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Chapter 47

Geometric measure theory

I offer a chapter on geometric measure theory, continuing from Chapter 26. The greater part of it is directed specifically at two topics: a version of the Divergence Theorem (475N) and the elementary theory of Newtonian capacity and potential (§479). I do not attempt to provide a balanced view of the subject, for which I must refer you to MATTILA 95, EVANS & GARIEPY 92 and FEDERER 69. However §472, at least, deals with something which must be central to any approach, Besicovitch's Density Theorem for Radon measures on $\mathbb{R}^r$ (472D). In §473 I examine Lipschitz functions, and give crude forms of some fundamental ineqalities relating integrals $\int \|\operatorname{grad} f\| d\mu$ with other measures of the variation of a function f (473H, 473K). In §474 I introduce perimeter measures λ^∂_E and outward-normal functions ψ_E as those for which the Divergence Theorem, in the form $\int_E \operatorname{div}\phi\, d\mu = \int \phi\,\boldsymbol{.}\,\psi_E\, d\lambda^\partial_E$, will be valid (474E), and give the geometric description of $\psi_E(x)$ as the Federer exterior normal to E at x (474R). In §475 I show that λ^∂_E can be identified with normalized Hausdorff $(r-1)$-dimensional measure on the essential boundary of E.

§471 is devoted to Hausdorff measures on general metric spaces, extending the ideas introduced in §264 for Euclidean space, up to basic results on densities (471P) and Howroyd's theorem (471S). In §476 I turn to a different topic, the problem of finding the subsets of $\mathbb{R}^r$ on which Lebesgue measure is most 'concentrated' in some sense. I present a number of classical results, the deepest being the Isoperimetric Theorem (476H): among sets with a given measure, those with the smallest perimeters are the balls.

The last three sections are different again. Classical electrostatics led to a vigorous theory of capacity and potential, based on the idea of 'harmonic function'. It turns out that 'Brownian motion' in $\mathbb{R}^r$ (§477) gives an alternative and very powerful approach to the subject. I have brought Brownian motion and Wiener measure to this chapter because I wish to use them to illuminate the geometry of $\mathbb{R}^r$; but much of §477 (in particular, the strong Markov property, 477G) is necessarily devoted to adapting ideas developed in the more general contexts of Lévy and Gaussian processes, as described in §§455-456. In §478 I give the most elementary parts of the theory of harmonic and superharmonic functions, building up to a definition of 'harmonic measures' based on Brownian motion (478P). In §479 I use these techniques to describe Newtonian capacity and its extension Choquet-Newton capacity (479C) on Euclidean space of three or more dimensions, and establish their basic properties (479E, 479F, 479N, 479P, 479U).

Version of 10.2.16

471 Hausdorff measures

I begin the chapter by returning to a class of measures which we have not examined in depth since Chapter 26. The primary importance of these measures is in studying the geometry of Euclidean space; in §265 I looked briefly at their use in describing surface measures, which will reappear in §475. Hausdorff measures are also one of the basic tools in the study of fractals, but for such applications I must refer you to FALCONER 90 and MATTILA 95. All I shall attempt to do here is to indicate some of the principal ideas which are applicable to general metric spaces, and to look at some special properties of Hausdorff measures related to the concerns of this chapter and of §261.

471A Definition Let (X,ρ) be a metric space and $r \in \,]0,\infty[$. For $\delta > 0$ and $A \subseteq X$, set

$$\theta_{r\delta}A = \inf\{\sum_{n=0}^{\infty}(\operatorname{diam} D_n)^r : \langle D_n\rangle_{n\in\mathbb{N}} \text{ is a sequence of subsets of } X \text{ covering } A,$$
$$\operatorname{diam} D_n \le \delta \text{ for every } n \in \mathbb{N}\}.$$

It will be useful to note that every $\theta_{r\delta}$ is an outer measure. Now set

$$\theta_r A = \sup_{\delta>0} \theta_{r\delta} A$$

for $A \subseteq X$; θ_r also is an outer measure on X; this is r**-dimensional Hausdorff outer measure** on X. Let μ_{Hr} be the measure defined by Carathéodory's method from θ_r; μ_{Hr} is r**-dimensional Hausdorff measure** on X.

471B Definition Let (X, ρ) be a metric space. An outer measure θ on X is a **metric outer measure** if $\theta(A \cup B) = \theta A + \theta B$ whenever A, $B \subseteq X$ and $\rho(A, B) > 0$.

471C Proposition Let (X, ρ) be a metric space and θ a metric outer measure on X. Let μ be the measure on X defined from θ by Carathéodory's method. Then μ is a topological measure.

471D Theorem Let (X, ρ) be a metric space and $r > 0$. Let μ_{Hr} be r-dimensional Hausdorff measure on X, and Σ its domain; write θ_r for r-dimensional Hausdorff outer measure on X.

(a) μ_{Hr} is a topological measure.

(b) For every $A \subseteq X$ there is a G_δ set $H \supseteq A$ such that $\mu_{Hr} H = \theta_r A$.

(c) θ_r is the outer measure defined from μ_{Hr}.

(d) Σ is closed under Souslin's operation.

(e) $\mu_{Hr} E = \sup\{\mu_{Hr} F : F \subseteq E \text{ is closed}\}$ whenever $E \in \Sigma$ and $\mu_{Hr} E < \infty$.

(f) If $A \subseteq X$ and $\theta_r A < \infty$ then A is separable and the set of isolated points of A is μ_{Hr}-negligible.

(g) μ_{Hr} is atomless.

(h) If μ_{Hr} is totally finite it is a quasi-Radon measure.

471E Corollary If (X, ρ) is a metric space, $r > 0$ and $Y \subseteq X$ then r-dimensional Hausdorff measure $\mu_{Hr}^{(Y)}$ on Y extends the subspace measure $(\mu_{Hr}^{(X)})_Y$ on Y induced by r-dimensional Hausdorff measure $\mu_{Hr}^{(X)}$ on X; and if either Y is measured by $\mu_{Hr}^{(X)}$ or Y has finite r-dimensional Hausdorff outer measure in X, then $\mu_{Hr}^{(Y)} = (\mu_{Hr}^{(X)})_Y$.

471F Corollary Let (X, ρ) be an analytic metric space, and write μ_{Hr} for r-dimensional Hausdorff measure on X. Suppose that ν is a locally finite indefinite-integral measure over μ_{Hr}. Then ν is a Radon measure.

471G Increasing Sets Lemma Let (X, ρ) be a metric space and $r > 0$.

(a) Suppose that $\delta > 0$ and that $\langle A_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of subsets of X with union A. Then $\theta_{r,6\delta}(A) \le (5^r + 2) \sup_{n\in\mathbb{N}} \theta_{r\delta} A_n$.

(b) Suppose that $\delta > 0$ and that $\langle A_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of subsets of X with union A. Then $\theta_{r\delta} A = \sup_{n\in\mathbb{N}} \theta_{r\delta} A_n$.

471H Corollary Let (X, ρ) be a metric space, and $r > 0$. For $A \subseteq X$, set

$$\theta_{r\infty} A = \inf\{\textstyle\sum_{n=0}^\infty (\operatorname{diam} D_n)^r : \langle D_n \rangle_{n\in\mathbb{N}} \text{ is a sequence of subsets of } X \text{ covering } A\}.$$

Then $\theta_{r\infty}$ is an outer regular Choquet capacity on X.

Remark $\theta_{r\infty}$ is r**-dimensional Hausdorff capacity** on X.

471I Theorem Let (X, ρ) be a metric space, and $r > 0$. Write μ_{Hr} for r-dimensional Hausdorff measure on X. If $A \subseteq X$ is analytic, then $\mu_{Hr} A$ is defined and equal to $\sup\{\mu_{Hr} K : K \subseteq A \text{ is compact}\}$.

471J Proposition Let (X, ρ) and (Y, σ) be metric spaces, and $f : X \to Y$ a γ-Lipschitz function, where $\gamma \ge 0$. If $r > 0$ and $\theta_r^{(X)}$, $\theta_r^{(Y)}$ are the r-dimensional Hausdorff outer measures on X and Y respectively, then $\theta_r^{(Y)} f[A] \le \gamma^r \theta_r^{(X)} A$ for every $A \subseteq X$.

471K Lemma Let (X,ρ) be a metric space, and $r>0$. Let μ_{Hr} be r-dimensional Hausdorff measure on X. If $A\subseteq X$, then $\mu_{Hr}A=0$ iff for every $\epsilon>0$ there is a countable family $\mathcal{D}$ of sets, covering A, such that $\sum_{D\in\mathcal{D}}(\operatorname{diam}D)^r\le\epsilon$.

471L Proposition Let (X,ρ) be a metric space and $0<r<s$. If $A\subseteq X$ is such that μ_{Hr}^*A is finite, then $\mu_{Hs}A=0$.

471M Definition If (X,ρ) is a metric space and $A\subseteq X$, write $A^\sim$ for $\{x:x\in X,\,\rho(x,A)\le 2\operatorname{diam}A\}$, where $\rho(x,A)=\inf_{y\in A}\rho(x,y)$. ($\emptyset^\sim=\emptyset$.)

471N Lemma Let (X,ρ) be a metric space. Let $\mathcal{F}$ be a family of subsets of X such that $\{\operatorname{diam}F:F\in\mathcal{F}\}$ is bounded. Set

$$Y=\bigcap_{\delta>0}\bigcup\{F:F\in\mathcal{F},\,\operatorname{diam}F\le\delta\}.$$

Then there is a disjoint family $\mathcal{I}\subseteq\mathcal{F}$ such that

(i) $\bigcup\mathcal{F}\subseteq\bigcup_{F\in\mathcal{I}}F^\sim$;

(ii) $Y\subseteq\overline{\bigcup\mathcal{J}}\cup\bigcup_{F\in\mathcal{I}\setminus\mathcal{J}}F^\sim$ for every $\mathcal{J}\subseteq\mathcal{I}$.

471O Lemma Let (X,ρ) be a metric space, and $r>0$. Suppose that A, $\mathcal{F}$ are such that

(i) $\mathcal{F}$ is a family of closed subsets of X such that $\sum_{n=0}^\infty(\operatorname{diam}F_n)^r$ is finite for every disjoint sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{F}$,

(ii) for every $x\in A$, $\delta>0$ there is an $F\in\mathcal{F}$ such that $x\in F$ and $0<\operatorname{diam}F\le\delta$.

Then there is a countable disjoint family $\mathcal{I}\subseteq\mathcal{F}$ such that $A\setminus\bigcup\mathcal{I}$ has zero r-dimensional Hausdorff measure.

471P Theorem Let (X,ρ) be a metric space, and $r>0$. Let μ_{Hr} be r-dimensional Hausdorff measure on X. Suppose that $A\subseteq X$ and $\mu_{Hr}^*A<\infty$.

(a) $\lim_{\delta\downarrow 0}\sup\{\frac{\mu_{Hr}^*(A\cap D)}{(\operatorname{diam}D)^r}:x\in D,\,0<\operatorname{diam}D\le\delta\}=1$ for μ_{Hr}-almost every $x\in A$.

(b) $\limsup_{\delta\downarrow 0}\frac{\mu_{Hr}^*(A\cap B(x,\delta))}{\delta^r}\ge 1$ for μ_{Hr}-almost every $x\in A$. So

$$2^{-r}\le\limsup_{\delta\downarrow 0}\frac{\mu_{Hr}^*(A\cap B(x,\delta))}{(\operatorname{diam}B(x,\delta))^r}\le 1$$

for μ_{Hr}-almost every $x\in A$.

(c) If A is measured by μ_{Hr}, then

$$\lim_{\delta\downarrow 0}\sup\{\frac{\mu_{Hr}^*(A\cap D)}{(\operatorname{diam}D)^r}:x\in D,\,0<\operatorname{diam}D\le\delta\}=0$$

for μ_{Hr}-almost every $x\in X\setminus A$.

471Q Lemma Let (X,ρ) be a metric space, and $r>0$, $\delta>0$. Suppose that $\theta_{r\delta}X$, as defined in 471A, is finite.

(a) There is a non-negative additive functional ν on $\mathcal{P}X$ such that $\nu X=5^{-r}\theta_{r\delta}X$ and $\nu A\le(\operatorname{diam}A)^r$ whenever $A\subseteq X$ and $\operatorname{diam}A\le\frac{1}{5}\delta$.

(b) If X is compact, there is a Radon measure μ on X such that $\mu X=5^{-r}\theta_{r\delta}X$ and $\mu G\le(\operatorname{diam}G)^r$ whenever $G\subseteq X$ is open and $\operatorname{diam}G\le\frac{1}{5}\delta$.

471R Lemma Let (X,ρ) be a compact metric space and $r>0$. Let μ_{Hr} be r-dimensional Hausdorff measure on X. If $\mu_{Hr}X>0$, there is a Borel set $H\subseteq X$ such that $0<\mu_{Hr}H<\infty$.

471S Theorem Let (X,ρ) be an analytic metric space, and $r>0$. Let μ_{Hr} be r-dimensional Hausdorff measure on X, and $\mathcal{B}$ the Borel σ-algebra of X. Then the Borel measure $\mu_{Hr}\restriction\mathcal{B}$ is semi-finite and tight.

471T Proposition Let (X,ρ) be a metric space, and $r>0$.

(a) If X is analytic and $\mu_{Hr}X>0$, then for every $s\in\,]0,r[$ there is a non-zero Radon measure μ on X such that $\iint\frac{1}{\rho(x,y)^s}\mu(dx)\mu(dy)<\infty$.

(b) If there is a non-zero topological measure μ on X such that $\iint\frac{1}{\rho(x,y)^r}\mu(dx)\mu(dy)$ is finite, then $\mu_{Hr}X=\infty$.

471Z Problems (a) Let $\mu_{H1}^{(2)}$, $\mu_{H,1/2}^{(1)}$ be one-dimensional Hausdorff measure on $\mathbb{R}^2$ and $\frac{1}{2}$-dimensional Hausdorff measure on $\mathbb{R}$ respectively, for their usual metrics. Are the measure spaces $(\mathbb{R}^2,\mu_{H1}^{(2)})$ and $(\mathbb{R},\mu_{H,1/2}^{(1)})$ isomorphic? (See 471Yj.)

(b) Let ρ be a metric on $\mathbb{R}^2$ inducing the usual topology, and $\mu_{H2}^{(\rho)}$ the corresponding 2-dimensional Hausdorff measure. Is it necessarily the case that $\mu_{H2}^{(\rho)}(\mathbb{R}^2)>0$? (See 471Yf.)

Version of 22.3.11

472 Besicovitch's Density Theorem

The first step in the program of the next few sections is to set out some very remarkable properties of Euclidean space. We find that in $\mathbb{R}^r$, for geometric reasons (472A), we have versions of Vitali's theorem (472B-472C) and Lebesgue's Density Theorem (472D) for arbitrary Radon measures. I add a version of the Hardy-Littlewood Maximal Theorem (472F).

Throughout the section, $r\ge 1$ will be a fixed integer. As usual, I write $B(x,\delta)$ for the closed ball with centre x and radius δ. $\|\;\|$ will represent the Euclidean norm, and $x\,.\,y$ the scalar product of x and y, so that $x\,.\,y=\sum_{i=1}^r\xi_i\eta_i$ if $x=(\xi_1,\dots,\xi_r)$ and $y=(\eta_1,\dots,\eta_r)$.

472A Besicovitch's Covering Lemma Suppose that $\epsilon>0$ is such that $(5^r+1)(1-\epsilon-\epsilon^2)^r>(5+\epsilon)^r$. Let $x_0,\dots,x_n\in\mathbb{R}^r$, $\delta_0,\dots,\delta_n>0$ be such that

$$\|x_i-x_j\|>\delta_i,\quad \delta_j\le(1+\epsilon)\delta_i$$

whenever $i<j\le n$. Then

$$\#(\{i:i\le n,\ \|x_i-x_n\|\le\delta_i+\delta_n\})\le 5^r.$$

472B Theorem Let $A\subseteq\mathbb{R}^r$ be a bounded set, and $\mathcal{I}$ a family of non-trivial closed balls in $\mathbb{R}^r$ such that every point of A is the centre of a member of $\mathcal{I}$. Then there is a family $\langle\mathcal{I}_k\rangle_{k<5^r}$ of countable subsets of $\mathcal{I}$ such that each $\mathcal{I}_k$ is disjoint and $\bigcup_{k<5^r}\mathcal{I}_k$ covers A.

472C Theorem Let λ be a Radon measure on $\mathbb{R}^r$, A a subset of $\mathbb{R}^r$ and $\mathcal{I}$ a family of non-trivial closed balls in $\mathbb{R}^r$ such that every point of A is the centre of arbitrarily small members of $\mathcal{I}$. Then

(a) there is a countable disjoint $\mathcal{I}_0\subseteq\mathcal{I}$ such that $\lambda(A\setminus\bigcup\mathcal{I}_0)=0$;

(b) for every $\epsilon>0$ there is a countable $\mathcal{I}_1\subseteq\mathcal{I}$ such that $A\subseteq\bigcup\mathcal{I}_1$ and $\sum_{B\in\mathcal{I}_1}\lambda B\le\lambda^*A+\epsilon$.

472D Besicovitch's Density Theorem Let λ be any Radon measure on $\mathbb{R}^r$. Then, for any locally λ-integrable real-valued function f,

(a) $f(y)=\lim_{\delta\downarrow 0}\frac{1}{\lambda B(y,\delta)}\int_{B(y,\delta)}f\,d\lambda$,

(b) $\lim_{\delta\downarrow 0}\frac{1}{\lambda B(y,\delta)}\int_{B(y,\delta)}|f(x)-f(y)|\lambda(dx)=0$

for λ-almost every $y\in\mathbb{R}^r$.

***472E Proposition** Let λ, λ' be Radon measures on $\mathbb{R}^r$, and $G\subseteq\mathbb{R}^r$ an open set. Let Z be the support of λ, and for $x\in Z\cap G$ set

$$M(x)=\sup\{\frac{\lambda' B}{\lambda B}:B\subseteq G \text{ is a non-trivial ball with centre } x\}.$$

Then

$$\lambda\{x:x\in Z,\,M(x)\ge t\}\le\frac{5^r}{t}\lambda' G$$

for every $t>0$.

