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Chapter 48

Gauge integrals

For the penultimate chapter of this volume I turn to a completely different approach to integration which has been developed in the last fifty years, following Kurzweil 57 and Henstock 63. This depends for its inspiration on a formulation of the Riemann integral (see 481Xe), and leads in particular to some remarkable extensions of the Lebesgue integral (§§483-484). While (in my view) it remains peripheral to the most important parts of measure theory, it has deservedly attracted a good deal of interest in recent years, and is entitled to a place here.

From the very beginning, in the definitions of §122, I have presented the Lebesgue integral in terms of almost-everywhere approximations by simple functions. Because the integral $\int \lim_{n\to\infty} f_n$ of a limit is *not* always the limit $\lim_{n\to\infty}\int f_n$ of the integrals, we are forced, from the start, to constrain ourselves by the ordering, and to work with monotone or dominated sequences. This almost automatically leads us to an 'absolute' integral, in which $|f|$ is integrable whenever f is, whether we start from measures (as in Chapter 11) or from linear functionals (as in §436). For four volumes now I have been happily developing the concepts and intuitions appropriate to such integrals. But if we return to one of the foundation stones of Lebesgue's theory, the Fundamental Theorem of Calculus, we find that it is easy to construct a differentiable function f such that the absolute value $|f'|$ of its derivative is not integrable (483Xd). It was observed very early (Perron 1914) that the Lebesgue integral can be extended to integrate the derivative of any function which is differentiable everywhere. The achievement of Henstock 63 was to find a formulation of this extension which was conceptually transparent enough to lend itself to a general theory, fragments of which I will present here.

The first step is to set out the essential structures on which the theory depends (§481), with a first attempt at a classification scheme. (One of the most interesting features of the Kurzweil-Henstock approach is that we have an extraordinary degree of freedom in describing our integrals, and apart from the Henstock integral itself it is not clear that we have yet found the right canonical forms to use.) In §482 I give a handful of general theorems showing what kinds of result can be expected and what difficulties arise. In §483, I work through the principal properties of the Henstock integral on the real line, showing, in particular, that it coincides with the Perron and special Denjoy integrals. Finally, in §484, I look at a very striking integral on $\mathbb{R}^r$, due to W.F.Pfeffer.

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481 Tagged partitions

I devote this section to establishing some terminology (481A-481B, 481E-481G) and describing a variety of examples (481I-481Q), some of which will be elaborated later. The clearest, simplest and most important example is surely Henstock's integral on a closed bounded interval (481J), so I recommend turning immediately to that paragraph and keeping it in mind while studying the notation here. It may also help you to make sense of the definitions here if you glance at the statements of some of the results in §482; in this section I give only the formula defining gauge integrals (481C), with some elementary examples of its use (481Xb-481Xh).

481A Tagged partitions and Riemann sums The common idea underlying all the constructions of this chapter is the following. We have a set X and a functional ν defined on some family $\mathcal{C}$ of subsets of X. We seek to define an integral $\int f d\nu$, for functions f with domain X, as a limit of *finite* **Riemann sums**

$\sum_{i=0}^{n} f(x_i)\nu C_i$, where $x_i \in X$ and $C_i \in \mathcal{C}$ for $i \le n$. There is no strict reason, at this stage, to forbid repetitions in the string $(x_0, C_0), \dots, (x_n, C_n)$, but also little to be gained from allowing them, and it will simplify some of the formulae below if I say from the outset that a **tagged partition** on X will be a finite subset $\boldsymbol{t}$ of $X \times \mathcal{P}X$.

So one necessary element of the definition will be a declaration of which tagged partitions $\{(x_0, C_0), \dots, (x_n, C_n)\}$ will be employed, in terms, for instance, of which sets C_i are permitted, whether they are allowed to overlap at their boundaries, whether they are required to cover the space, and whether each **tag** x_i is required to belong to the corresponding C_i. The next element of the definition will be a description of a filter $\mathcal{F}$ on the set T of tagged partitions, so that the integral will be the limit (when it exists) of the sums along the filter, as in 481C below.

In the formulations studied in this chapter, the C_i will generally be disjoint, but this is not absolutely essential, and it is occasionally convenient to allow them to overlap in 'small' sets, as in 481Ya. In some cases, we can restrict attention to families for which the C_i are non-empty and have union X, so that $\{C_0, \dots, C_n\}$ is a partition of X in the strict sense.

481B Notation Let me immediately introduce notations which will be in general use throughout the chapter.

(a) First, a shorthand to describe a particular class of sets of tagged partitions. If X is a set, a **straightforward set of tagged partitions** on X is a set of the form

$$T = \{\boldsymbol{t} : \boldsymbol{t} \in [Q]^{<\omega},\ C \cap C' = \emptyset \text{ whenever } (x, C),\ (x', C') \text{ are distinct members of } \boldsymbol{t}\}$$

where $Q \subseteq X \times \mathcal{P}X$; I will say that T is **generated** by Q. In this case, of course, Q can be recovered from T, since $Q = \bigcup T$. Note that no control is imposed on the tags at this point. It remains theoretically possible that a pair $(x, \emptyset)$ should belong to Q, though in many applications this will be excluded in one way or another.

(b) If X is a set and $\boldsymbol{t} \subseteq X \times \mathcal{P}X$ is a tagged partition, I write

$$W_{\boldsymbol{t}} = \bigcup\{C : (x, C) \in \boldsymbol{t}\}.$$

(c) If X is a set, $\mathcal{C}$ is a family of subsets of X, f and ν are real-valued functions, and $\boldsymbol{t} \in [X \times \mathcal{C}]^{<\omega}$ is a tagged partition, then

$$S_{\boldsymbol{t}}(f, \nu) = \sum_{(x,C)\in\boldsymbol{t}} f(x)\nu C$$

whenever $\boldsymbol{t} \subseteq \operatorname{dom} f \times \operatorname{dom} \nu$.

481C Proposition Let X be a set, $\mathcal{C}$ a family of subsets of X, $T \subseteq [X \times \mathcal{C}]^{<\omega}$ a non-empty set of tagged partitions and $\mathcal{F}$ a filter on T. For real-valued functions f and ν, set

$$I_\nu(f) = \lim_{\boldsymbol{t}\to\mathcal{F}} S_{\boldsymbol{t}}(f, \nu)$$

if this is defined in $\mathbb{R}$.

(a) I_ν is a linear functional defined on a linear subspace of $\mathbb{R}^X$.

(b) Now suppose that $\nu C \ge 0$ for every $C \in \mathcal{C}$. Then

(i) I_ν is a positive linear functional (definition: 351F);

(ii) if f, $g : X \to \mathbb{R}$ are such that $|f| \le g$ and $I_\nu(g)$ is defined and equal to 0, then $I_\nu(f)$ is defined and equal to 0.

proof (a) We have only to observe that if f, g are real-valued functions and $\alpha \in \mathbb{R}$, then

$$S_{\boldsymbol{t}}(f + g, \nu) = S_{\boldsymbol{t}}(f, \nu) + S_{\boldsymbol{t}}(g, \nu), \quad S_{\boldsymbol{t}}(\alpha f, \nu) = \alpha S_{\boldsymbol{t}}(f, \nu)$$

whenever the right-hand sides are defined, and apply 2A3Sf.

(b) If $g \ge 0$ in $\mathbb{R}^X$, that is, $g(x) \ge 0$ for every $x \in X$, then $S_{\boldsymbol{t}}(g, \nu) \ge 0$ for every $\boldsymbol{t} \in T$, so the limit $I_\nu(g)$, if defined, will also be non-negative. Next, if $|f| \le g$, then $|S_{\boldsymbol{t}}(f, \nu)| \le S_{\boldsymbol{t}}(g, \nu)$ for every $\boldsymbol{t}$, so if $I_\nu(g) = 0$ then $I_\nu(f)$ also is zero.

481D Remarks (a) Functionals $I_\nu = \lim_{\boldsymbol{t}\to\mathcal{F}} S_{\boldsymbol{t}}(.,\nu)$, as described above, are called **gauge integrals**.

(b) In fact even greater generality is possible at this point. There is no reason why f and ν should take real values. All we actually need is an interpretation of sums of products $f(x)\times\nu C$ in a space in which we can define limits. So for any linear spaces U, V and W with a bilinear functional $\phi : U\times V\to W$ (253A) and a Hausdorff linear space topology on W, we can set out to construct an integral of a function $f : X\to U$ with respect to a functional $\nu : \mathcal{C}\to V$ as a limit of sums $S_{\boldsymbol{t}}(f,\nu)=\sum_{(x,C)\in\boldsymbol{t}}\phi(f(x),\nu C)$ in W. I will not go farther along this path here. But it is worth noting that the constructions of this chapter lead the way to interesting vector integrals of many types.

(c) An extension which is, however, sometimes useful is to allow ν to be undefined (or take values outside $\mathbb{R}$, such as $\pm\infty$) on part of $\mathcal{C}$. In this case, set $\mathcal{C}_0=\nu^{-1}[\mathbb{R}]$. Provided that $T\cap[X\times\mathcal{C}_0]^{<\omega}$ belongs to $\mathcal{F}$, we can still define I_ν, and 481C will still be true.

481E Gauges The most useful method (so far) of defining filters on sets of tagged partitions is the following.

(a) If X is a set, a **gauge** on X is a subset δ of $X\times\mathcal{P}X$. For a gauge δ, a tagged partition $\boldsymbol{t}$ is **δ-fine** if $\boldsymbol{t}\subseteq\delta$. Now, for a set Δ of gauges and a non-empty set T of tagged partitions, we can seek to define a filter $\mathcal{F}$ on T as the filter generated by sets of the form $T_\delta=\{\boldsymbol{t}:\boldsymbol{t}\in T$ is δ-fine$\}$ as δ runs over Δ. Of course we shall need to establish that T and Δ are compatible in the sense that $\{T_\delta:\delta\in\Delta\}$ has the finite intersection property; this will ensure that there is indeed a filter containing every T_δ (4A1Ia).

In nearly all cases, Δ will be non-empty and downwards-directed (that is, for any δ_1, $\delta_2\in\Delta$ there will be a $\delta\in\Delta$ such that $\delta\subseteq\delta_1\cap\delta_2$); in this case, we shall need only to establish that T_δ is non-empty for every $\delta\in\Delta$. Note that the filter on T generated by $\{T_\delta:\delta\in\Delta\}$ depends only on T and the filter on $X\times\mathcal{P}X$ generated by Δ.

(b) The most important gauges (so far) are 'neighbourhood gauges'. If $(X,\mathfrak{T})$ is a topological space, a **neighbourhood gauge** on X is a set expressible in the form $\delta=\{(x,C):x\in X,\ C\subseteq G_x\}$ where $\langle G_x\rangle_{x\in X}$ is a family of open sets such that $x\in G_x$ for every $x\in X$. It is useful to note (i) that the family $\langle G_x\rangle_{x\in X}$ can be recovered from δ, since $G_x=\bigcup\{A:(x,A)\in\delta\}$ (ii) that $\delta_1\cap\delta_2$ is a neighbourhood gauge whenever δ_1 and δ_2 are. When (X,ρ) is a metric space, we can define a neighbourhood gauge δ_h from any function $h:X\to\,]0,\infty[$, setting

$$\delta_h=\{(x,C):x\in X,\ C\subseteq X,\ \rho(y,x)<h(x)\text{ for every }y\in C\}.$$

The set of gauges expressible in this form is coinitial with the set of all neighbourhood gauges and therefore defines the same filter on any compatible set T of tagged partitions. Specializing yet further, we can restrict attention to constant functions h, obtaining the **uniform metric gauges**

$$\delta_\eta=\{(x,C):x\in X,\ C\subseteq X,\ \rho(x,y)<\eta\text{ for every }y\in C\}$$

for $\eta>0$, used in the Riemann integral (481I). (The use of the letter 'δ' to represent a gauge has descended from its traditional appearance in the definition of the Riemann integral.)

(c) If X is a set and $\Delta\subseteq\mathcal{P}(X\times\mathcal{P}X)$ is a family of gauges on X, I will say that Δ is **countably full** if whenever $\langle\delta_n\rangle_{n\in\mathbb{N}}$ is a sequence in Δ, and $\phi:X\to\mathbb{N}$ is a function, then there is a $\delta\in\Delta$ such that $(x,C)\in\delta_{\phi(x)}$ whenever $(x,C)\in\delta$. I will say that Δ is **full** if whenever $\langle\delta_x\rangle_{x\in X}$ is a family in Δ, then there is a $\delta\in\Delta$ such that $(x,C)\in\delta_x$ whenever $(x,C)\in\delta$.

Of course a full set of gauges is countably full. Observe that if $(X,\mathfrak{T})$ is any topological space, the set of all neighbourhood gauges on X is full.

481F Residual sets The versatility and power of the methods being introduced here derives from the insistence on taking *finite* sums $\sum_{(x,C)\in\boldsymbol{t}}f(x)\nu C$, so that all questions about convergence are concentrated in the final limit $\lim_{\boldsymbol{t}\to\mathcal{F}}S_{\boldsymbol{t}}(f,\nu)$. Since in a given Riemann sum we can look at only finitely many sets C of finite measure, we cannot insist, even when ν is Lebesgue measure on $\mathbb{R}$, that $W_{\boldsymbol{t}}$ should always be X. There are many other cases in which it is impossible or inappropriate to insist that $W_{\boldsymbol{t}}=X$ for every tagged partition in T. We shall therefore need to add something to the definition of the filter $\mathcal{F}$ on T beyond what is possible in the language of 481E. In the examples below, the extra condition will always be of the following form. There will be a collection $\mathfrak{R}$ of **residual families** $\mathcal{R} \subseteq \mathcal{P}X$. It will help to have a phrase corresponding to the phrase 'δ-fine': if $\mathcal{R} \subseteq \mathcal{P}X$, and $\boldsymbol{t}$ is a tagged partition on X, I will say that $\boldsymbol{t}$ is **$\mathcal{R}$-filling** if $X \setminus W_{\boldsymbol{t}} \in \mathcal{R}$. Now, given a family $\mathfrak{R}$ of residual sets, and a family Δ of gauges on X, we can seek to define a filter $\mathcal{F}(T,\Delta,\mathfrak{R})$ on T as that generated by sets of the form T_δ, for $\delta \in \Delta$, and $T'_{\mathcal{R}}$, for $\mathcal{R} \in \mathfrak{R}$, where

$$T'_{\mathcal{R}} = \{\boldsymbol{t} : \boldsymbol{t} \in T \text{ is } \mathcal{R}\text{-filling}\}.$$

When there is such a filter, that is, the family $\{T_\delta : \delta \in \Delta\} \cup \{T'_{\mathcal{R}} : \mathcal{R} \in \mathfrak{R}\}$ has the finite intersection property, I will say that T is **compatible** with Δ and $\mathfrak{R}$.

It is important here to note that we shall *not* suppose that, for a typical residual family $\mathcal{R} \in \mathfrak{R}$, subsets of members of $\mathcal{R}$ again belong to $\mathcal{R}$; there will frequently be a restriction on the 'shape' of members of $\mathcal{R}$ as well as on their size. On the other hand, it will usually be helpful to arrange that $\mathfrak{R}$ is a filter base, so that (if Δ is also downwards-directed, and neither Δ nor $\mathfrak{R}$ is empty) we need only show that $T_\delta \cap T'_{\mathcal{R}}$ is always non-empty, and $\{T_{\delta\mathcal{R}} : \delta \in \Delta,\ \mathcal{R} \in \mathfrak{R}\}$ will be a base for $\mathcal{F}(T,\Delta,\mathfrak{R})$.

If the filter $\mathcal{F}$ is defined as in 481Ea, with no mention of a family $\mathfrak{R}$, we can still bring the construction into the framework considered here by setting $\mathfrak{R} = \emptyset$. If it is convenient to define T in terms which do not impose any requirement on the sets $W_{\boldsymbol{t}}$, but nevertheless we wish to restrict attention to sums $S_{\boldsymbol{t}}(f,\nu)$ for which the tagged partition covers the whole space X, we can do so by setting $\mathfrak{R} = \{\{\emptyset\}\}$.

481G Subdivisions When we come to analyse the properties of integrals constructed by the method of 481C, there is an important approach which depends on the following combination of features. I will say that $(X,T,\Delta,\mathfrak{R})$ is a **tagged-partition structure allowing subdivisions**, **witnessed by** $\mathcal{C}$, if

(i) X is a set.

(ii) Δ is a non-empty downwards-directed family of gauges on X.

(iii)(α) $\mathfrak{R}$ is a non-empty downwards-directed collection of families of subsets of X, all containing $\emptyset$;

(β) for every $\mathcal{R} \in \mathfrak{R}$ there is an $\mathcal{R}' \in \mathfrak{R}$ such that $A \cup B \in \mathcal{R}$ whenever A, $B \in \mathcal{R}'$ are disjoint.

(iv) $\mathcal{C}$ is a family of subsets of X such that whenever C, $C' \in \mathcal{C}$ then $C \cap C' \in \mathcal{C}$ and $C \setminus C'$ is expressible as the union of a disjoint finite subset of $\mathcal{C}$.

(v) Whenever $\mathcal{C}_0 \subseteq \mathcal{C}$ is finite and $\mathcal{R} \in \mathfrak{R}$, there is a finite set $\mathcal{C}_1 \subseteq \mathcal{C}$, including $\mathcal{C}_0$, such that $X \setminus \bigcup \mathcal{C}_1 \in \mathcal{R}$.

(vi) $T \subseteq [X \times \mathcal{C}]^{<\omega}$ is (in the language of 481Ba) a non-empty straightforward set of tagged partitions on X.

(vii) Whenever $C \in \mathcal{C}$, $\delta \in \Delta$ and $\mathcal{R} \in \mathfrak{R}$ there is a δ-fine tagged partition $\boldsymbol{t} \in T$ such that $W_{\boldsymbol{t}} \subseteq C$ and $C \setminus W_{\boldsymbol{t}} \in \mathcal{R}$.

481H Remarks (a) Conditions (ii) and (iii-α) of 481G are included primarily for convenience, since starting from any Δ and $\mathfrak{R}$ we can find non-empty directed sets leading to the same filter $\mathcal{F}(T,\Delta,\mathfrak{R})$. (iii-$\beta$), on the other hand, is saying something new.

(b) It is important to note, in (vii) of 481G, that the tags of $\boldsymbol{t}$ there are *not* required to belong to the set C.

(c) All the applications below will fall into one of two classes. In one type, the residual families $\mathcal{R} \in \mathfrak{R}$ will be families of 'small' sets, in some recognisably measure-theoretic sense, and, in particular, we shall have subsets of members of any $\mathcal{R}$ belonging to $\mathcal{R}$. In the other type, (vii) of 481G will be true because we can always find $\boldsymbol{t} \in T$ such that $W_{\boldsymbol{t}} = C$.

(d) The following elementary fact got left out of §136 and Chapter 31. Let $\mathfrak{A}$ be a Boolean algebra and $C \subseteq \mathfrak{A}$. Set

$$E = \{\sup C_0 : C_0 \subseteq C \text{ is finite and disjoint}\}.$$

If $c\cap c'$ and $c\setminus c'$ belong to E for all c, $c'\in C$, then E is a subring of $\mathfrak{A}$. **P** Write $\mathcal{D}$ for the family of finite disjoint subsets of C. (i) If C_0, $C_1\in\mathcal{D}$, then for $c\in C_0$, $c'\in C_1$ there is a $D_{cc'}\in\mathcal{D}$ with supremum $c\cap c'$. Now $D=\bigcup_{c\in C_0,c'\in C_1}D_{cc'}$ belongs to $\mathcal{D}$ and has supremum $(\sup C_0)\cap(\sup C_1)$. Thus $e\cap e'\in E$ for all e, $e'\in E$. (ii) Of course $e\cup e'\in E$ whenever e, $e'\in E$ and $e\cap e'=0$. (iii) Again suppose that C_0, $C_1\in\mathcal{D}$. Then $c\setminus c'\in E$ for all $c\in C_0$, $c'\in C_1$. By (i), $c\setminus\sup C_1\in E$ for every $c\in C_0$; by (ii), $(\sup C_0)\setminus(\sup C_1)\in E$. Thus $e\setminus e'\in E$ for all e, $e'\in E$. (iv) Putting (ii) and (iii) together, $e\triangle e'\in E$ for all e, $e'\in E$. (v) As $0=\sup\emptyset$ belongs to E, E is a subring of $\mathfrak{A}$. **Q**

In particular, if $\mathcal{C}\subseteq\mathcal{P}X$ has the properties in (iv) of 481G, then

$$\mathcal{E}=\{\bigcup\mathcal{C}_0:\mathcal{C}_0\subseteq\mathcal{C}\text{ is finite and disjoint}\}$$

is a ring of subsets of X.

(e) Suppose that X is a set and that $\mathfrak{R}\subseteq\mathcal{PP}X$ satisfies (iii) of 481G. Then for every $\mathcal{R}\in\mathfrak{R}$ there is a non-increasing sequence $\langle\mathcal{R}_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{R}$ such that $\bigcup_{i\le n}A_i\in\mathcal{R}$ whenever $A_i\in\mathcal{R}_i$ for $i\le n$ and $\langle A_i\rangle_{i\le n}$ is disjoint. **P** Take $\mathcal{R}_0\in\mathfrak{R}$ such that $\mathcal{R}_0\subseteq\mathcal{R}$ and $A\cup B\in\mathcal{R}$ for all disjoint A, $B\in\mathcal{R}_0$; similarly, for $n\in\mathbb{N}$, choose $\mathcal{R}_{n+1}\in\mathfrak{R}$ such that $\mathcal{R}_{n+1}\subseteq\mathcal{R}_n$ and $A\cup B\in\mathcal{R}_n$ for all disjoint A, $B\in\mathcal{R}_{n+1}$. Now, given that $A_i\in\mathcal{R}_i$ for $i\le n$ and $\langle A_i\rangle_{i\le n}$ is disjoint, we see by downwards induction on m that $\bigcup_{m<i\le n}A_i\in\mathcal{R}_m$ for each $m\le n$, so that $\bigcup_{i\le n}A_i\in\mathcal{R}$. **Q**

(f) If $(X,T,\Delta,\mathfrak{R})$ is a tagged-partition structure allowing subdivisions, then T is compatible with Δ and $\mathfrak{R}$ in the sense of 481F. **P** $\emptyset\in T$ so T is not empty. Take $\delta\in\Delta$ and $\mathcal{R}\in\mathfrak{R}$. Let $\langle\mathcal{R}_i\rangle_{i\in\mathbb{N}}$ be a sequence in $\mathfrak{R}$ such that $\bigcup_{i\le n}A_i\in\mathcal{R}$ whenever $A_i\in\mathcal{R}_i$ for $i\le n$ and $\langle A_i\rangle_{i\le n}$ is disjoint ((e) above). There is a finite set $\mathcal{C}_1\subseteq\mathcal{C}$ such that $X\setminus\bigcup\mathcal{C}_1\in\mathcal{R}_0$, by 481G(iv). By (d), there is a disjoint family $\mathcal{C}_0\subseteq\mathcal{C}$ such that $\bigcup\mathcal{C}_0=\bigcup\mathcal{C}_1$; enumerate $\mathcal{C}_0$ as $\langle C_i\rangle_{i<n}$. For each $i<n$, there is a δ-fine $\boldsymbol{t}_i\in T$ such that $W_{\boldsymbol{t}_i}\subseteq C_i$ and $C_i\setminus W_{\boldsymbol{t}_i}\in\mathcal{R}_{i+1}$, by 481G(vii). Set $\boldsymbol{t}=\bigcup_{i<n}\boldsymbol{t}_i$; then $\boldsymbol{t}\in T$ is δ-fine, and

$$X\setminus W_{\boldsymbol{t}}=(X\setminus\bigcup\mathcal{C}_1)\cup\bigcup_{i<n}(C_i\setminus W_{\boldsymbol{t}_i})\in\mathcal{R}$$

by the choice of $\langle\mathcal{R}_i\rangle_{i\in\mathbb{N}}$. Thus we have a δ-fine $\mathcal{R}$-filling member of T; as δ and $\mathcal{R}$ are arbitrary, T is compatible with Δ and $\mathfrak{R}$. **Q**

(g) For basic results which depend on 'subdivisions' as described in 481G(vii), see 482A-482B below. A hypothesis asserting the existence of a different sort of subdivision appears in 482G(iv).

481I I now run through some simple examples of these constructions, limiting myself for the moment to the definitions, the proofs that T is compatible with Δ and $\mathfrak{R}$, and (when appropriate) the proofs that the structures allow subdivisions.

The proper Riemann integral Fix a non-empty closed interval $X=[a,b]\subseteq\mathbb{R}$. Write $\mathcal{C}$ for the set of all intervals (open, closed or half-open, and allowing the empty set to count as an interval) included in $[a,b]$, and set $Q=\{(x,C):C\in C,\,x\in\overline{C}\}$; let T be the straightforward set of tagged partitions generated by Q. Let Δ be the set of uniform metric gauges on X, and $\mathfrak{R}=\{\{\emptyset\}\}$. Then $(X,T,\Delta,\mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$. If $a<b$, then Δ is not countably full.

proof (i), (iii) and (vi) of 481G are trivial and (ii), (iv) and (v) are elementary. As for (vii), given $\eta>0$ and $C\in\mathcal{C}$, take a disjoint family $\langle C_i\rangle_{i\in I}$ of non-empty intervals of length less than 2η covering C, and x_i to be the midpoint of C_i for $i\in I$; then $\boldsymbol{t}=\{(x_i,C_i):i\in I\}$ belongs to T and is δ_η-fine, in the language of 481E, and $W_{\boldsymbol{t}}=C$.

Of course (apart from the trivial case $a=b$) Δ is not countably full, since if we take δ_n to be the gauge $\{(x,C):|x-y|<2^{-n}\text{ for every }y\in C\}$ and any unbounded function $\phi:[a,b]\to\mathbb{N}$, there is no $\delta\in\Delta$ such that $(x,C)\in\delta_{\phi(x)}$ whenever $(x,C)\in\Delta$.

481J The Henstock integral on a bounded interval (Henstock 63) Take X, $\mathcal{C}$, T and $\mathfrak{R}$ as in 481I. This time, let Δ be the set of *all* neighbourhood gauges on $[a,b]$. Then $(X,T,\Delta,\mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$, and Δ is countably full.

proof Again, only 481G(vii) needs more than a moment's consideration. Take any $C\in\mathcal{C}$. If $C=\emptyset$, then $\boldsymbol{t}=\emptyset$ will suffice. Otherwise, set $a_0=\inf C$, $b_0=\sup C$ and let T_0 be the family of δ-fine partitions

$\boldsymbol{t} \in T$ such that $W_{\boldsymbol{t}}$ is a relatively closed initial subinterval of C, that is, is of the form $C \cap [a_0, y_{\boldsymbol{t}}]$ for some $y_{\boldsymbol{t}} \in [a_0, b_0]$. Set $A = \{y_{\boldsymbol{t}} : \boldsymbol{t} \in T_0\}$. I have to show that there is a $\boldsymbol{t} \in T_0$ such that $W_{\boldsymbol{t}} = C$, that is, that $b_0 \in A$.

Observe that there is an $\eta_0 > 0$ such that $(a_0, A) \in \delta$ whenever $A \subseteq [a, b] \cap [a_0 - \eta_0, a_0 + \eta_0]$, and now $\{(a_0, [a_0, a_0+\eta_0] \cap C)\}$ belongs to T_0, so $\min(a_0+\eta_0, b_0) \in A$ and A is a non-empty subset of $[a_0, b_0]$. It follows that $c = \sup A$ is defined in $[a_0, b_0]$. Let $\eta > 0$ be such that $(c, A) \in \delta$ whenever $A \subseteq [a, b] \cap [c - \eta, c + \eta]$. There is some $\boldsymbol{t} \in T_0$ such that $y_{\boldsymbol{t}} \geq c - \eta$. If $y_{\boldsymbol{t}} = b_0$, we can stop. Otherwise, set $C' = C \cap]y_{\boldsymbol{t}}, c + \eta]$. Then $(c, C') \in \delta$ and $c \in \overline{C'}$ and $C' \cap W_{\boldsymbol{t}}$ is empty, so $\boldsymbol{t}' = \boldsymbol{t} \cup \{(c, C')\}$ belongs to T_0 and $y_{\boldsymbol{t}'} = \min(c + \eta, b_0)$. Since $y_{\boldsymbol{t}'} \leq c$, this shows that $y_{\boldsymbol{t}'} = c = b_0$ and again $b_0 \in A$, as required.

Δ is full just because it is the family of neighbourhood gauges.

481K The Henstock integral on $\mathbb{R}$ This time, set $X = \mathbb{R}$ and let $\mathcal{C}$ be the family of all bounded intervals in $\mathbb{R}$. Let T be the straightforward set of tagged partitions generated by $\{(x, C) : C \in \mathcal{C},\ x \in \overline{C}\}$. Following 481J, let Δ be the set of all neighbourhood gauges on $\mathbb{R}$. This time, set $\mathfrak{R} = \{\mathcal{R}_{ab} : a \leq b \in \mathbb{R}\}$, where $\mathcal{R}_{ab} = \{\mathbb{R} \setminus [c, d] : c \leq a,\ d \geq b\} \cup \{\emptyset\}$. Then $(X, T, \Delta, \mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$.

proof This time we should perhaps take a moment to look at (iii) of 481G. But all we need to note is that $\mathcal{R}_{ab} \cap \mathcal{R}_{a'b'} = \mathcal{R}_{\min(a,a'),\max(b,b')}$, and that any two members of $\mathcal{R}_{ab}$ have non-empty intersection. Conditions (i), (ii), (iv), (v) and (vi) of 481G are again elementary, so once more we are left with (vii). But this can be dealt with by exactly the same argument as in 481J.

481L The symmetric Riemann-complete integral (cf. Carrington 72, chap. 3) Again take $X = \mathbb{R}$, and $\mathcal{C}$ the set of all bounded intervals in $\mathbb{R}$. This time, take T to be the straightforward set of tagged partitions generated by the set of pairs (x, C) where $C \in \mathcal{C} \setminus \{\emptyset\}$ and x is the *midpoint* of C. As in 481K, take Δ to be the set of all neighbourhood gauges on $\mathbb{R}$; but this time take $\mathfrak{R} = \{\mathcal{R}'_a : a \geq 0\}$, where $\mathcal{R}'_a = \{\mathbb{R} \setminus [-c, c] : c \geq a\} \cup \{\emptyset\}$. Then T is compatible with Δ and $\mathfrak{R}$.

proof Take $\delta \in \Delta$ and $\mathcal{R} \in \mathfrak{R}$. For $x \geq 0$, let $\theta(x) > 0$ be such that $(x, D) \in \delta$ whenever $D \subseteq [x - \theta(x), x + \theta(x)]$ and $(-x, D) \in \delta$ whenever $D \subseteq [-x - \theta(x), -x + \theta(x)]$. Write A for the set of those $a > 0$ such that there is a finite sequence $(a_0, \dots, a_n)$ such that $0 < a_0 < a_1 < \dots < a_n = a$, $a_0 \leq \theta(0)$ and $a_{i+1} - a_i \leq 2\theta(\frac{1}{2}(a_i + a_{i+1}))$ for $i < n$.

? Suppose, if possible, that A is bounded above. Then $c = \inf([0, \infty[\setminus \overline{A})$ is defined in $[0, \infty[$. Observe that if $0 < a \leq \theta(0)$, then the one-term sequence $<a>$ witnesses that $a \in A$. So $c \geq \theta(0) > 0$. Now there must be u, v such that $c < u < v < \min(c + \theta(c), 2c)$ and $]u, v[\cap A = \emptyset$; on the other hand, the interval $]2c - v, 2c - u[$ must contain a point x of A. Set $y = 2c - x$. Then we can find $a_0 < \dots < a_n = x$ such that $0 < a_0 \leq \theta(0)$ and $a_{i+1} - a_i \leq 2\theta(\frac{1}{2}(a_i + a_{i+1}))$ for $i < n$; setting $a_{n+1} = y$, we see that $\langle a_i \rangle_{i \leq n+1}$ witnesses that $y \in A$, though $y \in]u, v[$. **X**

This contradiction shows that A is unbounded above. So now suppose that $\mathcal{R} = \mathcal{R}_a$ where $a \geq 0$. Take $a_0, \dots, a_n$ such that $0 < a_0 < \dots < a_n$, $0 < a_0 \leq \theta(0)$ and $a_{i+1} - a_i \leq 2\theta(\frac{1}{2}(a_i + a_{i+1}))$ for $i < n$, and $a_n \geq a$. For $i < n$, set $x_i = \frac{1}{2}(a_i + a_{i+1})$, $C_i =]a_i, a_{i+1}]$, $x'_i = -x_i$, $C'_i = [-a_{i+1}, a_i[$. Then x_i, x'_i are the midpoints of C_i, C'_i and (by the choice of the function θ) $(x_i, C_i) \in \delta$, $(x'_i, C'_i) \in \delta$ for $i < n$. So if we set

$$\boldsymbol{t} = \{(x_i, C_i) : i < n\} \cup \{(x'_i, C'_i) : i < n\} \cup \{(0, [-a_0, a_0])\}$$

we shall obtain a δ-fine $\mathcal{R}$-filling member of T.

As Δ and $\mathfrak{R}$ are both downwards-directed, this is enough to show that T is compatible with Δ and $\mathfrak{R}$.

481M The McShane integral on an interval (McShane 73) As in 481J, take $X = [a, b]$ and let $\mathcal{C}$ be the family of subintervals of $[a, b]$. This time, take T to be the straightforward set of tagged partitions generated by $Q = X \times \mathcal{C}$, so that *no* condition is imposed relating the tags to their associated intervals. As in 481J, let Δ be the set of all neighbourhood gauges on X, and $\mathfrak{R} = \{\{\emptyset\}\}$. Proceed as before. Since the only change is that Q and T have been enlarged, $(X, T, \Delta, \mathfrak{R})$ is still a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$.

481N The McShane integral on a topological space (Fremlin 95) Now let $(X,\mathfrak{T},\Sigma,\mu)$ be any effectively locally finite τ-additive topological measure space, and take $\mathcal{C}=\{E:E\in\Sigma,\,\mu E<\infty\}$, $Q=X\times\mathcal{C}$; let T be the straightforward set of tagged partitions generated by Q. Again let Δ be the set of all neighbourhood gauges on X. This time, define $\mathfrak{R}$ as follows. For any set $E\in\Sigma$ of finite measure and $\eta>0$, let $\mathcal{R}_{E\eta}$ be the set $\{F:F\in\Sigma,\,\mu(F\cap E)\le\eta\}$, and set $\mathfrak{R}=\{\mathcal{R}_{E\eta}:\mu E<\infty,\,\eta>0\}$. Then $(X,T,\Delta,\mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$.

proof As usual, everything is elementary except perhaps 481G(vii). But if $C\in\mathcal{C}$, $\delta\in\Delta$, $E\in\Sigma$, $\mu E<\infty$ and $\eta>0$, take for each $x\in X$ an open set G_x containing x such that $(x,A)\in\delta$ whenever $A\subseteq G_x$. $\{G_x:x\in X\}$ is an open cover of X, so by 414Ea there is a finite family $\langle x_i\rangle_{i<n}$ in X such that $\mu(E\cap C\setminus\bigcup_{i<n}G_{x_i})\le\eta$; setting $C_i=C\cap G_{x_i}\setminus\bigcup_{j<i}G_{x_j}$ for $i<n$, we get a δ-fine tagged partition $\boldsymbol{t}=\{(x_i,C_i):i<n\}$ such that $C\setminus W_{\boldsymbol{t}}\in\mathcal{R}_{E\eta}$.

481O Convex partitions in $\mathbb{R}^r$ Fix $r\ge 1$. Let us say that a **convex polytope** in $\mathbb{R}^r$ is a non-empty bounded set expressible as the intersection of finitely many open or closed half-spaces; let $\mathcal{C}$ be the family of convex polytopes in $X=\mathbb{R}^r$, and T the straightforward set of tagged partitions generated by $\{(x,C):x\in\overline{C}\}$. Let Δ be the set of neighbourhood gauges on $\mathbb{R}^r$. For $a\ge 0$, let $\mathcal{C}_a$ be the set of closed convex polytopes $C\subseteq\mathbb{R}^r$ such that, for some $b\ge a$, $B(0,b)\subseteq C\subseteq B(0,2b)$, where $B(0,b)$ is the ordinary Euclidean ball with centre 0 and radius b; set $\mathcal{R}_a=\{\mathbb{R}^r\setminus C:C\in\mathcal{C}_a\}\cup\{\emptyset\}$, and $\mathfrak{R}=\{\mathcal{R}_a:a\ge 0\}$. Then $(X,T,\Delta,\mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$.

proof As usual, only 481G(vii) requires thought.

(a) We need a geometrical fact: if $C\in\mathcal{C}$, $x\in\overline{C}$ and $y\in C$, then $\alpha y+(1-\alpha)x\in C$ for every $\alpha\in\,]0,1]$. **P** The family of sets $C\subseteq\mathbb{R}^r$ with this property is closed under finite intersections and contains all half-spaces. **Q** It follows that if C_1, $C_2\in\mathcal{C}$ are not disjoint, then $\overline{C_1\cap C_2}=\overline{C}_1\cap\overline{C}_2$.

(b) Write $\mathcal{D}\subseteq\mathcal{C}$ for the family of products of non-empty bounded intervals in $\mathbb{R}$. The next step is to show that if $D\in\mathcal{D}$ and $\delta\in\Delta$, then there is a δ-fine tagged partition $\boldsymbol{t}\in T$ such that $W_{\boldsymbol{t}}=D$ and $\boldsymbol{t}\subseteq\mathbb{R}^r\times\mathcal{D}$. **P** Induce on r. For $r=1$ this is just 481J again. For the inductive step to $r+1$, suppose that $D\subseteq\mathbb{R}^{r+1}$ is a product of bounded intervals and that δ is a neighbourhood gauge on $\mathbb{R}^{r+1}$. Identifying $\mathbb{R}^{r+1}$ with $\mathbb{R}^r\times\mathbb{R}$, express D as $D'\times L$, where $D'\subseteq\mathbb{R}^r$ is a product of bounded intervals and $L\subseteq\mathbb{R}$ is a bounded interval. For $y\in\mathbb{R}^r$, $\alpha\in\mathbb{R}$ let $G(y,\alpha)$, $H(y,\alpha)$ be open sets containing y, α respectively such that $((y,\alpha),A)\in\delta$ whenever $A\subseteq G(y,\alpha)\times H(y,\alpha)$. For $y\in\mathbb{R}^r$, set $\delta_y=\{(\alpha,A):\alpha\in\mathbb{R},\,A\subseteq H(y,\alpha)\}$; then δ_y is a neighbourhood gauge on $\mathbb{R}$.

By the one-dimensional case there is a δ_y-fine tagged partition $\boldsymbol{s}_y\in T_1$ such that $W_{\boldsymbol{s}_y}=L$, where I write T_1 for the set of tagged partitions used in 481K. Set

$$\delta'=\{(y,A):y\in\mathbb{R}^r,\,A\subseteq G(y,\alpha)\text{ for every }(\alpha,F)\in\boldsymbol{s}_y\}.$$

δ' is a neighbourhood gauge on $\mathbb{R}^r$. By the inductive hypothesis, there is a δ'-fine tagged partition $\boldsymbol{u}\in T_r$ such that $W_{\boldsymbol{u}}=D'$, where here T_r is the set of tagged partitions on $\mathbb{R}^r$ corresponding to the r-dimensional version of this result. Consider the family

$$\boldsymbol{t}=\{((y,\alpha),E\times F):(y,E)\in\boldsymbol{u},\,(\alpha,F)\in\boldsymbol{s}_y\}.$$

For $(y,E)\in\boldsymbol{u}$, $(\alpha,F)\in\boldsymbol{s}_y$, we have

$$y\in\overline{E},\quad E\subseteq G(y,\alpha),\quad \alpha\in\overline{F},\quad F\subseteq H(y,\alpha),$$

so

$$(y,\alpha)\in\overline{E\times F},\quad E\times F\subseteq G(y,\alpha)\times H(y,\alpha),$$

and $((y,\alpha),E\times F)\in\delta$. If $((y,\alpha),E\times F)$, $((y',\alpha'),E'\times F')$ are distinct members of $\boldsymbol{t}$, then either $(y,E)\ne(y',E')$ so $E\cap E'=\emptyset$ and $(E\times F)\cap(E'\times F')$ is empty, or $y=y'$ and (α,F), (α',F') are distinct members of $\boldsymbol{s}_y$, so that $F\cap F'=\emptyset$ and again $E\times F$, $E'\times F'$ are disjoint. Thus $\boldsymbol{t}$ is a δ-fine member of T_{r+1}. Finally,

$$W_{\boldsymbol{t}}=\bigcup_{(y,E)\in\boldsymbol{u}}\bigcup_{(\alpha,F)\in\boldsymbol{s}_y}E\times F=\bigcup_{(y,E)\in\boldsymbol{u}}E\times L=D'\times L=D.$$

So the induction proceeds. **Q**

(c) Now suppose that C_0 is an arbitrary member of $\mathcal{C}$ and that δ is a neighbourhood gauge on $\mathbb{R}^r$. Set

$$\delta' = \delta \cap \{(x, A) : \text{either } x \in \overline{C}_0 \text{ or } A \cap \overline{C}_0 = \emptyset\}.$$

Then δ' is a neighbourhood gauge on $\mathbb{R}^r$, being the intersection of δ with the neighbourhood gauge associated with the family $\langle U_x\rangle_{x\in\mathbb{R}^r}$, where $U_x = \mathbb{R}^r$ if $x \in \overline{C}_0$, $\mathbb{R}^r \setminus \overline{C}_0$ otherwise. Let $D \in \mathcal{D}$ be such that $C_0 \subseteq D$. By (b), there is a δ'-fine tagged partition $\boldsymbol{t} \in T$ such that $W_{\boldsymbol{t}} = D$. Set $\boldsymbol{s} = \{(x, C\cap C_0) : (x, C) \in \boldsymbol{t},\, C\cap C_0 \ne \emptyset\}$. Since $\boldsymbol{t} \subseteq \delta'$, $x \in \overline{C}_0$ whenever $(x, C) \in \boldsymbol{t}$ and $C \cap C_0 \ne \emptyset$. By (a), $x \in \overline{C \cap C_0}$ for all such pairs (x, C); and of course $(x, C \cap C_0) \in \delta$ for every $(x, C) \in \boldsymbol{t}$. So $\boldsymbol{s}$ belongs to T, and $W_{\boldsymbol{s}} = W_{\boldsymbol{t}} \cap C_0 = C_0$. As C_0 and δ are arbitrary, 481G(vii) is satisfied.

481P Box products (cf. Muldowney 87, Prop. 1) Let $\langle(X_i, \mathfrak{T}_i)\rangle_{i\in I}$ be a non-empty family of non-empty compact metrizable spaces with product $(X, \mathfrak{T})$. Set $\pi_i(x) = x(i)$ for $x \in X$ and $i \in I$. For each $i \in I$, let $\mathcal{C}_i \subseteq \mathcal{P}X_i$ be such that (α) whenever E, $E' \in \mathcal{C}_i$ then $E \cap E' \in \mathcal{C}_i$ and $E \setminus E'$ is expressible as the union of a disjoint finite subset of $\mathcal{C}_i$ (β) $\mathcal{C}_i$ includes a base for $\mathfrak{T}_i$.

Let $\mathcal{C}$ be the set of subsets of X of the form

$$C = \{X \cap \bigcap_{i\in J} \pi_i^{-1}[E_i] : J \in [I]^{<\omega},\, E_i \in \mathcal{C}_i \text{ for every } i \in J\},$$

and let T be the straightforward set of tagged partitions generated by $\{(x, C) : C \in \mathcal{C},\, x \in \overline{C}\}$. Let Δ be the set of those neighbourhood gauges δ on X defined by families $\langle G_x\rangle_{x\in X}$ of open sets such that, for some countable $J \subseteq I$, every G_x is determined by coordinates in J (definition: 254M). Then $(X, T, \Delta, \{\{\emptyset\}\})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$. Δ is countably full; Δ is full iff $I' = \{i : \#(X_i) > 1\}$ is countable.

proof Conditions (i), (iii) and (vi) of 481G are trivial, and (ii), (iv) and (v) are elementary; so we are left with (vii), as usual. **?** Suppose, if possible, that $C \in \mathcal{C}$ and $\delta \in \Delta$ are such that there is no δ-fine $\boldsymbol{t} \in T$ with $W_{\boldsymbol{t}} = C$. Let $\langle G_x\rangle_{x\in X}$ be the family of open sets determining δ, and $J \subseteq I$ a non-empty countable set such that G_x is determined by coordinates in J for every $x \in X$. For $i \in J$, let $\mathcal{C}'_i \subseteq \mathcal{C}_i$ be a countable base for $\mathfrak{T}_i$ (4A2P(a-iii)), and take a sequence $\langle(i_n, E_n)\rangle_{n\in\mathbb{N}}$ running over $\{(i, E) : i \in J,\, E \in \mathcal{C}'_i\}$.

Write $\mathcal{D} = \{W_{\boldsymbol{t}} : \boldsymbol{t} \in T \text{ is } \delta\text{-fine}\}$. Note that if D_1, $D_2 \in \mathcal{D}$ are disjoint then $D_1 \cup D_2 \in \mathcal{D}$. So if $D \in \mathcal{C}\setminus\mathcal{D}$ and $C \in \mathcal{C}$, there must be some $D' \in \mathcal{C} \setminus \mathcal{D}$ such that either $D' \subseteq D \cap C$ or $D' \subseteq D \setminus C$, just because $\mathcal{C}$ satisfies 481G(iv). Now choose $\langle C_n\rangle_{n\in\mathbb{N}}$ inductively so that $C_0 = C$ and

$$C_n \in \mathcal{C} \setminus \mathcal{D},$$

$$\text{either } C_{n+1} \subseteq C_n \cap \pi_{i_n}^{-1}[E_n] \text{ or } C_{n+1} \subseteq C_n \setminus \pi_{i_n}^{-1}[E_n]$$

for every $n \in \mathbb{N}$. Because X, being a product of compact spaces, is compact, there is an $x \in \bigcap_{n\in\mathbb{N}} \overline{C_n}$. We know that G_x is determined by coordinates in J, so $G_x = \tilde\pi^{-1}[\tilde\pi[G_x]]$, where $\tilde\pi$ is the canonical map from X onto $Y = \prod_{i\in J} X_i$. $V = \tilde\pi[G_x]$ is open, so there must be a finite set $K \subseteq J$ and a family $\langle V_i\rangle_{i\in K}$ such that $x(i) \in V_i \in \mathfrak{T}_i$ for every $i \in K$ and $\{y : y \in Y,\, y(i) \in V_i \text{ for every } i \in K\}$ is included in V. This means that $\{z : z \in X,\, z(i) \subseteq V_i \text{ for every } i \in K\}$ is included in G_x. Now, for each $i \in K$, there is some $m \in \mathbb{N}$ such that $i = i_m$ and $x(i) \in E_m \subseteq V_i$. Because $x \in \overline{C}_{m+1}$, $\pi_{i_m}^{-1}[E_m]$ cannot be disjoint from C_{m+1}, and $C_{m+1} \subseteq \pi_{i_m}^{-1}[E_m] \subseteq \pi_i^{-1}[V_i]$.

But this means that, for any n large enough, $C_n \subseteq G_x$ and $\boldsymbol{t} = \{(x, C_n)\}$ is a δ-fine member of T with $W_{\boldsymbol{t}} = C_n$; contradicting the requirement that $C_n \notin \mathcal{D}$. **X**

This contradiction shows that 481G(vii) also is satisfied.

To see that Δ is countably full, note that if $\langle\delta_n\rangle_{n\in\mathbb{N}}$ is a sequence in Δ, we have for each $n \in \mathbb{N}$ a countable set $J_n \subseteq I$ and a family $\langle G_{nx}\rangle_{x\in X}$ of open sets, all determined by coordinates in J_n, such that $x \in G_{nx}$ and $(x, C) \in \delta_n$ whenever $x \in X$ and $C \subseteq G_{nx}$. Now, given $\phi : X \to \mathbb{N}$, set $\delta = \{(x, C) : x \in X,\, C \subseteq G_{\phi(x),x}\}$, and observe that $\delta \in \Delta$ and that $(x, C) \in \delta_{\phi(x)}$ whenever $(x, C) \in \Delta$.

If I' is countable, then Δ is the set of all neighbourhood gauges on X, so is full. If I' is uncountable, then for $j \in I'$ and $x \in X$ choose a proper open subset H_{jx} of X_j containing $x(j)$ and set $G_{jx} = \{y : y \in X,\, y(j) \in H_{jx}\}$. For $j \in I'$ set $\delta_j = \{(x, C) : C \subseteq G_{jx}\} \in \Delta$. Let $\phi : X \to I'$ be any function such that $\phi[X]$ is uncountable; then there is no $\delta \in \Delta$ such that $(x, C) \in \delta_{\phi(x)}$ whenever $(x, C) \in \delta$, so $\langle\delta_{\phi(x)}\rangle_{x\in X}$ witnesses that Δ is not full.

481Q The approximately continuous Henstock integral (GORDON 94, chap. 16) Let μ be Lebesgue measure on $\mathbb{R}$. As in 481K, let $\mathcal{C}$ be the family of non-empty bounded intervals in $\mathbb{R}$, T the straightforward set of tagged partitions generated by $\{(x,C) : C \in \mathcal{C},\ x \in \overline{C}\}$, and $\mathfrak{R} = \{\mathcal{R}_{ab} : a,\ b \in \mathbb{R}\}$, where $\mathcal{R}_{ab} = \{\mathbb{R} \setminus [c,d] : c \le a,\ d \ge b\} \cup \{\emptyset\}$ for $a,\ b \in \mathbb{R}$.

