bigcup_{\phi\in\mathbb{N}^{\mathbb{N}},\phi\le\psi} \bigcap_{n\ge1} E_{\phi\restriction n}) \le \epsilon.$$

496J Theorem Let X be a K-analytic Hausdorff space and μ a Maharam submeasure defined on the Borel σ-algebra of X. Then

$$\inf\{\mu(X \setminus K) : K \subseteq X \text{ is compact}\} = 0.$$

496K Proposition Let μ be a Maharam submeasure on the Borel σ-algebra of an analytic Hausdorff space X. Then the completion of μ is a totally finite Radon submeasure on X.

496L Free products of Maharam algebras If $\mathfrak{A}$, $\mathfrak{B}$ are Boolean algebras with submeasures μ, ν respectively, we have a submeasure $\mu \ltimes \nu$ on the free product $\mathfrak{A} \otimes \mathfrak{B}$. It is easy to see that if μ and ν are strictly positive so is $\mu \ltimes \nu$; moreover, if μ and ν are exhaustive so is $\mu \ltimes \nu$.

Now suppose that $\langle\mathfrak{A}_i\rangle_{i\in I}$ is a family of Maharam algebras, where I is a finite totally ordered set. Then we can take a strictly positive Maharam submeasure μ_i on each $\mathfrak{A}_i$, form an exhaustive submeasure λ on $\mathfrak{C}_I = \bigotimes_{i\in I} \mathfrak{A}_i$, and use λ to construct a metric completion $\widehat{\mathfrak{C}}_I$ which is a Maharam algebra. If we change each μ_i to μ'_i, where μ'_i is another strictly positive Maharam submeasure on $\mathfrak{A}_i$, then every μ'_i is absolutely continuous with respect to μ_i, so the corresponding λ' will be absolutely continuous with respect to λ, and vice versa; in which case the metrics on $\mathfrak{C}_I$ are uniformly equivalent and we get the same metric completion $\widehat{\mathfrak{C}}_I$ up to Boolean algebra isomorphism. We can therefore think of $\widehat{\mathfrak{C}}_I$ as 'the' **Maharam algebra free product** of the family $\langle\mathfrak{A}_i\rangle_{i\in I}$ of Boolean algebras; we shall have an isomorphism between $\widehat{\mathfrak{C}}_{J\cup K}$ and the Maharam algebra free product of $\widehat{\mathfrak{C}}_J$ and $\widehat{\mathfrak{C}}_K$ whenever J, $K \subseteq I$ and $j < k$ for every $j \in J$ and $k \in K$.

If $(\mathfrak{A}, \mu)$ and $(\mathfrak{B}, \nu)$ are probability algebras, then their Maharam algebra free product, regarded as a Boolean algebra, is isomorphic to their probability algebra free product.

496M Representing products of Maharam algebras: Theorem Let X and Y be sets, with σ-algebras Σ and T and Maharam submeasures μ and ν defined on Σ, T respectively. Let $\mathfrak{A}$, $\mathfrak{B}$ be their Maharam algebras and write $\bar{\mu}$, $\bar{\nu}$ for the strictly positive Maharam submeasures on $\mathfrak{A}$ and $\mathfrak{B}$ induced by μ and ν. Let $\Sigma\widehat{\otimes}\mathrm{T}$ be the σ-algebra of subsets of $X \times Y$ generated by $\{E \times F : E \in \Sigma,\ F \in \mathrm{T}\}$.

(a) Give $\mathfrak{B}$ its Maharam-algebra topology. If $W \in \Sigma\widehat{\otimes}\mathrm{T}$ then $W[\{x\}] \in \mathrm{T}$ for every $x \in X$ and the function $x \mapsto W[\{x\}]^\bullet : X \to \mathfrak{B}$ is Σ-measurable and has separable range. $x \mapsto \nu W[\{x\}] : X \to [0,\infty[$ is Σ-measurable.

