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Chapter 49

Further topics

I conclude the volume with notes on six almost unconnected special topics. In §491 I look at equidistributed sequences and the ideal $\mathcal{Z}$ of sets with asymptotic density zero. I give the principal theorems on the existence of equidistributed sequences in abstract topological measure spaces, and examine the way in which an equidistributed sequence can induce an embedding of a measure algebra in the quotient algebra $\mathcal{P}\mathbb{N}/\mathcal{Z}$. The next three sections are linked. In §492 I present some forms of 'concentration of measure' which echo ideas from §476 in combinatorial, rather than geometric, contexts, with theorems of Talagrand and Maurey on product measures and the Haar measure of a permutation group. In §493 I show how the ideas of §§449, 476 and 492 can be put together in the theory of 'extremely amenable' topological groups. Some of the important examples of extremely amenable groups are full groups of measure-preserving automorphisms of measure algebras, previously treated in §383; these are the subject of §494, where I look also at some striking algebraic properties of these groups. In §495, I move on to Poisson point processes, with notes on disintegrations and some special cases in which they can be represented by Radon measures. In §496, I revisit the Maharam submeasures of Chapter 39, showing that various ideas from the present volume can be applied in this more general context. In §497, I give a version of Tao's proof of Szemerédi's theorem on arithmetic progressions, based on a deep analysis of relative independence, as introduced in §458. Finally, in §498 I give a pair of simple, but perhaps surprising, results on subsets of sets of positive measure in product spaces.

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491 Equidistributed sequences

In many of the most important topological probability spaces, starting with Lebesgue measure (491Eb, 491Xg), there are sequences which are equidistributed in the sense that, in the limit, they spend the right proportion of their time in each part of the space (491Yi). I give the basic results on existence of equidistributed sequences in 491E-491H, 491Q and 491R. Investigating such sequences, we are led to some interesting properties of the asymptotic density ideal $\mathcal{Z}$ and the quotient algebra $\mathfrak{Z} = \mathcal{P}\mathbb{N}/\mathcal{Z}$ (491A, 491I-491K, 491P). For 'effectively regular' measures (491L-491M), equidistributed sequences lead to embeddings of measure algebras in $\mathfrak{Z}$ (491N).

491A The asymptotic density ideal (a) If I is a subset of $\mathbb{N}$, its **upper asymptotic density** is $d^*(I) = \limsup_{n\to\infty} \frac{1}{n}(I\cap n)$, and its **asymptotic density** is $d(I) = \lim_{n\to\infty} \frac{1}{n}\#(I\cap n)$ if this is defined. It is easy to check that d^* is a submeasure on $\mathcal{P}\mathbb{N}$ (definition: 392A), so that

$$\mathcal{Z} = \{I : I \subseteq \mathbb{N},\, d^*(I) = 0\} = \{I : I \subseteq \mathbb{N},\, d(I) = 0\}$$

is an ideal, the **asymptotic density ideal**.

(b) Note that

$$\mathcal{Z} = \{I : I \subseteq \mathbb{N},\, \lim_{n\to\infty} 2^{-n}\#(I\cap 2^{n+1}\setminus 2^n) = 0\}.$$

P If $I \subseteq \mathbb{N}$ and $d^*(I) = 0$, then

$$2^{-n}\#(I\cap 2^{n+1}\setminus 2^n) \le 2\cdot 2^{-n-1}\#(I\cap 2^{n+1}) \to 0$$

as $n\to\infty$. In the other direction, if $\lim_{n\to\infty}2^{-n}\#(I\cap 2^{n+1}\setminus 2^n)=0$, then for any $\epsilon>0$ there is an $m\in\mathbb{N}$ such that $\#(I\cap 2^{k+1}\setminus 2^k)\le 2^k\epsilon$ for every $k\ge m$. In this case, for $n\ge 2^m$, take k_n such that $2^{k_n}\le n<2^{k_n+1}$, and see that

$$\frac{1}{n}\#(I\cap n)\le 2^{-k_n}(\#(I\cap 2^m)+\sum_{k=m}^{k_n}2^k\epsilon)\le 2^{-k_n}\#(I\cap 2^m)+2\epsilon\to 2\epsilon$$

as $n\to\infty$, and $d^*(I)\le 2\epsilon$; as ϵ is arbitrary, $I\in\mathcal{Z}$. **Q**

(c) Writing $\mathcal{D}$ for the domain of d,

$$\begin{aligned}\mathcal{D}&=\{I:I\subseteq\mathbb{N},\ \limsup_{n\to\infty}\frac{1}{n}\#(I\cap n)=\liminf_{n\to\infty}\frac{1}{n}\#(I\cap n)\}\\&=\{I:I\subseteq\mathbb{N},\ d^*(I)=1-d^*(\mathbb{N}\setminus I)\},\end{aligned}$$

$$\mathbb{N}\in\mathcal{D},\quad\text{if } I,\ J\in\mathcal{D}\text{ and } I\subseteq J\text{ then } J\setminus I\in\mathcal{D},$$

$$\text{if } I,\ J\in\mathcal{D}\text{ and } I\cap J=\emptyset\text{ then } I\cup J\in\mathcal{D}\text{ and } d(I\cup J)=d(I)+d(J).$$

It follows that if $\mathcal{I}\subseteq\mathcal{D}$ and $I\cap J\in\mathcal{I}$ for all $I,\ J\in\mathcal{I}$, then the subalgebra of $\mathcal{P}\mathbb{N}$ generated by $\mathcal{I}$ is included in $\mathcal{D}$ (313Ga). But note that $\mathcal{D}$ itself is *not* a subalgebra of $\mathcal{P}\mathbb{N}$ (491Xa).

(d) The following elementary fact will be useful. If $\langle l_n\rangle_{n\in\mathbb{N}}$ is a strictly increasing sequence in $\mathbb{N}$ such that $\lim_{n\to\infty}l_{n+1}/l_n=1$, and $I\subseteq\mathbb{R}$, then

$$d^*(I)\le\limsup_{n\to\infty}\frac{1}{l_{n+1}-l_n}\#(I\cap l_{n+1}\setminus l_n).$$

P Set $\gamma=\limsup_{n\to\infty}\frac{1}{l_{n+1}-l_n}\#(I\cap l_{n+1}\setminus l_n)$, and take $\epsilon>0$. Let n_0 be such that $\#(I\cap l_{n+1}\setminus l_n)\le(\gamma+\epsilon)(l_{n+1}-l_n)$ and $l_{n+1}-l_n\le\epsilon l_n$ for every $n\ge n_0$, and write M for $\#(I\cap l_{n_0})$. If $m>l_{n_0}$, take k such that $l_k\le m<l_{k+1}$; then

$$\begin{aligned}\#(I\cap m)&\le M+\sum_{n=n_0}^{k-1}\#(I\cap l_{n+1}\setminus l_n)+(m-l_k)\\&\le M+\sum_{n=n_0}^{k-1}(\gamma+\epsilon)(l_{n+1}-l_n)+l_{k+1}-l_k\le M+m(\gamma+\epsilon)+\epsilon m,\end{aligned}$$

so

$$\frac{1}{m}\#(I\cap m)\le\frac{M}{m}+\gamma+2\epsilon.$$

Accordingly $d^*(I)\le\gamma+2\epsilon$; as ϵ is arbitrary, $d^*(I)\le\gamma$. **Q**

***(e)** The following remark will not be used directly in this section, but is one of the fundamental properties of the ideal $\mathcal{Z}$. If $\langle I_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\mathcal{Z}$, there is an $I\in\mathcal{Z}$ such that $I_n\setminus I$ is finite for every n. **P** Set $J_n=\bigcup_{j\le n}I_j$ for each n, so that $\langle J_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathcal{Z}$. Let $\langle l_n\rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence in $\mathbb{N}$ such that, for each n, $\#(J_n\cap k)\le 2^{-n}k$ for every $k\ge l_n$. Set $I=\bigcup_{n\in\mathbb{N}}J_n\setminus l_n$. Then $I_n\setminus I\subseteq l_n$ is finite for each n. Also, if $n\in\mathbb{N}$ and $l_n\le k<l_{n+1}$,

$$\#(I\cap k)\le\#(J_n\cap k)\le 2^{-n}k,$$

so $I\in\mathcal{Z}$. **Q**

491B Equidistributed sequences Let X be a topological space and μ a probability measure on X. I say that a sequence $\langle x_i\rangle_{i\in\mathbb{N}}$ in X is **(asymptotically) equidistributed** if $d^*(\{i:x_i\in F\})\le\mu F$ for every measurable closed set $F\subseteq X$; equivalently, if $\liminf_{n\to\infty}\frac{1}{n}\#(\{i:i<n,\ x_i\in G\})\ge\mu G$ for every measurable open set $G\subseteq X$.

Remark Equidistributed sequences are often called **uniformly distributed**. Traditionally, such sequences have been defined in terms of their action on continuous functions, as in 491Cf. I have adopted the definition here in order to deal both with Radon measures on spaces which are not completely regular (so that we cannot identify the measure with an integral) and with Baire measures (so that there may be closed sets which are not measurable). Note that we cannot demand that the sets $\{i : x_i \in F\}$ should have well-defined densities (491Ye).

491C I work through a list of basic facts. The technical details (if we do not specialize immediately to metrizable or compact spaces) are not quite transparent, so I set them out carefully.

Proposition Let X be a topological space, μ a probability measure on X and $\langle x_i\rangle_{i\in\mathbb{N}}$ a sequence in X.

(a) $\langle x_i\rangle_{i\in\mathbb{N}}$ is equidistributed iff $\liminf_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n f(x_i) \ge \int f d\mu$ for every measurable bounded lower semi-continuous function $f : X \to \mathbb{R}$.

(b) If μ measures every zero set and $\langle x_i\rangle_{i\in\mathbb{N}}$ is equidistributed, then $\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n f(x_i) = \int f d\mu$ for every $f \in C_b(X)$.

(c) Suppose that μ measures every zero set in X. If $\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n f(x_i) = \int f d\mu$ for every $f \in C_b(X)$, then $d^*(\{i : x_i \in F\}) \le \mu F$ for every zero set $F \subseteq X$.

(d) Suppose that X is normal and that μ measures every zero set and is inner regular with respect to the closed sets. If $\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n f(x_i) = \int f d\mu$ for every $f \in C_b(X)$, then $\langle x_i\rangle_{i\in\mathbb{N}}$ is equidistributed.

(e) Suppose that μ is τ-additive and there is a base $\mathcal{G}$ for the topology of X, consisting of measurable sets and closed under finite unions, such that $\liminf_{n\to\infty}\frac{1}{n+1}\#(\{i : i \le n,\, x_i \in G\}) \ge \mu G$ for every $G \in \mathcal{G}$. Then $\langle x_i\rangle_{i\in\mathbb{N}}$ is equidistributed.

(f) Suppose that X is completely regular and that μ measures every zero set and is τ-additive. Then $\langle x_i\rangle_{i\in\mathbb{N}}$ is equidistributed iff the limit $\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n f(x_i)$ is defined and equal to $\int f d\mu$ for every $f \in C_b(X)$.

(g) Suppose that X is metrizable and that μ is a topological measure. Then $\langle x_i\rangle_{i\in\mathbb{N}}$ is equidistributed iff the limit $\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n f(x_i)$ is defined and equal to $\int f d\mu$ for every $f \in C_b(X)$.

(h) Suppose that X is compact, Hausdorff and zero-dimensional, and that μ is a Radon measure on X. Then $\langle x_i\rangle_{i\in\mathbb{N}}$ is equidistributed iff $d(\{i : x_i \in G\}) = \mu G$ for every open-and-closed subset G of X.

proof (a)(i) Suppose that $\langle x_i\rangle_{i\in\mathbb{N}}$ is equidistributed. Let $f : X \to [0,1]$ be a measurable lower semi-continuous function and $k \ge 1$. For each $j \le k$ set $G_j = \{x : f(x) > \frac{j}{k}\}$. Then

$$\liminf_{n\to\infty}\frac{1}{n+1}\sum_{j=1}^n \chi G_j(x_i) = \liminf_{n\to\infty}\frac{1}{n+1}\#(\{i : i \le n,\, x_i \in G_j\}) \ge \mu G_j$$

because $\langle x_i\rangle_{i\in\mathbb{N}}$ is equidistributed and G_j is a measurable open set. Also $f - \frac{1}{k}\chi X \le \frac{1}{k}\sum_{j=1}^k \chi G_j \le f$, so

$$\begin{aligned}\int f d\mu - \frac{1}{k} &\le \frac{1}{k}\sum_{j=1}^k \mu G_j \le \frac{1}{k}\sum_{j=1}^k \liminf_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n \chi G_j(x_i)\\ &\le \frac{1}{k}\liminf_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n\sum_{j=1}^k \chi G_j(x_i) \le \liminf_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n f(x_i).\end{aligned}$$

As k is arbitrary, $\liminf_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n f(x_i) \ge \int f d\mu$.

The argument above depended on f taking values in $[0,1]$. But multiplying by an appropriate positive scalar we see that $\liminf_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n f(x_i) \ge \int f d\mu$ for every bounded measurable lower semi-continuous $f : X \to [0,\infty[$, and adding a multiple of χX we see that the same formula is valid for all bounded measurable lower semi-continuous $f : X \to \mathbb{R}$.

(ii) Conversely, if $\liminf_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n f(x_i) \ge \int f d\mu$ for every bounded measurable lower semi-continuous $f : X \to \mathbb{R}$, and $G \subseteq X$ is a measurable open set, then χG is lower semi-continuous, so $\liminf_{n\to\infty}\frac{1}{n+1}\#(\{i : i \le n,\, x_i \in G\}) \ge \mu G$. As G is arbitrary, $\langle x_i\rangle_{i\in\mathbb{N}}$ is equidistributed.

(b) Apply (a) to the lower semi-continuous functions f and $-f$. (Recall that if μ measures every zero set, then every bounded continuous real-valued function is integrable, by 4A3L.)

(c) Let $F \subseteq X$ be a zero set, and $\epsilon > 0$. Then there is a continuous $f : X \to \mathbb{R}$ such that $F = f^{-1}[\{0\}]$. Let $\delta > 0$ be such that $\mu\{x : 0 < |f(x)| \le \delta\} \le \epsilon$, and set $g = (\chi X - \frac{1}{\delta}|f|)^+$. Then $g : X \to [0,1]$ is continuous and $\chi F \le g$, so

$$\begin{aligned} d^*(\{i : x_i \in F\}) &\le \limsup_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} g(x_i) \\ &= \int g\,d\mu \le \mu\{x : |f(x)| \le \delta\} \le \mu F + \epsilon. \end{aligned}$$

As ϵ and F are arbitrary, we have the result.

(d) Let $F \subseteq X$ be a measurable closed set and $\epsilon > 0$. Because μ is inner regular with respect to the closed sets, there is a measurable closed set $F' \subseteq X \setminus F$ such that $\mu F' \ge \mu(X \setminus F) - \epsilon$. Because X is normal, there is a continuous function $f : X \to [0,1]$ such that $\chi F \le f \le \chi(X \setminus F')$. Now

$$d^*(\{i : x_i \in F\}) \le \limsup_{n\to\infty} \tfrac{1}{n+1}\textstyle\sum_{i=0}^{n} f(x_i) = \int f d\mu \le \mu(X \setminus F') \le \mu F + \epsilon.$$

As F and ϵ are arbitrary, $\langle x_i\rangle_{i\in\mathbb{N}}$ is equidistributed.

(e) Let $G \subseteq X$ be a measurable open set, and $\epsilon > 0$. Then $\mathcal{H} = \{H : H \in \mathcal{G}, H \subseteq G\}$ is upwards-directed and has union G; since μ is τ-additive, there is an $H \in \mathcal{H}$ such that $\mu H \ge \mu G - \epsilon$. Now

$$\begin{aligned} \mu G \le \epsilon + \mu H &\le \epsilon + \liminf_{n\to\infty} \frac{1}{n}\#(\{i : i < n,\, x_i \in H\}) \\ &\le \epsilon + \liminf_{n\to\infty} \frac{1}{n}\#(\{i : i < n,\, x_i \in G\}); \end{aligned}$$

as ϵ and G are arbitrary, $\langle x_i\rangle_{i\in\mathbb{N}}$ is equidistributed.

(f)(i) If $\langle x_i\rangle_{i\in\mathbb{N}}$ is equidistributed then (b) tells us that $\lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(x_i) = \int f d\mu$ for every $f \in C_b(X)$. (ii) Suppose that $\lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(x_i) = \int f d\mu$ for every $f \in C_b(X)$. If $G \subseteq X$ is a cozero set, we can apply (c) to its complement to see that $\liminf_{n\to\infty} \frac{1}{n+1}\#(\{i : i \le n,\, x_i \in G\}) \ge \mu G$. So applying (e) with $\mathcal{G}$ the family of cozero sets we see that $\langle x_i\rangle_{i\in\mathbb{N}}$ is equidistributed.

(g) Because every closed set is a zero set, this follows at once from (b) and (c).

(h) If $\langle x_i\rangle_{i\in\mathbb{N}}$ is equidistributed and $G \subseteq X$ is open-and-closed, then $d^*(\{i : x_i \in G\}) \le \mu G$ because G is closed and $d^*(\{i : x_i \notin G\}) \le 1 - \mu G$ because G is open; so $d(\{i : x_i \in G\}) = \mu G$. If the condition is satisfied, then (e) tells us that $\langle x_i\rangle_{i\in\mathbb{N}}$ is equidistributed.

491D The next lemma provides a useful general criterion for the existence of equidistributed sequences.

Lemma Let X be a topological space and μ a probability measure on X. Suppose that there is a sequence $\langle \nu_n\rangle_{n\in\mathbb{N}}$ of point-supported probability measures on X such that $\limsup_{n\to\infty} \nu_n F \le \mu F$ for every measurable closed set $F \subseteq X$. Then μ has an equidistributed sequence.

proof For each $n \in \mathbb{N}$, let $q_n : X \to [0,1]$ be such that $\nu_n E = \sum_{x\in E} q_n(x)$ for every $E \subseteq X$. Let $q'_n : X \to [0,1]$ be such that $\sum_{x\in X} q'_n(x) = 1$, $K_n = \{x : q'_n(x) > 0\}$ is finite, $q'_n(x)$ is rational for every x, and $\sum_{x\in X} |q_n(x) - q'_n(x)| \le 2^{-n}$; then $\limsup_{n\to\infty} \nu'_n F \le \mu F$ for every measurable closed F, where ν'_n is defined from q'_n. For each n, let $s_n \ge 1$ be such that $r_n(x) = q'_n(x)s_n$ is an integer for every $x \in K_n$. Let $\langle x_{ni}\rangle_{i<s_n}$ be a family in K_n such that $\#(\{i : i < s_n,\, x_{ni} = x\}) = r_n(x)$ for each $x \in K_n$; then $\nu'_n E = \frac{1}{s_n}\#(\{i : i < s_n,\, x_{ni} \in E\})$ for every $E \subseteq X$.

Let $\langle m_k\rangle_{k\in\mathbb{N}}$ be such that $s_{k+1} \le 2^{-k}\sum_{j=0}^{k} m_j s_j$ for each k. Set $l_0 = 0$. Given l_n, take the largest k such that $\sum_{j=0}^{k-1} m_j s_j \le l_n$; set $l_{n+1} = l_n + s_k$ and $x_i = x_{k,i-l_n}$ for $l_n \le i < l_{n+1}$; continue. By the choice of the m_k, $l_{n+1}/l_n \to 1$ as $n \to \infty$. For any $E \subseteq X$, $\#(\{i : l_n \le i < l_{n+1},\, x_i \in E\}) = \#(\{j : j < s_k,\, x_{kj} \in E\})$ whenever $\sum_{j=0}^{k-1} m_j s_j \le l_n < \sum_{j=0}^{k} m_j s_j$. So for any measurable closed set $F \subseteq X$,

$$d^*(\{i : x_i \in F\}) \le \limsup_{k\to\infty} \frac{1}{s_k}\#(\{j : j < s_k,\, x_{kj} \in F\})$$

(491Ad)

$$= \limsup_{k\to\infty} \nu'_k F \le \mu F.$$

As F is arbitrary, $\langle x_i\rangle_{i\in\mathbb{N}}$ is an equidistributed sequence for μ.

491E Proposition (a)(i) Suppose that X and Y are topological spaces, μ is a probability measure on X and $f : X \to Y$ is a continuous function. If $\langle x_i\rangle_{i\in\mathbb{N}}$ is a sequence in X which is equidistributed with respect to μ, then $\langle f(x_i)\rangle_{i\in\mathbb{N}}$ is equidistributed with respect to the image measure μf^{-1}.

(ii) Suppose that (X, μ) and (Y, ν) are topological probability spaces and $f : X \to Y$ is a continuous inverse-measure-preserving function. If $\langle x_i\rangle_{i\in\mathbb{N}}$ is a sequence in X which is equidistributed with respect to μ, then $\langle f(x_i)\rangle_{i\in\mathbb{N}}$ is equidistributed with respect to ν.

(b) Let X be a topological space and μ a probability measure on X, and suppose that X has a countable network consisting of sets measured by μ. Let λ be the ordinary product measure on $X^{\mathbb{N}}$. Then λ-almost every sequence in X is μ-equidistributed.

proof (a)(i) Let $F \subseteq Y$ be a closed set which is measured by μf^{-1}. Then $f^{-1}[F]$ is a closed set in X measured by μ. So

$$d^*(\{i : f(x_i) \in F\}) = d^*(\{i : x_i \in f^{-1}[F]\}) \le \mu f^{-1}[F].$$

(ii) Replace 'μf^{-1}' above by 'ν'.

(b) Let $\mathcal{A}$ be a countable network for the given topology $\mathfrak{S}$ of X consisting of measurable sets, and let $\mathcal{E}$ be the countable subalgebra of $\mathcal{P}X$ generated by $\mathcal{A}$. Let $\mathfrak{T} \supseteq \mathfrak{S}$ be the second-countable topology generated by $\mathcal{E}$; then μ is a τ-additive topological measure with respect to $\mathfrak{T}$ (4A2Nb, 414O), and $\mathcal{E}$ is a base for $\mathfrak{T}$ closed under finite unions. If $E \in \mathcal{E}$, then $d(\{i : x_i \in E\}) = \mu E$ for λ-almost every sequence $\langle x_i\rangle_{i\in\mathbb{N}}$ in X, by the strong law of large numbers (273J). So

$$d(\{i : x_i \in E\}) = \mu E \text{ for every } E \in \mathcal{E}$$

for λ-almost every $\langle x_i\rangle_{i\in\mathbb{N}}$. Now 491Ce tells us that any such sequence is equidistributed with respect to $\mathfrak{T}$ and therefore with respect to $\mathfrak{S}$.

491F Theorem Let $\langle(X_\alpha, \mathfrak{T}_\alpha, \Sigma_\alpha, \mu_\alpha)\rangle_{\alpha\in A}$ be a family of τ-additive topological probability spaces, each of which has an equidistributed sequence. If $\#(A) \le \mathfrak{c}$, then the τ-additive product measure λ on $X = \prod_{\alpha\in A} X_\alpha$ (definition: 417F) has an equidistributed sequence.

proof (a) For the time being (down to the end of (f)), let us suppose that $A = \mathcal{P}\mathbb{N}$ and that every μ_α is inner regular with respect to the Borel sets. (This will simplify the formulae and make it possible to use the theorems of §417, in particular 417G and 417J.)

For each $\alpha \subseteq \mathbb{N}$, let $\langle t_{\alpha i}\rangle_{i\in\mathbb{N}}$ be an equidistributed sequence in X_α; for $n \in \mathbb{N}$, let $\nu_{\alpha n}$ be the point-supported measure on X_α defined by setting $\nu_{\alpha n}E = \frac{1}{n+1}\#(\{i : i \le n,\, t_{\alpha i} \in E\})$ for $E \subseteq X_\alpha$. For each finite set $I \subseteq \mathcal{P}\mathbb{N}$, set $Y_I = \prod_{\alpha\in I} X_\alpha$ and $\pi_I(x) = x\restriction I \in Y_I$ for $x \in X$. Let λ_I be the τ-additive product of $\langle\mu_I\rangle_{\alpha\in I}$ and, for each n, let $\check\nu_{In}$ be the product of the measures $\langle\nu_{\alpha n}\rangle_{\alpha\in I}$. (Because I is finite, this is a point-supported probability measure, as in 251Xu. I do not say 'τ-additive product' here because I do not wish to assume that all singleton sets are Borel, so the $\nu_{\alpha n}$ may not be inner regular with respect to the Borel sets.)

(b) Suppose that $I \subseteq \mathcal{P}\mathbb{N}$ is finite and that $W \subseteq Y_I$ is an open set. Then $\lambda_I W \le \liminf_{n\to\infty} \check\nu_{In} W$. **P** Induce on $\#(I)$. If $I = \emptyset$, Y_I is a singleton and the result is trivial. For the inductive step, if $I \ne \emptyset$, take any $\alpha \in I$ and set $I' = I \setminus \{\alpha\}$. Then we can identify Y_I with $Y_{I'} \times X_\alpha$, λ_I with the τ-additive product of $\lambda_{I'}$ and μ_α (417J), and each $\check\nu_{In}$ with the product of $\check\nu_{I'n}$ and $\nu_{\alpha n}$.

Let $\mathcal{V}$ be the family of those subsets V of Y_I which are expressible as a finite union of sets of the form $U \times H$ where $U \subseteq Y_{I'}$ and $H \subseteq X_\alpha$ are open. Then $\mathcal{V}$ is a base for the topology of Y_I closed under finite unions. Let $\epsilon > 0$. Because λ_I is τ-additive, there is a $V \in \mathcal{V}$ such that $\lambda_I V \ge \lambda W - \epsilon$. The function $t \mapsto \lambda_{I'} V[\{t\}] : X_\alpha \to [0,1]$ is lower semi-continuous (417Ba), so 491Ca tells us that

$$\begin{aligned}\lambda_I V &= \int \lambda_{I'} V[\{t\}]\mu_\alpha(dt)\\ &\le \liminf_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} \lambda_{I'} V[\{t_{\alpha i}\}] = \liminf_{n\to\infty} \int \lambda_{I'} V[\{t\}]\nu_{\alpha n}(dt).\end{aligned}$$

At the same time, there are only finitely many sets of the form $V[\{t\}]$, and for each of these we have $\lambda_{I'} V[\{t\}] \le \liminf_{n\to\infty} \check{\nu}_{I'n} V[\{t\}]$, by the inductive hypothesis. So there is an $m \in \mathbb{N}$ such that $\lambda_{I'} V[\{t\}] \le \check{\nu}_{I'n} V[\{t\}] + \epsilon$ for every $n \ge m$ and every $t \in X_\alpha$. We must therefore have

$$\begin{aligned}\lambda_I W \le \lambda_I V + \epsilon &\le \liminf_{n\to\infty} \int \lambda_{\mathcal{I}'} V[\{t\}]\nu_{\alpha n}(dt) + \epsilon\\ &\le \liminf_{n\to\infty} \int \check{\nu}_{\mathcal{I}'n} V[\{t\}]\nu_{\alpha n}(dt) + 2\epsilon\\ &= \liminf_{n\to\infty} \check{\nu}_{In} V + 2\epsilon \le \liminf_{n\to\infty} \check{\nu}_{In} W + 2\epsilon.\end{aligned}$$

As ϵ and W are arbitrary, the induction proceeds. **Q**

(c) For $K \subseteq n \in \mathbb{N}$, set $A_{nK} = \{\alpha : \alpha \subseteq \mathbb{N},\, \alpha \cap n = K\}$ and $Z_{nK} = \prod_{\alpha \in A_{nK}} X_\alpha$. Then for each $n \in \mathbb{N}$ we can identify X with the finite product $\prod_{K \subseteq n} Z_{nK}$. For $K \subseteq n \in \mathbb{N}$ and $i \in \mathbb{N}$, define $z_{nKi} \in Z_{nK}$ by setting $z_{nKi}(\alpha) = t_{\alpha i}$ for $\alpha \in A_{mK}$; let $\tilde{\nu}_{nK}$ be the point-supported measure on Z_{nK} defined by setting $\tilde{\nu}_{nK} W = \frac{1}{n+1}\#(\{i : i \le n,\, z_{nKi} \in W\})$ for each $W \subseteq Z_{nK}$. For $n \in \mathbb{N}$ let $\tilde{\nu}_n$ be the measure on X which is the product of the measures $\tilde{\nu}_{nK}$ for $K \subseteq n$; this too is point-supported (251Xu(ii)).

(d) If $I \subseteq \mathcal{P}\mathbb{N}$ is finite, there is an $m \in \mathbb{N}$ such that $\pi_I : X \to Y_I$ is $(\tilde{\nu}_n, \check{\nu}_{In})$-inverse-measure-preserving for every $n \ge m$. **P** Let m be such that $\alpha \cap m \ne \alpha' \cap m$ for all distinct α, $\alpha' \in I$. If $n \ge m$, then $\tilde{\nu}_n$ is the product of the $\tilde{\nu}_{nK}$ for $K \subseteq n$. Now π_I, interpreted as a function from $\prod_{K \subseteq n} Z_{nK}$ onto Y_I, is of the form $\pi_I(\langle z_K \rangle_{K \subseteq n}) = \langle z_{\alpha \cap n}(\alpha) \rangle_{\alpha \in I}$. If $\alpha \in I$ and $E \subseteq X_\alpha$, then

$$\{z : z \in \textstyle\prod_{K \subseteq n} Z_{nK},\, \pi_I(z)(\alpha) \in E\} = \{z : z \in \textstyle\prod_{K \subseteq n} Z_{nK},\, z_{\alpha \cap n}(\alpha) \in E\},$$

so

$$\begin{aligned}\tilde{\nu}_n\{z : \pi_I(z)(\alpha) \in E\} &= \tilde{\nu}_{n,\alpha\cap n,n}\{y : y \in Z_{n,\alpha\cap n},\, y(\alpha) \in E\}\\ &= \frac{1}{n+1}\#(\{i : i \le n,\, z_{n,\alpha\cap n,i} \in E\})\\ &= \frac{1}{n+1}\#(\{i : i \le n,\, t_{\alpha i} \in E\}) = \nu_{\alpha n} E.\end{aligned}$$

If $W \subseteq Y_I$ is of the form $\{y : y(\alpha) \in E_\alpha$ for every $\alpha \in I\}$, where $E_\alpha \subseteq X_\alpha$ for each $\alpha \in I$, then

$$\begin{aligned}\tilde{\nu}_n \pi_I^{-1}[W] &= \tilde{\nu}_n(\bigcap_{\alpha \in I}\{z : z \in \prod_{K \subseteq n} Z_{nK},\, z(\alpha \cap n)(\alpha) \in E_\alpha\})\\ &= \prod_{\alpha \in I} \tilde{\nu}_n\{z : z(\alpha \cap n)(\alpha) \in E_\alpha\}\end{aligned}$$

(because $\{z : z(\alpha \cap n)(\alpha) \in E_\alpha\}$ is determined by coordinates in $\{\alpha \cap n\}$ for each $\alpha \in I$, and $\alpha \mapsto \alpha \cap n : I \to \mathcal{P}n$ is injective)

$$= \prod_{\alpha \in I} \nu_{\alpha n} E_\alpha = \check{\nu}_{In} W.$$

In particular, $\tilde{\nu}_n \pi_I^{-1}[\{y\}] = \check{\nu}_{In}\{y\}$ for every $y \in Y_I$. Consequently $\tilde{\nu}_n \pi_I^{-1}[D] = \check{\nu}_{In} D$ for any countable set $D \subseteq Y_I$. But if W is any subset of Y_I, there are countable subsets D, D' of W and Y_I respectively such that

$$\check\nu_{In}D=\check\nu_nW,\quad \check\nu_{In}D'=\check\nu_{In}(Y_I\setminus W),\quad \check\nu_{In}D+\check\nu_{In}D'=1$$

because $\check\nu_n$ is point-supported, and now

$$\pi_I^{-1}[D]\subseteq\pi_I^{-1}[W],\quad \pi_I^{-1}[D']\subseteq X\setminus\pi_I^{-1}[W],\quad \tilde\nu_n\pi_I^{-1}[D]+\tilde\nu_n\pi_I^{-1}[D']=1,$$

so $\tilde\nu_n\pi_I^{-1}[W]$ is defined and equal to $\tilde\nu_n\pi_I^{-1}[D]=\check\nu_{In}W$ because $\tilde\nu_n$ is complete. So π_I is $(\tilde\nu_n\pi_I^{-1},\check\nu_{In})$-inverse-measure-preserving. **Q**

(e) Let $\mathcal W$ be the family of those open sets $W\subseteq X$ expressible in the form $\pi_I^{-1}[W']$ for some finite $I\subseteq\mathcal P\mathbb N$ and some open $W'\subseteq Y_I$. If $W\in\mathcal W$, then $\lambda W\le\liminf_{n\in\mathbb N}\tilde\nu_nW$. **P** Take $I\in[\mathcal P\mathbb N]^{<\omega}$ and an open $W'\subseteq Y_I$ such that $W=\pi_I^{-1}[W']$. Then

$$\lambda W=\lambda_IW'$$

(417K)

$$\le\liminf_{n\to\infty}\check\nu_{In}W'$$

(by (b) above)

$$=\liminf_{n\to\infty}\tilde\nu_n\pi_I^{-1}[W']$$

(by (d))

$$=\liminf_{n\to\infty}\tilde\nu_nW.\ \mathbf{Q}$$

(f) If now $F\subseteq X$ is any closed set and $\epsilon>0$, then (because $\mathcal W$ is a base for the topology of X closed under finite unions) there is a $W\in\mathcal W$ such that $W\subseteq X\setminus F$ and $\lambda W\ge1-\lambda F-\epsilon$. In this case

$$\limsup_{n\to\infty}\tilde\nu_nF\le1-\liminf_{n\to\infty}\tilde\nu_nW\le1-\lambda W\le\lambda F+\epsilon.$$

As ϵ is arbitrary, $\limsup_{n\to\infty}\tilde\nu_nF\le\lambda F$; as F is arbitrary, 491D tells us that there is an equidistributed sequence in X.

(g) All this was done while assuming that $A=\mathcal P\mathbb N$ and every μ_α is inner regular with respect to the Borel sets. For the superficially more general case enunciated, given only that $\#(A)\le\mathfrak c$ and each μ_α is a τ-additive topological measure with an equidistributed sequence, we can of course take it that A is a subset of $\mathcal P\mathbb N$. Now let μ'_α be the restriction of μ_α to the Borel σ-algebra of X_α for each $\alpha\in A$, and for $\alpha\in\mathcal P\mathbb N\setminus A$ take X_α to be a singleton set, $\mathfrak T_\alpha$ its only topology and μ'_α the only probability measure on X_α. Every μ'_α is now τ-additive, and for $\alpha\in A$ any equidistributed sequence for μ_α is of course equidistributed for μ'_α, while for $\alpha\in\mathcal P\mathbb N\setminus A$ the only sequence in X_α is equidistributed for μ'_α. If we take λ' to be the τ-additive product of $\langle\mu'_\alpha\rangle_{\alpha\subseteq\mathbb N}$ on $X'=\prod_{\alpha\subseteq\mathbb N}X'_\alpha$, then (a)-(f) show that λ' has an equidistributed sequence $\langle x_i\rangle_{i\in\mathbb N}$ say.

Let $\pi_A:X'\to X$ be the restriction map $x\mapsto x\restriction A$. This is continuous, so $\langle\pi_A(x_i)\rangle_{i\in\mathbb N}$ is equidistributed with respect to $\lambda'\pi_A^{-1}$, by 491Ea. And $\lambda'\pi_A^{-1}$ agrees with λ on the open subsets of X. **P** If $I\subseteq A$ is finite and $H_\alpha\subseteq X_\alpha$ is open for $\alpha\in I$, then

$$\begin{aligned}(\lambda'\pi_A^{-1})\{x:x\in X,x(\alpha)\in H_\alpha\text{ for }\alpha\in I\}&\\ =\lambda'\{x:x\in X,\ x(\alpha)\in H_\alpha\text{ for }\alpha\in I\}&\\ =\prod_{\alpha\in I}\mu_\alpha H_\alpha=\lambda\{x:x\in X,\ x(\alpha)\in H_\alpha\text{ for }\alpha\in I\}.&\end{aligned}$$

So $\lambda'\pi_A^{-1}$ and λ agree on the family $\mathcal V_0$ of open cylinder subsets of X. But $\mathcal V_0$ is closed under finite intersections, so the probability measures $\lambda'\pi_A^{-1}$ and λ agree on the σ-algebra of subsets of X generated by $\mathcal V_0$, by the Monotone Class Theorem (136C). In particular, they agree on the family $\mathcal V_1$ of sets expressible as finite unions of members of $\mathcal V_0$, which is a base for the topology of X closed under finite unions. If $W\subseteq X$ is open, then $\{V:V\in\mathcal V_1,\ V\subseteq W\}$ is upwards-directed and has union W, so

$$(\lambda'\pi_A^{-1})W=\sup_{V\in\mathcal V_1,V\subseteq W}(\lambda'\pi_A^{-1})(V)$$

(because $\lambda'\pi_A^{-1}$ is τ-additive, by 411Gj)

$$= \sup_{V \in \mathcal{V}_1, V \subseteq W} \lambda V = \lambda W. \mathbf{Q}$$

At the same time, $\langle \pi_A(x_i) \rangle_{i \in \mathbb{N}}$ is equidistributed for $\lambda' \pi_A^{-1}$, by 491E(a-i). Directly from the definition in 491B, we see that $\langle \pi_A(x_i) \rangle_{i \in \mathbb{N}}$ is also equidistributed for λ, and λ has an equidistributed sequence in this case also.

491G Corollary The usual measure of $\{0,1\}^{\mathfrak{c}}$ has an equidistributed sequence.

proof The usual measure of $\{0,1\}$ of course has an equidistributed sequence (just set $x_i = 0$ for even i, $x_i = 1$ for odd i), so 491F gives the result at once.

491H Theorem (Veech 71) Any separable compact Hausdorff topological group has an equidistributed sequence for its Haar probability measure.

proof Let X be a separable compact Hausdorff topological group. Recall that X has exactly one Haar probability measure μ, which is both a left Haar measure and a right Haar measure (442Ic).

(a) We need some elementary facts about convolutions.

(i) If ν_1 and ν_2 are point-supported probability measures on X, then $\nu_1 * \nu_2$ is point-supported. $\mathbf{P}$ If $\nu_1 E = \sum_{x \in E} q_1(x)$ and $\nu_2 E = \sum_{x \in E} q_2(x)$ for every $E \subseteq X$, then

$$(\nu_1 * \nu_2)(E) = (\nu_1 \times \nu_2)\{(x,y) : xy \in E\}$$

(444A)

$$= \sum_{xy \in E} q_1(x) q_2(y) = \sum_{z \in E} q(z)$$

where $q(z) = \sum_{x \in X} q_1(x) q_2(x^{-1}z)$ for $z \in X$. $\mathbf{Q}$

(ii) Let ν, λ be Radon probability measures on X. Suppose that $f \in C(X)$, $\alpha \in \mathbb{R}$ and $\epsilon > 0$ are such that $|\int f(yxz)\nu(dx) - \alpha| \le \epsilon$ for every y, $z \in X$. Then $|\int f(yxz)(\lambda * \nu)(dx) - \alpha| \le \epsilon$ and $|\int f(yxz)(\lambda * \nu)(dx) - \alpha| \le \epsilon$ for every y, $z \in X$. $\mathbf{P}$

$$|\int f(yxz)(\lambda * \nu)(dx) - \alpha|$$

$$= |\iint f(ywxz)\nu(dx)\lambda(dw) - \alpha|$$

(444C)

$$\le \int |\int f(ywxz)\nu(dx) - \alpha|\lambda(dw) \le \int \epsilon\lambda(dw) = \epsilon,$$

$$|\int f(yxz)(\nu * \lambda)(dx) - \alpha|$$

$$= |\iint f(yxwz)\nu(dx)\lambda(dw) - \alpha|$$

$$\le \int |\int f(yxwz)\nu(dx) - \alpha|\lambda(dw) \le \int \epsilon\lambda(dw) = \epsilon. \mathbf{Q}$$

(b) Let $A \subseteq X$ be a countable dense set. Let N be the set of point-supported probability measures ν on X which are defined by functions q such that $\{x : q(x) > 0\}$ is a finite subset of A and $q(x)$ is rational for every x. Then N is countable. Now, for every $f \in C(X)$ and $\epsilon > 0$, there is a $\nu \in$ N such that $|\int f(yxz)\nu(dx) - \int f d\mu| \le \epsilon$ for all y, $z \in X$. $\mathbf{P}$ Because X is compact, f is uniformly continuous for the right uniformity of X (4A2Jf), so there is a neighbourhood U of the identity e such that $|f(x') - f(x)| \le \frac{1}{2}\epsilon$

whenever $x'x^{-1} \in U$. Next, again because X is compact, there is a neighbourhood V of e such that $yxy^{-1} \in U$ whenever $x \in V$ and $y \in X$ (4A5Ej). Because A is dense, $V^{-1}x \cap A \neq \emptyset$ for every $x \in X$, that is, $VA = X$; once more because X is compact, there are $x_0, \dots, x_n \in A$ such that $X = \bigcup_{i \le n} Vx_i$. Set $E_i = Vx_i \setminus \bigcup_{j<i} Vx_j$ for each $i \le n$. Let $\alpha_0, \dots, \alpha_n \in [0,1] \cap \mathbb{Q}$ be such that $\sum_{i=0}^n \alpha_i = 1$ and $\|f\|_\infty \sum_{i=0}^n |\alpha_i - \mu E_i| \le \frac{1}{2}\epsilon$, and define $\nu \in \mathrm{N}$ by setting $\nu E = \sum\{\alpha_i : i \le n,\, x_i \in E\}$ for every $E \subseteq X$.

Let y, $z \in X$. If $i \le n$ and $x \in E_i$, then $xx_i^{-1} \in V$ so $(yxz)(yx_iz)^{-1} = yxx_i^{-1}y^{-1} \in U$ and $|f(yxz) - f(yx_iz)| \le \frac{1}{2}\epsilon$. Accordingly

$$|\int f(yxz)\nu(dx) - \int f(x)\mu(dx)|$$

$$= |\sum_{i=0}^n \alpha_i f(yx_iz) - \int f(yx)\mu(dx)|$$

(441Ac)

$$= |\sum_{i=0}^n \alpha_i f(yx_iz) - \Delta(z)\int f(yxz)\mu(dx)|$$

(where Δ is the left modular function of X, by 442Kc)

$$= |\sum_{i=0}^n \alpha_i f(yx_iz) - \Delta(z)\int f(yxz)\mu(dx)|$$

(because X is unimodular, by 442Ic)

$$\le \sum_{i=0}^n |\alpha_i f(yx_iz) - \int_{E_i} f(yxz)\mu(dx)|$$

$$\le \sum_{i=0}^n |\alpha_i - \mu E_i||f(yx_iz)| + \sum_{i=0}^n |f(yx_iz)\mu E_i - \int_{E_i} f(yxz)\mu(dx)|$$

$$\le \|f\|_\infty \sum_{i=0}^n |\alpha_i - \mu E_i| + \sum_{i=0}^n \int_{E_i} |f(yxz) - f(yx_iz)|\mu(dx)$$

$$\le \frac{\epsilon}{2} + \sum_{i=0}^n \frac{\epsilon}{2}\mu E_i = \epsilon. \ \mathbf{Q}$$

(c) Let $\langle \nu_n \rangle_{n\in\mathbb{N}}$ be a sequence running over N, and set $\lambda_n = \nu_0 * \nu_1 * \dots * \nu_n$ for each n. (Recall from 444B that convolution is associative.) Then each λ_n is a point-supported probability measure on X, by (a-i). Also $\lim_{n\to\infty} \int f d\lambda_n = \int f d\mu$ for every $f \in C(X)$. **P** If $f \in C(X)$ and $\epsilon > 0$, then (b) tells us that there is an $m \in \mathbb{N}$ such that $|\int f(yxz)\nu_m(dx) - \int f d\mu| \le \epsilon$ for all y, $z \in X$. For any $n \ge m$, λ_n is of the form $\lambda' * \nu_m * \lambda''$. By (a-ii), used in both parts successively, $|\int f d\lambda_n - \int f d\mu| \le \epsilon$. As ϵ is arbitrary, we have the result. **Q**

(d) If $F \subseteq X$ is closed, then

$$\mu F = \inf\{\int f d\mu : \chi F \le f \in C(X)\}$$

$$= \inf_{\chi F \le f} \limsup_{n\to\infty} \int f d\lambda_n \ge \limsup_{n\to\infty} \lambda_n F.$$

By 491D, μ has an equidistributed sequence.

491I The quotient $\mathcal{P}\mathbb{N}/\mathcal{Z}$ I now return to the asymptotic density ideal $\mathcal{Z}$, moving towards a striking relationship between the corresponding quotient algebra and equidistributed sequences. Since $\mathcal{Z} \lhd \mathcal{P}\mathbb{N}$, we can form the quotient **asymptotic density algebra** $\mathfrak{Z} = \mathcal{P}\mathbb{N}/\mathcal{Z}$. The functional d^* descends naturally to $\mathfrak{Z}$ if we set

$$\bar{d}^*(I^\bullet) = d^*(I) \text{ for every } I \subseteq \mathbb{N}.$$

(a) $\bar{d}^*$ is a strictly positive submeasure on $\mathfrak{Z}$. **P** $\bar{d}^*$ is a submeasure on $\mathfrak{Z}$ because d^* is a submeasure on $\mathcal{P}\mathbb{N}$. $\bar{d}^*$ is strictly positive because $\mathcal{Z} \supseteq \{I : d^*(I) = 0\}$. **Q**

(b) Let $\bar{\rho}$ be the metric on $\mathfrak{Z}$ defined by saying that $\bar{\rho}(a,b) = \bar{d}^*(a \triangle b)$ for all a, $b \in \mathfrak{Z}$. Under $\bar{\rho}$, the Boolean operations $\cup$, $\cap$, $\triangle$ and $\setminus$ and the function $\bar{d}^* : \mathfrak{Z} \to [0,1]$ are uniformly continuous (392Hb), and $\mathfrak{Z}$ is complete. **P** Let $\langle c_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{Z}$ such that $\bar{\rho}(c_{n+1}, c_n) \le 2^{-n}$ for every $n \in \mathbb{N}$; then $\bar{\rho}(c_r, c_i) \le 2^{-i+1}$ for $i \le r$. For each $n \in \mathbb{N}$ choose $C_n \subseteq \mathbb{N}$ such that $C_n^\bullet = c_n$; then $d^*(C_r \triangle C_i) \le 2^{-i+1}$ for $i \le r$. Choose a strictly increasing sequence $\langle k_n\rangle_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that $k_{n+1} \ge 2k_n$ for every n and, for each $n \in \mathbb{N}$,

$$\frac{1}{m}\#((C_n \triangle C_i) \cap m) \le 2^{-i+2} \text{ whenever } i \le n,\ m \ge k_n.$$

Set $C = \bigcup_{n\in\mathbb{N}} C_n \cap k_{n+1} \setminus k_n$, and $c = C^\bullet \in \mathfrak{Z}$. If $n \in \mathbb{N}$ and $m \ge k_{n+1}$, then take $r > n$ such that $k_r \le m < k_{r+1}$; in this case $k_i \le 2^{i-r}m$ for $i \le r$, so

$$\begin{aligned}
\#((C \triangle C_n) \cap m) &\le k_n + \sum_{i=n}^{r-1} \#((C \triangle C_n) \cap k_{i+1} \setminus k_i) + \#((C \triangle C_n) \cap m \setminus k_r) \\
&= k_n + \sum_{i=n}^{r-1} \#((C_i \triangle C_n) \cap k_{i+1} \setminus k_i) + \#((C_r \triangle C_n) \cap m \setminus k_r) \\
&\le k_n + \sum_{i=n+1}^{r-1} \#((C_i \triangle C_n) \cap k_{i+1}) + \#((C_r \triangle C_n) \cap m) \\
&\le k_n + \sum_{i=n+1}^{r-1} 2^{-n+2}k_{i+1} + 2^{-n+2}m \\
&\le k_n + \sum_{i=n+1}^{r-1} 2^{-n+2}2^{i+1-r}m + 2^{-n+2}m \\
&\le k_n + 2^{-n+3}m + 2^{-n+2}m.
\end{aligned}$$

But this means that

$$\bar{\rho}(c, c_n) = d^*(C \triangle C_n) \le \lim_{m\to\infty} \frac{k_n}{m} + 2^{-n+3} + 2^{-n+2} \le 2^{-n+4}$$

for every n, and $\langle c_n\rangle_{n\in\mathbb{N}}$ converges to c in $\mathfrak{Z}$. **Q**

For the rest of this section, I will take it that $\mathfrak{Z}$ is endowed with the metric $\bar{\rho}$.

***(c)** If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{Z}$, there is an $a \in \mathfrak{Z}$ such that $a \subseteq a_n$ for every n and $\bar{d}^*(a) = \inf_{n\in\mathbb{N}} \bar{d}^*(a_n)$. **P** For each $n \in \mathbb{N}$, choose $I_n \subseteq \mathbb{N}$ such that $I_n^\bullet = a_n$; replacing I_n by $\bigcap_{j\le n} I_j$ if necessary, we can arrange that $I_{n+1} \subseteq I_n$ for every n. Set $\gamma = \inf_{n\in\mathbb{N}} \bar{d}^*(a_n) = \inf_{n\in\mathbb{N}} d^*(I_n)$. Let $\langle k_n\rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence in $\mathbb{N}$ such that $\#(I_n \cap k_n) \ge (\gamma - 2^{-n})k_n$ for every n. Set $I = \bigcup_{n\in\mathbb{N}} I_n \cap k_n$ and $a = I^\bullet \in \mathfrak{Z}$. Then $\#(I \cap k_n) \ge (\gamma - 2^{-n})k_n$ for every n, so $\bar{d}^*(a) = d^*(I) \ge \gamma$. Also $I \setminus I_n \subseteq k_n$ is finite, so $a \subseteq a_n$, for every n. Of course it follows at once that $\bar{d}^*(a) = \gamma$ exactly, as required. **Q**

***(d)** $\bar{d}^*$ is a Maharam submeasure on $\mathfrak{Z}$. (Immediate from (c).)

491J Lemma Let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $\mathfrak{Z} = \mathcal{P}\mathbb{N}/\mathcal{Z}$ such that $\lim_{n\to\infty} \bar{d}^*(a_n) + \bar{d}^*(1 \setminus a_n) = 1$. Then $\langle a_n\rangle_{n\in\mathbb{N}}$ is topologically convergent to a member a of $\mathfrak{Z}$; $a = \sup_{n\in\mathbb{N}} a_n$ in $\mathfrak{Z}$ and $d^*(a) + d^*(1 \setminus a) = 1$.

proof (a) The point is that if $m \le n$ then $\bar{d}^*(a_n \setminus a_m) \le \bar{d}^*(a_n) + \bar{d}^*(1 \setminus a_m) - 1$. **P** Let I, $J \subseteq \mathbb{N}$ be such that $I^\bullet = a_m$ and $J^\bullet = a_n$. For any $k \ge 1$,

$$\frac{1}{k}\#(k \cap J) + \frac{1}{k}\#(k \setminus I) = \frac{1}{k}\#(k \cap J \setminus I) + \frac{1}{k}\#(k \setminus (I \setminus J)),$$

so

$$\begin{aligned}
d^*(J) + d^*(\mathbb{N} \setminus I) &= \limsup_{k\to\infty} \frac{1}{k}\#(k \cap J) + \limsup_{k\to\infty} \frac{1}{k}\#(k \setminus I) \\
&\ge \limsup_{k\to\infty} \bigl(\frac{1}{k}\#(k \cap J) + \frac{1}{k}\#(k \setminus I)\bigr) \\
&= \limsup_{k\to\infty} \bigl(\frac{1}{k}\#(k \cap J \setminus I) + \frac{1}{k}\#(k \setminus (I \setminus J))\bigr) \\
&\ge \limsup_{k\to\infty} \frac{1}{k}\#(k \cap J \setminus I) + \liminf_{k\to\infty} \frac{1}{k}\#(k \setminus (I \setminus J)) = d^*(J \setminus I) + 1
\end{aligned}$$

because $a_m \subseteq a_n$, so $I \setminus J \in \mathcal{Z}$. But this means that

$$\bar{d}^*(a_n \setminus a_m) = d^*(J \setminus I) \le d^*(J) + d^*(\mathbb{N} \setminus I) - 1 = \bar{d}^*(a_n) + \bar{d}^*(1 \setminus a_m) - 1. \text{ } \mathbf{Q}$$

(b) Accordingly

$$\begin{aligned}
\limsup_{m\to\infty} \sup_{n\ge m} \bar{\rho}(a_m, a_n) &= \limsup_{m\to\infty} \sup_{n\ge m} \bar{d}^*(a_n \setminus a_m) \\
&\le \limsup_{m\to\infty} \bigl(\sup_{n\ge m} \bar{d}^*(a_n) + \bar{d}^*(1 \setminus a_m) - 1\bigr) \\
&= \limsup_{m\to\infty} \sup_{n\ge m} \bar{d}^*(a_n) - \bar{d}^*(a_m)
\end{aligned}$$

(because $\lim_{m\to\infty} \bar{d}^*(a_m) + \bar{d}^*(1 \setminus a_m) = 1$)

$$= 0,$$

and $\langle a_n \rangle_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathfrak{Z}$.

(c) Because $\mathfrak{Z}$ is complete, $a = \lim_{n\to\infty} a_n$ is defined in $\mathfrak{Z}$ (491Ib). For each $m \in \mathbb{N}$, $a_m \setminus a = \lim_{n\to\infty} a_m \setminus a_n = 0$ (because $\setminus$ is continuous), so $a_m \subseteq a$; thus a is an upper bound of $\{a_n : n \in \mathbb{N}\}$. If b is any upper bound of $\{a_n : n \in \mathbb{N}\}$, then $a \setminus b = \lim_{n\to\infty} a_n \setminus b = 0$; so $a = \sup_{n\in\mathbb{N}} a_n$. Finally,

$$\bar{d}^*(a) + \bar{d}^*(1 \setminus a) = \lim_{n\to\infty} \bar{d}^*(a_n) + \bar{d}^*(1 \setminus a_n) = 1.$$

491K Corollary Set $D = \{a : a \in \mathfrak{Z},\ \bar{d}^*(a) + \bar{d}^*(1 \setminus a) = 1\}$, and write $\bar{d}$ for $\bar{d}^*{\restriction}D$.

(a) If $I \subseteq \mathbb{N}$ then its asymptotic density $d(I)$ is defined iff $I^\bullet \in D$, and in this case $d(I) = \bar{d}(I^\bullet)$.

(b) If $a \in D$ then its complement $1 \setminus a$ in $\mathfrak{Z}$ belongs to D; if a, $b \in D$ and $a \cap b = 0$, then $a \cup b \in D$ and $\bar{d}(a \cup b) = \bar{d}(a) + \bar{d}(b)$; if a, $b \in D$ and $a \subseteq b$ then $b \setminus a \in D$ and $\bar{d}(b \setminus a) = \bar{d}(b) - \bar{d}(a)$.

(c) D is a topologically closed subset of $\mathfrak{Z}$.

(d) If $A \subseteq D$ is upwards-directed, then $\sup A$ is defined in $\mathfrak{Z}$ and belongs to D; moreover there is a sequence in A with the same supremum as A, and $\sup A$ belongs to the topological closure of A.

(e) Let $\mathfrak{B} \subseteq D$ be a subalgebra of $\mathfrak{Z}$. Then the following are equiveridical:

(i) $\mathfrak{B}$ is topologically closed in $\mathfrak{Z}$;

(ii) $\mathfrak{B}$ is order-closed in $\mathfrak{Z}$;

(iii) setting $\bar{\nu} = \bar{d}^*{\restriction}\mathfrak{B} = \bar{d}{\restriction}\mathfrak{B}$, $(\mathfrak{B}, \bar{\nu})$ is a probability algebra.

In this case, $\mathfrak{B}$ is regularly embedded in $\mathfrak{Z}$.

(f) If $I \subseteq D$ is closed under either $\cap$ or $\cup$, then the topologically closed subalgebra of $\mathfrak{Z}$ generated by I, which is also the order-closed subalgebra of $\mathfrak{Z}$ generated by I, is included in D.

proof (a)

$$\begin{aligned}
I^\bullet \in D &\iff \bar{d}^*(I^\bullet) + \bar{d}^*(1 \setminus I^\bullet) = 1 \\
&\iff d^*(I) + d^*(\mathbb{N} \setminus I) = 1 \iff d(I) \text{ is defined}
\end{aligned}$$

by 491Ac, and in this case

$$d(I) = d^*(I) = \bar{d}^*(I^\bullet) = \bar{d}(I^\bullet).$$

(b) These all follow directly from the corresponding results concerning $\mathcal{P}\mathbb{N}$ and d (491Ac).

(c) All we have to know is that $a \mapsto \bar{d}^*(a)$, $a \mapsto 1 \setminus a$ are continuous (392Hb); so that $\{a : \bar{d}^*(a) + \bar{d}^*(1 \setminus a) = 1\}$ is closed.

(d) Because A is upwards-directed, and $\bar{d}^*$ is a non-decreasing functional on $\mathfrak{Z}$, there is a non-decreasing sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ in A such that $\lim_{n\to\infty} \bar{d}^*(a_n) = \sup_{a\in A} \bar{d}^*(a) = \gamma$ say. By 491J, $b = \lim_{n\to\infty} a_n = \sup_{n\in\mathbb{N}} a_n$ is defined in $\mathfrak{Z}$ and belongs to D. If $a \in A$ and $\epsilon > 0$, there is an $n \in \mathbb{N}$ such that $\bar{d}^*(a_n) \ge \gamma - \epsilon$. Let $a' \in A$ be a common upper bound of a and a_n. Then

$$\bar{d}^*(a \setminus b) \le \bar{d}^*(a' \setminus a_n) = \bar{d}^*(a') - \bar{d}^*(a_n) \le \gamma - \bar{d}^*(a_n) \le \epsilon.$$

As ϵ is arbitrary, $a \subseteq b$; as a is arbitrary, b is an upper bound of A; as $b = \sup_{n\in\mathbb{N}} a_n$, b must be the supremum of A.

(e)(i)⇒(ii) Suppose that $\mathfrak{B}$ is topologically closed. If $A \subseteq \mathfrak{B}$ is a non-empty upwards-directed subset with supremum $b \in \mathfrak{Z}$, then (d) tells us that $b \in \overline{A} \subseteq \mathfrak{B}$. It follows that $\mathfrak{B}$ is order-closed in $\mathfrak{Z}$ (313E(a-i)).

(ii)⇒(iii) Suppose that $\mathfrak{B}$ is order-closed in $\mathfrak{Z}$. If $A \subseteq \mathfrak{B}$ is non-empty, then $A' = \{a_0 \cup \ldots \cup a_n : a_0, \ldots, a_n \in A\}$ is non-empty and upwards-directed, so has a supremum in $\mathfrak{Z}$, which must belong to $\mathfrak{B}$, and must be the least upper bound of A in $\mathfrak{B}$. Thus $\mathfrak{B}$ is Dedekind (σ-)complete. Now let $\langle a_n \rangle_{n\in\mathbb{N}}$ be a disjoint sequence in $\mathfrak{B}$ and set $b_n = \sup_{i\le n} a_i$ for each n. Then $\langle b_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in D so has a limit and supremum $b \in D$, and $b \in \mathfrak{B}$. Also $\bar{d}^*(b_n) = \sum_{i=0}^n \bar{d}^*(a_i)$ for each n (induce on n), so

$$\bar{\nu} b = \bar{d}^*(b) = \lim_{n\to\infty} \bar{d}^*(b_n) = \sum_{i=0}^\infty \bar{d}^*(a_i) = \sum_{i=0}^\infty \bar{\nu} a_i.$$

Since certainly $\bar{\nu}0 = 0$, $\bar{\nu}1 = 1$ and $\bar{\nu}b > 0$ whenever $b \in \mathfrak{B} \setminus \{0\}$, $(\mathfrak{B}, \bar{\nu})$ is a probability algebra.

(iii)⇒(i) Suppose that $(\mathfrak{B}, \bar{\nu})$ is a probability algebra. Then it is complete under its measure metric (323Gc), which agrees on $\mathfrak{B}$ with the metric $\bar{\rho}$ of $\mathfrak{Z}$; so $\mathfrak{B}$ must be topologically closed in $\mathfrak{Z}$.

We see also that $\mathfrak{B}$ is regularly embedded in $\mathfrak{Z}$. **P** (Compare 323H.) If $A \subseteq \mathfrak{B}$ is non-empty and downwards-directed and has infimum 0 in $\mathfrak{B}$, and $b \in \mathfrak{Z}$ is any lower bound of A in $\mathfrak{Z}$, then

$$\bar{d}^*(b) \le \inf_{a\in A} \bar{d}^*(a) = \inf_{a\in A} \bar{\nu} a = 0$$

(321F), so $b = 0$. Thus $\inf A = 0$ in $\mathfrak{Z}$. As A is arbitrary, this is enough to show that the identity map from $\mathfrak{B}$ to $\mathfrak{Z}$ is order-continuous (313Lb), that is, that $\mathfrak{B}$ is regularly embedded in $\mathfrak{Z}$. **Q**

(f) Let $\mathfrak{B}$ be the order-closed subalgebra of $\mathfrak{Z}$ generated by I. If I is closed under $\cap$, then (b), (d) and 313Gc tell us that $\mathfrak{B} \subseteq D$. If I is closed under $\cup$, then $I' = \{1 \setminus a : a \in I\}$ is a subset of D closed under $\cap$, while $\mathfrak{B}$ is the order-closed subalgebra generated by I', so again $\mathfrak{B} \subseteq D$. By (e), $\mathfrak{B}$ is in either case topologically closed. So we see that the topologically closed subalgebra generated by I is included in D; by (e) again, it is equal to $\mathfrak{B}$.

491L Effectively regular measures The examples 491Xf and 491Yf show that the definition in 491B is drawn a little too wide for comfort, and allows some uninteresting pathologies. These do not arise in the measure spaces we care most about, and the following definitions provide a fire-break. Let (X, Σ, μ) be a measure space, and $\mathfrak{T}$ a topology on X.

(a) I will say that a measurable subset K of X of finite measure is **regularly enveloped** if for every $\epsilon > 0$ there are an open measurable set G and a closed measurable set F such that $K \subseteq G \subseteq F$ and $\mu(F \setminus K) \le \epsilon$.

(b) Note that the family $\mathcal{K}$ of regularly enveloped measurable sets of finite measure is closed under finite unions and countable intersections. **P** (i) If K_1, $K_2 \in \mathcal{K}$ and $*$ is either $\cup$ or $\cap$, let $\epsilon > 0$. Take measurable open sets G_1, G_2 and measurable closed sets F_1, F_2 such that $K_i \subseteq G_i \subseteq F_i$ and $\mu(F_i \setminus K_i) \le \frac{1}{2}\epsilon$ for both i. Then $G_1 * G_2$ is a measurable open set, $F_1 * F_2$ is a measurable closed set, $K_1 * K_2 \subseteq G_1 * G_2 \subseteq F_1 * F_2$ and $\mu((F_1 * F_2) \setminus (K_1 * K_2)) \le \epsilon$. As ϵ is arbitrary, $K_1 * K_2 \in \mathcal{K}$. (ii) If $\langle K_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in

$\mathcal{K}$ with intersection K and $\epsilon > 0$, let $n \in \mathbb{N}$ be such that $\mu K_n < \mu K + \epsilon$. Then we can find a measurable open set G and a measurable closed set F such that $K_n \subseteq G \subseteq F$ and $\mu F \leq \mu K + \epsilon$. As ϵ is arbitrary, $K \in \mathcal{K}$. Together with (i), this is enough to show that $\mathcal{K}$ is closed under countable intersections. **Q**

(c) Now I say that μ is **effectively regular** if it is inner regular with respect to the regularly enveloped sets of finite measure.

491M Examples (a) Any totally finite Radon measure is effectively regular. **P** Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a totally finite Radon measure space. If $K \subseteq X$ is compact and $\epsilon > 0$, let $L \subseteq X \setminus K$ be a compact set such that $\mu L \geq \mu X - \mu K + \epsilon$. Let G, H be disjoint open sets including K, L respectively (4A2F(h-i)). Then $K \subseteq G \subseteq X \setminus H$, G is open, $X \setminus H$ is closed, both G and $X \setminus H$ are measurable, and $\mu((X \setminus H) \setminus K) \leq \epsilon$. This shows that every compact set is regularly enveloped, and μ is effectively regular. **Q**

(b) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space such that $\mathfrak{T}$ is a regular topology. Then μ is effectively regular. **P** Let $E \in \Sigma$ and take $\gamma < \mu E$. Choose sequences $\langle E_n \rangle_{n \in \mathbb{N}}$ and $\langle G_n \rangle_{n \in \mathbb{N}}$ inductively, as follows. $E_0 \subseteq E$ is to be any measurable set such that $\gamma < \mu E_0 < \infty$. Given that $\mu E_n > \gamma$, let G be an open set of finite measure such that $\mu(E_n \cap G) > \gamma$ (414Ea), and $F \subseteq G \setminus E_n$ a closed set such that $\mu F \geq \mu(G \setminus E_n) - 2^{-n}$. Let $\mathcal{H}$ be the family of open sets H such that $\overline{H} \subseteq G \setminus F$. Then $\mathcal{H}$ is upwards-directed and covers E_n (because $\mathfrak{T}$ is regular), so there is a $G_n \in \mathcal{H}$ such that $\mu(E_n \cap G_n) > \gamma$ (414Ea again). Now $\mu(\overline{G}_n \setminus E_n) \leq 2^{-n}$. Set $E_{n+1} = E_n \cap G_n$, and continue.

At the end of the induction, set $K = \bigcap_{n \in \mathbb{N}} E_n$. For each n, $K \subseteq G_n \subseteq \overline{G}_n$ and

$$\lim_{n \to \infty} \mu(\overline{G}_n \setminus K) \leq \lim_{n \to \infty} 2^{-n} + \mu(E_n \setminus K) = 0,$$

so K is regularly enveloped. At the same time, $K \subseteq E$ and $\mu K \geq \gamma$. As E and γ are arbitrary, μ is effectively regular. **Q**

(c) Any totally finite Baire measure is effectively regular. **P** Let μ be a totally finite Baire measure on a topological space X. If $F \subseteq X$ is a zero set, let $f : X \to \mathbb{R}$ be a continuous function such that $F = f^{-1}[\{0\}]$. For each $n \in \mathbb{N}$, set $G_n = \{x : |f(x)| < 2^{-n}\}$, $F_n = \{x : |f(x)| \leq 2^{-n}\}$; then G_n is a measurable open set, F_n is a measurable closed set, $F \subseteq G_n \subseteq F_n$ for every n and $\lim_{n \to \infty} \mu F_n = \mu F$ (because μ is totally finite). This shows that every zero set is regularly enveloped; as μ is inner regular with respect to the zero sets (412D), μ is effectively regular. **Q**

(d) A totally finite completion regular topological measure is effectively regular. (As in (c), all zero sets are regularly enveloped.)

491N Theorem Let X be a topological space and μ an effectively regular probability measure on X, with measure algebra $(\mathfrak{A}, \bar{\mu})$. Suppose that $\langle x_i \rangle_{i \in \mathbb{N}}$ is an equidistributed sequence in X. Then we have a unique order-continuous Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{Z} = \mathcal{P}\mathbb{N}/\mathcal{Z}$ such that $\pi G^\bullet \subseteq \{i : x_i \in G\}^\bullet$ for every measurable open set $G \subseteq X$, and $\bar{d}^*(\pi a) = \bar{\mu} a$ for every $a \in \mathfrak{A}$.

proof (a) Define $\theta : \mathcal{P}X \to \mathfrak{Z}$ by setting $\theta A = \{i : x_i \in A\}^\bullet$ for $A \subseteq X$; then θ is a Boolean homomorphism. If $F \subseteq X$ is closed and measurable, then $\bar{d}^*(\theta F) \leq \mu F$, because $\langle x_i \rangle_{i \in \mathbb{N}}$ is equidistributed. Write $\mathcal{K}$ for the family of regularly enveloped measurable sets.

If $K \in \mathcal{K}$, then $\pi_0 K = \inf\{\theta G : K \subseteq G \in \Sigma \cap \mathfrak{T}\}$ is defined in $\mathfrak{Z}$, $\bar{d}^*(\pi_0 K) = \mu K$ and $\pi_0 K \in D$ as defined in 491K. **P** For each $n \in \mathbb{N}$, let G_n, $F_n \in \Sigma$ be such that $K \subseteq G_n \subseteq F_n$, G_n is open, F_n is closed and $\mu(F_n \setminus K) \leq 2^{-n}$. Set $H_n = X \setminus \bigcap_{i \leq n} G_i$. Then

$$\begin{aligned}\bar{d}^*(\theta H_n) + \bar{d}^*(1 \setminus \theta H_n) &\leq \bar{d}^*(\theta H_n) + \bar{d}^*(\theta(\bigcap_{i \leq n} F_i)) \leq \mu H_n + \mu(\bigcap_{i \leq n} F_i) \\ &\leq \mu(X \setminus K) + \mu F_n \leq 1 + 2^{-n}.\end{aligned}$$

Also $\langle \theta H_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{Z}$. By 491J, $a = \lim_{n \to \infty} \theta H_n = \sup_{n \in \mathbb{N}} \theta H_n$ is defined in $\mathfrak{Z}$ and belongs to D. Set

$$b = 1 \setminus a = \lim_{n \to \infty} 1 \setminus \theta H_n = \lim_{n \to \infty} \theta(\bigcap_{i \leq n} G_i),$$

so that b also belongs to D. If $K \subseteq G \in \Sigma \cap \mathfrak{T}$, then

$$b \setminus \theta G = \lim_{n\to\infty} \theta(\bigcap_{i\le n} G_i) \setminus \theta G = \lim_{n\to\infty} \theta(\bigcap_{i\le n} G_i \setminus G)$$

and

$$\begin{aligned}\bar{d}^*(b \setminus \theta G) &= \lim_{n\to\infty} \bar{d}^*(\theta(\bigcap_{i\le n} G_i \setminus G)) \\ &\le \lim_{n\to\infty} \bar{d}^*(\theta(\bigcap_{i\le n} F_i \setminus G)) \le \lim_{n\to\infty} \mu(\bigcap_{i\le n} F_i \setminus G) = 0.\end{aligned}$$

This shows that $b \subseteq \theta G$ whenever $K \subseteq G \in \Sigma \cap \mathfrak{T}$. On the other hand, any lower bound of $\{\theta G : K \subseteq G \in \Sigma\cap\mathfrak{T}\}$ is also a lower bound of $\{\theta(\bigcap_{i\le n} G_i) : n \in \mathbb{N}\}$, so is included in b. Thus $b = \inf\{\theta G : K \subseteq G \in \Sigma\cap\mathfrak{T}\}$ and $\pi_0(K) = b$ is defined.

To compute $\bar{d}^*(b)$, observe first that $b \subseteq 1 \setminus \theta H_n \subseteq \theta F_n$ for every n, so

$$\bar{d}^*(b) \le \inf_{n\in\mathbb{N}} \bar{d}^*(\theta F_n) \le \inf_{n\in\mathbb{N}} \mu F_n = \mu K.$$

On the other hand,

$$\bar{d}^*(\theta(\bigcap_{i\le n} G_i)) \ge 1 - \bar{d}^*(\theta H_n) \ge 1 - \mu H_n \ge \mu K$$

for every n, so

$$\bar{d}^*(b) = \lim_{n\to\infty} \bar{d}^*(\theta(\bigcap_{i\le n} G_i)) \ge \mu K.$$

Accordingly $\bar{d}^*(b) = \mu K$, and $\pi_0 K$ has the required properties. **Q**

(b) If K, $L \in \mathcal{K}$, then $\pi_0(K \cap L) = \pi_0 K \cap \pi_0 L$. **P** We know that $K \cap L \in \mathcal{K}$ (491Lb). And

$$\begin{aligned}\pi_0 K \cap \pi_0 L &= \inf\{\theta G : K \subseteq G \in \mathfrak{T}\} \cap \inf\{\theta H : L \subseteq H \in \mathfrak{T}\} \\ &= \inf\{\theta G \cap \theta H : K \subseteq G \in \mathfrak{T},\, L \subseteq H \in \mathfrak{T}\} \\ &= \inf\{\theta(G \cap H) : K \subseteq G \in \mathfrak{T},\, L \subseteq H \in \mathfrak{T}\} \supseteq \pi_0(K \cap L).\end{aligned}$$

Now suppose that $U \supseteq K \cap L$ is a measurable open set and $\epsilon > 0$. Let G, G' be measurable open sets and F, F' measurable closed sets such that $K \subseteq G \subseteq F$, $L \subseteq G' \subseteq F'$, $\mu(F \setminus K) \le \epsilon$ and $\mu(F' \setminus L) \le \epsilon$. Then

$$\begin{aligned}\bar{d}^*(\pi_0 K \cap \pi_0 L \setminus \theta U) &\le \bar{d}^*(\theta G \cap \theta G' \setminus \theta U) = \bar{d}^*(\theta(G \cap G' \setminus U)) \\ &\le \bar{d}^*(\theta(F \cap F' \setminus U)) \le \mu(F \cap F' \setminus U) \le 2\epsilon.\end{aligned}$$

As ϵ is arbitrary, $\pi_0 K \cap \pi_0 L \subseteq \theta U$; as U is arbitrary, $\pi_0 K \cap \pi_0 L \subseteq \pi_0(K \cap L)$. **Q**

This means that $\{\pi_0 K : K \subseteq X$ is a regularly embedded measurable set$\}$ is a subset of D closed under $\cap$. By 491Kf, the topologically closed subalgebra $\mathfrak{B}$ of $\mathfrak{Z}$ generated by this family is included in D; by 491Ke, $\mathfrak{B}$ is order-closed and regularly embedded in $\mathfrak{Z}$, and $(\mathfrak{B}, \bar{d}^*\restriction\mathfrak{B})$ is a probability algebra.

(c) Now observe that if we set $Q = \{K^\bullet : K \in \mathcal{K}\} \subseteq \mathfrak{A}$, we have a function $\pi : Q \to \mathfrak{B}$ defined by setting $\pi K^\bullet = \pi_0 K$ whenever $K \in \mathcal{K}$. **P** Suppose that K, $L \in \mathcal{K}$ and $\mu(K \triangle L) = 0$. Then

$$\bar{d}^*(\pi_0 K \triangle \pi_0 L) = \bar{d}^*(\pi_0 K) + \bar{d}^*(\pi_0 L) - 2\bar{d}^*(\pi_0 K \cap \pi_0 L)$$

(because $\pi_0 K$ and $\pi_0 L$ belong to $\mathfrak{B} \subseteq D$)

$$\begin{aligned}&= \bar{d}^*(\pi_0 K) + \bar{d}^*(\pi_0 L) - 2\bar{d}^*(\pi_0(K \cap L)) \\ &= \mu K + \mu L - 2\mu(K \cap L) = 0.\end{aligned}$$

So $\pi_0 K = \pi_0 L$ and either can be used to define $\pi K^\bullet$. **Q** Next, the same formulae show that $\pi : Q \to \mathfrak{B}$ is an isometry when Q is given the measure metric of $\mathfrak{A}$, since if K, L belong to $\mathcal{K}$,

$$\bar{\rho}(\pi K^\bullet, \pi L^\bullet) = \bar{d}^*(\pi_0 K \triangle \pi_0 L) = \mu K + \mu L - 2\mu(K \cap L) = \mu(K \triangle L) = \bar{\mu}(K^\bullet \triangle L^\bullet).$$

As Q is dense in $\mathfrak{A}$ (412N), there is a unique extension of π to an isometry from $\mathfrak{A}$ to $\mathfrak{B}$.

(d) Because

$$\pi(K^\bullet \cap L^\bullet) = \pi((K \cap L)^\bullet) = \pi_0(K \cap L) = \pi_0 K \cap \pi_0 L = \pi K^\bullet \cap \pi L^\bullet$$

for all K, $L \in \mathcal{K}$, $\pi(a \cap a') = \pi a \cap \pi a'$ for all a, $a' \in \mathfrak{A}$. It follows that π is a Boolean homomorphism. **P** The point is that $\bar{d}^*(\pi a) = \bar{\mu} a$ for every $a \in Q$, and therefore for every $a \in \mathfrak{A}$. Now if $a \in \mathfrak{A}$, $\pi(1 \setminus a)$ must be disjoint from πa (since certainly $\pi 0 = 0$), and has the same measure as $1 \setminus \pi a$ (remember that we know that $(\mathfrak{B}, \bar{d}^* \restriction \mathfrak{B})$ is a measure algebra), so must be equal to $1 \setminus \pi a$. By 312H(ii), π is a Boolean homomorphism. **Q**

By 324G, π is order-continuous when regarded as a function from $\mathfrak{A}$ to $\mathfrak{B}$. Because $\mathfrak{B}$ is regularly embedded in $\mathfrak{Z}$, π is order-continuous when regarded as a function from $\mathfrak{A}$ to $\mathfrak{Z}$.

(e) Let $G \in \Sigma \cap \mathfrak{T}$. For any $\epsilon > 0$, there is a $K \in \mathcal{K}$ such that $K \subseteq G$ and $\mu(G \setminus K) \le \epsilon$. In this case, $\pi K^\bullet = \pi_0 K \subseteq \theta G$. So

$$\bar{d}^*(\pi G^\bullet \setminus \theta G) \le \bar{d}^*(\pi G^\bullet \setminus \pi K^\bullet) = \bar{\mu}(G^\bullet \setminus K^\bullet) = \mu(G \setminus K) \le \epsilon.$$

As ϵ is arbitrary, $\pi G^\bullet \subseteq \theta G$.

(f) This shows that we have a homomorphism π with the required properties. To see that π is unique, suppose that $\pi' : \mathfrak{A} \to \mathfrak{Z}$ is also a homomorphism of the same kind. In this case

$$\bar{d}^*(1 \setminus \pi' a) = \bar{d}^*(\pi'(1 \setminus a)) = \bar{\mu}(1 \setminus a) = 1 - \bar{\mu} a = 1 - \bar{d}^*(\pi' a),$$

so $\pi' a \in D$, for every $a \in \mathfrak{A}$. If $K \in \mathcal{K}$, then $\pi' K^\bullet \subseteq \theta G$ whenever $K \subseteq G \in \Sigma \cap \mathfrak{T}$, so $\pi' K^\bullet \subseteq \pi_0 K = \pi K^\bullet$. As both $\pi K^\bullet$ and $\pi' K^\bullet$ belong to D,

$$\bar{d}^*(\pi K^\bullet \setminus \pi' K^\bullet) = \bar{d}^*(\pi K^\bullet) - \bar{d}^*(\pi' K^\bullet) = \mu K - \mu K = 0,$$

and $\pi K^\bullet = \pi' K^\bullet$. As $\{K^\bullet : K \in \mathcal{K}\}$ is topologically dense in $\mathfrak{A}$, and both π and π' are continuous, they must be equal.

491O Proposition Let X be a topological space and μ an effectively regular probability measure on X which measures every zero set, and suppose that $\langle x_i \rangle_{i \in \mathbb{N}}$ is an equidistributed sequence in X. Let $\mathfrak{A}$ be the measure algebra of μ and $\pi : \mathfrak{A} \to \mathfrak{Z} = \mathcal{P}\mathbb{N}/\mathcal{Z}$ the regular embedding described in 491N; let $T_\pi : L^\infty(\mathfrak{A}) \to L^\infty(\mathfrak{Z})$ be the corresponding order-continuous Banach algebra embedding (363F). Let $S : \ell^\infty(X) \to \ell^\infty$ be the Riesz homomorphism defined by setting $(Sf)(i) = f(x_i)$ for $f \in \ell^\infty(X)$ and $i \in \mathbb{N}$, and $R : \ell^\infty \to L^\infty(\mathfrak{Z})$ the Riesz homomorphism corresponding to the Boolean homomorphism $I \mapsto I^\bullet : \mathcal{P}\mathbb{N} \to \mathfrak{Z}$. For $f \in \mathcal{L}^\infty(\mu)$ let $f^\bullet$ be the corresponding member of $L^\infty(\mu) \cong L^\infty(\mathfrak{A})$ (363I). Then $T_\pi(f^\bullet) = RSf$ for every $f \in C_b(X)$.

proof To begin with, suppose that $f : X \to [0,1]$ is continuous and $k \ge 1$. For each $i \le k$ set $G_i = \{x : f(x) > \frac{i}{k}\}$, $F_i = \{x : f(x) \ge \frac{i}{k}\}$. Then $\frac{1}{k}\sum_{i=1}^k \chi F_i \le f \le \frac{1}{k}\sum_{i=0}^k \chi G_i$. So

$$\tfrac{1}{k}\textstyle\sum_{i=1}^k \chi(\pi F_i^\bullet) \le T_\pi f^\bullet \le \tfrac{1}{k}\textstyle\sum_{i=0}^k \chi(\pi G_i^\bullet),$$

$$\tfrac{1}{k}\textstyle\sum_{i=1}^k \chi(\theta F_i) \le RSf \le \tfrac{1}{k}\textstyle\sum_{i=0}^k \chi(\theta G_i)$$

where $\theta : \mathcal{P}X \to \mathfrak{Z}$ is the Boolean homomorphism described in the proof of 491N, because $RS : \ell^\infty(X) \to L^\infty(\mathfrak{Z})$ is the Riesz homomorphism corresponding to θ (see 363Fa, 363Fg). Now 491N tells us that $\pi G^\bullet \subseteq \theta G$ for every cozero set $G \subseteq X$, so

$$\begin{aligned} T_\pi f^\bullet &\le \frac{1}{k}\sum_{i=0}^k \chi(\pi G_i^\bullet) \le \frac{1}{k}\sum_{i=0}^k \chi(\theta G_i) \\ &\le \frac{1}{k}\sum_{i=0}^k \chi(\theta F_i) = \frac{1}{k}e + \sum_{i=1}^k \chi(\theta F_i) \le \frac{1}{k}e + RSf \end{aligned}$$

where e is the standard order unit of the M-space $L^\infty(\mathfrak{Z})$. But looking at complements we see that we must have $\pi F^\bullet \supseteq \theta F$ for every zero set $F \subseteq X$, so

$$
\begin{aligned}
RSf &\le \frac{1}{k}\sum_{i=0}^{k}\chi(\theta G_i) \le \frac{1}{k}\sum_{i=0}^{k}\chi(\theta F_i)\\
&\le \frac{1}{k}\sum_{i=0}^{k}\chi(\pi F_i^\bullet) = \frac{1}{k}e + \sum_{i=1}^{k}\chi(\pi F_i^\bullet) \le \frac{1}{k}e + T_\pi f^\bullet.
\end{aligned}
$$

This means that $|T_\pi f^\bullet - RSf| \le \frac{1}{k}e$ for every $k \ge 1$, so that $T_\pi f^\bullet = RSf$. This is true whenever $f \in C_b(X)$ takes values in $[0,1]$; as all the operators here are linear, it is true for every $f \in C_b(X)$.

491P Proposition Any probability algebra $(\mathfrak{A}, \bar\mu)$ with cardinal at most $\mathfrak{c}$ can be regularly embedded as a subalgebra of $\mathfrak{Z} = \mathcal{P}\mathbb{N}/\mathcal{Z}$ in such a way that $\bar\mu$ is identified with the restriction of the submeasure $\bar d^*$ to the image of $\mathfrak{A}$.

proof The usual measure of $\{0,1\}^{\mathfrak{c}}$ is a totally finite Radon measure (416Ub), so is effectively regular (491Ma). It has an equidistributed sequence (491G), so its measure algebra $(\mathfrak{B}_{\mathfrak{c}}, \bar\nu_{\mathfrak{c}})$ can be regularly embedded in $\mathfrak{Z}$ in a way which matches $\bar\nu_{\mathfrak{c}}$ with $\bar d^*$ (491N). Now if $(\mathfrak{A}, \bar\mu)$ is any probability algebra with cardinal at most $\mathfrak{c}$, it can be regularly embedded (by a measure-preserving homomorphism) in $(\mathfrak{B}_{\mathfrak{c}}, \bar\nu_{\mathfrak{c}})$ (332N), and therefore in $(\mathfrak{Z}, \bar d^*)$.

491Q Corollary Every Radon probability measure on $\{0,1\}^{\mathfrak{c}}$ has an equidistributed sequence.

proof Let μ be a Radon probability measure on $\{0,1\}^{\mathfrak{c}}$, and $(\mathfrak{A}, \bar\mu)$ its measure algebra. For $\xi < \mathfrak{c}$ set $E_\xi = \{x : x \in \{0,1\}^{\mathfrak{c}}, x(\xi) = 1\}$ and $e_\xi = E_\xi^\bullet \in \mathfrak{A}$.

(a) $\#(\mathfrak{A}) \le \mathfrak{c}$. **P** The σ-algebra generated by $\{E_\xi : \xi < \mathfrak{c}\}$ is the Baire σ-algebra $\mathcal{B}\mathfrak{a}(\{0,1\}^{\mathfrak{c}})$ (4A3Na) and $E^\bullet$ belongs to the σ-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ generated by $\{e_\xi : \xi < \mathfrak{c}\}$ whenever $E \in \mathcal{B}\mathfrak{a}(\{0,1\}^{\mathfrak{c}})$. Now if $G \subseteq \{0,1\}^{\mathfrak{c}}$ is open, then $\mathcal{H} = \{H : H \subseteq G$ is determined by a finite set of coordinates$\}$ is an upwards-directed family of open sets with union G, while H is a Baire set (see 4A3Nb) and $H^\bullet \in \mathfrak{B}$ for every $H \in \mathcal{H}$. If $\epsilon > 0$, there is an $H \in \mathcal{G}$ such that $\mu G \le \mu H + \epsilon$, because μ is τ-additive; so we have a sequence $\langle H_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{H}$ such that $\mu(G \triangle \bigcup_{n\in\mathbb{N}} H_n) = 0$ and $G^\bullet = \sup_{n\in\mathbb{N}} H_n^\bullet \in \mathfrak{B}$. It follows that $F^\bullet \in \mathfrak{B}$ for every closed $F \subseteq \{0,1\}^{\mathfrak{c}}$. Next, if $E \subseteq \{0,1\}^{\mathfrak{c}}$ is measured by μ, there is a sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ of closed subsets of E such that $\mu(E \setminus \bigcup_{n\in\mathbb{N}} F_n) = 0$, because μ is inner regular with respect to the closed sets, so that $E^\bullet = \sup_{n\in\mathbb{N}} F_n^\bullet \in \mathfrak{B}$; thus $\mathfrak{A} = \mathfrak{B}$. But $\mathfrak{B}$ has cardinal at most $\mathfrak{c}$, by 4A1O. **Q**

(b) By 491P, there is a measure-preserving embedding $\pi : \mathfrak{A} \to \mathfrak{Z}$, and $\pi[\mathfrak{A}] \subseteq D$ as defined in 491K. For $\xi < \mathfrak{c}$ let $I_\xi \subseteq \mathbb{N}$ be such that $I_\xi^\bullet = \pi e_\xi$ in $\mathfrak{Z}$. Define $x_i(\xi)$, for $i \in \mathbb{N}$ and $\xi < \mathfrak{c}$, by setting $x_i(\xi) = 1$ if $i \in I_\xi$, 0 otherwise. Now suppose that $H \subseteq \{0,1\}^{\mathfrak{c}}$ is a basic open set of the form $\{x : x(\xi) = 1$ for $\xi \in K$, 0 for $\xi \in L\}$, where K, $L \subseteq \mathfrak{c}$ are finite. Set $b = \pi H^\bullet$ in $\mathfrak{Z}$,

$$I = \{i : x_i \in H\} = \mathbb{N} \cap \textstyle\bigcap_{\xi\in K} I_\xi \setminus \bigcup_{\xi\in L} I_\xi.$$

Then

$$
\begin{aligned}
b = \pi H^\bullet &= \pi(\inf_{\xi\in K} a_\xi \setminus \sup_{\xi\in L} a_\xi)\\
&= \inf_{\xi\in K} \pi a_\xi \setminus \sup_{\xi\in L} \pi a_\xi = \inf_{\xi\in K} I_\xi^\bullet \setminus \sup_{\xi\in L} I_\xi^\bullet = I^\bullet.
\end{aligned}
$$

Since $b \in D$, $d(I)$ is defined and is equal to $\bar d^*(b) = \bar\mu H^\bullet = \mu H$.

If we now take E to be an open-and-closed subset of $\{0,1\}^{\mathfrak{c}}$, it can be expressed as a disjoint union of finitely many basic open sets of the type just considered; because d is additive on disjoint sets, $d(\{i : x_i \in E\})$ is defined and equal to μE. But this is enough to ensure that $\langle x_i\rangle_{i\in\mathbb{N}}$ is equidistributed, by 491Ch.

491R In this section I have been looking at probability measures with equidistributed sequences. A standard line of investigation is to ask which of our ordinary constructions, applied to such measures, lead to others of the same kind, as in 491Ea and 491F. We find that the language developed here enables us to express another result of this type.

Proposition Let X be a topological space, μ an effectively regular topological probability measure on X which has an equidistributed sequence, and ν a probability measure on X which is an indefinite-integral measure over μ. Then ν has an equidistributed sequence.

proof Let $\mathcal{K}$ be the family of regularly enveloped measurable sets.