This time, define gauges as follows. Let E be the set of families $\boldsymbol{e} = \langle E_x\rangle_{x\in\mathbb{R}}$ where every E_x is a measurable set containing x such that x is a density point of E_x (definition: 223B). For $\boldsymbol{e} = \langle E_x\rangle_{x\in\mathbb{R}} \in \mathrm{E}$, set

$$\delta_{\boldsymbol{e}} = \{(x,C) : C \in \mathcal{C},\ x \in \overline{C},\ \inf C \in E_x \text{ and } \sup C \in E_x\}.$$

Set $\Delta = \{\delta_{\boldsymbol{e}} : \boldsymbol{e} \in \mathrm{E}\}$. Then $(X,T,\Delta,\mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$, and Δ is full.

proof (a) Turning to 481G, we find, as usual, that most of the conditions are satisfied for elementary reasons. Since we have here a new kind of gauge, we had better check 481G(ii); but if $\boldsymbol{e} = \langle E_x\rangle_{x\in\mathbb{R}}$ and $\boldsymbol{e}' = \langle E'_x\rangle_{x\in\mathbb{R}}$ both belong to E, so does $\boldsymbol{e} \wedge \boldsymbol{e}' = \langle E_x \cap E'_x\rangle_{x\in\mathbb{R}}$, because

$$\begin{aligned}\liminf_{\eta\downarrow 0} &\frac{1}{2\eta}\mu([x-\eta,x+\eta] \cap E_x \cap E'_x)\\ &\ge \lim_{\eta\downarrow 0}\frac{1}{2\eta}\big(\mu([x-\eta,x+\eta]\cap E_x) + \mu([x-\eta,x+\eta]\cap E'_x) - 2\eta\big) = 1\end{aligned}$$

for every x; and now $\delta_{\boldsymbol{e}} \cap \delta_{\boldsymbol{e}'} = \delta_{\boldsymbol{e}\wedge\boldsymbol{e}'}$ belongs to Δ. Everything else we have done before, except, of course, (vii).

(b) So take any $\delta \in \Delta$; express δ as $\delta_{\boldsymbol{e}}$, where $\boldsymbol{e} = \langle E_x\rangle_{x\in\mathbb{R}} \in \mathrm{E}$. For $x,\ y \in \mathbb{R}$, write $x \frown y$ if $x \le y$ and $E_x \cap E_y \cap [x,y] \ne \emptyset$; note that we always have $x \frown x$. For $x \in \mathbb{R}$, let $\eta_x > 0$ be such that $\mu(E_x \cap [x-\eta,x+\eta]) \ge \frac{5}{3}\eta$ whenever $0 \le \eta \le \eta_x$.

Fix $a < b$ in $\mathbb{R}$ for the moment. Say that a finite string $(x_0,\dots,x_n)$ is 'acceptable' if $a \le x_0 \frown \dots \frown x_n \le b$ and $\mu([x_0,x_n] \cap \bigcup_{i<n} E^+_{x_i}) \ge \frac{1}{2}(x_n - x_0)$, where $E^+_x = E_x \cap [x,\infty[$ for $x \in \mathbb{R}$. Observe that if $(x_0,\dots,x_m)$ and $(x_m,x_{m+1},\dots,x_n)$ are both acceptable, so is $(x_0,\dots,x_n)$. For $x \in [a,b]$, set

$$h(x) = \sup\{x_n : (x,x_1,\dots,x_n) \text{ is acceptable}\};$$

this is defined in $[x,b]$ because the string (x) is acceptable. If $a \le x < b$, then (x,y) is acceptable whenever $y \in E_x$ and $0 \le y \le \min(b,x+\eta_x)$, so $h(x) > x$. Now choose sequences $\langle x_i\rangle_{i\in\mathbb{N}}$, $\langle n_k\rangle_{k\in\mathbb{N}}$ inductively, as follows. $n_0 = 0$ and $x_0 = a$. Given that $x_i \frown x_{i+1}$ for $i < n_k$ and that $(x_{n_j},\dots,x_{n_k})$ is acceptable for every $j \le k$, let $n_{k+1} > n_k$, $(x_{n_k+1},\dots,x_{n_{k+1}})$ be such that $(x_{n_k},\dots,x_{n_{k+1}})$ is acceptable and $x_{n_{k+1}} \ge \frac{1}{2}(x_{n_k} + h(x_{n_k}))$; then $(x_{n_j},\dots,x_{n_{k+1}})$ is acceptable for any $j \le k+1$; continue.

At the end of the induction, set

$$c = \sup_{i\in\mathbb{N}} x_i = \sup_{k\in\mathbb{N}} x_{n_k}.$$

Then there are infinitely many i such that $x_i \frown c$. **P** For any $k \in \mathbb{N}$,

$$\begin{aligned}\mu([x_{n_k},c] \cap \bigcup_{i\ge n_k} E^+_{x_i}) &= \lim_{l\to\infty}\mu\big([x_{n_k},x_{n_l}] \cap \bigcup_{n_k\le i<n_l} E^+_{x_i}\big)\\ &\ge \lim_{l\to\infty}\frac{1}{2}(x_{n_l} - x_{n_k})\end{aligned}$$

(because $(x_{n_k},\dots,x_{n_l})$ is always an acceptable string)

$$= \frac{1}{2}(c - x_{n_k}).$$

But this means that if we take k so large that $x_{n_k} \ge c - \eta_c$, so that $\mu(E_c \cap [x_{n_k},c]) \ge \frac{2}{3}(c - x_{n_k})$, there must be some $z \in E_c \cap \bigcup_{i\ge n_k} E^+_i \cap [x_{n_k},c]$; and if $i \ge n_k$ is such that $z \in E^+_i$, then z witnesses that $x_i \frown c$. As k is arbitrarily large, we have the result. **Q**

? If $c < b$, take any $y \in E_c$ such that $c < y \le \min(c+\eta_c,b)$. Take $k \in \mathbb{N}$ such that $x_{n_k} \ge c - \frac{1}{3}(y-c)$, and $j \ge n_k$ such that $x_j \frown c$. In this case, $(x_{n_k},x_{n_k+1},\dots,x_j,c,y)$ is an acceptable string, because

$$\mu E_c^+ \cap [x_{n_k}, y] \ge \mu E_c \cap [c, y] \ge \tfrac{2}{3}(y - c) \ge \tfrac{1}{2}(y - x_{n_k}).$$

But this means that $h(x_{n_k}) \ge y$, so that

$$x_{n_{k+1}} \le c < \frac{1}{2}(x_{n_k} + h(x_{n_k})),$$

contrary to the choice of $x_{n_k+1}, \dots, x_{n_{k+1}}$. **X**

Thus $c = b$. We therefore have a $j \in \mathbb{N}$ such that $x_j \frown b$, and $a = x_0 \frown \dots \frown x_j \frown b$.

(c) Now suppose that $C \in \mathcal{C}$. Set $a = \inf C$ and $b = \sup C$. If $a = b$, then $\boldsymbol{t} = \{(a, C)\} \in T$ and $W_{\boldsymbol{t}} = C$. Otherwise, (b) tells us that we have $x_0, \dots, x_n$ such that $a = x_0 \frown \dots \frown x_n = b$. Choose $a_i \in [x_{i-1}, x_i] \cap E_{x_{i-1}} \cap E_{x_i}$ for $1 \le i \le n$. Set $C_i = [a_i, a_{i+1}[$ for $1 \le i < n$, $C_0 = [a, a_1[$, $C_n = [a_n, b]$; set $I = \{i : i \le n,\ C \cap C_i \ne \emptyset\}$; and check that $(x_i, C \cap C_i) \in \delta$ for $i \in I$, so that $\boldsymbol{t} = \{(x_i, C \cap C_i) : i \in I\}$ is a δ-fine member of T with $W_{\boldsymbol{t}} = C$. As C and δ are arbitrary, 481G(vii) is satisfied.

(d) Δ is full. **P** Let $\langle \delta'_x \rangle_{x \in \mathbb{R}}$ be a family in Δ. For each $x \in X$, there is a measurable set E_x such that x is a density point of E_x and $(x, C) \in \delta'_x$ whenever $C \in \mathcal{C}$, $x \in \overline{C}$ and both $\inf C$, $\sup C$ belong to E_x. Set $\boldsymbol{e} = \langle E_x \rangle_{x \in \mathbb{R}} \in \mathrm{E}$; then $(x, C) \in \delta'_x$ whenever $(x, C) \in \delta_{\boldsymbol{e}}$. **Q**

481X Basic exercises (a) Let X, $\mathcal{C}$, T and $\mathcal{F}$ be as in 481C. Show that if $f : X \to \mathbb{R}$, $\mu : \mathcal{C} \to \mathbb{R}$ and $\nu : \mathcal{C} \to \mathbb{R}$ are functions, then $I_{\mu+\nu}(f) = I_\mu(f) + I_\nu(f)$ whenever the right-hand side is defined.

>(b) Let I be any set. Set $T = \{\{(i, \{i\}) : i \in J\} : J \in [I]^{<\omega}\}$, $\delta = \{(i, \{i\}) : i \in I\}$, $\Delta = \{\delta\}$. For $J \in [I]^{<\omega}$ set $\mathcal{R}_J = \{I \setminus K : J \subseteq K \in [I]^{<\omega}\} \cup \{\emptyset\}$; set $\mathfrak{R} = \{\mathcal{R}_J : J \in [I]^{<\omega}\}$. Show that $(I, T, \Delta, \mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $[I]^{<\omega}$, and that Δ is full. Let $\nu : [I]^{<\omega} \to \mathbb{R}$ be any additive functional. Show that, for a function $f : I \to \mathbb{R}$, $\lim_{\boldsymbol{t} \to \mathcal{F}(T, \Delta, \mathfrak{R})} S_{\boldsymbol{t}}(f, \nu) = \sum_{i \in I} f(i)\nu(\{i\})$ if either exists in $\mathbb{R}$.

>(c) Set $T = \{\{(n, \{n\}) : n \in I\} : I \in [\mathbb{Z}]^{<\omega}\}$, $\delta = \{(n, \{n\}) : n \in \mathbb{Z}\}$, $\Delta = \{\delta\}$. For $I \in [\mathbb{Z}]^{<\omega}$ and m, $n \in \mathbb{N}$ set $R_{mn} = \mathbb{Z} \setminus \{i : -m \le i \le n\}$, $\mathcal{R}'_n = \{R_{kl} : k, l \ge n\} \cup \{\emptyset\}$, $\mathcal{R}''_n = \{R_{kk} : k \ge n\} \cup \{\emptyset\}$, $\mathfrak{R}' = \{\mathcal{R}'_n : n \in \mathbb{N}\}$, $\mathfrak{R}'' = \{\mathcal{R}''_n : n \in \mathbb{N}\}$. Show that $(\mathbb{Z}, T, \Delta, \mathfrak{R}')$ and $(\mathbb{Z}, T, \Delta, \mathfrak{R}'')$ are tagged-partition structures allowing subdivisions, witnessed by $[\mathbb{Z}]^{<\omega}$, and that Δ is full. Let μ be counting measure on $\mathbb{Z}$. Show that, for a function $f : \mathbb{Z} \to \mathbb{R}$, (i) $\lim_{\boldsymbol{t} \to \mathcal{F}(T, \Delta, \mathfrak{R}')} S_{\boldsymbol{t}}(f, \nu) = \lim_{m,n \to \infty} \sum_{i=-m}^n f(i)$ if either is defined in $\mathbb{R}$ (ii) $\lim_{\boldsymbol{t} \to \mathcal{F}(T, \Delta, \mathfrak{R}'')} S_{\boldsymbol{t}}(f, \mu) = \lim_{n \to \infty} \sum_{i=-n}^n f(i)$ if either is defined in $\mathbb{R}$.

(d) Set $X = \mathbb{N} \cup \{\infty\}$, and let T be the straightforward set of tagged partitions generated by $\{(n, \{n\}) : n \in \mathbb{N}\} \cup \{(\infty, X \setminus n) : n \in \mathbb{N}\}$ (interpreting a member of $\mathbb{N}$ as the set of its predecessors). For $n \in \mathbb{N}$ set $\delta_n = \{(i, \{i\}) : i \in \mathbb{N}\} \cup \{(\infty, A) : A \subseteq X \setminus n\}$; set $\Delta = \{\delta_n : n \in \mathbb{N}\}$. Show that $(X, T, \Delta, \{\{\emptyset\}\})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C} = [\mathbb{N}]^{<\omega} \cup \{X \setminus I : I \in [\mathbb{N}]^{<\omega}\}$, and that Δ is full. Let $h : \mathbb{N} \to \mathbb{R}$ be any function, and define $\nu : \mathcal{C} \to \mathbb{R}$ by setting $\nu I = \sum_{i \in I} h(i)$, $\nu(X \setminus I) = -\nu I$ for $I \in [\mathbb{N}]^{<\omega}$. Let $f : X \to \mathbb{R}$ be any function such that $f(\infty) = 0$. Show that $\lim_{\boldsymbol{t} \to \mathcal{F}(T, \Delta, \{\{\emptyset\}\})} S_{\boldsymbol{t}}(f, \mu) = \lim_{n \to \infty} \sum_{i=0}^n f(i)h(i)$ if either is defined in $\mathbb{R}$.

>(e) Take X, T, Δ and $\mathfrak{R}$ as in 481I. Show that if μ is Lebesgue measure on $[a, b]$ then the gauge integral $I_\mu = \lim_{\boldsymbol{t} \to \mathcal{F}(T, \Delta, \mathfrak{R})} S_{\boldsymbol{t}}(., \mu)$ is the ordinary Riemann integral $\mathcal{R}\!\!\int_a^b$ as described in 134K. (*Hint*: show first that they agree on step-functions.)

>(f) Let (X, Σ, μ) be a semi-finite measure space, and Σ^f the family of measurable sets of finite measure. Let T be the straightforward set of tagged partitions generated by $\{(x, E) : x \in E \in \Sigma^f\}$. For $E \in \Sigma^f$ and $\epsilon > 0$ set $\mathcal{R}_{E\epsilon} = \{F : F \in \Sigma,\ \mu(E \setminus F) \le \epsilon\}$; set $\mathfrak{R} = \{\mathcal{R}_{E\epsilon} : E \in \Sigma^f,\ \epsilon > 0\}$. Let $\mathfrak{E}$ be the family of countable partitions of X into measurable sets, and set $\delta_{\mathcal{E}} = \bigcup_{E \in \mathcal{E}} \{(x, A) : x \in E,\ A \subseteq E\}$ for $\mathcal{E} \in \mathfrak{E}$, $\Delta = \{\delta_{\mathcal{E}} : \mathcal{E} \in \mathfrak{E}\}$. Show that $(X, T, \Delta, \mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by Σ^f. In what circumstances is Δ full or countably full? Show that, for a function $f : X \to \mathbb{R}$, $\int f d\mu = \lim_{\boldsymbol{t} \to \mathcal{F}(T, \Delta, \mathfrak{R})} S_{\boldsymbol{t}}(f, \mu)$ if either is defined in $\mathbb{R}$. (*Hint*: when showing that if $I_\mu(f)$ is defined then f is μ-virtually measurable, you will need 413G or something similar; compare 482E.)

(g) Let (X,Σ,μ) be a totally finite measure space, and T the straightforward set of tagged partitions generated by $\{(x,E) : x \in E \in \Sigma\}$. Let $\mathfrak{E}$ be the family of finite partitions of X into measurable sets, and set $\delta_{\mathcal{E}} = \bigcup_{E\in\mathcal{E}}\{(x,A) : x \in E,\ A \subseteq E\}$ for $\mathcal{E} \in \mathfrak{E}$, $\Delta = \{\delta_{\mathcal{E}} : \mathcal{E} \in \mathfrak{E}\}$. Show that $(X,T,\Delta,\{\{\emptyset\}\})$ is a tagged-partition structure allowing subdivisions, witnessed by Σ. In what circumstances is Δ full or countably full? Show that, for a function $f : X \to \mathbb{R}$, $I_\mu(f) = \lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f,\mu)$ is defined iff $f \in \mathcal{L}^\infty(\mu)$ (definition: 243A), and that then $I_\mu(f) = \int f d\mu$.

(h) Let X be a zero-dimensional compact Hausdorff space and $\mathcal{E}$ the algebra of open-and-closed subsets of X. Let T be the straightforward set of tagged partitions generated by $\{(x,E) : x \in E \in \mathcal{E}\}$. Let Δ be the set of all neighbourhood gauges on X. Show that $(X,T,\Delta,\{\{\emptyset\}\})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{E}$. Now let $\nu : \mathcal{E} \to \mathbb{R}$ be an additive functional, and set $I_\nu(f) = \lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\{\{\emptyset\}\})} S_{\boldsymbol{t}}(f,\nu)$ when $f : X \to \mathbb{R}$ is such that the limit is defined. (i) Show that $I_\nu(\chi E) = \nu E$ for every $E \in \mathcal{E}$. (ii) Show that if ν is bounded then $I_\nu(f)$ is defined for every $f \in C(X)$, and is equal to $⨍ f d\nu$ as defined in 363L, if we identify X with the Stone space of $\mathfrak{A}$ and $C(X)$ with $L^\infty(\mathfrak{A})$.

(i) Let X be a set, Δ a set of gauges on X, $\mathfrak{R}$ a collection of families of subsets of X, and T a set of tagged partitions on X which is compatible with Δ and $\mathfrak{R}$. Let $H \subseteq X$ be such that there is a $\tilde{\delta} \in \Delta$ such that $H \cap A = \emptyset$ whenever $x \in X \setminus H$ and $(x,A) \in \tilde{\delta}$, and set $\delta_H = \{(x, A\cap H) : x \in H,\ (x,A) \in \delta\}$ for $\delta \in \Delta$, $\Delta_H = \{\delta_H : \delta \in \Delta\}$, $\mathfrak{R}_H = \{\{R \cap H : R \in \mathcal{R}\} : \mathcal{R} \in \mathfrak{R}\}$, $T_H = \{\{(x, C\cap H) : (x,C) \in \boldsymbol{t},\ x \in H\} : \boldsymbol{t} \in T\}$. Show that T_H is compatible with Δ_H and $\mathfrak{R}_H$.

(k) Let X be a set, Σ an algebra of subsets of X, and $\nu : \Sigma \to [0,\infty[$ an additive functional. Set $Q = \{(x,C) : x \in C \in \Sigma\}$ and let T be the straightforward set of tagged partitions generated by Q. Let $\mathbb{E}$ be the set of disjoint families $\mathcal{E} \subseteq \Sigma$ such that $\sum_{E\in\mathcal{E}} \nu E = \nu X$, and $\Delta = \{\delta_{\mathcal{E}} : \mathcal{E} \in \mathbb{E}\}$, where

$$\delta_{\mathcal{E}} = \{(x,C) : (x,C) \in Q \text{ and there is an } E \in \mathcal{E} \text{ such that } C \subseteq E\}$$

for $\mathcal{E} \in \mathbb{E}$. Set $\mathfrak{R} = \{\mathcal{R}_\epsilon : \epsilon > 0\}$ where $\mathcal{R}_\epsilon = \{E : E \in \Sigma,\ \nu E \le \epsilon\}$ for $\epsilon > 0$. Show that $(X,T,\Delta,\mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by Σ.

481Y Further exercises (a) Suppose that $[a,b]$, $\mathcal{C}$, T and Δ are as in 481J. Let $T' \subseteq [[a,b]\times\mathcal{C}]^{<\omega}$ be the set of tagged partitions $\boldsymbol{t} = \{(x_i,[a_i,a_{i+1}]) : i < n\}$ where $a = a_0 \le x_0 \le a_1 \le x_2 \le a_2 \le \ldots \le x_{n-1} \le a_n = b$. Show that T', as well as T, is compatible with Δ in the sense of 481Ea; let $\mathcal{F}'$, $\mathcal{F}$ be the corresponding filters on T' and T. Show that if $\nu : \mathcal{C} \to \mathbb{R}$ is a functional which is additive in the sense that $\nu(C\cup C') = \nu C + \nu C'$ whenever C, C' are disjoint members of $\mathcal{C}$ with union in $\mathcal{C}$, and if $\nu\{x\} = 0$ for every $x \in [a,b]$ and $f : [a,b] \to \mathbb{R}$ is any function, then $I'_\nu(f) = I_\nu(f)$ if either is defined, where

$$I_\nu(f) = \lim_{\boldsymbol{t}\to\mathcal{F}} S_{\boldsymbol{t}}(f,\nu), \quad I'_\nu(f) = \lim_{\boldsymbol{t}\to\mathcal{F}'} S_{\boldsymbol{t}}(f,\nu)$$

are the gauge integrals associated with $(T,\mathcal{F})$ and $(T',\mathcal{F}')$.

(b) Let us say that a family $\mathfrak{R}$ of residual families is 'the simple residual structure complementary to $\mathcal{H} \subseteq \mathcal{P}X$' if $\mathfrak{R} = \{\mathcal{R}_H : H \in \mathcal{H}\}$, where $\mathcal{R}_H = \{X \setminus H' : H \subseteq H' \in \mathcal{H}\} \cup \{\emptyset\}$ for each $H \in \mathcal{H}$. Suppose that, for each member i of a non-empty finite set I, $(X_i,T_i,\Delta_i,\mathfrak{R}_i)$ is a tagged-partition structure allowing subdivisions, witnessed by an upwards-directed family $\mathcal{C}_i \subseteq \mathcal{P}X_i$, where X_i is a topological space, Δ_i is the set of all neighbourhood gauges on X_i, and $\mathfrak{R}_i$ is the simple residual structure complementary to $\mathcal{C}_i$. Set $X = \prod_{i\in I} X_i$ and let Δ be the set of neighbourhood gauges on X; let $\mathcal{C}$ be $\{\prod_{i\in I} C_i : C_i \in \mathcal{C}_i$ for each $i \in I\}$, and $\mathfrak{R}$ the simple residual structure based on $\mathcal{C}$; and let T be the straightforward set of tagged partitions generated by $\{(\langle x_i\rangle_{i\in I}, \prod_{i\in I} C_i) : \{(x_i,C_i)\} \in T_i$ for every $i \in I\}$. Show that $(X,T,\Delta,\mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$.

(c) Give an example to show that, in 481Xi, $(X,T,\Delta,\mathfrak{R})$ can be a tagged-partition structure allowing subdivisions, while $(H,T_H,\Delta_H,\mathfrak{R}_H)$ is not.

481 Notes and comments In the examples above I have tried to give an idea of the potential versatility of the ideas here. Further examples may be found in Henstock 91. The goal of 481A-481F is the formula $I_\nu(f) = \lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f,\nu)$ (481C, 481E); the elaborate notation reflects the variety of the applications.

One of these is a one-step definition of the ordinary integral (481Xf). In §483 I will show that the Henstock integral (481J-481K) properly extends the Lebesgue integral on $\mathbb{R}$. In 481Xc I show how adjusting $\mathfrak{R}$ can change the class of integrable functions; in 481Xd I show how a similar effect can sometimes be achieved by adding a point at infinity and adjusting T and Δ. As will become apparent in later sections, one of the great strengths of gauge integrals is their ability to incorporate special limiting processes. Another is the fact that we don't need to assume that the functionals ν are countably additive; see 481Xd. In the formulae of this section, I don't even ask for finite additivity; but of course the functional I_ν is likely to have a rather small domain if ν behaves too erratically.

'Gauges', as I describe them here, have moved rather briskly forward from the metric gauges δ_h (481Eb), which have sufficed for most of the gauge integrals so far described. But the generalization affects only the notation, and makes it clear why so much of the theory of the ordinary Henstock integral applies equally well to the 'approximately continuous Henstock integral' (481Q), for instance. You will observe that the sets Q of 481Ba are 'gauges' in the wide sense used here. But (as the examples of this section show clearly) we generally use them in a different way.

In ordinary measure theory, we have a fairly straightforward theory of subspaces (§214) and a rather deeper theory of product spaces (chap. 25). For gauge integrals, there are significant difficulties in the theory of subspaces, some of which will appear in the next section (see 482G-482H). For closed subspaces, something can be done, as in 481Xi; but the procedure suggested there may lose some essential element of the original tagged-partition structure (481Yc). For products of gauge integrals, we do have a reasonably satisfying version of Fubini's theorem (482M); I offer 481O and 481P as alternative approaches. However, the example of the Pfeffer integral (§484) shows that other constructions may be more effective tools for geometric measure theory.

You will note the concentration on 'neighbourhood gauges' (481Eb) in the work above. This is partly because they are 'full' in the sense of 481Ec. As will appear repeatedly in the next section, this flexibility in constructing gauges is just what one needs when proving that functions are gauge-integrable.

While I have used such phrases as 'Henstock integral', 'symmetric Riemann-complete integral' above, I have not in fact discussed integrals here, except in the exercises; in most of the examples in 481I-481Q there is no mention of any functional ν from which a gauge integral I_ν can be defined. The essence of the method is that we can set up a tagged-partition structure quite independently of any set function, and it turns out that the properties of a gauge integral depend more on this structure than on the measure involved.

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482 General theory

I turn now to results which can be applied to a wide variety of tagged-partition structures. The first step is a 'Saks-Henstock' lemma (482B), a fundamental property of tagged-partition structures allowing subdivisions. In order to relate gauge integrals to the ordinary integrals treated elsewhere in this treatise, we need to know when gauge-integrable functions are measurable (482E) and when integrable functions are gauge-integrable (482F). There are significant difficulties when we come to interpret gauge integrals over subspaces, but I give a partial result in 482G. 482I, 482K and 482M are gauge-integral versions of the Fundamental Theorem of Calculus, B.Levi's theorem and Fubini's theorem, while 482H is a limit theorem of a new kind, corresponding to classical improper integrals.

Henstock's integral (481J-481K) remains the most important example and the natural test case for the ideas here; I will give the details in the next section, and you may wish to take the two sections in parallel.

482A Lemma Suppose that $(X,T,\Delta,\mathfrak{R})$ is a tagged-partition structure allowing subdivisions (481G), witnessed by $\mathcal{C}\subseteq\mathcal{P}X$.

(a) Whenever $\delta\in\Delta$, $\mathcal{R}\in\mathfrak{R}$ and E belongs to the subalgebra of $\mathcal{P}X$ generated by $\mathcal{C}$, there is a δ-fine $\boldsymbol{s}\in T$ such that $W_{\boldsymbol{s}}\subseteq E$ and $E\setminus W_{\boldsymbol{s}}\in\mathcal{R}$.

(b) Whenever $\delta\in\Delta$, $\mathcal{R}\in\mathfrak{R}$ and $\boldsymbol{t}\in T$ is δ-fine, there is a δ-fine $\mathcal{R}$-filling $\boldsymbol{t}'\in T$ including $\boldsymbol{t}$.

(c) Suppose that $f:X\to\mathbb{R}$, $\nu:\mathcal{C}\to\mathbb{R}$, $\delta\in\Delta$, $\mathcal{R}\in\mathfrak{R}$ and $\epsilon\ge 0$ are such that $|S_{\boldsymbol{t}}(f,\nu)-S_{\boldsymbol{t}'}(f,\nu)|\le\epsilon$ whenever $\boldsymbol{t}$, $\boldsymbol{t}'\in T$ are δ-fine and $\mathcal{R}$-filling. Then

(i) $|S_{\boldsymbol{t}}(f,\nu)-S_{\boldsymbol{t}'}(f,\nu)|\le\epsilon$ whenever $\boldsymbol{t}$, $\boldsymbol{t}'\in T$ are δ-fine and $W_{\boldsymbol{t}}=W_{\boldsymbol{t}'}$;

(ii) whenever $\boldsymbol{t}\in T$ is δ-fine, and $\delta'\in\Delta$, there is a δ'-fine $\boldsymbol{s}\in T$ such that $W_{\boldsymbol{s}}\subseteq W_{\boldsymbol{t}}$ and $|S_{\boldsymbol{s}}(f,\nu)-S_{\boldsymbol{t}}(f,\nu)|\le\epsilon$.

(d) Suppose that $f:X\to\mathbb{R}$ and $\nu:\mathcal{C}\to\mathbb{R}$ are such that $I_\nu(f)=\lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})}S_{\boldsymbol{t}}(f,\nu)$ is defined, where $\mathcal{F}(T,\Delta,\mathfrak{R})$ is the filter described in 481F. Then for any $\epsilon>0$ there is a $\delta\in\Delta$ such that $S_{\boldsymbol{t}}(f,\nu)\le I_\nu(f)+\epsilon$ for every δ-fine $\boldsymbol{t}\in T$.

(e) Suppose that $f:X\to\mathbb{R}$ and $\nu:\mathcal{C}\to\mathbb{R}$ are such that $I_\nu(f)=\lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})}S_{\boldsymbol{t}}(f,\nu)$ is defined. Then for any $\epsilon>0$ there is a $\delta\in\Delta$ such that $|S_{\boldsymbol{t}}(f,\nu)|\le\epsilon$ whenever $\boldsymbol{t}\in T$ is δ-fine and $W_{\boldsymbol{t}}=\emptyset$.

proof (a) By 481He, there is a non-increasing sequence $\langle\mathcal{R}_k\rangle_{k\in\mathbb{N}}$ in $\mathfrak{R}$ such that $\bigcup_{i\le k}A_i\in\mathcal{R}$ whenever $A_i\in\mathcal{R}_i$ for every $i\le k$ and $\langle A_i\rangle_{i\le k}$ is disjoint. Let $\mathcal{C}_0$ be a finite subset of $\mathcal{C}$ such that E belongs to the subalgebra of $\mathcal{P}X$ generated by $\mathcal{C}_0$, and let $\mathcal{C}_1\supseteq\mathcal{C}_0$ be a finite subset of $\mathcal{C}$ such that $X\setminus W\in\mathcal{R}_0$, where $W=\bigcup\mathcal{C}_1$ (481G(v)). Then either $E\subseteq W$ or $E\supseteq X\setminus W$. In either case, $E\cap W$ belongs to the ring generated by $\mathcal{C}$, so is expressible as $\bigcup_{i<n}C_i$ where $\langle C_i\rangle_{i<n}$ is a disjoint family in $\mathcal{C}$ (481Hd).

For each $i<n$, let $\boldsymbol{s}_i$ be a δ-fine member of T such that $W_{\boldsymbol{s}_i}\subseteq C_i$ and $C_i\setminus W_{\boldsymbol{s}_i}\in\mathcal{R}_{i+1}$ (481G(vii)). Set $\boldsymbol{s}=\bigcup_{i<n}\boldsymbol{s}_i$. Because $\langle W_{\boldsymbol{s}_i}\rangle_{i<n}$ is disjoint and T is a straightforward set of tagged partitions, $\boldsymbol{s}\in T$; $\boldsymbol{s}$ is δ-fine because every $\boldsymbol{s}_i$ is; $W_{\boldsymbol{s}}=\bigcup_{i<n}W_{\boldsymbol{s}_i}$ is included in E; and $E\setminus W_{\boldsymbol{s}}$ is either $\bigcup_{i<n}C_i\setminus W_{\boldsymbol{s}_i}$ or $(X\setminus W)\cup\bigcup_{i<n}(C_i\setminus W_{\boldsymbol{s}_i})$, and in either case belongs to $\mathcal{R}$, by the choice of $\langle\mathcal{R}_n\rangle_{n\in\mathbb{N}}$.

(b) Set $E=X\setminus W_{\boldsymbol{t}}$. By (a), there is a δ-fine $\boldsymbol{s}\in T$ such that $W_{\boldsymbol{s}}\subseteq E$ and $E\setminus W_{\boldsymbol{s}}\in\mathcal{R}$. Set $\boldsymbol{t}'=\boldsymbol{t}\cup\boldsymbol{s}$; this works.

(c)(i) As in (b), there is a δ-fine $\boldsymbol{s}\in T$ such that $W_{\boldsymbol{s}}\cap W_{\boldsymbol{t}}=\emptyset$ and $\boldsymbol{t}\cup\boldsymbol{s}$ is $\mathcal{R}$-filling. Now $W_{\boldsymbol{t}\cup\boldsymbol{s}}=W_{\boldsymbol{t}'\cup\boldsymbol{s}}$, so $\boldsymbol{t}'\cup\boldsymbol{s}$ also is $\mathcal{R}$-filling, and

$$|S_{\boldsymbol{t}}(f,\nu)-S_{\boldsymbol{t}'}(f,\nu)|=|S_{\boldsymbol{t}\cup\boldsymbol{s}}(f,\nu)-S_{\boldsymbol{t}'\cup\boldsymbol{s}}(f,\nu)|\le\epsilon.$$

(ii) Replacing δ' by a lower bound of $\{\delta,\delta'\}$ in Δ if necessary, we may suppose that $\delta'\subseteq\delta$. Enumerate $\boldsymbol{t}$ as $\langle(x_i,C_i)\rangle_{i<n}$. Let $\langle\mathcal{R}_k\rangle_{k\in\mathbb{N}}$ be a sequence in $\mathfrak{R}$ such that $\bigcup_{i\le k}A_i\in\mathcal{R}$ whenever $\langle A_i\rangle_{i\le k}$ is disjoint and $A_i\in\mathcal{R}_i$ for every $i\le k$. For each $i<n$, let $\boldsymbol{s}_i$ be a δ'-fine member of T such that $W_{\boldsymbol{s}_i}\subseteq C_i$ and $C_i\setminus W_{\boldsymbol{s}_i}\in\mathcal{R}_{i+1}$, and set $\boldsymbol{s}=\bigcup_{i<n}\boldsymbol{s}_i$, so that $\boldsymbol{s}\in T$ is δ'-fine. By (a), there is a δ-fine $\boldsymbol{u}\in T$ such that $W_{\boldsymbol{u}}\cap W_{\boldsymbol{t}}=\emptyset$ and $X\setminus(W_{\boldsymbol{t}}\cup W_{\boldsymbol{u}})\in\mathcal{R}_0$. Set $\boldsymbol{t}'=\boldsymbol{t}\cup\boldsymbol{u}$, $\boldsymbol{s}'=\boldsymbol{s}\cup\boldsymbol{u}$; then $\boldsymbol{t}'$ and $\boldsymbol{s}'$ are δ-fine and $\mathcal{R}$-filling, because

$$X\setminus W_{\boldsymbol{s}'}=(X\setminus(W_{\boldsymbol{t}}\cup W_{\boldsymbol{u}}))\cup\bigcup_{i<n}(C_i\setminus W_{\boldsymbol{s}_i})\in\mathcal{R},$$

by the choice of $\langle\mathcal{R}_k\rangle_{k\in\mathbb{N}}$. So

$$|S_{\boldsymbol{t}}(f,\nu)-S_{\boldsymbol{s}}(f,\nu)|=|S_{\boldsymbol{t}'}(f,\nu)-S_{\boldsymbol{s}'}(f,\nu)|\le\epsilon,$$

as required.

(d) There are $\delta\in\Delta$ and $\mathcal{R}\in\mathfrak{R}$ such that $|S_{\boldsymbol{t}}(f,\nu)-I_\nu(f)|\le\epsilon$ whenever $\boldsymbol{t}\in T$ is δ-fine and $\mathcal{R}$-filling. If $\boldsymbol{t}\in T$ is an arbitrary δ-fine tagged partition, there is a δ-fine $\mathcal{R}$-filling $\boldsymbol{t}'\supseteq\boldsymbol{t}$, by (b), so

$$S_{\boldsymbol{t}}(f,\nu)\le S_{\boldsymbol{t}'}(f,\nu)\le I_\nu(f)+\epsilon,$$

as claimed.

(e) Let $\delta\in\Delta$, $\mathcal{R}\in\mathfrak{R}$ be such that $|S_{\boldsymbol{s}}(f,\nu)-I_\nu(f)|\le\frac{1}{2}\epsilon$ whenever $\boldsymbol{s}\in T$ is δ-fine and $\mathcal{R}$-filling. If $\boldsymbol{t}\in T$ is δ-fine and $W_{\boldsymbol{t}}=\emptyset$, take any δ-fine $\mathcal{R}$-filling $\boldsymbol{s}\in T$, and consider $\boldsymbol{s}'=\boldsymbol{s}\setminus\boldsymbol{t}$, $\boldsymbol{s}''=\boldsymbol{s}\cup\boldsymbol{t}$. Because $W_{\boldsymbol{s}}\cap W_{\boldsymbol{t}}=\emptyset$, both $\boldsymbol{s}'$ and $\boldsymbol{s}''$ belong to T; both are δ-fine; and because $W_{\boldsymbol{s}'}=W_{\boldsymbol{s}''}=W_{\boldsymbol{s}}$, both are $\mathcal{R}$-filling. So

$$|S_{\boldsymbol{t}}(f,\nu)|=|S_{\boldsymbol{s}''}(f,\nu)-S_{\boldsymbol{s}'}(f,\nu)|\le|S_{\boldsymbol{s}''}(f,\nu)-I_\nu(f)|+|S_{\boldsymbol{s}'}(f,\nu)-I_\nu(f)|\le\epsilon,$$

as required.

482B Saks-Henstock Lemma Let $(X,T,\Delta,\mathfrak{R})$ be a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$, and $f:X\to\mathbb{R}$, $\nu:\mathcal{C}\to\mathbb{R}$ functions such that $I_\nu(f)=\lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})}S_{\boldsymbol{t}}(f,\nu)$ is defined. Let $\mathcal{E}$ be the algebra of subsets of X generated by $\mathcal{C}$. Then there is a unique additive functional $F:\mathcal{E}\to\mathbb{R}$ such that for every $\epsilon>0$ there are $\delta\in\Delta$ and $\mathcal{R}\in\mathfrak{R}$ such that

(α) $\sum_{(x,C)\in\boldsymbol{t}}|F(C)-f(x)\nu C|\le\epsilon$ for every δ-fine $\boldsymbol{t}\in T$,

(β) $|F(E)|\le\epsilon$ whenever $E\in\mathcal{E}\cap\mathcal{R}$.

Moreover, $F(X)=I_\nu(f)$.

proof (a) For $E\in\mathcal{E}$, write T_E for the set of those $\boldsymbol{t}\in T$ such that, for every $(x,C)\in\boldsymbol{t}$, either $C\subseteq E$ or $C\cap E=\emptyset$. For any $\delta\in\Delta$, $\mathcal{R}\in\mathfrak{R}$ and finite $\mathcal{D}\subseteq\mathcal{E}$ there is a δ-fine $\boldsymbol{t}\in\bigcap_{E\in\mathcal{D}}T_E$ such that $E\setminus W_{\boldsymbol{t}}\in\mathcal{R}$ for every $E\in\mathcal{D}$. **P** Let $\langle\mathcal{R}_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{R}$ such that whenever $A_i\in\mathcal{R}_i$ for $i\le n$ and $\langle A_i\rangle_{i\le n}$ is disjoint then $\bigcup_{i\le n}A_i\in\mathcal{R}$ (481He again). Let $\mathcal{E}_0$ be the subalgebra of $\mathcal{E}$ generated by $\mathcal{D}$, and enumerate the atoms of $\mathcal{E}_0$ as $\langle E_i\rangle_{i<n}$. By 482Aa, there is for each $i<n$ a δ-fine $\boldsymbol{s}_i\in T$ such that $W_{\boldsymbol{s}_i}\subseteq E_i$ and $E_i\setminus W_{\boldsymbol{s}_i}\in\mathcal{R}_i$. Set $\boldsymbol{t}=\bigcup_{i<n}\boldsymbol{s}_i$. If $E\in\mathcal{D}$ then $E=\bigcup_{i\in J}E_i$ for some $J\subseteq n$. For any $(x,C)\in\boldsymbol{t}$, there is some $i<n$ such that $C\subseteq E_i$, so that $C\subseteq E$ if $i\in J$, $C\cap E=\emptyset$ otherwise; thus $\boldsymbol{t}\in T_E$. Moreover, $E\setminus W_{\boldsymbol{t}}=\bigcup_{i\in J}(E_i\setminus W_{\boldsymbol{s}_i})$ belongs to $\mathcal{R}$. **Q**

(b) We therefore have a filter $\mathcal{F}^*$ on T generated by sets of the form

$$T_{E\delta\mathcal{R}}=\{\boldsymbol{t}:\boldsymbol{t}\in T_E\text{ is }\delta\text{-fine, }E\setminus W_{\boldsymbol{t}}\in\mathcal{R}\}$$

as δ runs over Δ, $\mathcal{R}$ runs over $\mathfrak{R}$ and E runs over $\mathcal{E}$. For $\boldsymbol{t}\in T$, $E\subseteq X$ set $\boldsymbol{t}_E=\{(x,C):(x,C)\in\boldsymbol{t},\,C\subseteq E\}$. Now $F(E)=\lim_{\boldsymbol{t}\to\mathcal{F}^*}S_{\boldsymbol{t}_E}(f,\nu)$ is defined for every $E\in\mathcal{E}$. **P** For any $\epsilon>0$, there are $\delta\in\Delta$, $\mathcal{R}\in\mathfrak{R}$ such that $|I_\nu(f)-S_{\boldsymbol{t}}(f,\nu)|\le\epsilon$ for every δ-fine $\mathcal{R}$-filling $\boldsymbol{t}\in T$. Let $\mathcal{R}'\in\mathfrak{R}$ be such that $A\cup B\in\mathcal{R}$ for all disjoint A, $B\in\mathcal{R}'$. If $\boldsymbol{t}$, $\boldsymbol{t}'$ belong to $T_{E,\delta,\mathcal{R}'}=T_{X\setminus E,\delta,\mathcal{R}'}$, then set

$$\boldsymbol{s}=\{(x,C):(x,C)\in\boldsymbol{t}',\,C\subseteq E\}\cup\{(x,C):(x,C)\in\boldsymbol{t},\,C\cap E=\emptyset\}.$$

Then $\boldsymbol{s}\in T_E$ is δ-fine, and also $E\setminus W_{\boldsymbol{s}}=E\setminus W_{\boldsymbol{t}'}$, $(X\setminus E)\setminus W_{\boldsymbol{s}}=(X\setminus E)\setminus W_{\boldsymbol{t}}$ both belong to $\mathcal{R}'$; so their union $X\setminus W_{\boldsymbol{s}}$ belongs to $\mathcal{R}$, and $\boldsymbol{s}$ is $\mathcal{R}$-filling. Accordingly

$$\begin{aligned}|S_{\boldsymbol{t}_E}(f,\nu)-S_{\boldsymbol{t}'_E}(f,\nu)|&=|S_{\boldsymbol{t}}(f,\nu)-S_{\boldsymbol{s}}(f,\nu)|\\&\le|S_{\boldsymbol{t}}(f,\nu)-I_\nu(f)|+|I_\nu(f)-S_{\boldsymbol{s}}(f,\nu)|\le2\epsilon.\end{aligned}$$

As ϵ is arbitrary, this is enough to show that $\liminf_{\boldsymbol{t}\to\mathcal{F}^*}S_{\boldsymbol{t}_E}(f,\nu)=\limsup_{\boldsymbol{t}\to\mathcal{F}^*}S_{\boldsymbol{t}_E}(f,\nu)$, so that the limit $\lim_{\boldsymbol{t}\to\mathcal{F}^*}S_{\boldsymbol{t}_E}(f,\nu)$ is defined (2A3Sf). **Q**

(c) $F(\emptyset)=0$. **P** Let $\epsilon>0$. By 482Ae, there is a $\delta\in\Delta$ such that $|S_{\boldsymbol{t}}(f,\nu)|\le\epsilon$ whenever $\boldsymbol{t}\in T$ is δ-fine and $W_{\boldsymbol{t}}=\emptyset$. Since $\{\boldsymbol{t}:\boldsymbol{t}\text{ is }\delta\text{-fine}\}$ belongs to $\mathcal{F}^*$,

$$|F(\emptyset)|=|\lim_{\boldsymbol{t}\to\mathcal{F}^*}S_{\boldsymbol{t}_\emptyset}(f,\nu)|\le\epsilon;$$

as ϵ is arbitrary, $F(\emptyset)=0$. **Q**

If E, $E'\in\mathcal{E}$, then

$$S_{\boldsymbol{t}_{E\cup E'}}(f,\nu)+S_{\boldsymbol{t}_{E\cap E'}}(f,\nu)=S_{\boldsymbol{t}_E}(f,\nu)+S_{\boldsymbol{t}_{E'}}(f,\nu)$$

for every $\boldsymbol{t}\in T_E\cap T_{E'}$; as $T_E\cap T_{E'}$ belongs to $\mathcal{F}^*$,

$$F(E\cup E')+F(E\cap E')=F(E)+F(E').$$

Since $F(\emptyset)=0$, $F(E\cup E')=F(E)+F(E')$ whenever $E\cap E'=\emptyset$, and F is additive.

(d) Now suppose that $\epsilon>0$. Let $\delta\in\Delta$, $\mathcal{R}^*\in\mathfrak{R}$ be such that $|I_\nu(f)-S_{\boldsymbol{t}}(f,\nu)|\le\frac{1}{4}\epsilon$ for every δ-fine, $\mathcal{R}^*$-filling $\boldsymbol{t}\in T$. Let $\mathcal{R}\in\mathfrak{R}$ be such that $A\cup B\in\mathcal{R}^*$ for all disjoint A, $B\in\mathcal{R}$.

(i) If $\boldsymbol{t}\in T$ is δ-fine, then $|F(W_{\boldsymbol{t}})-S_{\boldsymbol{t}}(f,\nu)|\le\frac{1}{2}\epsilon$. **P** For any $\eta>0$, there is a δ-fine $\boldsymbol{s}\in T$ such that

$|I_\nu(f)-S_{\boldsymbol{s}}(f,\nu)|\le\eta$,

for every $(x,C)\in\boldsymbol{s}$, either $C\subseteq W_{\boldsymbol{t}}$ or $C\cap W_{\boldsymbol{t}}=\emptyset$,

$(X\setminus W_{\boldsymbol{t}})\setminus W_{\boldsymbol{s}}\in\mathcal{R}$, $W_{\boldsymbol{t}}\setminus W_{\boldsymbol{s}}\in\mathcal{R}$,

$|F(W_{\boldsymbol{t}})-\sum_{(x,C)\in\boldsymbol{s},C\subseteq W_{\boldsymbol{t}}}f(x)\nu C|\le\eta$

because the set of $\boldsymbol{s}$ with these properties belongs to $\mathcal{F}^*$. Now, setting $\boldsymbol{s}_1=\{(x,C):(x,C)\in\boldsymbol{s},\,C\subseteq W_{\boldsymbol{t}}\}$ and $\boldsymbol{t}'=\boldsymbol{t}\cup(\boldsymbol{s}\setminus\boldsymbol{s}_1)$, $\boldsymbol{t}'$ is δ-fine and $\mathcal{R}^*$-filling, like $\boldsymbol{s}$, so

$$\begin{aligned}|F(W_{\boldsymbol{t}})-S_{\boldsymbol{t}}(f,\nu)|&\le|F(W_{\boldsymbol{t}})-S_{\boldsymbol{s}_1}(f,\nu)|+|S_{\boldsymbol{s}_1}(f,\nu)-S_{\boldsymbol{t}}(f,\nu)|\\&\le\eta+|S_{\boldsymbol{s}}(f,\nu)-S_{\boldsymbol{t}'}(f,\nu)|\\&\le\eta+|S_{\boldsymbol{s}}(f,\nu)-I_\nu(f)|+|I_\nu(f)-S_{\boldsymbol{t}'}(f,\nu)|\le\eta+\frac{1}{2}\epsilon.\end{aligned}$$

As η is arbitrary we have the result. **Q**

(ii) So if $\boldsymbol{t}\in T$ is δ-fine, $\sum_{(x,C)\in\boldsymbol{t}}|F(C)-f(x)\nu C|\le\epsilon$. **P** Set $\boldsymbol{t}'=\{(x,C):(x,C)\in\boldsymbol{t},\,F(C)\le f(x)\nu C\}$, $\boldsymbol{t}''=\boldsymbol{t}\setminus\boldsymbol{t}''$. Then both $\boldsymbol{t}'$ and $\boldsymbol{t}''$ are δ-fine, so

$$\begin{aligned}&\sum_{(x,C)\in\boldsymbol{t}}|F(C)-f(x)\nu C|\\&\qquad=|\sum_{(x,C)\in\boldsymbol{t}'}F(C)-f(x)\nu C|+|\sum_{(x,C)\in\boldsymbol{t}''}F(C)-f(x)\nu C|\le\epsilon.\ \textbf{Q}\end{aligned}$$

(iii) If $E\in\mathcal{E}\cap\mathcal{R}$, then $|F(E)|\le\epsilon$. **P** Let $\mathcal{R}'\in\mathfrak{R}$ be such that $A\cup B\in\mathcal{R}$ whenever A, $B\in\mathcal{R}'$ are disjoint. Let $\boldsymbol{t}$ be such that

$\boldsymbol{t}\in T_E$ is δ-fine,
$E\setminus W_{\boldsymbol{t}}$ and $(X\setminus E)\setminus W_{\boldsymbol{t}}$ both belong to $\mathcal{R}'$,
$|F(E)-S_{\boldsymbol{t}_E}(f,\nu)|\le\frac{1}{2}\epsilon$;

once again, the set of candidates belongs to $\mathcal{F}^*$, so is not empty. Then $\boldsymbol{t}$ and $\boldsymbol{t}_{X\setminus E}$ are both $\mathcal{R}^*$-filling and δ-fine, so

$$|F(E)|\le\frac{1}{2}\epsilon+|S_{\boldsymbol{t}_E}(f,\nu)|=\frac{1}{2}\epsilon+|S_{\boldsymbol{t}}(f,\nu)-S_{\boldsymbol{t}_{X\setminus E}}(f,\nu)|\le\epsilon.\ \textbf{Q}$$

As ϵ is arbitrary, this shows that F has all the required properties.

