Version of 9.5.11

Chapter 48

Gauge integrals

For the penultimate chapter of this volume I turn to a completely different approach to integration which has been developed in the last fifty years, following Kurzweil 57 and Henstock 63. This depends for its inspiration on a formulation of the Riemann integral (see 481Xe), and leads in particular to some remarkable extensions of the Lebesgue integral (§§483-484). While (in my view) it remains peripheral to the most important parts of measure theory, it has deservedly attracted a good deal of interest in recent years, and is entitled to a place here.

From the very beginning, in the definitions of §122, I have presented the Lebesgue integral in terms of almost-everywhere approximations by simple functions. Because the integral $\int\lim_{n\to\infty}f_n$ of a limit is *not* always the limit $\lim_{n\to\infty}\int f_n$ of the integrals, we are forced, from the start, to constrain ourselves by the ordering, and to work with monotone or dominated sequences. This almost automatically leads us to an 'absolute' integral, in which $|f|$ is integrable whenever f is, whether we start from measures (as in Chapter 11) or from linear functionals (as in §436). For four volumes now I have been happily developing the concepts and intuitions appropriate to such integrals. But if we return to one of the foundation stones of Lebesgue's theory, the Fundamental Theorem of Calculus, we find that it is easy to construct a differentiable function f such that the absolute value $|f'|$ of its derivative is not integrable (483Xd). It was observed very early (Perron 1914) that the Lebesgue integral can be extended to integrate the derivative of any function which is differentiable everywhere. The achievement of Henstock 63 was to find a formulation of this extension which was conceptually transparent enough to lend itself to a general theory, fragments of which I will present here.

The first step is to set out the essential structures on which the theory depends (§481), with a first attempt at a classification scheme. (One of the most interesting features of the Kurzweil-Henstock approach is that we have an extraordinary degree of freedom in describing our integrals, and apart from the Henstock integral itself it is not clear that we have yet found the right canonical forms to use.) In §482 I give a handful of general theorems showing what kinds of result can be expected and what difficulties arise. In §483, I work through the principal properties of the Henstock integral on the real line, showing, in particular, that it coincides with the Perron and special Denjoy integrals. Finally, in §484, I look at a very striking integral on $\mathbb{R}^r$, due to W.F.Pfeffer.

Version of 4.9.09/30.11.12

481 Tagged partitions

I devote this section to establishing some terminology (481A-481B, 481E-481G) and describing a variety of examples (481I-481Q), some of which will be elaborated later. The clearest, simplest and most important example is surely Henstock's integral on a closed bounded interval (481J), so I recommend turning immediately to that paragraph and keeping it in mind while studying the notation here. It may also help you to make sense of the definitions here if you glance at the statements of some of the results in §482; in this section I give only the formula defining gauge integrals (481C).

481A Tagged partitions and Riemann sums The common idea underlying this chapter is the following. We have a set X and a functional ν defined on some family $\mathcal{C}$ of subsets of X. We seek to define an integral $\int f d\nu$, for functions f with domain X, as a limit of *finite* **Riemann sums** $\sum_{i=0}^{n} f(x_i)\nu C_i$, where $x_i \in X$ and $C_i \in \mathcal{C}$ for $i \le n$. A **tagged partition** on X will be a finite subset $\boldsymbol{t}$ of $X \times \mathcal{P}X$.

The next element of the definition will be a description of a filter $\mathcal{F}$ on the set T of tagged partitions, so that the integral will be the limit (when it exists) of the sums along the filter.

481B Notation (a) If X is a set, a **straightforward set of tagged partitions** on X is a set of the form

$$T = \{\boldsymbol{t} : \boldsymbol{t} \in [Q]^{<\omega},\ C \cap C' = \emptyset \text{ whenever } (x, C),\ (x', C') \text{ are distinct members of } \boldsymbol{t}\}$$

where $Q \subseteq X \times \mathcal{P}X$; I will say that T is **generated** by Q.

(b) If X is a set and $\boldsymbol{t} \subseteq X \times \mathcal{P}X$ is a tagged partition,

$$W_{\boldsymbol{t}} = \bigcup\{C : (x, C) \in \boldsymbol{t}\}.$$

(c) If X is a set, $\mathcal{C}$ is a family of subsets of X, f and ν are real-valued functions, and $\boldsymbol{t} \in [X \times \mathcal{C}]^{<\omega}$ is a tagged partition, then

$$S_{\boldsymbol{t}}(f, \nu) = \sum_{(x,C)\in\boldsymbol{t}} f(x)\nu C$$

whenever $\boldsymbol{t} \subseteq \operatorname{dom} f \times \operatorname{dom} \nu$.

481C Proposition Let X be a set, $\mathcal{C}$ a family of subsets of X, $T \subseteq [X \times \mathcal{C}]^{<\omega}$ a non-empty set of tagged partitions and $\mathcal{F}$ a filter on T. For real-valued functions f and ν, set

$$I_\nu(f) = \lim_{\boldsymbol{t}\to\mathcal{F}} S_{\boldsymbol{t}}(f, \nu)$$

if this is defined in $\mathbb{R}$.

(a) I_ν is a linear functional defined on a linear subspace of $\mathbb{R}^X$.

(b) Now suppose that $\nu C \geq 0$ for every $C \in \mathcal{C}$. Then

(i) I_ν is a positive linear functional;

(ii) if f, $g : X \to \mathbb{R}$ are such that $|f| \leq g$ and $I_\nu(g)$ is defined and equal to 0, then $I_\nu(f)$ is defined and equal to 0.

481D Remarks (a) Functionals $I_\nu = \lim_{\boldsymbol{t}\to\mathcal{F}} S_{\boldsymbol{t}}(., \nu)$, as described above, are called **gauge integrals**.

(c) An extension which is sometimes useful is to allow ν to be undefined on part of $\mathcal{C}$. In this case, set $\mathcal{C}_0 = \nu^{-1}[\mathbb{R}]$. Provided that $T \cap [X \times \mathcal{C}_0]^{<\omega}$ belongs to $\mathcal{F}$, we can still define I_ν, and 481C will still be true.

481E Gauges(a) If X is a set, a **gauge** on X is a subset δ of $X \times \mathcal{P}X$. For a gauge δ, a tagged partition $\boldsymbol{t}$ is **δ-fine** if $\boldsymbol{t} \subseteq \delta$. Now, for a set Δ of gauges and a non-empty set T of tagged partitions, we can seek to define a filter $\mathcal{F}$ on T as the filter generated by sets of the form $T_\delta = \{\boldsymbol{t} : \boldsymbol{t} \in T \text{ is } \delta\text{-fine}\}$ as δ runs over Δ.

(b) If $(X, \mathfrak{T})$ is a topological space, a **neighbourhood gauge** on X is a set expressible in the form $\delta = \{(x, C) : x \in X,\ C \subseteq G_x\}$ where $\langle G_x\rangle_{x\in X}$ is a family of open sets such that $x \in G_x$ for every $x \in X$. It is useful to note (i) that the family $\langle G_x\rangle_{x\in X}$ can be recovered from δ (ii) that $\delta_1 \cap \delta_2$ is a neighbourhood gauge whenever δ_1 and δ_2 are. When (X, ρ) is a metric space, we have the **uniform metric gauges**

$$\delta_\eta = \{(x, C) : x \in X,\ C \subseteq X,\ \rho(x, y) < \eta \text{ for every } y \in C\}$$

for $\eta > 0$, used in the Riemann integral.

(c) If X is a set and $\Delta \subseteq \mathcal{P}(X \times \mathcal{P}X)$ is a family of gauges on X, I will say that Δ is **countably full** if whenever $\langle\delta_n\rangle_{n\in\mathbb{N}}$ is a sequence in Δ, and $\phi : X \to \mathbb{N}$ is a function, then there is a $\delta \in \Delta$ such that $(x, C) \in \delta_{\phi(x)}$ whenever $(x, C) \in \delta$. I will say that Δ is **full** if whenever $\langle\delta_x\rangle_{x\in X}$ is a family in Δ, then there is a $\delta \in \Delta$ such that $(x, C) \in \delta_x$ whenever $(x, C) \in \delta$.

Of course a full set of gauges is countably full. Observe that if $(X, \mathfrak{T})$ is any topological space, the set of all neighbourhood gauges on X is full.

