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Chapter 47

Geometric measure theory

I offer a chapter on geometric measure theory, continuing from Chapter 26. The greater part of it is directed specifically at two topics: a version of the Divergence Theorem (475N) and the elementary theory of Newtonian capacity and potential (§479). I do not attempt to provide a balanced view of the subject, for which I must refer you to MATTILA 95, EVANS & GARIEPY 92 and FEDERER 69. However §472, at least, deals with something which must be central to any approach, Besicovitch's Density Theorem for Radon measures on $\mathbb{R}^r$ (472D). In §473 I examine Lipschitz functions, and give crude forms of some fundamental ineqalities relating integrals $\int \|\operatorname{grad} f\| d\mu$ with other measures of the variation of a function f (473H, 473K). In §474 I introduce perimeter measures λ^{∂}_E and outward-normal functions ψ_E as those for which the Divergence Theorem, in the form $\int_E \operatorname{div}\phi\, d\mu = \int \phi \,\mathbf{.}\, \psi_E \, d\lambda^{\partial}_E$, will be valid (474E), and give the geometric description of $\psi_E(x)$ as the Federer exterior normal to E at x (474R). In §475 I show that λ^{∂}_E can be identified with normalized Hausdorff $(r-1)$-dimensional measure on the essential boundary of E.

§471 is devoted to Hausdorff measures on general metric spaces, extending the ideas introduced in §264 for Euclidean space, up to basic results on densities (471P) and Howroyd's theorem (471S). In §476 I turn to a different topic, the problem of finding the subsets of $\mathbb{R}^r$ on which Lebesgue measure is most 'concentrated' in some sense. I present a number of classical results, the deepest being the Isoperimetric Theorem (476H): among sets with a given measure, those with the smallest perimeters are the balls.

The last three sections are different again. Classical electrostatics led to a vigorous theory of capacity and potential, based on the idea of 'harmonic function'. It turns out that 'Brownian motion' in $\mathbb{R}^r$ (§477) gives an alternative and very powerful approach to the subject. I have brought Brownian motion and Wiener measure to this chapter because I wish to use them to illuminate the geometry of $\mathbb{R}^r$; but much of §477 (in particular, the strong Markov property, 477G) is necessarily devoted to adapting ideas developed in the more general contexts of Lévy and Gaussian processes, as described in §§455-456. In §478 I give the most elementary parts of the theory of harmonic and superharmonic functions, building up to a definition of 'harmonic measures' based on Brownian motion (478P). In §479 I use these techniques to describe Newtonian capacity and its extension Choquet-Newton capacity (479C) on Euclidean space of three or more dimensions, and establish their basic properties (479E, 479F, 479N, 479P, 479U).

Version of 10.2.16

471 Hausdorff measures

I begin the chapter by returning to a class of measures which we have not examined in depth since Chapter 26. The primary importance of these measures is in studying the geometry of Euclidean space; in §265 I looked briefly at their use in describing surface measures, which will reappear in §475. Hausdorff measures are also one of the basic tools in the study of fractals, but for such applications I must refer you to FALCONER 90 and MATTILA 95. All I shall attempt to do here is to indicate some of the principal ideas which are applicable to general metric spaces, and to look at some special properties of Hausdorff measures related to the concerns of this chapter and of §261.

471A Definition Let (X,ρ) be a metric space and $r \in \,]0,\infty[$. For $\delta > 0$ and $A \subseteq X$, set

$$\theta_{r\delta}A = \inf\{\sum_{n=0}^{\infty}(\operatorname{diam} D_n)^r : \langle D_n\rangle_{n\in\mathbb{N}} \text{ is a sequence of subsets of } X \text{ covering } A,$$
$$\operatorname{diam} D_n \le \delta \text{ for every } n \in \mathbb{N}\}.$$

(As in §264, take $\operatorname{diam}\emptyset = 0$ and $\inf\emptyset = \infty$.) It will be useful to note that every $\theta_{r\delta}$ is an outer measure. Now set

$$\theta_r A = \sup_{\delta>0} \theta_{r\delta} A$$

for $A \subseteq X$; θ_r also is an outer measure on X, as in 264B; this is **r-dimensional Hausdorff outer measure** on X. Let μ_{Hr} be the measure defined by Carathéodory's method from θ_r; μ_{Hr} is **r-dimensional Hausdorff measure** on X.

Notation It may help if I list some notation already used elsewhere. Suppose that (X, ρ) is a metric space. I write

$$B(x, \delta) = \{y : \rho(y, x) \le \delta\}, \quad U(x, \delta) = \{y : \rho(y, x) < \delta\}$$

for $x \in X$, $\delta \ge 0$; recall that $U(x, \delta)$ is open (2A3G). For $x \in X$ and A, $A' \subseteq X$ I write

$$\rho(x, A) = \inf_{y\in A} \rho(x, y), \quad \rho(A, A') = \inf_{y\in A, z\in A'} \rho(y, z);$$

for definiteness, take $\inf\emptyset$ to be ∞, as before.

471B Definition Let (X, ρ) be a metric space. An outer measure θ on X is a **metric outer measure** if $\theta(A \cup B) = \theta A + \theta B$ whenever A, $B \subseteq X$ and $\rho(A, B) > 0$.

471C Proposition Let (X, ρ) be a metric space and θ a metric outer measure on X. Let μ be the measure on X defined from θ by Carathéodory's method. Then μ is a topological measure.

proof (Compare 264E, part (b) of the proof.) Let $G \subseteq X$ be open, and A any subset of X such that $\theta A < \infty$. Set

$$A_n = \{x : x \in A,\ \rho(x, A \setminus G) \ge 2^{-n}\},$$

$$B_0 = A_0, \quad B_n = A_n \setminus A_{n-1} \text{ for } n > 1.$$

Observe that $A_n \subseteq A_{n+1}$ for every n and $\bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} B_n = A \cap G$. The point is that if m, $n \in \mathbb{N}$ and $n \ge m + 2$, and if $x \in B_m$ and $y \in B_n$, then there is a $z \in A \setminus G$ such that $\rho(y, z) < 2^{-n+1} \le 2^{-m-1}$, while $\rho(x, z)$ must be at least 2^{-m}, so $\rho(x, y) \ge \rho(x, z) - \rho(y, z) \ge 2^{-m-1}$. Thus $\rho(B_m, B_n) > 0$ whenever $n \ge m + 2$. It follows that for any $k \ge 0$

$$\sum_{m=0}^{k} \theta B_{2m} = \theta(\bigcup_{m\le k} B_{2m}) \le \theta(A \cap G) < \infty,$$

$$\sum_{m=0}^{k} \theta B_{2m+1} = \theta(\bigcup_{m\le k} B_{2m+1}) \le \theta(A \cap G) < \infty.$$

Consequently $\sum_{n=0}^{\infty} \theta B_n < \infty$.

But now, given $\epsilon > 0$, there is an m such that $\sum_{n=m}^{\infty} \theta B_m \le \epsilon$, so that

$$\theta(A \cap G) + \theta(A \setminus G) \le \theta A_m + \sum_{n=m}^{\infty} \theta B_n + \theta(A \setminus G)$$

$$\le \epsilon + \theta A_m + \theta(A \setminus G) = \epsilon + \theta(A_m \cup (A \setminus G))$$

(since $\rho(A_m, A \setminus G) \ge 2^{-m}$)

$$\le \epsilon + \theta A.$$

As ϵ is arbitrary, $\theta(A \cap G) + \theta(A \setminus G) \le \theta A$. As A is arbitrary, G is measured by μ; as G is arbitrary, μ is a topological measure.

471D Theorem Let (X, ρ) be a metric space and $r > 0$. Let μ_{Hr} be r-dimensional Hausdorff measure on X, and Σ its domain; write θ_r for r-dimensional Hausdorff outer measure on X, as defined in 471A.

(a) μ_{Hr} is a topological measure.

(b) For every $A \subseteq X$ there is a G_δ set $H \supseteq A$ such that $\mu_{Hr} H = \theta_r A$.

(c) θ_r is the outer measure defined from μ_{Hr} (that is, θ_r is a regular outer measure).

(d) Σ is closed under Souslin's operation.
(e) $\mu_{Hr}E = \sup\{\mu_{Hr}F : F \subseteq E$ is closed$\}$ whenever $E \in \Sigma$ and $\mu_{Hr}E < \infty$.
(f) If $A \subseteq X$ and $\theta_r A < \infty$ then A is separable and the set of isolated points of A is μ_{Hr}-negligible.
(g) μ_{Hr} is atomless.
(h) If μ_{Hr} is totally finite it is a quasi-Radon measure.

proof (a) The point is that θ_r, as defined in 471A, is a metric outer measure. **P** (Compare 264E, part (a) of the proof.) Let A, B be subsets of X such that $\rho(A, B) > 0$. Of course $\theta_r(A \cup B) \le \theta_r A + \theta_r B$, because θ_r is an outer measure. For the reverse inequality, we may suppose that $\theta_r(A \cup B) < \infty$, so that $\theta_r A$ and $\theta_r B$ are both finite. Let $\epsilon > 0$ and let δ_1, $\delta_2 > 0$ be such that

$$\theta_r A + \theta_r B \le \theta_{r\delta_1} A + \theta_{r\delta_2} B + \epsilon,$$

defining the $\theta_{r\delta_i}$ as in 471A. Set $\delta = \min(\delta_1, \delta_2, \frac{1}{2}\rho(A, B)) > 0$ and let $\langle D_n \rangle_{n \in \mathbb{N}}$ be a sequence of sets of diameter at most δ, covering $A \cup B$, and such that $\sum_{n=0}^{\infty} (\operatorname{diam} D_n)^r \le \theta_{r\delta}(A \cup B) + \epsilon$. Set

$$K = \{n : D_n \cap A \ne \emptyset\}, \quad L = \{n : D_n \cap B \ne \emptyset\}.$$

Because $\rho(x, y) > \operatorname{diam} D_n$ whenever $x \in A$, $y \in B$ and $n \in \mathbb{N}$, $K \cap L = \emptyset$; and of course $A \subseteq \bigcup_{n \in K} D_k$, $B \subseteq \bigcup_{n \in L} D_n$. Consequently

$$\begin{aligned}\theta_r A + \theta_r B &\le \epsilon + \theta_{r\delta_1} A + \theta_{r\delta_2} B \le \epsilon + \sum_{n \in K} (\operatorname{diam} D_n)^r + \sum_{n \in L} (\operatorname{diam} D_n)^r \\ &\le \epsilon + \sum_{n=0}^{\infty} (\operatorname{diam} D_n)^r \le 2\epsilon + \theta_{r\delta}(A \cup B) \le 2\epsilon + \theta_r(A \cup B).\end{aligned}$$

As ϵ is arbitrary, $\theta_r(A \cup B) \ge \theta_r A + \theta_r B$, and we therefore have equality. As A and B are arbitrary, θ_r is a metric outer measure. **Q**

Now 471C tells us that μ_{Hr} must be a topological measure.

(b) (Compare 264Fa.) If $\theta_r A = \infty$ this is trivial. Otherwise, for each $n \in \mathbb{N}$, let $\langle D_{ni} \rangle_{i \in \mathbb{N}}$ be a sequence of sets of diameter at most 2^{-n} such that $A \subseteq \bigcup_{i \in \mathbb{N}} D_i$ and $\sum_{i=0}^{\infty} (\operatorname{diam} D_{ni})^r \le \theta_{r,2^{-n}}(A) + 2^{-n}$, defining $\theta_{r,2^{-n}}$ as in 471A. Let $\eta_{ni} \in \left]0, 2^{-n}\right]$ be such that $(2\eta_{ni} + \operatorname{diam} D_{ni})^r \le 2^{-n-i} + (\operatorname{diam} D_{ni})^r$, and set $G_{ni} = \{x : \rho(x, D_{ni}) < \eta_{ni}\}$, for all n, $i \in \mathbb{N}$; then $G_{ni} = \bigcup_{x \in D_{ni}} U(x, \eta_{ni})$ is an open set including D_{ni} and $(\operatorname{diam} G_{ni})^r \le 2^{-n-i} + (\operatorname{diam} D_{ni})^r$. Set

$$H = \bigcap_{n \in \mathbb{N}} \bigcup_{i \in \mathbb{N}} G_{ni},$$

so that H is a G_δ set including A.

For any $\delta > 0$, there is an $n \in \mathbb{N}$ such that $3 \cdot 2^{-n} \le \delta$, so that $\operatorname{diam} G_{mi} \le \operatorname{diam} D_{mi} + 2\eta_{mi} \le \delta$ for every $i \in \mathbb{N}$ and $m \ge n$, and

$$\begin{aligned}\theta_{r\delta} H &\le \sum_{i=0}^{\infty} (\operatorname{diam} G_{mi})^r \le \sum_{i=0}^{\infty} 2^{-m-i} + (\operatorname{diam} D_{mi})^r \\ &\le 2^{-m+1} + \theta_{r,2^{-m}}(A) + 2^{-m} \le 2^{-m+2} + \theta_r A\end{aligned}$$

for every $m \ge n$. Accordingly $\theta_{r\delta} H \le \theta_r A$ for every $\delta > 0$, so $\theta_r H \le \theta_r A$. Of course this means that $\theta_r H = \theta_r A$; and since, by (a), μ_{Hr} measures H, we have $\mu_{Hr} H = \theta_r A$, as required.

(c) (Compare 264Fb.) If $A \subseteq X$,

$$\theta_r A \ge \mu_{Hr}^* A$$

(by (b))

$$= \inf\{\theta_r E : A \subseteq E \in \Sigma\} \ge \theta_r A.$$

(d) Use 431C.

(e) By (b), there is a Borel set $H \supseteq E$ such that $\mu_{Hr}H = \mu_{Hr}E$, and now there is a Borel set $H' \supseteq H \setminus E$ such that $\mu_{Hr}H' = \mu_{Hr}(H \setminus E) = 0$, so that $G = H \setminus H'$ is a Borel set included in E and $\mu_{Hr}G = \mu_{Hr}E$. Now G is a Baire set (4A3Kb), so is Souslin-F (421L), and $\mu_{Hr}G = \sup_{F\subseteq G \text{ is closed}} \mu_{Hr}F$, by 431E.

(f) For every $n \in \mathbb{N}$, there must be a sequence $\langle D_{ni}\rangle_{i\in\mathbb{N}}$ of sets of diameter at most 2^{-n} covering A; now if $D \subseteq A$ is a countable set which meets D_{ni} whenever i, $n \in \mathbb{N}$ and $A \cap D_{ni} \neq \emptyset$, D will be dense in A. If A_0 is the set of isolated points in A, it is still separable (4A2P(a-iv)); but as the only dense subset of A_0 is itself, it is countable. Since $\theta_{r\delta}\{x\} = (\operatorname{diam}\{x\})^r = 0$ for every $\delta > 0$, $\mu_{Hr}\{x\} = 0$ for every $x \in X$, and A_0 is negligible.

(g) In fact, if $A \subseteq X$ and $\theta_r A > 0$, there are disjoint A_0, $A_1 \subseteq A$ such that $\theta_r A_i > 0$ for both i. **P** (i) Suppose first that A is not separable. For each $n \in \mathbb{N}$, let $D_n \subseteq A$ be a maximal set such that $\rho(x, y) \geq 2^{-n}$ for all distinct x, $y \in D_n$; then $\bigcup_{n\in\mathbb{N}} D_n$ is dense in A, so there is some $n \in \mathbb{N}$ such that D_n is uncountable; if we take A_1, A_2 to be disjoint uncountable subsets of D_n, then $\theta_r A_1 = \theta_r A_2 = \infty$. (ii) If A is separable, then set $\mathcal{G} = \{G : G \subseteq X \text{ is open}, \theta_r(A \cap G) = 0\}$. Because A is hereditarily Lindelöf (4A2P(a-iii)), there is a countable subset $\mathcal{G}_0$ of $\mathcal{G}$ such that $A \cap \bigcup \mathcal{G} = A \cap \bigcup \mathcal{G}_0$ (4A2H(c-i)), so $A \cap \bigcup \mathcal{G}$ is negligible and $A \setminus \bigcup \mathcal{G}$ has at least two points x_0, x_1. If we set $A_i = A \cap U(x_i, \frac{1}{2}\rho(x_i, x_{1-i}))$ for each i, these are disjoint subsets of A of non-zero outer measure. **Q**

(h) If μ_{Hr} is totally finite, then it is inner regular with respect to the closed sets, by (e). Also, because X must be separable, by (f), therefore hereditarily Lindelöf, μ_{Hr} must be τ-additive (414O). Finally, μ_{Hr} is complete just because it is defined by Carathéodory's method. So μ_{Hr} is a quasi-Radon measure.

471E Corollary If (X, ρ) is a metric space, $r > 0$ and $Y \subseteq X$ then r-dimensional Hausdorff measure $\mu_{Hr}^{(Y)}$ on Y extends the subspace measure $(\mu_{Hr}^{(X)})_Y$ on Y induced by r-dimensional Hausdorff measure $\mu_{Hr}^{(X)}$ on X; and if either Y is measured by $\mu_{Hr}^{(X)}$ or Y has finite r-dimensional Hausdorff outer measure in X, then $\mu_{Hr}^{(Y)} = (\mu_{Hr}^{(X)})_Y$.

proof Write $\theta_r^{(X)}$ and $\theta_r^{(Y)}$ for the two r-dimensional Hausdorff outer measures.

If $A \subseteq Y$ and $\langle D_n\rangle_{n\in\mathbb{N}}$ is any sequence of subsets of X covering A, then $\langle D_n \cap Y\rangle_{n\in\mathbb{N}}$ is a sequence of subsets of Y covering A, and $\sum_{n=0}^{\infty}(\operatorname{diam}(D_n \cap Y))^r \leq \sum_{n=0}^{\infty}(\operatorname{diam} D_n)^r$; moreover, when calculating $\operatorname{diam}(D_n \cap Y)$, it doesn't matter whether we use the metric ρ on X or the subspace metric $\rho\restriction Y \times Y$ on Y. What this means is that, for any $\delta > 0$, $\theta_{r\delta}A$ is the same whether calculated in Y or in X, so that $\theta_r^{(Y)}A = \sup_{\delta>0} \theta_{r\delta}A = \theta_r^{(X)}A$.

Thus $\theta_r^{(Y)} = \theta_r^{(X)}\restriction\mathcal{P}Y$. Also, by 471Db, $\theta_r^{(X)}$ is a regular outer measure. So 214Hb gives the results.

471F Corollary Let (X, ρ) be an analytic metric space (that is, a metric space in which the topology is analytic in the sense of §423), and write μ_{Hr} for r-dimensional Hausdorff measure on X. Suppose that ν is a locally finite indefinite-integral measure over μ_{Hr}. Then ν is a Radon measure.

proof Since $\operatorname{dom}\nu \supseteq \operatorname{dom}\mu_{Hr}$, ν is a topological measure. Because X is separable, therefore hereditarily Lindelöf, ν is σ-finite and τ-additive, therefore locally determined and effectively locally finite. Next, it is inner regular with respect to the closed sets. **P** Let f be a Radon-Nikodým derivative of ν. If $\nu E > 0$, there is an $E' \subseteq E$ such that

$$0 < \nu E' = \int f \times \chi E' \, d\mu_{Hr} < \infty.$$

There is a μ_{Hr}-simple function g such that $g \leq f \times \chi E'$ μ_{Hr}-a.e. and $\int g \, d\mu_{Hr} > 0$; setting $H = E' \cap \{x : g(x) > 0\}$, $\nu_{Hr}H < \infty$. Now there is a closed set $F \subseteq H$ such that $\mu_{Hr}F > 0$, by 471De, and in this case $\nu F \geq \int_F g \, d\mu_{Hr} > 0$. By 412B, this is enough to show that ν is inner regular with respect to the closed sets. **Q**

Since ν is complete (234I), it is a quasi-Radon measure, therefore a Radon measure (434Jf, 434Jb).

471G Increasing Sets Lemma (Davies 70) Let (X, ρ) be a metric space and $r > 0$.

(a) Suppose that $\delta > 0$ and that $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of subsets of X with union A. Then $\theta_{r,6\delta}(A) \leq (5^r + 2)\sup_{n\in\mathbb{N}} \theta_{r\delta}A_n$.

(b) Suppose that $\delta > 0$ and that $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of subsets of X with union A. Then $\theta_{r\delta}A = \sup_{n\in\mathbb{N}}\theta_{r\delta}A_n$.

proof (a) If $\sup_{n\in\mathbb{N}}\theta_{r\delta}A_n = \infty$ this is trivial; suppose otherwise.

(i) Take any $\gamma > \gamma' > \sup_{n\in\mathbb{N}}\theta_{r\delta}A_n$. For each $i \in \mathbb{N}$, let $\zeta_i \in \left]0, \frac{1}{4}\delta\right]$ be such that $(\alpha + \zeta_i)^r \le \alpha^r + 2^{-i-1}(\gamma - \gamma')$ whenever $0 \le \alpha \le \delta$. For each $n \in \mathbb{N}$, there is a sequence $\langle C_{ni}\rangle_{i\in\mathbb{N}}$ of sets covering A_n such that $\operatorname{diam} C_{ni} \le \delta$ for every i and $\sum_{i=0}^{\infty}(\operatorname{diam} C_{ni})^r < \gamma'$; let $\langle\gamma_{ni}\rangle_{i\in\mathbb{N}}$ be such that $\operatorname{diam} C_{ni} \le \gamma_{ni} \le \delta$ and $\gamma_{ni} > 0$ for every i and $\sum_{i=0}^{\infty}\gamma_{ni}^r \le \gamma'$. Since $\sum_{i=0}^{\infty}\gamma_{ni}^r$ is finite, $\lim_{i\to\infty}\gamma_{ni} = 0$. Because $\gamma_{ni} > 0$ for every i, we may rearrange the sequences $\langle C_{ni}\rangle_{i\in\mathbb{N}}$, $\langle\gamma_{ni}\rangle_{i\in\mathbb{N}}$ in such a way that $\gamma_{ni} \ge \gamma_{n,i+1}$ for each i.

In this case, $\lim_{i\to\infty}\sup_{n\in\mathbb{N}}\gamma_{ni} = 0$. **P**

$$(i+1)\gamma_{ni}^r \le \textstyle\sum_{j=0}^{i}\gamma_{nj}^r \le \gamma$$

for every n, $i \in \mathbb{N}$. **Q**

(ii) By Ramsey's theorem (4A1G, with $n = 2$), there is an infinite set $I \subseteq \mathbb{N}$ such that

for all i, $j \in \mathbb{N}$ there is an $s \in \mathbb{N}$ such that either $C_{mi} \cap C_{nj} = \emptyset$ whenever m, $n \in I$ and $s \le m < n$ or $C_{mi} \cap C_{nj} \ne \emptyset$ whenever m, $n \in I$ and $s \le m < n$,

for each $i \in \mathbb{N}$, $\alpha_i = \lim_{n\in I,n\to\infty}\gamma_{ni}$ is defined in $\mathbb{R}$.