(e) I have still to show that F is unique. Suppose that $F':\mathcal{E}\to\mathbb{R}$ is another functional with the same properties, and take $E\in\mathcal{E}$ and $\epsilon>0$. Then there are δ, $\delta'\in\Delta$ and $\mathcal{R}$, $\mathcal{R}'\in\mathfrak{R}$ such that

$\sum_{(x,C)\in\boldsymbol{t}}|F(C)-f(x)\nu C|\le\epsilon$ for every δ-fine $\boldsymbol{t}\in T$,
$\sum_{(x,C)\in\boldsymbol{t}}|F'(C)-f(x)\nu C|\le\epsilon$ for every δ'-fine $\boldsymbol{t}\in T$,
$|F(R)|\le\epsilon$ whenever $R\in\mathcal{E}\cap\mathcal{R}$,
$|F'(R)|\le\epsilon$ whenever $R\in\mathcal{E}\cap\mathcal{R}'$.

Now taking $\delta''\in\Delta$ such that $\delta''\subseteq\delta\cap\delta'$, and $\mathcal{R}''\in\mathfrak{R}$ such that $\mathcal{R}''\subseteq\mathcal{R}\cap\mathcal{R}'$, there is a δ''-fine $\boldsymbol{t}\in T$ such that $E'=W_{\boldsymbol{t}}$ is included in E and $E\setminus E'\in\mathcal{R}''$. In this case

$$|F(E)-F'(E)|\le|F(E\setminus E')|+\sum_{(x,C)\in\boldsymbol{t}}|F(C)-F'(C)|+|F'(E\setminus E')|$$

(because F and F' are both additive)

$$\le2\epsilon+\sum_{(x,C)\in\boldsymbol{t}}|F(C)-f(x)\nu C|+\sum_{(x,C)\in\boldsymbol{t}}|F'(C)-f(x)\nu C|\le4\epsilon.$$

As ϵ and E are arbitrary, $F=F'$, as required.

(f) Finally, to calculate $F(X)$, take any $\epsilon>0$. Let $\delta\in\Delta$ and $\mathcal{R}\in\mathfrak{R}$ be such that $\sum_{(x,C)\in\boldsymbol{t}}|F(C)-f(x)\nu C|\le\epsilon$ for every δ-fine $\boldsymbol{t}\in T$ and $|F(E)|\le\epsilon$ whenever $E\in\mathcal{E}\cap\mathcal{R}$. Let $\boldsymbol{t}$ be any δ-fine $\mathcal{R}$-filling member of T such that $|S_{\boldsymbol{t}}(f,\nu)-I_\nu(f)|\le\epsilon$. Then, because F is additive,

$$\begin{aligned}|F(X)-I_\nu(f)|&\le|F(X)-F(W_{\boldsymbol{t}})|+|\sum_{(x,C)\in\boldsymbol{t}}F(C)-f(x)\nu C|+|S_{\boldsymbol{t}}(f,\nu)-I_\nu(f)|\\&\le3\epsilon.\end{aligned}$$

As ϵ is arbitrary, $F(X)=I_\nu(f)$.

482C Definition In the context of 482B, I will call the function F the **Saks-Henstock indefinite integral** of f; of course it depends on the whole structure $(X,T,\Delta,\mathfrak{R},\mathcal{C},f,\nu)$ and not just on (X,f,ν). You should *not* take it for granted that $F(E)=I_\nu(f\times\chi E)$ (482Ya); but see 482G.

482D The Saks-Henstock lemma characterizes the gauge integral, as follows.

Theorem Let $(X,T,\Delta,\mathfrak{R})$ be a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$, and $\nu:\mathcal{C}\to\mathbb{R}$ any function. Let $\mathcal{E}$ be the algebra of subsets of X generated by $\mathcal{C}$. If $f:X\to\mathbb{R}$ is any function, then the following are equiveridical:

(i) $I_\nu(f)=\lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})}S_{\boldsymbol{t}}(f,\nu)$ is defined in $\mathbb{R}$;

(ii) there is an additive functional $F:\mathcal{E}\to\mathbb{R}$ such that

(α) for every $\epsilon>0$ there is a $\delta\in\Delta$ such that $\sum_{(x,C)\in\boldsymbol{t}}|F(C)-f(x)\mu C|\le\epsilon$ for every δ-fine $\boldsymbol{t}\in T$,

(β) for every $\epsilon>0$ there is an $\mathcal{R}\in\mathfrak{R}$ such that $|F(E)|\le\epsilon$ for every $E\in\mathcal{E}\cap\mathcal{R}$;

(iii) there is an additive functional $F:\mathcal{E}\to\mathbb{R}$ such that

(α) for every $\epsilon>0$ there is a $\delta\in\Delta$ such that $|F(W_{\boldsymbol{t}})-\sum_{(x,C)\in\boldsymbol{t}}f(x)\mu C|\le\epsilon$ for every δ-fine $\boldsymbol{t}\in T$,

(β) for every $\epsilon>0$ there is an $\mathcal{R}\in\mathfrak{R}$ such that $|F(E)|\le\epsilon$ for every $E\in\mathcal{E}\cap\mathcal{R}$.

In this case, $F(X)=I_\nu(f)$.

proof (i)$\Rightarrow$(ii) is just 482B above, and (ii)$\Rightarrow$(iii) is elementary, because $F(W_{\boldsymbol{t}})=\sum_{(x,C)\in\boldsymbol{t}}F(C)$ whenever $F:\mathcal{E}\to\mathbb{R}$ is additive and $\boldsymbol{t}\in T$; so let us assume (iii) and seek to prove (i). Given $\epsilon>0$, take $\delta\in\Delta$ and $\mathcal{R}\in\mathfrak{R}$ such that (α) and (β) of (iii) are satisfied. Let $\boldsymbol{t}\in T$ be δ-fine and $\mathcal{R}$-filling. Then

$$|F(X)-S_{\boldsymbol{t}}(f,\mu)|\le|F(X\setminus W_{\boldsymbol{t}})|+|F(W_{\boldsymbol{t}})-\textstyle\sum_{(x,C)\in\boldsymbol{t}}f(x)\mu C|\le2\epsilon.$$

As ϵ is arbitrary, $I_\nu(f)$ is defined and equal to $F(X)$.

482E Theorem Let (X,ρ) be a metric space and μ a complete locally determined measure on X with domain Σ. Let $\mathcal{C}$, Q, T, Δ and $\mathfrak{R}$ be such that

(i) $\mathcal{C}\subseteq\Sigma$ and μC is finite for every $C\in\mathcal{C}$;

(ii) $Q\subseteq X\times\mathcal{C}$, and for each $C\in\mathcal{C}$, $(x,C)\in Q$ for almost every $x\in C$;

(iii) T is the straightforward set of tagged partitions generated by Q;

(iv) Δ is a downwards-directed family of gauges on X containing all the uniform metric gauges;

(v) if $\delta\in\Delta$, there are a negligible set $F\subseteq X$ and a neighbourhood gauge δ_0 on X such that $\delta\supseteq\delta_0\setminus(F\times\mathcal{P}X)$;

(vi) $\mathfrak{R}$ is a downwards-directed collection of families of subsets of X such that whenever $E\in\Sigma$, $\mu E<\infty$ and $\epsilon>0$, there is an $\mathcal{R}\in\mathfrak{R}$ such that $\mu^*(E\cap R)\le\epsilon$ for every $R\in\mathcal{R}$;

(vii) T is compatible with Δ and $\mathfrak{R}$.

Let $f:X\to\mathbb{R}$ be any function such that $I_\mu(f)=\lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})}S_{\boldsymbol{t}}(f,\mu)$ is defined. Then f is Σ-measurable.

proof ? Suppose, if possible, otherwise.

Because μ is complete and locally determined, there are a measurable set E of non-zero finite measure and $\alpha<\beta$ in $\mathbb{R}$ such that

$$\mu^*\{x:x\in E,\,f(x)\le\alpha\}=\mu^*\{x:x\in E,\,f(x)\ge\beta\}=\mu E$$

(413G). Let $\epsilon>0$ be such that $(\beta-\alpha)(\mu E-3\epsilon)>2\epsilon$. Let $\delta\in\Delta$, $\mathcal{R}\in\mathfrak{R}$ be such that $|S_{\boldsymbol{t}}(f,\mu)-I_\nu(f)|\le\epsilon$ whenever $\boldsymbol{t}\in T$ is δ-fine and $\mathcal{R}$-filling. By (v), there are a negligible set $F\subseteq X$ and a family $\langle G_x\rangle_{x\in X}$ of open sets such that $x\in G_x$ for every $x\in X$ and $\delta\supseteq\{(x,C):x\in X\setminus F,\,C\subseteq G_x\}$. For $m\ge1$, set

$$A_m=\{x:x\in E\setminus F,\,f(x)\le\alpha,\,U_{1/m}(x)\subseteq G_x\},$$

writing $U_{1/m}(x)$ for $\{y:\rho(y,x)<\frac{1}{m}\}$,

$$B_m=\{x:x\in E\setminus F,\,f(x)\ge\beta,\,U_{1/m}(x)\subseteq G_x\}.$$

Then there is some $m\ge1$ such that $\mu^*A_m\ge\mu E-\epsilon$ and $\mu^*B_m\ge\mu E-\epsilon$. By (iv), there is a $\delta'\in\Delta$ such that

$$\delta' \subseteq \delta \cap \{(x, C) : x \in C \subseteq U_{1/3m}(x)\}.$$

By (vi), there is an $\mathcal{R}' \in \mathfrak{R}$ such that $\mathcal{R}' \subseteq \mathcal{R}$ and $\mu^*(R \cap E) \leq \epsilon$ for every $R \in \mathcal{R}'$.

Let $\boldsymbol{t}$ be any δ'-fine $\mathcal{R}'$-filling member of T. Enumerate $\boldsymbol{t}$ as $\langle (x_i, C_i)\rangle_{i<n}$. Set

$$J = \{i : i < n,\, C_i \cap A_m \text{ is negligible}\}, \quad J' = \{i : i < n,\, C_i \cap B_m \text{ is negligible}\}.$$

Then

$$\mu(E \cap \bigcup_{i\in J} C_i) \leq \mu_*(E \setminus A_m) = \mu E - \mu^*(E \cap A_m) \leq \epsilon,$$

and similarly $\mu(E \cap \bigcup_{i\in J'} C_i) \leq \epsilon$. Also, because $X \setminus \bigcup_{i<n} C_i = X \setminus W_{\boldsymbol{t}}$ belongs to $\mathcal{R}'$, $\mu(E \setminus \bigcup_{i<n} C_i) \leq \epsilon$. So, setting $K = n \setminus (J \cup J')$, $\sum_{i\in K} \mu C_i \geq \mu E - 3\epsilon$.

For $i \in K$, $\mu^*(C_i \cap A_m) > 0$, while $\{x : x \in C_i,\, (x, C_i) \notin Q\}$ is negligible, by (ii), so we can find $x'_i \in C_i \cap A_m$ such that $(x'_i, C_i) \in Q$; similarly, there is an $x''_i \in C_i \cap B_m$ such that $(x''_i, C_i) \in Q$. For other $i < n$, set $x'_i = x''_i = x_i$. Now $\boldsymbol{s} = \{(x'_i, C_i) : i < n\}$ and $\boldsymbol{s}' = \{(x''_i, C_i) : i < n\}$ belong to T. Of course they are $\mathcal{R}'$-filling, therefore $\mathcal{R}$-filling, because $\boldsymbol{t}$ is. We also see that, because $(x_i, C_i) \in \delta'$, the diameter of C_i is at most $\frac{2}{3m}$ for each $i < n$, so that $C_i \subseteq G_{x'_i}$; as also $x'_i \in A_m \subseteq X \setminus F$, $(x'_i, C_i) \in \delta$, for each $i \in K$. But since surely $(x_i, C_i) \in \delta' \subseteq \delta$ for $i \in n \setminus K$, this means that $\boldsymbol{s}$ is δ-fine. Similarly, $\boldsymbol{s}'$ is δ-fine.

We must therefore have

$$|S_{\boldsymbol{s}'}(f, \mu) - S_{\boldsymbol{s}}(f, \mu)| \leq |S_{\boldsymbol{s}'}(f, \mu) - I_\mu(f)| + |S_{\boldsymbol{s}}(f, \mu) - I_\mu(f)| \leq 2\epsilon.$$

But

$$\begin{aligned} S_{\boldsymbol{s}'}(f, \mu) - S_{\boldsymbol{s}}(f, \mu) &= \sum_{i\in K} (f(x''_i) - f(x'_i))\mu C_i \\ &\geq (\beta - \alpha) \sum_{i\in K} \mu C_i \geq (\beta - \alpha)(\mu E - 3\epsilon) > 2\epsilon \end{aligned}$$

by the choice of ϵ. **X**

So we have the result.

482F Proposition Let X, Σ, μ, $\mathfrak{T}$, T, Δ and $\mathfrak{R}$ be such that

(i) (X, Σ, μ) is a measure space;

(ii) $\mathfrak{T}$ is a topology on X such that μ is inner regular with respect to the closed sets and outer regular with respect to the open sets;

(iii) $T \subseteq [X \times \Sigma]^{<\omega}$ is a set of tagged partitions such that $C \cap C'$ is empty whenever (x, C), (x', C') are distinct members of any $\boldsymbol{t} \in T$;

(iv) Δ is a set of gauges on X containing every neighbourhood gauge on X;

(v) $\mathfrak{R}$ is a collection of families of subsets of X such that whenever $\mu E < \infty$ and $\epsilon > 0$ there is an $\mathcal{R} \in \mathfrak{R}$ such that $\mu^*(E \cap R) \leq \epsilon$ for every $R \in \mathcal{R}$;

(vi) T is compatible with Δ and $\mathfrak{R}$.

Then $I_\mu(f) = \lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f, \mu)$ is defined and equal to $\int f d\mu$ for every μ-integrable function $f : X \to \mathbb{R}$.

proof (a) It is worth noting straight away that we can replace (X, Σ, μ) by its completion $(X, \hat{\Sigma}, \hat{\mu})$. **P** We need to check that $\hat{\mu}$ is inner and outer regular. But inner regularity is 412Ha, and outer regularity is equally elementary: if $\hat{\mu} E < \gamma$, there is an $E' \in \Sigma$ such that $E \subseteq E'$ and $\mu E' = \hat{\mu} E$ (212C), and now there is an open set $G \in \Sigma$ such that $E' \subseteq G$ and $\mu G \leq \gamma$, so that $E \subseteq G$ and $\hat{\mu} G \leq \gamma$. Since we are not changing T or Δ or $\mathfrak{R}$, $I_{\hat{\mu}}(f) = I_\mu(f)$ if either is defined; while also $\int f d\mu = \int f d\hat{\mu}$ if either is defined, by 212Fb. **Q**

So let us suppose that μ is actually complete.

(b) In this case, f is measurable. Suppose to begin with that it is non-negative. Let $\epsilon > 0$. For $m \in \mathbb{Z}$, set $E_m = \{x : x \in X,\, (1 + \epsilon)^m \leq f(x) < (1 + \epsilon)^{m+1}\}$. Then E_m is measurable and has finite measure, so there is a measurable open set $G_m \supseteq E_m$ such that $(1 + \epsilon)^{m+1}\mu(G_m \setminus E_m) \leq 2^{-|m|}\epsilon$.

Take a set H_0 of finite measure and $\eta_0 > 0$ such that $\int_E f d\mu \leq \epsilon$ whenever $E \in \Sigma$ and $\mu(E \cap H_0) \leq 2\eta_0$ (225A); replacing H_0 by $\{x : x \in H_0,\, f(x) > 0\}$ if necessary, we may suppose that $H_0 \subseteq \bigcup_{m\in\mathbb{Z}} E_m$. Let $F \subseteq H_0$ be a closed set such that $\mu(H_0 \setminus F) \leq \eta_0$.

Define $\langle V_x\rangle_{x\in X}$ by setting $V_x = G_m$ if $m \in \mathbb{Z}$ and $x \in E_m$, $V_x = X \setminus F$ if $f(x) = 0$. Let $\delta \in \Delta$ be the corresponding neighbourhood gauge $\{(x, C) : x \in X,\ C \subseteq V_x\}$. Let $\mathcal{R} \in \mathfrak{R}$ be such that $\mu^*(R \cap H_0) \le \eta_0$ for every $R \in \mathcal{R}$.

Suppose that $\boldsymbol{t}$ is any δ-fine $\mathcal{R}$-filling member of T. Enumerate $\boldsymbol{t}$ as $\langle(x_i, C_i)\rangle_{i<n}$. For each $m \in \mathbb{Z}$, set $J_m = \{i : i < n,\ x_i \in E_m\}$. Then $C_i \subseteq V_{x_i} \subseteq G_m$ for every $i \in J_m$, so

$$\begin{aligned} S_{\boldsymbol{t}}(f, \mu) &= \sum_{i<n} f(x_i)\mu C_i = \sum_{m\in\mathbb{Z}} \sum_{i\in J_m} f(x_i)\mu C_i \le \sum_{m\in\mathbb{Z}} (1+\epsilon)^{m+1}\mu G_m \\ &\le (1+\epsilon)\sum_{m\in\mathbb{Z}}(1+\epsilon)^m \mu E_m + \sum_{m\in\mathbb{Z}}(1+\epsilon)^{m+1}\mu(G_m \setminus E_m) \\ &\le (1+\epsilon)\int f d\mu + \sum_{m\in\mathbb{Z}} 2^{-|m|}\epsilon = (1+\epsilon)\int f d\mu + 3\epsilon. \end{aligned}$$

On the other hand, set $F' = F \cap \bigcup_{i<n} C_i$. Because $X \setminus \bigcup_{i<n} C_i \in \mathcal{R}$, $\mu(H_0 \setminus F') \le 2\eta_0$, and

$$\begin{aligned} S_{\boldsymbol{t}}(f, \mu) &\ge \sum_{m\in\mathbb{Z}} \sum_{i\in J_m} f(x_i)\mu(C_i \cap F) \\ &\ge \tfrac{1}{1+\epsilon} \sum_{m\in\mathbb{Z}} \sum_{i\in J_m} (1+\epsilon)^{m+1}\mu(C_i \cap F) \\ &\ge \tfrac{1}{1+\epsilon}\int_{F'} f d\mu \ge \tfrac{1}{1+\epsilon}(\int f d\mu - \epsilon). \end{aligned}$$

What this means is that

$$\{\boldsymbol{t} : \tfrac{1}{1+\epsilon}(\textstyle\int f d\mu - \epsilon) \le S_{\boldsymbol{t}}(f, \nu) \le (1+\epsilon)\int f d\mu + 3\epsilon\}$$

belongs to $\mathcal{F}(T, \Delta, \mathfrak{R})$, for any $\epsilon > 0$. So $\lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f, \mu)$ is defined and equal to $\int f d\mu$.

(c) In general, f is expressible as $f^+ - f^-$ where f^+ and f^- are non-negative integrable functions, so

$$I_\nu(f) = I_\nu(f^+) - I_\nu(f^-) = \textstyle\int f d\mu$$

by 481Ca.

482G Proposition Let $(X, T, \Delta, \mathfrak{R})$ be a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$. Suppose that

(i) $\mathfrak{T}$ is a topology on X, and Δ is the set of neighbourhood gauges on X;

(ii) $\nu : \mathcal{C} \to \mathbb{R}$ is a function which is additive in the sense that if $C_0, \dots, C_n \in \mathcal{C}$ are disjoint and have union $C \in \mathcal{C}$, then $\nu C = \sum_{i=0}^n \nu C_i$;

(iii) whenever $E \in \mathcal{C}$ and $\epsilon > 0$, there are closed sets $F \subseteq E$, $F' \subseteq X \setminus E$ such that $\sum_{(x,C)\in\boldsymbol{t}} |\nu C| \le \epsilon$ whenever $\boldsymbol{t} \in T$ and $W_{\boldsymbol{t}} \cap (F \cup F') = \emptyset$;

(iv) for every $E \in \mathcal{C}$ and $x \in X$ there is a neighbourhood G of x such that if $C \in \mathcal{C}$, $C \subseteq G$ and $\{(x, C)\} \in T$, there is a finite partition $\mathcal{D}$ of C into members of $\mathcal{C}$, each either included in E or disjoint from E, such that $\{(x, D)\} \in T$ for every $D \in \mathcal{D}$;

(v) for every $C \in \mathcal{C}$ and $\mathcal{R} \in \mathfrak{R}$, there is an $\mathcal{R}' \in \mathfrak{R}$ such that $C \cap A \in \mathcal{R}$ whenever $A \in \mathcal{R}'$.

Let $f : X \to \mathbb{R}$ be a function such that $I_\nu(f) = \lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f, \nu)$ is defined. Let $\mathcal{E}$ be the algebra of subsets of X generated by $\mathcal{C}$, and $F : \mathcal{E} \to \mathbb{R}$ the Saks-Henstock indefinite integral of f. Then $I_\nu(f \times \chi E)$ is defined and equal to $F(E)$ for every $E \in \mathcal{E}$.

proof (a) Because both F and I_ν are additive, and $F(X) = I_\nu(f)$, and either E or its complement is a finite disjoint union of members of $\mathcal{C}$ (see 481Hd), it is enough to consider the case in which $E \in \mathcal{C}$.

(b) Let $\epsilon > 0$. For each $x \in X$ let G_x be an open set containing x such that whenever $C \in \mathcal{C}$, $C \subseteq G$ and $\{(x, C)\} \in T$, there is a finite partition $\mathcal{D}$ of C into members of $\mathcal{C}$ such that $\{(x, D)\} \in T$ for every $D \in \mathcal{D}$ and every member of $\mathcal{D}$ is either included in E or disjoint from E. For each $n \in \mathbb{N}$, let $F_n \subseteq E$, $F'_n \subseteq X \setminus E$

be closed sets such that $\sum_{(x,C)\in\boldsymbol{t}}|\nu C|\le\frac{2^{-n}\epsilon}{n+1}$ whenever $\boldsymbol{t}\in T$ and $W_{\boldsymbol{t}}\cap(F_n\cup F'_n)=\emptyset$; now define G'_x, for $x\in X$, by saying that

$$\begin{aligned}G'_x&=G_x\setminus F'_n \text{ if } x\in E \text{ and } n\le|f(x)|<n+1,\\&=G_x\setminus F_n \text{ if } x\in X\setminus E \text{ and } n\le|f(x)|<n+1.\end{aligned}$$

Let $\delta_0\in\Delta$ be the neighbourhood gauge defined by the family $\langle G'_x\rangle_{x\in X}$. Let $\delta\in\Delta$ and $\mathcal{R}_1\in\mathfrak{R}$ be such that $\delta\subseteq\delta_0$, $\sum_{(x,C)\in\boldsymbol{t}}|F(C)-f(x)\nu C|\le\epsilon$ for every δ-fine $\boldsymbol{t}\in T$, and $|F(E)|\le\epsilon$ for every $E\in\mathcal{E}\cap\mathcal{R}_1$. Let $\mathcal{R}\in\mathfrak{R}$ be such that $R\cap E\in\mathcal{R}_1$ whenever $R\in\mathcal{R}$.

(c) As in the proof of 482B, let T_E be the set of those $\boldsymbol{t}\in T$ such that, for each $(x,C)\in\boldsymbol{t}$, either $C\subseteq E$ or $C\cap E=\emptyset$. The key to the proof is the following fact: if $\boldsymbol{t}\in T$ is δ-fine, then there is a δ-fine $\boldsymbol{s}\in T_E$ such that $W_{\boldsymbol{s}}=W_{\boldsymbol{t}}$ and $S_{\boldsymbol{s}}(g,\nu)=S_{\boldsymbol{t}}(g,\nu)$ for every $g:X\to\mathbb{R}$. **P** For each $(x,C)\in\boldsymbol{t}$, we know that $C\subseteq G'_x\subseteq G_x$, because $\delta\subseteq\delta_0$. Let $\mathcal{D}_{(x,C)}$ be a finite partition of C into members of $\mathcal{C}$, each either included in E or disjoint from E, such that $\{(x,D)\}\in T$ for every $D\in\mathcal{D}_{(x,C)}$. Then $\boldsymbol{s}=\{(x,D):(x,C)\in\boldsymbol{t},\ D\in\mathcal{D}_{(x,C)}\}$ belongs to T_E. Because δ is a neighbourhood gauge, $(x,D)\in\delta$ whenever $(x,C)\in\boldsymbol{t}$ and $D\in\mathcal{D}_{(x,C)}$, so $\boldsymbol{s}$ is δ-fine.

If $g:X\to\mathbb{R}$ is any function,

$$\begin{aligned}S_{\boldsymbol{s}}(g,\nu)&=\sum_{(x,C)\in\boldsymbol{t}}\sum_{D\in\mathcal{D}_{(x,C)}}g(x)\nu D\\&=\sum_{(x,C)\in\boldsymbol{t}}g(x)\sum_{D\in\mathcal{D}_{(x,C)}}\nu D=\sum_{(x,C)\in\boldsymbol{t}}g(x)\nu C\end{aligned}$$

(because ν is additive)

$$=S_{\boldsymbol{t}}(g,\nu).\ \mathbf{Q}$$

(d) Now suppose that $\boldsymbol{t}\in T$ is δ-fine and $\mathcal{R}$-filling. Let $\boldsymbol{s}\in T_E$ be as in (c), and set

$$\boldsymbol{s}^*=\{(x,D):(x,D)\in\boldsymbol{s},\ x\in E,\ D\subseteq E\},$$

$$\boldsymbol{s}'=\{(x,D):(x,D)\in\boldsymbol{s},\ x\notin E,\ D\subseteq E\},$$

$$\boldsymbol{s}''=\{(x,D):(x,D)\in\boldsymbol{s},\ x\in E,\ D\cap E=\emptyset\}.$$

Because $\boldsymbol{s}\in T_E$,

$$W_{\boldsymbol{s}^*\cup\boldsymbol{s}'}=E\cap W_{\boldsymbol{s}}=E\cap W_{\boldsymbol{t}}$$

and $E\setminus W_{\boldsymbol{s}^*\cup\boldsymbol{s}'}=E\setminus W_{\boldsymbol{t}}$ belongs to $\mathcal{R}_1$, by the choice of $\mathcal{R}$. Accordingly

$$|F(E)-S_{\boldsymbol{s}^*\cup\boldsymbol{s}'}(f,\nu)|\le|F(E)-F(W_{\boldsymbol{s}^*\cup\boldsymbol{s}'})|+|F(W_{\boldsymbol{s}^*\cup\boldsymbol{s}'})-S_{\boldsymbol{s}^*\cup\boldsymbol{s}'}(f,\nu)|\le2\epsilon$$

because $\boldsymbol{s}^*\cup\boldsymbol{s}'\subseteq\boldsymbol{s}$ is δ-fine.

For $n\in\mathbb{N}$ set

$$\boldsymbol{s}'_n=\{(x,D):(x,D)\in\boldsymbol{s}',\ n\le|f(x)|<n+1\},$$

$$\boldsymbol{s}''_n=\{(x,D):(x,D)\in\boldsymbol{s}'',\ n\le|f(x)|<n+1\}.$$

Then $W_{\boldsymbol{s}'_n}\subseteq E\setminus F_n$. **P** If $(x,D)\in\boldsymbol{s}'_n$, there is a $C\in\mathcal{C}$ such that $D\subseteq E\cap C$ and $(x,C)\in\boldsymbol{t}$, while $x\notin E$, so that $C\subseteq G'_x$ and $C\cap F_n=\emptyset$. **Q** Similarly, $W_{\boldsymbol{s}''_n}\subseteq(X\setminus E)\setminus F'_n$. Thus $W_{\boldsymbol{s}'_n\cup\boldsymbol{s}''_n}$ is disjoint from $F_n\cup F'_n$ and

$$\begin{aligned}|S_{\boldsymbol{s}'_n}(f,\nu)-S_{\boldsymbol{s}''_n}(f,\nu)|&=|\sum_{(x,D)\in\boldsymbol{s}'_n}f(x_i)\nu D-\sum_{(x,D)\in\boldsymbol{s}''_n}f(x_i)\nu D|\\&\le\sum_{(x,D)\in\boldsymbol{s}'_n\cup\boldsymbol{s}''_n}|f(x_i)||\nu D|\\&\le(n+1)\sum_{(x,D)\in\boldsymbol{s}'_n\cup\boldsymbol{s}''_n}|\nu D|\le2^{-n}\epsilon\end{aligned}$$

by the choice of F_n and F'_n.

Consequently,

$$|F(E) - S_{\boldsymbol{t}}(f \times \chi E, \nu)| = |F(E) - S_{\boldsymbol{s}}(f \times \chi E, \nu)| = |F(E) - S_{\boldsymbol{s}^* \cup \boldsymbol{s}''}(f, \nu)|$$

(because $\boldsymbol{s}^* \cup \boldsymbol{s}'' = \{(x, D) : (x, D) \in \boldsymbol{s},\ x \in E\}$)

$$\le |F(E) - S_{\boldsymbol{s}^* \cup \boldsymbol{s}'}(f, \nu)| + |S_{\boldsymbol{s}'}(f, \nu) - S_{\boldsymbol{s}''}(f, \nu)|$$

(because $\boldsymbol{s}^*$, $\boldsymbol{s}'$ and $\boldsymbol{s}''$ are disjoint subsets of $\boldsymbol{s}$)

$$\le 2\epsilon + |\sum_{n=0}^{\infty} S_{\boldsymbol{s}'_n}(f, \nu) - \sum_{n=0}^{\infty} S_{\boldsymbol{s}''_n}(f, \nu)|$$

(the infinite sums are well-defined because $\boldsymbol{s}$ is finite, so that all but finitely many terms are zero)

$$\le 2\epsilon + \sum_{n=0}^{\infty} |S_{\boldsymbol{s}'_n}(f, \nu) - S_{\boldsymbol{s}''_n}(f, \nu)|$$

$$\le 2\epsilon + \sum_{n=0}^{\infty} 2^{-n}\epsilon = 4\epsilon.$$

As ϵ is arbitrary, $I_\nu(f \times \chi E)$ is defined and equal to $F(E)$, as required.

482H Proposition Suppose that X, $\mathfrak{T}$, $\mathcal{C}$, ν, T, Δ and $\mathfrak{R}$ satisfy the conditions (i)-(v) of 482G, and that $f : X \to \mathbb{R}$, $\langle H_n\rangle_{n\in\mathbb{N}}$, H and γ are such that

(vi) $\langle H_n\rangle_{n\in\mathbb{N}}$ is a sequence of open subsets of X with union H,

(vii) $I_\nu(f \times \chi H_n)$ is defined for every $n \in \mathbb{N}$,

(viii) $\lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})} I_\nu(f \times \chi W_{\boldsymbol{t}\restriction H})$ is defined and equal to γ,

where $\boldsymbol{t}\restriction H = \{(x, C) : (x, C) \in \boldsymbol{t},\ x \in H\}$ for $\boldsymbol{t} \in T$. Then $I_\nu(f \times \chi H)$ is defined and equal to γ.

proof Let $\epsilon > 0$. For each $n \in \mathbb{N}$, let F_n be the Saks-Henstock indefinite integral of $f \times \chi H_n$. Let $\delta_n \in \Delta$ be such that

$$2^{-n}\epsilon \ge \sum_{(x,C)\in\boldsymbol{s}} |F_n(C) - (f \times \chi H_n)(x)\nu C|$$
$$\ge |F_n(W_{\boldsymbol{s}}) - S_{\boldsymbol{s}}(f \times \chi H_n, \nu)|$$

whenever $\boldsymbol{s} \in T$ is δ_n-fine. Set

$$\tilde{\delta} = \{(x, A) : x \in X \setminus H,\ A \subseteq X\}$$
$$\cup \bigcup_{n\in\mathbb{N}} \{(x, A) : x \in H_n \setminus \bigcup_{i<n} H_i,\ A \subseteq H_n,\ (x, A) \in \delta_n\},$$

so that $\tilde{\delta} \in \Delta$. Note that if $x \in H$ and $C \in \mathcal{C}$ and $(x, C) \in \tilde{\delta}$, then there is some $n \in \mathbb{N}$ such that $x \in H_n$ and $C \subseteq H_n$, so that

$$I_\nu(f \times \chi C) = I_\nu((f \times \chi H_n) \times \chi C) = F_n(C)$$

is defined, by 482G; this means that $I_\nu(f \times \chi W_{\boldsymbol{t}\restriction H})$ will be defined for every $\tilde{\delta}$-fine $\boldsymbol{t} \in T$. Let $\delta \in \Delta$, $\mathcal{R} \in \mathfrak{R}$ be such that $|\gamma - I_\nu(f \times \chi W_{\boldsymbol{t}\restriction H})| \le \epsilon$ whenever $\boldsymbol{t} \in T$ is δ-fine and $\mathcal{R}$-filling.

Let $\boldsymbol{t} \in T$ be $(\delta \cap \tilde{\delta})$-fine and $\mathcal{R}$-filling. For $n \in \mathbb{N}$, set $\boldsymbol{t}_n = \{(x, C) : (x, C) \in \boldsymbol{t},\ x \in H_n \setminus \bigcup_{i<n} H_i\}$. Then $\boldsymbol{t}\restriction H = \bigcup_{n\in\mathbb{N}} \boldsymbol{t}_n$, and $\boldsymbol{t}_n$ is δ_n-fine and $W_{\boldsymbol{t}_n} \subseteq H_n$ for every n. So

$$|\gamma - S_{\boldsymbol{t}}(f \times \chi H, \nu)| = |\gamma - \sum_{n=0}^{\infty} S_{\boldsymbol{t}_n}(f \times \chi H_n, \nu)|$$

$$\le |\gamma - I_\nu(f \times \chi W_{\boldsymbol{t}\restriction H})| + \sum_{n=0}^{\infty} |I_\nu(f \times \chi W_{\boldsymbol{t}_n}) - S_{\boldsymbol{t}_n}(f \times \chi H_n, \nu)|$$

(note that $\boldsymbol{t}_n = \emptyset$ for all but finitely many n, so that $I_\nu(f \times \chi W_{\boldsymbol{t}\restriction H}) = \sum_{n=0}^{\infty} I_\nu(f \times \chi W_{\boldsymbol{t}_n})$)

$$\leq \epsilon + \sum_{n=0}^{\infty} |I_\nu(f \times \chi H_n \times \chi W_{\boldsymbol{t}_n}) - S_{\boldsymbol{t}_n}(f \times \chi H_n, \nu)|$$

(because $\boldsymbol{t}$ is δ-fine and $\mathcal{R}$-filling, while $W_{\boldsymbol{t}_n} \subseteq H_n$ for each n)

$$= \epsilon + \sum_{n=0}^{\infty} |F_n(W_{\boldsymbol{t}_n}) - S_{\boldsymbol{t}_n}(f \times \chi H_n, \nu)|$$

(by 482G)

$$\leq \epsilon + \sum_{n=0}^{\infty} 2^{-n}\epsilon$$

(because every $\boldsymbol{t}_n$ is δ_n-fine)

$$= 3\epsilon.$$

As ϵ is arbitrary, $\gamma = I_\nu(f \times \chi H)$, as claimed.

Remark For applications of this result see 483Bd and 483N.

482I Integrating a derivative As will appear in the next two sections, the real strength of gauge integrals is in their power to integrate derivatives. I give an elementary general expression of this fact. In the formulae below, we can think of f as a 'derivative' of F if $\nu = \theta$ is strictly positive and additive and we rephrase condition (iii) as

$$\text{`}\lim_{C \to \mathcal{G}_x} \tfrac{F(C)}{\nu C} = f(x)\text{'},$$

where $\mathcal{G}_x$ is the filter on $\mathcal{C}$ generated by the sets $\{C : (x, C) \in \delta\}$ as δ runs over Δ.

Theorem Let X, $\mathcal{C} \subseteq \mathcal{P}X$, $\Delta \subseteq \mathcal{P}(X \times \mathcal{P}X)$, $\mathfrak{R} \subseteq \mathcal{P}\mathcal{P}X$, $T \subseteq [X \times \mathcal{C}]^{<\omega}$, $f : X \to \mathbb{R}$, $\nu : \mathcal{C} \to \mathbb{R}$, $F : \mathcal{C} \to \mathbb{R}$, $\theta : \mathcal{C} \to [0, 1]$ and $\gamma \in \mathbb{R}$ be such that

(i) T is a straightforward set of tagged partitions which is compatible with Δ and $\mathfrak{R}$,

(ii) Δ is a full set of gauges on X,

(iii) for every $x \in X$ and $\epsilon > 0$ there is a $\delta \in \Delta$ such that $|f(x)\nu C - F(C)| \leq \epsilon\theta C$ whenever $(x, C) \in \delta$,

(iv) $\sum_{i=0}^{n} \theta C_i \leq 1$ whenever $C_0, \dots, C_n \in \mathcal{C}$ are disjoint,

(v) for every $\epsilon > 0$ there is an $\mathcal{R} \in \mathfrak{R}$ such that $|\gamma - \sum_{C \in \mathcal{C}_0} F(C)| \leq \epsilon$ whenever $\mathcal{C}_0 \subseteq \mathcal{C}$ is a finite disjoint set and $X \setminus \bigcup \mathcal{C}_0 \in \mathcal{R}$.

Then $I_\nu(f) = \lim_{\boldsymbol{t} \to \mathcal{F}(T, \Delta, \mathfrak{R})} S_{\boldsymbol{t}}(f, \nu)$ is defined and equal to γ.

proof Let $\epsilon > 0$. For each $x \in X$ let $\delta_x \in \Delta$ be such that $|f(x)\nu C - F(C)| \leq \epsilon\theta C$ whenever $(x, C) \in \delta_x$. Because Δ is full, there is a $\delta \in \Delta$ such that $(x, C) \in \delta_x$ whenever $(x, C) \in \delta$. Let $\mathcal{R} \in \mathfrak{R}$ be as in (v). If $\boldsymbol{t} \in T$ is δ-fine and $\mathcal{R}$-filling, then

$$|S_{\boldsymbol{t}}(f, \nu) - \gamma| \leq |\gamma - \sum_{(x,C) \in \boldsymbol{t}} F(C)| + \sum_{(x,C) \in \boldsymbol{t}} |f(x)\nu C - F(C)|$$

$$\leq \epsilon + \sum_{(x,C) \in \boldsymbol{t}} \epsilon\theta C$$

(because $X \setminus \bigcup_{(x,C) \in \boldsymbol{t}} C \in \mathcal{R}$, while $(x, C) \in \delta_x$ whenever $(x, C) \in \boldsymbol{t}$)

$$\leq 2\epsilon$$

by condition (iv). As ϵ is arbitrary, we have the result.

482J Definition Let X be a set, $\mathcal{C}$ a family of subsets of X, $T \subseteq [X \times \mathcal{C}]^{<\omega}$ a family of tagged partitions, $\nu : \mathcal{C} \to [0, \infty[$ a function, and Δ a family of gauges on X. I will say that ν is **moderated** (with respect to T and Δ) if there are a $\delta \in \Delta$ and a function $h : X \to]0, \infty[$ such that $S_{\boldsymbol{t}}(h, \nu) \leq 1$ for every δ-fine $\boldsymbol{t} \in T$.

482K B.Levi's theorem Let $(X,T,\Delta,\mathfrak{R})$ be a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$, such that Δ is countably full, and $\nu:\mathcal{C}\to[0,\infty[$ a function which is moderated with respect to T and Δ.

Let $\langle f_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of functions from X to $\mathbb{R}$ with supremum $f:X\to\mathbb{R}$. If $\gamma=\lim_{n\to\infty}I_\nu(f_n)$ is defined in $\mathbb{R}$, then $I_\nu(f)$ is defined and equal to γ.

proof As in the proof of 123A, we may, replacing $\langle f_n\rangle_{n\in\mathbb{N}}$ by $\langle f_n-f_0\rangle_{n\in\mathbb{N}}$ if necessary, suppose that $f_n(x)\ge 0$ for every $n\in\mathbb{N}$ and $x\in X$.

(a) Take $\epsilon>0$. Then there is a $\delta\in\Delta$ such that $S_{\boldsymbol{t}}(f,\nu)\le\gamma+4\epsilon$ for every δ-fine $\boldsymbol{t}\in T$.
P Fix a strictly positive function $h:X\to]0,\infty[$ and a $\tilde\delta\in\Delta$ such that $S_{\boldsymbol{t}}(h,\nu)\le 1$ for every $\tilde\delta$-fine $\boldsymbol{t}\in T$. For each $n\in\mathbb{N}$ choose $\delta_n\in\Delta$ and $\mathcal{R}_n\in\mathfrak{R}$ such that $|I_\nu(f_n)-S_{\boldsymbol{t}}(f,\nu)|\le 2^{-n-1}\epsilon$ for every δ_n-fine $\mathcal{R}_n$-filling $\boldsymbol{t}\in T$. For each $x\in X$, take $r_x\in\mathbb{N}$ such that $f(x)\le f_{r_x}(x)+\epsilon h(x)$. Let $\delta\in\Delta$ be such that $(x,C)\in\tilde\delta\cap\delta_{r_x}$ for every $(x,C)\in\delta$.

Suppose that $\boldsymbol{t}\in T$ is δ-fine. Enumerate $\boldsymbol{t}$ as $\langle(x_i,C_i)\rangle_{i<n}$. Let $k\in\mathbb{N}$ be so large that $S_{\boldsymbol{t}}(f,\nu)\le S_{\boldsymbol{t}}(f_k,\nu)+\epsilon$ and $r_{x_i}\le k$ for every $i<n$. For $m\le k$, set $J_m=\{i:i<n,\ r_{x_i}=m\}$. For each $i\in J_m$, $(x_i,C_i)\in\delta_m$. By 482A(c-ii), there is a δ_k-fine $\boldsymbol{s}_m\in T$ such that $W_{\boldsymbol{s}_m}\subseteq\bigcup_{i\in J_m}C_i$ and $|S_{\boldsymbol{s}_m}(f_m,\nu)-\sum_{i\in J_m}f_m(x_i)\nu C_i|\le 2^{-m}\epsilon$.

Set $\boldsymbol{s}=\bigcup_{m\le k}\boldsymbol{s}_m$, so that $\boldsymbol{s}$ is a δ_k-fine member of T and

$$\sum_{m=0}^k\sum_{i\in J_m}f_m(x_i)\nu C_i\le\sum_{m=0}^k S_{\boldsymbol{s}_m}(f_m,\nu)+2^{-m}\epsilon\le\sum_{m=0}^k S_{\boldsymbol{s}_m}(f_k,\nu)+2^{-m}\epsilon$$
$$\le S_{\boldsymbol{s}}(f_k,\nu)+2\epsilon\le I_\nu(f_k)+3\epsilon$$

(because $\boldsymbol{s}$ extends to a δ_k-fine $\mathcal{R}_k$-filling member of T, by 482Ab)

$$\le\gamma+3\epsilon.$$

Now $\boldsymbol{t}$ is $\tilde\delta$-fine, so $S_{\boldsymbol{t}}(h,\nu)\le 1$. Accordingly

$$S_{\boldsymbol{t}}(f,\nu)=\sum_{i<n}f(x_i)\nu C_i=\sum_{m=0}^k\sum_{i\in J_m}f(x_i)\nu C_i$$
$$\le\sum_{m=0}^k\sum_{i\in J_m}(f_m(x_i)+\epsilon h(x_i))\nu C_i\le\gamma+3\epsilon+\epsilon S_{\boldsymbol{t}}(h,\nu)\le\gamma+4\epsilon,$$

as required. **Q**

As ϵ is arbitrary, $\limsup_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})}S_{\boldsymbol{t}}(f,\nu)$ is at most γ.

(b) On the other hand, given $\epsilon>0$, there is an $n\in\mathbb{N}$ such that $I_\nu(f_n)\ge\gamma-\epsilon$. So taking $\delta_n\in\Delta$, $\mathcal{R}_n\in\mathfrak{R}$ as in (a) above,

$$S_{\boldsymbol{t}}(f,\nu)\ge S_{\boldsymbol{t}}(f_n,\nu)\ge I_\nu(f_n)-\epsilon\ge\gamma-2\epsilon$$

for every δ_n-fine $\mathcal{R}_n$-filling $\boldsymbol{t}\in T$. So $\liminf_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})}S_{\boldsymbol{t}}(f,\nu)$ is at least γ, and $I_\nu(f)=\gamma$.

482L Lemma Let X be a set, $\mathcal{C}$ a family of subsets of X, Δ a countably full downwards-directed set of gauges on X, $\mathfrak{R}\subseteq\mathcal{PP}X$ a downwards-directed collection of residual families, and $T\subseteq[X\times\mathcal{C}]^{<\omega}$ a straightforward set of tagged partitions of X compatible with Δ and $\mathfrak{R}$. Suppose further that whenever $\delta\in\Delta$, $\mathcal{R}\in\mathfrak{R}$ and $\boldsymbol{t}\in T$ is δ-fine, there is a δ-fine $\mathcal{R}$-filling $\boldsymbol{t}'\in T$ including $\boldsymbol{t}$. (For instance, $(X,T,\Delta,\mathfrak{R})$ might be a tagged-partition structure allowing subdivisions, as in 482Ab.) If $\nu:\mathcal{C}\to[0,\infty[$ and $f:X\to[0,\infty[$ are such that $I_\nu(f)=0$, and $g:X\to\mathbb{R}$ is such that $g(x)=0$ whenever $f(x)=0$, then $I_\nu(g)=0$.

proof Let $\epsilon>0$. For each $n\in\mathbb{N}$, let $\delta_n\in\Delta$, $\mathcal{R}_n\in\mathfrak{R}$ be such that $S_{\boldsymbol{t}}(f,\nu)\le 2^{-n}\epsilon$ for every δ_n-fine $\mathcal{R}_n$-filling $\boldsymbol{t}\in T$. For $x\in X$, set $\phi(x)=\min\{n:|g(x)|\le nf(x)\}$; let $\delta\in\Delta$ be such that $(x,C)\in\delta_{\phi(x)}$ whenever $(x,C)\in\delta$.

Let $\boldsymbol{t}$ be any δ-fine member of T. Then $|S_{\boldsymbol{t}}(g,\nu)|\le 2\epsilon$. **P** Enumerate $\boldsymbol{t}$ as $\langle(x_i,C_i)\rangle_{i<n}$. For each $m\in\mathbb{N}$, set $K_m=\{i:i<n,\ \phi(x_i)=m\}$; then $\{(x_i,C_i):i\in K_m\}$ is a δ_m-fine member of T, so extends to a δ_m-fine $\mathcal{R}_m$-filling member $\boldsymbol{t}_m$ of T, and

$$\sum_{i\in K_m}|g(x_i)\nu C_i|\le m\sum_{i\in K_m}f(x_i)\nu C_i\le mS_{\boldsymbol{t}_m}(f,\nu)\le 2^{-m}m\epsilon.$$

Summing over m,

$$|S_{\boldsymbol{t}}(g,\nu)|\le\sum_{m=0}^{\infty}2^{-m}m\epsilon=2\epsilon.\ \mathbf{Q}$$

Because T is compatible with Δ and $\mathcal{R}$, this is enough to show that $I_\nu(g)$ is defined and equal to 0.

482M Fubini's theorem Suppose that, for $i=1$ and $i=2$, we have X_i, $\mathfrak{T}_i$, T_i, Δ_i, $\mathcal{C}_i$ and ν_i such that

(i) $(X_i,\mathfrak{T}_i)$ is a topological space;

(ii) Δ_i is the set of neighbourhood gauges on X_i;

(iii) $T_i\subseteq[X_i\times\mathcal{C}_i]^{<\omega}$ is a straightforward set of tagged partitions, compatible with Δ_i and $\{\{\emptyset\}\}$;

(iv) $\nu_i:\mathcal{C}_i\to[0,\infty[$ is a function;

(v) ν_1 is moderated with respect to T_1 and Δ_1;

(vi) whenever $\delta\in\Delta_1$ and $\boldsymbol{s}\in T_1$ is δ-fine, there is a δ-fine $\boldsymbol{s'}\in T_1$, including $\boldsymbol{s}$, such that $W_{\boldsymbol{s'}}=X_1$.

Write X for $X_1\times X_2$; Δ for the set of neighbourhood gauges on X; $\mathcal{C}$ for $\{C\times D:C\in\mathcal{C}_1,\,D\in\mathcal{C}_2\}$; Q for $\{((x,y),C\times D):\{(x,C)\}\in T_1,\,\{(y,D)\}\in T_2\}$; T for the straightforward set of tagged partitions generated by Q; and set $\nu(C\times D)=\nu_1C\cdot\nu_2D$ for $C\in\mathcal{C}_1$, $D\in\mathcal{C}_2$.

(a) T is compatible with Δ and $\{\{\emptyset\}\}$.

(b) Let I_{ν_1}, I_{ν_2} and I_ν be the gauge integrals defined by these structures as in 481C-481F. Suppose that $f:X\to\mathbb{R}$ is such that $I_\nu(f)$ is defined. Set $f_x(y)=f(x,y)$ for $x\in X_1$, $y\in X_2$. Let $g:X_1\to\mathbb{R}$ be any function such that $g(x)=I_{\nu_2}(f_x)$ whenever this is defined. Then $I_{\nu_1}(g)$ is defined and equal to $I_\nu(f)$.

proof (a) Let $\delta\in\Delta$; we seek a tagged partition $\boldsymbol{u}\in T$ such that $W_{\boldsymbol{u}}=X$. Let $\langle V_{xy}\rangle_{(x,y)\in X}$ be the family of open sets in X defining δ; choose open sets $G_{xy}\subseteq X_1$, $H_{xy}\subseteq X_2$ such that $(x,y)\in G_{xy}\times H_{xy}\subseteq V_{xy}$ for all $x\in X_1$, $y\in X_2$. For each $x\in X_1$, let δ_x be the neighbourhood gauge on X_2 defined from the family $\langle H_{xy}\rangle_{y\in X_2}$. Then there is a δ_x-fine tagged partition $\boldsymbol{t}_x\in T_2$ such that $W_{\boldsymbol{t}_x}=X_2$. Set $G_x=X_1\cap\bigcap_{(y,D)\in\boldsymbol{t}_x}G_{xy}$.