(a) Consider first the case in which ν has Radon-Nikodým derivative of the form $\frac{1}{\mu K}\chi K$ for some $K \in \mathcal{K}$ of non-zero measure. For each $m \in \mathbb{N}$, we have an open set $G_m \supseteq K$ such that $\mu(\overline{G}_m \setminus K) \le 2^{-m}$; of course we can arrange that $G_{m+1} \subseteq G_m$ for each m. Set $F_m = \overline{G}_m$ for each $m \in \mathbb{N}$. Let $\langle x_i \rangle_{i \in \mathbb{N}}$ be an equidistributed sequence for μ. Then there is an $I \subseteq \mathbb{N}$ such that $d(I) = \mu K$ and $\{i : i \in I,\, x_i \notin G_m\}$ is finite for every m. **P** For each $m \in \mathbb{N}$, set $I_m = \{i : x_i \in G_m\}$. We know that $\liminf_{n\to\infty} \frac{1}{n}\#(I_m \cap n) \ge \mu G_m \ge \mu K$ for each m, so we can find a strictly increasing sequence $\langle k_m \rangle_{m \in \mathbb{N}}$ such that $\frac{1}{n}\#(I_m \cap n) \ge \mu K - 2^{-m}$ whenever $m \in \mathbb{N}$ and $n > k_m$. Set

$$I = \bigcup_{m \in \mathbb{N}} I_m \cap k_{m+1} \subseteq \bigcap_{m \in \mathbb{N}} (I_m \cup k_{m+1}).$$

If $k_m < n \le k_{m+1}$,

$$\frac{1}{n}\#(I \cap n) \ge \frac{1}{n}\#(I_m \cap n) \ge \mu K - 2^{-m};$$

so $\liminf_{n\to\infty} \frac{1}{n}\#(I \cap n) \ge \mu K$. On the other hand, for any $m \in \mathbb{N}$,

$$\{i : i \in I,\, x_i \notin F_m\} \subseteq I \setminus I_m \subseteq k_{m+1}$$

is finite, so

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n}\#(I \cap n) &\le \limsup_{n\to\infty} \frac{1}{n}\#(\{i : i < n,\, x_i \in F_m\}) \\ &\le \mu F_m \le \mu K + 2^{-m}.\end{aligned}$$

Accordingly $\limsup_{n\to\infty} \frac{1}{n}\#(I \cap n) \le \mu K$ and $d(I)$ is defined and equal to μK. **Q**

Let $\langle j_n \rangle_{n \in \mathbb{N}}$ be the increasing enumeration of I, and set $y_n = x_{j_n}$ for each n. Then $\langle y_n \rangle_{n \in \mathbb{N}}$ is equidistributed for ν. **P** Note first that

$$\lim_{n\to\infty} \frac{n}{j_n} = \lim_{n\to\infty} \frac{1}{j_n}\#(I \cap j_n) = \mu K.$$

Let $F \subseteq X$ be closed. Then $\nu F = \frac{\mu(F \cap K)}{\mu K}$. On the other hand, for any $m \in \mathbb{N}$,

$$\begin{aligned}d^*(\{n : y_n \in F\}) &= \limsup_{n\to\infty} \frac{1}{n}\#(\{i : i < n,\, x_{j_i} \in F\}) \\ &= \limsup_{n\to\infty} \frac{1}{n}\#(\{i : i < j_n,\, i \in I,\, x_i \in F\}) \\ &= \limsup_{n\to\infty} \frac{1}{n}\#(\{i : i < j_n,\, i \in I,\, x_i \in F \cap G_m\})\end{aligned}$$

(because $\{i : i \in I,\, x_i \notin G_m\}$ is finite)

$$\begin{aligned}&\le \limsup_{n\to\infty} \frac{j_n}{n}\frac{1}{j_n}\#(\{i : i < j_n,\, x_i \in F \cap F_m\}) \\ &= \frac{1}{\mu K}\limsup_{n\to\infty} \frac{1}{j_n}\#(\{i : i < j_n,\, x_i \in F \cap F_m\}) \\ &\le \frac{1}{\mu K}\limsup_{n\to\infty} \frac{1}{n}\#(\{i : i < n,\, x_i \in F \cap F_m\}) \\ &\le \frac{1}{\mu K}\mu(F \cap F_m) \le \frac{1}{\mu K}(\mu(F \cap K) + 2^{-m}) = \nu F + \frac{1}{2^m \mu K};\end{aligned}$$

as m is arbitrary, $d^*(\{n : y_n \in F\}) \le \nu F$; as F is arbitrary, $\langle y_n\rangle_{n\in\mathbb{N}}$ is equidistributed for ν. **Q**

(b) Now turn to the general case. Let f be a Radon-Nikodým derivative of ν; we may suppose that f is measurable and non-negative and defined everywhere in X. Then there is a sequence $\langle K_m\rangle_{m\in\mathbb{N}}$ in $\mathcal{K}$ such that $f =_{\text{a.e.}} \sum_{m=0}^{\infty} \frac{1}{m+1}\chi K_m$. **P** Choose f_m, K_m inductively, as follows. $f_0 = f$. Given that $f_m \ge 0$ is measurable, set $E_m = \{x : f_m(x) \ge \frac{1}{m+1}\}$ and let $K_m \in \mathcal{K}$ be such that $K_m \subseteq E_m$ and $\mu(E_m \setminus K_m) \le 2^{-m}$; set $f_{m+1} = f_m - \frac{1}{m+1}\chi K_m$. Then $\langle f_m\rangle_{m\in\mathbb{N}}$ is non-increasing; set $g = \lim_{m\to\infty} f_m$. **?** If g is not zero almost everywhere, let $r \in \mathbb{N}$ be such that $\mu E > 2^{-r+1}$ where $E = \{x : g(x) \ge \frac{1}{r+1}\}$. Then $E \subseteq E_m$ for every $m \ge r$, so $\mu(E \setminus K_m) \le 2^{-m}$ for every $m \ge r$ and $F = E \cap \bigcap_{m\ge r} K_m$ is not empty. Take $x \in F$; then $f_{m+1}(x) \le f_m(x) - \frac{1}{m+1}$ for every $m \ge r$, which is impossible. **X** So $g = 0$ a.e. and $f =_{\text{a.e.}} \sum_{m=0}^{\infty} \frac{1}{m+1}\chi K_m$. **Q**

By (a), we have for each m a sequence $\langle y_{mn}\rangle_{n\in\mathbb{N}}$ in X such that

$$\mu(F \cap K_m) \ge \mu K_m \cdot \limsup_{n\to\infty} \frac{1}{n}\#(\{i : i < n,\, y_{mi} \in F\})$$

for every closed $F \subseteq X$. For $n \in \mathbb{N}$, let ν_n be the point-supported measure on X defined by setting

$$\nu_n A = \sum_{m=0}^{\infty} \frac{\mu K_m}{(n+1)(m+1)}\#(\{i : i \le n,\, y_{mi} \in A\})$$

for $A \subseteq X$; because $\sum_{m=0}^{\infty} \frac{\mu K_m}{m+1} = \int f d\mu = 1$, ν_n is a probability measure. If $F \subseteq X$ is closed,

$$\limsup_{n\to\infty} \nu_n F \le \sum_{m=0}^{\infty} \frac{\mu K_m}{m+1} \limsup_{n\to\infty} \frac{1}{n+1}\#(\{i : i \le n,\, y_{mi} \in F\})$$

(because $\sum_{m=0}^{\infty} \frac{\mu K_m}{m+1} < \infty$)

$$\le \sum_{m=0}^{\infty} \frac{1}{m+1}\mu(F \cap K_m) = \int_F f d\mu = \nu F.$$

So 491D tells us that there is an equidistributed sequence for ν, as required.

491S The asymptotic density filter Corresponding to the asymptotic density ideal, of course we have a filter. It is not surprising that convergence along this filter, in the sense of 2A3Sb, should be interesting and sometimes important.

(a) Set

$$\mathcal{F}_d = \{\mathbb{N} \setminus I : I \in \mathcal{Z}\} = \{I : I \subseteq \mathbb{N},\, \lim_{n\to\infty} \frac{1}{n}\#(I \cap n) = 1\}.$$

Then $\mathcal{F}_d$ is a filter on $\mathbb{N}$, the **(asymptotic) density filter**.

(b) For a bounded sequence $\langle \alpha_n\rangle_{n\in\mathbb{N}}$ in $\mathbb{C}$, $\lim_{n\to\mathcal{F}_d} \alpha_n = 0$ iff $\lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n} |\alpha_k| = 0$. **P** Set $M = \sup_{k\in\mathbb{N}} |\alpha_k|$, and for $\epsilon > 0$ set $I_\epsilon = \{n : |\alpha_n| \le \epsilon\}$. Then, for any $n \ge 1$,

$$\frac{\epsilon}{n+1}\#((n+1) \setminus I_\epsilon) \le \frac{1}{n+1}\sum_{k=0}^{n} |\alpha_k| \le \epsilon + \frac{M}{n+1}\#((n+1) \setminus I_\epsilon).$$

So if $\lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n} |\alpha_k| = 0$, then $\lim_{n\to\infty} \frac{1}{n+1}\#((n+1) \setminus I_\epsilon) = 0$, that is, $\mathbb{N} \setminus I_\epsilon \in \mathcal{Z}$ and $I_\epsilon \in \mathcal{F}_d$; as ϵ is arbitrary, $\lim_{n\to\mathcal{F}_d} \alpha_n = 0$. While if $\lim_{n\to\mathcal{F}_d} \alpha_n = 0$ then $\mathbb{N} \setminus I_\epsilon \in \mathcal{Z}$ and $\limsup_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n} |\alpha_k| \le \epsilon$; again, ϵ is arbitrary, so $\lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n} |\alpha_k| = 0$. **Q**

(c) For any $m \in \mathbb{N}$ and $A \subseteq \mathbb{N}$, $A+m \in \mathcal{F}_d$ iff $A \in \mathcal{F}_d$. **P** For any $n \ge m$, $\#(n\cap(A+m)) = \#((n-m)\cap A)$, so

$$d(A+m)=\lim_{n\to\infty}\frac{1}{n}\#(n\cap(A+m))=\lim_{n\to\infty}\frac{1}{n}\#(n\cap A)=d(A)$$

if either $d(A+m)$ or $d(A)$ is defined, in particular, if either $A+m$ or A belongs to $\mathcal{F}_d$. **Q** Hence, or otherwise, for any (real or complex) sequence $\langle\alpha_n\rangle_{n\in\mathbb{N}}$, $\lim_{n\to\mathcal{F}_d}\alpha_n=\lim_{n\to\mathcal{F}_d}\alpha_{m+n}$ if either is defined.

491X Basic exercises (a)(i) Show that if I, $J\in\mathcal{D}=\operatorname{dom}d$ as defined in 491A, then $I\cup J\in\mathcal{D}$ iff $I\cap J\in\mathcal{D}$ iff $I\setminus J\in\mathcal{D}$ iff $I\triangle J\in\mathcal{D}$. (ii) Show that if $\mathcal{E}\subseteq\mathcal{D}$ is an algebra of sets, then $d\restriction\mathcal{E}$ is additive. (iii) Find I, $J\in\mathcal{D}$ such that $I\cap J\notin\mathcal{D}$.

>(b) Suppose that $I\subseteq\mathbb{N}$ and that $f:\mathbb{N}\to\mathbb{N}$ is strictly increasing. Show that $d^*(f[I])\le d^*(I)d^*(f[\mathbb{N}])$, with equality if either I or $f[\mathbb{N}]$ has asymptotic density.

(c) Suppose that $I\subseteq\mathbb{N}$. (i) Show that there is a $J\subseteq I$ such that $d^*(J)=d^*(I\setminus J)=d^*(I)$. (ii) Show that if $0\le\alpha\le d^*(I)$ there is a $J\subseteq I$ such that $d^*(J)=\alpha$ and $d^*(I\setminus J)=d^*(I)-\alpha$. (iii) Show that if $d(I)$ is defined and $0\le\alpha\le d(I)$ there is a $J\subseteq I$ such that $d(J)$ is defined and equal to α.

(d)(i)(α) Show that if I, $K\subseteq\mathbb{N}$ are such that $d(I)$ and $d(K)$ are defined, there is a $J\subseteq\mathbb{N}$ such that $d(J)$ is defined and $d(J)=d^*(J\cap I)=d^*(J\cap K)=\min(d(I),d(K))$. ($\beta$) Let $I\subseteq\mathbb{N}$ be such that $d(J)=d^*(J\cap I)+d^*(J\setminus I)$ for every $J\subseteq\mathbb{N}$ such that $d(J)$ is defined. Show that either $I\in\mathcal{Z}$ or $\mathbb{N}\setminus I\in\mathcal{Z}$. (ii) Show that for every $\epsilon>0$ there is an $I\subseteq\mathbb{N}$ such that $d^*(I)=\epsilon$ but $d(J)=1$ whenever $J\supseteq I$ and $d(J)$ is defined.

(e) Let (X,Σ,μ) be a probability space and $\langle E_n\rangle_{n\in\mathbb{N}}$ a sequence in Σ. For $x\in X$, set $I_x=\{n:n\in\mathbb{N},\ x\in E_n\}$. Show that $\int d^*(I_x)\mu(dx)\ge\liminf_{n\to\infty}\mu E_n$.

(f) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a compact Radon probability space. Take any point ∞ not belonging to X, and give $X\cup\{\infty\}$ the topology generated by $\{G\cup\{\infty\}:G\in\mathfrak{T}\}$. Show that $X\cup\{\infty\}$ is compact and that the image measure μ_∞ of μ under the identity map from X to $X\cup\{\infty\}$ is a quasi-Radon measure, inner regular with respect to the compact sets. Show that if we set $x_i=\infty$ for every i, then $\langle x_i\rangle_{i\in\mathbb{N}}$ is equidistributed for μ_∞.

>(g)(i) Show that a sequence $\langle t_i\rangle_{i\in\mathbb{N}}$ in $[0,1]$ is equidistributed (with respect to Lebesgue measure) iff $\lim_{n\to\infty}\frac{1}{n+1}\#(\{i:i\le n,\ t_i\le\beta\})=\beta$ for every $\beta\in[0,1]$. (ii) Show that if $\alpha\in\mathbb{R}$ is irrational then the sequence $\langle<i\alpha>\rangle_{i\in\mathbb{N}}$ of fractional parts of multiples of α is equidistributed. (*Hint*: 281N.) (iii) Show that a function $f:[0,1]\to\mathbb{R}$ is Riemann integrable (134K) iff $\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n f(t_i)$ is defined in $\mathbb{R}$ for every equidistributed sequence $\langle t_i\rangle_{i\in\mathbb{N}}$ in $[0,1]$. (iv) Show that a sequence $\langle t_i\rangle_{i\in\mathbb{N}}$ in $[0,1]$ is equidistributed iff $\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n f(t_i)$ is defined and equal to $\mathcal{R}\!\!\int_0^1 f$ for every Riemann integrable function $f:[0,1]\to\mathbb{R}$.

>(h) Let X be a topological space, μ a probability measure on X measuring every zero set, and $\langle x_i\rangle_{i\in\mathbb{N}}$ an equidistributed sequence in X. Show that $\lim_{n\to\infty}\dfrac{1}{n+1}\sum_{i=0}^n f(x_i)$ is defined and equal to $\int f\,d\mu$ for every bounded $f:X\to\mathbb{R}$ which is continuous almost everywhere. (Cf. 134L.)

(i) Show that the usual measure on the split interval (419L) has an equidistributed sequence.

(j) Let X be a metrizable space, and μ a quasi-Radon probability measure on X. (i) Show that there is an equidistributed sequence for μ. (ii) Show that if the support of μ is not compact, and $\langle x_i\rangle_{i\in\mathbb{N}}$ is an equidistributed sequence for μ, then there is a continuous integrable function $f:X\to\mathbb{R}$ such that $\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n f(x_i)=\infty$.

(k) Let $\phi:\mathfrak{c}\to\mathcal{P}\mathbb{N}$ be an injective function. For each $n\in\mathbb{N}$ let λ_n be the uniform probability measure on $\mathcal{P}(\mathcal{P}n)$, giving measure 2^{-2^n} to each singleton. Define $\psi_n:\mathcal{P}(\mathcal{P}n)\to\{0,1\}^{\mathfrak{c}}$ by setting $\psi_n(\mathcal{I})(\xi)=1$ if $\phi(\xi)\cap n\in\mathcal{I}$, 0 otherwise, and let ν_n be the image measure $\lambda_n\psi_n^{-1}$. Show that ν_nE is the usual measure of E whenever $E\subseteq\{0,1\}^{\mathfrak{c}}$ is determined by coordinates in a finite set on which the map $\xi\mapsto\phi(\xi)\cap n$ is injective. Use this with 491D to prove 491G.

>(l)(i) Let Z be the Stone space of the measure algebra of Lebesgue measure on $[0,1]$, with its usual measure. Show that there is no equidistributed sequence in Z. (*Hint*: meager sets in Z have negligible closures.) (ii) Show that Dieudonné's measure on ω_1 (411Q) has no equidistributed sequence. (iii) Show that if $\#(I) > \mathfrak{c}$ then the usual measure on $\{0,1\}^I$ has no equidistributed sequence. (*Hint*: if $\langle x_i\rangle_{i\in\mathbb{N}}$ is any sequence in $\{0,1\}^I$, there is an infinite $J \subseteq I$ such that $\langle x_i(\eta)\rangle_{i\in\mathbb{N}} = \langle x_i(\xi)\rangle_{i\in\mathbb{N}}$ for all η, $\xi \in J$.) (iv) Show that if X is a topological group with a Haar probability measure μ, and X is not separable, then μ has no equidistributed sequence. (*Hint*: use 443D to show that every separable subset is negligible.)

(m) Let X be a compact Hausdorff abelian topological group and μ its Haar probability measure. Show that a sequence $\langle x_i\rangle_{i\in\mathbb{N}}$ in X is equidistributed for μ iff $\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^{n}\chi(x_i) = 0$ for every non-trivial character $\chi : X \to S^1$. (*Hint*: 281G.)

(n)(i) Let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $\mathfrak{Z} = \mathcal{P}\mathbb{N}/\mathcal{Z}$. Show that there is an $a \in \mathfrak{Z}$ such that $a_n \subseteq a$ for every $n \in \mathbb{N}$ and $\bar{d}^*(a) = \sup_{n\in\mathbb{N}}\bar{d}^*(a_n)$. (ii) Show that $\mathfrak{Z}$ is not Dedekind σ-complete. (*Hint*: 393Bc).

(o) Let $\mathfrak{Z}$, $\bar{d}^*$ and D be as in 491K. Show that if $a \in D\setminus\{0\}$ and $\mathfrak{Z}_a$ is the principal ideal of $\mathfrak{Z}$ generated by a, then $(\mathfrak{Z}_a, \bar{d}^*\restriction\mathfrak{Z}_a)$ is isomorphic, up to a scalar multiple of the submeasure, to $(\mathfrak{Z}, \bar{d}^*)$.

(p) Let (X,Σ,μ) be a semi-finite measure space and $\mathfrak{T}$ a topology on X. Show that μ is effectively regular iff whenever $E \in \Sigma$, $\mu E < \infty$ and $\epsilon > 0$ there are a measurable open set G and a measurable closed set $F \supseteq G$ such that $\mu(F\setminus E) + \mu(E\setminus G) \le \epsilon$. 52

(q) Let X be a normal topological space and μ a topological measure on X which is inner regular with respect to the closed sets and effectively locally finite. Show that μ is effectively regular.

(r) Let X be a topological space and μ an effectively regular measure on X. (i) Show that the completion and c.l.d. version of μ are also effectively regular. (ii) Show that if $Y \subseteq X$ then the subspace measure on Y is again effectively regular. (iii) Show that any totally finite indefinite-integral measure over μ is effectively regular.

(s)(i) Let X_1, X_2 be topological spaces with effectively regular measures μ_1, μ_2. Show that the c.l.d. product measure on $X_1 \times X_2$ is effectively regular with respect to the product topology. (*Hint*: 412R.) (ii) Let $\langle X_i\rangle_{i\in I}$ be a family of topological spaces and μ_i an effectively regular probability measure on X_i for each i. Show that the product probability measure on $\prod_{i\in I} X_i$ is effectively regular.

(t) Give $[0,1]$ the topology $\mathfrak{T}$ generated by the usual topology and $\{[0,1]\setminus A : A \subseteq \mathbb{Q}\}$. Let μ_L be Lebesgue measure on $[0,1]$, and Σ its domain. For $E \in \Sigma$ set $\mu E = \mu_L E + \#(E\cap\mathbb{Q})$ if $E\cap\mathbb{Q}$ is finite, ∞ otherwise. Show that μ is a σ-finite quasi-Radon measure with respect to the topology $\mathfrak{T}$, but is not effectively regular.

(u) Let $\mathfrak{A}$ be a countable Boolean algebra and ν a finitely additive functional on $\mathfrak{A}$ such that $\nu 1 = 1$. Show that there is a Boolean homomorphism $\pi : \mathfrak{A} \to \mathcal{P}\mathbb{N}$ such that $d(\pi a)$ is defined and equal to νa for every $a \in \mathfrak{A}$ (i) using 491Xc (ii) using 392H, 491P and 341Xc.

(v) Let X be a dyadic space. (i) Show that there is a Radon probability measure on X with support X. (ii) Show that the following are equiveridical: (α) $w(X) \le \mathfrak{c}$; (β) every Radon probability measure on X has an equidistributed sequence; (γ) X is separable. (*Hint*: 4A2Dd, 418L.)

(w) Give an example of a Radon probability space (X,μ) with a closed conegligible set $F \subseteq X$ such that μ has an equidistributed sequence but the subspace measure μ_F does not. (*Hint*: the Stone space of the measure algebra of Lebesgue measure embeds into $\{0,1\}^{\mathfrak{c}}$.)

(x) Let X be a topological space. A sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in X is called **statistically convergent** to $x \in X$ if $d(\{i : x_i \in G\}) = 1$ for every open set G containing x. (i) Show that if X is first-countable then $\langle x_n\rangle_{n\in\mathbb{N}}$ is statistically convergent to x iff there is a set $I \subseteq \mathbb{N}$ such that $d(I) = 1$ and $\langle x_n\rangle_{n\in I}$ converges to x in the ordinary sense that $\{n : n \in I,\ x_n \notin G\}$ is finite for every open set G containing x. (ii) Show that a bounded sequence $\langle \alpha_n\rangle_{n\in\mathbb{N}}$ in $\mathbb{R}$ is statistically convergent to α iff $\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}|\alpha_i - \alpha| = 0$.

491Y Further exercises (a) Show that every subset A of $\mathbb{N}$ is expressible in the form $I_A \triangle J_A$ where $d(I_A) = d(J_A) = \frac{1}{2}$ (i) by a direct construction, with $A \mapsto I_A$ a continuous function (ii) using 443D.

(b) (M.Elekes) Let (X, Σ, μ) be a σ-finite measure space and $\langle E_n \rangle_{n\in\mathbb{N}}$ a sequence in Σ such that $\bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} E_m$ is conegligible. Show that there is an $I \in \mathcal{Z}$ such that $\bigcap_{n\in\mathbb{N}} \bigcup_{m\in I\setminus n} E_m$ is conegligible.

(c) (cf. BERGELSON 85) Let $\mathfrak{A}$ be a Boolean algebra, $A \subseteq \mathfrak{A} \setminus \{0\}$ a non-empty set and $\alpha \in [0,1]$. Show that the following are equiveridical: (i) there is a finitely additive functional $\nu : \mathfrak{A} \to [0,1]$ such that $\nu a \geq \alpha$ for every $a \in A$ (ii) for every sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ in A there is a set $I \subseteq \mathbb{N}$ such that $d^*(I) \geq \alpha$ and $\inf_{i\in I\cap n} a_i \neq 0$ for every $n \in \mathbb{N}$.

(d) Let $\mathfrak{A}$ be a Boolean algebra, and $\nu : \mathfrak{A} \to [0,\infty]$ a submeasure. Show that ν is uniformly exhaustive iff whenever $\langle a_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}$ such that $\inf_{n\in\mathbb{N}} \nu a_n > 0$, there is a set $I \subseteq \mathbb{N}$ such that $d^*(I) > 0$ and $\inf_{i\in I\cap n} a_i \neq 0$ for every $n \in \mathbb{N}$.

>(e) Show that if X is a Hausdorff space and $f : \mathbb{N} \to X$ is injective, then there is an open set $G \subseteq X$ such that $f^{-1}[G]$ does not have asymptotic density.

(f) Find a topological space X with a τ-additive topological probability measure μ on X, a sequence $\langle x_i \rangle_{i\in\mathbb{N}}$ in X and a base $\mathcal{G}$ for the topology of X, consisting of measurable sets and closed under finite intersections, such that $\mu G \leq \liminf_{n\to\infty} \frac{1}{n+1}\#(\{i : i \leq n,\ x_i \in G\})$ for every $G \in \mathcal{G}$ but $\langle x_i \rangle_{i\in\mathbb{N}}$ is not equidistributed.

(g) Let X be a compact Hausdorff space on which every Radon probability measure has an equidistributed sequence. Show that the cylindrical σ-algebra of $C(X)$ is the σ-algebra generated by sets of the form $\{f : f \in C(X),\ f(x) > \alpha\}$ where $x \in X$ and $\alpha \in \mathbb{R}$.

(h) Give ω_1+1 and $[0,1]$ their usual compact Hausdorff topologies. Let $\langle t_i \rangle_{i\in\mathbb{N}}$ be a sequence in $[0,1]$ which is equidistributed for Lebesgue measure μ_L, and set $Q = \{t_i : i \in \mathbb{N}\}$, $X = (\omega_1 \times ([0,1] \setminus Q)) \cup (\{\omega_1\} \times Q)$, with the subspace topology inherited from $(\omega_1 + 1) \times [0,1]$. (i) Set $F = \{\omega_1\} \times Q$. Show that F is a closed Baire set in the completely regular Hausdorff space X. (ii) Show that if $f \in C_b(X)$ then there are a $g_f \in C([0,1])$ and a $\zeta < \omega_1$ such that such that $f(\xi, t) = g_f(t)$ whenever $(\xi, t) \in X$ and $\zeta \leq \xi \leq \omega_1$. (iii) Show that there is a Baire measure μ on X such that $\int f d\mu = \int g_f d\mu_L$ for every $f \in C_b(X)$. (iv) Show that $\mu F = 0$. (v) Show that $\int f d\mu = \lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^n f(\omega_1, t_i)$ for every $f \in C_b(X)$, but that $\langle (\omega_1, t_i) \rangle_{i\in\mathbb{N}}$ is not equidistributed with respect to μ.

(i) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a topological measure space. Let $\mathcal{E}$ be the Jordan algebra of X (411Yc). (i) Suppose that μ is a complete probability measure on X and $\langle x_i \rangle_{i\in\mathbb{N}}$ an equidistributed sequence in X. Show that the asymptotic density $d(\{i : x_i \in E\})$ is defined and equal to μE for every $E \in \mathcal{E}$. (ii) Suppose that μ is a probability measure on X and that $\langle x_i \rangle_{i\in\mathbb{N}}$ is a sequence in X such that $d(\{i : x_i \in E\})$ is defined and equal to μE for every $E \in \mathcal{E}$. Show that $\lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^n f(x_i) = \int f d\mu$ for every $f \in C_b(X)$.

(j) Show that a sequence $\langle x_i \rangle_{i\in\mathbb{N}}$ in $[0,1]$ is equidistributed for Lebesgue measure iff there is some $r_0 \in \mathbb{N}$ such that $\lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^n x_i^r = \frac{1}{r+1}$ for every $r \geq r_0$.

(k) Let Z, μ, $X = Z \times \{0,1\}$ and ν be as described in 439K, so that μ is a Radon probability measure on the compact metrizable space Z, X has a compact Hausdorff topology finer than the product topology and agreeing with the product topology on $Z \times \{0\}$, and ν is a measure on Z extending μ. (i) Show that if $f \in C(X)$, then $\{t : t \in Z,\ f(t,0) \neq f(t,1)\}$ is countable. (ii) Show that $\int f(t,0)\mu(dt) = \int f(t,1)\nu(dt)$ for every $f \in C(X)$. (iii) Let λ be the measure νg^{-1} on X, where $g(t,0) = g(t,1) = (t,1)$ for $t \in Zs$. Show that there is a sequence $\langle x_i \rangle_{i\in\mathbb{N}}$ in $Z \times \{0\}$ such that $\int f d\lambda = \lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^n f(x_i)$ for every $f \in C(X)$, but that $\langle x_i \rangle_{i\in\mathbb{N}}$ is not λ-equidistributed.

(l)(i) Show that a Radon probability measure on an extremally disconnected compact Hausdorff space has an equidistributed sequence iff it is point-supported. (ii) Show that there is a separable compact Hausdorff space with a Radon probability measure which has no equidistributed sequence.

(m) Show that there is a countable dense set $D \subseteq [0,1]^{\mathfrak{c}}$ such that no sequence in D is equidistributed for the usual measure on $[0,1]^{\mathfrak{c}}$.

(n) Let $\mathfrak{Z} = \mathcal{P}\mathbb{N}/\mathcal{Z}$ and $\bar{d}^* : \mathfrak{Z} \to [0,1]$ be as in 491I. Show that $\bar{d}^*$ is **order-continuous on the left** in the sense that whenever $A \subseteq \mathfrak{Z}$ is non-empty and upwards-directed and has a supremum $c \in \mathfrak{Z}$, then $\bar{d}^*(c) = \sup_{a\in A} \bar{d}^*(a)$.

(o)(i) Show that $\mathfrak{Z}$ is weakly (σ,∞)-distributive. (ii) Show that $\mathfrak{Z} \cong \mathfrak{Z}^{\mathbb{N}}$. (iii) Show that $\mathfrak{Z}$ has the σ-interpolation property, but is not Dedekind σ-complete. (iv) Show that $\mathfrak{Z}$ has many involutions in the sense of 382O.

(p) Let (X,ρ) be a separable metric space and μ a Borel probability measure on X. (i) Show that there is an equidistributed sequence in X. (ii) Show that if $\langle x_i\rangle_{i\in\mathbb{N}}$ is an equidistributed sequence in X, and $\langle y_i\rangle_{i\in\mathbb{N}}$ is a sequence in X such that $\lim_{i\to\infty} \rho(x_i,y_i) = 0$, then $\langle y_i\rangle_{i\in\mathbb{N}}$ is equidistributed. (iii) Show that if $f : X \to \mathbb{R}$ is a bounded function, then $\lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(x_i) - f(y_i) = 0$ for all equidistributed sequences $\langle x_i\rangle_{i\in\mathbb{N}}$, $\langle y_i\rangle_{i\in\mathbb{N}}$ in X iff $\{x : f$ is continuous at $x\}$ is conegligible, and in this case $\lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(x_i) = \int f d\mu$ for every equidistributed sequence $\langle x_i\rangle_{i\in\mathbb{N}}$ in X.

(q) Let $(X,\mathfrak{T})$ be a topological space, μ a probability measure on X, and $\phi : X \to X$ an inverse-measure-preserving function. (i) Suppose that $\mathfrak{T}$ has a countable network consisting of measurable sets, and that ϕ is ergodic. Show that $\langle\phi^n(x)\rangle_{n\in\mathbb{N}}$ is equidistributed for almost every $x \in X$. (ii) Suppose that μ is *either* inner regular with respect to the closed sets *or* effectively regular, and that $\{x : \langle\phi^n(x)\rangle_{n\in\mathbb{N}}$ is equidistributed$\}$ is not negligible. Show that ϕ is ergodic.

(r) Let $\langle X_\xi\rangle_{\xi<\mathfrak{c}}$ be a family of topological spaces with countable networks consisting of Borel sets, and μ a τ-additive topological probability measure on $X = \prod_{\xi<\mathfrak{c}} X_\xi$. Show that μ has an equidistributed sequence.

(s)(i) Show that there is a family $\langle a_\xi\rangle_{\xi<\mathfrak{c}}$ in $\mathfrak{Z}$ such that $\inf_{\xi\in I} a_\xi = 0$ and $\sup_{\xi\in I} a_\xi = 1$ for every infinite $I \subseteq \mathfrak{c}$. (ii) Show that if $B \subseteq \mathfrak{Z}\setminus\{0\}$ has cardinal less than $\mathfrak{c}$ then there is an $a \in \mathfrak{Z}$ such that $b\cap a$ and $b\setminus a$ are non-zero for every $b \in B$.

(t) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a τ-additive topological probability space. A sequence $\langle x_i\rangle_{i\in\mathbb{N}}$ in X is **completely equidistributed** if, for every $r \ge 1$, the sequence $\langle\langle x_{n+i}\rangle_{i<r}\rangle_{n\in\mathbb{N}}$ is equidistributed for some (therefore any) τ-additive extension of the c.l.d. product measure μ^r on X^r. (i) Show that if there is an equidistributed sequence in X, then there is a completely equidistributed sequence in X. (ii) Show that if $\mathfrak{T}$ is second-countable, then $\mu^{\mathbb{N}}$-almost every sequence in X is completely equidistributed. (iii) Show that if X has two disjoint open sets of non-zero measure, then no sequence which is well-distributed in the sense of 281Ym can be completely equidistributed.

(u) Suppose, in 491O, that μ is a topological measure. Show that $T_\pi f^\bullet \le RSf$ for every bounded lower semi-continuous $f : X \to \mathbb{R}$.

491Z Problem It is known that for almost every $x > 1$ the sequence $\langle{<}x^i{>}\rangle_{i\in\mathbb{N}}$ of fractional parts of powers of x is equidistributed for Lebesgue measure on $[0,1]$ (Kuipers & Niederreiter 74, p. 35). But is $\langle{<}(\frac{3}{2})^n{>}\rangle_{n\in\mathbb{N}}$ equidistributed?

491 Notes and comments The notations d^*, d (491A) are standard, and usefully suggestive. But coming from measure theory we have to remember that d^*, although a submeasure, is not an outer measure, the domain of d is not an algebra of sets (491Xa), and d and d^* are related by only one of the formulae we expect to connect a measure with an outer measure (491Ac, 491Xd). In 491C, the limit $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n} f(x_i)$, when it is defined, is the **Cesàro mean** of the sequence $\langle f(x_i)\rangle_{i\in\mathbb{N}}$. The delicacy of the arguments here arises from the fact that the family of (bounded) sequences with Cesàro means, although a norm-closed linear subspace of ℓ^∞, is neither a sublattice nor a subalgebra. When we turn to the quotient algebra $\mathfrak{Z} = \mathcal{P}\mathbb{N}/\mathcal{Z}$, we find ourselves with a natural submeasure to which we can apply ideas from §392 to good effect (491I;

see also 491Yn and 491Yo). What is striking is that equidistributed sequences induce regular embeddings of measure algebras in $\mathfrak{Z}$ which can be thought of as measure-preserving (491N).

Most authors have been content to define an 'equidistributed sequence' to be one such that the integrals of bounded continuous functions are correctly specified (491Cf, 491Cg); that is, that the point-supported measures $\frac{1}{n+1}\sum_{i=0}^{n}\delta_{x_i}$ converge to μ in the vague topology on an appropriate class of measures (437J). I am going outside this territory in order to cover some ideas I find interesting. 491Yk shows that it makes a difference; there are Borel measures on compact Hausdorff spaces which have sequences which give the correct Cesàro means for continuous functions, but lie within negligible closed sets; and the same can happen with Baire measures (491Yh). It seems to be difficult, in general, to determine whether a topological probability space – even a compact Radon probability space – has an equidistributed sequence. In the proofs of 491D-491G I have tried to collect the principal techniques for showing that spaces do have equidistributed sequences. In the other direction, it is obviously impossible for a space to have an equidistributed sequence if every separable subspace is negligible (491Xl). For an example of a separable compact Hausdorff space with a Radon measure which does not have an equidistributed sequence, we seem to have to go deeper (491Yl).

491Z is a famous problem. It is not clear that it is a problem in measure theory, and there is no reason to suppose that any of the ideas of this treatise beyond 491Xg are relevant. I mention it because I think everyone should know that it is there.

Version of 30.5.16

492 Combinatorial concentration of measure

'Concentration of measure' takes its most dramatic forms in the geometrically defined notions of concentration explored in §476. But the phenomenon is observable in many other contexts, if we can devise the right abstract geometries to capture it. In this section I present one of Talagrand's theorems on the concentration of measure in product spaces, using the Hamming metric (492D), and Maurey's theorem on concentration of measure in permutation groups (492H).

492A Lemma Let (X,Σ,μ) be a totally finite measure space, $\alpha<\beta$ in $\mathbb{R}$, $\phi:[\alpha,\beta]\to\mathbb{R}$ a convex function, and $f:X\to[\alpha,\beta]$ a Σ-measurable function. Then

$$\int\phi(f(x))\mu(dx)\le\frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\int fd\mu+\frac{\beta\phi(\alpha)-\alpha\phi(\beta)}{\beta-\alpha}\mu X.$$

proof If $t\in[\alpha,\beta]$ then $t=\frac{t-\alpha}{\beta-\alpha}\beta+\frac{\beta-t}{\beta-\alpha}\alpha$, so

$$\phi(t)\le\frac{t-\alpha}{\beta-\alpha}\phi(\beta)+\frac{\beta-t}{\beta-\alpha}\phi(\alpha)=\frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}t+\frac{\beta\phi(\alpha)-\alpha\phi(\beta)}{\beta-\alpha}.$$

Accordingly

$$\phi(f(x))\le\frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}f(x)+\frac{\beta\phi(\alpha)-\alpha\phi(\beta)}{\beta-\alpha}$$

for every $x\in X$; integrating with respect to x, we have the result.

492B Corollary Let (X,Σ,μ) be a probability space and $f:X\to[\alpha,1]$ a measurable function, where $0<\alpha\le1$. Then $\int\frac{1}{f}d\mu\cdot\int fd\mu\le\frac{(1+\alpha)^2}{4\alpha}$.

proof Set $\gamma=\int fd\mu$, so that $\alpha\le\gamma\le1$. By 492A, with $\phi(t)=\frac{1}{t}$,

$$\int\frac{1}{f}d\mu\le\frac{\gamma}{1-\alpha}(1-\frac{1}{\alpha})+\frac{1}{1-\alpha}(\frac{1}{\alpha}-\alpha)=\frac{1+\alpha-\gamma}{\alpha}.$$

Now $\frac{1+\alpha-\gamma}{\alpha}\cdot\gamma$ takes its maximum value $\frac{(1+\alpha)^2}{4\alpha}$ when $\gamma=\frac{1+\alpha}{2}$, so this is also a bound for $\int\frac{1}{f}\int f$.

492C Lemma $\frac{1}{2}(1+\cosh t) \le e^{t^2/4}$ for every $t \in \mathbb{R}$.

proof For $k \ge 1$, $4^k k! \le 2(2k)!$ (induce on k), so

$$1+\cosh t = 2+\sum_{k=1}^{\infty}\frac{t^{2k}}{(2k)!} \le 2+\sum_{k=1}^{\infty}\frac{2t^{2k}}{4^k k!} = 2e^{t^2/4}.$$

492D Theorem (Talagrand 95) Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i<n}$ be a non-empty finite family of probability spaces with product (X,Λ,λ). Let ρ be the **normalized Hamming metric** on X defined by setting $\rho(x,y)=\frac{1}{n}\#(\{i : i<n,\, x(i)\ne y(i)\})$ for x, $y \in X$. If $W \in \Lambda$ and $\lambda W > 0$, then

$$\overline{\int} e^{t\rho(x,W)}\lambda(dx) \le \frac{1}{\lambda W}e^{t^2/4n}$$

for every $t \ge 0$.

proof The formulae below will go much more smoothly if we work with the simple Hamming metric $\sigma(x,y)=\#(\{i : x(i)\ne y(i)\})$ instead of ρ. In this case, we can make sense of the case $n=0$, and this will be useful. In terms of σ, our target is to prove that if $W\in\Lambda$ and $\lambda W>0$, then

$$\overline{\int} e^{t\sigma(x,W)}\lambda(dx) \le \frac{1}{\lambda W}e^{nt^2/4}$$

for every $t \ge 0$.

(a) To begin with, suppose that every $X_i = Z = \{0,1\}^{\mathbb{N}}$, every μ_i is a Borel measure, and W is compact. Note that in this case λ is a Radon measure (because the X_i are compact and metrizable), and

$$\{x : \sigma(x,W)\le m\} = \bigcup_{I\subseteq n,\#(I)\le m}\{x : \exists\, y\in W,\, x{\restriction}n\setminus I = y{\restriction}n\setminus I\}$$

is compact for every m, so the function $x\mapsto\sigma(x,W)$ is measurable.

Induce on n. If $n=0$ we must have $W=X=\{\emptyset\}$ and $\sigma(x,W)=0$ for every x, so the result is trivial. For the inductive step to $n+1$, we have $W\subseteq X\times X_n$, where $X=\prod_{i<n}X_i$, and we are looking at $\iint e^{t\sigma((x,\xi),W)}\lambda(dx)\mu_n(d\xi)$. Now, setting $V_\xi=\{x : (x,\xi)\in W\}$ for $\xi\in X_n$,

$$V=\bigcup_{\xi\in X_n}V_\xi=\{x : \exists\,\xi\in X_n,\,(x,\xi)\in W\},$$

we have

$$\sigma((x,\xi),W)\le\min(\sigma(x,V_\xi),1+\sigma(x,V))$$

for all x and ξ, counting $\sigma(x,\emptyset)$ as ∞ if V_ξ is empty. So, for any $\xi\in X_n$,

$$\begin{aligned}\int e^{t\sigma((x,\xi),W)}\lambda(dx) &\le \min\Bigl(\int e^{t\sigma(x,V_\xi)}\lambda(dx), e^t\int e^{t\sigma(x,V)}\lambda(dx)\Bigr)\\ &\le e^{nt^2/4}\min\Bigl(\frac{1}{\lambda V_\xi},\frac{e^t}{\lambda V}\Bigr)\end{aligned}$$

by the inductive hypothesis, counting $\min(\frac{1}{0},\frac{e^t}{\lambda V})$ as $\frac{e^t}{\lambda V}$.

It follows that if we set $f(\xi)=\max(e^{-t},\frac{\lambda V_\xi}{\lambda V})$ for $\xi\in X_n$,

$$(\lambda\times\mu_n)(W)\cdot\int e^{t\sigma((x,\xi),W)}\lambda(dx)\mu_n(d\xi)$$

$$\le\int\lambda V_\xi\mu_n(d\xi)\cdot e^{nt^2/4}\int\min(\frac{1}{\lambda V_\xi},\frac{e^t}{\lambda V})\mu_n(d\xi)$$

$$=e^{nt^2/4}\int\frac{\lambda V_\xi}{\lambda V}\mu_n(d\xi)\cdot\int\min(e^t,\frac{\lambda V}{\lambda V_\xi})\mu_n(d\xi)$$

$$\le e^{nt^2/4}\int f(\xi)\mu_n(d\xi)\cdot\int\frac{1}{f(\xi)}\mu_n(d\xi)$$

$$\le e^{nt^2/4}\cdot\frac{(1+e^{-t})^2}{4e^{-t}}$$

(492B)

$$=\frac{1}{2}e^{nt^2/4}(1+\cosh t)\le e^{(n+1)t^2/4}$$

by 492C, and

$$\int e^{t\sigma((x,\xi),W)}\lambda(dx)\mu_n(d\xi)\le\frac{1}{(\lambda\times\mu_n)(W)}e^{(n+1)t^2/4},$$

so the induction continues.

(b) Now turn to the general case. If $W\in\Lambda$, there is a $W_1\subseteq W$ such that $W_1\in\widehat{\bigotimes}_{i<n}\Sigma_i$ and $\lambda W_1=\lambda W$ (251Wf). There must be countably-generated σ-subalgebras Σ'_i of Σ_i such that $W_1\in\widehat{\bigotimes}_{i<n}\Sigma'_i$. For each $i<n$, let $\langle E_{ik}\rangle_{k\in\mathbb{N}}$ be a sequence in Σ_i generating Σ'_i, and let $h_i:X_i\to Z$ be the corresponding Marczewski functional, so that $h_i(\xi)=\langle\chi E_{ik}(\xi)\rangle_{k\in\mathbb{N}}$ for $\xi\in X_i$. Let μ'_i be the Borel measure on Z defined by setting $\mu'_iF=\mu_ih_i^{-1}[F]$ for every Borel set $F\subseteq Z$, and let ν be the product of the measures μ'_i on $Y=Z^n$. If we set $h(x)=\langle h_i(x(i))\rangle_{i<n}$ for $x\in X$, then $h:X\to Y$ is inverse-measure-preserving for λ and ν (254H). Moreover, by the choice of the E_{ik}, $W_1=h^{-1}[V]$ for some Borel set $V\subseteq Y$.

Because Y is a compact metrizable space, ν is the completion of a Borel measure and is a Radon measure (433Cb). For each $I\subseteq n$, write ν_I for the product measure on Z^I, and set

$$V_I=\{u:u\in Z^{n\setminus I},\ \nu_I\{v:v\in Z^I,\ (u,v)\in V\}>0\},$$

$$V'_I=\{y:y\in Y,\ y{\restriction}n\setminus I\in V_I\}.$$

Then $\nu(V\setminus V'_I)=0$ for every $I\subseteq n$, so if we set $V'=\bigcap_{I\subseteq n}V'_I$ then $\nu V'=\nu V$. (Of course $V'\subseteq V'_\emptyset=V$.)

Take any $\gamma\in\,]0,\lambda W[\,=\,]0,\nu V'[$. Let $K\subseteq V'$ be a compact set such that $\nu K\ge\gamma$. Set $g(y)=e^{t\sigma(y,K)}$ for $y\in Y$, where I write σ for the Hamming metric on Y (regarded as a product of n factor spaces). Then $g:Y\to\mathbb{R}$ is Borel measurable and $gh:X\to\mathbb{R}$ is Λ-measurable. Also, for any $x\in X$, $\sigma(x,W)\le\sigma(h(x),K)$. **P** Take $y\in K$ such that $\sigma(h(x),y)=\sigma(h(x),K)$, and set

$$I=\{i:h(x)(i)\ne y(i)\},\quad u=h(x){\restriction}n\setminus I=y{\restriction}n\setminus I.$$

Because $y\in V'$, $u\in V_I$ and $\nu_IH>0$, where $H=\{v:v\in Z^I,\ (u,v)\in V\}$. But if we write λ_I for the product measure on $\prod_{i\in I}X_i$, and $h_I(z)=\langle h_i(z(i))\rangle_{i\in I}$ for $z\in\prod_{i\in I}X_i$, then h_I is inverse-measure-preserving for λ_I and ν_I; in particular, $h_I^{-1}[H]$ is non-empty. This means that we can find an $x'\in X$ such that $x'{\restriction}n\setminus I=x{\restriction}n\setminus I$ and $x'{\restriction}I\in h_I^{-1}[H]$. In this case, $h(x')\in V$, so $x'\in W_1\subseteq W$, and

$$\sigma(x,W)\le\sigma(x,x')\le\#(I)=\sigma(h(x),K).\ \mathbf{Q}$$

Accordingly

$$e^{t\sigma(x,W)}\le e^{t\sigma(h(x),K)}=g(h(x))$$

for every $x\in X$, and

$$\overline{\int}e^{t\sigma(x,W)}\lambda(dx)\le\int gh\,d\lambda=\int g\,d\nu$$

(because g is ν-integrable and h is inverse-measure-preserving, see 235G)

$$\leq \frac{1}{\nu K}e^{nt^2/4}$$

(by (a))

$$\leq \frac{1}{\gamma}e^{nt^2/4}.$$

As γ is arbitrary,

$$\overline{\int} e^{t\sigma(x,W)}\lambda(dx) \leq \frac{1}{\lambda W}e^{nt^2/4},$$

as claimed.

492E Corollary Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i<n}$ be a non-empty finite family of probability spaces with product (X,Λ,λ).

(a) Let ρ be the normalized Hamming metric on X. If $W \in \Lambda$ and $\lambda W > 0$, then

$$\lambda^*\{x : \rho(x,W) \geq \gamma\} \leq \frac{1}{\lambda W}e^{-n\gamma^2}$$

for every $\gamma \geq 0$.

(b) If W, $W' \in \Lambda$ and $\gamma > 0$ are such that $e^{-n\gamma^2} < \lambda W \cdot \lambda W'$ then there are $x \in W$, $x' \in W'$ such that $\#(\{i : i < n, \, x(i) \neq x'(i)\}) < n\gamma$.

proof (a) Set $t = 2n\gamma$. By 492D, there is a measurable function $f : X \to \mathbb{R}$ such that $f(x) \geq e^{t\rho(x,W)}$ for every $x \in X$ and $\int f d\lambda \leq \frac{1}{\lambda W}e^{t^2/4n}$. So

$$\lambda^*\{x : \rho(x,W) \geq \gamma\} \leq \lambda\{x : f(x) \geq e^{t\gamma}\} \leq e^{-t\gamma}\int f d\lambda$$
$$\leq \frac{1}{\lambda W}e^{-t\gamma+t^2/4n} = \frac{1}{\lambda W}e^{-n\gamma^2}.$$

(b) By (a), $\lambda^*\{x : \rho(x,W') \geq \gamma\} < \lambda W$, so there must be an $x \in W$ such that $\rho(x,W') < \gamma$.

492F The next theorem concerns concentration of measure in permutation groups. I approach this through a general result about slowly-varying martingales (492G).

Lemma $e^t \leq t + e^{t^2}$ for every $t \in \mathbb{R}$.

proof If $t \geq 1$ then $t \leq t^2$ so

$$e^t \leq e^{t^2} \leq t + e^{t^2}.$$

If $0 \leq t \leq 1$ then

$$e^t = 1 + t + \sum_{n=2}^{\infty}\frac{t^n}{n!} \leq 1 + t + \sum_{k=1}^{\infty}\Big(\frac{1}{(2k)!} + \frac{1}{(2k+1)!}\Big)t^{2k}$$
$$\leq 1 + t + \sum_{k=1}^{\infty}\frac{t^{2k}}{k!} = t + e^{t^2}.$$

If $t \leq 0$ then

$$e^t = 1 + t + \sum_{n=2}^{\infty}\frac{t^n}{n!} \leq 1 + t + \sum_{k=1}^{\infty}\frac{t^{2k}}{(2k)!}$$
$$\leq 1 + t + \sum_{k=1}^{\infty}\frac{t^{2k}}{k!} = t + e^{t^2}.$$

492G Lemma (Milman & Schechtman 86) Let (X, Σ, μ) be a probability space, and $\langle f_n\rangle_{n\in\mathbb{N}}$ a martingale on X. Suppose that $f_n \in \mathcal{L}^\infty(\mu)$ for every n, and that $\alpha_n \ge \operatorname{ess\,sup}|f_n - f_{n-1}|$ for $n \ge 1$. Then for any $n \ge 1$ and $\gamma \ge 0$,

$$\Pr(f_n - f_0 \ge \gamma) \le \exp(-\gamma^2/4\sum_{i=1}^n \alpha_i^2),$$

at least if $\sum_{i=1}^n \alpha_i^2 > 0$.

proof Let $\langle \Sigma_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of σ-subalgebras of Σ to which $\langle f_n\rangle_{n\in\mathbb{N}}$ is adapted.

(a) I show first that

$$\mathbb{E}(\exp(\lambda(f_n - f_0))) \le \exp(\lambda^2\sum_{i=1}^n \alpha_i^2)$$

for any $n \ge 0$ and any $\lambda > 0$. **P** Induce on n. For $n = 0$, interpreting $\sum_{i=1}^0$ as 0, this is trivial. For the inductive step to $n+1$, set $g = f_n - f_{n-1}$ and let g_1, g_2 be conditional expectations of $\exp(\lambda g)$ and $\exp(\lambda^2 g^2)$ on Σ_{n-1}. Because $|g| \le \alpha_n$ a.e., $\exp(\lambda^2 g^2) \le \exp(\lambda^2\alpha_n^2)$ a.e. and $g_2 \le \exp(\lambda^2\alpha_n^2)$ a.e. Because $\exp(\lambda g) \le \lambda g + \exp(\lambda^2 g^2)$ wherever g is defined (492F), and 0 is a conditional expectation of g on Σ_{n-1}, $g_1 \le g_2 \le \exp(\lambda^2\alpha_n^2)$ a.e.

Now observe that $f_{n-1} - f_0$ is Σ_{n-1}-measurable, so that $\exp(\lambda(f_{n-1} - f_0)) \times g_1$ is a conditional expectation of $\exp(\lambda(f_{n-1} - f_0)) \times \exp(\lambda g) =_{\text{a.e.}} \exp(\lambda(f_n - f_0))$ on Σ_{n-1} (233Eg). Accordingly

$$\begin{aligned}\mathbb{E}(\exp(\lambda(f_n - f_0))) &= \mathbb{E}(\exp(\lambda(f_{n-1} - f_0)) \times g_1)\\ &\le \operatorname{ess\,sup}|g_1| \cdot \mathbb{E}(\exp(\lambda(f_{n-1} - f_0)))\\ &\le \exp(\lambda^2\alpha_n^2)\exp(\lambda^2\sum_{i=1}^{n-1}\alpha_i^2)\end{aligned}$$

(by the inductive hypothesis)

$$= \exp(\lambda^2\sum_{i=1}^{n}\alpha_i^2)$$

and the induction continues. **Q**

(b) Now take $n \ge 1$ such that $\sum_{i=1}^n \alpha_i^2 > 0$ and $\gamma \ge 0$. Set $\lambda = \gamma/2\sum_{i=1}^n \alpha_i^2$. Then

$$\begin{aligned}\Pr(f_n - f_0 \ge \gamma) &= \Pr(\exp(\lambda(f_n - f_0)) \ge e^{\lambda\gamma})\\ &\le e^{-\lambda\gamma}\mathbb{E}(\exp(\lambda(f_n - f_0))) \le e^{-\lambda\gamma}\exp(\lambda^2\sum_{i=1}^n\alpha_i^2)\end{aligned}$$

(by (a) above)

$$= e^{-\lambda\gamma/2} = \exp(-\gamma^2/4\sum_{i=1}^n\alpha_i^2)$$

as claimed.

492H Theorem (Maurey 79) Let X be a non-empty finite set and G the group of all permutations of X with its discrete topology. For π, $\phi \in G$ set

$$\rho(\pi, \phi) = \frac{\#(\{x : x\in X, \pi(x)\ne\phi(x)\})}{\#(X)}.$$

Then ρ is a metric on G. Give G its Haar probability measure, and let $f : G \to \mathbb{R}$ be a 1-Lipschitz function. Then

$$\Pr(f - \mathbb{E}(f) \ge \gamma) \le \exp(-\frac{\gamma^2\#(X)}{16})$$

for any $\gamma \ge 0$.

proof (a) We may suppose that $X = n = \{0, \ldots, n-1\}$ where $n = \#(X)$. For $m \le n$, $p : m \to n$ set $A_p = \{\pi : \pi \in G, \pi{\restriction}m = p\}$, and let Σ_m be the subalgebra of $\mathcal{P}G$ generated by $\{A_p : p \in n^m\}$, and f_m the (unique) conditional expectation of f on Σ_m. Then

$$f_m(\pi) = \frac{1}{\#(A_p)}\sum\nolimits_{\phi\in A_p} f(\phi)$$

whenever $\pi \in G$ and $p = \pi{\restriction}m$, while

$$\{\emptyset, G\} = \Sigma_0 \subseteq \Sigma_1 \subseteq \ldots \subseteq \Sigma_{n-1} = \Sigma_n = \mathcal{P}G.$$

(b) $|f_m(\pi) - f_{m-1}(\pi)| \le \dfrac{2}{n}$ whenever $1 \le m \le n$ and $\pi \in G$. **P** Set $p = \pi{\restriction}m-1$ and $k = \pi(m-1)$. Set $J = p[m-1] = \{\pi(i) : i < m-1\}$, and for $j \in n \setminus J$ let $p_j = p^\frown{<}j{>}$ be that function from m to n which extends p and takes the value j at $m-1$; let α_j be the common value of $f_m(\phi)$ for $\phi \in A_{p_j}$, so that $f_m(\pi) = \alpha_k$. Now, for each $j \in n \setminus (J \cup \{k\})$, the function $\phi \mapsto (\overleftarrow{j\,k})\phi$ is a bijection from A_{p_k} to A_{p_j}, where $(\overleftarrow{j\,k}) \in G$ is the transposition which exchanges j and k. But this means that

$$\begin{aligned}|\alpha_j - \alpha_k| &= \Bigl|\frac{1}{(n-m)!}\sum_{\phi\in A_{p_j}} f(\phi) - \frac{1}{(n-m)!}\sum_{\phi\in A_{p_k}} f(\phi)\Bigr| \\ &= \frac{1}{(n-m)!}\Bigl|\sum_{\phi\in A_{p_j}} f(\phi) - f((\overleftarrow{j\,k})\phi)\Bigr| \\ &\le \sup_{\phi\in A_{p_j}} \bigl|f(\phi) - f((\overleftarrow{j\,k})\phi)\bigr| \le \frac{2}{n}\end{aligned}$$

because f is 1-Lipschitz and $\rho(\phi, (\overleftarrow{j\,k})\phi) = \frac{2}{n}$ for every ϕ. And this is true for every $j \in n \setminus (J \cup \{k\})$.

Accordingly

$$\begin{aligned}|f_m(\pi) - f_{m-1}(\pi)| &= \Bigl|\alpha_k - \frac{1}{(n-m+1)!}\sum_{\phi\in A_p} f(\phi)\Bigr| = \Bigl|\alpha_k - \frac{1}{n-m+1}\sum_{j\in n\setminus J}\alpha_j\Bigr| \\ &\le \frac{1}{n-m+1}\sum_{j\in n\setminus J}|\alpha_k - \alpha_j| \le \frac{1}{n-m+1}\sum_{j\in n\setminus J}\frac{2}{n} = \frac{2}{n},\end{aligned}$$

as claimed. **Q**

(c) Now observe that $f = f_{n-1}$ and that f_0 is the constant function with value $\mathbb{E}(f)$, so that

$$\Pr(f - \mathbb{E}(f) \ge \gamma) = \Pr(f_{n-1} - f_0 \ge \gamma) \le \exp(-\gamma^2/4\sum_{i=1}^{n-1}(\frac{2}{n})^2)$$

(492G)

$$= \exp(-\frac{n\gamma^2}{16}),$$

which is what we were seeking to prove.

492I Corollary Let X be a non-empty finite set, with $\#(X) = n$, and G the group of all permutations of X. Let μ be the Haar probability measure of G when given its discrete topology. Suppose that $A \subseteq G$ and $\mu A \ge \frac{1}{2}$. Then

$$\mu\{\pi : \pi \in G, \exists\,\phi \in A, \#(\{x : x \in X, \pi(x) \ne \phi(x)\}) \le k\} \ge 1 - \exp(-\frac{k^2}{64n})$$

for every $k \le n$.

proof If $\exp(-\frac{k^2}{64n}) \ge \frac{1}{2}$, this is trivial, since the left-hand-side of the inequality is surely at least $\frac{1}{2}$. Otherwise, set $g(\pi) = \frac{1}{n}\min_{\phi\in A}\#(\{x : x \in X,\, \pi(x) \ne \phi(x)\})$ for $\pi \in G$, so that g is 1-Lipschitz for the metric ρ of 492H. Applying 492H to $f = -g$, we see that

$$\Pr(\mathbb{E}(g) - g \ge \frac{k}{2n}) \le \exp(-\frac{k^2}{64n}) < \frac{1}{2},$$

and there must be some $\pi \in A$ such that $\mathbb{E}(g) - g(\pi) < \frac{k}{2n}$, so that $\mathbb{E}(g) < \frac{k}{2n}$. This means that

$$\begin{aligned}\mu\{\pi : \pi \in G,\, \exists\, \phi \in A, \#(\{x : x \in X,\, \pi(x) \ne \phi(x)\}) \le k\}\\ = 1 - \mu\{\pi : \pi \in G,\, g(\pi) > \frac{k}{n}\}\\ \ge 1 - \Pr(g - \mathbb{E}(g) \ge \frac{k}{2n}) \ge 1 - \exp(-\frac{k^2}{64n}),\end{aligned}$$

applying 492H to g itself.

492X Basic exercises (a) Let (X, Σ, μ) be a probability space, and $\langle f_n\rangle_{n\in\mathbb{N}}$ a martingale on X. Suppose that $f_n \in \mathcal{L}^\infty(\mu)$ for every n, and that $\sigma = \sqrt{\sum_{n=1}^\infty \alpha_n^2}$ is finite and not zero, where $\alpha_n = \operatorname{ess\,sup}|f_n - f_{n-1}|$ for $n \ge 1$. Show that $f = \lim_{n\to\infty} f_n$ is defined a.e., and that $\Pr(f - f_0 \ge \gamma) \le \exp(-\gamma^2/4\sigma^2)$ for every $\gamma \ge 0$. (*Hint*: show first that $\|f_n\|_1 \le \|f_n\|_2 \le \sigma + \|f_0\|_2$ for every n, so that we can apply 275G.)

(b) Let (X, ρ) be a metric space and μ a topological probability measure on X. Suppose that γ, $\epsilon > 0$ are such that $\Pr(f - \mathbb{E}(f) \ge \gamma) \le \epsilon$ whenever $f : X \to [0, 1]$ is 1-Lipschitz. Show that if $\mu F \ge \frac{1}{2}$ then $\mu\{x : \rho(x, F) \ge 2\gamma\} \le \epsilon$.

(c) Let (X, ρ) be a metric space and μ a topological probability measure on X. Suppose that γ, $\epsilon > 0$ are such that $\mu\{x : \rho(x, F) > \gamma\} \le \epsilon$ whenever $\mu F \ge \frac{1}{2}$. Show that if $f : X \to [0, 1]$ is a 1-Lipschitz function then $\Pr(f - \mathbb{E}(f) > 2\gamma + \epsilon) \le \epsilon$.

(d) Use 492G to show that if $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i<n}$ is a non-empty finite family of probability spaces with product (X, Λ, λ), and X is given its normalized Hamming metric, and $f \in \mathcal{L}^\infty(\lambda)$ is 1-Lipschitz, then $\Pr(f - \mathbb{E}(f) \ge \gamma) \le e^{-n\gamma^2/4}$ for every $\gamma \ge 0$. (*Hint*: if $\Sigma_k \subseteq \Lambda$ is the σ-algebra of subsets of X determined by coordinates in k, and f_k is a conditional expectation of f on Σ_k, then $\operatorname{ess\,sup}|f_{k+1} - f_k| \le \frac{1}{n}$.)

492 Notes and comments In metric spaces, we can say that a probability measure is 'concentrated' if every Lipschitz function f is almost constant in the sense that, for some α, the sets $\{x : |f(x) - \alpha| \ge \gamma\}$ have small measure. What is astonishing is that this does not mean that the measure itself is concentrated on a small set. In 492H, the measure is the Haar probability measure, spread as evenly as it well could be. Of course, when I say that $\{x : |f(x) - \alpha| \ge \gamma\}$ has 'small' measure, I have to let some other parameter – in 492H, the size of X – vary, while γ itself is fixed. Also the shapes of the formulae depend on which normalizations we choose (observe the effect of moving from ρ to σ in the proof of 492D). But the value of 492H is that it gives a strong bound which is independent of the particular function f, provided that it is 1-Lipschitz. This kind of concentration of measure can be described either in terms of the variation of Lipschitz functions from their means or in terms of the measures of neighbourhoods of sets of measure $\frac{1}{2}$ (492Xb-492Xc). The latter, in a more abstract context, is what is described by the concentration functions of measures on uniform spaces; there is an example of this in 493C.

The martingale method can be used to prove a version of 492E (492Xd). The method of 492D gives a better exponent ($e^{-n\gamma^2}$ in place of $e^{-n\gamma^2/4}$) and also information of a slightly different kind, in that it can be applied directly to sets W of small measure, at least provided that $\gamma > \frac{1}{\sqrt{n}}$ in 492E. We also need a little more measure theory here, since sets which are measured by product measures can be geometrically highly irregular, and our Lipschitz functions $x \mapsto \rho(x, W)$ need not be measurable.

In the proof of 492G we have an interesting application of the idea of 'martingale'. The inequality here is quite different from the standard martingale inequalities like 275D or 275F or 275Yd-275Ye. It gives a very strong inequality concerning the difference $f_n - f_0$, at the cost of correspondingly strong hypotheses on the differences $f_i - f_{i-1}$; but since we need control of $\sum_i \operatorname{ess\,sup}|f_i - f_{i-1}|^2$, not of $\sum_i \operatorname{ess\,sup}|f_i - f_{i-1}|$, there is scope for applications like 492H. What the inequality tells us is that most of the time the differences $f_i - f_{i-1}$ cancel out, just as in the Central Limit Theorem, and that once again we have a vaguely Gaussian sum $f_n - f_0$.

Concentration of measure, in many forms, has been studied intensively in the context of the geometry of normed spaces, as in Milman & Schechtman 86, from which 492F-492I are taken.

Version of 4.1.13

493 Extremely amenable groups

A natural variation on the idea of 'amenable group' (§449) is the concept of 'extremely amenable' group (493A). Expectedly, most of the ideas of 449C-449E can be applied to extremely amenable groups (493B); unexpectedly, we find not only that there are interesting extremely amenable groups, but that we need some of the central ideas of measure theory to study them. I give a criterion for extreme amenability of a group in terms of the existence of suitably concentrated measures (493C) before turning to three examples: measure algebras under symmetric difference (493D), L^0 spaces (493E) and isometry groups of spheres in infinite-dimensional Hilbert spaces (493G).

493A Definition Let G be a topological group. Then G is **extremely amenable** or has the **fixed point on compacta property** if every continuous action of G on a compact Hausdorff space has a fixed point.

493B Proposition (a) Let G and H be topological groups such that there is a continuous surjective homomorphism from G onto H. If G is extremely amenable, so is H.

(b) Let G be a topological group and suppose that there is a dense subset A of G such that every finite subset of A is included in an extremely amenable subgroup of G. Then G is extremely amenable.

(c) Let G be a topological group with an extremely amenable normal subgroup H such that G/H is extremely amenable. Then G is extremely amenable.

(d) The product of any family of extremely amenable topological groups is extremely amenable.

(e) Let G be a topological group. Then G is extremely amenable iff there is a point in the greatest ambit Z of G (definition: 449D) which is fixed by the action of G on Z.

(f) Let G be an extremely amenable topological group. Then every dense subgroup of G is extremely amenable.

proof We can use the same arguments as in 449C-449F, with some simplifications.

(a) As in 449Ca, let $\phi : G \to H$ be a continuous surjective homomorphism, X a non-empty compact Hausdorff space and $\bullet : H \times X \to X$ a continuous action. Let $\bullet_1$ be the continuous action of G on X defined by the formula $a\bullet_1 x = \phi(a)\bullet x$. Then any fixed point for $\bullet_1$ is a fixed point for $\bullet$.

(b) Let X be a non-empty compact Hausdorff space and $\bullet$ a continuous action of G on X. For $I \in [A]^{<\omega}$ let H_I be an extremely amenable subgroup of G including I. The restriction of the action to $H_I \times X$ is a continuous action of H_I on X, so

$$\{x : a\bullet x = x \text{ for every } a \in I\} \supseteq \{x : a\bullet x = x \text{ for every } a \in H_I\}$$

is closed and non-empty. Because X is compact, there is an $x \in X$ such that $a\bullet x = x$ for every $a \in A$. Now $\{a : a\bullet x = x\}$ includes the dense set A, so is the whole of G, and x is fixed under the action of G. As X and $\bullet$ are arbitrary, G is extremely amenable.

(c) Let X be a compact Hausdorff space and $\bullet$ a continuous action of G on X. Set $Q = \{x : x \in X,\ a\bullet x = x$ for every $a \in H\}$; then Q is a closed subset of X and, because H is extremely amenable, is non-empty. Next, $b\bullet x \in Q$ for every $x \in Q$ and $b \in G$. **P** If $a \in H$, then $b^{-1}ab \in H$ and

$$a\bullet(b\bullet x) = (ab)\bullet x = (bb^{-1}ab)\bullet x = b\bullet((b^{-1}ab)\bullet x) = b\bullet x.$$

As a is arbitrary, $b{\bullet}x \in Q$. **Q** Accordingly we have a continuous action of G on the compact Hausdorff space Q.

If $b \in G$, $a \in H$ and $x \in Q$, then $(ba){\bullet}x = b{\bullet}x$. So we have an action of G/H on Q defined by saying that $b^\bullet{\bullet}x = b{\bullet}x$ for every $b \in G$ and $x \in Q$, and this is continuous for the quotient topology on G/H, as in the proof of 449Cc. Because G/H is extremely amenable, there is a point x of Q which is fixed under the action of G/H. So $b{\bullet}x = b^\bullet{\bullet}x = x$ for every $b \in G$, and x is fixed under the action of G. As X and $\bullet$ are arbitrary, G is extremely amenable.

(d) By (c), the product of two extremely amenable topological groups is extremely amenable, since each can be regarded as a normal subgroup of the product. It follows that the product of finitely many extremely amenable topological groups is extremely amenable. Now let $\langle G_i\rangle_{i\in I}$ be any family of extremely amenable topological groups with product G. For finite $J \subseteq I$ let H_J be the set of those $a \in G$ such that $a(i)$ is the identity in G_i for every $i \in I \setminus J$. Then H_J is isomorphic (as topological group) to $\prod_{i\in J} G_i$, so is extremely amenable. Since $\{H_J : J \in [I]^{<\omega}\}$ is an upwards-directed family of subgroups of G with dense union, (b) tells us that G is extremely amenable.

(e) Repeat the arguments of 449E(i)$\Leftrightarrow$(ii), noting that if $z_0 \in Z$ is a fixed point under the action of G on Z, then its images under the canonical maps ϕ of 449Dd will be fixed for other actions.

(f) Again, the idea is to repeat the argument of 449F(a-ii). As there, let H be a dense subgroup of G, U the space of bounded real-valued functions on G which are uniformly continuous for the right uniformity, and V the space of bounded real-valued functions on H which are uniformly continuous for the right uniformity. As in 449F(a-ii), we have an extension operator $T : V \to U$ defined by saying that Tg is the unique continuous extension of g for every $g \in V$; and $b{\bullet_l}Tg = T(b{\bullet_l}g)$ for every $b \in H$ and $g \in V$. Now T is a Riesz homomorphism. So if $z \in Z$ is fixed by the action of G, that is, $z(a{\bullet_l}f) = z(f)$ for every $a \in G$ and $f \in U$, then $zT : V \to \mathbb{R}$ is a Riesz homomorphism, with $z(T\chi H) = 1$, and $z(T(b{\bullet_l}g)) = z(b{\bullet_l}Tg) = z(Tg)$ whenever $g \in V$ and $b \in H$. Thus zT is a fixed point of the greatest ambit of H, and H is extremely amenable.

493C Theorem Let G be a topological group and $\mathcal{B}$ its Borel σ-algebra. Suppose that for every $\epsilon > 0$, open neighbourhood V of the identity of G, finite set $I \subseteq G$ and finite family $\mathcal{E}$ of zero sets in G there is a finitely additive functional $\nu : \mathcal{B} \to [0,1]$ such that $\nu G = 1$ and

(i) $\nu(VF) \ge 1-\epsilon$ whenever $F \in \mathcal{E}$ and $\nu F \ge \frac{1}{2}$,

(ii) for every $a \in I$ there is a $b \in aV$ such that $|\nu(bF) - \nu F| \le \epsilon$ for every $F \in \mathcal{E}$.

Then G is extremely amenable.

proof (a) Write P for the set of finitely additive functionals $\nu : \mathcal{B} \to [0,1]$ such that $\nu G = 1$. If V is an open neighbourhood of the identity e of G, $\epsilon > 0$, $I \in [D]^{<\omega}$ and $\mathcal{E}$ is a finite family of zero sets in G, let $A(V,\epsilon,I,\mathcal{E})$ be the set of those $\nu \in P$ satisfying (i) and (ii) above. Our hypothesis is that none of these sets $A(V,\epsilon,I,\mathcal{E})$ are empty; since $A(V,\epsilon,I,\mathcal{E}) \subseteq A(V',\epsilon',I',\mathcal{E}')$ whenever $V \subseteq V'$, $\epsilon \le \epsilon'$, $I \supseteq I'$ and $\mathcal{E} \supseteq \mathcal{E}'$, there is an ultrafilter $\mathcal{F}$ on P containing all these sets.

Let U be the space of bounded real-valued functionals on G which are uniformly continuous for the right uniformity on G. If we identify $L^\infty(\mathcal{B})$ with the space of bounded Borel measurable real-valued functions on G (363H), then U is a Riesz subspace of $L^\infty(\mathcal{B})$. For each $\nu \in P$ we have a positive linear functional $⨍ d\nu : L^\infty(\mathcal{B}) \to \mathbb{R}$ (363L). For $f \in U$ set $z(f) = \lim_{\nu\to\mathcal{F}} ⨍ f d\nu$.

(b) $z : U \to \mathbb{R}$ is a Riesz homomorphism, and $z(\chi G) = 1$. **P** Of course z is a positive linear functional taking the value 1 at χG, just because all the integrals $⨍ d\nu$ are. Now suppose that $f_0, f_1 \in U$ and $f_0 \wedge f_1 = 0$. Take any $\epsilon > 0$. Then there is an open neighbourhood V of e such that $|f_i(x) - f_i(y)| \le \epsilon$ whenever $xy^{-1} \in V$ and $i \in \{0,1\}$. Set $F_i = \{x : f_i(x) = 0\}$, $E_i = VF_i$ for each i. Then $F_0 \cup F_1 = X$, so $\nu F_0 + \nu F_1 \ge 1$ for every $\nu \in P$, and there is a $j \in \{0,1\}$ such that $A_0 = \{\nu : \nu F_j \ge \frac{1}{2}\} \in \mathcal{F}$. Next,

$$A_1 = \{\nu\text{: if } \nu F_j \ge \frac{1}{2} \text{ then } \nu(VF_j) \ge 1-\epsilon\}$$

belongs to $\mathcal{F}$. Accordingly $\lim_{\nu\to\mathcal{F}} \nu E_j \ge 1-\epsilon$. As $f_j(x) \le \epsilon$ for every $x \in E_j$,

$$z(f_j) = \lim_{\nu\to\mathcal{F}} ⨍ f_j d\nu \le \epsilon(1 + \|f_j\|_\infty).$$

This shows that $\min(z(f_0), z(f_1)) \le \epsilon(1 + \|f_1\|_\infty + \|f_2\|_\infty)$. As ϵ is arbitrary, $\min(z(f_0), z(f_1)) = 0$; as f_0 and f_1 are arbitrary, z is a Riesz homomorphism (352G(iv)). **Q**

Thus z belongs to the greatest ambit Z of G.

(c) $\lim_{\nu\to\mathcal{F}} ⨍(a^{-1}\bullet_l f)d\nu = \lim_{\nu\to\mathcal{F}} ⨍ f d\nu$ for every non-negative $f \in U$ and $a \in G$. **P** Take any $\epsilon > 0$. Let V be an open neighbourhood of e such that $|f(x) - f(y)| \le \epsilon$ whenever $x \in Vy$; then

$$\|a^{-1}\bullet_l f - b^{-1}\bullet_l f\|_\infty = \sup_{x\in G} |f(ax) - f(bx)| \le \epsilon$$

whenever $b \in Va$. For $n \in \mathbb{N}$ set $F_n = \{x : x \in G,\ f(x) \ge n\epsilon\}$. Set $m = \lfloor \frac{1}{\epsilon}\|f\|_\infty \rfloor$, so that $F_n = \emptyset$ for every $n > m$. Set $\delta = \frac{1}{m+1}$,

$$\begin{aligned} A &= \{\nu :\ \text{there is a } b \in Va \text{ such that } |\nu(b^{-1}F_n) - \nu F_n| \le \delta \text{ for every } n \le m\} \\ &= \{\nu :\ \text{there is a } c \in a^{-1}V^{-1} \text{ such that } |\nu(cF_n) - \nu F_n| \le \delta \text{ for every } n \le m\} \in \mathcal{F}. \end{aligned}$$

Take any $\nu \in A$ and $b \in Va$ such that $|\nu(b^{-1}F_n) - \nu F_n| \le \delta$ for every $n \le m$. Then, setting $g = \sum_{n=1}^m \epsilon\chi F_n$, we have $g \in L^\infty(\mathcal{B})$ and $g \le f \le g + \epsilon\chi G$. Since $b^{-1}\bullet_l g$ (in the language of 4A5Cc) is just $\sum_{n=1}^m \epsilon\chi(b^{-1}F_n)$, we have

$$|⨍ a^{-1}\bullet_l f\, d\nu - ⨍ f\, d\nu| \le \epsilon + |⨍ b^{-1}\bullet_l f\, d\nu - ⨍ f\, d\nu| \le 3\epsilon + |⨍ b^{-1}\bullet_l g\, d\nu - ⨍ g\, d\nu|$$

(because $\|b^{-1}\bullet_l g - b^{-1}\bullet_l f\|_\infty = \|g - f\|_\infty \le \epsilon$)

$$\le 3\epsilon + \epsilon\sum_{n=1}^m |\nu F_n - \nu(b^{-1}F_n)| \le 3\epsilon + m\epsilon\delta \le 4\epsilon.$$

As $A \in \mathcal{F}$,

$$|\lim_{\nu\to\mathcal{F}} ⨍ a^{-1}\bullet_l f d\nu - ⨍ f d\nu| \le 5\epsilon;$$

as ϵ is arbitrary, $\lim_{\nu\to\mathcal{F}} ⨍ a^{-1}\bullet_l f\, d\nu = \lim_{\nu\to\mathcal{F}} ⨍ f d\nu$. **Q**

(d) Thus, for any $a \in G$,

$$(a\bullet z)(f) = z(a^{-1}\bullet_l f) = \lim_{\nu\to\mathcal{F}} ⨍ a^{-1}\bullet_l f\, d\nu = \lim_{\nu\to\mathcal{F}} ⨍ f\, d\nu = z(f)$$

for every non-negative $f \in U$ and therefore for every $f \in U$, and $a\bullet z = z$. So $z \in Z$ is fixed under the action of G on Z; by 493Ba, this is enough to ensure that G is extremely amenable.

493D I turn now to examples of extremely amenable groups. The first three are groups which we have already studied for other reasons.

Theorem Let $(\mathfrak{A}, \bar\mu)$ be an atomless measure algebra. Then $\mathfrak{A}$, with the group operation $\triangle$ and the measure-algebra topology (definition: 323A), is an extremely amenable group.

proof (a) To begin with let us suppose that $(\mathfrak{A}, \bar\mu)$ is a probability algebra; write σ for the measure metric of $\mathfrak{A}$, so that $\sigma(a, a') = \bar\mu(a \triangle a')$ for a, $a' \in \mathfrak{A}$. I seek to apply 493C.

(i) Let V be an open neighbourhood of 0 in $\mathfrak{A}$, $\epsilon \in]0, 3[$, $I \in [\mathfrak{A}]^{<\omega}$ and $\mathcal{E}$ a finite family of zero sets in $\mathfrak{A}$. Let $\gamma > 0$ be such that $V \supseteq \{a : \bar\mu a \le 2\gamma\}$. Let $\mathfrak{B}_0$ be the finite subalgebra of $\mathfrak{A}$ generated by I and B_0 the set of atoms in $\mathfrak{B}_0$. Set

$$t = \frac{1}{\gamma}\ln\frac{3}{\epsilon}, \quad n = \lceil \max(t^2, \sup_{b\in B_0}\frac{1}{\bar\mu b}) \rceil.$$

Because $\mathfrak{A}$ is atomless, we can split any member of $\mathfrak{A}\setminus\{0\}$ into two parts of equal measure (331C); if, starting from the disjoint set B_0, we successively split the largest elements until we have a disjoint set B with just n elements, then we shall have $\bar\mu b \le \dfrac{2}{n}$ for every $b \in B$. We have a natural identification between $\{0,1\}^B$ and the subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ generated by B, matching $x \in \{0,1\}^B$ with $f(x) = \sup\{b : b \in B,\ x(b) = 1\}$. Writing ρ for the normalized Hamming metric on $\{0,1\}^B$ (492D), we have $\sigma(f(x), f(y)) \le 2\rho(x, y)$ for all x, $y \in \{0,1\}^B$. **P** Set $J = \{b : b \in B,\ x(b) \ne y(b)\}$, so that

$$\sigma(f(x),f(y))=\bar\mu(f(x)\mathbin{\triangle}f(y))=\bar\mu(\sup J)=\sum_{b\in J}\bar\mu b\le\frac{2}{n}\#(J)=2\rho(x,y).\ \mathbf{Q}$$

(ii) Let ν_B be the usual measure on $\{0,1\}^B$ and set $\lambda E=\nu_Bf^{-1}[E]$ for every Borel set $E\subseteq\mathfrak{A}$. Then λ is a probability measure. Note that $f:\{0,1\}^B\to\mathfrak{B}$ is a group isomorphism if we give $\{0,1\}^B$ the addition $+_2$ corresponding to its identification with $\mathbb{Z}_2^B$, and $\mathfrak{B}$ the operation $\triangle$. Because ν_B is translation-invariant for $+_2$, its copy, the subspace measure $\lambda_{\mathfrak{B}}$ on the λ-conegligible finite set $\mathfrak{B}$, is translation-invariant for $\triangle$. But this means that $\lambda\{b\mathbin{\triangle}d:d\in F\}=\lambda F$ whenever $b\in\mathfrak{B}$ and $F\subseteq\mathfrak{B}$, and therefore that $\lambda\{b\mathbin{\triangle}d:d\in F\}=\lambda F$ whenever $b\in I$ and $F\in\mathcal{E}$. This shows that λ satisfies condition (ii) of 493C.

(iii) Now suppose that $F\in\mathcal{E}$ and that $\lambda F\ge\frac{1}{2}$. Set $W=f^{-1}[F]$, so that $\nu_BW\ge\frac{1}{2}$. By 492D,

$$\int e^{t\rho(x,W)}\nu_B(dx)\le 2e^{t^2/4n}\le 2e^{1/4}\le 3,$$

so

$$\nu_B\{x:\rho(x,W)\ge\gamma\}=\nu_B\{x:t\rho(x,W)\ge\ln\tfrac{3}{\epsilon}\}=\nu_B\{x:e^{t\rho(x,W)}\ge\tfrac{3}{\epsilon}\}\le\epsilon.$$

Accordingly

$$\begin{aligned}\lambda\{a\mathbin{\triangle}d:a\in V,\,d\in F\}&\ge\lambda\{a:\sigma(a,F)\le2\gamma\}=\nu_B\{x:\sigma(f(x),F)\le2\gamma\}\\&\ge\nu_B\{x:\sigma(f(x),f[W])\le2\gamma\}\ge\nu_B\{x:\rho(x,W)\le\gamma\}\end{aligned}$$

(because f is 2-Lipschitz)

$$\ge 1-\epsilon.$$

So λ also satisfies (i) of 493C.

(iv) Since V, ϵ, I and $\mathcal{E}$ are arbitrary, 493C tells us that $\mathfrak{A}$ is an extremely amenable group, at least when $(\mathfrak{A},\bar\mu)$ is an atomless probability algebra.

(b) For the general case, observe first that if $(\mathfrak{A},\bar\mu)$ is atomless and totally finite then $(\mathfrak{A},\triangle)$ is an extremely amenable group; this is trivial if $\mathfrak{A}=\{0\}$, and otherwise there is a probability measure on $\mathfrak{A}$ which induces the same topology, so we can apply (a). For a general atomless measure algebra $(\mathfrak{A},\bar\mu)$, set $\mathfrak{A}^f=\{c:c\in\mathfrak{A},\,\bar\mu c<\infty\}$ and for $c\in\mathfrak{A}^f$ let $\mathfrak{A}_c$ be the principal ideal generated by c. Then $\mathfrak{A}_c$ is a subgroup of $\mathfrak{A}$ and the measure-algebra topology of $\mathfrak{A}_c$, regarded as a measure algebra in itself, is the subspace topology induced by the measure-algebra topology of $\mathfrak{A}$. So $\{\mathfrak{A}_c:c\in\mathfrak{A}^f\}$ is an upwards-directed family of extremely amenable subgroups of $\mathfrak{A}$ with union which is dense in $\mathfrak{A}$, so $\mathfrak{A}$ itself is extremely amenable, by 493Bb. This completes the proof.

493E Theorem (PESTOV 02) Let (X,Σ,μ) be an atomless measure space. Then $L^0(\mu)$, with the group operation $+$ and the topology of convergence in measure, is an extremely amenable group.

proof It will simplify some of the formulae if we move at once to the space $L^0(\mathfrak{A})$, where $(\mathfrak{A},\bar\mu)$ is the measure algebra of (X,Σ,μ); for the identification of $L^0(\mathfrak{A})$ with $L^0(\mu)$ see 364Ic; for a note on convergence in measure in $L^0(\mathfrak{A})$, see 367L; of course $\mathfrak{A}$ is atomless if (X,Σ,μ) is (322Bg).

(a) I seek to prove that $S(\mathfrak{A})$, with the group operation of addition and the topology of convergence in measure, is extremely amenable. As in 493D, I start with the case in which $(\mathfrak{A},\bar\mu)$ is a probability algebra, and use 493C.

(i) Take an open neighbourhood V of 0 in $S(\mathfrak{A})$, an $\epsilon\in\,]0,3[$, a finite set $I\subseteq S(\mathfrak{A})$ and a finite family $\mathcal{E}$ of zero sets in $S(\mathfrak{A})$. Let $\gamma>0$ be such that $u\in V$ whenever $u\in S(\mathfrak{A})$ and $\bar\mu[\![u\ne0]\!]\le2\gamma$. Let $\mathfrak{B}_0$ be a finite subalgebra of $\mathfrak{A}$ such that I is included in the linear subspace of $S(\mathfrak{A})$ generated by $\{\chi b:b\in\mathfrak{B}_0\}$, and B_0 the set of atoms of $\mathfrak{B}_0$. As in the proof of 493D, set

$$t=\frac{1}{\gamma}\ln\frac{3}{\epsilon},\quad n=\lceil\max(t^2,\sup_{b\in B_0}\frac{1}{\bar\mu b})\rceil,$$

and let $B\subseteq\mathfrak{A}\setminus\{0\}$ be a partition of unity with n elements, refining B_0, such that $\bar\mu b\le\frac{2}{n}$ for every $b\in B$. We have a natural identification between $\mathbb{R}^B$ and the linear subspace of $S(\mathfrak{A})$ generated by $\{\chi b:b\in B\}$, matching $x \in \mathbb{R}^B$ with $f(x) = \sum_{b \in B} x(b)\chi b$, which is continuous if $\mathbb{R}^B$ is given its product topology. Writing ρ for the normalized Hamming metric on $\mathbb{R}^B$, we have

$$\bar{\mu}[\![f(x) \ne f(y)]\!] = \textstyle\sum_{x(b) \ne y(b)} \bar{\mu} b \le \frac{2}{n}\#(\{b : x(b) \ne y(b)\}) = 2\rho(x, y)$$

for all x, $y \in \mathbb{R}^B$.

(ii) Set $\beta = \sup_{v \in I} \|v\|_\infty$ (if $I = \emptyset$, take $\beta = 0$). Let $M > 0$ be so large that $(M + \beta)^n \le (1 + \frac{1}{2}\epsilon)M^n$. On $\mathbb{R}$, write μ_L for Lebesgue measure and μ_L' for the indefinite-integral measure over μ_L defined by the function $\frac{1}{2M}\chi[-M, M]$, so that $\mu_L' E = \frac{1}{2M}\mu_L(E \cap [-M, M])$ whenever $E \subseteq \mathbb{R}$ and $E \cap [-M, M]$ is Lebesgue measurable. Let λ, λ' be the product measures on $\mathbb{R}^B$ defined from μ_L and μ_L'. Let ν be the Borel probability measure on $S(\mathfrak{A})$ defined by setting $\nu F = \lambda' f^{-1}[F]$ for every Borel set $F \subseteq S(\mathfrak{A})$.

Now $|\nu(v + F) - \nu F| \le \epsilon$ for every $v \in I$ and Borel set $F \subseteq S(\mathfrak{A})$. **P** Because B refines B_0, v is expressible as $f(y)$ for some $y \in \mathbb{R}^B$; because $\|v\|_\infty \le \beta$, $|y(b)| \le \beta$ for every $b \in B$. Because $f : \mathbb{R}^B \to S(\mathfrak{A})$ is linear, $f^{-1}[v + F] = y + f^{-1}[F]$. Now

$$\begin{aligned}|\nu(v + F) - \nu F| &= |\lambda' f^{-1}[v + F] - \lambda' f^{-1}[F]| \\ &= \tfrac{1}{(2M)^n}\bigl|\lambda(f^{-1}[v + F] \cap [-M, M]^n) - \lambda(f^{-1}[F] \cap [-M, M]^n)\bigr|\end{aligned}$$

(use 253I, or otherwise)

$$\begin{aligned}&= \tfrac{1}{(2M)^n}\bigl|\lambda((y + f^{-1}[F]) \cap [-M, M]^n) - \lambda(f^{-1}[F] \cap [-M, M]^n)\bigr| \\ &= \tfrac{1}{(2M)^n}\bigl|\lambda(f^{-1}[F] \cap ([-M, M]^n - y)) - \lambda(f^{-1}[F] \cap [-M, M]^n)\bigr| \\ &\le \tfrac{1}{(2M)^n}\lambda(([-M, M]^n - y) \triangle [-M, M]^n) \\ &= \tfrac{2}{(2M)^n}\lambda(([-M, M]^n - y) \setminus [-M, M]^n) \\ &\le \tfrac{2}{(2M)^n}\lambda(([-M - \beta, M + \beta]^n \setminus [-M, M]^n) \\ &= \tfrac{2}{M^n}((M + \beta)^n - M^n) \le \epsilon. \quad \mathbf{Q}\end{aligned}$$

So ν satisfies (ii) of 493C.

(iii) Now suppose that $F \in \mathcal{E}$ and $\nu F \ge \frac{1}{2}$. Set $W = f^{-1}[F]$, so that $\lambda' W \ge \frac{1}{2}$. Just as in the proof of 493D, $\int e^{t\rho(x,W)}\lambda'(dx) \le 2e^{t^2/4n} \le 3$, so

$$\lambda'\{x : \rho(x, W) \ge \gamma\} = \lambda'\{x : e^{t\rho(x,W)} \ge \tfrac{3}{\epsilon}\} \le \epsilon,$$

and

$$\begin{aligned}\nu\{v + u : v \in V,\, u \in F\} &\ge \nu\{w : \exists\, u \in F,\, \bar{\mu}[\![u \ne w]\!] \le 2\gamma\} \\ &\ge \lambda'\{x : \rho(x, W) \le \gamma\} \ge 1 - \epsilon.\end{aligned}$$

So ν also satisfies (i) of 493C.

(iv) Since V, ϵ, I and $\mathcal{E}$ are arbitrary, 493C tells us that $S(\mathfrak{A})$ is an extremely amenable group, at least when $(\mathfrak{A}, \bar{\mu})$ is an atomless probability algebra.

(b) The rest of the argument is straightforward, as in 493D. First, $S(\mathfrak{A})$ is extremely amenable whenever $(\mathfrak{A}, \bar{\mu})$ is an atomless totally finite measure algebra. For a general atomless measure algebra $(\mathfrak{A}, \bar{\mu})$, set $\mathfrak{A}^f = \{c : \bar{\mu} c < \infty\}$. For each $c \in \mathfrak{A}^f$, let $\mathfrak{A}_c$ be the corresponding principal ideal of $\mathfrak{A}$. Then we can identify $S(\mathfrak{A}_c)$, as topological group, with the linear subspace of $L^0(\mathfrak{A})$ generated by $\{\chi a : a \in \mathfrak{A}_c\}$, and it is extremely amenable. Since $\{S(\mathfrak{A}_c) : c \in \mathfrak{A}^f\}$ is an upwards-directed family of extremely amenable subgroups of $L^0(\mathfrak{A})$ with dense union in $L^0(\mathfrak{A})$, $L^0(\mathfrak{A})$ itself is extremely amenable, by 493Bb, as before.

493F Returning to the ideas of §476, we find another remarkable example of an extremely amenable topological group. I recall the notation of 476I. Let X be a (real) inner product space. S_X will be the unit sphere $\{x : x \in X, \|x\| = 1\}$. Let H_X be the isometry group of S_X with its topology of pointwise convergence. When X is finite-dimensional, it is isomorphic, as inner product space, to $\mathbb{R}^r$, where $r = \dim X$. In this case S_X is compact, so (if $r \geq 1$) has a unique H_X-invariant Radon probability measure ν_X, which is strictly positive, and is a multiple of $(r-1)$-dimensional Hausdorff measure; also H_X is compact (441Gb), so has a unique Haar probability measure λ_X.