(b) For $W \in \Sigma\widehat{\otimes}\mathrm{T}$ set

$$\lambda W = \inf\{\epsilon : \epsilon > 0,\ \mu\{x : \nu W[\{x\}] > \epsilon\} \le \epsilon\}.$$

Then λ is a Maharam submeasure on $\Sigma\widehat{\otimes}\mathrm{T}$, and

$$\lambda^{-1}[\{0\}] = \{W : W \in \Sigma\widehat{\otimes}\mathrm{T},\ \{x : W[\{x\}] \notin \mathcal{N}(\nu)\} \in \mathcal{N}(\mu)\}.$$

(c) Let $\mathfrak{C}$ be the Maharam algebra of λ. Then $\mathfrak{A} \otimes \mathfrak{B}$ can be embedded in $\mathfrak{C}$ by mapping $E^\bullet \otimes F^\bullet$ to $(E \times F)^\bullet$ for all $E \in \Sigma$ and $F \in \mathrm{T}$.

(d) This embedding identifies $(\mathfrak{C}, \bar{\lambda})$ with the metric completion of $(\mathfrak{A} \otimes \mathfrak{B}, \bar{\mu} \ltimes \bar{\nu})$.

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497 Tao's proof of Szemerédi's theorem

Szemerédi's celebrated theorem on arithmetic progressions (497L) is not obviously part of measure theory. Remarkably, however, it has stimulated significant developments in the subject. The first was Furstenberg's multiple recurrence theorem. In this section I will give an account of an approach due to T.Tao which introduces another phenomenon of great interest from a measure-theoretic point of view.

497A Definitions (a) Let (X, Σ, μ) be a probability space, T a subalgebra of Σ and $\langle \Sigma_i \rangle_{i \in I}$ a family of σ-subalgebras of Σ. $\langle \Sigma_i \rangle_{i \in I}$ has **T-removable intersections** if whenever $J \subseteq I$ is finite and not empty, $E_i \in \Sigma_i$ for $i \in J$, $\mu(\bigcap_{i \in J} E_i) = 0$ and $\epsilon > 0$, there is a family $\langle F_i \rangle_{i \in J}$ such that $F_i \in \mathrm{T} \cap \Sigma_i$ and $\mu(E_i \setminus F_i) \le \epsilon$ for each $i \in J$, and $\bigcap_{i \in J} F_i = \emptyset$.

(b) If X is a set and Σ, Σ' are two σ-algebras of subsets of X, $\Sigma \vee \Sigma'$ will be the σ-algebra generated by $\Sigma \cup \Sigma'$. If $\langle \Sigma_i \rangle_{i \in I}$ is a family of σ-algebras of subsets of X, I will write $\bigvee_{i \in I} \Sigma_i$ for the σ-algebra generated by $\bigcup_{i \in I} \Sigma_i$.

(c) If (X, Σ, μ) is a probability space and $\mathcal{A} \subseteq \mathcal{E} \subseteq \Sigma$, $\mathcal{A}$ is **metrically dense** in $\mathcal{E}$ if for every $E \in \mathcal{E}$ and $\epsilon > 0$ there is an $F \in \mathcal{A}$ such that $\mu(E \triangle F) \le \epsilon$. Note that a subalgebra of Σ is metrically dense in the σ-algebra it generates.

497B Lemma Let (X, Σ, μ) be a probability space and T a subalgebra of Σ. Let $\langle \Sigma_i \rangle_{i \in I}$ be a family of σ-subalgebras of Σ.

(a) $\langle \Sigma_i \rangle_{i \in I}$ has T-removable intersections iff $\langle \Sigma_i \rangle_{i \in J}$ has T-removable intersections for every finite $J \subseteq I$.

(b) Suppose that $\langle \Sigma_i \rangle_{i \in I}$ has T-removable intersections and that $\mathrm{T} \cap \Sigma_i$ is metrically dense in Σ_i for every i. Let J be any set and $f : J \to I$ a function. Then $\langle \Sigma_{f(j)} \rangle_{j \in J}$ has T-removable intersections.

(c) Suppose that, for each $i \in I$, we are given a σ-subalgebra Σ_i' of Σ_i such that for every $E \in \Sigma_i$ there is an $E' \in \Sigma_i'$ such that $E \triangle E'$ is negligible. If $\langle \Sigma_i' \rangle_{i \in I}$ has T-removable intersections, so has $\langle \Sigma_i \rangle_{i \in I}$.