The family $\langle G_x\rangle_{x\in X_1}$ defines a neighbourhood gauge δ^* on X_1, and there is a δ^*-fine $\boldsymbol{s}\in T_1$ such that $W_{\boldsymbol{s}}=X_1$. Now consider

$$\boldsymbol{u}=\{((x,y),C\times D):(x,C)\in\boldsymbol{s},\,(y,D)\in\boldsymbol{t}_x\}.$$

Then it is easy to check (just as in part (b) of the proof of 481O) that $\boldsymbol{u}$ is a δ-fine member of T with $W_{\boldsymbol{u}}=X$.

(b)(i) Set $A=\{x:x\in X_1,\,I_{\nu_2}(f_x)\text{ is defined}\}$. Let $h:X_1\to\mathbb{R}$ be any function such that $h(x)=0$ for every $x\in A$. For $x\in X_1$, set

$$h_0(x)=\inf(\{1\}\cup\{\sup_{\boldsymbol{t},\boldsymbol{t'}\in F}S_{\boldsymbol{t}}(f_x,\nu_2)-S_{\boldsymbol{t'}}(f_x,\nu_2):F\in\mathcal{F}(T_2,\Delta_2,\{\{\emptyset\}\})\}).$$

(Thus $I_{\nu_2}(f_x)$ is defined iff $h_0(x)=0$.) Then $I_{\nu_1}(h_0)=0$. $\mathbf{P}$ Let $\epsilon>0$. Then there is a $\delta\in\Delta$ such that $S_{\boldsymbol{u}}(f,\nu)-S_{\boldsymbol{u'}}(f,\nu)\le\epsilon$ whenever $\boldsymbol{u},\,\boldsymbol{u'}\in T$ are δ-fine and $W_{\boldsymbol{u}}=W_{\boldsymbol{u'}}=X$. Define $\langle V_{xy}\rangle_{(x,y)\in X}$, $\langle G_{xy}\rangle_{(x,y)\in X}$, $\langle H_{xy}\rangle_{(x,y)\in X}$ and $\langle\delta_x\rangle_{x\in X_1}$ from δ as in (a) above. For each $x\in X_1$, we can find δ_x-fine partitions $\boldsymbol{t}_x,\,\boldsymbol{t}'_x\in T_2$ such that $W_{\boldsymbol{t}_x}=W_{\boldsymbol{t}'_x}=X_2$ and $S_{\boldsymbol{t}_x}(f_x,\nu_2)-S_{\boldsymbol{t}'_x}(f_x,\nu_2)\ge\frac{1}{2}h_0(x)$. Set $G_x=X_1\cap\bigcap_{(y,D)\in\boldsymbol{t}_x\cup\boldsymbol{t}'_x}G_{xy}$.

Let δ^* be the neighbourhood gauge on X_1 defined from $\langle G_x\rangle_{x\in X_1}$. Let $\boldsymbol{s}$ be any δ^*-fine member of T_1 with $W_{\boldsymbol{s}}=X_1$. Set

$$\boldsymbol{u}=\{((x,y),C\times D):(x,C)\in\boldsymbol{s},\,(y,D)\in\boldsymbol{t}_x\},$$

$$\boldsymbol{u'}=\{((x,y),C\times D):(x,C)\in\boldsymbol{s},\,(y,D)\in\boldsymbol{t}'_x\},$$

Then $\boldsymbol{u}$ and $\boldsymbol{u'}$ are δ-fine members of T with $W_{\boldsymbol{u}}=W_{\boldsymbol{u'}}=X$, so

$$\begin{aligned}S_{\boldsymbol{s}}(h_0,\nu_1)&=\sum_{(x,C)\in\boldsymbol{s}}h_0(x)\nu_1(C)\le 2\sum_{(x,C)\in\boldsymbol{s}}(S_{\boldsymbol{t}_x}(f_x,\nu_2)-S_{\boldsymbol{t}'_x}(f_x,\nu_2))\nu_1(C)\\&=2\big(\sum_{\substack{(x,C)\in\boldsymbol{s}\\(y,D)\in\boldsymbol{t}_x}}f(x,y)\nu_1(C)\nu_2(D)-\sum_{\substack{(x,C)\in\boldsymbol{s}\\(y,D)\in\boldsymbol{t}'_x}}f(x,y)\nu_1(C)\nu_2(D)\big)\\&=2\big(S_{\boldsymbol{u}}(f,\nu)-S_{\boldsymbol{u}'}(f,\nu)\big)\le 2\epsilon.\end{aligned}$$

As ϵ is arbitrary, $I_{\nu_1}(h_0)=0$. **Q**

By 482L, $I_{\nu_1}(h)$ also is zero.

(ii) Again take any $\epsilon>0$, and let $\delta\in\Delta$ be such that $|S_{\boldsymbol{u}}(f,\nu)-I_\nu(f)|\le\epsilon$ for every δ-fine $\boldsymbol{u}\in T$ such that $W_{\boldsymbol{u}}=X$. Define $\langle V_{xy}\rangle_{(x,y)\in X}$, $\langle G_{xy}\rangle_{(x,y)\in X}$, $\langle H_{xy}\rangle_{(x,y)\in X}$ and $\langle\delta_x\rangle_{x\in X_1}$ from δ as in (a) and (i) above. Let $\tilde\delta\in\Delta_1$, $\tilde h:X_1\to\,]0,\infty[$ be such that $S_{\boldsymbol{s}}(\tilde h,\nu_1)\le 1$ for every $\tilde\delta$-fine $\boldsymbol{s}\in T_1$.

For $x\in A$, let $\boldsymbol{t}_x\in T_2$ be δ_x-fine and such that $W_{\boldsymbol{t}_x}=X_2$ and $|S_{\boldsymbol{t}_x}(f_x,\nu_2)-I_{\nu_2}(f_x)|\le\epsilon\tilde h(x)$; for $x\in X_1\setminus A$, let $\boldsymbol{t}_x$ be any δ_x-fine member of T_2 such that $W_{\boldsymbol{t}_x}=X_2$. Set $G_x=X_1\cap\bigcap_{(y,D)\in\boldsymbol{t}_x}G_{xy}$ for every $x\in X_1$. Let δ^* be the neighbourhood gauge on X_1 defined by $\langle G_x\rangle_{x\in X_1}$.

Set $h_1(x)=g(x)-S_{\boldsymbol{t}_x}(f_x,\nu_2)$ for $x\in X_1\setminus A$, 0 for $x\in A$. Then $I_{\nu_1}(|h_1|)=0$, by (a). Let $\delta'\in\Delta_1$ be such that $\delta'\subseteq\delta^*\cap\tilde\delta$ and $S_{\boldsymbol{s}}(|h_1|,\nu_1)\le\epsilon$ for every δ'-fine $\boldsymbol{s}\in T_1$ such that $W_{\boldsymbol{s}}=X_1$.

Now suppose that $\boldsymbol{s}\in T_1$ is δ'-fine and that $W_{\boldsymbol{s}}=X_1$. Set

$$\boldsymbol{u}=\{((x,y),C\times D):(x,C)\in\boldsymbol{s},\ (y,D)\in\boldsymbol{t}_x\},$$

so that $\boldsymbol{u}\in T$ is δ-fine and $W_{\boldsymbol{u}}=X$. Then

$$\begin{aligned}|S_{\boldsymbol{s}}(g,\nu_1)-I_\nu(f)|&\le|S_{\boldsymbol{u}}(f,\nu)-I_\nu(f)|+|S_{\boldsymbol{s}}(g,\nu_1)-S_{\boldsymbol{u}}(f,\nu)|\\&\le\epsilon+\big|\sum_{(x,C)\in\boldsymbol{s}}\big(g(x)-\sum_{(y,D)\in\boldsymbol{t}_x}f(x,y)\nu_2D\big)\nu_1C\big|\\&=\epsilon+\big|\sum_{(x,C)\in\boldsymbol{s}}\big(g(x)-S_{\boldsymbol{t}_x}(f_x,\nu_2)\big)\nu_1C\big|\\&\le\epsilon+\sum_{(x,C)\in\boldsymbol{s}}|h_1(x)|\nu_1C\\&\qquad+\sum_{\substack{(x,C)\in\boldsymbol{s}\\x\in A}}|g(x)-\sum_{(y,D)\in\boldsymbol{t}_x}f(x,y)\nu_2D|\nu_1C\\&\le 2\epsilon+\sum_{\substack{(x,C)\in\boldsymbol{s}\\x\in A}}\epsilon\tilde h(x)\nu_1(C)\le 2\epsilon+\epsilon S_{\boldsymbol{s}}(\tilde h,\nu_1)\le 3\epsilon.\end{aligned}$$

As ϵ is arbitrary, $I_{\nu_1}(g)$ is defined and equal to $I_\nu(f)$, as claimed.

482X Basic exercises (a) Let $(X,T,\Delta,\mathfrak{R})$ be a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$, and $\mathcal{E}$ the algebra of subsets of X generated by $\mathcal{C}$. Write $\mathfrak{I}$ for the set of pairs (f,ν) such that $f:X\to\mathbb{R}$ and $\nu:\mathcal{C}\to\mathbb{R}$ are functions and $I_\nu(f)=\lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})}S_{\boldsymbol{t}}(f,\nu)$ is defined; for $(f,\nu)\in\mathfrak{I}$, let $F_{f\nu}:\mathcal{E}\to\mathbb{R}$ be the corresponding Saks-Henstock indefinite integral. Show that $(f,\nu)\mapsto F_{f\nu}$ is bilinear in the sense that

$$F_{f+g,\nu}=F_{f\nu}+F_{g\nu},\quad F_{\alpha f,\nu}=F_{f,\alpha\nu}=\alpha F_{f\nu},\quad F_{f,\mu+\nu}=F_{f\mu}+F_{f\nu}$$

whenever (f,ν), (g,ν) and (f,μ) belong to $\mathfrak{I}$ and $\alpha\in\mathbb{R}$.

(b) Let $(X,T,\Delta,\mathfrak{R})$ be a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$, and $\mathcal{E}$ the algebra of subsets of X generated by $\mathcal{C}$. Let $f:X\to\mathbb{R}$ and $\nu:\mathcal{C}\to[0,\infty[$ be functions such that $I_\nu(f)=\lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})}S_{\boldsymbol{t}}(f,\nu)$ is defined, and let $F:\mathcal{E}\to\mathbb{R}$ be the corresponding Saks-Henstock indefinite integral. Show that F is non-negative iff $I_\nu(f^-)=0$, where $f^-(x)=\max(0,-f(x))$ for every $x\in X$.

>(c) Let X be a zero-dimensional compact Hausdorff space, $\mathcal{E}$ the algebra of open-and-closed subsets of X, $Q = \{(x, E) : x \in E \in \mathcal{E}\}$, T the straightforward set of tagged partitions generated by Q, Δ the set of neighbourhood gauges on X and $\nu : \mathcal{E} \to \mathbb{R}$ a non-negative additive functional. Let μ be the corresponding Radon measure on X (416Qa) and I_ν the gauge integral defined by $(X, T, \Delta, \{\{\emptyset\}\})$ (cf. 481Xh). Show that, for $f : X \to \mathbb{R}$, $I_\nu(f) = \int f d\mu$ if either is defined in $\mathbb{R}$. (*Hint*: if f is measurable but not μ-integrable, take x_0 such that f is not integrable on any neighbourhood of x_0. Given $\delta \in \Delta$, fix a δ-fine partition containing (x_0, V_0) for some V_0; now replace (x_0, V_0) by refinements $\{(x_0, V_0'), (x_1, V_1), \dots, (x_n, V_n)\}$, where $\sum_{i=1}^n f(x_i)\nu V_i$ is large, to show that $S_{\boldsymbol{t}}(f, \nu)$ cannot be controlled by δ.)

>(d) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be an effectively locally finite τ-additive topological measure space in which μ is inner regular with respect to the closed sets and outer regular with respect to the open sets (see 412W). Let T be the straightforward set of tagged partitions generated by $X \times \{E : \mu E < \infty\}$, Δ the set of neighbourhood gauges on X, and $\mathfrak{R} = \{\mathcal{R}_{E\eta} : \mu E < \infty, \eta > 0\}$, where $\mathcal{R}_{E\eta} = \{F : \mu(F \cap E) \le \eta\}$, as in N. Show that if I_μ is the associated gauge integral, and $f : X \to \mathbb{R}$ is a function, then $I_\mu(f) = \int f d\mu$ if either is defined in $\mathbb{R}$.

(e) Let $\mathcal{C}$ be the set of non-empty subintervals of $X = [0, 1]$, T the straightforward tagged-partition structure generated by $[0, 1] \times \mathcal{C}$, and Δ the set of neighbourhood gauges on $[0, 1]$, as in 481M. Let μ be any Radon measure on $[0, 1]$, and I_μ the gauge integral defined from μ and the tagged-partition structure $([0, 1], T, \Delta, \{\{\emptyset\}\})$. Show that, for any $f : [0, 1] \to \mathbb{R}$, $I_\mu(f) = \int f d\mu$ if either is defined in $\mathbb{R}$.

(f) Let $\mathcal{C}$ be the set of non-empty subintervals of $[0, 1]$, T the straightforward tagged-partition structure generated by $\{(x, C) : C \in \mathcal{C}, x \in \overline{C}\}$, and Δ the set of neighbourhood gauges on X, as in 481J. Let $\mathcal{E}$ be the ring of subsets of $[0, 1]$ generated by $\mathcal{C}$, $\nu : \mathcal{E} \to \mathbb{R}$ a bounded additive functional, and I_ν the gauge integral defined from ν and $(X, T, \Delta, \{\{\emptyset\}\})$. Show that $I_\nu(\chi\,[a, b[) = \lim_{x \uparrow a, y \uparrow b} \nu([x, y])$ whenever $0 < a < b \le 1$.

(g) Let $\mathcal{C}$ be the set of non-empty subintervals of $X = [0, 1]$, T the straightforward tagged-partition structure generated by $\{(x, C) : C \in \mathcal{C}, x \in \overline{C}\}$, and Δ the set of uniform metric gauges on $[0, 1]$, as in 481I. Let μ be the Dirac measure on $[0, 1]$ concentrated at $\frac{1}{2}$, and let $I_\mu = \lim_{\boldsymbol{t} \to \mathcal{F}(T, \Delta, \{\{\emptyset\}\})} S_{\boldsymbol{t}}(\,, \mu)$ be the corresponding gauge integral. Show that $I_\mu(\chi[0, 1])$ is defined but that $I_\mu(\chi[0, \frac{1}{2}])$ is not.

>(h)(i) Show that the McShane integral on an interval $[a, b]$ as described in 481M coincides with the Lebesgue integral on $[a, b]$. (ii) Show that if $(X, \mathfrak{T}, \Sigma, \mu)$ is a quasi-Radon measure space and μ is outer regular with respect to the open sets then the McShane integral as described in 481N coincides with the usual integral.

(i) Explain how the results in 481Xb-481Xc can be regarded as special cases of 482H.

(j) Let $(X, \mathfrak{T})$ be a topological space, $\mathcal{C}$ a ring of subsets of X, $T \subseteq [X \times \mathcal{C}]^{<\omega}$ a straightforward set of tagged partitions, $\nu : \mathcal{C} \to [0, \infty[$ an additive function, and Δ the family of neighbourhood gauges on X. Suppose that there is a sequence $\langle G_n \rangle_{n \in \mathbb{N}}$ of open sets, covering X, such that $\sup\{\nu C : C \in \mathcal{C}, C \subseteq G_n\}$ is finite for every $n \in \mathbb{N}$. Show that ν is moderated with respect to T and Δ.

(k) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space. Let T be the straightforward tagged-partition structure generated by $\{(x, E) : \mu E < \infty, x \in E\}$ and Δ the set of all neighbourhood gauges on X. Show that μ is moderated with respect to T and Δ iff there is a sequence of open sets of finite measure covering X.

(l) Let $r \ge 1$ be an integer, and μ a Radon measure on $\mathbb{R}^r$. Let Q be the set of pairs (x, C) where $x \in \mathbb{R}^r$ and C is a closed ball with centre x, and T the straightforward set of tagged partitions generated by Q. Let Δ be the set of neighbourhood gauges on $\mathbb{R}^r$, and $\mathfrak{R} = \{\mathcal{R}_{E\eta} : \mu E < \infty, \eta > 0\}$, where $\mathcal{R}_{E\eta} = \{F : \mu(F \cap E) \le \eta\}$, as in 482Xd. (i) Show that T is compatible with Δ and $\mathfrak{R}$. (*Hint*: 472C.) (ii) Show that if I_μ is the associated gauge integral, and $f : \mathbb{R}^r \to \mathbb{R}$ is a function, then $I_\mu(f) = \int f d\mu$ if either is defined in $\mathbb{R}$.

(m) Let $(X, T, \Delta, \mathfrak{R})$, Σ and ν be as in 481Xj, so that $(X, T, \Delta, \mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by an algebra Σ of subsets of X, and $\nu : \Sigma \to [0, \infty[$ is additive. Let I_ν be the corresponding gauge integral, and $V \subseteq \mathbb{R}^X$ its domain. (i) Show that $\chi E \in V$ and $I_\nu(\chi E) = \nu E$ for every $E \in \Sigma$. (ii) Show that if $f \in \mathbb{R}^X$ then $f \in V$ iff for every $\epsilon > 0$ there is a disjoint family $\mathcal{E} \subseteq \Sigma$ such that $\sum_{E\in\mathcal{E}} \nu E = \nu X$ and $\sum_{E\in\mathcal{E}} \nu E \cdot \sup_{x,y\in E} |f(x) - f(y)| \le \epsilon$. (iii) Show that if $\langle f_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in V with supremum $f \in \mathbb{R}^X$, and $\gamma = \sup_{n\in\mathbb{N}} I_\nu(f_n)$ is finite, then $I_\nu(f)$ is defined and equal to γ. (iv) Show that I_ν extends $\oint d\nu$ as described in 363L, if we identify $L^\infty(\Sigma)$ with a space $\mathcal{L}^\infty$ of functions as in 363H. (v) Show that if Σ is a σ-algebra of sets then I_ν extends $\oint d\nu$ as described in 364Xj.

482Y Further exercises (a) Let X be the interval $[0,1]$, $\mathcal{C}$ the family of subintervals of X, Q the set $\{(x,C) : C \in \mathcal{C},\ x \in \overline{\operatorname{int} C}\}$, T the straightforward set of tagged partitions generated by Q, Δ the set of neighbourhood gauges on X, and $\mathfrak{R}$ the singleton $\{[X]^{<\omega}\}$. Show that $(X,T,\Delta,\mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$. For $C \in \mathcal{C}$ set $\nu C = 1$ if $0 \in \overline{\operatorname{int} C}$, 0 otherwise, and let f be $\chi\{0\}$. Show that $I_\nu(f) = \lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f,\nu)$ is defined and equal to 1. Let F be the Saks-Henstock indefinite integral of f. Show that $F(]0,1]) = 1$.

(b) Set $X = \mathbb{R}$ and let $\mathcal{C}$ be the family of non-empty bounded intervals in X; set $Q = \{(x,C) : C \in \mathcal{C},\ x = \inf C\}$, and let T be the straightforward set of tagged partitions generated by Q. Let $\mathfrak{S}$ be the Sorgenfrey right-facing topology on X (415Xc), and Δ the set of neighbourhood gauges for $\mathfrak{S}$. Set $\mathcal{R}_n = \{E : E \in \Sigma,\ \mu([-n,n] \cap E) \le 2^{-n}\}$ for $n \in \mathbb{N}$, where μ is Lebesgue measure on $\mathbb{R}$ and Σ its domain, and write $\mathfrak{R} = \{\mathcal{R}_n : n \in \mathbb{N}\}$. Show that $(X,T,\Delta,\mathfrak{R})$ is a tagged-partition structure allowing subdivisions. Show that if $f : X \to \mathbb{R}$ is such that $I_\mu(f)$ is defined, then f is Lebesgue measurable.

(c) Give an example of X, $\mathfrak{T}$, Σ, μ, T, Δ, $\mathcal{C}$, f and C such that $(X,\mathfrak{T},\Sigma,\mu)$ is a compact metrizable Radon probability space, Δ is the set of all neighbourhood gauges on X, $(X,T,\Delta,\{\{\emptyset\}\})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$, $f : X \to \mathbb{R}$ is a function such that $I_\mu(f) = \lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\{\{\emptyset\}\})} S_{\boldsymbol{t}}(f,\mu)$ is defined, $C \in \mathcal{C}$ is a closed set with negligible boundary, and $I_\mu(f \times \chi C)$ is not defined.

(d) Suppose that, for $i = 1$ and $i = 2$, we have a tagged-partition structure $(X_i,T_i,\Delta_i,\mathfrak{R}_i)$ allowing subdivisions, witnessed by a ring $\mathcal{C}_i \subseteq \mathcal{P}X_i$, where X_i is a topological space, Δ_i is the set of all neighbourhood gauges on X_i, and $\mathfrak{R}_i$ is the simple residual structure complementary to $\mathcal{C}_i$, as in 481Yb. Set $X = X_1 \times X_2$ and let Δ be the set of neighbourhood gauges on X; set $\mathcal{C} = \{C \times D : C \in \mathcal{C}_1,\ D \in \mathcal{C}_2\}$; let $\mathfrak{R}$ be the simple residual structure on X complementary to $\mathcal{C}$; and let T be the straightforward tagged-partition structure generated by $\{((x,y), C \times D) : \{(x,C)\} \in T_1,\ \{(y,D)\} \in T_2\}$. For each i, let $\nu_i : \mathcal{C}_i \to [0,\infty[$ be a function moderated with respect to T_i and Δ_i, and define $\nu : \mathcal{C} \to [0,\infty[$ by setting $\nu(C \times D) = \nu_1 C \cdot \nu_2 D$ for $C \in \mathcal{C}_1$, $D \in \mathcal{C}_2$. Show that T is compatible with Δ and $\mathfrak{R}$. Let I_{ν_1}, I_{ν_2}, I_ν be the gauge integrals defined by these structures. Suppose that $f : X \to \mathbb{R}$ is such that $I_\nu(f)$ is defined. Set $f_x(y) = f(x,y)$ for $x \in X_1$, $y \in X_2$. Let $g : X_1 \to \mathbb{R}$ be any function such that $g(x) = I_{\nu_2}(f_x)$ whenever this is defined. Show that $I_{\nu_1}(g)$ is defined and equal to $I_\nu(f)$.

482 Notes and comments In 482E, 482F and 482G the long lists of conditions reflect the variety of possible applications of these arguments. The price to be paid for the versatility of the constructions here is a theory which is rather weak in the absence of special hypotheses. As everywhere in this book, I try to set ideas out in maximal convenient generality; you may feel that in this section the generality is becoming inconvenient; but the theory of gauge integrals has not, to my eye, matured to the point that we can classify the systems here even as provisionally as I set out to classify topological measure spaces in Chapters 41 and 43.

Enthusiasts for gauge integrals offer two substantial arguments for taking them seriously, apart from the universal argument in pure mathematics, that these structures offer new patterns for our delight and new challenges to our ingenuity. First, they say, gauge integrals integrate more functions than Lebesgue-type integrals, and it is the business of a theory of integration to integrate as many functions as possible; and secondly, gauge integrals offer an easier path to the principal theorems. I have to say that I think the first argument is sounder than the second. It is quite true that the Henstock integral on $\mathbb{R}$ (481K) can be rigorously defined in fewer words, and with fewer concepts, than the Lebesgue integral. The style of Chapters 11 and 12 is supposed to be better adapted to the novice than the style of this chapter, but you will have no difficulty in putting the ideas of 481A, 481C, 481J and 481K together into an elementary definition of an integral for real functions in which the only non-trivial argument is that establishing the existence of enough tagged partitions (481J), corresponding I suppose to Proposition 114D. But the path I took in defining the integral in §122, though arduous at that point, made (I hope) the convergence theorems of §123 reasonably natural; the proof of 482K, on the other hand, makes significant demands on our technique. Furthermore, the particular clarity of the one-dimensional Henstock integral is not repeated in higher dimensions. Fubini's theorem, with exact statement and full proof, even for products of Lebesgue measures on Euclidean spaces, is a lot to expect of an undergraduate; but Lebesgue measure on $\mathbb{R}^r$ makes sense in a way that it is quite hard to repeat with gauge integrals. (For instance, Lebesgue measure is invariant under isometries; this is not particularly easy to prove – see 263A – but at least it is true; if we want a gauge integral which is invariant under isometries, then we have to use a construction such as 481O, which does not directly match any natural general definition of 'product gauge integral' along the lines of 481P, 482M or 482Yd.)

In my view, a stronger argument for taking gauge integrals seriously is their 'power', that is, their ability to provide us with integrals of many functions in consistent ways. 482E, 482F and 482I give us an idea of what to expect. If we start from a measure space (X,Σ,μ) and build a gauge integral I_μ from a set $T \subseteq [X \times \Sigma]^{<\omega}$ of tagged partitions, then we can hope that integrable functions will be gauge-integrable, with the right integrals (482F); while gauge-integrable functions will be measurable (482E). What this means is that for *non-negative* functions, the integrals will coincide. Any 'new' gauge-integrable functions f will be such that $\int f^+ = \int f^- = \infty$; the gauge integral will offer a process for cancelling the divergent parts of these integrals. On the other hand, we can hope for a large class of gauge-integrable derivatives. In the next two sections, I will explain how this works in the Henstock and Pfeffer integrals. For simple examples calling for such procedures, see the formulae of §§282 and 283; for radical applications of the idea, see Muldowney 87.

Against this, gauge integrals are not effective in 'general' measure spaces, and cannot be, because there is no structure in an abstract measure space (X,Σ,μ) which allows us to cancel an infinite integral $\int f^+ = \int_F f$ against $\int f^- = \int_{X\setminus F} f$. Put another way, if a tagged-partition structure is invariant under all automorphisms of the structure (X,Σ,μ), as in 481Xf-481Xg, we cannot expect anything better than the standard integral. In order to get something new, the most important step seems to be the specification of a family $\mathcal{C}$ of 'regular' sets, preliminary to describing a set T of tagged partitions. To get a 'powerful' gauge integral, we want a fine filter on T, corresponding to a small set $\mathcal{C}$ and a large set of gauges. The residual families of 481F are generally introduced just to ensure 'compatibility' in the sense described there; as a rule, we try to keep them simple. But even if we take the set of all neighbourhood gauges, as in the Henstock integral, this is not enough unless we also sharply restrict both the family $\mathcal{C}$ and the permissible tags (482Xc-482Xe). The most successful restrictions, so far, have been 'geometric', as in 481J and 481O, and 484F below. Further limitations on admissible pairs (x,C), as in 481L and 481Q, in which $\mathcal{C}$ remains the set of intervals, but fewer tags are permitted, also lead to very interesting results.

Another limitation in the scope of gauge integrals is the difficulty they have in dealing with spaces of infinite measure. Of course we expect to have to specify a limiting procedure if we are to calculate $I_\mu(f)$ from sums $S_{\boldsymbol{t}}(f,\mu)$ which necessarily consider only sets of finite measure, and this is one of the functions of the collections $\mathfrak{R}$ of residual families. But this is not yet enough. In B.Levi's theorem (482K) we already need to suppose that our set-function ν is 'moderated' in order to determine how closely $f_n(x)$ needs to approximate each $f(x)$. The condition

$$S_{\boldsymbol{t}}(h,\nu) \le 1 \text{ for every } \delta\text{-fine } \boldsymbol{t}$$

of 482J is very close to saying that $I_\mu(h) \le 1$. But it is *not* the same as saying that μ is σ-finite; it suggests rather that X should be covered by a sequence of open sets of finite measure (482Xk).

Because gauge integrals are not absolute – that is, we can have $I_\nu(f)$ defined and finite while $I_\nu(|f|)$ is not – we are bound to have difficulties with integrals $\int_H f$, even if we interpret these in the simplest way, as $I_\nu(f \times \chi H)$, so that we do not need a theory of subspaces as developed in §214. 482G(iii)-(v) are an attempt to find a reasonably versatile sufficient set of conditions. The 'multiplier problem', for a given gauge integral I_ν, is the problem of characterizing the functions g such that $I_\nu(f \times g)$ is defined whenever $I_\nu(f)$ is defined, and even for some intensively studied integrals remains challenging. In 484L I will give an important case which is not covered by 482G.

One of the striking features of gauge integrals is that there is no need to assume that the set-function ν is countably additive. We can achieve countable additivity of the integral – in the form of B.Levi's theorem, for instance – by requiring only that the set of gauges should be 'countably full' (482K, 482L; contrast 482Xg). If we watch our definitions carefully, we can make this match the rules for Stieltjes integrals (114Xa, 482Xf). In 481Db I have already remarked on the potential use of gauge integrals in vector integration.

It is important to recognise that a value $F(E)$ of a Saks-Henstock indefinite integral (482B-482C) cannot be identified with either $I_\nu(f \times \chi E)$ or with $I_{\nu \restriction \mathcal{P}E}(f \restriction E)$, because in the formula $F(E) = \lim_{\boldsymbol{t} \to \mathcal{F}^*} S_{\boldsymbol{t}_E}(f, \nu)$ used in the proof of 482B the tags of the partitions $\boldsymbol{t}_E$ need not lie in E. (See 482Ya.) The idea of 482G is to impose sufficient conditions to ensure that the contributions of 'straddling' elements (x, C), where either $x \in E$ and $C \not\subseteq E$ or $x \notin E$ and $C \cap E \neq \emptyset$, are insignificant. To achieve this we seem to need both a regularity condition on the functional ν (condition (iii) of 482G) and a geometric condition on the set $\mathcal{C}$ underlying T (482G(iv)). As usual, the regularity condition required is closer to *outer* than to *inner* regularity, in contexts in which there is a distinction.

I am not sure that I have the 'right' version of Proposition 482E. The hypothesis there is that we have a metric space. But in the principal non-metrizable cases the result is still valid (482Xc-482Xd, 482Yb), and the same happens in 482Xl, where condition 482E(ii) is not satisfied. Proposition 482H is a 'new' limit theorem; it shows that certain improper integrals from the classical theory can be represented as gauge integrals. The hypotheses seem, from where we are standing at the moment, to be exceedingly restrictive. In the leading examples in §483, however, the central requirement 482H(viii) is satisfied for straightforward geometric reasons.

Gauge integrals insist on finite functions defined everywhere. But since we have an effective theory of negligible sets (482L), we can easily get a consistent theory of integration for functions which are defined and real-valued almost everywhere if we say that

$$I_\nu(f) = I_\nu(g) \text{ whenever } g : X \to \mathbb{R} \text{ extends } f \restriction f^{-1}[\mathbb{R}]$$

whenever $I_\nu(g) = I_\nu(g')$ for all such extensions.

Version of 6.9.10

483 The Henstock integral

I come now to the original gauge integral, the 'Henstock integral' for real functions. The first step is to check that the results of §482 can be applied to show that this is an extension of both the Lebesgue integral and the improper Riemann integral (483B), coinciding with the Lebesgue integral for non-negative functions (483C). It turns out that any Henstock integrable function can be approximated in a strong sense by a sequence of Lebesgue integrable functions (483G). The Henstock integral can be identified with the Perron and special Denjoy integrals (483J, 483N, 483Yh). Much of the rest of the section is concerned with indefinite Henstock integrals. Some of the results of §482 on tagged-partition structures allowing subdivisions condense into a particularly strong Saks-Henstock lemma (483F). If f is Henstock integrable, it is equal almost everywhere to the derivative of its indefinite Henstock integral (483I). Finally, indefinite Henstock integrals can be characterized as continuous ACG$_*$ functions (483R).

483A Definition The following notation will apply throughout the section. Let $\mathcal{C}$ be the family of non-empty bounded intervals in $\mathbb{R}$, and let $T \subseteq [\mathbb{R} \times \mathcal{C}]^{<\omega}$ be the straightforward set of tagged partitions generated by $\{(x, C) : C \in \mathcal{C}, x \in \overline{C}\}$. Let Δ be the set of all neighbourhood gauges on $\mathbb{R}$. Set $\mathfrak{R} = \{\mathcal{R}_{ab} : a \le b \in \mathbb{R}\}$, where $\mathcal{R}_{ab} = \{\mathbb{R} \setminus [c, d] : c \le a, d \ge b\} \cup \{\emptyset\}$. Then $(\mathbb{R}, T, \Delta, \mathfrak{R})$ is a tagged-partition structure allowing subdivisions (481K), so T is compatible with Δ and $\mathfrak{R}$ (481Hf). The **Henstock integral** is the gauge integral defined by the process of 481E-481F from $(\mathbb{R}, T, \Delta, \mathfrak{R})$ and one-dimensional Lebesgue measure μ. For a function $f : \mathbb{R} \to \mathbb{R}$ I will say that f is **Henstock integrable**, and that $\oint f = \gamma$, if $\lim_{\boldsymbol{t} \to \mathcal{F}(T, \Delta, \mathfrak{R})} S_{\boldsymbol{t}}(f, \mu)$ is defined and equal to $\gamma \in \mathbb{R}$. For α, $\beta \in [-\infty, \infty]$ I will write $\oint_\alpha^\beta f$ for $\oint f \times \chi]\alpha, \beta[$ if this is defined in $\mathbb{R}$. I will use the symbol $\int$ for the ordinary integral, so that $\int f d\mu$ is the Lebesgue integral of f.

483B Tracing through the theorems of §482, we have the following.

Theorem (a) Every Henstock integrable function on $\mathbb{R}$ is Lebesgue measurable.

(b) Every Lebesgue integrable function $f:\mathbb{R}\to\mathbb{R}$ is Henstock integrable, with the same integral.

(c) If f is Henstock integrable so is $f\times\chi C$ for every interval $C\subseteq\mathbb{R}$.

(d) Suppose that $f:\mathbb{R}\to\mathbb{R}$ is any function, and $-\infty\le\alpha<\beta\le\infty$. Then

$$\not\!\!\int_\alpha^\beta f=\lim_{a\downarrow\alpha}\not\!\!\int_a^\beta f=\lim_{b\uparrow\beta}\not\!\!\int_\alpha^b f=\lim_{a\downarrow\alpha,b\uparrow\beta}\not\!\!\int_a^b f$$

if any of the four terms is defined in $\mathbb{R}$.

proof (a) Apply 482E.

(b) Apply 482F, referring to 134F to confirm that condition 482F(ii) is satisfied.

It will be useful to note at once that this shows that $\not\!\!\int f\times\chi\{a\}=0$ for every $f:\mathbb{R}\to\mathbb{R}$ and every $a\in\mathbb{R}$. Consequently $\not\!\!\int_a^b f=\not\!\!\int_a^c f+\not\!\!\int_c^b f$ whenever $a\le c\le b$ and the right-hand side is defined.

(c)-(d) In the following argument, f will always be a function from $\mathbb{R}$ to itself; when f is Henstock integrable, $F^{\textsc{sh}}$ will be its Saks-Henstock indefinite integral (482B-482C).

(i) The first thing to check is that the conditions of 482G are satisfied by $\mathbb{R}$, T, Δ, $\mathfrak{R}$, $\mathcal{C}$ and $\mu\restriction\mathcal{C}$. **P** 482G(i) is just 481K, and 482G(ii) is trivial. 482G(iii-α) and 482G(v) are elementary, and so is 482G(iii-β) — if you like, this is a special case of 412W(b-iii). As for 482G(iv), if $E\in\mathcal{C}$ is a singleton $\{x\}$, then whenever $(x,C)\in\mathcal{C}$ we can express C as a union of one or more of the sets $C\cap{]-\infty,x[}$, $C\cap\{x\}$ and $C\cap{]x,\infty[}$, and for any non-empty C' of these we have $\{(x,C')\}\in T$ and either $C'\subseteq E$ or $C'\cap E=\emptyset$. Otherwise, let $\eta>0$ be half the length of E, and let δ be the uniform metric gauge $\{(x,A):A\subseteq{]x-\eta,x+\eta[}\}$. Then if $x\in\partial E$ and $(x,C)\in T\cap\delta$, we can again express C as a union of one or more of the sets $C\cap{]-\infty,x[}$, $C\cap\{x\}$ and $C\cap{]x,\infty[}$, and these will witness that 482G(iv) is satisfied. **Q**

(ii) Now suppose that f is Henstock integrable. Then $\not\!\!\int_a^b f$ is defined whenever $a\le b$ in $\mathbb{R}$. **P** 482G tells us that $\not\!\!\int f\times\chi C$ is defined and equal to $F^{\textsc{sh}}(C)$ for every $C\in\mathcal{C}$; in particular, $\not\!\!\int_a^b f=\not\!\!\int f\times\chi\,]a,b[$ is defined whenever $a<b$ in $\mathbb{R}$. **Q** Note that because $\not\!\!\int f\times\chi\{c\}=0$ for every c, $\not\!\!\int f\times\chi C=\not\!\!\int_{\inf C}^{\sup C}f$ whenever $C\in\mathcal{C}$ is non-empty.

This proves (c) for bounded intervals; we shall come to unbounded intervals in (vii) below.

(iii) If f is Henstock integrable, then $\lim_{a\to-\infty,b\to\infty}\not\!\!\int_a^b f$ is defined and equal to $\not\!\!\int f$. **P** Given $\epsilon>0$, there is an $\mathcal{R}\in\mathfrak{R}$ such that $|F^{\textsc{sh}}(\mathbb{R}\setminus C)|\le\epsilon$ whenever $C\in\mathcal{C}$ and $\mathbb{R}\setminus C\in\mathcal{R}$; that is, there are $a_0\le b_0$ such that

$$|\not\!\!\int f-\not\!\!\int_a^b f|=|F^{\textsc{sh}}(\mathbb{R}\setminus{]a,b[})|=|F^{\textsc{sh}}(\mathbb{R}\setminus[a,b])|\le\epsilon$$

whenever $a\le a_0\le b_0\le b$. As ϵ is arbitrary, $\lim_{a\to-\infty,b\to\infty}\not\!\!\int_a^b f$ is defined and equal to $\not\!\!\int f$. **Q**

(iv) It follows that if f is Henstock integrable and $c\in\mathbb{R}$, $\lim_{b\to\infty}\not\!\!\int_c^b f$ is defined. **P** Let $\epsilon>0$. Then there are $a_0\le c$, $b_0\ge c$ such that $|\not\!\!\int_a^b f-\not\!\!\int f|\le\epsilon$ whenever $a\le a_0$ and $b\ge b_0$. But this means that

$$|\not\!\!\int_c^b f-\not\!\!\int_c^{b'} f|=|\not\!\!\int_{a_0}^b f-\not\!\!\int_{a_0}^{b'} f|\le 2\epsilon$$

whenever b, $b'\ge b_0$. As ϵ is arbitrary, $\lim_{b\to\infty}\int_c^b f$ is defined. **Q**

Similarly, $\lim_{a\to-\infty}\not\!\!\int_a^c f$ is defined.

(v) Moreover, if f is Henstock integrable and $a<b$ in $\mathbb{R}$, then $\lim_{c\uparrow b}\not\!\!\int_a^c f$ is defined and equal to $\int_a^b f$. **P** Let $\epsilon>0$. Then there is a $\delta\in\Delta$ such that $\sum_{(x,C)\in\boldsymbol{t}}|F^{\textsc{sh}}(C)-f(x)\mu C|\le\epsilon$ whenever $\boldsymbol{t}\in T$ is δ-fine. Let $\eta>0$ be such that $\eta|f(b)|\le\epsilon$ and $(b,[c,b[)\in\delta$ whenever $b-\eta\le c<b$. Then whenever $\max(a,b-\eta)\le c<b$,

$$\not\!\!\int_a^b f-\not\!\!\int_a^c f=F^{\textsc{sh}}([c,b[)\le|F^{\textsc{sh}}([c,b[)-f(b)\mu\,[c,b[\,|+|f(b)|\mu\,[c,b[\,\le 2\epsilon.$$

As ϵ is arbitrary, $\lim_{c\uparrow b}\not\!\!\int_a^c f=\not\!\!\int_a^b f$. **Q**

Similarly, $\lim_{c\downarrow a}\not\!\!\int_c^b f$ is defined and equal to $\not\!\!\int_a^b f$.

(vi) Now for a much larger step. If $-\infty \le \alpha < \beta \le \infty$ and $f : \mathbb{R} \to \mathbb{R}$ is such that $\lim_{a\downarrow\alpha, b\uparrow\beta} \fint_a^b f$ is defined and equal to γ, then $\fint_\alpha^\beta f$ is defined and equal to γ. **P** I seek to apply 482H, with $H =]\alpha, \beta[$ and $H_n =]a_n, b_n[$, where $\langle a_n \rangle_{n\in\mathbb{N}}$ is a strictly decreasing sequence with limit α, $\langle b_n \rangle_{n\in\mathbb{N}}$ is a strictly increasing sequence with limit β, and $a_0 < b_0$. We have already seen that the conditions of 482G are satisfied; 482H(vi) is elementary, and 482H(vii) is covered by (ii) above. So we are left with 482H(viii). Given $\epsilon > 0$, let $m \in \mathbb{N}$ be such that $|\gamma - \fint_a^b f| \le \epsilon$ whenever $\alpha < a \le a_m$ and $b_m \le b < \beta$. For $x \in \mathbb{R}$ let G_x be an open set, containing x, such that

$$\begin{aligned}\overline{G}_x &\subseteq H \text{ if } x \in H,\\ &\subseteq]-\infty, a_m[\text{ if } x < a_m,\\ &\subseteq]b_m, \infty[\text{ if } x > b_m,\end{aligned}$$

and let $\delta \in \Delta$ be the neighbourhood gauge corresponding to $\langle G_x \rangle_{x\in\mathbb{R}}$. Now suppose that $\boldsymbol{t} \in T$ is δ-fine and $\mathcal{R}_{a_m b_m}$-filling. Then $W_{\boldsymbol{t}}$ is a closed bounded interval including $[a_m, b_m]$. Since $\langle C \rangle_{(x,C)\in\boldsymbol{t}}$ is a disjoint family of intervals, $\boldsymbol{t}$ must have an enumeration $\langle (x_i, C_i) \rangle_{i\le r}$ where $x \le y$ whenever $i \le j \le r$, $x \in C_i$ and $y \in C_j$. As $H \subseteq \mathbb{R}$ is an interval, there are $i_0 \le i_1$ such that

$$\boldsymbol{t} \restriction H = \{(x_i, C_i) : i \le r,\, x_i \in H\} = \{(x_i, C_i) : i_0 \le i \le i_1\}.$$

Because $W_{\boldsymbol{t}}$ is an interval, so is $W_{\boldsymbol{t}\restriction H} = \bigcup_{i_0\le i\le i_1} C_i$; set $a = \inf W_{\boldsymbol{t}\restriction H}$ and $b = \sup W_{\boldsymbol{t}\restriction H}$; then $\fint f \times \chi W_{\boldsymbol{t}} = \fint_a^b f$ (see (ii)). Next, $\overline{W}_{\boldsymbol{t}\restriction H} \subseteq H$ (because $\overline{G}_x \subseteq H$ if $x \in H$) and $[a_m, b_m] \subseteq W_{\boldsymbol{t}\restriction H}$ (because $[a_m, b_m] \subseteq W_{\boldsymbol{t}}$ and $\overline{G}_x \cap [a_m, b_m] = \emptyset$ for $x \notin [a_m, b_m]$). So $\alpha < a \le a_m$, $b_m \le b < \beta$, and

$$|\gamma - \fint f \times \chi W_{\boldsymbol{t}\restriction H}| = |\gamma - \fint_a^b f| \le \epsilon.$$

As ϵ is arbitrary, this shows that

$$\lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})} \fint f \times \chi W_{\boldsymbol{t}\restriction H} = \gamma,$$

as required by 482H(viii). So $\fint_\alpha^\beta f = \fint f \times \chi H$ is defined and equal to γ, as claimed. **Q**

(vii) We are now in a position to confirm that if f is Henstock integrable then $\fint_c^\infty f = \lim_{b\to\infty} \fint_c^b f$ is defined for every $c \in \mathbb{R}$. **P** By (iii) and (iv), $\lim_{a\downarrow c} \fint_a^{c+1} f = \fint_c^{c+1} f$ and $\lim_{b\to\infty} \fint_{c+1}^b f$ are both defined. So

$$\begin{aligned}\lim_{a\downarrow c, b\to\infty} \fint_a^b f &= \lim_{a\downarrow c} \fint_a^{c+1} f + \lim_{b\to\infty} \fint_{c+1}^b f\\ &= \fint_c^{c+1} f + \lim_{b\to\infty} \fint_{c+1}^b = \lim_{b\to\infty} \fint_c^b f\end{aligned}$$

is defined, and is equal to $\fint_c^\infty f$, by (vi). **Q**

Similarly, $\fint_{-\infty}^c f = \lim_{a\to-\infty} \fint_a^c f$ is defined. So $\fint f \times \chi C$ is defined for sets C of the form $]c, \infty[$ or $]-\infty, c[$, and therefore for any unbounded interval, since the case $C = \mathbb{R}$ is immediate. So the proof of (c) is complete.

(viii) As for (d), (vi) has already given us part of it: if $\lim_{a\downarrow\alpha, b\uparrow\beta} \fint_a^b f$ is defined, this is $\fint_\alpha^\beta f$. In the other direction, if $\fint_\alpha^\beta f$ is defined, set $g = f \times \chi]\alpha, \beta[$, so that g is Henstock integrable, and take any $c \in]\alpha, \beta[$. Then $\lim_{b\uparrow\beta} \fint_c^b g$ is defined, and equal to $\fint_c^\beta g$, by (v) if β is finite and by (vii) if $\beta = \infty$. Similarly, $\lim_{a\downarrow\alpha} \fint_a^c g$ is defined and equal to $\fint_\alpha^c g$. Consequently

$$\lim_{a\downarrow\alpha, b\uparrow\beta} \fint_a^b f = \lim_{a\downarrow\alpha, b\uparrow\beta} \fint_a^b g = \lim_{a\downarrow\alpha} \fint_a^c g + \lim_{b\uparrow\beta} \fint_c^b g$$

is defined, and must be equal to $\fint_\alpha^\beta g = \fint_\alpha^\beta f$, while also

$$\lim_{a\downarrow\alpha} \fint_a^\beta f = \lim_{a\downarrow\alpha} \fint_a^\beta g = \lim_{a\downarrow\alpha} \fint_a^c g + \fint_c^\beta g = \fint_\alpha^c g + \fint_c^\beta g = \fint_\alpha^\beta g = \fint_\alpha^\beta f,$$

and similarly

$$\lim_{b\uparrow\beta} \not\!\!\int_\alpha^b f = \not\!\!\int_\alpha^\beta f.$$

(ix) Finally, we need to consider the case in which we are told that $\lim_{a\downarrow\alpha} \not\!\!\int_a^\beta f$ is defined. Taking any $c \in]\alpha,\beta[$, we know that $\not\!\!\int_c^\beta f$ is defined, by (ii) or (vii) applied to $f \times \chi]a,\beta[$ for some $a \le c$, and equal to $\lim_{b\uparrow\beta} \not\!\!\int_c^b f$, by (v) or (vii). But this means that

$$\lim_{a\downarrow\alpha,b\uparrow\beta} \not\!\!\int_a^b f = \lim_{a\downarrow\alpha} \not\!\!\int_a^c f + \lim_{b\uparrow\beta} \not\!\!\int_c^b f$$

is defined, so (vi) and (viii) tell us that $\not\!\!\int_\alpha^\beta f$ is defined and equal to $\lim_{a\downarrow\alpha} \not\!\!\int_a^\beta f$. The same argument, suitably inverted, deals with the case in which $\lim_{b\uparrow\beta} \not\!\!\int_\alpha^b f$ is defined.

483C Corollary The Henstock and Lebesgue integrals agree on non-negative functions, in the sense that if $f : \mathbb{R} \to [0,\infty[$ then $\not\!\!\int f = \int f d\mu$ if either is defined in $\mathbb{R}$.

proof If f is Lebesgue integrable, it is Henstock integrable, with the same integral, by 483Bb. If it is Henstock integrable, then it is measurable, by 483Ba, so that $\int f d\mu$ is defined in $[0,\infty]$; but

$$\int f d\mu = \sup\{\int g\, d\mu : g \le f \text{ is a non-negative simple function}\}$$

(213B)

$$= \sup\{\not\!\!\int g : g \le f \text{ is a non-negative simple function}\} \le \not\!\!\int f$$

(481Cb) is finite, so f is Lebesgue integrable.

483D Corollary If $f : \mathbb{R} \to \mathbb{R}$ is Henstock integrable, then $\alpha \mapsto \not\!\!\int_{-\infty}^\alpha f : [\infty,\infty] \to \mathbb{R}$ and $(\alpha,\beta) \mapsto \not\!\!\int_\alpha^\beta f : [-\infty,\infty]^2 \to \mathbb{R}$ are continuous and bounded.

proof Let F be the indefinite Henstock integral of f. Take any $x_0 \in \mathbb{R}$ and $\epsilon > 0$. By 483Bd, there is an $\eta_1 > 0$ such that $|\not\!\!\int_{-\infty}^x f - \not\!\!\int_{-\infty}^{x_0} f| \le \epsilon$ whenever $x_0 - \eta_1 \le x \le x_0$. By 483Bd again, there is an $\eta_2 > 0$ such that $|\not\!\!\int_x^\infty f - \not\!\!\int_{x_0}^\infty f| \le \epsilon$ whenever $x_0 \le x \le x_0 + \eta_2$. But this means that $|F(x) - F(x_0)| \le \epsilon$ whenever $x_0 - \eta_1 \le x \le x_0 + \eta_2$. As ϵ is arbitrary, F is continuous at x_0.