Lemma For any $m \in \mathbb{N}$ and any $\epsilon > 0$, there is an $r(m,\epsilon) \geq 1$ such that whenever X is a finite-dimensional inner product space over $\mathbb{R}$ of dimension at least $r(m,\epsilon)$, $x_0, \dots, x_{m-1} \in S_X$, Q_1, $Q_2 \subseteq H_X$ are closed sets and $\min(\lambda_X Q_1, \lambda_X Q_2) \geq \epsilon$, then there are $f_1 \in Q_1$, $f_2 \in Q_2$ such that $\|f_1(x_i) - f_2(x_i)\| \leq \epsilon$ for every $i < m$.

proof Induce on m. For $m = 0$, the result is trivial. For the inductive step to $m+1$, take $r(m+1,\epsilon) > r(m, \frac{1}{3}\epsilon)$ such that whenever $r(m+1,\epsilon) \leq \dim X < \omega$ and A_1, $A_2 \subseteq S_X$ and $\min(\nu_X^* A_1, \nu_X^* A_2) \geq \frac{1}{2}\epsilon$ then there are $x \in A_1$, $y \in A_2$ such that $\|x - y\| \leq \frac{1}{3}\epsilon$; this is possible by 476L.

Now take any inner product space X over $\mathbb{R}$ of finite dimension $r \geq r(m+1,\epsilon)$, closed sets Q_1, $Q_2 \subseteq H_X$ such that $\min(\lambda_X Q_1, \lambda_X Q_2) \geq \epsilon$, and $x_0, \dots, x_m \in S_X$. Let Y be the $(r-1)$-dimensional subspace $\{x : x \in X, (x|x_m) = 0\}$, so that $\dim Y \geq r(m, \frac{1}{3}\epsilon)$, and for $i < m$ let $y_i \in Y$ be a unit vector such that x_i is a linear combination of y_i and x_m. Set $H'_Y = \{f : f \in H_X, f(x_m) = x_m\}$; then $f \mapsto f{\restriction}S_Y$ is a topological group isomorphism from H'_Y to H_Y. **P** (i) If $f \in H'_Y$ and $x \in S_X$, then

$$x \in S_Y \iff (x|x_m) = 0 \iff (f(x)|f(x_m)) = 0 \iff f(x) \in S_Y,$$

so $f{\restriction}S_Y$ is a permutation of S_Y and belongs to H_Y. (ii) If $g \in H_Y$, we can define $f \in H'_Y$ by setting $f(\alpha x + \beta x_m) = \alpha g(x) + \beta x_m$ whenever $x \in S_Y$ and $\alpha^2 + \beta^2 = 1$, and $f{\restriction}S_Y = g$. (iii) Note that H'_Y is a closed subgroup of H_X, so in itself is a compact Hausdorff topological group. Since the map $f \mapsto f{\restriction}S_Y : H'_Y \to H_Y$ is a bijective continuous group homomorphism between compact Hausdorff topological groups, it is a topological group isomorphism. **Q**

Let λ'_Y be the Haar probability measure of H'_Y. Then $\lambda_X Q_1 = \int \lambda'_Y(H'_Y \cap f^{-1}Q_1)\lambda_X(df)$ (443Ue), so $\lambda_X Q'_1 \geq \frac{1}{2}\epsilon$, where $Q'_1 = \{f : \lambda'_Y(H'_Y \cap f^{-1}Q_1) \geq \frac{1}{2}\epsilon\}$. Similarly, setting $Q'_2 = \{f : \lambda'_Y(H'_Y \cap f^{-1}Q_2) \geq \frac{1}{2}\epsilon\}$, $\lambda_X Q'_2 \geq \frac{1}{2}\epsilon$. Next, setting $\theta(f) = f(x_m)$ for $f \in H_X$, $\lambda_X\theta^{-1}$ is an H_X-invariant Radon probability measure on S_X (443Ub), so must be equal to ν_X. Accordingly

$$\nu_X(\theta[Q'_j]) = \lambda_X(\theta^{-1}[\theta[Q'_j]]) \geq \lambda_X Q'_j \geq \frac{1}{2}\epsilon$$

for both j.

We chose $r(m+1,\epsilon)$ so large that we can be sure that there are $z_1 \in \theta[Q'_1]$, $z_2 \in \theta[Q'_2]$ such that $\|z_1 - z_2\| \leq \frac{1}{3}\epsilon$. Let $h_1 \in Q'_1$, $h_2 \in Q'_2$ be such that $h_1(x_m) = \theta(h_1) = z_1$ and $h_2(x_m) = z_2$. Let $h \in H_X$ be such that $h(z_1) = z_2$ and $\|h(x) - x\| \leq \frac{1}{3}\epsilon$ for every $x \in S_X$ (4A4Jg). Set $\tilde{h}_2 = hh_1$, so that $\tilde{h}_2(x_m) = z_2$ and $\|h_1(x) - \tilde{h}_2(x)\| \leq \frac{1}{3}\epsilon$ for every $x \in S_X$. Note that $\tilde{h}_2^{-1}h_2 \in H'_Y$, so that $\tilde{h}_2$ and h_2 belong to the same left coset of H'_Y, and

$$\lambda'_Y(H'_Y \cap \tilde{h}_2^{-1}Q_2) = \lambda'_Y(H'_Y \cap h_2^{-1}Q_2) \geq \frac{1}{2}\epsilon$$

by 443Qa.

At this point, recall that $\dim Y \geq r(m, \frac{1}{3}\epsilon)$, and that λ'_Y is a copy of λ_Y, the Haar probability measure on Y. So we have $g_1 \in H'_Y \cap h_1^{-1}Q_1$, $g_2 \in H'_Y \cap \tilde{h}_2^{-1}Q_2$ such that $\|g_1(y_i) - g_2(y_i)\| \leq \frac{1}{3}\epsilon$ for every $i < m$. We have $f_1 = h_1 g_1 \in Q_1$ and $f_2 = \tilde{h}_2 g_2 \in Q_2$. For any $i < m$,

$$\begin{aligned}\|f_1(y_i) - f_2(y_i)\| &\leq \|h_1 g_1(y_i) - h_1 g_2(y_i)\| + \|h_1 g_2(y_i) - \tilde{h}_2 g_2(y_i)\| \\ &\leq \|g_1(y_i) - g_2(y_i)\| + \frac{1}{3}\epsilon \leq \frac{2}{3}\epsilon.\end{aligned}$$

Also $g_1(x_m) = g_2(x_m) = x_m$, so

$$\|f_1(x_m) - f_2(x_m)\| = \|h_1(x_m) - \tilde{h}_2(x_m)\| \leq \frac{1}{3}\epsilon.$$

If $i<m$, then $x_i=(x_i|x_m)x_m+(x_i|y_i)y_i$, so $f_j(x_i)=(x_i|x_m)f_j(x_m)+(x_i|y_i)f_j(y_i)$ for both j (476J) and

$$\begin{aligned}\|f_1(x_i)-f_2(x_i)\| &\le \frac{1}{3}\epsilon|(x_i|x_m)|+\frac{2}{3}\epsilon|(x_i|y_i)| \\ &\le \sqrt{(\frac{1}{3}\epsilon)^2+(\frac{2}{3}\epsilon)^2}\sqrt{(x_i|x_m)^2+(x_i|y_i)^2}\le\epsilon.\end{aligned}$$

So f_1 and f_2 witness that the induction proceeds.

493G Theorem Let X be an infinite-dimensional inner product space over $\mathbb{R}$. Then the isometry group H_X of its unit sphere S_X, with its topology of pointwise convergence, is extremely amenable.

proof (a) Let $\mathcal{Y}$ be the family of finite-dimensional subspaces of X. For $Y\in\mathcal{Y}$, write $Y^\perp$ for the orthogonal complement of Y, so that $X=Y\oplus Y^\perp$ (4A4Jf). For $q\in H_Y$ define $\theta_Y(q):S_X\to S_X$ by saying that $\theta_Y(q)(\alpha y+\beta z)=\alpha q(y)+\beta z$ whenever $y\in S_Y$, $z\in S_{Y^\perp}$ and $\alpha^2+\beta^2=1$. Then $\theta_Y:H_Y\to H_X$ is a injective group homomorphism. Also it is continuous, because $q\mapsto\alpha q(y)+\beta z$ is continuous for all relevant α, β, y and z.

If Y, $W\in\mathcal{Y}$ and $Y\subseteq W$ then $\theta_Y[H_Y]\subseteq\theta_W[H_W]$. **P** For any $q\in H_Y$ we can define $q'\in H_W$ by saying that $q'(\alpha y+\beta x)=\alpha q(y)+\beta x$ whenever $y\in S_Y$, $x\in S_{W\cap Y^\perp}$ and $\alpha^2+\beta^2=1$. Now $\theta_Y(q)=\theta_W(q')\in\theta_W[H_W]$. **Q**

Set $G^*=\bigcup_{Y\in\mathcal{Y}}\theta_Y[H_Y]$, so that G^* is a subgroup of H_X.

(b) Let V be an open neighbourhood of the identity in G^* (with the subspace topology inherited from the topology of pointwise convergence on H_X), $\epsilon>0$ and $I\subseteq G^*$ a finite set. Then there is a Borel probability measure λ on G^* such that

(i) $\lambda(fQ)=\lambda Q$ for every $f\in I$ and every closed set $Q\subseteq G^*$,

(ii) $\lambda(VQ)\ge 1-\epsilon$ whenever $Q\subseteq G^*$ is closed and $\lambda Q\ge\frac{1}{2}$.

P We may suppose that $\epsilon\le\frac{1}{2}$. Let $J\in[S_X]^{<\omega}$ and $\delta>0$ be such that $f\in V$ whenever $f\in G^*$ and $\|f(x)-x\|\le\delta$ for every $x\in J$. We may suppose that J is non-empty; set $m=\#(J)$. Let $Y\in\mathcal{Y}$ be such that $J\subseteq Y$ and $I\subseteq\theta_Y[H_Y]$ and $\dim Y=r\ge r(m,\epsilon)$, as chosen in 493F. (This is where we need to know that X is infinite-dimensional.) Set $\lambda F=\lambda_Y\theta_Y^{-1}[F]$ for every Borel set $F\subseteq G^*$, where λ_Y is the Haar probability measure of H_Y, as before.

If $f\in I$ and $F\subseteq G^*$ is closed, then

$$\lambda(fF)=\lambda_Y\theta_Y^{-1}[fF]=\lambda_Y(\theta_Y^{-1}(f)\theta_Y^{-1}[F])=\lambda_Y\theta_Y^{-1}[F]=\lambda F.$$

So λ satisfies condition (i).

? Suppose, if possible, that $Q_1\subseteq G^*$ is a closed set such that $\lambda Q_1\ge\frac{1}{2}$ and $\lambda(VQ_1)<1-\epsilon$; set $Q_2=G^*\setminus VQ_1$. Then $\theta_Y^{-1}[Q_1]$ and $\theta_Y^{-1}[Q_2]$ are subsets of H_Y both of measure at least ϵ. Set $R_j=\{q:q\in H_Y,\,q^{-1}\in\theta_Y^{-1}[Q_j]\}$ for each j; because H_Y is compact, therefore unimodular,

$$\lambda_YR_j=\lambda_Y\theta_Y^{-1}[Q_j]=\lambda Q_j\ge\epsilon$$

for both j. Because $\dim Y\ge r(m,\epsilon)$, there are $q_1\in R_1$, $q_2\in R_2$ such that $\|q_1(x)-q_2(x)\|\le\epsilon$ for $x\in J$. Set $f=\theta_Y(q_2^{-1}q_1)$. If $x\in J$, then

$$\|f(x)-x\|=\|q_2^{-1}q_1(x)-x\|=\|q_1(x)-q_2(x)\|\le\epsilon.$$

As this is true whenever $x\in J$ and $f\in V$. On the other hand, $\theta_Y(q_1^{-1})\in Q_1$ and $\theta_Y(q_2^{-1})\in Q_2$ and $f\theta_Y(q_1^{-1})=\theta_Y(q_2^{-1})$, so $\theta_Y(q_2^{-1})\in VQ_1\cap Q_2$, which is impossible. **X**

Thus λ satisfies (ii). **Q**

(c) By 493C, G^* is extremely amenable. But G^* is dense in H_X. **P** If $f\in H_X$ and $I\subseteq S_X$ is finite and not empty, let Y_1 be the linear subspace of X generated by I, and let $(y_1,\dots,y_m)$ be an orthonormal basis of Y_1. Set $z_j=f(y_j)$ for each j, so that $(z_1,\dots,z_m)$ is orthonormal (476J); let Y be the linear subspace of X generated by $y_1,\dots,y_m,z_1,\dots,z_m$. Set $r=\dim Y$ and extend the orthonormal sets $(y_1,\dots,y_m)$ and $(z_1,\dots,z_m)$ to orthonormal bases $(y_1,\dots,y_r)$ and $(z_1,\dots,z_r)$ of Y. Then we have an isometric linear operator $T:Y\to Y$ defined by saying that $Ty_i=z_i$ for each i; set $q=T\restriction S_Y\in H_Y$. By 476J, $q(x)=f(x)$

for every $x \in I$, so $\theta_Y(q)$ agrees with f on I, while $\theta_Y(q) \in G^*$. As f and I are arbitrary, G^* is dense in G. **Q**

So 493Bb tells us that H_X is extremely amenable, and the proof is complete.

493H The following result shows why extremely amenable groups did not appear in Chapter 44.

Theorem (Veech 77) If G is a locally compact Hausdorff topological group with more than one element, it is not extremely amenable.

proof If G is compact, this is trivial, since the left action of G on itself has no fixed point; so let us assume henceforth that G is not compact.

(a) Let Z be the greatest ambit of G, $a \mapsto \hat{a} : G \to Z$ the canonical map, and U the space of bounded right-uniformly continuous real-valued functions on G. (I aim to show that the action of G on Z has no fixed point.) Take any $z^* \in Z$. Let V_0 be a compact neighbourhood of the identity e in G, and let $B_0 \subseteq G$ be a maximal set such that $V_0 b \cap V_0 c = \emptyset$ for all distinct b, $c \in B_0$. Then for any $a \in G$ there is a $b \in B_0$ such that $V_0 a \cap V_0 b \neq \emptyset$, that is, $a \in V_0^{-1}V_0B_0$. So if we set $Y_0 = \overline{\{\hat{b} : b \in B_0\}} \subseteq Z$, $\{a\bullet y : a \in V_0^{-1}V_0,\, y \in Y_0\}$ is a compact subset of Z including $\{\hat{a} : a \in G\}$, and is therefore the whole of Z (449Dc). Let $a_0 \in V_0^{-1}V_0$, $y_0 \in Y_0$ be such that $a_0\bullet y_0 = z^*$, and set $B_1 = a_0B_0$, $V_1 = a_0V_0a_0^{-1}$; then $z^* \in \overline{\{\hat{b} : b \in B_1\}}$ and $V_1b \cap V_1c = \emptyset$ for all distinct b, $c \in B_1$.

(b) Because V_1 is compact and G is not compact, there is an $a_1 \in G \setminus V_1$. Let $V_2 \subseteq V_1$ be a neighbourhood of e such that $a_1^{-1}V_2V_2^{-1}a_1 \subseteq V_1$. Then we can express B_1 as $D_0 \cup D_1 \cup D_2$ where $a_1D_i \cap V_2D_i = \emptyset$ for all i. **P** Consider $\{(b,c) : b,\, c \in B_1,\, a_1b \in V_2c\}$. Because $V_2c \cap V_2c' \subseteq V_1c \cap V_1c' = \emptyset$ for all distinct c, $c' \in B_1$, this is the graph of a function $h : D \to B_1$ for some $D \subseteq B_1$. **?** If h is not injective, we have distinct b, $c \in B_1$ and $d \in B_1$ such that a_1b and a_1c both belong to V_2d. But in this case b and c both belong to $a_1^{-1}V_2d$ and $bc^{-1} \in a_1^{-1}V_2dd^{-1}V_2^{-1}a_1 \subseteq V_1$ and $b \in V_1c$, which is impossible. **X** At the same time, if $b \in B_1$, then $a_1b \notin V_2b$ because $a_1 \notin V_2$, so $h(b) \neq b$ for every $b \in D$.

Let $D_0 \subseteq D$ be a maximal set such that $h[D_0] \cap D_0 = \emptyset$, and set $D_1 = h[D_0]$, $D_2 = B_1 \setminus (D_0 \cup D_1)$. Then $h[D_0] \cap D_0 = \emptyset$ by the choice of D_0; $h[D \cap D_1] \cap D_1 = \emptyset$ because h is injective and $D_1 \subseteq h[D \setminus D_1]$; and $h[D \cap D_2] \subseteq D_0$ because if $b \in D \cap D_2$ there must have been some reason why we did not put b into D_0, and it wasn't because $b \in h[D_0]$ or because $h(b) = b$. So $h[D_i] \cap D_i = \emptyset$ for all i, which is what was required. **Q**

(c) Since $z^* \in \overline{\{\hat{b} : b \in B_1\}}$, there must be some $j \leq 2$ such that $z^* \in \overline{\{\hat{b} : b \in D_j\}}$. Now recall that the right uniformity on G, like any uniformity, can be defined by some family of pseudometrics (4A2Ja). There is therefore a pseudometric ρ on G such that $W_\epsilon = \{(a,b) : a,\, b \in G,\, \rho(a,b) \leq \epsilon\}$ is a member of the right uniformity on G for every $\epsilon > 0$ and $W_1 \subseteq \{(a,b) : ab^{-1} \subseteq V_2\}$. If now we set

$$f(a) = \min(1, \rho(a, D_j)) = \min(1, \inf\{\rho(a,b) : b \in D_j\})$$

for $a \in G$, $f : G \to \mathbb{R}$ is bounded and uniformly continuous for the right uniformity, so belongs to U. On the other hand, if b, $c \in D_j$, then $a_1b \notin V_2c$, that is, $a_1bc^{-1} \notin V_2$ and $\rho(a_1b, c) > 1$; as c is arbitrary, $f(a_1b) = 1$.

(d) Now $\hat{b}(f) = f(b) = 0$ for every $b \in D_j$, so $z^*(f) = 0$. On the other hand, because $z \mapsto a_1\bullet z$ is continuous,

$$a_1\bullet z^* \in \overline{\{a_1\bullet\hat{b} : b \in D_j\}} = \overline{\{\widehat{a_1b} : b \in D_j\}},$$

so

$$(a_1\bullet z^*)(f) \geq \inf_{b \in D_j} \widehat{a_1b}(f) = \inf_{b \in D_j} f(a_1b) = 1,$$

and $a_1\bullet z^* \neq z^*$. As z^* is arbitrary, this shows that the action of G on Z has no fixed point, and G is not extremely amenable.

493X Basic exercises (a) Let G be a Hausdorff topological group, and $\widehat{G}$ its completion with respect to its bilateral uniformity. Show that G is extremely amenable iff $\widehat{G}$ is.

>(b) Let X be a set with more than one member and ρ the zero-one metric on X. Let G be the isometry group of X with the topology of pointwise convergence. Show that G is not extremely amenable. (*Hint*:

give X a total ordering $\le$, and let x, y be any two points of X. For $a \in G$ set $f(a) = 1$ if $a^{-1}(x) < a^{-1}(y)$, -1 otherwise. Show that, in the language of 449D, $f \in U$. Show that if $(\overleftarrow{x\,y})$ is the transposition exchanging x and y then $(\overleftarrow{x\,y}) \bullet_l f = -f$, while $|z(f)| = 1$ for every z in the greatest ambit of G.) (Compare 449Xh.)

(c) Show that under the conditions of 493C there is a finitely additive functional $\nu : \mathcal{B} \to [0,1]$ such that $\nu(aF) = \nu F$ for every $a \in G$ and every zero set $F \subseteq G$, while $\nu(VF) = 1$ whenever V is a neighbourhood of the identity, F is a zero set and $\nu F > \frac{1}{2}$.

(d) Prove 493G for infinite-dimensional inner product spaces over $\mathbb{C}$.

(e) Let X be any (real or complex) inner product space. Show that the isometry group of X, with its topology of pointwise convergence, is amenable. (*Hint*: 449Cd.)

(f) Let X be a separable Hilbert space. (i) Show that the isometry group G of its unit sphere, with its topology of pointwise convergence, is a Polish group. (ii) Show that if X is infinite-dimensional, then every countable discrete group can be embedded as a closed subgroup of G, so that G is an extremely amenable Polish group with a closed subgroup which is not amenable. (Cf. 449K.)

(g) If X is a (real or complex) Hilbert space, a bounded linear operator $T : X \to X$ is **unitary** if it is an invertible isometry. Show that the set of unitary operators on X, with its strong operator topology (3A5I), is an extremely amenable topological group.

(h) Let G be a topological group carrying Haar measures. Show that it is extremely amenable iff its topology is the indiscrete topology. (*Hint*: 443L.)

493Y Further exercises (a) For a Boolean algebra $\mathfrak{A}$ and a group G with identity e, write $S(\mathfrak{A}; G)$ for the set of partitions of unity $\langle a_g\rangle_{g\in G}$ in $\mathfrak{A}$ such that $\{g : a_g \ne 0\}$ is finite. For $\langle a_g\rangle_{g\in G}$, $\langle b_g\rangle_{g\in G} \in S(A)$, write $\langle a_g\rangle_{g\in G} \cdot \langle b_g\rangle_{g\in G} = \langle c_g\rangle_{g\in G}$ where $c_g = \sup\{a_h \cap b_{h^{-1}g} : h \in G\}$ for $g \in G$. (i) Show that under this operation $S(\mathfrak{A}; G)$ is a group. (ii) Show that if we write $h\chi a$ for the member $\langle a_g\rangle_{g\in G}$ of $S(\mathfrak{A}; G)$ such that $a_h = a$ and $a_g = 0$ for other $g \in G$, then $g\chi a \cdot h\chi b = (gh)\chi(a \cap b)$, and $S(\mathfrak{A}; G)$ is generated by $\{g\chi a : g \in G,\ a \in \mathfrak{A}\}$. (iii) Show that if $\mathfrak{A} = \Sigma/\mathcal{I}$ where Σ is an algebra of subsets of a set X and $\mathcal{I}$ is an ideal of Σ, then $S(\mathfrak{A}; G)$ can be identified with a space of equivalence classes in a suitable subgroup of G^X. (iv) Devise a universal mapping theorem for the construction $S(\mathfrak{A}; G)$ which matches 361F in the case $(G, \cdot) = (\mathbb{R}, +)$. (v) Now suppose that $(\mathfrak{A}, \bar\mu)$ is a measure algebra and that G is a topological group. Show that we have a topology on $S(\mathfrak{A}; G)$, making it a topological group, for which basic neighbourhoods of the identity $e\chi 1$ are of the form $V(c, \epsilon, U) = \{\langle a_g\rangle_{g\in G} : \bar\mu(c \cap \sup_{g\in G\setminus U} a_g) \le \epsilon\}$ with $\bar\mu c < \infty$, $\epsilon > 0$ and U a neighbourhood of the identity in G. (vi) Show that if G is an amenable locally compact Hausdorff group and $(\mathfrak{A}, \bar\mu)$ is an atomless measure algebra, then $S(\mathfrak{A}; G)$ is extremely amenable. (*Hint*: Pestov 02.) *(vi) Explore possible constructions of spaces $L^0(\mathfrak{A}; G)$. (See Hartman & Mycielski 58.)

493 Notes and comments In writing this section I have relied heavily on Pestov 99 and Pestov 02, where you may find many further examples of extremely amenable groups. It is a striking fact that while the theories of locally compact groups and extremely amenable groups are necessarily almost entirely separate (493H), both are dependent on measure theory. Curiously, what seems to have been the first non-trivial extremely amenable group to be described was found in the course of investigating the Control Measure Problem (Herer & Christensen 75).

The theory of locally compact groups has for seventy years now been a focal point for measure theory. Extremely amenable groups have not yet had such an influence. But they encourage us to look again at concentration-of-measure theorems, which are of the highest importance for quite separate reasons. In all the principal examples of this section, and again in the further example to come in §494, we need concentration of measure in product spaces (493D-493E and 494J), permutation groups (494I) or on spheres in Euclidean space (493G). 493D and 493E are special cases of a general result in Pestov 02 (493Ya(vi)) which itself extends an idea from Glasner 98. I note that 493D needs only concentration of measure in $\{0,1\}^I$, while 493E demands something rather closer to the full strength of Talagrand's theorem 492D.

I have expressed 493G as a theorem about the isometry groups of spheres in infinite-dimensional inner product spaces; of course these are isomorphic to the orthogonal groups of the whole spaces with their strong operator topologies (476Xd). Adapting the basic concentration-of-measure theorem 476K to the required lemma 493F involves an instructive application of ideas from §443.

Version of 17.5.13

494 Groups of measure-preserving automorphisms

I return to the study of automorphism groups of measure algebras, as in Chapter 38 of Volume 3, but this time with the intention of exploring possible topological group structures. Two topologies in particular have attracted interest, the 'weak' and 'uniform' topologies (494A). After a brief account of their basic properties (494B-494C) I begin work on the four main theorems. The first is the Halmos-Rokhlin theorem that if $(\mathfrak{A},\bar\mu)$ is the Lebesgue probability algebra the set of weakly mixing measure-preserving automorphisms of $\mathfrak{A}$ which are not mixing is comeager for the weak topology on $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ (494E). This depends on a striking characterization of weakly mixing automorphisms of a probability algebra in terms of eigenvectors of the corresponding operators on the complex Hilbert space $L^2_{\mathbb{C}}(\mathfrak{A},\bar\mu)$ (494D, 494Xj(i)). It turns out that there is an elegant example of a weakly mixing automorphism which is not mixing which can be described in terms of a Gaussian distribution of the kind introduced in §456, so I give it here (494F).

We need a couple of preliminary results on fixed-point subalgebras (494G-494H) before approaching the other three theorems. If $(\mathfrak{A},\bar\mu)$ is an atomless probability algebra, then $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is extremely amenable under its weak topology (494L); if $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is given its uniform topology, then every group homomorphism from $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ to a Polish group is continuous (494O); finally, there is no strictly increasing sequence of subgroups with union $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ (494Q). All these results have wide-ranging extensions to full subgroups of $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ subject to certain restrictions on the fixed-point subalgebras.

The work of this section will rely heavily on concepts and results from Volume 3 which have hardly been mentioned so far in the present volume. I hope that the cross-references, and the brief remarks in 494Ac-494Ad, will be adequate.

494A Definitions (Halmos 56) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ the group of measure-preserving automorphisms of $\mathfrak{A}$ (see §383). Write $\mathfrak{A}^f$ for $\{c : c\in\mathfrak{A},\ \bar\mu c<\infty\}$.

(a) I will say that the **weak topology** on $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is that generated by the pseudometrics $(\pi,\phi)\mapsto\bar\mu(\pi c\,\triangle\,\phi c)$ as c runs over $\mathfrak{A}^f$.

(b) I will say that the **uniform topology** on $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is that generated by the pseudometrics

$$(\pi,\phi)\mapsto\sup_{a\in\mathfrak{A}}\bar\mu(c\cap(\pi a\,\triangle\,\phi a))$$

as c runs over $\mathfrak{A}^f$.

(c) I recall some notation from Volume 3. For any Boolean algebra $\mathfrak{A}$ and $a\in\mathfrak{A}$, $\mathfrak{A}_a$ will be the principal ideal of $\mathfrak{A}$ generated by a (312D). I will generally use the symbol ι for the identity in the automorphism group $\mathrm{Aut}\,\mathfrak{A}$ of $\mathfrak{A}$. If $\pi\in\mathrm{Aut}\,\mathfrak{A}$ and $a\in\mathfrak{A}$, a supports π if $\pi d=d$ whenever $d\cap a=0$; the support

$$\operatorname{supp}\pi=\sup\{a\,\triangle\,\pi a : a\in\mathfrak{A}\}$$

of π is the smallest member of $\mathfrak{A}$ supporting π, if this is defined (381Bb, 381Ei). A subgroup G of $\mathrm{Aut}\,\mathfrak{A}$ is 'full' if $\phi\in G$ whenever $\phi\in\mathrm{Aut}\,\mathfrak{A}$ and there are $\langle a_i\rangle_{i\in I}$, $\langle\pi_i\rangle_{i\in I}$ such that $\langle a_i\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$ and $\pi_i\in G$ and $\phi d=\pi_i d$ whenever $i\in I$ and $d\subseteq a_i$ (381Be).

If a, $b\in\mathfrak{A}\setminus\{0\}$ are disjoint and $\pi\in\mathrm{Aut}\,\mathfrak{A}$ is such that $\pi a=b$, then $(\overleftarrow{a\,{}_\pi\,b})\in\mathrm{Aut}\,\mathfrak{A}$ will be the exchanging involution defined by saying that

$$\begin{aligned}(\overleftarrow{a\,{}_\pi\,b})(d)&=\pi d\text{ if }d\subseteq a,\\&=\pi^{-1}d\text{ if }d\subseteq b,\\&=d\text{ if }d\subseteq 1\setminus(a\cup b)\end{aligned}$$

(381R).

(d) In addition, I will repeatedly use the following ideas. Suppose that $(\mathfrak{A},\bar\mu)$ is a probability algebra (322Aa), $\mathfrak{C}$ is a closed subalgebra of $\mathfrak{A}$ (323I), and $L^\infty(\mathfrak{C})$ the M-space defined in §363. Then for each $a\in\mathfrak{A}$ we have a conditional expectation $u_a\in L^\infty(\mathfrak{C})$ of χa on $\mathfrak{C}$, so that $\int_c u_a=\bar\mu(a\cap c)$ for every $c\in\mathfrak{C}$ (365Q[1]).

If $\mathfrak{A}$ is relatively atomless over $\mathfrak{C}$ (331A), $a\in\mathfrak{A}$, and $v\in L^\infty(\mathfrak{C})$ is such that $0\le v\le u_a$, there is a $b\in\mathfrak{A}$ such that $b\subseteq a$ and $v=u_b$ (apply Maharam's lemma 331B to the functional $c\mapsto\int_c v:\mathfrak{C}\to[0,1]$). Elaborating on this, we see that if $\langle v_n\rangle_{n\in\mathbb{N}}$ is a sequence in $L^\infty(\mathfrak{C})^+$ and $\sum_{i=0}^n v_i\le u_a$ for every n, there are disjoint $b_0,\ldots\subseteq a$ such that $v_i=u_{b_i}$ for every i (choose the b_i inductively).

494B Proposition Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and give $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ its weak topology.

(a) $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is a topological group.

(b) $(\pi,a)\mapsto\pi a:\operatorname{Aut}_{\bar\mu}\mathfrak{A}\times\mathfrak{A}\to\mathfrak{A}$ is continuous for the weak topology on $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ and the measure-algebra topology on $\mathfrak{A}$.

(c) If $(\mathfrak{A},\bar\mu)$ is semi-finite (definition: 322Ad), $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is Hausdorff.

(d) If $(\mathfrak{A},\bar\mu)$ is localizable (definition: 322Ae), $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is complete under its bilateral uniformity.

(e) If $(\mathfrak{A},\bar\mu)$ is σ-finite (definition: 322Ac) and $\mathfrak{A}$ has countable Maharam type (definition: 331Fa), then $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is a Polish group.

proof (a) (Compare 441G.) Set $\rho_c(\pi,\phi)=\bar\mu(\pi c\,\triangle\,\phi c)$ for π, $\phi\in\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ and $c\in\mathfrak{A}^f$; it is elementary to check that ρ_c is always a pseudometric, so 494Aa is a proper definition of a topology. If π, $\phi\in\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ and $c\in\mathfrak{A}^f$, then for any π', $\phi'\in\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ we have

$$\begin{aligned}\rho_c(\pi'\phi',\pi\phi)&=\bar\mu(\pi'\phi'c\,\triangle\,\pi\phi c)\le\bar\mu(\pi'\phi'c\,\triangle\,\pi'\phi c)+\bar\mu(\pi'\phi c\,\triangle\,\pi\phi c)\\&=\bar\mu(\phi'c\,\triangle\,\phi c)+\rho_{\phi c}(\pi',\pi)=\rho_c(\phi',\phi)+\rho_{\phi c}(\pi',\pi);\end{aligned}$$

as c is arbitrary, $(\pi',\phi')\mapsto\pi'\phi'$ is continuous at (π,ϕ); thus multiplication is continuous. If $\phi\in\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ and $c\in\mathfrak{A}^f$, then for any $\pi\in\operatorname{Aut}_{\bar\mu}\mathfrak{A}$

$$\begin{aligned}\rho_c(\pi^{-1},\phi^{-1})&=\bar\mu(\pi^{-1}c\,\triangle\,\phi^{-1}c)=\bar\mu(c\,\triangle\,\pi\phi^{-1}c)\\&=\bar\mu(\phi\phi^{-1}c\,\triangle\,\pi\phi^{-1}c)=\rho_{\phi^{-1}c}(\pi,\phi);\end{aligned}$$

as c is arbitrary, $\pi\mapsto\pi^{-1}$ is continuous at ϕ; thus inversion is continuous and $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is a topological group.

(b) Suppose that $\phi\in\operatorname{Aut}_{\bar\mu}\mathfrak{A}$, $b\in\mathfrak{A}$ and that V is a neighbourhood of ϕb in $\mathfrak{A}$. Then there are $c\in\mathfrak{A}^f$ and $\epsilon>0$ such that V includes $\{d:\bar\mu(c\cap(d\,\triangle\,\phi b))\le 4\epsilon\}$. In this case, because inversion in $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is continuous,

$$U=\{\pi:\pi\in\operatorname{Aut}_{\bar\mu}\mathfrak{A},\ \bar\mu(\pi^{-1}c\,\triangle\,\phi^{-1}c)\le\epsilon,\ \bar\mu(\pi(\phi^{-1}c\cap b)\,\triangle\,\phi(\phi^{-1}c\cap b))\le\epsilon\},$$

$$V'=\{a:a\in\mathfrak{A},\ \bar\mu(\phi^{-1}c\cap(a\,\triangle\,b))\le\epsilon\}$$

are neighbourhoods of ϕ, b respectively. If $\pi\in U$ and $a\in V'$, then

$$\begin{aligned}\bar\mu(c\cap(\pi a\,\triangle\,\phi b))&\le\bar\mu(c\cap(\pi a\,\triangle\,\pi b))+\bar\mu(c\cap(\pi b\,\triangle\,\phi b))\\&=\bar\mu(\pi^{-1}c\cap(a\,\triangle\,b))+\bar\mu((c\cap\pi b)\,\triangle\,(c\cap\phi b))\\&\le\bar\mu(\pi^{-1}c\,\triangle\,\phi^{-1}c)+\bar\mu(\phi^{-1}c\cap(a\,\triangle\,b))+\bar\mu(\pi(\pi^{-1}c\cap b)\,\triangle\,\phi(\phi^{-1}c\cap b))\\&\le\epsilon+\epsilon+\bar\mu(\pi(\pi^{-1}c\cap b)\,\triangle\,\pi(\phi^{-1}c\cap b))\\&\qquad+\bar\mu(\pi(\phi^{-1}c\cap b)\,\triangle\,\phi(\phi^{-1}c\cap b))\\&\le 2\epsilon+\bar\mu((\pi^{-1}c\cap b)\,\triangle\,(\phi^{-1}c\cap b))+\epsilon\\&\le 3\epsilon+\bar\mu(\pi^{-1}c\,\triangle\,\phi^{-1}c)\le 4\epsilon,\end{aligned}$$

and $\pi a\in V$. As V, ϕ and b are arbitrary, $(\pi,a)\mapsto\pi a$ is continuous.

[1] Formerly 365R.

(c) Because $(\mathfrak{A},\bar\mu)$ is semi-finite, the measure-algebra topology on $\mathfrak{A}$ is Hausdorff (323Ga), so the product topology on $\mathfrak{A}^{\mathfrak{A}}$ is Hausdorff. Now $\pi\mapsto\langle\pi a\rangle_{a\in\mathfrak{A}}:\mathrm{Aut}_{\bar\mu}\mathfrak{A}\to\mathfrak{A}^{\mathfrak{A}}$ is injective, and by (b) it is continuous, so the topology of $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ must be Hausdorff.

(d)(i) For $c\in\mathfrak{A}^f$ and $\pi\in\mathrm{Aut}_{\bar\mu}\mathfrak{A}$, set $\theta_c(\pi)=\pi c$; then $\theta_c:\mathrm{Aut}_{\bar\mu}\mathfrak{A}\to\mathfrak{A}^f$ is uniformly continuous for the bilateral uniformity of $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ and the measure metric ρ of $\mathfrak{A}^f$ (323Ad). **P** We have $\rho(d,d')=\bar\mu(d\,\triangle\,d')$ for d, $d'\in\mathfrak{A}^f$. Let $\epsilon>0$; then $U=\{\pi:\rho_c(\pi,\iota)\le\epsilon\}$ is a neighbourhood of ι in $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$, and $W=\{(\pi,\phi):\phi^{-1}\pi\in U\}$ belongs to the bilateral uniformity. If $(\pi,\phi)\in W$, then

$$\rho(\theta_c(\pi),\theta_c(\phi))=\bar\mu(\pi c\,\triangle\,\phi c)=\bar\mu(\phi^{-1}\pi c\,\triangle\,c)=\rho_c(\phi^{-1}\pi,\iota)\le\epsilon;$$

as ϵ is arbitrary, θ_c is uniformly continuous. **Q**

(ii) Let $\mathcal{F}$ be a filter on $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ which is Cauchy for the bilateral uniformity on $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$. If $c\in\mathfrak{A}^f$, the image filter $\theta_c[[\mathcal{F}]]$ is Cauchy for the measure metric on $\mathfrak{A}^f$ (4A2Ji). Because $\mathfrak{A}^f$ is complete under its measure metric (323Mc), $\theta_c[[\mathcal{F}]]$ converges to ψ_0c say for the measure metric.

If c, $d\in\mathfrak{A}^f$ and $*$ is either of the Boolean operations $\cap$, $\triangle$, then

$$\psi_0(c*d)=\lim_{\pi\to\mathcal{F}}\pi(c*d)=\lim_{\pi\to\mathcal{F}}\pi c*\pi d=\lim_{\pi\to\mathcal{F}}\pi c*\lim_{\pi\to\mathcal{F}}\pi d$$

(because $*$ is continuous for the measure metric, see 323Ma)

$$=\psi_0c*\psi_0d.$$

So $\psi_0:\mathfrak{A}^f\to\mathfrak{A}^f$ is a ring homomorphism. Next, if $c\in\mathfrak{A}^f$, then

$$\bar\mu\psi_0c=\bar\mu(\lim_{\pi\to\mathcal{F}}\pi c)=\lim_{\pi\to\mathcal{F}}\bar\mu\pi c=\bar\mu c$$

because $\bar\mu:\mathfrak{A}^f\to[0,\infty[$ is continuous (323Mb).

Now recall that $\pi\mapsto\pi^{-1}:\mathrm{Aut}_{\bar\mu}\mathfrak{A}\to\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is uniformly continuous for the bilateral uniformity (4A5Hc). So if we set $\theta'_c(\pi)=\pi^{-1}c$ for $\pi\in\mathrm{Aut}_{\bar\mu}\mathfrak{A}$, we can apply the argument just above to θ' to find a ring homomorphism $\psi'_0:\mathfrak{A}^f\to\mathfrak{A}^f$ such that $\psi'_0c=\lim_{\pi\to\mathcal{F}}\pi^{-1}c$ for every $c\in\mathfrak{A}^f$. To relate ψ_0 and ψ'_0, we can argue as follows. Given $c\in\mathfrak{A}^f$,

$$\bar\mu(c\,\triangle\,\psi_0\psi'_0c)=\bar\mu(c\,\triangle\lim_{\phi\to\mathcal{F}}\phi\psi'_0c)=\lim_{\phi\to\mathcal{F}}\bar\mu(c\,\triangle\,\phi\psi'_0c)=\lim_{\phi\to\mathcal{F}}\bar\mu(\phi^{-1}c\,\triangle\,\psi'_0c)$$
$$=\bar\mu(\lim_{\phi\to\mathcal{F}}(\phi^{-1}c\,\triangle\,\psi'_0c))=\bar\mu((\lim_{\phi\to\mathcal{F}}\phi^{-1}c)\,\triangle\,\psi'_0c)=\bar\mu(\psi'_0c\,\triangle\,\psi'_0c)=0;$$

as c is arbitrary, $\psi_0\psi'_0$ is the identity on $\mathfrak{A}^f$. Similarly, $\psi'_0\psi_0$ is the identity on $\mathfrak{A}^f$. Thus ψ_0, ψ'_0 are the two halves of a measure-preserving ring isomorphism of $\mathfrak{A}^f$.

If we give $\mathfrak{A}$ its measure-algebra uniformity (323Ab), then ψ_0 is uniformly continuous for the induced uniformity on $\mathfrak{A}^f$. **P** If c, d_1, $d_2\in\mathfrak{A}^f$, then

$$\bar\mu(c\cap(\psi_0d_1\,\triangle\,\psi_0d_2))=\bar\mu(\psi_0^{-1}c\cap(d_1\,\triangle\,d_2)).\text{ **Q**}$$

Since $\mathfrak{A}^f$ is dense in $\mathfrak{A}$ for the measure-algebra topology on $\mathfrak{A}$ (323Bb), and $\mathfrak{A}$ is complete for the measure-algebra uniformity (323Gc), there is a unique extension of ψ_0 to a uniformly continuous function $\psi:\mathfrak{A}\to\mathfrak{A}$ (3A4G). Since the Boolean operations $\triangle$, $\cap$ on $\mathfrak{A}$ are continuous for the measure-algebra topology (323Ba), ψ is a ring homomorphism. Similarly, we have a unique continuous $\psi':\mathfrak{A}\to\mathfrak{A}$ extending ψ'_0; since $\psi\psi'$ and $\psi'\psi$ are continuous functions agreeing with the identity operator ι on $\mathfrak{A}^f$, they are both ι, and $\psi\in\mathrm{Aut}\,\mathfrak{A}$. To see that ψ is measure-preserving, note just that if $a\in\mathfrak{A}$ then

$$\bar\mu\psi a=\sup\{\bar\mu c:c\in\mathfrak{A}^f,\,c\subseteq\psi a\}=\sup\{\bar\mu\psi_0c:c\in\mathfrak{A}^f,\,\psi_0c\subseteq\psi a\}$$

(because ψ_0 is a permutation of $\mathfrak{A}^f$)

$$=\sup\{\bar\mu c:c\in\mathfrak{A}^f,\,\psi c\subseteq\psi a\}=\sup\{\bar\mu c:c\in\mathfrak{A}^f,\,c\subseteq a\}=\bar\mu a.$$

Thus $\psi\in\mathrm{Aut}_{\bar\mu}\mathfrak{A}$.

Finally, $\mathcal{F}\to\psi$. **P** If $c\in\mathfrak{A}^f$ and $\epsilon>0$, there is an $F\in\mathcal{F}$ such that $\bar\mu(\pi c\,\triangle\,\phi c)\le\epsilon$ whenever π, $\phi\in F$. We have

$$\bar\mu(\pi c\,\triangle\,\psi c)=\bar\mu(\pi c\,\triangle\,\lim_{\phi\to\mathcal{F}}\phi c)=\lim_{\phi\to\mathcal{F}}\bar\mu(\pi c\,\triangle\,\phi c)\le\epsilon$$

for every $\pi\in F$. As c and ϵ are arbitrary, $\mathcal{F}\to\psi$ for the weak topology on $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$. **Q** As $\mathcal{F}$ is arbitrary, the bilateral uniformity is complete.

(e)(i) The point is that $\mathfrak{A}^f$ is separable for the measure metric. **P** Because $\mathfrak{A}$ has countable Maharam type, there is a countable subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ which τ-generates $\mathfrak{A}$; by 323J, $\mathfrak{B}$ is dense in $\mathfrak{A}$ for the measure algebra topology. Next, there is a non-decreasing sequence $\langle c_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ with supremum 1. Set $D=\{b\cap c_n:b\in\mathfrak{B},\,n\in\mathbb{N}\}$. Then D is a countable subset of $\mathfrak{A}^f$. If $c\in\mathfrak{A}^f$ and $\epsilon>0$, there are an $n\in\mathbb{N}$ such that $\bar\mu(c\setminus c_n)\le\epsilon$, and a $b\in\mathfrak{B}$ such that $\bar\mu(c_n\cap(c\,\triangle\,b))\le\epsilon$. Now $d=b\cap c_n$ belongs to D, and

$$\begin{aligned}\bar\mu(c\,\triangle\,d)&\le\bar\mu(c\,\triangle\,(c\cap c_n))+\bar\mu((c\cap c_n)\,\triangle\,(b\cap c_n))\\&=\bar\mu(c\setminus c_n)+\bar\mu(c_n\cap(c\,\triangle\,b))\le 2\epsilon.\end{aligned}$$

As c and ϵ are arbitrary, D is dense in $\mathfrak{A}^f$ and $\mathfrak{A}^f$ is separable. **Q**

(ii) Let D be a countable dense subset of $\mathfrak{A}^f$, and $\mathcal{U}$ the family of sets of the form

$$\{\pi:\pi\in\mathrm{Aut}_{\bar\mu}\mathfrak{A},\ \bar\mu(d\,\triangle\,\pi d')<2^{-n}\}$$

where d, $d'\in D$ and $n\in\mathbb{N}$. All these sets are open for the weak topology. **P** If $U=\{\pi:\bar\mu(d\,\triangle\,\pi d')<2^{-n}\}$ and $\phi\in U$, set $\eta=\frac{1}{3}(2^{-n}-\bar\mu(d\,\triangle\,\phi d'))$. Then $V=\{\pi:\bar\mu(d\cap(\pi d'\,\triangle\,\phi d'))\le\eta\}$ is a neighbourhood of ϕ. If $\pi\in V$, then

$$\bar\mu(d\,\triangle\,\pi d')\le\bar\mu(d\,\triangle\,\phi d')+\bar\mu(\phi d'\,\triangle\,\pi d')<2^{-n}$$

and $\pi\in U$. Thus $\phi\in\operatorname{int}U$; as ϕ is arbitrary, U is open. **Q**

(iii) In fact $\mathcal{U}$ is a subbase for the weak topology on $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$. **P** If $W\subseteq\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is open and $\phi\in W$, there are $c_0,\dots,c_n\in\mathfrak{A}^f$ and $k\in\mathbb{N}$ such that W includes $\{\pi:\bar\mu(\pi c_i\,\triangle\,\phi c_i)\le 2^{-k}$ for every $i\le n\}$. Let $d_0,\dots,d_n,d'_0,\dots,d'_n\in D$ be such that $\bar\mu(d_i\,\triangle\,c_i)<2^{-k-2}$, $\bar\mu(d'_i\,\triangle\,\phi c_i)<2^{-k-2}$ for each $i\le n$. Set $U_i=\{\pi:\bar\mu(d'_i\,\triangle\,\pi d_i)<2^{-k-1}\}$; then $U_i\in\mathcal{U}$ and $\phi\in U_i$ for each $i\le n$, because

$$\bar\mu(d'_i\,\triangle\,\phi d_i)\le\bar\mu(d'_i\,\triangle\,\phi c_i)+\bar\mu(\phi c_i\,\triangle\,\phi d_i)=\bar\mu(d'_i\,\triangle\,\phi c_i)+\bar\mu(c_i\,\triangle\,d_i)<2^{-k-1}.$$

If $\pi\in U_i$, then

$$\bar\mu(\pi c_i\,\triangle\,\phi c_i)\le\bar\mu(\pi c_i\,\triangle\,\pi d_i)+\bar\mu(\pi d_i\,\triangle\,d'_i)+\bar\mu(d'_i\,\triangle\,\phi c_i)\le 2^{-k},$$

so $\bigcap_{i\le n}U_i\subseteq W$. As W and ϕ are arbitrary, $\mathcal{U}$ is a subbase for the topology. **Q**

(iv) Since $\mathcal{U}$ is countable, the weak topology is second-countable (4A2Oa). Since the weak topology is a group topology, it is regular (4A5Ha, or otherwise); by (c) above it is Hausdorff; so by 4A2Pb it is separable and metrizable. Accordingly the bilateral uniformity is metrizable (4A5Q(v)); by (d) above, $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is complete under the bilateral uniformity, so its topology is Polish.

494C Proposition Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and give $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ its uniform topology.

(a) $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is a topological group.

(b) For $c\in\mathfrak{A}^f$ and $\epsilon>0$, set

$$U(c,\epsilon)=\{\pi:\pi\in\mathrm{Aut}_{\bar\mu}\mathfrak{A},\ \pi\text{ is supported by an }a\in\mathfrak{A}\text{ such that }\bar\mu(c\cap a)\le\epsilon\}.$$

Then $\{U(c,\epsilon):c\in\mathfrak{A}^f,\ \epsilon>0\}$ is a base of neighbourhoods of ι.

(c) The set of periodic measure-preserving automorphisms of $\mathfrak{A}$ with supports of finite measure is dense in $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$.

(d) The weak topology on $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is coarser than the uniform topology.

(e) If $(\mathfrak{A},\bar\mu)$ is semi-finite, $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is Hausdorff.

(f) If $(\mathfrak{A},\bar\mu)$ is localizable, $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is complete under its bilateral uniformity.

(g) If $(\mathfrak{A},\bar\mu)$ is semi-finite and G is a full subgroup of $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$, then G is closed.

(h) If $(\mathfrak{A},\bar\mu)$ is σ-finite, then $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is metrizable.

(i) Suppose that $(\mathfrak{A},\bar{\mu})$ is σ-finite and $\mathfrak{A}$ has countable Maharam type. If $D\subseteq\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ is countable, then the full subgroup G of $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ generated by D, with its induced topology, is a Polish group.

proof (a) For π, $\phi\in\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ and $c\in\mathfrak{A}^f$, set $\rho'_c(\pi,\phi)=\sup_{a\in\mathfrak{A}}\bar{\mu}(c\cap(\pi a\,\triangle\,\phi a))$; as in part (a) of the proof of 494B, it is elementary that every ρ'_c is a pseudometric, so the uniform topology $\mathfrak{T}_u$ is properly defined. If π, ϕ, π', $\phi'\in\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$, $c\in\mathfrak{A}^f$ and $a\in\mathfrak{A}$, then

$$\begin{aligned}\bar{\mu}(c\cap(\pi'\phi'a\,\triangle\,\pi\phi a))&\le\bar{\mu}(c\cap(\pi'\phi'a\,\triangle\,\pi\phi'a))+\bar{\mu}(c\cap(\pi\phi'a\,\triangle\,\pi\phi a))\\&\le\rho'_c(\pi',\pi)+\bar{\mu}(\pi^{-1}c\cap(\phi'a\,\triangle\,\phi a))\\&\le\rho'_c(\pi',\pi)+\rho'_{\pi^{-1}c}(\phi',\phi);\end{aligned}$$

as a is arbitrary, $\rho'_c(\pi'\phi',\pi\phi)\le\rho'_c(\pi',\pi)+\rho'_{\pi^{-1}c}(\phi',\phi)$; as c is arbitrary, $(\pi',\phi')\mapsto\pi'\phi'$ is continuous at (π,ϕ); thus multiplication is continuous. If π, $\phi\in\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$, $c\in\mathfrak{A}^f$ and $a\in\mathfrak{A}$, then

$$\bar{\mu}(c\cap(\pi^{-1}a\,\triangle\,\phi^{-1}a))=\bar{\mu}(\phi c\cap(\phi\pi^{-1}a\,\triangle\,\pi\pi^{-1}a)\le\rho'_{\phi c}(\phi,\pi);$$

thus $\rho'_c(\pi^{-1},\phi^{-1})\le\rho'_{\phi c}(\pi,\phi)$, $\pi\mapsto\pi^{-1}$ is continuous at ϕ, and inversion is continuous. So once more we have a topological group.

(b)(i) If $c\in\mathfrak{A}^f$, $\pi\in\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ and $\epsilon>0$ are such that $\rho'_c(\pi,\iota)\le\frac{1}{3}\epsilon$, then $\pi\in U(c,\epsilon)$. **P** Consider $A=\{a:a\in\mathfrak{A}_c,\ a\cap\pi a=0\}$. Then

$$\bar{\mu}a\le\bar{\mu}(c\cap(\pi a\,\triangle\,a))\le\rho'_c(\pi,\iota)\le\frac{1}{3}\epsilon$$

for every $a\in A$. If $B\subseteq A$ is upwards-directed, then $b^*=\sup B$ is defined in $\mathfrak{A}$, and $\bar{\mu}b^*=\sup_{b\in B}\bar{\mu}b$ (321C). Now $\pi b^*=\sup_{b\in B}\pi b$, so $b^*\cap\pi b^*=\sup_{b\in B}b\cap\pi b=0$, and $b^*\in A$. By Zorn's Lemma, A has a maximal element a^*. Suppose that $d\in\mathfrak{A}_c$ is disjoint from $\pi^{-1}a^*\cup a^*\cup\pi a^*$. Then $a^*\cup(d\setminus\pi d)\in A$; by the maximality of a^*, $d\subseteq\pi d$ and $d=\pi d$ (because $\bar{\mu}d=\bar{\mu}\pi d<\infty$). Thus $(1\setminus c)\cup(\pi^{-1}a^*\cup a^*\cup\pi a^*)$ supports π and witnesses that $\pi\in U(c,\epsilon)$. **Q**

So every $U(c,\epsilon)$ is a $\mathfrak{T}_u$-neighbourhood of ι.

(ii) Conversely, if $c\in\mathfrak{A}^f$, $\epsilon>0$ and $\pi\in U(c,\epsilon)$, then $\rho'_c(\pi,\iota)\le\epsilon$. **P** Let $d\in\mathfrak{A}$ be such that π is supported by d and $\bar{\mu}(c\cap d)\le\epsilon$. Then, for any $a\in\mathfrak{A}$, $a\,\triangle\,\pi a\subseteq d$, so $\bar{\mu}(c\cap(a\,\triangle\,\pi a))\le\epsilon$; which is what we need to know. **Q**

So $\{U(c,\epsilon):c\in\mathfrak{A}^f,\ \epsilon>0\}$ is a base of neighbourhoods of ι for $\mathfrak{T}_u$.

(c) Take a non-empty open subset U of $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ and $\phi\in U$.

(i) By (b), there are a $c\in\mathfrak{A}^f$ and an $\epsilon>0$ such that $U(c,3\epsilon)U(c,3\epsilon)\subseteq U^{-1}\phi$. Now there is a $\psi\in\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ such that $\psi^{-1}\phi\in U(c,3\epsilon)$ and ψ is supported by $e=c\cup\phi c$. **P** By 332L, applied to $\mathfrak{A}_e$ and $\phi{\restriction}\mathfrak{A}_c$, there is a measure-preserving automorphism $\psi_0:\mathfrak{A}_e\to\mathfrak{A}_e$ agreeing with ϕ on $\mathfrak{A}_c$; now set

$$\psi a=\psi_0(a\cap e)\cup(a\setminus e)$$

for every $a\in\mathfrak{A}$ to get $\psi\in\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ agreeing with ϕ on $\mathfrak{A}_c$ and supported by e. As $\psi^{-1}\phi a=a$ for $a\subseteq c$, $\psi^{-1}\phi$ is supported by $1\setminus c$ and belongs to $U(c,3\epsilon)$. **Q**

(ii) By 381H, applied to $\psi{\restriction}\mathfrak{A}_e$, there is a partition $\langle c_m\rangle_{1\le m\le\omega}$ of unity in $\mathfrak{A}_e$ such that $\psi c_m\subseteq c_m$ for every m, $\psi{\restriction}\mathfrak{A}_{c_m}$ is periodic with period m for every $m\in\mathbb{N}\setminus\{0\}$, and $\psi{\restriction}\mathfrak{A}_{c_\omega}$ is aperiodic. Of course $\psi c_m=c_m$ for every m, just because $\bar{\mu}\psi c_m=\bar{\mu}c_m$. Let $n\ge1$ be such that $\bar{\mu}c_\omega\le n!\epsilon$ and $\bar{\mu}(\sup_{n<m<\omega}c_m)\le\epsilon$. By the Halmos-Rokhlin-Kakutani lemma (386C), applied to $\psi{\restriction}\mathfrak{A}_{c_\omega}$, there is a $b\subseteq c_\omega$ such that b, $\psi b,\dots,\psi^{n!-1}b$ are disjoint and $\bar{\mu}(c_\omega\setminus\sup_{i<n!}\psi^i b)\le\epsilon$. Note that $\bar{\mu}b$ is also at most ϵ.

Set

$$d=\sup_{n<m<\omega}c_m\cup(c_\omega\setminus\sup_{i<n!}\psi^i b),\quad d'=d\cup\psi^{n!-1}b,$$

and let $\pi:\mathfrak{A}\to\mathfrak{A}$ be the measure-preserving Boolean automorphism such that

$$\begin{aligned}\pi a &= \psi a \text{ if } 1 \le m \le n \text{ and } a \subseteq c_m,\\ &= \psi a \text{ if } 0 \le i \le n! - 2 \text{ and } a \subseteq \psi^i b,\\ &= \psi^{-n!+1} a \text{ if } a \subseteq \psi^{n!-1} b,\\ &= a \text{ if } a \subseteq d \cup (1 \setminus e).\end{aligned}$$

Then

$$\begin{aligned}\pi^{n!} a &= \psi^{n!} a = a \text{ if } 1 \le m \le n \text{ and } a \subseteq c_m,\\ &= \psi^{n!} \psi^{-n!} a = a \text{ if } 0 \le i < n! \text{ and } a \subseteq \psi^i b,\\ &= a \text{ if } a \subseteq d \cup (1 \setminus e),\end{aligned}$$

so $\pi^{n!} = \iota$ and π is periodic. Since π is supported by e, and $\mathfrak{A}_e$ is Dedekind complete, π has a support of finite measure. On the other hand $\pi a = \psi a$ whenever $a \cap d' = 0$, so $\pi^{-1}\psi$ is supported by d' and belongs to $U(c, \bar\mu d') \subseteq U(c, 3\epsilon)$.

Now

$$\pi^{-1}\phi = \pi^{-1}\psi\psi^{-1}\phi \in U(c, 3\epsilon)U(c, 3\epsilon) \subseteq U^{-1}\phi, \quad \pi \in U;$$

as U is arbitrary, the set of periodic automorphisms with supports of finite measure is dense in $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$.

(d) Let V be a neighbourhood of the identity ι for the weak topology $\mathfrak{T}_w$ on $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$. Then there are $c_0, \dots, c_k \in \mathfrak{A}^f$ and $\epsilon_0, \dots, \epsilon_k > 0$ such that

$$V \supseteq \{\pi : \bar\mu(c_i \triangle \pi c_i) \le \epsilon_i \text{ for every } i \le k\}.$$

Set $c = \sup_{i \le k} c_i$, $\epsilon = \frac{1}{2}\min_{i \le k} \epsilon_i$. If $\pi \in U(c, \epsilon)$ as defined in (b), there is an $a \in \mathfrak{A}$, supporting π, such that $\bar\mu(c \cap a) \le \epsilon$. In this case, for each $i \le k$, $c_i \setminus \pi c_i \subseteq c \cap a$, so

$$\bar\mu(c_i \triangle \pi c_i) = 2\bar\mu(c_i \setminus \pi c_i) \le 2\bar\mu(c \cap a) \le \epsilon_i.$$

Thus $V \supseteq U(c, \epsilon)$ and V is a neighbourhood of ι for $\mathfrak{T}_u$. As $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is a topological group under either topology, it follows that $\mathfrak{T}_u$ is finer than $\mathfrak{T}_w$ (4A5Fb).

(e) Because $(\mathfrak{A}, \bar\mu)$ is semi-finite, the weak topology is Hausdorff (494Bc), so the uniform topology, being finer, must also be Hausdorff.

(f) Let $\mathcal{F}$ be a Cauchy filter for the $\mathfrak{T}_u$-bilateral uniformity on $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$. Because the identity map from $(\operatorname{Aut}_{\bar\mu}\mathfrak{A}, \mathfrak{T}_u)$ to $(\operatorname{Aut}_{\bar\mu}\mathfrak{A}, \mathfrak{T}_w)$ is continuous ((d) above), it is uniformly continuous for the corresponding bilateral uniformities (4A5Hd), and $\mathcal{F}$ is Cauchy for the $\mathfrak{T}_w$-bilateral uniformity (4A2Ji). It follows that $\mathcal{F}$ has a $\mathfrak{T}_w$-limit ψ say (494Bd), in which case ψa is the limit $\lim_{\pi \to \mathcal{F}} \pi a$, for the measure-algebra topology of $\mathfrak{A}$, for every $a \in \mathfrak{A}$ (494Bb). But ψ is also the $\mathfrak{T}_u$-limit of $\mathcal{F}$. **P** Suppose that $c \in \mathfrak{A}^f$ and $\epsilon > 0$. Set $V(c, \epsilon) = \{\pi : \rho'_c(\pi, \iota) \le \epsilon\}$, where ρ'_c is defined as (a) above. Then $V(c, \epsilon)$ is a $\mathfrak{T}_u$-neighbourhood of ι, so $\{(\pi, \phi) : \phi\pi^{-1} \in V(c, \epsilon)\}$ belongs to the $\mathfrak{T}_u$-bilateral uniformity, and there is an $F \in \mathcal{F}$ such that $\phi\pi^{-1} \in V(c, \epsilon)$ whenever $\pi, \phi \in F$.

Now if $\phi \in F$ and $a \in \mathfrak{A}$,

$$\bar\mu(c \cap (\phi a \triangle \psi a)) = \bar\mu(c \cap (\phi a \triangle \lim_{\pi \to \mathcal{F}} \pi a)) = \lim_{\pi \to \mathcal{F}} \bar\mu(c \cap (\phi a \triangle \pi a))$$

(because $b \mapsto \bar\mu(c \cap (\phi a \triangle b))$ is continuous)

$$= \lim_{\pi \to \mathcal{F}} \bar\mu(c \cap (\phi\pi^{-1}\pi a \triangle \pi a)) \le \sup_{\pi \in F, b \in \mathfrak{A}} \bar\mu(c \cap (\phi\pi^{-1} b \triangle b)) \le \epsilon.$$

Thus $\rho'_c(\phi, \psi) \le \epsilon$ for every $\phi \in F$. As c and ϵ are arbitrary, $\mathcal{F}$ is $\mathfrak{T}_u$-convergent to ψ. **Q**

As $\mathcal{F}$ is arbitrary, $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is complete for the $\mathfrak{T}_u$-bilateral uniformity.

(g)(i) Suppose that ϕ belongs to the closure of G in $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$. Let B be the set of those $b \in \mathfrak{A}$ for which there is a $\pi \in G$ such that π and ϕ agree on the principal ideal $\mathfrak{A}_b$. Then B is order-dense in $\mathfrak{A}$. **P** Suppose that $a \in \mathfrak{A} \setminus \{0\}$. Because $(\mathfrak{A}, \bar\mu)$ is semi-finite, there is a non-zero $c \in \mathfrak{A}^f$ such that $c \subseteq a$. Take $\epsilon \in \,]0, \bar\mu c[$.

Then there is a $\pi \in G$ such that $\pi^{-1}\phi \in U(c,\epsilon)$. Let $d \in \mathfrak{A}$ be such that d supports $\pi^{-1}\phi$ and $\bar{\mu}(c \cap d) \le \epsilon$. Set $b = c \setminus d$. If $b' \subseteq b$, then $\pi^{-1}\phi b' = b'$, that is, $\phi b' = \pi b'$; so π and ϕ agree on $\mathfrak{A}_b$ and $b \in B$, while $0 \ne b \subseteq a$. **Q**

(ii) There is therefore a partition $\langle b_i \rangle_{i \in I}$ of unity consisting of members of B. For each $i \in I$ take $\pi_i \in G$ such that π_i and ϕ agree on $\mathfrak{A}_{b_i}$; because G is full, $\langle (b_i, \pi_i) \rangle_{i \in I}$ witnesses that $\phi \in G$. As ϕ is arbitrary, G is closed.

(h) Let $\langle c_n \rangle_{n \in \mathbb{N}}$ be a non-decreasing sequence in $\mathfrak{A}^f$ with supremum 1. Then $\{U(c_n, 2^{-n}) : n \in \mathbb{N}\}$ is a base of neighbourhoods of ι. **P** If $c \in \mathfrak{A}^f$ and $\epsilon > 0$, there is an $n \in \mathbb{N}$ such that $\bar{\mu}(c \setminus c_n) + 2^{-n} \le \epsilon$. If $\pi \in U(c_n, 2^{-n})$, there is an $a \in \mathfrak{A}$, supporting π, such that $\bar{\mu}(c_n \cap a) \le 2^{-n}$; in which case $\bar{\mu}(c \cap a) \le \epsilon$ and $\pi \in U(c,\epsilon)$. Thus we have found an n such that $U(c_n, 2^{-n}) \subseteq U(c,\epsilon)$. **Q**

By 4A5Q, $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ is metrizable.

(i)(*α*) By (h), $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ and therefore G are metrizable; the bilateral uniformity of $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ is therefore metrizable (4A5Q(v)). By (f), $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ is complete under its bilateral uniformity; by (g), G is closed, so is complete under the induced uniformity. So there is a metric on G, inducing its topology, under which G is complete, and all I have to show is that G is separable.

(*β*) Since the subgroup of $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ generated by D is again countable, we may suppose that D is itself a subgroup of $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$. Let $\langle c_n \rangle_{n \in \mathbb{N}}$ be a non-decreasing sequence in $\mathfrak{A}^f$ with supremum 1, and $\mathfrak{B}$ a countable subalgebra of $\mathfrak{A}$, which τ-generates $\mathfrak{A}$; by 323J again, $\mathfrak{B}$ is dense in $\mathfrak{A}$ for the measure-algebra topology. For m, $n \in \mathbb{N}$, $\pi_0, \dots, \pi_m \in D$ and $b_0, \dots, b_m \in \mathfrak{B}$, write $E(m,n,\pi_0,\dots,\pi_m,b_0,\dots,b_m)$ for

$$\{\pi : \pi \in G, \textstyle\sum_{i=0}^m \bar{\mu}(c_n \cap b_i \cap \mathrm{supp}(\pi^{-1}\pi_i)) \le 2^{-n}\}.$$

(The supports are defined because $\mathfrak{A}$ is Dedekind complete; see 381F.) Let $D' \subseteq G$ be a countable set such that $D' \cap E(m,n,\pi_0,\dots,\pi_m,b_0,\dots,b_m)$ is non-empty whenever m, $n \in \mathbb{N}$, $\pi_0, \dots, \pi_m \in D$ and $b_0, \dots, b_m \in \mathfrak{B}$ are such that $E(m,n,\pi_0,\dots,\pi_m,b_0,\dots,b_m)$ is non-empty.

Suppose that $\pi \in G$, $c \in \mathfrak{A}^f$ and $\epsilon > 0$. Let $n \in \mathbb{N}$ be such that $\bar{\mu}(c \setminus c_n) + 2^{-n+2} < \epsilon$. We have a family $\langle (a_j, \pi_j) \rangle_{j \in J}$ such that $\langle a_j \rangle_{j \in J}$ is a partition of unity in $\mathfrak{A}$ consisting of elements of finite measure, and, for each $j \in J$, $\pi_j \in D$ and π agrees with π_j on $\mathfrak{A}_{a_j}$ (381Ia), that is, $a_j \cap \mathrm{supp}(\pi^{-1}\pi_j) = 0$. Let $j_0, \dots, j_m \in J$ be such that $\bar{\mu}(c_n \setminus \sup_{i \le m} a_{j_i}) \le 2^{-n}$; for each $i \le m$, let $b_i \in \mathfrak{B}$ be such that $\bar{\mu}(c_n \cap (b_i \triangle a_{j_i})) \le \frac{2^{-n}}{m+1}$. In this case, $\pi \in E(m,n,\pi_{j_0},\dots,\pi_{j_m},b_0,\dots,b_m)$, so there is a $\tilde{\pi} \in D' \cap E(m,n,\pi_{j_0},\dots,\pi_{j_m},b_0,\dots,b_m)$. Consider $d = \mathrm{supp}(\pi^{-1}\tilde{\pi})$. If we set $d_i = \mathrm{supp}(\pi^{-1}\pi_{j_i}) \cup \mathrm{supp}(\tilde{\pi}^{-1}\pi_{j_i})$, then π and $\tilde{\pi}$ both agree with π_{j_i} on $1 \setminus d_i$, so $d \subseteq d_i$. Now

$$\begin{aligned}
\bar{\mu}(c_n \cap d) &\le \bar{\mu}(c_n \setminus \sup_{i \le m} b_i) + \sum_{i=0}^m \bar{\mu}(c_n \cap b_i \cap d) \\
&\le \bar{\mu}(c_n \setminus \sup_{i \le m} a_{j_i}) + \sum_{i=0}^m \bar{\mu}(a_{j_i} \setminus b_i) + \sum_{i=0}^m \bar{\mu}(c_n \cap b_i \cap d_i) \\
&\le 2^{-n} + 2^{-n} + \sum_{i=0}^m \bar{\mu}(c_n \cap b_i \cap \mathrm{supp}(\pi^{-1}\pi_{j_i})) + \sum_{i=0}^m \bar{\mu}(c_n \cap b_i \cap \mathrm{supp}(\tilde{\pi}^{-1}\pi_{j_i})) \\
&\le 4 \cdot 2^{-n} = 2^{-n+2},
\end{aligned}$$

and $\bar{\mu}(c \cap d) < \epsilon$. But this means that $\pi^{-1}\tilde{\pi} \in U(c,\epsilon)$; as c, ϵ and π are arbitrary, D' is dense in G and G is separable.

494D Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra and $\phi \in \mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$. Let $T = T_\phi : L^2_{\mathbb{C}} \to L^2_{\mathbb{C}}$ be the corresponding operator on the complex Hilbert space $L^2_{\mathbb{C}} = L^2_{\mathbb{C}}(\mathfrak{A}, \bar{\mu})$ (366M). Then the following are equiveridical:

(α) ϕ is weakly mixing (definition: 372Ob);

(β) $\inf_{k \in \mathbb{N}} |(T^k w|w)| < 1$ whenever $w \in L^2_{\mathbb{C}}$, $\|w\|_2 = 1$ and $\int w = 0$;

(γ) $\inf_{k \in \mathbb{N}} |(T^k w|w)| = 0$ whenever $w \in L^2_{\mathbb{C}}$, $\|w\|_2 = 1$ and $\int w = 0$.

proof (a) Regarding $\mathbb{Z}$, with addition and its discrete topology, as a topological group, its dual group is the circle group $S^1 = \{z : z \in \mathbb{C}, |z| = 1\}$ with multiplication and its usual topology (445Bb-445Bc); the duality being given by the functional $(k, z) \mapsto z^k : \mathbb{Z} \times S^1 \to S^1$. For $u \in L^2_{\mathbb{C}}$, define $h_u : \mathbb{Z} \to \mathbb{C}$ by setting $h_u(k) = (T^k u|u)$ for $k \in \mathbb{Z}$. Then h_u is positive definite in the sense of 445L. **P** If $\zeta_0, \ldots , \zeta_n \in \mathbb{C}$ and $m_0, \ldots , m_n \in \mathbb{Z}$, then

$$\sum_{j,k=0}^{n} \zeta_j\bar{\zeta}_k h_u(m_j - m_k) = \sum_{j,k=0}^{n} \zeta_j\bar{\zeta}_k(T^{m_j-m_k}u|u) = \sum_{j,k=0}^{n} \zeta_j\bar{\zeta}_k(T^{m_j}u|T^{m_k}u)$$

(366Me)

$$= (\sum_{j=0}^{n} \zeta_j T^{m_j}u|\sum_{k=0}^{n} \zeta_k T^{m_k}u) \ge 0. \text{ **Q**}$$

By Bochner's theorem (445N), there is a Radon probability measure ν_u on S^1 such that

$$\int z^k \nu_u(dz) = h_u(k) = (T^k u|u)$$

for every $k \in \mathbb{Z}$. Note that

$$\nu_u(S^1) = \int z^0 d\nu_u = (u|u) = \|u\|_2^2.$$

(b)(i) Let $P \subseteq C(S^1;\mathbb{C})$ be the set of functions which are expressible in the form

$$p(z) = \sum_{k\in\mathbb{Z}} \zeta_k z^k \text{ for every } z \in S^1$$

where $\zeta_k \in \mathbb{C}$ for every $k \in \mathbb{Z}$ and $\{k : \zeta_k \ne 0\}$ is finite. Then P is a linear subspace of the complex Banach space $C(S^1;\mathbb{C})$, closed under multiplication. Also, if $p \in P$, then $\bar{p} \in P$, where $\bar{p}(z) = \overline{p(z)}$ for every $z \in S^1$. **P** If $p(z) = \sum_{k\in\mathbb{Z}} \zeta_k z^k$, then

$$\bar{p}(z) = \sum_{k\in\mathbb{Z}} \bar{\zeta}_k z^{-k} = \sum_{k\in\mathbb{Z}} \bar{\zeta}_{-k} z^k$$

for every $z \in S^1$. **Q** Of course P contains the constant function $z \mapsto z^0$ and the identity function $z \mapsto z$, so by the Stone-Weierstrass theorem (281G) P is $\| \|_\infty$-dense in $C(S^1;\mathbb{C})$.

(ii) For any $p \in P$ the coefficients of the corresponding expression $p(z) = \sum_{k\in\mathbb{Z}} \zeta_k z^k$ are uniquely defined, since

$$\zeta_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikt}p(e^{it})dt$$

for every k. So we can define u_p, for every $u \in L^2_{\mathbb{C}}$, by saying that $u_p = \sum_{k\in\mathbb{Z}} \zeta_k T^k u$. Now we have

$$(u_p|u) = \sum_{k\in\mathbb{Z}} \zeta_k(T^k u|u) = \sum_{k\in\mathbb{Z}} \zeta_k \int z^k \nu_u(dz) = \int p\, d\nu_u.$$

We also see that

$$\int u_p = (u_p|\chi 1) = \sum_{k\in\mathbb{Z}} \zeta_k(T^k u|\chi 1)$$

$$= \sum_{k\in\mathbb{Z}} \zeta_k(u|T^{-k}\chi 1) = \sum_{k\in\mathbb{Z}} \zeta_k(u|\chi 1) = p(1)\int u.$$

It is elementary to check that if $p \in P$ and $q(z) = zp(z)$ for every $z \in S^1$, then $u_q = Tu_p$. Note also that $p \mapsto u_p : P \to L^2_{\mathbb{C}}$ is linear.

(iii) For any $p \in P$ and $u \in L^2_{\mathbb{C}}$, $\|u_p\|_2 \le \|u\|_2\|p\|_\infty$. **P** If $p(z) = \sum_{k\in\mathbb{Z}} \zeta_k z^k$ for $z \in S^1$, set

$$q(z) = p(z)\overline{p(z)} = \sum_{j,k\in\mathbb{Z}} \zeta_j\bar{\zeta}_k z^{j-k}$$

for $z \in S^1$. Then

$$\|u_p\|_2^2 = (\sum_{j\in\mathbb{Z}}\zeta_j T^j u|\sum_{k\in\mathbb{Z}}\zeta_k T^k u) = \sum_{j,k\in\mathbb{Z}}\zeta_j\bar\zeta_k(T^j u|T^k u) = \sum_{j,k\in\mathbb{Z}}\zeta_j\bar\zeta_k(T^{j-k}u|u)$$
$$= (u_q|u) = \int q\,d\nu_u \le \|q\|_\infty\nu_u(S^1) = \|p\|_\infty^2\|u\|_2^2. \ \mathbf{Q}$$

(c) Case 1 Suppose that $\nu_u\{z\}=0$ whenever $u\in L^2_{\mathbb{C}}$, $z\in S^1$ and $\int u=0$.

(i) If $u\in L^2_{\mathbb{C}}$ and $\int u=0$, then $\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^n|(T^k u|u)|^2=0$. **P** For any $n\in\mathbb{N}$,

$$\frac{1}{n+1}\sum_{k=0}^n|(T^k u|u)|^2 = \frac{1}{n+1}\sum_{k=0}^n(T^k u|u)(u|T^k u) = \frac{1}{n+1}\sum_{k=0}^n(T^k u|u)(T^{-k}u|u)$$
$$= \frac{1}{n+1}\sum_{k=0}^n\int z^k\nu_u(dz)\int w^{-k}\nu_u(dw)$$
$$= \frac{1}{n+1}\sum_{k=0}^n\int z^k w^{-k}\nu_u^2(d(z,w))$$

where ν_u^2 is the product measure on $(S^1)^2$. But observe that

$$|\frac{1}{n+1}\sum_{k=0}^n z^k w^{-k}|\le 1$$

for all z, $w\in S^1$, while for $w\ne z$ we have

$$\frac{1}{n+1}\sum_{k=0}^n z^k w^{-k} = \frac{1-(w^{-1}z)^{n+1}}{(n+1)(1-w^{-1}z)}\to 0.$$

Since

$$\nu_u^2\{(w,z):w=z\} = \int\nu_u\{z\}\nu_u(dz)=0,$$

Lebesgue's Dominated Convergence Theorem tells us that

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^n|(T^k u|u)|^2 = \lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^n\int z^k w^{-k}\nu_u^2(d(z,w))=0. \ \mathbf{Q}$$

(ii) Write $\mathcal{F}_d$ for the asymptotic density filter on $\mathbb{N}$ (491S). If $u\in L^2_{\mathbb{C}}$ and $\int u=0$, then $\lim_{k\to\mathcal{F}_d}|(T^k u|u)|^2=0$, by (i) above and 491Sb. It follows at once that $\lim_{k\to\mathcal{F}_d}(T^k u|u)=0$.

In fact $\lim_{k\to\mathcal{F}_d}(T^k u|v)=0$ whenever $\int u=\int v=0$. **P** We have

$$\lim_{k\to\mathcal{F}_d}(T^k u|v)+(T^k v|u) = \lim_{k\to\mathcal{F}_d}(T^k(u+v)|u+v)-(T^k u|v)-(T^k v|v)=0, \qquad (*)$$

and similarly

$$\lim_{k\to\mathcal{F}_d} i(T^k u|v)-i(T^k v|u) = \lim_{k\to\mathcal{F}_d}(T^k(iu)|v)+(T^k v|iu)=0,$$

so $\lim_{k\to\mathcal{F}_d}(T^k u|v)-(T^k v|u)=0$; adding this to (*), $\lim_{k\to\mathcal{F}}(T^k u|v)=0$. **Q**

(iii) Now take any a, $b\in\mathfrak{A}$ and set $u=\chi a-(\bar\mu a)\chi 1$, $v=\chi b-(\bar\mu b)\chi 1$. In this case, $\int u=\int v=0$ and

$$(T^k u|v) = (\chi(\phi^k a)-(\bar\mu a)\chi 1|\chi b-(\bar\mu b)\chi 1)$$
$$= \bar\mu(b\cap\phi^k a)-\bar\mu a\cdot\bar\mu b-\bar\mu(\phi^k a)\cdot\bar\mu b+\bar\mu a\cdot\bar\mu b = \bar\mu(b\cap\phi^k a)-\bar\mu a\cdot\bar\mu b$$

for every k, so $\lim_{k\to\mathcal{F}_d}\bar\mu(b\cap\phi^k a)-\bar\mu a\cdot\bar\mu b=0$ and

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^n|\bar\mu(b\cap\phi^k a)-\bar\mu a\cdot\bar\mu b|=0,$$

by 491Sb in the other direction. As a and b are arbitrary, ϕ is weakly mixing.

(d) Case 2 Suppose there are $u \in L^2_{\mathbb{C}}$ and $t \in \,]-\pi,\pi]$ such that $\int u = 0$ and $\nu_u\{e^{it}\} > 0$.

(i) For $n \in \mathbb{N}$, set set $f_n(z) = \max(0, 1 - 2^n|z - e^{it}|)$ for $z \in S^1$. Then

$$\begin{aligned} |zf_n(z) - e^{it}f_n(z)| &\le 2^{-n} \text{ if } |z - e^{it}| \le 2^{-n}, \\ &= 0 \text{ for other } z \in S^1. \end{aligned}$$

Because P is $\|\;\|_\infty$-dense in $C(S^1;\mathbb{C})$, there is a $p_n \in P$ such that $\|p_n - f_n\|_\infty \le 2^{-n}$, in which case

$$|zp_n(z) - e^{it}p_n(z)| \le 3 \cdot 2^{-n}$$

for every $z \in S^1$. Set $q_n(z) = zp_n(z)$ for $z \in S^1$; then

$$\|Tu_{p_n} - e^{-it}u_{p_n}\|_2 = \|u_{q_n} - e^{-it}u_{p_n}\|_2 \le \|u\|_2\|q_n - e^{-it}p_n\|_\infty$$

(by (b-iii))

$$\le 3 \cdot 2^{-n}\|u\|_2,$$

while

$$\|u_{p_n}\|_2 \le \|u\|_2\|p_n\|_\infty \le 2\|u\|_2.$$

(ii) Let $\mathcal{F}$ be any non-principal ultrafilter on $\mathbb{N}$. Then $v = \lim_{n\to\mathcal{F}} u_{p_n}$ is defined for the weak topology of the complex Hilbert space $L^2_{\mathbb{C}}$ (4A4Ka). Also

$$(v|u) = \lim_{n\to\mathcal{F}}(u_{p_n}|u) = \lim_{n\to\infty}\int p_n d\nu_u = \lim_{n\to\infty}\int f_n d\nu_u = \nu_u\{e^{it}\} > 0$$

so $v \ne 0$. But we also have

$$\int v = (v|\chi 1) = \lim_{n\to\mathcal{F}}(u_{p_n}|\chi 1) = \lim_{n\to\mathcal{F}} p_n(1)\int u = 0,$$

and, taking limits in the weak topology on $L^2_{\mathbb{C}}$,

$$Tv = \lim_{n\to\mathcal{F}} Tu_{p_n}$$

(because T is continuous for the weak topology, see 4A4Bd)

$$= \lim_{n\to\mathcal{F}} e^{it}u_{p_n} = e^{it}v.$$

Set $w = \frac{1}{\|v\|_2}v$; then $\|w\|_2 = 1$, $\int w = 0$,

$$\inf_{k\in\mathbb{N}} |(T^k w|w)| = \inf_{k\in\mathbb{N}} |e^{ikt}(w|w)| = 1$$

and (β) is false.

(e) Putting (c) and (d) together, we see that either (α) is true or (β) is false, that is, that (β) implies (α).

(f) On the other hand, (α) implies (γ). **P** Suppose that ϕ is weakly mixing. Then

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^n |\bar\mu(b \cap \phi^k a) - \bar\mu a \cdot \bar\mu b| = 0$$

for all a, $b \in \mathfrak{A}$; by 491Sb again,

$$\lim_{k\to\mathcal{F}_d} \bar\mu(b \cap \phi^k a) - \bar\mu a \cdot \bar\mu b = 0,$$

that is,

$$\lim_{k\to\mathcal{F}_d}(T^k\chi a|\chi b) = (\chi a|\chi 1)\cdot(\chi 1|\chi b),$$

whenever a, $b \in \mathfrak{A}$. Because $(\;|\;)$ is sesquilinear,

$$\lim_{k\to\mathcal{F}_d}(T^k u|v) = (u|\chi 1)\cdot(\chi 1|v)$$

whenever u, v belong to $S_{\mathbb{C}}=S_{\mathbb{C}}(\mathfrak{A})$, the complex linear span of $\{\chi a : a\in\mathfrak{A}\}$. Because $S_{\mathbb{C}}$ is norm-dense in $L^2_{\mathbb{C}}$ (366Mb), and $\{T^k u : k\in\mathbb{N}\}$ is norm-bounded, we shall have

$$\lim_{k\to\mathcal{F}_d}(u|T^{-k}v)=\lim_{k\to\mathcal{F}_d}(T^k u|v)=(u|\chi 1)\cdot(\chi 1|v)$$

whenever $u\in S_{\mathbb{C}}$ and $v\in L^2_{\mathbb{C}}$; now $\{T^{-k}v : k\in\mathbb{N}\}$ is norm-bounded, so

$$\lim_{k\to\mathcal{F}_d}(T^k u|v)=\lim_{k\to\mathcal{F}_d}(u|T^{-k}v)=(u|\chi 1)\cdot(\chi 1|v)$$

for all u, $v\in L^2_{\mathbb{C}}$. In particular, if $\|w\|_2=1$ and $\int w=0$,

$$\inf_{k\in\mathbb{N}}|(T^k w|w)|\le\lim_{k\to\mathcal{F}_d}|(T^k w|w)|=|(w|\chi 1)|^2=0,$$

as required. **Q**

(g) Since (γ) obviously implies (β), the three conditions are indeed equiveridical.

494E Theorem (Halmos 1944, Rokhlin 1948) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra, and give $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ its weak topology.

(a) If $\mathfrak{A}\ne\{0,1\}$, the set of mixing measure-preserving Boolean automorphisms is meager in $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$.

(b) If $\mathfrak{A}$ is atomless and homogeneous, the set of two-sided Bernouilli shifts on $(\mathfrak{A},\bar\mu)$ (definition: 385Qb) is dense in $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$.

(c) If $\mathfrak{A}$ has countable Maharam type, the set of weakly mixing measure-preserving Boolean automorphisms is a G_δ subset of $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$.

(d) If $\mathfrak{A}$ is atomless and has countable Maharam type, the set of weakly mixing measure-preserving Boolean automorphisms which are not mixing is comeager in $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$, and is not empty.

proof (a) Take $a\in\mathfrak{A}\setminus\{0,1\}$. Let $\delta>0$ be such that $\bar\mu a>\delta+(\bar\mu a)^2$, and consider

$$F_n=\{\pi:\pi\in\mathrm{Aut}_{\bar\mu}\mathfrak{A},\ \bar\mu(a\cap\pi^k a)\le\delta+(\bar\mu a)^2\text{ for every }k\ge n\}.$$

Because $\pi\mapsto\bar\mu(a\cap\pi^k a)$ is continuous for every k (494Bb), every F_n is closed. Because F_n cannot contain any periodic automorphism, $(\mathrm{Aut}_{\bar\mu}\mathfrak{A})\setminus F_n$ is dense for the uniform topology on $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ (494Cc) and therefore for the weak topology (494Cd). Accordingly $\bigcup_{n\in\mathbb{N}}F_n$ is meager; and every mixing measure-preserving automorphism belongs to $\bigcup_{n\in\mathbb{N}}F_n$.

(b) Suppose that $\phi\in\mathrm{Aut}_{\bar\mu}\mathfrak{A}$, $A\subseteq\mathfrak{A}$ is finite and $\epsilon>0$.

(i) By 494Cc, there is a periodic $\psi\in\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ such that $\bar\mu(\phi a\,\triangle\,\psi a)\le\epsilon$ for every $a\in\mathfrak{A}$. Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}$ generated by $\{\psi^k a : k\in\mathbb{Z},\ a\in A\}$; then $\mathfrak{B}$ is finite (because $\{\psi^k : k\in\mathbb{Z}\}$ is finite). Let B be the set of atoms of $\mathfrak{B}$. Since $\psi[\mathfrak{B}]=\mathfrak{B}$, $\psi[B]=B$ and $\psi\restriction B$ is a permutation of B. Let $B_0\subseteq B$ be such that B_0 meets each orbit of $\psi\restriction B$ in just one point; enumerate B_0 as $\langle b_j\rangle_{j<n}$.

Let $r\ge 1$ be such that $\#(B)+1\le\epsilon r$. For each $j<n$, let m_j be the size of the orbit of $\psi\restriction B$ containing b_j, and $p_j=\lceil r\bar\mu b_j\rceil-1$; set $M=\sum_{j=0}^{n-1}m_j p_j$. Because $\mathfrak{A}$ is atomless, we can find a disjoint family $\langle c_{jl}\rangle_{l<p_j}$ in $\mathfrak{A}_{b_j}$ such that $\bar\mu c_{jl}=\frac{1}{r}$ for every $l<p_j$. Because $\langle\psi^k b_j\rangle_{j<n,k<m_j}$ is disjoint, so is $\langle\psi^k c_{jl}\rangle_{j<n,l<p_j,k<m_j}$. Set

$$C=\{\psi^k c_{jl}:j<n,\ l<p_j,\ k<m_j\},\quad c=\sup C;$$

then

$$\begin{aligned}\bar\mu c=\frac{M}{r}&=\frac{1}{r}\sum_{j=0}^{n-1}p_j m_j\ge\frac{1}{r}\sum_{j=0}^{n-1}m_j(r\bar\mu b_j-1)\\&=1-\frac{1}{r}\sum_{j=0}^{n-1}m_j=1-\frac{\#(B)}{r}\ge 1-\epsilon.\end{aligned}$$

We shall need to know later that

$$\frac{M}{r}=\frac{1}{r}\sum_{j=0}^{n-1}p_j m_j<\frac{1}{r}\sum_{j-0}^{n-1}r m_j\bar\mu b_j=1.$$

(ii) Let $f:C\to C$ be the cyclic permutation defined by setting

$$\begin{aligned} f(\psi^k c_{jl}) &= \psi^{k+1}c_{jl} \text{ if } j < n,\ l < p_j,\ k \le m_j - 2,\\ &= c_{j,l+1} \text{ if } j < n,\ l \le p_j - 2,\ k = m_j - 1,\\ &= c_{j+1,0} \text{ if } j \le n-2,\ l = p_j - 1,\ k = m_j - 1,\\ &= c_{00} \text{ if } j = n-1,\ l = p_j - 1,\ k = m_j - 1. \end{aligned}$$

Set

$$C' = \{c : c \in C,\ f(c) \text{ and } \psi(c) \text{ are included in different members of } B\}.$$

Then $\#(C') \le n$. **P** If $c \in C'$, express it as $\psi^k c_{jl}$ where $j < n$, $l < p_j$ and $k < m_j$. We surely have $f(c) \ne \psi c$, so k must be $m_j - 1$. In this case,

$$\psi c = \psi^{m_j} c_{jl} \subseteq \psi^{m_j} b_j = b_j,$$

so $f(c) \not\subseteq b_j$ and l must be $p_j - 1$. Thus $c = \psi^{m_j-1}c_{j,p_j-1}$ for some $j < n$, and there are only n objects of this form. **Q**

(iii) We know that there is a two-sided Bernouilli shift π_0 on $(\mathfrak{A}, \bar\mu)$ (385Sb). Now π_0 is mixing (385Se), therefore ergodic (372Qb) and aperiodic (386D). We know that $\frac{M}{r} < 1$, so by 386C again there is a $d_0 \in \mathfrak{A}$ such that $d_0, \pi_0 d_0, \dots, \pi_0^{M-1}d_0$ are disjoint and $\bar\mu d_0 = \frac{1}{r}$. Because $\bar\mu f^i(c_{00}) = \bar\mu\pi_0^i d_0 = \frac{1}{r}$ for every $i < M$ and $\mathfrak{A}$ is homogeneous, there is a $\theta \in \mathrm{Aut}_{\bar\mu}\mathfrak{A}$ such that $\theta(\pi_0^i d_0) = f^i(c_{00})$ for every $i < M$. Set $\pi = \theta\pi_0\theta^{-1}$; then π is a two-sided Bernouilli shift (385Sg). Now

$$\pi f^i(c_{00}) = \theta\pi_0\theta^{-1}f^i(c_{00}) = \theta\pi_0\pi_0^i d_0 = f^{i+1}(c_{00})$$

whenever $i \le M - 2$. So

$$\begin{aligned} C'' &= \{c : c \in C,\ \pi c \text{ and } \psi(c) \text{ are included in different members of } B\}\\ &\subseteq C' \cup \{f^{M-1}(c_{00})\} \end{aligned}$$

has at most $n + 1$ members.