***472F Theorem** Let λ be a Radon measure on $\mathbb{R}^r$, and $f\in\mathcal{L}^p(\lambda)$ any function, where $1<p<\infty$. Let Z be the support of λ, and for $x\in Z$ set $f^*(x)=\sup_{\delta>0}\frac{1}{\lambda B(x,\delta)}\int_{B(x,\delta)}|f|d\lambda$. Then f^* is lower semi-continuous, and $\|f^*\|_p\le 2\bigl(\frac{5^r p}{p-1}\bigr)^{1/p}\|f\|_p$.

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473 Poincaré's inequality

In this section I embark on the main work of the first half of the chapter, leading up to the Divergence Theorem in §475. I follow the method in Evans & Gariepy 92. The first step is to add some minor results on differentiable and Lipschitz functions to those already set out in §262 (473B-473C). Then we need to know something about convolution products (473D), extending ideas in §§256 and 444; in particular, it will be convenient to have a fixed sequence $\langle\tilde h_n\rangle_{n\in\mathbb{N}}$ of smoothing functions with some useful special properties (473E).

The new ideas of the section begin with the Gagliardo-Nirenberg-Sobolev inequality, relating $\|f\|_{r/(r-1)}$ to $\int\|\operatorname{grad}f\|$. In its simplest form (473H) it applies only to functions with compact support; we need to work much harder to get results which we can use to estimate $\int_B|f|^{r/(r-1)}$ in terms of $\int_B\|\operatorname{grad}f\|$ and $\int_B|f|$ for balls B (473I, 473K).

473A Notation For the next three sections, $r\ge 2$ will be a fixed integer. For $x\in\mathbb{R}^r$ and $\delta\ge 0$, $B(x,\delta)=\{y:\|y-x\|\le\delta\}$ will be the closed ball with centre x and radius δ. I will write $\partial B(x,\delta)$ for the boundary of $B(x,\delta)$, the sphere $\{y:\|y-x\|=\delta\}$. $S_{r-1}=\partial B(\mathbf{0},1)$ will be the unit sphere. As in Chapter 26, I will use Greek letters to represent coordinates of vectors, so that $x=(\xi_1,\dots,\xi_r)$, etc.

μ will always be Lebesgue measure on $\mathbb{R}^r$. $\beta_r=\mu B(\mathbf{0},1)$ will be the r-dimensional volume of the unit ball, that is,

$$\begin{aligned}\beta_r&=\frac{\pi^k}{k!}\text{ if } r=2k \text{ is even,}\\&=\frac{2^{2k+1}k!\pi^k}{(2k+1)!}\text{ if } r=2k+1 \text{ is odd}\end{aligned}$$

(252Q). ν will be normalized Hausdorff $(r-1)$-dimensional measure on $\mathbb{R}^r$, that is, $\nu=2^{-r+1}\beta_{r-1}\mu_{H,r-1}$, where $\mu_{H,r-1}$ is $(r-1)$-dimensional Hausdorff measure on $\mathbb{R}^r$ as described in §264. Recall from 265F and 265H that $\nu S_{r-1}=2\pi\beta_{r-2}=r\beta_r$ (counting β_0 as 1).

473B Differentiable functions (a) Recall from §262 that a function ϕ from a subset of $\mathbb{R}^r$ to $\mathbb{R}^s$ (where $s\ge 1$) is differentiable at $x\in\mathbb{R}^r$, with derivative an $s\times r$ matrix T, if for every $\epsilon>0$ there is a $\delta>0$ such that $\|\phi(y)-\phi(x)-T(y-x)\|\le\epsilon\|y-x\|$ whenever $\|y-x\|\le\delta$; this includes the assertion that $B(x,\delta)\subseteq\operatorname{dom}\phi$. In this case, the coefficients of T are the partial derivatives $\frac{\partial\phi_j}{\partial\xi_i}(x)$ at x, where $\phi_1,\dots,\phi_s$ are the coordinate functions of ϕ, and $\frac{\partial}{\partial\xi_i}$ represents partial differentiation with respect to the ith coordinate.

(b) When $s=1$, I will write $(\operatorname{grad}f)(x)$ for the derivative of f at x, the **gradient** of f.

(c) Chain rule for functions of many variables Let $\phi : A \to \mathbb{R}^s$ and $\psi : B \to \mathbb{R}^p$ be functions, where $A \subseteq \mathbb{R}^r$ and $B \subseteq \mathbb{R}^s$. If $x \in A$ is such that ϕ is differentiable at x, with derivative S, and ψ is differentiable at $\phi(x)$, with derivative T, then the composition $\psi\phi$ is differentiable at x, with derivative TS.

(d) It follows that if f and g are real-valued functions defined on a subset of $\mathbb{R}^r$, and $x \in \operatorname{dom} f \cap \operatorname{dom} g$ is such that $(\operatorname{grad} f)(x)$ and $(\operatorname{grad} g)(x)$ are both defined, then $\operatorname{grad}(f \times g)(x)$ is defined and equal to $f(x) \operatorname{grad} g(x) + g(x) \operatorname{grad} f(x)$.

(e) Let D be a subset of $\mathbb{R}^r$ and $\phi : D \to \mathbb{R}^s$ any function. Set $D_0 = \{x : x \in D,\ \phi \text{ is differentiable at } x\}$. Then the derivative of ϕ, regarded as a function from D_0 to $\mathbb{R}^{rs}$, is (Lebesgue) measurable.

(f) If $G \subseteq \mathbb{R}^r$ is an open set, a function $\phi : G \to \mathbb{R}^s$ is **smooth** if it is differentiable arbitrarily often; that is, if all its repeated partial derivatives

$$\frac{\partial^m \phi_j}{\partial \xi_{i_1} \ldots \partial \xi_{i_m}}$$

are defined and continuous everywhere on G. I will write $\mathcal{D}$ for the family of real-valued functions from $\mathbb{R}^r$ to $\mathbb{R}$ which are smooth and have compact support.

473C Lipschitz functions (a) If f and g are bounded real-valued Lipschitz functions, defined on any subsets of $\mathbb{R}^r$, then $f \times g$, defined on $\operatorname{dom} f \cap \operatorname{dom} g$, is Lipschitz.

(b) Suppose that F_1, $F_2 \subseteq \mathbb{R}^r$ are closed sets with convex union C. Let $f : C \to \mathbb{R}$ be a function such that $f{\restriction}F_1$ and $f{\restriction}F_2$ are both Lipschitz. Then f is Lipschitz.

(c) Suppose that $f : \mathbb{R}^r \to \mathbb{R}$ is Lipschitz. $\operatorname{grad} f$ is defined almost everywhere. $\operatorname{grad} f$ is (Lebesgue) measurable on its domain. If γ is a Lipschitz constant for f, $\|\operatorname{grad} f(x)\| \le \gamma$ whenever $\operatorname{grad} f(x)$ is defined.

(d) Conversely, if $f : \mathbb{R}^r \to \mathbb{R}$ is differentiable and $\|\operatorname{grad} f(x)\| \le \gamma$ for every x, then γ is a Lipschitz constant for f.

(e) Note that if $f \in \mathcal{D}$ then f is Lipschitz as well as bounded.

(f)(i) If $D \subseteq \mathbb{R}^r$ is bounded and $f : D \to \mathbb{R}$ is Lipschitz, then there is a Lipschitz function $g : \mathbb{R}^r \to \mathbb{R}$, with compact support, extending f.

(ii) It follows that if $D \subseteq \mathbb{R}^r$ is bounded and $f : D \to \mathbb{R}^s$ is Lipschitz, then there is a Lipschitz function $g : \mathbb{R}^r \to \mathbb{R}^s$, with compact support, extending f.

473D Smoothing by convolution: Lemma Suppose that f and g are Lebesgue measurable real-valued functions defined μ-almost everywhere in $\mathbb{R}^r$.

(a) If f is integrable and g is essentially bounded, then their convolution $f * g$ is defined everywhere in $\mathbb{R}^r$ and uniformly continuous, and $\|f * g\|_\infty \le \|f\|_1 \operatorname{ess\,sup} |g|$.

(b) If f is locally integrable and g is bounded and has compact support, then $f * g$ is defined everywhere in $\mathbb{R}^r$ and is continuous.

(c) If f and g are defined everywhere in $\mathbb{R}^r$ and $x \in \mathbb{R}^r \setminus (\{y : f(y) \ne 0\} + \{z : g(z) \ne 0\})$, then $(f * g)(x)$ is defined and equal to 0.

(d) If f is integrable and g is bounded, Lipschitz and defined everywhere, then $f * \operatorname{grad} g$ and $\operatorname{grad}(f * g)$ are defined everywhere and equal, where $f * \operatorname{grad} g = (f * \frac{\partial g}{\partial \xi_1}, \ldots, f * \frac{\partial g}{\partial \xi_r})$. Moreover, $f * g$ is Lipschitz.

(e) If f is locally integrable, and $g \in \mathcal{D}$, then $f * g$ is defined everywhere and is smooth.

(f) If f is essentially bounded and $g \in \mathcal{D}$, then all the derivatives of $f * g$ are bounded, and $f * g$ is Lipschitz.

(g) If f is integrable and $\phi : \mathbb{R}^r \to \mathbb{R}^r$ is a bounded measurable function with components $\phi_1, \ldots, \phi_r$, and we write $(f * \phi)(x) = ((f * \phi_1)(x), \ldots, (f * \phi_r)(x))$, then $\|(f * \phi)(x)\| \le \|f\|_1 \sup_{y \in \mathbb{R}^r} \|\phi(y)\|$ for every $x \in \mathbb{R}^r$.

473E Lemma (a) Define $h:\mathbb{R}\to[0,1]$ by setting $h(t)=\exp(\frac{1}{t^2-1})$ for $|t|<1$, 0 for $|t|\ge 1$. Then h is smooth, and $h'(t)\le 0$ for $t\ge 0$.

(b) For $n\ge 1$, define $\tilde h_n:\mathbb{R}^r\to\mathbb{R}$ by setting

$$\alpha_n=\int h((n+1)^2\|x\|^2)\mu(dx),\quad \tilde h_n(x)=\frac{1}{\alpha_n}h((n+1)^2\|x\|^2)$$

for every $x\in\mathbb{R}^r$. Then $\tilde h_n\in\mathcal{D}$, $\tilde h_n(x)\ge 0$ for every x, $\tilde h_n(x)=0$ if $\|x\|\ge\frac{1}{n+1}$, and $\int\tilde h_n d\mu=1$.

(c) If $f\in\mathcal{L}^0(\mu)$, then $\lim_{n\to\infty}(f*\tilde h_n)(x)=f(x)$ for every $x\in\operatorname{dom}f$ at which f is continuous.

(d) If $f:\mathbb{R}^r\to\mathbb{R}$ is uniformly continuous (in particular, if it is either Lipschitz or a continuous function with compact support), then $\lim_{n\to\infty}\|f-f*\tilde h_n\|_\infty=0$.

(e) If $f\in\mathcal{L}^0(\mu)$ is locally integrable, then $f(x)=\lim_{n\to\infty}(f*\tilde h_n)(x)$ for μ-almost every $x\in\mathbb{R}^r$.

(f) If $f\in\mathcal{L}^p(\mu)$, where $1\le p<\infty$, then $\lim_{n\to\infty}\|f-f*\tilde h_n\|_p=0$.

473F Lemma For any measure space (X,Σ,λ) and any non-negative $f_1,\dots,f_k\in\mathcal{L}^0(\lambda)$,

$$\int\prod_{i=1}^k f_i^{1/k}d\lambda\le\prod_{i=1}^k\Bigl(\int f_i d\lambda\Bigr)^{1/k}.$$

473G Lemma Let (X,Σ,λ) be a σ-finite measure space and $k\ge 2$ an integer. Write λ_k for the product measure on X^k. For $x=(\xi_1,\dots,\xi_k)\in X^k$, $t\in X$ and $1\le i\le k$ set $S_i(x,t)=(\xi'_1,\dots,\xi'_k)$ where $\xi'_i=t$ and $\xi'_j=\xi_j$ for $j\ne i$. Then if $h\in\mathcal{L}^1(\lambda_k)$ is non-negative, and we set $h_i(x)=\int h(S_i(x,t))\lambda(dt)$ whenever this is defined in $\mathbb{R}$, we have

$$\int(\prod_{i=1}^k h_i)^{1/(k-1)}d\lambda_k\le(\int h\,d\lambda_k)^{k/(k-1)}.$$

473H Gagliardo-Nirenberg-Sobolev inequality Suppose that $f:\mathbb{R}^r\to\mathbb{R}$ is a Lipschitz function with compact support. Then $\|f\|_{r/(r-1)}\le\int\|\operatorname{grad}f\|d\mu$.

473I Lemma For any Lipschitz function $f:B(\mathbf{0},1)\to\mathbb{R}$,

$$\int_{B(\mathbf{0},1)}|f|^{r/(r-1)}d\mu\le\Bigl(2^{r+4}\sqrt{r}\int_{B(\mathbf{0},1)}\|\operatorname{grad}f\|+|f|d\mu\Bigr)^{r/(r-1)}.$$

473J Lemma Let $f:\mathbb{R}^r\to\mathbb{R}$ be a Lipschitz function. Then

$$\int_{B(y,\delta)}|f(x)-f(z)|\mu(dx)\le\frac{2^r}{r}\delta^r\int_{B(y,\delta)}\|\operatorname{grad}f(x)\|\|x-z\|^{1-r}\mu(dx)$$

whenever $y\in\mathbb{R}^r$, $\delta>0$ and $z\in B(y,\delta)$.

473K Poincaré's inequality for balls Let $B\subseteq\mathbb{R}^r$ be a non-trivial closed ball, and $f:B\to\mathbb{R}$ a Lipschitz function. Set $\gamma=\frac{1}{\mu B}\int_B f d\mu$. Then

$$\Bigl(\int_B|f-\gamma|^{r/(r-1)}d\mu\Bigr)^{(r-1)/r}\le c\int_B\|\operatorname{grad}f\|d\mu,$$

where $c=2^{r+4}\sqrt{r}(1+2^{r+1})$.

473L Corollary Let $B\subseteq\mathbb{R}^r$ be a non-trivial closed ball, and $f:B\to[0,1]$ a Lipschitz function. Set

$$F_0=\{x:x\in B,\ f(x)\le\tfrac14\},\quad F_1=\{x:x\in B,\ f(x)\ge\tfrac34\}.$$

Then

$$\bigl(\min(\mu F_0,\mu F_1)\bigr)^{(r-1)/r}\le 4c\int_B\|\operatorname{grad}f\|d\mu,$$

where $c=2^{r+4}\sqrt{r}(1+2^{r+1})$.

473M The case $r=1$ In this case, B is just a closed interval, and

$$\|f\times\chi B-\gamma\chi B\|_\infty=\sup_{x\in B}|f(x)-\gamma|\le\sup_{x,y\in B}|f(x)-f(y)|\le\int_B|f'|d\mu,$$

giving a version of 473K for $r=1$. As for 473L, if $\int_B|f'|<\frac{1}{2}$ then at least one of F_0, F_1 must be empty.

Version of 17.11.12

474 The distributional perimeter

The next step is a dramatic excursion, defining (for appropriate sets E) a perimeter *measure* for which a version of the Divergence Theorem is true (474E). I begin the section with elementary notes on the divergence of a vector field (474B-474C). I then define 'locally finite perimeter' (474D), 'perimeter measure' and 'outward normal' (474F) and 'reduced boundary' (474G). The definitions rely on the Riesz representation theorem, and we have to work very hard to relate them to any geometrically natural idea of 'boundary'. Even half-spaces (474I) demand some attention. From Poincaré's inequality we can prove isoperimetric inequalities for perimeter measures (474L). With some effort we can locate the reduced boundary as a subset of the topological boundary, and obtain asymptotic inequalities on the perimeter measures of small balls (474N). With much more effort we can find a geometric description of outward normal functions in terms of 'Federer exterior normals' (474R), and get a tight asymptotic description of the perimeter measures of small balls (474S). I end with the Compactness Theorem for sets of bounded perimeter (474T).

474B The divergence of a vector field (a) For a function ϕ from a subset of $\mathbb{R}^r$ to $\mathbb{R}^r$, write $\operatorname{div}\phi=\sum_{i=1}^r\frac{\partial\phi_i}{\partial\xi_i}$, where $\phi=(\phi_1,\dots,\phi_r)$; for definiteness, let us take the domain of $\operatorname{div}\phi$ to be the set of points at which ϕ is differentiable. $\operatorname{div}\phi\in\mathcal{D}$ for every $\phi\in\mathcal{D}_r$.

(b) If $f:\mathbb{R}^r\to\mathbb{R}$ and $\phi:\mathbb{R}^r\to\mathbb{R}^r$ are functions, then $\operatorname{div}(f\times\phi)=\phi\centerdot\operatorname{grad}f+f\times\operatorname{div}\phi$ at any point at which f and ϕ are both differentiable.

(c) If $\phi:\mathbb{R}^r\to\mathbb{R}^r$ is a Lipschitz function with compact support, then $\int\operatorname{div}\phi\,d\mu=0$.

(d) If $\phi:\mathbb{R}^r\to\mathbb{R}^r$ and $f:\mathbb{R}^r\to\mathbb{R}$ are Lipschitz functions, one of which has compact support, then $f\times\phi$ is Lipschitz.

It follows that

$$\int\phi\centerdot\operatorname{grad}f\,d\mu+\int f\times\operatorname{div}\phi\,d\mu=0.$$

(e) If $f\in\mathcal{L}^\infty(\mu)$, $g\in\mathcal{L}^1(\mu)$ is even and $\phi:\mathbb{R}^r\to\mathbb{R}^r$ is a Lipschitz function with compact support, then $\int(f*g)\times\operatorname{div}\phi=\int f\times\operatorname{div}(g*\phi)$.