481F Residual sets Let $\mathfrak{R}$ be a collection of **residual families** $\mathcal{R} \subseteq \mathcal{P}X$. It will help to have a phrase corresponding to the phrase 'δ-fine': if $\mathcal{R} \subseteq \mathcal{P}X$, and $\boldsymbol{t}$ is a tagged partition on X, $\boldsymbol{t}$ is **$\mathcal{R}$-filling** if $X \setminus W_{\boldsymbol{t}} \in \mathcal{R}$. Now, given a family $\mathfrak{R}$ of residual sets, and a family Δ of gauges on X, we can seek to define a filter $\mathcal{F}(T, \Delta, \mathfrak{R})$ on T as that generated by sets of the form T_δ, for $\delta \in \Delta$, and $T'_{\mathcal{R}}$, for $\mathcal{R} \in \mathfrak{R}$, where

$$T'_{\mathcal{R}} = \{\boldsymbol{t} : \boldsymbol{t} \in T \text{ is } \mathcal{R}\text{-filling}\}.$$

When there is such a filter, I will say that T is **compatible** with Δ and $\mathfrak{R}$.

481G Subdivisions I will say that $(X,T,\Delta,\mathfrak{R})$ is a **tagged-partition structure allowing subdivisions**, **witnessed by** $\mathcal{C}$, if

(i) X is a set.

(ii) Δ is a non-empty downwards-directed family of gauges on X.

(iii)(α) $\mathfrak{R}$ is a non-empty downwards-directed collection of families of subsets of X, all containing $\emptyset$;

(β) for every $\mathcal{R}\in\mathfrak{R}$ there is an $\mathcal{R}'\in\mathfrak{R}$ such that $A\cup B\in\mathcal{R}$ whenever A, $B\in\mathcal{R}'$ are disjoint.

(iv) $\mathcal{C}$ is a family of subsets of X such that whenever C, $C'\in\mathcal{C}$ then $C\cap C'\in\mathcal{C}$ and $C\setminus C'$ is expressible as the union of a disjoint finite subset of $\mathcal{C}$.

(v) Whenever $\mathcal{C}_0\subseteq\mathcal{C}$ is finite and $\mathcal{R}\in\mathfrak{R}$, there is a finite set $\mathcal{C}_1\subseteq\mathcal{C}$, including $\mathcal{C}_0$, such that $X\setminus\bigcup\mathcal{C}_1\in\mathcal{R}$.

(vi) $T\subseteq[X\times\mathcal{C}]^{<\omega}$ is a non-empty straightforward set of tagged partitions on X.

(vii) Whenever $C\in\mathcal{C}$, $\delta\in\Delta$ and $\mathcal{R}\in\mathfrak{R}$ there is a δ-fine tagged partition $\boldsymbol{t}\in T$ such that $W_{\boldsymbol{t}}\subseteq C$ and $C\setminus W_{\boldsymbol{t}}\in\mathcal{R}$.

481H Remarks (d) Let $\mathfrak{A}$ be a Boolean algebra and $C\subseteq\mathfrak{A}$. Set

$$E=\{\sup C_0 : C_0\subseteq C \text{ is finite and disjoint}\}.$$

If $c\cap c'$ and $c\setminus c'$ belong to E for all c, $c'\in C$, then E is a subring of $\mathfrak{A}$.

In particular, if $\mathcal{C}\subseteq\mathcal{P}X$ has the properties in (iv) of 481G, then

$$\mathcal{E}=\{\bigcup\mathcal{C}_0 : \mathcal{C}_0\subseteq\mathcal{C} \text{ is finite and disjoint}\}$$

is a ring of subsets of X.

(e) Suppose that X is a set and that $\mathfrak{R}\subseteq\mathcal{P}\mathcal{P}X$ satisfies (iii) of 481G. Then for every $\mathcal{R}\in\mathfrak{R}$ there is a non-increasing sequence $\langle\mathcal{R}_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{R}$ such that $\bigcup_{i\le n}A_i\in\mathcal{R}$ whenever $A_i\in\mathcal{R}_i$ for $i\le n$ and $\langle A_i\rangle_{i\le n}$ is disjoint.

(f) If $(X,T,\Delta,\mathfrak{R})$ is a tagged-partition structure allowing subdivisions, then T is compatible with Δ and $\mathfrak{R}$.

481I The proper Riemann integral Fix a non-empty closed interval $X=[a,b]\subseteq\mathbb{R}$. Write $\mathcal{C}$ for the set of all intervals (open, closed or half-open, and allowing the empty set to count as an interval) included in $[a,b]$, and set $Q=\{(x,C) : C\in\mathcal{C},\ x\in\overline{C}\}$; let T be the straightforward set of tagged partitions generated by Q. Let Δ be the set of uniform metric gauges on X, and $\mathfrak{R}=\{\{\emptyset\}\}$. Then $(X,T,\Delta,\mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$. If $a<b$, then Δ is not countably full.

481J The Henstock integral on a bounded interval Take X, $\mathcal{C}$, T and $\mathfrak{R}$ as in 481I. This time, let Δ be the set of *all* neighbourhood gauges on $[a,b]$. Then $(X,T,\Delta,\mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$, and Δ is countably full.

481K The Henstock integral on $\mathbb{R}$ This time, set $X=\mathbb{R}$ and let $\mathcal{C}$ be the family of all bounded intervals in $\mathbb{R}$. Let T be the straightforward set of tagged partitions generated by $\{(x,C) : C\in\mathcal{C},\ x\in\overline{C}\}$. Following 481J, let Δ be the set of all neighbourhood gauges on $\mathbb{R}$. This time, set $\mathfrak{R}=\{\mathcal{R}_{ab} : a\le b\in\mathbb{R}\}$, where $\mathcal{R}_{ab}=\{\mathbb{R}\setminus[c,d] : c\le a,\ d\ge b\}\cup\{\emptyset\}$. Then $(X,T,\Delta,\mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$.

481L The symmetric Riemann-complete integral Again take $X=\mathbb{R}$, and $\mathcal{C}$ the set of all bounded intervals in $\mathbb{R}$. This time, take T to be the straightforward set of tagged partitions generated by the set of pairs (x,C) where $C\in\mathcal{C}\setminus\{\emptyset\}$ and x is the *midpoint* of C. As in 481K, take Δ to be the set of all neighbourhood gauges on $\mathbb{R}$; but this time take $\mathfrak{R}=\{\mathcal{R}'_a : a\ge 0\}$, where $\mathcal{R}'_a=\{\mathbb{R}\setminus[-c,c] : c\ge a\}\cup\{\emptyset\}$. Then T is compatible with Δ and $\mathfrak{R}$.

481M The McShane integral on an interval As in 481J, take $X = [a, b]$ and let $\mathcal{C}$ be the family of subintervals of $[a, b]$. This time, take T to be the straightforward set of tagged partitions generated by $Q = X \times \mathcal{C}$. As in 481J, let Δ be the set of all neighbourhood gauges on X, and $\mathfrak{R} = \{\{\emptyset\}\}$. $(X, T, \Delta, \mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$.

481N The McShane integral on a topological space Now let $(X, \mathfrak{T}, \Sigma, \mu)$ be any effectively locally finite τ-additive topological measure space, and take $\mathcal{C} = \{E : E \in \Sigma,\ \mu E < \infty\}$, $Q = X \times \mathcal{C}$; let T be the straightforward set of tagged partitions generated by Q. Again let Δ be the set of all neighbourhood gauges on X. For any set $E \in \Sigma$ of finite measure and $\eta > 0$, let $\mathcal{R}_{E\eta}$ be the set $\{F : F \in \Sigma,\ \mu(F \cap E) \le \eta\}$, and set $\mathfrak{R} = \{\mathcal{R}_{E\eta} : \mu E < \infty,\ \eta > 0\}$. Then $(X, T, \Delta, \mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$.

481O Convex partitions in $\mathbb{R}^r$ Fix $r \ge 1$. Let us say that a **convex polytope** in $\mathbb{R}^r$ is a non-empty bounded set expressible as the intersection of finitely many open or closed half-spaces; let $\mathcal{C}$ be the family of convex polytopes in $X = \mathbb{R}^r$, and T the straightforward set of tagged partitions generated by $\{(x, C) : x \in \overline{C}\}$. Let Δ be the set of neighbourhood gauges on $\mathbb{R}^r$. For $a \ge 0$, let $\mathcal{C}_a$ be the set of closed convex polytopes $C \subseteq \mathbb{R}^r$ such that, for some $b \ge a$, $B(0, b) \subseteq C \subseteq B(0, 2b)$, where $B(0, b)$ is the ordinary Euclidean ball with centre 0 and radius b; set $\mathcal{R}_a = \{\mathbb{R}^r \setminus C : C \in \mathcal{C}_a\} \cup \{\emptyset\}$, and $\mathfrak{R} = \{\mathcal{R}_a : a \ge 0\}$. Then $(X, T, \Delta, \mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$.