Because B is disjoint, $e = \sup_{b\in B} \pi b \triangle \psi b$ is disjoint from

$$\sup_{b\in B} \pi b \cap \psi b \supseteq \sup(C \setminus C'')$$

and has measure at most

$$\bar\mu(\sup C'') + \bar\mu(1 \setminus c) \le \frac{n+1}{r} + \epsilon \le 2\epsilon.$$

If $a \in A$, then a is the supremum of the members of B it includes, so $\pi a \triangle \psi a \subseteq e$ and

$$\bar\mu(\pi a \triangle \phi a) \le \bar\mu(\pi a \triangle \psi a) + \bar\mu(\psi a \triangle \phi a) \le 3\epsilon.$$

(iv) Thus, given $\phi \in \mathrm{Aut}_{\bar\mu}\mathfrak{A}$, $A \in [\mathfrak{A}]^{<\omega}$ and $\epsilon > 0$, we can find a two-sided Bernouilli shift π such that $\bar\mu(\pi a \triangle \phi a) \le 3\epsilon$ for every $a \in A$; as ϕ, A and ϵ are arbitrary, the two-sided Bernouilli shifts are dense in $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$.

(c)(i) The point is that $L^2_{\mathbb{C}} = L^2_{\mathbb{C}}(\mathfrak{A}, \bar\mu)$ is separable in its norm topology. **P** By 331O, there is a countable set $A \subseteq \mathfrak{A}$ which is dense for the measure-algebra topology of $\mathfrak{A}$. Let C be a countable dense subset of $\mathbb{C}$, and

$$D = \{\sum_{j=0}^n \zeta_j\chi a_j : \zeta_0, \dots, \zeta_n \in C,\ a_0, \dots, a_n \in A\},$$

so that D is a countable subset of $L^2_{\mathbb{C}}$. Because the function

$$(\zeta_0, \dots, \zeta_n, a_0, \dots, a_n) \mapsto \sum_{j=0}^n \zeta_j\chi a_j : \mathbb{C}^{n+1} \times \mathfrak{A}^{n+1} \to L^2_{\mathbb{C}}$$

is continuous for each n, $\overline{D}$ contains $\sum_{j=0}^n \zeta_j\chi a_j$ whenever $\zeta_0, \dots, \zeta_n \in \mathbb{C}$ and $a_0, \dots, a_n \in \mathfrak{A}$, that is, $S_{\mathbb{C}} = S_{\mathbb{C}}(\mathfrak{A}) \subseteq \overline{D}$. But $S_{\mathbb{C}}$ is norm-dense in $L^2_{\mathbb{C}}$, so D also is dense and $L^2_{\mathbb{C}}$ is separable. **Q**

(ii) For $\pi \in \mathrm{Aut}_{\bar\mu}\mathfrak{A}$, let $T_\pi : L^2_{\mathbb{C}} \to L^2_{\mathbb{C}}$ be the corresponding linear operator, as in 494D. We need to know that the function $\pi \mapsto T_\pi v : \mathrm{Aut}_{\bar\mu}\mathfrak{A} \to L^2_{\mathbb{C}}$ is continuous for every $v \in L^2$. **P** It is elementary to check that $a \mapsto \chi a : \mathfrak{A} \to L^2_{\mathbb{C}}$ is continuous for the measure-algebra topology on $\mathfrak{A}$, so $(\pi, a) \mapsto T_\pi \chi a = \chi \pi a$ is continuous (494Ba-494Bb), and $\pi \mapsto T_\pi \chi a$ is continuous, for every $a \in \mathfrak{A}$. Because addition and scalar multiplication are continuous on $L^2_{\mathbb{C}}$, $\pi \mapsto T_\pi v$ is continuous for every $v \in S_{\mathbb{C}}$. Now if v is any member of $L^2_{\mathbb{C}}$, $\phi \in \mathrm{Aut}_{\bar\mu}\mathfrak{A}$ and $\epsilon > 0$, there is a $v' \in S_{\mathbb{C}}$ such that $\|v - v'\|_2 \le \epsilon$, in which case

$$\{\pi : \|T_\pi v - T_\phi v\|_2 \le 3\epsilon\} \supseteq \{\pi : \|T_\pi v' - T_\phi v'\| \le \epsilon\}$$

is a neighbourhood of ϕ. Thus $\pi \mapsto T_\pi v$ is continuous for arbitrary $v \in L^2_{\mathbb{C}}$. **Q**

(iii) It follows from (i) that the set $V = \{v : v \in L^2_{\mathbb{C}}, \|v\|_2 = 1, \int v = 0\}$ is separable (4A2P(a-iv)). Let D' be a countable dense subset of V. For $v \in D'$, set

$$F_v = \{\pi : |(T^k_\pi v|v)| \ge \frac{1}{2} \text{ for every } k \in \mathbb{N}\}.$$

Since the maps

$$\pi \mapsto \pi^k \mapsto T_{\pi^k} v = T^k_\pi v \mapsto (T^k_\pi v|v)$$

are all continuous (494Ba and (ii) just above), F_v is closed. Consider $E = \mathrm{Aut}_{\bar\mu}\mathfrak{A} \setminus \bigcup_{v \in D'} F_v$. If $\pi \in \mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is weakly mixing, then $(\alpha)\Rightarrow(\gamma)$ of 494D tells us that $\pi \in E$. On the other hand, if $\pi \in \mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is not weakly mixing, $(\beta)\Rightarrow(\alpha)$ of 494D tells us that there is a $w \in V$ such that $\inf_{k\in\mathbb{N}} |(T^k_\pi w|w)| \ge 1$. Let $v \in D'$ be such that $\|v - w\|_2 \le \frac{1}{4}$. Then, for any $k \in \mathbb{N}$,

$$|(T^k_\pi v|v)| \ge |(T^k_\pi w|v)| - \|T^k_\pi w - T^k_\pi v\|_2 \|v\|_2 \ge |(T^k_\pi w|v)| - \frac{1}{4}$$

$$\ge |(T^k_\pi w|w)| - \|T^k_\pi w\|_2 \|v - w\|_2 - \frac{1}{4} \ge \frac{1}{2}.$$

So $\pi \in F_v \subseteq (\mathrm{Aut}_{\bar\mu}\mathfrak{A}) \setminus E$. Thus the set of weakly mixing automorphisms is precisely E, and is a G_δ set.

(d) We know that every two-sided Bernouilli shift is weakly mixing (385Se, 372Qb), so the set E of weakly mixing automorphisms is dense, by (b) here, and G_δ, by (c), therefore comeager. By (a), the set E' of weakly mixing automorphisms which are not mixing is also comeager. By 494Be, $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is a Polish space, so E' is non-empty.

494F 494Ed tells us that 'many' automorphisms of the Lebesgue probability algebra are weakly mixing but not mixing. It is another matter to give an explicit description of one. Bare-handed constructions (e.g., Chacon 69) demand ingenuity and determination. I prefer to show you an example taken from Tao l08, Lecture 12, Exercises 5 and 8, although it will take some pages in the style of this book, as it gives practice in using ideas already presented.

Example (a) There is a Radon probability measure ν on $\mathbb{R}$, zero on singletons, such that

$$\int \cos(2\pi \cdot 3^j t)\, \nu(dt) = \int \cos 2\pi t\, \nu(dt) > 0$$

for every $j \in \mathbb{N}$.

(b) Set $\sigma_{jk} = \int \cos 2\pi(k - j)t\, \nu(dt)$ for j, $k \in \mathbb{Z}$. Then there is a centered Gaussian distribution μ on $X = \mathbb{R}^{\mathbb{Z}}$ with covariance matrix $\langle\sigma_{jk}\rangle_{j,k\in\mathbb{Z}}$.

(c) Let $S : X \to X$ be the shift operator defined by saying that $(Sx)(j) = x(j + 1)$ for $x \in X$ and $j \in \mathbb{Z}$. Then S is an automorphism of (X, μ).

(d) Let $(\mathfrak{A}, \bar\mu)$ be the measure algebra of μ and $\phi \in \mathrm{Aut}_{\bar\mu}\mathfrak{A}$ the automorphism represented by S. Then ϕ is not mixing.

(e) ϕ is weakly mixing.

proof (a)(i) Let $\tilde\nu$ be the usual measure on $\mathcal{P}\mathbb{N}$ (254Jb, 464A). Define $h : \mathcal{P}\mathbb{N} \to \mathbb{R}$ by setting

$$h(I) = \frac{2}{3}\sum_{j\in I} 3^{-j}$$

for $I \subseteq \mathbb{N}$. Then h is continuous, so the image measure $\nu = \tilde\nu h^{-1}$ is a Radon probability measure on $\mathbb{R}$ (418I). Also h is injective, so ν, like $\tilde\nu$, is zero on singletons.

(ii) The function $t \mapsto \mathord{<}3t\mathord{>} = 3t - \lfloor 3t \rfloor$ is inverse-measure-preserving for ν. **P** Set $\psi_0(I) = \{j : j+1 \in I\}$ for $I \subseteq \mathbb{N}$, $\psi_1(t) = \mathord{<}3t\mathord{>}$ for $t \in \mathbb{R}$. Then $\psi_0 : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ is inverse-measure-preserving for $\tilde{\nu}$, because

$$\tilde{\nu}\{I : \psi_0(I) \cap J = K\} = \tilde{\nu}\{I : I \cap (J+1) = K+1\} = 2^{-\#(J)}$$

whenever $K \subseteq J \in [\mathbb{N}]^{<\omega}$. Next, for any $I \in \mathcal{P}\mathbb{N} \setminus \{\mathbb{N}, \mathbb{N} \setminus \{0\}\}$,

$$\psi_1(h(I)) = \mathord{<}2\sum_{j\in I} 3^{-j}\mathord{>} = 2\sum_{j\in I\setminus\{0\}} 3^{-j} = 2\sum_{j+1\in I} 3^{-j-1} = h(\psi_0(I)).$$

So $\psi_1 h =_{\text{a.e.}} h\psi_0$, and

$$\nu\psi_1^{-1} = \tilde{\nu}h^{-1}\psi_1^{-1} = \tilde{\nu}\psi_0^{-1}h^{-1} = \tilde{\nu}h^{-1} = \nu. \textbf{ Q}$$

Similarly, if we set

$$\begin{aligned} \theta(t) &= \frac{1}{3} - t \text{ if } 0 \le t \le \frac{1}{3}, \\ &= \frac{5}{3} - t \text{ if } \frac{2}{3} \le t \le 1, \\ &= t \text{ otherwise,} \end{aligned}$$

then $\theta h(I) = h(I \triangle (\mathbb{N} \setminus \{0\}))$ for every $I \subseteq \mathbb{N}$, and $\nu\theta^{-1} = \nu$.

(iii) Consequently, for any $m \in \mathbb{N}$,

$$\int \cos(2\pi \cdot 3mt)\,\nu(dt) = \int \cos(2\pi m\mathord{<}3t\mathord{>})\,\nu(dt) = \int \cos 2\pi mt\,\nu(dt)$$

(235G). Inducing on j, we see that

$$\int \cos(2\pi \cdot 3^j t)\,\nu(dt) = \int \cos 2\pi t\,\nu(dt)$$

for every $j \in \mathbb{N}$.

(iv) As for $\int \cos 2\pi t\,\nu(dt)$, this is equal to $\int \cos 2\pi\theta(t)\,\nu(dt)$. Now

$$\begin{aligned} \cos 2\pi t + \cos 2\pi\theta(t) &= \cos 2\pi t + \cos 2\pi(\frac{1}{3} - t) \\ &\qquad = 2\cos\frac{\pi}{3}\cos 2\pi(t - \frac{1}{6}) > 0 \text{ if } 0 \le t \le \frac{1}{3}, \\ &= \cos 2\pi t + \cos 2\pi(\frac{5}{3} - t) \\ &\qquad = 2\cos\frac{5\pi}{3}\cos 2\pi(t - \frac{5}{6}) > 0 \text{ if } \frac{2}{3} \le t \le 1; \end{aligned}$$

but $h[\mathcal{P}\mathbb{N}] \subseteq [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, so $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ is ν-conegligible, and $\cos 2\pi t + \cos 2\pi\theta(t) > 0$ for ν-almost every t. Accordingly

$$\int \cos 2\pi t\,\nu(dt) = \frac{1}{2}\int \cos 2\pi t + \cos 2\pi\theta(t)\nu(dt) > 0.$$

(b) $\sigma_{jk} = \sigma_{kj}$ for all j, $k \in \mathbb{Z}$. If $J \subseteq \mathbb{Z}$ is finite and $\langle\gamma_j\rangle_{j\in J} \in \mathbb{R}^J$, then

$$\begin{aligned}\sum_{j,k\in J}\gamma_j\gamma_k\sigma_{jk} &= \sum_{j,k\in J}\gamma_j\gamma_k\int\cos 2\pi(k-j)t\,\nu(dt)\\ &= \sum_{j,k\in J}\gamma_j\gamma_k\int\cos 2\pi kt\cos 2\pi jt+\sin 2\pi kt\sin 2\pi jt\,\nu(dt)\\ &= \int\sum_{j,k\in J}\gamma_j\gamma_k\cos 2\pi kt\cos 2\pi jt\,\nu(dt)\\ &\qquad\qquad +\int\sum_{j,k\in J}\gamma_j\gamma_k\sin 2\pi kt\sin 2\pi jt\,\nu(dt)\\ &= \int\sum_{j\in J}\gamma_j\cos 2\pi jt\sum_{k\in J}\gamma_k\cos 2\pi kt\,\nu(dt)\\ &\qquad\qquad +\int\sum_{j\in J}\gamma_j\sin 2\pi jt\sum_{k\in J}\gamma_k\sin 2\pi kt\,\nu(dt)\\ &\ge 0.\end{aligned}$$

By 456C(iv), we have a Gaussian distribution of the right kind.

(c) Of course S is linear, and $\mathbb{Z}$ is countable, so the image measure μS^{-1} is a centered Gaussian distribution (456Ba). Since

$$\begin{aligned}\int x(j)x(k)(\mu S^{-1})(dx) &= \int(Sx)(j)(Sx)(k)\mu(dx)\\ &= \int x(j+1)x(k+1)\mu(dx)=\sigma_{j+1,k+1}=\sigma_{jk}\end{aligned}$$

for all j, $k\in\mathbb{Z}$, μS^{-1} and μ have the same covariance matrix, and are equal (456Bb). Thus the bijection S is an automorphism of (X,μ).

(d) Write L^2 for $L^2(\mathfrak{A},\bar\mu)$, and $T_\phi:L^2\to L^2$ for the linear operator associated with the automorphism ϕ. For $k\in\mathbb{Z}$, set $f_k(x)=x(k)$ for $x\in X$ and $u_k=f_k^\bullet\in L^2$. Then $f_kS=f_{k+1}$ so $T_\phi u_k=u_{k+1}$, by 364Qd. Consider

$$\begin{aligned}(T_\phi^{3^j}u_0|u_0) &= \int u_{3^j}\times u_0=\int x(3^j)x(0)\mu(dx)\\ &= \sigma_{3^j,0}=\int\cos(2\pi\cdot 3^jt)\nu(dt)=\int\cos 2\pi t\,\nu(dt)\ne 0,\end{aligned}$$

for every j, while

$$\int u_0=\int x(0)\mu(dx)=0.$$

By 372Q(a-iv), π is not mixing.

(e)(i) $\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^n|\int e^{2\pi ikt}\nu(dt)|^2=0$. **P** For any $n\in\mathbb{N}$,

$$\begin{aligned}\frac{1}{n+1}\sum_{k=0}^n|\int e^{2\pi ikt}\nu(dt)|^2 &= \frac{1}{n+1}\sum_{k=0}^n\int e^{2\pi iks}\nu(ds)\int e^{-2\pi ikt}\nu(dt)\\ &= \int\frac{1}{n+1}\sum_{k=0}^n e^{2\pi ik(s-t)}\nu^2(d(s,t))\end{aligned}$$

where ν^2 is the product measure on $\mathbb{R}^2$. Now, for any s, $t\in\mathbb{R}$, $|\frac{1}{n+1}\sum_{k=0}^n e^{2\pi ik(s-t)}|\le 1$ for every n, while if $s-t$ is not an integer,

$$\frac{1}{n+1}\sum_{k=0}^{n} e^{2\pi ik(s-t)} = \frac{1-\exp(2\pi i(n+1)(s-t))}{(n+1)(1-\exp(2\pi i(s-t)))} \to 0$$

as $n \to \infty$. As ν is zero on singletons,

$$\nu^2\{(s,t) : s-t \in \mathbb{Z}\} = \int \nu\{s : s \in t+\mathbb{Z}\}\nu(ds) = 0.$$

So

$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n} |\int e^{2\pi ikt}\nu(dt)|^2 = \lim_{n\to\infty} \int \frac{1}{n+1}\sum_{k=0}^{n} e^{2\pi ik(s-t)}\nu^2(d(s,t)) = 0$$

by Lebesgue's dominated convergence theorem. **Q**

Consequently, as in (c-ii) of the proof of 494D,

$$\begin{aligned}0 = \lim_{k\to\mathcal{F}_d} |\int e^{2\pi ikt}\nu(dt)|^2 &= \lim_{k\to\mathcal{F}_d} \int e^{2\pi ikt}\nu(dt)\\ = \lim_{k\to\mathcal{F}_d} \mathcal{R}\mathrm{e} \int e^{2\pi ikt}\nu(dt) &= \lim_{k\to\mathcal{F}_d} \int \cos(2\pi kt)\nu(dt).\end{aligned}$$

By 491Sc,

$$\lim_{k\to\mathcal{F}_d} \sigma_{jk} = \lim_{k\to\mathcal{F}_d} \sigma_{j,j+k} = \lim_{k\to\mathcal{F}_d} \int \cos(2\pi kt)\nu(dt) = 0.$$

(ii) Suppose that f, $g : X \to \mathbb{R}$ are functions such that, for some finite $J \subseteq \mathbb{Z}$, there are continuous bounded functions f_0, $g_0 : \mathbb{R}^J \to \mathbb{R}$ such that $f(x) = f_0(x\restriction J)$ and $g(x) = g_0(x\restriction J)$ for every $x \in \mathbb{R}^X$. Then $\lim_{n\to\mathcal{F}_d} \int fS^n \times g\,d\mu = \int f d\mu \int g\,d\mu$.

P For any $n \in \mathbb{N}$, define $R_n : X \to \mathbb{R}^{J\times\{0,1\}}$ by setting

$$R_n(x)(j,0) = x(j), \quad R_n(x)(j,1) = x(j+n)$$

for $x \in X$ and $j \in J$; then R_n is linear, so μR_n^{-1} is a centered Gaussian distribution on $\mathbb{R}^{J\times\{0,1\}}$. The covariance matrix $\sigma^{(n)}$ of μR_n^{-1} is given by

$$\begin{aligned}\sigma^{(n)}_{(j,\epsilon),(k,\epsilon')} &= \int z(j,\epsilon)z(k,\epsilon')\mu R_n^{-1}(dz) = \int (R_n x)(j,\epsilon)(R_n x)(k,\epsilon')\mu(dx)\\ &= \int x(j)x(k)\mu(dx) = \sigma_{jk} \text{ if } \epsilon = \epsilon' = 0,\\ &= \int x(j)x(k+n)\mu(dx) = \sigma_{j,k+n} \text{ if } \epsilon = 0,\ \epsilon' = 1,\\ &= \int x(j+n)x(k)\mu(dx) = \sigma_{j+n,k} = \sigma_{k,j+n} \text{ if } \epsilon = 1,\ \epsilon' = 0,\\ &= \int x(j+n)x(j+n)\mu(dx) = \sigma_{j+n,k+n} = \sigma_{jk} \text{ if } \epsilon = \epsilon' = 1\end{aligned}$$

for all j, $k \in J$. So

$$\begin{aligned}\lim_{n\to\mathcal{F}_d} \sigma^{(n)}_{(j,\epsilon),(k,\epsilon')} &= \sigma_{jk} \text{ if } \epsilon = \epsilon',\\ &= 0 \text{ if } \epsilon \ne \epsilon'.\end{aligned}$$

Let $\tilde\mu$ be the centered Gaussian distribution $\tilde\mu$ on $\mathbb{R}^{J\times\{0,1\}}$ with covariance matrix τ where

$$\begin{aligned}\tau_{(j,\epsilon),(k,\epsilon')} &= \sigma_{jk} \text{ if } \epsilon = \epsilon',\\ &= 0 \text{ if } \epsilon \ne \epsilon'\end{aligned}$$

for any j, $k \in J$. By 456Q, there is such a distribution and $\tilde\mu = \lim_{n\to\mathcal{F}_d} \mu R_n^{-1}$ for the narrow topology.

Next observe that, for $x \in X$ and $z \in \mathbb{R}^{J\times\{0,1\}}$,

$$
\begin{aligned}
R_n(x) = z &\implies x(j+n) = z(j,1) \text{ for every } j \in J \\
&\iff (S^n x)(j) = z(j,1) \text{ for every } j \in J \\
&\implies f(S^n x) = f_0'(z), \\
R_n(x) = z &\implies x(j) = z(j,0) \text{ for every } r < m \\
&\implies g(x) = g_0'(z),
\end{aligned}
$$

where we set

$$f_0'(z) = f_0(\langle z(j,1)\rangle_{j\in J}), \quad g_0'(z) = g_0(\langle z(j,0)\rangle_{j\in J}$$

for $z \in \mathbb{R}^{J\times\{0,1\}}$. So $fS^n = f_0'R_n$, $g = g_0'R_n$,

$$\int fS^n \times g\,d\mu = \int (f_0'R_n) \times (g_0'R_n)d\mu = \int f_0' \times g_0' d(\mu R_n^{-1})$$

for every n, and

$$\lim_{n\to\mathcal{F}_d} \int fS^n \times g\,d\mu = \int f_0' \times g_0' d\tilde{\mu}$$

because $f_0' \times g_0'$ is a bounded continuous function (437Mb).

Since $\tau_{(j,0),(k,1)} = 0$ whenever j, $k \in J$, the σ-algebras Σ_0, Σ_1 generated by coordinates in $J\times\{0\}$, $J\times\{1\}$ respectively are $\tilde{\mu}$-independent (456Eb). Since f_0' is Σ_0-measurable and g_0' is Σ_1-measurable,

$$\lim_{n\to\mathcal{F}_d} \int fS^n \times g\,d\mu = \int f_0' \times g_0'\,d\tilde{\mu} = \int f_0' d\tilde{\mu} \int g_0' d\tilde{\mu}$$

(272D, 272R)

$$
\begin{aligned}
&= \lim_{n\to\mathcal{F}_d} \int f_0' d(\mu R_n^{-1}) \cdot \lim_{n\to\mathcal{F}_d} \int g_0' d(\mu R_n^{-1}) \\
&= \lim_{n\to\mathcal{F}_d} \int f_0' R_n d\mu \int g_0' R_n d\mu \\
&= \lim_{n\to\mathcal{F}_d} \int fS^n d\mu \int g\,d\mu \\
&= \lim_{n\to\mathcal{F}_d} \int f d\mu \int g\,d\mu = \int f d\mu \cdot \int g\,d\mu,
\end{aligned}
$$

as required. **Q**

(iii) If F, $F' \subseteq X$ are compact, then $\lim_{n\to\mathcal{F}_d} \mu(S^{-n}[F] \cap F') = \mu F \cdot \mu F'$. **P** Let $\epsilon > 0$. For $k \in \mathbb{N}$, set $J_k = \{j : j \in \mathbb{Z}, |j| \le k\}$ and $F_k = \{x\restriction J_k : x \in F\}$. Set $f_k^{(0)}(z) = \max(0, 1 - 2^k \rho_k(z, F_k))$ for $z \in \mathbb{R}^{J_k}$, where ρ_k is Euclidean distance in $\mathbb{R}^{J_k}$, and $f_k(x) = f_k^{(0)}(x\restriction J_k)$ for $x \in X$. Then $\langle f_k(x)\rangle_{k\in\mathbb{N}} \to \chi F(x)$ for every $x \in X$. So there is a $k \in \mathbb{N}$ such that $\int |f_k - \chi F| d\mu \le \epsilon$. Set $f = f_k$; then f is a continuous function from X to $[0,1]$, $\int |f - \chi F| \le \epsilon$, and f factors through the continuous function $f_k^{(0)} : \mathbb{R}^{J_k} \to [0,1]$.

Similarly, there is a continuous function $g : X \to [0,1]$ such that $\int |g - \chi F'| d\mu \le \epsilon$ and g factors through a continuous function on $\mathbb{R}^{J_l}$ for some l. Setting $J = J_k \cup J_l$, we see that f and g satisfy the conditions of (ii) and

$$\lim_{n\to\mathcal{F}_d} \int fS^n \times g\,d\mu = \int f d\mu \int g\,d\mu.$$

But for every $n \in \mathbb{N}$,

$$
\begin{aligned}
|\mu(S^{-n}[F] \cap F') - \int fS^n \times g| &\le \int |fS^n - \chi S^{-n}[F]| + |g - \chi F'| d\mu \\
&= \int |f - \chi F| + |g - \chi F'| d\mu \le 2\epsilon,
\end{aligned}
$$

$$|\mu F \cdot \mu F' - \int f d\mu \int g\,d\mu| \le \int |f - \chi F| + |g - \chi F'| d\mu \le 2\epsilon,$$

so

$$\limsup_{n\to\mathcal{F}_d} |\mu(S^{-n}[F]\cap F') - \mu F\cdot\mu F'|$$

$$\le 4\epsilon + \lim_{n\to\mathcal{F}_d} |\int fS^n \times g\,d\mu - \int f d\mu \int g\,d\mu| = 4\epsilon.$$

As ϵ is arbitrary, $\lim_{n\to\mathcal{F}_d}\mu(S^{-n}[F]\cap F') = \mu F\cdot\mu F'$. **Q**

(iv) Now suppose that a, $b\in\mathfrak{A}$ and $\epsilon>0$. Because μ is a Radon measure (454J(iii)), there are compact sets F_0, $F_1\subseteq X$ such that $\bar\mu(a\triangle F_0^\bullet)+\bar\mu(b\triangle F_1^\bullet)\le\epsilon$. Now, for any $n\in\mathbb{N}$,

$$\begin{aligned}|\bar\mu(\phi^n a\cap b) - \mu(S^{-n}[F_0]\cap F_1)| &= |\bar\mu(\phi^n a\cap b) - \bar\mu(\phi^n F_0^\bullet\cap F_1^\bullet)|\\ &\le \bar\mu(\phi^n a\triangle\phi^n F_0^\bullet) + \bar\mu(b\triangle F_1^\bullet)\\ &= \bar\mu(a\triangle F_0^\bullet)+\bar\mu(b\triangle F_1^\bullet)\le\epsilon,\end{aligned}$$

$$|\bar\mu a\cdot\bar\mu b - \mu F_0\cdot\mu F_1| \le |\bar\mu a - \mu F_0| + |\bar\mu b - \mu F_1| \le \epsilon.$$

So

$$\limsup_{n\to\mathcal{F}_d} |\bar\mu(\phi^n a\cap b) - \bar\mu a\cdot\bar\mu b|$$

$$\le 2\epsilon + \lim_{n\to\mathcal{F}_d} |\mu(S^{-n}[F_0]\cap F_1) - \mu F_0\cdot\mu F_1| = 2\epsilon$$

by (iii). As ϵ, a and b are arbitrary, ϕ is weakly mixing (using 491Sb once more).

Remark Of course the measure ν of part (a) is Cantor measure (256Hc, 256Xk).

494G Proposition Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and G a full subgroup of $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$, with fixed-point subalgebra $\mathfrak{C}$ (definition: 395Ga).

(a) If $a\in\mathfrak{A}^f$ and $\pi\in G$, there is a $\phi\in G$, supported by $a\cup\pi a$, such that $\phi d=\pi d$ for every $d\subseteq a$.

(b) If $(\mathfrak{A},\bar\mu)$ is localizable and a, $b\in\mathfrak{A}^f$, then the following are equiveridical:

(i) there is a $\pi\in G$ such that $\pi a\subseteq b$;

(ii) $\bar\mu(a\cap c)\le\bar\mu(b\cap c)$ for every $c\in\mathfrak{C}$.

(c) If $(\mathfrak{A},\bar\mu)$ is localizable and a, $b\in\mathfrak{A}^f$, then the following are equiveridical:

(i) there is a $\pi\in G$ such that $\pi a=b$;

(ii) $\bar\mu(a\cap c)=\bar\mu(b\cap c)$ for every $c\in\mathfrak{C}$.

(d) If $(\mathfrak{A},\bar\mu)$ is totally finite (definition: 322Ab) and $\langle a_i\rangle_{i\in I}$, $\langle b_i\rangle_{i\in I}$ are disjoint families in $\mathfrak{A}$ such that $\bar\mu(a_i\cap c)=\bar\mu(b_i\cap c)$ for every $i\in I$ and $c\in\mathfrak{C}$, there is a $\pi\in G$ such that $\pi a_i=b_i$ for every $i\in I$.

(e) If $(\mathfrak{A},\bar\mu)$ is localizable and $H=\{\pi:\pi\in\operatorname{Aut}_{\bar\mu}\mathfrak{A},\ \pi c=c$ for every $c\in\mathfrak{C}\}$, then H is the closure of G for the weak topology of $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$.

proof (a) Let $\langle(a_i,n_i,b_i)\rangle_{i\in I}$ be a maximal family such that

— $\langle a_i\rangle_{i\in I}$ is a disjoint family in $\mathfrak{A}_{\pi a\setminus a}\setminus\{0\}$,

— $\langle b_i\rangle_{i\in I}$ is a disjoint family in $\mathfrak{A}_{a\setminus\pi a}$;

— for every $i\in I$, $n_i\in\mathbb{Z}$ and $\pi^{n_i}a_i=b_i$.

Because $\bar\mu a<\infty$, I is countable. Set

$$a'=(\pi a\setminus a)\setminus\sup_{i\in I}a_i,\quad b'=(a\setminus\pi a)\setminus\sup_{i\in I}b_i;$$

then

$$\bar\mu a'=\bar\mu\pi a-\bar\mu(a\cap\pi a)-\textstyle\sum_{i\in I}\bar\mu a_i=\bar\mu a-\bar\mu(a\cap\pi a)-\sum_{i\in I}\bar\mu b_i=\bar\mu b'.$$

? If $a'\ne0$, set $c=\sup_{n\in\mathbb{Z}}\pi^n a'$. Then $\pi c=c$, so

$$\bar\mu(c\cap b_i)=\bar\mu(c\cap\pi^{n_i}a_i)=\bar\mu(\pi^{n_i}(c\cap a_i))=\bar\mu(c\cap a_i)$$

for every $i\in I$, and

$$\begin{aligned}\bar\mu(c\cap b')&=\bar\mu(c\cap a\setminus\pi a)-\sum_{i\in I}\bar\mu(c\cap b_i)=\bar\mu(c\cap a)-\bar\mu(c\cap a\cap\pi a)-\sum_{i\in I}\bar\mu(c\cap b_i)\\&=\bar\mu(c\cap\pi a)-\bar\mu(c\cap a\cap\pi a)-\sum_{i\in I}\bar\mu(c\cap a_i)=\bar\mu(c\cap\pi a\setminus a)-\sum_{i\in I}\bar\mu(c\cap a_i)\\&=\bar\mu(c\cap a')=\bar\mu a'>0,\end{aligned}$$

and $c\cap b'\ne 0$. There is therefore an $n\in\mathbb{Z}$ such that $\pi^n a'\cap b'\ne 0$. But now, setting $d=a'\cap\pi^{-n}b'$, $d\ne 0$ and we ought to have added $(d,n,\pi^n d)$ to $\langle(a_i,n_i,b_i)\rangle_{i\in I}$. **X**

Thus $\sup_{i\in I}a_i=\pi a\setminus a$ and $\sup_{i\in I}b_i=a\setminus\pi a$. Now we can define $\phi\in\operatorname{Aut}\mathfrak{A}$ by the formula

$$\begin{aligned}\phi d&=\pi d\text{ if }d\subseteq a,\\&=\pi^{n_i}d\text{ if }i\in I\text{ and }d\subseteq a_i,\\&=d\text{ if }d\cap(a\cup\pi a)=0\end{aligned}$$

(381C, because I is countable and $\mathfrak{A}$ is Dedekind σ-complete). Because G is full, $\phi\in G$; ϕ is supported by $a\cup\pi a$, and ϕ agrees with π on $\mathfrak{A}_a$, as required.

(b)(i)⇒(ii) If $\pi a\subseteq b$ and $c\in\mathfrak{C}$, then

$$\bar\mu(a\cap c)=\bar\mu\pi(a\cap c)=\bar\mu(\pi a\cap\pi c)\le\bar\mu(b\cap c).$$

(ii)⇒(i) Now suppose that $\bar\mu(a\cap c)\le\bar\mu(b\cap c)$ for every $c\in\mathfrak{C}$.

(α) Consider first the case in which $a\cap b=0$. Let $\langle(a_i,\pi_i,b_i)\rangle_{i\in I}$ be a maximal family such that
— $\langle a_i\rangle_{i\in I}$ is a disjoint family in $\mathfrak{A}_a\setminus\{0\}$,
— $\langle b_i\rangle_{i\in I}$ is a disjoint family in $\mathfrak{A}_b$,
— for every $i\in I$, $\pi_i\in G$ and $\pi_i a_i=b_i$.
Set $a'=\sup_{i\in I}a_i$, $b'=\sup_{i\in I}b_i$ and

$$c=\operatorname{upr}(b\setminus b',\mathfrak{C})=\sup_{\pi\in G}\pi(b\setminus b')\in\mathfrak{C}$$

(395G, because $\mathfrak{A}$ is Dedekind complete).

$a\cap c=a'\cap c$. **P?** Otherwise, $a\setminus a'$ meets c, so there is a $\pi\in G$ such that $(a\setminus a')\cap\pi(b\setminus b')\ne 0$, in which case we ought to have added

$$((a\setminus a')\cap\pi(b\setminus b'),\pi^{-1},\pi^{-1}(a\setminus a')\cap b\setminus b')$$

to our family $\langle(a_i,\pi_i,b_i)\rangle_{i\in I}$. **XQ**

Now note that $1\setminus c\in\mathfrak{C}$, so

$$\begin{aligned}\bar\mu(a\setminus c)\le\bar\mu(b\setminus c)&=\bar\mu(b'\setminus c)=\sum_{i\in I}\bar\mu(b_i\setminus c)\\&=\sum_{i\in I}\bar\mu\pi_i(a_i\setminus c)=\sum_{i\in I}\bar\mu(a_i\setminus c)=\bar\mu(a'\setminus c),\end{aligned}$$

so $a\setminus c=a'\setminus c$ and $a=a'$.

Accordingly we can define $\pi\in\operatorname{Aut}\mathfrak{A}$ by setting

$$\begin{aligned}\pi d&=\pi_i d\text{ if }i\in I\text{ and }d\subseteq a_i,\\&=\pi_i^{-1}d\text{ if }i\in I\text{ and }d\subseteq b_i,\\&=d\text{ if }d\subseteq 1\setminus(a\cup b')\end{aligned}$$

(381C again). Because G is full, $\pi\in G$, and

$$\pi a=\pi(\sup_{i\in I}a_i)=\sup_{i\in I}\pi a_i=\sup_{i\in I}b_i=b'\subseteq b.$$

(β) For the general case, we have

$$\bar\mu(c\cap a\setminus b)=\bar\mu(c\cap a)-\bar\mu(c\cap a\cap b)\le\bar\mu(c\cap b)-\bar\mu(c\cap a\cap b)=\bar\mu(c\cap b\setminus a)$$

for every $c \in \mathfrak{C}$, so (α) tells us that there is a $\pi_0 \in G$ such that $\pi_0(a \setminus b) \subseteq b \setminus a$. Now if we set $\pi = \overleftarrow{(a \setminus b_{\pi_0}\, \pi_0(a \setminus b))}$, $\pi \in G$ (because G is full, see 381Sd), and $\pi a \subseteq b$, as required.

(c) If $\pi \in G$ and $\pi a = b$, then $\pi a \subseteq b$ and $\pi^{-1} b \subseteq a$, so (b) tells us that $\bar\mu(a \cap c) = \bar\mu(b \cap c)$ for every $c \in \mathfrak{C}$. If $\bar\mu(a \cap c) = \bar\mu(b \cap c)$ for every $c \in \mathfrak{C}$, then (b) tells us that there is a $\pi \in G$ such that $\pi a \subseteq b$; but as $\bar\mu \pi a = \bar\mu a = \bar\mu b$, we have $\pi a = b$.

(d) Let j be any object not belonging to I and set $a_j = 1 \setminus \sup_{i \in I} a_i$, $b_j = 1 \setminus \sup_{i \in I} b_i$. Then

$$\bar\mu(a_j \cap c) = \bar\mu c - \sum_{i \in I} \bar\mu(a_i \cap c) = \bar\mu c - \sum_{i \in I} \bar\mu(b_i \cap c) = \bar\mu(b_j \cap c)$$

for every $c \in \mathfrak{C}$. Set $J = I \cup \{j\}$. By (c), there is for each $i \in J$ a $\pi_i \in G$ such that $\pi_i a_i = b_i$. Now $\langle a_i \rangle_{i \in J}$ and $\langle b_i \rangle_{i \in J}$ are partitions of unity in $\mathfrak{A}$, so there is a $\pi \in \operatorname{Aut} \mathfrak{A}$ such that $\pi d = \pi_i d$ whenever $i \in J$ and $d \subseteq a_j$; because G is full, $\pi \in G$, and has the property we seek.

(e)(i) If $a \in \mathfrak{A}$, then $U = \{\pi : \pi \in \operatorname{Aut}_{\bar\mu} \mathfrak{A}, \pi a \not\subseteq a\}$ is open for the weak topology. **P** The functions

$$\pi \mapsto \pi a : \operatorname{Aut}_{\bar\mu} \mathfrak{A} \to \mathfrak{A}, \quad b \mapsto b \setminus a : \mathfrak{A} \to \mathfrak{A}$$

are continuous (494Bb and 323Ba), and $c \mapsto \bar\mu c : \mathfrak{A} \to [0, \infty]$ is lower semi-continuous (323Cb, because $\mathfrak{A}$ is semi-finite), so $\pi \mapsto \bar\mu(\pi a \setminus a)$ is lower semi-continuous (4A2B(d-ii)) and $U = \{\pi : \bar\mu(\pi a \setminus a) > 0\}$ is open. **Q**

Consequently, $\{\pi : \pi c \subseteq c$ for every $c \in \mathfrak{C}\}$ is closed. But if $\pi c \subseteq c$ for every $c \in \mathfrak{C}$, then $\pi c = c$ for every $c \in \mathfrak{C}$. So H is closed. Of course H includes G, so $\overline{G} \subseteq H$.

(ii) Suppose that $\pi \in H$ and that U is an open neighbourhood of π. Then there are $a_0, \dots, a_n \in \mathfrak{A}^f$ and $\delta > 0$ such that U includes $\{\phi : \phi \in \operatorname{Aut}_{\bar\mu} \mathfrak{A}, \bar\mu(\pi a_i \triangle \phi a_i) \le \delta$ for every $i \le n\}$. Set $e = \sup_{i \le n} a_i$; let $\mathfrak{B}$ be the finite subalgebra of $\mathfrak{A}_e$ generated by $\{e \cap a_i : i \le n\}$, and B the set of its atoms (definition: 316K). If $b \in B$, then $\bar\mu(\pi b \cap c) = \bar\mu(b \cap c)$ for every $c \in \mathfrak{C}$, so there is a $\phi_b \in G$ such that $\phi_b b = \pi b$, by (c) above. Equally, there is a $\phi \in G$ such that $\phi e = \pi e$. Now we can define $\psi \in \operatorname{Aut}_{\bar\mu} \mathfrak{A}$ by saying that

$$\begin{aligned} \psi d &= \phi_b d \text{ if } b \in B \text{ and } d \subseteq b, \\ &= \phi d \text{ if } d \subseteq 1 \setminus e; \end{aligned}$$

as usual, $\psi \in G$, while $\psi b = \pi b$ for every $b \in B$. But this means that $\psi a_i = \pi a_i$ for every $i \le n$, so $\psi \in G \cap U$. As U is arbitrary, $\pi \in \overline{G}$; as π is arbitrary, G is dense in H and $H = \overline{G}$.

494H Proposition Let $\mathfrak{A}$ be a Boolean algebra, G a full subgroup of $\operatorname{Aut} \mathfrak{A}$, and $a \in \mathfrak{A}$. Set $G_a = \{\pi : \pi \in G, \pi$ is supported by $a\}$, $H_a = \{\pi \restriction \mathfrak{A}_a : \pi \in G_a\}$.

(a) G_a is a full subgroup of $\operatorname{Aut} \mathfrak{A}$ and H_a is a full subgroup of $\operatorname{Aut} \mathfrak{A}_a$, for every $a \in \mathfrak{A}$.

(b) Suppose that $\mathfrak{A}$ is Dedekind complete, and that the fixed-point subalgebra of G is $\mathfrak{C}$. Then the fixed-point subalgebra of H_a is $\{a \cap c : c \in \mathfrak{C}\}$.

proof (a)(i) By 381Eb and 381Eh, G_a is a subgroup of G, and $\pi \mapsto \pi \restriction \mathfrak{A}_a$ is a group homomorphism from G_a to $\operatorname{Aut} \mathfrak{A}_a$, so its image H_a is a subgroup of $\operatorname{Aut} \mathfrak{A}_a$.

(ii) Suppose that $\phi \in \operatorname{Aut} \mathfrak{A}$ and that $\langle (a_i, \pi_i) \rangle_{i \in I}$ is a family in $\mathfrak{A} \times G_a$ such that $\langle a_i \rangle_{i \in I}$ is a partition of unity in $\mathfrak{A}$ and $\pi_i d = \phi d$ whenever $i \in I$ and $d \subseteq a_i$. Then $\phi \in G$, because G is full; and

$$\phi d = \sup_{i \in I} \pi_i(d \cap a_i) = \sup_{i \in I} d \cap a_i = d$$

whenever $d \cap a = 0$, so ϕ is supported by a and belongs to G_a.

(iii) Suppose that $\phi \in \operatorname{Aut} \mathfrak{A}_a$ and that $\langle (a_i, \pi_i) \rangle_{i \in I}$ is a family in $\mathfrak{A}_a \times H_a$ such that $\langle a_i \rangle_{i \in I}$ is a partition of unity in $\mathfrak{A}_a$ and $\pi_i d = \phi d$ whenever $i \in I$ and $d \subseteq a_i$. For each $i \in I$, there is a $\pi'_i \in G_a$ such that $\pi_i = \pi'_a \restriction \mathfrak{A}_a$. Take $j \notin I$ and set $J = I \cup \{j\}$, $a_j = 1 \setminus a$, $\pi'_j = \iota$; define $\psi \in \operatorname{Aut} \mathfrak{A}$ by setting $\psi d = \phi d$ for $d \subseteq a$, d for $d \subseteq 1 \setminus a$. Then $\langle a_j \rangle_{j \in J}$ is a partition of unity in $\mathfrak{A}$ and $\psi d = \pi'_j d$ whenever $j \in J$ and $d \subseteq a_j$, so $\psi \in G$. Also ψ is supported by a, so $\phi = \psi \restriction \mathfrak{A}_a$ belongs to H_a. As ϕ and $\langle (a_i, \pi_i) \rangle_{i \in I}$ are arbitrary, H_a is full.

(b)(i) If $c \in \mathfrak{C}$, then $\pi(a \cap c) = \pi a \cap \pi c = a \cap c$ whenever $\pi \in G$ and $\pi a = a$, so $a \cap c$ belongs to the fixed-point subalgebra of H_a.

(ii) In the other direction, take any b in the fixed-point subalgebra of H_a. Set $c = \text{upr}(b, \mathfrak{C}) = \sup_{\pi \in G} \pi b$ (395G once more). Of course $b \subseteq a \cap c$. **?** If $b \neq a \cap c$, set $e = a \cap c \setminus b$. Then there is a $\pi \in G$ such that $e_1 = e \cap \pi b \neq 0$; set $e_2 = \pi^{-1} e_1 \subseteq b$ and $\phi = (\overleftarrow{e_2 \, {}_\pi \, e_1})$. Then $\phi \in G$ (381Sd again) and ϕ is supported by $e_1 \cup e_2 \subseteq a$, so $\phi \restriction \mathfrak{A}_a \in H_a$; but $\phi b \neq b$, so this is impossible. **X** Thus b is expressed as the intersection of a with a member of $\mathfrak{C}$, as required.

494I I take the proof of the next theorem in a series of lemmas, the first being the leading special case.

Lemma (Giordano & Pestov 92) Let $(\mathfrak{A}, \bar{\mu})$ be an atomless homogeneous probability algebra (definitions: 316Kb, 316N). Then $\text{Aut}_{\bar{\mu}} \mathfrak{A}$, with its weak topology, is extremely amenable.

proof I seek to apply 493C.

(a) Take $\epsilon > 0$, a neighbourhood V of the identity in $\text{Aut}_{\bar{\mu}} \mathfrak{A}$, a finite set $I \subseteq \text{Aut}_{\bar{\mu}} \mathfrak{A}$ and a finite family $\mathcal{A}$ of zero sets in $\text{Aut}_{\bar{\mu}} \mathfrak{A}$. Let $\delta > 0$ and $K \in [\mathfrak{A}]^{<\omega}$ be such that $\pi \in V$ whenever $\pi \in \text{Aut}_{\bar{\mu}} \mathfrak{A}$ and $\bar{\mu}(a \triangle \pi a) \leq \delta$ for every $a \in K$. Let C be the set of atoms of the finite subalgebra $\mathfrak{C}$ of $\mathfrak{A}$ generated by K, and D the set of atoms of the subalgebra $\mathfrak{D}$ generated by $K \cup \bigcup_{\pi \in I} \pi[K]$; set $k = \#(C)$ and $k' = \#(D)$. Let $m \in \mathbb{N}$ be so large that $2kk' \leq m\delta$ and $(m\delta - 1)^2 \geq 64m \ln \frac{1}{\epsilon}$; set $r = \lfloor m\delta \rfloor$, so that $\exp(-\frac{r^2}{64m}) \leq \epsilon$.

(b) For each $d \in D$ let E_d be a maximal disjoint family in $\mathfrak{A}_d$ such that $\bar{\mu} e = \frac{1}{m}$ for every $e \in E_d$; let E be a partition of unity in $\mathfrak{A}$, including $\bigcup_{d \in D} E_d$, such that $\bar{\mu} e = \frac{1}{m}$ for every $e \in E$. Let H be the group of permutations of E. Then we have a group homomorphism $\theta : H \to \text{Aut}_{\bar{\mu}} \mathfrak{A}$ such that $\theta(\psi) \restriction E = \psi$ for every $\psi \in H$. **P** Fix $e_0 \in E$. Then for each $e \in E$ there is a measure-preserving isomorphism $\phi_e : \mathfrak{A}_{e_0} \to \mathfrak{A}_e$, because $\mathfrak{A}$ is homogeneous (331I). For $\psi \in H_E$, E and $\langle \psi e \rangle_{e \in E}$ are partitions of unity in $\mathfrak{A}$, so we can define $\theta(\psi) \in \text{Aut}_{\bar{\mu}} \mathfrak{A}$ by the formula

$$\theta(\psi)(a) = \phi_{\psi e} \phi_e^{-1} a \text{ whenever } a \subseteq e \in E.$$

It is easy to see that $\theta(\psi)(e) = \psi e$ for every $e \in E$. If ψ, $\psi' \in H_E$, then

$$\begin{aligned} \theta(\psi\psi')(a) &= \phi_{\psi\psi' e} \phi_e^{-1} a \\ &= \phi_{\psi\psi' e} \phi_{\psi' e}^{-1} \phi_{\psi' e} \phi_e^{-1} a = \theta(\psi)\theta(\psi')(a) \end{aligned}$$

whenever $a \subseteq e \in E$; so θ is a group homomorphism. **Q**

Write G for $\theta[H]$, so that G is a subgroup of $\text{Aut}_{\bar{\mu}} \mathfrak{A}$.

(c) $I \subseteq GV^{-1}$. **P** Take $\pi \in I$. For $c \in C$, set

$$E'_c = \bigcup \{E_d : d \in D, \, d \subseteq c\}, \quad E''_c = \bigcup \{E_d : d \in D, \, d \subseteq \pi c\}.$$

Since $\sup E_d \subseteq d$ and $\bar{\mu}(d \setminus \sup E_d) \leq \frac{1}{m}$ for every $d \in D$, $\sup E'_c \subseteq c$ and $\bar{\mu}(c \setminus \sup E'_c) \leq \frac{k'}{m}$; so $m\bar{\mu}c - k' \leq \#(E'_c) \leq m\bar{\mu}c$. Similarly, $\sup E''_c \subseteq \pi c$ and

$$m\bar{\mu}c - k' = m\bar{\mu}\pi c - k' \leq \#(E''_c) \leq m\bar{\mu}\pi c = m\bar{\mu}c.$$

Let $\tilde{E}'_c \subseteq E'_c$, $\tilde{E}''_c \subseteq E''_c$ be sets of size $\min(\#(E'_c), \#(E''_c)) \geq m\bar{\mu}c - k'$. Setting $c' = \sup \tilde{E}'_c$ and $c'' = \sup \tilde{E}''_c$ we have

$$c' \subseteq c, \quad \bar{\mu}(c \setminus c') = \frac{1}{m}(m\bar{\mu}c - \#(\tilde{E}'_c)) \leq \frac{k'}{m},$$

and similarly $c'' \subseteq \pi c$ and $\bar{\mu}(\pi c \setminus c'') \leq \frac{k'}{m}$.

Because $\langle \tilde{E}'_c \rangle_{c \in C}$ and $\langle \tilde{E}''_c \rangle_{c \in C}$ are both disjoint, there is a $\psi \in H$ such that $\psi[\tilde{E}'_c] = \tilde{E}''_c$ for every $c \in C$. Set $\phi = \theta(\psi)$; then $\phi \in G$ and $\phi c' = c''$ for every $c \in C$. Now this means that

$$\begin{aligned} \bar{\mu}(c \triangle \pi^{-1}\phi c) &= \bar{\mu}(\pi c \triangle \phi c) \leq \bar{\mu}(\pi c \triangle c'') + \bar{\mu}(c'' \triangle \phi c) \\ &= \bar{\mu}(\pi c \triangle c'') + \bar{\mu}(c' \triangle c) \leq \frac{2k'}{m} \end{aligned}$$

for every $c \in C$. Consequently

$$\bar{\mu}(a \triangle \pi^{-1}\phi a) \leq \frac{2kk'}{m} \leq \delta$$

for every $a \in K$, and $\pi^{-1}\phi \in V$. Accordingly $\pi \in \phi V^{-1} \subseteq GV^{-1}$; as π is arbitrary, $I \subseteq GV^{-1}$. **Q**

(d) I am ready to introduce the functional ν demanded by the hypotheses of 493C. Let λ be the Haar probability measure on the finite group H, and ν the image measure $\lambda\theta^{-1}$, regarded as a measure on $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$. If $\pi \in I$, then (c) tells us that there is a $\psi \in H$ such that $\phi = \theta(\psi)$ belongs to πV. In this case, for any $A \in \mathcal{A}$,

$$\nu(\phi A) = \lambda\theta^{-1}[\phi A] = \lambda(\psi\theta^{-1}[A]) = \lambda\theta^{-1}[A] = \nu A.$$

So ν satisfies condition (ii) of 493C.

(e) As for condition (i) of 493C, consider $W = \{\psi : \psi \in H, \#(\{e : e \in E, \psi e \neq e\}) \leq r\}$. Then $\theta[W] \subseteq V$. **P** If $\psi \in W$, then $\theta(\psi)(d) = d$ whenever $d \subseteq e \in E$ and $\psi e = e$. So $\theta(\psi)$ is supported by $b = \sup\{e : e \in E, \psi e \neq e\}$. Now $\bar{\mu}b \leq \frac{r}{m} \leq \delta$. So $\bar{\mu}(a \triangle \phi a) \leq \bar{\mu}b \leq \delta$ for every $a \in \mathfrak{A}$, and $\phi \in V$. **Q**

Now suppose that $F \subseteq \mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ and $\nu F \geq \frac{1}{2}$. Then

$$\nu(VF) = \lambda\theta^{-1}[VF] \geq \lambda(W\theta^{-1}[F])$$

(because $\theta[W] \subseteq V$)

$$\geq 1 - \exp(-\frac{r^2}{64m})$$

(by 492I, because $\lambda\theta^{-1}[F] = \nu F \geq \frac{1}{2}$)

$$\geq 1 - \epsilon.$$

So ν satisfies the first condition in 493C.

(f) As ϵ, V, I and $\mathcal{A}$ are arbitrary, $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ is extremely amenable.

494J Lemma Let $(\mathfrak{C}, \bar{\lambda})$ be a totally finite measure algebra, $(\mathfrak{B}, \bar{\nu})$ a probability algebra, and $(\mathfrak{A}, \bar{\mu})$ the localizable measure algebra free product $(\mathfrak{C}, \bar{\lambda})\widehat{\otimes}(\mathfrak{B}, \bar{\nu})$ (325E). Give $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ its weak topology, and let G be the subgroup $\{\pi : \pi \in \mathrm{Aut}_{\bar{\mu}}\mathfrak{A}, \pi(c \otimes 1) = c \otimes 1$ for every $c \in \mathfrak{C}\}$. Suppose that $\mathfrak{B}$ is either finite, with all its atoms of the same measure, or homogeneous. Then G is amenable, and if either $\mathfrak{B}$ is homogeneous or $\mathfrak{C}$ is atomless, G is extremely amenable.

proof (a) Let $\mathcal{E}$ be the family of finite partitions of unity in $\mathfrak{C}$ not containing $\{0\}$. Then for any $E \in \mathcal{E}$ we have a function $\theta_E : (\mathrm{Aut}_{\bar{\nu}}\mathfrak{B})^E \to G$ defined by saying that

$$\theta_E(\boldsymbol{\phi})(c \otimes b) = \sup_{e \in E}(c \cap e) \otimes \phi_e b$$

whenever $\boldsymbol{\phi} = \langle \phi_e \rangle_{e \in E} \in (\mathrm{Aut}_{\bar{\nu}}\mathfrak{B})^E$, $c \in \mathfrak{C}$ and $b \in \mathfrak{B}$. **P** For each $e \in E$, the defining universal mapping theorem 325Da tells us that there is a unique measure-preserving Boolean homomorphism $\psi_e : \mathfrak{A} \to \mathfrak{A}$ such that $\psi_e(c \otimes 1) = c \otimes 1$ and $\psi_e(1 \otimes b) = 1 \otimes \phi_e b$ for all $b \in \mathfrak{B}$ and $c \in \mathfrak{C}$. To see that ψ_e is surjective, note that $\psi_e[\mathfrak{A}]$ must be a closed subalgebra including $\mathfrak{C} \otimes \mathfrak{B}$ (324Kb), which is dense (325D(c-i)). So $\psi_e \in \mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$. Now $\langle e \otimes 1 \rangle_{e \in E}$ is a partition of unity in $\mathfrak{A}$, and $\psi_e(e \otimes 1) = e \otimes 1$ for every e, so we have a $\pi \in \mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ defined by saying that $\pi a = \sup_{e \in E} \psi_e(a \cap e)$ for every $a \in \mathfrak{A}$. Because G is full, $\pi \in G$. So we can set $\theta_E(\boldsymbol{\phi}) = \pi$. Of course π is the only automorphism satisfying the given formula for $\theta_E(\boldsymbol{\phi})$. **Q**

(b)(i) It is easy to check that if $E \in \mathcal{E}$ then θ_E is a group homomorphism from $(\mathrm{Aut}_{\bar{\nu}}\mathfrak{B})^E$ to G; write G_E for its set of values. Because $0 \notin E$, θ_E is injective, and G_E is a subgroup of G isomorphic to the group $(\mathrm{Aut}_{\bar{\nu}}\mathfrak{B})^E$. Give $\mathrm{Aut}_{\bar{\nu}}\mathfrak{B}$ its weak topology, $(\mathrm{Aut}_{\bar{\nu}}\mathfrak{B})^E$ the product topology and G the topology induced by the weak topology of $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$.

(ii) θ_E is continuous. **P** If U is a neighbourhood of the identity in G_E, there are $a_0, \dots, a_n \in \mathfrak{A}$ and $\epsilon > 0$ such that U includes $\{\pi : \pi \in G_E, \bar{\mu}(a_i \triangle \pi a_i) \leq 3\epsilon$ for every $i \leq n\}$. For each $i \leq n$, there is an $a'_i \in \mathfrak{C} \otimes \mathfrak{B}$ such that $\bar{\mu}(a_i \triangle a'_i) \leq \epsilon$. Let $\mathfrak{B}_0$ be a finite subalgebra of $\mathfrak{B}$ such that $a'_i \in \mathfrak{C} \otimes \mathfrak{B}_0$ for every $i \leq n$. Let $\delta > 0$ be such that $\delta\bar{\lambda}1 \leq \epsilon$. Then there is a neighbourhood V of the identity in $\mathrm{Aut}_{\bar{\nu}}\mathfrak{B}$ such that $\bar{\nu}(b \triangle \phi b) \le \delta$ whenever $\phi \in V$ and $b \in \mathfrak{B}_0$. If now $\boldsymbol{\phi} = \langle \phi_e \rangle_{e \in E}$ belongs to V^E, then for each $i \le n$ we can express a'_i as $\sup_{j \le m_i} c_{ij} \otimes b_{ij}$ where $\langle c_{ij} \rangle_{j \le m_i}$ is a partition of unity in $\mathfrak{C}$ and $b_{ij} \in \mathfrak{B}_0$ for every $j \le m_i$ (315Oa). So

$$\bar{\mu}(a'_i \triangle \theta_E(\boldsymbol{\phi}) a'_i) \le \sum_{\substack{j \le m_i \\ e \in E}} \bar{\mu}(((c_{ij} \cap e) \otimes b_{ij}) \triangle \theta_E(\boldsymbol{\phi})((c_{ij} \cap e) \otimes b_{ij}))$$

(because $\langle (c_{ij} \cap e) \otimes b_{ij} \rangle_{j \le m_i, e \in E}$ is a disjoint family with supremum a'_i)

$$= \sum_{\substack{j \le m_i \\ e \in E}} \bar{\mu}(((c_{ij} \cap e) \otimes b_{ij}) \triangle ((c_{ij} \cap e) \otimes \phi_e b_{ij}))$$

$$= \sum_{\substack{j \le m_i \\ e \in E}} \bar{\mu}((c_{ij} \cap e) \otimes (b_{ij} \triangle \phi_e b_{ij}))$$

$$= \sum_{\substack{j \le m_i \\ e \in E}} \bar{\lambda}(c_{ij} \cap e) \cdot \bar{\nu}(b_{ij} \triangle \phi_e b_{ij}) \le \delta \sum_{\substack{j \le m_i \\ e \in E}} \bar{\lambda}(c_{ij} \cap e) \le \epsilon,$$

and

$$\bar{\mu}(a_i \triangle \theta_E(\boldsymbol{\phi}) a_i) \le \bar{\mu}(a_i \triangle a'_i) + \bar{\mu}(a'_i \triangle \theta_E(\boldsymbol{\phi}) a'_i) + \bar{\mu}(\theta_E(\boldsymbol{\phi}) a'_i \triangle \theta_E(\boldsymbol{\phi}) a_i) \le 3\epsilon.$$

This is true for every $i \le n$, so $\theta_E(\boldsymbol{\phi}) \in U$ whenever $\boldsymbol{\phi} \in V^E$. As U is arbitrary, θ_E is continuous. **Q**

(iii) θ_E^{-1} is continuous. **P** Let V be a neighbourhood of the identity in $\operatorname{Aut}_{\bar{\nu}} \mathfrak{B}$. Then there are $\epsilon > 0$ and $b_0, \dots, b_n \in \mathfrak{B}$ such that $\phi \in V$ whenever $\phi \in \operatorname{Aut}_{\bar{\nu}} \mathfrak{B}$ and $\bar{\nu}(b_i \triangle \phi b_i) \le \epsilon$ for every $i \le n$. Let $\delta > 0$ be such that $\delta \le \epsilon \bar{\lambda} e$ for every $e \in E$, and let U be

$$\{\pi : \pi \in G_E,\ \bar{\mu}((e \otimes b_i) \triangle \pi(e \otimes b_i)) \le \delta \text{ whenever } e \in E \text{ and } i \le n\}.$$

Then U is a neighbourhood of the identity in G_E. If $\boldsymbol{\phi} = \langle \phi_e \rangle_{e \in E} \in (\operatorname{Aut}_{\bar{\nu}} \mathfrak{B})^E$ is such that $\theta_E(\boldsymbol{\phi}) \in U$, then for every $e \in E$ and $i \le n$ we have

$$\bar{\nu}(b_i \triangle \phi_e b_i) = \frac{1}{\bar{\lambda} e} \bar{\mu}((e \otimes b_i) \triangle \theta_E(\boldsymbol{\phi})(e \otimes b_i)) \le \frac{\delta}{\bar{\lambda} e} \le \epsilon,$$

so $\boldsymbol{\phi} \in V^E$. As V is arbitrary, θ_E^{-1} is continuous. **Q**

(iv) Putting these together, θ_E is a topological group isomorphism.

(c) The next step is to show that $\bigcup_{E \in \mathcal{E}} G_E$ is dense in G.

(i) Note first that there is an upwards-directed family $\mathbb{D}$ of finite subalgebras $\mathfrak{D}$ of $\mathfrak{B}$ such that if $\mathfrak{D} \in \mathbb{D}$ then every atom of $\mathfrak{D}$ has the same measure, and $\bigcup \mathbb{D}$ is dense in $\mathfrak{B}$ (for the measure-algebra topology of $\mathfrak{B}$). **P** If $\mathfrak{B}$ is finite, with all its atoms of the same measure, this is trivial; take $\mathbb{D} = \{\mathfrak{B}\}$. Otherwise, because $\mathfrak{B}$ is homogeneous, $(\mathfrak{B}, \bar{\nu})$ must be isomorphic to the measure algebra $(\mathfrak{B}_\kappa, \bar{\nu}_\kappa)$ of the usual measure on $\{0,1\}^\kappa$ for some infinite cardinal κ, and we can take $\mathbb{D}$ to be the family of subalgebras determined by finite subsets of κ. **Q**

(ii) Suppose that $\pi \in G$, $a_0, \dots, a_n \in \mathfrak{A}$ and $\epsilon > 0$. Let $\mathfrak{A}_0$ be the subalgebra of $\mathfrak{A}$ generated by $a_0, \dots, a_n$ and A the set of its atoms; let $\eta > 0$ be such that $12\eta\#(A) \le \epsilon$. Consider subalgebras of $\mathfrak{A}$ of the form $\mathfrak{C}_0 \otimes \mathfrak{D}$ where $\mathfrak{C}_0$ is a finite subalgebra of $\mathfrak{C}$ and $\mathfrak{D} \in \mathbb{D}$. This is an upwards-directed family of subalgebras, and the closure of its union includes $c \otimes b$ whenever $c \in \mathfrak{C}$ and $b \in \mathfrak{B}$, so is the whole of $\mathfrak{A}$. There must therefore be a finite subalgebra $\mathfrak{C}_0$ of $\mathfrak{C}$, a $\mathfrak{D} \in \mathbb{D}$, and a', $a'' \in \mathfrak{C}_0 \otimes \mathfrak{D}$, for each $a \in A$, such that $\bar{\mu}(a \triangle a') \le \eta$ and $\bar{\mu}(\pi a \triangle a'') \le \eta$ for every $a \in A$. Note that this implies that

$$|\bar{\mu} a' - \bar{\mu} a''| \le |\bar{\mu} a' - \bar{\mu} a| + |\bar{\mu} a - \bar{\mu} a''| \le 2\eta$$

for every $a \in A$.

(iii) Let $E \in \mathcal{E}$ be the set of atoms of $\mathfrak{C}_0$, D the set of atoms of $\mathfrak{D}$, and γ the common measure of the members of D. For $e \in E$ and $a \in A$, set

$$D'_{ea} = \{d : d \in D,\, \bar\mu((e \otimes d) \cap a) > \frac{1}{2}\bar\mu(e \otimes d)\}, \quad b'_{ea} = \sup D'_{ea},$$

$$D''_{ea} = \{d : d \in D,\, \bar\mu((e \otimes d) \cap \pi a) > \frac{1}{2}\bar\mu(e \otimes d)\}, \quad b''_{ea} = \sup D''_{ea}.$$

Note that as A is disjoint, $\langle D'_{ea}\rangle_{a\in A}$ is disjoint, for each e; and similarly $\langle D''_{ea}\rangle_{a\in A}$ is disjoint for each e, because $\langle \pi a\rangle_{a\in A}$ is disjoint. Next, for $a \in A$ and $e \in E$, set $D_{ea} = \{d : d \in D,\ e \otimes d \subseteq a'\}$. Then, for each $a \in A$,

$$\begin{aligned}
\bar\mu(a' \mathbin{\triangle} \sup_{e\in E} e \otimes b'_{ea}) &= \sum_{\substack{e\in E\\ d\in D'_{ea}\setminus D_{ea}}} \bar\mu(e \otimes d) + \sum_{\substack{e\in E\\ d\in D_{ea}\setminus D'_{ea}}} \bar\mu(e \otimes d)\\
&\le \sum_{\substack{e\in E\\ d\in D'_{ea}\setminus D_{ea}}} 2\bar\mu((e \otimes d) \cap a) + \sum_{\substack{e\in E\\ d\in D_{ea}\setminus D'_{ea}}} 2\bar\mu((e \otimes d) \setminus a)\\
&\le \sum_{\substack{e\in E\\ d\in D\setminus D_{ea}}} 2\bar\mu((e \otimes d) \cap a) + \sum_{\substack{e\in E\\ d\in D_{ea}}} 2\bar\mu((e \otimes d) \setminus a)\\
&= 2\bar\mu(a \setminus a') + 2\bar\mu(a' \setminus a) = 2\bar\mu(a \mathbin{\triangle} a') \le 2\eta,
\end{aligned}$$

and $\bar\mu(a \mathbin{\triangle} \sup_{e\in E} e \otimes b'_{ea}) \le 3\eta$. Similarly, passing through a'' in place of a', we see that $\bar\mu(\pi a \mathbin{\triangle} \sup_{e\in E} e \otimes b''_{ea}) \le 3\eta$.

Consequently, for any $a \in A$,

$$\begin{aligned}
\sum_{e\in E} \bar\lambda e \cdot \gamma|\#(D'_{ea}) - \#(D''_{ea})| &= \sum_{e\in E} |\bar\mu(e \otimes b'_{ea}) - \bar\mu(e \otimes b''_{ea})|\\
&\le \sum_{e\in E} \bar\mu((e \otimes b'_{ea}) \mathbin{\triangle} ((e \otimes 1) \cap a))\\
&\qquad + |\bar\mu((e \otimes 1) \cap a) - \bar\mu((e \otimes 1) \cap \pi a)|\\
&\qquad + \bar\mu(((e \otimes 1) \cap \pi a) \mathbin{\triangle} (e \otimes b''_{ea}))\\
&= \sum_{e\in E} \bar\mu((e \otimes b'_{ea}) \mathbin{\triangle} ((e \otimes 1) \cap a))\\
&\qquad + \bar\mu(((e \otimes 1) \cap \pi a) \mathbin{\triangle} (e \otimes b''_{ea}))
\end{aligned}$$

(because $\pi \in G$, so $(e \otimes 1) \cap \pi a = \pi((e \otimes 1) \cap a)$ for every e)

$$= \bar\mu(a \mathbin{\triangle} \sup_{e\in E} e \otimes b'_{ea}) + \bar\mu(\pi a \mathbin{\triangle} \sup_{e\in E} e \otimes b''_{ea}) \le 6\eta.$$

(iv) Fix $e \in E$ for the moment. For each $a \in A$, take $\tilde D'_{ea} \subseteq D'_{ea}$, $\tilde D''_{ea} \subseteq D''_{ea}$ such that $\#(\tilde D'_{ea}) = \#(\tilde D''_{ea}) = \min(\#(D'_{ea}), \#(D''_{ea}))$. As $\langle \tilde D'_{ea}\rangle_{a\in A}$ and $\langle \tilde D''_{ea}\rangle_{a\in A}$ are always disjoint families, there is a permutation $\psi_e : D \to D$ such that $\psi_e[\tilde D'_{ea}] = \tilde D''_{ea}$ for every $a \in A$. Because $(\mathfrak{B}, \bar\nu)$ is homogeneous, there is a $\phi_e \in \mathrm{Aut}_{\bar\nu}\mathfrak{B}$ such that $\phi_e d = \psi_e d$ for every $d \in D$.

(v) This gives us a family $\boldsymbol{\phi} = \langle \phi_e\rangle_{e\in E}$. Consider $\theta_E(\boldsymbol{\phi})$. For each $a \in A$,

$$\begin{aligned}\bar\mu(\pi a\bigtriangleup\theta_E(\boldsymbol{\phi})(a)) &\le \bar\mu(\pi a\bigtriangleup\sup_{e\in E}e\otimes b''_{ea})+\bar\mu((\sup_{e\in E}e\otimes b''_{ea})\bigtriangleup\theta_E(\boldsymbol{\phi})(\sup_{e\in E}e\otimes b'_{ea}))\\ &\qquad+\bar\mu(\theta_E(\boldsymbol{\phi})(\sup_{e\in E}e\otimes b'_{ea})\bigtriangleup\theta_E(\boldsymbol{\phi})(a))\\ &\le 3\eta+\sum_{e\in E}\bar\mu((e\otimes b''_{ea})\bigtriangleup(e\otimes\phi_e b'_{ea}))+\bar\mu((\sup_{e\in E}e\otimes b'_{ea})\bigtriangleup a)\\ &\le 3\eta+\sum_{e\in E}\bar\lambda e\cdot\gamma\#(D''_{ea}\triangle\psi_e[D'_{ea}])+3\eta\\ &\le 6\eta+\sum_{e\in E}\bar\lambda e\cdot\gamma(\#(D''_{ea}\setminus\tilde D''_{ea})+\#(D'_{ea}\setminus\tilde D'_{ea}))\\ &=6\eta+\sum_{e\in E}\bar\lambda e\cdot\gamma|\#(D''_{ea})-\#(D'_{ea})|\le 12\eta.\end{aligned}$$

(vi) Now, for each $i\le n$, set $A_i=\{a:a\in A,\ a\subseteq a_i\}$; then

$$\bar\mu(\pi a_i\bigtriangleup\theta_E(\boldsymbol{\phi})(a_i))\le\sum_{a\in A_i}\bar\mu(\pi a\bigtriangleup\theta_E(\boldsymbol{\phi})(a))\le 12\eta\#(A)\le\epsilon,$$

while $\theta_E\boldsymbol{\phi}\in G_E$. As ϕ, $a_0,\dots,a_n$ and ϵ are arbitrary, $\bigcup_{E\in\mathcal{E}}G_E$ is dense in G.

(vii) Note also that if E, $E'\in\mathcal{E}$, there is an $F\in\mathcal{E}$ such that $G_F\supseteq G_E\cup G_{E'}$. **P** Set $F=\{e\cap e':e\in E,\ e'\in E'\}\setminus\{0\}$. If $\boldsymbol{\phi}=\langle\phi_e\rangle_{e\in E}\in(\mathrm{Aut}_{\bar\nu}\mathfrak{B})^E$, define $\langle\psi_f\rangle_{f\in F}\in(\mathrm{Aut}_{\bar\nu}\mathfrak{B})^F$ by saying that $\psi_f=\phi_e$ whenever $f\in F$, $e\in E$ and $f\subseteq e$. Then it is easy to check that $\theta_F(\langle\psi_f\rangle_{f\in F})=\theta_E(\boldsymbol{\phi})$. So $G_F\supseteq G_E$; similarly, $G_F\supseteq G_{E'}$. **Q**

So $\{G_E:E\in\mathcal{E}\}$ is an upwards-directed family of subgroups of G with dense union in G.

(d) At this point, we start looking at the rest of the hypotheses.

(i) Suppose that $\mathfrak{B}$ is atomless. Then 494I tells us that $\mathrm{Aut}_{\bar\nu}\mathfrak{B}$ is extremely amenable. So all the products $(\mathrm{Aut}_{\bar\nu}\mathfrak{B})^E$ are extremely amenable (493Bd), all the G_E are extremely amenable, and G is extremely amenable by (c) and 493Bb.

(ii) Suppose that $\mathfrak{B}$ is finite. Then $\mathrm{Aut}_{\bar\nu}\mathfrak{B}$ is finite, therefore amenable (449Cg); all the products $(\mathrm{Aut}_{\bar\nu}\mathfrak{B})^E$ are amenable (449Ce), and G is amenable (449Cb).

(e) I have still to finish the case in which $\mathfrak{C}$ is atomless and $\mathfrak{B}$ is finite. If $\mathfrak{B}=\{0\}$ then of course $G=\{\iota\}$ is extremely amenable, so we may take it that $\bar\lambda 1>0$.

(i) Take $\epsilon>0$, a neighbourhood V of the identity in G, a finite set $I\subseteq G$ and a finite family $\mathcal{A}$ of zero sets in G. Let V_1 be a neighbourhood of the identity in G such that $V_1^2\subseteq V^{-1}$. By (c), there is an $E'\in\mathcal{E}$ such that $I\subseteq G_{E'}V_1$. Set $k=\#(E')$. V_1 is a neighbourhood of the identity for the uniform topology on G (494Cd), so there is a $\delta>0$ such that $\pi\in V_1$ whenever $\pi\in G$ and the support of π has measure at most δ (494Cb). Let m be so large that $m\delta\ge k\bar\lambda 1$ and $(\frac{m\delta}{\bar\lambda 1}-1)^2\ge m\ln(\frac{2}{\epsilon})$; set $r=\lfloor\frac{m\delta}{\bar\lambda 1}\rfloor$, so that $2\exp(-\frac{r^2}{m})\le\epsilon$.

(ii) For each $e\in E'$ let D_e be a maximal disjoint set of elements of measure $\frac{1}{m}\bar\lambda 1$ in $\mathfrak{C}_e$; let $E\supseteq\bigcup_{e\in E'}D_e$ be a maximal disjoint set of elements of measure $\frac{1}{m}\bar\lambda 1$ in $\mathfrak{C}$. Note that $c=1\setminus\sup_{e\in E'}\sup D_e$ has measure at most $\frac{k}{m}\bar\lambda 1\le\delta$. Consequently $G_{E'}\subseteq G_EV_1$. **P** If $\pi'\in G_{E'}$, express $\theta_{E'}^{-1}(\pi')$ as $\langle\phi'_e\rangle_{e\in E'}$. Let $\langle\phi_e\rangle_{e\in E}\in(\mathrm{Aut}_{\bar\nu}\mathfrak{B})^E$ be such that $\phi_e=\phi'_{e'}$ whenever $e'\in E'$ and $e\in D_{e'}$, and set $\pi=\theta_E(\langle\phi_e\rangle_{e\in E})$. Then $\pi a=\pi' a$ for every $a\subseteq(1\setminus c)\otimes 1$, so $\pi^{-1}\pi'$ is supported by $c\otimes 1$ and belongs to V_1. Thus $\pi'\in\pi V_1$; as π' is arbitrary, $G_{E'}\subseteq G_EV_1$. **Q** It follows that $I\subseteq G_EV_1^2\subseteq G_EV^{-1}$.

(iii) Set $H=\mathrm{Aut}_{\bar\nu}\mathfrak{B}$, and let λ_0 be the Haar probability measure on H, that is, the uniform probability measure. Let λ be the product measure on H^E, so that λ is the Haar probability measure on H^E. Let ν be the image measure $\lambda\theta_E^{-1}$ on G. If $\pi\in I$, then G_E meets πV, so there is a $\boldsymbol{\phi}\in H^E$ such that $\theta_E(\boldsymbol{\phi})\in\pi V$; now

$$\nu(\theta_E(\boldsymbol{\phi})F)=\lambda\theta_E^{-1}[\theta_E(\boldsymbol{\phi})F]=\lambda(\boldsymbol{\phi}\theta_E^{-1}[F])=\lambda\theta_E^{-1}[F]=\nu F$$

for every $F\subseteq G$, and in particular for every $F\in\mathcal{A}$. Thus ν satisfies condition (ii) of 493C.

(iv) Set

$$U = \{\langle \phi_e \rangle_{e\in E} : \phi_e \in \mathrm{Aut}_{\bar\nu}\,\mathfrak{B} \text{ for every } e \in E,\ \#(\{e : \phi_e \text{ is not the identity}\}) \le r\}.$$

Then $\theta_E[U] \subseteq V$. **P** If $\boldsymbol{\phi} = \langle \phi_e \rangle_{e\in E}$ belongs to U, then $b = \sup\{e : e \in E,\ \phi_e \text{ is not the identity}\}$ has measure at most $\frac{r}{m}\bar\lambda 1 \le \delta$, while b supports $\theta_E(\boldsymbol{\phi})$. So $\bar\lambda(a \mathbin{\triangle} \theta_E(\boldsymbol{\phi})(a)) \le \delta$ for every $a \in \mathfrak{A}$, and $\theta_E(\boldsymbol{\phi}) \in V$. **Q**

Let ρ be the normalized Hamming metric on H^E (492D). If $\boldsymbol{\phi} = \langle \phi_e \rangle_{e\in E}$ and $\boldsymbol{\psi} = \langle \psi_e \rangle_{e\in E}$ belong to H^E and $\rho(\boldsymbol{\phi}, \boldsymbol{\psi}) \le \frac{r}{m}$, then $\{e : \phi_e\psi_e^{-1} \text{ is not the identity}\}$ has at most r members, and $\boldsymbol{\phi}\boldsymbol{\psi}^{-1} \in U$, that is, $\boldsymbol{\phi} \in U\boldsymbol{\psi}$. So if $W \subseteq H^E$ is such that $\lambda W \ge \frac{1}{2}$,

$$\lambda(UW) \ge \lambda\{\boldsymbol{\phi} : \rho(\boldsymbol{\phi}, W) \le \frac{r}{m}\} \ge 1 - 2\exp(-m(\frac{r}{m})^2)$$

(492Ea)

$$\ge 1 - \epsilon$$

by the choice of m and r. Transferring this to G, remembering that $\theta_E : H^E \to G$ is an injective homomorphism, we get

$$\nu(VF) = \lambda\theta_E^{-1}[VF] \ge \lambda(U\theta_E^{-1}[F]) \ge 1 - \epsilon$$

whenever $F \subseteq G$ and $\nu F \ge \frac{1}{2}$. So ν satisfies the first condition of 493C.

(v) As ϵ, V, I and $\mathcal{A}$ were arbitrary, 493C tells us that G is extremely amenable. This completes the proof.

494K Lemma Let $(\mathfrak{A}, \bar\mu)$ be a totally finite measure algebra, and give $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ its weak topology. Let G be a subgroup of $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$, and suppose that $G = \{\pi : \pi \in \mathrm{Aut}_{\bar\mu}\mathfrak{A},\ \pi c = c \text{ for every } c \in \mathfrak{C}\}$. Then G is amenable, and if every atom of $\mathfrak{A}$ belongs to $\mathfrak{C}$, then G is extremely amenable.

proof (a) We need the structure theorems of §333; the final one 333R is the best adapted to our purposes here. I repeat some of the special notation used in that theorem. For $n \in \mathbb{N}$, set $\mathfrak{B}_n = \mathcal{P}(n+1)$ and let $\bar\nu_n$ be the uniform probability measure on $n+1$, so that $\mathfrak{B}_n$ has $n+1$ atoms of the same measure; for an infinite cardinal κ, let $(\mathfrak{B}_\kappa, \bar\nu_\kappa)$ be the measure algebra of the usual measure on $\{0,1\}^\kappa$. Then 333R tells us that there are a partition of unity $\langle c_i \rangle_{i\in I}$ in $\mathfrak{C}$, where I is a countable set of cardinals, and a measure-preserving isomorphism $\theta : \mathfrak{A} \to \mathfrak{A}' = \prod_{i\in I} \mathfrak{C}_{c_i} \widehat{\otimes} \mathfrak{B}_i$ such that $\theta c = \langle (c \cap c_i) \otimes 1 \rangle_{i\in I}$ for every $c \in \mathfrak{C}$. In particular, for any $i \in I$,

$$\begin{aligned}(\theta c_i)(j) &= c_i \otimes 1 \text{ if } j = i,\\ &= 0 \text{ otherwise,}\end{aligned}$$

that is, $\theta[\mathfrak{A}_{c_i}]$ is just the principal ideal of $\mathfrak{A}'$ corresponding to the factor $\mathfrak{C}_{c_i} \widehat{\otimes} \mathfrak{B}_i$. Thus we have an isomorphism $\theta_i : \mathfrak{A}_{c_i} \to \mathfrak{C}_{c_i} \widehat{\otimes} \mathfrak{B}_i$ such that $\theta_i c = c \otimes 1$ for every $c \in \mathfrak{C}_{c_i}$.

(b) For each $i \in I$, set $H_i = \{\pi \restriction \mathfrak{A}_{c_i} : \pi \in G\}$. Because $\pi c_i = c_i$ for every $\pi \in G$, H_i is a subgroup of $\mathrm{Aut}\,\mathfrak{A}_{c_i}$, and $\pi \mapsto \pi \restriction \mathfrak{A}_{c_i}$ is a group homomorphism from G to H_i. Set $\Theta(\pi) = \langle \pi \restriction \mathfrak{A}_{c_i} \rangle_{i\in I}$ for $\pi \in G$. Then $\Theta : G \to \prod_{i\in I} H_i$ is a group homomorphism. Because $\langle c_i \rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$, Θ is injective. In the other direction, suppose that $\boldsymbol{\phi} = \langle \phi_i \rangle_{i\in I}$ is such that every ϕ_i is a measure-preserving automorphism of $\mathfrak{A}_{c_i}$ and $\phi_i c = c$ for every $c \in \mathfrak{C}_{c_i}$. Then we have a $\pi \in \mathrm{Aut}_{\bar\mu}\mathfrak{A}$ such that $\pi a = \phi_i a$ whenever $i \in I$ and $a \subseteq c_i$; it is easy to check that $\pi \in G$ and now $\Theta(\pi) = \boldsymbol{\phi}$. Thus

$$H_i = \{\phi : \phi \in \mathrm{Aut}\,\mathfrak{A}_{c_i} \text{ is measure-preserving, } \phi c = c \text{ for every } c \in \mathfrak{C}_{c_i}\}$$

for each i, and Θ is a group isomorphism between G and $\prod_{i\in I} H_i$.

(c) As in part (b) of the proof of 494J, the next step is to confirm that Θ is a homeomorphism for the weak topologies. The argument is very similar.

(i) If U is a neighbourhood of the identity in G, then there are a finite set $K \subseteq A$ and an $\epsilon > 0$ such that U includes $\{\pi : \pi \in G,\, \bar{\mu}(a \triangle \pi a) \le 2\epsilon$ for every $a \in K\}$. Let $J \subseteq I$ be a finite set such that $\sum_{i \in I \setminus J} \bar{\mu} c_i \le \epsilon$, and set

$$V = \{\langle \phi_i \rangle_{i \in I} : \phi_i \in H_i \text{ for every } i \in I,$$
$$\bar{\mu}((a \cap c_i) \triangle \phi_i(a \cap c_i)) \le \tfrac{\epsilon}{1+\#(J)} \text{ for every } i \in J \text{ and } a \in K\}.$$

Then V is a neighbourhood of the identity in $\prod_{i \in I} H_i$. If $\boldsymbol{\phi} = \langle \phi_i \rangle_{i \in I}$ belongs to V, and $\pi = \Theta^{-1}(\boldsymbol{\phi})$, then, for $a \in K$,

$$\begin{aligned}\bar{\mu}(a \triangle \pi a) &= \sum_{i \in I} \bar{\mu}((a \cap c_i) \triangle \pi(a \cap c_i)) = \sum_{i \in I} \bar{\mu}((a \cap c_i) \triangle \phi_i(a \cap c_i)) \\ &\le \sum_{i \in J} \bar{\mu}((a \cap c_i) \triangle \phi_i(a \cap c_i)) + \sum_{i \in I \setminus J} \bar{\mu} c_i \le \tfrac{\epsilon \#(J)}{\#(J)+1} + \epsilon \le 2\epsilon,\end{aligned}$$

and $\pi \in U$. As U is arbitrary, Θ^{-1} is continuous.

(ii) If V is a neighbourhood of the identity in $\prod_{i \in I} H_i$, then there are a finite $J \subseteq I$, finite sets $K_j \subseteq \mathfrak{A}_{c_j}$ for $j \in J$, and an $\epsilon > 0$ such that $\boldsymbol{\phi} = \langle \phi_i \rangle_{i \in I}$ belongs to V if $\boldsymbol{\phi} \in \prod_{i \in I} H_i$ and $\bar{\mu}(a \triangle \phi_j a) \le \epsilon$ whenever $j \in J$ and $a \in K_j$. In this case.

$$U = \{\pi : \pi \in G,\, \bar{\mu}(a \triangle \pi a) \le \epsilon \text{ whenever } a \in \textstyle\bigcup_{j \in J} K_j\}$$

is a neighbourhood of the identity in G, and $\Theta(\pi) \in V$ whenever $\pi \in U$. As V is arbitrary, Θ is continuous.

(d) Now observe that under the isomorphism θ_i the group H_i corresponds to the group of measure-preserving automorphisms of $\mathfrak{C}_{c_i} \widehat{\otimes} \mathfrak{B}_i$ fixing $c \otimes 1$ for every $c \in \mathfrak{C}_{c_i}$. By 494J, H_i is amenable. By (b)-(c) and 449Ce, G is amenable.

(e) Finally, suppose that every atom of $\mathfrak{A}$ belongs to $\mathfrak{C}$, and look more closely at the algebras $\mathfrak{C}_{c_i} \widehat{\otimes} \mathfrak{B}_i$ and the groups H_i. If $i \in I$ is an infinite cardinal, then $\mathfrak{B}_i$ is homogeneous and 494J tells us that H_i is extremely amenable. If $0 \in I$, then $\mathfrak{B}_0 = \{0, 1\}$ and $\mathfrak{C}_{c_0} \widehat{\otimes} \mathfrak{B}_0$ is isomorphic to $\mathfrak{C}_{c_0}$; in this case, H_0 consists of the identity alone, and is surely extremely amenable. If $i \in I$ is finite and not 0, then $\mathfrak{B}_i$ is finite; and also $\mathfrak{C}_{c_i}$ is atomless. **P?** If $c \in \mathfrak{C}_{c_i}$ is an atom, take an atom b of $\mathfrak{B}_i$; then $c \otimes b$ is an atom of $\mathfrak{C}_{c_i} \widehat{\otimes} \mathfrak{B}_i$, and $\theta^{-1}(c \otimes b)$ is an atom of $\mathfrak{A}$ not belonging to $\mathfrak{C}$. **XQ** So in this case again, 494J tells us that H_i is extremely amenable. Thus G is isomorphic to a product of extremely amenable groups and is extremely amenable (493Bd).

494L Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra and G a full subgroup of $\mathrm{Aut}_{\bar{\mu}} \mathfrak{A}$, with the topology induced by the weak topology of $\mathrm{Aut}_{\bar{\mu}} \mathfrak{A}$. Then G is amenable. If every atom of $\mathfrak{A}$ with finite measure belongs to the fixed-point subalgebra of G, then G is extremely amenable.

proof (a) To begin with, suppose that $(\mathfrak{A}, \bar{\mu})$ is totally finite. Let $\mathfrak{C}$ be the fixed-point subalgebra of G, and $G' \supseteq G$ the subgroup $\{\pi : \pi \in \mathrm{Aut}_{\bar{\mu}} \mathfrak{A},\, \pi c = c$ for every $c \in \mathfrak{C}\}$. Then G is dense in G', by 494Ge. $\mathfrak{C}$ is of course the fixed-point subalgebra of G', so G' is amenable (494K) and G is amenable (449F(a-ii)). If every atom of $\mathfrak{A}$ belongs to $\mathfrak{C}$, then G' and G are extremely amenable, by 494K and 493Bf.

(b) Now for the general case.

(i) For each $a \in \mathfrak{A}^f$, set

$$G_a = \{\pi : \pi \in G,\, \pi \text{ is supported by } a\}, \quad H_a = \{\pi \restriction \mathfrak{A}_a : \pi \in G_a\}.$$

Then H_a is a full subgroup of $\mathrm{Aut}_{\bar{\mu} \restriction \mathfrak{A}_a} \mathfrak{A}_a$ (494Ha), and is isomorphic to G_a; moreover, the isomorphism is a homeomorphism for the weak topologies. **P** Set $\theta(\pi) = \pi \restriction \mathfrak{A}_a$ for $\pi \in G_a$. (α) If V is a neighbourhood of the identity in H_a, let $\delta > 0$ and $K \in [\mathfrak{A}_a]^{<\omega}$ be such that

$$V \supseteq \{\phi : \phi \in H_a,\, \bar{\mu}(b \triangle \phi b) \le \delta \text{ for every } b \in K\};$$

then

$$U = \{\pi : \pi \in G_a,\ \bar\mu(b \mathbin{\triangle} \pi b) \le \delta \text{ for every } b \in K\}$$

is a neighbourhood of the identity in G_a, and $\theta(\pi) \in V$ for every $\pi \in U$. So θ is continuous. (β) If U is a neighbourhood of the identity in G_a, let $\delta > 0$ and $K \in [\mathfrak{A}]^{<\omega}$ be such that

$$U \supseteq \{\pi : \pi \in G_a,\ \bar\mu(b \mathbin{\triangle} \pi b) \le \delta \text{ for every } b \in K\};$$

then

$$V = \{\phi : \phi \in G_a,\ \bar\mu((b \cap a) \mathbin{\triangle} \phi(b \cap a)) \le \delta \text{ for every } b \in K\}$$

is a neighbourhood of the identity in G_a, and $\theta^{-1}(\phi) \in U$ for every $\phi \in V$, because

$$\theta^{-1}(\phi)(b) = \phi(b \cap a) \cup (b \setminus a), \quad b \mathbin{\triangle} \theta^{-1}(\phi)(b) = (b \cap a) \mathbin{\triangle} \phi(b \cap a)$$

for every $\phi \in G_a$ and $b \in \mathfrak{A}$. Thus θ^{-1} is continuous. **Q**

By (a), H_a, and therefore G_a, is amenable.