474C Invariance under isometries: Proposition Suppose that $T:\mathbb{R}^r\to\mathbb{R}^r$ is an isometry, and that ϕ is a function from a subset of $\mathbb{R}^r$ to $\mathbb{R}^r$. Then

$$\operatorname{div}(T^{-1}\phi T)=(\operatorname{div}\phi)T.$$

474D Locally finite perimeter: Definition Let $E\subseteq\mathbb{R}^r$ be a Lebesgue measurable set. Its **perimeter** $\operatorname{per}E$ is

$$\sup\{|\int_E\operatorname{div}\phi\,d\mu|:\phi:\mathbb{R}^r\to B(\mathbf{0},1)\text{ is a Lipschitz function with compact support}\}$$

(allowing ∞). A set $E\subseteq\mathbb{R}^r$ has **locally finite perimeter** if it is Lebesgue measurable and

$$\sup\{|\int_E\operatorname{div}\phi\,d\mu|:\phi:\mathbb{R}^r\to\mathbb{R}^r\text{ is a Lipschitz function, }\|\phi\|\le\chi B(\mathbf{0},n)\}$$

is finite for every $n\in\mathbb{N}$. Of course a Lebesgue measurable set with finite perimeter also has locally finite perimiter.

474E Theorem Suppose that $E \subseteq \mathbb{R}^r$ has locally finite perimeter.

(i) There are a Radon measure λ^∂_E on $\mathbb{R}^r$ and a Borel measurable function $\psi : \mathbb{R}^r \to S_{r-1}$ such that

$$\int_E \operatorname{div}\phi\,d\mu = \int \phi\,\boldsymbol{.}\,\psi\,d\lambda^\partial_E$$

for every Lipschitz function $\phi : \mathbb{R}^r \to \mathbb{R}^r$ with compact support.

(ii) This formula uniquely determines λ^∂_E, which can also be defined by saying that

$$\lambda^\partial_E(G) = \sup\{|\int_E \operatorname{div}\phi\,d\mu| : \phi : \mathbb{R}^r \to \mathbb{R}^r \text{ is Lipschitz, } \|\phi\| \le \chi G\}$$

whenever $G \subseteq \mathbb{R}^r$ is open.

(iii) If $\hat\psi$ is another function defined λ^∂_E-a.e. and satisfying the formula in (i), then $\hat\psi$ and ψ are equal λ^∂_E-almost everywhere.

474F Definitions In the context of 474E, I will call λ^∂_E the **perimeter measure** of E, and if ψ is a function from a λ^∂_E-conegligible subset of $\mathbb{R}^r$ to S_{r-1} which has the property in (i) of the theorem, I will call it an **outward-normal** function for E.

Observe that if E has locally finite perimeter, then $\operatorname{per}E = \lambda^\partial_E(\mathbb{R}^r)$. The definitions in 474D-474E also make it clear that if E, $F \subseteq \mathbb{R}$ are Lebesgue measurable and $\mu(E \triangle F) = 0$, then $\operatorname{per}E = \operatorname{per}F$ and E has locally finite perimeter iff F has; and in this case $\lambda^\partial_E = \lambda^\partial_F$ and an outward-normal function for E is an outward-normal function for F.

474G The reduced boundary Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter; let λ^∂_E be its perimeter measure and ψ an outward-normal function for E. The **reduced boundary** $\partial^\$E$ is the set of those $y \in \mathbb{R}^r$ such that, for some $z \in S_{r-1}$,

$$\lim_{\delta\downarrow 0} \frac{1}{\lambda^\partial_E B(y,\delta)} \int_{B(y,\delta)} \|\psi(x) - z\|\,\lambda^\partial_E(dx) = 0.$$

Note that, writing $\psi = (\psi_1, \dots, \psi_r)$ and $z = (\zeta_1, \dots, \zeta_r)$, we must have

$$\zeta_i = \lim_{\delta\downarrow 0} \frac{1}{\lambda^\partial_E B(y,\delta)} \int_{B(y,\delta)} \psi_i d\lambda^\partial_E,$$

so that z is uniquely defined; call it $\psi_E(y)$. $\partial^\$E$ and ψ_E are determined entirely by the set E.

$$\lim_{\delta\downarrow 0} \frac{1}{\lambda^\partial_E B(x,\delta)} \int_{B(x,\delta)} |\psi_i(x) - \psi_i(y)|\lambda^\partial_E(dx) = 0$$

for every $i \le r$, for λ^∂_E-almost every $y \in \mathbb{R}^r$; and for any such y, $\psi_E(y)$ is defined and equal to $\psi(y)$. $\partial^\$E$ is λ^∂_E-conegligible and ψ_E is an outward-normal function for E. I will call $\psi_E : \partial^\$E \to S_{r-1}$ the **canonical outward-normal function** of E.

Once again, we see that if E, $F \subseteq \mathbb{R}^r$ are sets with locally finite perimeter and $E \triangle F$ is Lebesgue negligible, then they have the same reduced boundary and the same canonical outward-normal function.

474H Invariance under isometries: Proposition Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter. Let λ^∂_E be its perimeter measure, and ψ_E its canonical outward-normal function. If $T : \mathbb{R}^r \to \mathbb{R}^r$ is any isometry, then $T[E]$ has locally finite perimeter, $\lambda^\partial_{T[E]}$ is the image measure $\lambda^\partial_E T^{-1}$, the reduced boundary $\partial^\$T[E]$ is $T[\partial^\$E]$, and the canonical outward-normal function of $T[E]$ is $S\psi_E T^{-1}$, where S is the derivative of T.

474I Half-spaces: Proposition Let $H \subseteq \mathbb{R}^r$ be a half-space $\{x : x\,\boldsymbol{.}\,v \le \alpha\}$, where $v \in S^{r-1}$. Then H has locally finite perimeter; its perimeter measure λ^∂_H is defined by saying

$$\lambda^\partial_H(F) = \nu(F \cap \partial H)$$

whenever $F \subseteq \mathbb{R}^r$ is such that ν measures $F \cap \partial H$, and the constant function with value v is an outward-normal function for H.

474J Lemma Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter. Let λ^∂_E be the perimeter measure of E, and ψ_E its canonical outward-normal function. Then $\mathbb{R}^r \setminus E$ also has locally finite perimeter; its perimeter measure is λ^∂_E, its reduced boundary is $\partial^\$ E$, and its canonical outward-normal function is $-\psi_E$.

474K Lemma Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter; let λ^∂_E be its perimeter measure, and ψ an outward-normal function for E. Let $\phi : \mathbb{R}^r \to \mathbb{R}^r$ be a Lipschitz function with compact support, and $g \in \mathcal{D}$ an even function. Then

$$\int \phi \,.\, \mathrm{grad}(g * \chi E) d\mu + \int (g * \phi) \,.\, \psi \, d\lambda^\partial_E = 0.$$

474L Two isoperimetric inequalities: Theorem Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter, and λ^∂_E its perimeter measure.

(a) If E is bounded, then $(\mu E)^{(r-1)/r} \le \mathrm{per}\, E$.

(b) If $B \subseteq \mathbb{R}^r$ is a closed ball, then

$$\min(\mu(B \cap E), \mu(B \setminus E))^{(r-1)/r} \le 2c\lambda^\partial_E(\mathrm{int}\, B),$$

where $c = 2^{r+4}\sqrt{r}(1 + 2^{r+1})$.

474M Lemma Suppose that $E \subseteq \mathbb{R}^r$ has locally finite perimeter, with perimeter measure λ^∂_E and an outward-normal function ψ. Then for any $y \in \mathbb{R}^r$ and any Lipschitz function $\phi : \mathbb{R}^r \to \mathbb{R}^r$,

$$\int_{E \cap B(y,\delta)} \mathrm{div}\, \phi \, d\mu = \int_{B(y,\delta)} \phi \,.\, \psi \, d\lambda^\partial_E + \int_{E \cap \partial B(y,\delta)} \phi(x) \,.\, \tfrac{1}{\delta}(x - y) \, \nu(dx)$$

for almost every $\delta > 0$.

474N Lemma Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter, and λ^∂_E its perimeter measure. Then, for any $y \in \partial^\$ E$,

(i) $\liminf_{\delta \downarrow 0} \frac{\mu(B(y,\delta) \cap E)}{\delta^r} \ge \frac{1}{(3r)^r}$;

(ii) $\liminf_{\delta \downarrow 0} \frac{\mu(B(y,\delta) \setminus E)}{\delta^r} \ge \frac{1}{(3r)^r}$;

(iii) $\liminf_{\delta \downarrow 0} \frac{\lambda^\partial_E B(y,\delta)}{\delta^{r-1}} \ge \frac{1}{2c(3r)^{r-1}}$,

where $c = 2^{r+4}\sqrt{r}(1 + 2^{r+1})$;

(iv) $\limsup_{\delta \downarrow 0} \frac{\lambda^\partial_E B(y,\delta)}{\delta^{r-1}} \le 4\pi\beta_{r-2}$.

474O Definition Let $A \subseteq \mathbb{R}^r$ be any set, and $y \in \mathbb{R}^r$. A **Federer exterior normal to A at y** is a $v \in S_{r-1}$ such that,

$$\lim_{\delta \downarrow 0} \frac{\mu^*((H \triangle A) \cap B(y,\delta))}{\mu B(y,\delta)} = 0,$$

where H is the half-space $\{x : (x - y) \,.\, v \le 0\}$.

474P Lemma If $A \subseteq \mathbb{R}^r$ and $y \in \mathbb{R}^r$, there can be at most one Federer exterior normal to A at y.

474Q Lemma Set $c' = 2^{r+3}\sqrt{r-1}(1 + 2^r)$. Suppose that c^*, ϵ and δ are such that

$$c^* \ge 0, \quad \delta > 0, \quad 0 < \epsilon < \frac{1}{\sqrt{2}}, \quad c^*\epsilon^3 < \tfrac{1}{4}\beta_{r-1}, \quad 4c'\epsilon \le \tfrac{1}{8}\beta_{r-1}.$$

Set $V_\delta = \{z : z \in \mathbb{R}^{r-1}, \|z\| \le \delta\}$ and $C_\delta = V_\delta \times [-\delta, \delta]$, regarded as a cylinder in $\mathbb{R}^r$. Let $f \in \mathcal{D}$ be such that

$$\int_{C_\delta} \|\,\mathrm{grad}_{r-1} f\| + \max(\frac{\partial f}{\partial \xi_r}, 0) d\mu \le c^*\epsilon^3\delta^{r-1},$$

where $\mathrm{grad}_{r-1} f = (\frac{\partial f}{\partial \xi_1}, \dots, \frac{\partial f}{\partial \xi_{r-1}}, 0)$. Set

$$F = \{x : x \in C_\delta,\ f(x) \ge \frac{3}{4}\}, \quad F' = \{x : x \in C_\delta,\ f(x) \le \frac{1}{4}\}.$$

and for $\gamma \in \mathbb{R}$ set $H_\gamma = \{x : x \in \mathbb{R}^r,\ \xi_r \le \gamma\}$. Then there is a $\gamma \in \mathbb{R}$ such that

$$\mu(F \triangle (H_\gamma \cap C_\delta)) \le 9\mu(C_\delta \setminus (F \cup F')) + (c^*\beta_{r-1} + 16c')\epsilon\delta^r.$$

474R Theorem Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter, ψ_E its canonical outward-normal function, and y any point of its reduced boundary $\partial^\$ E$. Then $\psi_E(y)$ is the Federer exterior normal to E at y.

474S Corollary Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter, and λ_E^∂ its perimeter measure. Let y be any point of the reduced boundary of E. Then

$$\lim_{\delta \downarrow 0} \frac{\lambda_E^\partial B(y,\delta)}{\beta_{r-1}\delta^{r-1}} = 1.$$

474T The Compactness Theorem Let Σ be the algebra of Lebesgue measurable subsets of $\mathbb{R}^r$, and give it the topology $\mathfrak{T}_m$ of convergence in measure defined by the pseudometrics $\rho_H(E,F) = \mu((E \triangle F) \cap H)$ for measurable sets H of finite measure. Then

(a) $\mathrm{per} : \Sigma \to [0,\infty]$ is lower semi-continuous;

(b) for any $\gamma < \infty$, $\{E : E \in \Sigma,\ \mathrm{per}\, E \le \gamma\}$ is compact.

Version of 24.1.13

475 The essential boundary

The principal aim of this section is to translate Theorem 474E into geometric terms. We have already identified the Federer exterior normal as an outward-normal function, so we need to find a description of perimeter measures. Most remarkably, these turn out, in every case considered in 474E, to be just normalized Hausdorff measures (475G). This description needs the concept of 'essential boundary' (475B). In order to complete the programme, we need to be able to determine which sets have 'locally finite perimeter'; there is a natural criterion in the same language (475L). We now have all the machinery for a direct statement of the Divergence Theorem (for Lipschitz functions) which depends on nothing more advanced than the definition of Hausdorff measure (475N). (The definitions, at least, of 'Federer exterior normal' and 'essential boundary' are elementary.)

This concludes the main work of the first part of this chapter. But since we are now within reach of a reasonably direct proof of a fundamental fact about the $(r-1)$-dimensional Hausdorff measure of the boundaries of subsets of $\mathbb{R}^r$ (475Q), I continue up to Cauchy's Perimeter Theorem and the Isoperimetric Theorem for convex sets (475S, 475T).

475B The essential boundary Let $A \subseteq \mathbb{R}^r$ be any set. The **essential closure** of A is the set

$$\mathrm{cl}^*A = \{x : \limsup_{\delta \downarrow 0} \frac{\mu^*(B(x,\delta) \cap A)}{\mu B(x,\delta)} > 0\}$$

Similarly, the **essential interior** of A is the set

$$\mathrm{int}^*A = \{x : \liminf_{\delta \downarrow 0} \frac{\mu_*(B(x,\delta) \cap A)}{\mu B(x,\delta)} = 1\}.$$

Finally, the **essential boundary** ∂^*A of A is $\mathrm{cl}^*A \setminus \mathrm{int}^*A$.

475C Lemma Let $A, A' \subseteq \mathbb{R}^r$.

(a)

$$\mathrm{int}\, A \subseteq \mathrm{int}^*A \subseteq \mathrm{cl}^*A \subseteq \overline{A}, \quad \partial^*A \subseteq \partial A,$$

$$\mathrm{cl}^*A = \mathbb{R}^r \setminus \mathrm{int}^*(\mathbb{R}^r \setminus A), \quad \partial^*(\mathbb{R}^r \setminus A) = \partial^*A.$$

(b) If $A \setminus A'$ is negligible, then $\mathrm{cl}^*A \subseteq \mathrm{cl}^*A'$ and $\mathrm{int}^*A \subseteq \mathrm{int}^*A'$; if A itself is negligible, cl^*A, int^*A and ∂^*A are all empty.

(c) int^*A, cl^*A and ∂^*A are Borel sets.

(d) $\mathrm{cl}^*(A\cup A') = \mathrm{cl}^*A\cup\mathrm{cl}^*A'$ and $\mathrm{int}^*(A\cap A') = \mathrm{int}^*A\cap\mathrm{int}^*A'$, so $\partial^*(A\cup A')$, $\partial^*(A\cap A')$ and $\partial^*(A\triangle A')$ are all included in $\partial^*A\cup\partial^*A'$.

(e) $\mathrm{cl}^*A\cap\mathrm{int}^*A' \subseteq \mathrm{cl}^*(A\cap A')$, $\partial^*A\cap\mathrm{int}^*A' \subseteq \partial^*(A\cap A')$ and $\partial^*A\setminus\mathrm{cl}^*A' \subseteq \partial^*(A\cup A')$.

(f) $\partial^*(A\cap A') \subseteq (\mathrm{cl}^*A'\cap\partial A)\cup(\partial^*A'\cap\mathrm{int}\,A)$.

(g) If $E \subseteq \mathbb{R}^r$ is Lebesgue measurable, then $E\triangle\mathrm{int}^*E$, $E\triangle\mathrm{cl}^*E$ and ∂^*E are Lebesgue negligible.

(h) A is Lebesgue measurable iff ∂^*A is Lebesgue negligible.

475D Lemma Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter. Then $\partial^{\$}E \subseteq \partial^*E$ and $\nu(\partial^*E\setminus\partial^{\$}E) = 0$.

475E Lemma Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter.

(a) If $A \subseteq \partial^{\$}E$, then $\nu^*A \le (\lambda^\partial_E)^*A$.

(b) If $A \subseteq \mathbb{R}^r$ and $\nu A = 0$, then $\lambda^\partial_E A = 0$.

475F Lemma Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter, and $\epsilon > 0$. Then λ^∂_E is inner regular with respect to the family $\mathcal{E} = \{F : F \subseteq \mathbb{R}^r \text{ is Borel}, \lambda^\partial_E F \le (1+\epsilon)\nu F\}$.

475G Theorem Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter. Then $\lambda^\partial_E = \nu \llcorner \partial^*E$.

475H Proposition Let $V \subseteq \mathbb{R}^r$ be a hyperplane, and $T : \mathbb{R}^r \to V$ the orthogonal projection. Suppose that $A \subseteq \mathbb{R}^r$ is such that νA is defined and finite, and for $u \in V$ set

$$\begin{aligned} f(u) &= \#(A\cap T^{-1}[\{u\}]) \text{ if this is finite,} \\ &= \infty \text{ otherwise.} \end{aligned}$$

Then $\int_V f(u)\nu(du)$ is defined and at most νA.

475I Lemma Let $\mathcal{K}$ be the family of compact sets $K \subseteq \mathbb{R}^r$ such that $K = \mathrm{cl}^*K$. Then μ is inner regular with respect to $\mathcal{K}$.

475J Lemma Let E be a Lebesgue measurable subset of $\mathbb{R}^r$, identified with $\mathbb{R}^{r-1}\times\mathbb{R}$. For $u \in \mathbb{R}^{r-1}$, set $G_u = \{t : (u,t) \in \mathrm{int}^*E\}$, $H_u = \{t : (u,t) \in \mathrm{int}^*(\mathbb{R}^r\setminus E)\}$ and $D_u = \{t : (u,t) \in \partial^*E\}$, so that G_u, H_u and D_u are disjoint and cover $\mathbb{R}$.

(a) There is a μ_{r-1}-conegligible set $Z \subseteq \mathbb{R}^{r-1}$ such that whenever $u \in Z$, $t < t'$ in $\mathbb{R}$, $t \in G_u$ and $t' \in H_u$, there is an $s \in D_u \cap \,]t,t'[$.