We know also that $\lim_{x\to\infty} F(x) = \not\!\!\int f$ is defined in $\mathbb{R}$; while

$$\lim_{x\to-\infty} F(x) = \not\!\!\int f - \lim_{x\to-\infty} \not\!\!\int_x^\infty f = 0$$

is also defined, by 483Bd once more. So F is continuous at $\pm\infty$.

Now writing $G(\alpha,\beta) = \not\!\!\int_\alpha^\beta f$, we have $G(\alpha,\beta) = F(\beta) - F(\alpha)$ if $\alpha \le \beta$ and zero if $\beta \le \alpha$. So G also is continuous. F and G are bounded because $[-\infty,\infty]$ is compact.

483E Definition If $f : \mathbb{R} \to \mathbb{R}$ is Henstock integrable, then its **indefinite Henstock integral** is the function $F : \mathbb{R} \to \mathbb{R}$ defined by saying that $F(x) = \not\!\!\int_{-\infty}^x f$ for every $x \in \mathbb{R}$.

483F In the present context, the Saks-Henstock lemma can be sharpened, as follows.

Theorem Let $f : \mathbb{R} \to \mathbb{R}$ and $F : \mathbb{R} \to \mathbb{R}$ be functions. Then the following are equiveridical:

(i) f is Henstock integrable and F is its indefinite Henstock integral;

(ii)(α) F is continuous,

(β) $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x)$ is defined in $\mathbb{R}$,

(γ) for every $\epsilon > 0$ there are a gauge $\delta \in \Delta$ and a non-decreasing function $\phi : \mathbb{R} \to [0,\epsilon]$ such that $|f(x)(b-a) - F(b) + F(a)| \le \phi(b) - \phi(a)$ whenever $a \le x \le b$ in $\mathbb{R}$ and $(x,[a,b]) \in \delta$.

proof (i)⇒(ii) (α)-(β) are covered by 483D. As for (γ), 482G tells us that we can identify the Saks-Henstock indefinite integral of f with $E \mapsto \not\!\!\int f \times \chi E : \mathcal{E} \to \mathbb{R}$, where $\mathcal{E}$ is the algebra generated by $\mathcal{C}$. Let $\epsilon > 0$. Then there is a $\delta \in \Delta$ such that $\sum_{(x,C)\in\boldsymbol{t}} |f(x)\mu C - \not\!\!\int f \times \chi C| \le \epsilon$ for every δ-fine $\boldsymbol{t} \in T$. Set

$$\phi(a)=\sup_{\boldsymbol{t}\in T\cap\delta}\sum_{(x,C)\in\boldsymbol{t},C\subseteq]-\infty,a]}|f(x)\mu C-\fint f\times\chi C|,$$

so that $\phi:\mathbb{R}\to[0,\epsilon]$ is a non-decreasing function. Now suppose that $a\le y\le b$ and that $(y,[a,b])\in\delta$. In this case, whenever $\boldsymbol{t}\in T\cap\delta$, $\boldsymbol{s}=\{(x,C):(x,C)\in\boldsymbol{t},\ C\subseteq\,]-\infty,a]\}\cup\{(y,]a,b[)\}$ also belongs to $T\cap\delta$. Now $F(b)-F(a)=\fint_a^b f$, so

$$\begin{aligned}\phi(b)&\ge\sum_{(x,C)\in\boldsymbol{s}}|f(x)\mu C-\fint f\times\chi C|\\&=\sum_{(x,C)\in\boldsymbol{t},C\subseteq]-\infty,a]}|f(x)\mu C-\fint f\times\chi C|+|f(y)(b-a)-F(b)+F(a)|.\end{aligned}$$

As $\boldsymbol{t}$ is arbitrary,

$$\phi(b)\ge\phi(a)+|f(y)(b-a)-F(b)+F(a)|,$$

that is, $|f(y)(b-a)-F(b)+F(a)|\le\phi(b)-\phi(a)$, as called for by (γ).

(ii)⇒(i) Assume (ii). Set $\gamma=\lim_{x\to\infty}F(x)$. Let $\epsilon>0$. Let $a\le b$ be such that $|F(x)|\le\epsilon$ for every $x\le a$ and $|F(x)-\gamma|\le\epsilon$ for every $x\ge b$. Let $\delta\in\Delta$, $\phi:\mathbb{R}\to[0,\epsilon]$ be such that ϕ is non-decreasing and $|f(x)(\beta-\alpha)-F(\beta)+F(\alpha)|\le\phi(\beta)-\phi(\alpha)$ whenever $\alpha\le x\le\beta$ and $(x,[\alpha,\beta])\in\delta$. Let $\delta'\in\Delta$ be such that $(x,\overline{A})\in\delta$ whenever $(x,A)\in\delta'$. For $C\in\mathcal{C}$ set $\lambda C=F(\sup C)-F(\inf C)$, $\nu C=\phi(\sup C)-\phi(\inf C)$; then if $(x,C)\in\delta'$, $(x,[\inf C,\sup C])\in\delta$, so $|f(x)\mu C-\lambda C|\le\nu C$. Note that λ and ν are both additive in the sense that $\lambda(C\cup C')=\lambda C+\lambda C'$, $\nu(C\cup C')=\nu C+\nu C'$ whenever C, C' are disjoint members of $\mathcal{C}$ such that $C\cup C'\in\mathcal{C}$ (cf. 482G(iii-α)).

Let $\boldsymbol{t}\in T$ be δ'-fine and $\mathcal{R}_{ab}$-filling. Then $W_{\boldsymbol{t}}$ is of the form $[c,d]$ where $c\le a$ and $b\le d$. So

$$\begin{aligned}|S_{\boldsymbol{t}}(f,\mu)-\gamma|&\le2\epsilon+|S_{\boldsymbol{t}}(f,\mu)-F(d)+F(c)|=2\epsilon+|S_{\boldsymbol{t}}(f,\mu)-\lambda[c,d]|\\&=2\epsilon+|\sum_{(x,C)\in\boldsymbol{t}}f(x)\mu C-\lambda C|\le2\epsilon+\sum_{(x,C)\in\boldsymbol{t}}\nu C\\&=2\epsilon+\nu[c,d]\le3\epsilon.\end{aligned}$$

As ϵ is arbitrary, f is Henstock integrable, with integral γ.

I still have to check that F is the indefinite integral of f. Set $F_1(x)=\fint_{-\infty}^x f$ for $x\in\mathbb{R}$, and $G=F-F_1$. Then (ii) applies equally to the pair (f,F_1), because (i)⇒(ii). So, given $\epsilon>0$, we have δ, $\delta_1\in\Delta$ and non-decreasing functions ϕ, $\phi_1:\mathbb{R}\to[0,\epsilon]$ such that

$|f(x)(b-a)-F(b)+F(a)|\le\phi(b)-\phi(a)$ whenever $a\le x\le b$ in $\mathbb{R}$ and $(x,[a,b])\in\delta$,

$|f(x)(b-a)-F_1(b)+F_1(a)|\le\phi_1(b)-\phi_1(a)$ whenever $a\le x\le b$ in $\mathbb{R}$ and $(x,[a,b])\in\delta_1$.

Putting these together,

$|G(b)-G(a)|\le\psi(b)-\psi(a)$ whenever $a\le x\le b$ in $\mathbb{R}$ and $(x,[a,b])\in\delta\cap\delta_1$,

where $\psi=\phi+\phi_1$. But if $a\le b$ in $\mathbb{R}$, there are $a_0\le x_0\le a_1\le x_1\le\ldots\le x_{n-1}\le a_n$ such that $a=a_0$, $a_n=b$ and $(x_i,[a_i,a_{i+1}])\in\delta$ for $i<n$ (481J), so that

$$\begin{aligned}|G(b)-G(a)|&\le\sum_{i=0}^{n-1}|G(a_{i+1})-G(a_i)|\\&\le\sum_{i=0}^{n-1}\psi(a_{i+1})-\psi(a_i)=\psi(b)-\psi(a)\le2\epsilon.\end{aligned}$$

As ϵ is arbitrary, G is constant. As $\lim_{x\to-\infty}F(x)=\lim_{x\to-\infty}F_1(x)=0$, $F=F_1$, as required.

483G Theorem Let $f:\mathbb{R}\to\mathbb{R}$ be a Henstock integrable function. Then there is a countable cover $\mathcal{K}$ of $\mathbb{R}$ by compact sets such that $f\times\chi K$ is Lebesgue integrable for every $K\in\mathcal{K}$.

proof (a) For $n\in\mathbb{N}$ set $E_n=\{x:|x|\le n,\ |f(x)|\le n\}$. By 483Ba, f is Lebesgue measurable, so for each $n\in\mathbb{N}$ we can find a compact set $K_n\subseteq E_n$ such that $\mu(E_n\setminus K_n)\le2^{-n}$; f is Lebesgue integrable over K_n, and $Y=\mathbb{R}\setminus\bigcup_{n\in\mathbb{N}}K_n$ is Lebesgue negligible.

Let F be the indefinite Henstock integral of f, and take a gauge $\delta_0 \in \Delta$ and a non-decreasing function $\phi : \mathbb{R} \to [0,1]$ such that $|f(x)(b-a) - F(b) + F(a)| \le \phi(b) - \phi(a)$ whenever $a \le x \le b$ and $(x, [a,b]) \in \delta_0$ (483F). Because $I\!\!\!\int |f| \times \chi Y = \int_Y |f| d\mu = 0$ (483Bb), there is a $\delta_1 \in \Delta$ such that $S_{\boldsymbol{t}}(|f| \times \chi Y, \mu) \le 1$ whenever $\boldsymbol{t} \in T$ is δ_1-fine (482Ad). For $C \in \mathcal{C}$, set $\lambda C = F(\sup C) - F(\inf C)$, $\nu C = \phi(\sup C) - \phi(\inf C)$. Set

$$D_n = \{x : x \in E_n \cap Y,\ (x, [a,b]) \in \delta_0 \cap \delta_1 \text{ whenever } x - 2^{-n} \le a \le x \le b \le x + 2^{-n}\},$$

so that $\bigcup_{n\in\mathbb{N}} D_n = Y$; set $K'_n = \overline{D}_n$, so that K'_n is compact and $\bigcup_{n\in\mathbb{N}} K'_n \supseteq Y$.

(b) The point is that $f \times \chi K'_n$ is Lebesgue integrable for each n. **P** For $k \in \mathbb{N}$, let A_k be the set of points $x \in K'_n$ such that $\left]x, x+2^{-k}\right] \cap K'_n = \emptyset$; then A_k is finite, because $K'_n \subseteq [-n,n]$ is bounded. Similarly, if $A'_k = \{x : x \in K'_n,\ \left[x - 2^{-k}, x\right[\cap K'_n = \emptyset\}$, A'_k is finite. Set $B = K'_n \setminus \bigcup_{k\in\mathbb{N}}(A_k \cup A'_k)$, so that $K'_n \setminus B$ is countable.

Set

$$\delta = \delta_0 \cap \delta_1 \cap \{(x,A) : x \in \mathbb{R},\ A \subseteq \left]x - 2^{-n-1}, x + 2^{-n-1}\right[\},$$

so that $\delta \in \Delta$. Note that if $C \in \mathcal{C}$, $x \in B \cap \overline{C}$, $(x,C) \in \delta$ and $\mu C > 0$, then $\operatorname{int} C$ meets K'_n (because there are points of K'_n arbitrarily close to x on both sides) so $\operatorname{int} C$ meets D_n; and if $y \in D_n \cap \operatorname{int} C$ then $(y,C) \in \delta_0 \cap \delta_1$, because $\operatorname{diam} C \le 2^{-n}$. This means that if $\boldsymbol{t} \in T$ is δ-fine and $\boldsymbol{t} \subseteq B \times \mathcal{C}$, then there is a $\delta_0 \cap \delta_1$-fine $\boldsymbol{s} \in T$ such that $\boldsymbol{s} \subseteq D_n \times \mathcal{C}$, $W_{\boldsymbol{s}} \subseteq W_{\boldsymbol{t}}$ and whenever $(x,C) \in \boldsymbol{t}$ and C is not a singleton, there is a y such that $(y,C) \in \boldsymbol{s}$. Accordingly

$$\begin{aligned}\sum_{(x,C)\in\boldsymbol{t}} |\lambda C| &\le \sum_{(y,C)\in\boldsymbol{s}} |\lambda C| \le \sum_{(y,C)\in\boldsymbol{s}} |f(y)\mu C - \lambda C| + \sum_{(y,C)\in\boldsymbol{s}} |f(y)|\mu C \\ &\le \sum_{(y,C)\in\boldsymbol{s}} \nu C + S_{\boldsymbol{s}}(|f| \times \chi Y, \mu) \le 2.\end{aligned}$$

But this means that if $\boldsymbol{t} \in T$ is δ-fine,

$$S_{\boldsymbol{t}}(|f \times \chi B|, \mu) = \sum_{(x,C)\in\boldsymbol{t}\restriction B} |f(x)\mu C|$$

(where $\boldsymbol{t}\restriction B = \boldsymbol{t} \cap (B \times \mathcal{C})$)

$$\begin{aligned}&\le \sum_{(x,C)\in\boldsymbol{t}\restriction B} |f(x)\mu C - \lambda C| + \sum_{(x,C)\in\boldsymbol{t}\restriction B} |\lambda C| \\ &\le \sum_{(x,C)\in\boldsymbol{t}\restriction B} \nu C + 2 \le 3.\end{aligned}$$

It follows that if g is a μ-simple function and $0 \le g \le |f \times \chi B|$,

$$\begin{aligned}\int g\,d\mu = I\!\!\!\int g &\le \sup_{\boldsymbol{t}\in T \text{ is } \delta\text{-fine}} S_{\boldsymbol{t}}(g, \mu) \\ &\le \sup_{\boldsymbol{t}\in T \text{ is } \delta\text{-fine}} S_{\boldsymbol{t}}(|f \times \chi B|, \mu) \le 3,\end{aligned}$$

and $|f \times \chi B|$ is μ-integrable, by 213B, so $f \times \chi B$ is μ-integrable, by 122P. As $K'_n \setminus B$ is countable, therefore negligible, $f \times \chi K'_n$ is μ-integrable. **Q**

(c) So if we set $\mathcal{K} = \{K_n : n \in \mathbb{N}\} \cup \{K'_n : n \in \mathbb{N}\}$, we have a suitable family.

483H Upper and lower derivates: Definition Let $F : \mathbb{R} \to \mathbb{R}$ be any function. For $x \in \mathbb{R}$, set

$$(\overline{D}F)(x) = \limsup_{y\to x} \frac{F(y)-F(x)}{y-x}, \quad (\underline{D}F)(x) = \liminf_{y\to x} \frac{F(y)-F(x)}{y-x}$$

in $[-\infty, \infty]$, that is, $(\overline{D}F)(x) = \max((\overline{D}^+F)(x), (\overline{D}^-F)(x))$ and $(\underline{D}F)(x) = \min((\underline{D}^+F)(x), (\underline{D}^-F)(x))$ as defined in 222J.

483I Theorem Suppose that $f:\mathbb{R}\to\mathbb{R}$ is Henstock integrable, and F is its indefinite Henstock integral. Then $F'(x)$ is defined and equal to $f(x)$ for almost every $x\in\mathbb{R}$.

proof For $n\in\mathbb{N}$, set $A_n=\{x:|x|\le n,\ (\overline{D}F)(x)>f(x)+2^{-n}\}$. Then $\mu^*A_n\le 2^{-n+1}$. **P** Let $\delta\in\Delta$ and $\phi:\mathbb{R}\to[0,4^{-n}]$ be such that ϕ is non-decreasing and $|f(x)(b-a)-F(b)+F(a)|\le\phi(b)-\phi(a)$ whenever $a\le x\le b$ and $(x,[a,b])\in\delta$ (483F). Let $\mathcal{I}$ be the set of non-trivial closed intervals $[a,b]\subseteq\mathbb{R}$ such that, for some $x\in[a,b]\cap A_n$, $(x,[a,b])\in\delta$ and $\frac{F(b)-F(a)}{b-a}\ge f(x)+2^{-n}$. By Vitali's theorem (221A) we can find a countable disjoint family $\mathcal{I}_0\subseteq\mathcal{I}$ such that $A_n\setminus\bigcup\mathcal{I}_0$ is negligible; so we have a finite family $\mathcal{I}_1\subseteq\mathcal{I}_0$ such that $\mu^*(A_n\setminus\bigcup\mathcal{I}_1)\le 2^{-n}$. Enumerate $\mathcal{I}_1$ as $\langle[a_i,b_i]\rangle_{i<m}$, and for each $i<m$ take $x_i\in[a_i,b_i]\cap A_n$ such that $(x_i,[a_i,b_i])\in\delta$ and $F(b_i)-F(a_i)\ge(b_i-a_i)(f(x_i)+2^{-n})$. Then

$$\phi(b_i)-\phi(a_i)\ge|f(x_i)(b_i-a_i)-F(b_i)+F(a_i)|\ge 2^{-n}(b_i-a_i)$$

for each $i<m$, so

$$\mu(\bigcup\mathcal{I}_1)=\sum_{i<m}b_i-a_i\le 2^n\sum_{i<m}\phi(b_i)-\phi(a_i)\le 2^{-n},$$

and $\mu^*A_n\le 2^{-n+1}$. **Q**

Accordingly $\{x:(\overline{D}F)(x)>f(x)\}=\bigcup_{m\in\mathbb{N}}\bigcap_{n\ge m}A_n$ is negligible. Similarly, or applying the argument to $-f$, $\{x:(\underline{D}F)(x)<f(x)\}$ is negligible. So $\overline{D}F\le_{\text{a.e.}}f\le_{\text{a.e.}}\underline{D}F$. Since $\underline{D}F\le\overline{D}F$ everywhere, $\overline{D}F=_{\text{a.e.}}\underline{D}F=_{\text{a.e.}}f$. But $F'(x)=f(x)$ whenever $(\overline{D}F)(x)=(\underline{D}F)(x)=f(x)$, so we have the result.

483J Theorem Let $f:\mathbb{R}\to\mathbb{R}$ be a function. Then the following are equiveridical:

(i) f is Henstock integrable;

(ii) for every $\epsilon>0$ there are functions F_1, $F_2:\mathbb{R}\to\mathbb{R}$, with finite limits at both $-\infty$ and ∞, such that $(\overline{D}F_1)(x)\le f(x)\le(\underline{D}F_2)(x)$ and $0\le F_2(x)-F_1(x)\le\epsilon$ for every $x\in\mathbb{R}$.

proof (i)⇒(ii) Suppose that f is Henstock integrable and that $\epsilon>0$. Let F be the indefinite Henstock integral of f. Let $\delta\in\Delta$, $\phi:\mathbb{R}\to[0,\frac{1}{2}\epsilon]$ be such that ϕ is non-decreasing and $|f(x)(b-a)-F(b)+F(a)|\le\phi(b)-\phi(a)$ whenever $a\le x\le b$ and $(x,[a,b])\in\delta$ (483F). Set $F_1=F-\phi$, $F_2=F+\phi$; then $F_1(x)\le F_2(x)\le F_1(x)+\epsilon$ for every $x\in\mathbb{R}$, and the limits at $\pm\infty$ are defined because F and ϕ both have limits at both ends. If $x\in\mathbb{R}$, there is an $\eta>0$ such that $(x,A)\in\delta$ whenever $A\subseteq[x-\eta,x+\eta]$. So if $x-\eta\le a\le x\le b\le x+\eta$ and $a<b$,

$$\Bigl|\frac{F(b)-F(a)}{b-a}-f(x)\Bigr|\le\frac{\phi(b)-\phi(a)}{b-a},$$

and

$$\frac{F_1(b)-F_1(a)}{b-a}\le f(x)\le\frac{F_2(b)-F_2(a)}{b-a}.$$

In particular, this is true whenever $x-\eta\le a<x=b$ or $x=a<b\le x+\eta$. So $(\overline{D}F_1)(x)\le f(x)\le(\underline{D}F_2)(x)$. As x is arbitrary, we have a suitable pair F_1, F_2.

(ii)⇒(i) Suppose that (ii) is true. Take any $\epsilon>0$. Let F_1, $F_2:\mathbb{R}\to\mathbb{R}$ be as in the statement of (ii).

(α) We need to know that F_2-F_1 is non-decreasing. **P** Set $G=F_2-F_1$. Then

$$\liminf_{y\to x}\frac{G(y)-G(x)}{y-x}=\liminf_{y\to x}\frac{F_2(y)-F_2(x)}{y-x}-\frac{F_1(y)-F_1(x)}{y-x}$$

$$\ge\liminf_{y\to x}\frac{F_2(y)-F_2(x)}{y-x}-\limsup_{y\to x}\frac{F_1(y)-F_1(x)}{y-x}$$

(2A3Sf)

$$=(\underline{D}F_2)(x)-(\overline{D}F_1)(x)\ge 0$$

for any $x\in\mathbb{R}$. **?** If $a<b$ and $G(a)>G(b)$, set $\gamma=\frac{G(a)-G(b)}{2(b-a)}$, and choose $\langle a_n\rangle_{n\in\mathbb{N}}$, $\langle b_n\rangle_{n\in\mathbb{N}}$ inductively as follows. $a_0=a$ and $b_0=b$. Given that $a_n<b_n$ and $G(a_n)-G(b_n)>\gamma(b_n-a_n)$, set $c=\frac{1}{2}(a_n+b_n)$;

then either $G(a_n)-G(c)>\gamma(c-a_n)$ or $G(c)-G(b_n)\ge\gamma(b_n-c)$; in the former case, take $a_{n+1}=a_n$ and $b_{n+1}=c$; in the latter, take $a_{n+1}=c$ and $b_{n+1}=b_n$. Set $x=\lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n$. Then for each n, either $G(a_n)-G(x)>\gamma(x-a_n)$ or $G(x)-G(b_n)>\gamma(b_n-x)$. In either case, we have a y such that $0<|y-x|\le 2^{-n}(b-a)$ and $\frac{G(y)-G(x)}{y-x}<-\gamma$. So $(\underline{D}G)(x)\le-\gamma<0$, which is impossible. **X**

Thus G is non-decreasing, as required. **Q**

(β) Let $a\le b$ be such that $|F_1(x)-F_1(a)|\le\epsilon$ whenever $x\le a$ and $|F_1(x)-F_1(b)|\le\epsilon$ whenever $x\ge b$. Let $h:\mathbb{R}\to\,]0,\infty[$ be a strictly positive integrable function such that $\int h\,d\mu\le\epsilon$. Then $\rlap{\,/}\!\!\int h\le\epsilon$, by 483Bb, so there is a $\delta_0\in\Delta$ such that $S_{\boldsymbol{t}}(h,\mu)\le 2\epsilon$ for every δ_0-fine $\boldsymbol{t}\in T$ (482Ad). For $x\in\mathbb{R}$ let $\eta_x>0$ be such that

$$\frac{F_1(y)-F_1(x)}{y-x}\le f(x)+h(x),\quad \frac{F_2(y)-F_2(x)}{y-x}\ge f(x)-h(x)$$

whenever $0<|y-x|\le 2\eta_x$; set $\delta=\{(x,A):(x,A)\in\delta_0,\ A\subseteq\,]x-\eta_x,x+\eta_x[\}$, so that $\delta\in\Delta$. Note that if $x\in\mathbb{R}$ and $x-\eta_x\le\alpha\le x\le\beta\le x+\eta_x$, then

$$F_1(\beta)-F_1(x)\le(\beta-x)(f(x)+h(x)),\quad F_1(x)-F_1(\alpha)\le(x-\alpha)(f(x)+h(x)),$$

so that $F_1(\beta)-F_1(\alpha)\le(\beta-\alpha)(f(x)+h(x))$; and similarly $F_2(\beta)-F_2(\alpha)\ge(\beta-\alpha)(f(x)-h(x))$.

For $C\in\mathcal{C}$, set

$$\lambda_1C=F_1(\sup C)-F_1(\inf C),\quad \lambda_2C=F_2(\sup C)-F_2(\inf C).$$

Then if $C\in\mathcal{C}$, $x\in\overline{C}$ and $(x,C)\in\delta$,

$$\lambda_1C\le(f(x)+h(x))\mu C,\quad \lambda_2C\ge(f(x)-h(x))\mu C.$$

Suppose that $\boldsymbol{t}\in T$ is δ-fine and $\mathcal{R}_{ab}$-filling. Then $W_{\boldsymbol{t}}=[\alpha,\beta]$ for some $\alpha\le a$ and $\beta\ge b$, so that

$$\begin{aligned}S_{\boldsymbol{t}}(f,\mu)&=\sum_{(x,C)\in\boldsymbol{t}}f(x)\mu C\le\sum_{(x,C)\in\boldsymbol{t}}\lambda_2C+h(x)\mu C=\lambda_2[\alpha,\beta]+S_{\boldsymbol{t}}(h,\mu)\\&\le F_2(\beta)-F_2(\alpha)+2\epsilon\le F_1(\beta)-F_1(\alpha)+3\epsilon\le F_1(b)-F_1(a)+5\epsilon.\end{aligned}$$

Similarly,

$$\begin{aligned}S_{\boldsymbol{t}}(f,\mu)&=\sum_{(x,C)\in\boldsymbol{t}}f(x)\mu C\ge\sum_{(x,C)\in\boldsymbol{t}}\lambda_1C-h(x)\mu C\\&=\lambda_1[\alpha,\beta]-S_{\boldsymbol{t}}(h,\mu)\ge F_1(\beta)-F_1(\alpha)-2\epsilon\ge F_1(b)-F_1(a)-4\epsilon.\end{aligned}$$

But this means that if $\boldsymbol{t}$, $\boldsymbol{t}'$ are two δ-fine $\mathcal{R}_{ab}$-filling members of T, $|S_{\boldsymbol{t}}(f,\mu)-S_{\boldsymbol{t}'}(f,\mu)|\le 9\epsilon$. As ϵ is arbitrary,

$$\lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})}S_{\boldsymbol{t}}(f,\mu)=\rlap{\,/}\!\!\int f$$

is defined.

Remark The formulation (ii) above is a version of the method of integration described by Perron 1914.

483K Proposition Let $f:\mathbb{R}\to\mathbb{R}$ be a Henstock integrable function, and F its indefinite Henstock integral. Then $F[E]$ is Lebesgue negligible for every Lebesgue negligible set $E\subseteq\mathbb{R}$.

proof Let $\epsilon>0$. By 483C and 482Ad, as usual, together with 483F, there are a $\delta\in\Delta$ and a non-decreasing $\phi:\mathbb{R}\to[0,\epsilon]$ such that

$$S_{\boldsymbol{t}}(|f|\times\chi E,\mu)\le\epsilon,\quad |f(x)(b-a)-F(b)+F(a)|\le\phi(b)-\phi(a)$$

whenever $\boldsymbol{t}\in T$ is δ-fine, $a\le x\le b$ and $(x,[a,b])\in\delta$. For $n\in\mathbb{N}$ and $i\in\mathbb{Z}$, set

$$E_{ni}=\{x:x\in E\cap[2^{-n}i,2^{-n}(i+1)[\,,\ (x,A)\in\delta\text{ whenever }A\subseteq[x-2^{-n},x+2^{-n}]\}.$$

Set $J_n=\{i:i\in\mathbb{Z},\ -4^n<i\le 4^n,\ E_{ni}\ne\emptyset\}$. Observe that

$$E=\bigcup_{n\in\mathbb{N}}\bigcap_{m\ge n}\bigcup_{i\in J_m}E_{mi}.$$

For $i \in J_n$, take x_{ni}, $y_{ni} \in E_{ni}$ such that $x_{ni} \le y_{ni}$ and

$$\min(F(x_{ni}), F(y_{ni})) \le \inf F[E_{ni}] + 4^{-n}\epsilon,$$

$$\max(F(x_{ni}), F(y_{ni})) \ge \sup F[E_{ni}] - 4^{-n}\epsilon,$$

so that $\mu^*F[E_{ni}] \le |F(y_{ni}) - F(x_{ni})| + 2^{-2n+1}\epsilon$. Now, for each $i \in J_n$, $(x_{ni}, [x_{ni}, y_{ni}]) \in \delta$, while $[x_{ni}, y_{ni}] \subseteq [2^{-n}i, 2^{-n}(i+1)[$, so $\boldsymbol{t} = \{(x_{ni}, [x_{ni}, y_{ni}]) : i \in J_n\}$ is a δ-fine member of T, and

$$\begin{aligned}\mu^*F[\bigcup_{i\in J_n} E_{ni}] &\le \sum_{i\in J_n} \mu^*F[E_{ni}] \le \sum_{i\in J_n} 2^{-2n+1}\epsilon + |F(y_{ni}) - F(x_{ni})| \\ &\le 4\epsilon + \sum_{i\in J_n} |f(x_{ni})(y_{ni} - x_{ni})| + \sum_{i\in J_n} \phi(y_{ni}) - \phi(x_{ni}) \\ &\le 4\epsilon + S_{\boldsymbol{t}}(|f| \times \chi E, \mu) + \epsilon \le 6\epsilon.\end{aligned}$$

Since this is true for every $n \in \mathbb{N}$,

$$\begin{aligned}\mu^*F[E] &= \mu^*F[\bigcup_{n\in\mathbb{N}} \bigcap_{m\ge n} \bigcup_{i\in J_m} E_i] \\ &= \mu^*(\bigcup_{n\in\mathbb{N}} F[\bigcap_{m\ge n} \bigcup_{i\in J_m} E_i]) = \sup_{n\in\mathbb{N}} \mu^*F[\bigcap_{m\ge n} \bigcup_{i\in J_m} E_i]\end{aligned}$$

(132Ae)

$$\le 6\epsilon.$$

As ϵ is arbitrary, $F[E]$ is negligible, as claimed.

Remark Compare 225M.

483L Definition If $f : \mathbb{R} \to \mathbb{R}$ is Henstock integrable, I write $\|f\|_H$ for $\sup_{a\le b} |\,\oint_a^b f|$. It is elementary to check that this is a seminorm on the linear space of all Henstock integrable functions. (It is finite-valued by 483D.)

483M Proposition (a) If $f : \mathbb{R} \to \mathbb{R}$ is Henstock integrable, then $|\oint f| \le \|f\|_H$, and $\|f\|_H = 0$ iff $f = 0$ a.e.

(b) Write $\mathcal{HL}^1$ for the linear space of all Henstock integrable real-valued functions on $\mathbb{R}$, and HL^1 for $\{f^\bullet : f \in \mathcal{HL}^1\} \subseteq L^0(\mu)$ (§241). If we write $\|f^\bullet\|_H = \|f\|_H$ for every $f \in \mathcal{HL}^1$, then HL^1 is a normed space. The ordinary space $L^1(\mu)$ of equivalence classes of Lebesgue integrable functions is a linear subspace of HL^1, and $\|u\|_H \le \|u\|_1$ for every $u \in L^1(\mu)$.

(c) We have a linear operator $T : HL^1 \to C_b(\mathbb{R})$ defined by saying that $T(f^\bullet)$ is the indefinite Henstock integral of f for every $f \in \mathcal{HL}^1$, and $\|T\| = 1$.

proof (a) Of course

$$|\oint f| = \lim_{a\to-\infty, b\to\infty} |\oint_a^b f| \le \|f\|_H$$

(using 483Bd). Let F be the indefinite Henstock integral of f, so that $F(b) - F(a) = \oint_a^b f$ whenever $a \le b$. If $f = 0$ a.e., then $F(x) = \int_{-\infty}^x f d\mu = 0$ for every x, by 483Bb, so $\|f\|_H = 0$. If $\|f\|_H = 0$, then F is constant, so $f = F' = 0$ a.e., by 483I.

(b) That HL^1 is a normed space follows immediately from (a). (Compare the definitions of the norms $\|\;\|_p$ on L^p, for $1 \le p \le \infty$, in §§242-244.) By 483Bb, $L^1(\mu) \subseteq HL^1$, and

$$\|u\|_H \le \|u^+\|_H + \|u^-\|_H = \|u^+\|_1 + \|u^-\|_1 = \|u\|_1$$

for every $u \in L^1(\mu)$, writing u^+ and u^- for the positive and negative parts of u, as in Chapter 24.

(c) If f, $g \in \mathcal{HL}^1$ and $f^\bullet = g^\bullet$, then f and g have the same indefinite Henstock integral, by 483Bb or otherwise; so T is defined as a function from HL^1 to $\mathbb{R}^{\mathbb{R}}$. By 483F, Tu is continuous and bounded for every $u \in HL^1$, and by 481Ca T is linear. If $f \in \mathcal{HL}^1$ and $Tf^\bullet = F$, then $\|f\|_H = \sup_{x,y\in\mathbb{R}} |F(y) - F(x)|$; since $\lim_{x\to-\infty} F(x) = 0$, $\|f\|_H \ge \|F\|_\infty$; as f is arbitrary, $\|T\| \le 1$. On the other hand, for any non-negative Lebesgue integrable function f, $\|Tf^\bullet\|_\infty = \|f\|_1 = \|f\|_H$, so $\|T\| = 1$.

483N Proposition Suppose that $\langle I_m\rangle_{m\in M}$ is a disjoint family of open intervals in $\mathbb{R}$ with union G, and that $f : \mathbb{R} \to \mathbb{R}$ is a function such that $f_m = f \times \chi I_m$ is Henstock integrable for every $m \in M$. If $\sum_{m\in M} \|f_m\|_H < \infty$, then $f \times \chi G$ is Henstock integrable, and $\Join\!\!\!\!\!\int f \times \chi G = \sum_{m\in M} \Join\!\!\!\!\!\int f_m$.

proof I seek to apply 482H again. We have already seen, in the proof of 483Bc, that the conditions of 482G are satisfied by $\mathbb{R}$, T, Δ, $\mathfrak{R}$, $\mathcal{C}$, $\mathfrak{T}$ and μ. Of course $G = \bigcup_{m\in M} I_m$ is the union of a sequence of open sets over which f is Henstock integrable. So we have only to check 482H(viii).

Set

$$\delta_0 = \bigcup_{m\in M}\{(x, A) : x \in I_m,\ A \subseteq I_m\} \cup \{(x, A) : x \in \mathbb{R} \setminus G,\ A \subseteq \mathbb{R}\},$$

so that $\delta_0 \in \Delta$. For each $m \in M$ let F^{SH}_m be the Saks-Henstock indefinite integral of f_m. Let $\epsilon > 0$. Then there is a finite set $M_0 \subseteq M$ such that $\sum_{m\in M\setminus M_0} \|f_m\|_H \le \epsilon$. Next, there must be $\delta_1 \in \Delta$ and $\mathcal{R} \in \mathfrak{R}$ such that

$$\textstyle\sum_{m\in M_0} | \Join\!\!\!\!\!\int f_m - S_{\boldsymbol{t}}(f_m, \mu)| \le \epsilon$$

for every δ_1-fine $\mathcal{R}$-filling $\boldsymbol{t} \in T$, and $\delta_2 \in \Delta$ such that $\sum_{m\in M_0} |S_{\boldsymbol{t}}(f_m, \mu) - F^{\text{SH}}_m(W_{\boldsymbol{t}})| \le \epsilon$ for every δ_2-fine $\boldsymbol{t} \in T$.

Now let $\boldsymbol{t} \in T$ be $(\delta_0\cap\delta_1\cap\delta_2)$-fine and $\mathcal{R}$-filling. For each $m \in M$ set $\boldsymbol{t}_m = \boldsymbol{t}\restriction I_m$, so that $\boldsymbol{t}\restriction G = \bigcup_{m\in M} \boldsymbol{t}_m$. Because $W_{\boldsymbol{t}}$ is an interval, each $W_{\boldsymbol{t}_m}$ must be an interval, as in part (c)-(d)(vi) of the proof of 483B, and $W_{\boldsymbol{t}_m}$ is a subinterval of I_m because $\boldsymbol{t}$ is δ_0-fine. So (using 482G)

$$|F^{\text{SH}}_m(W_{\boldsymbol{t}_m})| = | \Join\!\!\!\!\!\int f_m \times \chi W_{\boldsymbol{t}_m}| \le \|f_m\|_H.$$

Also

$$\begin{aligned}\sum_{m\in M_0} | \Join\!\!\!\!\!\int f_m - \Join\!\!\!\!\!\int f \times \chi W_{\boldsymbol{t}_m}| &\le \sum_{m\in M_0} | \Join\!\!\!\!\!\int f_m - S_{\boldsymbol{t}}(f_m, \mu)| + \sum_{m\in M_0} |S_{\boldsymbol{t}}(f_m, \mu) - F^{\text{SH}}_m(W_{\boldsymbol{t}_m}| \\ &\le 2\epsilon.\end{aligned}$$

On the other hand,

$$\textstyle\sum_{m\in M\setminus M_0}| \Join\!\!\!\!\!\int f_m - \Join\!\!\!\!\!\int f \times \chi W_{\boldsymbol{t}_m}| \le 2\sum_{m\in M\setminus M_0} \|f_m\|_H \le 2\epsilon.$$

Putting these together,

$$| \Join\!\!\!\!\!\int f \times \chi W_{\boldsymbol{t}\restriction G} - \sum_{m\in M} \Join\!\!\!\!\!\int f_m| = | \sum_{m\in M} \Join\!\!\!\!\!\int f \times \chi W_{\boldsymbol{t}_m} - \sum_{m\in M} \Join\!\!\!\!\!\int f_m|$$

(because $\boldsymbol{t}$ is finite, so all but finitely many terms in the sum $\sum_{m\in M} f \times \chi W_{\boldsymbol{t}_m}$ are zero)

$$\le \sum_{m\in M} | \Join\!\!\!\!\!\int f \times \chi W_{\boldsymbol{t}_m} - \Join\!\!\!\!\!\int f_m| \le 4\epsilon.$$

As ϵ is arbitrary, condition 482H(viii) is satisfied, with

$$\lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})} \Join\!\!\!\!\!\int f \times \chi W_{\boldsymbol{t}\restriction G} = \textstyle\sum_{m\in M} \Join\!\!\!\!\!\int f_m,$$

and 482H gives the result we seek.

483O Definitions (a) For any real-valued function F, write $\omega(F)$ for $\sup_{x,y\in\operatorname{dom} F} |F(x) - F(y)|$, the **oscillation** of F. (Interpret $\sup\emptyset$ as 0, so that $\omega(\emptyset) = 0$.)

(b) Let $F:\mathbb{R}\to\mathbb{R}$ be a function. For $A\subseteq\mathbb{R}$, we say that F is **AC$_*$ on** A if for every $\epsilon>0$ there is an $\eta>0$ such that $\sum_{I\in\mathcal{I}}\omega(F\restriction\overline{I})\le\epsilon$ whenever $\mathcal{I}$ is a disjoint family of open intervals with endpoints in A and $\sum_{I\in\mathcal{I}}\mu I\le\eta$. Note that whether F is AC$_*$ on A is *not* determined by $F\restriction A$, since it depends on the behaviour of F on intervals with endpoints in A.

(c) Finally, F is **ACG$_*$** if it is continuous and there is a countable family $\mathcal{A}$ of sets, covering $\mathbb{R}$, such that F is AC$_*$ on every member of $\mathcal{A}$.

483P Elementary results (a)(i) If F, $G:\mathbb{R}\to\mathbb{R}$ are functions and $A\subseteq B\subseteq\mathbb{R}$, then $\omega(F+G\restriction A)\le\omega(F\restriction A)+\omega(G\restriction A)$ and $\omega(F\restriction A)\le\omega(F\restriction B)$.

(ii) If F is the indefinite Henstock integral of $f:\mathbb{R}\to\mathbb{R}$ and $C\subseteq\mathbb{R}$ is an interval, then $\|f\times\chi C\|_H=\omega(F\restriction C)$.

(iii) If $F:\mathbb{R}\to\mathbb{R}$ is continuous, then $(a,b)\mapsto\omega(F\restriction[a,b]):\mathbb{R}^2\to\mathbb{R}$ is continuous, and $\omega(F\restriction\overline{A})=\omega(F\restriction A)$ for every set $A\subseteq\mathbb{R}$.

(b)(i) If $F:\mathbb{R}\to\mathbb{R}$ is AC$_*$ on $A\subseteq\mathbb{R}$, it is AC$_*$ on every subset of A.

(ii) If $F:\mathbb{R}\to\mathbb{R}$ is continuous and is AC$_*$ on $A\subseteq\mathbb{R}$, it is AC$_*$ on $\overline{A}$. **P** Let $\epsilon>0$. Let $\eta>0$ be such that $\sum_{I\in\mathcal{I}}\omega(F\restriction\overline{I})\le\epsilon$ whenever $\mathcal{I}$ is a disjoint family of open intervals with endpoints in A and $\sum_{I\in\mathcal{I}}\mu I\le\eta$. Let $\mathcal{I}$ be a disjoint family of open intervals with endpoints in $\overline{A}$ and $\sum_{I\in\mathcal{I}}\mu I\le\frac{1}{2}\eta$. Let $\mathcal{I}_0\subseteq\mathcal{I}$ be a non-empty finite set; then we can enumerate $\mathcal{I}_0$ as $\langle]a_i,b_i[\rangle_{i\le n}$ where $a_0,b_0,\dots,a_n,b_n\in\overline{A}$ and $a_0\le b_0\le a_1\le b_1\le\ldots\le a_n\le b_n$. Because $(a,b)\mapsto\omega(F\restriction[a,b])$ is continuous, as noted in (a-ii) above, we can find $a_0',\dots,b_n'\in A$ such that $a_0'\le b_0'\le a_1'\le\ldots\le a_n'\le b_n'$, $\sum_{i=0}^n b_i'-a_i'\le\eta$, and $\sum_{i=0}^n|\omega(F\restriction[a_i',b_i'])-\omega(F\restriction[a_i,b_i])|\le\epsilon$; so that

$$\sum_{I\in\mathcal{I}_0}\omega(F\restriction\overline{I})\le\sum_{i=0}^n\omega(F\restriction[a_i',b_i'])\le 2\epsilon.$$

As $\mathcal{I}_0$ is arbitrary, $\sum_{I\in\mathcal{I}}\omega(F\restriction\overline{I})\le 2\epsilon$; as ϵ is arbitrary, F is AC$_*$ on $\overline{A}$. **Q**

483Q Lemma Let $F:\mathbb{R}\to\mathbb{R}$ be a continuous function, and $K\subseteq\mathbb{R}$ a non-empty compact set such that F is AC$_*$ on K. Write $\mathcal{I}$ for the family of non-empty bounded open intervals, disjoint from K, with endpoints in K.

(a) $\sum_{I\in\mathcal{I}}\omega(F\restriction I)$ is finite.

(b) Write a^* for $\inf K=\min K$. Then there is a Lebesgue integrable function $g:\mathbb{R}\to\mathbb{R}$, zero off K, such that

$$F(x)-F(a^*)=\int_{a^*}^x g+\sum_{J\in\mathcal{I},J\subseteq[a^*,x]}F(\sup J)-F(\inf J)$$

for every $x\in K$.

proof (a) Let $\eta>0$ be such that $\sum_{I\in\mathcal{J}}\omega(F\restriction\overline{I})\le 1$ whenever $\mathcal{J}$ is a disjoint family of open intervals with endpoints in K and $\sum_{I\in\mathcal{J}}\mu I\le\eta$. Let m_0, $m_1\in\mathbb{Z}$ be such that $K\subseteq[m_0\eta,m_1\eta]$. For integers m between m_0 and m_1, let $\mathcal{I}_m$ be the set of intervals in $\mathcal{I}$ included in $]m\eta,(m+1)\eta[$. Then $\sum_{I\in\mathcal{I}_m}\mu I\le\eta$, so $\sum_{I\in\mathcal{I}_m}\omega(F\restriction I)\le 1$ for each m. Also every member of $\mathcal{J}=\mathcal{I}\setminus\bigcup_{m_0\le m<m_1}\mathcal{I}_m$ contains $m\eta$ for some m between m_0 and m_1, so $\#(\mathcal{J})\le m_1-m_0$. Accordingly

$$\begin{aligned}\sum_{I\in\mathcal{I}}\omega(F\restriction\overline{I})&\le\sum_{I\in\mathcal{J}}\omega(F\restriction\overline{I})+\sum_{m=m_0}^{m_1-1}\sum_{I\in\mathcal{I}_m}\omega(F\restriction\overline{I})\\&\le\sum_{I\in\mathcal{J}}\omega(F\restriction\overline{I})+m_1-m_0<\infty\end{aligned}$$

because F is continuous, therefore bounded on every bounded interval.

(b)(i) Set $b^*=\sup K=\max K$. Define $G:[a^*,b^*]\to\mathbb{R}$ by setting

$$G(x) = F(x) \text{ if } x \in K,$$
$$= \tfrac{F(b)(x-a)+F(a)(b-x)}{b-a} \text{ if } x \in \,]a,b[\, \in \mathcal{I}.$$

Then G is absolutely continuous. **P** G is continuous because F is. Let $\epsilon > 0$. Let $\eta_1 > 0$ be such that $\sum_{I\in\mathcal{J}} \omega(F\restriction\overline{I}) \le \epsilon$ whenever $\mathcal{J}$ is a disjoint family of open intervals with endpoints in K and $\sum_{I\in\mathcal{J}} \mu I \le \eta_1$. Let $\mathcal{I}_0 \subseteq \mathcal{I}$ be a finite set such that $\sum_{I\in\mathcal{I}\setminus\mathcal{I}_0} \omega(F\restriction\overline{I}) \le \epsilon$, and take $M > 0$ such that $|F(b)-F(a)| \le M(b-a)$ whenever $]a,b[\, \in \mathcal{I}_0$; set $\eta = \min(\eta_1, \frac{\epsilon}{M}) > 0$.

Let $\mathcal{J}^*$ be the set of non-empty open subintervals J of $[a^*,b^*]$ such that either $J\cap K = \emptyset$ or both endpoints of J belong to K. Let $\mathcal{J} \subseteq \mathcal{J}^*$ be a disjoint family such that $\sum_{I\in\mathcal{J}} \mu I \le \eta$. Set $\mathcal{J}' = \{J : J \in \mathcal{J},\ J\cap K = \emptyset\}$. Then

$$\begin{aligned}\sum_{J\in\mathcal{J}\setminus\mathcal{J}'} |G(\sup J) - G(\inf J)| &= \sum_{J\in\mathcal{J}\setminus\mathcal{J}'} |F(\sup J) - F(\inf J)| \\ &\le \sum_{J\in\mathcal{J}\setminus\mathcal{J}'} \omega(F\restriction\overline{J}) \le \epsilon.\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{J\in\mathcal{J}'} |G(\sup J) - G(\inf J)| &= \sum_{I\in\mathcal{I}_0} \sum_{\substack{J\in\mathcal{J}\\ J\subseteq I}} |G(\sup J) - G(\inf J)| \\ &\quad + \sum_{I\in\mathcal{I}\setminus\mathcal{I}_0} \sum_{\substack{J\in\mathcal{J}\\ J\subseteq I}} |G(\sup J) - G(\inf J)| \\ &\le M \sum_{I\in\mathcal{I}_0} \sum_{\substack{J\in\mathcal{J}\\ J\subseteq I}} \mu J + \sum_{I\in\mathcal{I}\setminus\mathcal{I}_0} |F(\sup I) - F(\inf I)|\end{aligned}$$

(because G is monotonic on $\overline{I}$ for each $I \in \mathcal{I}$)

$$\le M\eta + \epsilon \le 2\epsilon,$$

so $\sum_{J\in\mathcal{J}} |G(\sup J) - G(\inf J)| \le 3\epsilon$.

Generally, if J is any non-empty open subinterval of $[a^*,b^*]$, we can split it into at most three intervals belonging to $\mathcal{J}^*$. So if $\mathcal{J}$ is any disjoint family of non-empty open subintervals of $[a^*,b^*]$ with $\sum_{J\in\mathcal{J}} \mu J \le \eta$, we can find a family $\tilde{\mathcal{J}} \subseteq \mathcal{J}^*$ with $\sum_{J\in\tilde{\mathcal{J}}} \mu J = \sum_{J\in\mathcal{J}} \mu J$ and $\sum_{J\in\tilde{\mathcal{J}}} |G(\sup J) - G(\inf J)| \ge \sum_{J\in\mathcal{J}} |G(\sup J) - G(\inf J)|$. But this means that $\sum_{J\in\mathcal{J}} |G(\sup J) - G(\inf J)| \le 3\epsilon$. As ϵ is arbitrary, G is absolutely continuous. **Q**

(ii) By 225E, G' is Lebesgue integrable and $G(x) = G(a^*) + \int_{a^*}^x G'$ for every $x \in [a^*,b^*]$. Set $g(x) = G'(x)$ when $x \in K$ and $G'(x)$ is defined, 0 for other $x \in \mathbb{R}$, so that $g : \mathbb{R} \to \mathbb{R}$ is Lebesgue integrable. Now take any $x \in K$. Then

$$\begin{aligned}F(x) = G(x) = G(a^*) + \int_{a^*}^x G' &= F(a^*) + \int_{a^*}^x g + \int_{[a^*,x]\setminus K} G' \\ &= F(a^*) + \int_{a^*}^x g + \sum_{\substack{I\in\mathcal{I}\\ I\subseteq[a^*,x]}} \int_I G'\end{aligned}$$

(because $\mathcal{I}$ is a disjoint countable family of measurable sets, and $\bigcup_{I\in\mathcal{I}, I\subseteq[a^*,x]} I = [a^*,x] \setminus K$)

$$= F(a^*) + \int_{a^*}^x g + \sum_{\substack{I\in\mathcal{I}\\ I\subseteq[a^*,x]}} G(\sup I) - G(\inf I)$$

(note that this sum is absolutely summable)

$$= F(a^*) + \int_{a^*}^{x} g + \sum_{\substack{I \in \mathcal{I} \\ I \subseteq [a^*, x]}} F(\sup I) - F(\inf I)$$

as required.