(Apply 4A1Fb with the families

$$\begin{aligned}\mathcal{J}_{ij} = \{J : J \in [\mathbb{N}]^\omega,\ &\text{either } C_{mi} \cap C_{nj} = \emptyset \text{ whenever } m,\, n \in J \text{ and } m < n\\ &\text{or } C_{mi} \cap C_{nj} \ne \emptyset \text{ whenever } m,\, n \in J \text{ and } m < n\}\\ \mathcal{J}'_{iq} = \{J : J \in [\mathbb{N}]^\omega,\ &\text{either } \gamma_{ni} \le q \text{ for every } n \in J\\ &\text{or } \gamma_{ni} \ge q \text{ for every } n \in J\}\end{aligned}$$

for i, $j \in \mathbb{N}$ and $q \in \mathbb{Q}$.)

Of course $\alpha_j \le \alpha_i \le \delta$ whenever $i \le j$, because $\gamma_{nj} \le \gamma_{ni} \le \delta$ for every n. Set $D_{ni} = \{x : \rho(x, C_{ni}) \le 2\alpha_i + 2\zeta_i\}$ for all n, $i \in \mathbb{N}$, and $D_i = \bigcup_{s\in\mathbb{N}}\bigcap_{n\in I\setminus s} D_{ni}$ for $i \in \mathbb{N}$. (I am identifying each $s \in \mathbb{N}$ with the set of its predecessors.) Note that if $i \in \mathbb{N}$ and x, $y \in D_i$, then there is an $s \in \mathbb{N}$ such that x, y both belong to D_{ni} for every $n \in I \setminus s$, so $\rho(x, y) \le \operatorname{diam} D_{ni}$ for every $n \in I \setminus s$ and $\rho(x, y) \le \liminf_{n\in I,n\to\infty}\operatorname{diam} D_{ni}$. Accordingly $\operatorname{diam} D_i \le \liminf_{n\in I,n\to\infty}\operatorname{diam} D_{ni}$.

(iii) Set

$$L = \{(i, j) : i,\, j \in \mathbb{N},\ \forall\, s \in \mathbb{N}\ \exists\, m,\, n \in I,\ s \le m < n \text{ and } C_{mi} \cap C_{nj} \ne \emptyset\}.$$

If $(i, j) \in L$ then there is an $s \in \mathbb{N}$ such that $C_{mi} \subseteq D_{\min(i,j)}$ whenever $m \in I$ and $m \ge s$. **P** By the choice of I, we know that there is an $s_0 \in \mathbb{N}$ such that $C_{mi} \cap C_{nj} \ne \emptyset$ whenever m, $n \in I$ and $s_0 \le m < n$. Let $s_1 \ge s_0$ be such that

$$\gamma_{mi} \le \alpha_i + \min(\zeta_i, \zeta_j), \qquad \gamma_{mj} \le \alpha_j + \min(\zeta_i, \zeta_j)$$

whenever $m \in I$ and $m \ge s_1$. Take $m_0 \in I$ such that $m_0 \ge s_1$, and set $s = m_0 + 1$. Let $m \in I$ be such that $m \ge s$.

(α) Suppose that $i \le j$ and $x \in C_{mi}$. Take any $n \in I$ such that $m \le n$. Then there is an $n' \in I$ such that $n < n'$. We know that $C_{mi} \cap C_{n'j}$ and $C_{ni} \cap C_{n'j}$ are both non-empty. So

$$\rho(x, C_{ni}) \le \operatorname{diam} C_{mi} + \operatorname{diam} C_{n'j} \le \gamma_{mi} + \gamma_{n'j} \le \alpha_i + \zeta_i + \alpha_j + \zeta_i \le 2\alpha_i + 2\zeta_i$$

and $x \in D_{ni}$. This is true for all $n \in I$ such that $n \ge m$, so $x \in D_i$. As x is arbitrary, $C_{mi} \subseteq D_i$.

(β) Suppose that $j \le i$ and $x \in C_{mi}$. Take any $n \in I$ such that $n > m$. Then $C_{mi} \cap C_{nj}$ is not empty, so

$$\rho(x, C_{nj}) \le \operatorname{diam} C_{mi} \le \gamma_i \le \alpha_i + \zeta_j \le \alpha_j + \zeta_j$$

and $x \in D_{nj}$. As x and n are arbitrary, $C_{mi} \subseteq D_j$.

Thus $C_{mi} \subseteq D_{\min(i,j)}$ in both cases. **Q**

(iv) Set

$$D=\bigcup_{i\in\mathbb{N}}D_i,\quad J=\{i:i\in\mathbb{N},\ \exists\, s\in\mathbb{N},\ C_{ni}\subseteq D \text{ whenever } n\in I \text{ and } n\ge s\}.$$

If $i\in\mathbb{N}\setminus J$ and $j\in\mathbb{N}$, then (iii) tells us that $(i,j)\notin L$, so there is some $s\in\mathbb{N}$ such that $C_{mi}\cap C_{nj}=\emptyset$ whenever m, $n\in I$ and $s\le m<n$.

(v) For $l\in\mathbb{N}$, $\mu^*_{Hr}(A_l\setminus D)\le 2\gamma$. **P** Let $\epsilon>0$. Then there is a $k\in\mathbb{N}$ such that $\gamma_{ni}\le\epsilon$ whenever $n\in\mathbb{N}$ and $i>k$. Next, there is an $s\in\mathbb{N}$ such that

$$C_{ni}\subseteq D \text{ whenever } i\le k,\ i\in J,\ n\in I \text{ and } s\le n,$$

$$C_{mi}\cap C_{nj}=\emptyset \text{ whenever } i,\ j\le k,\ i\notin J,\ m,\ n\in I \text{ and } s\le m<n.$$

Take m, $n\in I$ such that $\max(l,s)\le m<n$. Then

$$\begin{aligned}
A_l\setminus D&=\bigcup_{i\in\mathbb{N}}A_l\cap C_{mi}\setminus D\\
&\subseteq\bigcup_{i\le k}(A_n\cap C_{mi}\setminus D)\cup\bigcup_{i>k}C_{mi}\\
&\subseteq\bigcup_{i\le k,i\notin J}(A_n\cap C_{mi})\cup\bigcup_{i>k}C_{mi}\\
&\subseteq\bigcup_{i\le k,i\notin J,j\le k}(C_{mi}\cap C_{nj})\cup\bigcup_{j>k}C_{nj}\cup\bigcup_{i>k}C_{mi}\\
&=\bigcup_{j>k}C_{nj}\cup\bigcup_{i>k}C_{mi}.
\end{aligned}$$

Since $\operatorname{diam}C_{nj}\le\gamma_j\le\epsilon$ and $\operatorname{diam}C_{mi}\le\gamma_i\le\epsilon$ for all i, $j>k$,

$$\theta_{r\epsilon}(A_l\setminus D)\le\sum_{i=k+1}^\infty\gamma_{ni}^r+\sum_{i=k+1}^\infty\gamma_{mi}^r\le 2\gamma.$$

This is true for every $\epsilon>0$, so $\mu^*_{Hr}(A_l\setminus D)\le 2\gamma$, as claimed. **Q**

(vi) This is true for each $l\in\mathbb{N}$. But this means that $\mu^*_{Hr}(A\setminus D)\le 2\gamma$ (132Ae). Now $\theta_{r,6\delta}D\le 5^r\gamma$. **P** For each $i\in\mathbb{N}$,

$$\begin{aligned}
\operatorname{diam}D_i&\le\liminf_{n\in I,n\to\infty}\operatorname{diam}D_{ni}\le\liminf_{n\in I,n\to\infty}\operatorname{diam}C_{ni}+4\alpha_i+4\zeta_i\\
&\le\lim_{n\in I,n\to\infty}\gamma_{ni}+4\alpha_i+4\zeta_i=5\alpha_i+4\zeta_i\le 6\delta.
\end{aligned}$$

Next, for any $k\in\mathbb{N}$,

$$\sum_{i=0}^k\alpha_i^r=\lim_{n\in I,n\to\infty}\sum_{i=0}^k\gamma_{ni}^r\le\gamma',$$

so

$$\sum_{i=0}^k(\operatorname{diam}D_i)^r\le 5^r\sum_{i=0}^k(\alpha_i+\zeta_i)^r\le 5^r(\sum_{i=0}^k\alpha_i^r+2^{-i-1}(\gamma-\gamma'))$$

(by the choice of the ζ_i)

$$\le 5^r(\gamma'+\sum_{i=0}^k 2^{-i-1}(\gamma-\gamma'))\le 5^r\gamma.$$

Letting $k\to\infty$,

$$\theta_{r,6\delta}D\le\sum_{i=0}^\infty(\operatorname{diam}D_i)^r\le 5^r\gamma.\ \mathbf{Q}$$

Putting these together,

$$\theta_{r,6\delta}A\le\theta_{r,6\delta}D+\theta_{r,6\delta}(A\setminus D)\le\theta_{r,6\delta}D+\mu^*_{Hr}(A\setminus D)\le(5^r+2)\gamma.$$

As γ is arbitrary, we have the preliminary result (a).

(b) Now let us turn to the sharp form (b). Once again, we may suppose that $\sup_{n\in\mathbb{N}}\theta_{r\delta}A_n$ is finite.

(i) Take γ such that $\sup_{n\in\mathbb{N}}\theta_{r\delta}A_n<\gamma$. As in (a)(i) above, we can find a family $\langle C_{ni}\rangle_{n,i\in\mathbb{N}}$ such that

$$A_n\subseteq\bigcup_{i\in\mathbb{N}}C_{ni},$$

$$\operatorname{diam}C_{ni}\le\delta\text{ for every }i\in\mathbb{N},$$

$$\sum_{i=0}^{\infty}(\operatorname{diam}C_{ni})^r\le\gamma$$

for each n, and

$$\lim_{i\to\infty}\sup_{n\in\mathbb{N}}\operatorname{diam}C_{ni}=0.$$

Replacing each C_{ni} by its closure if necessary, we may suppose that every C_{ni} is a Borel set.

Let $Q\subseteq X$ be a countable set which meets C_{ni} whenever n, $i\in\mathbb{N}$ and C_{ni} is not empty. This time, let $I\subseteq\mathbb{N}$ be an infinite set such that

$$\alpha_i=\lim_{n\in I,n\to\infty}\operatorname{diam}C_{ni}\text{ is defined in }[0,\delta]\text{ for every }i\in\mathbb{N},$$

$$\lim_{n\in I,n\to\infty}\rho(z,C_{ni})\text{ is defined in }[0,\infty]\text{ for every }i\in\mathbb{N}\text{ and every }z\in Q.$$

(Take $\rho(z,\emptyset)=\infty$ if any of the C_{ni} are empty.) It will be helpful to note straight away that the limit $\lim_{n\in I,n\to\infty}\rho(x,C_{ni})$ is defined in $[0,\infty]$ for every $i\in\mathbb{N}$ and $x\in\overline{Q}$. **P** If $\lim_{n\in I,n\to\infty}\rho(y,C_{ni})=\infty$ for some $y\in Q$, then $\lim_{n\in I,n\to\infty}\rho(x,C_{ni})=\infty$, and we can stop. Otherwise, for any $\epsilon>0$, there are a $z\in Q$ such that $\rho(x,z)\le\epsilon$ and an $s\in\mathbb{N}$ such that C_{mi} is not empty and $|\rho(z,C_{mi})-\rho(z,C_{ni})|\le\epsilon$ whenever m, $n\in I\setminus s$; in which case $|\rho(x,C_{mi})-\rho(x,C_{ni})|\le3\epsilon$ whenever m, $n\in I\setminus s$. As ϵ is arbitrary, $\lim_{n\in I,n\to\infty}\rho(x,C_{ni})$ is defined in $\mathbb{R}$. **Q**

Let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbb{N}$ containing I, and for $i\in\mathbb{N}$ set

$$D_i=\{x:\lim_{n\to\mathcal{F}}\rho(x,C_{ni})=0\}.$$

Set $D=\bigcup_{i\in\mathbb{N}}\overline{D}_i$. (Actually it is easy to check that every D_i is closed.)

(ii) Set

$$A^*=\bigcup_{m\in\mathbb{N}}\bigcap_{n\in I\setminus m}\bigcup_{i\in\mathbb{N}}C_{ni}\setminus D;$$

note that A^* is a Borel set. For k, $m\in\mathbb{N}$, set

$$A^*_{km}=\bigcap_{n\in I\setminus m}\bigcup_{i\ge k}C_{ni}.$$

For fixed k, $\langle A^*_{km}\rangle_{m\in\mathbb{N}}$ is a non-decreasing sequence of sets. Also its union includes A^*. **P** Take $x\in A^*$.

(α)? If $x\notin\overline{Q}$, let $\epsilon>0$ be such that $Q\cap B(x,\epsilon)=\emptyset$. Let $l\in\mathbb{N}$ be such that $\operatorname{diam}C_{ni}\le\epsilon$ whenever $n\in\mathbb{N}$ and $i\ge l$; then $x\notin C_{ni}$ whenever $n\in\mathbb{N}$ and $i\ge l$. Let $m\in\mathbb{N}$ be such that $x\in\bigcup_{i\in\mathbb{N}}C_{ni}$ whenever $n\in I$ and $n\ge m$. Then $x\in\bigcup_{i<l}C_{ni}$ whenever $n\in I$ and $n\ge m$. But this means that there must be some $i<l$ such that $\{n:x\in C_{ni}\}\in\mathcal{F}$ and $x\in D_i\subseteq D$; which is impossible. **X**

(β) Thus $x\in\overline{Q}$, so $\lim_{n\in I,n\to\infty}\rho(x,C_{ni})$ is defined for each i (see (i) above), and must be greater than 0, since $x\notin D_i$. In particular, there is an $s\in\mathbb{N}$ such that $x\notin C_{ni}$ whenever $i<k$ and $n\in I\setminus s$; there is also an $m\in\mathbb{N}$ such that $x\in\bigcap_{n\in I\setminus m}\bigcup_{i\in\mathbb{N}}C_{ni}$; so that $x\in A^*_{k,\max(s,m)}$. As x is arbitrary, $A^*\subseteq\bigcup_{m\in\mathbb{N}}A^*_{km}$. **Q**

(iii) $\mu_{Hr}A^*$ is finite. **P** Take any $\epsilon>0$. Let $k\in\mathbb{N}$ be such that $\operatorname{diam}C_{ni}\le\epsilon$ whenever $i\ge k$ and $n\in\mathbb{N}$. For any $m\in I$, $\theta_{r\epsilon}A^*_{km}\le\sum_{i=k}^{\infty}(\operatorname{diam}C_{mi})^r\le\gamma$. By (a),

$$\begin{aligned}\theta_{r,6\epsilon}A^*&\le(5^r+2)\sup_{m\in\mathbb{N}}\theta_{r\epsilon}A^*_{km}\\&=(5^r+2)\sup_{m\in I}\theta_{r\epsilon}A^*_{km}\le(5^r+2)\gamma.\end{aligned}$$

As ϵ is arbitrary, $\mu_{Hr}A^*\le(5^r+2)\gamma<\infty$. **Q**

(iv) Actually, $\mu_{Hr}A^* \le \gamma - \sum_{i=0}^{\infty} \alpha_i^r$. **P?** Suppose, if possible, otherwise. Take β such that $\gamma - \sum_{i=0}^{\infty} \alpha_i^r < \beta < \mu_{Hr}A^*$. For $x \in A^*$ and $k \in \mathbb{N}$, set $f_k(x) = \min\{n : n \in I,\ x \in A^*_{kn}\}$; then $\langle f_k \rangle_{k\in\mathbb{N}}$ is a non-decreasing sequence of Borel measurable functions from A^* to $\mathbb{N}$. Choose $\langle s_k \rangle_{k\in\mathbb{N}}$ inductively so that

$$\mu_{Hr}\{x : x \in A^*,\ f_j(x) \le s_j \text{ for every } j \le k\} > \beta$$

for every $k \in \mathbb{N}$. Set $\tilde{A} = \{x : x \in A^*,\ f_j(x) \le s_j \text{ for every } j \in \mathbb{N}\}$; because $\mu_{Hr}A^*$ is finite, $\mu_{Hr}\tilde{A} \ge \beta$. Take $\epsilon > 0$ such that $\theta_{r\epsilon}\tilde{A} > \gamma - \sum_{i=0}^{\infty} \alpha_i^r$. Let $k \in \mathbb{N}$ be such that $\theta_{r\epsilon}\tilde{A} + \sum_{i=0}^{k-1} \alpha_i^r > \gamma$ and $\operatorname{diam} C_{ni} \le \epsilon$ whenever $n \in \mathbb{N}$ and $i \ge k$. Take $n \in I$ such that $n \ge s_j$ for every $j \le k$ and $\theta_{r\epsilon}\tilde{A} + \sum_{i=0}^{k-1} (\operatorname{diam} C_{ni})^r > \gamma$. If $x \in \tilde{A}$, then

$$f_k(x) \le s_k \le n, \quad x \in A^*_{kn} \subseteq \bigcup_{i\ge k} C_{ni},$$

so $\theta_{r\epsilon}\tilde{A} \le \sum_{i=k}^{\infty} (\operatorname{diam} C_{ni})^r$; but this means that $\sum_{i=0}^{\infty} (\operatorname{diam} C_{ni})^r > \gamma$, contrary to the choice of the C_{ni}. **XQ**

(v) Now observe that

$$A \subseteq \bigcup_{m\in\mathbb{N}} \bigcap_{n\ge m} \bigcup_{i\in\mathbb{N}} C_{ni} \subseteq A^* \cup D.$$

Moreover, for any $i \in \mathbb{N}$, $\operatorname{diam} \overline{D}_i \le \alpha_i \le \delta$. **P** If x, $y \in D_i$ then for every $\epsilon > 0$

$$\rho(x, C_{ni}) \le \epsilon, \quad \rho(y, C_{ni}) \le \epsilon, \quad \operatorname{diam} C_{ni} \le \alpha_i + \epsilon$$

for all but finitely many $n \in I$. So $\rho(x, y) \le \alpha_i + 3\epsilon$. As x, y and ϵ are arbitrary, $\operatorname{diam} \overline{D}_i = \operatorname{diam} D_i \le \alpha_i$. Of course $\alpha_i \le \delta$ because $\operatorname{diam} C_{ni} \le \delta$ for every n. **Q**

Now

$$\theta_{r\delta}D \le \sum_{i=0}^{\infty} (\operatorname{diam} \overline{D}_i)^r \le \sum_{i=0}^{\infty} \alpha_i^r.$$

Putting this together with (iv),

$$\theta_{r\delta}A \le \theta_{r\delta}D + \theta_{r\delta}A^* \le \theta_{r\delta}D + \mu_{Hr}A^* \le \gamma.$$

As γ is arbitrary,

$$\theta_{r\delta}A \le \sup_{n\in\mathbb{N}} \theta_{r\delta}A_n;$$

as $\theta_{r\delta}$ is an outer measure, we have equality.

471H Corollary Let (X, ρ) be a metric space, and $r > 0$. For $A \subseteq X$, set

$$\theta_{r\infty}A = \inf\{\sum_{n=0}^{\infty} (\operatorname{diam} D_n)^r : \langle D_n \rangle_{n\in\mathbb{N}} \text{ is a sequence of subsets of } X \text{ covering } A\}.$$

Then $\theta_{r\infty}$ is an outer regular Choquet capacity on X.

proof (a) Of course $0 \le \theta_{r\infty}A \le \theta_{r\infty}B$ whenever $A \subseteq B \subseteq X$.

(b) Suppose that $\langle A_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of subsets of A with union A. By (a), $\gamma = \lim_{n\to\infty} \theta_{r\infty}A_n$ is defined and less than or equal to $\theta_{r\infty}A$. If $\gamma = \infty$, of course it is equal to $\theta_{r\infty}A$. Otherwise, take $\beta = (\gamma + 1)^{1/r}$. For n, $k \in \mathbb{N}$ there is a sequence $\langle D_{nki} \rangle_{i\in\mathbb{N}}$ of sets, covering A_n, such that $\sum_{i=0}^{\infty} (\operatorname{diam} D_{nki})^r \le \gamma + 2^{-k}$. But in this case $\operatorname{diam} D_{nki} \le \beta$ for all n, k and i, so the D_{nki} witness that $\theta_{r\beta}A_n \le \gamma$. By 471Gb, $\gamma \ge \theta_{r\beta}A \ge \theta_{r\infty}A$ and again we have $\gamma = \theta_{r\infty}A$.

(c) Let $A \subseteq X$ be any set, and suppose that $\gamma > \theta_{r\infty}A$. Let $\langle D_n \rangle_{n\in\mathbb{N}}$ be a sequence of sets, covering A, such that $\sum_{n=0}^{\infty} (\operatorname{diam} D_n)^r < \gamma$. Let $\langle \epsilon_n \rangle_{n\in\mathbb{N}}$ be a sequence of strictly positive real numbers such that $\sum_{n=0}^{\infty} (\operatorname{diam} D_n + 2\epsilon_n)^r \le \gamma$. Set $G_n = \{x : \rho(x, D_n) < \epsilon_n\}$ for each n; then G_n is open and $\operatorname{diam} G_n \le \operatorname{diam} D_n + 2\epsilon_n$. So $G = \bigcup_{n\in\mathbb{N}} G_n$ is an open set including A, and $\langle G_n \rangle_{n\in\mathbb{N}}$ witnesses that $\theta_{r\infty}G \le \gamma$. As A and γ are arbitrary, the condition of 432Jb is satisfied and $\theta_{r\infty}$ is an outer regular Choquet capacity.

Remark $\theta_{r\infty}$ is **r-dimensional Hausdorff capacity** on X.