(b) There is a μ_{r-1}-conegligible set $Z_1 \subseteq \mathbb{R}^{r-1}$ such that whenever $u \in Z_1$, t, $t' \in \mathbb{R}$, $t \in G_u$ and $t' \in H_u$, there is a member of D_u between t and t'.

(c) If E has locally finite perimeter, there is a conegligible set $Z_2 \subseteq Z_1$ such that, for every $u \in Z_2$, $D_u \cap [-n,n]$ is finite for every $n \in \mathbb{N}$, G_u and H_u are open, and $D_u = \partial G_u = \partial H_u$, so that the constituent intervals of $\mathbb{R}\setminus D_u$ lie alternately in G_u and H_u.

475K Lemma Suppose that $h : \mathbb{R}^r \to [-1,1]$ is a Lipschitz function with compact support; let $n \in \mathbb{N}$ be such that $h(x) = 0$ for $\|x\| \ge n$. Suppose that $E \subseteq \mathbb{R}^r$ is a Lebesgue measurable set. Then

$$|\int_E \frac{\partial h}{\partial \xi_r} d\mu| \le 2\Bigl(\beta_{r-1}n^{r-1} + \nu(\partial^*E\cap B(\mathbf{0},n))\Bigr).$$

475L Theorem Suppose that $E \subseteq \mathbb{R}^r$. Then E has locally finite perimeter iff $\nu(\partial^*E\cap B(\mathbf{0},n))$ is finite for every $n \in \mathbb{N}$.

475M Corollary (a) The family of sets with locally finite perimeter is a subalgebra of the algebra of Lebesgue measurable subsets of $\mathbb{R}^r$.

(b) A set $E \subseteq \mathbb{R}^r$ is Lebesgue measurable and has finite perimeter iff $\nu(\partial^* E) < \infty$, and in this case $\nu(\partial^* E)$ is the perimeter of E.

(c) If $E \subseteq \mathbb{R}^r$ has finite measure, then $\operatorname{per} E = \liminf_{\alpha\to\infty} \operatorname{per}(E \cap B(\mathbf{0},\alpha))$.

475N The Divergence Theorem Let $E \subseteq \mathbb{R}^r$ be such that $\nu(\partial^* E \cap B(\mathbf{0},n))$ is finite for every $n \in \mathbb{N}$.

(a) E is Lebesgue measurable.

(b) For ν-almost every $x \in \partial^* E$, there is a Federer exterior normal v_x of E at x.

(c) For every Lipschitz function $\phi : \mathbb{R}^r \to \mathbb{R}^r$ with compact support,

$$\int_E \operatorname{div} \phi \, d\mu = \int_{\partial^* E} \phi(x) \,\boldsymbol{.}\, v_x \, \nu(dx).$$

475O Lemma Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter, and ψ_E its canonical outward-normal function. Let v be the unit vector $(0,\dots,0,1)$. Identify $\mathbb{R}^r$ with $\mathbb{R}^{r-1}\times\mathbb{R}$. Then we have sequences $\langle F_n\rangle_{n\in\mathbb{N}}$, $\langle g_n\rangle_{n\in\mathbb{N}}$ and $\langle g'_n\rangle_{n\in\mathbb{N}}$ such that

(i) for each $n \in \mathbb{N}$, F_n is a Lebesgue measurable subset of $\mathbb{R}^{r-1}$, and g_n, $g'_n : F_n \to [-\infty,\infty]$ are Lebesgue measurable functions such that $g_n(u) < g'_n(u)$ for every $u \in F_n$;

(ii) if m, $n \in \mathbb{N}$ are distinct and $u \in F_m \cap F_n$, then $[g_m(u), g'_m(u)] \cap [g_n(u), g'_n(u)] = \emptyset$;

(iii) $\sum_{n=0}^{\infty} \int_{F_n} g'_n - g_n d\mu_{r-1} = \mu E$;

(iv) for any continuous function $h : \mathbb{R}^r \to \mathbb{R}$ with compact support,

$$\int_{\partial^* E} h(x) v \,\boldsymbol{.}\, \psi_E(x) \, \nu(dx) = \sum_{n=0}^{\infty} \int_{F_n} h(u, g'_n(u)) - h(u, g_n(u)) \mu_{r-1}(du),$$

where we interpret $h(u,\infty)$ and $h(u,-\infty)$ as 0 if necessary;

(v) for μ_{r-1}-almost every $u \in \mathbb{R}^{r-1}$,

$$\begin{aligned}\{t : (u,t) \in \partial^* E\} = \{g_n(u) : n \in \mathbb{N},\, u \in F_n,\, g_n(u) \ne -\infty\} \\ \cup \{g'_n(u) : n \in \mathbb{N},\, u \in F_n,\, g'_n(u) \ne \infty\}.\end{aligned}$$

475P Lemma Let $v \in S_{r-1}$ be any unit vector, and $V \subseteq \mathbb{R}^r$ the hyperplane $\{x : x \,\boldsymbol{.}\, v = 0\}$. Let $T : \mathbb{R}^r \to V$ be the orthogonal projection. If $E \subseteq \mathbb{R}^r$ is any set with locally finite perimeter and canonical outward-normal function ψ_E, then

$$\int_{\partial^* E} |v \,\boldsymbol{.}\, \psi_E| d\nu = \int_V \#(\partial^* E \cap T^{-1}[\{u\}]) \nu(du),$$

interpreting $\#(\partial^* E \cap T^{-1}[\{u\}])$ as ∞ if $\partial^* E \cap T^{-1}[\{u\}]$ is infinite.

475Q Theorem (a) Let $E \subseteq \mathbb{R}^r$ be a set with finite perimeter. For $v \in S_{r-1}$ write V_v for $\{x : x \,\boldsymbol{.}\, v = 0\}$, and let $T_v : \mathbb{R}^r \to V_v$ be the orthogonal projection. Then

$$\begin{aligned}\operatorname{per} E = \nu(\partial^* E) &= \frac{1}{2\beta_{r-1}} \int_{S_{r-1}} \int_{V_v} \#(\partial^* E \cap T_v^{-1}[\{u\}]) \nu(du) \nu(dv) \\ &= \lim_{\delta\downarrow 0} \frac{1}{2\beta_{r-1}\delta} \int_{S_{r-1}} \mu(E \triangle (E + \delta v)) \nu(dv).\end{aligned}$$

(b) Suppose that $E \subseteq \mathbb{R}^r$ is Lebesgue measurable. Set

$$\gamma = \sup_{x\in\mathbb{R}^r\setminus\{0\}} \frac{1}{\|x\|} \mu(E \triangle (E + x)).$$

Then $\gamma \le \operatorname{per} E \le \frac{r\beta_r\gamma}{2\beta_{r-1}}$.

475R Convex sets in $\mathbb{R}^r$: Lemma Let $C \subseteq \mathbb{R}^r$ be a convex set.

(a) If $x\in C$ and $y\in\operatorname{int}C$, then $ty+(1-t)x\in\operatorname{int}C$ for every $t\in\,]0,1]$.
(b) $\overline{C}$ and $\operatorname{int}C$ are convex.
(c) If $\operatorname{int}C\ne\emptyset$ then $\overline{C}=\overline{\operatorname{int}C}$.
(d) If $\operatorname{int}C=\emptyset$ then C lies within some hyperplane.
(e) $\operatorname{int}\overline{C}=\operatorname{int}C$.

475S Corollary: Cauchy's Perimeter Theorem Let $C\subseteq\mathbb{R}^r$ be a bounded convex set with non-empty interior. For $v\in S_{r-1}$ write V_v for $\{x:x\textbf{.}v=0\}$, and let $T_v:\mathbb{R}^r\to V_v$ be the orthogonal projection. Then

$$\nu(\partial C)=\frac{1}{\beta_{r-1}}\int_{S_{r-1}}\nu(T_v[C])\nu(dv).$$

475T Corollary: the Convex Isoperimetric Theorem If $C\subseteq\mathbb{R}^r$ is a bounded convex set, then $\nu(\partial C)\le r\beta_r(\frac{1}{2}\operatorname{diam}C)^{r-1}$.

Version of 29.7.21

476 Concentration of measure

Among the myriad special properties of Lebesgue measure, a particularly interesting one is 'concentration of measure'. For a set of given measure in the plane, it is natural to feel that it is most 'concentrated' if it is a disk. There are many ways of defining 'concentration', and I examine three of them in this section (476F, 476G and 476H); all lead us to Euclidean balls as the 'most concentrated' shapes. On the sphere the same criteria lead us to caps (476K).

All the main theorems of this section will be based on the fact that semi-continuous functions on compact spaces attain their bounds. The compact spaces in question will be spaces of subsets, and I begin with some general facts concerning the topologies introduced in 4A2T (476A-476B). The particular geometric properties of Euclidean space which make all these results possible are described in 476D-476E, where I describe concentrating operators based on reflections. The actual theorems 476F-476H and 476K can now almost be mass-produced.

476A Proposition Let X be a topological space, $\mathcal{C}$ the family of closed subsets of X, $\mathcal{K}\subseteq\mathcal{C}$ the family of closed compact sets and μ a topological measure on X.

(a)(i) If μ is outer regular with respect to the open sets then $\mu\restriction\mathcal{C}:\mathcal{C}\to[0,\infty[$ is upper semi-continuous with respect to the Vietoris topology on $\mathcal{C}$.

(ii) If μ is locally finite and inner regular with respect to the closed sets then $\mu\restriction\mathcal{K}$ is upper semi-continuous with respect to the Vietoris topology.

(iii) If μ is inner regular with respect to the closed sets and f is a non-negative μ-integrable real-valued function then $F\mapsto\int_F f d\mu:\mathcal{C}\to\mathbb{R}$ is upper semi-continuous with respect to the Vietoris topology.

(b) Suppose that μ is tight.

(i) If μ is totally finite then $\mu\restriction\mathcal{C}$ is upper semi-continuous with respect to the Fell topology on $\mathcal{C}$.

(ii) If f is a non-negative μ-integrable real-valued function then $F\mapsto\int_F f d\mu:\mathcal{C}\to\mathbb{R}$ is upper semi-continuous with respect to the Fell topology.

(c) Suppose that X is metrizable, and that ρ is a metric on X defining its topology; let $\tilde\rho$ be the Hausdorff metric on $\mathcal{C}\setminus\{\emptyset\}$.

(i) If μ is totally finite, then $\mu\restriction\mathcal{C}\setminus\{\emptyset\}$ is upper semi-continuous with respect to $\tilde\rho$.

(ii) If μ is locally finite, then $\mu\restriction\mathcal{K}\setminus\{\emptyset\}$ is upper semi-continuous with respect to $\tilde\rho$.

(iii) If f is a non-negative μ-integrable real-valued function, then $F\mapsto\int_F f d\mu:\mathcal{C}\setminus\{\emptyset\}\to\mathbb{R}$ is upper semi-continuous with respect to $\tilde\rho$.

476B Lemma Let (X,ρ) be a metric space, and $\mathcal{C}$ the family of closed subsets of X, with its Fell topology. For $\epsilon > 0$, set $U(A,\epsilon) = \{x : x \in X,\ \rho(x,A) < \epsilon\}$ if $A \subseteq X$ is not empty; set $U(\emptyset,\epsilon) = \emptyset$. Then for any τ-additive topological measure μ on X, the function

$$(F,\epsilon) \mapsto \mu U(F,\epsilon) : \mathcal{C} \times]0,\infty[\to [0,\infty]$$

is lower semi-continuous.

476C Proposition Let (X,ρ) be a non-empty compact metric space, and suppose that its isometry group G acts transitively on X. Then X has a unique G-invariant Radon probability measure μ, which is strictly positive.

476D Concentration by partial reflection (a) Let X be an inner product space. For any unit vector $e \in X$ and any $\alpha \in \mathbb{R}$, write $R = R_{e\alpha} : X \to X$ for the reflection in the hyperplane $V = V_{e\alpha} = \{x : x \in X,\ (x|e) = \alpha\}$, so that $R(x) = x + 2(\alpha - (x|e))e$ for every $x \in X$. Next, for any $A \subseteq X$, define $\psi(A) = \psi_{e\alpha}(A)$ by setting

$$\begin{aligned}\psi(A) = \{x : x \in A,\ (x|e) \ge \alpha\} &\cup \{x : x \in A, (x|e) < \alpha,\ R(x) \in A\}\\ &\cup \{x : x \in \mathbb{R}^r \setminus A,\ (x|e) \ge \alpha,\ R(x) \in A\}\\ = (W \cap (A \cup R[A])) &\cup (A \cap R[A]),\end{aligned}$$

where $W = W_{e\alpha}$ is the half-space $\{x : (x|e) \ge \alpha\}$.

(b)(i) If $A \subseteq B \subseteq X$, $\psi(A) \subseteq \psi(B)$. (ii) For any $A \subseteq X$, $\psi(R[A]) = \psi(A)$. (iii) If $F \subseteq X$ is closed, then $\psi(F)$ is closed.

(c) If $x \in X \setminus W$ and $y \in W$ then $\|x - R(y)\| \le \|x - y\|$.

(d) For non-empty $A \subseteq X$ and $\epsilon > 0$, set $U(A,\epsilon) = \{x : \rho(x,A) < \epsilon\}$, where ρ is the standard metric on X. $U(\psi(A),\epsilon) \subseteq \psi(U(A,\epsilon))$.

476E Lemma Let X be an inner product space, $e \in X$ a unit vector and $\alpha \in \mathbb{R}$. Let $R = R_{e\alpha} : X \to X$ be the reflection operator, and $\psi = \psi_{e\alpha} : \mathcal{P}X \to \mathcal{P}X$ the associated transformation, as described in 476D. For $x \in A \subseteq X$, define

$$\begin{aligned}\phi_A(x) &= x \text{ if } (x|e) \ge \alpha,\\ &= x \text{ if } (x|e) < \alpha \text{ and } R(x) \in A,\\ &= R(x) \text{ if } (x|e) < \alpha \text{ and } R(x) \notin A.\end{aligned}$$

Let ν be a topological measure on X which is R-invariant.

(a) For any $A \subseteq X$, $\phi_A : A \to \psi(A)$ is a bijection. If $\alpha < 0$, then $\|\phi_A(x)\| \le \|x\|$ for every $x \in A$, with $\|\phi_A(x)\| < \|x\|$ iff $(x|e) < \alpha$ and $R(x) \notin A$.

(b)(i) If $E \subseteq X$ is measured by ν, then $\psi(E)$ is measured by ν, $\nu\psi(E) = \nu E$ and ϕ_E is a measure space isomorphism for the subspace measures on E and $\psi(E)$ induced by ν.

(ii) For any $A \subseteq X$, $\nu^*\psi(A) \le \nu^* A \le 2\nu^*\psi(A)$.

(c) If $\alpha < 0$ and $E \subseteq X$ is measured by ν, then $\int_E \|x\|\nu(dx) \ge \int_{\psi(E)} \|x\|\nu(dx)$, with equality iff $\{x : x \in E,\ (x|e) < \alpha,\ R(x) \notin E\}$ is negligible.

(d) Suppose that X is separable. Let λ be the c.l.d. product measure of ν with itself on $X \times X$. If $E \subseteq X$ is measured by ν, then

$$\int_{E\times E} \|x - y\|\lambda(d(x,y)) \ge \int_{\psi(E)\times\psi(E)} \|x - y\|\lambda(d(x,y)).$$

(e) Now suppose that $X = \mathbb{R}^r$. Then $\nu(\partial^*\psi(A)) \le \nu(\partial^* A)$ for every $A \subseteq \mathbb{R}^r$, where $\partial^* A$ is the essential boundary of A.

476F Theorem Let $r \ge 1$ be an integer, and let μ be Lebesgue measure on $\mathbb{R}^r$. For non-empty $A \subseteq \mathbb{R}^r$ and $\epsilon > 0$, write $U(A,\epsilon)$ for $\{x : \rho(x,A) < \epsilon\}$, where ρ is the Euclidean metric on $\mathbb{R}^r$. If $\mu^* A$ is finite, then $\mu U(A,\epsilon) \ge \mu U(B_A,\epsilon)$, where B_A is the closed ball with centre $\mathbf{0}$ and measure $\mu^* A$.

476G Theorem Let $r \geq 1$ be an integer, and let μ be Lebesgue measure on $\mathbb{R}^r$; write λ for the product measure on $\mathbb{R}^r \times \mathbb{R}^r$. For any measurable set $E \subseteq \mathbb{R}^r$ of finite measure, write B_E for the closed ball with centre $\mathbf{0}$ and the same measure as E. Then

$$\int_{E\times E} \|x-y\|\lambda(d(x,y)) \geq \int_{B_E\times B_E} \|x-y\|\lambda(d(x,y)).$$

476H The Isoperimetric Theorem Let $r \geq 1$ be an integer, and let μ be Lebesgue measure on $\mathbb{R}^r$. If $E \subseteq \mathbb{R}^r$ is a measurable set of finite measure, then $\operatorname{per} E \geq \operatorname{per} B_E$, where B_E is the closed ball with centre $\mathbf{0}$ and the same measure as E.

476I Spheres in inner product spaces For the rest of the section I will use the following notation. Let X be a (real) inner product space. S_X will be the unit sphere $\{x : x \in X, \|x\| = 1\}$. Let H_X be the isometry group of S_X with its topology of pointwise convergence.

A **cap** in S_X will be a set of the form $\{x : x \in S_X, (x|e) \geq \alpha\}$ where $e \in S_X$ and $-1 \leq \alpha \leq 1$.

When X is finite-dimensional, it is isomorphic to $\mathbb{R}^r$, where $r = \dim X$. If $r \geq 1$, S_X has a unique H_X-invariant Radon probability measure ν_X, which is strictly positive. If $r \geq 1$ is an integer, the $(r-1)$-dimensional Hausdorff measure of the sphere $S_{\mathbb{R}^r}$ is finite and non-zero. $(r-1)$-dimensional Hausdorff measure on $S_{\mathbb{R}^r}$ is a multiple of the normalized invariant measure $\nu_{\mathbb{R}^r}$. The same is therefore true in any r-dimensional inner product space.