483R Theorem Let $F : \mathbb{R} \to \mathbb{R}$ be a function. Then F is an indefinite Henstock integral iff it is ACG_*, $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x)$ is defined in $\mathbb{R}$.

proof (a) Suppose that F is the indefinite Henstock integral of $f : \mathbb{R} \to \mathbb{R}$.

(i) By 483F, F is continuous, with limit zero at $-\infty$ and finite at ∞. So I have just to show that there is a sequence of sets, covering $\mathbb{R}$, on each of which F is AC_*. Recall that there is a sequence $\langle K_m \rangle_{m\in\mathbb{N}}$ of compact sets, covering $\mathbb{R}$, such that $f \times K_m$ is Lebesgue integrable for every $m \in \mathbb{N}$ (483G). By the arguments of (i)$\Rightarrow$(ii) in the proof of 483J, there is a function $F_2 \geq F$ such that $\underline{D}F_2 \geq f$ and $F_2 - F$ is non-decreasing and takes values between 0 and 1. For $n \in \mathbb{N}$, $j \in \mathbb{Z}$ set

$$I_{nj} = [2^{-n}j, 2^{-n}(j+1)],$$

$$B_{nj} = \{x : x \in I_{nj},\ \tfrac{F_2(y)-F_2(x)}{y-x} \geq -n \text{ whenever } y \in I_{nj} \setminus \{x\}\}.$$

Observe that $\bigcup_{n\in\mathbb{N}, j\in\mathbb{Z}} B_{nj} = \mathbb{R}$, so that $\{B_{nj} \cap K_m : m,\, n \in \mathbb{N},\, j \in \mathbb{Z}\}$ is a countable cover of $\mathbb{R}$. It will therefore be enough to show that F is AC_* on every $B_{nj} \cap K_m$.

(ii) Fix m, $n \in \mathbb{N}$ and $j \in \mathbb{Z}$, and set $A = B_{nj} \cap K_m$. If $A = \emptyset$, then of course F is AC_* on A; suppose that A is not empty. Set $G(x) = F_2(x) + nx$ for $x \in I_{nj}$, so that $F = G - (F_2 - F) - H$ on I_{nj}, where $H(x) = nx$. Whenever a, $b \in B_{nj}$ and $a \leq x \leq b$, then $G(a) \leq G(x) \leq G(b)$, because $x \in I_{nj}$. So if $a_0, b_0, a_1, b_1, \ldots, a_k, b_k \in A$ and $a_0 \leq b_0 \leq a_1 \leq b_1 \leq \ldots \leq a_k \leq b_k$,

$$\textstyle\sum_{i=0}^{k} \omega(G{\restriction}[a_i, b_i]) = \sum_{i=0}^{k} G(b_i) - G(a_i) \leq G(b_k) - G(a_0) \leq \omega(G{\restriction}I_{nj}),$$

and

$$\begin{aligned}\sum_{i=0}^{k} \omega(F{\restriction}[a_i, b_i]) &\leq \sum_{i=0}^{k} \omega(G{\restriction}[a_i, b_i]) + \sum_{i=0}^{k} \omega(F_2 - F{\restriction}[a_i, b_i]) + \sum_{i=0}^{k} \omega(H{\restriction}[a_i, b_i]) \\ &\leq \omega(G{\restriction}I_{nj}) + \omega(F_2 - F{\restriction}I_{nj}) + \omega(H{\restriction}I_{nj})\end{aligned}$$

(because $F_2 - F$ and H are monotonic)

$$\begin{aligned}&\leq \omega(F{\restriction}I_{nj}) + 2\omega(F_2 - F{\restriction}I_{nj}) + 2\omega(H{\restriction}I_{nj}) \\ &\leq \omega(F{\restriction}I_{nj}) + 2(1 + n\mu I_{nj}) = M\end{aligned}$$

say, which is finite, because F is bounded.

(iii) By 2A2I (or 4A2Rj), the open set $\mathbb{R} \setminus \overline{A}$ is expressible as a countable union of disjoint non-empty open intervals. Two of these are unbounded; let $\mathcal{I}$ be the set consisting of the rest, so that $\overline{A} \cup \bigcup \mathcal{I} = [a^*, b^*]$ is a closed interval included in I_{nj}. If I, I' are distinct members of $\mathcal{I}$ and $\inf I \leq \inf I'$, then $\sup I \leq \inf I'$, because $I \cap I' = \emptyset$, and there must be a point of $\overline{A}$ in the interval $[\sup I, \inf I']$; so in fact there must be a point of A in this interval, since A does not meet either I or I'. It follows that $\sum_{I\in\mathcal{I}} \omega(F{\restriction}I) \leq M$. **P** If $\mathcal{I}_0 \subseteq \mathcal{I}$ is finite and non-empty, we can enumerate it as $\langle I_i \rangle_{i\leq k}$ where $\sup I_i \leq \inf I_{i'}$ whenever $i < i' \leq k$. We can find $a_0, \ldots, a_{k+1} \in A$ such that $a_0 \leq \inf I_0$, $\sup I_i \leq a_{i+1} \leq \inf I_{i+1}$ for every $i < k$, and $\sup I_k \leq a_{k+1}$; so that

$$\textstyle\sum_{I\in\mathcal{I}_0} \omega(F{\restriction}I) = \sum_{i=0}^{k} \omega(F{\restriction}I_i) \leq \sum_{i=0}^{k} \omega(F{\restriction}[a_i, a_{i+1}]) \leq M.$$

As $\mathcal{I}_0$ is arbitrary, this gives the result. **Q**

(iv) Whenever $a^* \leq x \leq y \leq b^*$,

$$|F(y) - F(x)| \leq \textstyle\int_{\overline{A}\cap[x,y]} |f| d\mu + \sum_{I\in\mathcal{I}} \omega(F{\restriction}I \cap\,]x, y[).$$

P For each $I\in\mathcal{I}$,

$$\|f\times\chi(I\cap]x,y[)\|_H=\omega(F\restriction I\cap]x,y[)\le\omega(F\restriction I)$$

(483Pa). So

$$\sum_{I\in\mathcal{I}}\|f\times\chi(I\cap]x,y[)\|_H\le\sum_{I\in\mathcal{I}}\omega(F\restriction I)$$

is finite. Writing $H=]x,y[\cap\bigcup\mathcal{I}$, $\int\!\!\!\!\!\!H\, f\times\chi H$ is defined and equal to $\sum_{I\in\mathcal{I}}\int\!\!\!\!\!\!H\, f\times\chi(I\cap]x,y[)$, by 483N. On the other hand,

$$]x,y[\setminus H\subseteq\overline{A}\subseteq K_m,$$

so $f\times\chi(]x,y[\setminus H)$ is Lebesgue integrable. Accordingly

$$\begin{aligned}|F(y)-F(x)|&=|\int\!\!\!\!\!\!H\, f\times\chi\,]x,y[\,|\\&\le|\int\!\!\!\!\!\!H\, f\times\chi H|+|\int f\times\chi(]x,y[\setminus H)d\mu|\\&\le\sum_{I\in\mathcal{I}}|\int\!\!\!\!\!\!H\, f\times\chi(I\cap]x,y[)|+\int_{\overline{A}\cap[x,y]}|f|d\mu\\&\le\sum_{I\in\mathcal{I}}\omega(F\restriction I\cap]x,y[)+\int_{\overline{A}\cap[x,y]}|f|d\mu,\end{aligned}$$

as claimed. **Q**

(v) So if $a^*\le a\le b\le b^*$,

$$\omega(F\restriction[a,b])\le\sum_{I\in\mathcal{I}}\omega(F\restriction I\cap[a,b])+\int_{\overline{A}\cap[a,b]}|f|d\mu.$$

P We have only to observe that if $a\le x\le y\le b$, then $\omega(F\restriction I\cap]x,y[)\le\omega(F\restriction I\cap[a,b])$ for every $I\in\mathcal{I}$, and $\int_{\overline{A}\cap]x,y[}|f|d\mu\le\int_{\overline{A}\cap[a,b]}|f|d\mu$. **Q**

(vi) Now let $\epsilon>0$. Setting $\tilde F(x)=\int_{a^*}^x|f|\times\chi\overline{A}\,d\mu$ for $x\in[a^*,b^*]$, $\tilde F$ is absolutely continuous (225E), and there is an $\eta_0>0$ such that $\sum_{i=0}^k\tilde F(b_i)-\tilde F(a_i)\le\epsilon$ whenever $a^*\le a_0\le b_0\le a_1\le b_1\le\ldots\le a_k\le b_k\le b^*$ and $\sum_{i=0}^k b_i-a_i\le\eta_0$. Take $\mathcal{I}_0\subseteq\mathcal{I}$ to be a finite set such that $\sum_{I\in\mathcal{I}\setminus\mathcal{I}_0}\omega(F\restriction I)\le\epsilon$, and let $\eta>0$ be such that $\eta\le\eta_0$ and $\eta<\operatorname{diam}I$ for every $I\in\mathcal{I}_0$.

Suppose that $a_0,b_0,\ldots,a_k,b_k\in A$ are such that $a_0\le b_0\le\ldots\le a_k\le b_k$ and $\sum_{i=0}^k b_i-a_i\le\eta$. Then no member of $\mathcal{I}_0$ can be included in any interval $[a_i,b_i]$, and therefore, because no a_i or b_i can belong to $\bigcup\mathcal{I}$, no member of $\mathcal{I}_0$ meets any $[a_i,b_i]$. Also, of course, $a^*\le a_0$ and $b_k\le b^*$. We therefore have

$$\begin{aligned}\sum_{i=0}^k\omega(F\restriction[a_i,b_i])&\le\sum_{i=0}^k\Bigl(\sum_{I\in\mathcal{I}}\omega(F\restriction I\cap[a_i,b_i])+\int_{\overline{A}\cap[a_i,b_i]}|f|d\mu\Bigr)\\&=\sum_{i=0}^k\sum_{I\in\mathcal{I}\setminus\mathcal{I}_0}\omega(F\restriction I\cap[a_i,b_i])+\sum_{i=0}^k\tilde F(b_i)-\tilde F(a_i)\\&\le\sum_{i=0}^k\sum_{\substack{I\in\mathcal{I}\setminus\mathcal{I}_0\\ I\subseteq[a_i,b_i]}}\omega(F\restriction I)+\epsilon\end{aligned}$$

(because if $I\in\mathcal{I}$ meets $[a_i,b_i]$, it is included in it)

$$\le\sum_{I\in\mathcal{I}\setminus\mathcal{I}_0}\omega(F\restriction I)+\epsilon\le 2\epsilon.$$

As ϵ is arbitrary, F is AC$_*$ on A. This completes the proof that F is ACG$_*$ and therefore satisfies the conditions given.

(b) Now suppose that F satisfies the conditions. Set $F(-\infty)=0$ and $F(\infty)=\lim_{x\to\infty}F(x)$, so that $F:[-\infty,\infty]\to\mathbb{R}$ is continuous. For $x\in\mathbb{R}$, set $f(x)=F'(x)$ if this is defined, 0 otherwise. Let $\mathcal{J}$ be the family of all non-empty intervals $C\subseteq\mathbb{R}$ such that $\mathrm{H}\!\!\int f\times\chi C$ is defined and equal to $F(\sup C)-F(\inf C)$, and let $\mathcal{I}$ be the set of non-empty open intervals I such that every non-empty subinterval of I belongs to $\mathcal{J}$. I seek to show that $\mathbb{R}$ belongs to $\mathcal{I}$.

(i) Of course singleton intervals belong to $\mathcal{J}$. If C_1, $C_2\in\mathcal{J}$ are disjoint and $C=C_1\cup C_2$ is an interval, then

$$\begin{aligned}\mathrm{H}\!\!\int f\times\chi C&=\mathrm{H}\!\!\int f\times\chi C_1+\mathrm{H}\!\!\int f\times\chi C_2\\&=F(\sup C_1)-F(\inf C_1)+F(\sup C_2)-F(\inf C_2)\\&=F(\sup C)-F(\inf C)\end{aligned}$$

and $C\in\mathcal{J}$. If I_1, $I_2\in\mathcal{I}$ and $I_1\cap I_2$ is non-empty, then $I_1\cup I_2$ is an interval; also any subinterval C of $I_1\cup I_2$ is either included in one of the I_j or is expressible as a disjoint union $C_1\cup C_2$ where C_j is a subinterval of I_j for each j; so $C\in\mathcal{J}$ and $I\in\mathcal{I}$. If I_1, $I_2\in\mathcal{I}$ and $\sup I_1=\inf I_2$, then $I=I_1\cup I_2\cup\{\sup I_1\}\in\mathcal{I}$, because any subinterval of I is expressible as the disjoint union of at most three intervals in $\mathcal{J}$.

(ii) If $\mathcal{I}_0\subseteq\mathcal{I}$ is non-empty and upwards-directed, then $\bigcup\mathcal{I}_0\in\mathcal{I}$. **P** This is a consequence of 483Bd. If we take a non-empty open subinterval J of $\bigcup\mathcal{I}_0$ and express it as $]\alpha,\beta[$, where $-\infty\le\alpha<\beta\le\infty$, then whenever $\alpha<a<b<\beta$ there are members of $\mathcal{I}_0$ containing a and b, and therefore a member of $\mathcal{I}_0$ containing both, so that $[a,b]\in\mathcal{J}$. Accordingly $\mathrm{H}\!\!\int_a^b f$ is defined and equal to $F(b)-F(a)$. Since F is continuous, $\lim_{a\downarrow\alpha,b\uparrow\beta}\mathrm{H}\!\!\int_a^b f$ is defined and equal to $F(\beta)-F(\alpha)$; by 483Bd, $\mathrm{H}\!\!\int_\alpha^\beta f$ is defined and equal to $F(\beta)-F(\alpha)$, so that $J\in\mathcal{J}$. I wrote this out for open intervals, for convenience; but any non-empty subinterval of $\bigcup\mathcal{I}_0$ is either a singleton or expressible as an open interval with at most two points added, so belongs to $\mathcal{J}$. Accordingly $\bigcup\mathcal{I}_0\in\mathcal{I}$. **Q**

(iii) It follows that every member of $\mathcal{I}$ is included in a maximal member of $\mathcal{I}$. Let $\mathcal{I}^*$ be the set of maximal members of $\mathcal{I}$. By (i), these are all disjoint, so no endpoint of any member of $\mathcal{I}^*$ can belong to $\bigcup\mathcal{I}^*$.

? Suppose, if possible, that $\mathbb{R}\notin\mathcal{I}$. Then $\bigcup\mathcal{I}=\bigcup\mathcal{I}^*$ cannot be $\mathbb{R}$, and $V=\mathbb{R}\setminus\bigcup\mathcal{I}$ is a non-empty closed set. By (i), no two distinct members of $\mathcal{I}^*$ can share a boundary point, so V has no isolated points.

We are supposing that F is ACG$_*$, so there is a countable family $\mathcal{A}$ of sets, covering $\mathbb{R}$, such that F is AC$_*$ on A for every $A\in\mathcal{A}$. By Baire's theorem (3A3G or 4A2Ma), applied to the locally compact Polish space V, $V\setminus\overline{A}$ cannot be dense in V for every $A\in\mathcal{A}$, so there are an $A\in\mathcal{A}$ and a bounded open interval $\tilde J$ such that $\emptyset\ne V\cap\tilde J\subseteq\overline{A}$. Set $K=\overline{\tilde J\cap A}$; by 483Pb, F is AC$_*$ on K, and $V\cap\tilde J\subseteq K$. Because V has no isolated points, $V\cap\tilde J$ is infinite, so, setting $a^*=\min K$ and $b^*=\max K$, $V\cap\,]a^*,b^*[$ is non-empty.

Let $\mathcal{I}_0$ be the family of non-empty bounded open intervals, disjoint from K, with endpoints in K. By 483Q, $\sum_{I\in\mathcal{I}_0}\omega(F\restriction I)$ is finite, and there is a Lebesgue integrable function $g:\mathbb{R}\to\mathbb{R}$ such that $g(x)=0$ for $x\in\mathbb{R}\setminus K$ and

$$F(x)=F(a^*)+\int_{a^*}^x g+\sum_{I\in\mathcal{I}_0,I\subseteq[a^*,x]}F(\sup I)-F(\inf I)$$

for $x\in K$. Since every member of $\mathcal{I}_0$ is disjoint from V, it is included in some member of $\mathcal{I}^*$ and belongs to $\mathcal{J}$, so $F(\sup I)-F(\inf I)=\mathrm{H}\!\!\int f\times\chi I$ for every $I\in\mathcal{I}_0$. If $I\in\mathcal{I}_0$, then

$$\begin{aligned}\|f\times\chi I\|_H&=\sup_{C\in\mathcal{C}}|\mathrm{H}\!\!\int f\times\chi I\times\chi C|=\sup_{C\in\mathcal{C},C\subseteq I}|\mathrm{H}\!\!\int f\times\chi C|\\&=\sup_{C\in\mathcal{C},C\subseteq I}|F(\sup C)-F(\inf C)|=\omega(F\restriction I)\end{aligned}$$

because every non-empty subinterval of I belongs to $\mathcal{J}$. So $\sum_{I\in\mathcal{I}_0}\|f\times\chi I\|_H$ is finite, and $f\times\chi H$ is Henstock integrable, where $H=\bigcup\mathcal{I}_0$, by 483N. Moreover, if $x\in K$, then

$$\begin{aligned}\int_{a^*}^{x}\!\!\!\!\!\!\!\mathrm{H}\; f\times\chi H &= \int\!\!\!\!\!\!\mathrm{H}\; f\times\chi(\bigcup\{I : I\in\mathcal{I}_0,\, I\subseteq[a^*,x]\})\\ &= \sum_{I\in\mathcal{I}_0, I\subseteq[a^*,x]} \int\!\!\!\!\!\!\mathrm{H}\; f\times\chi I = \sum_{I\in\mathcal{I}_0, I\subseteq[a^*,x]} F(\sup I)-F(\inf I).\end{aligned}$$

But this means that if $y\in[a^*,b^*]$, and $x=\max(K\cap[a^*,y])$, so that $]x,y[\,\subseteq H$, then

$$\begin{aligned}F(y) &= F(x)+F(y)-F(x)\\ &= F(a^*)+\int_{a^*}^{x} g+\sum_{I\in\mathcal{I}_0, I\subseteq[a^*,x]}(F(\sup I)-F(\inf I))+\int\!\!\!\!\!\!\mathrm{H}\; f\times\chi\,]x,y[\\ &= F(a^*)+\int g\times\chi(K\cap[a^*,y[)+\int_{a^*}^{x}\!\!\!\!\!\!\!\mathrm{H}\; f\times\chi H+\int\!\!\!\!\!\!\mathrm{H}\; f\times\chi\,]x,y[\\ &= F(a^*)+\int_{-\infty}^{y}\!\!\!\!\!\!\!\!\mathrm{H}\;\; h\end{aligned}$$

where $h=f\times\chi H+g\times\chi K$ is Henstock integrable because $f\times\chi H$ is Henstock integrable and $g\times\chi K$ is Lebesgue integrable.

Accordingly, if C is any non-empty subinterval of $[a^*,b^*]$, $F(\sup C)-F(\inf C)=\int\!\!\!\!\!\!\mathrm{H}\; h\times\chi C$. But we also know that $\frac{d}{dy}\int_{-\infty}^{y}\!\!\!\!\!\!\!\mathrm{H}\; h=h(y)$ for almost every y, by 483I. So $F'(y)$ is defined and equal to $h(y)$ for almost every $y\in[a^*,b^*]$, and $h=f$ a.e. on $[a^*,b^*]$. This means that $F(\sup C)-F(\inf C)=\int\!\!\!\!\!\!\mathrm{H}\; f\times\chi C$ for any non-empty subinterval C of $[a^*,b^*]$, and $]a^*,b^*[\,\in\mathcal{I}$. But $]a^*,b^*[$ meets V, so this is impossible. **X**

(iv) This contradiction shows that $\mathbb{R}\in\mathcal{I}$ and that F is an indefinite Henstock integral, as required.

483X Basic exercises >(a) Let $[a,b]\subseteq\mathbb{R}$ be a non-empty closed interval, and let I_μ be the gauge integral on $[a,b]$ defined from Lebesgue measure and the tagged-partition structure defined in 481J. Show that, for $f:\mathbb{R}\to\mathbb{R}$, $I_\mu(f\restriction[a,b])=\int\!\!\!\!\!\!\mathrm{H}\; f\times\chi[a,b]$ if either is defined.

>(b) Extract ideas from the proofs of 482G and 482H to give a direct proof of 483B(c)-(d).

(c) Set $f(0)=0$ and $f(x)=\frac{\sin x}{x}$ for other real x. Show that f is Henstock integrable, and that $\int_0^\infty\!\!\!\!\!\!\!\mathrm{H}\; f=\frac{\pi}{2}$. (*Hint*: 283Da.)

(d) Set $f(x)=\dfrac{1}{x}\cos(\dfrac{1}{x^2})$ for $0<x\le 1$, 0 for other real x. Show that f is Henstock integrable but not Lebesgue integrable. (*Hint*: by considering $\dfrac{d}{dx}(x^2\sin\dfrac{1}{x^2})$, show that $\lim_{a\downarrow 0}\int_a^1 f$ is defined.)

(e) Let $f:\mathbb{R}\to\mathbb{R}$ be a Henstock integrable function. Show that there is a finitely additive functional $\lambda:\mathcal{P}\mathbb{R}\to\mathbb{R}$ such that for every $\epsilon>0$ there are a gauge $\delta\in\Delta$ and a Radon measure ν on $\mathbb{R}$ such that $\nu\mathbb{R}\le\epsilon$ and $|S_{\boldsymbol{t}}(f,\mu)-\lambda W_{\boldsymbol{t}}|\le\nu W_{\boldsymbol{t}}$ for every δ-fine $\boldsymbol{t}\in T$.

>(f) Let $f:\mathbb{R}\to\mathbb{R}$ be a Henstock integrable function. Show that $\bigcup\{G : G\subseteq\mathbb{R}$ is open, f is Lebesgue integrable over $G\}$ is dense. (*Hint*: 483G.)

>(g) Let $f:\mathbb{R}\to\mathbb{R}$ be a Henstock integrable function and F its indefinite Henstock integral. Show that f is Lebesgue integrable iff F is of bounded variation on $\mathbb{R}$. (*Hint*: 224I.)

>(h) Let $F:\mathbb{R}\to\mathbb{R}$ be a continuous function such that $\lim_{x\to-\infty}F(x)=0$, $\lim_{x\to\infty}F(x)$ is defined in $\mathbb{R}$, and $F'(x)$ is defined for all but countably many $x\in\mathbb{R}$. Show that F is the indefinite Henstock integral of any function $f:\mathbb{R}\to\mathbb{R}$ extending F'. (*Hint*: in 483J, take F_1 and F_2 differing from F by saltus functions.)

(i) Let $f:\mathbb{R}\to\mathbb{R}$ be a Henstock integrable function, and $\langle I_n\rangle_{n\in\mathbb{N}}$ a disjoint sequence of intervals in $\mathbb{R}$. Show that $\lim_{n\to\infty}\|f\times\chi I_n\|_H=0$.

(j) Show that HL^1 is not a Banach space. (*Hint*: there is a continuous function which is nowhere differentiable (477K).)

>(k) Use 483N to replace part of the proof of 483Bd.

(l) Let $F:\mathbb{R}\to\mathbb{R}$ be such that $\overline{D}F$ and $\underline{D}F$ are both finite everywhere. Show that $F(b)-F(a) = ⨎\overline{D}F\times\chi[a,b]$ whenever $a\le b$ in $\mathbb{R}$. (*Hint*: F is AC_* on $\{x : |F(y)-F(x)|\le n|y-x|$ whenever $|y-x|\le 2^{-n}\}$.)

(m) For integers $r\ge 1$, write $\mathcal{C}_r$ for the family of subsets of $\mathbb{R}^r$ of the form $\prod_{i<r}C_i$ where $C_i\subseteq\mathbb{R}$ is a bounded interval for each $i<r$. Set $Q_r=\{(x,C): C\in\mathcal{C}_r,\ x\in\overline{C}\}$; let T_r be the straightforward set of tagged partitions generated by Q_r, Δ_r the set of neighbourhood gauges on $\mathbb{R}^r$, and $\mathfrak{R}_r=\{\mathcal{R}_C : C\in\mathcal{C}_r\}$ where $\mathcal{R}_C=\{\mathbb{R}^r\setminus C' : C\subseteq C'\in\mathcal{C}_r\}\cup\{\emptyset\}$ for $C\in\mathcal{C}_r$. Let ν_r be the restriction of r-dimensional Lebesgue measure to $\mathcal{C}_r$. (i) Show that $(\mathbb{R}^r,T_r,\Delta_r,\mathfrak{R}_r)$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}_r$. (ii) For a function $f:\mathbb{R}^r\to\mathbb{R}$ write $⨎ f(x)dx$ for the gauge integral $I_{\nu_r}(f)$ associated with this structure when it is defined. Show that if r, $s\ge 1$ are integers, $f:\mathbb{R}^{r+s}\to\mathbb{R}$ has compact support and $⨎ f(z)dz$ is defined, then, identifying $\mathbb{R}^{r+s}$ with $\mathbb{R}^r\times\mathbb{R}^s$, $⨎ g(x)dx$ is defined and equal to $⨎ f(x,y)d(x,y)$ whenever $g:\mathbb{R}^r\to\mathbb{R}$ is such that $g(x)=⨎ f(x,y)dy$ for every $x\in\mathbb{R}^r$ for which this is defined.

483Y Further exercises (a) Let us say that a **Lebesgue measurable neighbourhood gauge** on $\mathbb{R}$ is a neighbourhood gauge of the form $\{(x,A): x\in\mathbb{R},\ A\subseteq\,]x-\eta_x,x+\eta_x[\}$ where $x\mapsto\eta_x$ is a Lebesgue measurable function from $\mathbb{R}$ to $]0,\infty[$. Let $\tilde\Delta$ be the set of Lebesgue measurable neighbourhood gauges. Show that the gauge integral defined by the tagged-partition structure $(\mathbb{R},T,\tilde\Delta,\mathfrak{R})$ and μ is the Henstock integral.

(b) Show that if $\Delta_0\subseteq\Delta$ is any set of cardinal at most $\mathfrak{c}$, then the gauge integral defined by $(\mathbb{R},T,\Delta_0,\mathfrak{R})$ and μ does not extend the Lebesgue integral, so is not the Henstock integral.

(c) Let $f:\mathbb{R}\to\mathbb{R}$ be a Henstock integrable function with indefinite Henstock integral F, and ν a totally finite Radon measure on $\mathbb{R}$. Set $G(x)=\nu\,]-\infty,x]$ for $x\in\mathbb{R}$. Show that $f\times G$ is Henstock integrable, with indefinite Henstock integral H, where $H(x)=F(x)G(x)-\int_{]-\infty,x]}F\,d\nu$ for $x\in\mathbb{R}$.

(d) Let ν be any Radon measure on $\mathbb{R}$, I_ν the gauge integral defined from ν and the tagged-partition structure of 481K and this section, and $f:\mathbb{R}\to\mathbb{R}$ a function.

(i) Show that if $I_\nu(f)$ is defined, then f is $\operatorname{dom}\nu$-measurable.

(ii) Show that if $\int f d\nu$ is defined in $\mathbb{R}$, then $I_\nu(f)$ is defined and equal to $\int f d\nu$.

(iii) Show that if $\alpha\in\,]-\infty,\infty]$ then $I_\nu(f\times\chi\,]-\infty,\alpha[)=\lim_{\beta\uparrow\alpha}I_\nu(f\times\chi\,]-\infty,\beta[)$ if either is defined in $\mathbb{R}$.

(iv) Suppose that $I_\nu(f)$ is defined. (α) Let F^{SH} be the Saks-Henstock indefinite integral of f with respect to ν. Show that for any $\epsilon>0$ there are a Radon measure ζ on $\mathbb{R}$ and a $\delta\in\Delta$ such that $\zeta\mathbb{R}\le\delta$ and $|F^{\text{SH}}(W_{\boldsymbol{t}})-S_{\boldsymbol{t}}(f,\nu)|\le\zeta W_{\boldsymbol{t}}$ whenever $\boldsymbol{t}\in T$ is δ-fine. (β) Show that there is a countable cover of $\mathbb{R}$ by compact sets K such that $\int_K|f|d\nu<\infty$.

(e) Let $f:\mathbb{R}\to\mathbb{R}$ be a Henstock integrable function, and $G:\mathbb{R}\to\mathbb{R}$ a function of bounded variation. Show that $f\times G$ is Henstock integrable, and

$$⨎ f\times G\le(\lim_{x\to\infty}|G(x)|+\operatorname{Var}_{\mathbb{R}}G)\sup_{x\in\mathbb{R}}|⨎_{-\infty}^{x}f|.$$

(Compare 224J.)

(f) Let $f:\mathbb{R}\to\mathbb{R}$ be a Henstock integrable function and $g:\mathbb{R}\to\mathbb{R}$ a Lebesgue integrable function; let F and G be their indefinite (Henstock) integrals. Show that $⨎ f\times G+\int g\times F\,d\mu$ is defined and equal to $\lim_{x\to\infty}F(x)G(x)$.

(g) Let $f:\mathbb{R}\to\mathbb{R}$ be a function. Show that f is Lebesgue integrable iff $f\times g$ is Henstock integrable for every bounded continuous function $g:\mathbb{R}\to\mathbb{R}$.

(h) Let U be a linear subspace of $\mathbb{R}^{\mathbb{R}}$ and $\phi : U \to \mathbb{R}$ a linear functional such that (i) $f \in U$ and $\phi f = \int f d\mu$ for every Lebesgue integrable function $f : \mathbb{R} \to \mathbb{R}$ (ii) $f \times \chi C \in U$ whenever $f \in U$ and $C \in \mathcal{C}$ (iii) whenever $f \in \mathbb{R}^{\mathbb{R}}$ and $\mathcal{I}$ is a disjoint family of non-empty open intervals such that $f \times \chi I \in U$ for every $I \in \mathcal{I}$ and $\sum_{I \in \mathcal{I}} \sup_{C \subseteq I, C \in \mathcal{C}} |\phi(f \times \chi C)| < \infty$, then $f \times \chi(\bigcup \mathcal{I}) \in U$ and $\phi(f \times \chi(\bigcup \mathcal{I})) = \sum_{I \in \mathcal{I}} \phi(f \times \chi I)$. Show that if $f : \mathbb{R} \to \mathbb{R}$ is any Henstock integrable function, then $f \in U$ and $\phi(f) = ⨎ f$. (*Hint*: use the argument of part (b) of the proof of 483R.)

(i) (Bongiorno Piazza & Preiss 00) Let $\mathcal{C}$ be the set of non-empty subintervals of a closed interval $[a, b] \subseteq \mathbb{R}$, and T the straightforward set of tagged partitions generated by $[a, b] \times \mathcal{C}$. Let $\Delta_{[a,b]}$ be the set of neighbourhood gauges on $[a, b]$. For $\alpha \ge 0$ set

$$T_\alpha = \{\boldsymbol{t} : \boldsymbol{t} \in T, \ \textstyle\sum_{(x,C) \in \boldsymbol{t}} \rho(x, C) \le \alpha\},$$

writing $\rho(x, C) = \inf_{y \in C} |x - y|$ as usual. Show that T_α is compatible with $\Delta_{[a,b]}$ and $\mathfrak{R} = \{\{\emptyset\}\}$ in the sense of 481F. Show that if I_α is the gauge integral defined from $[a, b]$, T_α, $\Delta_{[a,b]}$, $\mathfrak{R}$ and Lebesgue measure, then I_α extends the ordinary Lebesgue integral and $I_\alpha(F') = F(b) - F(a)$ whenever $F : [a, b] \to \mathbb{R}$ is differentiable relative to its domain.

(j) Let V be a Banach space and $f : \mathbb{R} \to V$ a function. For $\boldsymbol{t} \in T$, set $S_{\boldsymbol{t}}(f, \mu) = \sum_{(x,C) \in \boldsymbol{t}} \mu C \cdot f(x)$. We say that f is **Henstock integrable**, with **Henstock integral** $v = ⨎ f \in V$, if $v = \lim_{\boldsymbol{t} \to \mathcal{F}(T, \Delta, \mathfrak{R})} S_{\boldsymbol{t}}(f, \mu)$.

(i) Show that the set $\mathcal{HL}^1_V$ of Henstock integrable functions from $\mathbb{R}$ to V is a linear subspace of $V^{\mathbb{R}}$ including the space $\mathcal{L}^1_V$ of Bochner integrable functions (253Yf), and that $⨎ : \mathcal{HL}^1_V \to V$ is a linear operator extending the Bochner integral.

(ii) Show that if $f : \mathbb{R} \to V$ is Henstock integrable, so is $f \times \chi C$ for every interval $C \subseteq \mathbb{R}$, and that $(a, b) \mapsto ⨎ f \times \chi\,]a, b[$ is continuous. Set $\|f\|_H = \sup_{C \in \mathcal{C}} \|⨎ f \times \chi C\|$.

(iii) Show that if $\mathcal{I}$ is a disjoint family of open intervals in $\mathbb{R}$, and $f : \mathbb{R} \to V$ is such that $f \times \chi I \in \mathcal{HL}^1_V$ for every $I \in \mathcal{I}$ and $\sum_{I \in \mathcal{I}} \|f \times \chi I\|_H$ is finite, then $⨎ f \times \chi(\bigcup \mathcal{I})$ is defined and equal to $\sum_{I \in \mathcal{I}} ⨎ f \times \chi I$.

(iv) Define $f : \mathbb{R} \to \ell^\infty([0, 1])$ by setting $f(x) = \chi([0, 1] \cap]-\infty, x])$ for $x \in \mathbb{R}$. Show that f is Henstock integrable, but that if $F(x) = ⨎ f \times \chi\,]-\infty, x[$ for $x \in \mathbb{R}$, then $\lim_{y \to x} \frac{1}{y-x}(F(y) - F(x))$ is not defined in $\ell^\infty([0, 1])$ for any $x \in [0, 1]$.

(k) Find a function $f : \mathbb{R}^2 \to \mathbb{R}$, with compact support, such that $⨎ f$ is defined in the sense of 483Xm, but $⨎ fT$ is not defined, where $T(x, y) = \frac{1}{\sqrt{2}}(x + y, x - y)$ for x, $y \in \mathbb{R}$.

(l) Show that for a function $g : \mathbb{R} \to \mathbb{R}$ the following are equiveridical: (i) there is a function $h : \mathbb{R} \to \mathbb{R}$, of bounded variation, such that $g =_{\text{a.e.}} h$ (ii) g is a **multiplier** for the Henstock integral, that is, $f \times g$ is Henstock integrable for every Henstock integrable $f : \mathbb{R} \to \mathbb{R}$.

483 Notes and comments I hope that the brief account here (largely taken from Gordon 94) will give an idea of the extraordinary power of gauge integrals. While what I am calling the 'Henstock integral', regarded as a linear functional on a space of real functions, was constructed long ago by Perron and Denjoy, the gauge integral approach makes it far more accessible, and gives clear pathways to corresponding Stieltjes and vector integrals (483Yd, 483Yj).

Starting from our position in the fourth volume of a book on measure theory, it is natural to try to describe the Henstock integral in terms of the Lebesgue integral, as in 483C (they agree on non-negative functions) and 483Yh (offering an extension process to generate the Henstock integral from the Lebesgue integral); on the way, we see that Henstock integrable functions are necessarily Lebesgue integrable over many intervals (483Xf). Alternatively, we can set out to understand indefinite Henstock integrals and their derivatives, just as Lebesgue integrable functions can be characterized as almost everywhere equal to derivatives of absolutely continuous functions (222E, 225E), because if f is Henstock integrable then it is equal almost everywhere to the derivative of its indefinite integral (483I). Any differentiable function (indeed, any continuous function differentiable except on a countable set) is an indefinite Henstock integral (483Xh). Recall that the Cantor function (134H) is continuous and differentiable almost everywhere but is not an indefinite integral, so we have to look for a characterization which can exclude such cases. For this we have to work quite hard, but we find that 'ACG$_*$ functions' are the appropriate class (483R).

Gauge integrals are good at integrating derivatives (see 483Xh), but bad at integrating over subspaces. Even to show that $f\times\chi\,[0,\infty[$ is Henstock integrable whenever f is (483Bc) involves us in some unexpected manoeuvres. I give an argument which is designed to show off the general theory of §482, and I recommend you to look for short cuts (483Xb), but any method must depend on careful examination of the exact classes $\mathcal{C}$ and $\mathfrak{R}$ chosen for the definition of the integral. We do however have a new kind of convergence theorem in 482H and 483N.

One of the incidental strengths of the Henstock integral is that it includes the improper Riemann integral (483Bd, 483Xc); so that, for instance, Carleson's theorem (286U) can be written in the form

$$\hat{f}(y)=\frac{1}{\sqrt{2\pi}}\, ⨎ e^{-ixy}f(x)dx \text{ for almost every } y \text{ if } f:\mathbb{R}\to\mathbb{C} \text{ is square-integrable.}$$

But to represent the many expressions of the type $\lim_{a\to\infty}\int_{-a}^{a}f$ in §283 (e.g., 283F, 283I and 283L) directly in the form $I_\mu(f)$ we need to change $\mathfrak{R}$, as in 481L or 481Xc.

Version of 21.1.10

484 The Pfeffer integral

I give brief notes on what seems at present to be the most interesting of the multi-dimensional versions of the Henstock integral, leading to Pfeffer's Divergence Theorem (484N).

484A Notation This section will depend heavily on Chapter 47, and will use much of the same notation. $r\geq 2$ will be a fixed integer, and μ will be Lebesgue measure on $\mathbb{R}^r$, while μ_{r-1} is Lebesgue measure on $\mathbb{R}^{r-1}$. As in §§473-475, let ν be 'normalized' $(r-1)$-dimensional Hausdorff measure on $\mathbb{R}^r$, as described in §265; that is, $\nu=2^{-r+1}\beta_{r-1}\mu_{H,r-1}$, where $\mu_{H,r-1}$ is $(r-1)$-dimensional Hausdorff measure on $\mathbb{R}^r$ as described in §264, and

$$\beta_{r-1}=\frac{2^{2k}k!\pi^{k-1}}{(2k)!} \text{ if } r=2k \text{ is even,}$$
$$=\frac{\pi^k}{k!} \text{ if } r=2k+1 \text{ is odd}$$

is the Lebesgue measure of a ball of radius 1 in $\mathbb{R}^{r-1}$ (264I). For this section only, let us say that a subset of $\mathbb{R}^r$ is **thin** if it is of the form $\bigcup_{n\in\mathbb{N}}A_n$ where ν^*A_n is finite for every n. Note that every thin set is μ-negligible (471L). For $A\subseteq\mathbb{R}^r$, write ∂A for its ordinary topological boundary. If $x\in\mathbb{R}^r$ and $\epsilon>0$, $B(x,\epsilon)$ will be the closed ball $\{y:\|y-x\|\leq\epsilon\}$.

I will use the term **dyadic cube** for sets of the form $\prod_{i<r}[2^{-m}k_i,2^{-m}(k_i+1)[$ where m, $k_0,\dots,k_{r-1}\in\mathbb{Z}$; write $\mathcal{D}$ for the set of dyadic cubes in $\mathbb{R}^r$. Note that if D, $D'\in\mathcal{D}$, either $D\subseteq D'$ or $D'\subseteq D$ or $D\cap D'=\emptyset$; so if $\mathcal{D}_0\subseteq\mathcal{D}$, the maximal members of $\mathcal{D}_0$ are disjoint.

It will be helpful to have an abbreviation for the following expression: set

$$\alpha^*=\min(\frac{1}{r^{r/2}},\frac{2^{r-2}}{r\beta_r^{(r-1)/r}}).$$

(As will become apparent, the actual value of this constant is of no importance; but the strict logic of the arguments below depends on α^* being small enough.)

As in §475, I write int^*A, cl^*A and ∂^*A for the essential interior, essential closure and essential boundary of a set $A\subseteq\mathbb{R}^r$ (475B). Recall that a set $A\subseteq\mathbb{R}^r$ has finite perimeter in the sense of 474D iff $\nu(\partial^*A)$ is finite, and then

$$\nu(\partial^*A)=\lambda_A^\partial(\mathbb{R}^r)=\operatorname{per}A$$

is the perimeter of A (475M); we shall also need to remember that A is necessarily Lebesgue measurable.

$\mathcal{C}$ will be the family of subsets of $\mathbb{R}^r$ with locally finite perimeter, and $\mathcal{V}$ the family of bounded sets in $\mathcal{C}$, that is, the family of bounded sets with finite perimeter. Note that $\mathcal{C}$ is an algebra of subsets of $\mathbb{R}^r$ (475Ma), and that $\mathcal{V}$ is an ideal in $\mathcal{C}$.

484B Theorem (Tamanini & Giacomelli 89) Let $E \subseteq \mathbb{R}^r$ be a Lebesgue measurable set of finite measure and perimeter, and $\epsilon > 0$. Then there is a Lebesgue measurable set $G \subseteq E$ such that $\operatorname{per} G \le \operatorname{per} E$, $\mu(E \setminus G) \le \epsilon$ and $\mathrm{cl}^*G = \overline{G}$.

proof (Pfeffer 91b) **(a)** Set $\alpha = \frac{1}{\epsilon}\operatorname{per} E$. For measurable sets $G \subseteq E$ set $q(G) = \operatorname{per} G - \alpha\mu G$. Then there is a self-supporting measurable set $G \subseteq E$ such that $q(G) \le q(G')$ whenever $G' \subseteq E$ is measurable.

P Write Σ for the family of Lebesgue measurable subsets of $\mathbb{R}^r$; give Σ the topology of convergence in measure defined by the pseudometrics $\rho_H(G, G') = \mu((G \triangle G') \cap H)$ for measurable sets H of finite measure (cf. 474T). Extend q to Σ by setting $q(G) = \operatorname{per}(E \cap G) - \alpha\mu(E \cap G)$ for every $G \in \Sigma$. Because $\operatorname{per} : \Sigma \to [0, \infty]$ is lower semi-continuous for the topology of convergence in measure (474Ta), and $G \mapsto E \cap G$, $G \mapsto \mu(E \cap G)$ are continuous, $q : \Sigma \to [0, \infty[$ is lower semi-continuous (4A2Bd). Next, $\mathcal{K} = \{G : G \in \Sigma, \operatorname{per} G \le \operatorname{per} E\}$ is compact (474Tb), while $\mathcal{L} = \{G : \mu(G \setminus E) = 0\}$ is closed, so there is a $G_0 \in \mathcal{L} \cap \mathcal{K}$ such that $q(G_0) = \inf_{G \in \mathcal{L} \cap \mathcal{K}} q(G)$ (4A2Gl). Since $G_0 \in \mathcal{L}$, $\operatorname{per}(G_0 \cap E) = \operatorname{per}(G_0)$ and $\mu(G_0 \cap E) = \mu G_0$, so we may suppose that $G_0 \subseteq E$. Moreover, there is a self-supporting set $G \subseteq G_0$ such that $G_0 \setminus G$ is negligible (414F), and we still have $q(G) = q(G_0)$. Of course $q(G) \le q(E)$, just because $E \in \mathcal{L} \cap \mathcal{K}$.

? If there is a measurable set $G' \subseteq E$ such that $q(G') < q(G)$, then

$$\operatorname{per} G' = q(G') + \alpha\mu G' \le q(E) + \alpha\mu E = \operatorname{per} E,$$

so $G' \in \mathcal{K}$; but this means that $G' \in \mathcal{L} \cap \mathcal{K}$ and $q(G) = q(G_0) \le q(G')$. **X** So G has the required properties. **Q**

(b) Since $q(G) \le q(E)$, we must have

$$\operatorname{per} G + \alpha\mu(E \setminus G) = q(G) + \alpha\mu E \le q(E) + \alpha\mu E = \operatorname{per} E = \alpha\epsilon.$$

So $\mu(E \setminus G) \le \epsilon$.

(c) Next, $\overline{G} \subseteq \mathrm{cl}^*G$. **P** Let $x \in \overline{G}$. For every $t > 0$, set $U_t = \{y : \|y - x\| < t\}$; then

$$\begin{aligned}\operatorname{per}(G \cap U_t) + \operatorname{per}(G \setminus U_t) &= \nu(\partial^*(G \cap U_t)) + \nu(\partial^*(G \setminus U_t))\\ &\le \nu(\partial^*G \cap U_t) + \nu(\mathrm{cl}^*G \cap \partial U_t)\\ &\qquad + \nu(\partial^*G \setminus U_t) + \nu(\mathrm{cl}^*G \cap \partial U_t)\end{aligned}$$

(475Cf, because $\partial(\mathbb{R}^r \setminus U_t) = \partial U_t$)

$$= \nu(\partial^*G) + 2\nu(\mathrm{cl}^*G \cap \partial U_t) = \operatorname{per} G + 2\nu(G \cap \partial U_t)$$

for almost every $t > 0$, because

$$\int_0^\infty \nu((G \triangle \mathrm{cl}^*G) \cap \partial U_t)dt = \mu(G \triangle \mathrm{cl}^*G) = 0$$

(265G). So, for almost every t,

$$\mu(G \cap U_t)^{(r-1)/r} \le \operatorname{per}(G \cap U_t)$$

(474La)

$$\begin{aligned}&\le \operatorname{per} G + 2\nu(G \cap \partial U_t) - \operatorname{per}(G \setminus U_t)\\ &= q(G) + \alpha\mu(G \cap U_t) + 2\nu(G \cap \partial U_t) - q(G \setminus U_t)\\ &\le \alpha\mu(G \cap U_t) + 2\nu(G \cap \partial U_t)\end{aligned}$$

because $q(G)$ is minimal.

For $t > 0$, set

$$g(t) = \mu(G \cap U_t) = \int_0^t \nu(G \cap \partial U_s)ds,$$

so that

$$g'(t) = \nu(G \cap \partial U_t) \ge \frac{1}{2}(g(t)^{(r-1)/r} - \alpha g(t))$$

for almost every t. Because G is self-supporting and U_t is open and $G \cap U_t \ne \emptyset$, $g(t) > 0$ for every $t > 0$; and $\lim_{t\downarrow 0} g(t) = 0$.

Set

$$h(t) = \frac{d}{dt}g(t)^{1/r} = \frac{g'(t)}{rg(t)^{(r-1)/r}} \ge \frac{1}{2r}(1 - \alpha g(t)^{1/r})$$

for almost every t. Then

$$\limsup_{t\downarrow 0} \frac{g(t)^{1/r}}{t} = \limsup_{t\downarrow 0} \frac{1}{t}\int_0^t h \ge \frac{1}{2r},$$

and

$$\limsup_{t\downarrow 0} \frac{\mu(G\cap B(x,t))}{\mu B(x,t))} = \limsup_{t\downarrow 0} \frac{g(t)}{\beta_r t^r} > 0.$$

Thus $x \in \mathrm{cl}^*G$. As x is arbitrary, $\overline{G} \subseteq \mathrm{cl}^*G$. **Q**

Since we certainly have $\mathrm{cl}^*G \subseteq \overline{G}$, this G serves.

484C Lemma Let $E \in \mathcal{V}$ and $l \in \mathbb{N}$ be such that $\max(\mathrm{per}\,E, \mathrm{diam}\,E) \le l$. Then E is expressible as $\bigcup_{i<n} E_i$ where $\langle E_i\rangle_{i<n}$ is disjoint, $\mathrm{per}\,E_i \le 1$ for each $i < n$ and n is at most $2^r(l+1)^r(4r(2l^2+1))^{r/(r-1)} + 2^{r+1}l^2$.

proof For $D \in \mathcal{D}$, write $\mathcal{D}_D$ for $\{D' : D' \in \mathcal{D},\ D' \subseteq D,\ \mathrm{diam}\,D' = \frac{1}{2}\mathrm{diam}\,D\}$, the family of the 2^r dyadic subcubes of D at the next level down.