471I Theorem Let (X, ρ) be a metric space, and $r > 0$. Write μ_{Hr} for r-dimensional Hausdorff measure on X. If $A \subseteq X$ is analytic, then $\mu_{Hr}A$ is defined and equal to $\sup\{\mu_{Hr}K : K \subseteq A \text{ is compact}\}$.

proof (a) Before embarking on the main line of the proof, it will be convenient to set out a preliminary result. For $\delta > 0$, $n \in \mathbb{N}$, $B \subseteq X$ set

$$\theta^{(n)}_{r\delta}(B) = \inf\{\sum_{i=0}^{n}(\operatorname{diam} D_i)^r : B \subseteq \bigcup_{i\le n} D_i,\ \operatorname{diam} D_i \le \delta \text{ for every } i \le n\},$$

taking $\inf \emptyset = \infty$ as usual. Then $\theta_{r\delta}B \le \theta^{(n)}_{r\delta}(B)$ for every n. Now the point is that $\theta^{(n)}_{r\delta}(B) = \sup\{\theta^{(n)}_{r\delta}(I) : I \subseteq B \text{ is finite}\}$. **P** Set $\gamma = \sup_{I\in[B]^{<\omega}} \theta^{(n)}_{r\delta}(I)$. Of course $\gamma \le \theta^{(n)}_{r\delta}(B)$. If $\gamma = \infty$ there is nothing more to say. Otherwise, take any $\gamma' > \gamma$. For each $I \in [B]^{<\omega}$, we have a function $f_I : I \to \{0, \dots, n\}$ such that $\sum_{i\in J} \rho(x_i, y_i)^r \le \gamma'$ whenever $J \subseteq \{0, \dots, n\}$ and x_i, $y_i \in I$ and $f_I(x_i) = f_I(y_i) = i$ for every $i \in J$, while $\rho(x, y) \le \delta$ whenever x, $y \in I$ and $f_I(x) = f_I(y)$. Let $\mathcal{F}$ be an ultrafilter on $[B]^{<\omega}$ such that $\{I : x \in I \in [B]^{<\omega}\} \in \mathcal{F}$ for every $x \in B$ (4A1Ia). Then for every $x \in B$ there is an $f(x) \in \{0, \dots, n\}$ such that $\{I : x \in I \in [B]^{<\omega},\ f_I(x) = f(x)\} \in \mathcal{F}$. Set $D_i = f^{-1}[\{i\}]$ for $i \le n$. If x, $y \in B$ and $f(x) = f(y)$, there is an $I \in [B]^{<\omega}$ containing both x and y such that $f_I(x) = f(x) = f(y) = f_I(y)$, so that $\rho(x, y) \le \delta$; thus $\operatorname{diam} D_i \le \delta$ for each i. If $J \subseteq \{0, \dots, n\}$ and for each $i \in J$ we take x_i, $y_i \in D_i$, then there is an $I \in [B]^{<\omega}$ such that $f_I(x_i) = f_I(y_i) = i$ for every $i \in J$, so $\sum_{i\in J} \rho(x_i, y_i)^r \le \gamma'$. This means that $\sum_{i\le n}(\operatorname{diam} D_i)^r \le \gamma'$, so that $\theta^{(n)}_{r\delta}(B) \le \gamma'$. As γ' is arbitrary, $\theta^{(n)}_{r\delta}(B) \le \gamma$, as claimed. **Q**

(b) Now let us turn to the set A. Because A is Souslin-F (422Ha), μ_{Hr} measures A (471Da, 471Dd). Set $\gamma = \sup\{\mu_{Hr}K : K \subseteq A \text{ is compact}\}$.

? Suppose, if possible, that $\mu_{Hr}A > \gamma$. Take $\gamma' \in \,]\gamma, \mu_{Hr}A[$. Let $\delta > 0$ be such that $\gamma' < \theta_{r\delta}A$. Let $f : \mathbb{N}^{\mathbb{N}} \to A$ be a continuous surjection. For $\sigma \in S = \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$, set

$$F_\sigma = \{\phi : \phi \in \mathbb{N}^{\mathbb{N}},\ \phi(i) \le \sigma(i) \text{ for every } i < \#(\sigma)\},$$

so that $f[F_\emptyset] = A$. Now choose $\psi \in \mathbb{N}^{\mathbb{N}}$ and a sequence $\langle I_n\rangle_{n\in\mathbb{N}}$ of finite subsets of $\mathbb{N}^{\mathbb{N}}$ inductively, as follows. Given that $I_j \subseteq F_{\psi\restriction n}$ for every $j < n$ and that $\theta_{r\delta}(f[F_{\psi\restriction n}]) > \gamma'$, then $\theta^{(n)}_{r\delta}(f[F_{\psi\restriction n}]) > \gamma'$, so by (a) above there is a finite subset I_n of $F_{\psi\restriction n}$ such that $\theta^{(n)}_{r\delta}(f[I_n]) \ge \gamma'$. Next,

$$\lim_{i\to\infty} \theta_{r\delta} f[F_{(\psi\restriction n)^\frown<i>}] = \theta_{r\delta}(\bigcup_{i\in\mathbb{N}} f[F_{(\psi\restriction n)^\frown<i>}])$$

(by 471G)

$$= \theta_{r\delta} f[F_{\psi\restriction n}] > \gamma',$$

so we can take $\psi(n)$ such that $\bigcup_{j\le n} I_j \subseteq F_{\psi\restriction n+1}$ and $\theta_{r\delta} f[F_{\psi\restriction n+1}] > \gamma'$, and continue.

At the end of the induction, set $K = \{\phi : \phi \le \psi\}$. Then $f[K]$ is a compact subset of A, and $I_n \subseteq K$ for every $n \in \mathbb{N}$, so

$$\theta^{(n)}_{r\delta}(f[K]) \ge \theta^{(n)}_{r\delta}(f[I_n]) \ge \gamma'$$

for every $n \in \mathbb{N}$. On the other hand, $\mu_{Hr}(f[K]) \le \gamma$, so there is a sequence $\langle D_i\rangle_{i\in\mathbb{N}}$ of sets, covering $f[K]$, all of diameter less than δ, such that $\sum_{i=0}^{\infty}(\operatorname{diam} D_i)^r < \gamma'$. Enlarging the D_i slightly if need be, we may suppose that they are all open. But in this case there is some finite n such that $K \subseteq \bigcup_{i\le n} D_i$, and $\theta^{(n)}_{r\delta}(K) \le \sum_{i=0}^{n}(\operatorname{diam} D_i)^r < \gamma'$; which is impossible. **X**

This contradiction shows that $\mu_{Hr}A = \gamma$, as required.

471J Proposition Let (X, ρ) and (Y, σ) be metric spaces, and $f : X \to Y$ a γ-Lipschitz function, where $\gamma \ge 0$. If $r > 0$ and $\theta^{(X)}_r$, $\theta^{(Y)}_r$ are the r-dimensional Hausdorff outer measures on X and Y respectively, then $\theta^{(Y)}_r f[A] \le \gamma^r \theta^{(X)}_r A$ for every $A \subseteq X$.

proof (Compare 264G.) Let $\delta > 0$. Set $\eta = \delta/(1+\gamma)$ and consider $\theta^{(X)}_{r\eta} : \mathcal{P}X \to [0, \infty]$, defined as in 471A. We know that $\theta^{(X)}_r A \ge \theta^{(X)}_{r\eta} A$, so there is a sequence $\langle D_n\rangle_{n\in\mathbb{N}}$ of sets, all of diameter at most η, covering A, with $\sum_{n=0}^{\infty}(\operatorname{diam} D_n)^r \le \theta^{(X)}_r A + \delta$. Now $f[A] \subseteq \bigcup_{n\in\mathbb{N}} f[D_n]$ and

$$\operatorname{diam} f[D_n] \le \gamma \operatorname{diam} D_n \le \gamma\eta \le \delta$$

for every n. Consequently

$$\theta_{r\delta}^{(Y)}(f[A]) \le \sum_{n=0}^{\infty}(\operatorname{diam} f[D_n])^r \le \sum_{n=0}^{\infty}\gamma^r(\operatorname{diam} D_n)^r \le \gamma^r(\theta_r^{(X)}A+\delta),$$

and

$$\theta_r^{(Y)}(f[A]) = \lim_{\delta\downarrow 0}\theta_{r\delta}^{(Y)}(f[A]) \le \gamma^r\theta_r^{(X)}A,$$

as claimed.

471K Lemma Let (X,ρ) be a metric space, and $r>0$. Let μ_{Hr} be r-dimensional Hausdorff measure on X. If $A\subseteq X$, then $\mu_{Hr}A=0$ iff for every $\epsilon>0$ there is a countable family $\mathcal{D}$ of sets, covering A, such that $\sum_{D\in\mathcal{D}}(\operatorname{diam} D)^r\le\epsilon$.

proof If $\mu_{Hr}A=0$ and $\epsilon>0$, then, in the language of 471A, $\theta_{r1}A\le\epsilon$, so there is a sequence $\langle D_n\rangle_{n\in\mathbb{N}}$ of sets covering A such that $\sum_{n=0}^{\infty}(\operatorname{diam} D_n)^r\le\epsilon$.

If the condition is satisfied, then for any ϵ, $\delta>0$ there is a countable family $\mathcal{D}$ of sets, covering A, such that $\sum_{D\in\mathcal{D}}(\operatorname{diam} D)^r\le\min(\epsilon,\delta^r)$. If $\mathcal{D}$ is infinite, enumerate it as $\langle D_n\rangle_{n\in\mathbb{N}}$; if it is finite, enumerate it as $\langle D_n\rangle_{n<m}$ and set $D_n=\emptyset$ for $n\ge m$. Now $A\subseteq\bigcup_{n\in\mathbb{N}}D_n$ and $\operatorname{diam} D_n\le\delta$ for every $n\in\mathbb{N}$, so $\theta_{r\delta}A\le\sum_{n=0}^{\infty}(\operatorname{diam} D_n)^r\le\epsilon$. As ϵ is arbitrary, $\theta_{r\delta}A=0$; as δ is arbitrary, $\theta_rA=0$; it follows at once that $\mu_{Hr}A$ is defined and is zero (113Xa).

471L Proposition Let (X,ρ) be a metric space and $0<r<s$. If $A\subseteq X$ is such that μ_{Hr}^*A is finite, then $\mu_{Hs}A=0$.

proof Let $\epsilon>0$. Let $\delta>0$ be such that $\delta^{s-r}(1+\mu_{Hr}^*A)\le\epsilon$. Then there is a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of sets of diameter at most δ such that $A\subseteq\bigcup_{n\in\mathbb{N}}A_n$ and $\sum_{n=0}^{\infty}(\operatorname{diam} A_n)^r\le 1+\mu_{Hr}^*A$. But now, by the choice of δ, $\sum_{n=0}^{\infty}(\operatorname{diam} A_n)^s\le\epsilon$. As ϵ is arbitrary, $\mu_{Hs}A=0$, by 471K.

471M There is a generalization of the density theorems of §§223 and 261 for general Hausdorff measures, which (as one expects) depends on a kind of Vitali theorem. I will use the following notation for the next few paragraphs.

Definition If (X,ρ) is a metric space and $A\subseteq X$, write $A^\sim$ for $\{x:x\in X,\ \rho(x,A)\le 2\operatorname{diam} A\}$, where $\rho(x,A)=\inf_{y\in A}\rho(x,y)$. (Following the conventions of 471A, $\emptyset^\sim=\emptyset$.)

471N Lemma Let (X,ρ) be a metric space. Let $\mathcal{F}$ be a family of subsets of X such that $\{\operatorname{diam} F:F\in\mathcal{F}\}$ is bounded. Set

$$Y=\bigcap_{\delta>0}\bigcup\{F:F\in\mathcal{F},\ \operatorname{diam} F\le\delta\}.$$

Then there is a disjoint family $\mathcal{I}\subseteq\mathcal{F}$ such that

(i) $\bigcup\mathcal{F}\subseteq\bigcup_{F\in\mathcal{I}}F^\sim$;

(ii) $Y\subseteq\overline{\bigcup\mathcal{J}}\cup\bigcup_{F\in\mathcal{I}\setminus\mathcal{J}}F^\sim$ for every $\mathcal{J}\subseteq\mathcal{I}$.

proof (a) Let γ be an upper bound for $\{\operatorname{diam} F:F\in\mathcal{F}\}$. Choose $\langle\mathcal{I}_n\rangle_{n\in\mathbb{N}}$, $\langle\mathcal{J}_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. $\mathcal{I}_0=\emptyset$. Given $\mathcal{I}_n$, set $\mathcal{F}_n'=\{F:F\in\mathcal{F},\ \operatorname{diam} F\ge 2^{-n}\gamma,\ F\cap\bigcup\mathcal{I}_n=\emptyset\}$, and let $\mathcal{J}_n\subseteq\mathcal{F}_n'$ be a maximal disjoint set; now set $\mathcal{I}_{n+1}=\mathcal{I}_n\cup\mathcal{J}_n$, and continue.

At the end of the induction, set

$$\mathcal{I}'=\bigcup_{n\in\mathbb{N}}\mathcal{I}_n,\quad \mathcal{I}=\mathcal{I}'\cup\{\{x\}:x\in F\setminus\bigcup\mathcal{I}',\ \{x\}\in\mathcal{F}\}.$$

The construction ensures that every $\mathcal{I}_n$ is a disjoint subset of $\mathcal{F}$, so $\mathcal{I}'$ and $\mathcal{I}$ are also disjoint subfamilies of $\mathcal{F}$.

(b) ? Suppose, if possible, that there is a point x in $\bigcup\mathcal{F}\setminus\bigcup_{F\in\mathcal{I}}F^\sim$. Let $F\in\mathcal{F}$ be such that $x\in F$. Since $x\notin\bigcup\mathcal{I}'$ and $\{x\}\notin\mathcal{I}$, $\{x\}\notin\mathcal{F}$, and $\operatorname{diam} F>0$; let $n\in\mathbb{N}$ be such that $2^{-n}\gamma\le\operatorname{diam} F\le 2^{-n+1}\gamma$. If $F\notin\mathcal{F}_n'$, there is a $D\in\mathcal{I}_n$ such that $F\cap D\ne\emptyset$; otherwise, since $\mathcal{J}_n$ is maximal and $F\notin\mathcal{J}_n$, there is a $D\in\mathcal{J}_n$ such that $F\cap D\ne\emptyset$. In either case, we have a $D\in\mathcal{I}$ such that $F\cap D\ne\emptyset$ and $\operatorname{diam} F\le 2\operatorname{diam} D$. But in this case $\rho(x,D)\le\operatorname{diam} F\le 2\operatorname{diam} D$ and $x\in D^\sim$, which is impossible. **X**

(c) ? Suppose, if possible, that there are a point $x\in Y$ and a set $\mathcal{J}\subseteq\mathcal{I}$ such that $x\notin\overline{\bigcup\mathcal{J}}\cup\bigcup_{F\in\mathcal{I}\setminus\mathcal{J}}F^\sim$. Then there is an $F\in\mathcal{F}$ such that $x\in F$ and $\operatorname{diam} F<\rho(x,\bigcup\mathcal{J})$, so that $F\cap\bigcup\mathcal{J}=\emptyset$. As in (b), F

cannot be $\{x\}$, and there must be an $n \in \mathbb{N}$ such that $2^{-n}\gamma < \operatorname{diam} F \le 2^{-n+1}\gamma$. As in (b), there must be a $D \in \mathcal{I}_{n+1}$ such that $F \cap D \ne \emptyset$, so that $x \in D^\sim$; and as D cannot belong to $\mathcal{J}$, we again have a contradiction. **X**

471O Lemma Let (X,ρ) be a metric space, and $r > 0$. Suppose that A, $\mathcal{F}$ are such that

(i) $\mathcal{F}$ is a family of closed subsets of X such that $\sum_{n=0}^\infty (\operatorname{diam} F_n)^r$ is finite for every disjoint sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{F}$,

(ii) for every $x \in A$, $\delta > 0$ there is an $F \in \mathcal{F}$ such that $x \in F$ and $0 < \operatorname{diam} F \le \delta$.

Then there is a countable disjoint family $\mathcal{I} \subseteq \mathcal{F}$ such that $A \setminus \bigcup \mathcal{I}$ has zero r-dimensional Hausdorff measure.

proof Replacing $\mathcal{F}$ by $\{F : F \in \mathcal{F},\ 0 < \operatorname{diam} F \le 1\}$ if necessary, we may suppose that $\sup_{F\in\mathcal{F}} \operatorname{diam} F$ is finite and that $\operatorname{diam} F > 0$ for every $F \in \mathcal{F}$. Take a disjoint family $\mathcal{I} \subseteq \mathcal{F}$ as in 471N. If $\mathcal{I}$ is finite, then $A \subseteq Y \subseteq \bigcup \mathcal{I}$, where Y is defined as in 471N, so we can stop. Otherwise, hypothesis (i) tells us that $\{F : F \in \mathcal{I},\ \operatorname{diam} F \ge \delta\}$ is finite for every $\delta > 0$, so $\mathcal{I}$ is countable; enumerate it as $\langle F_n\rangle_{n\in\mathbb{N}}$; we must have $\sum_{n=0}^\infty (\operatorname{diam} F_n)^r < \infty$. Since $\operatorname{diam} F_n^\sim \le 5 \operatorname{diam} F_n$ for every n, $\sum_{n=0}^\infty (\operatorname{diam} F_n^\sim)^r$ is finite, and $\inf_{n\in\mathbb{N}} \sum_{i=n}^\infty (\operatorname{diam} F_i^\sim)^r = 0$. But now observe that the construction ensures that $A \setminus \bigcup \mathcal{I} \subseteq \bigcup_{i\ge n} F_i^\sim$ for every $n \in \mathbb{N}$. By 471K, $\mu_{Hr}(A \setminus \bigcup \mathcal{I}) = 0$, as required.

471P Theorem Let (X,ρ) be a metric space, and $r > 0$. Let μ_{Hr} be r-dimensional Hausdorff measure on X. Suppose that $A \subseteq X$ and $\mu_{Hr}^*A < \infty$.

(a) $\lim_{\delta\downarrow 0} \sup\{\frac{\mu_{Hr}^*(A\cap D)}{(\operatorname{diam} D)^r} : x \in D,\ 0 < \operatorname{diam} D \le \delta\} = 1$ for μ_{Hr}-almost every $x \in A$.

(b) $\limsup_{\delta\downarrow 0} \frac{\mu_{Hr}^*(A\cap B(x,\delta))}{\delta^r} \ge 1$ for μ_{Hr}-almost every $x \in A$. So

$$2^{-r} \le \limsup_{\delta\downarrow 0} \frac{\mu_{Hr}^*(A\cap B(x,\delta))}{(\operatorname{diam} B(x,\delta))^r} \le 1$$

for μ_{Hr}-almost every $x \in A$.

(c) If A is measured by μ_{Hr}, then

$$\lim_{\delta\downarrow 0} \sup\{\frac{\mu_{Hr}^*(A\cap D)}{(\operatorname{diam} D)^r} : x \in D,\ 0 < \operatorname{diam} D \le \delta\} = 0$$

for μ_{Hr}-almost every $x \in X \setminus A$.

proof (a)(i) Note first that as the quantities

$$\sup\{\frac{\mu_{Hr}^*(A\cap D)}{(\operatorname{diam} D)^r} : x \in D,\ 0 < \operatorname{diam} D \le \delta\}$$

decrease with δ, the limit is defined in $[0,\infty]$ for every $x \in X$. Moreover, since $\operatorname{diam} D = \operatorname{diam} \overline{D}$ and $\mu_{Hr}^*(A \cap \overline{D}) \ge \mu_{Hr}^*(A \cap D)$ for every D,

$$\sup\{\frac{\mu_{Hr}^*(A\cap D)}{(\operatorname{diam} D)^r} : x \in D \subseteq X,\ 0 < \operatorname{diam} D \le \delta\}$$
$$= \sup\{\frac{\mu_{Hr}^*(A\cap F)}{(\operatorname{diam} F)^r} : F \subseteq X \text{ is closed},\ x \in F,\ 0 < \operatorname{diam} F \le \delta\}$$

for every x and δ.

(ii) Fix ϵ for the moment, and set

$$A_\epsilon = \{x : x \in A,\ \lim_{\delta\downarrow 0} \sup\{\frac{\mu_{Hr}^*(A\cap D)}{(\operatorname{diam} D)^r} : x \in D,\ 0 < \operatorname{diam} D \le \delta\} > 1 + \epsilon\}.$$

Then $\theta_{r\eta}(A) \le \mu_{Hr}^*A - \frac{\epsilon}{1+\epsilon}\mu_{Hr}^*A_\epsilon$ for every $\eta > 0$, where $\theta_{r\eta}$ is defined in 471A. **P** Let $\mathcal{F}$ be the family

$$\{F : F \subseteq X \text{ is closed},\ 0 < \operatorname{diam} F \le \eta,\ (1+\epsilon)(\operatorname{diam} F)^r \le \mu_{Hr}^*(A \cap F)\}.$$

Then every member of A_ϵ belongs to sets in $\mathcal{F}$ of arbitrarily small diameter. Also, if $\langle F_n\rangle_{n\in\mathbb{N}}$ is any disjoint sequence in $\mathcal{F}$,

$$\sum_{n=0}^{\infty}(\operatorname{diam} F_n)^r \le \sum_{n=0}^{\infty}\mu_{Hr}^*(A\cap F_n) \le \mu_{Hr}^*A < \infty$$

because every F_n, being closed, is measured by μ_{Hr}. (If you like, $F_n \cap A$ is measured by the subspace measure on A for every n.) So 471O tells us that there is a countable disjoint family $\mathcal{I} \subseteq \mathcal{F}$ such that $A_\epsilon \setminus \bigcup\mathcal{I}$ is negligible, and $\mu_{Hr}^*A_\epsilon = \mu_{Hr}^*(A_\epsilon \cap \bigcup\mathcal{I})$.

Because $\theta_{r\eta}$ is an outer measure and $\theta_{r\eta} \le \mu_{Hr}^*$,

$$\theta_{r\eta}A \le \theta_{r\eta}(A\cap\bigcup\mathcal{I}) + \theta_{r\eta}(A\setminus\bigcup\mathcal{I}) \le \sum_{F\in\mathcal{I}}(\operatorname{diam} F)^r + \mu_{Hr}^*(A\setminus\bigcup\mathcal{I})$$

(because $\mathcal{I}$ is countable)

$$\le \tfrac{1}{1+\epsilon}\mu_{Hr}^*(A\cap\bigcup\mathcal{I}) + \mu_{Hr}^*(A\setminus\bigcup\mathcal{I}) = \mu_{Hr}^*A - \tfrac{\epsilon}{1+\epsilon}\mu_{Hr}^*(A\cap\bigcup\mathcal{I})$$

$$\le \mu_{Hr}^*A - \tfrac{\epsilon}{1+\epsilon}\mu_{Hr}^*(A_\epsilon\cap\bigcup\mathcal{I}) = \mu_{Hr}^*A - \tfrac{\epsilon}{1+\epsilon}\mu_{Hr}^*A_\epsilon,$$

as claimed. **Q**

(iii) Taking the supremum as $\eta \downarrow 0$, $\mu_{Hr}^*A \le \mu_{Hr}^*A - \frac{\epsilon}{1+\epsilon}\mu_{Hr}^*A_\epsilon$ and $\mu_{Hr}A_\epsilon = 0$.