476J Lemma Let X be a real inner product space and $f \in H_X$. Then $(f(x)|f(y)) = (x|y)$ for all x, $y \in S_X$. $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ whenever x, $y \in S_X$ and α, $\beta \in \mathbb{R}$ are such that $\alpha x + \beta y \in S_X$.

476K Theorem Let X be a finite-dimensional inner product space of dimension at least 2, S_X its unit sphere and ν_X the invariant Radon probability measure on S_X. For a non-empty set $A \subseteq S_X$ and $\epsilon > 0$, write $U(A, \epsilon) = \{x : \rho(x, A) < \epsilon\}$, where ρ is the usual metric of X. Then there is a cap $C \subseteq S_X$ such that $\nu_X C = \nu_X^* A$, and $\nu_X(S_X \cap U(A, \epsilon)) \geq \nu_X(S_X \cap U(C, \epsilon))$ for any such C and every $\epsilon > 0$.

476L Corollary For any $\epsilon > 0$, there is an $r_0 \geq 1$ such that whenever X is a finite-dimensional inner product space of dimension at least r_0, A_1, $A_2 \subseteq S_X$ and $\min(\nu_X^* A_1, \nu_X^* A_2) \geq \epsilon$, then there are $x \in A_1$, $y \in A_2$ such that $\|x - y\| \leq \epsilon$.

Version of 4.1.08/2.1.10

477 Brownian motion

I presented §455 with an extraordinary omission: the leading example of a Lévy process, and the inspiration for the whole project, was relegated to an anonymous example (455Xg). In this section I will take the subject up again. The theorem that the sum of independent normally distributed random variables is again normally distributed (274B), when translated into the language of this volume, tells us that we have a family $\langle \lambda_t \rangle_{t>0}$ of centered normal distributions such that $\lambda_{s+t} = \lambda_s * \lambda_t$ for all s, $t > 0$. Consequently we have a corresponding example of a Lévy process on $\mathbb{R}$, and this is the process which we call 'Brownian motion' (477A). This is special in innumerable ways, but one of them is central: we can represent it in such a way that sample paths are continuous (477B), that is, as a Radon measure on the space of continuous paths starting at 0. In this form, it also appears as a limit, for the narrow topology, of interpolations of random walks (477C).

For the geometric ideas of §479, we need Brownian motion in three dimensions; the r-dimensional theory of 477D-477G gives no new difficulties. The simplest expression of Brownian motion in $\mathbb{R}^r$ is just to take a product measure (477Da), but in order to apply the results of §455, and match the construction with the ideas of §456, a fair bit of explanation is necessary. The geometric properties of Brownian motion begin with the invariant transformations of 477E. As for all Lévy processes, we have a strong Markov property, and Theorem 455U translates easily into the new formulation (477G), as does the theory of hitting times (477I). I conclude with a classic result on maximal values which will be useful later (477J), and with proofs that almost all Brownian paths are nowhere differentiable (477K) and have zero two-dimensional Hausdorff measure (477L).

477A Brownian motion: Theorem There are a probability space (Ω, Σ, μ) and a family $\langle X_t \rangle_{t \ge 0}$ of real-valued random variables on Ω such that

(i) $X_0 = 0$ almost everywhere;

(ii) whenever $0 \le s < t$ then $X_t - X_s$ is normally distributed with expectation 0 and variance $t - s$;

(iii) $\langle X_t \rangle_{t \ge 0}$ has independent increments.

477B Theorem Let $\langle X_t \rangle_{t \ge 0}$ be as in 477A, and $\hat{\mu}$ the distribution of the process $\langle X_t \rangle_{t \ge 0}$. Let $C([0, \infty[)_0$ be the set of continuous functions $\omega : [0, \infty[\to \mathbb{R}$ such that $\omega(0) = 0$. Then $C([0, \infty[)_0$ has full outer measure for $\hat{\mu}$, and the subspace measure μ_W on $C([0, \infty[)_0$ induced by $\hat{\mu}$ is a Radon measure when $C([0, \infty[)_0$ is given the topology $\mathfrak{T}_c$ of uniform convergence on compact sets.

***477C Theorem** For $\alpha > 0$, define $f_\alpha : \mathbb{R}^{\mathbb{N}} \to \Omega = C([0, \infty[)_0$ by setting $f_\alpha(z)(t) = \sqrt{\alpha}(\sum_{i<n} z(i) + \frac{1}{\alpha}(t - n\alpha) z(n))$ when $z \in \mathbb{R}^{\mathbb{N}}$, $n \in \mathbb{N}$ and $n\alpha \le t \le (n+1)\alpha$. Give Ω its topology $\mathfrak{T}_c$ of uniform convergence on compact sets, and $\mathbb{R}^{\mathbb{N}}$ its product topology; then f_α is continuous. For a Radon probability measure ν on $\mathbb{R}$, let $\mu_{\nu\alpha}$ be the image Radon measure $\nu^{\mathbb{N}} f_\alpha^{-1}$ on Ω, where $\nu^{\mathbb{N}}$ is the product measure on $\mathbb{R}^{\mathbb{N}}$. Let μ_W be the Radon measure of 477B, and U a neighbourhood of μ_W in the space $P_{\mathrm{R}}(\Omega)$ of Radon probability measures on Ω for the narrow topology. Then there is a $\delta > 0$ such that $\mu_{\nu\alpha} \in U$ whenever $\alpha \in]0, \delta]$ and ν is a Radon probability measure on $\mathbb{R}$ with mean $0 = \int x \, \nu(dx)$ and variance $1 = \int x^2 \nu(dx)$ and

$$\int_{\{x : |x| \ge \delta/\sqrt{\alpha}\}} x^2 \nu(dx) \le \delta. \qquad (\dagger)$$

477D Multidimensional Brownian motion Fix an integer $r \ge 1$.

(a) Let μ_{W1} be the Radon probability measure on $\Omega_1 = C([0, \infty[)_0$ described in 477B; I will call it **one-dimensional Wiener measure**. We can identify the power Ω_1^r with $\Omega = C([0, \infty[; \mathbb{R}^r)_0$, the space of continuous functions $\omega : [0, \infty[\to \mathbb{R}^r$ such that $\omega(0) = 0$, with the topology of uniform convergence on compact sets; note that Ω_1 is Polish, so Ω_1^r also is. Because Ω_1 is separable and metrizable, the c.l.d. product measure μ_{W1}^r measures every Borel set, while it is inner regular with respect to the compact sets, so it is a Radon measure. I will say that $\mu_W = \mu_{W1}^r$, interpreted as a measure on $C([0, \infty[; \mathbb{R}^r)_0$, is r**-dimensional Wiener measure**.

μ_{W1} is the subspace measure on Ω_1 induced by the distribution $\hat{\mu}$ of the process $\langle X_t \rangle_{t \ge 0}$ in 477A. μ_W here, regarded as a measure on $C([0, \infty[)_0^r$, is the subspace measure induced by the measure $\hat{\mu}^r$ on $(\mathbb{R}^{[0,\infty[})^r \cong \mathbb{R}^{[0,\infty[\times r}$.

(b) For $\omega \in \Omega$, $t \ge 0$ and $i < r$, set $X_t^{(i)}(\omega) = \omega(t)(i)$. Then $\langle X_t^{(i)} \rangle_{t \ge 0, i < r}$ is a centered Gaussian process, with covariance matrix

$$\begin{aligned} \mathbb{E}(X_s^{(i)} \times X_t^{(j)}) &= 0 \text{ if } i \ne j, \\ &= \min(s, t) \text{ if } i = j. \end{aligned}$$

(c)(i) μ_W is the only Radon probability measure on Ω such that the process $\langle X_t^{(i)} \rangle_{t \ge 0, i < r}$ described in (b) is a Gaussian process with the covariance matrix there.

(ii) Another way of looking at the family $\langle X_t^{(i)} \rangle_{i<r, t \ge 0}$ is to write $X_t(\omega) = \omega(t)$ for $t \ge 0$, so that $\langle X_t \rangle_{t \ge 0}$ is now a family of $\mathbb{R}^r$-valued random variables defined on Ω. We can describe its distribution in terms matching those of 455Q and 477A, which become

(i) $X_0 = 0$ everywhere;

(ii) whenever $0 \le s < t$ then $\frac{1}{\sqrt{t-s}}(X_t - X_s)$ has the standard Gaussian distribution μ_G^r;

(iii) whenever $0 \le t_1 < \ldots < t_n$, then $X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent.

Note that these properties also determine the Radon measure μ_W.

477E Invariant transformations of Wiener measure: Proposition Let $r \ge 1$ be an integer, and μ_W Wiener measure on $\Omega = C([0,\infty[\,;\mathbb{R}^r)_0$. Let $\hat\mu^r$ be the product measure on $(\mathbb{R}^{[0,\infty[})^r$ as described in 477D.

(a) Suppose that $f : (\mathbb{R}^{[0,\infty[})^r \to (\mathbb{R}^{[0,\infty[})^r$ is inverse-measure-preserving for $\hat\mu^r$, and that $\Omega_0 \subseteq \Omega$ is a μ_W-conegligible set such that $f[\Omega_0] \subseteq \Omega_0$. Then $f{\restriction}\Omega_0$ is inverse-measure-preserving for the subspace measure induced by μ_W on Ω_0.

(b) Suppose that $\hat T : \mathbb{R}^{r\times[0,\infty[} \to \mathbb{R}^{r\times[0,\infty[}$ is a linear operator such that, for i, $j < r$ and s, $t \ge 0$,

$$\begin{aligned}\int(\hat T\omega)(i,s)(\hat T\omega)(j,t)\hat\mu^r(d\omega) &= \min(s,t) \text{ if } i=j,\\ &= 0 \text{ if } i \ne j.\end{aligned}$$

Then, identifying $\mathbb{R}^{r\times[0,\infty[}$ with $(\mathbb{R}^{[0,\infty[})^r$, $\hat T$ is inverse-measure-preserving for $\hat\mu^r$.

(c) Suppose that $t \ge 0$. Define $S_t : \Omega \to \Omega$ by setting $(S_t\omega)(s) = \omega(s+t) - \omega(s)$ for $s \ge 0$ and $\omega \in \Omega$. Then S_t is inverse-measure-preserving for μ_W.

(d) Let $T : \mathbb{R}^r \to \mathbb{R}^r$ be an orthogonal transformation. Define $\tilde T : \Omega \to \Omega$ by setting $(\tilde T\omega)(t) = T(\omega(t))$ for $t \ge 0$ and $\omega \in \Omega$. Then $\tilde T$ is an automorphism of (Ω, μ_W).

(e) Suppose that $\alpha > 0$. Define $U_\alpha : \Omega \to \Omega$ by setting $U_\alpha(\omega)(t) = \frac{1}{\sqrt{\alpha}}\omega(\alpha t)$ for $t \ge 0$ and $\omega \in \Omega$. Then U_α is an automorphism of (Ω, μ_W).

(f) Set

$$\Omega_0 = \{\omega : \omega \in \Omega,\ \lim_{t\to\infty}\frac{1}{t}\omega(t) = 0\},$$

and define $R : \Omega_0 \to \Omega_0$ by setting

$$\begin{aligned}(R\omega)(t) &= t\omega(\tfrac{1}{t}) \text{ if } t > 0,\\ &= 0 \text{ if } t = 0.\end{aligned}$$

Then Ω_0 is μ_W-conegligible and R is an automorphism of Ω_0 with its subspace measure.

(g) Suppose that $1 \le r' \le r$, and that $\mu_W^{(r')}$ is Wiener measure on $C([0,\infty[\,;\mathbb{R}^{r'})_0$. Define $P : \Omega \to C([0,\infty[\,;\mathbb{R}^{r'})_0$ by setting $(P\omega)(t)(i) = \omega(t)(i)$ for $t \ge 0$, $i < r'$ and $\omega \in \Omega$. Then $\mu_W^{(r')}$ is the image measure $\mu_W P^{-1}$.

477F Proposition Let $r \ge 1$ be an integer. Then Wiener measure on $\Omega = C([0,\infty[\,;\mathbb{R}^r)_0$ is strictly positive for the topology $\mathfrak{T}_c$ of uniform convergence on compact sets.

477G The strong Markov property: Theorem Suppose that $r \ge 1$, μ_W is Wiener measure on $\Omega = C([0,\infty[\,;\mathbb{R}^r)_0$ and Σ is its domain. For $t \ge 0$ let Σ_t be

$$\{F : F \in \Sigma,\ \omega' \in F \text{ whenever } \omega \in F,\ \omega' \in \Omega \text{ and } \omega'{\restriction}[0,t] = \omega{\restriction}[0,t]\},$$

$$\Sigma_t^+ = \bigcap\nolimits_{s>t}\Sigma_s,$$

and let $\tau : \Omega \to [0,\infty]$ be a stopping time adapted to the family $\langle\Sigma_t^+\rangle_{t\ge0}$. Define $\phi_\tau : \Omega\times\Omega \to \Omega$ by saying that

$$\begin{aligned}\phi_\tau(\omega,\omega')(t) &= \omega(t) \text{ if } t \le \tau(\omega),\\ &= \omega(\tau(\omega)) + \omega'(t-\tau(\omega)) \text{ if } t \ge \tau(\omega).\end{aligned}$$

Then ϕ_τ is inverse-measure-preserving for $\mu_W \times \mu_W$ and μ_W.

477H Some families of σ-algebras: Proposition Let $r \ge 1$ be an integer, μ_W r-dimensional Wiener measure on $\Omega = C([0,\infty[\,;\mathbb{R}^r)_0$ and Σ its domain. Set $X_t^{(i)}(\omega) = \omega(t)(i)$ for $t \ge 0$ and $i < r$. For $I \subseteq [0,\infty[$, let T_I be the σ-algebra of subsets of Ω generated by $\{X_s^{(i)} - X_t^{(i)} : s,\, t \in I,\, i < r\}$, and $\hat{\mathrm{T}}_I$ the σ-algebra $\{E \triangle F : E \in \mathrm{T}_I,\, \mu_W F = 0\}$.

(a) $\mathrm{T}_{[0,\infty[}$ is the Borel σ-algebra of Ω either for the topology of pointwise convergence inherited from $(\mathbb{R}^r)^{[0,\infty[}$ or $\mathbb{R}^{r\times[0,\infty[}$, or for the topology of uniform convergence on compact sets.

(b) If $\mathcal{I}$ is a family of subsets of $[0,\infty[$ such that for all distinct I, $J \in \mathcal{I}$ either $\sup I \le \inf J$ or $\sup J \le \inf I$ (counting $\inf\emptyset$ as ∞ and $\sup\emptyset$ as 0), then $\langle\hat{\mathrm{T}}_I\rangle_{I\in\mathcal{I}}$ is an independent family of σ-algebras.

(c) For $t \ge 0$, let Σ_t be the σ-algebra of sets $F \in \Sigma$ such that $\omega' \in F$ whenever $\omega \in F$, $\omega' \in \Omega$ and $\omega' \restriction [0,t] = \omega \restriction [0,t]$, and $\Sigma_t^+ = \bigcap_{s>t} \Sigma_s$. Write $\hat{\mathrm{T}}^+_{[0,t]}$ for $\bigcap_{s>t} \hat{\mathrm{T}}_{[0,s]}$. Then, for any $t \ge 0$,

$$\mathrm{T}_{[0,t]} \subseteq \Sigma_t \subseteq \Sigma_t^+ \subseteq \hat{\mathrm{T}}^+_{[0,t]} = \hat{\mathrm{T}}_{[0,t]} = \hat{\mathrm{T}}_{[0,t[}.$$

(d) On the tail σ-algebra $\bigcap_{t\ge0} \hat{\mathrm{T}}_{[t,\infty[}$, μ_W takes only the values 0 and 1.

477I Hitting times Take $r \ge 1$, and let μ_W be Wiener measure on $\Omega = C([0,\infty[;\mathbb{R}^r)_0$ and Σ its domain; for $t \ge 0$ define Σ_t^+ and $\mathrm{T}_{[0,t]}$ as in 477G and 477H. Give Ω its topology of uniform convergence on compact sets.

(a) Suppose that $A \subseteq \mathbb{R}^r$. For $\omega \in \Omega$ set $\tau(\omega) = \inf\{t : t \in [0,\infty[,\ \omega(t) \in A\}$, counting $\inf\emptyset$ as ∞. I will call τ the **Brownian hitting time** to A, or the **Brownian exit time** from $\mathbb{R}^r \setminus A$. I will say that the **Brownian hitting probability** of A, or the **Brownian exit probability** of $\mathbb{R}^r \setminus A$, is $\mathrm{hp}(A) = \mu_W\{\omega : \tau(\omega) < \infty\}$ if this is defined. More generally, I will write

$$\mathrm{hp}^*(A) = \mu_W^*\{\omega : \tau(\omega) < \infty\} = \mu_W^*\{\omega : \omega^{-1}[A] \ne \emptyset\},$$

the **outer Brownian hitting probability**, for any $A \subseteq \mathbb{R}^r$.

(b) If $A \subseteq \mathbb{R}^r$ is analytic, the Brownian hitting time to A is a stopping time adapted to the family $\langle\Sigma_t^+\rangle_{t\ge0}$.

In particular, there is a well-defined Brownian hitting probability of A.

(c) Let $F \subseteq \mathbb{R}^r$ be a closed set, and τ the Brownian hitting time to F.

(i) If $\tau(\omega) < \infty$, then

$$\tau(\omega) = \inf \omega^{-1}[F] = \min \omega^{-1}[F]$$

because ω is continuous. If $0 \notin F$ and $\tau(\omega) < \infty$, then $\omega(\tau(\omega)) \in \partial F$.

(ii) τ is lower semi-continuous.

(iii) τ is adapted to $\langle\mathrm{T}_{[0,t]}\rangle_{t\ge0}$.

In the language of 477G, we have $\mathrm{T}_{[0,t]} \subseteq \Sigma_t$ for every $t \ge 0$, so τ must also be adapted to $\langle\Sigma_t\rangle_{t\ge0}$.