481P Box products Let $\langle (X_i, \mathfrak{T}_i) \rangle_{i \in I}$ be a non-empty family of non-empty compact metrizable spaces with product $(X, \mathfrak{T})$. Set $\pi_i(x) = x(i)$ for $x \in X$ and $i \in I$. For each $i \in I$, let $\mathcal{C}_i \subseteq \mathcal{P}X_i$ be such that (α) whenever E, $E' \in \mathcal{C}_i$ then $E \cap E' \in \mathcal{C}_i$ and $E \setminus E'$ is expressible as the union of a disjoint finite subset of $\mathcal{C}_i$ (β) $\mathcal{C}_i$ includes a base for $\mathfrak{T}_i$.

Let $\mathcal{C}$ be the set of subsets of X of the form

$$C = \{X \cap \bigcap_{i \in J} \pi_i^{-1}[E_i] : J \in [I]^{<\omega},\ E_i \in \mathcal{C}_i \text{ for every } i \in J\},$$

and let T be the straightforward set of tagged partitions generated by $\{(x, C) : C \in \mathcal{C},\ x \in \overline{C}\}$. Let Δ be the set of those neighbourhood gauges δ on X defined by families $\langle G_x \rangle_{x \in X}$ of open sets such that, for some countable $J \subseteq I$, every G_x is determined by coordinates in J. Then $(X, T, \Delta, \{\{\emptyset\}\})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$. Δ is countably full; Δ is full iff $I' = \{i : \#(X_i) > 1\}$ is countable.

481Q The approximately continuous Henstock integral Let μ be Lebesgue measure on $\mathbb{R}$. As in 481K, let $\mathcal{C}$ be the family of non-empty bounded intervals in $\mathbb{R}$, T the straightforward set of tagged partitions generated by $\{(x, C) : C \in \mathcal{C},\ x \in \overline{C}\}$, and $\mathfrak{R} = \{\mathcal{R}_{ab} : a,\ b \in \mathbb{R}\}$, where $\mathcal{R}_{ab} = \{\mathbb{R} \setminus [c, d] : c \le a,\ d \ge b\} \cup \{\emptyset\}$ for a, $b \in \mathbb{R}$.

Let E be the set of families $\boldsymbol{e} = \langle E_x \rangle_{x \in \mathbb{R}}$ where every E_x is a measurable set containing x such that x is a density point of E_x. For $\boldsymbol{e} = \langle E_x \rangle_{x \in \mathbb{R}} \in \mathrm{E}$, set

$$\delta_{\boldsymbol{e}} = \{(x, C) : C \in \mathcal{C},\ x \in \overline{C},\ \inf C \in E_x \text{ and } \sup C \in E_x\}.$$

Set $\Delta = \{\delta_{\boldsymbol{e}} : \boldsymbol{e} \in \mathrm{E}\}$. Then $(X, T, \Delta, \mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$, and Δ is full.

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482 General theory

I turn now to results which can be applied to a wide variety of tagged-partition structures. The first step is a 'Saks-Henstock' lemma (482B), a fundamental property of tagged-partition structures allowing subdivisions. In order to relate gauge integrals to the ordinary integrals treated elsewhere in this treatise, we need to know when gauge-integrable functions are measurable (482E) and when integrable functions are gauge-integrable (482F). There are significant difficulties when we come to interpret gauge integrals over subspaces, but I give a partial result in 482G. 482I, 482K and 482M are gauge-integral versions of the Fundamental Theorem of Calculus, B.Levi's theorem and Fubini's theorem, while 482H is a limit theorem of a new kind, corresponding to classical improper integrals.

Henstock's integral (481J-481K) remains the most important example and the natural test case for the ideas here; I will give the details in the next section, and you may wish to take the two sections in parallel.

482A Lemma Suppose that $(X, T, \Delta, \mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C} \subseteq \mathcal{P}X$.

(a) Whenever $\delta \in \Delta$, $\mathcal{R} \in \mathfrak{R}$ and E belongs to the subalgebra of $\mathcal{P}X$ generated by $\mathcal{C}$, there is a δ-fine $\boldsymbol{s} \in T$ such that $W_{\boldsymbol{s}} \subseteq E$ and $E \setminus W_{\boldsymbol{s}} \in \mathcal{R}$.

(b) Whenever $\delta \in \Delta$, $\mathcal{R} \in \mathfrak{R}$ and $\boldsymbol{t} \in T$ is δ-fine, there is a δ-fine $\mathcal{R}$-filling $\boldsymbol{t}' \in T$ including $\boldsymbol{t}$.

(c) Suppose that $f : X \to \mathbb{R}$, $\nu : \mathcal{C} \to \mathbb{R}$, $\delta \in \Delta$, $\mathcal{R} \in \mathfrak{R}$ and $\epsilon \geq 0$ are such that $|S_{\boldsymbol{t}}(f, \nu) - S_{\boldsymbol{t}'}(f, \nu)| \leq \epsilon$ whenever $\boldsymbol{t}$, $\boldsymbol{t}' \in T$ are δ-fine and $\mathcal{R}$-filling. Then

(i) $|S_{\boldsymbol{t}}(f, \nu) - S_{\boldsymbol{t}'}(f, \nu)| \leq \epsilon$ whenever $\boldsymbol{t}$, $\boldsymbol{t}' \in T$ are δ-fine and $W_{\boldsymbol{t}} = W_{\boldsymbol{t}'}$;

(ii) whenever $\boldsymbol{t} \in T$ is δ-fine, and $\delta' \in \Delta$, there is a δ'-fine $\boldsymbol{s} \in T$ such that $W_{\boldsymbol{s}} \subseteq W_{\boldsymbol{t}}$ and $|S_{\boldsymbol{s}}(f, \nu) - S_{\boldsymbol{t}}(f, \nu)| \leq \epsilon$.

(d) Suppose that $f : X \to \mathbb{R}$ and $\nu : \mathcal{C} \to \mathbb{R}$ are such that $I_\nu(f) = \lim_{\boldsymbol{t} \to \mathcal{F}(T, \Delta, \mathfrak{R})} S_{\boldsymbol{t}}(f, \nu)$ is defined. Then for any $\epsilon > 0$ there is a $\delta \in \Delta$ such that $S_{\boldsymbol{t}}(f, \nu) \leq I_\nu(f) + \epsilon$ for every δ-fine $\boldsymbol{t} \in T$.

(e) Suppose that $f : X \to \mathbb{R}$ and $\nu : \mathcal{C} \to \mathbb{R}$ are such that $I_\nu(f) = \lim_{\boldsymbol{t} \to \mathcal{F}(T, \Delta, \mathfrak{R})} S_{\boldsymbol{t}}(f, \nu)$ is defined. Then for any $\epsilon > 0$ there is a $\delta \in \Delta$ such that $|S_{\boldsymbol{t}}(f, \nu)| \leq \epsilon$ whenever $\boldsymbol{t} \in T$ is δ-fine and $W_{\boldsymbol{t}} = \emptyset$.

482B Saks-Henstock Lemma Let $(X, T, \Delta, \mathfrak{R})$ be a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$, and $f : X \to \mathbb{R}$, $\nu : \mathcal{C} \to \mathbb{R}$ functions such that $I_\nu(f) = \lim_{\boldsymbol{t} \to \mathcal{F}(T, \Delta, \mathfrak{R})} S_{\boldsymbol{t}}(f, \nu)$ is defined. Let $\mathcal{E}$ be the algebra of subsets of X generated by $\mathcal{C}$. Then there is a unique additive functional $F : \mathcal{E} \to \mathbb{R}$ such that for every $\epsilon > 0$ there are $\delta \in \Delta$ and $\mathcal{R} \in \mathfrak{R}$ such that

(α) $\sum_{(x,C) \in \boldsymbol{t}} |F(C) - f(x)\nu C| \leq \epsilon$ for every δ-fine $\boldsymbol{t} \in T$,

(β) $|F(E)| \leq \epsilon$ whenever $E \in \mathcal{E} \cap \mathcal{R}$.

Moreover, $F(X) = I_\nu(f)$.