This is true for any $\epsilon > 0$. But

$$\{x : x \in A,\ \lim_{\delta\downarrow 0}\sup\{\tfrac{\mu_{Hr}^*(A\cap D)}{(\operatorname{diam} D)^r} : x \in D,\ 0 < \operatorname{diam} D \le \delta\} > 1\}$$

is just $\bigcup_{n\in\mathbb{N}} A_{2^{-n}}$, so is negligible.

(iv) Next, for $0 < \epsilon \le 1$, set

$$A'_\epsilon = \{x : x \in A,\ \mu_{Hr}^*(A\cap D) \le (1-\epsilon)(\operatorname{diam} D)^r \text{ whenever } x \in D \text{ and } 0 < \operatorname{diam} D \le \epsilon\}.$$

Then A'_ϵ is negligible. **P** Let $\langle D_n\rangle_{n\in\mathbb{N}}$ be any sequence of sets of diameter at most ϵ covering A'_ϵ. Set $K = \{n : D_n \cap A'_\epsilon \ne \emptyset\}$. Then

$$\mu_{Hr}^*A'_\epsilon \le \sum_{n\in K}\mu_{Hr}^*(A\cap D_n)$$

$$\le (1-\epsilon)\sum_{n\in K}(\operatorname{diam} D_n)^r \le (1-\epsilon)\sum_{n=0}^{\infty}(\operatorname{diam} D_n)^r.$$

As $\langle D_n\rangle_{n\in\mathbb{N}}$ is arbitrary,

$$\mu_{Hr}^*A'_\epsilon \le (1-\epsilon)\theta_{r\epsilon}A'_\epsilon \le (1-\epsilon)\mu_{Hr}^*A'_\epsilon,$$

and $\mu_{Hr}^*A'_\epsilon$ (being finite) must be zero. **Q**

This means that

$$\{x : x \in A,\ \lim_{\delta\downarrow 0}\sup\{\tfrac{\mu_{Hr}^*(A\cap D)}{(\operatorname{diam} D)^r} : x \in D,\ 0 < \operatorname{diam} D \le \delta\} < 1\} \subseteq \bigcup_{n\in\mathbb{N}} A'_{2^{-n}}$$

is also negligible, and we have the result.

(b) We need a slight modification of the argument in (a)(iv). This time, for $0 < \epsilon \le 1$, set

$$\tilde{A}'_\epsilon = \{x : x \in A,\ \mu_{Hr}^*(A\cap B(x,\delta)) \le (1-\epsilon)\delta^r \text{ whenever } 0 < \delta \le \epsilon\}.$$

Then $\mu_{Hr}^*\tilde{A}'_\epsilon \le \epsilon$. **P** Note first that, as $\mu_{Hr}\{x\} = 0$ for every x, $\mu_{Hr}^*(A\cap B(x,\delta)) \le (1-\epsilon)\delta^r$ whenever $x \in \tilde{A}'_\epsilon$ and $0 \le \delta \le \epsilon$. Let $\langle D_n\rangle_{n\in\mathbb{N}}$ be a sequence of sets of diameter at most ϵ covering $\tilde{A}'_\epsilon$. Set $K = \{n : D_n \cap \tilde{A}'_\epsilon \ne \emptyset\}$, and for $n \in K$ choose $x_n \in D_n \cap \tilde{A}'_\epsilon$ and set $\delta_n = \operatorname{diam} D_n$. Then $D_n \subseteq B(x_n,\delta_n)$ and $\delta_n \le \epsilon$ for each n, so $\tilde{A}'_\epsilon \subseteq \bigcup_{n\in K} B(x_n,\delta_n)$ and

$$\begin{aligned}\mu_{Hr}^*\tilde{A}'_\epsilon &\le \sum_{n\in K}\mu_{Hr}^*(\tilde{A}'_\epsilon\cap B(x_n,\delta_n))\\ &\le \sum_{n\in K}(1-\epsilon)\delta_n^r \le (1-\epsilon)\sum_{n=0}^{\infty}(\operatorname{diam} D_n)^r.\end{aligned}$$

As $\langle D_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mu_{Hr}^*\tilde{A}'_\epsilon \le (1-\epsilon)\mu_{Hr}^*\tilde{A}'_\epsilon$ and $\tilde{A}'_\epsilon$ must be negligible. **Q**

Now

$$\{x : x\in A,\ \limsup_{\delta\downarrow 0}\tfrac{\mu_{Hr}^*(A\cap B(x,\delta))}{\delta^r} < 1\} = \bigcup_{n\in\mathbb{N}}\tilde{A}'_{2^{-n}}$$

is negligible. As for the second formula, we need note only that $\operatorname{diam} B(x,\delta)\le 2\delta$ for every $x\in X$, $\delta>0$ to obtain the first inequality, and apply (a) to obtain the second.

(c) Let $\epsilon>0$. This time, write $\tilde{A}_\epsilon$ for

$$\{x : x\in X,\ \lim_{\delta\downarrow 0}\sup\{\tfrac{\mu_{Hr}^*(A\cap D)}{(\operatorname{diam} D)^r} : x\in D,\ 0<\operatorname{diam} D\le\delta\} > \epsilon\}.$$

Let $E\subseteq A$ be a closed set such that $\mu(A\setminus E)\le\epsilon^2$ (471De). For $\eta>0$, let $\mathcal{F}_\eta$ be the family

$$\{F : F\subseteq X\setminus E \text{ is closed, } 0<\operatorname{diam} F\le\eta,\ \mu_{Hr}(A\cap F)\ge\epsilon(\operatorname{diam} F)^r\}.$$

Just as in (a) above, every point in $\tilde{A}_\epsilon\setminus E$ belongs to members of $\mathcal{F}_\eta$ of arbitrarily small diameter. If $\langle F_i\rangle_{i\in I}$ is a countable disjoint family in $\mathcal{F}_\eta$,

$$\textstyle\sum_{i\in I}(\operatorname{diam} F_i)^r \le \frac{1}{\epsilon}\mu_{Hr}(A\setminus E)\le\epsilon$$

is finite. There is therefore a countable disjoint family $\mathcal{I}_\eta\subseteq\mathcal{F}_\eta$ such that $\mu_{Hr}((\tilde{A}_\epsilon\setminus E)\setminus\bigcup\mathcal{I}_\eta)=0$. If $\theta_{r\eta}$ is the outer measure defined in 471A, we have

$$\begin{aligned}\theta_{r\eta}(\tilde{A}_\epsilon\setminus A) &\le \theta_{r\eta}(\bigcup\mathcal{I}_\eta)+\theta_{r\eta}(\tilde{A}_\epsilon\setminus(E\cup\bigcup\mathcal{I}_\eta))\\ &\le \sum_{F\in\mathcal{I}_\eta}(\operatorname{diam} F)^r+\mu_{Hr}^*(\tilde{A}_\epsilon\setminus(E\cup\bigcup\mathcal{I}_\eta))\le\epsilon.\end{aligned}$$

As η is arbitrary, $\mu_{Hr}^*(\tilde{A}_\epsilon\setminus A)\le\epsilon$. But now

$$\{x : x\in X\setminus A,\ \lim_{\delta\downarrow 0}\sup\{\tfrac{\mu_{Hr}^*(A\cap D)}{(\operatorname{diam} D)^r} : x\in D,\ 0<\operatorname{diam} D\le\delta\} > 0\}$$

is $\bigcap_{n\in\mathbb{N}}\tilde{A}_{2^{-n}}\setminus A$, and is negligible.

471Q I now come to a remarkable fact about Hausdorff measures on analytic spaces: their Borel versions are semi-finite (471S). We need some new machinery.

Lemma Let (X,ρ) be a metric space, and $r>0$, $\delta>0$. Suppose that $\theta_{r\delta}X$, as defined in 471A, is finite.

(a) There is a non-negative additive functional ν on $\mathcal{P}X$ such that $\nu X = 5^{-r}\theta_{r\delta}X$ and $\nu A\le(\operatorname{diam} A)^r$ whenever $A\subseteq X$ and $\operatorname{diam} A\le\frac{1}{5}\delta$.

(b) If X is compact, there is a Radon measure μ on X such that $\mu X = 5^{-r}\theta_{r\delta}X$ and $\mu G\le(\operatorname{diam} G)^r$ whenever $G\subseteq X$ is open and $\operatorname{diam} G\le\frac{1}{5}\delta$.

proof (a) I use 391E. If $\theta_{r\delta}X=0$ the result is trivial. Otherwise, set $\gamma=5^r/\theta_{r\delta}X$ and define $\phi:\mathcal{P}X\to[0,1]$ by setting $\phi A=\min(1,\gamma(\operatorname{diam} A)^r)$ if $\operatorname{diam} A\le\frac{1}{5}\delta$, 1 for other $A\subseteq X$. Now

> whenever $\langle A_i\rangle_{i\in I}$ is a finite family of subsets of X, $m\in\mathbb{N}$ and $\sum_{i\in I}\chi A_i\ge m\chi X$, then $\sum_{i\in I}\phi A_i\ge m$.

P Discarding any A_i for which $\phi A_i=1$, if necessary, we may suppose that $\operatorname{diam} A_i\le\frac{1}{5}\delta$ and $\phi A_i=\gamma(\operatorname{diam} A_i)^r$ for every i. Choose $\langle I_j\rangle_{j\le m}$, $\langle J_j\rangle_{j<m}$ inductively, as follows. $I_0=I$. Given that $j<m$ and that $I_j\subseteq I$ is such that $\sum_{i\in I_j}\chi A_i\ge(m-j)\chi X$, apply 471N to $\{A_i : i\in I_j\}$ to find $J_j\subseteq I_j$ such that

$\langle A_i\rangle_{i\in J_j}$ is disjoint and $\bigcup_{i\in I_j} A_i \subseteq \bigcup_{i\in J_j} A_j^\sim$. Set $I_{j+1} = I_j \setminus J_j$. Observe that $\sum_{i\in J_j}\chi A_j \le \chi X$, so $\sum_{i\in I_{j+1}}\chi A_i \ge (m-j-1)\chi X$ and the induction proceeds.

Now note that, for each $j < m$, $\langle A_i^\sim\rangle_{i\in J_j}$ is a cover of $\bigcup_{i\in I_j} A_i = X$ by sets of diameter at most δ. So $\sum_{i\in J_j}(\operatorname{diam} A_i^\sim)^r \ge \theta_{r\delta}X$ for each j, and $\sum_{i\in I}(\operatorname{diam} A_i^\sim)^r \ge m\theta_{r\delta}X$. Accordingly

$$\begin{aligned}\sum_{i\in I}\phi A_i = \gamma\sum_{i\in I}(\operatorname{diam} A_i)^r &\ge 5^{-r}\gamma\sum_{i\in I}(\operatorname{diam} A_i^\sim)^r\\ &\ge 5^{-r}m\gamma\theta_{r\delta}X = m. \quad \mathbf{Q}\end{aligned}$$

By 391E, there is an additive functional $\nu_0 : \mathcal{P}X \to [0,1]$ such that $\nu_0 X = 1$ and $\nu_0 A \le \phi A$ for every $A \subseteq X$. Setting $\nu = 5^{-r}\theta_{r\delta}X\nu_0$, we have the result.

(b) Now suppose that X is compact. By 416K, there is a Radon measure μ on X such that $\mu K \ge \nu K$ for every compact $K \subseteq X$ and $\mu G \le \nu G$ for every open $G \subseteq X$. Because X itself is compact, $\mu X = \nu X = 5^{-r}\theta_{r\delta}X$. If G is open and $\operatorname{diam} G \le \frac{1}{5}\delta$,

$$\mu G \le \nu G \le (\operatorname{diam} G)^r,$$

as required.

471R Lemma (Howroyd 95) Let (X,ρ) be a compact metric space and $r > 0$. Let μ_{Hr} be r-dimensional Hausdorff measure on X. If $\mu_{Hr}X > 0$, there is a Borel set $H \subseteq X$ such that $0 < \mu_{Hr}H < \infty$.

proof (a) Let $\delta > 0$ be such that $\theta_{r,5\delta}(X) > 0$, where $\theta_{r,5\delta}$ is defined as in 471A. Then there is a family $\mathcal{V}$ of open subsets of X such that (i) $\operatorname{diam} V \le \delta$ for every $V \in \mathcal{V}$ (ii) $\{V : V \in \mathcal{V}, \operatorname{diam} V \ge \epsilon\}$ is finite for every $\epsilon > 0$ (iii) whenever $A \subseteq X$ and $0 < \operatorname{diam} A < \frac{1}{4}\delta$ there is a $V \in \mathcal{V}$ such that $A \subseteq V$ and $\operatorname{diam} V \le 8\operatorname{diam} A$. **P** For each $k \in \mathbb{N}$, let I_k be a finite subset of X such that $X = \bigcup_{x\in I_k} B(x, 2^{-k-2}\delta)$; now set $\mathcal{V} = \{U(x, 2^{-k-1}\delta) : k \in \mathbb{N}, x \in I_k\}$. Then $\mathcal{V}$ is a family of open sets and (i) and (ii) are satisfied. If $A \subseteq X$ and $0 < \operatorname{diam} A < \frac{1}{4}\delta$, let $k \in \mathbb{N}$ be such that $2^{-k-3}\delta \le \operatorname{diam} A < 2^{-k-2}\delta$. Take $x \in I_k$ such that $B(x, 2^{-k-2}\delta) \cap A \ne \emptyset$; then $A \subseteq U(x, 2^{-k-1}\delta) \in \mathcal{V}$ and $\operatorname{diam} U(x, 2^{-k-1}\delta) \le 2^{-k}\delta \le 8\operatorname{diam} A$. **Q**

In particular, $\{V : V \in \mathcal{V}, \operatorname{diam} V \le \epsilon\}$ covers X for every $\epsilon > 0$.

(b) Set

$$P = \{\mu : \mu \text{ is a Radon measure on } X, \mu V \le (\operatorname{diam} V)^r \text{ for every } V \in \mathcal{V}\}.$$

P is non-empty (it contains the zero measure, for instance). Now if $G \subseteq X$ is open, $\mu \mapsto \mu G$ is lower semi-continuous for the narrow topology (437Jd), so P is a closed set in the narrow topology on the set of Radon measures on X, which may be identified with a subset of $C(X)^*$ with its weak* topology (437Kc). Moreover, since there is a finite subfamily of $\mathcal{V}$ covering X, $\gamma = \sup\{\mu X : \mu \in P\}$ is finite, and P is compact (437Pb/437Rf). Because $\mu \mapsto \mu X$ is continuous, $P_0 = \{\mu : \mu \in P, \mu X = \gamma\}$ is non-empty. Of course P and P_0 are both convex, and P_0, like P, is compact. By the Kreĭn-Mil'man theorem (4A4Gb), applied in $C(X)^*$, P has an extreme point ν say.

Note next that $\theta_{r,5\delta}(X)$ is certainly finite, again because X is compact. By 471Qb, $\gamma > 0$, and ν is non-trivial. For any $\epsilon > 0$, there is a finite cover of X by sets in $\mathcal{V}$ of diameter at most ϵ, which have measure at most ϵ^r (for ν); so ν is atomless. In particular, $\nu\{x\} = 0$ for every $x \in X$.

(c) For $\epsilon > 0$, set

$$G_\epsilon = \bigcup\{V : V \in \mathcal{V}, 0 < \operatorname{diam} V \le \epsilon \text{ and } \nu V \ge \tfrac{1}{2}(\operatorname{diam} V)^r\}.$$

Then G_ϵ is ν-conegligible. **P?** Otherwise, $\nu(X \setminus G_\epsilon) > 0$. Because $\mathcal{V}'_\epsilon = \{V : V \in \mathcal{V}, \operatorname{diam} V > \epsilon\}$ is finite, there is a Borel set $E \subseteq X \setminus G_\epsilon$ such that $\nu E > 0$ and, for every $V \in \mathcal{V}'_\epsilon$, either $E \subseteq V$ or $E \cap V = \emptyset$. Because ν is atomless, there is a measurable set $E_0 \subseteq E$ such that $\nu E_0 = \frac{1}{2}\nu E$ (215D); set $E_1 = E \setminus E_0$.

Define Radon measures ν_0, ν_1 on X by setting

$$\nu_i(F) = 2\nu(F \cap E_i) + \nu(F \setminus E)$$

whenever ν measures $F \setminus E_{1-i}$, for each i (use 416S if you feel the need to check that this defines a Radon measure on the definitions of this book). If $V \in \mathcal{V}$, then, by the choice of E,

either $E \subseteq V$ and $\nu_i V = \nu V \le (\operatorname{diam} V)^r$
or $E \cap V = \emptyset$ and $\nu_i V = \nu V \le (\operatorname{diam} V)^r$
or $0 < \operatorname{diam} V \le \epsilon$ and $\nu V < \frac{1}{2}(\operatorname{diam} V)^r$, in which case $\nu_i V \le 2\nu V \le (\operatorname{diam} V)^r$
or $\operatorname{diam} V = 0$ and $\nu_i V = \nu V = 0 = (\operatorname{diam} V)^r$.

So both ν_i belong to P and therefore to P_0, since $\nu_i X = \nu X = \gamma$. But $\nu = \frac{1}{2}(\nu_0 + \nu_1)$ and $\nu_0 \ne \nu_1$, so this is impossible, because ν is supposed to be an extreme point of P_0. **XQ**

(d) Accordingly, setting $H = \bigcap_{n\in\mathbb{N}} G_{2^{-n}}$, $\nu H = \nu X = \gamma$. Now examine $\mu_{Hr} H$.

(i) $\mu_{Hr} H \ge 8^{-r}\gamma$. **P** Let $\langle A_n\rangle_{n\in\mathbb{N}}$ be a sequence of sets covering H with $\operatorname{diam} A_n \le \frac{1}{8}\delta$ for every n. Set $K = \{n : \operatorname{diam} A_n > 0\}$, $H' = H \cap \bigcup_{n\in K} A_n$; then $H \setminus H'$ is countable, so $\nu H' = \nu H$. For each $n \in K$, let $V_n \in \mathcal{V}$ be such that $A_n \subseteq V_n$ and $\operatorname{diam} V_n \le 8 \operatorname{diam} A_n$ ((a) above). Then

$$\begin{aligned}\sum_{n=0}^{\infty}(\operatorname{diam} A_n)^r &= \sum_{n\in K}(\operatorname{diam} A_n)^r \ge 8^{-r}\sum_{n\in K}(\operatorname{diam} V_n)^r \\ &\ge 8^{-r}\sum_{n\in K}\nu V_n \ge 8^{-r}\nu H' = 8^{-r}\gamma.\end{aligned}$$

As $\langle A_n\rangle_{n\in\mathbb{N}}$ is arbitrary,

$$8^{-r}\gamma \le \theta_{r,\delta/8}(H) \le \mu_{Hr}^* H = \mu_{Hr} H. \text{ } \mathbf{Q}$$

(ii) $\mu_{Hr} H \le 2\gamma$. **P** Let $\eta > 0$. Set $\mathcal{F} = \{\overline{V} : V \in \mathcal{V},\ 0 < \operatorname{diam} V \le \eta,\ \nu V \ge \frac{1}{2}(\operatorname{diam} V)^r\}$. Then $\mathcal{F}$ is a family of closed subsets of X, and (by the definition of G_ϵ) every member of H belongs to members of $\mathcal{F}$ of arbitrarily small diameter. Also $\nu F \ge \frac{1}{2}(\operatorname{diam} F)^r$ for every $F \in \mathcal{F}$, so

$$\textstyle\sum_{n=0}^{\infty}(\operatorname{diam} F_n)^r \le 2\sum_{n=0}^{\infty}\nu F_n < \infty$$

for any disjoint sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{F}$. By 471O, there is a countable disjoint family $\mathcal{I} \subseteq \mathcal{F}$ such that $\mu_{Hr}(H \setminus \bigcup\mathcal{I}) = 0$. Accordingly

$$\textstyle\theta_{r\eta}(H) \le \sum_{F\in\mathcal{I}}(\operatorname{diam} F)^r + \theta_{r\eta}(H \setminus \bigcup\mathcal{I}) \le \sum_{F\in\mathcal{I}} 2\nu F \le 2\gamma.$$

As η is arbitrary, $\mu_{Hr} H = \mu_{Hr}^* H \le 2\gamma$. **Q**

(e) But this means that we have found a Borel set H with $0 < \mu_{Hr} H < \infty$, as required.

471S Theorem (Howroyd 95) Let (X,ρ) be an analytic metric space, and $r > 0$. Let μ_{Hr} be r-dimensional Hausdorff measure on X, and $\mathcal{B}$ the Borel σ-algebra of X. Then the Borel measure $\mu_{Hr}\restriction\mathcal{B}$ is semi-finite and tight (that is, inner regular with respect to the closed compact sets).

proof Suppose that $E \in \mathcal{B}$ and $\mu_{Hr} E > 0$. Since E is analytic (423Eb), 471I above tells us that there is a compact set $K \subseteq E$ such that $\mu_{Hr} K > 0$. Next, by 471R, there is a Borel set $H \subseteq K$ such that $0 < \mu_{Hr} H < \infty$. (Strictly speaking, $\mu_{Hr} H$ here should be calculated as the r-dimensional Hausdorff measure of H defined by the subspace metric $\rho\restriction K \times K$ on K. By 471E we do not need to distinguish between this and the r-dimensional measure calculated from ρ itself.) By 471I again (applied to the subspace metric on H), there is a compact set $L \subseteq H$ such that $\mu_{Hr} L > 0$.

Thus E includes a non-negligible compact set of finite measure. As E is arbitrary, this is enough to show both that $\mu_{Hr}\restriction\mathcal{B}$ is semi-finite and that it is tight.

471T Proposition Let (X,ρ) be a metric space, and $r > 0$.

(a) If X is analytic and $\mu_{Hr} X > 0$, then for every $s \in \,]0,r[$ there is a non-zero Radon measure μ on X such that $\iint \frac{1}{\rho(x,y)^s}\mu(dx)\mu(dy) < \infty$.