497C Lemma Let (X, Σ, μ) be a probability space and T a subalgebra of Σ. Let I be a set, A an upwards-directed set, and $\langle \Sigma_{\alpha i} \rangle_{\alpha \in A, i \in I}$ a family of σ-subalgebras of Σ such that, setting $\Sigma_i = \bigvee_{\alpha \in A} \Sigma_{\alpha i}$ for each i,

(i) $\Sigma_{\alpha i} \subseteq \Sigma_{\beta i}$ whenever $i \in I$ and $\alpha \le \beta$ in A,

(ii) $\langle \Sigma_{\alpha i} \rangle_{i \in I}$ has T-removable intersections for every $\alpha \in A$,

(iii) Σ_i and $\bigvee_{j \in I} \Sigma_{\alpha j}$ are relatively independent over $\Sigma_{\alpha i}$ for every $i \in I$ and $\alpha \in A$.

Then $\langle \Sigma_i \rangle_{i \in I}$ has T-removable intersections.

497D Lemma Let (X,Σ,μ) be a probability space, T a subalgebra of Σ, and $\langle\Sigma_i\rangle_{i\in I}$ a finite family of σ-subalgebras of Σ which has T-removable intersections; suppose that $\mathrm{T}\cap\Sigma_i$ is metrically dense in Σ_i for each i. Set $\Sigma^*=\bigvee_{i\in I}\Sigma_i$. Suppose that we have a finite set Γ, a function $g:\Gamma\to I$ and a family $\langle\Lambda_\gamma\rangle_{\gamma\in\Gamma}$ of σ-subalgebras of Σ such that

$\langle\Lambda_\gamma\rangle_{\gamma\in\Gamma}$ is relatively independent over Σ^*,

for each $\gamma\in\Gamma$, Λ_γ and Σ^* are relatively independent over $\Sigma_{g(\gamma)}$,

for each $\gamma\in\Gamma$, $\mathrm{T}\cap\Lambda_\gamma$ is metrically dense in Λ_γ.

Let A be a finite set and $f:A\to I$, $\phi:A\to\mathcal{P}\Gamma$ functions such that $\Sigma_{g(\gamma)}\subseteq\Sigma_{f(\alpha)}$ whenever $\alpha\in A$ and $\gamma\in\phi(\alpha)$. Suppose that

for each $\alpha\in A$, $\bigvee_{\gamma\in\phi(\alpha)}\Lambda_\gamma$ and $\Sigma^*\vee\bigvee_{\gamma\in\Gamma\setminus\phi(\alpha)}\Lambda_\gamma$ are relatively independent over $\Sigma_{f(\alpha)}$.

Set $\tilde\Sigma_\alpha=\Sigma_{f(\alpha)}\vee\bigvee_{\gamma\in\phi(\alpha)}\Lambda_\gamma$ for $\alpha\in A$. Then $\langle\tilde\Sigma_\alpha\rangle_{\alpha\in A}$ has T-removable intersections.

497E Theorem Let (X,Σ,μ) be a probability space, and T a subalgebra of Σ. Let Γ be a partially ordered set such that $\gamma\wedge\delta=\inf\{\gamma,\delta\}$ is defined in Γ for all γ, $\delta\in\Gamma$, and $\langle\Sigma_\gamma\rangle_{\gamma\in\Gamma}$ a family of σ-subalgebras of Σ such that

(i) $\mathrm{T}\cap\Sigma_\gamma$ is metrically dense in Σ_γ for every $\gamma\in\Gamma$,

(ii) if γ, $\delta\in\Gamma$ and $\gamma\le\delta$ then $\Sigma_\gamma\subseteq\Sigma_\delta$,

(iii) if $\gamma\in\Gamma$ and Δ, Δ' are finite subsets of Γ such that $\delta\wedge\gamma\in\Delta'$ for every $\delta\in\Delta$, then Σ_γ and $\bigvee_{\delta\in\Delta}\Sigma_\delta$ are relatively independent over $\bigvee_{\delta\in\Delta'}\Sigma_\delta$.

Then $\langle\Sigma_\gamma\rangle_{\gamma\in\Gamma}$ has T-removable intersections.