482C Definition In the context of 482B, I will call the function F the **Saks-Henstock indefinite integral** of f.

482D Theorem Let $(X, T, \Delta, \mathfrak{R})$ be a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$, and $\nu : \mathcal{C} \to \mathbb{R}$ any function. Let $\mathcal{E}$ be the algebra of subsets of X generated by $\mathcal{C}$. If $f : X \to \mathbb{R}$ is any function, then the following are equiveridical:

(i) $I_\nu(f) = \lim_{\boldsymbol{t} \to \mathcal{F}(T, \Delta, \mathfrak{R})} S_{\boldsymbol{t}}(f, \nu)$ is defined in $\mathbb{R}$;

(ii) there is an additive functional $F : \mathcal{E} \to \mathbb{R}$ such that

(α) for every $\epsilon > 0$ there is a $\delta \in \Delta$ such that $\sum_{(x,C) \in \boldsymbol{t}} |F(C) - f(x)\mu C| \leq \epsilon$ for every δ-fine $\boldsymbol{t} \in T$,

(β) for every $\epsilon > 0$ there is an $\mathcal{R} \in \mathfrak{R}$ such that $|F(E)| \leq \epsilon$ for every $E \in \mathcal{E} \cap \mathcal{R}$;

(iii) there is an additive functional $F : \mathcal{E} \to \mathbb{R}$ such that

(α) for every $\epsilon > 0$ there is a $\delta \in \Delta$ such that $|F(W_{\boldsymbol{t}}) - \sum_{(x,C) \in \boldsymbol{t}} f(x)\mu C| \leq \epsilon$ for every δ-fine $\boldsymbol{t} \in T$,

(β) for every $\epsilon > 0$ there is an $\mathcal{R} \in \mathfrak{R}$ such that $|F(E)| \leq \epsilon$ for every $E \in \mathcal{E} \cap \mathcal{R}$.

In this case, $F(X) = I_\nu(f)$.

482E Theorem Let (X, ρ) be a metric space and μ a complete locally determined measure on X with domain Σ. Let $\mathcal{C}$, Q, T, Δ and $\mathfrak{R}$ be such that

(i) $\mathcal{C} \subseteq \Sigma$ and μC is finite for every $C \in \mathcal{C}$;

(ii) $Q \subseteq X \times \mathcal{C}$, and for each $C \in \mathcal{C}$, $(x, C) \in Q$ for almost every $x \in C$;

(iii) T is the straightforward set of tagged partitions generated by Q;

(iv) Δ is a downwards-directed family of gauges on X containing all the uniform metric gauges;

(v) if $\delta \in \Delta$, there are a negligible set $F \subseteq X$ and a neighbourhood gauge δ_0 on X such that $\delta \supseteq \delta_0 \setminus (F \times \mathcal{P}X)$;

(vi) $\mathfrak{R}$ is a downwards-directed collection of families of subsets of X such that whenever $E \in \Sigma$, $\mu E < \infty$ and $\epsilon > 0$, there is an $\mathcal{R} \in \mathfrak{R}$ such that $\mu^*(E \cap R) \le \epsilon$ for every $R \in \mathcal{R}$;

(vii) T is compatible with Δ and $\mathfrak{R}$.

Let $f : X \to \mathbb{R}$ be any function such that $I_\mu(f) = \lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f,\mu)$ is defined. Then f is Σ-measurable.

482F Proposition Let X, Σ, μ, $\mathfrak{T}$, T, Δ and $\mathfrak{R}$ be such that

(i) (X, Σ, μ) is a measure space;

(ii) $\mathfrak{T}$ is a topology on X such that μ is inner regular with respect to the closed sets and outer regular with respect to the open sets;

(iii) $T \subseteq [X \times \Sigma]^{<\omega}$ is a set of tagged partitions such that $C \cap C'$ is empty whenever (x, C), (x', C') are distinct members of any $\boldsymbol{t} \in T$;

(iv) Δ is a set of gauges on X containing every neighbourhood gauge on X;

(v) $\mathfrak{R}$ is a collection of families of subsets of X such that whenever $\mu E < \infty$ and $\epsilon > 0$ there is an $\mathcal{R} \in \mathfrak{R}$ such that $\mu^*(E \cap R) \le \epsilon$ for every $R \in \mathcal{R}$;

(vi) T is compatible with Δ and $\mathfrak{R}$.

Then $I_\mu(f) = \lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f,\mu)$ is defined and equal to $\int f d\mu$ for every μ-integrable function $f : X \to \mathbb{R}$.

482G Proposition Let $(X, T, \Delta, \mathfrak{R})$ be a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$. Suppose that

(i) $\mathfrak{T}$ is a topology on X, and Δ is the set of neighbourhood gauges on X;

(ii) $\nu : \mathcal{C} \to \mathbb{R}$ is a function which is additive in the sense that if $C_0, \dots, C_n \in \mathcal{C}$ are disjoint and have union $C \in \mathcal{C}$, then $\nu C = \sum_{i=0}^n \nu C_i$;

(iii) whenever $E \in \mathcal{C}$ and $\epsilon > 0$, there are closed sets $F \subseteq E$, $F' \subseteq X \setminus E$ such that $\sum_{(x,C)\in\boldsymbol{t}} |\nu C| \le \epsilon$ whenever $\boldsymbol{t} \in T$ and $W_{\boldsymbol{t}} \cap (F \cup F') = \emptyset$;

(iv) for every $E \in \mathcal{C}$ and $x \in X$ there is a neighbourhood G of x such that if $C \in \mathcal{C}$, $C \subseteq G$ and $\{(x, C)\} \in T$, there is a finite partition $\mathcal{D}$ of C into members of $\mathcal{C}$, each either included in E or disjoint from E, such that $\{(x, D)\} \in T$ for every $D \in \mathcal{D}$;

(v) for every $C \in \mathcal{C}$ and $\mathcal{R} \in \mathfrak{R}$, there is an $\mathcal{R}' \in \mathfrak{R}$ such that $C \cap A \in \mathcal{R}$ whenever $A \in \mathcal{R}'$.

Let $f : X \to \mathbb{R}$ be a function such that $I_\nu(f) = \lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f,\nu)$ is defined. Let $\mathcal{E}$ be the algebra of subsets of X generated by $\mathcal{C}$, and $F : \mathcal{E} \to \mathbb{R}$ the Saks-Henstock indefinite integral of f. Then $I_\nu(f \times \chi E)$ is defined and equal to $F(E)$ for every $E \in \mathcal{E}$.

482H Proposition Suppose that X, $\mathfrak{T}$, $\mathcal{C}$, ν, T, Δ and $\mathfrak{R}$ satisfy the conditions (i)-(v) of 482G, and that $f : X \to \mathbb{R}$, $\langle H_n \rangle_{n\in\mathbb{N}}$, H and γ are such that

(vi) $\langle H_n \rangle_{n\in\mathbb{N}}$ is a sequence of open subsets of X with union H,

(vii) $I_\nu(f \times \chi H_n)$ is defined for every $n \in \mathbb{N}$,

(viii) $\lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})} I_\nu(f \times \chi W_{\boldsymbol{t}\restriction H})$ is defined and equal to γ,

where $\boldsymbol{t}\restriction H = \{(x, C) : (x, C) \in \boldsymbol{t},\ x \in H\}$ for $\boldsymbol{t} \in T$. Then $I_\nu(f \times \chi H)$ is defined and equal to γ.

482I Integrating a derivative: Theorem Let X, $\mathcal{C} \subseteq \mathcal{P}X$, $\Delta \subseteq \mathcal{P}(X \times \mathcal{P}X)$, $\mathfrak{R} \subseteq \mathcal{P}\mathcal{P}X$, $T \subseteq [X \times \mathcal{C}]^{<\omega}$, $f : X \to \mathbb{R}$, $\nu : \mathcal{C} \to \mathbb{R}$, $F : \mathcal{C} \to \mathbb{R}$, $\theta : \mathcal{C} \to [0, 1]$ and $\gamma \in \mathbb{R}$ be such that

(i) T is a straightforward set of tagged partitions which is compatible with Δ and $\mathfrak{R}$,

(ii) Δ is a full set of gauges on X,

(iii) for every $x \in X$ and $\epsilon > 0$ there is a $\delta \in \Delta$ such that $|f(x)\nu C - F(C)| \le \epsilon\theta C$ whenever $(x, C) \in \delta$,

(iv) $\sum_{i=0}^n \theta C_i \le 1$ whenever $C_0, \dots, C_n \in \mathcal{C}$ are disjoint,

(v) for every $\epsilon > 0$ there is an $\mathcal{R} \in \mathfrak{R}$ such that $|\gamma - \sum_{C\in\mathcal{C}_0} F(C)| \le \epsilon$ whenever $\mathcal{C}_0 \subseteq \mathcal{C}$ is a finite disjoint set and $X \setminus \bigcup \mathcal{C}_0 \in \mathcal{R}$.