(b) If there is a non-zero topological measure μ on X such that $\iint \frac{1}{\rho(x,y)^r}\mu(dx)\mu(dy)$ is finite, then $\mu_{Hr} X = \infty$.

proof (a) By 471S, there is a compact set $K\subseteq X$ such that $\mu_{Hr}K>0$. Set $\delta=5\operatorname{diam}K$ and define $\theta_{r\delta}$ as in 471A. Then $\theta_{r\delta}K>0$, by 471K, and $\theta_{r\delta}K\le(\operatorname{diam}K)^r<\infty$. By 471Qb, there is a Radon measure ν on K such that $\nu K>0$ and $\nu G\le(\operatorname{diam}G)^r$ whenever $G\subseteq K$ is relatively open; consequently $\nu^*A\le(\operatorname{diam}A)^r$ for every $A\subseteq K$. Now, for any $y\in X$,

$$\begin{aligned}\int_K\tfrac{1}{\rho(x,y)^s}\nu(dx)&=\int_0^\infty\nu\{x:x\in K,\,\tfrac{1}{\rho(x,y)^s}\ge t\}dt=\int_0^\infty\nu\{x:x\in K,\,\rho(x,y)\le\tfrac{1}{t^{1/s}}\}dt\\&=\int_0^\infty\nu(K\cap B(y,\tfrac{1}{t^{1/s}}))dt\le\int_0^\infty(\operatorname{diam}(K\cap B(y,\tfrac{1}{t^{1/s}})))^r dt\\&\le\int_0^\infty(\min(\operatorname{diam}K,\tfrac{2}{t^{1/s}}))^r dt\le 2^r\int_0^\infty\min((\operatorname{diam}K)^r,\tfrac{1}{t^{r/s}})dt<\infty\end{aligned}$$

because $r>s$. It follows at once that $\int_K\int_K\frac{1}{\rho(x,y)^s}\nu(dx)\nu(dy)$ is finite. Taking μ to be the extension of ν to a Radon measure on X for which $X\setminus K$ is negligible, we have an appropriate μ.

(b)(i) We can suppose that X is separable (471Df). Since the integrand is strictly positive, μ must be σ-finite, so that there is no difficulty with the repeated integral. Replacing μ by $\mu\llcorner F$ for some set F of non-zero finite measure, we can suppose that μ is totally finite; and replacing μ by a scalar multiple of itself, we can suppose that it is a probability measure.

(ii) Let $\epsilon>0$. Let H be the conegligible set $\{y:\int\frac{1}{\rho(x,y)^r}\mu(dx)<\infty\}$. For any $y\in X$, $\mu\{y\}=0$, so

$$\lim_{\delta\downarrow0}\int_{B(y,\delta)}\frac{1}{\rho(x,y)^r}\mu(dx)=0$$

for every $y\in H$. For each $\delta>0$,

$$(x,y)\mapsto\frac{\chi B(y,\delta)(x)}{\rho(x,y)^r}:X\times X\to[0,\infty]$$

is Borel measurable, so

$$y\mapsto\int_{B(y,\delta)}\frac{1}{\rho(x,y)^r}\mu(dx):X\to[0,\infty]$$

is Borel measurable (252P, applied to the restriction of μ to the Borel σ-algebra of X). There is therefore a $\delta>0$ such that $E=\{y:y\in H,\,\int_{B(y,\delta)}\frac{1}{\rho(x,y)^r}\mu(dy)\le\epsilon\}$ has measure $\mu E\ge\frac12$. Note that if $C\subseteq X$ has diameter less than or equal to δ and meets E then $\mu\overline{C}\le\epsilon(\operatorname{diam}C)^r$. **P** Set $\gamma=\operatorname{diam}C$ and take $y\in C\cap E$. If $C=\{y\}$ then $\mu\overline{C}=0$. Otherwise,

$$\mu\overline{C}\le\mu B(y,\gamma)\le\gamma^r\int_{B(y,\delta)}\frac{1}{\rho(x,y)^r}\mu(dx)\le\gamma^r\epsilon.\ \mathbf{Q}$$

Now suppose that $E\subseteq\bigcup_{i\in I}C_i$ where $\operatorname{diam}C_i\le\delta$ for every i, and each C_i is either empty or meets E. Then

$$\frac12\le\mu E\le\sum_{i=0}^\infty\mu\overline{C}_i\le\sum_{i=0}^\infty\epsilon(\operatorname{diam}C_i)^r.$$

As $\langle C_i\rangle_{i\in\mathbb{N}}$ is arbitrary, $\epsilon\mu_{Hr}E\ge\frac12$ and $\mu_{Hr}X\ge\frac{1}{2\epsilon}$. As ϵ is arbitrary, $\mu_{Hr}X=\infty$.

Remark See 479Cb below.

471X Basic exercises (a) Define a metric ρ on $X=\{0,1\}^{\mathbb{N}}$ by setting $\rho(x,y)=2^{-n}$ if $x\restriction n=y\restriction n$ and $x(n)\ne y(n)$. Show that the usual measure μ on X is one-dimensional Hausdorff measure. (*Hint*: $\operatorname{diam}F\ge\mu F$ for every closed set $F\subseteq X$.)

(b) Suppose that $g:\mathbb{R}\to\mathbb{R}$ is continuous and non-decreasing, and that ν is the corresponding Lebesgue-Stieltjes measure (114Xa). Define $\rho(x,y)=|x-y|+\sqrt{|g(x)-g(y)|}$ for x, $y\in\mathbb{R}$. Show that ρ is a metric on $\mathbb{R}$ defining the usual topology. Show that ν is 2-dimensional Hausdorff measure for the metric ρ.

(c) Let $r \ge 1$ be an integer, and give $\mathbb{R}^r$ the metric $((\xi_1,\dots,\xi_r),(\eta_1,\dots,\eta_r)) \mapsto \max_{i\le r}|\xi_i-\eta_i|$. Show that Lebesgue measure on $\mathbb{R}^r$ is Hausdorff r-dimensional measure for this metric.

(d) Let (X,ρ) be a metric space and $r>0$; let μ_{Hr}, $\theta_{r\infty}$ be r-dimensional Hausdorff measure and capacity on X. (i) Show that, for $A \subseteq X$, $\mu_{Hr}A=0$ iff $\theta_{r\infty}A=0$. (ii) Suppose that $E \subseteq X$ and $\delta>0$ are such that $\delta\mu_{Hr}E<\theta_{r\infty}E$. Show that there is a closed set $F \subseteq E$ such that $\mu_{Hr}F>0$ and $\delta\mu_{Hr}(F\cap G) \le (\operatorname{diam} G)^r$ whenever μ_{Hr} measures G. (*Hint*: show that $\{G^\bullet : \theta_{r\infty}G<\delta\mu_{Hr}G\}$ cannot be order-dense in the measure algebra of μ_{Hr}. This is a version of 'Frostman's Lemma'.) (iii) Let $\mathcal{C}$ be the family of closed subsets of X, with its Vietoris topology. Show that $\theta_{r\infty}\restriction\mathcal{C}$ is upper semi-continuous.

(e) Show that all the outer measures $\theta_{r\delta}$ described in 471A are outer regular Choquet capacities.

(f) Let (X,ρ) be an analytic metric space, (Y,σ) a metric space, and $f:X\to Y$ a Lipschitz function. Show that if $r>0$ and $A\subseteq X$ is measured by Hausdorff r-dimensional measure on X, with finite measure, then $f[A]$ is measured by Hausdorff r-dimensional measure on Y.

(g) Let (X,ρ) be a metric space and $r>0$. Show that a set $A\subseteq X$ is negligible for Hausdorff r-dimensional measure on X iff there is a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of subsets of X such that $\sum_{n=0}^{\infty}(\operatorname{diam} A_n)^r$ is finite and $A\subseteq\bigcap_{n\in\mathbb{N}}\bigcup_{m\ge n}A_m$.

(h) Let (X,ρ) be a metric space. (i) Show that there is a unique $\dim_H(X)\in[0,\infty]$ such that the r-dimensional Hausdorff measure of X is infinite if $0<r<\dim_H(X)$, zero if $r>\dim_H(X)$. ($\dim_H(X)$ is the **Hausdorff dimension** of X.) (ii) Show that if $\langle A_n\rangle_{n\in\mathbb{N}}$ is any sequence of subsets of X, then $\dim_H(\bigcup_{n\in\mathbb{N}}A_n)=\sup_{n\in\mathbb{N}}\dim_H(A_n)$.

(i) Let (X,ρ) be a metric space, and μ any topological measure on X. Suppose that $E\subseteq X$ and that μE is defined and finite. (i) Show that $(x,\delta)\mapsto\mu(E\cap B(x,\delta)):X\times[0,\infty[\,\to\mathbb{R}$ is upper semi-continuous. (ii) Show that $x\mapsto\limsup_{\delta\downarrow0}\frac{1}{\delta^r}\mu(E\cap B(x,\delta)):X\to[0,\infty]$ is Borel measurable, for every $r\ge0$. (iii) Show that if X is separable, then $\mu B(x,\delta)>0$ for every $\delta>0$, for μ-almost every $x\in X$.

(j) Give $\mathbb{R}$ its usual metric. Let $C\subseteq\mathbb{R}$ be the Cantor set, and $r=\ln 2/\ln 3$. Show that

$$\liminf_{\delta\downarrow0}\frac{\mu_{Hr}(C\cap B(x,\delta))}{(\operatorname{diam} B(x,\delta))^r}\le 2^{-r}$$

for every $x\in\mathbb{R}$.

(k) Let (X,ρ) be a metric space and $r>0$. Let μ_{Hr} be r-dimensional Hausdorff measure on X and $\tilde\mu_{Hr}$ its c.l.d. version (213D-213E). Show that $\tilde\mu_{Hr}$ is inner regular with respect to the closed sets, and that $\tilde\mu_{Hr}A=\mu_{Hr}A$ for every analytic set $A\subseteq X$.

471Y Further exercises (a) The next few exercises (down to 471Yd) will be based on the following. Let (X,ρ) be a metric space and $\psi:\mathcal{P}X\to[0,\infty]$ a function such that $\psi\emptyset=0$ and $\psi A\le\psi A'$ whenever $A\subseteq A'\subseteq X$. Set

$$\theta_{\psi\delta}A=\inf\{\sum_{n=0}^{\infty}\psi D_n:\langle D_n\rangle_{n\in\mathbb{N}}\text{ is a sequence of subsets of }X\text{ covering }A,$$
$$\operatorname{diam} D_n\le\delta\text{ for every }n\in\mathbb{N}\}$$

for $\delta>0$, and $\theta_\psi A=\sup_{\delta>0}\theta_{\psi\delta}A$ for $A\subseteq X$. Show that θ_ψ is a metric outer measure. Let μ_ψ be the measure defined from θ_ψ by Carathéodory's method.

(b) Suppose that $\psi A=\inf\{\psi E:E\text{ is a Borel set including }A\}$ for every $A\subseteq X$. Show that $\theta_\psi=\mu_\psi^*$ and that $\mu_\psi E=\sup\{\mu_\psi F:F\subseteq E\text{ is closed}\}$ whenever $\mu_\psi E<\infty$.

(c) Suppose that X is separable and that $\beta>0$ is such that $\psi A^\sim\le\beta\psi A$ for every $A\subseteq X$, where $A^\sim$ is defined in 471M. (i) Suppose that $A\subseteq X$ and $\mathcal{F}$ is a family of closed subsets of X such that $\sum_{n=0}^{\infty}\psi F_n$

is finite for every disjoint sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{F}$ and for every $x \in A$, $\delta > 0$ there is an $F \in \mathcal{F}$ such that $x \in F$ and $0 < \operatorname{diam} F \le \delta$. Show that there is a disjoint family $\mathcal{I} \subseteq \mathcal{F}$ such that $\mu_\psi(A \setminus \bigcup\mathcal{I}) = 0$. (ii) Suppose that $\delta > 0$ and that $\theta_{\psi\delta}(X) < \infty$. Show that there is a non-negative additive functional ν on $\mathcal{P}X$ such that $\nu X = \frac{1}{\beta}\theta_{\psi\delta}(X)$ and $\nu A \le \psi A$ whenever $A \subseteq X$ and $\operatorname{diam} A \le \frac{1}{5}\delta$. (iii) Now suppose that for every $x \in X$ and $\epsilon > 0$ there is a $\delta > 0$ such that $\psi B(x,\delta) \le \epsilon$. Show that if X is compact and $\mu_\psi X > 0$ there is a compact set $K \subseteq X$ such that $0 < \mu_\psi K < \infty$.

(d) State and prove a version of 471P appropriate to this context.

(e) Give an example of a set $A \subseteq \mathbb{R}^2$ which is measured by Hausdorff 1-dimensional measure on $\mathbb{R}^2$ but is such that its projection onto the first coordinate is not measured by Hausdorff 1-dimensional measure on $\mathbb{R}$.

(f) Let ρ be a metric on $\mathbb{R}$ inducing the usual topology. Show that the corresponding Hausdorff dimension of $\mathbb{R}$ is at least 1.

(g) Show that the space (X,ρ) of 471Xa can be isometrically embedded as a subset of a metric space (Y,σ) in such a way that (i) $\operatorname{diam} B(y,\delta) = 2\delta$ for every $y \in Y$ and $\delta \ge 0$ (ii) $Y \setminus X$ is countable (iii) if μ_{H1} is one-dimensional Hausdorff measure on Y, then $\mu_{H1}B(y,\delta) \in \{0,\delta\}$ for every $y \in Y$ and $\delta \ge 0$, so that

$$\lim_{\delta\downarrow 0}\frac{\mu_{H1}B(y,\delta)}{\operatorname{diam} B(x,\delta)} \in \{0,\tfrac{1}{2}\}$$

for every $y \in Y$.

(h) Let $\langle k_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathbb{N}\setminus\{0,1,2,3\}$ such that $\sum_{n=0}^\infty \frac{1}{k_n} < \infty$. Set $X = \prod_{n\in\mathbb{N}} k_n$. Set $m_0 = 1$, $m_{n+1} = k_0k_1\ldots k_n$ for $n \in \mathbb{N}$. Define a metric ρ on X by saying that

$$\begin{aligned}\rho(x,y) &= 1/2m_n \text{ if } n = \min\{i : x(i) \ne y(i)\} \text{ and } \min(x(n),y(n)) = 0,\\ &= 1/m_n \text{ if } n = \min\{i : x(i) \ne y(i)\} \text{ and } \min(x(n),y(n)) > 0.\end{aligned}$$

Let ν be the product measure on X obtained by giving each factor k_n the uniform probability measure in which each singleton set has measure $1/k_n$. (i) Show that if $A \subseteq X$ then $\nu^*A \le \operatorname{diam} A$. (ii) Show that ν is one-dimensional Hausdorff measure on X. (iii) Set $E = \bigcup_{n\in\mathbb{N}}\{x : x \in X,\ x(n) = 0\}$. Show that $\nu E < 1$. (iv) Show that

$$\limsup_{\delta\downarrow 0}\frac{\nu(E\cap B(x,\delta))}{\nu B(x,\delta)} \ge \frac{1}{2}$$

for every $x \in X$. (v) Show that there is a family $\mathcal{F}$ of closed balls in X such that every point of X is the centre of arbitrarily small members of $\mathcal{F}$, but $\nu(\bigcup\mathcal{I}) < 1$ for any disjoint subfamily $\mathcal{I}$ of $\mathcal{F}$.

(i) Let (X,ρ) be a metric space and $0 < r < s$. Suppose that there is an analytic set $A \subseteq X$ such that $\mu_{Hs}A > 0$. Show that there is a Borel surjection $f : X \to \mathbb{R}$ such that $\mu_{Hr}f^{-1}[\{\alpha\}] = \infty$ for every $\alpha \in \mathbb{R}$.

(j) Let ρ be the metric on $\{0,1\}^{\mathbb{N}}$ defined in 471Xa. (i) Show that for any integer $k \ge 1$ there are a $\gamma_k > 0$ and a bijection $f : [0,1]^k \to \{0,1\}^{\mathbb{N}}$ such that whenever $0 < r \le 1$, $\mu_{H,rk}$ is Hausdorff rk-dimensional measure on $[0,1]^k$ (for its usual metric) and $\tilde\mu_{Hr}$ is Hausdorff r-dimensional measure on $\{0,1\}^{\mathbb{N}}$, then $\mu^*_{H,rk}A \le \gamma_k\tilde\mu^*_{Hr}f[A] \le \gamma_k^2\mu^*_{H,rk}A$ for every $A \subseteq [0,1]^k$. (ii) Show that in this case $\mu_{H,rk}$ and the image measure $\tilde\mu_{Hr}f^{-1}$ have the same measurable sets, the same negligible sets and the same sets of finite measure.

(k) Let (X,ρ) be a metric space, and $r > 0$. Give $X \times \mathbb{R}$ the metric σ where $\sigma((x,\alpha),(y,\beta)) = \max(\rho(x,y),|\alpha-\beta|)$. Write μ_L, μ_r and μ_{r+1} for Lebesgue measure on $\mathbb{R}$, r-dimensional Hausdorff measure on (X,ρ) and $(r+1)$-dimensional Hausdorff measure on $(X\times\mathbb{R},\sigma)$ respectively. Let λ be the c.l.d. product of μ_r and μ_L. (i) Show that if $W \subseteq X\times\mathbb{R}$ then $\overline{\int}\mu_r^*W^{-1}[\{\alpha\}]d\alpha \le \mu_{r+1}^*W$. (ii) Show that if $I \subseteq \mathbb{R}$ is a bounded interval, $A \subseteq X$ and μ_r^*A is finite, then $\mu_{r+1}^*(A\times I) = \mu_r^*A\cdot\mu_L I$. (iii) Give an example in which there is a compact set $K \subseteq X\times\mathbb{R}$ such that $\mu_{r+1}K = 1$ and $\lambda K = 0$. (iv) Show that if μ_r is σ-finite then $\mu_{r+1} = \lambda$. (*Hint*: Federer 69, 2.10.45.)

471Z Problems (a) Let $\mu_{H1}^{(2)}$, $\mu_{H,1/2}^{(1)}$ be one-dimensional Hausdorff measure on $\mathbb{R}^2$ and $\frac{1}{2}$-dimensional Hausdorff measure on $\mathbb{R}$ respectively, for their usual metrics. Are the measure spaces $(\mathbb{R}^2, \mu_{H1}^{(2)})$ and $(\mathbb{R}, \mu_{H,1/2}^{(1)})$ isomorphic? (See 471Yj.)

(b) Let ρ be a metric on $\mathbb{R}^2$ inducing the usual topology, and $\mu_{H2}^{(\rho)}$ the corresponding 2-dimensional Hausdorff measure. Is it necessarily the case that $\mu_{H2}^{(\rho)}(\mathbb{R}^2) > 0$? (See 471Yf.)

471 Notes and comments In the exposition above, I have worked throughout with simple r-dimensional measures for $r > 0$. As noted in 264Db, there are formulae in which it is helpful to interpret μ_{H0} as counting measure. More interestingly, when we come to use Hausdorff measures to give us information about the geometric structure of an object (e.g., in the observation that the Cantor set has $\ln 2/\ln 3$-dimensional Hausdorff measure 1, in 264J), it is sometimes useful to refine the technique by using other functionals than $A \mapsto (\operatorname{diam} A)^r$ in the basic formulae of 264A or 471A. The most natural generalization is to functionals of the form $\psi A = h(\operatorname{diam} A)$ where $h : [0, \infty] \to [0, \infty]$ is a non-decreasing function (264Yo). But it is easy to see that many of the arguments are valid in much greater generality, as in 471Ya-471Yc. For more in these directions see Rogers 70 and Federer 69.

In the context of this book, the most conspicuous peculiarity of Hausdorff measures is that they are often very far from being semi-finite. (This is trivial for non-separable spaces, by 471Df. That Hausdorff one-dimensional measure on a subset of $\mathbb{R}^2$ can be purely infinite is not I think obvious; I gave an example in 439H.) The response I ordinarily recommend in such cases is to take the c.l.d. version. But then of course we need to know just what effect this will have. In geometric applications, one usually begins by checking that the sets one is interested in have σ-finite measure, and that therefore no problems arise; but it is a striking fact that Hausdorff measures behave relatively well on analytic sets, even when not σ-finite, provided we ask exactly the right questions (471I, 471S, 471Xk).

The geometric applications of Hausdorff measures, naturally, tend to rely heavily on density theorems; it is therefore useful to know that we have effective versions of Vitali's theorem available in this context (471N-471O), leading to a general density theorem (471P) similar to that in 261D; see also 472D below. I note that 471P is useful only after we have been able to concentrate our attention on a set of finite measure. And traps remain. For instance, the formulae of 261C-261D cannot be transferred to the present context without re-evaluation (471Yh).

This section, and indeed the chapter as a whole, is devoted to calculations involving metrics, which is why the phrase 'metric space' is constantly repeated while the word 'metrizable' does not appear. But of course topological ideas are omnipresent. See 471Yf for an interesting elementary fact with an obvious implied challenge (471Zb). There is a less elementary fact in 471Yj, which shows that much of the measure space structure, if not the geometry, of Hausdorff r-dimensional measure on $\mathbb{R}^k$ is determined by the ratio r/k. (See 471Za.)

Version of 22.3.11

472 Besicovitch's Density Theorem

The first step in the program of the next few sections is to set out some very remarkable properties of Euclidean space. We find that in $\mathbb{R}^r$, for geometric reasons (472A), we have versions of Vitali's theorem (472B-472C) and Lebesgue's Density Theorem (472D) for arbitrary Radon measures. I add a version of the Hardy-Littlewood Maximal Theorem (472F).

Throughout the section, $r \ge 1$ will be a fixed integer. As usual, I write $B(x, \delta)$ for the closed ball with centre x and radius δ. $\| \ \|$ will represent the Euclidean norm, and $x \textbf{.} y$ the scalar product of x and y, so that $x \textbf{.} y = \sum_{i=1}^r \xi_i \eta_i$ if $x = (\xi_1, \dots, \xi_r)$ and $y = (\eta_1, \dots, \eta_r)$.

472A Besicovitch's Covering Lemma Suppose that $\epsilon > 0$ is such that $(5^r + 1)(1 - \epsilon - \epsilon^2)^r > (5 + \epsilon)^r$. Let $x_0, \dots, x_n \in \mathbb{R}^r$, $\delta_0, \dots, \delta_n > 0$ be such that

$$\|x_i - x_j\| > \delta_i, \quad \delta_j \le (1 + \epsilon)\delta_i$$

whenever $i < j \le n$. Then

$$\#(\{i : i \le n,\ \|x_i - x_n\| \le \delta_i + \delta_n\}) \le 5^r.$$

proof Set $I = \{i : i \le n,\ \|x_i - x_n\| \le \delta_n + \delta_i\}$.