(ii) $H = \bigcup_{a \in \mathfrak{A}^f} G_a$ is dense in G. **P** If $\pi \in G$, $a_0, \dots, a_n \in \mathfrak{A}^f$ and $\epsilon > 0$, set $a = \sup_{i \le n} a_i$ and $b = a \cup \pi a$. Then there is a $\phi \in G$ such that ϕ agrees with π on $\mathfrak{A}_a$ and ϕ is supported by b (494Ga). In this case, $\phi \in G_b$ and $\bar\mu(\pi a_i \mathbin{\triangle} \phi a_i) = 0 \le \epsilon$ for every $i \le n$. **Q**

Since $\langle G_a \rangle_{a \in \mathfrak{A}^f}$ is upwards-directed, and every G_a is amenable, G is amenable (449Cb).

(iii) If every atom of $\mathfrak{A}$ of finite measure is fixed under the action of G, then every atom of $\mathfrak{A}_a$ is fixed under the action of H_a, for every $a \in \mathfrak{A}^f$. So every H_a and every G_a is extremely amenable, and G is extremely amenable, by 493Bb.

494M Lemma Let $\mathfrak{A}$ be a Boolean algebra, G a full subgroup of $\operatorname{Aut}\mathfrak{A}$, and $V \subseteq G$ a symmetric set. Let $\sim_G$ be the orbit equivalence relation on $\mathfrak{A}$ induced by the action of G, so that $a \sim_G b$ iff there is a $\phi \in G$ such that $\phi a = b$. Suppose that $a \in \mathfrak{A}$ and π, $\pi' \in G$ are such that

$\pi = (\overleftarrow{b\,{}_{\pi}\,c})$ and $\pi' = (\overleftarrow{b'\,{}_{\pi'}\,c'})$ are exchanging involutions supported by a,
$b \sim_G b'$ and $a \setminus (b \cup c) \sim_G a \setminus (b' \cup c')$,
$\pi \in V$,
whenever $\phi \in G$ is supported by a there is a $\psi \in V$ agreeing with ϕ on $\mathfrak{A}_a$.

Then $\pi' \in V^3$.

proof (a) There is a $\phi \in G$, supported by a, such that $\phi\pi' = \pi\phi$. **P** We know that there are ϕ_0, $\phi_1 \in G$ such that $\phi_0(a \setminus (b' \cup c')) = a \setminus (b \cup c)$ and $\phi_1 b' = b$. Set $\phi_2 = \pi\phi_1\pi'$; then $\phi_2 \in G$, $\pi\phi_2 = \phi_1\pi'$, $\phi_2\pi' = \pi\phi_1$ and $\phi_2 c' = c$. Because $(a \setminus (b' \cup c'), b', c', 1 \setminus a)$ and $(a \setminus (b \cup c), b, c, 1 \setminus a)$ are partitions of unity in $\mathfrak{A}$, there is a $\phi \in \operatorname{Aut}\mathfrak{A}$ such that

$$\begin{aligned} \phi d &= \phi_0 d \text{ if } d \subseteq a \setminus (b' \cup c'), \\ &= \phi_1 d \text{ if } d \subseteq b', \\ &= \phi_2 d \text{ if } d \subseteq c', \\ &= d \text{ if } d \subseteq 1 \setminus a \end{aligned}$$

(381C once more); because G is full, $\phi \in G$. Of course ϕ is supported by a. Now

$$\phi\pi' d = \phi d = \pi\phi d \text{ if } d \subseteq a \setminus (b' \cup c')$$

(because $\phi d = \phi_0 d$ is disjoint from $b \cup c$),

$$\begin{aligned} &= \phi_2\pi' d = \pi\phi_1 d = \pi\phi d \text{ if } d \subseteq b', \\ &= \phi_1\pi' d = \pi\phi_2 d = \pi\phi d \text{ if } d \subseteq c', \\ &= \phi d = d = \pi\phi d \text{ if } d \subseteq 1 \setminus a. \end{aligned}$$

So $\phi\pi' = \pi\phi$. **Q**

(b) By our hypothesis, there is a $\psi \in V$ agreeing with ϕ on $\mathfrak{A}_a$. In this case,

$$\psi\pi' d = \phi\pi' d = \pi\phi d = \pi\psi d \text{ if } d \subseteq a,$$
$$= \psi d = \pi\psi d \text{ if } d \subseteq 1 \setminus a.$$

So $\psi\pi' = \pi\psi$ and

$$\pi' = \psi^{-1}\pi\psi \in V^3$$

because V is symmetric and π, $\psi \in V$.

494N Lemma Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra and $G \subseteq \mathrm{Aut}_{\bar\mu}\mathfrak{A}$ a full subgroup with fixed-point subalgebra $\mathfrak{C}$. Suppose that $\mathfrak{A}$ is relatively atomless over $\mathfrak{C}$. For $a \in \mathfrak{A}$, let $u_a \in L^\infty(\mathfrak{C})$ be the conditional expectation of χa on $\mathfrak{C}$, and let G_a be $\{\pi : \pi \in G \text{ is supported by } a\}$. Suppose that $a \subseteq e$ in $\mathfrak{A}$ and $V \subseteq G$ are such that

V is symmetric, that is, $V = V^{-1}$,

for every $\phi \in G_e$ there is a $\psi \in V$ such that ϕ and ψ agree on $\mathfrak{A}_e$,

there is an involution in V with support a,

$u_a \le \frac{2}{3}u_e$.

Then $G_a \subseteq V^{18} = \{\pi_1 \dots \pi_{18} : \pi_1, \dots, \pi_{18} \in V\}$.

proof (a)(i) Note that 494Gb, in the language of conditional expectations, tells us that if b, $c \in \mathfrak{A}$ then $b \sim_G c$ in the notation of 494M iff $u_b = u_c$. Similarly, 494Gd tells us that if $\langle b_i\rangle_{i\in I}$ and $\langle c_i\rangle_{i\in I}$ are disjoint families in $\mathfrak{A}$ and $u_{b_i} = u_{c_i}$ for every $i \in I$, there is a $\pi \in G$ such that $\pi b_i = c_i$ for every i.

(ii) It follows that every non-zero $b \in \mathfrak{A}$ is the support of an involution in G. **P** Because $\mathfrak{A}$ is relatively atomless over $\mathfrak{C}$, there is a $c \subseteq b$ such that $u_c = \frac{1}{2}u_b$ (494Ad); now there is a $\phi \in G$ such that $\phi c = b \setminus c$, and $\pi = (\overleftarrow{c\,{}_{\phi}\,b \setminus c})$ is an involution, belonging to G (381Sd once more), with support b. **Q**

(b) If $\pi' \in G$ is an involution with support $a' \subseteq e$ and $u_{a'} = u_a$, then $\pi' \in V^3$. **P** Let $\pi_0 \in V$ be an involution with support a. Because $\mathfrak{A}$ is Dedekind complete, π_0 is an exchanging involution (382Fa); express it as $(\overleftarrow{b_0\,{}_{\pi_0}\,c_0})$ and π' as $(\overleftarrow{b'\,{}_{\pi'}\,c'})$. Because $\pi b_0 = c_0$, $u_{b_0} = u_{c_0}$, while $u_{b_0} + u_{c_0} = u_{b_0 \cup c_0} = u_a$; so $u_{b_0} = \frac{1}{2}u_a$. Similarly, $u_{b'} = \frac{1}{2}u_{a'} = u_{b_0}$. On the other hand,

$$u_{e\setminus a} = u_e - u_a = u_e - u_{a'} = u_{e\setminus a'}$$

and $e \setminus a \sim_G e \setminus a'$. So the conditions of 494M are satisfied and $\pi' \in V^3$. **Q**

(c) Now suppose that π is any involution in G with support included in a. Then $\pi \in V^6$. **P** Let b, c be such that $\pi = (\overleftarrow{b\,{}_{\pi}\,c})$. Once again,

$$u_b = u_c \le \frac{1}{2}u_a, \quad u_{e\setminus a} = u_e - u_a \ge \frac{1}{2}u_a,$$

so we can find $d \subseteq e \setminus a$ such that $u_d = u_c$, while there is also a $b_1 \subseteq b$ such that $u_{b_1} = \frac{1}{2}u_b$. Set

$$c_1 = \pi b_1, \quad b_2 = b \setminus b_1, \quad c_2 = \pi b_2, \quad \pi_1 = (\overleftarrow{b_1\,{}_{\pi}\,c_1}), \quad \pi_2 = (\overleftarrow{b_2\,{}_{\pi}\,c_2});$$

then π_1 and π_2 are involutions, with supports $b_1 \cup c_1$ and $b_2 \cup c_2$ respectively, belonging to G and such that $\pi = \pi_1\pi_2$. Next, (a-iii) tells us that there are involutions π_3, $\pi_4 \in G$ with supports d and $a \setminus (b \cup c)$ (if $a = b \cup c$, set $\pi_4 = \iota$). Since π_1, π_2, π_3 and π_4 have disjoint supports, they commute (381Ef). Consequently $\pi_1\pi_3\pi_4$, $\pi_2\pi_3\pi_4$ are involutions, belonging to G, with supports $a_1 = b_1 \cup c_1 \cup (a \setminus (b \cup c)) \cup d$, $a_2 = b_2 \cup c_2 \cup (a \setminus (b \cup c)) \cup d$ respectively. But now observe that

$$u_{a_1} = u_{b_1} + u_{c_1} + u_a - u_b - u_c + u_d = u_b + u_a - u_b = u_a,$$

and similarly $u_{a_2} = u_a$. By (b), both $\pi_1\pi_3\pi_4$ and $\pi_2\pi_3\pi_4$ belong to V^3. But this means that

$$\pi = \pi_1\pi_2 = \pi_1\pi_2\pi_3^2\pi_4^2 = \pi_1\pi_3\pi_4\pi_2\pi_3\pi_4$$

belongs to V^6, as claimed. **Q**

(d) By 382N, every member of G_a is expressible as the product of at most three involutions belonging to G_a, so belongs to V^{18}.

494O Theorem (Kittrell & Tsankov 09) Suppose that $(\mathfrak{A}, \bar{\mu})$ is an atomless probability algebra and $G \subseteq \mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ is a full ergodic subgroup (definition: 395Ge), with the topology induced by the uniform topology of $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$.

(a) If $V \subseteq G$ is symmetric and G can be covered by countably many left translates of V in G, then $V^{38} = \{\pi_1\pi_2 \dots \pi_{38} : \pi_1, \dots , \pi_{38} \in V\}$ is a neighbourhood of the identity in G.

(b) If H is a topological group such that for every neighbourhood W of the identity in H there is a countable set $D \subseteq H$ such that $H = DW$, and $\theta : G \to H$ is a group homomorphism, then θ is continuous.

proof (a)(i) Let $\langle\psi_n\rangle_{n\in\mathbb{N}}$ be a sequence in G such that $G = \bigcup_{n\in\mathbb{N}} \psi_n V$. It may help if I note straight away that $\iota \in V^2$. **P** There is an $n \in \mathbb{N}$ such that $\iota \in \psi_n V$, that is, $\psi_n^{-1} \in V$; as V is symmetric, $\psi_n \in V$ and $\iota = \psi_n\psi_n^{-1}$ belongs to V^2. **Q**

(ii) As before, set $G_a = \{\pi : \pi \in G,\ \pi \text{ is supported by } a\}$ for $a \in \mathfrak{A}$. Now there is a non-zero $e \in \mathfrak{A}$ such that for every $\pi \in G_e$ there is a $\phi \in V^2$ agreeing with π on $\mathfrak{A}_e$. **P** Because $\mathfrak{A}$ is atomless, there is a disjoint sequence $\langle b_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A} \setminus \{0\}$. **?** Suppose, if possible, that for every $n \in \mathbb{N}$ there is a $\pi_n \in G_{b_n}$ such that there is no $\phi \in V^2$ agreeing with π_n on $\mathfrak{A}_{b_n}$. If $n \in \mathbb{N}$, then $V^2 = (\psi_n V)^{-1}\psi_n V$ and $\pi_n = \iota^{-1}\pi_n$, so there must be a $\pi'_n \in G_{b_n}$, either ι or π_n, not agreeing with ϕ on $\mathfrak{A}_{b_n}$ for any $\phi \in \psi_n V$. Define $\psi \in \mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ by the formula

$$\psi d = \pi'_n d \text{ if } n \in \mathbb{N} \text{ and } d \subseteq b_n,$$
$$= d \text{ if } d \cap \sup_{n\in\mathbb{N}} b_n = 0.$$

Because G is full, $\psi \in G$ and there is an $m \in \mathbb{N}$ such that $\psi \in \psi_m V$. But now π'_m agrees with ψ on $\mathfrak{A}_{b_m}$, contrary to the choice of π'_m. **X** So one of the b_n will serve for e. **Q**

(iii) There is an involution $\pi \in V^2$, supported by e, such that $\bar{\mu}(\operatorname{supp}\pi) \le \frac{2}{3}\bar{\mu}e$. **P** Take disjoint b, $b' \subseteq e$ such that $\bar{\mu}b = \bar{\mu}b' = \frac{1}{2}\bar{\mu}e$. Because G is full and ergodic, there is a $\phi \in G$ such that $\phi b = b'$. (By 395Gf, the fixed-point subalgebra of G is $\{0, 1\}$, so we can apply 494Gc.) For every $d \in \mathfrak{A}_b$, set $\phi_d = (\overleftarrow{d \,{}_{\phi}\, \phi d})$. Because G is full, $\phi_d \in G$. Observe that

$$\phi_c\phi_d = \phi_{c\setminus d}\phi_{c\cap d}\phi_{c\cap d}\phi_{d\setminus c} = \phi_{c\setminus d}\phi_{d\setminus c} = \phi_{c\triangle d}$$

for all c, $d \subseteq b$. Set $A_n = \{d : d \in \mathfrak{A}_b,\ \phi_d \in \psi_n V\}$ for each $n \in \mathbb{N}$. Since $\mathfrak{A}_b$ is complete under its measure metric, there is an $n \in \mathbb{N}$ such that A_n is non-meager; because $\mathfrak{A}$ is atomless, $\mathfrak{A}_b$ has no isolated points; so there are d_0, $d_1 \in A_n$ such that $0 < \bar{\mu}(d_0 \triangle d_1) \le \frac{1}{3}\bar{\mu}e$. Set $d = d_0 \triangle d_1$. Then

$$\phi_d = \phi_{d_0}\phi_{d_1} = \phi_{d_0}^{-1}\phi_{d_1} \in V^{-1}\psi_n^{-1}\psi_n V = V^2,$$

and we can take ϕ_d for π. **Q**

(iv) Taking $a = \operatorname{supp}\pi$ in (iv), a and e satisfy the conditions of 494N with respect to V^2 and $\mathfrak{C} = \{0, 1\}$, so $G_a \subseteq (V^2)^{18} = V^{36}$.

(v) Finally, there is a $\delta > 0$ such that, in the language of 494C, $G \cap U(1, \delta) \subseteq V^{38}$. **P?** Otherwise, we can find for each $n \in \mathbb{N}$ a $\pi_n \in G \cap U(1, 2^{-n-1}\bar{\mu}a) \setminus V^{38}$. Set $\pi'_n = \psi_n\pi_n\psi_n^{-1}$, $b_n = \operatorname{supp}\pi'_n$; then $\bar{\mu}b_n = \bar{\mu}(\operatorname{supp}\pi_n) \le 2^{-n-1}\bar{\mu}a$ for each n (381Gd). So $b = \sup_{n\in\mathbb{N}} b_n$ has measure at most $\bar{\mu}a$, and there is a $\phi \in G$ such that $\phi b \subseteq a$. In this case, there is an $n \in \mathbb{N}$ such that $\phi^{-1} \in \psi_n V$, that is, $\phi\psi_n \in V^{-1} = V$. Now $\pi = \phi\psi_n\pi_n\psi_n^{-1}\phi^{-1}$ has support $\phi b_n \subseteq a$, so belongs to V^{36}. But this means that $\pi_n = \psi_n^{-1}\phi^{-1}\pi\phi\psi_n$ belongs to V^{38}, contrary to the choice of π_n. **XQ**

So V^{38} is a neighbourhood of ι in G, as claimed.

(b) Let W be a neighbourhood of the identity in H. Then there is a symmetric neighbourhood W_1 of the identity in H such that $W_1^{38} \subseteq W$. Set $V = \theta^{-1}[W_1]$. Let W_2 be a neighbourhood of the identity in H such that $W_2^{-1}W_2 \subseteq W_1$, and $\langle y_n\rangle_{n\in\mathbb{N}}$ a sequence in H such that $H = \bigcup_{n\in\mathbb{N}} y_n W_2$. For each $n \in \mathbb{N}$, choose $\psi_n \in G$ such that $\theta(\psi_n) \in y_n W_2$ whenever $\theta[G]$ meets $y_n W_2$. If $\pi \in G$, there is an $n \in \mathbb{N}$ such that $\theta(\pi) \in y_n W_2$; in this case, $\theta(\psi_n) \in y_n W_2$, so

$$\theta(\psi_n^{-1}\pi) \in W_2^{-1}y_n^{-1}y_n W_2 \subseteq W_1$$

and $\psi_n^{-1}\pi \in V$. Thus $\pi \in \psi_n V$; as π is arbitrary, $G = \bigcup_{n\in\mathbb{N}} \psi_n V$. By (a), V^{38} is a neighbourhood of ι; but $V^{38} \subseteq \theta^{-1}[W_1^{38}] \subseteq \theta^{-1}[W]$, so $\theta^{-1}[W]$ is a neighbourhood of ι. As W is arbitrary, θ is continuous (4A5Fa).

494P Remark Note that if a topological group H is either Lindelöf or ccc, it satisfies the condition of (b) above. **P** Let W be an open neighbourhood of the identity in H. (α) If H is Lindelöf, the result follows immediately from the fact that $\{yW : y \in H\}$ is an open cover of H. (β) If H is ccc, let W_1 be an open neighbourhood of the identity such that $W_1W_1^{-1} \subseteq W$, and $D \subseteq H$ a maximal set such that $\langle yW_1\rangle_{y\in D}$ is disjoint. Then D is countable. If $x \in H$, there is a $y \in D$ such that $xW_1 \cap yW_1 \ne \emptyset$, that is, $x \in yW_1W_1^{-1} \subseteq yW$; thus $H = DW$. **Q** See also 494Yh.

494Q Some of the same ideas lead to an interesting group-theoretic property of the automorphism groups here.

Theorem (see Droste Holland & Ulbrich 08) Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra and G a full subgroup of $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ such that $\mathfrak{A}$ is relatively atomless over the fixed-point subalgebra $\mathfrak{C}$ of G. Let $\langle V_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of subsets of G such that $V_n^2 \subseteq V_{n+1}$ for every n and $G = \bigcup_{n\in\mathbb{N}} V_n$. Then there is an $n \in \mathbb{N}$ such that $G = V_n$.

proof (a) For the time being (down to the end of (e) below), suppose that every V_n is symmetric. As in 494N, for each $a \in \mathfrak{A}$ write u_a for the conditional expectation of χa on $\mathfrak{C}$, and set $G_a = \{\pi : \pi \in G, \pi$ is supported by $a\}$.

(b) There are an $\alpha_1 > 0$, an $a_1 \in \mathfrak{A}$ and an $n_0 \in \mathbb{N}$ such that $u_{a_1} = \alpha_1\chi 1$ and for every $\pi \in G_{a_1}$ there is a $\phi \in V_{n_0}$ agreeing with π on $\mathfrak{A}_{a_1}$. **P?** Otherwise, because $\mathfrak{A}$ is relatively atomless, we can choose inductively a disjoint sequence $\langle b_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $u_{b_n} = 2^{-n-1}\chi 1$ for each n (use 494Ad). For each $n \in \mathbb{N}$ there must be a $\pi_n \in G_{b_n}$ such that there is no $\phi \in V_n$ agreeing with π_n on $\mathfrak{A}_{b_n}$. Because G is full, there is a $\phi \in G$ agreeing with π_n on $\mathfrak{A}_{b_n}$ for every n. But now $\phi \notin \bigcup_{n\in\mathbb{N}} V_n = G$. **XQ**

(c) There is an $a_0 \subseteq a_1$ such that $u_{a_0} = \frac{2}{3}\alpha_1\chi 1$ and there is an involution $\pi \in G$ with support a_0. **P** Take disjoint a, $a' \subseteq a_1$ such that $u_a = u_{a'} = \frac{1}{3}\alpha_1\chi 1$ (494Ad again). Set $a_0 = a \cup a'$, so that $u_{a_0} = \frac{2}{3}\alpha_1\chi 1$. There is a $\phi \in G$ such that $\phi a = a'$, and $\pi = (\overleftarrow{a\ {}_{\phi}\ a'})$ is an involution in G with support a_0. **Q** Let $n_1 \ge n_0$ be such that $\pi \in V_{n_1}$.

(d) By 494N, $G_{a_0} \subseteq V_{n_1}^{18} \subseteq V_{n_1+5}$. Taking $k \ge \frac{3}{\alpha_1}$, 494Ad once more gives us a disjoint family $\langle d_i\rangle_{i<k}$ in $\mathfrak{A}$ such that $u_{d_i} = \frac{1}{k}\chi 1$ for every $i < k$; since $\sum_{i=0}^{k-1} \bar\mu d_i = 1$, $\langle d_i\rangle_{i<k}$ is a partition of unity, while $u_{d_i} \le \frac{1}{3}\alpha_1\chi 1$ for every i. For i, $j < k$, let $\phi_{ij} \in G$ be such that $\phi_{ij}(d_i \cup d_j) \subseteq a_0$ (494Gc). Let $n_2 \ge n_1 + 5$ be such that $\phi_{ij} \in V_{n_2}$ for all i, $j < k$. Then any involution in G belongs to $V_{n_2}^{3k^2}$. **P** Let $\pi \in G$ be an involution; by 382Fa again, we can express it as $(\overleftarrow{e\ {}_{\pi}\ e'})$. For i, $j < k$, set $e_{ij} = e \cap d_i \cap \pi d_j$, $e'_{ij} = \pi e_{ij} = e' \cap \pi d_i \cap d_j$; set $\pi_{ij} = (\overleftarrow{e_{ij}\ {}_{\pi}\ e'_{ij}})$. In this case, because all the e_{ij} and e'_{ij} are disjoint, $\langle \pi_{ij}\rangle_{i,j<k}$ is a commuting family, and we can talk of $\prod_{i,j<k} \pi_{ij}$, which of course is equal to π. Now, for each i, $j < k$,

$$\phi_{ij}\pi_{ij}\phi_{ij}^{-1} = (\overleftarrow{\phi_{ij}e_{ij}\ {}_{\phi_{ij}\pi\phi_{ij}^{-1}}\ \phi_{ij}e'_{ij}})$$

(381Sb) belongs to $G_{a_0} \subseteq V_{n_1+5} \subseteq V_{n_2}$. So

$$\pi_{ij} = \phi_{ij}^{-1}\phi_{ij}\pi_{ij}\phi_{ij}^{-1}\phi_{ij}$$

belongs to $V_{n_2}^3$ and $\pi = \prod_{i,j<k} \pi_{ij}$ belongs to $V_{n_2}^{3k^2}$. **Q**

(e) Since, by 382N again, every member of G is expressible as a product of at most three involutions belonging to G, $G \subseteq V_{n_2}^{9k^2} \subseteq V_n$, where $n = n_2 + \lceil \log_2(9k^2)\rceil$.

(f) This completes the proof on the assumption that every V_n is symmetric. For the general case, set $W_n = V_n \cap V_n^{-1}$ for every n. Then $\langle W_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of symmetric sets with union G, and

$$W_n^2 \subseteq V_n^2 \cap V_n^{-2} = V_n^2 \cap (V_n^2)^{-1} \subseteq V_{n+1} \cap V_{n+1}^{-1} = W_{n+1}$$

for every n, so there is an $n \in \mathbb{N}$ such that $G = W_n = V_n$.

494R There are many alternative versions of 494Q; see, for instance, 494Xm. Rather than attempt a portmanteau result to cover them all, I give one which can be applied to the measure algebra of Lebesgue measure on $\mathbb{R}$ and indicates some of the new techniques required.

Theorem Let $(\mathfrak{A},\bar\mu)$ be an atomless localizable measure algebra, and G a full ergodic subgroup of $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$. Let $\langle V_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of subsets of G, covering G, with $V_n^2\subseteq V_{n+1}$ for every n. Then there is an $n\in\mathbb{N}$ such that $G=V_n$.

proof (a) My aim is to mimic the proof of 494Q. We have a simplification because G is ergodic, but 494M will be applied in a different way. As before, it will be enough to consider the case in which every V_n is symmetric; as before, I will write G_a for $\{\pi:\pi\in G,\ \pi$ is supported by $a\}$.

Because G is ergodic, $\mathfrak{A}$ must be quasi-homogeneous (374G); as it is also atomless, there is an infinite cardinal κ such that $\mathfrak{A}_a$ is homogeneous, with Maharam type κ, for every $a\in\mathfrak{A}^f\setminus\{0\}$ (374H). If $(\mathfrak{A},\bar\mu)$ is totally finite, then the result is immediate from 494Q, normalizing the measure if necessary. So I will assume that $(\mathfrak{A},\bar\mu)$ is not totally finite. In this case, the orbits of G can be described in terms of 'magnitude' (332Ga). If $a\in\mathfrak{A}^f$, $\operatorname{mag}a=\bar\mu a$; otherwise, $\operatorname{mag}a$ is the cellularity of $\mathfrak{A}_a$, and there will be a disjoint family in $\mathfrak{A}_a$ of this cardinality (332F). Set $\lambda=\operatorname{mag}1\ge\omega$; then whenever $a\in\mathfrak{A}$ and $\operatorname{mag}a=\lambda$, there is a partition of unity $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}_a$ such that $\operatorname{mag}a_n=\lambda$ for every n.

(b) The key fact, corresponding to 494Gd, is as follows: if $\langle a_i\rangle_{i\in I}$ and $\langle b_i\rangle_{i\in I}$ are partitions of unity in $\mathfrak{A}$ such that $\operatorname{mag}a_i=\operatorname{mag}b_i$ for every $i\in I$, then there is a $\phi\in G$ such that $\phi a_i=b_i$ for every $i\in I$.

P (i) Consider first the case in which all the a_i, b_i have finite measure. In this case, let $\langle(c_j,\pi_j,d_j)\rangle_{j\in J}$ be a maximal family such that

— $\langle c_j\rangle_{j\in J}$ is a disjoint family in $\mathfrak{A}^f\setminus\{0\}$,

— $\langle d_j\rangle_{j\in J}$ is a disjoint family in $\mathfrak{A}$,

— for every $j\in J$, $\pi_j\in G$, $\pi_jc_j=d_j$ and there is an $i\in I$ such that $c_j\subseteq a_i$ and $d_j\subseteq b_i$.

Set $a=1\setminus\sup_{j\in J}c_j$ and $b=1\setminus\sup_{j\in J}d_j$. **?** If $a\ne0$, there is an $i\in I$ such that $a\cap a_i\ne0$. In this case,

$$\begin{aligned}\sum_{j\in J}\bar\mu(b_i\cap d_j)&=\sum_{j\in J,d_j\subseteq b_i}\bar\mu d_j=\sum_{j\in J,d_j\subseteq b_i}\bar\mu\pi_j^{-1}d_j\\&=\sum_{j\in J,c_j\subseteq a_i}\bar\mu c_j<\bar\mu a_i=\bar\mu b_i,\end{aligned}$$

and $b\cap b_i\ne0$. Because G is ergodic, there is a $\pi\in G$ such that $\pi(a\cap a_i)\cap(b\cap b_i)\ne0$. Setting $d=a\cap a_i\cap\pi^{-1}(b\cap b_i)$, we ought to have added $(d,\pi,\pi d)$ to $\langle(c_j,\pi_j,d_j)\rangle_{j\in J}$. **X**

Thus $a=0$; similarly, $b=0$ and $\langle c_j\rangle_{j\in J}$, $\langle d_j\rangle_{j\in J}$ are partitions of unity in $\mathfrak{A}$. Because $\mathfrak{A}$ is Dedekind complete, there is a $\phi\in\operatorname{Aut}\mathfrak{A}$ such that $\phi d=\pi_jd$ whenever $j\in J$ and $d\subseteq c_j$, and now $\phi\in G$ and $\phi a_i=b_i$ for every $i\in I$.

(ii) For the general case, refine the partitions $\langle a_i\rangle_{i\in I}$ and $\langle b_i\rangle_{i\in I}$ as follows. For each $i\in I$, if $\bar\mu a_i=\bar\mu b_i$ is finite, take $\lambda_i=1$, $c_{i0}=a_i$ and $d_{i0}=b_i$; otherwise, take $\lambda_i=\operatorname{mag}a_i=\operatorname{mag}b_i$, and let $\langle c_{i\xi}\rangle_{\xi<\lambda_i}$, $\langle d_{i\xi}\rangle_{\xi<\lambda_i}$ be partitions of unity in $\mathfrak{A}_{a_i}$, $\mathfrak{A}_{b_i}$ respectively with $\bar\mu c_{i\xi}=\bar\mu d_{i\xi}=1$ for every $\xi<\lambda_i$ (332I). Now (i) tells us that there is a $\phi\in G$ such that $\phi c_{i\xi}=d_{i\xi}$ whenever $i\in I$ and $\xi<\lambda_i$, in which case ϕa_i will be equal to b_i for every $i\in I$. **Q**

(c) There are an $a_1\in\mathfrak{A}$ and an $n_0\in\mathbb{N}$ such that $\operatorname{mag}a_1=\operatorname{mag}(1\setminus a_1)=\lambda$ and whenever $\pi\in G_{a_1}$ there is a $\phi\in V_{n_0}$ agreeing with π on $\mathfrak{A}_{a_1}$. **P?** Otherwise, let $\langle b_n\rangle_{n\in\mathbb{N}}$ be a partition of unity in $\mathfrak{A}$ such that $\operatorname{mag}b_n=\lambda$ for every n. For each $n\in\mathbb{N}$ there must be a $\pi_n\in G_{b_n}$ such that there is no $\phi\in V_n$ agreeing with π_n on $\mathfrak{A}_{b_n}$. Now there is a $\phi\in G$ agreeing with π_n on $\mathfrak{A}_{b_n}$ for every n. But in this case $\phi\notin\bigcup_{n\in\mathbb{N}}V_n=G$. **XQ**

(d) There is an $a_0\subseteq a_1$ such that $\operatorname{mag}a_0=\operatorname{mag}(a_1\setminus a_0)=\lambda$ and there is an involution $\pi_0\in G$ with support a_0. **P** Take disjoint a, a', $a''\subseteq a_1$ all of magnitude λ; by (b), there is a $\phi\in G$ such that $\phi a=a'$, and $\pi_0=\overleftarrow{(a\,{}_{\phi}\,a')}$ is an involution with support $a_0=a\cup a'$ of magnitude λ, while $a_1\setminus a_0\supseteq a''$ also has magnitude λ. **Q** Let $n_1\ge n_0$ be such that $\pi_0\in V_{n_1}$.

(e) If $\pi \in G$ is an involution with support $b_1 \subseteq a_1$ and $\operatorname{mag} b_1 = \operatorname{mag}(a_1 \setminus b_1) = \lambda$, then $\pi \in V_{n_1}^3$. **P** Express π_0 and π as $(\overleftarrow{a_{\pi_0} a'})$ and $(\overleftarrow{b_{\pi} b'})$ respectively, and set $\tilde a = a_1 \setminus (a \cup a')$, $\tilde b = a_1 \setminus b_1$; then a, a', b, b', $\tilde a$ and $\tilde b$ must all have magnitude λ. By (b), there is a $\phi_0 \in G$ such that

$$\phi_0 b = a, \quad \phi_0 b' = a', \quad \phi_0 \tilde b = \tilde a, \quad \phi_0(1 \setminus a_1) = 1 \setminus a_1.$$

In particular, $a \sim_G b$ and $\tilde a \sim_G \tilde b$, in the language of 494M, and (c) tells us that the final hypothesis of 494M is satisfied; so $\pi \in V_{n_1}^3$. **Q**

(f) Now suppose that π is any involution in G_{a_0}. Then $\pi \in V_{n_1}^6$. **P** Let b, b' be such that $\pi = (\overleftarrow{b_{\pi} b'})$. Next take disjoint c, $c' \subseteq a_1 \setminus a_0$ such that $\operatorname{mag} c = \operatorname{mag} c' = \operatorname{mag}(a_1 \setminus (a_0 \cup c \cup c')) = \lambda$. Then there there is an involution $\pi' \in G$ exchanging c and c', and π', $\pi\pi'$ are both involutions in G_{a_1} satisfying the conditions of (e). So both belong to $V_{n_1}^3$ and $\pi = \pi\pi'\pi' \in V_{n_1}^6$. **Q**

(g) Set $d_0 = a_0$, $d_1 = a_1 \setminus a_0$ and $d_2 = 1 \setminus a_1$. Then d_0, d_1 and d_2 all have magnitude λ, so for all i, $j < 3$ there is a $\phi_{ij} \in G$ such that $\phi_{ij}(d_i \cup d_j) = d_0$. Let $n_2 \ge n_1 + 3$ be such that $\phi_{ij} \in V_{n_2}$ for all i, $j < 3$. Then any involution in G belongs to $V_{n_2}^{27}$. **P** Let $\pi \in G$ be an involution; express it as $(\overleftarrow{e_{\pi} e'})$. For i, $j < 3$, set $e_{ij} = e \cap d_i \cap \pi d_j$, $e'_{ij} = \pi e_{ij} = e' \cap \pi d_i \cap d_j$; set $\pi_{ij} = (\overleftarrow{e_{ij\,\pi} e'_{ij}})$. In this case, because all the e_{ij} and e'_{ij} are disjoint, $\langle \pi_{ij} \rangle_{i,j<3}$ is a commuting family, and we can talk of $\prod_{i,j<3} \pi_{ij}$, which of course is equal to π. Now, for each pair i, $j < 3$,

$$\phi_{ij}\pi_{ij}\phi_{ij}^{-1} = (\overleftarrow{\phi_{ij} e_{ij}\,{}_{\phi_{ij}\pi\phi_{ij}^{-1}}\,\phi_{ij} e'_{ij}})$$

is an involution in G_{a_0}, so belongs to $V_{n_1}^6 \subseteq V_{n_1+3} \subseteq V_{n_2}$. So

$$\pi_{ij} = \phi_{ij}^{-1}\phi_{ij}\pi_{ij}\phi_{ij}^{-1}\phi_{ij}$$

belongs to $V_{n_2}^3$ and $\pi = \prod_{i,j<3} \pi_{ij}$ belongs to $V_{n_2}^{27}$. **Q**

(h) Since every member of G is expressible as a product of at most three involutions in G (382N once more), $G = V_{n_2}^{81} = V_{n_2+7}$.

494X Basic exercises (a) Let $(\mathfrak{A}, \bar\mu)$ be a semi-finite measure algebra. Show that the natural action of $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ on $\mathfrak{A}^f$ identifies $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ with a subgroup of the isometry group G of $\mathfrak{A}^f$ when $\mathfrak{A}^f$ is given its measure metric, and that the weak topology on $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ corresponds to the topology of pointwise convergence on G as described in 441G.

(b) Let $(\mathfrak{A}, \bar\mu)$ be a semi-finite measure algebra. Show that the following are equiveridical: (i) $\mathfrak{A}$ is purely atomic and has at most finitely many atoms of any fixed measure; (ii) $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is locally compact in its weak topology; (iii) $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is compact in its uniform topology; (iv) $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ has a Haar measure for its weak topology.

(c) Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra. Show that the weak topology on $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is that induced by the product topology on $\mathfrak{A}^{\mathfrak{A}}$ if $\mathfrak{A}$ is given its measure-algebra topology.

(d) Let $(\mathfrak{A}, \bar\mu)$ be a semi-finite measure algebra, G a subgroup of $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ and $\overline{G}$ its closure for the weak topology on $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$. Show that G is ergodic iff $\overline{G}$ is ergodic.

(e) Let I be a set, ν_I the usual measure on $\{0,1\}^I$, and $(\mathfrak{B}_I, \bar\nu_I)$ its measure algebra. Let Ψ be the group of measure space automorphisms g of $\{0,1\}^I$ for which there is a countable set $J \subseteq I$ such that for every $x \in \{0,1\}^I$ there is a finite set $K \subseteq J$ such that $g(x)(i) = x(i)$ for every $i \in I \setminus K$. For $g \in \Psi$, let $\pi_g \in \operatorname{Aut}_{\bar\nu_I}\mathfrak{B}_I$ be the corresponding automorphism defined by saying that $\pi_g(E^\bullet) = g^{-1}[E]^\bullet$ whenever ν_I measures E. (i) Show that $G = \{\pi_g : g \in \Psi\}$ is a full subgroup of $\operatorname{Aut}_{\bar\nu_I}\mathfrak{B}_I$. (ii) Show that G is ergodic and dense in $\operatorname{Aut}_{\bar\nu_I}\mathfrak{B}_I$ for the weak topology on $\operatorname{Aut}_{\bar\nu_I}\mathfrak{B}_I$.

(f) Let $(\mathfrak{A}, \bar\mu)$ be the measure algebra of Lebesgue measure on $[0,1]$, and give $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ its weak topology. (i) Show that the entropy function h of 385M is Borel measurable. (*Hint*: 385Xj.) (ii) Show that the set of ergodic measure-preserving automorphisms is a dense G_δ set. (*Hint*: let $D \subseteq \mathfrak{A}$ be a countable dense set. Show that $\pi \in \operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is ergodic iff $\inf_{n\in\mathbb{N}} \|\frac{1}{n+1}\sum_{i=0}^{n} \chi(\pi^i d) - \bar\mu d \cdot \chi 1\|_1 = 0$ for every $d \in D$.)

(g) Let $(\mathfrak{A},\bar\mu)$ be an atomless semi-finite measure algebra. (i) Show that $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is metrizable under its weak topology iff $(\mathfrak{A},\bar\mu)$ is σ-finite and has countable Maharam type. (ii) Show that $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is metrizable under its uniform topology iff $(\mathfrak{A},\bar\mu)$ is σ-finite.

(h) Let $(\mathfrak{A},\bar\mu)$ be a localizable measure algebra. Show that $\operatorname{supp}:\operatorname{Aut}_{\bar\mu}\mathfrak{A}\to\mathfrak{A}$ is continuous for the uniform topology on $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ and the measure-algebra topology on $\mathfrak{A}$.

(i) Let $(\mathfrak{A},\bar\mu)$ be an atomless homogeneous probability algebra. Show that there is a weakly mixing measure-preserving automorphism of $\mathfrak{A}$ which is not mixing. (*Hint*: 372Yj).

(j) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra, $\pi\in\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ and T the corresponding operator on $L^2_{\mathbb{C}}=L^2_{\mathbb{C}}(\mathfrak{A},\bar\mu)$. (i) Show that π is not ergodic iff there is a non-zero $v\in L^2_{\mathbb{C}}$ such that $\int v=0$ and $Tv=v$. (ii) Show that π is not weakly mixing iff there is a non-zero $v\in L^2_{\mathbb{C}}$ such that $\int v=0$ and Tv is a multiple of v. (iii) Let $(\mathfrak{A}\widehat{\otimes}\mathfrak{A},\bar\lambda)$ be the probability algebra free product of $(\mathfrak{A},\bar\mu)$ with itself (definition: 325K), and $\tilde\pi\in\operatorname{Aut}_{\bar\lambda}(\mathfrak{A}\widehat{\otimes}\mathfrak{A})$ the automorphism such that $\tilde\pi(a\otimes b)=\pi a\otimes\pi b$ for all a, $b\in\mathfrak{A}$. Show that π is weakly mixing iff $\tilde\pi$ is ergodic iff $\tilde\pi$ is weakly mixing. (*Hint*: consider $T_{\tilde\pi}(v\otimes\bar v)$.)

(k) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. For $\pi\in\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ let T_π be the corresponding operator on $L^2_{\mathbb{C}}=L^2_{\mathbb{C}}(\mathfrak{A},\bar\mu)$. Show that $(\pi,v)\mapsto T_\pi v:\operatorname{Aut}_{\bar\mu}\mathfrak{A}\times L^2_{\mathbb{C}}\to L^2_{\mathbb{C}}$ is continuous if $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is given its weak topology and $L^2_{\mathbb{C}}$ its norm topology.

(l) Let $(\mathfrak{A},\bar\mu)$ be an atomless probability algebra; give $\mathfrak{A}$ its measure metric. Show that the isometry group of $\mathfrak{A}$, with its topology of pointwise convergence, is extremely amenable. (*Hint*: every isometry of $\mathfrak{A}$ is of the form $a\mapsto c\,\triangle\,\pi a$, where $c\in\mathfrak{A}$ and $\pi\in\operatorname{Aut}_{\bar\mu}\mathfrak{A}$; now use 493Bc.)

(m) Let $\mathfrak{A}$ be a homogeneous Dedekind complete Boolean algebra, and $\langle V_n\rangle_{n\in\mathbb{N}}$ a non-decreasing sequence of subsets of $\operatorname{Aut}\mathfrak{A}$, covering $\operatorname{Aut}\mathfrak{A}$, with $V_n^2\subseteq V_{n+1}$ for every n. Show that there is an $n\in\mathbb{N}$ such that $\operatorname{Aut}\mathfrak{A}=V_n$.

494Y Further exercises (a) Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra. For $\pi\in\operatorname{Aut}_{\bar\mu}\mathfrak{A}$, let $T_\pi: L^0(\mathfrak{A})\to L^0(\mathfrak{A})$ be the Riesz space automorphism such that $T_\pi(\chi a)=\chi(\pi a)$ for every $a\in\mathfrak{A}$ (364P). Take any $p\in[1,\infty[$ and write $L^p_{\bar\mu}$ for $L^p(\mathfrak{A},\bar\mu)$ as defined in 366A. Set $G_p=\{T_\pi\restriction L^p_{\bar\mu}:\pi\in\operatorname{Aut}_{\bar\mu}\mathfrak{A}\}$. (i) Show that $\pi\mapsto T_\pi\restriction L^p_{\bar\mu}$ is a topological group isomorphism between $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$, with its weak topology, and G_p, with the strong operator topology from $\mathsf{B}(L^p_{\bar\mu};L^p_{\bar\mu})$ (3A5I). (ii) Show that G_p is closed in $\mathsf{B}(L^p_{\bar\mu};L^p_{\bar\mu})$. (iiI) Show that if $(\mathfrak{A},\bar\mu)$ is totally finite, then $\pi\mapsto T_\pi\restriction L^1_{\bar\mu}$ is a topological group isomorphism between $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$, with its uniform topology, and G_1 with the topology of uniform convergence on weakly compact subsets of $L^1_{\bar\mu}$.

(b) Let $\mathfrak{A}$ be any Boolean algebra. For $I\subseteq\mathfrak{A}$, set $U_I=\{\pi:\pi\in\operatorname{Aut}\mathfrak{A},\ \pi a=a$ for every $a\in I\}$. (i) Show that $\{U_I:I\in[\mathfrak{A}]^{<\omega}\}$ is a base of neighbourhoods of the identity for a Hausdorff topology on $\operatorname{Aut}\mathfrak{A}$ under which $\operatorname{Aut}\mathfrak{A}$ is a topological group. (ii) Show that if $\mathfrak{A}$ is countable then $\operatorname{Aut}\mathfrak{A}$, with this topology, is a Polish group.

(c) Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra. Show that $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$, with its weak topology, is weakly α-favourable.

(d) Let $\bar\mu$ be counting measure on $\mathbb{N}$. (i) Show that if we identify $\operatorname{Aut}_{\bar\mu}\mathcal{P}\mathbb{N}$ with the set of permutations on $\mathbb{N}$, the weak topology of $\operatorname{Aut}_{\bar\mu}\mathcal{P}\mathbb{N}$ is the topology induced by the usual topology of $\mathbb{N}^{\mathbb{N}}$. (ii) Show that there is a comeager conjugacy class in $\operatorname{Aut}_{\bar\mu}\mathcal{P}\mathbb{N}$.

(e) (Rosendal 09) Let $(\mathfrak{A},\bar\mu)$ be the measure algebra of Lebesgue measure on $[0,1]$, and give $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ its weak topology. Let $\mathcal{V}$ be a countable base of open neighbourhoods of ι in $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$. (i) Show that if $I\subseteq\mathbb{N}$ is infinite and $V\in\mathcal{V}$, then $\{\pi:\pi\in\operatorname{Aut}_{\bar\mu}\mathfrak{A},\ \pi^n=\iota$ for some $n\in I\}$ is dense in $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$, and that $B(I,V)=\{\pi:\pi^n\in V$ for some $n\in I\}$ is dense and open. (ii) Show that if $I\subseteq\mathbb{N}$ is infinite then $C(I)=\bigcap_{V\in\mathcal{V}}B(I,V)$ is comeager, and is a union of conjugacy classes. (iii) Show that $\bigcap\{C(I):I\in[\mathbb{N}]^\omega\}=\{\iota\}$. (iv) Show that every conjugacy class in $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is meager.

(f) Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra. Suppose that $\pi \in \mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ is aperiodic. Show that the set of conjugates of π in $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ is dense for the weak topology on $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$.

(g) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra. Show that the set of weakly mixing automorphisms, with the subspace topology inherited from the weak topology of $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$, is weakly α-favourable.

(h) Let G be a Hausdorff topological group. Show that the following are equiveridical: (i) for every neighbourhood V of the identity in G there is a countable set $D \subseteq G$ such that $G = DV$; (ii) there is a family $\langle H_i \rangle_{i \in I}$ of Polish groups such that G is isomorphic, as topological group, to a subgroup of $\prod_{i \in I} H_i$.

(i) Let $(\mathfrak{A}, \bar{\mu})$ be an atomless σ-finite measure algebra, and G a full ergodic subgroup of $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$. Let $V \subseteq \mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ be a symmetric set such that countably many left translates of V cover $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$. Show that V^{228} is a neighbourhood of ι for the uniform topology on $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$.

(j) Let $(\mathfrak{A}, \bar{\mu})$ be a purely atomic probability algebra with two atoms of measure 2^{-n-2} for each $n \in \mathbb{N}$; give $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ its uniform topology. (i) Show that $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A} \cong \mathbb{Z}_2^{\mathbb{N}}$ is compact, therefore not extremely amenable, and can be regarded as a linear space over the field $\mathbb{Z}_2$. (ii) Show that there is a strictly increasing sequence of subgroups of $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ with union $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$. (iii) Show that there is a subgroup V of $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$, not open, such that $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ is covered by countably many translates of V. (iv) Show that there is a discontinuous homomorphism from $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ to a Polish group.

(k) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Show that it is not the union of a strictly increasing sequence of subalgebras.

494Z Problems For $k \in \mathbb{N}$, say that a topological group G is k-**Steinhaus** if whenever $V \subseteq G$ is a symmetric set, containing the identity, such that countably many left translates of V cover G, then V^k is a neighbourhood of the identity. For your favourite groups, determine the smallest k, if any, for which they are k-Steinhaus. (See Rosendal & Solecki 07.)

494 Notes and comments In 494B-494C I run through properties of the weak and uniform topologies of $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ in parallel. The effect is to emphasize their similarities, but they are of course very different – for instance, consider 494Xg, or the contrast between 494Cg and 494Ge. Both have expressions in terms of standard topologies on spaces of linear operators (494Ya), and the weak topology corresponds to the pointwise topology of an isometry group (494Xa). There are other more or less natural topologies which can be considered (e.g., that of 494Yb), but at present the two examined in this section seem to be the most important. I spell out 494Be and 494Ci to show that the groups here provide interesting examples of Polish groups with striking properties.

The formulation of 494D is specifically designed for the application in the proof of 494E(b-ii); the version in 494Xj(ii) is much closer to the real strength of the idea, and takes us directly to one of the important reasons for being interested in weakly mixing automorphisms in 494Xj(iii). The proof of 494D through Bochner's theorem saves space here, but fails to signal the concept of 'spectral resolution' of a unitary operator on a Hilbert space (Riesz & Sz.-Nagy 55, §109), which is an important tool in understanding operators T_π and hence automorphisms π.

While 494H and 494G are of some interest in themselves, their function here is to prepare the way to 494L, 494O and 494Q. The first belongs to the series in §493; like the results in that section, it depends on concentration-of-measure theorems, quoted in part (e) of the proof of 494I and again in part (e) of the proof of 494J. In addition, for the generalization from ergodic full groups to arbitrary full groups, we need the structure theory for closed subalgebras developed in §333.

494O and 494Q-494R break new ground. The former, following Kittrell & Tsankov 09, examines a curious phenomenon identified by Rosendal & Solecki 07 in the course of a search for automatic-continuity results. We cannot dispense entirely with the hypotheses that $\mathfrak{A}$ should be atomless and G ergodic (494Yj), though perhaps they can be relaxed. Many examples are now known of k-Steinhaus groups (494O, 494Yi), but as far as I am aware there are no non-trivial cases in which the critical value of k has been determined (494Z). The automatic-continuity corollary in 494Ob is really a result about homomorphisms into Polish groups (see 494Yh), but applies in many other cases (494P).

The phenomenon of 494Q, which we might call a (negative) 'algebraic cofinality' result, has attracted attention with regard to many algebraic structures, starting with Bergman 06. Apart from the variations of 494Q in 494R and 494Xm, there is a simple example in 494Yk. 494Yj again indicates one of the limits of the result.

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495 Poisson point processes

A classical challenge in probability theory is to formulate a consistent notion of 'random set'. Simple geometric considerations lead us to a variety of measures which are both interesting and important. All these are manifestly special constructions. Even in the most concrete structures, we have to make choices which come to seem arbitrary as soon as we are conscious of the many alternatives. There is however one construction which has a claim to pre-eminence because it is both robust under the transformations of abstract measure theory and has striking properties when applied to familiar measures (to the point, indeed, that it is relevant to questions in physics and chemistry). This gives the 'Poisson point processes' of 495D-495E. In this section I give a brief introduction to the measure-theoretic aspects of this construction.

495A Poisson distributions We need a little of the elementary theory of Poisson distributions.

(a) The **Poisson distribution** with parameter $\gamma > 0$ is the point-supported Radon probability measure ν_γ on $\mathbb{R}$ such that $\nu_\gamma\{n\} = \frac{\gamma^n}{n!}e^{-\gamma}$ for every $n \in \mathbb{N}$. (See 285Q and 285Xr.) Its expectation is $\sum_{n=1}^\infty \frac{\gamma^n}{(n-1)!}e^{-\gamma} = \gamma$. Since $\nu_\gamma\mathbb{N} = 1$, ν_γ can be identified with the corresponding subspace measure on $\mathbb{N}$. It will be convenient to allow $\gamma = 0$, so that the Dirac measure on $\mathbb{R}$ or $\mathbb{N}$ concentrated at 0 becomes a 'Poisson distribution with expectation 0'.

(b) The convolution of two Poisson distributions is a Poisson distribution. **P** If α, $\beta > 0$ then

$$(\nu_\alpha * \nu_\beta)(\{n\}) = \int \nu_\beta(\{n\} - t)\nu_\alpha(dt)$$

(444A)

$$= \sum_{i=0}^n \frac{\beta^{n-i}}{(n-i)!}e^{-\beta} \cdot \frac{\alpha^i}{i!}e^{-\alpha}$$

$$= \frac{1}{n!}\sum_{i=0}^n \frac{n!}{i!(n-i)!}\alpha^i\beta^{n-i}e^{-\alpha-\beta} = \frac{(\alpha+\beta)^n}{n!}e^{-\alpha-\beta}$$

for every $n \in \mathbb{N}$, so $\nu_\alpha * \nu_\beta = \nu_{\alpha+\beta}$. **Q** So if f and g are independent random variables with Poisson distributions then $f + g$ has a Poisson distribution (272T[2]).

(c) If $\langle f_i\rangle_{i\in I}$ is a countable independent family of random variables with Poisson distributions, and $\alpha = \sum_{i\in I}\mathbb{E}(f_i)$ is finite, then $f = \sum_{i\in I} f_i$ is defined a.e. and has a Poisson distribution with expectation α. **P** For finite I we can induce on $\#(I)$, using (b) (and 272L) for the inductive step. For the infinite case we can suppose that $I = \mathbb{N}$. In this case $f_i \ge 0$ a.e. for each i so $f = \sum_{i=0}^\infty f_i$ is defined a.e. and has expectation α, by B.Levi's theorem. Setting $g_n = \sum_{i=0}^n f_i$ for each n, so that g_n has a Poisson distribution with expectation $\beta_n = \sum_{i=0}^n \alpha_i$, we have

$$\Pr(f \le \gamma) = \lim_{n\to\infty}\Pr(g_n \le \gamma) = \lim_{n\to\infty}\sum_{i=0}^{\lfloor\gamma\rfloor}\frac{\beta_n^i}{i!}e^{-\beta_n} = \sum_{i=0}^{\lfloor\gamma\rfloor}\frac{\alpha^i}{i!}e^{-\alpha}$$

for every $\gamma \ge 0$, so f has a Poisson distribution with expectation α. **Q**

[2]Formerly 272S.

(d) I find myself repeatedly calling on the simple fact that $1-e^{-\gamma}(1+\gamma)=\nu_\gamma(\mathbb{N}\setminus\{0,1\})$ is at most $\frac{1}{2}\gamma^2$ for every $\gamma\ge 0$; this is because $\frac{d}{dt}(\frac{1}{2}t^2+e^{-t}(1+t))=t(1-e^{-t})\ge 0$ for $t\ge 0$.

495B Theorem Let (X,Σ,μ) be a measure space. Set $\Sigma^f=\{E:E\in\Sigma,\,\mu E<\infty\}$. Then for any $\gamma>0$ there are a probability space (Ω,Λ,λ) and a family $\langle g_E\rangle_{E\in\Sigma^f}$ of random variables on Ω such that

(i) for every $E\in\Sigma^f$, g_E has a Poisson distribution with expectation $\gamma\mu E$;

(ii) whenever $\langle E_i\rangle_{i\in I}$ is a disjoint family in Σ^f, then $\langle g_{E_i}\rangle_{i\in I}$ is stochastically independent;

(iii) whenever $\langle E_i\rangle_{i\in\mathbb{N}}$ is a disjoint sequence in Σ^f with union $E\in\Sigma^f$, then $g_E=_{\text{a.e.}}\sum_{i=0}^\infty g_{E_i}$.

proof (a) Let $\mathcal{H}\subseteq\{H:H\in\Sigma,\,0<\mu H<\infty\}$ be a maximal family such that $H\cap H'$ is negligible for all distinct H, $H'\in\mathcal{H}$. For $H\in\mathcal{H}$, let μ'_H be the normalized subspace measure defined by setting $\mu'_H E=\mu E/\mu H$ for $E\in\Sigma\cap\mathcal{P}H$, and λ_H the corresponding product probability measure on $H^{\mathbb{N}}$. Next, for $H\in\mathcal{H}$, let ν_H be the Poisson distribution with expectation $\gamma\mu H$, regarded as a probability measure on $\mathbb{N}$. Let λ be the product measure on $\Omega=\prod_{H\in\mathcal{H}}(\mathbb{N}\times H^{\mathbb{N}})$, giving each $\mathbb{N}\times H^{\mathbb{N}}$ the product measure $\nu_H\times\lambda_H$. For $\omega\in\Omega$, write $m_H(\omega)$, $x_{Hj}(\omega)$ for its coordinates, so that $\omega=\langle(m_H(\omega),\langle x_{Hj}(\omega)\rangle_{j\in\mathbb{N}})\rangle_{H\in\mathcal{H}}$.

(b) For $H\in\mathcal{H}$ and $E\in\Sigma$, set $g_{HE}(\omega)=\#(\{j:j<m_H(\omega),\,x_{Hj}(\omega)\in E\})$ when this is finite. Then g_{HE} is measurable and has a Poisson distribution with expectation $\gamma\mu(H\cap E)$; moreover, if $E_0,\dots,E_r\in\Sigma$ are disjoint, then $g_{HE_0},\dots,g_{HE_r}$ are independent. **P** It is enough to examine the case in which the E_i cover X. Then for any $n_0,\dots,n_r\in\mathbb{N}$ with sum n,

$$\begin{aligned}
&\lambda\{\omega:g_{HE_i}(\omega)=n_i\text{ for every }i\le r\}\\
&\quad=\lambda\{\omega:\#(\{j:j<m_H(\omega),\,x_{Hj}\in E_i\})=n_i\text{ for every }i\le r\}\\
&\quad=\lambda\{\omega:m_H(\omega)=n,\,\#(\{j:j<n,\,x_{Hj}\in E_i\})=n_i\text{ for every }i\le r\}\\
&\quad=\sum_{\substack{J_0,\dots,J_r\text{ partition }n\\ \#(J_i)=n_i\text{ for each }i\le r}}\lambda\{\omega:m_H(\omega)=n,\,x_{Hj}\in E_i\text{ whenever }i\le r,\,j\in J_i\}\\
&\quad=\sum_{\substack{J_0,\dots,J_r\text{ partition }n\\ \#(J_i)=n_i\text{ for each }i\le r}}\frac{(\gamma\mu H)^n}{n!}e^{-\gamma\mu H}\prod_{i=0}^r\Bigl(\frac{\mu(H\cap E_i)}{\mu H}\Bigr)^{n_i}\\
&\quad=\frac{n!}{n_0!\dots n_r!}\frac{1}{n!}e^{-\gamma\mu H}\prod_{i=0}^r(\gamma\mu(H\cap E_i))^{n_i}\\
&\quad=\prod_{i=0}^r\frac{(\gamma\mu(H\cap E_i))^{n_i}}{n_i!}e^{-\gamma\mu(H\cap E_i)},
\end{aligned}$$

which is just what we wanted to know. **Q**

Obviously $g_{HE}=\sum_{i=0}^\infty g_{HE_i}$ whenever $\langle E_i\rangle_{i\in\mathbb{N}}$ is a disjoint sequence in Σ with union E, and $g_{HE}=0$ a.e. if $\mu(H\cap E)=0$.

(c) Suppose that $H_0,\dots,H_m\in\mathcal{H}$ are distinct and $E_0,\dots,E_r\in\Sigma$ are disjoint. Then the random variables $g_{H_jE_i}$ are independent. **P** For each $j\le m$, $g_{H_jE_i}$ is Λ_{H_j}-measurable, where Λ_{H_j} is the σ-algebra of subsets of Ω which are measured by λ and determined by the single coordinate H_j in the product $\prod_{H\in\mathcal{H}}(\mathbb{N}\times H^{\mathbb{N}})$. Now the σ-algebras Λ_{H_j} are independent (272Ma). So if we have any family $\langle n_{ij}\rangle_{i\le r,j\le m}$ in $\mathbb{N}$,

$$\begin{aligned}
\lambda\{\omega:g_{H_jE_i}(\omega)=n_{ij}\text{ for every }i\le r,j\le m\}
&=\prod_{j=0}^m\lambda\{\omega:g_{H_jE_i}(\omega)=n_{ij}\text{ for every }i\le r\}\\
&=\prod_{j=0}^m\prod_{i=0}^r\lambda\{\omega:g_{H_jE_i}(\omega)=n_{ij}\}
\end{aligned}$$

by (b); and this is what we need to know. **Q**

(d) For $E \in \Sigma^f$, set $\mathcal{H}_E = \{H : H \in \mathcal{H},\ \mu(E \cap H) > 0\}$; then $\mathcal{H}_E$ is countable, because $\mathcal{H}$ is almost disjoint, and $\mu E = \sum_{H \in \mathcal{H}_E} \mu(H \cap E)$, because $\mathcal{H}$ is maximal. Set $g_E(\omega) = \sum_{H \in \mathcal{H}_E} g_{HE}(\omega)$ when this is finite. Then g_E is defined a.e. and has a Poisson distribution with expectation $\gamma\mu E$ (495Ac). Also $\langle g_{E_i}\rangle_{i \in I}$ are independent whenever $\langle E_i\rangle_{i \in I}$ is a disjoint family in Σ^f. **P** It is enough to deal with the case of finite I (272Bb). Set $\mathcal{H}^* = \bigcup_{i \in I} \mathcal{H}_{E_i}$, so that $\mathcal{H}^*$ is countable, and for $i \in I$ set $g'_i = \sum_{H \in \mathcal{H}^*} g_{HE_i}$. Then each g'_i is equal almost everywhere to the corresponding g_{E_i}, and $\langle g'_i\rangle_{i \in I}$ is independent, by 272K. (The point is that each g'_i is Λ^*_i-measurable, where Λ^*_i is the σ-algebra generated by $\{g_{HE_i} : H \in \mathcal{H}\}$, and 272K, with (c) above, assures us that the Λ^*_i are independent.) It follows at once that $\langle g_{E_i}\rangle_{i \in I}$ is independent (272H). **Q** This proves (i) and (ii).

(e) Similarly, if $\langle E_i\rangle_{i \in \mathbb{N}}$ is a disjoint sequence in Σ^f with union $E \in \Sigma^f$, set $\mathcal{H}^* = \mathcal{H}_E \cup \bigcup_{i \in \mathbb{N}} \mathcal{H}_{E_i}$. For each $i \in \mathbb{N}$, set $g'_i = \sum_{H \in \mathcal{H}^*} g_{HE_i}$; then $g'_i =_{\text{a.e.}} g_{E_i}$. Now

$$\sum_{i=0}^{\infty} g_{E_i} =_{\text{a.e.}} \sum_{i=0}^{\infty} g'_i = \sum_{H \in \mathcal{H}^*} \sum_{i=0}^{\infty} g_{HE_i} = \sum_{H \in \mathcal{H}^*} g_{HE} =_{\text{a.e.}} g_E,$$

as required by (iii).

495C Lemma Let X be a set and $\mathcal{E}$ a subring of the Boolean algebra $\mathcal{P}X$. Let $\mathcal{H}$ be the family of sets of the form

$$\{S : S \subseteq X,\ \#(S \cap E_i) = n_i \text{ for every } i \in I\}$$

where $\langle E_i\rangle_{i \in I}$ is a finite disjoint family in $\mathcal{E}$ and $n_i \in \mathbb{N}$ for every $i \in I$. Then the Dynkin class $\mathrm{T} \subseteq \mathcal{P}(\mathcal{P}X)$ generated by $\mathcal{H}$ (136A) is the σ-algebra of subsets of $\mathcal{P}X$ generated by $\mathcal{H}$.

proof Let Q be the set of functions q from finite subsets of $\mathcal{E}$ to $\mathbb{N}$, and for $q \in Q$ set

$$H_q = \{S : S \subseteq X,\ \#(S \cap E) = q(E) \text{ for every } E \in \operatorname{dom} q\}.$$

Our family $\mathcal{H}$ is just $\{H_q : q \in Q,\ \operatorname{dom} q \text{ is disjoint}\}$.

If $q \in Q$ and $\operatorname{dom} q$ is a subring of $\mathcal{E}$, then $H_q \in \mathrm{T}$. **P** Being a finite Boolean ring, $\operatorname{dom} q$ is a Boolean algebra; let $\mathcal{A}$ be the set of its atoms. Then H_q is either empty or equal to $H_{q\restriction\mathcal{A}}$; in either case it belongs to T. **Q**

If q is any member of Q, then $H_q \in \mathrm{T}$. **P** Let $\mathcal{E}'$ be the subring of $\mathcal{P}X$ generated by $\operatorname{dom} q$. Then $H_q = \bigcup_{q \subseteq q' \in Q, \operatorname{dom} q' = \mathcal{E}'} H_{q'}$ is the union of a countable disjoint family in T, so belongs to T. **Q**

Now observe that $\mathcal{H}_1 = \{H_q : q \in Q\} \cup \{\emptyset\}$ is a subset of T closed under finite intersections, so by the Monotone Class Theorem (136B) T includes the σ-algebra generated by $\mathcal{H}_1$, and must be precisely the σ-algebra generated by $\mathcal{H}$.

495D Theorem Let (X, Σ, μ) be an atomless measure space. Set $\Sigma^f = \{E : E \in \Sigma,\ \mu E < \infty\}$; for $E \in \Sigma^f$, set $f_E(S) = \#(S \cap E)$ when $S \subseteq X$ meets E in a finite set. Let T be the σ-algebra of subsets of $\mathcal{P}X$ generated by sets of the form $\{S : f_E(S) = n\}$ where $E \in \Sigma^f$ and $n \in \mathbb{N}$. Then for any $\gamma > 0$ there is a unique probability measure ν with domain T such that

(i) for every $E \in \Sigma^f$, f_E is measurable and has a Poisson distribution with expectation $\gamma\mu E$;

(ii) whenever $\langle E_i\rangle_{i \in I}$ is a disjoint family in Σ^f, then $\langle f_{E_i}\rangle_{i \in I}$ is stochastically independent.

proof (a) Let $\mathcal{H}$, $\langle \nu_H\rangle_{H \in \mathcal{H}}$, $\langle \mu_H\rangle_{H \in \mathcal{H}}$, $\langle \mu'_H\rangle_{H \in \mathcal{H}}$, $\langle \lambda_H\rangle_{H \in \mathcal{H}}$, Ω, λ, $\langle \mathcal{H}_E\rangle_{E \in \Sigma^f}$ and $\langle g_E\rangle_{E \in \Sigma^f}$ be as in the proof of 495B. Note that all the μ'_H are atomless (234Nf[3]). Define $\phi : \Omega \to \mathcal{P}X$ by setting

$$\phi(\omega) = \{x_{Hj}(\omega) : H \in \mathcal{H},\ j < m_H(\omega)\}$$

for $\omega \in \Omega$.

(b) For $E \in \Sigma^f$, let A_E be the set of those $\omega \in \Omega$ such that

either there are $H \in \mathcal{H} \setminus \mathcal{H}_E$, $j \in \mathbb{N}$ such that $x_{Hj}(\omega) \in E$

[3]Formerly 234F.

or there are distinct H, $H' \in \mathcal{H}_E$ and $j \in \mathbb{N}$ such that $x_{Hj}(\omega) \in H'$

or there is an $H \in \mathcal{H}$ such that the $x_{Hj}(\omega)$, for $j \in \mathbb{N}$, are not all distinct.

Then for any sequence $\langle E_i\rangle_{i\in\mathbb{N}}$ in Σ^f, $\lambda_*(\bigcup_{i\in\mathbb{N}} A_{E_i}) = 0$. **P** Set $\mathcal{H}^* = \bigcup_{i\in\mathbb{N}} \mathcal{H}_{E_i}$, so that $\mathcal{H}^*$ is a countable subset of $\mathcal{H}$. For $H \in \mathcal{H}$, set

$$F_H = H \setminus (\bigcup\{E_i : i \in \mathbb{N},\, H \cap E_i \text{ is negligible}\} \cup \bigcup\{H' : H' \in \mathcal{H}^*,\, H' \neq H\}),$$

$$W_H = \{\boldsymbol{x} : \boldsymbol{x} \in H^{\mathbb{N}} \text{ is injective}\},$$

so that F_H is μ'_H-conegligible and W_H is λ_H-conegligible (because μ'_H is atomless, see 254V). Now

$$\Omega \setminus \bigcup_{k\in\mathbb{N}} A_{E_k} \supseteq \prod_{H\in\mathcal{H}}(\mathbb{N} \times (W_H \cap F_H^{\mathbb{N}}))$$

has full outer measure in Ω, by 254Lb, and its complement has zero inner measure (413Ec). **Q**

It follows that there is a probability measure $\tilde\lambda$ on Ω, extending λ, such that $\tilde\lambda A_E = 0$ for every $E \in \Sigma^f$ (417A). Let ν_0 be the image measure $\tilde\lambda\phi^{-1}$.

(c) If $E \in \Sigma^f$ and $\omega \in \Omega \setminus A_E$, then $f_E(\phi(\omega)) = g_E(\omega)$ if either is defined. **P** If $H \in \mathcal{H}$, then all the $x_{Hj}(\omega)$ are distinct; if $H \in \mathcal{H} \setminus \mathcal{H}_E$, no $x_{Hj}(\omega)$ can belong to E; if H, $H' \in \mathcal{H}_E$ are distinct, then no $x_{Hj}(\omega)$ can belong to H'. So all the $x_{Hj}(\omega)$, $x_{H'k}(\omega)$ for H, $H' \in \mathcal{H}_E$ and j, $k \in \mathbb{N}$ must be distinct, and

$$\begin{aligned} f_E(\phi(\omega)) &= \#(\{x_{Hj}(\omega) : H \in \mathcal{H},\, j < m_H(\omega),\, x_{Hj}(\omega) \in E\}) \\ &= \#(\{(H,j) : H \in \mathcal{H}_E,\, j < m_H(\omega),\, x_{Hj}(\omega) \in E\}) \\ &= \sum_{H\in\mathcal{H}_E} g_{HE}(\omega) = g_E(\omega) \end{aligned}$$

if any of these is finite. **Q** It follows at once that if $E_0, \dots, E_r \in \Sigma^f$ are disjoint, then $\{\omega : f_{E_i}(\phi(\omega)) = g_{E_i}(\omega)$ for every $i \le r\}$ is $\tilde\lambda$-conegligible, so that if $n_0, \dots, n_r \in \mathbb{N}$ then

$$\begin{aligned} \nu_0\{S : f_{E_i}(S) = n_i \text{ for every } i \le r\} &= \tilde\lambda\{\omega : f_{E_i}(\phi(\omega)) = n_i \text{ for every } i \le r\} \\ &= \tilde\lambda\{\omega : g_{E_i}(\omega) = n_i \text{ for every } i \le r\} \\ &= \lambda\{\omega : g_{E_i}(\omega) = n_i \text{ for every } i \le r\} \\ &= \prod_{i=0}^{r} \frac{(\gamma\mu E_i)^{n_i}}{n_i!} e^{-\gamma\mu E_i}. \end{aligned}$$

Thus every f_{E_i} is finite ν_0-a.e., belongs to $\mathcal{L}^0(\nu_0)$ and has a Poisson distribution with the appropriate expectation, and they are independent.

(d) As T is defined to be the σ-algebra generated by the family $\{f_E : E \in \Sigma^f\}$, it is included in the domain of ν_0. Set $\nu = \nu_0 \restriction \mathrm{T}$; then ν has the properties (i) and (ii). To see that it is unique, observe that if ν' also has these properties, then $\{A : \nu A = \nu' A\}$ is a Dynkin class containing every set of the form

$$\{S : f_{E_i}(S) = n_i \text{ for } i \le r\}$$

where $E_0, \dots, E_r \in \Sigma^f$ are disjoint and $n_0, \dots, n_r \in \mathbb{N}$. By 495C it contains the σ-algebra generated by this family, which is T. So ν and ν' agree on T, and are equal.

495E Definition In the context of 495D, I will call the completion of ν the **Poisson point process** on X with **intensity** or **density** γ.

Note that the Poisson point process on (X, μ) with intensity $\gamma > 0$ is identical with the Poisson point process on $(X, \gamma\mu)$ with intensity 1. There would therefore be no real loss of generality in the main theorems of this section if I spoke only of point processes with intensity 1. I retain the extra parameter because applications frequently demand it, and the formulae will be more useful with the γs in their proper places; moreover, there are important ideas associated with variations in γ, as in 495Xe.

495F Proposition Let (X, Σ, μ) be an atomless measure space, $\langle X_i\rangle_{i\in I}$ a countable partition of X into measurable sets and $\gamma > 0$. Let ν be the Poisson point process of (X, Σ, μ) with intensity γ; for $i \in I$ let ν_i

be the Poisson point process of (X_i,Σ_i,μ_i) with intensity γ, where μ_i is the subspace measure on X_i and Σ_i its domain. For $S\subseteq X$ set $\phi(S)=\langle S\cap X_i\rangle_{i\in I}\in\prod_{i\in I}\mathcal{P}X_i$. Then ϕ is an isomorphism between ν and the product measure $\lambda=\prod_{i\in I}\nu_i$ on $Z=\prod_{i\in I}\mathcal{P}X_i$.

proof (a) Because $\langle X_i\rangle_{i\in I}$ is a partition of X, ϕ is a bijection.

(b) For $E\in\Sigma^f$ and $S\subseteq X$, set $f_E(S)=\#(S\cap E)$ if this is finite, ∞ otherwise. As in 495D, take T to be the σ-algebra of subsets of X generated by $\{f_E:E\in\Sigma^f\}$; similarly, take T_i to be the σ-algebra of subsets of X_i generated by $\{f_F\restriction\mathcal{P}X_i:F\in\Sigma_i^f\}$ for each $i\in I$.

If $E\in\Sigma^f$, $f_E(S)=\sum_{i\in I}f_{E\cap X_i}(S\cap X_i)$ for every $S\subseteq X$. Consequently f_E is measurable with respect to the σ-algebra T' generated by $\bigcup_{i\in I}\{f_F:F\in\Sigma_i^f\}$; as $\mathrm{T}'\subseteq\mathrm{T}$ and E is arbitrary, $\mathrm{T}'=\mathrm{T}$. Next, the σ-algebra $\widehat{\bigotimes}_{i\in I}\mathrm{T}_i\subseteq\mathcal{P}Z$ is generated by the family of subsets of Z of the form

$$W_{jFm}=\{\langle S_i\rangle_{i\in I}:S_i\subseteq X_i\text{ for every }i\in I,\ f_F(S_j)=m\}$$

with $j\in I$, $F\in\Sigma_j^f$ and $m\in\mathbb{N}$. Since

$$\phi^{-1}[W_{jFm}]=\{S:S\subseteq X,\ f_F(S)=m\}\in\mathrm{T}.$$

ϕ is $(\mathrm{T},\widehat{\bigotimes}_{i\in I}\mathrm{T}_i)$-measurable. On the other hand, the σ-algebra of subsets of $\mathcal{P}X$ generated by $\{\phi^{-1}[W_{jFm}]:j\in I,\ F\in\Sigma_j,\ m\in\mathbb{N}\}$ is $\mathrm{T}'=\mathrm{T}$. So ϕ is actually an isomorphism between $(\mathcal{P}X,\mathrm{T})$ and $(Z,\widehat{\bigotimes}_{i\in I}\mathrm{T}_i)$.

(c) I repeat the idea of 495C in a more complex form. This time, let Q be the set of functions from finite subsets of $\{E:E\in\Sigma^f,\ \{i:E\cap X_i\ne\emptyset\}\text{ is finite}\}$ to $\mathbb{N}$, and for $q\in Q$ set

$$H_q=\{S:S\subseteq X,\ \#(S\cap E)=q(E)\text{ for every }E\in\operatorname{dom}q\};$$

let $\mathcal{H}$ be

$$\{H_q:q\in Q,\ \operatorname{dom}q\subseteq\textstyle\bigcup_{i\in I}\Sigma_i^f\text{ is disjoint}\},$$

and $\mathrm{T}''\subseteq\mathcal{P}(\mathcal{P}X)$ the Dynkin class generated by $\mathcal{H}$.

If $q\in Q$ then $H_q\in\mathrm{T}''$. **P** $J=\bigcup_{E\in\operatorname{dom}q}\{i:E\cap X_i\ne\emptyset\}$ is finite, and $F=\bigcup\operatorname{dom}q$ belongs to Σ^f and is included in $\bigcup_{i\in J}X_i$. Let $\mathcal{A}$ be the set of atoms of the subring $\mathcal{E}$ generated by $\operatorname{dom}q\cup\{F\cap X_i:i\in J\}$. If $q'\in Q$ and $\operatorname{dom}q'=\mathcal{E}$ then $H_{q'}$ is either empty or equal to $H_{q'\restriction\mathcal{A}}\in\mathcal{H}$. Now

$$H_q=\textstyle\bigcup_{q\subseteq q'\in Q,\operatorname{dom}q'=\mathcal{E}}H_{q'}$$

is the union of a countable disjoint family in T'', so belongs to T''. **Q**

Accordingly $\mathcal{H}_1=\{H_q:q\in Q\}\cup\{\emptyset\}$ is a subset of T'' closed under finite intersections so T'' includes the σ-algebra generated by $\mathcal{H}_1$ and is the σ-algebra generated by $\mathcal{H}$.

(d) In (c), if $q\in Q$ and $\operatorname{dom}q\subseteq\bigcup_{i\in I}\Sigma_i^f$ is disjoint, then $\lambda\phi[H_q]=\nu H_q$. **P** If $\emptyset\in\operatorname{dom}q$ and $q(\emptyset)\ne0$ then H_q is empty and we can stop. Otherwise, set $\mathcal{A}=\operatorname{dom}q\setminus\{\emptyset\}$; then $H_q=\bigcap_{E\in\mathcal{A}}\{S:S\subseteq X,\ f_E(S)=q(E)\}$, so

$$\nu H_q=\textstyle\prod_{E\in\mathcal{A}}\frac{(\gamma\mu E)^{q(E)}}{q(E)!}e^{-\gamma\mu E}$$

because $\mathcal{A}$ is disjoint. On the other hand, setting $\mathcal{A}_i=\{E:E\in\mathcal{A},\ E\subseteq X_i\}$ and $H_{qi}=X_i\cap\bigcap_{E\in\mathcal{A}_i}\{S:S\subseteq X_i,\ f_E(S)=q(E)\}$ for $i\in I$, $\phi[H_q]=\prod_{i\in I}H_{qi}$ and

$$\begin{aligned}\lambda\phi[H_q]&=\prod_{i\in I}\nu_iH_{qi}=\prod_{i\in I}\prod_{E\in\mathcal{A}_i}\frac{(\gamma\mu E)^{q(E)}}{q(E)!}e^{-\gamma\mu E}\\&=\prod_{E\in\mathcal{A}}\frac{(\gamma\mu E)^{q(E)}}{q(E)!}e^{-\gamma\mu E}=\nu H_q.\ \mathbf{Q}\end{aligned}$$

(e) Of course $\{H:H\in\mathrm{T},\ \lambda\phi[H]=\nu H\}$ is a Dynkin class; as it includes $\mathcal{H}$, it includes T''. But $\mathcal{H}$ contains $\{S:S\subseteq X,\ \#(S\cap E)=m\}$ whenever $E\in\Sigma^f$ is included in some X_i and $m\in\mathbb{N}$. Because I is countable and T'' is a σ-algebra, $\{S:S\subseteq X,\ \#(S\cap E)=m\}$ belongs to T'' whenever $E\in\Sigma^f$ and $m\in\mathbb{N}$, just because $f_E(S)=\sum_{i\in I}f_{E\cap X_i}(S)$ is a countable sum of T''-measurable functions. But now we see that $\mathrm{T}''=\mathrm{T}$ and ϕ is an isomorphism between $(\mathcal{P}X,\mathrm{T},\nu\restriction\mathrm{T})$ and $(Z,\lambda,\lambda\restriction\widehat{\bigotimes}_{i\in I}\mathrm{T}_i)$.

(f) Finally, ν was defined as the completion of $\nu\restriction\mathrm{T}$, while λ is the completion of $\lambda\restriction\widehat{\bigotimes}_{i\in I}\mathrm{T}_i$ because ν_i is always the completion of $\nu\restriction\mathrm{T}_i$. So ϕ is an isomorphism between $(\mathcal{P}X,\nu)$ and (Z,λ), as required.

495G Proposition Let (X,Σ,μ) be a perfect atomless measure space, and $\gamma>0$. Then the Poisson point process on X with intensity γ is a perfect probability measure.

proof I refer to the construction in 495B-495D. In (b) of the proof of 495D, use the construction set out in the proof of 417A, so that the domain $\tilde{\Lambda}$ of $\tilde{\lambda}$ is precisely the family of sets of the form $W\triangle A$ where W belongs to the domain Λ of the product measure λ and A belongs to the σ-ideal $\mathcal{A}^*$ generated by $\{A_E:E\in\Sigma^f\}$. Then $\tilde{\lambda}$ is perfect. **P** Let $h:\Omega\to\mathbb{R}$ be a $\tilde{\Lambda}$-measurable function and $W\in\tilde{\Lambda}$ a set of non-zero measure. Then there are a $W'\in\Lambda$ and an $A\in\mathcal{A}^*$ such that $W\triangle W'\subseteq A$ and $h\restriction\Omega\setminus A$ is Λ-measurable; let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a sequence in Σ^f such that $A\subseteq\bigcup_{n\in\mathbb{N}}A_{E_n}$, and $h_1:\Omega\to\mathbb{R}$ a Λ-measurable function agreeing with h on $\Omega\setminus A$. Set $\mathcal{H}^*=\bigcup_{n\in\mathbb{N}}\mathcal{H}_{E_n}$, so that $\mathcal{H}^*$ is countable. As in the proof of 495D, set

$$F_H=H\setminus(\bigcup\{E_n:n\in\mathbb{N},\,H\cap E_n\text{ is negligible}\}\cup\bigcup\{H':H'\in\mathcal{H}^*,\,H'\ne H\}),$$

$$W_H=\{\boldsymbol{x}:\boldsymbol{x}\in H^{\mathbb{N}}\text{ is injective}\},$$

for $H\in\mathcal{H}$, so that $W'_H=W_H\cap F_H^{\mathbb{N}}$ is λ_H-conegligible. Set $\Omega'=\prod_{H\in\mathcal{H}}(\mathbb{N}\times W'_H)$. This is disjoint from every A_{E_n} (as in 495D) and therefore from A. The subspace measure $\lambda_{\Omega'}$ on Ω' induced by λ is just the product of the measures on $\mathbb{N}\times W'_H$ (254La). All of these are perfect (451Jc, 451Dc), so $\lambda_{\Omega'}$ also is perfect (451Jc again). Now

$$\lambda_{\Omega'}(W\cap\Omega')=\lambda_{\Omega'}(W'\cap\Omega')=\lambda W'=\tilde{\lambda}W>0.$$

It follows that there is a compact set $K\subseteq h_1[W\cap\Omega']$ such that $\lambda_{\Omega'}(h_1^{-1}[K]\cap\Omega')>0$. As h and h_1 agree on Ω', $K\subseteq h[W]$, while

$$\tilde{\lambda}h^{-1}[K]=\tilde{\lambda}h_1^{-1}[K]$$

(because $\tilde{\lambda}A=0$)

$$=\lambda h_1^{-1}[K]=\lambda_{\Omega'}(h^{-1}[K]\cap\Omega')>0.$$

As W and h are arbitrary, $\tilde{\lambda}$ is perfect. **Q**

It follows at once that the image measure $\tilde{\lambda}\phi^{-1}$ and its restriction to T are perfect (451Ea); finally, the completion is perfect, by 451Gc.

495H Lemma Let (X,Σ,μ) be an atomless σ-finite measure space, and $\gamma>0$; let ν be the Poisson point process on X with intensity γ. Suppose that $f:X\to\mathbb{R}$ is a Σ-measurable function such that $\mu f^{-1}[\{\alpha\}]=0$ for every $\alpha\in\mathbb{R}$. Then $\nu\{S:S\subseteq X,\,f\restriction S\text{ is injective}\}=1$.

proof If $E\subseteq X$ has finite measure then $\{S:S\subseteq X,\,f\restriction S\cap E\text{ is injective}\}$ is ν-conegligible. **P** Set $\beta=\mu E$. For $\alpha\in[-\infty,\infty]$ set $h(\alpha)=\mu\{x:x\in E,\,f(x)\le\alpha\}$; then $h:[-\infty,\infty]\to[0,\beta]$ is non-decreasing, continuous and surjective. Take any $m\ge1$; then there are $\alpha_0<\ldots<\alpha_m$ in $[-\infty,\infty]$ such that $\alpha_0=-\infty$, $\alpha_m=\infty$ and $h(\alpha_{i+1})-h(\alpha_i)=\dfrac{\beta}{m}$ for each $i<m$. Set $F_{ni}=\{x:\alpha_i\le f(x)<\alpha_{i+1}\}$ for $i<m$. Then

$$\begin{aligned}\nu_*\{S:S\subseteq X,\,f\restriction S\cap E\text{ is injective}\}&\ge\nu\{S:\#(S\cap F_{ni})\le1\text{ for every }i<m\}\\&=\prod_{i<m}\nu\{S:\#(S\cap F_{ni})\le1\}\\&=\prod_{i<m}e^{-\gamma\beta/m}(1+\frac{\gamma\beta}{m})\\&=e^{-\gamma\beta}(1+\frac{\gamma\beta}{m})^m\to1\end{aligned}$$

as $m \to \infty$. So $\{S : f{\restriction}S \cap E$ is injective$\}$ has inner measure 1 and is conegligible. **Q**

Now we are supposing that there is a non-decreasing sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ of measurable sets of finite measure covering X, so that

$$\{S : f{\restriction}S \text{ is injective}\} = \bigcap_{n\in\mathbb{N}}\{S : f{\restriction}S \cap E_n \text{ is injective}\}$$

has measure $\lim_{n\to\infty} \nu\{S : f{\restriction}S \cap E_n$ is injective$\} = 1$.

495I Proposition Let (X, Σ, μ) be an atomless countably separated measure space (definition: 343D) and $\gamma > 0$. Let ν' be a complete probability measure on $\mathcal{P}X$ such that $\nu'\{S : S \subseteq X,\ S \cap E = \emptyset\}$ is defined and equal to $e^{-\gamma\mu E}$ whenever $E \in \Sigma$ has finite measure. Then ν' extends the Poisson point process ν on X with intensity γ.

proof (a) Write T for $\operatorname{dom}\nu$, and for $E \in \Sigma^f$, write $A(E)$ for $\{S : S \subseteq X,\ S \cap E \neq \emptyset\} \in$ T. If $E_0, \dots, E_n \in \Sigma^f$ are disjoint, then $A(E_0), \dots, A(E_n)$ are ν'-independent. **P** For $I \subseteq \{0, \dots, n\}$,

$$\begin{aligned}\nu'(\mathcal{P}X \cap \bigcap_{i\in I} A(E_i)) &= 1 - \nu'\{S : S \cap \bigcup_{i\in I} E_i = \emptyset\} = \exp(-\gamma\mu(\bigcup_{i\in I} E_i))\\ &= \prod_{i\in I} \exp(-\gamma\mu E_i) = \prod_{i\in I} \nu' A(E_i). \ \mathbf{Q}\end{aligned}$$

(b) If $E_0, \dots, E_k \in \Sigma^f$ are disjoint and $n_0, \dots, n_k \in \mathbb{N}$, then $\nu'\{S : S \subseteq X,\ \#(S \cap E_j) = n_j$ for every $j \le k\}$ is defined and equal to $\nu\{S : S \subseteq X,\ \#(S \cap E_j) = n_j$ for every $j \le k\}$. **P** Let $\langle F_m\rangle_{m\in\mathbb{N}}$ be a sequence in Σ separating the points of X. For each $m \in \mathbb{N}$ let $\mathcal{E}_m$ be the subalgebra of $\mathcal{P}X$ generated by $\{E_j : j \le k\} \cup \{F_i : i \le m\}$, $\mathcal{A}_m$ the set of atoms of $\mathcal{E}_m$ included in $E = \bigcup_{j\le k} E_j$,

$$\mathfrak{I}_m = \{\mathcal{I} : \mathcal{I} \subseteq \mathcal{A}_m,\ \#(\{A : A \in \mathcal{I},\ A \subseteq E_j\}) = n_j \text{ for every } j \le k\}$$

and

$$C_m = \{S : S \subseteq X,\ \{A : A \in \mathcal{A}_m,\ S \cap A \neq \emptyset\} \in \mathfrak{I}_m\}.$$

Then

$$\begin{aligned}\nu' C_m &= \sum_{\mathcal{I}\in\mathfrak{I}_m} \nu'\{S : S \cap A \neq \emptyset \text{ for } A \in \mathcal{I}, S \cap E \setminus \bigcup\mathcal{I} = \emptyset\}\\ &= \sum_{\mathcal{I}\in\mathfrak{I}_m} e^{-\gamma\mu(E\setminus\bigcup\mathcal{I})} \prod_{A\in\mathcal{I}} (1 - e^{-\gamma\mu(E\cap A)})\end{aligned}$$

(by (a))

$$\begin{aligned}&= \sum_{\mathcal{I}\in\mathfrak{I}_m} \nu\{S : S \cap A \neq \emptyset \text{ for } A \in \mathcal{I}, S \cap E \setminus \bigcup\mathcal{I} = \emptyset\}\\ &= \nu C_m.\end{aligned}$$

Now, for $S \subseteq X$,

$$\begin{aligned}\lim_{m\to\infty} \chi C_m(S) &= 1 \text{ if } \#(S \cap E_j) = n_j \text{ for every } j \le k,\\ &= 0 \text{ otherwise}\end{aligned}$$

because $\{F_m : m \in \mathbb{N}\}$ separates the points of X. By Lebesgue's Dominated Convergence Theorem (123C), $\nu'\{S : \#(S \cap E_j) = n_j$ for every $j \le k\}$ is defined and equal to

$$\lim_{m\to\infty} \nu' C_m = \lim_{m\to\infty} \nu C_m = \nu\{S : \#(S \cap E_j) = n_j \text{ for every } j \le k\}. \ \mathbf{Q}$$

(c) Now

$$\{H : H \subseteq \mathcal{P}X,\ \nu' H \text{ and } \nu H \text{ are defined and equal}\}$$

is a Dynkin class including the family $\mathcal{H}$ of 495C, so includes the σ-algebra T of subsets of $\mathcal{P}X$ generated by $\{\{S : \#(S \cap E) = n\} : E \in \Sigma^f,\ n \in \mathbb{N}\}$. Because ν is defined to be the completion of $\nu{\restriction}$T and ν' is complete, $\nu' H$ is defined and equal to νH whenever H is measured by ν.

495J Proposition Let (X_1,Σ_1,μ_1) and (X_2,Σ_2,μ_2) be atomless measure spaces, and $f:X_1\to X_2$ an inverse-measure-preserving function. Let $\gamma>0$, and let ν_1, ν_2 be the Poisson point processes on X_1, X_2 respectively with intensity γ. Then $S\mapsto f[S]:\mathcal{P}X_1\to\mathcal{P}X_2$ is inverse-measure-preserving for ν_1 and ν_2; in particular, $\mathcal{P}A$ has full outer measure for ν_2 whenever $A\subseteq X_2$ has full outer measure for μ_2.

proof Set $\psi(S)=f[S]$ for $S\subseteq X_1$.