(a) If $l \le 1$ the result is trivial, so let us suppose that $l \ge 2$. Let $m \in \mathbb{N}$ be minimal subject to $4r(2l^2+1) \le 2^{m(r-1)}$, so that $2^m \le 2(4r(2l^2+1))^{1/(r-1)}$. Then we can cover E by a family $\mathcal{L}_0$ of dyadic cubes of side 2^{-m} with

$$\#(\mathcal{L}_0) \le (2^m l + 1)^r \le 2^{mr}(l+1)^r \le 2^r(l+1)^r(4r(2l^2+1))^{r/(r-1)}.$$

(b) Let $\mathcal{L}_1$ be the set of those $D \in \mathcal{D}$ such that $\lfloor 2\nu(D' \cap \partial^*E)\rfloor < \lfloor 2\nu(D \cap \partial^*E)\rfloor$ for every $D' \in \mathcal{D}_D$. Then $\#(\mathcal{L}_1) \le 2l^2$. **P** For $k \ge 1$, set

$$\mathcal{L}_1^{(k)} = \{F : D \in \mathcal{L}_1,\ \lfloor 2\nu(D \cap \partial^*E)\rfloor = k\}.$$

If D, $D' \in \mathcal{L}_1^{(k)}$ are distinct, neither can be included in the other, so they are disjoint. Accordingly $k\#(\mathcal{L}_1^{(k)}) \le 2\nu(\partial^*E) \le 2l$ and $\#(\mathcal{L}_1^{(k)}) \le 2l$. Since $\mathcal{L}_1 = \bigcup_{1\le k\le l}\mathcal{L}_1^{(k)}$, $\#(\mathcal{L}_1) \le 2l^2$. **Q**

(c) For $D \in \mathcal{D}$, set $\tilde{D} = D \setminus \bigcup\{D' : D' \in \mathcal{L}_1,\ D' \subseteq D\}$. Then $\nu(\tilde{D} \cap \partial^*E) \le \frac{1}{2}$. **P?** Otherwise, set $j = \lfloor 2\nu(D \cap \partial^*E)\rfloor \ge 1$, and choose $\langle D_i\rangle_{i\in\mathbb{N}}$ inductively, as follows. $D_0 = D$. Given that $D_i \in \mathcal{D}$, $D_i \subseteq D$ and $\lfloor 2\nu(D_i \cap \partial^*E)\rfloor = j$, $\nu((D \setminus D_i) \cap \partial^*E) < \frac{1}{2}$ and $D_i \cap \tilde{D}$ is non-empty, so $D_i \notin \mathcal{L}_1$ and there must be a $D_{i+1} \in \mathcal{D}_{D_i}$ such that $\lfloor 2\nu(D_{i+1} \cap \partial^*E)\rfloor = j$. Continue. This gives us a strictly decreasing sequence $\langle D_i\rangle_{i\in\mathbb{N}}$ in $\mathcal{D}$ such that $\nu(D_i \cap \partial^*E) \ge \frac{j}{2}$ for every i. But (because $\mathrm{per}\,E$ is finite) this means that, writing x for the unique member of $\bigcap_{i\in\mathbb{N}}\overline{D}_i$, $\nu\{x\} \ge \frac{j}{2}$, which is absurd. **XQ**

(d) Set

$$\mathcal{L}_1' = \{D : D \in \mathcal{L}_1 \text{ is included in some member of } \mathcal{L}_0\},$$

$$\mathcal{L}_2 = \mathcal{L}_0 \cup \bigcup\{\mathcal{D}_D : D \in \mathcal{L}_1'\}, \quad \mathcal{K} = \{\tilde{D} : D \in \mathcal{L}_2\}.$$

(i)

$$\begin{aligned}\#(\mathcal{K}) \le \#(\mathcal{L}_2) &\le \#(\mathcal{L}_0) + 2^r\#(\mathcal{L}_1)\\ &\le 2^r(l+1)^r(4r(2l^2+1))^{r/(r-1)} + 2^{r+1}l^2.\end{aligned}$$

(ii) $\bigcup\mathcal{K} \supseteq E$. **P** If $x \in E$, there is a smallest member D of $\mathcal{L}_2$ containing it, because certainly $x \in \bigcup\mathcal{L}_0$. But now x cannot belong to any member of $\mathcal{L}_1$ included in D, so $x \in \tilde{D}$. **Q**

(iii) $\mathcal{K}$ is disjoint. **P** If D_1, $D_2 \in \mathcal{L}_2$ are disjoint, then of course $\tilde{D}_1 \cap \tilde{D}_2 = \emptyset$. If $D_1 \subset D_2$, then $D_1 \notin \mathcal{L}_0$, so there is a $D \in \mathcal{L}'_1$ such that $D_1 \in \mathcal{D}_D$; in this case $D \subseteq D_2$ so $\tilde{D}_2 \subseteq D_2 \setminus D$ is disjoint from D_1. **Q**

(iv) $\operatorname{per}(D \cap E) \le 1$ for every $D \in \mathcal{K}$. **P** Take $D_0 \in \mathcal{L}_2$ such that $D = \tilde{D}_0$; then

$$\nu(\partial D) \le \nu(\partial D_0) + \sum_{D' \in \mathcal{L}'_1} \nu(\partial D') \le 2r(2l^2+1)2^{-m(r-1)} \le \frac{1}{2}$$

by the choice of m. So

$$\operatorname{per}(D \cap E) \le \nu(\partial D) + \nu(D \cap \partial^* E) \le \frac{1}{2} + \frac{1}{2} = 1$$

by 475Cf. **Q**

(e) So if we take $\langle E_i \rangle_{i<n}$ to be an enumeration of $\{E \cap D : D \in \mathcal{K}\}$, we shall have the required result.

484D Definitions The gauge integrals of this section will be based on the following residual families. Let H be the family of strictly positive sequences $\eta = \langle \eta(i) \rangle_{i\in\mathbb{N}}$ in $\mathbb{R}$. For $\eta \in \mathrm{H}$, write $\mathcal{M}_\eta$ for the set of disjoint sequences $\langle E_i \rangle_{i\in\mathbb{N}}$ of measurable subsets of $\mathbb{R}^r$ such that $\mu E_i \le \eta(i)$ and $\operatorname{per} E_i \le 1$ for every $i \in \mathbb{N}$, and E_i is empty for all but finitely many i. For $\eta \in \mathrm{H}$ and $V \in \mathcal{V}$ set

$$\mathcal{R}_\eta = \{\bigcup_{i\in\mathbb{N}} E_i : \langle E_i \rangle_{i\in\mathbb{N}} \in \mathcal{M}_\eta\} \subseteq \mathcal{C}, \quad \mathcal{R}_\eta^{(V)} = \{R : R \subseteq \mathbb{R}^r,\ R \cap V \in \mathcal{R}_\eta\};$$

finally, set $\mathfrak{R} = \{\mathcal{R}_\eta^{(V)} : V \in \mathcal{V},\ \eta \in \mathrm{H}\}$.

484E Lemma (a)(i) For every $\mathcal{R} \in \mathfrak{R}$, there is an $\eta \in \mathrm{H}$ such that $\mathcal{R}_\eta \subseteq \mathcal{R}$.

(ii) If $\mathcal{R} \in \mathfrak{R}$ and $C \in \mathcal{C}$, there is an $\mathcal{R}' \in \mathfrak{R}$ such that $C \cap R \in \mathcal{R}$ whenever $R \in \mathcal{R}'$.

(b)(i) If $\eta \in \mathrm{H}$ and $\gamma \ge 0$, there is an $\epsilon > 0$ such that $R \in \mathcal{R}_\eta$ whenever $\mu R \le \epsilon$, $\operatorname{diam} R \le \gamma$ and $\operatorname{per} R \le \gamma$.

(ii) If $\mathcal{R} \in \mathfrak{R}$ and $\gamma \ge 0$, there is an $\epsilon > 0$ such that $R \in \mathcal{R}$ whenever $\mu R \le \epsilon$ and $\operatorname{per} R \le \gamma$.

(c) If $\mathcal{R} \in \mathfrak{R}$ there is an $\mathcal{R}' \in \mathfrak{R}$ such that $R \cup R' \in \mathcal{R}$ whenever R, $R' \in \mathcal{R}'$ and $R \cap R' = \emptyset$.

(d)(i) If $\eta \in \mathrm{H}$ and $A \subseteq \mathbb{R}^r$ is a thin set, then there is a set $\mathcal{D}_0 \subseteq \mathcal{D}$ such that every point of A belongs to the interior of $\bigcup\mathcal{D}_1$ for some finite $\mathcal{D}_1 \subseteq \mathcal{D}_0$, and $\bigcup\mathcal{D}_1 \in \mathcal{R}_\eta$ for every finite set $\mathcal{D}_1 \subseteq \mathcal{D}_0$.

(ii) If $\mathcal{R} \in \mathfrak{R}$ and $A \subseteq \mathbb{R}^r$ is a thin set, then there is a set $\mathcal{D}_0 \subseteq \mathcal{D}$ such that every point of A belongs to the interior of $\bigcup\mathcal{D}_1$ for some finite set $\mathcal{D}_1 \subseteq \mathcal{D}_0$, and $\bigcup\mathcal{D}_1 \in \mathcal{R}$ for every finite set $\mathcal{D}_1 \subseteq \mathcal{D}_0$.

proof (a)(i) Express $\mathcal{R}$ as $\mathcal{R}_{\eta'}^{(V)}$ where $\eta' \in \mathrm{H}$ and $V \in \mathcal{V}$. Let $l \in \mathbb{N}$ be such that $\max(\operatorname{diam} V, 1 + \operatorname{per} V) \le l$, and take $n \ge 2^r(l+1)^r(4r(2l^2+1))^{r/(r-1)} + 2^{r+1}l^2$. Set $\eta(i) = \min\{\eta'(j) : ni \le j < n(i+1)\}$ for every $i \in \mathbb{N}$, so that $\eta \in \mathrm{H}$. If $R \in \mathcal{R}_\eta$, express it as $\bigcup_{i\in\mathbb{N}} E_i$ where $\langle E_i \rangle_{i\in\mathbb{N}} \in \mathcal{M}_\eta$. For each i, $\max(\operatorname{diam}(E_i \cap V), \operatorname{per}(E_i \cap V)) \le l$, so by 484C we can express $E_i \cap V$ as $\bigcup_{ni\le j<n(i+1)} E'_j$, where $\langle E'_j \rangle_{ni\le j<n(i+1)}$ is disjoint and $\operatorname{per} E'_j \le 1$ for each j. Now

$$\mu E'_j \le \mu E_i \le \eta(i) \le \eta'(j)$$

for $ni \le j < n(i+1)$. Also $\{j : E'_j \ne \emptyset\}$ is finite because $\{j : E_j = \emptyset\}$ is finite. So $\langle E'_j \rangle_{j\in\mathbb{N}} \in \mathcal{M}_{\eta'}$ and $R \cap V = \bigcup_{j\in\mathbb{N}} E'_j$ belongs to $\mathcal{R}_{\eta'}$, that is, $R \in \mathcal{R}$. As R is arbitrary, $\mathcal{R}_\eta \subseteq \mathcal{R}$.

(ii) Express $\mathcal{R}$ as $\mathcal{R}_\eta^{(V)}$, where $V \in \mathcal{V}$ and $\eta \in \mathrm{H}$. By (i), there is an $\eta' \in \mathrm{H}$ such that $\mathcal{R}_{\eta'} \subseteq \mathcal{R}_\eta^{(C\cap V)}$. Set $\mathcal{R}' = \mathcal{R}_{\eta'}^{(V)} \in \mathfrak{R}$. If $R \in \mathcal{R}'$, then $R \cap V \in \mathcal{R}_{\eta'}$, so $C \cap R \cap V \in \mathcal{R}_\eta$ and $C \cap R \in \mathcal{R}$.

(b)(i) Take $l \ge \gamma$ and $n \ge 2^r(l+1)^r(4r(2l^2+1))^{r/(r-1)} + 2^{r+1}l^2$, and set $\epsilon = \min_{i<n} \eta(i)$. If $\mu R \le \epsilon$, $\operatorname{diam} R \le l$ and $\operatorname{per} R \le l$, then R is expressible as $\bigcup_{i<n} E_i$ where $\langle E_i \rangle_{i<n}$ is disjoint and $\operatorname{per} E_i \le 1$ for each $i < n$. Since $\mu E_i \le \mu R \le \eta(i)$ for each i, $R \in \mathcal{R}_\eta$.

(ii) Express $\mathcal{R}$ as $\mathcal{R}_\eta^{(V)}$. By (i), there is an $\epsilon > 0$ such that $R \in \mathcal{R}_\eta$ whenever $\operatorname{diam} R \le \operatorname{diam} V$, $\operatorname{per} R \le \gamma + \operatorname{per} V$ and $\mu R \le \epsilon$; and this ϵ serves.

(c) Express $\mathcal{R}$ as $\mathcal{R}^{(V)}_\eta$. Set $\eta'(i)=\min(\eta(2i),\eta(2i+1))$ for every i; then $\eta'\in\mathrm{H}$ and if $\langle E_i\rangle_{i\in\mathbb{N}}$, $\langle E'_i\rangle_{i\in\mathbb{N}}$ belong to $\mathcal{M}_{\eta'}$ and have disjoint unions, then $(E_0,E'_0,E_1,E'_1,\dots)\in\mathcal{M}_\eta$; this is enough to show that $R\cup R'\in\mathcal{R}_\eta$ whenever R, $R'\in\mathcal{R}_{\eta'}$ are disjoint, so that $R\cup R'\in\mathcal{R}$ whenever R, $R'\in\mathcal{R}^{(V)}_{\eta'}$ are disjoint.

(d)(i)($\boldsymbol{\alpha}$) We can express A as $\bigcup_{i\in\mathbb{N}}A_i$ where $\mu^*_{H,r-1}A_i<2^{-r}/2r$ for every i. **P** Because A is thin, it is the union of a sequence of sets of finite outer measure for ν, and therefore for $\mu_{H,r-1}$. On each of these the subspace measure is atomless (471E, 471Dg), so that the set can be dissected into finitely many sets of measure less than $2^{-r}/2r$ (215D). **Q**

($\boldsymbol{\beta}$) For each $i\in\mathbb{N}$, we can cover A_i by a sequence $\langle A_{ij}\rangle_{j\in\mathbb{N}}$ of sets such that $\operatorname{diam}A_{ij}<\eta(i)$ for every j and $\sum_{j=0}^\infty(\operatorname{diam}A_{ij})^{r-1}<2^{-r}/2r$; enlarging the A_{ij} slightly if need be, we can suppose that they are all open. Now we can cover each A_{ij} by 2^r cubes $D_{ijk}\in\mathcal{D}$ in such a way that the side length of each D_{ijk} is at most the diameter of A_{ij}.

Setting $\mathcal{D}_0=\{D_{ijk}:i,\,j\in\mathbb{N},\,k<2^r\}$, we see that every point of A belongs to an open set A_{ij} which is covered by a finite subset of $\mathcal{D}_0$. If $\mathcal{D}_1\subseteq\mathcal{D}_0$ is finite, let $\mathcal{D}'_1$ be the family of maximal elements of $\mathcal{D}_1$, so that $\mathcal{D}'_1$ is disjoint and $\bigcup\mathcal{D}'_1=\bigcup\mathcal{D}_1$. Express $\mathcal{D}'_1$ as $\{D_{ijk}:(i,j,k)\in I\}$ where $I\subseteq\mathbb{N}\times\mathbb{N}\times 2^r$ is finite and $\langle D_{ijk}\rangle_{(i,j,k)\in I}$ is disjoint. Set $I_i=\{(j,k):(i,j,k)\in I\}$ and $E_i=\bigcup_{(j,k)\in I_i}D_{ijk}$ for $i\in\mathbb{N}$. Then $\langle E_i\rangle_{i\in\mathbb{N}}$ is disjoint, $E_i=\emptyset$ for all but finitely many i, and for each $i\in\mathbb{N}$

$$\begin{aligned}\mu E_i&\le\sum_{j=0}^\infty\sum_{k=0}^{2^r-1}\mu D_{ijk}\le\sum_{j=0}^\infty 2^r(\operatorname{diam}A_{ij})^r\\&\le 2^r\sum_{j=0}^\infty\eta(i)(\operatorname{diam}A_{ij})^{r-1}\le\eta(i),\\ \nu(\partial E_i)&\le\sum_{(j,k)\in I_i}\nu(\partial D_{ijk})\le\sum_{j=0}^\infty\sum_{k=0}^{2^r-1}\nu(\partial D_{ijk})\\&\le\sum_{j=0}^\infty 2^r\cdot 2r(\operatorname{diam}A_{ij})^{r-1}\le 1.\end{aligned}$$

So $\bigcup\mathcal{D}_1=\bigcup_{i\in\mathbb{N}}E_i$ belongs to $\mathcal{R}_\eta$.

(ii) By (a-i), there is an $\eta\in\mathrm{H}$ such that $\mathcal{R}_\eta\subseteq\mathcal{R}$. By (i) here, there is a $\mathcal{D}_0\subseteq\mathcal{D}$ such that every point of A belongs to the interior of $\bigcup\mathcal{D}_1$ for some finite $\mathcal{D}_1\subseteq\mathcal{D}_0$, and $\bigcup\mathcal{D}_1\in\mathcal{R}_\eta\subseteq\mathcal{R}$ for every finite set $\mathcal{D}_1\subseteq\mathcal{D}_0$.

484F A family of tagged-partition structures For $\alpha>0$, let $\mathcal{C}_\alpha$ be the family of those $C\in\mathcal{V}$ such that $\mu C\ge\alpha(\operatorname{diam}C)^r$ and $\alpha\operatorname{per}C\le(\operatorname{diam}C)^{r-1}$, and let T_α be the straightforward set of tagged partitions generated by the set

$$\{(x,C):C\in\mathcal{C}_\alpha,\ x\in\operatorname{cl}^*C\}.$$

Let Θ be the set of functions $\theta:\mathbb{R}^r\to[0,\infty[$ such that $\{x:\theta(x)=0\}$ is thin (definition: 484A), and set $\Delta=\{\delta_\theta:\theta\in\Theta\}$, where $\delta_\theta=\{(x,A):x\in\mathbb{R}^r,\ \theta(x)>0,\ \|y-x\|<\theta(x)\text{ for every }y\in A\}$.

Then whenever $0<\alpha<\alpha^*$, $(\mathbb{R}^r,T_\alpha,\Delta,\mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$.

proof (a) We had better look again at all the conditions in 481G.

(i) and (vi) really are trivial. (iv) and (v) are true because $\mathcal{C}$ is actually an algebra of sets and $\emptyset\in\mathcal{R}$ for every $\mathcal{R}\in\mathfrak{R}$.

For (ii), we have to observe that the union of two thin sets is thin, so that $\theta\wedge\theta'\in\Theta$ for all θ, $\theta'\in\Theta$; since $\delta_\theta\cap\delta_{\theta'}=\delta_{\theta\wedge\theta'}$, this is all we need.

(iii)(α) follows from 484Ea: given V, $V'\in\mathcal{V}$ and η, $\eta'\in\mathrm{H}$, take $\tilde\eta$, $\tilde\eta'\in\mathrm{H}$ such that $\mathcal{R}_{\tilde\eta}\subseteq\mathcal{R}^{(V)}_\eta$ and $\mathcal{R}_{\tilde\eta'}\subseteq\mathcal{R}^{(V')}_{\eta'}$. Then $V\cup V'\in\mathcal{V}$ and $\mathcal{R}^{(V)}_\eta\cap\mathcal{R}^{(V')}_{\eta'}\supseteq\mathcal{R}^{(V\cup V')}_{\tilde\eta\wedge\tilde\eta'}$. ($\beta$) is just 484Ec.

(b) Now let us turn to 481G(vii).

(i) Fix $C \in \mathcal{C}$, $\delta \in \Delta$ and $\mathcal{R} \in \mathfrak{R}$. Express δ as δ_θ where $\theta \in \Theta$. Let $\mathcal{R}' \in \mathfrak{R}$ be such that $A \cup A' \in \mathcal{R}$ whenever A, $A' \in \mathcal{R}'$ are disjoint. Express $\mathcal{R}'$ as $\mathcal{R}_\eta^{(V)}$ where $V \in \mathcal{V}$ and $\eta \in \mathrm{H}$. By 484E(b-ii), there is an $\epsilon > 0$ such that $R \in \mathcal{R}'$ whenever $\operatorname{per} R \le 2\operatorname{per}(C \cap V)$ and $\mu R \le \epsilon$.

By 484B, there is an $E \subseteq C \cap V$ such that $\operatorname{per} E \le \operatorname{per}(C \cap V)$, $\mu((C \cap V) \setminus E) \le \epsilon$ and $\operatorname{cl}^* E = \overline{E}$. In this case,

$$\operatorname{per}(C \cap V \setminus E) \le \operatorname{per}(C \cap V) + \operatorname{per} E \le 2\operatorname{per}(C \cap V),$$

so $C \cap V \setminus E \in \mathcal{R}'$, by the choice of ϵ. By 484E(a-ii), there is an $\mathcal{R}'' \in \mathfrak{R}$ such that $E \cap R \in \mathcal{R}'$ whenever $R \in \mathcal{R}''$.

Now consider

$$A = \{x : \theta(x) = 0\} \cup \partial^* E \cup \{x : \limsup_{\zeta \downarrow 0} \sup_{x \in G, 0 < \operatorname{diam} G \le \zeta} \tfrac{\nu^*(G \cap \partial^* E)}{(\operatorname{diam} G)^{r-1}} > 0\}.$$

Because $\operatorname{per} E < \infty$ and

$$\{x : x \in \mathbb{R}^r \setminus \partial^* E, \limsup_{\zeta \downarrow 0} \sup_{x \in G, 0 < \operatorname{diam} G \le \zeta} \tfrac{\nu^*(G \cap \partial^* E)}{(\operatorname{diam} G)^{r-1}} > 0\}$$

is ν-negligible (471Pc), A is thin. By 484E(d-ii), there is a set $\mathcal{D}_0 \subseteq \mathcal{D}$ such that

$$A \subseteq \bigcup\{\operatorname{int}(\bigcup \mathcal{D}_1) : \mathcal{D}_1 \in [\mathcal{D}_0]^{<\omega}\}, \quad \bigcup \mathcal{D}_1 \in \mathcal{R}'' \text{ for every } \mathcal{D}_1 \in [\mathcal{D}_0]^{<\omega}.$$

(ii) Write T' for the set of those δ-fine $\boldsymbol{t} \in T_\alpha$ such that every member of $\boldsymbol{t}$ is of the form $(x, D \cap E)$ for some $D \in \mathcal{D}$. If $x \in \operatorname{int}^* E \setminus A$, there is an $h(x) > 0$ such that $h(x) < \theta(x)$ and $\{(x, D \cap E)\} \in T'$ whenever $D \in \mathcal{D}$, $x \in \overline{D}$ and $\operatorname{diam} D \le h(x)$. **P** Let $\epsilon_1 > 0$ be such that $r^{r/2}\epsilon_1 \le \frac{1}{2}$ and

$$\alpha \le \tfrac{1 - r^{r/2}\epsilon_1}{r^{r/2}}, \quad \frac{1}{\alpha} \ge (2r + r^{r/2}\epsilon_1) \cdot \frac{1}{2^{r-1}} \big(\tfrac{\beta_r}{1 - r^{r/2}\epsilon_1}\big)^{(r-1)/r};$$

this is where we need to know that $\alpha < \alpha^*$. Because $x \notin A$, $\theta(x) > 0$; let $h(x) \in \,]0, \theta(x)[$ be such that (α) $\nu^*(D \cap \partial^* E) \le \epsilon_1 (\operatorname{diam} D)^{r-1}$ whenever $x \in \overline{D}$ and $\operatorname{diam} D \le h(x)$ (β) $\mu(B(x,t) \setminus E) \le \epsilon_1 t^r$ whenever $0 \le t \le h(x)$. Now suppose that $D \in \mathcal{D}$ and $x \in \overline{D}$ and $\operatorname{diam} D \le h(x)$. Then, writing γ for the side length of D,

$$\begin{aligned}\mu(D \cap E) &\ge \mu D - \mu(B(x, \operatorname{diam} D) \setminus E) \ge \gamma^r - \epsilon_1 (\operatorname{diam} D)^r \\ &= \gamma^r (1 - r^{r/2}\epsilon_1) \ge \alpha \gamma^r r^{r/2} = \alpha (\operatorname{diam} D)^r \ge \alpha \operatorname{diam}(D \cap E)^r.\end{aligned}$$

Using 264H, we see also that

$$\operatorname{diam}(D \cap E) \ge \big(\tfrac{2^r}{\beta_r}\mu(D \cap E)\big)^{1/r} \ge 2\gamma \big(\tfrac{1 - r^{r/2}\epsilon_1}{\beta_r}\big)^{1/r}.$$

Next,

$$\begin{aligned}\operatorname{per}(D \cap E) &\le \nu(\partial D) + \nu(D \cap \partial^* E) \\ &\qquad \text{(475Cf)} \\ &\le 2r\gamma^{r-1} + \epsilon_1 (\operatorname{diam} D)^{r-1} \le \gamma^{r-1}(2r + r^{r/2}\epsilon_1) \\ &\le (2r + r^{r/2}\epsilon_1) \cdot \tfrac{1}{2^{r-1}} \big(\tfrac{\beta_r}{1 - \epsilon_1 r^{r/2}}\big)^{(r-1)/r} \operatorname{diam}(D \cap E)^{r-1} \\ &\le \frac{1}{\alpha} \operatorname{diam}(D \cap E)^{r-1}.\end{aligned}$$

So $D \cap E \in \mathcal{C}_\alpha$. Also, for every $s > 0$, there is a $D' \in \mathcal{D}$ such that $D' \subseteq D$ and $x \in \overline{D'}$ and $\operatorname{diam} D' \le s$. In this case

$$\mu(B(x, \operatorname{diam} D') \setminus E) \le \epsilon_1 (\operatorname{diam} D')^r = \epsilon_1 r^{r/2} \mu D' \le \frac{1}{2}\mu D',$$

so

$$\mu(D \cap E \cap B(x, \operatorname{diam} D')) \ge \mu D' - \mu(B(x, \operatorname{diam} D') \setminus E)$$
$$\ge \frac{1}{2}\mu D' = \frac{1}{2\beta_r r^{r/2}}\mu B(x, \operatorname{diam} D').$$

As $\operatorname{diam} D'$ is arbitrarily small, $x \in \operatorname{cl}^*(D \cap E)$ and $\boldsymbol{t} = \{(x, D \cap E)\} \in T_\alpha$. Finally, since $\operatorname{diam} D < \theta(x)$, $(x, D \cap E) \in \delta$ and $\boldsymbol{t} \in T'$. **Q**

(iii) Let $\mathcal{H}$ be the set of those $H \subseteq \mathbb{R}^r$ such that $W_{\boldsymbol{t}} \subseteq E \cap H \subseteq W_{\boldsymbol{t}} \cup \bigcup \mathcal{D}_1$ for some $\boldsymbol{t} \in T'$ and finite $\mathcal{D}_1 \subseteq \mathcal{D}_0$. Then $H \cup H' \in \mathcal{H}$ whenever H, $H' \in \mathcal{H}$ are disjoint. If $\langle D_n \rangle_{n \in \mathbb{N}}$ is any strictly decreasing sequence in $\mathcal{D}$, then some D_n belongs to $\mathcal{H}$. **P** Let x be the unique point of $\bigcap_{n \in \mathbb{N}} \overline{D_n}$.

case 1 If $x \in A$, then there is a finite subset $\mathcal{D}_1$ of $\mathcal{D}_0$ whose union is a neighbourhood of x, and therefore includes D_n for some n; so $\boldsymbol{t} = \emptyset$ and $\mathcal{D}_1$ witness that $D_n \in \mathcal{H}$.

case 2 If $x \in E \setminus A$, then $x \in \operatorname{int}^* E$, so $h(x) > 0$ and there is some $n \in \mathbb{N}$ such that $\operatorname{diam} D_n \le h(x)$. In this case $\boldsymbol{t} = \{(x, D_n \cap E)\}$ belongs to T', by the choice of $h(x)$, so that $\boldsymbol{t}$ and $\emptyset$ witness that $D_n \in \mathcal{H}$.

case 3 Finally, if $x \notin E \cup A$, then $x \notin \operatorname{cl}^* E$ so $x \notin \overline{E}$ and there is some n such that $D_n \cap E = \emptyset$, in which case $\boldsymbol{t} = \mathcal{D}_1 = \emptyset$ witness that $D_n \in \mathcal{H}$. **Q**

(iv) In fact $\mathbb{R}^r \in \mathcal{H}$. **P?** Otherwise, because $E \subseteq V$ is bounded, it can be covered by a finite disjoint family in $\mathcal{D}$, and there must be some $D_0 \in \mathcal{D} \setminus \mathcal{H}$. Now we can find $\langle D_n \rangle_{n \ge 1}$ in $\mathcal{D} \setminus \mathcal{H}$ such that $D_n \subseteq D_{n-1}$ and $\operatorname{diam} D_n = \frac{1}{2} \operatorname{diam} D_{n-1}$ for every n. But this contradicts (iii).**XQ**

(v) We therefore have a $\boldsymbol{t} \in T'$ and a finite set $\mathcal{D}_1 \subseteq \mathcal{D}_0$ such that $W_{\boldsymbol{t}} \subseteq E \subseteq W_{\boldsymbol{t}} \cup \bigcup \mathcal{D}_1$. Now we can find $\boldsymbol{t}' \subseteq \boldsymbol{t}$ and $\mathcal{D}'_1 \subseteq \mathcal{D}_1$ such that $W_{\boldsymbol{t}'} \cap \bigcup \mathcal{D}'_1 = \emptyset$ and $E \subseteq W_{\boldsymbol{t}'} \cup \bigcup \mathcal{D}'_1$. **P** Express $\boldsymbol{t}$ as $\langle (x_i, D_i \cap E) \rangle_{i \in I}$ where $D_i \in \mathcal{D}$ for each i. Then $E \subseteq \bigcup_{i \in I} D_i \cup \bigcup \mathcal{D}_1$. Set $\mathcal{D}'_1 = \{D : D \in \mathcal{D}_1,\ D \not\subseteq D_i$ for every $i \in I\}$, $J = \{i : i \in I,\ D_i \not\subseteq D$ for every $D \in \mathcal{D}'_1\}$, $\boldsymbol{t}' = \{(x_i, D_i \cap E) : i \in J\}$. **Q**

By the choice of $\mathcal{D}_0$, $\bigcup \mathcal{D}'_1 \in \mathcal{R}''$; by the choice of $\mathcal{R}''$, $E \setminus W_{\boldsymbol{t}'} = E \cap \bigcup \mathcal{D}'_1$ belongs to $\mathcal{R}'$. But we know also that $C \cap V \setminus E \in \mathcal{R}'$, that is, $C \setminus E \in \mathcal{R}'$, because $\mathcal{R}' = \mathcal{R}_\eta^{(V)}$. By the choice of $\mathcal{R}'$, $C \setminus W_{\boldsymbol{t}'} \in \mathcal{R}$. And $\boldsymbol{t}' \in T'$ is a δ-fine member of T_α. As C, δ and $\mathcal{R}$ are arbitrary, 481G(vii) is satisfied, and the proof is complete.

484G The Pfeffer integral (a) For $\alpha \in]0, \alpha^*[$, write I_α for the linear functional defined by setting

$$I_\alpha(f) = \lim_{\boldsymbol{t} \to \mathcal{F}(T_\alpha, \Delta, \mathfrak{R})} S_{\boldsymbol{t}}(f, \mu)$$

whenever $f : \mathbb{R}^r \to \mathbb{R}$ is such that the limit is defined. (See 481F for the notation $\mathcal{F}(T_\alpha, \Delta, \mathfrak{R})$.) Then if $0 < \beta \le \alpha < \alpha^*$ and $I_\beta(f)$ is defined, so is $I_\alpha(f)$, and the two are equal. **P** All we have to observe is that $\mathcal{C}_\alpha \subseteq \mathcal{C}_\beta$ so that $T_\alpha \subseteq T_\beta$, while $\mathcal{F}(T_\alpha, \Delta, \mathfrak{R})$ is just $\{A \cap T_\alpha : A \in \mathcal{F}(T_\beta, \Delta, \mathfrak{R})\}$. **Q**

(b) Let $f : \mathbb{R}^r \to \mathbb{R}$ be a function. I will say that it is **Pfeffer integrable**, with **Pfeffer integral** $\text{Pf}\!\!\int f$, if

$$\text{Pf}\!\!\int f = \lim_{\alpha \downarrow 0} I_\alpha(f)$$

is defined; that is to say, if $I_\alpha(f)$ is defined whenever $0 < \alpha < \alpha^*$.

484H The first step is to work through the results of §482 to see which ideas apply directly to the limit integral $\text{Pf}\!\!\int$.

Proposition (a) The domain of $\text{Pf}\!\!\int$ is a linear space of functions, and $\text{Pf}\!\!\int$ is a positive linear functional.

(b) If f, $g : \mathbb{R}^r \to \mathbb{R}$ are such that $|f| \le g$ and $\text{Pf}\!\!\int g = 0$, then $\text{Pf}\!\!\int f$ is defined and equal to 0.

(c) If $f : \mathbb{R}^r \to \mathbb{R}$ is Pfeffer integrable, then there is a unique additive functional $F : \mathcal{C} \to \mathbb{R}$ such that whenever $\epsilon > 0$ and $0 < \alpha < \alpha^*$ there are $\delta \in \Delta$ and $\mathcal{R} \in \mathfrak{R}$ such that

$$\textstyle\sum_{(x,C) \in \boldsymbol{t}} |F(C) - f(x)\mu C| \le \epsilon \text{ for every } \delta\text{-fine } \boldsymbol{t} \in T_\alpha,$$

$$|F(E)| \le \epsilon \text{ whenever } E \in \mathcal{C} \cap \mathcal{R}.$$

Moreover, $F(\mathbb{R}^r) = \text{Pf}\!\!\int f$.

(d) Every Pfeffer integrable function is Lebesgue measurable.

(e) Every Lebesgue integrable function is Pfeffer integrable, with the same integral.

(f) A non-negative function is Pfeffer integrable iff it is Lebesgue integrable.

proof (a)-(b) Immediate from 481C.

(c) For each $\alpha \in]0, \alpha^*[$ let F_α be the Saks-Henstock indefinite integral corresponding to the the structure $(\mathbb{R}^r, T_\alpha, \Delta, \mathfrak{R}, \mu)$. Then all the F_α coincide. **P** Suppose that $0 < \beta \le \alpha < \alpha^*$. Then, for any $\epsilon > 0$, there are $\delta \in \Delta$, $\mathcal{R} \in \mathfrak{R}$ such that

$$\sum_{(x,C)\in\boldsymbol{t}} |F_\beta(C) - f(x)\mu C| \le \epsilon \text{ for every } \delta\text{-fine } \boldsymbol{t} \in T_\beta,$$

$$|F_\beta(E)| \le \epsilon \text{ whenever } E \in \mathcal{R}.$$

Since $T_\alpha \subseteq T_\beta$, this means that

$$\sum_{(x,C)\in\boldsymbol{t}} |F_\beta(C) - f(x)\mu C| \le \epsilon \text{ for every } \delta\text{-fine } \boldsymbol{t} \in T_\alpha.$$

And this works for any $\epsilon > 0$. By the uniqueness assertion in 482B, F_β must be exactly the same as F_α. **Q**

So we have a single functional F; and 482B also tells us that

$$F(\mathbb{R}^r) = I_\alpha(f) = \text{Pf}\!\!\int f$$

for every α.

(d) In fact if there is any α such that $I_\alpha(f)$ is defined, f must be Lebesgue measurable. **P** We have only to check that the conditions of 482E are satisfied by μ, $\mathcal{C}_\alpha$, $\{(x, C) : C \in \mathcal{C}_\alpha,\ x \in \mathrm{cl}^*C\}$, T_α, Δ and $\mathfrak{R}$. (i), (iii) and (v) are built into the definitions above, and (iv) and (vii) are covered by 484F. 482E(ii) is true because $C \setminus \mathrm{cl}^*C$ is negligible for every $C \in \mathcal{C}$ (475Cg).

As for 482E(vi), this is true because if $\mu E < \infty$ and $\epsilon > 0$, there are $n \in \mathbb{N}$ and $\eta \in \mathrm{H}$ such that $\mu(E \setminus B(\mathbf{0}, n)) \le \frac{1}{2}\epsilon$ and $\sum_{i=0}^\infty \eta(i) \le \frac{1}{2}\epsilon$, so that

$$\mu(E \cap R) \le \mu(E \setminus B(\mathbf{0}, n)) + \mu(R \cap B(\mathbf{0}, n)) \le \epsilon$$

for every $R \in \mathcal{R}_\eta^{(B(\mathbf{0},n))}$. **Q**

(e) This time, we have to check that the conditions of 482F are satisfied by T_α, Δ and $\mathfrak{R}$ whenever $0 < \alpha < \alpha^*$. **P** Of course μ is inner regular with respect to the closed sets and outer regular with respect to the open sets (134F). Condition 482F(v) just repeats 482E(v), verified in (d) above. **Q**

(f) If $f \ge 0$ is integrable in the ordinary sense, then it is Pfeffer integrable, by (e). If it is Pfeffer integrable, then it is measurable; but also $\int g\,d\mu = \text{Pf}\!\!\int g \le \text{Pf}\!\!\int f$ for every simple function $g \le f$, so f is integrable (213B).

484I Definition If $f : \mathbb{R}^r \to \mathbb{R}$ is Pfeffer integrable, I will call the function $F : \mathcal{C} \to \mathbb{R}$ defined in 484Hc the **Saks-Henstock indefinite integral** of f.

484J In fact 484Hc characterizes the Pfeffer integral, just as the Saks-Henstock lemma can be used to define general gauge integrals based on tagged-partition structures allowing subdivisions.

Proposition Suppose that $f : \mathbb{R}^r \to \mathbb{R}$ and $F : \mathcal{C} \to \mathbb{R}$ are such that

(i) F is additive,

(ii) whenever $0 < \alpha < \alpha^*$ and $\epsilon > 0$ there is a $\delta \in \Delta$ such that $\sum_{(x,C)\in\boldsymbol{t}} |F(C) - f(x)\mu C| \le \epsilon$ for every δ-fine $\boldsymbol{t} \in T_\alpha$,

(iii) for every $\epsilon > 0$ there is an $\mathcal{R} \in \mathfrak{R}$ such that $|F(E)| \le \epsilon$ for every $E \in \mathcal{C} \cap \mathcal{R}$.

Then f is Pfeffer integrable and F is the Saks-Henstock indefinite integral of f.

proof By 482D, the gauge integral $I_\alpha(f)$ is defined and equal to $F(\mathbb{R}^r)$ for every $\alpha \in]0, \alpha^*[$. So f is Pfeffer integrable. Now 484Hc tells us that F must be its Saks-Henstock indefinite integral.

484K Lemma Suppose that $\alpha > 0$ and $0 < \alpha' < \alpha \min(\frac{1}{2}, 2^{r-1}(\frac{\alpha}{2\beta_r})^{(r-1)/r})$. If $E \in \mathcal{C}$ is such that $E \subseteq \mathrm{cl}^*E$, then there is a $\delta \in \Delta$ such that $\{(x, C \cap E)\} \in T_{\alpha'}$ whenever $(x, C) \in \delta$, $x \in E$ and $\{(x, C)\} \in T_\alpha$.

proof Take $\epsilon > 0$ such that $\beta_r\epsilon \le \frac{1}{2}\alpha$ and

$$\frac{1}{\alpha} + 2^{r-1}\epsilon \le \frac{2^{r-1}}{\alpha'}\bigl(\frac{\alpha}{2\beta_r}\bigr)^{(r-1)/r}.$$

Set

$$A = \partial^*E \cup \{x : \lim_{\zeta\downarrow 0}\sup_{x\in G, 0<\operatorname{diam} G\le\zeta} \frac{\nu^*(G\cap\partial^*E)}{(\operatorname{diam} G)^{r-1}} > 0\},$$

so that A is a thin set, as in (b-i) of the proof of 484F. (Of course ∂^*E is thin because $\nu(\partial^*E \cap B(\mathbf{0}, n))$ is finite for every $n \in \mathbb{N}$.)

For $x \in E \setminus A$, we have $x \in \operatorname{int}^*E$ (because $E \subseteq \operatorname{cl}^*E$), so there is a $\theta(x) > 0$ such that

$$\mu(B(x,\zeta) \setminus E) \le \epsilon\mu B(x,\zeta), \quad \nu(\partial^*E \cap B(x,\zeta)) \le \epsilon(2\zeta)^{r-1}$$

whenever $0 < \zeta \le 2\theta(x)$. If we set $\theta(x) = 0$ for $x \in E \cap A$ and $\theta(x) = 1$ for $x \in \mathbb{R}^r \setminus E$, then $\theta \in \Theta$ and $\delta_\theta \in \Delta$.

Now suppose that $x \in E$, $(x, C) \in \delta_\theta$ and $\{(x, C)\} \in T_\alpha$, that is, that $C \in \mathcal{C}_\alpha$ and $x \in (E \cap \operatorname{cl}^*C) \setminus A$ and $\|x - y\| < \theta(x)$ for every $y \in C$. Set $\gamma = \operatorname{diam} C \le 2\theta(x)$. Then

$$\mu(C \setminus E) \le \mu(B(x,\gamma) \setminus E) \le \epsilon\mu B(x,\gamma) = \beta_r\epsilon\gamma^r,$$

so

$$\begin{aligned}\mu(C \cap E) &\ge \mu C - \beta_r\epsilon\gamma^r \ge (\alpha - \beta_r\epsilon)\gamma^r \\ &\ge \frac{1}{2}\alpha\gamma^r \ge \alpha'\gamma^r \ge \alpha' \operatorname{diam}(C \cap E)^r.\end{aligned}$$

Next,

$$\begin{aligned}\operatorname{per}(C \cap E) &= \nu(\partial^*(C \cap E)) \le \nu(B(x,\gamma) \cap (\partial^*C \cup \partial^*E)) \\ &\le \nu(\partial^*C) + \nu(B(x,\gamma) \cap \partial^*E) \le \frac{1}{\alpha}\gamma^{r-1} + \epsilon(2\gamma)^{r-1} = (\frac{1}{\alpha} + 2^{r-1}\epsilon)\gamma^{r-1}.\end{aligned}$$

Moreover,

$$2^{-r}\beta_r \operatorname{diam}(C \cap E)^r \ge \mu(C \cap E) \ge \frac{1}{2}\alpha\gamma^r$$

(264H), so $\operatorname{diam}(C \cap E) \ge 2(\frac{\alpha}{2\beta_r})^{1/r}\gamma$ and

$$\operatorname{per}(C \cap E) \le (\frac{1}{\alpha} + 2^{r-1}\epsilon) \cdot \frac{1}{2^{r-1}}\bigl(\frac{2\beta_r}{\alpha}\bigr)^{(r-1)/r} \operatorname{diam}(C \cap E)^{r-1} \le \frac{1}{\alpha'} \operatorname{diam}(C \cap E)^{r-1}.$$

Putting these together, we see that $C \in \mathcal{C}_{\alpha'}$.

Finally, because $x \in \operatorname{cl}^*C \cap \operatorname{int}^*E \subseteq \operatorname{cl}^*(C \cap E)$ (475Ce), $\{(x, C \cap E)\} \in T_{\alpha'}$.

484L Proposition Suppose that $f : \mathbb{R}^r \to \mathbb{R}$ is Pfeffer integrable, and that $F : \mathcal{C} \to \mathbb{R}$ is its Saks-Henstock indefinite integral. Then $\oint f \times \chi E$ is defined and equal to $F(E)$ for every $E \in \mathcal{C}$.

proof (a) To begin with (down to the end of (d) below), suppose that $E \in \mathcal{C}$ is such that $\operatorname{int}^*E \subseteq E \subseteq \operatorname{cl}^*E$. For $C \in \mathcal{C}$ set $F_1(C) = F(C \cap E)$. I seek to show that F_1 satisfies the conditions of 484J.

Of course $F_1 : \mathcal{C} \to \mathbb{R}$ is additive. If $\epsilon > 0$, there is an $\mathcal{R} \in \mathfrak{R}$ such that $|F(G)| \le \epsilon$ whenever $G \in \mathcal{R}$, by 484H; now there is an $\mathcal{R}' \in \mathfrak{R}$ such that $G \cap E \in \mathcal{R}$ for every $G \in \mathcal{R}'$ (484E(a-ii)), so that $|F_1(G)| \le \epsilon$ for every $G \in \mathcal{C} \cap \mathcal{R}'$. Thus F_1 satisfies (iii) of 484J.

(b) Take $\alpha \in \,]0, \alpha^*[$ and $\epsilon > 0$. Take α' such that $0 < \alpha' < \alpha\min(\frac{1}{2}, 2^{r-1}(\frac{\alpha}{2\beta_r})^{(r-1)/r})$. Applying 484K to E and its complement, and appealing to the definition of F, we see that there is a $\delta \in \Delta$ such that

(α) $\{(x, C \cap E)\} \in T_{\alpha'}$ whenever $(x, C) \in \delta$, $x \in E$ and $\{(x, C)\} \in T_\alpha$,

(β) $\{(x, C \setminus E)\} \in T_{\alpha'}$ whenever $(x, C) \in \delta$, $x \in \mathbb{R}^r \setminus E$ and $\{(x, C)\} \in T_\alpha$,

(γ) $\sum_{(x,C)\in\boldsymbol{t}} |F(C) - f(x)\mu C| \le \epsilon$ for every δ-fine $\boldsymbol{t} \in T_{\alpha'}$.

(For (α), we need to know that $E \subseteq \mathrm{cl}^*E$ and for (β) we need $\mathrm{int}^*E \subseteq E$.) Next, choose for each $n \in \mathbb{N}$ closed sets $H_n \subseteq E$, $H'_n \subseteq \mathbb{R}^r \setminus E$ such that $\mu(E \setminus H_n) \le 2^{-n}\epsilon$ and $\mu((\mathbb{R}^r \setminus E) \setminus H'_n) \le 2^{-n}\epsilon$. Define $\theta : \mathbb{R}^r \to]0, \infty[$ by setting

$$\begin{aligned}\theta(x) &= \min(1, \frac{1}{2}\rho(x, H'_n)) \text{ if } x \in E \text{ and } n \le |f(x)| < n+1,\\ &= \min(1, \frac{1}{2}\rho(x, H_n)) \text{ if } x \in \mathbb{R}^r \setminus E \text{ and } n \le |f(x)| < n+1,\end{aligned}$$

writing $\rho(x, A) = \inf_{y \in A} \|x - y\|$ if $A \subseteq \mathbb{R}^r$ is non-empty, ∞ if $A = \emptyset$. Then $\delta_\theta \in \Delta$. Let $\mathcal{R}' \in \mathfrak{R}$ be such that $A \cap E \in \mathcal{R}$ whenever $A \in \mathcal{R}'$.

(c) Write f_E for $f \times \chi E$. Then $\sum_{(x,C) \in \boldsymbol{t}} |F_1(C) - f_E(x)\mu(C)| \le 11\epsilon$ whenever $\boldsymbol{t} \in T_\alpha$ is $(\delta \cap \delta_\theta)$-fine. **P** Set

$$\boldsymbol{t}' = \{(x, C \cap E) : (x, C) \in \boldsymbol{t},\, x \in E\}.$$

By clause (α) of the choice of δ, $\boldsymbol{t}' \in T_{\alpha'}$, and of course it is δ-fine. So

$$\sum_{(x,C) \in \boldsymbol{t}, x \in E} |F(C \cap E) - f(x)\mu(C \cap E)| \le \epsilon$$

by clause (γ) of the choice of δ. Next,

$$\begin{aligned}\sum_{(x,C) \in \boldsymbol{t}, x \in E} |f(x)\mu(C \setminus E)| &= \sum_{n=0}^{\infty} \sum_{\substack{(x,C) \in \boldsymbol{t}, x \in E\\ n \le |f(x)| < n+1}} |f(x)|\mu(C \setminus E)\\ &\le \sum_{n=0}^{\infty} (n+1)\mu((\mathbb{R}^r \setminus E) \setminus H'_n)\end{aligned}$$

(because $\operatorname{diam} C \le \theta(x)$, so $C \cap H'_n = \emptyset$ whenever $(x, C) \in \boldsymbol{t}$, $x \in E$ and $n \le |f(x)| < n+1$)

$$\le \sum_{n=0}^{\infty} 2^{-n}(n+1)\epsilon = 4\epsilon,$$

and

$$\sum_{(x,C) \in \boldsymbol{t}, x \in E} |F(C \cap E) - f(x)\mu C| \le 5\epsilon.$$

Similarly,

$$\sum_{(x,C) \in \boldsymbol{t}, x \notin E} |F(C \setminus E) - f(x)\mu C)| \le 5\epsilon.$$

But as

$$\sum_{(x,C) \in \boldsymbol{t}, x \notin E} |F(C) - f(x)\mu C| \le \epsilon$$

(because surely $\boldsymbol{t}$ itself belongs to $T_{\alpha'}$), we have

$$\sum_{(x,C) \in \boldsymbol{t}, x \notin E} |F(C \cap E)| \le 6\epsilon.$$

Putting these together,

$$\begin{aligned}\sum_{(x,C) \in \boldsymbol{t}} |F_1(C) - f_E(x)\mu C| &= \sum_{\substack{(x,C) \in \boldsymbol{t}\\ x \in E}} |F(C \cap E) - f(x)\mu C| + \sum_{\substack{(x,C) \in \boldsymbol{t}\\ x \notin E}} |F(C \cap E)|\\ &\le 5\epsilon + 6\epsilon = 11\epsilon. \textbf{ Q}\end{aligned}$$

(d) As α and ϵ are arbitrary, condition (ii) of 484J is satisfied by F_1 and f_E, so $\mathcal{P}\!\!\!\int f_E = F_1(\mathbb{R}^r) = F(E)$.

(e) This completes the proof when $\mathrm{int}^*E \subseteq E \subseteq \mathrm{cl}^*E$. For a general set $E \in \mathcal{C}$, set $E_1 = (E \cup \mathrm{int}^*E) \cap \mathrm{cl}^*E$. Then $E \triangle E_1$ is negligible, so

$$\mathrm{int}^*E_1 = \mathrm{int}^*E \subseteq E_1 \subseteq \mathrm{cl}^*E = \mathrm{cl}^*E_1.$$

Also $\fint f \times \chi(E \setminus E_1) = \int f \times \chi(E \setminus E_1) d\mu$, $\fint f \times \chi(E_1 \setminus E) = \int f \times \chi(E \setminus E_1) d\mu$ are both zero, and

$$\fint f \times \chi E = \fint f \times \chi E_1 = F(E_1) = F(E).$$

(To see that $F(E_1) = F(E)$, note that $E \setminus E_1$ and $E_1 \setminus E$, being negligible sets, have empty essential boundary and zero perimeter, so belong to every member of $\mathfrak{R}$, by 484E(b-ii), or otherwise.)