(a) It will simplify the formulae of the main argument if we suppose for the time being that $\delta_n = 1$; in this case $1 \le (1+\epsilon)\delta_i$, so that $\delta_i \ge \frac{1}{1+\epsilon}$ for every $i \le n$, while we still have $\delta_i < \|x_i - x_n\|$ for every $i < n$, and $\|x_i - x_n\| \le 1 + \delta_i$ for every $i \in I$.

For $i \in I$, define x'_i by saying that

– if $\|x_i - x_n\| \le 2 + \epsilon$, $x'_i = x_i$;

– if $\|x_i - x_n\| > 2 + \epsilon$, x'_i is to be that point of the closed line segment from x_n to x_i which is at distance $2 + \epsilon$ from x_n.

(b) The point is that $\|x'_i - x'_j\| > 1 - \epsilon - \epsilon^2$ whenever i, j are distinct members of I. **P** We may suppose that $i < j$.

case 1 Suppose that $\|x_i - x_n\| \le 2 + \epsilon$ and $\|x_j - x_n\| \le 2 + \epsilon$. In this case

$$\|x'_i - x'_j\| = \|x_i - x_j\| \ge \delta_i \ge \tfrac{1}{1+\epsilon} \ge 1 - \epsilon.$$

case 2 Suppose that $\|x_i - x_n\| \ge 2 + \epsilon \ge \|x_j - x_n\|$. In this case

$$\begin{aligned}\|x'_i - x'_j\| &= \|x'_i - x_j\| \ge \|x_i - x_j\| - \|x_i - x'_i\| \\ &\ge \delta_i - \|x_i - x_n\| + 2 + \epsilon \ge \delta_i - \delta_i - 1 + 2 + \epsilon = 1 + \epsilon.\end{aligned}$$

case 3 Suppose that $\|x_i - x_n\| \le 2 + \epsilon \le \|x_j - x_n\|$. Then

$$\begin{aligned}\|x'_i - x'_j\| &= \|x_i - x'_j\| \ge \|x_i - x_j\| - \|x_j - x'_j\| > \delta_i - \|x_j - x_n\| + 2 + \epsilon \\ &\ge \delta_i - \delta_j - 1 + 2 + \epsilon \ge \delta_i - \delta_i(1+\epsilon) + 1 + \epsilon \ge 1 + \epsilon - \epsilon(2+\epsilon)\end{aligned}$$

(because $\delta_i < \|x_i - x_n\| \le 2 + \epsilon$)

$$= 1 - \epsilon - \epsilon^2.$$

case 4 Suppose that $2 + \epsilon \le \|x_j - x_n\| \le \|x_i - x_n\|$. Let y be the point on the line segment between x_i and x_n which is the same distance from x_n as x_j. In this case

$$\|y - x_j\| \ge \|x_i - x_j\| - \|x_i - y\| \ge \delta_i - \|x_i - x_n\| + \|x_j - x_n\| \ge \|x_j - x_n\| - 1.$$

Because the triangles (x_n, y, x_j) and (x_n, x'_i, x'_j) are similar,

$$\|x'_i - x'_j\| = \tfrac{2+\epsilon}{\|x_j - x_n\|}\|y - x_j\| \ge (2+\epsilon)\tfrac{\|x_j - x_n\| - 1}{\|x_j - x_n\|} \ge 1 + \epsilon$$

because $\|x_j - x_n\| \ge 2 + \epsilon$.

case 5 Suppose that $2 + \epsilon \le \|x_i - x_n\| \le \|x_j - x_n\|$. This time, let y be the point on the line segment from x_n to x_j which is the same distance from x_n as x_i is. We now have

$$\begin{aligned}\|y - x_i\| &\ge \|x_i - x_j\| - \|x_j - y\| > \delta_i - \|x_j - x_n\| + \|x_i - x_n\| \\ &\ge \delta_j - \epsilon\delta_i - (\delta_j + 1) + \|x_i - x_n\| \\ &= \|x_i - x_n\| - 1 - \epsilon\delta_i \ge \|x_i - x_n\|(1-\epsilon) - 1,\end{aligned}$$

so that

$$\begin{aligned}\|x'_i - x'_j\| &= \tfrac{2+\epsilon}{\|x_i - x_n\|}\|y - x_i\| > (2+\epsilon)\tfrac{\|x_i - x_n\|(1-\epsilon) - 1}{\|x_i - x_n\|} \\ &\ge (2+\epsilon)\tfrac{(2+\epsilon)(1-\epsilon) - 1}{2+\epsilon} = 1 - \epsilon - \epsilon^2.\end{aligned}$$

So we have the required inequality in all cases. **Q**

(c) Now consider the balls $B(x'_i, \frac{1-\epsilon-\epsilon^2}{2})$ for $i \in I$. These are disjoint, all have Lebesgue measure $2^{-r}\beta_r(1-\epsilon-\epsilon^2)^r$ where β_r is the measure of the unit ball $B(\mathbf{0}, 1)$, and are all included in the ball $B(x_n, 2+\epsilon+\frac{1-\epsilon}{2})$, which has measure $2^{-r}\beta_r(5+\epsilon)^r$. So we must have

$$2^{-r}\beta_r(1-\epsilon-\epsilon^2)^r\#(I) \le 2^{-r}\beta_r(5+\epsilon)^r.$$

But ϵ was declared to be so small that this implies that $\#(I) \le 5^r$, as claimed.

(d) This proves the lemma in the case $\delta_n = 1$. For the general case, replace each x_i by $\delta_n^{-1}x_i$ and each δ_i by δ_i/δ_n; the change of scale does not affect the hypotheses or the set I.

472B Theorem Let $A \subseteq \mathbb{R}^r$ be a bounded set, and $\mathcal{I}$ a family of non-trivial closed balls in $\mathbb{R}^r$ such that every point of A is the centre of a member of $\mathcal{I}$. Then there is a family $\langle \mathcal{I}_k\rangle_{k<5^r}$ of countable subsets of $\mathcal{I}$ such that each $\mathcal{I}_k$ is disjoint and $\bigcup_{k<5^r}\mathcal{I}_k$ covers A.

proof (a) For each $x \in A$ let $\delta_x > 0$ be such that $B(x, \delta_x) \in \mathcal{I}$. If either A is empty or $\sup_{x\in A}\delta_x = \infty$, the result is trivial. (In the latter case, take $x \in A$ such that $\delta_x \ge \operatorname{diam} A$ and set $\mathcal{I}_0 = \{B(x, \delta_x)\}$, $\mathcal{I}_k = \emptyset$ for $k > 0$.) So let us suppose henceforth that $\{\delta_x : x \in A\}$ is bounded in $\mathbb{R}$. In this case, $C = \bigcup_{x\in A} B(x, \delta_x)$ is bounded in $\mathbb{R}^r$.

Fix $\epsilon > 0$ such that $(5^r + 1)(1-\epsilon-\epsilon^2)^r > (5+\epsilon)^r$.

(b) Choose inductively a sequence $\langle B_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{I} \cup \{\emptyset\}$ as follows. Given $\langle B_i\rangle_{i<n}$, then if $A \subseteq \bigcup_{i<n} B_i$ set $B_n = \emptyset$. Otherwise, set $\alpha_n = \sup\{\delta_x : x \in A\setminus\bigcup_{i<n} B_i\}$, choose $x_n \in A\setminus\bigcup_{i<n} B_i$ such that $(1+\epsilon)\delta_{x_n} \ge \alpha_n$, set $B_n = B(x_n, \delta_{x_n})$ and continue.

Now whenever $n \in \mathbb{N}$, $I_n = \{i : i < n,\ B_i \cap B_n \ne \emptyset\}$ has fewer than 5^r members. **P** We may suppose that $B_n \ne \emptyset$, in which case $B_i = B(x_i, \delta_{x_i})$ for every $i \le n$, and the x_i, δ_{x_i} are such that, whenever $i < j \le n$,

$$x_j \notin B_i, \text{ i.e., } \|x_i - x_j\| > \delta_{x_i},$$

$$\delta_{x_j} \le \alpha_i \le (1+\epsilon)\delta_{x_i}.$$

But now 472A gives the result at once. **Q**

(c) We may therefore define a function $f : \mathbb{N} \to \{0, 1, \dots, 5^r - 1\}$ by setting

$$f(n) = \min\{k : 0 \le k < 5^r,\ f(i) \ne k \text{ for every } i \in I_n\}$$

for every $n \in \mathbb{N}$. Set $\mathcal{I}_k = \{B_i : i \in \mathbb{N},\ f(i) = k,\ B_i \ne \emptyset\}$ for each $k < 5^r$. By the choice of f, $i \notin I_j$, so that $B_i \cap B_j = \emptyset$, whenever $i < j$ and $f(i) = f(j)$; thus every $\mathcal{I}_k$ is disjoint. Since $B_i \subseteq C$ for every i, $\sum\{\mu B_i : f(i) = k\} \le \mu^*C$ for every $k < 5^r$, and $\sum_{i=0}^{\infty}\mu B_i \le 5^r\mu^*C$ is finite.

(d) ? Suppose, if possible, that

$$A \not\subseteq \bigcup_{k<5^r}\bigcup\mathcal{I}_k = \bigcup_{n\in\mathbb{N}} B_n.$$

Take $x \in A\setminus\bigcup_{n\in\mathbb{N}} B_n$. Then, first, $A \not\subseteq \bigcup_{i<n} B_i$ for every n, so that α_n is defined; next, $\alpha_n \ge \delta_x$, so that $(1+\epsilon)\delta_{x_n} \ge \delta_x$ for every n. But this means that $\mu B_n \ge \beta_r\bigl(\frac{\delta_x}{1+\epsilon}\bigr)^r$ for every n, and $\sum_{n=0}^{\infty}\mu B_n = \infty$; which is impossible. **X**

(e) Thus $A \subseteq \bigcup_{k<5^r}\bigcup\mathcal{I}_k$, as required.

472C Theorem Let λ be a Radon measure on $\mathbb{R}^r$, A a subset of $\mathbb{R}^r$ and $\mathcal{I}$ a family of non-trivial closed balls in $\mathbb{R}^r$ such that every point of A is the centre of arbitrarily small members of $\mathcal{I}$. Then

(a) there is a countable disjoint $\mathcal{I}_0 \subseteq \mathcal{I}$ such that $\lambda(A\setminus\bigcup\mathcal{I}_0) = 0$;

(b) for every $\epsilon > 0$ there is a countable $\mathcal{I}_1 \subseteq \mathcal{I}$ such that $A \subseteq \bigcup\mathcal{I}_1$ and $\sum_{B\in\mathcal{I}_1}\lambda B \le \lambda^*A + \epsilon$.

proof (a)(i) The first step is to show that if $A' \subseteq A$ is bounded then there is a finite disjoint set $\mathcal{J} \subseteq \mathcal{I}$ such that $\lambda^*(A' \cap \bigcup\mathcal{J}) \ge 6^{-r}\lambda^*A'$. **P** If $\lambda^*A' = 0$ take $\mathcal{J} = \emptyset$. Otherwise, by 472B, there is a family $\langle\mathcal{J}_k\rangle_{k<5^r}$ of disjoint countable subsets of $\mathcal{I}$ such that $\bigcup_{k<5^r}\mathcal{J}_k$ covers A'. Accordingly

$$\lambda^* A' \le \sum_{k=0}^{5^r-1} \lambda^*(A' \cap \bigcup \mathcal{J}_k)$$

and there is some $k < 5^r$ such that $\lambda^*(A' \cap \bigcup \mathcal{J}_k) \ge 5^{-r}\lambda^* A'$. Let $\langle B_i \rangle_{i\in\mathbb{N}}$ be a sequence running over $\mathcal{J}_k$; then

$$\lim_{n\to\infty} \lambda^*(A' \cap \bigcup_{i\le n} B_i) = \lambda^*(A' \cap \bigcup \mathcal{J}_k) \ge 5^{-r}\lambda^* A',$$

so there is some $n \in \mathbb{N}$ such that $\lambda^*(A' \cap \bigcup_{i\le n} B_i) \ge 6^{-r}\lambda^* A'$, and we can take $\mathcal{J} = \{B_i : i \le n\}$. **Q**

(ii) Now choose $\langle \mathcal{K}_n \rangle_{n\in\mathbb{N}}$ inductively, as follows. Start by fixing on a sequence $\langle m_n \rangle_{n\in\mathbb{N}}$ running over $\mathbb{N}$ with cofinal repetitions. Take $\mathcal{K}_0 = \emptyset$. Given that $\mathcal{K}_n$ is a finite disjoint subset of $\mathcal{I}$, set $\mathcal{I}' = \{B : B \in \mathcal{I},\, B \cap \bigcup \mathcal{K}_n = \emptyset\}$, $A_n = A \cap B(\mathbf{0}, m_n) \setminus \bigcup \mathcal{K}_n$. Because every point of A is the centre of arbitrarily small members of $\mathcal{I}$, and $\bigcup \mathcal{K}_n$ is closed, every member of A_n is the centre of (arbitrarily small) members of $\mathcal{I}'$, and (i) tells us that there is a finite disjoint set $\mathcal{J}_n \subseteq \mathcal{I}'$ such that $\lambda^*(A_n \cap \bigcup \mathcal{J}_n) \ge 6^{-r}\lambda^* A_n$. Set $\mathcal{K}_{n+1} = \mathcal{K} \cup \mathcal{J}_n$, and continue. At the end of the induction, set $\mathcal{I}_0 = \bigcup_{n\in\mathbb{N}} \mathcal{K}_n$; because $\langle \mathcal{K}_n \rangle_{n\in\mathbb{N}}$ is non-decreasing and every $\mathcal{K}_n$ is disjoint, $\mathcal{I}_0$ is disjoint, and of course $\mathcal{I}_0 \subseteq \mathcal{I}$.

The effect of this construction is to ensure that

$$\lambda^*(A \cap B(\mathbf{0}, m_n) \setminus \bigcup \mathcal{K}_{n+1}) = \lambda^*(A_n \setminus \bigcup \mathcal{J}_n) = \lambda^* A_n - \lambda^*(A_n \cap \bigcup \mathcal{J}_n)$$

(because $\bigcup \mathcal{J}_n$ is a closed set, therefore measured by λ)

$$\le (1 - 6^{-r})\lambda^* A_n$$
$$= (1 - 6^{-r})\lambda^*(A \cap B(\mathbf{0}, m_n) \setminus \bigcup \mathcal{K}_n)$$

for every n. So, for any $m \in \mathbb{N}$,

$$\lambda^*(A \cap B(\mathbf{0}, m) \setminus \bigcup \mathcal{K}_n) \le \lambda^*(A \cap B(\mathbf{0}, m))(1 - 6^{-r})^{\#(\{j : j<n, m_j = m\})} \to 0$$

as $n \to \infty$, and $\lambda^*(A \cap B(\mathbf{0}, m) \setminus \bigcup \mathcal{I}_0) = 0$. As m is arbitrary, $\lambda^*(A \setminus \bigcup \mathcal{I}_0) = 0$, as required.

(b)(i) Let $E \supseteq A$ be such that $\lambda E = \lambda^* A$, and $H \supseteq E$ an open set such that $\lambda H \le \lambda E + \frac{1}{2}\epsilon$ (256Bb). Set $\mathcal{I}' = \{B : B \in \mathcal{I},\, B \subseteq H\}$. Then every point of A is the centre of arbitrarily small members of $\mathcal{I}'$, so by (a) there is a disjoint family $\mathcal{I}_0 \subseteq \mathcal{I}'$ such that $\lambda(A \setminus \bigcup \mathcal{I}_0) = 0$. Of course

$$\sum_{B\in\mathcal{I}_0} \lambda B = \lambda(\bigcup \mathcal{I}_0) \le \lambda H \le \lambda^* A + \tfrac{1}{2}\epsilon.$$

(ii) For $m \in \mathbb{N}$ set $A_m = A \cap B(\mathbf{0}, m) \setminus \bigcup \mathcal{I}_0$. Then there is a $\mathcal{J}_m \subseteq \mathcal{I}$, covering A_m, such that $\sum_{B\in\mathcal{J}_m} \lambda B \le 2^{-m-2}\epsilon$. **P** There is an open set $G \supseteq A_m$ such that $\lambda G \le 5^{-r}2^{-m-2}\epsilon$. Now $\mathcal{I}'' = \{B : B \in \mathcal{I},\, B \subseteq G\}$ covers A_m, so there is a family $\langle \mathcal{J}_{mk} \rangle_{k<5^r}$ of disjoint countable subfamilies of $\mathcal{I}''$ such that $\mathcal{J}_m = \bigcup_{k<5^r} \mathcal{J}_{mk}$ covers A_m. For each k,

$$\sum_{B\in\mathcal{J}_{mk}} \lambda B = \lambda(\bigcup \mathcal{J}_{mk}) \le \lambda G,$$

so

$$\sum_{B\in\mathcal{J}_m} \lambda B \le 5^r \lambda G \le 2^{-m-2}\epsilon. \textbf{ Q}$$

(iii) Setting $\mathcal{I}_1 = \mathcal{I}_0 \cup \bigcup_{m\in\mathbb{N}} \mathcal{J}_m$ we have a cover of A by members of $\mathcal{I}$, and

$$\sum_{B\in\mathcal{I}_1} \lambda B \le \sum_{B\in\mathcal{I}_0} \lambda B + \sum_{m=0}^{\infty} \sum_{B\in\mathcal{J}_m} \lambda B$$
$$\le \lambda^* A + \frac{1}{2}\epsilon + \sum_{m=0}^{\infty} 2^{-m-2}\epsilon = \lambda^* A + \epsilon.$$

472D Besicovitch's Density Theorem Let λ be any Radon measure on $\mathbb{R}^r$. Then, for any locally λ-integrable real-valued function f,

(a) $f(y) = \lim_{\delta\downarrow 0} \frac{1}{\lambda B(y,\delta)} \int_{B(y,\delta)} f d\lambda$,

(b) $\lim_{\delta\downarrow 0}\frac{1}{\lambda B(y,\delta)}\int_{B(y,\delta)}|f(x)-f(y)|\lambda(dx)=0$
for λ-almost every $y\in\mathbb{R}^r$.

Remark The theorem asserts that, for λ-almost every y, limits of the form $\lim_{\delta\downarrow 0}\frac{1}{\lambda B(y,\delta)}\ldots$ are defined; in my usage, this includes the assertion that $\lambda B(y,\delta)\ne 0$ for all sufficiently small $\delta>0$.

proof (Compare 261C and 261E.)

(a) Let Z be the support of λ (411Nd); then Z is λ-conegligible and $\lambda B(y,\delta)>0$ whenever $y\in Z$ and $\delta>0$. For $q<q'$ in $\mathbb{Q}$ and $n\in\mathbb{N}$ set

$$A_{nqq'}=\{y:y\in Z\cap\operatorname{dom}f,\ \|y\|<n,\ f(y)\le q,\ \limsup_{\delta\downarrow 0}\frac{1}{\lambda B(y,\delta)}\int_{B(y,\delta)}fd\lambda>q'\}.$$

Then $\lambda A_{nqq'}=0$. **P** Let $\epsilon>0$. Then there is an $\eta\in\,]0,\epsilon]$ such that $\int_F|f|d\lambda\le\epsilon$ whenever $F\subseteq B(\mathbf{0},n)$ and $\lambda F\le\eta$ (225A). Let E be a measurable envelope of $A_{nqq'}$ included in $\{y:y\in Z\cap\operatorname{dom}f,\ f(y)\le q,\ \|y\|<n\}$, and take an open set $G\supseteq E$ such that $G\subseteq B(\mathbf{0},n)$ and $\lambda(G\setminus E)\le\eta$ (256Bb again). Let $\mathcal{I}$ be the family of non-singleton closed balls $B\subseteq G$ such that $\int_B f\ge q'\lambda B$. Then every point of $A_{nqq'}$ is the centre of arbitrarily small members of $\mathcal{I}$, so there is a disjoint family $\mathcal{I}_0\subseteq\mathcal{I}$ such that $\lambda(A_{nqq'}\setminus\bigcup\mathcal{I}_0)=0$ (472C). Now $\lambda(E\setminus\bigcup\mathcal{I}_0)=0$ and $\lambda((\bigcup\mathcal{I}_0)\setminus E)\le\eta\le\epsilon$, so

$$\begin{aligned}q'\lambda E&\le q'\lambda(\bigcup\mathcal{I}_0)+\epsilon|q'|=\sum_{B\in\mathcal{I}_0}q'\lambda B+\epsilon|q'|\\&\le\sum_{B\in\mathcal{I}_0}\int_B fd\lambda+\epsilon|q'|=\int_{\bigcup\mathcal{I}_0}fd\lambda+\epsilon|q'|\\&\le\int_E fd\lambda+\epsilon(1+|q'|)\le q\lambda E+\epsilon(1+|q'|),\end{aligned}$$

and

$$(q'-q)\lambda^*A_{nqq'}=(q'-q)\lambda E\le(1+|q'|)\epsilon.$$

As ϵ is arbitrary, $\lambda^*A_{nqq'}=0$. **Q**

As n, q and q' are arbitrary,

$$\limsup_{\delta\downarrow 0}\frac{1}{\lambda B(y,\delta)}\int_{B(y,\delta)}f\le f(y)$$

for λ-almost every $y\in Z$, therefore for λ-almost every $y\in\mathbb{R}^r$. Similarly, or applying the same argument to $-f$,

$$\liminf_{\delta\downarrow 0}\frac{1}{\lambda B(y,\delta)}\int_{B(y,\delta)}f\ge f(y)$$

for λ-almost every y, and

$$\lim_{\delta\downarrow 0}\frac{1}{\lambda B(y,\delta)}\int_{B(y,\delta)}f\text{ exists }=f(y)$$

for λ-almost every y.