(d) If $A \subseteq \mathbb{R}^r$ is any set, then

$$\mathrm{hp}^*(A) = \min\{\mathrm{hp}(B) : B \supseteq A \text{ is an analytic set}\} = \min\{\mathrm{hp}(E) : E \supseteq A \text{ is a } \mathrm{G}_\delta \text{ set}\}.$$

(e) If $A \subseteq \mathbb{R}^r$ is analytic, then $\mathrm{hp}(A) = \sup\{\mathrm{hp}(K) : K \subseteq A \text{ is compact}\}$.

477J Proposition Let μ_W be Wiener measure on $\Omega = C([0,\infty[)_0$. Set $X_t(\omega) = \omega(t)$ for $\omega \in \Omega$. Then

$$\Pr(\max_{s\le t} X_s \ge \alpha) = 2\Pr(X_t \ge \alpha) = \frac{2}{\sqrt{2\pi}}\int_{\alpha/\sqrt{t}}^\infty e^{-u^2/2}du$$

whenever $t > 0$ and $\alpha \ge 0$.

477K Typical Brownian paths: Proposition Let μ_W be Wiener measure on $\Omega = C([0,\infty[)_0$. Then μ_W-almost every element of Ω is nowhere differentiable.

477L Theorem Let $r \ge 1$ be an integer, and μ_W Wiener measure on $\Omega = C([0,\infty[;\mathbb{R}^r)_0$; for $s > 0$ let μ_{Hs} be s-dimensional Hausdorff measure on $\mathbb{R}^r$.

(a) $\{\omega(t) : t \in [0, \infty[\}$ is μ_{H2}-negligible for μ_W-almost every ω.

(b) Now suppose that $r \geq 2$. For $\omega \in \Omega$, let F_ω be the compact set $\{\omega(t) : t \in [0,1]\}$. Then for μ_W-almost every $\omega \in \Omega$, $\mu_{Hs}F_\omega = \infty$ for every $s \in]0,2[$.

Version of 4.6.09/7.8.20

478 Harmonic functions

In this section and the next I will attempt an introduction to potential theory. This is an enormous subject and my choice of results is necessarily somewhat arbitrary. My principal aim is to give the most elementary properties of Newtonian capacity, which will appear in §479. It seems that this necessarily involves a basic understanding of harmonic and superharmonic functions. I approach these by the 'probabilistic' route, using Brownian motion as described in §477.

The first few paragraphs, down to 478J, do not in fact involve Brownian motion; they rely on multidimensional advanced calculus and on the Divergence Theorem. Defining 'harmonic function' in terms of average values over concentric spherical shells (478B), the first step is to identify this with the definition in terms of the Laplacian differential operator (478E). An essential result is a formula for a harmonic function inside a sphere in terms of its values on the boundary and the 'Poisson kernel' (478Ib), and we also need to understand the effects of smoothing by convolution with appropriate functions (478J). I turn to Brownian motion with Dynkin's formula (478K), relating the expected value of $f(X_\tau)$ for a stopped Brownian process X_τ to an integral in terms of $\nabla^2 f$. This is already enough to deal with the asymptotic behaviour of Brownian paths, which depends in a striking way on the dimension of the space (478M).

We can now approach Dirichlet's problem. If we have a bounded open set $G \subseteq \mathbb{R}^r$, there is a family $\langle \mu_x \rangle_{x \in G}$ of probability measures such that whenever $f : \overline{G} \to \mathbb{R}$ is continuous and $f \restriction G$ is harmonic, then $f(x) = \int f d\mu_x$ for every $x \in G$ (478Pc). So this family of 'harmonic measures' gives a formula continuously extending a function on ∂G to a harmonic function on G, whenever such an extension exists (478S). The method used expresses μ_x in terms of the distribution of points at which Brownian paths starting at x strike ∂G, and relies on Dynkin's formula through Theorem 478O. The strong Markov property of Brownian motion now enables us to relate harmonic measures associated with different sets (478R).

478A Notation $r \geq 1$ will be an integer; if you find it easier to focus on one dimensionality at a time, you should start with $r = 3$, because $r = 1$ is too easy and $r = 2$ is exceptional. μ will be Lebesgue measure on $\mathbb{R}^r$, and $\| \;\|$ the Euclidean norm on $\mathbb{R}^r$; ν will be normalized $(r-1)$-dimensional Hausdorff measure on $\mathbb{R}^r$. In the elementary case $r = 1$, ν will be counting measure on $\mathbb{R}$.

$$\begin{aligned}\beta_r &= \frac{1}{k!}\pi^k \text{ if } r = 2k \text{ is even,}\\ &= \frac{2^{2k+1}k!}{(2k+1)!}\pi^k \text{ if } r = 2k+1 \text{ is odd.}\end{aligned}$$

$$\begin{aligned}\nu(\partial B(\mathbf{0}, 1)) = r\beta_r &= \frac{2}{(k-1)!}\pi^k \text{ if } r = 2k \text{ is even,}\\ &= \frac{2^{2k+1}k!}{(2k)!}\pi^k \text{ if } r = 2k+1 \text{ is odd.}\end{aligned}$$

In the formulae below, there are repeated expressions of the form $\frac{1}{\|x-y\|^{r-1}}$, $\frac{1}{\|x-y\|^{r-2}}$; in these, it will often be convenient to interpret $\frac{1}{0}$ as ∞.

It will be convenient to do some calculations in the one-point compactification $\mathbb{R}^r \cup \{\infty\}$ of $\mathbb{R}^r$. For a set $A \subseteq \mathbb{R}^r$

$$\overline{A}^\infty = \overline{A}, \quad \partial^\infty A = \partial A$$

if A is bounded, and

$$\overline{A}^\infty = \overline{A} \cup \{\infty\}, \quad \partial^\infty A = \partial A \cup \{\infty\}$$

if A is unbounded. $\overline{A}^\infty$ and $\partial^\infty A$ are always compact. In this context I will take $x + \infty = \infty$ for every $x \in \mathbb{R}^r$.

μ_W will be r-dimensional Wiener measure on $\Omega = C([0,\infty[\,;\mathbb{R}^r)_0$, the space of continuous functions ω from $[0,\infty[$ to $\mathbb{R}^r$ such that $\omega(0)=0$, endowed with the topology of uniform convergence on compact sets; Σ will be the domain of μ_W. μ_W^2 will be the product measure on $\Omega\times\Omega$. I will write $X_t(\omega)=\omega(t)$ for $t\in[0,\infty[$ and $\omega\in\Omega$, and if $\tau:\Omega\to[0,\infty]$ is a function, I will write $X_\tau(\omega)=\omega(\tau(\omega))$ whenever $\omega\in\Omega$ and $\tau(\omega)$ is finite.

I will write Σ_t for the σ-algebra of sets $F\in\Sigma$ such that $\omega'\in F$ whenever $\omega\in F$, $\omega'\in\Omega$ and $\omega'\restriction[0,t]=\omega\restriction[0,t]$, and $\Sigma_t^+=\bigcap_{s>t}\Sigma_s$. $\mathrm{T}_{[0,t]}$ will be the σ-algebra of subsets of Ω generated by $\{X_s : s\le t\}$.

478B Harmonic and superharmonic functions Let $G\subseteq\mathbb{R}^r$ be an open set and $f:G\to[-\infty,\infty]$ a function.

(a) f is **superharmonic** if $\frac{1}{\nu(\partial B(x,\delta))}\int_{\partial B(x,\delta)} f d\nu$ is defined in $[-\infty,\infty]$ and less than or equal to $f(x)$ whenever $x\in G$, $\delta>0$ and $B(x,\delta)\subseteq G$.

(b) f is **subharmonic** if $-f$ is superharmonic, that is, $\frac{1}{\nu(\partial B(x,\delta))}\int_{\partial B(x,\delta)} f d\nu$ is defined in $[-\infty,\infty]$ and greater than or equal to $f(x)$ whenever $x\in G$, $\delta>0$ and $B(x,\delta)\subseteq G$.

(c) f is **harmonic** if it is both superharmonic and subharmonic, that is, $\frac{1}{\nu(\partial B(x,\delta))}\int_{\partial B(x,\delta)} f d\nu$ is defined and equal to $f(x)$ whenever $x\in G$, $\delta>0$ and $B(x,\delta)\subseteq G$.

478C Elementary facts Let $G\subseteq\mathbb{R}^r$ be an open set.

(a) If $f:G\to[-\infty,\infty]$ is a function, then f is superharmonic iff $-f$ is subharmonic.

(b) If f, $g:G\to[-\infty,\infty[$ are superharmonic functions, then $f+g$ is superharmonic.

(c) If f, $g:G\to[-\infty,\infty]$ are superharmonic functions, then $f\wedge g$ is superharmonic.

(d) Let $f:G\to\mathbb{R}$ be a harmonic function which is locally integrable with respect to Lebesgue measure on G. Then

$$f(x)=\frac{1}{\mu B(x,\delta)}\int_{B(x,\delta)} f d\mu$$

whenever $x\in G$, $\delta>0$ and $B(x,\delta)\subseteq G$. So f is continuous.

478D Maximal principle: Proposition Let $G\subseteq\mathbb{R}^r$ be a non-empty open set. Suppose that $g:\overline{G}^\infty\to\,]-\infty,\infty]$ is lower semi-continuous, $g(y)\ge 0$ for every $y\in\partial^\infty G$, and $g\restriction G$ is superharmonic. Then $g(x)\ge0$ for every $x\in G$.

478E Theorem Let $G\subseteq\mathbb{R}^r$ be an open set and $f:G\to\mathbb{R}$ a function with continuous second derivative. Write $\nabla^2 f$ for its Laplacian $\operatorname{div}\operatorname{grad} f=\sum_{i=1}^r\frac{\partial^2 f}{\partial\xi_i^2}$.

(a) f is superharmonic iff $\nabla^2 f\le 0$ everywhere in G.
(b) f is subharmonic iff $\nabla^2 f\ge 0$ everywhere in G.
(c) f is harmonic iff $\nabla^2 f=0$ everywhere in G.

478F Basic examples (a) For any y, $z\in\mathbb{R}^r$,

$$x\mapsto\frac{1}{\|x-z\|^{r-2}},\quad x\mapsto\frac{(x-z)\,.\,y}{\|x-z\|^r},$$

$$x\mapsto\frac{\|y-z\|^2-\|x-y\|^2}{\|x-z\|^r}=2\frac{(x-z)\,.\,(y-z)}{\|x-z\|^r}-\frac{1}{\|x-z\|^{r-2}}$$

are harmonic on $\mathbb{R}^r\setminus\{z\}$.

(b) For any $z\in\mathbb{R}^2$,

$$x\mapsto\ln\|x-z\|$$

is harmonic on $\mathbb{R}^2\setminus\{z\}$.

478G Lemma (a) $\frac{1}{\nu(\partial B(\mathbf{0},\delta))}\int_{\partial B(\mathbf{0},\delta)}\frac{1}{\|x-z\|^{r-2}}\nu(dz)=\frac{1}{\max(\delta,\|x\|)^{r-2}}$ whenever $x\in\mathbb{R}^r$ and $\delta>0$.

(b) $\frac{1}{\nu(\partial B(\mathbf{0},\delta))}\int_{\partial B(\mathbf{0},\delta)}\frac{|\delta^2-\|x\|^2|}{\|x-z\|^r}\nu(dz)=\frac{1}{\max(\delta,\|x\|)^{r-2}}$ whenever $x\in\mathbb{R}^r$, $\delta>0$ and $\|x\|\ne\delta$.

(c) $\int_{B(\mathbf{0},\delta)}\frac{1}{\|x-z\|^{r-2}}\mu(dz)\le\frac{1}{2}r\beta_r\delta^2$ whenever $x\in\mathbb{R}^r$ and $\delta>0$.

478H Corollary If $r\ge 2$, then $x\mapsto\frac{1}{\|x-z\|^{r-2}}:\mathbb{R}^r\to[0,\infty]$ is superharmonic for any $z\in\mathbb{R}^r$.

478I Theorem Suppose that $y\in\mathbb{R}^r$ and $\delta>0$; let $S=\partial B(y,\delta)$.

(a) Let ζ be a totally finite Radon measure on S, and define f on $\mathbb{R}^r\setminus S$ by setting

$$f(x)=\frac{1}{r\beta_r\delta}\int_S\frac{|\delta^2-\|x-y\|^2|}{\|x-z\|^r}\zeta(dz)$$

for $x\in\mathbb{R}^r\setminus S$. Then f is continuous and harmonic.

(b) Let $g:S\to\mathbb{R}$ be a ν_S-integrable function, where ν_S is the subspace measure on S induced by ν, and define $f:\mathbb{R}^r\to\mathbb{R}$ by setting

$$\begin{aligned}f(x)&=\frac{1}{r\beta_r\delta}\int_S g(z)\frac{|\delta^2-\|x-y\|^2|}{\|x-z\|^r}\nu(dz)\text{ if }x\in\mathbb{R}^r\setminus S,\\&=g(x)\text{ if }x\in S.\end{aligned}$$

(i) f is continuous and harmonic in $\mathbb{R}^r\setminus S$.

(ii) If $r\ge 2$, then

$$\liminf_{z\in S,z\to z_0}g(x)=\liminf_{x\to z_0}f(x),\quad\limsup_{x\to z_0}f(x)=\limsup_{z\in S,z\to z_0}g(x)$$

for every $z_0\in S$.

(iii) f is continuous at any point of S where g is continuous, and if g is lower semi-continuous then f also is.

(iv) $\sup_{x\in\mathbb{R}^r}f(x)=\sup_{z\in S}g(z)$ and $\inf_{x\in\mathbb{R}^r}f(x)=\inf_{z\in S}g(z)$.

478J Convolutions and smoothing: Proposition (a) Suppose that $f:\mathbb{R}^r\to[0,\infty]$ is Lebesgue measurable, and $G\subseteq\mathbb{R}^r$ an open set such that $f\restriction G$ is superharmonic. Let $h:\mathbb{R}^r\to[0,\infty]$ be a Lebesgue integrable function, and $f*h$ the convolution of f and h. If $H\subseteq G$ is an open set such that $H-\{z:h(z)\ne 0\}\subseteq G$, then $(f*h)\restriction H$ is superharmonic.

(b) Suppose, in (a), that $h(y)=h(z)$ whenever $\|y\|=\|z\|$ and that $\int_{\mathbb{R}^r}h\,d\mu\le 1$. If $x\in G$ and $\gamma>0$ are such that $B(x,\gamma)\subseteq G$ and $h(y)=0$ whenever $\|y\|\ge\gamma$, then $(f*h)(x)\le f(x)$.

(c) Let $f:\mathbb{R}^r\to[0,\infty]$ be a lower semi-continuous function, and $\langle\tilde h_n\rangle_{n\in\mathbb{N}}$ the sequence of 473E. If $G\subseteq\mathbb{R}^r$ is an open set such that $f\restriction G$ is superharmonic, then $f(x)=\lim_{n\to\infty}(f*\tilde h_n)(x)$ for every $x\in G$.

(d) Let $G\subseteq\mathbb{R}^r$ be an open set, $K\subseteq G$ a compact set and $g:G\to\mathbb{R}$ a smooth function. Then there is a smooth function $f:G\to\mathbb{R}$ with compact support included in G such that f agrees with g on an open set including K.

478K Dynkin's formula: Lemma Let μ_W be r-dimensional Wiener measure on $\Omega=C([0,\infty[\,;\mathbb{R}^r)_0$; set $X_t(\omega)=\omega(t)$ for $\omega\in\Omega$ and $t\ge 0$. Let $f:\mathbb{R}^r\to\mathbb{R}$ be a three-times-differentiable function such that f and its first three derivatives are continuous and bounded.

(a) $\mathbb{E}(f(X_t))=f(0)+\frac{1}{2}\mathbb{E}(\int_0^t(\nabla^2 f)(X_s)ds)$ for every $t\ge 0$.

(b) If $\tau:\Omega\to[0,\infty[$ is a stopping time adapted to $\langle\Sigma_t^+\rangle_{t\ge 0}$ and $\mathbb{E}(\tau)$ is finite, then

$$\mathbb{E}(f(X_\tau))=f(0)+\frac{1}{2}\mathbb{E}(\int_0^\tau(\nabla^2 f)(X_s)ds).$$

478L Theorem Let μ_W be r-dimensional Wiener measure on $\Omega=C([0,\infty[\,;\mathbb{R}^r)_0$, $f:\mathbb{R}^r\to[0,\infty]$ a lower semi-continuous superharmonic function, and $\tau:\Omega\to[0,\infty]$ a stopping time adapted to $\langle\Sigma_t^+\rangle_{t\ge 0}$. Set $H=\{\omega:\omega\in\Omega,\ \tau(\omega)<\infty\}$. Then

$$f(x) \geq \int_H f(x+\omega(\tau(\omega)))\mu_W(d\omega)$$

for every $x \in \mathbb{R}^r$.

478M Proposition (a) If $r = 1$, then $\{\omega(t) : t \geq 0\} = \mathbb{R}$ for almost every $\omega \in \Omega$.
(b) If $r \leq 2$, then $\{\omega(t) : t \geq 0\}$ is dense in $\mathbb{R}^2$ for almost every $\omega \in \Omega$.
(c) If $r \geq 2$, then for every $z \in \mathbb{R}^2$, $z \notin \{\omega(t) : t > 0\}$ for almost every $\omega \in \Omega$.
(d) If $r \geq 3$, then $\lim_{t\to\infty} \|\omega(t)\| = \infty$ for almost every $\omega \in \Omega$.

478N Wandering paths Let $G \subseteq \mathbb{R}^r$ be an open set, and for $x \in G$ set

$$F_x(G) = \{\omega : \text{either } \tau_x(\omega) < \infty \text{ or } \lim_{t\to\infty} \|\omega(t)\| = \infty\}$$

where τ_x is the Brownian exit time from $G - x$. I will say that G has **few wandering paths** if $F_x(G)$ is conegligible for every $x \in G$. In this case we can be sure that, if $x \in G$, then for almost every ω *either* $\lim_{t\to\infty} \|\omega(t)\| = \infty$ *or* $\omega(t) \notin G-x$ for some t. So we can speak of $X_{\tau_x}(\omega) = \omega(\tau_x(\omega))$, taking this to be ∞ if $\omega \in F_x(G)$ and $\tau_x(\omega) = \infty$; and ω will be continuous on $[0, \tau_x(\omega)]$ for every $\omega \in F_x(G)$. $X_{\tau_x} : \Omega \to \partial^\infty(G-x)$ is Borel measurable.