Then $I_\nu(f) = \lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f,\nu)$ is defined and equal to γ.

482J Definition Let X be a set, $\mathcal{C}$ a family of subsets of X, $T \subseteq [X \times \mathcal{C}]^{<\omega}$ a family of tagged partitions, $\nu : \mathcal{C} \to [0, \infty[$ a function, and Δ a family of gauges on X. I will say that ν is **moderated** (with respect to T and Δ) if there are a $\delta \in \Delta$ and a function $h : X \to]0, \infty[$ such that $S_{\boldsymbol{t}}(h, \nu) \le 1$ for every δ-fine $\boldsymbol{t} \in T$.

482K B.Levi's theorem Let $(X, T, \Delta, \mathfrak{R})$ be a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$, such that Δ is countably full, and $\nu : \mathcal{C} \to [0, \infty[$ a function which is moderated with respect to T and Δ.

Let $\langle f_n \rangle_{n \in \mathbb{N}}$ be a non-decreasing sequence of functions from X to $\mathbb{R}$ with supremum $f : X \to \mathbb{R}$. If $\gamma = \lim_{n\to\infty} I_\nu(f_n)$ is defined in $\mathbb{R}$, then $I_\nu(f)$ is defined and equal to γ.

482L Lemma Let X be a set, $\mathcal{C}$ a family of subsets of X, Δ a countably full downwards-directed set of gauges on X, $\mathfrak{R} \subseteq \mathcal{PP}X$ a downwards-directed collection of residual families, and $T \subseteq [X \times \mathcal{C}]^{<\omega}$ a straightforward set of tagged partitions of X compatible with Δ and $\mathfrak{R}$. Suppose further that whenever $\delta \in \Delta$, $\mathcal{R} \in \mathfrak{R}$ and $\boldsymbol{t} \in T$ is δ-fine, there is a δ-fine $\mathcal{R}$-filling $\boldsymbol{t}' \in T$ including $\boldsymbol{t}$. If $\nu : \mathcal{C} \to [0, \infty[$ and $f : X \to [0, \infty[$ are such that $I_\nu(f) = 0$, and $g : X \to \mathbb{R}$ is such that $g(x) = 0$ whenever $f(x) = 0$, then $I_\nu(g) = 0$.

482M Fubini's theorem Suppose that, for $i = 1$ and $i = 2$, we have X_i, $\mathfrak{T}_i$, T_i, Δ_i, $\mathcal{C}_i$ and ν_i such that

(i) $(X_i, \mathfrak{T}_i)$ is a topological space;

(ii) Δ_i is the set of neighbourhood gauges on X_i;

(iii) $T_i \subseteq [X_i \times \mathcal{C}_i]^{<\omega}$ is a straightforward set of tagged partitions, compatible with Δ_i and $\{\{\emptyset\}\}$;

(iv) $\nu_i : \mathcal{C}_i \to [0, \infty[$ is a function;

(v) ν_1 is moderated with respect to T_1 and Δ_1;

(vi) whenever $\delta \in \Delta_1$ and $\boldsymbol{s} \in T_1$ is δ-fine, there is a δ-fine $\boldsymbol{s}' \in T_1$, including $\boldsymbol{s}$, such that $W_{\boldsymbol{s}'} = X_1$.

Write X for $X_1 \times X_2$; Δ for the set of neighbourhood gauges on X; $\mathcal{C}$ for $\{C \times D : C \in \mathcal{C}_1,\ D \in \mathcal{C}_2\}$; Q for $\{((x, y), C \times D) : \{(x, C)\} \in T_1,\ \{(y, D)\} \in T_2\}$; T for the straightforward set of tagged partitions generated by Q; and set $\nu(C \times D) = \nu_1 C \cdot \nu_2 D$ for $C \in \mathcal{C}_1$, $D \in \mathcal{C}_2$.

(a) T is compatible with Δ and $\{\{\emptyset\}\}$.

(b) Let I_{ν_1}, I_{ν_2} and I_ν be the gauge integrals defined by these structures. Suppose that $f : X \to \mathbb{R}$ is such that $I_\nu(f)$ is defined. Set $f_x(y) = f(x, y)$ for $x \in X_1$, $y \in X_2$. Let $g : X_1 \to \mathbb{R}$ be any function such that $g(x) = I_{\nu_2}(f_x)$ whenever this is defined. Then $I_{\nu_1}(g)$ is defined and equal to $I_\nu(f)$.

Version of 6.9.10

483 The Henstock integral

I come now to the original gauge integral, the 'Henstock integral' for real functions. The first step is to check that the results of §482 can be applied to show that this is an extension of both the Lebesgue integral and the improper Riemann integral (483B), coinciding with the Lebesgue integral for non-negative functions (483C). It turns out that any Henstock integrable function can be approximated in a strong sense by a sequence of Lebesgue integrable functions (483G). The Henstock integral can be identified with the Perron and special Denjoy integrals (483J, 483N). Much of the rest of the section is concerned with indefinite Henstock integrals. Some of the results of §482 on tagged-partition structures allowing subdivisions condense into a particularly strong Saks-Henstock lemma (483F). If f is Henstock integrable, it is equal almost everywhere to the derivative of its indefinite Henstock integral (483I). Finally, indefinite Henstock integrals can be characterized as continuous ACG_* functions (483R).

483A Definition The following notation will apply throughout the section. Let $\mathcal{C}$ be the family of non-empty bounded intervals in $\mathbb{R}$, and let $T \subseteq [\mathbb{R} \times \mathcal{C}]^{<\omega}$ be the straightforward set of tagged partitions generated by $\{(x, C) : C \in \mathcal{C},\ x \in \overline{C}\}$. Let Δ be the set of all neighbourhood gauges on $\mathbb{R}$. Set $\mathfrak{R} = \{\mathcal{R}_{ab} : a \le b \in \mathbb{R}\}$, where $\mathcal{R}_{ab} = \{\mathbb{R} \setminus [c,d] : c \le a,\, d \ge b\} \cup \{\emptyset\}$. Then $(\mathbb{R}, T, \Delta, \mathfrak{R})$ is a tagged-partition structure allowing subdivisions, so T is compatible with Δ and $\mathfrak{R}$. The **Henstock integral** is the gauge integral defined from $(\mathbb{R}, T, \Delta, \mathfrak{R})$ and one-dimensional Lebesgue measure μ. For a function $f : \mathbb{R} \to \mathbb{R}$ I will say that f is **Henstock integrable**, and that $\int f = \gamma$, if $\lim_{\boldsymbol{t} \to \mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f, \mu)$ is defined and equal to $\gamma \in \mathbb{R}$. For $\alpha,\, \beta \in [-\infty, \infty]$ I will write $\int_\alpha^\beta f$ for $\int f \times \chi\,]\alpha, \beta[$ if this is defined in $\mathbb{R}$. I will use the symbol $\int$ for the ordinary integral, so that $\int f d\mu$ is the Lebesgue integral of f.

483B Theorem (a) Every Henstock integrable function on $\mathbb{R}$ is Lebesgue measurable.

(b) Every Lebesgue integrable function $f : \mathbb{R} \to \mathbb{R}$ is Henstock integrable, with the same integral.

(c) If f is Henstock integrable so is $f \times \chi C$ for every interval $C \subseteq \mathbb{R}$.

(d) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is any function, and $-\infty \le \alpha < \beta \le \infty$. Then

$$\int_\alpha^\beta f = \lim_{a \downarrow \alpha} \int_a^\beta f = \lim_{b \uparrow \beta} \int_\alpha^b f = \lim_{a \downarrow \alpha, b \uparrow \beta} \int_a^b f$$

if any of the four terms is defined in $\mathbb{R}$.

483C Corollary The Henstock and Lebesgue integrals agree on non-negative functions, in the sense that if $f : \mathbb{R} \to [0, \infty[$ then $\int f = \int f d\mu$ if either is defined in $\mathbb{R}$.