(b) Now, for each $q\in\mathbb{Q}$ set $g_q(x)=|f(x)-q|$ for $x\in\operatorname{dom}f$. By (a), we have a λ-conegligible set D such that

$$\lim_{\delta\downarrow 0}\frac{1}{\lambda B(y,\delta)}\int_{B(y,\delta)}g_q d\lambda=g_q(y)$$

for every $y\in D$ and $q\in\mathbb{Q}$. Now, if $y\in D$ and $\epsilon>0$, there is a $q\in\mathbb{Q}$ such that $|f(y)-q|\le\epsilon$, and a $\delta_0>0$ such that

$$\Bigl|\frac{1}{\lambda B(y,\delta)}\int_{B(y,\delta)}g_q d\lambda-g_q(y)\Bigr|\le\epsilon$$

whenever $0<\delta\le\delta_0$. But in this case

$$\frac{1}{\lambda B(y,\delta)}\int_{B(y,\delta)}|f(x)-f(y)|\lambda(dx)$$
$$\le\frac{1}{\lambda B(y,\delta)}\int_{B(y,\delta)}|f(x)-q|\lambda(dx)+\frac{1}{\lambda B(y,\delta)}\int_{B(y,\delta)}|q-f(y)|\lambda(dx)$$
$$\le 3\epsilon.$$

As ϵ is arbitrary,

$$\lim_{\delta\downarrow 0}\frac{1}{\lambda B(y,\delta)}\int_{B(y,\delta)}|f(x)-f(y)|\lambda(dx)=0;$$

as this is true for every $y\in D$, the theorem is proved.

***472E Proposition** Let λ, λ' be Radon measures on $\mathbb{R}^r$, and $G\subseteq\mathbb{R}^r$ an open set. Let Z be the support of λ, and for $x\in Z\cap G$ set

$$M(x)=\sup\{\frac{\lambda' B}{\lambda B}:B\subseteq G\text{ is a non-trivial ball with centre }x\}.$$

Then

$$\lambda\{x:x\in Z,\,M(x)\ge t\}\le\frac{5^r}{t}\lambda' G$$

for every $t>0$.

proof The function $M:Z\to[0,\infty]$ is lower semi-continuous. **P** If $M(x)>t\ge 0$, there is a $\delta>0$ such that $B(x,\delta)\subseteq G$ and $\lambda' B(x,\delta)>t\lambda B(x,\delta)$. Because λ is a Radon measure, there is an open set $V\supseteq B(x,\delta)$ such that $V\subseteq G$ and $\lambda' B(x,\delta)>t\lambda V$; because $B(x,\delta)$ is compact, there is an $\eta>0$ such that $B(x,\delta+2\eta)\subseteq V$. Now if $y\in Z$ and $\|y-x\|\le\eta$,

$$B(x,\delta)\subseteq B(y,\delta+\eta)\subseteq V,$$

so $\lambda' B(y,\delta+\eta)>t\lambda B(y,\delta+\eta)$ and $M(y)>t$. **Q**

In particular, $H_t=\{x:x\in Z\cap G,\,M(x)>t\}$ is always measured by λ. Now, given $t>0$, let $\mathcal{I}$ be the set of non-trivial closed balls $B\subseteq G$ such that $\lambda' B>t\lambda B$. By 472B, there is a family $\langle\mathcal{I}_k\rangle_{k<5^r}$ of countable disjoint subsets of $\mathcal{I}$ such that $\bigcup_{k<5^r}\mathcal{I}_k$ covers H_t. So

$$\lambda H_t\le\sum_{k=0}^{5^r-1}\sum_{B\in\mathcal{I}_k}\lambda B\le\frac{1}{t}\sum_{k=0}^{5^r-1}\sum_{B\in\mathcal{I}_k}\lambda' B\le\frac{5^r}{t}\lambda' G,$$

as claimed.

***472F Theorem** Let λ be a Radon measure on $\mathbb{R}^r$, and $f\in\mathcal{L}^p(\lambda)$ any function, where $1<p<\infty$. Let Z be the support of λ, and for $x\in Z$ set $f^*(x)=\sup_{\delta>0}\frac{1}{\lambda B(x,\delta)}\int_{B(x,\delta)}|f|d\lambda$. Then f^* is lower semi-continuous, and $\|f^*\|_p\le 2\bigl(\frac{5^r p}{p-1}\bigr)^{1/p}\|f\|_p$.

proof (a) Replacing f by $|f|$ if necessary, we may suppose that $f\ge 0$. Z is λ-conegligible, so that f^* is defined λ-almost everywhere. Next, f^* is lower semi-continuous. **P** I repeat an idea from the proof of 472E. If $f^*(x)>t\ge 0$, there is a $\delta>0$ such that $\int_{B(x,\delta)}|f|d\lambda>t\lambda B(x,\delta)$. Because λ is a Radon measure, there is an open set $V\supseteq B(x,\delta)$ such that $\int_{B(x,\delta)}|f|d\lambda>t\lambda V$; because $B(x,\delta)$ is compact, there is an $\eta>0$ such that $B(x,\delta+2\eta)\subseteq V$; and now $f^*(y)>t$ for every $y\in Z\cap B(x,\eta)$. **Q**

(b) For $t>0$, set $H_t=\{x:x\in Z,\,f^*(x)>t\}$ and $F_t=\{x:x\in\operatorname{dom}f,\,f(x)\ge t\}$. Then

$$\lambda H_t\le\frac{2\cdot 5^r}{t}\int_{F_{t/2}}fd\lambda.$$

P Set $g=f\times\chi F_{t/2}$. Because $(\frac{t}{2})^p\lambda F_{t/2}\le\|f\|_p^p$ is finite, $\lambda F_{t/2}$ is finite, $\chi F_{t/2}\in\mathcal{L}^q(\lambda)$ (where $\frac{1}{p}+\frac{1}{q}=1$) and $g\in\mathcal{L}^1(\lambda)$ (244Eb). Let λ' be the indefinite-integral measure defined by g over λ (234J); then λ' is totally finite, and is a Radon measure (416Sa). Set

$$M(x) = \sup\{\frac{\lambda' B}{\lambda B} : B \subseteq \mathbb{R}^r \text{ is a non-trivial ball with centre } x\}$$

for $x \in Z$. Then $f^*(x) \le M(x) + \frac{t}{2}$ for every $x \in Z$, just because

$$\int_B f d\lambda \le \frac{t}{2}\lambda B + \int_B g\,d\lambda = \frac{t}{2}\lambda B + \lambda' B$$

for every closed ball B. Accordingly

$$\lambda H_t \le \lambda\{x : M(x) > \frac{t}{2}\} \le \frac{2\cdot 5^r}{t}\lambda'\mathbb{R}^r$$

(by 472E)

$$= \frac{2\cdot 5^r}{t}\int_{F_{t/2}} f d\lambda. \text{ } \mathbf{Q}$$

(c) As in part (c) of the proof of 286A, we now have

$$\begin{aligned}\int (f^*)^p d\lambda &= \int_0^\infty \lambda\{x : f^*(x)^p > t\}dt = p\int_0^\infty t^{p-1}\lambda\{x : f^*(x) > t\}dt \\ &\le 2\cdot 5^r p\int_0^\infty t^{p-2}\int_{F_{t/2}} f d\lambda dt = 2\cdot 5^r p\int_{\mathbb{R}^r} f(x)\int_0^{2f(x)} t^{p-2}dt\lambda(dx) \\ &= 2\cdot 5^r p\int_{\mathbb{R}^r}\frac{2^{p-1}}{p-1}f(x)^p\lambda(dx) = \frac{2^p 5^r p}{p-1}\int f^p d\lambda.\end{aligned}$$

Taking pth roots, we have the result.

472X Basic exercises (a) Show that if λ, λ' are Radon measures on $\mathbb{R}^r$ which agree on closed balls, they are equal. (Cf. 466Xj.)

(b) Let λ be a Radon measure on $\mathbb{R}^r$. Let $A \subseteq \mathbb{R}^r$ be a non-empty set, and $\epsilon > 0$. Show that there is a sequence $\langle B_n\rangle_{n\in\mathbb{N}}$ of closed balls in $\mathbb{R}^r$, all of radius at most ϵ and with centres in A, such that $A \subseteq \bigcup_{n\in\mathbb{N}} B_n$ and $\sum_{n=0}^\infty \lambda B_n \le \lambda^* A + \epsilon$.

(c) Let λ be a non-zero Radon measure on $\mathbb{R}^r$ and Z its support. Show that we have a lower density ϕ (definition: 341C) for the subspace measure λ_Z defined by setting $\phi E = \{x : x \in Z, \lim_{\delta\downarrow 0}\frac{\lambda(E\cap B(x,\delta))}{\lambda B(x,\delta)} = 1\}$ whenever λ_Z measures E.

(d) Let λ be a Radon measure on $\mathbb{R}^r$, and f a locally λ-integrable function. Show that $E = \{y : g(y) = \lim_{\delta\downarrow 0}\frac{1}{\lambda B(y,\delta)}\int_{B(y,\delta)} f\,d\lambda$ is defined in $\mathbb{R}\}$ is a Borel set, and that $g : E \to \mathbb{R}$ is Borel measurable.

472Y Further exercises (a)(i) Let $\mathcal{I}$ be a finite family of intervals (open, closed or half-open) in $\mathbb{R}$. Show that there are subfamilies $\mathcal{I}_0$, $\mathcal{I}_1 \subseteq \mathcal{I}$, both disjoint, such that $\mathcal{I}_0 \cup \mathcal{I}_1$ covers $\bigcup\mathcal{I}$. (*Hint*: induce on $\#(\mathcal{I})$.) Show that this remains true if any totally ordered set is put in place of $\mathbb{R}$. (ii) Show that if $\mathcal{I}$ is any family of non-empty intervals in $\mathbb{R}$ such that none contains the centre of any other, then $\mathcal{I}$ is expressible as $\mathcal{I}_0 \cup \mathcal{I}_1$ where both $\mathcal{I}_0$ and $\mathcal{I}_1$ are disjoint.

(b) Let $m = m(r)$ be the largest number such that there are $u_1, \dots, u_m \in \mathbb{R}^r$ such that $\|u_i\| = 1$ for every i and $\|u_i - u_j\| \ge 1$ for all $i \ne j$. Let $A \subseteq \mathbb{R}^r$ be a bounded set and $x \mapsto \delta_x : A \to]0,\infty[$ a bounded function; set $B_x = B(x,\delta_x)$ for $x \in A$. (i) Show that $m < 3^r$. (ii) Show that there is an $\epsilon \in \left]0, \frac{1}{10}\right]$ such that whenever $\|u_0\| = \ldots = \|u_m\| = 1$ there are distinct i, $j \le m$ such that $u_i \cdot u_j > \frac{1}{2}(1+\epsilon)$. (iii) Suppose that u, $v \in \mathbb{R}^r$ are such that $\frac{1}{3} \le \|u\| \le 1$, $\|v\| \le 1+\epsilon$ and $\|u - v\| > 1$. Show that the angle $u\widehat{\mathbf{0}}v$ has cosine at most $\frac{1}{2}(1+\epsilon)$. (*Hint*: maximise $\frac{a^2+b^2-c^2}{2ab}$ subject to $\frac{1}{3} \le a \le 1$, $b \le 1+\epsilon$ and $c \ge 1$.) (iv) Suppose that $\langle x_n\rangle_{n\in\mathbb{N}}$ is a sequence in A such that $x_n\notin B_{x_i}$ for $i<n$ and $(1+\epsilon)\delta_{x_n}\ge\sup\{\delta_x : x\in A\setminus\bigcup_{i<n}B_{x_i}\}$ for every n. Show that $A\subseteq\bigcup_{n\in\mathbb{N}}B_{x_n}$. (v) Take $y\in\mathbb{R}^r$. Show that there is at most one n such that $\|y-x_n\|\le\frac{1}{3}\delta_{x_n}$. (vi) Show that if $i<j$, $\frac{1}{3}\delta_{x_i}\le\|y-x_i\|\le\delta_{x_i}$ and $\|y-x_j\|\le\delta_j$ then the cosine of the angle $x_i\widehat{y}x_j$ is at most $\frac{1}{2}(1+\epsilon)$. (vii) Show that $\#(\{i : y\in B_{x_i}\})\le m+1$.

Hence show that if $\mathcal{I}$ is any family of non-trivial closed balls such that every point of A is the centre of some member of $\mathcal{I}$, then there is a countable $\mathcal{I}_0\subseteq\mathcal{I}$, covering A, such that no point of $\mathbb{R}^r$ belongs to more than 3^r members of $\mathcal{I}_0$.

(c) Use 472Yb to prove an alternative version of 472B, but with the constant 9^r+1 in place of 5^r.

(d) Let $A\subseteq\mathbb{R}^r$ be a bounded set, and $\mathcal{I}$ a family of non-trivial closed balls in $\mathbb{R}^r$ such that whenever $x\in A$ and $\epsilon>0$ there is a ball $B(y,\delta)\in\mathcal{I}$ such that $\|x-y\|\le\epsilon\delta$. Show that there is a family $\langle\mathcal{I}_k\rangle_{k<5^r}$ of subsets of $\mathcal{I}$ such that each $\mathcal{I}_k$ is disjoint and $\bigcup_{k<5^r}\mathcal{I}_k$ covers A.

(e) Give an example of a strictly positive Radon probability measure μ on a compact metric space (X,ρ) for which there is a Borel set $E\subseteq X$ such that

$$\liminf_{\delta\downarrow 0}\frac{\mu(E\cap B(x,\delta))}{\mu B(x,\delta)}=0,\quad \limsup_{\delta\downarrow 0}\frac{\mu(E\cap B(x,\delta))}{\mu B(x,\delta)}=1$$

for every $x\in X$.

(f) Let λ be a Radon measure on $\mathbb{R}^r$, and f a λ-integrable real-valued function. Show that $\sup_{\delta>0}\frac{1}{\lambda B(x,\delta)}\int_{B(x,\delta)}|f|d\lambda$ is defined and finite for λ-almost every $x\in\mathbb{R}^r$.

(g) Let λ, λ' be Radon measures on $\mathbb{R}^r$. (i) Show that $g(x)=\lim_{\delta\downarrow 0}\frac{\lambda' B(x,\delta)}{\lambda B(x,\delta)}$ is defined in $\mathbb{R}$ for λ-almost every x. (ii) Setting $\lambda_0=\sup_{n\in\mathbb{N}}\lambda'\wedge n\lambda$ in the cone of Radon measures on $\mathbb{R}^r$ (437Yi), show that g is a Radon-Nikodým derivative of λ_0 with respect to λ. (*Hint*: show that if λ and λ' are mutually singular then $g=0$ λ-a.e.)

472 Notes and comments I gave primacy to the 'weak' Vitali's theorem in 261B because I think it is easier than the 'strong' form in 472C, it uses the same ideas as the original one-dimensional theorem in 221A, and it is adequate for the needs of Volume 2. Any proper study of general measures on $\mathbb{R}^r$, however, will depend on the ideas in 472A-472C. You will see that in 472B, as in other forms of Vitali's theorem, there is a key step in which a sequence is chosen greedily. This time we must look much more carefully at the geometry of $\mathbb{R}^r$ because we can no longer rely on a measure to tell us what is happening. (Though you will observe that I still use the elementary properties of Euclidean volume in the argument of 472A.) Once we have reached 472C, however, we are in a position to repeat all the arguments of 261C-261E in much greater generality (472D), and, as a bonus, can refine 261F (472Xb). For more in this direction see Mattila 95 and Federer 69, §2.8.

It is natural to ask whether the constant '5^r' in 472B is best possible. The argument of 472A is derived from Sullivan 94, where a more thorough analysis is given. It seems that even for $r=2$ the best constant is unknown. (For $r=1$, the best constant is 2; see 472Ya.) Note that even for finite families $\mathcal{I}$ we should have to find the colouring number of a graph (counting two balls as linked if they intersect), so it may well be a truly difficult problem. The method in 472B amounts to using the greedy colouring algorithm after ordering the balls by size, and one does not expect such approaches to give exact colouring numbers. Of course the questions addressed here depend only on the existence of *some* function of r to do the job.

An alternative argument runs through a kind of pointwise version of 472A (472Yb-472Yc). It gives a worse constant but is attractive in other ways. For many of the applications of 472C, the result of 472Yb is already sufficient.

The constant $2\bigl(\frac{5^r p}{p-1}\bigr)^{1/p}$ in 472F makes no pretence to be 'best', or even 'good'. The only reason for giving a formula at all is to emphasize the remarkable fact that it does not depend on the measure λ. The theorems of this section are based on the metric geometry of Euclidean space, not on any special properties of Lebesgue measure. The constants *do* depend on the dimension, so that even in Hilbert space (for instance) we cannot expect any corresponding results.

Version of 25.7.11/7.8.20

473 Poincaré's inequality

In this section I embark on the main work of the first half of the chapter, leading up to the Divergence Theorem in §475. I follow the method in Evans & Gariepy 92. The first step is to add some minor results on differentiable and Lipschitz functions to those already set out in §262 (473B-473C). Then we need to know something about convolution products (473D), extending ideas in §§256 and 444; in particular, it will be convenient to have a fixed sequence $\langle\tilde{h}_n\rangle_{n\in\mathbb{N}}$ of smoothing functions with some useful special properties (473E).

The new ideas of the section begin with the Gagliardo-Nirenberg-Sobolev inequality, relating $\|f\|_{r/(r-1)}$ to $\int\|\operatorname{grad}f\|$. In its simplest form (473H) it applies only to functions with compact support; we need to work much harder to get results which we can use to estimate $\int_B|f|^{r/(r-1)}$ in terms of $\int_B\|\operatorname{grad}f\|$ and $\int_B|f|$ for balls B (473I, 473K).

473A Notation For the next three sections, $r\ge 2$ will be a fixed integer. For $x\in\mathbb{R}^r$ and $\delta\ge 0$, $B(x,\delta)=\{y:\|y-x\|\le\delta\}$ will be the closed ball with centre x and radius δ. I will write $\partial B(x,\delta)$ for the boundary of $B(x,\delta)$, the sphere $\{y:\|y-x\|=\delta\}$. $S_{r-1}=\partial B(\mathbf{0},1)$ will be the unit sphere. As in Chapter 26, I will use Greek letters to represent coordinates of vectors, so that $x=(\xi_1,\dots,\xi_r)$, etc.

μ will always be Lebesgue measure on $\mathbb{R}^r$. $\beta_r=\mu B(\mathbf{0},1)$ will be the r-dimensional volume of the unit ball, that is,

$$\begin{aligned}\beta_r&=\frac{\pi^k}{k!}\text{ if }r=2k\text{ is even,}\\&=\frac{2^{2k+1}k!\pi^k}{(2k+1)!}\text{ if }r=2k+1\text{ is odd}\end{aligned}$$

(252Q). ν will be normalized Hausdorff $(r-1)$-dimensional measure on $\mathbb{R}^r$, that is, $\nu=2^{-r+1}\beta_{r-1}\mu_{H,r-1}$, where $\mu_{H,r-1}$ is $(r-1)$-dimensional Hausdorff measure on $\mathbb{R}^r$ as described in §264. Recall from 265F and 265H that $\nu S_{r-1}=2\pi\beta_{r-2}=r\beta_r$ (counting β_0 as 1).

473B Differentiable functions (a) Recall from §262 that a function ϕ from a subset of $\mathbb{R}^r$ to $\mathbb{R}^s$ (where $s\ge 1$) is differentiable at $x\in\mathbb{R}^r$, with derivative an $s\times r$ matrix T, if for every $\epsilon>0$ there is a $\delta>0$ such that $\|\phi(y)-\phi(x)-T(y-x)\|\le\epsilon\|y-x\|$ whenever $\|y-x\|\le\delta$; this includes the assertion that $B(x,\delta)\subseteq\operatorname{dom}\phi$. In this case, the coefficients of T are the partial derivatives $\frac{\partial\phi_j}{\partial\xi_i}(x)$ at x, where $\phi_1,\dots,\phi_s$ are the coordinate functions of ϕ, and $\frac{\partial}{\partial\xi_i}$ represents partial differentiation with respect to the ith coordinate (262Ic).

(b) When $s=1$, so that we have a real-valued function f defined on a subset of $\mathbb{R}^r$, I will write $(\operatorname{grad}f)(x)$ for the derivative of f at x, the **gradient** of f. If we strictly adhere to the language of (a), $\operatorname{grad}f$ is a $1\times r$ matrix $\begin{pmatrix}\frac{\partial f}{\partial\xi_1}&\dots&\frac{\partial f}{\partial\xi_r}\end{pmatrix}$; but it is convenient to treat it as a vector, so that $\operatorname{grad}f(x)$ (when defined) belongs to $\mathbb{R}^r$, and we can speak of $y\,\boldsymbol{.}\operatorname{grad}f(x)$ rather than $(\operatorname{grad}f(x))(y)$, etc.

(c) Chain rule for functions of many variables I find that I have not written out the following basic fact. Let $\phi:A\to\mathbb{R}^s$ and $\psi:B\to\mathbb{R}^p$ be functions, where $A\subseteq\mathbb{R}^r$ and $B\subseteq\mathbb{R}^s$. If $x\in A$ is such that ϕ is differentiable at x, with derivative S, and ψ is differentiable at $\phi(x)$, with derivative T, then the composition $\psi\phi$ is differentiable at x, with derivative TS.