From 478M, we see that if $r \geq 3$ then any open set in $\mathbb{R}^r$ will have few wandering paths, while if $r \leq 2$ then G will have few wandering paths whenever it is not dense in $\mathbb{R}^r$. Note that if $G \subseteq \mathbb{R}^r$ is open, H is a component of G, and $x \in H$, then the exit times from $H - x$ and $G - x$ are the same, and $F_x(G) = F_x(H)$. It follows that if G has more than one component then it has few wandering paths.

478O Theorem Let $G \subseteq \mathbb{R}^r$ be an open set with few wandering paths and $f : \overline{G}^\infty \to \mathbb{R}$ a bounded lower semi-continuous function such that $f{\restriction}G$ is superharmonic. Take $x \in G$ and let $\tau : \Omega \to [0, \infty]$ be the Brownian exit time from $G - x$. Then $f(x) \geq \mathbb{E}(f(x + X_\tau))$.

478P Harmonic measures (a) Let $A \subseteq \mathbb{R}^r$ be an analytic set and $x \in \mathbb{R}^r$. Let $\tau : \Omega \to [0, \infty]$ be the Brownian hitting time to $A - x$. Then τ is Σ-measurable, where Σ is the domain of μ_W. Setting $H = \{\omega : \tau(\omega) < \infty\}$, $X_\tau : H \to \mathbb{R}^r$ is Σ-measurable.

Consider the function $\omega \mapsto x+\omega(\tau(\omega)) : H \to \mathbb{R}^r$. This induces a Radon image measure μ_x on $\mathbb{R}^r$ defined by saying that

$$\mu_x F = \mu_W\{\omega : \omega \in H,\, x + \omega(\tau(\omega)) \in F\} = \Pr(x + X_\tau \in F)$$

whenever this is defined. $X_\tau(\omega) \in \partial(A - x)$ for every $\omega \in H$, and ∂A is conegligible for μ_x. I will call μ_x the **harmonic measure for arrivals in** A **from** x. Of course $\mu_x\mathbb{R}^r$ is the Brownian hitting probability of A.

Note that if $F \subseteq \mathbb{R}^r$ is closed and $x \in \mathbb{R}^r \setminus F$, then the Brownian hitting time to $F - x$ is the same as the Brownian hitting time to $\partial F - x$, so that the harmonic measure for arrivals in F from x coincides with the harmonic measure for arrivals in ∂F from x.

(b) Let $A \subseteq \mathbb{R}^r$ be an analytic set, $x \in \mathbb{R}^r$, and μ_x the harmonic measure for arrivals in A from x. If $f : \mathbb{R}^r \to [0, \infty]$ is a lower semi-continuous superharmonic function, $f(x) \geq \int f d\mu_x$.

(c) We can re-interpret 478O in this language. Let $G \subseteq \mathbb{R}^r$ be an open set with few wandering paths, and $x \in G$. Let μ_x be the harmonic measure for arrivals in $\mathbb{R}^r \setminus G$ from x. In this case, taking τ to be the Brownian exit time from $G - x$ and $H = \{\omega : \tau(\omega) < \infty\}$, $\lim_{t\to\infty} \|\omega(t)\| = \infty$ for almost every $\omega \in \Omega \setminus H$. If $f : \partial^\infty G \to [-\infty, \infty]$ is a function, then

$$\mathbb{E}(f(x + X_\tau)) = \int_H f(x + X_\tau(\omega))\mu_W(d\omega) + f(\infty)\mu_W(\Omega \setminus H)$$

(counting $f(\infty)$ as zero if G is bounded)

$$= \int f d\mu_x + f(\infty)(1 - \mu_x\mathbb{R}^r)$$

if either integral is defined in $[-\infty, \infty]$. In particular, if $f : \overline{G}^\infty \to \mathbb{R}$ is a bounded lower semi-continuous function and $f{\restriction}G$ is superharmonic, then $f(x) \geq \int f d\mu_x + f(\infty)(1 - \mu_x\mathbb{R}^r)$. Similarly, if $f : \overline{G}^\infty \to \mathbb{R}$ is continuous and $f{\restriction}G$ is harmonic, then $f(x) = \int f d\mu_x + f(\infty)(1 - \mu_x\mathbb{R}^r)$ for every $x \in G$.

(d) Suppose that $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of analytic subsets of $\mathbb{R}^r$, with union A. For $x\in\mathbb{R}^r$, let $\mu_x^{(n)}$, μ_x be the harmonic measures for arrivals in A_n, A respectively from x. Then μ_x is the limit $\lim_{n\to\infty}\mu_x^{(n)}$ for the narrow topology on the space of totally finite Radon measures on $\mathbb{R}^r$.

478Q Proposition Let S be the sphere $\partial B(y,\delta)$, where $y\in\mathbb{R}^r$ and $\delta>0$. For $x\in\mathbb{R}^r\setminus S$, let ζ_x be the indefinite-integral measure over ν defined by the function

$$\begin{aligned} z &\mapsto \tfrac{|\delta^2-\|x-y\|^2|}{r\beta_r\delta\|x-z\|^r} \text{ if } z\in S,\\ &\mapsto 0 \text{ if } z\in\mathbb{R}^r\setminus S. \end{aligned}$$

(a) If $x\in\operatorname{int}B(y,\delta)$, then the harmonic measure μ_x for arrivals in S from x is ζ_x.

(b) In particular, the harmonic measure μ_y for arrivals in S from y is $\dfrac{1}{\nu S}\nu\llcorner S$.

(c) Suppose that $r\ge 2$. If $x\in\mathbb{R}^r\setminus B(y,\delta)$, then the harmonic measure μ_x for arrivals in S from x is ζ_x. $\mu_x\mathbb{R}^r=\frac{\delta^{r-2}}{\|x-y\|^{r-2}}$.

478R Theorem Let A, $B\subseteq\mathbb{R}^r$ be analytic sets with $A\subseteq B$. For $x\in\mathbb{R}^r$, let $\mu_x^{(A)}$, $\mu_x^{(B)}$ be the harmonic measures for arrivals in A, B respectively from x. Then, for any $x\in\mathbb{R}^r$, $\langle\mu_y^{(A)}\rangle_{y\in\mathbb{R}^r}$ is a disintegration of $\mu_x^{(A)}$ over $\mu_x^{(B)}$.

478S Corollary Let $A\subseteq\mathbb{R}^r$ be an analytic set, and $f:\partial A\to\mathbb{R}$ a bounded universally measurable function. For $x\in\mathbb{R}^r\setminus\overline{A}$ set $g(x)=\int f d\mu_x$, where μ_x is the harmonic measure for arrivals in A from x. Then g is harmonic.

478T Corollary Let $A\subseteq\mathbb{R}^r$ be an analytic set, and for $x\in\mathbb{R}^r$ let μ_x be the harmonic measure for arrivals in A from x. Then $x\mapsto\mu_x$ is continuous on $\mathbb{R}^r\setminus\overline{A}$ for the total variation metric on the set of totally finite Radon measures on $\mathbb{R}^r$.

478U Proposition Suppose that $A\subseteq\mathbb{R}^r$ and that 0 belongs to the essential closure of A. Then the outer Brownian hitting probability $\operatorname{hp}^*(A)$ of A is 1.

***478V Theorem** (a) Let $G\subseteq\mathbb{R}^r$ be an open set with few wandering paths and $f:\overline{G}^\infty\to\mathbb{R}$ a continuous function such that $f\restriction G$ is harmonic. For $x\in\mathbb{R}^r$ let $\tau_x:\Omega\to[0,\infty]$ be the Brownian exit time from $G-x$. Set

$$\begin{aligned} g_{\tau_x}(\omega) &= f(x+\omega(\tau_x(\omega))) \text{ if } \tau_x(\omega)<\infty,\\ &= f(\infty) \text{ if } \lim_{t\to\infty}\|\omega(t)\|=\infty \text{ and } \tau_x(\omega)=\infty. \end{aligned}$$

Then $f(x)=\mathbb{E}(g_{\tau_x})$.

(b) Now suppose that σ is a stopping time adapted to $\langle\Sigma_t\rangle_{t\ge0}$ such that $\sigma(\omega)\le\tau_x(\omega)$ for every ω. Set

$$\begin{aligned} g_\sigma(\omega) &= g_{\tau_x}(\omega) \text{ if } \sigma(\omega)=\tau_x(\omega)=\infty,\\ &= f(x+\omega(\sigma(\omega))) \text{ otherwise.} \end{aligned}$$

As in 455Lc, set $\Sigma_\sigma=\{E:E\in\operatorname{dom}\mu_W,\ E\cap\{\omega:\sigma(\omega)\le t\}\in\Sigma_t \text{ for every } t\ge0\}$. Then g_σ is a conditional expectation of g_{τ_x} on Σ_σ.

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479 Newtonian capacity

I end the chapter with a sketch of fragments of the theory of Newtonian capacity. I introduce equilibrium measures as integrals of harmonic measures (479B); this gives a quick definition of capacity (479C), with a substantial number of basic properties (479D, 479E), including its extendability to a Choquet capacity (479Ed). I give sufficient fragments of the theory of Newtonian potentials (479F, 479J) and harmonic analysis (479G, 479I) to support the classical definitions of capacity and equilibrium measures in terms of potential and energy (479K, 479N). The method demands some Fourier analysis extending that of Chapter 28 (479H). 479P is a portmanteau theorem on generalized equilibrium measures and potentials with an exact description of the latter in terms of outer Brownian hitting probabilities. I continue with notes on capacity and Hausdorff measure (479Q), self-intersecting Brownian paths (479R) and an example of a discontinuous equilibrium potential (479S). Yet another definition of capacity, for compact sets, can be formulated in terms of gradients of potential functions (479U); this leads to a simple inequality relating capacity to Lebesgue measure (479V). The section ends with an alternative description of capacity in terms of a measure on the family of closed subsets of $\mathbb{R}^r$ (479W).

479A Notation In this section, unless otherwise stated, r will be a fixed integer greater than or equal to 3. μ will be Lebesgue measure on $\mathbb{R}^r$, and β_r the measure of $B(\mathbf{0},1)$; ν will be normalized $(r-1)$-dimensional Hausdorff measure on $\mathbb{R}^r$.

Recall that if ζ is a measure on a space X, and $E \in \operatorname{dom}\zeta$, then $\zeta \llcorner E$ is defined by saying that $(\zeta \llcorner E)(F) = \zeta(E \cap F)$ whenever $F \subseteq X$ and ζ measures $E \cap F$. If ζ is a Radon measure, so is $\zeta \llcorner E$.

Ω will be $C([0,\infty[\,;\mathbb{R}^r)_0$, with the topology of uniform convergence on compact sets; μ_W will be Wiener measure on Ω. Recall that the Brownian hitting probability $\operatorname{hp}(D)$ of a set $D \subseteq \mathbb{R}^r$ is $\mu_W\{\omega : \omega^{-1}[D] \ne \emptyset\}$ if this is defined, and that for any $D \subseteq \mathbb{R}^r$ the outer Brownian hitting probability is $\operatorname{hp}^*(D) = \mu_W^*\{\omega : \omega^{-1}[D] \ne \emptyset\}$.

If $x \in \mathbb{R}^r$ and $A \subseteq \mathbb{R}^r$ is an analytic set, $\mu_x^{(A)}$ will be the harmonic measure for arrivals in A from x; note that $\mu_x^{(A)}(\mathbb{R}^r) = \mu_x^{(A)}(\partial A) = \operatorname{hp}(A - x)$.

I will write ρ_{tv} for the total variation metric on the space $M_{\mathrm{R}}^+(\mathbb{R}^r)$ of totally finite Radon measures on $\mathbb{R}^r$, so that

$$\rho_{\mathrm{tv}}(\lambda,\zeta) = \sup_{E,F\subseteq\mathbb{R}^r \text{ are Borel}} \lambda E - \zeta E - \lambda F + \zeta F$$

for λ, $\zeta \in M_{\mathrm{R}}^+(\mathbb{R}^r)$.

479B Theorem Let $A \subseteq \mathbb{R}^r$ be a bounded analytic set.

(i) There is a Radon measure λ_A on $\mathbb{R}^r$, with support included in ∂A, defined by saying that $\langle \frac{1}{r\beta_r\gamma}\mu_x^{(A)}\rangle_{x\in\partial B(\mathbf{0},\gamma)}$ is a disintegration of λ_A over the subspace measure $\nu_{\partial B(\mathbf{0},\gamma)}$ whenever $\gamma > 0$ and $\overline{A} \subseteq \operatorname{int} B(\mathbf{0},\gamma)$.

(ii) λ_A is the limit $\lim_{\|x\|\to\infty} \|x\|^{r-2}\mu_x^{(A)}$ for the total variation metric on $M_{\mathrm{R}}^+(\mathbb{R}^r)$.

479C Definitions (a)(i) In the context of 479B, I will call λ_A the **equilibrium measure** of the bounded analytic set A, and $\lambda_A\mathbb{R}^r$ the **Newtonian capacity** $\operatorname{cap} A$ of A.

(ii) For any $D \subseteq \mathbb{R}^r$, its **Choquet-Newton capacity** will be

$$c(D) = \inf_{G\supseteq D \text{ is open}} \sup_{K\subseteq G \text{ is compact}} \operatorname{cap} K.$$

Sets with zero Choquet-Newton capacity are called **polar**.

(b) If ζ is a Radon measure on $\mathbb{R}^r$, the **Newtonian potential** associated with ζ is the function $W_\zeta : \mathbb{R}^r \to [0,\infty]$ defined by the formula

$$W_\zeta(x) = \int_{\mathbb{R}^r} \frac{1}{\|y-x\|^{r-2}}\zeta(dy)$$

for $x \in \mathbb{R}^r$. The **energy** of ζ is now

$$\text{energy}(\zeta)=\int W_\zeta d\zeta=\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\frac{1}{\|x-y\|^{r-2}}\zeta(dy)\zeta(dx).$$

If A is a bounded analytic subset of $\mathbb{R}^r$, the potential $\tilde W_A=W_{\lambda_A}$ is the **equilibrium potential** of A.

(c) If ζ is a Radon measure on $\mathbb{R}^r$, I will write U_ζ for the **$(r-1)$-potential** of ζ, defined by saying that $U_\zeta(x)=\int_{\mathbb{R}^r}\frac{1}{\|x-y\|^{r-1}}\zeta(dy)\in[0,\infty]$ for $x\in\mathbb{R}^r$.

479D Proposition (a) For any $\gamma>0$ and $z\in\mathbb{R}^r$, the Newtonian capacity of $B(z,\gamma)$ is γ^{r-2}, the equilibrium measure of $B(z,\gamma)$ is $\frac{1}{r\beta_r\gamma}\nu\llcorner\partial B(z,\gamma)$, and the equilibrium potential of $B(z,\gamma)$ is given by

$$\tilde W_{B(z,\gamma)}(x)=\min(1,\frac{\gamma^{r-2}}{\|x-z\|^{r-2}})$$

for every $x\in\mathbb{R}^r$.

(b) Let $A\subseteq\mathbb{R}^r$ be a bounded analytic set with equilibrium measure λ_A and equilibrium potential $\tilde W_A$.

(i) $\tilde W_A(x)\le 1$ for every $x\in\mathbb{R}^r$.

(ii) If $B\subseteq A$ is another analytic set, $\tilde W_B\le\tilde W_A$.

(iii) $\tilde W_A(x)=1$ for every $x\in\operatorname{int}A$.

(c) Let A, $B\subseteq\mathbb{R}^r$ be bounded analytic sets.

(i) $\lambda_{A\cup B}\le\lambda_A+\lambda_B$.

(ii) $\lambda_A B\le\operatorname{cap}B$.

479E Theorem (a) Newtonian capacity cap is submodular.

(b) Suppose that $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of analytic subsets of $\mathbb{R}^r$ with bounded union A.

(i) The equilibrium measure λ_A is the limit $\lim_{n\to\infty}\lambda_{A_n}$ for the narrow topology on the space $M_{\mathrm{R}}^+(\mathbb{R}^r)$ of totally finite Radon measures on $\mathbb{R}^r$.

(ii) $\operatorname{cap}A=\lim_{n\to\infty}\operatorname{cap}A_n$.

(iii) The equilibrium potential $\tilde W_A$ is $\lim_{n\to\infty}\tilde W_{A_n}=\sup_{n\in\mathbb{N}}\tilde W_{A_n}$.

(c) Suppose that $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of bounded analytic subsets of $\mathbb{R}^r$ such that $\bigcap_{n\in\mathbb{N}}A_n=\bigcap_{n\in\mathbb{N}}\overline{A_n}=A$ say.

(i) λ_A is the limit $\lim_{n\to\infty}\lambda_{A_n}$ for the narrow topology on $M_{\mathrm{R}}^+(\mathbb{R}^r)$.

(ii) $\operatorname{cap}A=\lim_{n\to\infty}\operatorname{cap}A_n$.

(d)(i) Choquet-Newton capacity $c:\mathcal{P}\mathbb{R}^r\to[0,\infty]$ is the unique outer regular Choquet capacity on $\mathbb{R}^r$ extending cap.

(ii) c is submodular.

(iii) $c(A)=\sup\{\operatorname{cap}K:K\subseteq A\text{ is compact}\}$ for every analytic set $A\subseteq\mathbb{R}^r$.

479F Theorem Let ζ be a totally finite Radon measure on $\mathbb{R}^r$, and set $G=\mathbb{R}^r\setminus\operatorname{supp}\zeta$. Let W_ζ be the Newtonian potential associated with ζ.

(a) $W_\zeta:\mathbb{R}^r\to[0,\infty]$ is lower semi-continuous, and $W_\zeta\restriction G:G\to[0,\infty[$ is continuous.

(b) W_ζ is superharmonic, and $W_\zeta\restriction G$ is harmonic.

(c) W_ζ is locally μ-integrable; in particular, it is finite μ-a.e.

(d) If ζ has compact support, then $\zeta\mathbb{R}^r=\lim_{\|x\|\to\infty}\|x\|^{r-2}W_\zeta(x)$.