484M Lemma Let G, $H \in \mathcal{C}$ be disjoint and $\phi : \mathbb{R}^r \to \mathbb{R}^r$ a continuous function. If *either* $G \cup H$ is bounded *or* ϕ has compact support,

$$\int_{\partial^*(G\cup H)} \phi \mathbin{.} \psi_{G\cup H}\, d\nu = \int_{\partial^* G} \phi \mathbin{.} \psi_G\, d\nu + \int_{\partial^* H} \phi \mathbin{.} \psi_G\, d\nu,$$

where ψ_G, ψ_H and $\psi_{G\cup H}$ are the canonical outward-normal functions (474G).

proof (a) Suppose first that ϕ is a Lipschitz function with compact support. Then 475N tells us that

$$\begin{aligned}\int_{\partial^*(G\cup H)} \phi \mathbin{.} \psi_{G\cup H}\, d\nu &= \int_{G\cup H} \operatorname{div}\phi\, d\mu = \int_G \operatorname{div}\phi\, d\mu + \int_H \operatorname{div}\phi\, d\mu \\ &= \int_{\partial^* G} \phi \mathbin{.} \psi_G\, d\nu + \int_{\partial^* H} \phi \mathbin{.} \psi_G\, d\nu.\end{aligned}$$

(Recall from 474R that we can identify canonical outward-normal functions with Federer exterior normals, as in the statement of 475N.)

(b) Now suppose that ϕ is a continuous function with compact support. Let $\langle \tilde h_n\rangle_{n\in\mathbb{N}}$ be the smoothing sequence of 473E. Then all the functions $\phi * \tilde h_n$ are Lipschitz and $\langle \phi * \tilde h_n\rangle_{n\in\mathbb{N}}$ converges uniformly to ϕ (473Df, 473Ed). So

$$\begin{aligned}\int_{\partial^*(G\cup H)} \phi \mathbin{.} \psi_{G\cup H}\, d\nu &= \lim_{n\to\infty} \int_{\partial^*(G\cup H)} (\phi * \tilde h_n) \mathbin{.} \psi_{G\cup H}\, d\nu \\ &= \lim_{n\to\infty} \int_{\partial^* G} (\phi * \tilde h_n) \mathbin{.} \psi_G\, d\nu + \lim_{n\to\infty} \int_{\partial^* H} (\phi * \tilde h_n) \mathbin{.} \psi_H\, d\nu \\ &= \int_{\partial^* G} \phi \mathbin{.} \psi_G\, d\nu + \int_{\partial^* H} \phi \mathbin{.} \psi_G\, d\nu.\end{aligned}$$

(c) If, on the other hand, ϕ is continuous and G and H are bounded, then we can find a continuous function $\tilde\phi$ with compact support agreeing with ϕ on $\overline{G\cup H}$ (4A2G(e-i), or otherwise); applying (b) to $\tilde\phi$, we get the required result for ϕ.

484N Pfeffer's Divergence Theorem Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter, and $\phi : \mathbb{R}^r \to \mathbb{R}^r$ a continuous function with compact support such that $\{x : x \in \mathbb{R}^r,\ \phi \text{ is not differentiable at } x\}$ is thin. Let v_x be the Federer exterior normal to E at any point x where the normal exists. Then $\fint \operatorname{div}\phi \times \chi E$ is defined and equal to $\int_{\partial^* E} \phi(x) \mathbin{.} v_x \nu(dx)$.

proof (a) Let n be such that $\phi(x) = \mathbf{0}$ for $\|x\| \ge n$. For $C \in \mathcal{C}$, set $F(C) = \int_{\partial^* C} \phi \mathbin{.} \psi_C \nu(dx)$, where ψ_C is the canonical outward-normal function; recall that $\psi_E(x) = v_x$ for ν-almost every $x \in \partial^* E$ (474R, 475D). By 484M, F is additive.

(b) If $0 < \alpha < \alpha^*$ and $\epsilon > 0$ and $x \in \mathbb{R}^r$ is such that ϕ is differentiable at x, there is a $\gamma > 0$ such that $|F(C) - \operatorname{div}\phi(x)\mu C| \le \epsilon\mu C$ whenever $C \in \mathcal{C}_\alpha$, $x \in \overline{C}$ and $\operatorname{diam} C \le \gamma$. **P** Let T be the derivative of ϕ at x. Let $\gamma > 0$ be such that $\|\phi(y) - \phi(x) - T(y - x)\| \le \alpha^2\epsilon\|y - x\|$ whenever $\|y - x\| \le \gamma$. Let $\tilde\phi : \mathbb{R}^r \to \mathbb{R}^r$ be a Lipschitz function with compact support such that $\tilde\phi(y) = \phi(x) + (y - x)$ whenever $\|y - x\| \le \gamma$ (473Cf). If $C \in \mathcal{C}_\alpha$ has diameter at most γ and $x \in \overline{C}$, then

$$|F(C) - \operatorname{div}\phi(x)\mu C| = |\int_{\partial^* C} \phi \mathbin{.} \psi_C \nu(dx) - \int_C \operatorname{div}\tilde\phi\, d\mu|$$

(because T is the derivative of $\tilde\phi$ everywhere on $B(x, \gamma)$, so $\operatorname{div}\tilde\phi(y) = \operatorname{div}\phi(x)$ for every $y \in C$)

$$= |\int_{\partial^* C}(\phi-\tilde\phi)\,\boldsymbol{.}\,\psi_C\nu(dx)|$$

(applying the Divergence Theorem 475N to $\tilde\phi$)

$$\le \nu(\partial^* C)\sup_{y\in C}\|\phi(y)-\tilde\phi(y)\|$$

$$\le \alpha^2\epsilon\operatorname{diam}C\operatorname{per}C \le \alpha\epsilon(\operatorname{diam}C)^r \le \epsilon\mu C$$

because $C\in\mathcal{C}_\alpha$. **Q**

(c) If $\epsilon>0$ and $\alpha\in\,]0,\alpha^*[$, there is a $\delta\in\Delta$ such that $\sum_{(x,C)\in\boldsymbol{t}}|F(C)-\operatorname{div}\phi(x)\mu C|\le\epsilon$ whenever $\boldsymbol{t}\in T_\alpha$ is δ-fine. **P** Let $\zeta>0$ be such that $\zeta\mu B(\mathbf{0},n+2)\le\epsilon$. Set $A=\{x:x\in\mathbb{R}^r,\ \phi$ is not differentiable at $x\}$, and for $x\in\mathbb{R}^r\setminus A$ let $\theta(x)\in\left]0,\frac12\right]$ be such that $|F(C)-\operatorname{div}\phi(x)\mu C|\le\zeta\mu C$ whenever $C\in\mathcal{C}_\alpha$, $x\in\overline{C}$ and $\operatorname{diam}C\le\theta(x)$; for $x\in A$ set $\theta(x)=0$. Now suppose that $\boldsymbol{t}\in T_\alpha$ is δ_θ-fine. Then

$$\sum_{(x,C)\in\boldsymbol{t}}|F(C)-\operatorname{div}\phi(x)\mu C| = \sum_{\substack{(x,C)\in\boldsymbol{t}\\ x\in B(\mathbf{0},n+1)}}|F(C)-\operatorname{div}\phi(x)\mu C|$$

(because $\operatorname{diam}C\le 1$ whenever $(x,C)\in\boldsymbol{t}$, so if $\|x\|>n+1$ then $F(C)=\operatorname{div}\phi(x)=0$)

$$\le\sum_{\substack{(x,C)\in\boldsymbol{t}\\ x\in B(\mathbf{0},n+1)}}\zeta\mu C\le\zeta\mu B(\mathbf{0},n+2)\le\epsilon.$$ **Q**

(d) Because ϕ is a continuous function with compact support, it is uniformly continuous (apply 4A2Jf to each of the coordinates of ϕ). For $\zeta>0$, let $\gamma(\zeta)>0$ be such that $\|\phi(x)-\phi(y)\|\le\zeta$ whenever $\|x-y\|\le\gamma(\zeta)$.

If $C\in\mathcal{C}$ and $\operatorname{per}C\le 1$ and $\mu C\le\zeta\gamma(\zeta)$, where $\zeta>0$, then $|F(C)|\le r\zeta(2\|\phi\|_\infty+\frac12)$, writing $\|\phi\|_\infty$ for $\sup_{x\in\mathbb{R}^r}\|\phi(x)\|$. **P** For $1\le i\le r$, let $\phi_i:\mathbb{R}^r\to\mathbb{R}$ be the ith component of ϕ, and v_i the ith unit vector $(0,\dots,1,\dots,0)$; write

$$\alpha_i=\int_{\partial^* C}\phi_i(x)(v_i|\psi_E(x))\nu(dx),$$

so that $F(C)=\sum_{i=1}^r\alpha_i$. I start by examining α_r. By 475O, we have sequences $\langle H_n\rangle_{n\in\mathbb{N}}$, $\langle g_n\rangle_{n\in\mathbb{N}}$ and $\langle g'_n\rangle_{n\in\mathbb{N}}$ such that

(i) for each $n\in\mathbb{N}$, H_n is a Lebesgue measurable subset of $\mathbb{R}^{r-1}$, and g_n, $g'_n:H_n\to[-\infty,\infty]$ are Lebesgue measurable functions such that $g_n(u)<g'_n(u)$ for every $u\in H_n$;

(ii) if m, $n\in\mathbb{N}$ then $g_m(u)\ne g'_n(u)$ for every $u\in H_m\cap H_n$;

(iii) $\sum_{n=0}^\infty\int_{H_n}g'_n-g_n d\mu_{r-1}=\mu C$;

(iv)

$$\alpha_r=\sum_{n=0}^\infty\int_{H_n}\phi_r(u,g'_n(u))-\phi_r(u,g_n(u))\mu_{r-1}(du),$$

where we interpret $\phi_r(u,\infty)$ and $\phi_r(u,-\infty)$ as 0 if necessary;

(v) for μ_{r-1}-almost every $u\in\mathbb{R}^{r-1}$,

$$\{t:(u,t)\in\partial^* C\}=\{g_n(u):n\in\mathbb{N},\ u\in H_n,\ g_n(u)\ne-\infty\}\cup\{g'_n(u):n\in\mathbb{N},\ u\in H_n,\ g'_n(u)\ne\infty\}.$$

From (iii) we see that g'_n and g_n are both finite almost everywhere on H_n, for every n. Consequently, by (v) and 475H,

$$2\sum_{n=0}^\infty\mu_{r-1}H_n=\int\#(\{t:(u,t)\in\partial^* C\})\mu_{r-1}(du)\le\nu(\partial^* C)\le 1.$$

For each n, set

$$H'_n=\{u:u\in H_n,\ g'_n(u)-g_n(u)>\gamma(\zeta)\}.$$

Then $\gamma(\zeta)\sum_{n=0}^\infty\mu H'_n\le\mu C$ so $\sum_{n=0}^\infty\mu H'_n\le\zeta$ and

$$|\sum_{n=0}^{\infty}\int_{H'_n}\phi_r(u,g'_n(u))-\phi_r(u,g_n(u))\mu_{r-1}(du)|\le 2\|\phi\|_\infty\sum_{n=0}^{\infty}\mu H'_n\le 2\zeta\|\phi\|_\infty.$$

On the other hand, for $n\in\mathbb{N}$ and $u\in H_n\setminus H'_n$, $|\phi_r(u,g'_n(u))-\phi_r(u,g_n(u))|\le\zeta$, so

$$|\sum_{n=0}^{\infty}\int_{H_n\setminus H'_n}\phi_r(u,g'_n(u))-\phi_r(u,g_n(u))\mu_{r-1}(du)|\le\sum_{n=0}^{\infty}\zeta\mu_{r-1}(H_n\setminus H'_n)\le\frac{1}{2}\zeta.$$

Putting these together,

$$\alpha_r\le 2\zeta\|\phi\|_\infty+\frac{1}{2}\zeta.$$

But of course the same arguments apply to all the α_i, so

$$|F(C)|\le\sum_{i=1}^{r}|\alpha_i|\le r\zeta(2\|\phi\|_\infty+\frac{1}{2}),$$

as claimed. **Q**

(e) If $\epsilon>0$ there is an $\mathcal{R}\in\mathfrak{R}$ such that $|F(E)|\le\epsilon$ for every $E\in\mathcal{C}\cap\mathcal{R}$. **P** Let $\langle\epsilon_i\rangle_{i\in\mathbb{N}}$ be a sequence of strictly positive real numbers such that $r(2\|\phi\|_\infty+\frac{1}{2})\sum_{i=0}^{\infty}\epsilon_i\le\epsilon$. For each $i\in\mathbb{N}$, set $\eta(i)=\epsilon_i\gamma(\epsilon_i)>0$. Set $V=B(\mathbf{0},n+1)$, and take any $E\in\mathcal{C}\cap\mathcal{R}_\eta^{(V)}$. Then $F(E\setminus V)=0$, so $F(E)=F(E\cap V)$, while $E\cap V\in\mathcal{R}_\eta$. Express $E\cap V$ as $\bigcup_{i\le n}E_i$, where $\langle E_i\rangle_{i\le n}$ is disjoint, $\operatorname{per}E_i\le 1$ and $\mu E_i\le\eta(i)$ for each i. Then $|F(E_i)|\le r\epsilon_i(2\|\phi\|_\infty+\frac{1}{2})$ for each i, by (d), so

$$|F(E)|=|F(E\cap V)|\le\sum_{i=0}^{n}|F(E_i)|\le\epsilon,$$

as required. **Q**

(f) By 484J, $\operatorname{div}\phi$ is Pfeffer integrable. Moreover, by the uniqueness assertion in 484Hc, its Saks-Henstock indefinite integral is just the function F here. By 484L, $F(E)=\oint\operatorname{div}\phi\times\chi E$ for every $E\in\mathcal{C}$, as required.

484O Differentiating the indefinite integral: Theorem Let $f:\mathbb{R}^r\to\mathbb{R}$ be a Pfeffer integrable function, and F its Saks-Henstock indefinite integral. Then whenever $0<\alpha<\alpha^*$,

$$\begin{aligned}f(x)&=\lim_{\zeta\downarrow 0}\sup\{\frac{F(C)}{\mu C}:C\in\mathcal{C}_\alpha,\,x\in\overline{C},\,0<\operatorname{diam}C\le\zeta\}\\&=\lim_{\zeta\downarrow 0}\inf\{\frac{F(C)}{\mu C}:C\in\mathcal{C}_\alpha,\,x\in\overline{C},\,0<\operatorname{diam}C\le\zeta\}\end{aligned}$$

for μ-almost every $x\in\mathbb{R}^r$.

proof (a) It will be useful to know the following: if $C\in\mathcal{C}_\alpha$, $\operatorname{diam}C>0$, $x\in\overline{C}$ and $\epsilon>0$, then for any sufficiently small $\zeta>0$, $C\cup B(x,\zeta)\in\mathcal{C}_{\alpha/2}$ and $|F(C\cup B(x,\zeta))-F(C)|\le\epsilon$. **P** Let $\mathcal{R}\in\mathfrak{R}$ be such that $|F(R)|\le\epsilon$ whenever $R\in\mathcal{C}\cap\mathcal{R}$, let $\mathcal{R}'\in\mathfrak{R}$ be such that $(\mathbb{R}^r\setminus C)\cap R\in\mathcal{R}$ whenever $R\in\mathcal{R}'$ (484E(a-ii)), and let $\eta\in\mathrm{H}$ be such that $\mathcal{R}_\eta\subseteq\mathcal{R}'$ (484E(a-i)). Then for all sufficiently small $\zeta>0$, we shall have $\operatorname{per}B(x,\zeta)\le 1$ and $\mu B(x,\zeta)\le\eta(0)$, so that $B(x,\zeta)\in\mathcal{R}_\eta$, $B(x,\zeta)\setminus C\in\mathcal{R}$ and

$$|F(C\cup B(x,\zeta))-F(C)|=|F(B(x,\zeta)\setminus C)|\le\epsilon.$$

Next, for all sufficiently small $\zeta>0$,

$$\begin{aligned}\mu(C\cup B(x,\zeta))&\ge\mu C\ge\alpha(\operatorname{diam}C)^r\\&\ge\frac{\alpha}{2}(\zeta+\operatorname{diam}C)^r\ge\frac{\alpha}{2}\operatorname{diam}(C\cup B(x,\zeta))^r\end{aligned}$$

(because $x\in\overline{C}$) and

$$\operatorname{per}(C \cup B(x,\zeta)) \le \operatorname{per} C + \operatorname{per} B(x,\zeta) \le \frac{1}{\alpha}(\operatorname{diam} C)^{r-1} + \operatorname{per} B(x,\zeta)$$
$$\le \frac{2}{\alpha}(\operatorname{diam} C)^{r-1} \le \frac{2}{\alpha}\operatorname{diam}(C \cup B(x,\zeta))^{r-1},$$

so that $C \cup B(x,\zeta) \in \mathcal{C}_{\alpha/2}$. **Q**

(b) For $x \in \mathbb{R}$, set

$$g(x) = \lim_{\zeta\downarrow 0} \sup\{\tfrac{F(C)}{\mu C} : C \in \mathcal{C}_\alpha,\, x \in \overline{C},\, \operatorname{diam} C \le \zeta\}.$$

? Suppose, if possible, that there are rational numbers $q < q'$ and $n \in \mathbb{N}$ such that $A = \{x : \|x\| \le n,\ f(x) \le q < q' < g(x)\}$ is not μ-negligible. Set

$$\epsilon = \tfrac{(q'-q)\alpha}{4\beta_r}\mu^* A > 0.$$

Let $\theta \in \Theta$ be such that

$$\sum_{(x,C)\in\boldsymbol{t}} |F(C) - f(x)\mu C| \le \epsilon$$

for every δ_θ-fine $\boldsymbol{t} \in T_{\alpha/2}$. Let $\mathcal{I}$ be the family of all balls $B(x,\zeta)$ where $x \in A$, $0 < \zeta \le \theta(x)$ and there is a $C \in \mathcal{C}_\alpha$ such that $x \in \overline{C}$, $\operatorname{diam} C = \zeta$ and $\dfrac{F(C)}{\mu C} > q'$. Then every member of $A_1 = A \setminus \theta^{-1}[\{0\}]$ is the centre of arbitrarily small members of $\mathcal{I}$, so by 472C there is a countable disjoint family $\mathcal{J}_0 \subseteq \mathcal{I}$ such that

$$\mu(\bigcup \mathcal{J}_0) > \frac{1}{2}\mu^* A_1 = \frac{1}{2}\mu^* A.$$

There is therefore a finite family $\mathcal{J}_1 \subseteq \mathcal{J}_0$ such that $\mu(\bigcup \mathcal{J}_1) > \frac{1}{2}\mu^* A$; enumerate $\mathcal{J}_1$ as $\langle B(x_i,\zeta_i)\rangle_{i\le n}$ where, for each $i \le n$, $x_i \in A$, $0 < \zeta_i \le \theta_i(x)$ and there is a $C_i \in \mathcal{C}_\alpha$ such that $x_i \in \overline{C_i}$, $\operatorname{diam} C_i = \zeta_i$ and $F(C_i) > q'\mu C_i$. By (a), we can enlarge C_i by adding a sufficiently small ball around x_i to form a $C_i' \in \mathcal{C}_{\alpha/2}$ such that $x_i \in \operatorname{int} C_i'$, $C_i' \subseteq B(x_i,\zeta_i)$ and $F(C_i') \ge q'\mu C_i'$.

Consider $\boldsymbol{t} = \{(x_i, C_i') : i \le n\}$. Then, because the balls $B(x_i,\zeta_i)$ are disjoint, and $x_i \in \operatorname{int} C_i' \subseteq \operatorname{cl}^* C_i'$ for every i, $\boldsymbol{t}$ is a δ_θ-fine member of $T_{\alpha/2}$. So $\sum_{i=0}^n F(C_i') \le \epsilon + \sum_{i=0}^n f(x_i)\mu C_i'$. But as $F(C_i') \ge q'\mu C_i'$ and $f(x_i) \le q$ for every i, this means that $(q'-q)\sum_{i=0}^n \mu C_i' \le \epsilon$.

But now remember that $\operatorname{diam} C_i' \ge \operatorname{diam} C_i = \zeta_i$ and that $C_i' \in \mathcal{C}_{\alpha/2}$ for each i. This means that

$$\mu C_i' \ge \frac{\alpha}{2}\zeta_i^r \ge \frac{\alpha}{2\beta_r}\mu B(x_i,\zeta_i)$$

for each i, and

$$\epsilon \ge (q'-q)\sum_{i=0}^n \mu C_i' \ge \tfrac{(q'-q)\alpha}{2\beta_r}\sum_{i=0}^n \mu B(x_i,\zeta_i)$$
$$> \tfrac{(q'-q)\alpha}{4\beta_r}\mu^* A = \epsilon$$

which is absurd. **X**

(c) Since q, q' and n are arbitrary, this means that $g \le_{\text{a.e.}} f$. Similarly (or applying (b) to $-f$ and $-F$)

$$f(x) \le \lim_{\zeta\downarrow 0} \inf\{\tfrac{F(C)}{\mu C} : C \in \mathcal{C}_\alpha,\, x \in \overline{C},\, 0 < \operatorname{diam} C \le \zeta\}$$

for almost all x, as required.

484P Lemma Let $\phi : \mathbb{R}^r \to \mathbb{R}^r$ be an injective Lipschitz function, and H the set of points at which it is differentiable; for $x \in H$, write $T(x)$ for the derivative of ϕ at x and $J(x)$ for $|\det T(x)|$. Then, for μ-almost every $x \in \mathbb{R}^r$,

$$J(x)=\lim_{\zeta\downarrow 0}\sup\{\frac{\mu\phi[C]}{\mu C}:C\in\mathcal{C}_\alpha,\,x\in\overline{C},\,0<\operatorname{diam}C\le\zeta\}$$
$$=\lim_{\zeta\downarrow 0}\inf\{\frac{\mu\phi[C]}{\mu C}:C\in\mathcal{C}_\alpha,\,x\in\overline{C},\,0<\operatorname{diam}C\le\zeta\}$$

for every $\alpha>0$.

proof By Rademacher's theorem (262Q), H is conegligible. Let H' be the Lebesgue set of J, so that H' also is conegligible (261E). Take any $x\in H'$ and $\epsilon>0$. Then there is a $\zeta_0>0$ such that $\int_{B(x,\zeta)}|J(y)-J(x)|\mu(dy)\le\epsilon\mu B(x,\zeta)$ for every $\zeta\in[0,\zeta_0]$. Now suppose that $C\in\mathcal{C}_\alpha$, $x\in\overline{C}$ and $0<\operatorname{diam}C\le\zeta_0$. Then $\mu(C\setminus H)=0$ so $\mu\phi[C\setminus H]=0$ (262D), and

$$\mu\phi[C]=\mu\phi[C\cap H]=\int_{C\cap H}J\,d\mu$$

(263D(iv)). So

$$|\mu\phi[C]-J(x)\mu C|=|\int_{C\cap H}J\,d\mu-J(x)\mu(C\cap H)|\le\int_{C\cap H}|J(y)-J(x)|d\mu$$
$$\le\int_{B(x,\operatorname{diam}C)}|J(y)-J(x)|d\mu\le\epsilon\mu B(x,\operatorname{diam}C)$$
$$=\beta_r\epsilon(\operatorname{diam}C)^r\le\frac{\beta_r\epsilon}{\alpha}\mu C.$$

Thus $|\frac{\mu\phi[C]}{\mu C}-J(x)|\le\frac{\beta_r}{\alpha}\epsilon$ whenever $C\in\mathcal{C}_\alpha$, $x\in\overline{C}$ and $0<\operatorname{diam}C\le\zeta_0$; as ϵ is arbitrary,

$$J(x)=\lim_{\zeta\downarrow 0}\sup\{\frac{\mu\phi[C]}{\mu C}:C\in\mathcal{C}_\alpha,\,x\in\overline{C},\,0<\operatorname{diam}C\le\zeta\}$$
$$=\lim_{\zeta\downarrow 0}\inf\{\frac{\mu\phi[C]}{\mu C}:C\in\mathcal{C}_\alpha,\,x\in\overline{C},\,0<\operatorname{diam}C\le\zeta\}.$$

And this is true for μ-almost every x.

484Q Definition If (X,ρ) and (Y,σ) are metric spaces, a function $\phi:X\to Y$ is a **lipeomorphism** if it is bijective and both ϕ and ϕ^{-1} are Lipschitz. Of course a lipeomorphism is a homeomorphism.

484R Lemma Let $\phi:\mathbb{R}^r\to\mathbb{R}^r$ be a lipeomorphism.

(a) For any set $A\subseteq\mathbb{R}^r$,

$$\mathrm{cl}^*(\phi[A])=\phi[\mathrm{cl}^*A],\quad \mathrm{int}^*(\phi[A])=\phi[\mathrm{int}^*A],\quad \partial^*(\phi[A])=\phi[\partial^*A].$$

(b) $\phi[C]\in\mathcal{C}$ for every $C\in\mathcal{C}$, and $\phi[V]\in\mathcal{V}$ for every $V\in\mathcal{V}$.

(c) For any $\alpha>0$ there is an $\alpha'\in\,]0,\alpha]$ such that $\phi[C]\in\mathcal{C}_{\alpha'}$ for every $C\in\mathcal{C}_\alpha$ and $\{(\phi(x),\phi[C]):(x,C)\in\boldsymbol{t}\}$ belongs to $T_{\alpha'}$ for every $\boldsymbol{t}\in T_\alpha$.

(d) For any $\mathcal{R}\in\mathfrak{R}$ there is an $\mathcal{R}'\in\mathfrak{R}$ such that $\phi[R]\in\mathcal{R}$ for every $R\in\mathcal{R}'$.

(e) $\theta\phi:\mathbb{R}^r\to[0,\infty[$ belongs to Θ for every $\theta\in\Theta$.

proof Let γ be so large that it is a Lipschitz constant for both ϕ and ϕ^{-1}. Observe that in this case

$$\phi^{-1}[B(\phi(x),\frac{\zeta}{\gamma})]\subseteq B(x,\zeta),\quad \phi[B(x,\zeta)]\supseteq B(\phi(x),\frac{\zeta}{\gamma})$$

for every $x\in\mathbb{R}^r$ and $\zeta\ge 0$, while

$$\mu^*A=\mu^*\phi^{-1}[\phi[A]]\le\gamma^r\mu^*\phi[A],\quad \nu^*A\le\gamma^{r-1}\nu^*\phi[A]$$

for every $A\subseteq\mathbb{R}^r$ (471J).

(a) If $A\subseteq\mathbb{R}^r$ and $x\in\mathrm{cl}^*A$, set

$$\epsilon=\frac{1}{2}\limsup_{\zeta\downarrow 0}\frac{\mu^*(B(x,\zeta)\cap A)}{\mu B(x,\zeta)}>0.$$

Take any $\zeta_0 > 0$. Then there is a ζ such that $0 < \zeta \le \zeta_0$ and $\mu^*(B(x,\frac{\zeta}{\gamma}) \cap A) \ge \epsilon\mu B(x,\frac{\zeta}{\gamma})$, so that

$$\mu^*(B(\phi(x),\zeta) \cap \phi[A]) \ge \mu^*\phi[B(x,\frac{\zeta}{\gamma}) \cap A] \ge \frac{1}{\gamma^r}\mu^*(B(x,\frac{\zeta}{\gamma}) \cap A)$$
$$\ge \frac{\epsilon}{\gamma^r}\mu B(x,\frac{\zeta}{\gamma}) = \frac{\epsilon}{\gamma^{2r}}\mu B(x,\zeta).$$

As ζ_0 is arbitrary,

$$\limsup_{\zeta\downarrow 0}\frac{\mu^*(B(\phi(x),\zeta)\cap\phi[A]}{\mu B(\phi(x),\zeta)} \ge \frac{\epsilon}{\gamma^{2r}} > 0,$$

and $\phi(x) \in \mathrm{cl}^*(\phi[A])$.

This shows that $\phi[\mathrm{cl}^*A] \subseteq \mathrm{cl}^*(\phi[A])$. The same argument applies to ϕ^{-1} and $\phi[A]$, so that $\phi[\mathrm{cl}^*A]$ must be equal to $\mathrm{cl}^*(\phi[A])$. Taking complements, $\phi[\mathrm{int}^*A] = \mathrm{int}^*(\phi[A])$, so that $\phi[\partial^*A] = \partial^*(\phi[A])$.

(b) Take $C \in \mathcal{C}$. Then, for any $n \in \mathbb{N}$, $\phi^{-1}[B(\mathbf{0},n)]$ is bounded, so is included in $B(\mathbf{0},m)$ for some m. Now

$$\nu(\partial^*\phi[C] \cap B(\mathbf{0},n)) = \nu(\phi[\partial^*C] \cap B(\mathbf{0},n))$$
$$\le \nu(\phi[\partial^*C \cap B(\mathbf{0},m)]) \le \gamma^{r-1}\nu(\partial^*C \cap B(\mathbf{0},m))$$

is finite. This shows that $\phi[C]$ has locally finite perimeter and belongs to $\mathcal{C}$. Since $\phi[V]$ is bounded whenever V is bounded, $\phi[V] \in \mathcal{V}$ whenever $V \in \mathcal{V}$.

(c) Set $\alpha' = \gamma^{-2r}\alpha$. Note that as γ^2 is a Lipschitz constant for the identity map, $\gamma \ge 1$, and $\alpha' \le \min(\alpha,\gamma^{2-2r}\alpha)$. If $C \in \mathcal{C}_\alpha$, then

$$\mu\phi[C] \ge \frac{1}{\gamma^r}\mu C \ge \frac{\alpha}{\gamma^r}(\mathrm{diam}\,C)^r \ge \alpha\,\mathrm{diam}\,\phi[C])^r \ge \alpha'(\mathrm{diam}\,\phi[C])^r,$$

$$\mathrm{per}\,\phi[C] = \nu(\partial^*(\phi[C])) = \nu(\phi[\partial^*C]) \le \gamma^{r-1}\nu(\partial^*C)$$
$$\le \frac{\gamma^{r-1}}{\alpha}(\mathrm{diam}\,C)^{r-1} \le \frac{\gamma^{r-1}}{\alpha}(\gamma\,\mathrm{diam}\,\phi[C])^{r-1} \le \frac{1}{\alpha'}(\mathrm{diam}\,\phi[C])^{r-1}.$$

So $C \in \mathcal{C}_{\alpha'}$.

If now $\boldsymbol{t} \in T_\alpha$, then, for any $(x,C) \in \boldsymbol{t}$, $\phi[C] \in \mathcal{C}_{\alpha'}$ and $\phi(x) \in \mathrm{cl}^*\phi[C]$; also, because ϕ is injective, $\langle\phi[C]\rangle_{(x,C)\in\boldsymbol{t}}$ is disjoint, so $\{(\phi(x),\phi[C]) : (x,C) \in \boldsymbol{t}\} \in T_{\alpha'}$.

(d) Express $\mathcal{R}$ as $\mathcal{R}^{(V)}_\eta$ where $V \in \mathcal{V}$ and $\eta \in \mathrm{H}$, so that $R \in \mathcal{R}$ whenever $R \cap V \in \mathcal{R}$. By 484Ec and 481He, there is a sequence $\langle\mathcal{Q}_i\rangle_{i\in\mathbb{N}}$ in $\mathfrak{R}$ such that $\bigcup_{i\le n}A_i \in \mathcal{R}$ whenever $n \in \mathbb{N}$, $\langle A_i\rangle_{i\le n}$ is disjoint and $A_i \in \mathcal{Q}_i$ for every i. By 484E(b-ii), there is an $\eta' \in \mathrm{H}$ such that $R \in \mathcal{Q}_i$ whenever $i \in \mathbb{N}$ and R is such that $\mu R \le \gamma^r\eta'(i)$ and $\mathrm{per}\,R \le \gamma^{r-1}$. Try $\mathcal{R}' = \mathcal{R}^{(\phi^{-1}[V])}_{\eta'} \in \mathfrak{R}$. If $R \in \mathcal{R}'$, we can express $R \cap \phi^{-1}[V]$ as $\bigcup_{i\le n}E_i$ where $\mathrm{per}\,E_i \le 1$ and $\mu E_i \le \eta'_i$ for each $i \le n$, and $\langle E_i\rangle_{i\le n}$ is disjoint. So $\phi[R] \cap V = \bigcup_{i\le n}\phi[E_i]$ and $\langle\phi[E_i]\rangle_{i\le n}$ is disjoint. Now, for each i,

$$\mu\phi[E_i] \le \gamma^r\mu E_i \le \gamma^r\eta'(i), \quad \mathrm{per}\,\phi[E_i] \le \gamma^{r-1}\,\mathrm{per}\,E_i \le \gamma^{r-1},$$

so $\phi[E_i] \in \mathcal{Q}_i$. By the choice of $\langle\mathcal{Q}_i\rangle_{i\in\mathbb{N}}$, $\phi[R] \cap V \in \mathcal{R}$ and $\phi[R] \in \mathcal{R}$. So $\mathcal{R}'$ has the property we need.

(e) We have only to observe that if A is the thin set $\theta^{-1}[\{0\}]$, then $(\theta\phi)^{-1}[\{0\}] = \phi^{-1}[A]$ is also thin. **P** If $A = \bigcup_{n\in\mathbb{N}}A_n$ where ν^*A_n is finite for every n, then $\phi^{-1}[A] = \bigcup_{n\in\mathbb{N}}\phi^{-1}[A_n]$, while $\nu^*\phi^{-1}[A_n] \le \gamma^{r-1}\nu^*A_n$ is finite for every $n \in \mathbb{N}$. **Q**

484S Theorem Let $\phi : \mathbb{R}^r \to \mathbb{R}^r$ be a lipeomorphism. Let H be the set of points at which ϕ is differentiable. For $x \in H$, write $T(x)$ for the derivative of ϕ at x; set $J(x) = |\det T(x)|$ for $x \in H$, 0 for $x \in \mathbb{R}^r \setminus H$. Then, for any function $f : \mathbb{R}^r \to \mathbb{R}^r$,

$$\mathrm{Pf}\!\!\int f = \mathrm{Pf}\!\!\int J \times f\phi$$

if either is defined in $\mathbb{R}$.

proof (a) Let $H' \subseteq H$ be a conegligible set such that

$$J(x) = \lim_{\zeta\downarrow 0} \sup\{\frac{\mu\phi[C]}{\mu C} : C \in \mathcal{C}_\alpha,\ x \in \overline{C},\ 0 < \operatorname{diam} C \le \zeta\}$$
$$= \lim_{\zeta\downarrow 0} \inf\{\frac{\mu\phi[C]}{\mu C} : C \in \mathcal{C}_\alpha,\ x \in \overline{C},\ 0 < \operatorname{diam} C \le \zeta\}$$

for every $\alpha > 0$ and every $x \in H'$ (484P). To begin with (down to the end of (c)), suppose that f is Pfeffer integrable and that $f\phi(x) = 0$ for every $x \in \mathbb{R}^r \setminus H'$. Let F be the Saks-Henstock indefinite integral of f, and define $G : \mathcal{C} \to \mathbb{R}$ by setting $G(C) = F(\phi[C])$ for every $C \in \mathcal{C}$ (using 484Rb to see that this is well-defined).

(b) G and $J \times f\phi$ satisfy the conditions of 484J.

P(i) Of course G is additive, because F is.

(ii) Suppose that $0 < \alpha < \alpha^*$ and $\epsilon > 0$. Let $\alpha' \in]0, \alpha^*[$ be such that $\{(\phi(x), \phi[C]) : (x, C) \in \boldsymbol{t}\} \in T_{\alpha'}$ whenever $\boldsymbol{t} \in T_\alpha$ (484Rc). Let $\theta_1 \in \Theta$ be such that $\sum_{(x,C)\in\boldsymbol{t}} |F(C) - f(x)\mu C| \le \frac{1}{2}\epsilon$ for every δ_{θ_1}-fine $\boldsymbol{t} \in T_{\alpha'}$. Let $\theta_2 : \mathbb{R}^r \to]0, 1]$ be such that whenever $x \in H'$, $n \le \|x\| + |f\phi(x)| < n + 1$, $C \in \mathcal{C}_{\alpha'}$, $x \in \overline{C}$ and $\operatorname{diam} C \le 2\theta_2(x)$ then

$$|\mu\phi[C] - J(x)\mu C| \le \frac{\epsilon\mu C}{2^{n+2}\beta_r(n+2)^r(n+1)}.$$

Set $\theta(x) = \min(\frac{1}{\gamma}\theta_1\phi(x), \theta_2(x))$ for $x \in \mathbb{R}^r$, where $\gamma > 0$ is a Lipschitz constant for ϕ, so that $\theta \in \Theta$ (484Re).

If $\boldsymbol{t} \in T_\alpha$ is δ_θ-fine, set $\boldsymbol{t}' = \{(\phi(x), \phi[C]) : (x, C) \in \boldsymbol{t}\}$. Then $\boldsymbol{t}' \in T_{\alpha'}$, by the choice of α'. If $(x, C) \in \boldsymbol{t}$, then $\theta(x) > 0$ so $\theta_1\phi(x) > 0$; also, for any $y \in \phi[C]$,

$$\|\phi(x) - y\| \le \gamma\|x - \phi^{-1}(y)\| < \gamma\theta(x) \le \theta_1\phi(x).$$

This shows that $\boldsymbol{t}'$ is δ_{θ_1}-fine. We therefore have

$$\sum_{(x,C)\in\boldsymbol{t}} |G(C) - J(x)f(\phi(x))\mu C| \le \sum_{(x,C)\in\boldsymbol{t}} |F(\phi[C]) - f(\phi(x))\mu\phi[C]|$$
$$+ \sum_{(x,C)\in\boldsymbol{t}} |f(\phi(x))||\mu\phi[C] - J(x)\mu C|$$
$$\le \sum_{(x,C)\in\boldsymbol{t}'} |F(C) - f(x)\mu C|$$
$$+ \sum_{\substack{(x,C)\in\boldsymbol{t}\\ x\in H'}} |f(\phi(x))||\mu\phi[C] - J(x)\mu C|$$

(because if $x \notin H'$ then $f(\phi(x)) = 0$)

$$\le \frac{1}{2}\epsilon + \sum_{n=0}^{\infty} \sum_{\substack{(x,C)\in\boldsymbol{t},x\in H'\\ n\le\|x\|+|f(\phi(x))|<n+1}} \frac{(n+1)\epsilon\mu C}{2^{n+2}\beta_r(n+2)^r(n+1)}$$
$$\le \frac{1}{2}\epsilon + \sum_{n=0}^{\infty} \frac{\epsilon\mu B(\mathbf{0}, n+2)}{2^{n+2}\beta_r(n+2)^r}$$

(remembering that $\theta_2(x) \le 1$, so $C \subseteq B(\mathbf{0}, n + 2)$ whenever $(x, C) \in \boldsymbol{t}$ and $\|x\| < n + 1$)

$$= \epsilon.$$

As $\boldsymbol{t}$ is arbitrary, this shows that G and $J \times f\phi$ satisfy (ii) of 484J.

(iii) Given $\epsilon > 0$, there is an $\mathcal{R} \in \mathfrak{R}$ such that $|F(C)| \le \epsilon$ for every $C \in \mathcal{C} \cap \mathcal{R}$. Now by 484Rd there is an $\mathcal{R}' \in \mathfrak{R}$ such that $\phi[R] \in \mathcal{R}$ for every $R \in \mathcal{R}'$, so that $|G(C)| \le \epsilon$ for every $C \in \mathcal{C} \cap \mathcal{R}'$. Thus G satisfies (iii) of 484J. **Q**

(c) This shows that $J \times f\phi$ is Pfeffer integrable, with Saks-Henstock indefinite integral G; so, in particular,

$$\mathrm{Pf}\!\!\int f \times \phi = G(\mathbb{R}^r) = F(\mathbb{R}^r) = \mathrm{Pf}\!\!\int f.$$

(d) Now suppose that f is an arbitrary Pfeffer integrable function. In this case set $f_1 = f \times \chi\phi[H']$. Because $\mathbb{R}^r \setminus H'$ is μ-negligible, so is $\phi[\mathbb{R}^r \setminus H']$, and $f_1 = f$ μ-a.e. Also, of course, $f\phi = f_1\phi$ μ-a.e. Because the Pfeffer integral extends the Lebesgue integral (484He),

$$\mathrm{Pf}\!\!\int J \times f\phi = \mathrm{Pf}\!\!\int J \times f_1\phi = \mathrm{Pf}\!\!\int f_1 = \mathrm{Pf}\!\!\int f.$$

(e) All this has been on the assumption that f is Pfeffer integrable. If $g = J \times f\phi$ is Pfeffer integrable, consider $\tilde{J} \times g\phi^{-1}$, where $\tilde{J}(x) = |\det \tilde{T}(x)|$ whenever the derivative $\tilde{T}(x)$ of ϕ^{-1} at x is defined, and otherwise is zero. Now

$$\tilde{J}(x)g(\phi^{-1}(x)) = \tilde{J}(x) \cdot J(\phi^{-1}(x))f(x)$$

for every x. But, for μ-almost every x,

$$\tilde{J}(x)J(\phi^{-1}(x)) = |\det \tilde{T}(x)||\det T(\phi^{-1}(x))| = |\det \tilde{T}(x)T(\phi^{-1}(x))| = 1$$

because $\tilde{T}(x)T(\phi^{-1}(x))$ is (whenever it is defined) the derivative at $\phi^{-1}(x)$ of the identity function $\phi^{-1}\phi$, by 473Bc. (I see that we need to know that $\{x : \phi$ is differentiable at $\phi^{-1}(x)\} = \phi[H]$ is conegligible.) So $\tilde{J} \times g\phi^{-1} = f$ μ-a.e., and f is Pfeffer integrable. This completes the proof.

484X Basic exercises >(a) Show that for every $\mathcal{R} \in \mathfrak{R}$ there are $\eta \in \mathrm{H}$ and $n \in \mathbb{N}$ such that $\mathcal{R}_\eta^{(B(\mathbf{0},n))} \subseteq \mathcal{R}$.

>(b) (PFEFFER 91A) For $\alpha > 0$ let $\mathcal{C}'_\alpha$ be the family of bounded Lebesgue measurable sets C such that $\mu C \ge \alpha \operatorname{diam} C \operatorname{per} C$. Show that $\mathcal{C}_{\sqrt{\alpha}} \subseteq \mathcal{C}'_\alpha \subseteq \mathcal{C}_{\min(\alpha,\alpha^r)}$. (*Hint*: 474La.)

>(c) For $\alpha > 0$, let $\mathcal{C}''_\alpha$ be the family of bounded convex sets $C \subseteq \mathbb{R}^r$ such that $\mu C \ge \alpha(\operatorname{diam} C)^r$. Show that if $0 < \alpha < 1/2r$ then $\mathcal{C}''_\alpha \subseteq \mathcal{C}_\alpha$ (*hint*: 475T) and $T_\alpha \cap [\mathbb{R}^r \times \mathcal{C}''_\alpha]^{<\omega}$ is compatible with Δ and $\mathfrak{R}$. (*Hint*: use the argument of 484F, but in part (b) take $C = \mathbb{R}^r$, $E = V$ a union of members of $\mathcal{D}$.)

(d) Describe a suitable filter $\mathcal{F}$ to express the Pfeffer integral directly in the form considered in 481C.

(e) Let $f : \mathbb{R}^r \to \mathbb{R}$ be a Pfeffer integrable function. Show that there is some $n \in \mathbb{N}$ such that $\int_{\mathbb{R}^r \setminus B(\mathbf{0},n)} |f| d\mu$ is finite.

(f) (Here take $r = 2$.) Let $\langle \delta_n \rangle_{n\in\mathbb{N}}$ be a strictly decreasing summable sequence in $]0,1]$. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by saying that $f(x) = \frac{(-1)^n}{(n+1)(\delta_n^2 - \delta_{n+1}^2)}$ if $n \in \mathbb{N}$ and $\delta_{n+1} \le \|x\| < \delta_n$, 0 otherwise. Show that $\lim_{\delta\downarrow 0} \int_{\mathbb{R}^2 \setminus B(\mathbf{0},\delta)} f d\mu$ is defined, but that f is not Pfeffer integrable. (*Hint*: 484J.)

(g) (Again take $r = 2$.) Show that there are a Lebesgue integrable $f_1 : \mathbb{R} \to \mathbb{R}$ and a Henstock integrable $f_2 : \mathbb{R} \to \mathbb{R}$, both with bounded support, such that $(\xi_1, \xi_2) \mapsto f_1(\xi_1)f_2(\xi_2) : \mathbb{R}^2 \to \mathbb{R}$ is not Pfeffer integrable.

(h) Let $E \subseteq \mathbb{R}^r$ be a bounded set with finite perimeter, and $\phi : \mathbb{R}^r \to \mathbb{R}^r$ a differentiable function. Let v_x be the Federer exterior normal to E at any point x where the normal exists. Show that $\mathrm{Pf}\!\!\int \operatorname{div} \phi \times \chi E$ is defined and equal to $\int_{\partial^* E} \phi(x) \cdot v_x \nu(dx)$.

(i) Show that there is a Lipschitz function $f : \mathbb{R}^r \to [0,1]$ such that $\mathbb{R}^r \setminus \operatorname{dom} f'$ is not thin. (*Hint*: there is a Lipschitz function $f : \mathbb{R} \to [0,1]$ not differentiable at any point of the Cantor set.)

484Y Further exercises (a) Let $E \subseteq \mathbb{R}^r$ be any Lebesgue measurable set, and $\epsilon > 0$. Show that there is a Lebesgue measurable set $G \subseteq E$ such that $\operatorname{per} G \le \operatorname{per} E$, $\mu(E \setminus G) \le \epsilon$ and $\mathrm{cl}^* G = \overline{G}$.

(b) Give an example of a compact set $K \subseteq \mathbb{R}^2$ with zero one-dimensional Hausdorff measure such that whenever $\theta : K \to]0, \infty[$ is a strictly positive function, and $\gamma \in \mathbb{R}$, there is a disjoint family $\langle B(x_i, \zeta_i) \rangle_{i\le n}$ of balls such that $x_i \in K$ and $\zeta_i \le \theta(x_i)$ for every i, while $\operatorname{per}(\bigcup_{i\le n} B(x_i, \zeta_i)) \ge \gamma$.

484 Notes and comments Listing the properties of the Pfeffer integral as developed above, we have

expected relations with Lebesgue measure and integration (484Hd-484Hf);
Saks-Henstock indefinite integrals (484H-484J);
integration over suitable subsets (484L);
a divergence theorem (484N);
a density theorem (484O);
a change-of-variable theorem for lipeomorphisms (484S).

The results on indefinite integrals and integration over subsets are restricted in comparison with what we have for the Lebesgue integral, since we can deal only with sets with locally finite perimeter; and 484S is similarly narrower in scope than 263D(v). Pfeffer's Divergence Theorem, on the other hand, certainly applies to many functions ϕ for which $\operatorname{div}\phi$ is not Lebesgue integrable, though it does not entirely cover 475N (see 484Xi). In comparison with the one-dimensional case, the Pfeffer integral does not share the most basic property of the special Denjoy integral (483Bd, 484Xf), but 484N is a step towards the Perron integral (483J). 484O is a satisfactory rendering of the idea in 483I, and even for Lebesgue integrable functions adds something to 261C. Throughout, I have written on the assumption that $r \geq 2$. It would be possible to work through the same arguments with $r = 1$, but in this case we should find that 'thin' sets became countable, therefore easily controllable by neighbourhood gauges, making the methods here inappropriate.

The whole point of 'gauge integrals' is that we have an enormous amount of freedom within the framework of §§481-482. There is a corresponding difficulty in making definitive choices. The essential ideology of the Pfeffer integral is that we take an intersection of a family of gauge integrals, each determined by a family $\mathcal{C}_\alpha$ of sets which are 'Saks regular' in the sense that their measures, perimeters and diameters are linked (compare 484Xb). Shrinking $\mathcal{C}_\alpha$ and T_α, while leaving Δ and $\mathfrak{R}$ unchanged, of course leads to a more 'powerful' integral (supposing, at least, that we do not go so far that T_α is no longer compatible with Δ and $\mathfrak{R}$), so that Pfeffer's Divergence Theorem will remain true. One possibility is to turn to convex sets (484Xc), though we could not then expect invariance under lipeomorphisms.

The family $\Delta = \{\delta_\theta : \theta \in \Theta\}$ of gauges is designed to permit the exclusion of tags from thin sets; apart from this refinement, we are looking at neighbourhood gauges, just as with the Henstock integral. This feature, or something like it, seems to be essential when we come to the identification $F(E) = \mathrm{P}\!\!\!\int\!\mathrm{f}\, f \times \chi E$ in 484L, which is demanded by the formula in the target theorem 484N. In order to make our families T_α compatible in the sense of 481F, we are then forced to allow non-trivial residual families; with some effort (484C, 484Ed), we can get tagged-partition structures allowing subdivisions (484F). Note that this is one of the cases in which our residual families $\mathcal{R}_\eta^{(V)}$ are defined by 'shape' as well as by 'size'. In the indefinite-integral characterization of the Pfeffer integral (484J), we certainly cannot demand 'for every $\epsilon > 0$ there is an $\mathcal{R} \in \mathfrak{R}$ such that $|F(E)| \leq \epsilon$ whenever $E \in \mathcal{C}$ is included in a member of $\mathcal{R}$', since all small balls belong to $\mathcal{R}$, and we should immediately be driven to the Lebesgue integral. However I use the construction $\mathcal{R}_\eta^{(V)} = \{R : R \cap V \in \mathcal{R}_\eta\}$ (484D) as a quick method of eliminating any difficulties at infinity (484Xe). We do not of course need to look at arbitrary sets $V \in \mathcal{V}$ here (484Xa).

Observe that 484B can be thought of as a refinement of 475I. As usual, the elaborate formula in the statement of 484C is there only to emphasize that we have a bound depending only on l and r. Note that 484S depends much more on the fact that the Pfeffer integral can be characterized in the language of 484J, than on the exact choices made in forming $\mathfrak{R}$ and the $\mathcal{C}_\alpha$. For a discussion of integrals *defined* by Saks-Henstock lemmas, see Pfeffer 01.

It would be agreeable to be able to think of the Pfeffer integral as a product in some sense, so we naturally look for Fubini-type theorems. I give 484Xg to indicate one of the obstacles.

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