483D Corollary If $f : \mathbb{R} \to \mathbb{R}$ is Henstock integrable, then $\alpha \mapsto \int_{-\infty}^\alpha f : [\infty, \infty] \to \mathbb{R}$ and $(\alpha, \beta) \mapsto \int_\alpha^\beta f : [-\infty, \infty]^2 \to \mathbb{R}$ are continuous and bounded.

483E Definition If $f : \mathbb{R} \to \mathbb{R}$ is Henstock integrable, then its **indefinite Henstock integral** is the function $F : \mathbb{R} \to \mathbb{R}$ defined by saying that $F(x) = \int_{-\infty}^x f$ for every $x \in \mathbb{R}$.

483F Theorem Let $f : \mathbb{R} \to \mathbb{R}$ and $F : \mathbb{R} \to \mathbb{R}$ be functions. Then the following are equiveridical:

(i) f is Henstock integrable and F is its indefinite Henstock integral;

(ii)(α) F is continuous,

(β) $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x)$ is defined in $\mathbb{R}$,

(γ) for every $\epsilon > 0$ there are a gauge $\delta \in \Delta$ and a non-decreasing function $\phi : \mathbb{R} \to [0, \epsilon]$ such that $|f(x)(b - a) - F(b) + F(a)| \le \phi(b) - \phi(a)$ whenever $a \le x \le b$ in $\mathbb{R}$ and $(x, [a, b]) \in \delta$.

483G Theorem Let $f : \mathbb{R} \to \mathbb{R}$ be a Henstock integrable function. Then there is a countable cover $\mathcal{K}$ of $\mathbb{R}$ by compact sets such that $f \times \chi K$ is Lebesgue integrable for every $K \in \mathcal{K}$.

483H Upper and lower derivates: Definition Let $F : \mathbb{R} \to \mathbb{R}$ be any function. For $x \in \mathbb{R}$, set

$$(\overline{D}F)(x) = \limsup_{y \to x} \frac{F(y) - F(x)}{y - x}, \quad (\underline{D}F)(x) = \liminf_{y \to x} \frac{F(y) - F(x)}{y - x}$$

in $[-\infty, \infty]$.

483I Theorem Suppose that $f : \mathbb{R} \to \mathbb{R}$ is Henstock integrable, and F is its indefinite Henstock integral. Then $F'(x)$ is defined and equal to $f(x)$ for almost every $x \in \mathbb{R}$.

483J Theorem Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then the following are equiveridical:

(i) f is Henstock integrable;

(ii) for every $\epsilon > 0$ there are functions F_1, $F_2 : \mathbb{R} \to \mathbb{R}$, with finite limits at both $-\infty$ and ∞, such that $(\overline{D}F_1)(x) \le f(x) \le (\underline{D}F_2)(x)$ and $0 \le F_2(x) - F_1(x) \le \epsilon$ for every $x \in \mathbb{R}$.

483K Proposition Let $f : \mathbb{R} \to \mathbb{R}$ be a Henstock integrable function, and F its indefinite Henstock integral. Then $F[E]$ is Lebesgue negligible for every Lebesgue negligible set $E \subseteq \mathbb{R}$.

483L Definition If $f : \mathbb{R} \to \mathbb{R}$ is Henstock integrable, I write $\|f\|_H$ for $\sup_{a \le b} |\int_a^b f|$. It is elementary to check that this is a seminorm on the linear space of all Henstock integrable functions.

483M Proposition (a) If $f:\mathbb{R}\to\mathbb{R}$ is Henstock integrable, then $|\oint f|\le\|f\|_H$, and $\|f\|_H=0$ iff $f=0$ a.e.

(b) Write $\mathcal{HL}^1$ for the linear space of all Henstock integrable real-valued functions on $\mathbb{R}$, and HL^1 for $\{f^\bullet:f\in\mathcal{HL}^1\}\subseteq L^0(\mu)$. If we write $\|f^\bullet\|_H=\|f\|_H$ for every $f\in\mathcal{HL}^1$, then HL^1 is a normed space. The ordinary space $L^1(\mu)$ of equivalence classes of Lebesgue integrable functions is a linear subspace of HL^1, and $\|u\|_H\le\|u\|_1$ for every $u\in L^1(\mu)$.

(c) We have a linear operator $T:HL^1\to C_b(\mathbb{R})$ defined by saying that $T(f^\bullet)$ is the indefinite Henstock integral of f for every $f\in\mathcal{HL}^1$, and $\|T\|=1$.

483N Proposition Suppose that $\langle I_m\rangle_{m\in M}$ is a disjoint family of open intervals in $\mathbb{R}$ with union G, and that $f:\mathbb{R}\to\mathbb{R}$ is a function such that $f_m=f\times\chi I_m$ is Henstock integrable for every $m\in M$. If $\sum_{m\in M}\|f_m\|_H<\infty$, then $f\times\chi G$ is Henstock integrable, and $\oint f\times\chi G=\sum_{m\in M}\oint f_m$.

483O Definitions (a) For any real-valued function F, write $\omega(F)$ for $\sup_{x,y\in\operatorname{dom}F}|F(x)-F(y)|$, the **oscillation** of F. (Interpret $\sup\emptyset$ as 0, so that $\omega(\emptyset)=0$.)

(b) Let $F:\mathbb{R}\to\mathbb{R}$ be a function. For $A\subseteq\mathbb{R}$, we say that F is **AC$_*$ on** A if for every $\epsilon>0$ there is an $\eta>0$ such that $\sum_{I\in\mathcal{I}}\omega(F\restriction\overline{I})\le\epsilon$ whenever $\mathcal{I}$ is a disjoint family of open intervals with endpoints in A and $\sum_{I\in\mathcal{I}}\mu I\le\eta$.

(c) F is **ACG$_*$** if it is continuous and there is a countable family $\mathcal{A}$ of sets, covering $\mathbb{R}$, such that F is AC$_*$ on every member of $\mathcal{A}$.

483P Elementary results (a)(i) If F, $G:\mathbb{R}\to\mathbb{R}$ are functions and $A\subseteq B\subseteq\mathbb{R}$, then $\omega(F+G\restriction A)\le\omega(F\restriction A)+\omega(G\restriction A)$ and $\omega(F\restriction A)\le\omega(F\restriction B)$.

(ii) If F is the indefinite Henstock integral of $f:\mathbb{R}\to\mathbb{R}$ and $C\subseteq\mathbb{R}$ is an interval, then $\|f\times\chi C\|_H=\omega(F\restriction C)$.

(iii) If $F:\mathbb{R}\to\mathbb{R}$ is continuous, then $(a,b)\mapsto\omega(F\restriction[a,b]):\mathbb{R}^2\to\mathbb{R}$ is continuous, and $\omega(F\restriction\overline{A})=\omega(F\restriction A)$ for every set $A\subseteq\mathbb{R}$.

(b)(i) If $F:\mathbb{R}\to\mathbb{R}$ is AC$_*$ on $A\subseteq\mathbb{R}$, it is AC$_*$ on every subset of A.

(ii) If $F:\mathbb{R}\to\mathbb{R}$ is continuous and is AC$_*$ on $A\subseteq\mathbb{R}$, it is AC$_*$ on $\overline{A}$.

483Q Lemma Let $F:\mathbb{R}\to\mathbb{R}$ be a continuous function, and $K\subseteq\mathbb{R}$ a non-empty compact set such that F is AC$_*$ on K. Write $\mathcal{I}$ for the family of non-empty bounded open intervals, disjoint from K, with endpoints in K.

(a) $\sum_{I\in\mathcal{I}}\omega(F\restriction I)$ is finite.

(b) Write a^* for $\inf K=\min K$. Then there is a Lebesgue integrable function $g:\mathbb{R}\to\mathbb{R}$, zero off K, such that

$$F(x)-F(a^*)=\int_{a^*}^x g+\sum_{J\in\mathcal{I},J\subseteq[a^*,x]}F(\sup J)-F(\inf J)$$

for every $x\in K$.

483R Theorem Let $F:\mathbb{R}\to\mathbb{R}$ be a function. Then F is an indefinite Henstock integral iff it is ACG$_*$, $\lim_{x\to-\infty}F(x)=0$ and $\lim_{x\to\infty}F(x)$ is defined in $\mathbb{R}$.

Version of 21.1.10

484 The Pfeffer integral

I give brief notes on what seems at present to be the most interesting of the multi-dimensional versions of the Henstock integral, leading to Pfeffer's Divergence Theorem (484N).