(a) If $F\in\Sigma_2^f$, then $\{S:S\subseteq X_1,\ f{\restriction}f^{-1}[F]\cap S$ is not injective$\}$ is ν_1-negligible. **P** Let $n\in\mathbb{N}$. Set $\alpha=\frac{1}{n+1}\mu_2F$. Because μ_2 is atomless, we can find a partition of F into sets $F_0,\dots,F_n$ of measure α. Now

$$\{S:f{\restriction}f^{-1}[F]\cap S\text{ is not injective}\}\subseteq\bigcup_{i\le n}\{S:\#(S\cap f^{-1}[F_i])>1\}$$

has ν_1-outer measure at most

$$(n+1)(1-e^{-\gamma\alpha}(1+\gamma\alpha))\le\frac{1}{2}(n+1)\alpha^2\gamma^2=\frac{1}{2(n+1)}(\gamma\mu_2F)^2.$$

As n is arbitrary, $\{S:f{\restriction}f^{-1}[F]\cap S$ is not injective$\}$ is negligible. **Q**

(b) It follows that, for any $F\in\Sigma_2^f$ and $n\in\mathbb{N}$,

$$\{S:\#(f[S]\cap F)=n\}\triangle\{S:\#(S\cap f^{-1}[F])=n\}$$

is ν_1-negligible and $\{S:\#(f[S]\cap F)=n\}$ is measured by ν_1. So if T_2 is the σ-algebra of subsets of $\mathcal{P}X_2$ generated by sets of the form $\{T:\#(F\cap T)=n\}$ for $F\in\Sigma_2^f$ and $n\in\mathbb{N}$, then ν_1 measures $\psi^{-1}[H]$ for every $H\in\mathrm{T}_2$. Next, if $\langle F_i\rangle_{i\in I}$ is a finite disjoint family in Σ_2^f and $n_i\in\mathbb{N}$ for $i\in I$,

$$\begin{aligned}\nu_1\{S:\#(f[S]\cap F_i)=n_i\text{ for every }i\in I\}&\\ &=\nu_1\{S:\#(S\cap f^{-1}[F_i])=n_i\text{ for every }i\in I\}\\ &=\prod_{i\in I}\frac{(\gamma\mu_1f^{-1}[F_i])^n}{n!}e^{-\gamma\mu_1f^{-1}[F_i]}\end{aligned}$$

(because $\langle f^{-1}[F_i]\rangle_{i\in I}$ is a disjoint family in Σ_1^f)

$$=\prod_{i\in I}\frac{(\gamma\mu_2F_i)^n}{n!}e^{-\gamma\mu_2F_i}.$$

So the image measure $\nu_1\psi^{-1}$ satisfies (i) and (ii) of 495D, and must agree with ν_2 on T_2; that is, ψ is inverse-measure-preserving for ν_1 and $\nu_2{\restriction}\mathrm{T}_2$. As ν_1 is complete, ψ is inverse-measure-preserving for ν_1 and ν_2 (234Ba[4]).

(c) If $A\subseteq X_2$ has full outer measure, then we can take μ_1 to be the subspace measure on $X_1=A$ and $f(x)=x$ for $x\in A$. In this case, $\mathcal{P}A=\psi[\mathcal{P}A]$ must have full outer measure for ν_2.

495K Lemma Let $(\tilde X,\tilde\Sigma,\tilde\mu)$ be an atomless σ-finite measure space, and $\gamma>0$. Write μ_L for Lebesgue measure on $[0,1]$, μ' for the product measure on $X'=\tilde X\times[0,1]$, and λ' for the product measure on $\Omega'=[0,1]^{\tilde X}$. Let $\tilde\nu$, ν' be the Poisson point processes on $\tilde X$, X' respectively with intensity γ. For $T\subseteq\tilde X$ define $\psi_T:\Omega'\to\mathcal{P}X'$ by setting $\psi_T(z)=\{(t,z(t)):t\in T\}$ for $z\in\Omega'$; let ν'_T be the image measure $\lambda'\psi_T^{-1}$ on $\mathcal{P}X'$. Then $\langle\nu'_T\rangle_{T\subseteq\tilde X}$ is a disintegration of ν' over $\tilde\nu$ (definition: 452E).

proof (a) Let $E\subseteq X'$ be a measurable set with finite measure, and write $H_E=\{S:S\cap E\ne\emptyset\}$. Then $\nu'H_E=1-e^{-\gamma\mu'E}\le\gamma\mu'E$; but also $\overline{\int}\nu'_T(H_E)\tilde\nu(dT)\le2\gamma\mu'E$. **P** We know that $\int\mu_LE[\{t\}]\tilde\mu(dt)=\mu'E$ (252D). Let $Y\subseteq\tilde X$ be a conegligible set such that $E[\{t\}]$ is measurable for every $t\in Y$ and $t\mapsto\mu_LE[\{t\}]:Y\to[0,1]$ is measurable. Set $F_i=\{t:t\in Y,\ 2^{-i-1}<\mu_LE[\{t\}]\le2^{-i}\}$ for each $i\in\mathbb{N}$; let $\langle F'_i\rangle_{i\in\mathbb{N}}$ be a sequence of sets of finite measure with union $\tilde X\setminus\bigcup_{i\in\mathbb{N}}F_i$. Let W be the set of those $T\subseteq\tilde X$ such that $T\cap(\tilde X\setminus Y)$ is empty and $T\cap F_i$, $T\cap F'_i$ are finite for every $i\in\mathbb{N}$; then W is $\tilde\nu$-conegligible.

For any $T\in W$,

[4]Formerly 235Hc.

$$\begin{aligned}\psi_T^{-1}[H_E] &= \{z : \psi_T(z) \cap E \neq \emptyset\}\\ &= \bigcup_{i\in\mathbb{N}} \bigcup_{t\in T\cap F_i} \{z : z(t) \in E[\{t\}]\} \cup \bigcup_{i\in\mathbb{N}} \bigcup_{t\in T\cap F'_i} \{z : z(t) \in E[\{t\}]\}\end{aligned}$$

is measured by λ' and has measure at most $\sum_{i=0}^{\infty} 2^{-i}\#(T\cap F_i)$, because $\mu_L E[\{t\}]$ has measure at most 2^{-i} if $t \in T \cap F_i$, and is zero if $t \in T \cap F'_i$. So

$$\begin{aligned}\overline{\int}\nu'_T(H_E)\tilde{\nu}(dT) &= \overline{\int}\lambda'\psi_T^{-1}[H_E]\tilde{\nu}(dT) \le \int\sum_{i=0}^{\infty} 2^{-i}\#(T\cap F_i)\tilde{\nu}(dT)\\ &= \sum_{i=0}^{\infty} 2^{-i}\int \#(T\cap F_i)\tilde{\nu}(dT) = \sum_{i=0}^{\infty} 2^{-i}\gamma\tilde{\mu}F_i\end{aligned}$$

(because $T \mapsto \#(T\cap F_i)$ has expectation $\gamma\tilde{\mu}F_i$)

$$\le 2\gamma\int \mu_L E[\{t\}]\tilde{\nu}(dt) = 2\gamma\mu' E. \ \mathbf{Q}$$

(b) Suppose that $\langle F_j\rangle_{j<s}$, $\langle C_{ij}\rangle_{i<r,j<s}$ and $\langle n_{ij}\rangle_{i<r,j<s}$ are such that

r, $s \in \mathbb{N}$,

$n_{ij} \in \mathbb{N}$ for $i < r$, $j < s$,

$\langle F_j\rangle_{j<s}$ is a disjoint family of subsets of $\tilde{X}$ with finite measure,

for each $j < s$, $\langle C_{ij}\rangle_{i<r}$ is a disjoint family of measurable subsets of $[0,1]$.

Set $E_{ij} = F_j \times C_{ij}$ for $i < r$ and $j < s$, and $H = \{S : S \subseteq X', \#(S\cap E_{ij}) = n_{ij}$ for every $i < r$, $j < s\}$. Then $\int \nu'_T(H)\tilde{\nu}(dT) = \nu' H$.

P (i) To begin with, suppose that $\bigcup_{i<r} C_{ij} = [0,1]$ for every j. Set $n_j = \sum_{i=0}^{r-1} n_{ij}$ for each j, and let W be the set of those $T \subseteq \tilde{X}$ such that $\#(T\cap F_j) = n_j$ for every j. Then $\tilde{\nu}W = \prod_{j=0}^{s-1}\frac{(\gamma\tilde{\mu}F_j)^{n_j}}{n_j!}e^{-\gamma\tilde{\mu}F_j}$. If $T \subseteq \tilde{X}$ and $z \in \psi_T^{-1}[H]$, then for each $j < s$ we must have

$$\begin{aligned}\#(T\cap F_j) &= \#(\{t : t \in T\cap F_j,\ z(t) \in \bigcup_{i<r} C_{ij}\})\\ &= \sum_{i=0}^{r-1}\#(\{t : t\in T\cap F_j,\ z(t)\in C_{ij}\})\\ &= \sum_{i=0}^{r-1}\#(\psi_T(z)\cap E_{ij}) = \sum_{i=0}^{r-1} n_{ij} = n_j.\end{aligned}$$

Turning this round, we see that if $T \notin W$ then $\psi_T^{-1}[H] = \emptyset$ and $\nu'_T H = 0$.

If $T \in W$, let Q be the set of all $q = \langle q(i,j)\rangle_{i<r,j<s}$ such that $\langle q(i,j)\rangle_{i<r}$ is a disjoint family of subsets of $T\cap F_j$ for each j and $\#(q(i,j)) = n_{ij}$ for all i and j. Then $\#(Q) = \prod_{j=0}^{s-1}\frac{n_j!}{\prod_{i=0}^{r-1} n_{ij}!}$. Accordingly

$$\begin{aligned}\nu'_T H &= \lambda'\{z : \psi_T(z) \in H\}\\ &= \lambda'\{z : \#(\{t : t\in T\cap F_j,\ z(t)\in C_{ij}\}) = n_{ij} \text{ for all } i, j\}\\ &= \sum_{q\in Q}\lambda\{z : z(t)\in C_{ij} \text{ whenever } i<r,\ j<s \text{ and } t\in q(i,j)\}\\ &= \sum_{q\in Q}\prod_{\substack{i<r,j<s\\ t\in q(i,j)}}\mu_L C_{ij} = \sum_{q\in Q}\prod_{i<r,j<s}(\mu_L C_{ij})^{n_{ij}} = \prod_{j=0}^{s-1}(n_j!\prod_{i=0}^{r-1}\frac{(\mu_L C_{ij})^{n_{ij}}}{n_{ij}!}).\end{aligned}$$

It follows that

$$\begin{aligned}\int \nu'_T(H)\tilde{\nu}(dT) &= \tilde{\nu}W \cdot \prod_{j=0}^{s-1}(n_j!\prod_{i=0}^{r-1}\frac{(\mu_L C_{ij})^{n_{ij}}}{n_{ij}!}) \\ &= \prod_{j=0}^{s-1}\frac{(\gamma\tilde{\mu}F_j)^{n_j}}{n_j!}e^{-\gamma\tilde{\mu}F_j}\cdot\prod_{j=0}^{s-1}(n_j!\prod_{i=0}^{r-1}\frac{(\mu_L C_{ij})^{n_{ij}}}{n_{ij}!}) \\ &= \prod_{j=0}^{s-1}((\gamma\tilde{\mu}F_j)^{n_j}e^{-\gamma\tilde{\mu}F_j}\prod_{i=0}^{r-1}\frac{(\mu_L C_{ij})^{n_{ij}}}{n_{ij}!}) \\ &= \prod_{j=0}^{s-1}\prod_{i=0}^{r-1}e^{-\gamma\mu' E_{ij}}\frac{(\gamma\mu' E_{ij})^{n_{ij}}}{n_{ij}!} = \nu' H,\end{aligned}$$

as required.

(ii) For the general case, set $C_{rj} = [0,1]\setminus\bigcup_{i<r}C_{ij}$, $E_{rj} = F_j\times C_{rj}$ for each $j<s$. For $\sigma\in\mathbb{N}^{(r+1)\times s}$, set

$$H_\sigma = \{S : S\subseteq X',\ \#(S\cap E_{ij}) = \sigma(i,j) \text{ for every } i\le r \text{ and } j<s\}.$$

By (i), we have $\nu' H_\sigma = \int\nu'_T(H_\sigma)\tilde{\nu}(dT)$ for every $\sigma\in\mathbb{N}^{(r+1)\times s}$.

Set

$$J = \{\sigma : \sigma\in\mathbb{N}^{(r+1)\times s},\ \sigma(i,j) = n_{ij} \text{ for } i<r,\ j<s\},\quad K = \mathbb{N}^{(r+1)\times s}\setminus J,$$

$$H'_1 = \bigcup_{\sigma\in J}H_\sigma,\quad H'_2 = \bigcup_{\sigma\in K}H_\sigma.$$

Then $H'_1\subseteq H$, $H'_2\cap H = \emptyset$ and

$$H'_1\cup H'_2 = \{S : S\cap E_{ij} \text{ is finite for all } i\le r,\ j<s\}$$

is ν'-conegligible. Accordingly we have

$$\begin{aligned}\underline{\int}(\nu'_T)_*(H)\tilde{\nu}(dT) &\ge \underline{\int}(\nu'_T)_*(H'_1)\tilde{\nu}(dT) \ge \sum_{\sigma\in J}\int\nu'_T(H_\sigma)\tilde{\nu}(dT) \\ &= \sum_{\sigma\in J}\nu' H_\sigma = 1 - \sum_{\sigma\in K}\nu' H_\sigma\end{aligned}$$

(because $\bigcup_{\sigma\in J\cup K}H_\sigma$ is ν'-conegligible)

$$\begin{aligned}&= 1 - \sum_{\sigma\in K}\int\nu'_T(H_\sigma)\tilde{\nu}(dT) = \int 1 - \sum_{\sigma\in K}\nu'_T(H_\sigma)\,\tilde{\nu}(dT) \\ &= \int\nu'_T(\mathcal{P}X'\setminus H'_2)\tilde{\nu}(dT) \ge \overline{\int}(\nu'_T)^*(H)\tilde{\nu}(dT).\end{aligned}$$

But this means, first, that $(\nu'_T)_*(H) = (\nu'_T)^*(H)$ for $\tilde{\nu}$-almost every T; since ν'_T, being an image of the complete measure λ', is always complete, $\nu'_T(H)$ is defined for $\tilde{\nu}$-almost every T. Finally,

$$\nu' H = \nu' H'_1 = \sum_{\sigma\in J}\nu' H_\sigma = \int\nu'_T(H)\tilde{\nu}(dT),$$

as required. **Q**

(c) Suppose that $\langle E_i\rangle_{i<r}$ is a disjoint family in $\tilde{\Sigma}\otimes\Sigma_L$ such that all the projections of the E_i onto $\tilde{X}$ have finite measure, and $n_i\in\mathbb{N}$ for each $i<r$. Set $H = \{S : S\subseteq X',\ \#(S\cap E_i) = n_i \text{ for every } i<r\}$. Then $\int\nu'_T(H)\tilde{\nu}(dT) = \nu' H$.

P Let $\mathcal{E}$ be a finite subalgebra of $\tilde{\Sigma}$ such that every E_i belongs to $\mathcal{E}\otimes\Sigma_L$, and let $\langle F_j\rangle_{j<s}$ enumerate the atoms of $\mathcal{E}$ of finite measure; extend this to an enumeration $\langle F_j\rangle_{j<s'}$ of all the atoms of $\mathcal{E}$. Then we

can express each E_i as $\bigcup_{j<s'} F_j \times C_{ij}$ where each $C_{ij} \in \Sigma_L$; but as the projection of E_i has finite measure, C_{ij} must be empty for every $j \ge s$, so $E_i = \bigcup_{j<s} F_j \times C_{ij}$. Let Q be the set of all $q \in \mathbb{N}^{r\times s}$ such that $\sum_{j=0}^{s-1} q(i,j) = n_i$ for every $i < r$. For $q \in Q$ set

$$H_q = \{S : S \subseteq X', \#(S \cap (F_j \times C_{ij})) = q(i,j) \text{ for every } i < r,\ j < s\}.$$

Then $\langle H_q\rangle_{q\in Q}$ is disjoint and has union H, so

$$\int \nu'_T(H)\tilde\nu(dT) = \sum_{q\in Q}\int \nu'_T(H_q)\tilde\nu(dT) = \sum_{q\in Q}\nu' H_q = \nu' H,$$

using (b) for the middle equality. **Q**

(d) Now let $\langle E_i\rangle_{i<r}$ be a finite disjoint family of subsets of X' of finite measure, and $\langle n_i\rangle_{i<r}$ a family in $\mathbb{N}$. Set $H = \{S : S \subseteq X', \#(S\cap E_i) = n_i \text{ for every } i < r\}$. Then $\int \nu'_T(H)\tilde\nu(dT)$ is defined and equal to $\nu' H$.

P Let $\epsilon > 0$. For each $i < r$ we can find an $E'_i \in \tilde\Sigma \hat\otimes \Sigma_L$ such that $\mu'(E_i \triangle E'_i) \le \epsilon$ (251Ie). Discarding a negligible set from E'_i if necessary, we may suppose that the projection of E'_i on $\tilde X$ has finite measure. Set $\hat E_i = E'_i \setminus \bigcup_{k<i} E'_k$ for each i, so that $\langle \hat E_i\rangle_{i<r}$ is a disjoint family in $\tilde\Sigma \hat\otimes \Sigma_L$, and the projections of the $\hat E_i$ are still of finite measure. Set $\hat H = \{S : S \subseteq X', \#(S\cap \hat E_i) = n_i \text{ for every } i < r\}$. Then (c) tells us that $\int \nu'_T(\hat H)\tilde\nu(dT) = \nu'\hat H$.

Set $E = \bigcup_{i<r}(E_i \triangle E'_i)$. Then $\mu' E \le r\epsilon$, while E includes $E_i \triangle \hat E_i$ for every i, so

$$\hat H \setminus H_E \subseteq H \subseteq \hat H \cup H_E,$$

where $H_E = \{S : S \cap E \ne \emptyset\}$ as in (a). Accordingly

$$\nu' H - 3r\gamma\epsilon \le \nu'\hat H - 2r\gamma\epsilon$$

(by (a))

$$= \int \nu'_T(\hat H)\tilde\nu(dT) - 2r\gamma\epsilon$$

(by (c))

$$\le \int \nu'_T(\hat H) - \nu'_T(H_E)\tilde\nu(dT)$$

(by the other part of (a))

$$\le \underline{\int}(\nu'_T)_*(H)\tilde\nu(dT) \le \overline{\int}(\nu'_T)^*(H)\tilde\nu(dT)$$

$$\le \int \nu'_T(\hat H) + \nu'_T(H_E)\tilde\nu(dT) \le \nu'\hat H + 2r\gamma\epsilon \le \nu' H + 3r\gamma\epsilon.$$

As ϵ is arbitrary,

$$\nu' H = \underline{\int}(\nu'_T)_*(H)\tilde\nu(dT) = \overline{\int}(\nu'_T)^*(H)\tilde\nu(dT).$$

As in (c-ii) above, it follows that $\int \nu'_T(H)\tilde\nu(dT)$ is defined and equal to $\nu' H$. **Q**

(e) So if we write $\mathcal{H}$ for the family of subsets H of $\mathcal{P}X'$ such that $\int \nu'_T(H)\tilde\nu(dT)$ is defined and equal to $\nu' H$, and $\mathcal{H}_0$ for the family of sets of the form $H = \{S : S \subseteq X', \#(S\cap E_i) = n_i \text{ for every } i < r\}$ where $\langle E_i\rangle_{i<r}$ is a disjoint family of sets of finite measure and $n_i \in \mathbb{N}$ for $i < r$, we have $\mathcal{H} \supseteq \mathcal{H}_0$. But $\mathcal{H}$ is a Dynkin class, so includes the σ-algebra T' generated by $\mathcal{H}_0$, by 495C. Since every ν'-negligible set is included in a ν'-negligible member of T', $\mathcal{H}$ contains every ν'-negligible set, and therefore every set measured by ν'; which is what we need to know.

495L Theorem Let (X,Σ,μ) and $(\tilde X,\tilde\Sigma,\tilde\mu)$ be atomless σ-finite measure spaces and $\gamma > 0$. Let ν, $\tilde\nu$ be the Poisson point processes on X, $\tilde X$ respectively with intensity γ. Suppose that $f : X \to \tilde X$ is inverse-measure-preserving and that $\langle \mu_t\rangle_{t\in\tilde X}$ is a disintegration of μ over $\tilde\mu$ consistent with f (definition: 452E) such that every μ_t is a probability measure. Write λ for the product measure $\prod_{t\in\tilde X}\mu_t$ on $\Omega = X^{\tilde X}$, and for

$T \subseteq \tilde{X}$ define $\phi_T : \Omega \to \mathcal{P}X$ by setting $\phi_T(z) = z[T]$ for $z \in \Omega$; let ν_T be the image measure $\lambda\phi_T^{-1}$ on $\mathcal{P}X$. Then $\langle \nu_T \rangle_{T \subseteq \tilde{X}}$ is a disintegration of ν over $\tilde{\nu}$. Moreover

(i) setting $\tilde{f}(S) = f[S]$ for $S \subseteq X$, $\langle \nu_T \rangle_{T \subseteq \tilde{X}}$ is consistent with $\tilde{f} : \mathcal{P}X \to \mathcal{P}\tilde{X}$;

(ii) if $\langle \mu_t \rangle_{t \in \tilde{X}}$ is strongly consistent with f, then $\langle \nu_T \rangle_{T \subseteq \tilde{X}}$ is strongly consistent with $\tilde{f}$.

proof (a) For $T \subseteq \tilde{X}$ let V_T be the set of those $z \in \Omega$ such that $fz{\restriction}T$ is injective. We need to know that

$$W = \{T : T \subseteq \tilde{X},\ T \text{ is countable},\ V_T \text{ is } \lambda\text{-conegligible}\}$$

is $\tilde{\nu}$-conegligible. **P** Write $\tilde{\Sigma}^f = \{F : F \in \tilde{\Sigma},\ \tilde{\mu}F < \infty\}$. Because $\tilde{\mu}$ is atomless and σ-finite, there is a countable subalgebra $\mathcal{E}$ of $\tilde{\Sigma}$ such that for every $\epsilon > 0$ there is a cover of $\tilde{X}$ by members of $\mathcal{E}$ of measure at most ϵ. Set

$$Y = \{t : t \in \tilde{X},\ \mu_t f^{-1}[F] = (\chi F)(t) \text{ for every } F \in \mathcal{E}\},$$

so that Y is $\tilde{\mu}$-conegligible and $\mathcal{P}Y$ is $\tilde{\nu}$-conegligible. For $F \in \mathcal{E}$, let W_F be the set of those $T \subseteq Y$ such that for every $t \in T \cap F$ there is an $F' \in \mathcal{E}$ such that $T \cap F' = \{t\}$. Now, given $F \in \mathcal{E} \cap \tilde{\Sigma}^f$ and $\epsilon > 0$, there is a partition $\langle F_i \rangle_{i \in I}$ of F into members of $\mathcal{E}$ of measure at most ϵ. Then

$$\begin{aligned}\tilde{\nu}^*(\mathcal{P}Y \setminus W_F) &\le \tilde{\nu}\{T : \#(T \cap F_i) > 1 \text{ for some } i \in I\}\\ &\le \sum_{i \in I} 1 - e^{-\gamma\tilde{\mu}F_i}(1 + \gamma\tilde{\mu}F_i) \le \sum_{i \in I} \frac{1}{2}(\gamma\tilde{\mu}F_i)^2\\ &\le \frac{1}{2}\epsilon\gamma^2 \sum_{i \in I} \tilde{\mu}F_i = \frac{1}{2}\epsilon\gamma^2\tilde{\mu}F.\end{aligned}$$

As ϵ is arbitary, W_F is $\tilde{\nu}$-conegligible; accordingly $W' = \bigcap\{W_F : F \in \mathcal{E} \cap \tilde{\Sigma}^f\}$ is $\tilde{\nu}$-conegligible.

Now suppose that $T \in W'$. Because $\tilde{X}$ is covered by $\mathcal{E} \cap \tilde{\Sigma}^f$, we see that for every $t \in T$ there is an $F \in \mathcal{E} \cap \tilde{\Sigma}^f$ containing t, and now there is an $F' \in \mathcal{E}$ such that $T \cap F' = \{t\}$. In particular, T is countable, so

$$U_T = \{z : z(t) \in f^{-1}[F] \text{ whenever } F \in \mathcal{E} \text{ and } t \in T \cap F\}$$

is λ-conegligible. Take $z \in U_T$. If t, t' are distinct points of T, there is an $F \in \mathcal{E}$ containing t but not t', and now F contains $f(z(t))$ but not $f(z(t'))$. So $fz{\restriction}T$ is injective. Thus $U_T \subseteq V_T$ and V_T is λ-conegligible. This is true for every $T \in W'$, so $W \supseteq W'$ is $\tilde{\nu}$-conegligible. **Q**

(b) Suppose that $\langle E_i \rangle_{i<r}$ is a disjoint family of subsets of X with finite measure, and $n_i \in \mathbb{N}$ for $i < r$. Set $H = \{S : S \subseteq X,\ \#(S \cap E_i) = n_i \text{ for every } i < r\}$. Then $\int \nu_T(H)\tilde{\nu}(dT)$ is defined and equal to νH. **P** As in 495K, set $X' = \tilde{X} \times [0,1]$ with the product measure μ', and write λ' for the product measure on $\Omega' = [0,1]^{\tilde{X}}$. Let ν' be the Poisson point process on X' with intensity γ. For $T \subseteq \tilde{X}$ define $\psi_T : \Omega' \to \mathcal{P}X'$ by setting $\psi_T(z) = \{(t, z(t)) : t \in T\}$ for $z \in \Omega'$ and let ν'_T be the image measure $\lambda'\psi_T^{-1}$ on $\mathcal{P}X'$. By 495K, $\langle \nu'_T \rangle_{T \subseteq \tilde{X}}$ is a disintegration of ν' over $\tilde{\nu}$.

For each $i < r$, $\int \mu_t(E_i)\tilde{\mu}(dt) = \mu E_i$; set $Y_1 = \{t : \mu_t E_i \text{ is defined for every } i < r\}$, so that $Y_1 \subseteq \tilde{X}$ is $\tilde{\nu}$-conegligible. Set $g_i(t) = \sum_{j<i} \mu_t E_j$ for $t \in Y_1$ and $i \le r$, and

$$E'_i = \{(t, \alpha) : t \in Y_1,\ g_i(t) \le \alpha < g_{i+1}(t)\}$$

for $i < r$. Then $\mu' E'_i = \int g_{i+1} - g_i \, d\tilde{\nu} = \mu E_i$ for each i, by 252N.

Set

$$H' = \{S : S \subseteq X',\ \#(S' \cap E'_i) = n_i \text{ for every } i < r\},$$

$$W_1 = \{T : T \in W,\ T \subseteq Y_1,\ \nu'_T H' \text{ is defined}\},$$

so H' is measured by ν' and W_1 is $\tilde{\nu}$-conegligible. Let T be any member of W_1. Let Q be the set of partitions $q = \langle q(i) \rangle_{i \le r}$ of T such that $\#(q(i)) = n_i$ for every $i < r$; because T is countable, so is Q. Set $E_r = X \setminus \bigcup_{i<r} E_i$ and $E'_r = X' \setminus \bigcup_{i<r} E'_i$. Then

$$\mu_t E_r = 1 - \textstyle\sum_{i=0}^{r-1} \mu_t E_i = 1 - g_r(t) = \mu_L E'_r[\{t\}]$$

for every $t\in Y_1$. Now

$$\begin{aligned}
\nu'_T H' &= \lambda'\{z : z\in\Omega',\, \psi_T(z)\in H'\}\\
&= \lambda'\{z : z\in\Omega',\, \#(\{t : t\in T,\, z(t)\in E'_i[\{t\}]\}) = n_i \text{ for every } i<r\}\\
&= \sum_{q\in Q}\lambda'\{z : z\in\Omega',\, z(t)\in E'_i[\{t\}] \text{ whenever } i\le r \text{ and } t\in q(i)\}\\
&= \sum_{q\in Q}\prod_{i=0}^{r}\prod_{t\in q(i)}\mu_L E'_i[\{t\}] = \sum_{q\in Q}\prod_{i=0}^{r}\prod_{t\in q(i)}\mu_t E_i\\
&= \sum_{q\in Q}\lambda\{z : z\in\Omega,\, z(t)\in E_i \text{ whenever } i\le r \text{ and } t\in q(i)\}\\
&= \lambda\{z : z\in\Omega,\, \#(\{t : t\in T,\, z(t)\in E_i\}) = n_i \text{ for every } i<r\}\\
&= \lambda\{z : z\in V_T,\, \#(\{t : t\in T,\, z(t)\in E_i\}) = n_i \text{ for every } i<r\}\\
&= \lambda\{z : z\in V_T,\, \#(z[T]\cap E_i) = n_i \text{ for every } i<r\}\\
&= \lambda\{z : z\in V_T,\, \phi_T(z)\in H\} = \nu_T H.
\end{aligned}$$

Since this is true for $\tilde\nu$-almost every T,

$$\begin{aligned}
\int \nu_T(H)\tilde\nu(dT) &= \int \nu'_T(H')\tilde\nu(dT) = \nu' H'\\
&= \prod_{i<r}\frac{(\gamma\mu' E'_i)^{n_i}}{n_i!}e^{-\gamma\mu' E'_i} = \prod_{i<r}\frac{(\gamma\mu E_i)^{n_i}}{n_i!}e^{-\gamma\mu E_i} = \nu H. \text{ **Q**}
\end{aligned}$$

Now, just as in part (e) of the proof of 495K, 495C tells us that $\int \nu_T(H)\tilde\nu(dT) = \nu H$ whenever ν measures H, so that $\langle\nu_T\rangle_{T\subseteq\tilde X}$ is a disintegration of ν over $\tilde\nu$.

(c) Let $\langle F_i\rangle_{i<r}$ be a disjoint family in $\tilde\Sigma^f$, and take $n_i\in\mathbb{N}$ for $i<r$. Set

$$F_r = \tilde X\setminus\bigcup_{i<r}F_i,$$

$$Y_2 = \{t : t\in\tilde X,\, \mu_t f^{-1}[F_i] = (\chi F_i)(t) \text{ for every } i\le r\},$$

$$W_2 = \{T : T\in W,\, T\subseteq Y_2\},$$

so that Y_2 is $\tilde\mu$-conegligible and W_2 is $\tilde\nu$-conegligible. Set

$$\tilde H = \{T : T\in W_2,\, \#(T\cap F_i) = n_i \text{ for every } i<r\},$$

$$H = \tilde f^{-1}[\tilde H] = \{S : S\subseteq X,\, f[S]\in\tilde H\}.$$

Then $\nu_T H = \chi\tilde H(T)$ for every $T\in\tilde H$. **P** For $i\le r$ and $t\in T\cap F_i$, we have

$$\lambda\{z : z(t)\in f^{-1}[F_i]\} = \mu_t f^{-1}[F_i] = 1$$

because $T\subseteq Y_2$. So

$$V = \{z : z\in V_T,\, f(z(t))\in F_i \text{ whenever } i\le r \text{ and } t\in T\cap F_i\}$$

is λ-conegligible. But if $z\in V$ then $fz\restriction T$ is injective, so

$$\#(f[z[T]]\cap F_i) = \#(T\cap(fz)^{-1}[F_i]) = \#(T\cap F_i)$$

for every $i<r$, and $z[T]\in H$ iff $T\in\tilde H$. Thus

$$\begin{aligned}
\nu_T H = \lambda\{z : z[T]\in H\} &= \lambda V = 1 \text{ if } T\in\tilde H,\\
&= \lambda(\Omega\setminus V) = 0 \text{ otherwise. **Q**}
\end{aligned}$$

Setting

$$\mathcal{H} = \{\tilde{H} : \tilde{H} \subseteq \tilde{X},\ \nu_T \tilde{f}^{-1}[\tilde{H}] = \chi\tilde{H}(T) \text{ for } \tilde{\nu}\text{-almost every } T \in \tilde{H}\},$$

it is easy to check that $\mathcal{H}$ is a Dynkin class containing all sets of the form $\{T : T \subseteq \tilde{X},\ \#(T \cap F_i) = n_i$ for every $i < r\}$ where $\langle F_i \rangle_{i<r}$ is a disjoint family in $\tilde{\Sigma}^f$, and therefore including the σ-algebra generated by such sets, by 495C. But as $\mathcal{H}$ also contains any subset of a negligible set belonging to $\mathcal{H}$ (remember that all the ν_T are complete probability measures, like λ), it includes the domain of $\tilde{\nu}$, and $\langle \nu_T \rangle_{T \subseteq \tilde{X}}$ is consistent with $\tilde{f}$.

(d) Now suppose that $\langle \mu_t \rangle_{t \in \tilde{X}}$ is strongly consistent with f. Set $Y_3 = \{t : \mu_t f^{-1}[\{t\}] = 1\}$ and $W_3 = \{T : T \in W,\ T \subseteq Y_3\}$. Then $\nu_T \tilde{f}^{-1}[\{T\}] = 1$ for every $T \in W_3$. **P** Set $V'_T = \{z : z \in \Omega,\ f(z(t)) = t$ for every $t \in T\}$. For each $t \in T$,

$$\lambda\{z : f(z(t)) = t\} = \mu_t f^{-1}[\{t\}] = 1,$$

because $T \subseteq W_3$. As T is countable, V'_T is λ-conegligible. But now

$$\nu_T \tilde{f}^{-1}[\{T\}] = \lambda\{z : f[z[T]] = T\} \ge \lambda V'_T = 1. \ \mathbf{Q}$$

As W_3 is $\tilde{\nu}$-conegligible, $\langle \nu_T \rangle_{T \subseteq \tilde{X}}$ is strongly consistent with $\tilde{f}$.

495M Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, and $\gamma > 0$. Then there are a probability algebra $(\mathfrak{B}, \bar{\lambda})$ and a function $\theta : \mathfrak{A} \to \mathfrak{B}$ such that

(i) $\theta(\sup A) = \sup \theta[A]$ for every non-empty $A \subseteq \mathfrak{A}$ such that $\sup A$ is defined in $\mathfrak{A}$;

(ii) $\bar{\lambda}\theta(a) = 1 - e^{-\gamma\bar{\mu}a}$ for every $a \in \mathfrak{A}$, interpreting $e^{-\infty}$ as 0;

(iii) whenever $\langle a_i \rangle_{i \in I}$ is a disjoint family in $\mathfrak{A}$ and $\mathfrak{C}_i$ is the closed subalgebra of $\mathfrak{B}$ generated by $\{\theta(a) : a \subseteq a_i\}$ for each i, then $\langle \mathfrak{C}_i \rangle_{i \in I}$ is stochastically independent.

proof (a) We may suppose that $(\mathfrak{A}, \bar{\mu})$ is the measure algebra of a measure space (X, Σ, μ) (321J). Set $\Sigma^f = \{E : E \in \Sigma,\ \mu E < \infty\}$ and $\mathfrak{A}^f = \{a : a \in \mathfrak{A},\ \bar{\mu}a < \infty\}$. Let $(\Omega, \Lambda, \lambda)$ and $\langle g_E \rangle_{E \in \Sigma^f}$ be as in 495B, and take $(\mathfrak{B}, \bar{\lambda})$ to be the measure algebra of $(\Omega, \Lambda, \lambda)$. Note that if E, $F \in \Sigma^f$ and $\mu(E \triangle F) = 0$, then $g_{E \setminus F}$ and $g_{F \setminus E}$ have Poisson distributions with expectation 0, so are zero almost everywhere, while $g_E =_{\text{a.e.}} g_{E \cap F} + g_{E \setminus F}$ and $g_F =_{\text{a.e.}} g_{E \cap F} + g_{F \setminus E}$; so that $g_E =_{\text{a.e.}} g_F$. This means that we can define $\theta : \mathfrak{A}^f \to \mathfrak{B}$ by setting $\theta(E^\bullet) = \{\omega : g_E(\omega) \ne 0\}^\bullet$ whenever $E \in \Sigma^f$, and we shall have $\bar{\lambda}(\theta a) = 1 - e^{-\gamma\bar{\mu}a}$ because g_E has a Poisson distribution with expectation $\bar{\mu}a$ whenever $E \in \Sigma^f$ and $E^\bullet = a$. For $a \in \mathfrak{A} \setminus \mathfrak{A}^f$ set $\theta(a) = 1_{\mathfrak{B}}$.

(b) If a, $b \in \mathfrak{A}^f$ are disjoint, they can be represented as $E^\bullet$, $F^\bullet$ where E, $F \in \Sigma^f$ are disjoint. In this case, $g_{E \cup F} =_{\text{a.e.}} g_E + g_F$, so $\theta(a \cup b) = \theta(a) \cup \theta(b)$. Of course the same is true if a, $b \in \mathfrak{A}$ are disjoint and either has infinite measure. It follows at once that for any a, $b \in \mathfrak{A}$,

$$\theta(a \cup b) = \theta(a \setminus b) \cup \theta(a \cap b) \cup \theta(b \setminus a) = \theta(a) \cup \theta(b).$$

Consequently $\theta(\sup A) = \sup \theta[A]$ for any finite set $A \subseteq \mathfrak{A}$. If $A \subseteq \mathfrak{A}$ is an infinite set with supremum a^*, then $A' = \{\sup B : B \in [A]^{<\omega}\}$ is an upwards-directed set with supremum a^*, so there is a non-decreasing sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ in A' such that $\lim_{n\to\infty} \bar{\mu}a_n = \bar{\mu}a^*$ (321D). In this case, $b^* = \sup_{n \in \mathbb{N}} \theta(a_n)$ is defined in $\mathfrak{B}$ and

$$\bar{\lambda}b^* = \lim_{n\to\infty} \bar{\lambda}\theta(a_n) = \lim_{n\to\infty} 1 - e^{-\gamma\bar{\mu}a_n} = 1 - e^{-\gamma\bar{\mu}a^*} = \bar{\lambda}\theta(a^*).$$

So $b^* = \theta(a^*)$; since $\theta(a^*)$ is certainly an upper bound of $\theta[A']$, it must actually be the supremum of $\theta[A']$ and therefore (because θ preserves finite suprema) of $\theta[A]$.

(c) Thus θ satisfies (i) and (ii). As for (iii), note first that if $\langle a_i \rangle_{i \in I}$ is a finite disjoint family in $\mathfrak{A}$, then $\bar{\lambda}(\inf_{i \in I} \theta(a_i)) = \prod_{i \in I} \bar{\lambda}\theta(a_i)$. **P** Set $J = \{i : i \in I,\ \bar{\mu}a_i < \infty\}$. For $i \in J$, represent a_i as $E_i^\bullet$ where $\langle E_i \rangle_{i \in J}$ is a disjoint family in Σ^f. Then $\langle g_{E_i} \rangle_{i \in J}$ is independent, so

$$\begin{aligned}\bar{\lambda}(\inf_{i \in I} \theta(a_i)) = \bar{\lambda}(\inf_{i \in J} \theta(a_i)) &= \lambda(\Omega \cap \bigcap_{i \in J} \{\omega : g_{E_i}(\omega) = 0\}) \\ &= \prod_{i \in J} \lambda\{\omega : g_{E_i}(\omega) = 0\} = \prod_{i \in J} \bar{\lambda}\theta(a_i) = \prod_{i \in I} \bar{\lambda}\theta(a_i). \ \mathbf{Q}\end{aligned}$$

Now suppose that $\langle a_i\rangle_{i\in I}$ is a finite disjoint family in $\mathfrak{A}$ and that $\mathfrak{D}_i$ is the subalgebra of $\mathfrak{B}$ generated by $D_i = \{\theta(a) : a \subseteq a_i\}$ for each i. We know that each D_i is closed under $\cup$ (by (i)) and that $\bar\lambda(\inf_{i\in J} d_i) = \prod_{i\in J} \bar\lambda d_i$ whenever $J \subseteq I$ and $d_i \in D_i$ for each $i \in J$, that is, that $\langle d_i\rangle_{i\in I}$ is stochastically independent whenever $d_i \in D_i$ for each i. Setting $D'_i = \{1 \setminus d : d \in D_i\} \cup \{0\}$, we see that D'_i is closed under $\cap$ and that $\langle d_i\rangle_{i\in I}$ is stochastically independent whenever $d_i \in D'_i$ for each i (as in 272F). An induction on $\#(J)$, using 313Ga for the inductive step, shows that if $J \subseteq I$, $d_i \in \mathfrak{D}_i$ for $i \in J$, and $d_i \in D'_i$ for $i \in I \setminus J$, then $\bar\lambda(\inf_{i\in I} d_i) = \prod_{i\in I} \bar\lambda d_i$. At the end of the induction, we see that $\bar\lambda(\inf_{i\in I} d_i) = \prod_{i\in I} \bar\lambda d_i$ whenever $d_i \in \mathfrak{D}_i$ for each i, and therefore whenever d_i belongs to the topological closure of $\mathfrak{D}_i$ for each i, where $\mathfrak{B}$ is given its measure-algebra topology (§323).

Finally, suppose that $\langle a_i\rangle_{i\in I}$ is any disjoint family in $\mathfrak{A}$, and $\mathfrak{C}_i$ is the closed subalgebra of $\mathfrak{B}$ generated by $D_i = \{\theta(a) : a \subseteq a_i\}$ for each i. Take a finite set $J \subseteq I$ and $c_i \in \mathfrak{C}_i$ for each $i \in J$. By 323J, $\mathfrak{C}_i$ is the topological closure of the subalgebra $\mathfrak{D}_i$ of $\mathfrak{B}$ generated by $\{\theta(a) : a \subseteq a_i\}$; so $\bar\lambda(\inf_{i\in J} c_i) = \prod_{i\in J} \bar\lambda c_i$. As $\langle c_i\rangle_{i\in J}$ is arbitrary, $\langle \mathfrak{C}_i\rangle_{i\in I}$ is independent.

495N Proposition Let U be any L-space. Then there are a probability space $(\Omega, \Lambda, \lambda)$ and a positive linear operator $T : U \to L^1(\lambda)$ such that $\|Tu\|_1 = \|u\|_1$ whenever $u \in L^1(\mu)^+$ and $\langle Tu_i\rangle_{i\in I}$ is stochastically independent in $L^0(\lambda)$ whenever $\langle u_i\rangle_{i\in I}$ is a disjoint family in $L^1(\mu)$.

Remarks Recall that a family $\langle u_i\rangle_{i\in I}$ in a Riesz space is 'disjoint' if $|u_i| \wedge |u_j| = 0$ for all distinct i, $j \in I$ (352C). A family $\langle v_i\rangle_{i\in I}$ in $L^0(\lambda)$ is 'independent' if $\langle g_i\rangle_{i\in I}$ is an independent family of random variables whenever $g_i \in \mathcal{L}^0(\lambda)$ represents v_i for each i; compare 367W.

proof (a) By Kakutani's theorem, there is a measure algebra $(\mathfrak{A}, \bar\mu)$ such that U is isomorphic, as Banach lattice, to $L^1(\mathfrak{A}, \bar\mu)$; now $(\mathfrak{A}, \bar\mu)$ can be represented as the measure algebra of a measure space (X, Σ, μ), and we can identify U and $L^1(\mathfrak{A}, \bar\mu)$ with $L^1(\mu)$ (365B). Set $\Sigma^f = \{E : E \in \Sigma, \mu E < \infty\}$ and $\mathfrak{A}^f = \{a : a \in \mathfrak{A}, \bar\mu a < \infty\}$ as usual. Take $(\Omega, \Lambda, \lambda)$ and $\langle g_E\rangle_{E\in\Sigma^f}$ from 495B, with $\gamma = 1$. As in the proof of 495M, we have $g_E =_{\text{a.e.}} g_F$ whenever E, $F \in \Sigma^f$ and $\mu(E \triangle F) = 0$; consequently we can define $\psi : \mathfrak{A}^f \to L^1(\lambda)$ by setting $\psi a = g_E^\bullet$ whenever $E \in \Sigma^f$ and $E^\bullet = a$. Again as in 495M, $g_{E\cup F} =_{\text{a.e.}} g_E + g_F$ whenever E, $F \in \Sigma^f$ are disjoint, so ψ is additive. Also

$$\|\psi a\|_1 = \int g_E d\lambda = \mu E = \bar\mu a$$

whenever $E \in \Sigma^f$ represents $a \in \mathfrak{A}^f$. By 365I, there is a unique bounded linear operator $T : L^1(\mathfrak{A}, \bar\mu) \to L^1(\lambda)$ such that $T(\chi a) = \psi a$ for every $a \in \mathfrak{A}^f$. By 365Ja[5], T is a positive operator. The set $\{u : u \in L^1(\mathfrak{A}, \bar\mu)^+, \|Tu\|_1 = \|u\|_1\}$ is closed under addition, norm-closed and contains $\alpha\chi a$ for every $a \in \mathfrak{A}^f$ and $\alpha \geq 0$, so is the whole of $L^1(\mathfrak{A}, \bar\mu)^+$, by 365F.

Note that if $\langle a_i\rangle_{i\in I}$ is a disjoint family in $\mathfrak{A}^f$, then $\langle \psi a_i\rangle_{i\in I}$ is stochastically independent, by 495B(ii).

(b) Now let $\langle u_i\rangle_{i\in I}$ be a disjoint family in $L^1(\mathfrak{A}, \bar\mu)$. Then $\langle Tu_i\rangle_{i\in I}$ is independent. **P?** Otherwise, there are a finite set $J \subseteq I$ and a family $\langle V_i\rangle_{i\in J}$ such that V_i is a neighbourhood of Tu_i in the topology of convergence in measure on $L^0(\mu)$ for each $i \in J$, and $\langle v_i\rangle_{i\in J}$ is not independent whenever $v_i \in V_i$ for each i (367W). Because the embedding $L^1(\lambda) \subsetneq L^0(\lambda)$ is continuous for the norm topology on $L^1(\lambda)$ and the topology of convergence in measure (245G), there is a $\delta > 0$ such that $Tu'_i \in V_i$ whenever $i \in J$, $u'_i \in L^1(\mathfrak{A}, \bar\mu)$ and $\|u'_i - u_i\|_1 \leq \delta$. Now we can find such $u'_i \in S(\mathfrak{A}^f)$ with $|u'_i| \leq |u_i|$ (365F).

Express each u'_i as $\sum_{k=0}^{n_i} \alpha_{ik}\chi a_{ik}$ where $\langle a_{ik}\rangle_{k\leq n_i}$ is a disjoint family in $\mathfrak{A}^f$ and no α_{ik} is zero (361Eb). In this case, all the a_{ik}, for $i \in J$ and $k \leq n_i$, are disjoint, so all the $\psi(a_{ik})$ are independent. But this means that $\langle Tu'_i\rangle_{i\in J} = \langle \sum_{k=0}^{n_i} \alpha_{ik}\psi(a_{ik})\rangle_{i\in J}$ is independent (272K); which is impossible, because $Tu'_i \in V_i$ for every $i \in J$. **XQ**

So T, regarded as a function from U to $L^1(\lambda)$, has the required properties.

495O The following is a more concrete expression of the same ideas.

Proposition Let (X, Σ, μ) be an atomless measure space, and ν the Poisson point process on X with intensity $\gamma > 0$.

[5] Formerly 365Ka.

(a) If $f \in \mathcal{L}^1(\mu)$, $Q_f(S) = \sum_{x \in S \cap \operatorname{dom} f} f(x)$ is defined and finite for ν-almost every $S \subseteq X$, and $\int Q_f d\nu = \gamma \int f d\mu$.

(b) If $f \in \mathcal{L}^1(\mu) \cap \mathcal{L}^2(\mu)$, $\int Q_f^2 d\nu$ is defined and equal to $\gamma \int f^2 d\mu + (\gamma \int f d\mu)^2$.

(c) We have a positive linear operator $T : L^1(\mu) \to L^1(\nu)$ defined by setting $T(f^\bullet) = Q_f^\bullet$ for every $f \in \mathcal{L}^1(\mu)$.

(d) $\|Tu\|_1 = \gamma \|u\|_1$ whenever $u \in L^1(\mu)^+$ and $\langle Tu_i \rangle_{i \in I}$ is stochastically independent in $L^0(\lambda)$ whenever $\langle u_i \rangle_{i \in I}$ is a disjoint family in $L^1(\mu)$.

proof (a) In the language of 495D, $Q_{\chi E} = f_E$ for every $E \in \Sigma^f$. So $Q_{\chi E} \in \mathcal{L}^1(\nu)$ and $\int Q_{\chi E} d\nu = \gamma \mu E$ for every $E \in \Sigma^f$. If $f = \sum_{i=0}^r \alpha_i \chi E_i$ is a simple function on X, then $Q_f =_{\text{a.e.}} \sum_{i=0}^r \alpha_i Q_{\chi E_i} \in \mathcal{L}^1(\nu)$ and $\int Q_f d\nu = \gamma \int f d\mu$. If $f \in \mathcal{L}^1(\mu)$ is zero a.e., then $\{S : S \subseteq f^{-1}[\{0\}]\}$ is ν-conegligible, so $Q_f = 0$ a.e. It follows that if $f \in \mathcal{L}^1(\mu)$ is non-negative, and $\langle f_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence of simple functions converging to f almost everywhere, then $Q_f =_{\text{a.e.}} \lim_{n\to\infty} Q_{f_n}$, while $Q_{f_n} \le_{\text{a.e.}} Q_{f_{n+1}}$ for every n; so Q_f is ν-integrable and

$$\int Q_f d\nu = \lim_{n\to\infty} \int Q_{f_n} d\nu = \lim_{n\to\infty} \gamma \int f_n d\mu = \gamma \int f d\mu.$$

(b) We can use the same ideas, with some further twists.

(i) If $E \in \Sigma^f$, then

$$\begin{aligned}
\int f_E^2 d\nu - \gamma\mu E &= \int f_E^2 d\nu - \gamma \int f_E d\nu \\
&= \sum_{n=0}^\infty (n^2 - n)\nu\{S : \#(S \cap E) = n\} \\
&= \sum_{n=2}^\infty n(n-1)\tfrac{(\gamma\mu E)^n}{n!} e^{-\gamma\mu E} \\
&= (\gamma\mu E)^2 \sum_{n=0}^\infty \tfrac{(\gamma\mu E)^n}{n!} e^{-\gamma\mu E} = (\gamma\mu E)^2
\end{aligned}$$

so $\int f_E^2 d\nu = \gamma\mu E + (\gamma\mu E)^2$.

(ii) If $f = \sum_{i=0}^n \alpha_i \chi E_i$ where $E_0, \dots, E_n \in \Sigma^f$ are disjoint and $\alpha_0, \dots, \alpha_n \in \mathbb{R}$, then $f_{E_0}, \dots, f_{E_n}$ are ν-independent (495D), so

$$\begin{aligned}
\int Q_f^2 d\nu &= \int (\sum_{i=0}^n \alpha_i f_{E_i})^2 d\nu \\
&= \sum_{i,j \le n} \alpha_i \alpha_j \int f_{E_i} \times f_{E_j} d\nu \\
&= \sum_{i=0}^n \alpha_i^2 \int f_{E_i}^2 d\nu + \sum_{i,j \le n, i \ne j} \alpha_i \alpha_j \int f_{E_i} d\nu \cdot \int f_{E_j} d\nu
\end{aligned}$$

(272R)

$$\begin{aligned}
&= \sum_{i=0}^n \alpha_i^2 (\gamma\mu E_i + (\gamma\mu E_i)^2) + (\int Q_f d\nu)^2 - \sum_{i=0}^n \alpha_i^2 (\int f_{E_i} d\nu)^2 \\
&= \gamma \sum_{i=0}^n \alpha_i^2 \mu E_i + \sum_{i=0}^n \alpha_i^2 (\gamma\mu E_i)^2 + (\gamma \int f d\mu)^2 - \sum_{i=0}^n \alpha_i^2 (\gamma\mu E_i)^2 \\
&= \gamma \int f^2 d\mu + (\gamma \int f d\mu)^2.
\end{aligned}$$

(iii) If $f \in \mathcal{L}^1(\mu) \cap \mathcal{L}^2(\mu)$ is non-negative, there is a non-decreasing sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ of non-negative simple functions converging μ-almost everywhere to f. Now $\langle Q_{f_n} \rangle_{n\in\mathbb{N}}$ is non-decreasing and converges ν-almost everywhere to Q_f, and

$$\begin{aligned}\int Q_f^2 d\nu &= \lim_{n\to\infty} \int Q_{f_n}^2 d\nu = \lim_{n\to\infty} \gamma \int f_n^2 d\mu + (\gamma \int f_n d\mu)^2 \\ &= \gamma \int f^2 d\mu + (\gamma \int f d\mu)^2.\end{aligned}$$

(iv) Generally, if $f \in \mathcal{L}^1(\mu)\cap\mathcal{L}^2(\mu)$, it is equal almost everywhere to a difference $f_1 - f_2$ of non-negative functions in $\mathcal{L}^1(\mu) \cap \mathcal{L}^2(\mu)$ such that there is an $E \in \Sigma$ for which $f_2(x) = 0$ whenever $x \in E$ and $f_1(x) = 0$ whenever $x \in X \setminus E$. In this case, identifying $\mathcal{P}X$ with $\mathcal{P}E \times \mathcal{P}(X \setminus E)$, Q_{f_1} depends on the first coordinate and Q_{f_2} on the second, so they are ν-independent (495F). Consequently

$$\begin{aligned}\int Q_f^2 d\nu &= \int Q_{f_1}^2 d\nu - 2\int Q_{f_1} \times Q_{f_2} d\nu + \int Q_{f_2}^2 d\nu \\ &= \gamma \int f_1^2 d\mu + (\gamma \int f_1 d\mu)^2 - \int Q_{f_1} d\nu \cdot \int Q_{f_2} d\mu + \gamma \int f_2^2 d\mu + (\gamma \int f_2 d\mu)^2 \\ &= \gamma \int f_1^2 d\mu + (\gamma \int f_1 d\mu)^2 - \gamma^2 \int f_1 d\mu \cdot \int f_2 d\mu + \gamma \int f_2^2 d\mu + (\gamma \int f_2 d\mu)^2 \\ &= \gamma \int f^2 d\mu + (\gamma \int f d\mu)^2,\end{aligned}$$

as claimed.

(c) Since $Q_f =_{\text{a.e.}} Q_{f'}$ whenever $f =_{\text{a.e.}} f'$ in $\mathcal{L}^1(\mu)$, we can define $T : L^1(\mu) \to L^1(\nu)$ by setting $T(f^\bullet) = (Q_f)^\bullet$ for every $f \in \mathcal{L}^1(\mu)$; because $Q_{\alpha f} =_{\text{a.e.}} \alpha Q_f$ and $Q_{f+f'} =_{\text{a.e.}} Q_f + Q_{f'}$ whenever f, $f' \in \mathcal{L}^1(\mu)$ and $\alpha \in \mathbb{R}$, T is linear; because $Q_f \ge 0$ a.e. whenever $f \ge 0$ a.e., T is positive.

(d) Because $\int Q_f d\nu = \gamma \int f\, d\mu$ for every $f \in \mathcal{L}^1(\mu)$, and T is positive, $\|Tu\|_1 = \gamma\|u\|_1$ for every $u \in L^1(\mu)^+$. Finally, if $\langle u_i \rangle_{i\in I}$ is a finite disjoint family in $L^1(\mu)$, we can find a family $\langle f_i \rangle_{i\in I}$ of measurable functions from X to $\mathbb{R}$ such that $f_i^\bullet = u_i$ for each i and the sets $E_i = \{x : f_i(x) \ne 0\}$ are disjoint. For each $i \in I$, let T_i be the σ-algebra of subsets of $\mathcal{P}X$ generated by sets of the form $\{S : \#(S \cap E) = n\}$ where $E \subseteq E_i$ has finite measure and $n \in \mathbb{N}$. Then $\langle \mathrm{T}_i \rangle_{i\in I}$ is independent (as in part (c) of the proof of 495M), and each Q_{f_i} is T_i-measurable, so $\langle Q_{f_i} \rangle_{i\in I}$ is independent and $\langle Tu_i \rangle_{i\in I}$ is independent.

495P We can identify the characteristic functions of the random variables Q_f as defined above.

Proposition Let (X, Σ, μ) be an atomless measure space, and ν the Poisson point process on X with intensity $\gamma > 0$. For $f \in \mathcal{L}^1(\mu)$ set $Q_f(S) = \sum_{x\in S\cap\operatorname{dom} f} f(x)$ when $S \subseteq X$ and the sum is defined in $\mathbb{R}$. Then

$$\int_{\mathcal{P}X} e^{iyQ_f} d\nu = \exp\Bigl(\gamma \int_X (e^{iyf} - 1) d\mu\Bigr)$$

for any $y \in \mathbb{R}$.

proof Note that Q_f is defined ν-almost everywhere, by 495Oa.

(a) Consider first the case in which f is a simple function, expressed as $\sum_{j=0}^n \alpha_j \chi F_j$ where $\langle F_j \rangle_{j\le n}$ is a disjoint family of sets of finite measure and $\alpha_j \in \mathbb{R}$ for each j. Then $Q_f(S) = \sum_{j=0}^n \alpha_j \#(S \cap F_j)$ for ν-almost every S, so

$$\int e^{iyQ_f} d\nu = \int \prod_{j=0}^n e^{iy\alpha_j \#(S\cap F_j)} \nu(dS) = \prod_{j=0}^n \int e^{iy\alpha_j \#(S\cap F_j)} \nu(dS)$$

(because the functions $S \mapsto \#(S \cap F_j)$ are independent)

$$= \prod_{j=0}^{n} \sum_{k=0}^{\infty} \frac{(\gamma\mu F_j)^k}{k!} e^{-\gamma\mu F_j} e^{iy\alpha_j k} = \prod_{j=0}^{n} e^{-\gamma\mu F_j} \sum_{k=0}^{\infty} \frac{(e^{iy\alpha_j}\gamma\mu F_j)^k}{k!}$$

$$= \prod_{j=0}^{n} \exp\bigl((e^{iy\alpha_j} - 1)\gamma\mu F_j)\bigr) = \exp\bigl(\gamma \sum_{j=0}^{n}(e^{iy\alpha_j} - 1)\mu F_j\bigr)$$

$$= \exp\bigl(\gamma \int (e^{iyf} - 1)d\mu\bigr).$$

(b) Now suppose that f is any integrable function. Then there is a sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ of simple functions such that $|f_n| \le_{\text{a.e.}} |f|$ for every n and $\lim_{n\to\infty} f_n =_{\text{a.e.}} f$. Write q_n, q for Q_{f_n}, Q_f. Set

$$D = \{x : x \in \operatorname{dom} f,\ |f_n(x)| \le |f(x)| \text{ for every } n \text{ and } \lim_{n\to\infty} f_n(x) = f(x)\},$$

so that D is μ-conegligible. If $S \subseteq D$ and $Q_{|f|}(S)$ is defined, then $q(S) = \lim_{n\to\infty} q_n(S)$, and this is true for ν-almost every S; so $\int e^{iyq}d\nu = \lim_{n\to\infty} e^{iyq_n}d\nu$, by Lebesgue's Dominated Convergence Theorem. On the other hand,

$$|e^{i\alpha} - 1| = |\int_0^\alpha \frac{1}{i}e^{it}dt| \le \alpha, \quad |e^{-i\alpha} - 1| = |\int_0^\alpha \frac{1}{i}e^{-it}dt| \le \alpha$$

for every $\alpha \ge 0$. So if we set $g(x) = e^{iyf(x)} - 1$, $g_n(x) = e^{iyf_n(x)} - 1$ when these are defined, we have $|g_n| \le_{\text{a.e.}} |yf_n| \le_{\text{a.e.}} |yf|$ for every n. Accordingly

$$\int(e^{iyf} - 1)d\mu = \int \lim_{n\to\infty}(e^{iyf_n} - 1)d\mu = \lim_{n\to\infty} \int(e^{iyf_n} - 1)d\mu$$

by Lebesgue's theorem again. It follows that

$$\int e^{iyQ_f}d\nu = \lim_{n\to\infty} \int e^{iyq_n}d\nu = \lim_{n\to\infty} \exp\bigl(\gamma \int (e^{iyf_n} - 1)d\mu\bigr)$$

(by (a))

$$= \exp\bigl(\gamma \lim_{n\to\infty} \int (e^{iyf_n} - 1)d\mu\bigr) = \exp\bigl(\gamma \int (e^{iyf} - 1)d\mu\bigr),$$

as claimed.

Remark Recall that a Poisson random variable with expectation γ has characteristic function $y \mapsto \exp(\gamma(e^{iy} - 1))$ (part (a) of the proof of 285Q), corresponding to the case $f = \chi F$ where $\mu F = 1$. The random variables Q_f have **compound Poisson** distributions.

495Q If our underlying measure is a Radon measure, we can look for Radon measures on $\mathcal{P}X$ to represent the Poisson point processes on X. There seem to be difficulties in general, but I can offer the following. See also 495Yd.

Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space such that μ is outer regular with respect to the open sets, and $\gamma > 0$. Give the space $\mathcal{C}$ of closed subsets of X its Fell topology (4A2T).

(a) There is a unique quasi-Radon probability measure $\tilde{\nu}$ on $\mathcal{C}$ such that

$$\tilde{\nu}\{C : \#(C \cap E) = \emptyset\} = e^{-\gamma\mu E}$$

whenever $E \subseteq X$ is a measurable set of finite measure.

(b) If $E_0, \dots, E_r$ are disjoint sets of finite measure, none including any singleton set of non-zero measure, and $n_i \in \mathbb{N}$ for $i \le r$, then

$$\tilde{\nu}\{C : \#(C \cap E_i) = n_i \text{ for every } i \le r\} = \prod_{i=0}^{r} \frac{(\gamma\mu E_i)^{n_i}}{n_i!} e^{-\gamma\mu E_i}.$$

(c) Suppose that μ is atomless and ν is the Poisson point process on X with intensity γ.

(i) $\mathcal{C}$ has full outer measure for ν, and $\tilde{\nu}$ extends the subspace measure $\nu_{\mathcal{C}}$.

(ii) If moreover μ is σ-finite, then $\mathcal{C}$ is ν-conegligible.

(d) If X is locally compact then $\tilde{\nu}$ is a Radon measure.

(e) If X is second-countable and μ is atomless then $\tilde{\nu} = \nu_{\mathcal{C}}$.

proof (a)(i) Set $\Sigma^f = \{E : \mu E < \infty\}$. There is a disjoint family $\mathcal{H}$ of non-empty self-supporting measurable subsets of X of finite measure such that $\mu E = \sum_{H\in\mathcal{H}} \mu(E \cap H)$ for every $E \in \Sigma$ (412I); so if $G \subseteq X$ is an open set of finite measure, $\{H : H \in \mathcal{H},\, G \cap H \neq \emptyset\}$ is countable. If E is any set of finite measure, it is included in an open set of finite measure, because μ is outer regular with respect to the open sets; so once again $\{H : H \in \mathcal{H},\, E \cap H \neq \emptyset\}$ is countable.

Build $\Omega = \prod_{H\in\mathcal{H}} \mathbb{N} \times H^{\mathbb{N}}$, $\langle g_{HE}\rangle_{H\in\mathcal{H},E\in\Sigma}$ and the product measure λ on Ω as in the proof of 495B; as in the proof of 495D, set

$$\phi(\omega) = \{x_{Hj}(\omega) : H \in \mathcal{H},\, j < m_H(\omega)\}$$

for $\omega \in \Omega$.

(ii) If $E \in \Sigma^f$, $\lambda\{\omega : E \cap \phi(\omega) = \emptyset\} = e^{-\gamma\mu E}$. **P** $\mathcal{H}' = \{H : H \in \mathcal{H},\, E \cap H \neq \emptyset\}$ is countable. Now

$$\{\omega : E \cap \phi(\omega) = \emptyset\} = \bigcap_{H\in\mathcal{H}'} \{\omega : x_{Hj} \notin E \text{ for every } j < m_H(\omega)\}$$

has measure

$$\prod_{H\in\mathcal{H}'} \lambda\{\omega : g_{HE}(\omega) = 0\} = \prod_{H\in\mathcal{H}'} e^{-\gamma\mu(H\cap E)} = e^{-\gamma\mu E}$$

because $\mu E = \sum_{H\in\mathcal{H}'} \mu(H \cap E)$. **Q**

Let T_0 be the σ-algebra of subsets of $\mathcal{P}X$ generated by sets of the form $\{S : S \cap E = \emptyset\}$ where $E \in \Sigma^f$. By the Monotone Class Theorem (136B), λ measures $\phi^{-1}[W]$ for every $W \in \mathrm{T}_0$; set $\nu_0 W = \lambda\phi^{-1}[W]$ for $W \in \mathrm{T}_0$, so that if $E \in \Sigma$ then $\nu_0\{S : S \cap E = \emptyset\} = e^{-\gamma\mu E}$.

(iii) Give $\mathcal{P}X$ the topology $\mathfrak{S}$ generated by sets of the form

$$\{S : S \cap G \neq \emptyset\}, \quad \{S : S \cap K = \emptyset\}$$

for open sets $G \subseteq X$ and compact sets $K \subseteq X$. (Thus the Fell topology on $\mathcal{C}$ is the subspace topology induced by $\mathfrak{S}$.) Then $\mathcal{P}X$ is compact. **P** Follow the proof of 4A2T(b-iii) word for word, but replacing every $\mathcal{C}$ with $\mathcal{P}X$. **Q**

(iv) ν_0 is inner regular with respect to the $\mathfrak{S}$-closed sets. **P** Write $\mathcal{L}$ for the family of $\mathfrak{S}$-closed sets belonging to T_0. Of course $\mathcal{L}$ is closed under finite unions and countable intersections.

(α) Suppose that $E \in \Sigma^f$ and $W = \{S : S \cap E \neq \emptyset\}$. Let $\epsilon > 0$. Then there is a compact set $K \subseteq E$ such that $\mu(E \setminus K) \le \epsilon$. Set $V = \{S : S \cap K \neq \emptyset\}$; then $V \in \mathcal{L}$, $V \subseteq W$ and

$$\nu_0(W \setminus V) \le \nu_0\{S : S \cap E \setminus K \neq \emptyset\} \le 1 - e^{-\gamma\epsilon}.$$

As ϵ is arbitrary, $\nu_0 W = \sup\{\nu_0 V : V \in \mathcal{L},\, V \subseteq W\}$.

(β) Suppose that $E \in \Sigma^f$ and $W = \{S : S \cap E = \emptyset\}$. Let $\epsilon > 0$. Then there is an open set $G \supseteq E$ such that $\mu(G \setminus E) \le \epsilon$. Set $V = \{S : S \cap G = \emptyset\}$; then $V \in \mathcal{L}$, $V \subseteq W$ and

$$\nu_0(W \setminus V) \le \nu_0\{S : S \cap G \setminus E \neq \emptyset\} \le 1 - e^{-\gamma\epsilon}.$$

As ϵ is arbitrary, $\nu_0 W = \sup\{\nu_0 V : V \in \mathcal{L},\, V \subseteq W\}$.

(γ) By 412C, ν_0 is inner regular with respect to the $\mathfrak{S}$-closed sets. **Q**

(v) Since $\mathfrak{S}$ is a compact topology, the family of $\mathfrak{S}$-closed sets is a compact class, so 413P tells us that ν_0 has an extension to a complete topological measure $\tilde{\nu}_0$ on $\mathcal{P}X$, inner regular with respect to the closed sets. Of course $\tilde{\nu}_0$, being a probability measure, is effectively locally finite and locally determined, so it is a quasi-Radon measure with respect to the topology $\mathfrak{S}$. Consequently the subspace measure $\tilde{\nu}$ on $\mathcal{C}$ is a quasi-Radon measure for the Fell topology on $\mathcal{C}$ (415B).

(vi) $\mathcal{C}$ has full outer measure for $\tilde{\nu}_0$. **P?** Otherwise, there is a non-empty closed set $V \subseteq \mathcal{P}X \setminus \mathcal{C}$. Consider the family $\mathcal{U}$ of subsets of $\mathcal{P}X$ of the form

$$\{S : S \cap K = \emptyset,\, S \cap G_i \neq \emptyset \text{ for } i < r\}$$

where $K \subseteq X$ is compact and $G_i \subseteq X$ is an open set of finite measure for every $i < r$. Because μ is locally finite, this is a base for $\mathfrak{S}$. So $\mathcal{U}' = \{U : U \in \mathcal{U},\, U \cap W = \emptyset\}$ is a cover of $\mathcal{P}X \setminus W \supseteq \mathcal{C}$. Of course $U \cap \mathcal{C}$ is open in the Fell topology for every $U \in \mathcal{U}$; because $\mathcal{C}$ is compact, there are $U_0, \dots, U_m \in \mathcal{U}'$ covering $\mathcal{C}$.

Express each U_j as $\{S : S\cap K_j=\emptyset,\, S\cap G_{ji}\ne\emptyset$ for $i<r_j\}$, where the K_j are all compact and the G_{ji} are all open. Because $\bigcup_{j\le m}U_j$ is disjoint from V, there is an $S\subseteq\mathcal{P}X$ which does not belong to any U_j. Let $\mathcal{E}$ be the finite algebra of subsets of X generated by $\{K_j : j\le m\}\cup\{G_{ji} : j\le m,\, i<r_j\}$; then there is a finite set $C\subseteq S$ such that $C\cap E\ne\emptyset$ whenever $E\in\mathcal{E}$ and $S\cap E\ne\emptyset$. In this case, $C\in\mathcal{C}\setminus\bigcup_{j\le m}U_j$; which is supposed to be impossible. **XQ**

(vii) Consequently $\tilde\nu$ is a probability measure. If $E\in\Sigma^f$, then

$$\begin{aligned}\tilde\nu\{C : C\in\mathcal{C},\, C\cap E=\emptyset\}&=\tilde\nu(\mathcal{C}\cap\{S : S\subseteq X,\, S\cap E=\emptyset\})\\&=\tilde\nu_0\{S : S\cap E=\emptyset\}=\nu_0\{S : S\cap E=\emptyset\}=e^{-\gamma\mu E}.\end{aligned}$$

(viii) To see that $\tilde\nu$ is uniquely defined, let $\tilde\nu'$ be another quasi-Radon probability measure on $\mathcal{C}$ with the same property.

(α) Suppose that $E_0,\dots,E_r\subseteq X$ are disjoint measurable sets of finite measure, and

$$W=\{C : C\in\mathcal{C},\, C\cap E_0=\emptyset,\, C\cap E_i\ne\emptyset \text{ for } 1\le i\le r\}.$$

Then $\tilde\nu W=\tilde\nu' W$. **P** Induce on r. If $r=0$ the result is immediate. For the inductive step to $r\ge 1$, consider $\{C : C\cap E_0=\emptyset,\, C\cap E_i\ne\emptyset$ for $1\le i<r\}$ and $\{C : C\cap(E_0\cup E_r)=\emptyset,\, C\cap E_i\ne\emptyset$ for $1\le i<r\}$. By the inductive hypothesis, $\tilde\nu$ and $\tilde\nu'$ agree on these two sets, and therefore on their difference $\{C : C\in\mathcal{C},$ $C\cap E_0=\emptyset,\, C\cap E_i\ne\emptyset$ for $1\le i\le r\}$. **Q**

(β) Suppose that we have a compact set $K\subseteq X$ and open sets $G_i\subseteq X$ of finite measure, for $i<r$, and set

$$V=\{C : C\in\mathcal{C},\, C\cap K=\emptyset,\, C\cap G_i\ne\emptyset \text{ for every } i<r\}.$$

Then $\tilde\nu V=\tilde\nu' V$. **P** Let $\mathcal{E}$ be the finite subalgebra of $\mathcal{P}X$ generated by $\{G_i : i<r\}\cup\{K\}$, and $\mathcal{A}$ the set of atoms of $\mathcal{E}$ included in $K\cup\bigcup_{i<r}G_i$. For $\mathcal{I}\subseteq\mathcal{A}$ set

$$V_{\mathcal{I}}=\{C : C\in\mathcal{C},\, C\cap E\ne\emptyset \text{ for } E\in\mathcal{I},\, C\cap E=\emptyset \text{ for } E\in\mathcal{A}\setminus\mathcal{I}\}.$$

Then $V=\bigcup_{\mathcal{I}\in\mathfrak{I}}V_{\mathcal{I}}$, where

$$\begin{aligned}\mathfrak{I}=\{\mathcal{I} : \mathcal{I}\subseteq\mathcal{A},\, &A\cap K=\emptyset \text{ for every } A\in\mathcal{I},\\&\text{for every } i<r \text{ there is an } A\in\mathcal{I} \text{ such that } A\subseteq G_i\}.\end{aligned}$$

Now (α) shows that $\tilde\nu V_{\mathcal{I}}=\tilde\nu' V_{\mathcal{I}}$ for every $\mathcal{I}\subseteq\mathcal{A}$, so that $\tilde\nu V=\tilde\nu' V$. Since sets V of the type described form a base for the Fell topology closed under finite intersections, $\tilde\nu=\tilde\nu'$ (415H(v)). **Q**

This completes the proof of (a).

(b)(i) In the construction of 495B and (a-i) above, all the normalized subspace measures μ'_H are Radon measures (416Rb), while of course all the Poisson distributions ν_H are Radon measures, so the product measure λ on $\Omega=\prod_{H\in\mathcal{H}}\mathbb{N}\times H^{\mathbb{N}}$ has an extension to a Radon measure $\tilde\lambda$ (417Q). Let $\mathcal{W}$ be the family of those sets $W\subseteq\mathcal{P}X$ such that $\tilde\nu_0 W$ and $\tilde\lambda\phi^{-1}[E]$ are defined and equal. Then $\mathcal{W}$ is a Dynkin class. So if $\mathcal{W}_0\subseteq\mathcal{W}$ is closed under finite intersections, the σ-algebra of subsets of $\mathcal{P}X$ generated by $\mathcal{W}_0$ is included in $\mathcal{W}$. By (a-ii), $\mathrm{T}_0\subseteq\mathcal{W}$.

(ii) Let $\mathfrak{S}_0$ be the topology on $\mathcal{P}X$ generated by sets of the form $\{S : S\cap G\ne\emptyset\}$ where $G\subseteq X$ is open. (So $\mathfrak{S}_0$ is coarser than the topology $\mathfrak{S}$ of (a-iii) above.) Then $\phi:\Omega\to\mathcal{P}X$ is continuous for the product topology $\mathfrak{U}$ on Ω and $\mathfrak{S}_0$ on $\mathcal{P}X$. **P** If $G\subseteq X$ is open, then

$$\phi^{-1}[\{S : S\cap G\ne\emptyset\}]=\Omega\cap\bigcap_{i<r}\bigcup_{H\in\mathcal{H},j\in\mathbb{N}}\{\omega : j<m_H(\omega),\, x_{Hj}(\omega)\in G_i\}$$

is open; by 4A2B(a-ii), this is enough. **Q**

(iii) $\mathfrak{S}_0\subseteq\mathcal{W}$. **P** Because μ is locally finite, the family $\mathcal{U}$ of sets of the form

$$\{S : S\cap G_i\ne\emptyset \text{ for } i<r\},$$

where $G_i\subseteq X$ is an open set of finite measure for each $i<r$, is a base for $\mathfrak{S}_0$; and $\mathcal{U}\subseteq\mathrm{T}_0\subseteq\mathcal{W}$. So if $W\in\mathfrak{S}_0$, $\mathcal{V}=\{V : V\in\mathfrak{S}_0\cap\mathrm{T}_0,\, V\subseteq W\}$ is an upwards-directed family of sets with union W. Since $\tilde\nu_0$ and $\tilde\lambda$ are both τ-additive, and $\phi^{-1}[V]$ is open for every $V\in\mathcal{V}$,

$$\tilde{\lambda}\phi^{-1}[W] = \sup_{V\in\mathcal{V}} \tilde{\lambda}\phi^{-1}[V] = \sup_{V\in\mathcal{V}} \tilde{\nu}_0 V = \tilde{\nu}_0 W,$$

and $W \in \mathcal{W}$. **Q**

(iv) If $G \subseteq X$ is open and $n \in \mathbb{N}$, $W = \{S : \#(S \cap G) \ge n\}$ belongs to $\mathfrak{S}_0$. **P**

$$W = \bigcup\{\{S : S \cap G_i \ne \emptyset \text{ for every } i < n\} : \langle G_i\rangle_{i<n} \text{ is a disjoint family of open subsets of } G \text{ of finite measure}\}. \text{ } \mathbf{Q}$$

(v) If $E_0, \dots , E_r \subseteq X$ are sets of finite measure, and $n_0, \dots , n_r \in \mathbb{N}$, then

$$V = \{S : \#(S \cap E_i) \ge n_i \text{ for } i \le r\}$$

belongs to $\mathcal{W}$. **P** Let $\epsilon > 0$. Then there is a $\delta > 0$ such that $1 - e^{-\gamma\delta} \le \epsilon$. Let $G_0, \dots , G_r$ be open sets such that $E_i \subseteq G_i$ for $i \le r$ and $\sum_{i=0}^{r} \mu(G_i \setminus E_i) \le \delta$. Set

$$W = \{S : \#(S \cap G_i) \ge n_i \text{ for } i \le r\}, \quad W_0 = \{S : S \cap \bigcup_{i\le r} G_i \setminus E_i \ne \emptyset\};$$

then $W \in \mathfrak{S}_0$ and $W_0 \in \mathrm{T}_0$, so both belong to $\mathcal{W}$, while

$$\tilde{\nu}_0 W_0 = \tilde{\lambda}\phi^{-1}[W_0] = 1 - \exp(-\gamma\mu(\bigcup_{i\le r} G_i \setminus E_i)) \le \epsilon.$$

Now

$$W \setminus W_0 \subseteq V \subseteq W, \quad \phi^{-1}[W] \setminus \phi^{-1}[W_0] \subseteq \phi^{-1}[V] \subseteq \phi^{-1}[W].$$

So

$$\tilde{\nu}_0^* V - (\tilde{\nu}_0)_* V \le \epsilon, \quad \tilde{\lambda}^*(\phi^{-1}[V]) - \tilde{\lambda}_*(\phi^{-1}[V]) \le \epsilon, \quad |\tilde{\nu}_0^* V - \tilde{\lambda}^*(\phi^{-1}[V])| \le \epsilon.$$

As ϵ is arbitrary (and $\tilde{\nu}_0$, $\tilde{\lambda}$ are complete), V is measured by $\tilde{\nu}_0$, $\phi^{-1}[V]$ is measured by $\tilde{\lambda}$, and

$$|\tilde{\nu}_0 V - \tilde{\lambda}\phi^{-1}[V]| = |\tilde{\nu}_0^* V - \tilde{\lambda}^*(\phi^{-1}[V])| = 0. \text{ } \mathbf{Q}$$

(vi) If $E_0, \dots , E_r \subseteq X$ are sets of finite measure, $n_0, \dots , n_r \in \mathbb{N}$ and $j \le r$, then

$$\{S : \#(S \cap E_i) = n_i \text{ for } i < j, \#(S \cap E_i) \ge n_i \text{ for } j \le i \le r\}$$

belongs to $\mathcal{W}$. **P** Induce on j. For $j = 0$ we just have the case of (v). For the inductive step to $j + 1$, we have

$$\begin{aligned}\{S : \#(S \cap E_i) = n_i &\text{ for } i \le j, \#(S \cap E_i) \ge n_i \text{ for } j < i \le r\}\\ &= \{S : \#(S \cap E_i) = n_i \text{ for } i < j, \#(S \cap E_i) \ge n_i \text{ for } j \le i \le r\}\\ &\qquad \setminus \{S : \#(S \cap E_i) = n_i \text{ for } i < j, \#(S \cap E_j) \ge n_j + 1,\\ &\qquad\qquad \#(S \cap E_i) \ge n_i \text{ for } j < i \le r\}\\ &\in \mathcal{W}\end{aligned}$$

because $\mathcal{W}$ is a Dynkin class. **Q**

(vii) If $E \in \Sigma$ has finite measure and does not include any non-negligible singleton, then $\#(E\cap\phi(\omega)) = g_E(\omega)$, as defined in 495B, for λ-almost every $\omega \in \Omega$. **P** Let A_E be the set of those $\omega \in \Omega$ such that

either there are an $H \in \mathcal{H}$ and $j \in \mathbb{N}$ such that $\mu(H \cap E) = 0$ and $x_{Hj}(\omega) \in E$

or there are an $H \in \mathcal{H}$ and distinct i, $j \in \mathbb{N}$ such that $x_{Hi}(\omega) = x_{Hj}(\omega) \in E$.

As observed in (a-i) above, $\{H : H \in \mathcal{H}, H \cap E \ne \emptyset\}$ is countable; while for any $H \in \mathcal{H}$ and distinct i, $j \in \mathbb{N}$ the set $\{\omega : x_{Hi}(\omega) = x_{Hj}(\omega) \in E\}$ is negligible because the subspace measure on E is atomless (414G/416Xa), so the diagonal $\{(x, x) : x \in E\}$ is negligible in X^2. Consequently $\lambda A_E = 0$. But $\#(E \cap \phi(\omega)) = g_E(\omega)$ for every $\omega \in \Omega \setminus A_E$. **Q**

(viii) Now suppose that $E_0, \dots , E_r \subseteq X$ are disjoint sets of finite measure, none including any non-negligible singleton, and $n_0, \dots , n_r \in \mathbb{N}$. Then

$$V = \{S : S \subseteq X, \#(S \cap E_i) = n_i \text{ for every } i \le r\}$$

belongs to $\mathcal{W}$, by (vi). Next,

$$\phi^{-1}[V] = \{\omega : \#(E_i \cap \phi(\omega)) = n_i \text{ for every } i \le r\},$$

so

$$\tilde{\nu}_0 V = \tilde{\lambda}\phi^{-1}[V] = \tilde{\lambda}\{\omega : g_{E_i}(\omega) = n_i \text{ for every } i \le r\}$$

(by (vii))

$$= \prod_{i=0}^{r} \tilde{\lambda}\{\omega : g_{E_i}(\omega) = n_i\} = \prod_{i=0}^{r} \frac{(\gamma\mu E_i)^{n_i}}{n_i!} e^{-\gamma\mu E_i}.$$

Finally, because $\tilde{\nu}_0^*\mathcal{C} = 1$ and $\tilde{\nu}$ is the subspace measure on $\mathcal{C}$,

$$\tilde{\nu}\{C : C \in \mathcal{C},\, \#(C \cap E_i) = n_i \text{ for every } i \le r\} = \tilde{\nu}(V \cap \mathcal{C}) = \tilde{\nu}_0 V$$
$$= \prod_{i=0}^{r} \frac{(\gamma\mu E_i)^{n_i}}{n_i!} e^{-\gamma\mu E_i}.$$

This completes the proof of (b).

(c)(i) Taking $\mathrm{T} \supseteq \mathrm{T}_0$ to be the σ-algebra of subsets of $\mathcal{P}X$ generated by sets of the form $\{S : \#(S \cap E) = n\}$ where $E \in \Sigma^f$ and $n \in \mathbb{N}$, (b-viii) tells us that $\tilde{\nu}_0{\restriction}\mathrm{T}$ satisfies the conditions of 495D, so its completion ν is the Poisson point process as defined in 495E. Because $\tilde{\nu}_0$ is complete, it extends ν. (The identity map from $\mathcal{P}X$ to itself is inverse-measure-preserving for $\tilde{\nu}_0$ and $\tilde{\nu}_0{\restriction}\mathrm{T}$, therefore also for their completions $\tilde{\nu}_0$ and ν.) Since $\mathcal{C}$ has full outer measure for $\tilde{\nu}_0$, by (a-v), it has full outer measure for ν, and

$$\nu_{\mathcal{C}}(V \cap \mathcal{C}) = \nu V = \tilde{\nu}_0 V = \tilde{\nu}(V \cap \mathcal{C})$$

whenever ν measures V, so $\tilde{\nu}$ extends $\nu_{\mathcal{C}}$.

(ii) If μ is σ-finite, then there is a sequence $\langle H_n \rangle_{n \in \mathbb{N}}$ of open sets of finite measure covering X. For each $n \in \mathbb{N}$, $\{S : S \subseteq X,\, S \cap H_n \text{ is finite}\}$ is ν-conegligible. So $W = \{S : S \cap H_n \text{ is finite for every } n\}$ is ν-conegligible. But $W \subseteq \mathcal{C}$, so $\mathcal{C}$ is ν-conegligible.

(d) If X is locally compact then $\mathcal{C}$ is Hausdorff (4A2T(e-ii)); so $\tilde{\nu}$, being a quasi-Radon probability measure on a compact Hausdorff space, is a Radon measure (416G).

(e) Now suppose that X is second-countable.

(i) $\mathcal{C}$ has a countable network consisting of sets in $\mathrm{T}_{\mathcal{C}}$, the subspace σ-algebra induced by the σ-algebra T of (c-i). **P** Let $\mathcal{U}$ be a countable base for $\mathfrak{T}$, closed under finite unions, consisting of sets of finite measure. For $U_0 \in \mathcal{U}$ and finite $\mathcal{U}_0 \subseteq \mathcal{U}$, set

$$V(U_0, \mathcal{U}_0) = \{C : C \in \mathcal{C},\, C \cap U_0 = \emptyset,\, C \cap U \ne \emptyset \text{ for every } U \in \mathcal{U}\} \in \mathrm{T}_{\mathcal{C}}.$$

If $W \subseteq \mathcal{C}$ is open for the Fell topology and $C_0 \in W$, there are a compact set $K \subseteq X$ and a finite family $\mathcal{G} \subseteq \mathfrak{T}$ such that

$$C_0 \in \{C : C \in \mathcal{C},\, C \cap K = \emptyset,\, C \cap G \ne \emptyset \text{ for every } G \in \mathcal{G}\}.$$

For $G \in \mathcal{G}$ let y_G be a point of $C_0 \cap G$. Now there are a $U_0 \in \mathcal{U}$ such that $K \subseteq U_0 \subseteq X \setminus C$ and a family $\langle U_G \rangle_{G \in \mathcal{G}}$ in $\mathcal{U}$ such that $x_G \in U_G \subseteq G$ for every $G \in \mathcal{G}$. In this case,

$$C_0 \in V(U_0, \{U_G : G \in \mathcal{G}\}) \subseteq W.$$

As C_0 and W are arbitrary, the countable set $\{V(U_0, \mathcal{U}_0) : U_0 \in \mathcal{U}, \mathcal{U}_0 \in [\mathcal{U}]^{<\omega}\}$ is a network for the topology of $\mathcal{C}$. **Q**

(ii) Since $\nu_{\mathcal{C}}$ measures every set in this countable network, it is a topological measure. Since it is also complete, and $\tilde{\nu}$, being a quasi-Radon probability measure, is the completion of its restriction to the Borel σ-algebra of $\mathcal{C}$, $\nu_{\mathcal{C}}$ extends $\tilde{\nu}$, and the two must be equal.

495R Proposition Let $(X, \mathfrak{T})$ be a σ-compact locally compact Hausdorff space and $M_{\mathrm{R}}^{\infty+}(X)$ the set of Radon measures on X. Give $M_{\mathrm{R}}^{\infty+}(X)$ the topology generated by sets of the form $\{\mu : \mu G > \alpha\}$ and

$\{\mu : \mu K < \alpha\}$ where $G \subseteq X$ is open, $K \subseteq X$ is compact and $\alpha \in \mathbb{R}$. Let $\mathcal{C}$ be the space of closed subsets of X with its Fell topology, and $P_{\mathrm{R}}(\mathcal{C})$ the set of Radon probability measures on $\mathcal{C}$ with its narrow topology (definition: 437Jd). For $\mu \in M_{\mathrm{R}}^{\infty+}(X)$ and $\gamma > 0$ let $\tilde{\nu}_{\mu,\gamma}$ be the Radon measure on $\mathcal{C}$ defined from μ and γ as in 495Q. Then the function $(\mu, \gamma) \mapsto \tilde{\nu}_{\mu,\gamma} : M_{\mathrm{R}}^{\infty+}(X) \times]0, \infty[\to P_{\mathrm{R}}(\mathcal{C})$ is continuous.

proof (a) Note that because X is σ-compact, every Radon measure on X is σ-finite, therefore outer regular with respect to the open sets (412Wb), and we can apply 495Q to build the measures $\tilde{\nu}_{\mu,\gamma}$. Just as in 495E for ordinary Poisson point processes, the uniqueness assertion in 495Qa assures us that $\tilde{\nu}_{\mu,\gamma} = \tilde{\nu}_{\gamma\mu,1}$ for all γ and μ. Of course the sets

$$\{(\mu, \gamma) : \gamma\mu G > \alpha\}, \quad \{(\mu, \gamma) : \gamma\mu K < \alpha\}$$

where $G \subseteq X$ is open, $K \subseteq X$ is compact and $\alpha \in \mathbb{R}$, are all open in $M_{\mathrm{R}}^{\infty+}(X) \times]0, \infty[$; so the map $(\mu, \gamma) \mapsto \gamma\mu$ is continuous. It will therefore be enough to show that the map $\mu \mapsto \tilde{\nu}_{\mu,1} : M_{\mathrm{R}}^{\infty+}(X) \to P_{\mathrm{R}}(\mathcal{C})$ is continuous. Write $\tilde{\nu}_\mu$ for $\tilde{\nu}_{\mu,1}$.

(b) Fix an open set $W_0 \subseteq \mathcal{C}$, $\alpha_0 > 0$ and $\mu_0 \in M_{\mathrm{R}}^{\infty+}(X)$ such that $\tilde{\nu}_{\mu_0} W_0 > \alpha_0$. Let $\mathcal{E}$ be the family of relatively compact Borel subsets E of X such that $\mu_0(\partial E) = 0$. Then $\mathcal{E}$ is a subring of $\mathcal{P}X$ (4A2Bi). Also $\mu \mapsto \mu E : M_{\mathrm{R}}^{\infty+}(X) \to [0, \infty[$ is continuous at μ_0 for every $E \in \mathcal{E}$. **P** If $E \in \mathcal{E}$ and $\epsilon > 0$, then

$$\begin{aligned} &\{\mu : \mu_0 E - \epsilon < \mu E < \mu_0 E + \epsilon\} \\ &\qquad \supseteq \{\mu : \mu(\operatorname{int} E) > \mu_0(\operatorname{int} E) - \epsilon,\ \mu\overline{E} < \mu_0\overline{E} + \epsilon\} \end{aligned}$$

is a neighbourhood of μ_0. **Q**

(c) Next, $\mathcal{U} = \mathcal{E} \cap \mathfrak{T}$ is a base for $\mathfrak{T}$ (411Gi). It follows that the family $\mathcal{V}$ of sets of the form

$$\{C : C \in \mathcal{C},\ C \cap U_i \neq \emptyset \text{ for } i < r,\ C \cap \overline{U} = \emptyset\},$$

where $U, U_0, \ldots \in \mathcal{U}$, is a base for the Fell topology on $\mathcal{C}$. **P** If $W \subseteq \mathcal{C}$ is open for the Fell topology and $C_0 \in W$, there are $r \in \mathbb{N}$, open sets $G_i \subseteq X$ for $i < r$ and a compact set $K \subseteq X$ such that

$$C_0 \in \{C : C \cap G_i \neq \emptyset \text{ for each } i < r,\ C \cap K = \emptyset\} \subseteq W.$$

For each $i < r$ choose $x_i \in C_0 \cap G_i$ and $U_i \in \mathcal{U}$ such that $x_i \in U_i \subseteq G_i$. Because X is locally compact and Hausdorff, it is regular, so every point of K belongs to a member of $\mathcal{U}$ with closure disjoint from C_0; because $\mathcal{U}$ is closed under finite unions, there is a $U \in \mathcal{U}$ such that $K \subseteq U$ and $C_0 \cap \overline{U} = \emptyset$. Now

$$\{C : C \in \mathcal{C},\ C \cap U_i \neq \emptyset \text{ for } i < r,\ C \cap \overline{U} = \emptyset\}$$

belongs to $\mathcal{V}$, contains C_0 and is included in W. As C_0 and W are arbitrary, $\mathcal{V}$ is a base for the Fell topology on $\mathcal{C}$. **Q**

(d) If $V \in \mathcal{V}$, then $\mu \mapsto \tilde{\nu}_\mu V : M_{\mathrm{R}}^{\infty+}(X) \to [0, 1]$ is continuous at μ_0. **P** Express V as $\{C : C \in \mathcal{C},\ C \cap U_i \neq \emptyset$ for $i < r$, $C \cap \overline{U} = \emptyset\}$, where $U_i, U \in \mathcal{U}$. Let $\mathcal{A}$ be the set of atoms of the finite subring of $\mathcal{E}$ generated by $\{U_i : i < r\} \cup \{\overline{U}\}$. For $\mathcal{I} \subseteq \mathcal{A}$ set

$$V_{\mathcal{I}} = \{C : C \in \mathcal{C},\ \mathcal{I} = \{A : A \in \mathcal{A},\ C \cap A \neq \emptyset\}\}.$$

Let $\mathfrak{I}$ be the set of those $\mathcal{I} \subseteq \mathcal{A}$ such that $A \cap \overline{U} = \emptyset$ for every $A \in \mathcal{I}$ and for every $i < r$ there is an $A \in \mathcal{I}$ such that $A \subseteq U_i$. Then $\langle V_{\mathcal{I}} \rangle_{\mathcal{I} \in \mathfrak{I}}$ is a partition of V. Moreover, for any $\mu \in M_{\mathrm{R}}^{\infty+}(X)$ and $\mathcal{I} \subseteq \mathcal{A}$,

$$\tilde{\nu}_\mu V_{\mathcal{I}} = \prod_{A \in \mathcal{A} \setminus \mathcal{I}} e^{-\mu A} \cdot \prod_{A \in \mathcal{I}} (1 - e^{-\mu A}).$$

Since each $\mu \mapsto \mu A$ is continuous at μ_0, by (a), so are the functionals $\mu \mapsto \tilde{\nu}_\mu V_{\mathcal{I}}$, for $\mathcal{I} \subseteq \mathcal{A}$, and $\mu \mapsto \tilde{\nu}_\mu V = \sum_{\mathcal{I} \in \mathfrak{I}} \tilde{\nu}_\mu V_{\mathcal{I}}$. **Q**

(e) Let $\mathcal{V}^*$ be the family of Borel subsets V of $\mathcal{C}$ such that $\mu \mapsto \tilde{\nu}_\mu V : M_{\mathrm{R}}^{\infty+}(X) \to [0, \infty[$ is continuous at μ_0. Then $\mathcal{V} \subseteq \mathcal{V}^*$ (by (c)), $\mathcal{C} \in \mathcal{V}^*$ and $V \setminus V' \in \mathcal{V}^*$ whenever $V, V' \in \mathcal{V}^*$ and $V' \subseteq V$. Because $\mathcal{V}$ is closed under finite intersections, it follows that $\mathcal{V}^*$ includes the algebra of subsets of $\mathcal{C}$ generated by $\mathcal{V}$ (313Ga); in particular, any finite union of members of $\mathcal{V}$ belongs to $\mathcal{V}^*$.