497F Invariant measures on $\mathcal{P}([I]^{<\omega})$ (a) Let I be a set. Then $\mathcal{P}([I]^{<\omega})$ is a compact Hausdorff space, if we give it its usual topology, generated by sets of the form $\{R:a\in R\subseteq[I]^{<\omega},\ b\notin R\}$ for finite sets a, $b\subseteq I$. Let G_I be the set of permutations of I, and for $\phi\in G_I$, $R\subseteq[I]^{<\omega}$ set

$$\phi\bullet R=\{\phi[a]:a\in R\}=\{a:a\in[I]^{<\omega},\ \phi^{-1}[a]\in R\},$$

so that $\bullet$ is an action of G_I on $\mathcal{P}([I]^{<\omega})$, and $R\mapsto\phi\bullet R$ is a homeomorphism for every $\phi\in G_I$. Let P_I be the set of Radon probability measures on $\mathcal{P}([I]^{<\omega})$. Then we have an action of G_I on P_I defined by saying that

$$\phi\bullet E=\{\phi\bullet R:R\in E\}$$

for $\phi\in G_I$ and $E\subseteq\mathcal{P}([I]^{<\omega})$, and

$$(\phi\bullet\mu)(E)=\mu(\phi^{-1}\bullet E)$$

for $\phi\in G_I$, $\mu\in P_I$ and Borel sets $E\subseteq\mathcal{P}([I]^{<\omega})$. Because $R\mapsto\phi\bullet R$ is a homeomorphism, the map $\mu\mapsto\phi\bullet\mu$ is a homeomorphism when P_I is given its narrow topology.

(b) If $\mu\in P_I$, μ is **permutation-invariant** if $\mu=\phi\bullet\mu$ for every $\phi\in G_I$.

(c) For $R\subseteq[I]^{<\omega}$ and $J\subseteq I$ write $R\lceil J$ for the trace $R\cap\mathcal{P}J\subseteq[J]^{<\omega}$ of R on J. Let $\mathcal{V}$ be the family of sets of the form $\{R:R\subseteq[I]^{<\omega},\ R\lceil J=S\}$ where $J\subseteq I$ is finite and $S\subseteq\mathcal{P}J$. If μ, $\nu\in P_I$ agree on $\mathcal{V}$, they are equal.

(d) If I, J are sets and $f:I\to J$ is a function, define $\tilde f:\mathcal{P}([J]^{<\omega})\to\mathcal{P}([I]^{<\omega})$ by setting $\tilde f(R)=\{a:a\in[I]^{<\omega},\ f[a]\in R\}$ for $R\subseteq[J]^{<\omega}$. $\tilde f$ is continuous. If $I\subseteq J$ and f is the identity function, then $\tilde f(R)=R\lceil I$ for every $R\subseteq[J]^{<\omega}$. Observe that when $\phi\in G_I$ and $R\subseteq[I]^{<\omega}$ then $\tilde\phi(R)=\phi^{-1}\bullet R$.

497G Theorem Let I be an infinite set and $\mathcal{J}$ a filter on I not containing any finite set. Let T be the algebra of open-and-closed subsets of $\mathcal{P}([I]^{<\omega})$, and $\mu\in P_I$ a permutation-invariant measure. For $J\subseteq I$, write Σ_J for the σ-algebra of subsets of $\mathcal{P}([I]^{<\omega})$ generated by sets of the form $E_a=\{R:a\in R\subseteq[I]^{<\omega}\}$ where $a\in[J]^{<\omega}$. Then $\langle\Sigma_J\rangle_{J\in\mathcal{J}}$ has T-removable intersections with respect to μ.

497H Construction Suppose we are given a sequence $\langle(m_n,T_n)\rangle_{n\in\mathbb{N}}$ and a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$ such that

(α) $\langle m_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathbb{N}\setminus\{0\}$ and $\lim_{n\to\mathcal{F}} m_n=\infty$,
(β) $T_n\subseteq\mathcal{P}m_n$ for each n.
Then for any set I there is a permutation-invariant $\mu\in P_I$ such that

$$\mu\{R:R\restriction K=S\}=\lim_{n\to\mathcal{F}}\frac{1}{m_n^{\#(K)}}\#(\{z:z\in m_n^K,\,\tilde z(T_n)=S\})$$

whenever $K\subseteq I$ is finite and $S\subseteq\mathcal{P}K$.