484A Notation $r \geq 2$ will be a fixed integer, and μ will be Lebesgue measure on $\mathbb{R}^r$. As in §§473-475, let ν be 'normalized' $(r-1)$-dimensional Hausdorff measure on $\mathbb{R}^r$, and

$$\beta_{r-1} = \frac{2^{2k}k!\pi^{k-1}}{(2k)!} \text{ if } r = 2k \text{ is even,}$$
$$= \frac{\pi^k}{k!} \text{ if } r = 2k+1 \text{ is odd}$$

the Lebesgue measure of a ball of radius 1 in $\mathbb{R}^{r-1}$. For this section only, let us say that a subset of $\mathbb{R}^r$ is **thin** if it is of the form $\bigcup_{n\in\mathbb{N}} A_n$ where $\nu^* A_n$ is finite for every n.

I will use the term **dyadic cube** for sets of the form $\prod_{i<r} [2^{-m}k_i, 2^{-m}(k_i+1)[$ where $m, k_0, \dots, k_{r-1} \in \mathbb{Z}$; write $\mathcal{D}$ for the set of dyadic cubes in $\mathbb{R}^r$. Note that if $\mathcal{D}_0 \subseteq \mathcal{D}$, the maximal members of $\mathcal{D}_0$ are disjoint.

It will be helpful to have an abbreviation for the following expression: set

$$\alpha^* = \min(\frac{1}{r^{r/2}}, \frac{2^{r-2}}{r\beta_r^{(r-1)/r}}).$$

$\mathcal{C}$ will be the family of subsets of $\mathbb{R}^r$ with locally finite perimeter, and $\mathcal{V}$ the family of bounded sets in $\mathcal{C}$.

484B Theorem Let $E \subseteq \mathbb{R}^r$ be a Lebesgue measurable set of finite measure and perimeter, and $\epsilon > 0$. Then there is a Lebesgue measurable set $G \subseteq E$ such that $\operatorname{per} G \leq \operatorname{per} E$, $\mu(E \setminus G) \leq \epsilon$ and $\mathrm{cl}^* G = \overline{G}$.

484C Lemma Let $E \in \mathcal{V}$ and $l \in \mathbb{N}$ be such that $\max(\operatorname{per} E, \operatorname{diam} E) \leq l$. Then E is expressible as $\bigcup_{i<n} E_i$ where $\langle E_i \rangle_{i<n}$ is disjoint, $\operatorname{per} E_i \leq 1$ for each $i < n$ and n is at most $2^r(l+1)^r(4r(2l^2+1))^{r/(r-1)} + 2^{r+1}l^2$.

484D Definitions Let H be the family of strictly positive sequences $\eta = \langle \eta(i) \rangle_{i\in\mathbb{N}}$ in $\mathbb{R}$. For $\eta \in \mathrm{H}$, write $\mathcal{M}_\eta$ for the set of disjoint sequences $\langle E_i \rangle_{i\in\mathbb{N}}$ of measurable subsets of $\mathbb{R}^r$ such that $\mu E_i \leq \eta(i)$ and $\operatorname{per} E_i \leq 1$ for every $i \in \mathbb{N}$, and E_i is empty for all but finitely many i. For $\eta \in \mathrm{H}$ and $V \in \mathcal{V}$ set

$$\mathcal{R}_\eta = \{\bigcup_{i\in\mathbb{N}} E_i : \langle E_i \rangle_{i\in\mathbb{N}} \in \mathcal{M}_\eta\} \subseteq \mathcal{C}, \quad \mathcal{R}_\eta^{(V)} = \{R : R \subseteq \mathbb{R}^r, R \cap V \in \mathcal{R}_\eta\};$$

finally, set $\mathfrak{R} = \{\mathcal{R}_\eta^{(V)} : V \in \mathcal{V}, \eta \in \mathrm{H}\}$.

484E Lemma (a)(i) For every $\mathcal{R} \in \mathfrak{R}$, there is an $\eta \in \mathrm{H}$ such that $\mathcal{R}_\eta \subseteq \mathcal{R}$.

(ii) If $\mathcal{R} \in \mathfrak{R}$ and $C \in \mathcal{C}$, there is an $\mathcal{R}' \in \mathfrak{R}$ such that $C \cap R \in \mathcal{R}$ whenever $R \in \mathcal{R}'$.

(b)(i) If $\eta \in \mathrm{H}$ and $\gamma \geq 0$, there is an $\epsilon > 0$ such that $R \in \mathcal{R}_\eta$ whenever $\mu R \leq \epsilon$, $\operatorname{diam} R \leq \gamma$ and $\operatorname{per} R \leq \gamma$.

(ii) If $\mathcal{R} \in \mathfrak{R}$ and $\gamma \geq 0$, there is an $\epsilon > 0$ such that $R \in \mathcal{R}$ whenever $\mu R \leq \epsilon$ and $\operatorname{per} R \leq \gamma$.

(c) If $\mathcal{R} \in \mathfrak{R}$ there is an $\mathcal{R}' \in \mathfrak{R}$ such that $R \cup R' \in \mathcal{R}$ whenever $R, R' \in \mathcal{R}'$ and $R \cap R' = \emptyset$.

(d)(i) If $\eta \in \mathrm{H}$ and $A \subseteq \mathbb{R}^r$ is a thin set, then there is a set $\mathcal{D}_0 \subseteq \mathcal{D}$ such that every point of A belongs to the interior of $\bigcup \mathcal{D}_1$ for some finite $\mathcal{D}_1 \subseteq \mathcal{D}_0$, and $\bigcup \mathcal{D}_1 \in \mathcal{R}_\eta$ for every finite set $\mathcal{D}_1 \subseteq \mathcal{D}_0$.

(ii) If $\mathcal{R} \in \mathfrak{R}$ and $A \subseteq \mathbb{R}^r$ is a thin set, then there is a set $\mathcal{D}_0 \subseteq \mathcal{D}$ such that every point of A belongs to the interior of $\bigcup \mathcal{D}_1$ for some finite set $\mathcal{D}_1 \subseteq \mathcal{D}_0$, and $\bigcup \mathcal{D}_1 \in \mathcal{R}$ for every finite set $\mathcal{D}_1 \subseteq \mathcal{D}_0$.

484F A family of tagged-partition structures For $\alpha > 0$, let $\mathcal{C}_\alpha$ be the family of those $C \in \mathcal{V}$ such that $\mu C \geq \alpha(\operatorname{diam} C)^r$ and $\alpha \operatorname{per} C \leq (\operatorname{diam} C)^{r-1}$, and let T_α be the straightforward set of tagged partitions generated by the set

$$\{(x, C) : C \in \mathcal{C}_\alpha, x \in \mathrm{cl}^* C\}.$$

Let Θ be the set of functions $\theta : \mathbb{R}^r \to [0, \infty[$ such that $\{x : \theta(x) = 0\}$ is thin, and set $\Delta = \{\delta_\theta : \theta \in \Theta\}$, where $\delta_\theta = \{(x, A) : x \in \mathbb{R}^r, \theta(x) > 0, \|y - x\| < \theta(x) \text{ for every } y \in A\}$.

Then whenever $0 < \alpha < \alpha^*$, $(\mathbb{R}^r, T_\alpha, \Delta, \mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$.

484G The Pfeffer integral (a) For $\alpha \in]0, \alpha^*[$, write I_α for the linear functional defined by setting

$$I_\alpha(f) = \lim_{\boldsymbol{t} \to \mathcal{F}(T_\alpha, \Delta, \mathfrak{R})} S_{\boldsymbol{t}}(f, \mu)$$

whenever $f : \mathbb{R}^r \to \mathbb{R}$ is such that the limit is defined. (See 481F for the notation $\mathcal{F}(T_\alpha, \Delta, \mathfrak{R})$.) Then if $0 < \beta \le \alpha < \alpha^*$ and $I_\beta(f)$ is defined, so is $I_\alpha(f)$, and the two are equal.

(b) Let $f : \mathbb{R}^r \to \mathbb{R}$ be a function. I will say that it is **Pfeffer integrable**, with **Pfeffer integral** $\mathrm{Pf}\!\!\int f$, if

$$\mathrm{Pf}\!\!\int f = \lim_{\alpha \downarrow 0} I_\alpha(f)$$

is defined; that is to say, if $I_\alpha(f)$ is defined whenever $0 < \alpha < \alpha^*$.

484H Proposition (a) The domain of $\mathrm{Pf}\!\!\int$ is a linear space of functions, and $\mathrm{Pf}\!\!\int$ is a positive linear functional.