(f) Let us return to the open set $W_0 \subseteq \mathcal{C}$ and the $\alpha_0 \in \mathbb{R}$ of part (a). Because $\tilde{\nu}_{\mu_0}$ is τ-additive and $\mathcal{V}$ is a base for the topology of $\mathcal{C}$ ((b) above), there is a finite family $\mathcal{V}_0 \subseteq \mathcal{V}$ such that $V_0 = \bigcup \mathcal{V}_0$ is included in W_0 and $\tilde{\nu}_{\mu_0} V_0 > \alpha_0$. But this means that

$$\{\mu : \mu \in M_{\mathrm{R}}^{\infty+}(X),\, \tilde{\nu}_\mu W_0 > \alpha_0\} \supseteq \{\mu : \mu \in M_{\mathrm{R}}^{\infty+}(X),\, \tilde{\nu}_\mu V_0 > \alpha_0\}$$

is a neighbourhood of μ_0. As μ_0 is arbitrary, $\{\mu : \tilde{\nu}_\mu W_0 > \alpha_0\}$ is open; as W_0 and α_0 are arbitrary, $\mu \mapsto \tilde{\nu}_\mu$ is continuous.

495S There are many constructions which, in particular cases, can be used as an alternative to the method of 495B-495D in setting up Poisson point processes. I give one which applies to the half-line $[0,\infty[$ with Lebesgue measure.

Theorem Let $\gamma > 0$, and let ν be the Poisson point process on $[0,\infty[$, with Lebesgue measure, with intensity γ. Let λ_0 be the exponential distribution with expectation $1/\gamma$, regarded as a Radon probability measure on $]0,\infty[$, and λ the corresponding product measure on $]0,\infty[^{\mathbb{N}}$. Define $\phi :]0,\infty[^{\mathbb{N}} \to \mathcal{P}([0,\infty[)$ by setting $\phi(x) = \{\sum_{i=0}^{n} x(i) : n \in \mathbb{N}\}$ for $x \in]0,\infty[^{\mathbb{N}}$. Then ϕ is a measure space isomorphism between $]0,\infty[^{\mathbb{N}}$ and a ν-conegligible subset of $\mathcal{P}([0,\infty[)$.

Remark As I seem not to have mentioned exponential distributions earlier in this treatise, I remark now that the **exponential distribution** with parameter γ has distribution function

$$F(t) = 0 \text{ if } t < 0,\ 1 - e^{-\gamma t} \text{ if } t \geq 0,$$

and probability density function

$$f(t) = 0 \text{ if } t \leq 0,\ \gamma e^{-\gamma t} \text{ if } t > 0;$$

its expectation is

$$\int_0^\infty \gamma t e^{-\gamma t} dt = -\int_0^\infty \frac{d}{dt}\Bigl(\frac{\gamma t+1}{\gamma} e^{-\gamma t}\Bigr) dt = \frac{1}{\gamma}.$$

Because (when regarded as a Radon probability measure on $\mathbb{R}$, following my ordinary rule set out in §271) it gives measure zero to $]-\infty,0]$, it can be identified with the subspace measure on $]0,\infty[$, as here.

proof (a) For each $n \in \mathbb{N}$, $\#(S \cap [0,n])$ is finite for ν-almost every S; so the set

$$Q_0 = \{S : S \subseteq [0,\infty[,\ \#(S \cap [0,n]) \text{ is finite for every } n\}$$

is ν-conegligible. Next, the sets $\{S : S \cap [n,n+1[\, \neq \emptyset\}$ are ν-independent and have measure $1 - e^{-\gamma} > 0$, so

$$\{S : S \cap [n,n+1[\, \neq \emptyset \text{ for infinitely many } n\}$$

is ν-conegligible (273K). Finally, $\nu\{S : 0 \in S\} = 0$, so $Q = \{S : S \in Q_0,\ 0 \notin S,\ S \text{ is infinite}\}$ is ν-conegligible. For $S \in Q$, let $\langle g_n(S)\rangle_{n\in\mathbb{N}}$ be the increasing enumeration of S. Let T be the σ-algebra of subsets of $\mathcal{P}([0,\infty[)$ generated by sets of the form $\{S : \#(S \cap E) = n\}$ where $E \subseteq [0,\infty[$ has finite measure and $n \in \mathbb{N}$. Then, for $n \in \mathbb{N}$ and $\alpha \geq 0$, $\{S : g_n(S) \leq \alpha\} = \{S : \#(S \cap [0,\alpha]) \geq n+1\}$ belongs to T, so g_n is T-measurable. Set $h_0(S) = g_0(S)$, $h_n(S) = g_n(S) - g_{n-1}(S)$ for $n \geq 1$, and $h(S) = \langle h_n(S)\rangle_{n\in\mathbb{N}}$; then $h : Q \to]0,\infty[^{\mathbb{N}}$ is a bijection, and its inverse is ϕ.

(b) For each $k \in \mathbb{N}$, $I_S = \{i : i \in \mathbb{N},\ S \cap [2^{-k}i, 2^{-k}(i+1)[\, \neq \emptyset\}$ is infinite for every $S \in Q$. So we can define $g_{kn} : Q \to]0,\infty[$, for each n, by taking $g_{kn}(S) = 2^{-k}(j+1)$ if $j \in I_S$ and $\#(I_S \cap j) = n$. Because all the sets $\{S : j \in I_S\}$ belong to T, each g_{kn} is T-measurable, and $\langle g_{kn}\rangle_{k\in\mathbb{N}}$ is a non-increasing sequence with limit g_n. Set $h_{k0}(S) = g_{k0}(S)$ and $h_{kn}(S) = g_{kn}(S) - g_{k,n-1}(S)$ for $n \geq 1$. Then $h_n = \lim_{k\to\infty} h_{kn}$.

(c) For any $n \in \mathbb{N}$, $j_0, \ldots, j_n \in \mathbb{N}$, $k \in \mathbb{N}$, set $j'_r = \sum_{i=0}^{r} j_i$ for $r \leq n$. Then

$$
\begin{aligned}
\nu\{S : S \in Q,\, h_{ki}(S) = 2^{-k}(j_i+1) \text{ for every } i \le n\}
&= \nu\{S : S \in Q,\, g_{kr}(S) = 2^{-k}(r+1+j'_r) \text{ for every } r \le n\} \\
&= \nu\{S : S \cap \left[2^{-k}(r+j'_r), 2^{-k}(r+1+j'_r)\right[\ne \emptyset \text{ for every } r \le n, \\
&\qquad S \cap \left[0, 2^{-k}j_0\right[= \emptyset, \\
&\qquad S \cap \left[2^{-k}(r+1+j'_r), 2^{-k}(r+1+j'_{r+1})\right[= \emptyset \text{ for every } r < n\} \\
&= (1-\exp(-2^{-k}\gamma))^{n+1}\exp(-2^{-k}\gamma j_0)\prod_{r<n}\exp(-2^{-k}\gamma(j'_{r+1}-j'_r)) \\
&= \prod_{i=0}^{n}(1-\exp(-2^{-k}\gamma))\exp(-2^{-k}\gamma j_i).
\end{aligned}
$$

This means that the h_{ki}, for $i \in \mathbb{N}$ are independent, with

$$\Pr(h_{ki} = 2^{-k}(j+1)) = (1-e^{-2^{-k}\gamma})e^{-2^{-k}\gamma j}$$

for each j. Since $h_{ki} \to h_i$ ν-a.e. for each i, $\langle h_i\rangle_{i\in\mathbb{N}}$ is also independent (367W). Now, for any $\alpha > 0$,

$$
\begin{aligned}
\Pr(h_{ki} \le \alpha) &= \sum_{2^{-k}(j+1)\le\alpha}(1-\exp(-2^{-k}\gamma))\exp(-2^{-k}\gamma j) \\
&= 1-\exp(-2^{-k}\gamma\lfloor 2^k\alpha\rfloor) \to 1-e^{-\gamma\alpha}
\end{aligned}
$$

as $k \to \infty$. So

$$
\begin{aligned}
\Pr(h_i \le \alpha) &= \inf_{\beta>\alpha}\liminf_{k\to\infty}\Pr(h_{ki} \le \beta) \\
\text{(271L)} \qquad &= \inf_{\beta>\alpha} 1-e^{-\gamma\beta} = 1-e^{-\gamma\alpha}
\end{aligned}
$$

for every $\alpha \ge 0$ and every $i \in \mathbb{N}$.

(d) Accordingly $\langle h_i\rangle_{i\in\mathbb{N}}$ is an independent sequence of random variables, each exponentially distributed with expectation $1/\gamma$. It follows that $h : Q \to \left]0,\infty\right[^{\mathbb{N}}$ is inverse-measure-preserving for the subspace measure ν_Q and λ (254G).

Observe next that if $E \subseteq [0,\infty[$ is Lebesgue measurable and $n \in \mathbb{N}$, then

$$\{x : x \in \left]0,\infty\right[^{\mathbb{N}},\, \#(\phi(x)\cap E) = n\} = \bigcup_{I\in[\mathbb{N}]^n}\{x : \textstyle\sum_{i=0}^{j} x(i) \in E \iff j \in I\} \in \Lambda,$$

writing Λ for the domain of λ. So ϕ is (Λ, T)-measurable. Now, for any $W \in \mathrm{T}$,

$$\lambda\phi^{-1}[W] = \nu(h^{-1}[\phi^{-1}[W]]) = \nu(W \cap Q) = \nu W.$$

So ϕ is inverse-measure-preserving for λ and for $\nu\restriction\mathrm{T}$. Since λ is complete and ν is defined as the completion of its restriction to T, ϕ is inverse-measure-preserving for λ and ν. Thus ϕ and h are the two halves of an isomorphism between $(\left]0,\infty\right[^{\mathbb{N}}, \lambda)$ and the subspace (Q, ν_Q), as claimed.

495X Basic exercises

(b) Let (X, Σ, μ) be an atomless measure space, and ν a Poisson point process on X. (i) Show that $[X]^{\le\omega}$ has full outer measure for ν. (ii) Show that if μ is semi-finite then $[X]^{\le\omega}$ is conegligible iff μ is σ-finite. (iii) Show that if μ is semi-finite, then $[X]^{<\omega}$ is non-negligible iff $[X]^{<\omega}$ is conegligible iff μ is totally finite.

(c) Let (X,Σ,μ) be a strictly localizable measure space and $\langle X_i\rangle_{i\in I}$ a decomposition of X. Let ν be the Poisson point process on X with intensity 1, and for each $i\in I$ let ν_i be the Poisson point process on X_i with intensity 1 corresponding to the subspace measure μ_{X_i} on X_i. Let λ be the product of the family $\langle\nu_i\rangle_{i\in I}$. Show that the map $S\mapsto\langle S\cap X_i\rangle_{i\in I}:\mathcal{P}X\to\prod_{i\in I}\mathcal{P}X_i$ is inverse-measure-preserving for ν and λ.

>(d) Let (X,Σ,μ) be an atomless measure space and $\mathfrak{T}$ a topology on X such that X is covered by a sequence of open sets of finite outer measure. Let ν be a Poisson point process on X. Show that ν-almost every set $S\subseteq X$ is locally finite in the sense that X is covered by the open sets meeting S in finite sets; in particular, if X is T_1, then ν-almost every subset of X is closed.

>(e)(i) Let (X,Σ,μ) be an atomless measure space, and for $\gamma>0$ let ν_γ be the Poisson point process on X with intensity γ. Show that for any γ, $\delta>0$ the map $(S,T)\mapsto S\cup T:\mathcal{P}X\times\mathcal{P}X\to\mathcal{P}X$ is inverse-measure-preserving for the product measure $\nu_\gamma\times\nu_\delta$ and $\nu_{\gamma+\delta}$. (ii) Let X be a set, Σ a σ-algebra of subsets of X, and $\langle\mu_i\rangle_{i\in I}$ a countable family of measures with domain Σ such that $\mu=\sum_{i\in I}\mu_i$ is atomless. Let ν, ν_i be the Poisson point processes with intensity 1 corresponding to the measures μ, μ_i. Show that the map $\langle S_i\rangle_{i\in I}\mapsto\bigcup_{i\in I}S_i:(\mathcal{P}X)^I\to\mathcal{P}X$ is inverse-measure-preserving for the product measure $\prod_{i\in I}\nu_i$ and ν. (iii) Compare with 495Xc(i).

(f) Let (X,Σ,μ) be an atomless semi-finite measure space, and ν the Poisson point process on X with intensity 1. Show that ν is perfect iff μ is.

(g) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. Show that there is a probability measure λ on $\mathbb{R}^{\mathfrak{A}^f}$ such that (i) for every $a\in\mathfrak{A}^f$ the corresponding marginal measure on $\mathbb{R}$ is the Poisson distribution with expectation $\bar\mu a$ (ii) whenever $a_0,\dots,a_n\in\mathfrak{A}^f$ are disjoint, the functions $z\mapsto z(a_i):\mathbb{R}^{\mathfrak{A}^f}\to\mathbb{R}$ are stochastically independent with respect to λ. (*Hint*: prove the result for finite $\mathfrak{A}$ and use 454D.) Use this to prove 495M.

(h) Let U be a Hilbert space. Show that there are a probability algebra $(\mathfrak{B},\bar\lambda)$ and a linear operator $T:U\to L^2(\mathfrak{B})$ such that (i) for every $u\in U$, Tu has a normal distribution with expectation 0 and variance $\|u\|_2^2$ (ii) if $\langle u_i\rangle_{i\in I}$ is an orthogonal family in U then $\langle Tu_i\rangle_{i\in I}$ is $\bar\lambda$-independent. (*Hint*: see the proof of 456K.)

(i) Let ν be the Poisson point process with intensity 1 on $[0,\infty[$ with Lebesgue measure. Set $Q_0=\{S:S\subseteq[0,\infty[,\ S\cap[0,n]$ is finite for every $n\}$ and for $S\in Q_0$ set $\psi(S)(t)=\#(S\cap[0,t])$ for $t\in[0,\infty[$. Show that ψ is inverse-measure-preserving for the subspace measure ν_{Q_0} and the distribution on $\mathbb{R}^{[0,\infty[}$ corresponding to the Poisson process of 455Xh.

(j) Let (Y,T,ν) be a probability space, and λ_0 the exponential distribution with expectation 1, regarded as a Radon measure on $]0,\infty[$. Let λ be the product measure $\lambda_0^{\mathbb{N}}\times\nu^{\mathbb{N}}$ on $]0,\infty[^{\mathbb{N}}\times Y^{\mathbb{N}}$. Set $\phi(x,y)=\{(\sum_{i=0}^n x(i),y(n)):n\in\mathbb{N}\}$ for $x\in\,]0,\infty[^{\mathbb{N}}$ and $y\in Y^{\mathbb{N}}$. Show that $\phi:]0,\infty[^{\mathbb{N}}\times Y^{\mathbb{N}}\to\mathcal{P}(]0,\infty[\times Y)$ is a measure space isomorphism between $(]0,\infty[^{\mathbb{N}}\times Y^{\mathbb{N}},\lambda)$ and a conegligible set for the Poisson point process on $]0,\infty[\times Y$ with intensity 1 for the c.l.d. product measure $\mu_L\times\nu$, where μ_L is Lebesgue measure.

(k) Let $\mathcal{C}$ be the family of closed subsets of $[0,\infty[$. Let ρ be the usual metric on $[0,\infty[$ and $\tilde\rho$ the corresponding Hausdorff metric on $\mathcal{C}\setminus\{\emptyset\}$ (4A2T). Let ν be the Poisson point process on $[0,\infty[$ with intensity 1 over Lebesgue measure. Show that every member of $\mathcal{C}\setminus\{\emptyset\}$ has a ν-negligible $\tilde\rho$-neighbourhood.

(l) Show that the topology on $M_{\mathrm{R}}^+(X)$ described in 495R is just the topology induced by the natural embedding of $M_{\mathrm{R}}(X)$ into $C_k(X)^\sim$ (436J) and the weak topology $\mathfrak{T}_s(C_k(X)^\sim,C_k(X))$, where $C_k(X)$ is the Riesz space of continuous real-valued functions on X with compact support.

(m) Let $\mathcal{C}$ be the set of closed subsets of $[0,\infty[$ with its Fell topology. For $\delta\in\,]0,1]$ let λ_δ be the measure on $\{0,1\}^{\mathbb{N}}$ which is the product of copies of the measure on $\{0,1\}$ in which $\{1\}$ is given measure δ. Define $\phi_\delta:\{0,1\}^{\mathbb{N}}\to\mathcal{C}$ by setting $\phi_\delta(x)=\{n\delta:n\in\mathbb{N},\ x(n)=1\}$, and let $\tilde\nu_\delta$ be the Radon measure $\lambda_\delta\phi_\delta^{-1}$ on $\mathcal{C}$. Show that the Radon measure on $\mathcal{C}$ representing the Poisson point process on $[0,\infty[$ with intensity 1 over Lebesgue measure is the limit $\lim_{\delta\downarrow0}\tilde\nu_\delta$ for the narrow topology on the space of Radon probability measures on $\mathcal{C}$.

(n) Show that the standard gamma distribution with expectation 1 is the exponential distribution with expectation 1.

(o) Let $r \ge 1$ be an integer; let μ be Lebesgue measure on $\mathbb{R}^r$ and β_r the volume of the unit ball in $\mathbb{R}^r$. Set $\psi(t) = (t/\beta_r)^{1/r}$ for $t \ge 0$, so that the volume of a ball of radius $\psi(t)$ is t. Let S_{r-1} be the unit sphere in $\mathbb{R}^r$ and θ the invariant Radon probability measure on S_{r-1}, so that θ is a multiple of $(r-1)$-dimensional Hausdorff measure (see 476I). Let λ_0 be the exponential distribution with expectation 1, regarded as a Radon probability measure on $]0,\infty[$, and λ the product measure $\lambda_0^{\mathbb{N}} \times \theta^{\mathbb{N}}$ on $]0,\infty[^{\mathbb{N}} \times S_{r-1}^{\mathbb{N}}$. Set

$$\phi(x,z) = \{\psi(\sum_{i=0}^{n} x(i))z(n) : n \in \mathbb{N}\}$$

for $x \in]0,\infty[^{\mathbb{N}}$ and $z \in S_{r-1}^{\mathbb{N}}$. Show that $\phi :]0,\infty[^{\mathbb{N}} \times S_{r-1}^{\mathbb{N}} \to \mathcal{P}(\mathbb{R}^r)$ is a measure space isomorphism between $]0,\infty[^{\mathbb{N}} \times S_{r-1}^{\mathbb{N}}$ and a conegligible set for the Poisson point process on $\mathbb{R}^r$ with intensity 1.

495Y Further exercises (a) Let U be an L-space. Show that there are a probability algebra $(\mathfrak{B},\bar\lambda)$ and a linear operator $T : U \to L^0(\mathfrak{B})$ such that (i) for every $u \in U$, Tu has a Cauchy distribution with centre 0 and scale parameter $\|u\|$ (ii) if $\langle u_i\rangle_{i\in I}$ is a disjoint family in U then $\langle Tu_i\rangle_{i\in I}$ is $\bar\lambda$-independent.

(b) Let U be an L-space. Show that there are a probability algebra $(\mathfrak{B},\bar\lambda)$ and a linear operator $T : U \to L^1(\mathfrak{B},\bar\lambda)$ such that (i) for every $u \in U^+$, Tu has a standard gamma distribution (definition: 455Xj) with expectation $\|u\|$ (ii) if $\langle u_i\rangle_{i\in I}$ is a disjoint family in U then $\langle Tu_i\rangle_{i\in I}$ is $\bar\lambda$-independent.

(c) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. For α, $y \in \mathbb{R}$ set $h_y(\alpha) = e^{iy\alpha}$, and let $\bar h_y : L^0(\mathfrak{A}) \to L^0_{\mathbb{C}}(\mathfrak{A})$ (definition: 366M[6]) be the corresponding operator (to be defined, following the ideas of 364H[7] or otherwise). Show that there are a probability algebra $(\mathfrak{B},\bar\lambda)$ and a positive linear operator $T : L^1(\mathfrak{A},\bar\mu) \to L^1(\mathfrak{B},\bar\lambda)$ such that (i) $\|Tu\|_1 = \|u\|_1$ whenever $u \in L^1(\mathfrak{A},\bar\mu)^+$ (ii) $\langle Tu_i\rangle_{i\in I}$ is $\bar\lambda$-independent in $L^0(\mathfrak{B})$ whenever $\langle u_i\rangle_{i\in I}$ is a disjoint family in $L^1(\mathfrak{A},\bar\mu)$ (iii) $\int \bar h_y(Tu)d\bar\lambda = \exp(\int(\bar h_y(u) - \chi 1)d\bar\mu)$ for every $u \in L^1(\mathfrak{A},\bar\mu)$ and $y \in \mathbb{R}$.

(d) Let (X,ρ) be a totally bounded metric space, μ a Radon measure on X and $\gamma > 0$. Let $\mathcal{C}$ be the set of closed subsets of X, and $\tilde\nu$ the quasi-Radon measure of 495Q; let $\tilde\rho$ be the Hausdorff metric on $\mathcal{C} \setminus \{\emptyset\}$. Show that the subspace measure on $\mathcal{C} \setminus \{\emptyset\}$ induced by $\tilde\nu$ is a Radon measure for the topology induced by $\tilde\rho$.

495 Notes and comments The underlying fact on which this section relies is that the Poisson distributions form a one-parameter semigroup of infinitely divisible distributions, with $\nu_\alpha * \nu_\beta = \nu_{\alpha+\beta}$ for all α, $\beta > 0$. Other well-known families with this property are normal distributions, Cauchy distributions and gamma distributions; for each of these we have results corresponding to 495B and 495N (495Xh, 495Ya, 495Yb). The same distributions appeared, for the same reason, in the Lévy processes of §455. Observe that the version for the normal distribution is related to the Gaussian processes of §456. The 'compound Poisson' distributions of 495P provide further examples, which approach the general form of infinitely divisible distributions (Loève 77, §23, or Fristedt & Gray 97, §16.3).

The special feature of the Poisson point process, in this context, is the fact that (for atomless measure spaces (X,μ)) it can be represented by a measure on $\mathcal{P}X$ rather than on some abstract auxiliary space (495D); so that we have a notion of 'random subset', and can discuss the expected topological properties of subsets of X (495Xb, 495Xd). In Euclidean spaces the geometric properties of these random subsets are also of great interest; see Meester & Roy 96. Here I look at the relations between this construction and others which have been prominent in this book, such as inverse-measure-preserving functions (495J) and disintegrations (495K-495L). In the latter we find ourselves in an interesting difficulty. If, as in 495K, we have a measure space $X = \tilde X \times [0,1]$, where $\tilde X$ is an atomless measure space, then it is natural to suppose that our Poisson process on X can be represented by picking a random subset T of $\tilde X$ and then, for each $t \in T$, a random $(t,\alpha) \in X$. The obvious model for this idea is the map $(T,z) \mapsto \{(t,z(t)) : t \in T\} : \mathcal{P}\tilde X \times [0,1]^{\tilde X} \to \mathcal{P}X$. The problem with this model is that the map is simply not measurable for the standard σ-algebras on $\mathcal{P}\tilde X$, $\mathcal{P}\tilde X \times [0,1]^{\tilde X}$ and $\mathcal{P}X$. When we have a canonical ordering in order type ω of almost every subset of $\tilde X$

[6] Formerly 364Yn.
[7] Formerly 364I.

('almost every' with respect to the Poisson point process on $\tilde{X}$, of course), as in 495Xo, there can be a way around this, cutting $[0,1]^{\tilde{X}}$ down to a countable product and re-inventing the representation of pairs (T,z) as subsets of X. But in the general case it seems that we have to set up a disintegration of the Poisson point process on X over the Poisson point process on $\tilde{X}$ which does not correspond to any measure on a product $\mathcal{P}\tilde{X}\times\Omega$.

Following my usual custom, I have expressed the theorems of this section in terms of arbitrary (atomless) measure spaces. The results are not quite without interest when applied to totally finite measures, but their natural domain is the class of non-totally-finite σ-finite measures, as in 495Q-495S. There is an unavoidable obstacle if we wish to extend the ideas to measure spaces which are not atomless. The functions $S\mapsto\#(S\cap E)$ may no longer have Poisson distributions, since if E is a singleton of positive measure then we shall have a non-trivial two-valued random variable. In 495Q-495R I take one of the possible resolutions of this, with measures $\tilde{\nu}$ on spaces of subsets for which at least the sets $\{S:S\cap E=\emptyset\}$, for disjoint E, are independent. An alternative which is sometimes appropriate is to work with functions $h:X\to\mathbb{N}$ and $\sum_{x\in E}h(x)$ in place of subsets S of X and $\#(S\cap E)$; see Fristedt & Gray 97, §29.

In 495M-495O we have a little cluster of results which are relevant to rather different questions, to which I will return in Chapter 52 of Volume 5. The objective here is to connect the structure of a measure algebra or Banach lattice of arbitrarily large cellularity with something which can be realized in a probability space. In each case, disjointness is transformed into stochastic independence. Once again, the special feature of the Poisson point process is that we have a concrete representation of a linear operator which can also be described in a more abstract way (495O).

The construction of 495B-495D seems to be the most straightforward way to generate Poisson point processes. It fails however to give a direct interpretation of one of the most important approaches to these processes, as limits of purely atomic processes in which sets are chosen by including or excluding individual points independently (495Xm). In order to make sense of the limit here it seems that we need to put some further structure onto the underlying measure space, and 'σ-finite locally compact Radon measure space' is sufficient to give a positive result (495R).

Version of 27.5.09

496 Maharam submeasures

The old problem of characterizing measurable algebras led, among other things, to the concepts of 'Maharam submeasure' and 'Maharam algebra' (§393). It is known that these can be very different from measures (§394), but the differences are not well understood. In this section I will continue the work of §393 by showing that some, at least, of the ways in which topologies and measures interact apply equally to Maharam submeasures. The most important of these interactions are associated with the concept of 'Radon measure', so the first step is to find a corresponding notion of 'Radon submeasure' (496C, 496Y). In 496D-496K I run through a handful of theorems which parallel results in §§416 and 431-433. Products of submeasures remain problematic, but something can be done (496L-496M).

496A Definitions As we have hardly had 'submeasures' before in this volume, I repeat the essential definitions from Chapter 39. If $\mathfrak{A}$ is a Boolean algebra, a **submeasure** on $\mathfrak{A}$ is a functional $\mu:\mathfrak{A}\to[0,\infty]$ such that $\mu 0=0$ and $\mu a\le\mu(a\cup b)\le\mu a+\mu b$ for all a, $b\in\mathfrak{A}$ (392A). μ is **strictly positive** if $\mu a>0$ for every $a\in\mathfrak{A}\setminus\{0\}$ (392Ba), **exhaustive** if $\lim_{n\to\infty}\mu a_n=0$ for every disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ (392Bb), **totally finite** if $\mu 1<\infty$ (392Bd), a **Maharam submeasure** if it is totally finite and $\lim_{n\to\infty}\mu a_n=0$ for every non-increasing sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ with zero infimum (393A). A Maharam submeasure is sequentially order-continuous (393Ba). If μ and ν are two submeasures on a Boolean algebra $\mathfrak{A}$, then μ is **absolutely continuous** with respect to ν if for every $\epsilon>0$ there is a $\delta>0$ such that $\mu a\le\epsilon$ whenever $\nu a\le\delta$ (392Bg). A **Maharam algebra** is a Dedekind σ-complete Boolean algebra which carries a strictly positive Maharam submeasure (393E).

496B Basic facts I list some elementary ideas for future reference.

(a) Let μ be a submeasure on a Boolean algebra $\mathfrak{A}$.

(i) Set $I=\{a:a\in\mathfrak{A},\ \mu a=0\}$. Clearly I is an ideal of $\mathfrak{A}$; write $\mathfrak{C}$ for the quotient Boolean algebra $\mathfrak{A}/I$. Then we have a strictly positive submeasure $\bar\mu$ on $\mathfrak{C}$ defined by setting $\bar\mu a^\bullet=\mu a$ for every $a\in\mathfrak{A}$. **P** If $a^\bullet=b^\bullet$ then

$$\mu(a\setminus b)=\mu(b\setminus a)=\mu(a\triangle b)=0,\quad \mu a=\mu(a\cap b)=\mu b;$$

so $\bar\mu$ is well-defined. The formulae defining 'submeasure' transfer directly from μ to $\bar\mu$. If $\bar\mu a^\bullet=0$ then $\mu a=0$, $a\in I$ and $a^\bullet=0$, so $\bar\mu$ is strictly positive. **Q**

(ii) If μ is exhaustive, so is $\bar\mu$. **P** If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}$ such that $\langle a_n^\bullet\rangle_{n\in\mathbb{N}}$ is disjoint in $\mathfrak{A}/I$, set $b_n=a_n\setminus\sup_{i<n}a_i$ for each n; then $\langle b_n\rangle_{n\in\mathbb{N}}$ is disjoint so

$$\lim_{n\to\infty}\bar\mu a_n^\bullet=\lim_{n\to\infty}\bar\mu b_n^\bullet=\lim_{n\to\infty}\mu b_n=0;$$

thus $\bar\mu$ is exhaustive. **Q**

(iii) If $\mathfrak{A}$ is Dedekind σ-complete and μ is a Maharam submeasure, then $\mathfrak{C}$ is a Maharam algebra. **P** As μ is sequentially order-continuous, I is a σ-ideal and $\mathfrak{C}$ is Dedekind σ-complete (314C). Now suppose that $\langle a_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}$ such that $\langle a_n^\bullet\rangle_{n\in\mathbb{N}}$ is non-increasing and has zero infimum in $\mathfrak{C}$. Set $b_n=\inf_{i\le n}a_i$ for each n, and $a=\inf_{n\in\mathbb{N}}a_n$; then $a^\bullet=0$ so $\mu a=0$ and (again because μ is sequentially order-continuous)

$$\lim_{n\to\infty}\bar\mu a_n^\bullet=\lim_{n\to\infty}\bar\mu b_n^\bullet=\lim_{n\to\infty}\mu b_n=\mu a=0.$$

Since we already know that $\bar\mu$ is a strictly positive submeasure, it is a strictly positive Maharam submeasure and $\mathfrak{C}$ is a Maharam algebra. **Q**

In this context I will say that $\mathfrak{C}$ is **the Maharam algebra of** μ.

(b) If μ is a strictly positive totally finite submeasure on a Boolean algebra $\mathfrak{A}$, there is an associated metric $(a,b)\mapsto\mu(a\triangle b)$ (392H); the corresponding metric completion $\widehat{\mathfrak{A}}$ admits a continuous extension of μ to a strictly positive submeasure $\hat\mu$ on $\widehat{\mathfrak{A}}$. If μ is exhaustive, then $\hat\mu$ is a Maharam submeasure and $\widehat{\mathfrak{A}}$ is a Maharam algebra (393H). A Maharam algebra is ccc, therefore Dedekind complete, and weakly (σ,∞)-distributive (393Eb).

(c) If μ is a submeasure defined on an algebra Σ of subsets of a set X, I will say that the **null ideal** $\mathcal{N}(\mu)$ of μ is the ideal of subsets of X generated by $\{E:E\in\Sigma,\ \mu E=0\}$. If $\mathcal{N}(\mu)\subseteq\Sigma$ I will say that μ is **complete**. Generally, the **completion** of μ is the functional $\hat\mu$ defined by saying that $\hat\mu(E\triangle A)=\mu E$ whenever $E\in\Sigma$ and $A\in\mathcal{N}(\mu)$; it is elementary to check that $\hat\mu$ is a complete submeasure.

(d) If $\mathfrak{A}$ is a Maharam algebra, and μ, ν are two strictly positive Maharam submeasures on $\mathfrak{A}$, then each is absolutely continuous with respect to the other (393F). Consequently the metrics associated with them are uniformly equivalent, and induce the same topology, the **Maharam-algebra topology** of $\mathfrak{A}$ (393G).

496C Radon submeasures Let X be a Hausdorff space. A **totally finite Radon submeasure** on X is a complete totally finite submeasure μ defined on a σ-algebra Σ of subsets of X such that (i) Σ contains every open set (ii) $\inf\{\mu(E\setminus K):K\subseteq E\text{ is compact}\}=0$ for every $E\in\Sigma$.

In this context I will say that a set $E\in\Sigma$ is **self-supporting** if $\mu(E\cap G)>0$ whenever $G\subseteq X$ is open and $G\cap E\ne\emptyset$.

496D Proposition Let μ be a totally finite Radon submeasure on a Hausdorff space X with domain Σ.

(a) μ is a Maharam submeasure.

(b) $\inf\{\mu(G\setminus E):G\supseteq E\text{ is open}\}=0$ for every $E\in\Sigma$.

(c) If $E\in\Sigma$ there is a relatively closed $F\subseteq E$ such that F is self-supporting and $\mu(E\setminus F)=0$.

(d) If $E\in\Sigma$ and $\epsilon>0$ there is a compact self-supporting $K\subseteq E$ such that $\mu(E\setminus K)\le\epsilon$.

proof (a) Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a non-increasing sequence in Σ with empty intersection. **?** If $\inf_{n\in\mathbb{N}}\mu E_n=\epsilon>0$, then for each $n\in\mathbb{N}$ choose a compact set $K_n\subseteq E_n$ such that $\mu(E_n\setminus K_n)\le 2^{-n-2}\epsilon$. For each $n\in\mathbb{N}$,

$$\mu(E_n\setminus\bigcap_{i\le n}K_i)\le\sum_{i=0}^n\mu(E_i\setminus K_i)<\epsilon\le\mu E_n,$$

so $\bigcap_{i\le n}K_i\ne\emptyset$. There is therefore a point in $\bigcap_{n\in\mathbb{N}}K_n\subseteq\bigcap_{n\in\mathbb{N}}E_n$. **X** As $\langle E_n\rangle_{n\in\mathbb{N}}$ is arbitrary, μ is a Maharam submeasure.

(b) We have only to observe that

$$\inf\{\mu(G\setminus E):G\supseteq E\text{ and }G\text{ is open}\}$$
$$\le\inf\{\mu((X\setminus E)\setminus K):K\subseteq X\setminus E\text{ is compact}\}=0.$$

(c) Let $\mathcal{G}$ be the family of open subsets G of X such that $\mu(E\cap G)=0$, and $H=\bigcup\mathcal{G}$. Then $\mathcal{G}$ is upwards-directed. If $\epsilon>0$ there is a compact set $K\subseteq E\cap H$ such that $\mu(E\cap H\setminus K)\le\epsilon$; now there is a $G\in\mathcal{G}$ such that $K\subseteq G$ and $\mu(E\cap H)\le\epsilon+\mu K=\epsilon$. As ϵ is arbitrary, $H\in\mathcal{G}$; set $F=E\setminus H$.

(d) There is a compact $K_0\subseteq E$ such that $\mu(E\setminus K_0)\le\epsilon$; by (c), there is a closed self-supporting $K\subseteq K_0$ such that $\mu(K_0\setminus K)=0$.

496E Theorem Let X be a Hausdorff space and $\mathcal{K}$ the family of compact subsets of X. Let $\phi:\mathcal{K}\to[0,\infty[$ be a bounded functional such that

(α) $\phi\emptyset=0$ and $\phi K\le\phi(K\cup L)\le\phi K+\phi L$ for all K, $L\in\mathcal{K}$;

(β) whenever $K\in\mathcal{K}$ and $\epsilon>0$ there is an $L\in\mathcal{K}$ such that $L\subseteq X\setminus K$ and $\phi K'\le\epsilon$ whenever $K'\in\mathcal{K}$ is disjoint from $K\cup L$;

(γ) whenever K, $L\in\mathcal{K}$ and $K\subseteq L$ then $\phi L\le\phi K+\sup\{\phi K':K'\in\mathcal{K},\ K'\subseteq L\setminus K\}$.

Then there is a unique totally finite Radon submeasure on X extending ϕ.

proof (a) For $A\subseteq X$ write $\phi_*A=\sup\{\phi K:K\subseteq A\text{ is compact}\}$. Then ϕ_* extends ϕ, by (α). Also $\phi_*(\bigcup_{n\in\mathbb{N}}G_n)\le\sum_{n=0}^{\infty}\phi_*G_n$ for every sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ of open subsets of X. **P** If $K\subseteq\bigcup_{n\in\mathbb{N}}G_n$ is compact, it is expressible as $\bigcup_{i\le n}K_i$ where $n\in\mathbb{N}$ and $K_i\subseteq G_i$ is compact for every $i\le n$ (4A2Fj). Now

$$\phi K\le\sum_{i=0}^{n}\phi K_i\le\sum_{i=0}^{\infty}\phi_*G_i.$$

As K is arbitrary, $\phi_*(\bigcup_{n\in\mathbb{N}}G_n)\le\sum_{n=0}^{\infty}\phi_*G_n$. **Q** In particular, because $\phi\emptyset=0$, $\phi_*(G\cup H)\le\phi_*G+\phi_*H$ for all open G, $H\subseteq X$.

(b) Let Σ be the family of subsets E of X such that for every $\epsilon>0$ there is a $K\subseteq X$ such that $K\cap E$ and $K\setminus E$ are both compact and $\phi_*(X\setminus K)\le\epsilon$. Then Σ is an algebra of subsets of X including $\mathcal{K}$. **P** (i) Of course $X\setminus E\in\Sigma$ whenever $E\in\Sigma$. (ii) If E, $F\in\Sigma$ and $\epsilon>0$, let K, $L\subseteq X$ be such that $K\cap E$, $K\setminus E$, $L\cap F$ and $L\setminus F$ are all compact and $\phi_*(X\setminus K)$, $\phi_*(X\setminus L)$ are both at most $\frac{1}{2}\epsilon$. Then $(K\cap L)\cap(E\cup F)$ and $(K\cap L)\setminus(E\cup F)$ are both compact, and

$$\phi_*(X\setminus(K\cap L))\le\phi_*(X\setminus K)+\phi_*(X\setminus L)\le\epsilon.$$

As ϵ is arbitrary, $E\cup F\in\Sigma$. (iii) By hypothesis (β), $\mathcal{K}\subseteq\Sigma$. **Q**

(c) Σ is a σ-algebra of subsets of X. **P** Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a sequence in Σ with intersection E, and $\epsilon>0$. For each $n\in\mathbb{N}$ let $K_n\subseteq X$ be such that $K_n\cap E_n$ and $K_n\setminus E_n$ are compact and $\phi_*(X\setminus K_n)\le 2^{-n}\epsilon$; set $K=\bigcap_{n\in\mathbb{N}}K_n$. Set $L=\bigcap_{n\in\mathbb{N}}K_n\cap E_n$, so that $L\subseteq E$ is compact, and let $L'\subseteq X\setminus L$ be a compact set such that $\phi_*(X\setminus(L\cup L'))\le\epsilon$; set $K'=K\cap(L\cup L')$. Then $\phi_*(X\setminus K')\le 3\epsilon$. As $L'\cap L=\emptyset$ there is an $n\in\mathbb{N}$ such that $L'\cap\bigcap_{i\le n}K_i\cap E_i$ is empty. Now

$$K\cap L'\subseteq\bigcup_{i\le n}(X\setminus(K_i\cap E_i))\cap\bigcap_{i\le n}K_i\subseteq\bigcup_{i\le n}X\setminus E_i\subseteq X\setminus E,$$

so $K'\cap E=K\cap L$ and $K'\setminus E=K\cap L'$ are compact. As ϵ is arbitrary, $E\in\Sigma$. **Q**

(d) Set $\mu=\phi_*\restriction\Sigma$. Then μ is subadditive. **P** Suppose that E, $F\in\Sigma$ and $K\subseteq E\cup F$ is compact. Let $\epsilon>0$. Then there are L_1, $L_2\in\mathcal{K}$ such that $L_1\cap E$, $L_1\setminus E$, $L_2\cap F$ and $L_2\setminus F$ are all compact, while $\phi_*(X\setminus L_1)$ and $\phi_*(X\setminus L_2)$ are both at most ϵ. Set $K_1=L_1\cap E$ and $K_2=L_2\cap F$, so that

$$\phi K\le\phi(K\cup K_1\cup K_2)\le\phi(K_1\cup K_2)+\phi_*(K\setminus(K_1\cup K_2))$$

(by hypothesis (γ))

$$\le\phi K_1+\phi K_2+\phi_*(X\setminus(L_1\cap L_2))\le\phi_*E+\phi_*F+2\epsilon.$$

As ϵ and K are arbitrary, $\phi_*(E\cup F)\le\phi_*E+\phi_*F$. **Q**

(e) Every open set belongs to Σ. **P** Let $G\subseteq X$ be open, and $\epsilon>0$. Applying (β) with $K=\emptyset$ we have an $L\in\mathcal{K}$ such that $\phi_*(X\setminus L)\le\epsilon$. Next, there is an $L'\in\mathcal{K}$, disjoint from $L\setminus G$, such that $\phi_*(X\setminus((L\setminus G)\cup L'))\le\epsilon$. Set $L''=L\cap((L\setminus G)\cup L')$. Then $L''\cap G=L\cap L'$ and $L''\setminus G=L\setminus G$ are compact and $\phi_*(X\setminus L'')\le 2\epsilon$. **Q**

(f) If $E\subseteq F\in\Sigma$ and $\mu F=0$ then $E\in\Sigma$. **P** Let $\epsilon>0$. Let $K\subseteq X$ be such that $K\cap F$ and $K\setminus F$ are both compact and $\phi_*(X\setminus K)\le\epsilon$. Then $(K\setminus F)\cap E$ and $(K\setminus F)\setminus E$ are both compact, and

$$\phi_*(X\setminus(K\setminus F))=\mu(X\setminus(K\setminus F))\le\mu(X\setminus K)+\mu F=\phi_*(X\setminus K)\le\epsilon.$$

As ϵ is arbitrary, $E\in\Sigma$. **Q**

(g) If $E\in\Sigma$ and $\epsilon>0$, there is a compact $K\subseteq E$ such that $\mu(E\setminus K)\le\epsilon$. **P** Let $K_0\subseteq X$ be such that $K_0\cap E$ and $K_0\setminus E$ are both compact and $\phi_*(X\setminus K_0)\le\epsilon$. Set $K=E\cap K_0$. If $L\in\mathcal{K}$ and $L\subseteq E\setminus K$ then $\phi L\le\phi_*(X\setminus K_0)\le\epsilon$; so $\mu(E\setminus K)\le\epsilon$. **Q**

(h) So μ is a totally finite Radon submeasure. To see that it is unique, let μ' be another totally finite Radon submeasure with the same properties, and Σ' its domain. By condition (ii) of 496C, $\mu'=\phi_*\restriction\Sigma'$. If $E\in\Sigma$ there are sequences $\langle K_n\rangle_{n\in\mathbb{N}}$, $\langle L_n\rangle_{n\in\mathbb{N}}$ of compact sets such that $K_n\subseteq E$, $L_n\subseteq X\setminus E$ and $\mu(E\setminus K_n)+\mu((X\setminus E)\setminus L_n)\le 2^{-n}$ for every n. Set $F=\bigcup_{n\in\mathbb{N}}K_n$ and $F'=\bigcup_{n\in\mathbb{N}}L_n$; then $F\cup F'$ belongs to $\Sigma\cap\Sigma'$ and

$$\begin{aligned}\mu'(X\setminus(F\cup F'))&=\phi_*(X\setminus(F\cup F'))=\mu(X\setminus(F\cup F'))\\&\le\inf_{n\in\mathbb{N}}\mu(X\setminus(K_n\cup L_n))=0.\end{aligned}$$

Consequently $E\setminus F\in\Sigma'$ and $E\in\Sigma'$.

The same works with μ and μ' interchanged, so $\Sigma=\Sigma'$ and $\mu'=\phi_*\restriction\Sigma=\mu$.

496F Theorem Let X be a zero-dimensional compact Hausdorff space and $\mathcal{E}$ the algebra of open-and-closed subsets of X. Let $\nu:\mathcal{E}\to[0,\infty[$ be an exhaustive submeasure. Then there is a unique totally finite Radon submeasure on X extending ν.

proof (a) Let $\mathcal{K}$ be the family of compact subsets of X and for $K\in\mathcal{K}$ set $\phi K=\inf\{\nu E:K\subseteq E\in\mathcal{E}\}$. Then ϕ satisfies the conditions of 496E.

P$(\boldsymbol{\alpha})$ Of course $\phi\emptyset=0$ and $\phi K\le\phi L$ whenever $K\subseteq L$ in $\mathcal{K}$. If $K\subseteq E\in\mathcal{E}$ and $L\subseteq F\in\mathcal{E}$, then $K\cup L\subseteq E\cup F\in\mathcal{E}$ and $\nu(E\cup F)\le\nu E+\nu F$, so ϕ is subadditive.

$(\boldsymbol{\beta})$ The point is that for every $K\in\mathcal{K}$ and $\epsilon>0$ there is an $E\in\mathcal{E}$ such that $K\subseteq E$ and $\nu F\le\epsilon$ whenever $F\in\mathcal{E}$ and $F\subseteq E\setminus K$; since otherwise we could find a disjoint sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$ with $\nu F_n\ge\epsilon$ for every n. But now $L=X\setminus E$ is compact and disjoint from K, and every compact subset of $X\setminus(K\cup L)=E\setminus K$ is included in a member of $\mathcal{E}$ included in $E\setminus K$; so $\sup\{\phi K':K'\subseteq X\setminus(K\cup L)$ is compact$\}\le\epsilon$.

$(\boldsymbol{\gamma})$ If K and L are compact and $K\subseteq L$ and $\epsilon>0$, take $E\in\mathcal{E}$ such that $K\subseteq E$ and $\nu E\le\phi K+\epsilon$. Set $K'=L\setminus E$. If $F\in\mathcal{E}$ and $F\supseteq K'$, then $E\cup F\supseteq L$, so

$$\phi L\le\nu(E\cup F)\le\nu E+\nu F\le\phi K+\epsilon+\nu F.$$

As F is arbitrary, $\phi L\le\phi K+\phi K'+\epsilon$. **Q**

There is therefore a totally finite Radon submeasure μ extending ϕ and ν.

(b) If μ' is another totally finite Radon submeasure extending ν, then $\mu'\restriction\mathcal{K}=\phi$. **P** Of course $\mu'K\le\phi K$ for every $K\in\mathcal{K}$. **?** If $K\in\mathcal{K}$ and $\epsilon>0$ and $\mu'K+\epsilon<\phi K$, let $E\in\mathcal{E}$ be such that $K\subseteq E$ and $\phi L\le\epsilon$ whenever $L\subseteq E\setminus K$ is compact, as in (a-β) above. Then

$$\begin{aligned}\mu'(E\setminus K)&=\sup\{\mu'L:L\subseteq E\setminus K\text{ is compact}\}\\&\le\sup\{\phi L:L\subseteq E\setminus K\text{ is compact}\}\le\epsilon\end{aligned}$$

and

$$\nu E=\mu'E\le\epsilon+\mu'K<\mu K\le\mu E=\nu E.\ \mathbf{XQ}$$

By the guarantee of uniqueness in 496E, $\mu'=\mu$.

496G Theorem Let $\mathfrak{A}$ be a Maharam algebra, and μ a strictly positive Maharam submeasure on $\mathfrak{A}$. Let Z be the Stone space of $\mathfrak{A}$, and write $\widehat{a}$ for the open-and-closed subset of Z corresponding to each $a \in \mathfrak{A}$. Then there is a unique totally finite Radon submeasure ν on Z such that $\nu\widehat{a} = \mu a$ for every $a \in \mathfrak{A}$. The domain of ν is the Baire-property algebra $\widehat{\mathcal{B}}$ of Z, and the null ideal of ν is the nowhere dense ideal of Z.

proof Let $\mathcal{E}$ be the algebra of open-and-closed subsets of Z, and $\mathcal{M}$ the ideal of meager subsets of Z. Because $\mathfrak{A}$ is Dedekind complete (393Eb/496Bb), $\mathcal{E}$ is the regular open algebra of Z (314S). By 496R(b-ii), $\widehat{\mathcal{B}} = \{E\triangle F : E \in \mathcal{E},\, F \in \mathcal{M}\}$.

For $a \in \mathfrak{A}$, let $\widehat{a}$ be the corresponding member of $\mathcal{E}$. By 314M, we have an isomorphism $\theta : \mathfrak{A} \to \widehat{\mathcal{B}}/\mathcal{M}$ defined by setting $\theta(a) = \widehat{a}^\bullet$ for every $a \in \mathfrak{A}$. For $E \in \widehat{\mathcal{B}}$, set $\nu E = \mu(\theta^{-1}E^\bullet)$. Because $E \mapsto E^\bullet$ is a sequentially order-continuous Boolean homomorphism (313P(b-ii)), ν is a Maharam submeasure on $\widehat{\mathcal{B}}$. Because μ is strictly positive, the null ideal of ν is $\mathcal{M}$.

Because $\mathfrak{A}$ is weakly (σ,∞)-distributive (393Eb/496Bb), $\mathcal{M}$ is the ideal of nowhere dense subsets of Z (316I). If $E \in \widehat{\mathcal{B}}$, consider $B = \{b : b \in \mathfrak{A},\, \widehat{b} \subseteq E\}$; set $a = \sup B$ in $\mathfrak{A}$. Now $E \setminus \widehat{a}$ is nowhere dense. **P?** Otherwise, there is a non-zero $c \in \mathfrak{A}$ such that $F = \widehat{c} \setminus (E \setminus \widehat{a})$ is nowhere dense. In this case, the non-empty open set $\widehat{c} \setminus \overline{F}$ is included in $E \setminus \widehat{a}$ and there is a non-zero $b \in \mathfrak{A}$ such that $\widehat{b} \subseteq E \setminus \widehat{a}$. But in this case $b \in B$ and $\widehat{b} \subseteq \widehat{a}$, which is absurd. **XQ**

Set $D = \{a \setminus b : b \in B\}$. Then D is downwards-directed and has infimum 0. Because μ is sequentially order-continuous and $\mathfrak{A}$ is ccc, μ is order-continuous (316Fc), and $\inf_{d\in D} \mu d = 0$. Accordingly

$$\begin{aligned}\inf\{\nu(E \setminus K) : K \subseteq E \text{ is compact}\} &\le \inf_{b\in B} \nu(E \setminus \widehat{b}) = \inf_{b\in B} \nu(\widehat{a} \setminus \widehat{b}) \\ &= \inf_{b\in B} \mu(a \setminus b) = 0.\end{aligned}$$

Thus condition (ii) of 496C is satisfied and ν is a totally finite Radon measure.

By 496F, ν is unique.

496H Theorem Let X be a Hausdorff space, Σ_0 an algebra of subsets of X, and $\mu_0 : \Sigma_0 \to [0,\infty[$ an exhaustive submeasure such that $\inf\{\mu_0(E \setminus K) : K \in \Sigma_0 \text{ is compact}, K \subseteq E\} = 0$ for every $E \in \Sigma_0$. Then μ_0 has an extension to a totally finite Radon submeasure μ_1 on X.

proof (a) Let P be the set of all submeasures μ, defined on algebras of subsets of X, which extend μ_0, and have the properties

(α) $\inf\{\mu(E \setminus K) : K \in \operatorname{dom}\mu \text{ is compact}, K \subseteq E\} = 0$ for every $E \in \operatorname{dom}\mu$,

(*) for every $E \in \operatorname{dom}\mu$ and $\epsilon > 0$ there is an $F \in \Sigma_0$ such that $\mu(E\triangle F) \le \epsilon$.

Order P by extension of functions, so that P is a partially ordered set.

(b) If $\mu \in P$, then μ is exhaustive. **P?** Otherwise, let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a disjoint sequence in $\operatorname{dom}\mu$ such that $\epsilon = \inf_{n\in\mathbb{N}} \mu E_n$ is greater than 0. For each $n \in \mathbb{N}$, let $F_n \in \Sigma_0$ be such that $\mu(E_n\triangle F_n) \le 2^{-n-2}\epsilon$; set $G_n = F_n \setminus \bigcup_{i<n} F_i$ for each n. Then

$$E_n \subseteq G_n \cup \bigcup_{i\le n}(E_i\triangle F_i), \quad \epsilon \le \mu G_n + \sum_{i=0}^n 2^{-i-2}\epsilon \le \mu_0 G_n + \frac{1}{2}\epsilon$$

and $\mu_0 G_n \ge \frac{1}{2}\epsilon$ for every n. But $\langle G_n\rangle_{n\in\mathbb{N}}$ is disjoint and μ_0 is supposed to be exhaustive. **XQ**

(c) Suppose that $\mu \in P$ has domain Σ, and that $V \subseteq X$ is such that

$\ddagger(V,\mu)$: for every $\epsilon > 0$ there is a $K \in \Sigma$ such that $K \cap V$ is compact and $\mu(X \setminus K) \le \epsilon$.

(i) Set $\mathcal{H} = \{H : V \subseteq H \in \Sigma\}$. Then $\mathcal{H}$ is downwards-directed. If $\epsilon > 0$ there is an $H \in \mathcal{H}$ such that $\mu(H \setminus H') \le \epsilon$ for every $H' \in \mathcal{H}$. **P?** Otherwise, there would be a non-increasing sequence $\langle H_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{H}$ such that $\mu(H_n \setminus H_{n+1}) \ge \epsilon$ for every n; but μ is exhaustive, by (b). **XQ**

(ii) Let $\mathcal{F}$ be the filter on $\mathcal{H}$ generated by sets of the form $\{H' : H' \in \mathcal{H},\, H' \subseteq H\}$ for $H \in \mathcal{H}$. Then $\lim_{H\to\mathcal{F}} \mu((E \cap H) \cup (F \setminus H))$ is defined for all $E, F \in \Sigma$. **P** Given $\epsilon > 0$, then (i) tells us that there is an $H_0 \in \mathcal{H}$ such that $\mu(H\triangle H') \le \mu(H_0 \setminus (H \cap H')) \le \epsilon$ whenever $H, H' \in \mathcal{H}$ are included in H_0. Now, for such H and H',

$$((E \cap H) \cup (F \setminus H)) \triangle ((E \cap H') \cup (F \setminus H')) \subseteq H \triangle H',$$

so

$$|\mu((E \cap H) \cup (F \setminus H)) - \mu((E \cap H') \cup (F \setminus H'))| \le \epsilon. \text{ } \mathbf{Q}$$

(iii) If E, F, E', $F' \in \Sigma$ and $(E \cap V) \cup (F \setminus V) = (E' \cap V) \cup (F' \setminus V)$, then

$$\lim_{H \to \mathcal{F}} \mu((E \cap H) \cup (F \setminus H)) = \lim_{H \to \mathcal{F}} \mu((E' \cap H) \cup (F' \setminus H)).$$

P Given $\epsilon > 0$, there is an $H_0 \in \mathcal{H}$ such that $\mu G \le \epsilon$ whenever $G \in \Sigma$ and $G \subseteq H_0 \setminus V$, by (i). Now if $H \in \mathcal{H}$ and $H \subseteq H_0$,

$$G = ((E \cap H) \cup (F \setminus H)) \triangle ((E' \cap H) \cup (F' \setminus H)) \subseteq H \setminus V,$$

so

$$|\mu((E \cap H) \cup (F \setminus H)) - \mu((E' \cap H) \cup (F' \setminus H))| \le \mu G \le \epsilon.$$

As ϵ is arbitrary, the limits are equal. **Q**

(iv) Consequently, taking Σ' to be the algebra $\{(E \cap V) \cup (F \setminus V) : E, F \in \Sigma\}$ of subsets of X generated by $\Sigma \cup \{V\}$, we have a functional $\mu' : \Sigma' \to [0, \infty[$ defined by saying that

$$\mu'((E \cap V) \cup (F \setminus V)) = \lim_{H \to \mathcal{F}} \mu((E \cap H) \cup (F \setminus H))$$

whenever E, $F \in \Sigma$.

(v) μ' is a submeasure extending μ. **P** If $E \in \Sigma$, then

$$\mu' E = \mu'((E \cap V) \cup (E \setminus V)) = \lim_{H \to \mathcal{F}} \mu((E \cap H) \cup (E \setminus H)) = \mu E,$$

so μ' extends μ. If E_1, E_2, F_1, $F_2 \in \Sigma$, set $E = E_1 \cup E_2$, $F = F_1 \cup F_2$; then

$$((E_1 \cap A) \cup (F_1 \setminus A)) \cup ((E_2 \cap A) \cup (F_2 \setminus A)) = ((E \cap A) \cup (F \setminus A))$$

for every set A, so

$$\begin{aligned}
\mu'\bigl(((E_1 \cap V) \cup (F_1 \setminus V)) &\cup ((E_2 \cap V) \cup (F_2 \setminus V))\bigr) \\
&= \mu'(((E \cap V) \cup (F \setminus V)) \\
&= \lim_{H \to \mathcal{F}} \mu((E \cap H) \cup (F \setminus H)) \\
&= \lim_{H \to \mathcal{F}} \mu\bigl(((E_1 \cap H) \cup (F_1 \setminus H)) \cup ((E_2 \cap H) \cup (F_2 \setminus H))\bigr) \\
&\le \lim_{H \to \mathcal{F}} \mu((E_1 \cap H) \cup (F_1 \setminus H)) + \mu((E_2 \cap H) \cup (F_2 \setminus H)) \\
&= \mu'((E_1 \cap V) \cup (F_1 \setminus V)) + \mu'((E_2 \cap V) \cup (F_2 \setminus V)).
\end{aligned}$$

Thus μ' is subadditive; monotonicity is easier. **Q**

(vi) μ' has the property (α). **P** Suppose that E, $F \in \Sigma$ and that $\epsilon > 0$. Let $H_0 \in \mathcal{H}$ be such that $\mu(H_0 \setminus H) \le \epsilon$ whenever $H \in \mathcal{H}$ and $H \subseteq H_0$. Let $K_0 \in \Sigma$ be such that $\mu(X \setminus K_0) \le \epsilon$ and $K_0 \cap V$ is compact. Let $K_1 \subseteq E$ and $K_2 \subseteq F \setminus H_0$ be compact sets, belonging to Σ, such that $\mu(E \setminus K_1) \le \epsilon$ and $\mu((F \setminus H_0) \setminus K_2) \le \epsilon$. Set $K = (K_1 \cap K_0 \cap V) \cup K_2$, so that K is a compact set belonging to Σ' and $K \subseteq (E \cap V) \cup (F \setminus V)$. Now if $H \in \mathcal{H}$ and $H \subseteq H_0$,

$$\begin{aligned}
\mu\bigl(((E \setminus (K_1 \cap K_0)) \cap H) &\cup ((F \setminus K_2) \setminus H)\bigr) \\
&\le \mu(E \setminus K_1) + \mu(X \setminus K_0) + \mu((F \setminus H_0) \setminus K_2) + \mu(H_0 \setminus H) \le 4\epsilon.
\end{aligned}$$

Taking the limit along $\mathcal{F}$,

$$\mu'\bigl(((E \cap V) \cup (F \setminus V)) \setminus K\bigr) = \mu'\bigl(((E \setminus (K_1 \cap K_0)) \cap V) \cup ((F \setminus K_2) \setminus V)\bigr) \le 4\epsilon.$$

As E, F and ϵ are arbitrary, we have the result. **Q**

(vii) μ' has the property (*). **P** Suppose that E, $F \in \Sigma$ and that $\epsilon > 0$. Let $H_0 \in \mathcal{H}$ be such that $\mu(H_0 \setminus H) \le \epsilon$ whenever $H \in \mathcal{H}$ and $H \subseteq H_0$. Set $G = (E \cap H_0) \cup (F \setminus H_0) \in \Sigma$. Then

$$((E \cap V) \cup (F \setminus V)) \triangle G \subseteq H_0 \setminus V,$$

so

$$\mu'\big(((E \cap V) \cup (F \setminus V)) \triangle G\big) \le \mu'(H_0 \setminus V) = \lim_{H \to \mathcal{F}} \mu(H_0 \setminus H) \le \epsilon. \text{ } \mathbf{Q}$$

(d)(i) If $\mu \in P$ and $V \in \mathcal{N}(\mu)$, then $\ddagger(V, \mu)$ is true. **P** Let $\epsilon > 0$. There is an $E \in \operatorname{dom} \mu$, including V, such that $\mu E = 0$; now there is a compact $K \in \operatorname{dom} \mu$, included in $X \setminus E$, such that

$$\epsilon \ge \mu((X \setminus E) \setminus K) = \mu(X \setminus K),$$

while $K \cap V = \emptyset$ is compact. **Q**

(ii) If $\mu \in P$ and $V \subseteq X$ is closed, then $\ddagger(V, \mu)$ is true. **P** For every $\epsilon > 0$, there is a compact $K \in \operatorname{dom} \mu$ such that $\mu(X \setminus K) \le \epsilon$, and now $K \cap V$ is compact. **Q**

(iii) Now suppose that $\mu \in P$ is such that every compact subset of X belongs to the domain Σ of μ, and that $\langle E_n \rangle_{n \in \mathbb{N}}$ is a sequence in Σ with intersection V. Then $\ddagger(V, \mu)$ is true. **P** Let $\epsilon > 0$. For each $n \in \mathbb{N}$, there are compact sets $K_n \subseteq E_n$, $K'_n \subseteq X \setminus E_n$ such that

$$\mu(E_n \setminus K_n) + \mu((X \setminus E_n) \setminus K'_n) \le 2^{-n-1}\epsilon.$$

Set $K = \bigcap_{n \in \mathbb{N}} K_n \cup K'_n$; then K is compact, so belongs to Σ. If $L \subseteq X \setminus K$ is compact, then there is an $n \in \mathbb{N}$ such that $L \cap \bigcap_{i \le n} K_i \cup K'_i$ is empty, so that

$$\mu L \le \sum_{i=0}^{n} \mu(X \setminus (K_i \cup K'_i)) \le \sum_{i=0}^{n} 2^{-n-1}\epsilon \le \epsilon.$$

As L is arbitrary, $\mu(X \setminus K) \le \epsilon$. Finally,

$$K \cap V = \bigcap_{n \in \mathbb{N}} (K_n \cup K'_n) \cap E_n = \bigcap_{n \in \mathbb{N}} K_n$$

is compact. **Q**

(e) If $Q \subseteq P$ is a non-empty totally ordered subset of P, $\bigcup Q \in P$. So P has a maximal element μ_1, which is a submeasure, satisfying (α), and extending μ_0. Setting $\Sigma_1 = \operatorname{dom} \mu_1$, (c) tells us that $V \in \Sigma_1$ whenever $V \subseteq X$ and $\ddagger(V, \mu_1)$ is true. By (d-i), $\mathcal{N}(\mu_1) \subseteq \Sigma_1$ and μ_1 is complete. By (d-ii), every closed set, and therefore every open set, belongs to Σ_1. So (d-iii) tells us that $\bigcap_{n \in \mathbb{N}} E_n \in \Sigma_1$ for every sequence $\langle E_n \rangle_{n \in \mathbb{N}}$ in Σ_1, and Σ_1 is a σ-algebra. Putting these together, all the conditions of 496C are satisfied, and μ_1 is a totally finite Radon submeasure.

496I Theorem Let X be a set, Σ a σ-algebra of subsets of X, and μ a complete Maharam submeasure on Σ.

(a) Σ is closed under Souslin's operation.

(b) If A is the kernel of a Souslin scheme $\langle E_\sigma \rangle_{\sigma \in S}$ in Σ, and $\epsilon > 0$, there is a $\psi \in \mathbb{N}^{\mathbb{N}}$ such that

$$\mu(A \setminus \bigcup_{\phi \in \mathbb{N}^{\mathbb{N}}, \phi \le \psi} \bigcap_{n \ge 1} E_{\phi \restriction n}) \le \epsilon.$$

proof (a) Let $\mathcal{N}(\mu)$ be the null ideal of μ. Because μ is exhaustive, every disjoint sequence in $\Sigma \setminus \mathcal{N}(\mu)$ is countable, so 431G tells us that Σ is closed under Souslin's operation.

(b) The argument of 431D applies, with trifling modifications in its expression. For $\sigma \in S = \bigcup_{k \in \mathbb{N}} \mathbb{N}^k$, set $A_\sigma = \bigcup_{\sigma \subseteq \phi \in \mathbb{N}^{\mathbb{N}}} \bigcap_{n \ge 1} E_{\phi \restriction n}$; then $A_\sigma \in \Sigma$, by (a). Given $\epsilon > 0$, let $\langle \epsilon_\sigma \rangle_{\sigma \in S}$ be a family of strictly positive real numbers such that $\sum_{\sigma \in S} \epsilon_\sigma \le \epsilon$. For each $\sigma \in S$, let m_σ be such that $\mu(A_\sigma \setminus \bigcup_{i \le m_\sigma} A_{\sigma^\frown <i>}) \le \epsilon_\sigma$. Set

$$\psi(k) = \max\{m_\sigma : \sigma \in \mathbb{N}^k,\ \sigma(i) \le \psi(i) \text{ for every } i < k\}$$

for $k \in \mathbb{N}$; then

$$A \setminus \bigcup_{\phi \in \mathbb{N}^{\mathbb{N}}, \phi \le \psi} \bigcap_{n \ge 1} E_{\phi \restriction n} \subseteq \bigcup_{\sigma \in S} \big(A_\sigma \setminus \bigcup_{i \le m_\sigma} A_{\sigma^\frown <i>}\big)$$

has submeasure at most ϵ.

496J Theorem Let X be a K-analytic Hausdorff space and μ a Maharam submeasure defined on the Borel σ-algebra of X. Then

$$\inf\{\mu(X \setminus K) : K \subseteq X \text{ is compact}\} = 0.$$

proof Again, we have only to re-use the ideas of 432B. Let $\hat{\mu}$ be the completion of μ (496A) and Σ the domain of $\hat{\mu}$. Let $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ be an usco-compact relation such that $R[\mathbb{N}^{\mathbb{N}}] = X$. For $\sigma \in S^* = \bigcup_{n\ge 1} \mathbb{N}^n$ set $I_\sigma = \{\phi : \sigma \subseteq \phi \in \mathbb{N}^{\mathbb{N}}\}$, $F_\sigma = \overline{R[I_\sigma]}$; then X is the kernel of the Souslin scheme $\langle F_\sigma\rangle_{\sigma\in S^*}$. Now, given $\epsilon > 0$, 496Ib tells us that there is a $\psi \in \mathbb{N}^{\mathbb{N}}$ such that $\mu(X \setminus F) \le \epsilon$, where $F = \bigcup_{\phi\in\mathbb{N}^{\mathbb{N}},\phi\le\psi} \bigcap_{n\ge 1} F_{\phi\restriction n}$; but $F = R[K]$ where K is the compact set $\{\phi : \phi \in \mathbb{N}^{\mathbb{N}}, \phi \le \psi\}$, so F is compact.

496K Proposition Let μ be a Maharam submeasure on the Borel σ-algebra of an analytic Hausdorff space X. Then the completion of μ is a totally finite Radon submeasure on X.

proof If $E \subseteq X$ is Borel, then it is K-analytic (423Eb); applying 496J to $\mu\restriction\mathcal{P}E$, we see that $\inf\{\mu(E\setminus K) : K \subseteq E \text{ is compact}\} = 0$. Consequently, writing Σ for the domain of the completion $\hat{\mu}$ of μ, $\inf\{\hat{\mu}(E \setminus K) : K \subseteq E \text{ is compact}\} = 0$ for every $E \in \Sigma$. Condition (i) of the definition 496C is surely satisfied by $\hat{\mu}$, so $\hat{\mu}$ is a totally finite Radon submeasure.

496L Free products of Maharam algebras If $\mathfrak{A}$, $\mathfrak{B}$ are Boolean algebras with submeasures μ, ν respectively, we have a submeasure $\mu \ltimes \nu$ on the free product $\mathfrak{A} \otimes \mathfrak{B}$ (392K). It is easy to see, in 392K, that if μ and ν are strictly positive so is $\mu \ltimes \nu$; moreover, if μ and ν are exhaustive so is $\mu \ltimes \nu$ (392Ke).

Now suppose that $\langle\mathfrak{A}_i\rangle_{i\in I}$ is a family of Maharam algebras, where I is a finite totally ordered set. Then we can take a strictly positive Maharam submeasure μ_i on each $\mathfrak{A}_i$, form an exhaustive submeasure λ on $\mathfrak{C}_I = \bigotimes_{i\in I} \mathfrak{A}_i$, and use λ to construct a metric completion $\widehat{\mathfrak{C}}_I$ which is a Maharam algebra, as in 393H/496Bb. (If $I = \{i_0, \ldots , i_n\}$ where $i_0 < \ldots < i_n$, then $\lambda = (.(\mu_{i_0} \ltimes \mu_{i_1}) \ltimes \ldots) \ltimes \mu_{i_n}$ (392Kf). By 392Kc, the product is associative, so the arrangement of the brackets is immaterial.) If we change each μ_i to μ_i', where μ_i' is another strictly positive Maharam submeasure on $\mathfrak{A}_i$, then every μ_i' is absolutely continuous with respect to μ_i (393F/496Bd), so the corresponding λ' will be absolutely continuous with respect to λ, and vice versa (392Kd); in which case the metrics on $\mathfrak{C}_I$ are uniformly equivalent and we get the same metric completion $\widehat{\mathfrak{C}}_I$ up to Boolean algebra isomorphism. We can therefore think of $\widehat{\mathfrak{C}}_I$ as 'the' **Maharam algebra free product** of the family $\langle\mathfrak{A}_i\rangle_{i\in I}$ of Boolean algebras; as in 392Kf, we shall have an isomorphism between $\widehat{\mathfrak{C}}_{J\cup K}$ and the Maharam algebra free product of $\widehat{\mathfrak{C}}_J$ and $\widehat{\mathfrak{C}}_K$ whenever J, $K \subseteq I$ and $j < k$ for every $j \in J$ and $k \in K$.

From 392Kg we see that if $(\mathfrak{A}, \mu)$ and $(\mathfrak{B}, \nu)$ are probability algebras, then their Maharam algebra free product, regarded as a Boolean algebra, is isomorphic to their probability algebra free product as defined in §325.

496M Representing products of Maharam algebras: Theorem Let X and Y be sets, with σ-algebras Σ and T and Maharam submeasures μ and ν defined on Σ, T respectively. Let $\mathfrak{A}$, $\mathfrak{B}$ be their Maharam algebras and write $\bar{\mu}$, $\bar{\nu}$ for the strictly positive Maharam submeasures on $\mathfrak{A}$ and $\mathfrak{B}$ induced by μ and ν as in 496Ba above. Let $\Sigma\widehat{\otimes}\mathrm{T}$ be the σ-algebra of subsets of $X \times Y$ generated by $\{E \times F : E \in \Sigma, F \in \mathrm{T}\}$.

(a) (Compare 418T.) Give $\mathfrak{B}$ its Maharam-algebra topology (393G/496Bd). If $W \in \Sigma\widehat{\otimes}\mathrm{T}$ then $W[\{x\}] \in \mathrm{T}$ for every $x \in X$ and the function $x \mapsto W[\{x\}]^\bullet : X \to \mathfrak{B}$ is Σ-measurable and has separable range. Consequently $x \mapsto \nu W[\{x\}] : X \to [0,\infty[$ is Σ-measurable.

(b) For $W \in \Sigma\widehat{\otimes}\mathrm{T}$ set

$$\lambda W = \inf\{\epsilon : \epsilon > 0,\ \mu\{x : \nu W[\{x\}] > \epsilon\} \le \epsilon\}.$$

Then λ is a Maharam submeasure on $\Sigma\widehat{\otimes}\mathrm{T}$, and

$$\lambda^{-1}[\{0\}] = \{W : W \in \Sigma\widehat{\otimes}\mathrm{T},\ \{x : W[\{x\}] \notin \mathcal{N}(\nu)\} \in \mathcal{N}(\mu)\}.$$

(c) Let $\mathfrak{C}$ be the Maharam algebra of λ. Then $\mathfrak{A} \otimes \mathfrak{B}$ can be embedded in $\mathfrak{C}$ by mapping $E^\bullet \otimes F^\bullet$ to $(E \times F)^\bullet$ for all $E \in \Sigma$ and $F \in \mathrm{T}$.

(d) This embedding identifies $(\mathfrak{C}, \bar{\lambda})$ with the metric completion of $(\mathfrak{A} \otimes \mathfrak{B}, \bar{\mu} \ltimes \bar{\nu})$.

proof (a) Write $\mathcal{W}$ for the set of those $W \subseteq X \times Y$ such that $W[\{x\}] \in \mathrm{T}$ for every $x \in X$ and $x \mapsto W[\{x\}]^\bullet : X \to \mathfrak{B}$ is Σ-measurable and has separable range. Then $\Sigma \otimes \mathrm{T}$ (identified with the algebra of subsets of $X \times Y$ generated by $\{E \times F : E \in \Sigma, F \in \mathrm{T}\}$) is included in $\mathcal{W}$.

If $\langle W_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathcal{W}$ with union W, then $W \in \mathcal{W}$. **P** Of course $W[\{x\}] = \bigcup_{n\in\mathbb{N}} W_n[\{x\}]$ belongs to T for every $x \in X$. Set $f_n(x) = W_n[\{x\}]^\bullet$ for $n \in \mathbb{N}$ and $x \in X$. For each $x \in X$, $W[\{x\}] \setminus W_n[\{x\}]$ is a non-increasing sequence with empty intersection, so $\lim_{n\to\infty} \nu(W[\{x\}] \setminus W_n[\{x\}]) = 0$ and $\langle f_n(x)\rangle_{n\in\mathbb{N}}$ converges to $f(x) = W[\{x\}]^\bullet$ in $\mathfrak{B}$. By 418Ba, f is measurable. Also $D = \overline{\{f_n(x) : x \in X, n \in \mathbb{N}\}}$ is a separable subspace of $\mathfrak{B}$ including $f[X]$. So $W \in \mathcal{W}$. **Q**

Similarly, $\bigcap_{n\in\mathbb{N}} W_n \in \mathcal{W}$ for any non-increasing sequence $\langle W_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{W}$. $\mathcal{W}$ therefore includes the σ-algebra generated by $\Sigma \otimes \mathrm{T}$ (136G), which is $\Sigma\widehat{\otimes}\mathrm{T}$.

Now $x \mapsto \nu W[\{x\}] = \bar\nu W[\{x\}]^\bullet$ is measurable because $\bar\nu : \mathfrak{B} \to \mathbb{R}$ is continuous.

(b) Of course $\lambda\emptyset = 0$ and $\lambda W \le \lambda W'$ if $W, W' \in \Sigma\widehat{\otimes}\mathrm{T}$ and $W \subseteq W'$. If $W_1, W_2 \in \Sigma\widehat{\otimes}\mathrm{T}$ have union W, $\lambda W_1 = \alpha_1$ and $\lambda W_2 = \alpha_2$, then

$$\{x : \nu W[\{x\}] > \alpha_1 + \alpha_2\} \subseteq \{x : \nu W_1[\{x\}] > \alpha_1\} \cup \{x : \nu W_2[\{x\}] > \alpha_2\},$$

so, setting $\alpha = \alpha_1 + \alpha_2$,

$$\mu\{x : \nu W[\{x\}] > \alpha\} \le \mu\{x : \nu W_1[\{x\}] > \alpha_1\} + \mu\{x : \nu W_2[\{x\}] > \alpha_2\} \le \alpha_1 + \alpha_2 = \alpha,$$

and $\lambda W \le \alpha$. Thus λ is monotonic and subadditive.

If now $\langle W_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\Sigma\widehat{\otimes}\mathrm{T}$ with empty intersection, and $\epsilon > 0$, set $E_n = \{x : \nu W_n[\{x\}] \ge \epsilon\}$ for each n. Then $\langle E_n\rangle_{n\in\mathbb{N}}$ is non-increasing; moreover, for any $x \in X$, $\langle W_n[\{x\}]\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in T with empty intersection, so $\lim_{n\to\infty} \nu W_n[\{x\}] = 0$ and $x \notin \bigcap_{n\in\mathbb{N}} E_n$. There is therefore an n such that $\mu E_n \le \epsilon$ and $\lambda W_n \le \epsilon$. As $\langle W_n\rangle_{n\in\mathbb{N}}$ and ϵ are arbitrary, λ is a Maharam submeasure.

Finally, for $W \in \Sigma\widehat{\otimes}\mathrm{T}$,

$$\begin{aligned}
\lambda W = 0 &\iff \mu\{x : \nu W[\{x\}] \ge 2^{-n}\} \le 2^{-n} \text{ for every } n \in \mathbb{N} \\
&\iff \mu\{x : \nu W[\{x\}] \ge 2^{-m}\} \le 2^{-n} \text{ for every } m, n \in \mathbb{N} \\
&\iff \mu\{x : \nu W[\{x\}] > 0\} \le 2^{-n} \text{ for every } n \in \mathbb{N} \\
&\iff \mu\{x : \nu W[\{x\}] > 0\} = 0 \iff \{x : W[\{x\}] \notin \mathcal{N}(\nu)\} \in \mathcal{N}(\mu).
\end{aligned}$$

(c) If $E \in \Sigma$, then $\lambda(E \times F) = \min(\mu E, \nu F)$ for all $E \in \Sigma$ and $F \in \mathrm{T}$. So $\lambda(E \times F) = 0$ iff $E^\bullet \otimes F^\bullet = 0$ in $\mathfrak{A} \otimes \mathfrak{B}$. Consequently we have injective Boolean homomorphisms from $\mathfrak{A}$ to $\mathfrak{C}$ and from $\mathfrak{B}$ to $\mathfrak{C}$ defined by the formulae

$$E^\bullet \mapsto (E \times Y)^\bullet \text{ for } E \in \Sigma, \quad F^\bullet \mapsto (X \times F)^\bullet \text{ for } F \in \mathrm{T};$$

by 315J and 315Kb[8], we have an injective Boolean homomorphism from $\mathfrak{A} \otimes \mathfrak{B}$ to $\mathfrak{C}$ which maps $E^\bullet \otimes F^\bullet$ to $(E \times F)^\bullet$ whenever $E \in \Sigma$ and $F \in \mathrm{T}$.

(d) $\bar\lambda(\phi e) = (\mu \ltimes \nu)(e)$ for every $e \in \mathfrak{A} \otimes \mathfrak{B}$. **P** Express e as $\sup_{i\in I} a_i \otimes b_i$ where $\langle a_i\rangle_{i\in I}$ is a finite partition of unity in $\mathfrak{A}$ and $b_i \in \mathfrak{B}$ for each i. For each i, we can express a_i, b_i as $E_i^\bullet$, $F_i^\bullet$ where $E_i \in \Sigma$ and $F_i \in \mathrm{T}$; moreover, we can do this in such a way that $\langle E_i\rangle_{i\in I}$ is a partition of X. In this case, $\phi e = W^\bullet$ where $W = \bigcup_{i\in I} E_i \times F_i$, so that, for $\epsilon > 0$,

$$\mu\{x : \nu W[\{x\}] > \epsilon\} = \mu(\bigcup\{E_i : i \in I, \nu F_i > \epsilon\}) = \bar\mu(\sup\{a_i : i \in I, \bar\nu b_i > \epsilon\}).$$

Accordingly

$$\begin{aligned}
(\mu \ltimes \nu)(e) &= \inf\{\epsilon : \bar\mu(\sup\{a_i : i \in I, \bar\nu b_i > \epsilon\}) \le \epsilon\} \\
&= \inf\{\epsilon : \mu\{x : \nu W[\{x\}] > \epsilon\} \le \epsilon\} = \lambda W = \bar\lambda W^\bullet = \bar\lambda(\phi e). \ \mathbf{Q}
\end{aligned}$$

Next, $\phi[\mathfrak{A} \otimes \mathfrak{B}]$ is dense in $\mathfrak{C}$ for the metric induced by $\bar\lambda$. **P** Let $\mathfrak{D}$ be the metric closure of $\phi[\mathfrak{A} \otimes \mathfrak{B}]$ and set $\mathcal{V} = \{V : V \in \Sigma \otimes \mathrm{T}, V^\bullet \in \mathfrak{D}\}$. Then $\mathcal{V}$ includes $\Sigma \otimes \mathrm{T}$ and is closed under unions and intersections

[8] Formerly 315I-315J.

of monotonic sequences, so is the whole of $\Sigma\widehat{\otimes}\mathrm{T}$, and $\mathfrak{D} = \mathfrak{C}$, as required. **Q** But this means that we can identify $\mathfrak{C}$ with the metric completions of $\phi[\mathfrak{A} \otimes \mathfrak{B}]$ and $\mathfrak{A} \otimes \mathfrak{B}$.

496X Basic exercises (a) Let X and Y be Hausdorff spaces, μ a totally finite Radon submeasure on X, and $f : X \to Y$ a function which is almost continuous in the sense that for every $\epsilon > 0$ there is a compact $K \subseteq X$ such that $f{\restriction}K$ is continuous and $\mu(X \setminus K) \le \epsilon$. Show that the image submeasure μf^{-1}, defined on $\{F : F \subseteq Y,\ f^{-1}[F] \in \operatorname{dom}\mu\}$, is a totally finite Radon submeasure on Y.

(b) Let X be a Hausdorff space and μ a totally finite Radon submeasure on X. For $A \subseteq X$, set $\mu^*A = \inf\{\mu E : A \subseteq E \in \operatorname{dom}\mu\}$. Show that μ^* is an outer regular Choquet capacity on X.

(c) Let X and Y be compact Hausdorff spaces, $f : X \to Y$ a continuous surjection, and ν a totally finite Radon submeasure on Y. Show that there is a totally finite Radon submeasure μ on X such that ν is the image submeasure μf^{-1}.

(d) Let X be a regular K-analytic Hausdorff space, and μ a Maharam submeasure on the Borel σ-algebra of X which is τ-additive in the sense that whenever $\mathcal{G}$ is a non-empty upwards-directed family of open sets in X with union H, then $\inf_{G\in\mathcal{G}} \mu(H \setminus G) = 0$. Show that the completion of μ is a totally finite Radon submeasure on X. (*Hint*: let Σ_0 be the algebra of subsets of X generated by the compact sets; show that there is a totally finite Radon submeasure extending $\mu{\restriction}\Sigma_0$.)

496Y Further exercises In the following exercises, I will say that a **Radon submeasure** is a complete submeasure μ on a Hausdorff space X such that (i) the domain Σ of μ is a σ-algebra of subsets of X containing every open set (ii) every point of X belongs to an open set G such that $\mu G < \infty$ (iii)(α) $\mu E = \sup\{\mu K : K \subseteq E \text{ is compact}\}$ for every $E \in \Sigma$ (β) $\inf\{\mu(E \setminus K) : K \subseteq E \text{ is compact}\} = 0$ whenever $E \in \Sigma$ and $\mu E < \infty$ (iv) if $E \subseteq X$ is such that $E \cap K \in \Sigma$ for every compact $K \subseteq X$, then $E \in \Sigma$.

(a) Let μ be a Radon submeasure with domain Σ and null ideal $\mathcal{N}(\mu)$. Show that $\Sigma/\mathcal{N}(\mu)$ is Dedekind complete.

(b) Let X be a Hausdorff space, Y a metrizable space, μ a Radon submeasure on X with domain Σ, and $f : X \to Y$ a Σ-measurable function. Let $\mathcal{H}$ be the family of those $H \in \Sigma$ such that $f{\restriction}H$ is continuous. Show that (α) $\mu E = \sup\{\mu H : H \in \mathcal{H},\ H \subseteq E\}$ for every $E \in \Sigma$ (β) $\inf\{\mu(E \setminus H) : H \in \mathcal{H},\ H \subseteq E\} = 0$ whenever $E \in \Sigma$ and $\mu E < \infty$.

(c) Let X and Y be Hausdorff spaces, μ a Radon submeasure on X with domain Σ, and $f : X \to Y$ a function. Let $\mathcal{F}$ be the family of those $F \in \Sigma$ such that $f{\restriction}F$ is continuous, and suppose that (α) $\mu E = \sup\{\mu F : F \in \mathcal{F},\ F \subseteq E\}$ for every $E \in \Sigma$ (β) $\inf\{\mu(E \setminus F) : F \in \mathcal{F},\ F \subseteq E\} = 0$ whenever $E \in \Sigma$ and $\mu E < \infty$. (i) Show that the image submeasure $\nu = \mu f^{-1}$, defined on $\{F : F \subseteq Y,\ f^{-1}[F] \in \Sigma\}$, is a submeasure on Y defined on a σ-algebra of sets containing every open subset of Y. (ii) Show that if ν is locally finite in the sense that $Y = \bigcup\{H : H \subseteq Y \text{ is open},\ \nu H < \infty\}$, then ν is a Radon submeasure.

(d) Let X be a Hausdorff space and μ a Radon submeasure on X which is either submodular or supermodular. Show that there is a Radon measure on X with the same domain and null ideal as μ. (*Hint*: 413Yh.)

(e) Let X be a topological space, $\mathcal{G}$ the family of cozero subsets of X, $\mathcal{B}\mathfrak{a}$ the Baire σ-algebra of X and $\psi : \mathcal{G} \to [0,\infty[$ a functional. Show that ψ can be extended to a Maharam submeasure with domain $\mathcal{B}\mathfrak{a}$ iff

(α) $\psi G \le \psi H$ whenever $G,\ H \in \mathcal{G}$ and $G \subseteq H$,

(β) $\psi(\bigcup_{n\in\mathbb{N}} G_n) \le \sum_{n=0}^{\infty} \psi G_n$ for every sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{G}$,

(γ) $\lim_{n\to\infty} \psi G_n = 0$ for every non-increasing sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{G}$ with empty intersection,

and that in this case the extension is unique. (*Hint*: consider the family of sets $E \subseteq X$ such that for every $\epsilon > 0$ there are a cozero set $G \supseteq E$ and a zero set $F \subseteq E$ such that $\psi(G \setminus F) \le \epsilon$.)

(f) Let X be a Hausdorff space and $\mathcal{K}$ the family of compact subsets of X. Let $\phi : \mathcal{K} \to [0,\infty[$ be a functional such that

(α) $\phi\emptyset = 0$ and $\phi K \le \phi(K \cup L) \le \phi K + \phi L$ for all K, $L \in \mathcal{K}$;

(β) for every disjoint sequence $\langle K_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$, either $\lim_{n\to\infty} \phi K_n = 0$ or $\lim_{n\to\infty} \phi(\bigcup_{i\le n} K_i) = \infty$;

(γ) whenever K, $L \in \mathcal{K}$ and $K \subseteq L$ then $\phi L \le \phi K + \sup\{\phi K' : K' \in \mathcal{K},\ K' \subseteq L \setminus K\}$;

(δ) for every $x \in X$ there is an open set G containing x such that $\sup\{\phi K : K \in \mathcal{K},\ K \subseteq G\}$ is finite.

Show that there is a unique Radon submeasure on X extending ϕ.

496 Notes and comments 'Submeasures' turn up in all sorts of places, if you are looking out for them; so, as always, I have tried to draw my definitions as wide as practicable. When we come to 'Maharam' and 'Radon' submeasures, however, we certainly want to begin with results corresponding to the familiar properties of totally finite measures, and the new language is complex enough without troubling with infinite submeasures. For the main part of this section, therefore, I look only at totally finite submeasures.

I have tried here to give a sample of the ideas from the present volume which can be applied to submeasures as well as to measures. I think they go farther than most of us would take for granted. One key point concerns the definition of inner regularity: to the familiar '$\mu E = \sup\{\mu K : K \in \mathcal{K},\ K \subseteq E\}$' we need to add 'if μE is finite, then $\inf\{\mu(E \setminus K) : K \in \mathcal{K},\ K \subseteq E\} = 0$' (496C, 496Y; see also condition (β) of 496Yf). Using this refinement, we can repeat a good proportion of the arguments of measure theory which are based on topology and orderings rather than on arithmetic identities.

Version of 7.12.10

497 Tao's proof of Szemerédi's theorem

Szemerédi's celebrated theorem on arithmetic progressions (497L) is not obviously part of measure theory. Remarkably, however, it has stimulated significant developments in the subject. The first was Furstenberg's multiple recurrence theorem (Furstenberg 77, Furstenberg 81, Furstenberg & Katznelson 85). In this section I will give an account of an approach due to T.Tao (Tao 07) which introduces another phenomenon of great interest from a measure-theoretic point of view.