497I Definition If I, J are sets, $R\subseteq\mathcal{P}I$ and $S\subseteq\mathcal{P}J$, an **embedding** of (I,R) in (J,S) is an injective function $f:I\to J$ such that $f[a]\in S$ for every $a\in R$.

497J Theorem Let L be a finite set with r members, and $T\subseteq\mathcal{P}L$. Then for every $\epsilon>0$ there is a $\delta>0$ such that whenever I is a non-empty finite set, $R\subseteq\mathcal{P}I$ and the number of embeddings of (L,T) in (I,R) is at most $\delta\#(I)^r$, there is an $S\subseteq\mathcal{P}I$ such that $\#(S\cap[I]^k)\le\epsilon\#(I)^k$ for every k and there is no embedding of (L,T) in $(I,R\setminus S)$.

497K Corollary: the Hypergraph Removal Lemma For every $\epsilon>0$ and $r\ge1$ there is a $\delta>0$ such that whenever I is a finite set, $R\subseteq[I]^r$ and $\#(\{J:J\in[I]^{r+1},\,[J]^r\subseteq R\})\le\delta\#(I)^{r+1}$, there is an $S\subseteq[I]^r$ such that $\#(S)\le\epsilon\#(I)^r$ and there is no $J\in[I]^{r+1}$ such that $[J]^r\subseteq R\setminus S$.

497L Corollary: Szemerédi's Theorem For every $\epsilon>0$ and $r\ge2$ there is an $n_0\in\mathbb{N}$ such that whenever $n\ge n_0$, $A\subseteq n$ and $\#(A)\ge\epsilon n$ there is an arithmetic progression of length $r+1$ in A.

497M Lemma (cf. Solymosi 03) Suppose that $r\ge1$ and $n\in\mathbb{N}$. For $0\le j,\,k<r$ set $e_j(k)=1$ if $k=j$, 0 otherwise. For $z\in n^r$ and $C\subseteq n^r$ write

$$\Delta(z,C)=\{k:k\in\mathbb{Z},\,z+ke_i\in C\text{ for every }i<r\},\quad q(z,C)=\#(\Delta(z,C)).$$

Then for every $\epsilon>0$ there is a $\delta>0$ such that $\#(\{z:z\in n^r,\,q(z,C)\ge\delta n\})\ge\delta n^r$ whenever $n\in\mathbb{N}$, $C\subseteq n^r$ and $\#(C)\ge\epsilon n^r$.

497N Theorem Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and $\langle\pi_i\rangle_{i<r}$ a non-empty finite commuting family of measure-preserving Boolean homomorphisms from $\mathfrak{A}$ to itself. If $a\in\mathfrak{A}\setminus\{0\}$, there is an $\eta>0$ such that

$$\sum_{k=0}^{n-1}\bar\mu(\inf_{i<r}\pi_i^k a)\ge\eta n$$

for every $n\in\mathbb{N}$.

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498 Cubes in product spaces

I offer a brief note on a special property of (Radon) product measures.

498A Proposition Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra with its measure-algebra topology. Suppose that $A\subseteq\mathfrak{A}$ is an uncountable analytic set. Then there is a compact set $L\subseteq A$, homeomorphic to $\{0,1\}^{\mathbb{N}}$, such that $\inf L\ne0$ in $\mathfrak{A}$.

498B Proposition Let $(X,\mathfrak{T},\Sigma,\mu)$ be an atomless Radon measure space, $(Y,\mathfrak{S},\mathrm{T},\nu)$ an effectively locally finite τ-additive topological measure space and $\tilde\lambda$ the τ-additive product measure on $X\times Y$. Then if $W\subseteq X\times Y$ is closed and $\tilde\lambda W>0$ there are a non-scattered compact set $K\subseteq X$ and a closed set $F\subseteq Y$ of positive measure such that $K\times F\subseteq W$.

498C Proposition Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a countable family of atomless Radon probability spaces, and $\tilde\lambda$ the product Radon probability measure on $X=\prod_{i\in I}X_i$. If $W\subseteq X$ and $\tilde\lambda W>0$, there is a family $\langle K_i\rangle_{i\in I}$ such that $K_i\subseteq X_i$ is a non-scattered compact set for each $i\in I$ and $\prod_{i\in I}K_i\subseteq W$.