(b) If f, $g : \mathbb{R}^r \to \mathbb{R}$ are such that $|f| \le g$ and $\mathrm{Pf}\!\!\int g = 0$, then $\mathrm{Pf}\!\!\int f$ is defined and equal to 0.

(c) If $f : \mathbb{R}^r \to \mathbb{R}$ is Pfeffer integrable, then there is a unique additive functional $F : \mathcal{C} \to \mathbb{R}$ such that whenever $\epsilon > 0$ and $0 < \alpha < \alpha^*$ there are $\delta \in \Delta$ and $\mathcal{R} \in \mathfrak{R}$ such that

$$\sum_{(x,C) \in \boldsymbol{t}} |F(C) - f(x)\mu C| \le \epsilon \text{ for every } \delta\text{-fine } \boldsymbol{t} \in T_\alpha,$$

$$|F(E)| \le \epsilon \text{ whenever } E \in \mathcal{C} \cap \mathcal{R}.$$

Moreover, $F(\mathbb{R}^r) = \mathrm{Pf}\!\!\int f$.

(d) Every Pfeffer integrable function is Lebesgue measurable.

(e) Every Lebesgue integrable function is Pfeffer integrable, with the same integral.

(f) A non-negative function is Pfeffer integrable iff it is Lebesgue integrable.

484I Definition If $f : \mathbb{R}^r \to \mathbb{R}$ is Pfeffer integrable, I will call the function $F : \mathcal{C} \to \mathbb{R}$ defined in 484Hc the **Saks-Henstock indefinite integral** of f.

484J Proposition Suppose that $f : \mathbb{R}^r \to \mathbb{R}$ and $F : \mathcal{C} \to \mathbb{R}$ are such that

(i) F is additive,

(ii) whenever $0 < \alpha < \alpha^*$ and $\epsilon > 0$ there is a $\delta \in \Delta$ such that $\sum_{(x,C) \in \boldsymbol{t}} |F(C) - f(x)\mu C| \le \epsilon$ for every δ-fine $\boldsymbol{t} \in T_\alpha$,

(iii) for every $\epsilon > 0$ there is an $\mathcal{R} \in \mathfrak{R}$ such that $|F(E)| \le \epsilon$ for every $E \in \mathcal{C} \cap \mathcal{R}$.

Then f is Pfeffer integrable and F is the Saks-Henstock indefinite integral of f.

484K Lemma Suppose that $\alpha > 0$ and $0 < \alpha' < \alpha \min(\frac{1}{2}, 2^{r-1}(\frac{\alpha}{2\beta_r})^{(r-1)/r})$. If $E \in \mathcal{C}$ is such that $E \subseteq \mathrm{cl}^*E$, then there is a $\delta \in \Delta$ such that $\{(x, C \cap E)\} \in T_{\alpha'}$ whenever $(x, C) \in \delta$, $x \in E$ and $\{(x, C)\} \in T_\alpha$.

484L Proposition Suppose that $f : \mathbb{R}^r \to \mathbb{R}$ is Pfeffer integrable, and that $F : \mathcal{C} \to \mathbb{R}$ is its Saks-Henstock indefinite integral. Then $\mathrm{Pf}\!\!\int f \times \chi E$ is defined and equal to $F(E)$ for every $E \in \mathcal{C}$.

484M Lemma Let G, $H \in \mathcal{C}$ be disjoint and $\phi : \mathbb{R}^r \to \mathbb{R}^r$ a continuous function. If *either* $G \cup H$ is bounded *or* ϕ has compact support,

$$\int_{\partial^*(G \cup H)} \phi \centerdot \psi_{G \cup H} \, d\nu = \int_{\partial^* G} \phi \centerdot \psi_G \, d\nu + \int_{\partial^* H} \phi \centerdot \psi_G \, d\nu,$$

where ψ_G, ψ_H and $\psi_{G \cup H}$ are the canonical outward-normal functions.

484N Pfeffer's Divergence Theorem Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter, and $\phi : \mathbb{R}^r \to \mathbb{R}^r$ a continuous function with compact support such that $\{x : x \in \mathbb{R}^r, \phi \text{ is not differentiable at } x\}$ is thin. Let v_x be the Federer exterior normal to E at any point x where the normal exists. Then $\mathrm{Pf}\!\!\int \operatorname{div} \phi \times \chi E$ is defined and equal to $\int_{\partial^* E} \phi(x) \centerdot v_x \nu(dx)$.

484O Differentiating the indefinite integral: Theorem Let $f : \mathbb{R}^r \to \mathbb{R}$ be a Pfeffer integrable function, and F its Saks-Henstock indefinite integral. Then whenever $0 < \alpha < \alpha^*$,

$$\begin{aligned} f(x) &= \lim_{\zeta\downarrow 0} \sup\{\frac{F(C)}{\mu C} : C \in \mathcal{C}_\alpha,\, x \in \overline{C},\, 0 < \operatorname{diam} C \le \zeta\} \\ &= \lim_{\zeta\downarrow 0} \inf\{\frac{F(C)}{\mu C} : C \in \mathcal{C}_\alpha,\, x \in \overline{C},\, 0 < \operatorname{diam} C \le \zeta\} \end{aligned}$$

for μ-almost every $x \in \mathbb{R}^r$.

484P Lemma Let $\phi : \mathbb{R}^r \to \mathbb{R}^r$ be an injective Lipschitz function, and H the set of points at which it is differentiable; for $x \in H$, write $T(x)$ for the derivative of ϕ at x and $J(x)$ for $|\det T(x)|$. Then, for μ-almost every $x \in \mathbb{R}^r$,

$$\begin{aligned} J(x) &= \lim_{\zeta\downarrow 0} \sup\{\frac{\mu\phi[C]}{\mu C} : C \in \mathcal{C}_\alpha,\, x \in \overline{C},\, 0 < \operatorname{diam} C \le \zeta\} \\ &= \lim_{\zeta\downarrow 0} \inf\{\frac{\mu\phi[C]}{\mu C} : C \in \mathcal{C}_\alpha,\, x \in \overline{C},\, 0 < \operatorname{diam} C \le \zeta\} \end{aligned}$$

for every $\alpha > 0$.

484Q Definition If (X, ρ) and (Y, σ) are metric spaces, a function $\phi : X \to Y$ is a **lipeomorphism** if it is bijective and both ϕ and ϕ^{-1} are Lipschitz.

484R Lemma Let $\phi : \mathbb{R}^r \to \mathbb{R}^r$ be a lipeomorphism.

(a) For any set $A \subseteq \mathbb{R}^r$,

$$\mathrm{cl}^*(\phi[A]) = \phi[\mathrm{cl}^*A], \quad \mathrm{int}^*(\phi[A]) = \phi[\mathrm{int}^*A], \quad \partial^*(\phi[A]) = \phi[\partial^*A].$$

(b) $\phi[C] \in \mathcal{C}$ for every $C \in \mathcal{C}$, and $\phi[V] \in \mathcal{V}$ for every $V \in \mathcal{V}$.

(c) For any $\alpha > 0$ there is an $\alpha' \in \,]0, \alpha]$ such that $\phi[C] \in \mathcal{C}_{\alpha'}$ for every $C \in \mathcal{C}_\alpha$ and $\{(\phi(x), \phi[C]) : (x, C) \in \boldsymbol{t}\}$ belongs to $T_{\alpha'}$ for every $\boldsymbol{t} \in T_\alpha$.

(d) For any $\mathcal{R} \in \mathfrak{R}$ there is an $\mathcal{R}' \in \mathfrak{R}$ such that $\phi[R] \in \mathcal{R}$ for every $R \in \mathcal{R}'$.

(e) $\theta\phi : \mathbb{R}^r \to [0, \infty[$ belongs to Θ for every $\theta \in \Theta$.

484S Theorem Let $\phi : \mathbb{R}^r \to \mathbb{R}^r$ be a lipeomorphism. Let H be the set of points at which ϕ is differentiable. For $x \in H$, write $T(x)$ for the derivative of ϕ at x; set $J(x) = |\det T(x)|$ for $x \in H$, 0 for $x \in \mathbb{R}^r \setminus H$. Then, for any function $f : \mathbb{R}^r \to \mathbb{R}^r$,

$$\mathrm{Pf}\!\!\!\int f = \mathrm{Pf}\!\!\!\int J \times f\phi$$

if either is defined in $\mathbb{R}$.