(e) If $W_\zeta\restriction\operatorname{supp}\zeta$ is continuous then W_ζ is continuous.

(f) If K is a compact set such that $W_\zeta\restriction K$ is continuous and finite-valued then $W_{\zeta\llcorner K}$ is continuous.

(g) If W_ζ is finite ζ-a.e. and $f:\mathbb{R}^r\to[0,\infty]$ is a lower semi-continuous superharmonic function such that $f\ge W_\zeta$ ζ-a.e., then $f\ge W_\zeta$.

(h) If ζ' is another Radon measure on $\mathbb{R}^r$ and $\zeta'\le\zeta$, then $W_{\zeta'}\le W_\zeta$ and $\text{energy}(\zeta')\le\text{energy}(\zeta)$.

479G Lemma (In this result, r may be any integer greater than or equal to 1.) For $\alpha\in\mathbb{R}$, set $k_\alpha(x)=\frac{1}{\|x\|^\alpha}$ for $x\in\mathbb{R}^r\setminus\{0\}$. If $\alpha<r$, $\beta<r$ and $\alpha+\beta>r$, then $k_{\alpha+\beta-r}$ is a constant multiple of the convolution $k_\alpha*k_\beta$.

Remark If $r\ge 3$, I will take $c_r>0$ to be the constant such that $c_rk_{r-2}=k_{r-1}*k_{r-1}$.

479H Theorem (In this result, r may be any integer greater than or equal to 1.) Let ζ be a totally finite Radon measure on $\mathbb{R}^r$ and $\hat{\zeta}$ its Fourier transform.

(a) If $f \in \mathcal{L}^1_{\mathbb{C}}(\mu)$, then $\zeta * f$ is μ-integrable and $(\zeta * f)^\wedge = (\sqrt{2\pi})^r \hat{\zeta} \times \hat{f}$.

(b) If ζ has compact support and $h : \mathbb{R}^r \to \mathbb{C}$ is a rapidly decreasing test function, then $\zeta * h$ and $h \times \hat{\zeta}$ are rapidly decreasing test functions.

(c) Suppose that f is a tempered function on $\mathbb{R}^r$. If *either* ζ has compact support *or* f is expressible as the sum of a μ-integrable function and a bounded function, then $\zeta * f$ is defined μ-almost everywhere and is a tempered function.

(d) Suppose that f, g are tempered functions on $\mathbb{R}^r$ such that g represents the Fourier transform of f. If *either* ζ has compact support *or* f is expressible as the sum of a bounded function and a μ-integrable function, then $(\sqrt{2\pi})^r \hat{\zeta} \times g$ represents the Fourier transform of $\zeta * f$.

479I Proposition (In this result, r may be any integer greater than or equal to 1.)

(a) Suppose that $0 < \alpha < r$.

(i) There is a tempered function representing the Fourier transform of k_α.

(ii) There is a measurable function g_0, defined almost everywhere on $[0, \infty[$, such that $y \mapsto g_0(\|y\|)$ represents the Fourier transform of k_α.

(iii) In (ii),

$$2^{\alpha/2}\Gamma(\tfrac{\alpha}{2})\int_0^\infty t^{r-1} g_0(t) e^{-\epsilon t^2} dt = 2^{(r-\alpha)/2}\Gamma(\tfrac{r-\alpha}{2})\int_0^\infty t^{\alpha-1} e^{-\epsilon t^2} dt$$

for every $\epsilon > 0$.

(iv) $2^{\alpha/2}\Gamma(\frac{\alpha}{2}) g_0(t) = 2^{(r-\alpha)/2}\Gamma(\frac{r-\alpha}{2}) t^{\alpha-r}$ for almost every $t > 0$.

(v) $2^{(r-\alpha)/2}\Gamma(\frac{r-\alpha}{2}) k_{r-\alpha}$ represents the Fourier transform of $2^{\alpha/2}\Gamma(\frac{\alpha}{2}) k_\alpha$.

(b) Suppose that ζ_1, ζ_2 are totally finite Radon measures on $\mathbb{R}^r$, and $0 < \alpha < r$. If $\zeta_1 * k_\alpha = \zeta_2 * k_\alpha$ μ-a.e., then $\zeta_1 = \zeta_2$.

479J Lemma (a) Let ζ be a totally finite Radon measure on $\mathbb{R}^r$. Let U_ζ be the $(r-1)$-potential of ζ and W_ζ the Newtonian potential of ζ; let k_{r-1} and k_{r-2} be the Riesz kernels. Then $U_\zeta =_{\text{a.e.}} \zeta * k_{r-1}$ and $W_\zeta =_{\text{a.e.}} \zeta * k_{r-2}$.

(b) Let ζ, ζ_1 and ζ_2 be totally finite Radon measures on $\mathbb{R}^r$.

(i) $\int_{\mathbb{R}^r} W_{\zeta_1} d\zeta_2 = \int_{\mathbb{R}^r} W_{\zeta_2} d\zeta_1 = \frac{1}{c_r}\int_{\mathbb{R}^r} U_{\zeta_1} \times U_{\zeta_2} d\mu$.

(ii) The energy energy(ζ) of ζ is $\frac{1}{c_r}\|U_\zeta\|_2^2$, counting $\|U_\zeta\|_2$ as ∞ if $U_\zeta \notin \mathcal{L}^2(\mu)$.

(iii) If $\zeta = \zeta_1 + \zeta_2$ then $U_\zeta = U_{\zeta_1} + U_{\zeta_2}$ and $W_\zeta = W_{\zeta_1} + W_{\zeta_2}$; similarly, $U_{\alpha\zeta} = \alpha U_\zeta$ and $W_{\alpha\zeta} = \alpha W_\zeta$ for $\alpha \ge 0$.

(iv) If $U_{\zeta_1} = U_{\zeta_2}$ μ-a.e., then $\zeta_1 = \zeta_2$.

(v) If $W_{\zeta_1} = W_{\zeta_2}$ μ-a.e., then $\zeta_1 = \zeta_2$.

(vi) $\zeta\mathbb{R}^r = \lim_{\gamma\to\infty} \frac{1}{r\beta_r\gamma}\int_{\partial B(0,\gamma)} W_\zeta d\nu$.

(c) Let $M_{\mathrm{R}}^+(\mathbb{R}^r)$ be the set of totally finite Radon measures on $\mathbb{R}^r$, with its narrow topology. Then energy $: M_{\mathrm{R}}^+(\mathbb{R}^r) \to [0, \infty]$ is lower semi-continuous.

479K Lemma Let $K \subseteq \mathbb{R}^r$ be a compact set, with equilibrium measure λ_K. Then $\lambda_K K = \operatorname{cap} K = \text{energy}(\lambda_K)$, and if ζ is any Radon measure on $\mathbb{R}^r$ such that $\zeta K \ge \operatorname{cap} K \ge \text{energy}(\zeta)$, $\zeta = \lambda_K$.

479L Corollary Let $K \subseteq \mathbb{R}^r$ be a compact set with equilibrium potential $\tilde{W}_K$.

(a) If ζ is any Radon measure on $\mathbb{R}^r$ with finite energy, then $\tilde{W}_K(x) = 1$ for ζ-almost every $x \in K$.

(b) If ζ is a Radon measure on $\mathbb{R}^r$ such that $W_\zeta \le 1$ everywhere on K, $\zeta K \le \operatorname{cap} K$.

(c) $\tilde{W}_K(x) \le \operatorname{hp}(K - x)$ for every $x \in \mathbb{R}^r \setminus K$.

479M Lemma Let $A \subseteq \mathbb{R}^r$ be an analytic set with finite Choquet-Newton capacity $c(A)$.

(a) $\lim_{\gamma\to\infty} c(A \setminus B(\mathbf{0}, \gamma)) = 0$.

(b) $\lambda_A = \lim_{\gamma\to\infty} \lambda_{A\cap B(\mathbf{0},\gamma)}$ is defined for the total variation metric on the space $M_{\mathrm{R}}^+(\mathbb{R}^r)$ of totally finite Radon measures on $\mathbb{R}^r$.

(c)(i) $\lambda_A\mathbb{R}^r = c(A)$.

(ii) $\operatorname{supp}(\lambda_A) \subseteq \partial A$.

(iii) If $B \subseteq \mathbb{R}^r$ is another analytic set such that $c(B) < \infty$, then $\lambda_{A\cup B} \le \lambda_A + \lambda_B$.

(d)(i) $\tilde W_A = W_{\lambda_A}$ is the limit $\lim_{\gamma\to\infty} \tilde W_{A\cap B(\mathbf{0},\gamma)} = \sup_{\gamma\ge 0} \tilde W_{A\cap B(\mathbf{0},\gamma)}$.

(ii) $\tilde W_A(x) \le 1$ for every $x \in \mathbb{R}^r$.

(iii) If ζ is any Radon measure on $\mathbb{R}^r$ with finite energy, $\tilde W_A(x) = 1$ for ζ-almost every $x \in A$.

(iv) $\operatorname{energy}(\lambda_A) = c(A)$.

479N Theorem Let $A \subseteq \mathbb{R}^r$ be an analytic set with finite Choquet-Newton capacity $c(A)$.

(a) Writing W_ζ for the Newtonian potential of a Radon measure ζ on $\mathbb{R}^r$,

$$c(A) = \sup\{\zeta A : \zeta \text{ is a Radon measure on } \mathbb{R}^r,\ W_\zeta(x) \le 1 \text{ for every } x \in \mathbb{R}^r\};$$

if A is closed, the supremum is attained.

(b) $c(A) = \inf\{\operatorname{energy}(\zeta) : \zeta$ is a Radon measure on $\mathbb{R}^r$, $\zeta A \ge c(A)\}$; if A is closed, the infimum is attained.

(c) If $A \ne \emptyset$, $c(A) = \sup\{\frac{1}{\operatorname{energy}(\zeta)} : \zeta$ is a Radon measure on $\mathbb{R}^r$ such that $\zeta A = 1\}$, counting $\frac{1}{\infty}$ as zero; if A is closed, the supremum is attained.

479O Polar sets: Proposition For a set $D \subseteq \mathbb{R}^r$, the following are equiveridical:

(i) D is polar;

(ii) there is a totally finite Radon measure ζ on $\mathbb{R}^r$ such that its Newtonian potential W_ζ is infinite at every point of D;

(iii) there is an analytic set $E \supseteq D$ such that $\zeta E = 0$ whenever ζ is a Radon measure on $\mathbb{R}^r$ with finite energy.

479P Theorem Let $D \subseteq \mathbb{R}^r$ be a set with finite Choquet-Newton capacity $c(D)$.

(a) There is a totally finite Radon measure λ_D on $\mathbb{R}^r$ such that $\lambda_D = \lambda_A$, as defined in 479Mb, whenever $A \supseteq D$ is analytic and $c(A) = c(D)$.

(b) Write $\tilde W_D = W_{\lambda_D}$ for the equilibrium potential corresponding to the equilibrium measure λ_D. Then $\tilde W_D(x) = \operatorname{hp}^*((D \setminus \{x\}) - x)$ for every $x \in \mathbb{R}^r$.

(c)(i)(α) $\lambda_D\mathbb{R}^r = c(D)$;

(β) if ζ is any Radon measure on $\mathbb{R}^r$ with finite energy, $\tilde W_D(x) = 1$ for ζ-almost every $x \in D$;

(γ) $\operatorname{energy}(\lambda_D) = c(D)$;

(δ) if $D' \subseteq D$ and $c(D') = c(D)$, then $\lambda_{D'} = \lambda_D$.

(ii) $\operatorname{supp}(\lambda_D) \subseteq \partial D$.

(iii) For any $D' \subseteq \mathbb{R}^r$ such that $c(D') < \infty$,

(α) $\lambda_D^*(D') \le c(D')$;

(β) $\lambda_{D\cup D'} \le \lambda_D + \lambda_{D'}$;

(γ) $\tilde W_{D\cap D'} + \tilde W_{D\cup D'} \le \tilde W_D + \tilde W_{D'}$;

(δ) $\rho_{\mathrm{tv}}(\lambda_D, \lambda_{D'}) \le 2c(D \triangle D')$.

(iv) If $\langle D_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of sets with union D, then

(α) $\tilde W_D = \lim_{n\to\infty} \tilde W_{D_n} = \sup_{n\in\mathbb{N}} \tilde W_{D_n}$;

(β) $\langle\lambda_{D_n}\rangle_{n\in\mathbb{N}} \to \lambda_D$ for the narrow topology on $M_{\mathrm{R}}^+(\mathbb{R}^r)$.

(v) $c(D) = \inf\{\zeta\mathbb{R}^r : \zeta$ is a Radon measure on $\mathbb{R}^r$, $W_\zeta \ge \chi D\}$

$= \inf\{\operatorname{energy}(\zeta) : \zeta$ is a Radon measure on $\mathbb{R}^r$, $W_\zeta \ge \chi D\}$.

(vi) Writing cl^*D for the essential closure of D, $c(\mathrm{cl}^*D) \le c(D)$ and $\tilde W_{\mathrm{cl}^*D} \le \tilde W_D$.

(vii) Suppose that $f : D \to \mathbb{R}^r$ is γ-Lipschitz, where $\gamma \ge 0$. Then $c(f[D]) \le \gamma^{r-2}c(D)$.

479Q Hausdorff measure: Theorem For $s \in \,]0,\infty[$ let μ_{Hs} be Hausdorff s-dimensional measure on $\mathbb{R}^r$. Let D be any subset of $\mathbb{R}^r$.

(a) If the Choquet-Newton capacity $c(D)$ is non-zero, then $\mu_{H,r-2}^*D = \infty$.

(b) If $s > r-2$ and $\mu_{Hs}^*D > 0$, then $c(D) > 0$.

479R Proposition (a) Suppose that $r=3$. Then almost every $\omega\in\Omega$ is not injective.

(b) If $r\ge 4$, then almost every $\omega\in\Omega$ is injective.

479S Example Suppose that $e\in\mathbb{R}^r$ is a unit vector. Then there is a sequence $\langle\delta_n\rangle_{n\in\mathbb{N}}$ of strictly positive real numbers such that the equilibrium potential $\tilde W_K$ is discontinuous at e whenever $K\subseteq B(\mathbf{0},1)$ is compact, $e\in\overline{\operatorname{int}K}$ and $\|x-te\|\le\delta_n$ whenever $n\in\mathbb{N}$, $t\in[1-2^{-n},1]$, $x\in K$ and $\|x\|=t$.

***479T Lemma** (a) If $g:\mathbb{R}^r\to\mathbb{R}$ is a smooth function with compact support,

$$\int_{\mathbb{R}^r}\frac{1}{\|x-y\|^{r-2}}\nabla^2 g\,d\mu=-r(r-2)\beta_r g(x)$$

for every $x\in\mathbb{R}^r$.

(b) Let g, $h:\mathbb{R}^r\to\mathbb{R}$ be smooth functions with compact support. Then

$$\int_{\mathbb{R}^r}h\times\nabla^2 g\,d\mu=\int_{\mathbb{R}^r}g\times\nabla^2 h=-\int_{\mathbb{R}^r}\operatorname{grad}h\,\boldsymbol{.}\,\operatorname{grad}g\,d\mu.$$

(c) Let ζ be a totally finite Radon measure on $\mathbb{R}^r$, and $W_\zeta:\mathbb{R}^r\to[0,\infty]$ the associated Newtonian potential. Then $\int_{\mathbb{R}^r}W_\zeta\times\nabla^2 g\,d\mu=-r(r-2)\beta_r\int_{\mathbb{R}^r}g\,d\zeta$ for every smooth function $g:\mathbb{R}^r\to\mathbb{R}$ with compact support.

(d) Let ζ be a totally finite Radon measure on $\mathbb{R}^r$ such that W_ζ is finite-valued everywhere and Lipschitz. Then $\int_{\mathbb{R}^r}\operatorname{grad}f\,\boldsymbol{.}\,\operatorname{grad}W_\zeta d\mu=r(r-2)\beta_r\int_{\mathbb{R}^r}f\,d\zeta$ for every Lipschitz function $f:\mathbb{R}^r\to\mathbb{R}$ with compact support.

(e) Let $K\subseteq\mathbb{R}^r$ be a compact set, and $\epsilon>0$. Then there is a Radon measure ζ on $\mathbb{R}^r$, with support included in $K+B(\mathbf{0},\epsilon)$, such that W_ζ is a smooth function with compact support, $W_\zeta\ge\chi K$, $\zeta\mathbb{R}^r\le\operatorname{cap}K+\epsilon$ and

$$\int_{\mathbb{R}^r}\|\operatorname{grad}W_\zeta\|^2d\mu=r(r-2)\beta_r\operatorname{energy}(\zeta)\le r(r-2)\beta_r\zeta\mathbb{R}^r.$$

***479U Theorem** Let $K\subseteq\mathbb{R}^r$ be compact, and let Φ be the set of Lipschitz functions $g:\mathbb{R}^r\to\mathbb{R}$ such that $g(x)\ge 1$ for every $x\in K$ and $\lim_{\|x\|\to\infty}g(x)=0$. Then

$$\begin{aligned}r(r-2)\beta_r\operatorname{cap}K&=\inf\{\int_{\mathbb{R}^r}\|\operatorname{grad}g\|^2d\mu:g\in\Phi\text{ is smooth and has compact support}\}\\&=\inf\{\int_{\mathbb{R}^r}\|\operatorname{grad}g\|^2d\mu:g\in\Phi\}.\end{aligned}$$

***479V Theorem** Let $D\subseteq\mathbb{R}^r$ be a set of finite outer Lebesgue measure, and B_D the closed ball with centre 0 and the same outer measure as D. Then the Choquet-Newton capacity $c(D)$ of D is at least $\operatorname{cap}B_D$.

***479W Theorem** Let $\mathcal{C}^+$ be the family of non-empty closed subsets of $\mathbb{R}^r$, with its Fell topology. Then there is a unique Radon measure θ on $\mathcal{C}^+$ such that $\theta^*\{C:C\in\mathcal{C}^+,\,D\cap C\ne\emptyset\}$ is the Choquet-Newton capacity $c(D)$ of D for every $D\subseteq\mathbb{R}^r$.