497A Definitions (a) Let (X, Σ, μ) be a probability space, T a subalgebra of Σ (*not* necessarily a σ-subalgebra) and $\langle \Sigma_i \rangle_{i\in I}$ a family of σ-subalgebras of Σ. I will say that $\langle \Sigma_i \rangle_{i\in I}$ has **T-removable intersections** if whenever $J \subseteq I$ is finite and not empty, $E_i \in \Sigma_i$ for $i \in J$, $\mu(\bigcap_{i\in J} E_i) = 0$ and $\epsilon > 0$, there is a family $\langle F_i \rangle_{i\in J}$ such that $F_i \in \mathrm{T} \cap \Sigma_i$ and $\mu(E_i \setminus F_i) \le \epsilon$ for each $i \in J$, and $\bigcap_{i\in J} F_i = \emptyset$. (This is a stronger version of what Tao 07 calls the 'uniform intersection property'.)

(b) If X is a set and Σ, Σ' are two σ-algebras of subsets of X, $\Sigma \vee \Sigma'$ will be the σ-algebra generated by $\Sigma \cup \Sigma'$. If $\langle \Sigma_i \rangle_{i\in I}$ is a family of σ-algebras of subsets of X, I will write $\bigvee_{i\in I} \Sigma_i$ for the σ-algebra generated by $\bigcup_{i\in I} \Sigma_i$.

(c) If (X, Σ, μ) is a probability space and $\mathcal{A} \subseteq \mathcal{E} \subseteq \Sigma$, I will say that $\mathcal{A}$ is **metrically dense** in $\mathcal{E}$ if for every $E \in \mathcal{E}$ and $\epsilon > 0$ there is an $F \in \mathcal{A}$ such that $\mu(E \triangle F) \le \epsilon$; that is, if $\{F^\bullet : F \in A\}$ is dense in $\{E^\bullet : E \in \mathcal{E}\}$ for the measure-algebra topology on the measure algebra of μ (323A). Note that a subalgebra of Σ is metrically dense in the σ-algebra it generates (compare 323J).

497B Lemma Let (X, Σ, μ) be a probability space and T a subalgebra of Σ. Let $\langle \Sigma_i \rangle_{i\in I}$ be a family of σ-subalgebras of Σ.

(a) $\langle \Sigma_i \rangle_{i\in I}$ has T-removable intersections iff $\langle \Sigma_i \rangle_{i\in J}$ has T-removable intersections for every finite $J \subseteq I$.

(b) Suppose that $\langle \Sigma_i \rangle_{i\in I}$ has T-removable intersections and that $\mathrm{T} \cap \Sigma_i$ is metrically dense in Σ_i for every i. Let J be any set and $f : J \to I$ a function. Then $\langle \Sigma_{f(j)} \rangle_{j\in J}$ has T-removable intersections.

(c) Suppose that, for each $i \in I$, we are given a σ-subalgebra Σ'_i of Σ_i such that for every $E \in \Sigma_i$ there is an $E' \in \Sigma'_i$ such that $E \triangle E'$ is negligible. If $\langle \Sigma'_i \rangle_{i\in I}$ has T-removable intersections, so has $\langle \Sigma_i \rangle_{i\in I}$.

proof (a) is trivial.

(b) Suppose that $K \subseteq J$ is finite and not empty, that $\langle E_j \rangle_{j\in K} \in \prod_{j\in K} \Sigma_{f(j)}$ is such that $\mu(\bigcap_{j\in K} E_j) = 0$, and $\epsilon > 0$. Set $n = \#(K)$ and $\eta = \frac{\epsilon}{n+2} > 0$. Set $E'_i = \bigcap_{j\in K, f(j)=i} E_j \in \Sigma_i$ for $i \in f[K]$; then $\bigcap_{i\in f[K]} E'_i = \bigcap_{j\in K} E_j$ is negligible, so we have $F'_i \in \mathrm{T} \cap \Sigma_i$, for $i \in f[K]$, such that $\bigcap_{i\in f[K]} F'_i = \emptyset$ and $\mu(E'_i \setminus F'_i) \le \eta$ for every $i \in f[K]$. As $\mathrm{T} \cap \Sigma_i$ is metrically dense in Σ_i for each i, we can find $G_j \in \mathrm{T} \cap \Sigma_{f(j)}$ such that $\mu(E_j \triangle G_j) \le \eta$ for each $j \in K$. Set $G'_i = \bigcap_{j\in K, f(j)=i} G_j$ for $i \in f[K]$. Then

$$\mu(G'_i \setminus F'_i) \le \mu(G'_i \setminus E'_i) + \mu(E'_i \setminus F'_i) \le \textstyle\sum_{j\in K, f(j)=i} \mu(G_j \setminus E_j) + \eta \le (n+1)\eta.$$

Note that $G'_i \in \mathrm{T} \cap \Sigma_i$ for each i. Now set $F_j = G_j \setminus (G'_{f(j)} \setminus F'_{f(j)})$ for $j \in K$. Then $F_j \in \mathrm{T} \cap \Sigma_{f(j)}$ and

$$\mu(E_j \setminus F_j) \le \mu(E_j \setminus G_j) + \mu(G'_{f(j)} \setminus F'_{f(j)}) \le (n+2)\eta = \epsilon.$$

Also

$$\begin{aligned}\bigcap_{j\in K} F_j &= \bigcap_{i\in f[K]} \ \bigcap_{\substack{j\in K\\ f(j)=i}} G_j \setminus (G'_i \setminus F'_i)\\ &= \bigcap_{i\in f[K]} G'_i \setminus (G'_i \setminus F'_i) \subseteq \bigcap_{i\in f[K]} F'_i = \emptyset.\end{aligned}$$

As $\langle E_j \rangle_{j\in K}$ and ϵ are arbitrary, $\langle \Sigma_{f(j)} \rangle_{j\in J}$ has T-removable intersections.

(c) If $J \subseteq I$ is finite and not empty, $\langle E_j \rangle_{j\in J} \in \prod_{j\in J} \Sigma_j$, $\bigcap_{j\in J} E_j$ is negligible and $\epsilon > 0$, then for each $j \in J$ let $E'_j \in \Sigma'_j$ be such that $E'_j \triangle E_j$ is negligible. In this case, $\bigcap_{j\in J} E'_j$ is negligible, so there are $F_j \in \mathrm{T} \cap \Sigma'_j$, for $j \in J$, such that $\mu(E'_j \setminus F_j) \le \epsilon$ for every $j \in J$ and $\bigcap_{j\in J} F_j$ is empty. Now $\mu(E_j \setminus F_j) \le \epsilon$ for every j. As $\langle E_j \rangle_{j\in J}$ and ϵ are arbitrary, $\langle \Sigma_i \rangle_{i\in I}$ has T-removable intersections.

497C Lemma Let (X, Σ, μ) be a probability space and T a subalgebra of Σ. Let I be a set, A an upwards-directed set, and $\langle \Sigma_{\alpha i} \rangle_{\alpha\in A, i\in I}$ a family of σ-subalgebras of Σ such that, setting $\Sigma_i = \bigvee_{\alpha\in A} \Sigma_{\alpha i}$ for each i,

(i) $\Sigma_{\alpha i} \subseteq \Sigma_{\beta i}$ whenever $i \in I$ and $\alpha \le \beta$ in A,
(ii) $\langle \Sigma_{\alpha i} \rangle_{i\in I}$ has T-removable intersections for every $\alpha \in A$,
(iii) Σ_i and $\bigvee_{j\in I} \Sigma_{\alpha j}$ are relatively independent over $\Sigma_{\alpha i}$ for every $i \in I$ and $\alpha \in A$.

Then $\langle \Sigma_i \rangle_{i\in I}$ has T-removable intersections.

proof Take a non-empty finite set $J \subseteq I$, a family $\langle E_i \rangle_{i\in J}$ such that $E_i \in \Sigma_i$ for every $i \in J$ and $\bigcap_{i\in J} E_i$ is negligible, and $\epsilon > 0$. Set $\delta = \frac{1}{\#(J)+1}$, $\eta = \delta\sqrt{\frac{\epsilon}{2}} > 0$. For each $i \in J$ there are an $\alpha \in A$ and an $E'_i \in \Sigma_{\alpha i}$ such that $\mu(E_i \triangle E'_i) \le \eta^2$ (because A is upwards-directed, so $\bigcup_{\alpha\in A} \Sigma_{\alpha i}$ is a subalgebra of Σ and is metrically dense in Σ_i); we can suppose that it is the same α for each i. Let $g_i : X \to [0,1]$ be a $\Sigma_{\alpha i}$-measurable function which is a conditional expectation of χE_i on $\Sigma_{\alpha i}$; then

$$\|\chi E_i - g_i\|_2 \le \|\chi E_i - \chi E'_i\|_2 \le \eta$$

(cf. 244Nb). Set $E''_i = \{x : g_i(x) \ge 1 - \delta\} \in \Sigma_{\alpha i}$; then

$$\mu(E_i \setminus E''_i) = \mu\{x : \chi E_i(x) - g_i(x) > \delta\} \le \frac{\eta^2}{\delta^2} = \frac{\epsilon}{2}.$$

Set $E = \bigcap_{i\in J} E''_i$. Then $\mu E = 0$. **P** Since $\mu(\bigcap_{i\in J} E_i) = 0$, $\mu E \le \sum_{i\in J} \mu(E \setminus E_i)$. For $i \in J$, set $H_i = X \cap \bigcap_{j\in J\setminus\{i\}} E''_j$ and let h_i be a conditional expectation of χH_i on $\Sigma_{\alpha i}$. Then $X \setminus E_i \in \Sigma_i$ and $H_i \in \bigvee_{j\in I} \Sigma_{\alpha j}$ are relatively independent over $\Sigma_{\alpha i}$, while $\chi X - g_i$ is a conditional expectation of $\chi(X \setminus E_i)$ on $\Sigma_{\alpha i}$, so

$$\mu(E \setminus E_i) = \mu((E''_i \setminus E_i) \cap H_i) = \int_{E''_i} \chi(X \setminus E_i) \times \chi H_i d\mu = \int_{E''_i} (\chi X - g_i) \times h_i d\mu$$

(by the definition of 'relative independence', 458Aa)

$$= \int (\chi X - g_i) \times \chi E_i'' \times h_i d\mu \le \int \delta \chi E_i'' \times h_i d\mu$$

(by the definition of E_i'')

$$= \delta \int_{E_i''} h_i d\mu = \delta\mu(E_i'' \cap H_i) = \delta\mu E.$$

Summing, we have

$$\mu E \le \delta\#(J)\mu E;$$

but $\delta\#(J) < 1$, so $\mu E = 0$. **Q**

Because $\langle\Sigma_{\alpha i}\rangle_{i\in I}$ has T-removable intersections, there are $F_i \in \mathrm{T} \cap \Sigma_{\alpha i} \subseteq \mathrm{T} \cap \Sigma_i$, for $i \in J$, such that $\bigcap_{i\in I} F_i = \emptyset$ and $\mu(E_i'' \setminus F_i) \le \frac{\epsilon}{2}$ for each i; in which case $\mu(E_i \setminus F_i) \le \epsilon$ for each i. As $\langle E_i\rangle_{i\in J}$ and ϵ are arbitrary, $\langle\Sigma_i\rangle_{i\in I}$ has T-removable intersections.

497D Lemma Let (X, Σ, μ) be a probability space, T a subalgebra of Σ, and $\langle\Sigma_i\rangle_{i\in I}$ a finite family of σ-subalgebras of Σ which has T-removable intersections; suppose that $\mathrm{T} \cap \Sigma_i$ is metrically dense in Σ_i for each i. Set $\Sigma^* = \bigvee_{i\in I} \Sigma_i$. Suppose that we have a finite set Γ, a function $g : \Gamma \to I$ and a family $\langle\Lambda_\gamma\rangle_{\gamma\in\Gamma}$ of σ-subalgebras of Σ such that

$\langle\Lambda_\gamma\rangle_{\gamma\in\Gamma}$ is relatively independent over Σ^*,

for each $\gamma \in \Gamma$, Λ_γ and Σ^* are relatively independent over $\Sigma_{g(\gamma)}$,

for each $\gamma \in \Gamma$, $\mathrm{T} \cap \Lambda_\gamma$ is metrically dense in Λ_γ.

Let A be a finite set and $f : A \to I$, $\phi : A \to \mathcal{P}\Gamma$ functions such that $\Sigma_{g(\gamma)} \subseteq \Sigma_{f(\alpha)}$ whenever $\alpha \in A$ and $\gamma \in \phi(\alpha)$. Suppose that

for each $\alpha \in A$, $\bigvee_{\gamma\in\phi(\alpha)} \Lambda_\gamma$ and $\Sigma^* \vee \bigvee_{\gamma\in\Gamma\setminus\phi(\alpha)} \Lambda_\gamma$ are relatively independent over $\Sigma_{f(\alpha)}$.

Set $\tilde\Sigma_\alpha = \Sigma_{f(\alpha)} \vee \bigvee_{\gamma\in\phi(\alpha)} \Lambda_\gamma$ for $\alpha \in A$. Then $\langle\tilde\Sigma_\alpha\rangle_{\alpha\in A}$ has T-removable intersections.

proof Of course we can suppose that A is non-empty, and that $\Gamma = \bigcup_{\alpha\in A} \phi(\alpha)$.

(a) To begin with, suppose that every Λ_γ is actually a finite subalgebra of T.

(i) Take a non-empty set $B \subseteq A$, a family $\langle E_\alpha\rangle_{\alpha\in B} \in \prod_{\alpha\in B} \tilde\Sigma_\alpha$ such that $\bigcap_{\alpha\in B} E_\alpha$ is negligible, and $\epsilon > 0$. Set $\Delta = \bigcup_{\alpha\in B} \phi(\alpha)$. Let $\mathcal{A}$ be the set of atoms of $\bigvee_{\gamma\in\Delta} \Lambda_\gamma$ and set $\eta = \frac{\epsilon}{\#(\mathcal{A})} > 0$.

(ii) For each $H \in \mathcal{A}$ and $\alpha \in B$, let $C(H, \alpha)$ be the atom of $\bigvee_{\gamma\in\phi(\alpha)} \Lambda_\gamma$ including H. Then there is a family $\langle F_{H\alpha}\rangle_{\alpha\in B}$, with empty intersection, such that $F_{H\alpha} \in \mathrm{T} \cap \Sigma_\alpha$ and $\mu(E_\alpha \cap C(H, \alpha) \setminus F_{H\alpha}) \le \eta$ for each $\alpha \in B$. **P** For each $\gamma \in \Delta$, let H_γ be the atom of Λ_γ including H, $h_\gamma : X \to [0, 1]$ a $\Sigma_{g(\gamma)}$-measurable function which is a conditional expectation of χH_γ on $\Sigma_{g(\gamma)}$, and $G_\gamma = \{x : h_\gamma(x) > 0\}$. Note that

$$C(H, \alpha) = X \cap \textstyle\bigcap_{\gamma\in\phi(\alpha)} H_\gamma \text{ for every } \alpha \in B,$$

$$H = X \cap \textstyle\bigcap_{\gamma\in\Delta} H_\gamma = \bigcap_{\alpha\in B} C(H, \alpha).$$

Because Λ_γ and Σ^* are relatively independent over $\Sigma_{g(\gamma)}$, and $\Sigma_{g(\gamma)} \subseteq \Sigma^*$, h_γ is a conditional expectation of χH_γ on Σ^* for each γ (458Fb). Because $\langle\Lambda_\gamma\rangle_{\gamma\in\Delta}$ is relatively independent over Σ^*, $h = \prod_{\gamma\in\Delta} h_\gamma$ is a conditional expectation of $\chi H = \chi(X \cap \bigcap_{\gamma\in\Delta} H_\gamma)$ on Σ^*. (For the trivial case in which $\Delta = \emptyset$, take $h = \chi X$.) For each $\alpha \in B$ we have $E_\alpha \in \Sigma_{f(\alpha)} \vee \bigvee_{\gamma\in\phi(\alpha)} \Lambda_\gamma$, so there is an an $E_\alpha' \in \Sigma_{f(\alpha)}$ such that $E_\alpha \cap C(H, \alpha) = E_\alpha' \cap C(H, \alpha)$. Now $\bigcap_{\alpha\in B} E_\alpha' \in \Sigma^*$, so

$$\int_{\bigcap_{\alpha\in B} E_\alpha'} h = \mu(\textstyle\bigcap_{\alpha\in B} E_\alpha' \cap \bigcap_{\gamma\in\Delta} H_\gamma) \le \mu(\bigcap_{\alpha\in B}(E_\alpha \cap C(H, \alpha))) = 0;$$

accordingly $\bigcap_{\alpha\in B} E_\alpha' \cap \bigcap_{\gamma\in\Delta} G_\gamma$ is negligible. Set $E_\alpha'' = E_\alpha' \cap \bigcap_{\gamma\in\phi(\alpha)} G_\gamma$ for each $\alpha \in B$; then $E_\alpha'' \in \Sigma_{f(\alpha)}$, because $G_\gamma \in \Sigma_{g(\gamma)} \subseteq \Sigma_{f(\alpha)}$ whenever $\gamma \in \phi(\alpha)$. Also $\bigcap_{\alpha\in B} E_\alpha''$ is negligible.

Because $\langle\Sigma_i\rangle_{i\in I}$ has T-removable intersections and $\mathrm{T} \cap \Sigma_i$ is metrically dense in Σ_i for each i, $\langle\Sigma_{f(\alpha)}\rangle_{\alpha\in B}$ has T-removable intersections (497Bb). So we have $F_{H\alpha} \in \mathrm{T} \cap \Sigma_{f(\alpha)}$, for $\alpha \in B$, such that $\bigcap_{\alpha\in A} F_{H\alpha} = \emptyset$ and $\mu(E_\alpha'' \setminus F_{H\alpha}) \le \eta$ for every α.

If $\alpha \in B$ and $\gamma \in \phi(\alpha)$,

$$0 = \int_{E'_\alpha \setminus G_\gamma} h_\gamma = \mu(H_\gamma \cap E'_\alpha \setminus G_\gamma)$$

because h_γ is a conditional expectation of χH_γ on Σ^*. So if $\alpha \in B$,

$$\begin{aligned}\mu(E_\alpha \cap C(H,\alpha) \setminus F_{H\alpha}) &= \mu(E'_\alpha \cap C(H,\alpha) \setminus F_{H\alpha}) \\ &\le \mu(E''_\alpha \setminus F_{H\alpha}) + \sum_{\gamma \in \phi(\alpha)} \mu(E'_\alpha \cap H_\gamma \setminus G_\gamma)\end{aligned}$$

(because $C(H,\alpha) = X \cap \bigcap_{\gamma \in \phi(\alpha)} H_\gamma$)

$$\le \eta,$$

as required. **Q**

(iii) For $\alpha \in B$ let $\mathcal{A}_\alpha$ be the set of atoms of $\bigvee_{\gamma \in \phi(\alpha)} \Lambda_\gamma$ and set

$$F_\alpha = \bigcup_{G \in \mathcal{A}_\alpha} \bigcap_{H \in \mathcal{A}, H \subseteq G} F_{H\alpha}.$$

Then $F_\alpha \in \mathrm{T} \cap \tilde{\Sigma}_\alpha$ and

$$\begin{aligned}\mu(E_\alpha \setminus F_\alpha) &= \sum_{G \in \mathcal{A}_\alpha} \mu(E_\alpha \cap G \setminus F_\alpha) \\ &\le \sum_{G \in \mathcal{A}_\alpha} \sum_{\substack{H \in \mathcal{A} \\ H \subseteq G}} \mu(E_\alpha \cap G \setminus F_{H\alpha}) \\ &= \sum_{H \in \mathcal{A}} \mu(E_\alpha \cap C(H,\alpha) \setminus F_{H\alpha}) \le \eta\#(\mathcal{A}) = \epsilon.\end{aligned}$$

If $H \in \mathcal{A}$ then $H \subseteq C(H,\alpha) \in \mathcal{A}_\alpha$ and

$$H \cap F_\alpha \subseteq F_\alpha \cap C(H,\alpha) \subseteq F_{H\alpha},$$

for each α. So $H \cap \bigcap_{\alpha \in A} F_\alpha$ is empty. But $X = \bigcup \mathcal{A}$ so $\bigcap_{\alpha \in A} F_\alpha = \emptyset$. As $\langle E_\alpha \rangle_{\alpha \in B}$ and ϵ are arbitrary, $\langle \tilde{\Sigma}_\alpha \rangle_{\alpha \in A}$ has T-removable intersections.

(b) Next, suppose that each Λ_γ is the σ-algebra generated by $\mathrm{T} \cap \Lambda_\gamma$.

(i) For $L \in [\mathrm{T}]^{<\omega}$, $\gamma \in \Gamma$ and $\alpha \in A$ write $\Lambda_{L\gamma}$ for the algebra σ-generated by $\Lambda_\gamma \cap L$ and $\tilde{\Sigma}_{L\alpha} = \Sigma_{f(\alpha)} \vee \bigvee_{\gamma \in \phi(\alpha)} \Lambda_{L\gamma}$. Then

$$\tilde{\Sigma}_{\Delta\alpha} \subseteq \tilde{\Sigma}_{L\alpha} \text{ whenever } \alpha \in A \text{ and } \Delta \subseteq L \in [\mathrm{T}]^{<\omega},$$

$$\bigvee_{L \in [\mathrm{T}]^{<\omega}} \tilde{\Sigma}_{L\alpha} = \Sigma_{f(\alpha)} \vee \bigvee_{\gamma \in \phi(\alpha)} \bigvee_{L \in [\mathrm{T}]^{<\omega}} \Lambda_{L\gamma} = \Sigma_{f(\alpha)} \vee \bigvee_{\gamma \in \phi(\alpha)} \Lambda_\gamma = \tilde{\Sigma}_\alpha$$

because each Λ_γ is the σ-algebra generated by $\mathrm{T} \cap \Lambda_\gamma = \bigcup_{L \in [\mathrm{T}]^{<\omega}} \Lambda_{L\gamma}$. By (a), $\langle \tilde{\Sigma}_{L\alpha} \rangle_{\alpha \in A}$ has T-removable intersections for every $L \in [\mathrm{T}]^{<\omega}$.

(ii) Suppose that $\alpha \in A$ and $L \in [\mathrm{T}]^{<\omega}$. Then $\bigvee_{\gamma \in \phi(\alpha)} \Lambda_\gamma$ and $\Sigma^* \vee \bigvee_{\gamma \in \Gamma \setminus \phi(\alpha)} \Lambda_\gamma$ are relatively independent over $\Sigma_{f(\alpha)}$, by hypothesis. So $\tilde{\Sigma}_\alpha = \Sigma_{f(\alpha)} \vee \bigvee_{\gamma \in \phi(\alpha)} \Lambda_\gamma$ and $\Sigma^* \vee \bigvee_{\gamma \in \Gamma \setminus \phi(\alpha)} \Lambda_\gamma$ are relatively independent over $\Sigma_{f(\alpha)}$ (458Db). Because $\Sigma_{f(\alpha)} \subseteq \tilde{\Sigma}_{L\alpha} \subseteq \tilde{\Sigma}_\alpha$, $\tilde{\Sigma}_\alpha$ and $\Sigma^* \vee \bigvee_{\gamma \in \Gamma \setminus \phi(\alpha)} \Lambda_\gamma$ are relatively independent over $\tilde{\Sigma}_{L\alpha}$ (458Dc). Because $\bigvee_{\gamma \in \phi(\alpha)} \Lambda_{L\gamma} \subseteq \tilde{\Sigma}_{L\alpha}$, $\tilde{\Sigma}_\alpha$ and $\Sigma^* \vee \bigvee_{\gamma \in \Gamma} \Lambda_{L\gamma}$ are relatively independent over $\tilde{\Sigma}_{L\alpha}$ (458Db again). So $\tilde{\Sigma}_\alpha$ and

$$\bigvee_{\beta \in A} \tilde{\Sigma}_{L\beta} = \bigvee_{\beta \in A} (\Sigma_{f(\beta)} \vee \bigvee_{\gamma \in \phi(\beta)} \Lambda_{L\gamma}) \subseteq \Sigma^* \vee \bigvee_{\gamma \in \Gamma} \Lambda_{L\gamma}$$

are relatively independent over $\tilde{\Sigma}_{L\alpha}$.

(iii) With (i), this shows that the family $\langle \tilde{\Sigma}_{L\alpha} \rangle_{L \in [\mathrm{T}]^{<\omega}, \alpha \in A}$ satisfies the conditions of 497C, and $\langle \tilde{\Sigma}_\alpha \rangle_{\alpha \in A}$ has T-removable intersections.

(c) Finally, for the general case, let Λ'_γ be the σ-algebra generated by $\Lambda_\gamma \cap \mathrm{T}$ for $\gamma \in \Gamma$, and $\Sigma'_\alpha = \Sigma_{f(\alpha)} \vee \bigvee_{\gamma \in \phi(\alpha)} \Lambda'_\gamma$ for $\alpha \in A$. If $\gamma \in \Gamma$ and $F \in \Lambda_\gamma$, there is an $F' \in \Lambda'_\gamma$ such that $F \triangle F'$ is negligible; so if $\alpha \in A$ and $E \in \tilde\Sigma_\alpha$, there is an $E' \in \Sigma'_\alpha$ such that $E \triangle E'$ is negligible. By (b), $\langle \Sigma'_\alpha \rangle_{\alpha \in A}$ has T-removable intersections; by 497Bc, $\langle \tilde\Sigma_\alpha \rangle_{\alpha \in A}$ has T-removable intersections.

497E Theorem (TAO 07) Let (X, Σ, μ) be a probability space, and T a subalgebra of Σ. Let Γ be a partially ordered set such that $\gamma \wedge \delta = \inf\{\gamma, \delta\}$ is defined in Γ for all γ, $\delta \in \Gamma$, and $\langle \Sigma_\gamma \rangle_{\gamma \in \Gamma}$ a family of σ-subalgebras of Σ such that

(i) $\mathrm{T} \cap \Sigma_\gamma$ is metrically dense in Σ_γ for every $\gamma \in \Gamma$,

(ii) if γ, $\delta \in \Gamma$ and $\gamma \le \delta$ then $\Sigma_\gamma \subseteq \Sigma_\delta$,

(iii) if $\gamma \in \Gamma$ and Δ, Δ' are finite subsets of Γ such that $\delta \wedge \gamma \in \Delta'$ for every $\delta \in \Delta$, then Σ_γ and $\bigvee_{\delta \in \Delta} \Sigma_\delta$ are relatively independent over $\bigvee_{\delta \in \Delta'} \Sigma_\delta$.

Then $\langle \Sigma_\gamma \rangle_{\gamma \in \Gamma}$ has T-removable intersections.

proof (a) To begin with (down to the end of (d) below) suppose that Γ is finite. In this case, we have a rank function $r : \Gamma \to \mathbb{N}$ such that $r(\gamma) = \min\{n : n \in \mathbb{N},\ r(\delta) < n$ for every $\delta < \gamma\}$ for each $\gamma \in \Gamma$. For $a \subseteq \Gamma$ set $\tilde\Sigma_a = \bigvee_{\gamma \in a} \Sigma_\gamma$; note that $\mathrm{T} \cap \tilde\Sigma_a$ is always metrically dense in $\tilde\Sigma_a$.

Let A be the family of those sets $a \subseteq \Gamma$ such that $\gamma \in a$ whenever $\gamma \le \delta \in a$. For $n \in \mathbb{N}$ set $\Gamma_n = \{\gamma : r(\gamma) = n\}$ and $A_n = \{a : a \in A,\ r(\gamma) < n$ for every $\gamma \in a\}$

(b) Suppose that a, b, c are subsets of Γ and that $\gamma \wedge \delta \in c$ whenever $\gamma \in a$ and $\delta \in b \cup (a \setminus \{\gamma\})$. Then

(i) $\langle \Sigma_\gamma \rangle_{\gamma \in a}$ is relatively independent over $\tilde\Sigma_c$,

(ii) $\tilde\Sigma_a$ and $\tilde\Sigma_b$ are relatively independent over $\tilde\Sigma_c$.

P Induce on $\#(a)$. If $a = \emptyset$ then $\tilde\Sigma_a = \{\emptyset, X\}$ and the result is trivial. For the inductive step, take $\gamma_0 \in a$ and set $a' = a \setminus \{\gamma_0\}$. Then the inductive hypothesis tells us that $\langle \Sigma_\gamma \rangle_{\gamma \in a'}$ is relatively independent over $\tilde\Sigma_c$ and that $\tilde\Sigma_{a'}$ and $\tilde\Sigma_b$ are relatively independent over $\tilde\Sigma_c$. We also see that $\gamma_0 \wedge \delta \in c$ whenever $\delta \in a'$, so that Σ_{γ_0} and $\tilde\Sigma_{a'}$ are relatively independent over $\tilde\Sigma_c$, by condition (iii) of this theorem. But this means that $\langle \Sigma_\gamma \rangle_{\gamma \in a}$ is relatively independent over $\tilde\Sigma_c$ (458Hb). Similarly, because in fact $\gamma_0 \wedge \delta \in c$ for every $\delta \in a' \cup b$, Σ_{γ_0} and $\tilde\Sigma_{a'} \vee \tilde\Sigma_b$ are relatively independent over $\tilde\Sigma_c$; so the triple Σ_{γ_0}, $\tilde\Sigma_{a'}$ and $\tilde\Sigma_b$ are relatively independent over $\tilde\Sigma_c$ (458Hb again), and $\tilde\Sigma_a = \Sigma_{\gamma_0} \vee \tilde\Sigma_{a'}$ and $\tilde\Sigma_b$ are relatively independent over $\tilde\Sigma_c$ (458Ha). Thus the induction continues. **Q**

(c) For each $n \in \mathbb{N}$, $\langle \tilde\Sigma_a \rangle_{a \in A_n}$ has T-removable intersections. **P** Induce on n. If $n = 0$ then $A_n = \{\emptyset\}$ and the result is trivial. For the inductive step to $n + 1 \ge 1$, apply 497D, as follows. The inductive hypothesis tells us that $\langle \tilde\Sigma_a \rangle_{a \in A_n}$ has T-removable intersections, and we know that $\mathrm{T} \cap \tilde\Sigma_a$ is always metrically dense in $\tilde\Sigma_a$. Set

$$\Sigma^* = \bigvee_{a \in A_n} \tilde\Sigma_a = \tilde\Sigma_d$$

where $d = \bigcup_{m<n} \Gamma_m$ is the largest member of A_n. Define $g : \Gamma_n \to A_n$ by setting $g(\gamma) = \{\delta : \delta < \gamma\}$. Then $\gamma \wedge \delta \in d$ for all distinct γ, $\delta \in \Gamma_n$, so $\langle \Sigma_\gamma \rangle_{\gamma \in \Gamma_n}$ is relatively independent over Σ^*, by (b-i) just above. If $\gamma \in \Gamma_n$ and $\delta \in d$, $\gamma \wedge \delta \in g(\gamma)$, so $\Sigma_\gamma = \tilde\Sigma_{\{\gamma\}}$ and $\Sigma^* = \tilde\Sigma_d$ are relatively independent over $\tilde\Sigma_{g(\gamma)}$, by (b-ii). Of course $\mathrm{T} \cap \Sigma_\gamma$ is metrically dense in Σ_γ for every $\gamma \in \Gamma_n$.

For $a \in A_{n+1}$, set $\phi(a) = a \cap \Gamma_n$ and

$$f(a) = a \setminus \phi(a) = a \cap \bigcup_{m<n} \Gamma_m \in A_n.$$

If $\gamma \in \phi(a)$ then $g(\gamma) \subseteq a$, by the definition of A, so $g(\gamma) \subseteq f(a)$ and $\tilde\Sigma_{g(\gamma)} \subseteq \tilde\Sigma_{f(a)}$. Finally, by (b-ii), $\bigvee_{\gamma \in \phi(a)} \Sigma_\gamma$ and $\Sigma^* \vee \bigvee_{\gamma \in \Gamma_n \setminus \phi(a)} \Sigma_\gamma$ are relatively independent over $\tilde\Sigma_{f(a)}$, because if $\gamma \in \phi(a)$ and $\delta \in d \cup (\Gamma_n \setminus \phi(a))$ then $\gamma \wedge \delta \in g(\gamma) \subseteq f(a)$.

So all the hypotheses of 497D are satisfied, and

$$\langle \tilde\Sigma_{f(a)} \vee \bigvee_{\gamma \in \phi(a)} \Sigma_\gamma \rangle_{a \in A_{n+1}} = \langle \tilde\Sigma_a \rangle_{a \in A_{n+1}}$$

has T-removable intersections. Thus the induction proceeds. **Q**

(d) Because Γ is finite, there is some n such that $A = A_n$. Now, for each $\gamma \in \Gamma$, set $e_\gamma = \{\delta : \delta \le \gamma\}$; then $e_\gamma \in A$ and $\Sigma_\gamma = \tilde\Sigma_{e_\gamma}$. By 497Bb, or otherwise, $\langle \Sigma_\gamma \rangle_{\gamma \in \Gamma}$ has T-removable intersections, as required.

(e) Thus the theorem is true when Γ is finite. For the general case, take any finite $\Gamma_0 \subseteq \Gamma$ and set $\Gamma' = \{\inf a : a \subseteq \Gamma_0 \text{ is non-empty}\}$. Then Γ' is finite and closed under $\wedge$, and $\langle\Sigma_\gamma\rangle_{\gamma\in\Gamma'}$ satisfies the conditions of the theorem. So $\langle\Sigma_\gamma\rangle_{\gamma\in\Gamma'}$ and $\langle\Sigma_\gamma\rangle_{\gamma\in\Gamma_0}$ have T-removable intersections. As Γ_0 is arbitrary, $\langle\Sigma_\gamma\rangle_{\gamma\in\Gamma}$ has T-removable intersections (497Ba), and the proof is complete.

497F Invariant measures on $\mathcal{P}([I]^{<\omega})$ (a) Let I be a set. Then $\mathcal{P}([I]^{<\omega})$ is a compact Hausdorff space, if we give it its usual topology, generated by sets of the form $\{R : a \in R \subseteq [I]^{<\omega},\ b \notin R\}$ for finite sets a, $b \subseteq I$. (You should perhaps fix on the case $I = \mathbb{N}$ for the first reading of this paragraph, so that $[I]^{<\omega}$ will be a relatively familiar countable set, and you can remember that $\mathcal{P}([I]^{<\omega})$ is homeomorphic to the Cantor set.) Let G_I be the set of permutations of I, and for $\phi \in G_I$, $R \subseteq [I]^{<\omega}$ set

$$\phi\bullet R = \{\phi[a] : a \in R\} = \{a : a \in [I]^{<\omega},\ \phi^{-1}[a] \in R\},$$

so that $\bullet$ is an action of G_I on $\mathcal{P}([I]^{<\omega})$, and $R \mapsto \phi\bullet R$ is a homeomorphism for every $\phi \in G_I$. Let P_I be the set of Radon probability measures on $\mathcal{P}([I]^{<\omega})$. Then we have an action of G_I on P_I defined by saying that

$$\phi\bullet E = \{\phi\bullet R : R \in E\}$$

for $\phi \in G_I$ and $E \subseteq \mathcal{P}([I]^{<\omega})$, and

$$(\phi\bullet\mu)(E) = \mu(\phi^{-1}\bullet E)$$

for $\phi \in G_I$, $\mu \in P_I$ and Borel sets $E \subseteq \mathcal{P}([I]^{<\omega})$. Because $R \mapsto \phi\bullet R$ is a homeomorphism, the map $\mu \mapsto \phi\bullet\mu$ is a homeomorphism when P_I is given its narrow topology, corresponding to the weak* topology on $C(\mathcal{P}([I]^{<\omega}))^*$ (437J, 437Kc).

(b) If $\mu \in P_I$, I will say that μ is **permutation-invariant** if $\mu = \phi\bullet\mu$ for every $\phi \in G_I$.

(c) For $R \subseteq [I]^{<\omega}$ and $J \subseteq I$ I write $R\restriction J$ for the trace $R \cap \mathcal{P}J \subseteq [J]^{<\omega}$ of R on J. Let $\mathcal{V}$ be the family of sets of the form $V_{JS} = \{R : R \subseteq [I]^{<\omega},\ R\restriction J = S\}$ where $J \subseteq I$ is finite and $S \subseteq \mathcal{P}J$. If μ, $\nu \in P_I$ agree on $\mathcal{V}$, they are equal. **P** If $E \subseteq \mathcal{P}([I])^{<\omega}$ is open-and-closed, it is determined by coordinates in some finite subset $\mathcal{K}$ of $[I]^{<\omega}$, in the sense that if $R \in E$, $R' \subseteq [I]^{<\omega}$ and $R \cap \mathcal{K} = R' \cap \mathcal{K}$, then $R' \in E$. Let $J \subseteq I$ be a finite set such that $\mathcal{K} \subseteq [J]^{<\omega}$, and set $\mathcal{S} = \{R\restriction J : R \in E\}$. Now $\langle V_{JS}\rangle_{S\in\mathcal{S}}$ is a disjoint family in $\mathcal{V}$ with union E, so

$$\mu E = \sum_{S\in\mathcal{S}} \mu V_{JS} = \nu E.$$

As E is arbitrary, $\mu = \nu$ (416Qa). **Q**

(d) If I, J are sets and $f : I \to J$ is a function, I define $\tilde f : \mathcal{P}([J]^{<\omega}) \to \mathcal{P}([I]^{<\omega})$ by setting $\tilde f(R) = \{a : a \in [I]^{<\omega},\ f[a] \in R\}$ for $R \subseteq [J]^{<\omega}$. Note that $\tilde f$ is continuous, since $\{R : a \in \tilde f(R)\} = \{R : f[a] \in R\}$ is a basic open-and-closed set in $\mathcal{P}([J]^{<\omega})$ for every $a \in [I]^{<\omega}$. If $I \subseteq J$ and f is the identity function, then $\tilde f(R) = R\restriction I$ for every $R \subseteq [J]^{<\omega}$. Observe that when $\phi \in G_I$ and $R \subseteq [I]^{<\omega}$ then $\tilde\phi(R) = \phi^{-1}\bullet R$.

497G Theorem (Tao 07) Let I be an infinite set and $\mathcal{J}$ a filter on I not containing any finite set. Let T be the algebra of open-and-closed subsets of $\mathcal{P}([I]^{<\omega})$, and $\mu \in P_I$ a permutation-invariant measure. For $J \subseteq I$, write Σ_J for the σ-algebra of subsets of $\mathcal{P}([I]^{<\omega})$ generated by sets of the form $E_a = \{R : a \in R \subseteq [I]^{<\omega}\}$ where $a \in [J]^{<\omega}$. Then $\langle\Sigma_J\rangle_{J\in\mathcal{J}}$ has T-removable intersections with respect to μ.

proof I seek to apply 497E with $\Gamma = \mathcal{J}$, ordered by $\subseteq$. If $J \in \mathcal{J}$ and $a \in [J]^{<\omega}$ then $\{R : a \in R \subseteq [I]^{<\omega}\}$ belongs to $\mathrm{T} \cap \Sigma_J$; accordingly Σ_J is the σ-algebra generated by $\mathrm{T} \cap \Sigma_J$ and $\mathrm{T} \cap \Sigma_J$ is metrically dense in Σ_J. Condition (ii) of 497E is obviously satisfied. As for condition (iii), we can use 459I, as follows. Taking $X = \mathcal{P}([I]^{<\omega})$, we have the action $\bullet$ of G_I on X described in 497Fa, and $R \mapsto \phi\bullet R$ is inverse-measure-preserving for each ϕ because μ is permutation-invariant. Now we see easily that

— for every $J \subseteq I$, $\bigcup_{K\in[J]^{<\omega}} \Sigma_K$ contains E_a for every $a \in [J]^{<\omega}$, so σ-generates Σ_J;

— if $a \in [I]^{<\omega}$ and $\phi \in G_I$,

$$E_{\phi[a]} = \{R : \phi[a] \in R\} = \{\phi\bullet R : \phi[a] \in \phi\bullet R\} = \{\phi\bullet R : a \in R\} = \phi\bullet E_a;$$

— if $J \subseteq I$, then $\{E : \phi\bullet E \in \Sigma_{\phi[J]}\}$ is a σ-algebra of sets containing $\phi^{-1}\bullet E_{\phi[a]} = E_a$ whenever $a \in [J]^{<\omega}$, so it includes Σ_J, and $\phi\bullet E \in \Sigma_{\phi[J]}$ for every $E \in \Sigma_J$;

— if $J \subseteq I$ and $\phi \in G_I$ is such that $\phi(i) = i$ for every $i \in J$, then $\{E : \phi\bullet E = E\}$ is a σ-algebra of sets containing E_a for every $a \in [J]^{<\omega}$, so it includes Σ_J, and $\phi\bullet E = E$ for every $E \in \Sigma_J$.

Thus the conditions of 459I are satisfied. So if $J \in \mathcal{J}$ and $\mathcal{K}$, $\mathcal{K}'$ are (finite) subsets of $\mathcal{J}$ such that $J \cap K \in \mathcal{K}'$ for every $K \in \mathcal{K}$, 459I tells us that Σ_J and $\bigvee_{K\in\mathcal{K}} \Sigma_K$ are relatively independent over $\bigvee_{K\in\mathcal{K}'} \Sigma_K$, as required by (iii) of 497E.

So 497E gives the result we seek.

497H I come now to the next essential ingredient of the proof.

Construction Suppose we are given a sequence $\langle(m_n, T_n)\rangle_{n\in\mathbb{N}}$ and a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$ such that

(α) $\langle m_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathbb{N}\setminus\{0\}$ and $\lim_{n\to\mathcal{F}} m_n = \infty$,

(β) $T_n \subseteq \mathcal{P}m_n$ for each n.

Then for any set I there is a permutation-invariant $\mu \in P_I$ such that

$$\mu\{R : R\lceil K = S\} = \lim_{n\to\mathcal{F}} \frac{1}{m_n^{\#(K)}}\#(\{z : z \in m_n^K,\ \tilde z(T_n) = S\})$$

whenever $K \subseteq I$ is finite and $S \subseteq \mathcal{P}K$.

proof (a) For each $n \in \mathbb{N}$ let ν_n be the usual measure on m_n^I, the product of I copies of the uniform probability measure on the finite set m_n. The function $w \mapsto \tilde w(T_n) : m_n^I \to \mathcal{P}([I]^{<\omega})$ is continuous, since for any $a \in [I]^{<\omega}$ the set $\{w : a \in \tilde w(T_n)\} = \{w : w[a] \in T_n\}$ is determined by coordinates in the finite set a. So we have a corresponding Radon probability measure μ_n on $\mathcal{P}([I]^{<\omega})$ defined by saying that $\mu_n E = \nu_n\{w : \tilde w(T_n) \in E\}$ for every set $E \subseteq \mathcal{P}([I]^{<\omega})$ such that ν_n measures $\{w : \tilde w(T_n) \in E\}$ (418I). If $K \subseteq I$ is finite and $w \in m_n^I$, then

$$\tilde w(T_n)\lceil K = \{a : a \subseteq K,\ w[a] \in T_n\} = \{a : a \subseteq K,\ (w\restriction K)[a] \in T_n\} = (w\restriction K)^{\sim}(T_n).$$

So if $S \subseteq \mathcal{P}K$, then

$$\begin{aligned}\mu_n\{R : R\lceil K = S\} &= \nu_n\{w : w \in m_n^I,\ \tilde w(T_n)\lceil K = S\}\\ &= \nu_n\{w : w \in m_n^I,\ (w\restriction K)^{\sim}(T_n) = S\}\\ &= \frac{1}{m_n^{\#(K)}}\#(\{z : z \in m_n^K,\ \tilde z(T_n) = S\}).\end{aligned}$$

Let μ be the limit $\lim_{n\to\mathcal{F}} \mu_n$ in the narrow topology on P_I; then

$$\mu\{R : R\lceil K = S\} = \lim_{n\to\mathcal{F}} \mu_n\{R : R\lceil K = S\}$$

(because $\{R : R\lceil K = S\}$ is open-and-closed; see 437Jf)

$$= \lim_{n\to\mathcal{F}} \frac{1}{m_n^{\#(K)}}\#(\{z : z \in m_n^K,\ \tilde z(T_n) = S\})$$

whenever $K \subseteq I$ is finite and $S \subseteq \mathcal{P}K$.

(b) Now let $\phi : I \to I$ be any permutation and $\tilde\phi : \mathcal{P}([I]^{<\omega}) \to \mathcal{P}([I]^{<\omega})$ the corresponding permutation. Then for any finite $K \subseteq I$,

$$\begin{aligned}\tilde\phi(R)\lceil K &= \{a : a \subseteq K,\ a \in \tilde\phi(R)\}\\ &= \{a : a \subseteq K,\ \phi[a] \in R\} = \{a : a \in [I]^{<\omega},\ \phi[a] \in R\lceil\phi[K]\}.\end{aligned}$$

Fix $n \in \mathbb{N}$ for the moment. If $K \subseteq I$ is finite, and $S \subseteq \mathcal{P}K$, then

$$\begin{aligned}
\mu_n\tilde{\phi}^{-1}\{R : R\lceil K = S\} &= \mu_n\{R : \tilde{\phi}(R)\lceil K = S\}\\
&= \mu_n\{R : S = \{a : a \in [I]^{<\omega},\ \phi[a] \in R\lceil\phi[K]\}\}\\
&= \mu_n\{R : R\lceil\phi[K] = \{\phi[a] : a \in S\}\}\\
&= \nu_n\{w : \tilde{w}(T_n)\lceil\phi[K] = \{\phi[a] : a \in S\}\}\\
&= \nu_n\{w : \{a : a \subseteq \phi[K],\ w[a] \in T_n\} = \{\phi[a] : a \in S\}\}\\
&= \nu_n\{w : \{\phi[a] : a \subseteq K,\ w[\phi[a]] \in T_n\} = \{\phi[a] : a \in S\}\}\\
&= \nu_n\{w : \{a : a \subseteq K,\ (w\phi)[a] \in T_n\} = S\}\\
&= \nu_n\{w : \{a : a \subseteq K,\ w[a] \in T_n\} = S\}
\end{aligned}$$

(because $w \mapsto w\phi : m_n^I \to m_n^I$ is an automorphism for the measure ν_n)

$$= \nu_n\{w : \tilde{w}(T_n)\lceil K = S\} = \mu_n\{R : R\lceil K = S\}.$$

So μ_n and $\mu_n\tilde{\phi}^{-1}$ agree on the family $\mathcal{V}$ of basic open-and-closed sets described in 497F. As this is true for every n, we also have $\mu V = \mu\tilde{\phi}^{-1}[V]$ for every $V \in \mathcal{V}$, and $\mu = \mu\tilde{\phi}^{-1}$. As ϕ is arbitrary, μ is permutation-invariant.

497I Definition If I, J are sets, $R \subseteq \mathcal{P}I$ and $S \subseteq \mathcal{P}J$, I will say for the purposes of the next two results that an **embedding** of (I, R) in (J, S) is an injective function $f : I \to J$ such that $f[a] \in S$ for every $a \in R$, that is (when $S \subseteq [J]^{<\omega}$), $R \subseteq \tilde{f}(S)$.

497J Theorem (Nagle Rödl & Schacht 06) Let L be a finite set with r members, and $T \subseteq \mathcal{P}L$. Then for every $\epsilon > 0$ there is a $\delta > 0$ such that whenever I is a non-empty finite set, $R \subseteq \mathcal{P}I$ and the number of embeddings of (L, T) in (I, R) is at most $\delta\#(I)^r$, there is an $S \subseteq \mathcal{P}I$ such that $\#(S \cap [I]^k) \le \epsilon\#(I)^k$ for every k and there is no embedding of (L, T) in $(I, R \setminus S)$.

proof (Tao 07) **?** Suppose, if possible, otherwise.

(a) We have a sequence $\langle(m_n, T_n)\rangle_{n\in\mathbb{N}}$ such that

> $m_n \in \mathbb{N}\setminus\{0\}$, $T_n \subseteq \mathcal{P}m_n$; the number of embeddings of (L, T) in (m_n, T_n) is at most $2^{-n}m_n^r$; if $S \subseteq \mathcal{P}m_n$ and $\#(S\cap[m_n]^k) \le \epsilon m_n^k$ for every k then there is an embedding of (L, T) in $(m_n, T_n\setminus S)$

for every $n \in \mathbb{N}$. Of course (L, T) always has at least one embedding in (m_n, T_n) so $\lim_{n\to\infty} m_n = \infty$. Let I be an infinite set including L and $\mathcal{F}$ a non-principal ultrafilter on $\mathbb{N}$. Let $\mu \in P_I$ be the permutation-invariant measure defined from $\langle(m_n, T_n)\rangle_{n\in\mathbb{N}}$ and $\mathcal{F}$ by the process of 497H.

(b) For $c \subseteq L$ set $J_c = c \cup (I \setminus L)$, so that Σ_{J_c}, in the notation of 497G, is the σ-algebra of subsets of $\mathcal{P}([I]^{<\omega})$ generated by sets of the form $E_a = \{R : a \in R \subseteq [I]^{<\omega}\}$ where $a \in [c\cup(I\setminus L)]^{<\omega}$. Note that every member of Σ_{J_c} is determined by coordinates in $\mathcal{P}J_c$, in the sense that if $R \in E \in \Sigma_{J_c}$, $R' \subseteq \mathcal{P}([I]^{<\omega})$ and $R \cap \mathcal{P}J_c) = R' \cap \mathcal{P}J_c$, then $R' \in E$.

By 497G, applied to the filter $\mathcal{J}$ on I generated by $\{I \setminus L\}$, $\langle\Sigma_{J_c}\rangle_{c\subseteq L}$ has T-removable intersections with respect to μ, where T is the algebra of open-and-closed subsets of $\mathcal{P}([I]^{<\omega})$. *A fortiori*, $\langle\Sigma_{J_c}\rangle_{c\in T}$ has T-removable intersections with respect to μ.

(c) $E_c \in \Sigma_{J_c}$ for every $c \in T$, and

$$\begin{aligned}\mu(\bigcap_{c\in T}E_c)=\mu\{R:T\subseteq R\}&=\sum_{T\subseteq T'\subseteq\mathcal{P}L}\mu\{R:R\restriction L=T'\}\\
&=\sum_{T\subseteq T'\subseteq\mathcal{P}L}\lim_{n\to\mathcal{F}}\frac{1}{m_n^r}\#(\{z:z\in m_n^L,\,\tilde z(T_n)=T'\})\\
&=\lim_{n\to\mathcal{F}}\frac{1}{m_n^r}\sum_{T\subseteq T'\subseteq\mathcal{P}L}\#(\{z:z\in m_n^L,\,\tilde z(T_n)=T'\})\\
&=\lim_{n\to\mathcal{F}}\frac{1}{m_n^r}\#(\{z:z\in m_n^L,\,T\subseteq\tilde z(T_n)\})\\
&=\lim_{n\to\mathcal{F}}\frac{1}{m_n^r}\#(\{z:z\in m_n^L\text{ is injective},\,T\subseteq\tilde z(T_n)\})\end{aligned}$$

(because $\lim_{n\to\infty}\dfrac{k_n}{m_n^r}=0$, where $k_n=m_n^r-\dfrac{m_n!}{(m_n-r)!}$ is the number of non-injective functions from L to m_n)

$$=\lim_{n\to\mathcal{F}}\frac{1}{m_n^r}\#(\{z:z\text{ is an embedding of }(L,T)\text{ in }(m_n,T_n)\})=0.$$

(d) Take $\eta>0$ such that $2\eta\#(T)<\epsilon$, and $\langle F_c\rangle_{c\in T}$ such that $\bigcap_{c\in T}F_c=\emptyset$ and $F_c\in\mathrm{T}\cap\Sigma_{J_c}$ and $\mu(E_c\setminus F_c)\le\eta$ for every $c\in T$. Every F_c is open-and-closed, so there is an $M\in[I]^{<\omega}$ such that $L\subseteq M$ and every F_c is determined by coordinates in $\mathcal{P}M$. In this case, each F_c is determined by coordinates in $\mathcal{P}M\cap\mathcal{P}J_c=\mathcal{P}(c\cup(M\setminus L))$. Setting

$$F'_c=\{R\restriction M:R\in F_c\},\quad E'_c=\{R\restriction M:R\in E_c\}=\{R:c\in R\subseteq\mathcal{P}M\},$$

we have

$$F_c=\{R:R\subseteq[I]^{<\omega},\,R\restriction M\in F'_c\},\quad E_c=\{R:R\subseteq[I]^{<\omega},\,R\restriction M\in E'_c\},$$

while both E'_c and F'_c, and therefore $E'_c\setminus F'_c$, regarded as subsets of $\mathcal{P}(\mathcal{P}M)$, are determined by coordinates in $\mathcal{P}(c\cup(M\setminus L))$. Because $\bigcap_{c\in T}F_c$ is empty, so is $\bigcap_{c\in T}F'_c$.

(e) Let $n\ge r$ be such that

$$\begin{aligned}\tfrac{1}{m_n^{\#(M)}}\#(\{z:z\in m_n^M,\,\tilde z(T_n)\in E'_c\setminus F'_c\})&\le\eta+\mu\{R:R\restriction M\in E'_c\setminus F'_c\}\\
&=\eta+\mu\{R:R\in E_c\setminus F_c\}\le 2\eta\end{aligned}$$

for every $c\in T$. For $c\in T$ set

$$Q_c=\{z:z\in m_n^M,\,\tilde z(T_n)\in E'_c\setminus F'_c\},$$

so that $\#(Q_c)\le 2\eta m_n^{\#(M)}$. Since

$$\begin{aligned}\sum_{\substack{c\in T\\ w\in m_n^{M\setminus L}}}\#(\{z:w\subseteq z\in Q_c\})&=\sum_{c\in T}\#(Q_c)\le 2\eta\#(T)m_n^{\#(M)}\\
&\le\epsilon m_n^{\#(M)}=\epsilon\#(m_n^{M\setminus L})m_n^r,\end{aligned}$$

there must be a $w\in m_n^{M\setminus L}$ such that

$$\textstyle\sum_{c\in T,w\in m_n^{M\setminus L}}\#(\{z:w\subseteq z\in Q_c\})\le\epsilon m_n^r;$$

set

$$Q'_c=\{z:w\subseteq z\in Q_c,\,z\restriction c\text{ is injective}\}$$

for $c\in T$, so that $\sum_{c\in T}\#(Q'_c)\le\epsilon m_n^r$.

If $c\in T$ and $\#(c)=k$, then

$$\#(\{z[c]:z\in Q'_c\})=\tfrac{1}{m_n^{r-k}}\#(Q'_c).$$

P If z, $z'\in m_n^M$ and $z\restriction(c\cup(M\setminus L))=z'\restriction(c\cup(M\setminus L))$, then $\tilde z(T_n)\lceil(c\cup(M\setminus L))=\tilde z'(T_n)\lceil(c\cup(M\setminus L))$, so $\tilde z(T_n)\in E'_c\setminus F'_c$ iff $\tilde z'(T_n)\in E'_c\setminus F'_c$, that is, $z\in Q_c$ iff $z'\in Q_c$. So if $a=z[c]$ for some $z\in Q'_c$, then

$$\{z':z'\in Q'_c,\,z'[c]=a\}=\{z':z'\in m_n^M,\,z'\restriction(c\cup(M\setminus L))=z\restriction(c\cup(M\setminus L))\}$$

has just $\#(m_n^{L\setminus c})=m_n^{r-k}$ members. **Q**

(f) Consider

$$S=\{z[c]:c\in T,\,z\in Q'_c\}.$$

Then

$$\begin{aligned}\#(S\cap[m_n]^k)&=\#([m_n]^k\cap\{z[c]:c\in T,\,z\in Q'_c\})\\&=\#(\{z[c]:c\in T\cap[L]^k,\,z\in Q'_c\})\end{aligned}$$

(because every member of Q'_c is injective on c)

$$\le\sum_{\substack{c\in T\\ \#(c)=k}}\frac{1}{m_n^{r-k}}\#(Q'_c)$$

(by the last remark in (e))

$$\le\frac{1}{m_n^{r-k}}\epsilon m_n^r=\epsilon m_n^k$$

for every k. So by the choice of (m_n,T_n) there is an embedding v of (L,T) in $(m_n,T_n\setminus S)$; take $z=v\cup w$, so that $w\subseteq z\in m_n^M$ and $z\restriction L=v$ is injective and $z[c]\notin S$ for every $c\in T$. However, there is some $c\in T$ such that $\tilde z(T_n)\notin F'_c$. As $c\in\tilde z(T_n)$, $\tilde z(T_n)\in E'_c$. But now $z\in Q'_c$ and $z[c]\in S$. **X**

This contradiction proves the theorem.

497K Corollary: the Hypergraph Removal Lemma For every $\epsilon>0$ and $r\ge1$ there is a $\delta>0$ such that whenever I is a finite set, $R\subseteq[I]^r$ and $\#(\{J:J\in[I]^{r+1},\,[J]^r\subseteq R\})\le\delta\#(I)^{r+1}$, there is an $S\subseteq[I]^r$ such that $\#(S)\le\epsilon\#(I)^r$ and there is no $J\in[I]^{r+1}$ such that $[J]^r\subseteq R\setminus S$.

proof In 497J, take L to be a set with $r+1$ members, and set $T=[L]^r$ in 497J. Then there is a $\delta_0>0$ such that whenever I is a finite set, $R\subseteq[I]^r$ and the number of embeddings of $(L,[L]^r)$ in (I,R) is at most $\delta_0\#(I)^{r+1}$, there is an $S\subseteq[I]^r$ such that $\#(S)\le\epsilon\#(I)^r$ and there is no embedding of $(L,[L^r])$ in $(I,R\setminus S)$. Try $\delta=\frac{1}{(r+1)!}\delta_0$. If I is finite, $R\subseteq[I]^r$ and $\mathcal J=\{J:J\in[I]^{r+1},\,[J]^r\subseteq R\}$ has at most $\delta\#(I)^{r+1}$ members, then an embedding of $(L,[L]^r)$ in (I,R) is an injective function $f:L\to I$ such that $f[J]\in R$ for every $J\in[L]^r$, that is, $f[L]\in\mathcal J$. So the number of such embeddings is $(r+1)!\#(\mathcal J)\le\delta_0\#(I)^{r+1}$. There is therefore an $S\subseteq[I]^r$ such that $\#(S)\le\epsilon\#(I)^r$ and there is no embedding of $(L,[L]^r)$ in $(I,R\setminus S)$, that is, there is no $J\in[I]^{r+1}$ such that $[J]^r\subseteq R\setminus S$.

497L Corollary: Szemerédi's Theorem (Szemerédi 75) For every $\epsilon>0$ and $r\ge2$ there is an $n_0\in\mathbb N$ such that whenever $n\ge n_0$, $A\subseteq n$ and $\#(A)\ge\epsilon n$ there is an arithmetic progression of length $r+1$ in A.

proof (Frankl & Rödl 02) Set $\eta=\frac{1}{r!}(\frac{\epsilon}{2r!})^r$. Take $\delta>0$ such that whenever I is a finite set, $R\subseteq[I]^r$ and $\#(\{J:J\in[I]^{r+1},\,[J]^r\subseteq R\})\le\delta\#(I)^{r+1}$, there is an $S\subseteq[I]^r$ such that $\#(S)\le\frac{\eta}{2(r+1)^r}\#(I)^r$ and there is no $J\in[I]^{r+1}$ such that $[J]^r\subseteq R\setminus S$. Let n_0 be such that $\epsilon n\ge2r\cdot r!$ and $n(r+1)^{r+1}\delta\ge1$ whenever $n\ge n_0$. Take $n\ge n_0$ and $A\subseteq n$ such that $\#(A)\ge\epsilon n$.

Let $C\subseteq n^r$ be the set

$$\{(i_0,i_1,\dots,i_{r-1}):\textstyle\sum_{j=0}^{r-1}(j+1)i_j\in A\}.$$

Then $\#(C) \ge \eta n^r$.[9] **P** For $m < r!$ set $A_m = \{i : i \in A,\ i \equiv m \mod r!\}$. Then there is an m such that $\#(A_m) \ge \frac{\epsilon n}{r!}$. Now we have an injection $\phi : [A_m]^r \to C$ given by saying that if $l_0 < \ldots < l_{r-1}$ in A_m then

$$\begin{aligned}\phi(\{l_0, \ldots , l_{r-1}\})(j) &= l_0 \text{ if } j = 0\\ &= \tfrac{1}{j+1}(l_j - l_{j-1}) \text{ if } 0 < j < r.\end{aligned}$$

So

$$\#(C) \ge \#([A_m]^r) \ge \frac{1}{r!}(\frac{\epsilon n}{r!} - r)^r \ge \frac{1}{r!}(\frac{\epsilon n}{2r!})^r = \eta n^r. \text{ } \mathbf{Q}$$

Let I be $n \times (r + 1)$ and for $c = (i_0, \ldots , i_{r-1}) \in C$ set

$$J_c = \{(i_j, j) : j < r\} \cup \{(\textstyle\sum_{j=0}^{r-1} i_j, r)\} \in [I]^{r+1}.$$

Observe that if c, $c' \in C$ are distinct, then $[J_c]^r \cap [J_{c'}]^r = \emptyset$, since given any face of the r-simplex J_c we can read off all but at most one of the coordinates of c and calculate the last. Set $R = \bigcup_{c\in C}[J_c]^r \subseteq [I]^r$.

? Suppose, if possible, that the only r-simplices $J \in [I]^{r+1}$ such that $[J]^r \subseteq R$ are of the form J_c for some $c \in C$. Then there are at most

$$\#(C) \le n^r \le n^r \cdot n(r + 1)^{r+1}\delta = \delta\#(I)^{r+1}$$

such simplices; by the choice of δ, there is an $S \subseteq [I]^r$ such that $R \setminus S$ covers no r-simplices and

$$\#(S) \le \frac{\eta}{2(r+1)^r}\#(I)^r = \frac{\eta}{2}n^r < \#(C).$$

But every J_c must have a face in S, and no two J_c share a face, so this is impossible. **X**

So we have an r-simplex $J \in [I]^{r+1}$, which is not of the form J_c where $c \in C$, such that $[J]^r \subseteq R$. Now since the only faces put into R come from the J_c, and therefore meet each of the $r + 1$ levels $n \times \{k\}$ in at most one point, J must be of the form $\{(i_j, j) : j < r\} \cup \{(l, r)\}$. Since $\{(i_j, j) : j < r\}$ is a face of some J_c, $c = (i_0, \ldots , i_{r-1}) \in C$. Set $l' = i_0 + \ldots + i_{r-1}$; then $l' \ne l$ because $J \ne J_c$. For each $k < r$, $J \setminus \{(i_k, k)\}$ is a face of J and therefore of $J_{c'}$ for some $c' \in C$; now $J_{c'}$ must be

$$(J \setminus \{(i_k, k)\}) \cup \{(l - \textstyle\sum_{j<r,j\ne k} i_j, k)\} = (J \setminus \{(i_k, k)\}) \cup \{(i_k + l - l', k)\}$$

and

$$\textstyle\sum_{j=0}^{r}(j + 1)i_j + (k + 1)(l - l')$$

belongs to A. Since this is true for every $k < r$, and we also have $\sum_{j=0}^{r}(j+1)i_j \in A$ because $c \in C$, we have an arithmetic progression in A of length $r + 1$, as required.

497M For a full-strength version of the multiple recurrence theorem it seems that the ideas described above are inadequate; for an adaptation which goes farther, see Austin 10a and Austin 10b. However the methods here can reach the following.

Lemma (cf. Solymosi 03) Suppose that $r \ge 1$ and $n \in \mathbb{N}$. For $0 \le j$, $k < r$ set $e_j(k) = 1$ if $k = j$, 0 otherwise. For $z \in n^r$ and $C \subseteq n^r$ write

$$\Delta(z, C) = \{k : k \in \mathbb{Z},\ z + ke_i \in C \text{ for every } i < r\}, \quad q(z, C) = \#(\Delta(z, C)).$$

Then for every $\epsilon > 0$ there is a $\delta > 0$ such that $\#(\{z : z \in n^r,\ q(z, C) \ge \delta n\}) \ge \delta n^r$ whenever $n \in \mathbb{N}$, $C \subseteq n^r$ and $\#(C) \ge \epsilon n^r$.

proof We can use some of the same ideas as in 497L. Let $\epsilon' > 0$ be such that $2^r r^r \epsilon' < \epsilon$. Let $\delta > 0$ be such that whenever I is finite, $R \subseteq [I]^r$ and $\#(\{J : J \in [I]^{r+1},\ [J]^r \subseteq R\}) \le \delta\#(I)^{r+1}$ there is an $S \subseteq R$ such that $\#(S) \le \epsilon'\#(I)^r$ and there is no $J \in [I]^{r+1}$ such that $[J]^r \subseteq R \setminus S$ (497K).

Take $n \in \mathbb{N}$ and $C \subseteq n^r$ such that $\#(C) \ge \epsilon n^r$. Set $I = (n \times r) \cup (nr \times \{r\})$, so that $\#(I) = 2nr$. For $c \in C$ set

[9] For the rest of this proof, and also in 497M and 497N below, I will use the formula n^r both for the set of functions from $r = \{0, \ldots , r - 1\}$ to $n = \{0, \ldots , n - 1\}$ and for its cardinal interpreted as a real number; I trust that this will not lead to any confusion.

$$J_c = \{(c(i), i) : i < r\} \cup \{(\sum_{i=0}^{r-1} c(i), r)\} \in [I]^{r+1};$$

set $R = \bigcup_{c \in C} [J_c]^r$. Observe that if c, $c' \in C$ are distinct then $[J_c]^r$ and $[J_{c'}]^r$ are disjoint. If $S \subseteq R$ and $\#(S) \le \epsilon' \#(I)^r$, then $\#(S) < \epsilon n^r$ and there must be a $c \in C$ such that $[J_c]^r \cap S = \emptyset$ and $[J_c]^r \subseteq R \setminus S$. Consequently

$$\mathcal{K} = \{K : K \in [I]^{r+1},\ [K]^r \subseteq R\}$$

must have more than $\delta \#(I)^{r+1} \ge 2\delta n^{r+1}$ members, by the choice of δ.

Next, $\#(\mathcal{K}) = \sum_{z \in n^r} q(z, C)$. **P** Set $B = \{(z, k) : z \in n^r,\ k \in \mathbb{Z},\ z + ke_i \in C$ for every $i < r\}$; then $\#(B) = \sum_{z \in n^r} q(z, C)$. For any $K \in \mathcal{K}$, there must be a $c_K \in C$ such that $(c_K(i), i) \in K$ for every $i < r$ while $(k_K + \sum_{i=0}^{r-1} c_K(i), r) \in K$ for some k_K; in this case, $c_K + k_K e_i \in C$ for every $i < r$ and $(c_K, k_K) \in B$. Conversely, starting from $(z, k) \in B$, $\{(z(i), i) : i < r\} \cup \{(k + \sum_{i=0}^{r-1} z(i), r)\}$ belongs to K. So $K \mapsto (c_K, k_K)$ is a bijection from $\mathcal{K}$ to B and $\#(\mathcal{K}) = \#(B)$. **Q**

Thus $\sum_{z \in n^r} q(z, C) \ge 2\delta n^{r+1}$. Of course

$$q(z, C) \le \#(\{k : z + ke_0 \in n^r\} \le n$$

for every $z \in n^r$. So setting $D = \{z : z \in n^r,\ q(z, C) \ge \delta n\}$, we have

$$2\delta n^{r+1} \le n\#(D) + \delta n \cdot n^r \le n\#(D) + \delta n^{r+1}$$

and $\#(D) \ge \delta n^r$, as claimed.

497N Theorem (FURSTENBURG 81) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra and $\langle \pi_i \rangle_{i<r}$ a non-empty finite commuting family of measure-preserving Boolean homomorphisms from $\mathfrak{A}$ to itself. If $a \in \mathfrak{A} \setminus \{0\}$, there is an $\eta > 0$ such that

$$\sum_{k=0}^{n-1} \bar{\mu}(\inf_{i<r} \pi_i^k a) \ge \eta n$$

for every $n \in \mathbb{N}$.

proof (a) To being with, suppose that every π_i is an automorphism. Doubling r if necessary, we can suppose that for every $i < r$ there is a $j < r$ such that $\pi_j = \pi_i^{-1}$. Let (Z, Σ, μ) be the Stone space of $(\mathfrak{A}, \bar{\mu})$ (321K), and set $E = \widehat{a}$, the open-and-closed subset of Z corresponding to $a \in \mathfrak{A}$. For each $i < r$ let $T_i : Z \to Z$ be the homeomorphism corresponding to $\pi_i : \mathfrak{A} \to \mathfrak{A}$, so that $T_i^{-1}[\widehat{b}] = \widehat{\pi_i b}$ for every $b \in \mathfrak{A}$ (312Q[10]); note that $T_i T_j$ corresponds to $\pi_j \pi_i$ (312R[11]) and $T_i T_j = T_j T_i$ (because the representations in 312Q are unique), for all i, $j < r$.

In 497M, set $\epsilon = \frac{1}{2}\mu E = \frac{1}{2}\bar{\mu} a$ and take a corresponding $\delta > 0$; set $\eta = \frac{1}{2}\delta^2 \epsilon$. Now, given $n \ge 1$, then for $z \in n^r$ set $\tilde{T}_z = \prod_{i<r} T_i^{z(i)}$. (We can speak of the product without inhibitions because the T_i commute.) Consider the set $W = \{(x, z) : z \in n^r,\ \tilde{T}_z(x) \in E\}$. Then $W^{-1}[\{z\}]$ has measure μE for every z, so if we set $F = \{x : x \in E,\ \#(W[\{x\}]) \ge \epsilon n^r\}$ we have

$$n^r \mu E \le n^r \mu F + \epsilon n^r, \quad \mu F \ge \epsilon.$$

In the notation of 497M, set

$$V = \{(x, z) : (x, z) \in W,\ q(z, W[\{x\}]) \ge \delta n\};$$

then for any $x \in F$ we have $\#(V[\{x\}]) \ge \delta n^r$, by the choice of δ. There must therefore be a $z \in n^r$ such that $\mu V^{-1}[\{z\}] \ge \delta \mu F \ge \delta\epsilon$. Take any $x \in V^{-1}[\{z\}]$ and $k \in \Delta(z, W[\{x\}])$. Setting $e_i(i) = 1$ and $e_i(j) = 0$ for $i < r$ and $j \in r \setminus \{i\}$, $z + ke_i \in W[\{x\}]$, that is, $T_i^k \tilde{T}_z(x) \in E$, for every $i < r$. Also $|k| < n$. Set $G = \tilde{T}_z[V^{-1}[\{z\}]]$, so that $\mu G \ge \delta\epsilon$ and for every $y \in G$ we have $\#(\{k : |k| < n,\ T_i^k(y) \in E$ for every $i < r\}) \ge \delta n$. But this means that

[10] Formerly 312P.
[11] Formerly 312Q.

$$\sum_{k=0}^{n-1}\bar{\mu}(\inf_{i<r}\pi_i^k a)=\sum_{k=0}^{n-1}\mu\{x:T_i^k(x)\in E \text{ for every } i<r\}$$
$$\ge\frac{1}{2}\sum_{|k|<n}\mu\{x:T_i^k(x)\in E \text{ for every } i<r\}$$

(because if $T_i^k(x)\in E$ for every $i<r$ then $T_i^{|k|}(x)\in E$ for every $i<r$)

$$=\frac{1}{2}\int\#(\{k:|k|<n,\, T_i^k(x)\in E \text{ for every } i<r\})\mu(dx)$$
$$\ge\frac{1}{2}\delta n\mu G\ge\frac{1}{2}\delta^2\epsilon n=\eta n,$$

as required.

(b) For the general case, 328J tells us that there are a probability algebra $(\mathfrak{C},\bar{\lambda})$, a measure-preserving Boolean homomorphism $\pi:\mathfrak{A}\to\mathfrak{C}$ and a commuting family $\langle\tilde{\pi}_i\rangle_{i<r}$ of measure-preserving automorphisms of $\mathfrak{C}$ such that $\tilde{\pi}_i\pi=\pi\pi_i$ for every $i<r$. Now $\pi a\in\mathfrak{C}\setminus\{0\}$, so there is an $\eta>0$ such that

$$\eta n\le\sum_{k=0}^{n-1}\bar{\lambda}(\inf_{i<r}\tilde{\pi}_i^k\pi a)=\sum_{k=0}^{n-1}\bar{\lambda}(\inf_{i<r}\pi\pi_i^k a)=\sum_{k=0}^{n-1}\bar{\mu}(\inf_{i<r}\pi_i^k a)$$

for every $n\in\mathbb{N}$.

497X Basic exercises (a) Let (X,Σ,μ) be a probability space, $\langle\Sigma_i\rangle_{i\in I}$ an independent family of σ-subalgebras of Σ, and T a subalgebra of Σ such that $\mathrm{T}\cap\Sigma_i$ is metrically dense in Σ_i for every $i\in I$. For $J\subseteq I$ set $\tilde{\Sigma}_J=\bigvee_{i\in J}\Sigma_i$. Show that $\langle\tilde{\Sigma}_J\rangle_{J\subseteq I}$ has T-removable intersections.

(b) Let I be a set, G_I the group of permutations of I with its topology of pointwise convergence (441G, 449Xh), and $\bullet$ the action of G_I on $\mathcal{P}([I]^{<\omega})$ described in 497F. Show that $\bullet$ is continuous.

(c) In 497F, show that $\{\mu:\mu\in P_I \text{ is permutation-invariant}\}$ is a closed subset of P_I.

497Y Further exercises (a) (i) Show that if $A\subseteq\mathbb{N}$ has non-zero upper asymptotic density then there is a translation-invariant additive functional $\nu:\mathcal{P}\mathbb{Z}\to[0,1]$ such that $\nu A>0$. (ii) Consider the statement

(†) If $\epsilon>0$ and $A\subseteq\mathbb{N}$ are such that $\#(A\cap n)\ge\epsilon n$ for every n then A includes arithmetic progressions of all finite lengths.

Use Theorem 497N to prove (†). (iii) Find a direct proof that (†) implies Szemerédi's theorem.

497 Notes and comments I am grateful to T.D.Austin for introducing me to a preprint of Tao 07, on which this section is based.

Regarded as a proof of Szemerédi's theorem, the argument above has the virtues of reasonable brevity and (I hope) of completeness and correctness. It depends, of course, on non-trivial ideas from measure theory, which for anyone except a measure theorist will compromise the claim of 'brevity'; and even measure theorists may find that the proofs here demand close attention. There are further, more significant, defects. The outstanding problem associated with Szemerédi's theorem is the estimation of n_0 as a function of r and ϵ; and while in a theoretical sense it must be possible to trace through the arguments above to establish rigorous bounds, the methods are not well adapted to such an exercise, and one would not expect the bounds obtained to be good. There is also the point that I have made uninhibited use of the axiom of choice. The ultrafilter in 497J can easily be replaced by an appropriate sequence, but all standard treatments of measure theory assume at least the countable axiom of choice, and Szemerédi's theorem is clearly true in significantly weaker theories than ordinary ZF.

The first 'measure-theoretic' proof of Szemerédi's theorem was due to Furstenburg 77, and relied on a deep analysis of the structure of measure-preserving transformations. While the methods described here do not seem to give us any information on this structure, it is apparently a folklore result that the hypergraph removal lemma provides a quick proof of the basic theorem used in Furstenburg's approach (497N, 497Ya).

The value of the work here, therefore, lies less in its applications to the hypergraph removal lemma and Szemerédi's theorem, than in the idea of 'removable intersections', where Theorems 497E and 497G give us two remarkable results, and useful exercises in the theory of relative independence from §458. We also have an instructive example of a more general phenomenon. Given a sequence of finite objects with quantititative aspects, it is often profitable to seek a measure μ reflecting the asymptotic behaviour of this sequence; this is the idea of the construction in 497H. The 'quantitative aspects' here, as developed in 497J, are the proportion of functions from L to m_n which are embeddings of (L,T) in (m_n,T_n), and the proportion of simplices in $[m_n]^k$ which must be removed from T_n in order to destroy all these embeddings. The measure μ is set up to describe the limits of these proportions as measures of appropriate sets.

Returning to the definition 497Aa, most of its clauses can be expressed in terms of the measure algebra of the measure μ; but the final '$\bigcap_{i\in J}F_i=\emptyset$' has to be taken literally, and makes sense only in terms of the measure space itself. In the key application (part (d) of the proof of 497J), the original sets E_c, with negligible intersection, already belong to the algebra T, but the adjustment to sets F_c with empty intersection is still non-trivial, because of the requirement that each F_c must belong to the prescribed σ-algebra Σ_{J_c}.

I said in 497F that you could note that $\mathcal{P}([\mathbb{N}]^{<\omega})$ is homeomorphic to the Cantor set, so that $P_{\mathbb{N}}$ is isomorphic to the space of Radon probability measures on $\{0,1\}^{\mathbb{N}}$. However the point of the construction there is that we are looking at a particular action of the symmetric group $G_{\mathbb{N}}$ on $\mathcal{P}([\mathbb{N}]^{<\omega})$; and this has very little to do with the natural actions of $G_{\mathbb{N}}$ on $\mathcal{P}\mathbb{N}$ or $\{0,1\}^{\mathbb{N}}$, as studied in 459E and 459H, for instance. In particular, permutation-invariant measures, in the sense of 497Fb, will not normally be invariant under the much larger group derived from all permutations of $[\mathbb{N}]^{<\omega}$ rather than just those corresponding to permutations of $\mathbb{N}$.

I express 497N in terms of measure-preserving automorphisms of probability algebras in order to connect it with the treatment of ergodic theory in Chapter 38, but you will observe that the proof presented immediately shifts to a more traditional formulation in terms of probability spaces. This is only one of many multiple recurrence theorems, some of them much stronger (and, it seems, deeper) than 497N or, indeed, 497J.

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498 Cubes in product spaces

I offer a brief note on a special property of (Radon) product measures.

498A Proposition Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra with its measure-algebra topology (323A). Suppose that $A\subseteq\mathfrak{A}$ is an uncountable analytic set. Then there is a compact set $L\subseteq A$, homeomorphic to $\{0,1\}^{\mathbb{N}}$, such that $\inf L\ne 0$ in $\mathfrak{A}$.

proof $A\setminus\{0\}$ is still an uncountable analytic subset of $\mathfrak{A}$. By 423K, it has a subset homeomorphic to $\{0,1\}^{\mathbb{N}}\cong\mathcal{P}\mathbb{N}$; let $f:\mathcal{P}\mathbb{N}\to A\setminus\{0\}$ be an injective continuous function. Because $(\mathfrak{A},\bar\mu)$ is semi-finite, there is an $a\subseteq f(\emptyset)$ such that $0<\bar\mu a<\infty$; set $\delta=\frac{1}{2}\bar\mu a$. Note that $(I,J)\mapsto\bar\mu(a\cap f(I)\setminus f(J)):(\mathcal{P}\mathbb{N})^2\to\mathbb{R}$ is continuous. Choose a sequence $\langle k_n\rangle_{n\in\mathbb{N}}$ in $\mathbb{N}$ inductively, as follows. Given $\langle k_i\rangle_{i<n}$, set $K_n=\{k_i:i<n\}$. For each $J\subseteq K_n$ we have $\lim_{r\to\infty}\bar\mu(f(J)\setminus f(J\cup\{r\}))=0$, so there is a k_n, greater than k_i for every $i<n$, such that $\bar\mu(f(J)\setminus f(J\cup\{k_n\})\le 2^{-2n-1}\delta$ for every $J\subseteq K_n$; continue.

Now

$$\bar\mu(a\cap\inf_{J\subseteq K_n}f(J))\ge\delta(1+2^{-n})$$

for every $n\in\mathbb{N}$. **P** Induce on n. If $n=0$, then $a\cap\inf_{J\subseteq K_n}f(J)=a\cap f(\emptyset)$ has measure $2\delta=\delta(1+2^{-0})$. For the inductive step to $n+1\ge 1$, observe that

$$\begin{aligned}\bar\mu(a\cap\inf_{J\subseteq K_{n+1}}f(J))&=\bar\mu(a\cap\inf_{J\subseteq K_n}f(J)\cap f(J\cup\{k_n\}))\\&\ge\bar\mu(a\cap\inf_{J\subseteq K_n}f(J))-\sum_{J\subseteq K_n}\bar\mu(f(J)\setminus f(J\cup\{k_n\}))\\&\ge\delta(1+2^{-n})-\sum_{J\subseteq K_n}2^{-2n-1}\delta\end{aligned}$$

(by the inductive hypothesis and the choice of k_n)

$$=\delta(1+2^{-n-1}).$$

So the induction proceeds. **Q**

Set $K=\{k_i:i\in\mathbb{N}\}$, $c=\inf\{f(J):J\subseteq K\text{ is finite}\}$. Then

$$\bar\nu(a\cap c)=\inf_{n\in\mathbb{N}}\bar\mu(a\cap\inf_{J\subseteq K_n}f(J))\ge\delta,$$

and $c\ne0$. But now observe that $L=f[\mathcal{P}K]$ is a subset of A homeomorphic to $\mathcal{P}K$ and therefore to $\{0,1\}^{\mathbb{N}}$. Also $\{b:b\supseteq c\}$ is closed (323D(d-i)), so $C=\{J:f(J)\supseteq c\}$ is closed in $\mathcal{P}K$; as it includes the dense set $[K]^{<\omega}$, $C=\mathcal{P}K$ and $\inf L\supseteq c$ is non-zero.

498B Proposition (see Brodskiĭ 1949, Eggleston 54) Let $(X,\mathfrak{T},\Sigma,\mu)$ be an atomless Radon measure space, $(Y,\mathfrak{S},\mathrm{T},\nu)$ an effectively locally finite τ-additive topological measure space and $\tilde\lambda$ the τ-additive product measure on $X\times Y$ (417C, 417F). Then if $W\subseteq X\times Y$ is closed and $\tilde\lambda W>0$ there are a non-scattered compact set $K\subseteq X$ and a closed set $F\subseteq Y$ of positive measure such that $K\times F\subseteq W$.

proof (a) To begin with (down to the end of (c)), let us suppose that both μ and ν are totally finite. Let $(\mathfrak{B},\bar\nu)$ be the measure algebra of ν. Writing λ for the c.l.d. product measure on $X\times Y$, there is a $W'\supseteq W$ such that $\lambda W'$ is defined and equal to $\tilde\lambda W$ (apply 417C(b-v) to the complement of W). By 418Tb, there is a μ-conegligible set X_0 such that $W'[\{x\}]\in\mathrm{T}$ for every $x\in X_0$, $B=\{W'[\{x\}]^\bullet:x\in X_0\}$ is separable for the measure-algebra topology of $\mathfrak{B}$, and $x\mapsto W'[\{x\}]^\bullet:X_0\to\mathfrak{B}$ is measurable. Now Fubini's theorem, applied in the form of 252D to λ and in the form of 417Ga to $\tilde\lambda$, tells us that

$$\int\nu W'[\{x\}]\mu(dx)=\lambda W'=\tilde\lambda W=\int\nu W[\{x\}]\mu(dx).$$

So $X_1=\{x:x\in X_0,\ W'[\{x\}]^\bullet=W[\{x\}]^\bullet\}$ is μ-conegligible. Since the topology of $\mathfrak{B}$ is metrizable (323Ad or 323Gb), B is separable and metrizable, and $x\mapsto W[\{x\}]^\bullet:X_1\to B$ is almost continuous (418J, applied to the subspace measure on X_1). Let $K^*\subseteq X_1$ be a compact set of non-zero measure such that $x\mapsto W[\{x\}]^\bullet:K^*\to\mathfrak{B}$ is continuous.

(b) There is a non-zero $c\in\mathfrak{B}$ such that $K_c=\{x:x\in K^*,\ c\subseteq W[\{x\}]^\bullet\}$ is compact and not scattered. **P** Because $x\mapsto W[\{x\}]^\bullet$ is continuous on K^*, $B^*=\{W[\{x\}]^\bullet:x\in K^*\}$ is compact and every K_c is compact. (i) If B^* is countable, then $K^*=\bigcup_{b\in B^*}K_b$, so there is some $c\in B^*$ such that $\mu K_c>0$. Let E be a non-negligible self-supporting subset of K_c; then (because μ is atomless, therefore zero on singletons) E has no isolated points. So K_c is not scattered. (ii) If B^* is uncountable, then by 498A there is a set $D\subseteq B^*$, homeomorphic to $\{0,1\}^{\mathbb{N}}$, with a non-zero lower bound c in $\mathfrak{B}$. Now $\{W[\{x\}]^\bullet:x\in K_c\}$ includes D, so $\{0,1\}^{\mathbb{N}}$ and therefore $[0,1]$ are continuous images of closed subsets of K_c and K_c is not scattered (4A2G(j-iv)). **Q**

(c) Set $K=K_c$. Then 414Ac tells us that

$$(\bigcap_{x\in K}W[\{x\}])^\bullet=\inf_{x\in K}W[\{x\}]^\bullet\supseteq c$$

is non-zero, so $F=\bigcap_{x\in K}W[\{x\}]$ is non-negligible; while $K\times F\subseteq W$. Since every section $W[\{x\}]$ is closed, so is F. So we have found appropriate sets K and F, at least when μ and ν are totally finite.

(d) For the general case, we need observe only that by 417C(b-iii) there are $X'\in\Sigma$ and $Y'\in\mathrm{T}$, both of finite measure, such that $\tilde\lambda(W\cap(X'\times Y'))>0$. Now the subspace measure $\mu_{X'}$ on X' is atomless and Radon (214Ka, 416Rb), the subspace measure $\nu_{Y'}$ on Y' is τ-additive (414K), and the τ-additive product of $\mu_{X'}$ and $\nu_{Y'}$ is the subspace measure on $X'\times Y'$ induced by $\tilde\lambda$ (417I), while $W'=W\cap(X'\times Y')$ is relatively closed in $X'\times Y'$. So we can apply (a)-(c) to $\mu_{X'}$ and $\nu_{Y'}$ to see that there are a non-scattered compact set $K\subseteq X'$ and a non-negligible relatively closed set $F'\subseteq Y'$ such that $K\times F'\subseteq W'$. Now the closure $F=\overline{F'}$ of F' in Y is closed and $K\times F=\overline{K\times F'}\subseteq W$.

498C Proposition (see Ciesielski & Pawlikowski 03) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a countable family of atomless Radon probability spaces, and $\tilde\lambda$ the product Radon probability measure on $X=\prod_{i\in I}X_i$ (417Q, 417R). If $W\subseteq X$ and $\tilde\lambda W>0$, there is a family $\langle K_i\rangle_{i\in I}$ such that $K_i\subseteq X_i$ is a non-scattered compact set for each $i\in I$ and $\prod_{i\in I}K_i\subseteq W$.

proof (a) To begin with, let us suppose that $I=\mathbb{N}$. As $\tilde\lambda$ is inner regular with respect to the closed sets, it is enough to deal with the case in which W is closed. For each $n\in\mathbb{N}$, set $Y_n=\prod_{i\ge n}X_i$ and let $\tilde\lambda_n$ be the product Radon probability measure on Y_n, so that $\tilde\lambda_0=\tilde\lambda$ and $\tilde\lambda_n$ can be identified with the Radon product of μ_n and $\tilde\lambda_{n+1}$ (417J). Using 498B repeatedly, we can find non-scattered compact sets $K_n\subseteq X_n$ and closed non-negligible sets $W_n\subseteq Y_n$ such that $W_0=W$ and $K_n\times W_{n+1}\subseteq W_n$ for every n. In this case, $\prod_{i<n}K_i\times W_n\subseteq W_0$ for every n. If $x\in\prod_{i\in\mathbb{N}}K_i$, then there is for each $n\in\mathbb{N}$ an $x_n\in(\prod_{i<n}K_i)\times W_n$ such that $x_n{\restriction}n=x{\restriction}n$, just because W_n is not empty. But now every x_n belongs to W and so does $x=\lim_{n\to\infty}x_n$. x is arbitrary, $\prod_{i\in\mathbb{N}}K_i\subseteq W$.

(b) For the general case, we may suppose that $I\subseteq\mathbb{N}$. For $i\in\mathbb{N}\setminus I$, take $(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)$ to be $[0,1]$ with Lebesgue measure. Set $\tilde W=\{x:x\in\prod_{i\in\mathbb{N}}X_i,\ x{\restriction}I\in W\}$. By (a), there are non-scattered compact sets $K_i\subseteq X_i$ such that $\prod_{i\in\mathbb{N}}K_i\subseteq\tilde W$, in which case $\prod_{i\in I}K_i\subseteq W$, as required.

498X Basic exercises (a) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon measure space, $(Y,\mathfrak{S},\mathrm{T},\nu)$ an effectively locally finite τ-additive topological measure space, and $\tilde\lambda$ the τ-additive product measure on $X\times Y$. Show that if $W\subseteq X\times Y$ is closed and $\tilde\lambda W>0$ there are a compact set $K\subseteq X$ and a closed set $F\subseteq Y$ of positive measure such that $K\times F\subseteq W$ and K is either non-scattered or non-negligible.

(b) Let $(X,\mathfrak{T},\Sigma,\mu)$ be an atomless Radon measure space, (Y,T,ν) any measure space, and λ the c.l.d. product measure on $X\times Y$. Show that if $W\subseteq X\times Y$ and $\lambda W>0$ there are a non-scattered compact set $K\subseteq X$ and a set $F\subseteq Y$ of positive measure such that $K\times F\subseteq W$. (*Hint*: reduce to the case in which ν is totally finite and T is countably generated, so that the completion of ν is a quasi-Radon measure for an appropriate second-countable topology.)

(c) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be any family of atomless Radon probability spaces, and λ the ordinary product measure on $X=\prod_{i\in I}X_i$. Show that if $W\subseteq X$ and $\lambda W>0$ then there are non-scattered compact sets $K_i\subseteq X_i$ for $i\in I$ such that $\prod_{i\in I}K_i\subseteq W$.

(d) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a countable family of atomless Radon probability spaces, and $W\subseteq\prod_{i\in I}X_i$ a set with positive measure for the Radon product of $\langle\mu_i\rangle_{i\in I}$. Show that there are atomless Radon probability measures ν_i on X_i such that W is conegligible for the Radon product of $\langle\nu_i\rangle_{i\in I}$. (*Hint*: 439Xh(vii).)

498Y Further exercises (a) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of atomless perfect probability spaces, and λ the ordinary product measure on $X=\prod_{i\in I}X_i$. Show that if $W\subseteq X$ and $\lambda W>0$ then there are sets $K_i\subseteq X_i$ for $i\in I$, all with cardinal $\mathfrak{c}$, such that $\prod_{i\in I}K_i\subseteq W$.

498 Notes and comments I have previously noted (325Yd) that a set W of positive measure in a product space need not include the product of two sets of positive measure; this fact is also the basis of 419E. Here, however, we see that if one of the factors is a Radon measure space then W does include the product of a non-trivial compact set and a set of positive measure. There are many possible variations on the result, corresponding to different product measures (498B, 498Xb) and different notions of 'non-trivial' (498Xa, 498Ya). The most important of the latter seems to be the idea of a 'non-scattered' compact set K; this is a quick way of saying that $[0,1]$ is a continuous image of K, which is a little stronger than saying that $\#(K)\ge\mathfrak{c}$, and arises naturally from the proof of 498B.

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