P Recall that if we regard S and T as linear operators, they have finite norms (262H). Given $\epsilon>0$, let $\eta>0$ be such that $\eta\|T\|+\eta(\|S\|+\eta)\le\epsilon$. Let $\delta_1,\delta_2>0$ be such that $\phi(y)$ is defined and $\|\phi(y)-\phi(x)-S(y-x)\|\le\eta\|y-x\|$ whenever $\|y-x\|\le\delta_1$, and $\psi(z)$ is defined and $\|\psi(z)-\psi\phi(x)-T(z-\phi(x))\|\le\eta\|z-\phi(x)\|$ whenever $\|z-\phi(x)\|\le\delta_2$. Set $\delta=\min(\delta_1,\frac{\delta_2}{\eta+\|S\|})>0$. If $\|y-x\|\le\delta$, then $\phi(y)$ is defined and

$$\|\phi(y)-\phi(x)\|\le\|S(y-x)\|+\|\phi(y)-\phi(x)-S(y-x)\|\le(\|S\|+\eta)\|y-x\|\le\delta_2,$$

so $\psi\phi(y)$ is defined and

$$\begin{aligned}\|\psi\phi(y) - \psi\phi(x) - TS(y-x)\| &\le \|\psi\phi(y) - \psi\phi(x) - T(\phi(y) - \phi(x))\| + \|T\|\|\phi(y) - \phi(x) - S(y-x)\| \\ &\le \eta\|\phi(y) - \phi(x)\| + \|T\|\eta\|y-x\| \\ &\le \eta(\|S\| + \eta)\|y-x\| + \|T\|\eta\|y-x\| \le \epsilon\|y-x\|;\end{aligned}$$

as ϵ is arbitrary, $\psi\phi$ is differentiable at x with derivative TS. **Q**

(d) It follows that if f and g are real-valued functions defined on a subset of $\mathbb{R}^r$, and $x \in \operatorname{dom} f \cap \operatorname{dom} g$ is such that $(\operatorname{grad} f)(x)$ and $(\operatorname{grad} g)(x)$ are both defined, then $\operatorname{grad}(f \times g)(x)$ is defined and equal to $f(x)\operatorname{grad} g(x) + g(x)\operatorname{grad} f(x)$. **P** Set $\phi(y) = \begin{pmatrix} f(y) \\ g(y) \end{pmatrix}$ for $y \in \operatorname{dom} f \cap \operatorname{dom} g$; then ϕ is differentiable at x with derivative the $2 \times r$ matrix $\begin{pmatrix} \operatorname{grad} f(x) \\ \operatorname{grad} g(x) \end{pmatrix}$ (262Ib). Set $\psi(z) = \zeta_1\zeta_2$ for $z = (\zeta_1, \zeta_2) \in \mathbb{R}^2$; then ψ is differentiable everywhere, with derivative the 1×2 matrix $(\,\zeta_2 \quad \zeta_1\,)$. So $f \times g = \psi\phi$ is differentiable at x with derivative

$$(\,g(x) \quad f(x)\,)\begin{pmatrix} \operatorname{grad} f(x) \\ \operatorname{grad} g(x) \end{pmatrix} = g(x)\operatorname{grad} f(x) + f(x)\operatorname{grad} g(x).\ \mathbf{Q}$$

(e) Let D be a subset of $\mathbb{R}^r$ and $\phi : D \to \mathbb{R}^s$ any function. Set $D_0 = \{x : x \in D,\ \phi \text{ is differentiable at } x\}$. Then the derivative of ϕ, regarded as a function from D_0 to $\mathbb{R}^{rs}$, is (Lebesgue) measurable. **P** Use 262P; the point is that, writing $T(x)$ for the derivative of ϕ at x, $T(x)$ is surely a derivative of $\phi\restriction D_0$, relative to D_0, at every point of D_0. **Q** (See also 473Ya.)

(f) If $G \subseteq \mathbb{R}^r$ is an open set, a function $\phi : G \to \mathbb{R}^s$ is **smooth** if it is differentiable arbitrarily often; that is, if all its repeated partial derivatives

$$\frac{\partial^m \phi_j}{\partial\xi_{i_1}\ldots\partial\xi_{i_m}}$$

are defined and continuous everywhere on G. I will write $\mathcal{D}$ for the family of real-valued functions from $\mathbb{R}^r$ to $\mathbb{R}$ which are smooth and have compact support.

473C Lipschitz functions (a) If f and g are bounded real-valued Lipschitz functions, defined on any subsets of $\mathbb{R}^r$, then $f \times g$, defined on $\operatorname{dom} f \cap \operatorname{dom} g$, is Lipschitz. **P** Let γ_f, M_f, γ_g and M_g be such that $|f(x)| \le M_f$ and $|f(x)-f(y)| \le \gamma_f\|x-y\|$ for all x, $y \in \operatorname{dom} f$, while $|g(x)| \le M_g$ and $|g(x)-g(y)| \le \gamma_g\|x-y\|$ for all x, $y \in \operatorname{dom} g$. Then for any x, $y \in \operatorname{dom} f \cap \operatorname{dom} g$,

$$\begin{aligned}|f(x)g(x) - f(y)g(y)| &\le |f(x)||g(x) - g(y)| + |g(y)||f(x) - f(y)| \\ &\le (M_f\gamma_g + M_g\gamma_f)\|x-y\|.\end{aligned}$$

So $M_f\gamma_g + M_g\gamma_f$ is a Lipschitz constant for $f \times g$. **Q**

(b) Suppose that F_1, $F_2 \subseteq \mathbb{R}^r$ are closed sets with convex union C. Let $f : C \to \mathbb{R}$ be a function such that $f\restriction F_1$ and $f\restriction F_2$ are both Lipschitz. Then f is Lipschitz. **P** For each j, let γ_j be a Lipschitz constant for $f\restriction F_j$, and set $\gamma = \max(\gamma_1, \gamma_2)$, so that γ is a Lipschitz constant for both $f\restriction F_1$ and $f\restriction F_2$. Take any x, $y \in C$. If both belong to the same F_j, then $|f(x) - f(y)| \le \gamma\|x-y\|$. If $x \in F_j$ and $y \notin F_j$, then y must belong to F_{3-j}, and $(1-t)x + ty \in F_1 \cup F_2$ for every $t \in [0,1]$, because C is convex. Set $t_0 = \sup\{t : t \in [0,1],\ (1-t)x + ty \in F_j\}$, $z = (1-t_0)x + t_0y$; then $z \in F_1 \cap F_2$, because both are closed, so

$$|f(x) - f(y)| \le |f(x) - f(z)| + |f(z) - f(y)| \le \gamma\|x-z\| + \gamma\|z-y\| = \gamma\|x-y\|.$$

As x and y are arbitrary, γ is a Lipschitz constant for f. **Q**

(c) Suppose that $f : \mathbb{R}^r \to \mathbb{R}$ is Lipschitz. Recall that by Rademacher's theorem (262Q), $\operatorname{grad} f$ is defined almost everywhere. All the partial derivatives of f are (Lebesgue) measurable, by 473Be, so $\operatorname{grad} f$

is (Lebesgue) measurable on its domain. If γ is a Lipschitz constant for f, $\|\operatorname{grad} f(x)\| \le \gamma$ whenever $\operatorname{grad} f(x)$ is defined. **P** If $z \in \mathbb{R}^r$, then

$$\lim_{t\downarrow 0}\frac{1}{t}|f(x+tz)-f(x)-tz\,.\operatorname{grad} f(x)|=0,$$

so

$$|z\,.\operatorname{grad} f(x)|=\lim_{t\downarrow 0}\frac{1}{t}|f(x+tz)-f(x)|\le\gamma\|z\|;$$

as z is arbitrary, $\|\operatorname{grad} f(x)\| \le \gamma$. **Q**

(d) Conversely, if $f:\mathbb{R}^r\to\mathbb{R}$ is differentiable and $\|\operatorname{grad} f(x)\|\le\gamma$ for every x, then γ is a Lipschitz constant for f. **P** Take $x, y \in \mathbb{R}^r$. Set $g(t)=f((1-t)x+ty)$ for $t\in\mathbb{R}$. The function $t\mapsto(1-t)x+ty:\mathbb{R}\to\mathbb{R}^r$ is everywhere differentiable, with constant derivative $y-x$, so by 473Bc g is differentiable, with derivative $g'(t)=(y-x)\,.\operatorname{grad} f((1-t)x+ty)$ for every t; in particular, $|g'(t)|\le\gamma\|y-x\|$ for every t. Now, by the Mean Value Theorem, there is a $t\in[0,1]$ such that $g(1)-g(0)=g'(t)$, so that $|f(y)-f(x)|=|g'(t)|\le\gamma\|y-x\|$. As x and y are arbitrary, f is γ-Lipschitz. **Q**

(e) Note that if $f\in\mathcal{D}$ then all its partial derivatives are continuous functions with compact support, so are bounded (436Ia), and f is Lipschitz as well as bounded, by (d) here.

(f)(i) If $D\subseteq\mathbb{R}^r$ is bounded and $f:D\to\mathbb{R}$ is Lipschitz, then there is a Lipschitz function $g:\mathbb{R}^r\to\mathbb{R}$, with compact support, extending f. **P** By 262Bb there is a Lipschitz function $f_1:\mathbb{R}^r\to\mathbb{R}$ which extends f. Let $\gamma>0$ be such that $D\subseteq B(\mathbf{0},\gamma)$ and γ is a Lipschitz constant for f_1; set $M=|f_1(0)|+\gamma^2$; then $|f_1(x)|\le M$ for every $x\in D$, so if we set $f_2(x)=\operatorname{med}(-M,f_1(x),M)$ for $x\in\mathbb{R}^r$, f_2 is a bounded Lipschitz function extending f. Set $f_3(x)=\operatorname{med}(0,1+\gamma-\|x\|,1)$ for $x\in\mathbb{R}^r$; then f_3 is a bounded Lipschitz function with compact support. By (a), $g=f_3\times f_2$ is Lipschitz, and $g:\mathbb{R}^r\to\mathbb{R}$ is a function with compact support extending f. **Q**

(ii) It follows that if $D\subseteq\mathbb{R}^r$ is bounded and $f:D\to\mathbb{R}^s$ is Lipschitz, then there is a Lipschitz function $g:\mathbb{R}^r\to\mathbb{R}^s$, with compact support, extending f. **P** By 262Ba, we need only apply (i) to each coordinate of f. **Q**

473D Smoothing by convolution We shall need a miscellany of facts, many of them special cases of results in §§255 and 444, concerning convolutions on $\mathbb{R}^r$. Recall that I write $(f*g)(x)=\int f(y)g(x-y)\mu(dy)$ whenever f and g are real-valued functions defined almost everywhere in $\mathbb{R}^r$ and the integral is defined, and that $f*g=g*f$ (255Fb, 444Og). Now we have the following.

Lemma Suppose that f and g are Lebesgue measurable real-valued functions defined μ-almost everywhere in $\mathbb{R}^r$.

(a) If f is integrable and g is essentially bounded, then their convolution $f*g$ is defined everywhere in $\mathbb{R}^r$ and uniformly continuous, and $\|f*g\|_\infty\le\|f\|_1\operatorname{ess\,sup}|g|$.

(b) If f is locally integrable and g is bounded and has compact support, then $f*g$ is defined everywhere in $\mathbb{R}^r$ and is continuous.

(c) If f and g are defined everywhere in $\mathbb{R}^r$ and $x\in\mathbb{R}^r\setminus(\{y:f(y)\ne0\}+\{z:g(z)\ne0\})$, then $(f*g)(x)$ is defined and equal to 0.

(d) If f is integrable and g is bounded, Lipschitz and defined everywhere, then $f*\operatorname{grad} g$ and $\operatorname{grad}(f*g)$ are defined everywhere and equal, where $f*\operatorname{grad} g=(f*\frac{\partial g}{\partial\xi_1},\dots,f*\frac{\partial g}{\partial\xi_r})$. Moreover, $f*g$ is Lipschitz.

(e) If f is locally integrable, and $g\in\mathcal{D}$, then $f*g$ is defined everywhere and is smooth.

(f) If f is essentially bounded and $g\in\mathcal{D}$, then all the derivatives of $f*g$ are bounded, and $f*g$ is Lipschitz.

(g) If f is integrable and $\phi:\mathbb{R}^r\to\mathbb{R}^r$ is a bounded measurable function with components $\phi_1,\dots,\phi_r$, and we write $(f*\phi)(x)=((f*\phi_1)(x),\dots,(f*\phi_r)(x))$, then $\|(f*\phi)(x)\|\le\|f\|_1\sup_{y\in\mathbb{R}^r}\|\phi(y)\|$ for every $x\in\mathbb{R}^r$.

proof (a) See 255K.

(b) Suppose that $g(y) = 0$ when $\|y\| \geq n$. Given $x \in \mathbb{R}^r$, set $\tilde{f} = f \times \chi B(x, n+1)$. Then $\tilde{f} * g$ is defined everywhere and continuous, by (a), while $(f * g)(z) = (\tilde{f} * g)(z)$ whenever $z \in B(x, 1)$; so $f * g$ is defined everywhere in $B(x, 1)$ and is continuous at x.

(c) We have only to note that $f(y)g(x - y) = 0$ for every y.

(d) Let γ be a Lipschitz constant for g. We know that $\operatorname{grad} g$ is defined almost everywhere, is measurable, and that $\|\operatorname{grad} g(x)\| \leq \gamma$ whenever it is defined (473Cc); so $(f * \operatorname{grad} g)(x)$ is defined for every x, by (a) here. Fix $x \in \mathbb{R}^r$. If y, $z \in \mathbb{R}^r$ set

$$\theta(y, z) = \frac{1}{\|z\|}\bigl(g(x - y + z) - g(x - y) - z \mathbin{.} \operatorname{grad} g(x - y)\bigr)$$

whenever this is defined. Then $|\theta(y, z)| \leq 2\gamma$ whenever it is defined. Now suppose that $\langle z_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}^r \setminus \{\mathbf{0}\}$ converging to $\mathbf{0}$. Then $\lim_{n\to\infty} \theta(y, z_n) = 0$ whenever $\operatorname{grad} g(x - y)$ is defined, which almost everywhere. So $\lim_{n\to\infty} \int f(y)\theta(y, z_n)\mu(dy) = 0$, by Lebesgue's Dominated Convergence Theorem. But this means that

$$\frac{1}{\|z_n\|}\bigl((f * g)(x + z_n) - (f * g)(x) - ((f * \operatorname{grad} g)(x)) \mathbin{.} z_n\bigr) \to 0$$

as $n \to \infty$. As $\langle z_n \rangle_{n\in\mathbb{N}}$ is arbitrary, $\operatorname{grad}(f * g)(x)$ is defined and is equal to $(f * \operatorname{grad} g)(x)$.

Now $\operatorname{grad} g$ is bounded, because g is Lipschitz, so $\operatorname{grad}(f * g) = f * \operatorname{grad} g$ also is bounded, by (a), and $f * g$ must be Lipschitz (473Cd).

(e) By (b), $f * g$ is defined everywhere and is continuous. Now, for any $i \leq r$, $\frac{\partial}{\partial \xi_i}(f * g) = f * \frac{\partial g}{\partial \xi_i}$ everywhere. **P** Let $n \in \mathbb{N}$ be such that $g(y) = 0$ if $\|y\| \geq n$. Given $x \in \mathbb{R}^r$, set $\tilde{f} = f \times \chi B(x, n+1)$. Then $(f * g)(z) = (\tilde{f} * g)(z)$ for every $z \in B(x, 1)$, so that

$$\frac{\partial(f*g)}{\partial \xi_i}(x) = \frac{\partial(\tilde{f}*g)}{\partial \xi_i}(x) = (\tilde{f} * \frac{\partial g}{\partial \xi_i})(x)$$

(by (d))

$$= (f * \frac{\partial g}{\partial \xi_i})(x)$$

(because of course $\frac{\partial g}{\partial \xi_i}$ is also zero outside $B(\mathbf{0}, n)$). **Q** Inducing on k,

$$\frac{\partial^k}{\partial \xi_{i_1} \ldots \partial \xi_{i_k}}(f * g)(x) = (f * \frac{\partial^k g}{\partial \xi_{i_1} \ldots \partial \xi_{i_k}})(x)$$

for every $x \in \mathbb{R}^r$ and every $i_1, \ldots, i_k$; so we have the result.

(f) The point is just that all the partial derivatives of g, being smooth functions with compact support, are integrable, and that

$$|\frac{\partial}{\partial \xi_i}(f * g)(x)| = |(f * \frac{\partial g}{\partial \xi_i})(x)| \leq \|f\|_\infty \|\frac{\partial g}{\partial \xi_i}\|_1$$

for every x and every $i \leq r$. Inducing on the order of D, we see that $D(f * g) = f * Dg$ and $\|D(f * g)\|_\infty \leq \|f\|_\infty \|Dg\|_1$, so that $D(f * g)$ is bounded, for any partial differential operator D. In particular, $\operatorname{grad}(f * g)$ is bounded, so that $f * g$ is Lipschitz, by 473Cd.

(g) If x, $z \in \mathbb{R}^r$, then

$$\begin{aligned} z \mathbin{.} (f * \phi)(x) &= \sum_{i=1}^r \zeta_i (f * \phi_i)(x) = \int f(y) \sum_{i=1}^r \zeta_i \phi_i(x - y)\mu(dy) \\ &\leq \int |f(y)| |\sum_{i=1}^r \zeta_i \phi_i(x - y)| \mu(dy) \\ &\leq \int |f(y)| \|z\| \|\phi(x - y)\| \mu(dy) \leq \|z\| \|f\|_1 \sup_{y\in\mathbb{R}^r} \|\phi(y)\|. \end{aligned}$$

As z is arbitrary, $\|(f*\phi)(x)\| \le \|f\|_1 \sup_{y\in\mathbb{R}^r}\|\phi(y)\|$.

473E Lemma (a) Define $h:\mathbb{R}\to[0,1]$ by setting $h(t)=\exp(\frac{1}{t^2-1})$ for $|t|<1$, 0 for $|t|\ge 1$. Then h is smooth, and $h'(t)\le 0$ for $t\ge 0$.

(b) For $n\ge 1$, define $\tilde h_n:\mathbb{R}^r\to\mathbb{R}$ by setting

$$\alpha_n=\int h((n+1)^2\|x\|^2)\mu(dx),\qquad \tilde h_n(x)=\frac{1}{\alpha_n}h((n+1)^2\|x\|^2)$$

for every $x\in\mathbb{R}^r$. Then $\tilde h_n\in\mathcal{D}$, $\tilde h_n(x)\ge 0$ for every x, $\tilde h_n(x)=0$ if $\|x\|\ge\frac{1}{n+1}$, and $\int\tilde h_n d\mu=1$.

(c) If $f\in\mathcal{L}^0(\mu)$, then $\lim_{n\to\infty}(f*\tilde h_n)(x)=f(x)$ for every $x\in\operatorname{dom}f$ at which f is continuous.

(d) If $f:\mathbb{R}^r\to\mathbb{R}$ is uniformly continuous (in particular, if it is either Lipschitz or a continuous function with compact support), then $\lim_{n\to\infty}\|f-f*\tilde h_n\|_\infty=0$.

(e) If $f\in\mathcal{L}^0(\mu)$ is locally integrable, then $f(x)=\lim_{n\to\infty}(f*\tilde h_n)(x)$ for μ-almost every $x\in\mathbb{R}^r$.

(f) If $f\in\mathcal{L}^p(\mu)$, where $1\le p<\infty$, then $\lim_{n\to\infty}\|f-f*\tilde h_n\|_p=0$.

proof (a) Set $h_0(t)=\exp(-\frac{1}{t})$ for $t>0$, 0 for $t\le 0$. A simple induction on n shows that the nth derivative $h_0^{(n)}$ of h_0 is of the form

$$\begin{aligned}h_0^{(n)}(t)&=q_n(\tfrac{1}{t})\exp(-\tfrac{1}{t})\text{ for }t>0\\&=0\text{ for }t\le 0,\end{aligned}$$

where each q_n is a polynomial of degree $2n$; the inductive hypothesis depends on the fact that $\lim_{s\to\infty}q(s)e^{-s}=0$ for every polynomial q. So h_0 is smooth. Now $h(t)=h_0(1-t^2)$ so h also is smooth. If $0\le t<1$ then

$$h'(t)=-\exp(\tfrac{1}{t^2-1})\cdot\tfrac{2t}{(t^2-1)^2}<0;$$

if $t>1$ then $h'(t)=0$; since h' is continuous, $h'(t)\le 0$ for every $t\ge 0$.

(b) We need only observe that

$$x\mapsto(n+1)^2\|x\|^2=(n+1)^2\sum_{i=1}^r\xi_i^2$$

is smooth and that the composition of smooth functions is smooth (using 473Bc).

(c) If f is continuous at x and $\epsilon>0$, let $n_0\in\mathbb{N}$ be such that $|f(y)-f(x)|\le\epsilon$ whenever $y\in\operatorname{dom}f$ and $\|y-x\|\le\frac{1}{n_0+1}$. Then for any $n\ge n_0$,

$$\begin{aligned}|(f*\tilde h_n)(x)-f(x)|&=|\int f(x-y)\tilde h_n(y)\mu(dy)-\int f(x)\tilde h_n(y)\mu(dy)|\\&\le\int|f(x-y)-f(x)|\tilde h_n(y)\mu(dy)\le\int\epsilon\tilde h_n(y)\mu(dy)=\epsilon.\end{aligned}$$

As ϵ is arbitrary, we have the result.

(d) Repeat the argument of (c), but 'uniformly in x'; that is, given $\epsilon>0$, take n_0 such that $|f(y)-f(x)|\le\epsilon$ whenever $x,\,y\in\mathbb{R}^r$ and $\|y-x\|\le\frac{1}{n_0+1}$, and see that $|(f*\tilde h_n)(x)-f(x)|\le\epsilon$ for every $n\ge n_0$ and every x.

(e) We know from 472Db or 261E that, for almost every $x\in\mathbb{R}^r$,

$$\lim_{\delta\downarrow 0}\frac{1}{\mu B(x,\delta)}\int_{B(x,\delta)}|f(y)-f(x)|\mu(dy)=0.$$

Take any such x. Set $\gamma=f(x)$, Set $g(y)=|f(y)-\gamma|$ for every $y\in\operatorname{dom}f$. Let $\epsilon>0$. Then there is some $\delta>0$ such that $\frac{q(t)}{\beta_r t^r}\le\epsilon$ whenever $0<t\le\delta$, where

$$q(t) = \int_{B(x,t)} g\,d\mu = \int_0^t \int_{\partial B(x,s)} g(y)\nu(dy)dt$$

by 265G, so $q'(t) = \int_{\partial B(y,t)} g\,d\nu$ for almost every $t \in [0,\delta]$, by 222E. If $n+1 \ge \frac{1}{\delta}$, then

$$(g * \tilde{h}_n)(x) = \int g(y)\tilde{h}_n(x-y)\mu(dy) = \int_{B(x,\delta)} g(y)\tilde{h}_n(x-y)\mu(dy)$$

$$= \frac{1}{\alpha_n}\int_0^\delta \int_{\partial B(x,t)} g(y)h((n+1)^2t^2)\nu(dy)dt$$

(265G again)

$$= \frac{1}{\alpha_n}\int_0^\delta h((n+1)^2t^2)q'(t)dt$$

$$= -\frac{1}{\alpha_n}\int_0^\delta 2(n+1)^2th'((n+1)^2t^2)q(t)dt$$

(integrating by parts (225F), because $q(0) = h((n+1)^2\delta^2) = 0$ and both q and h are absolutely continuous)

$$\le -\frac{\epsilon}{\alpha_n}\int_0^\delta 2(n+1)^2th'((n+1)^2t^2)\beta_r t^r dt$$

(because $0 \le q(t) \le \epsilon\beta_r t^r$ and $h'((n+1)^2t^2) \le 0$ for $0 \le t \le \delta$)

$$= \epsilon$$

(applying the same calculations with $\chi\mathbb{R}^r$ in place of g). But now, since $(\gamma\chi\mathbb{R}^r * \tilde{h}_n)(x) = \gamma$ for every n,

$$|(f * \tilde{h}_n)(x) - \gamma| = |\int (f(y)-\gamma)\tilde{h}_n(x-y)\mu(dy)| \le \int |f(y)-\gamma|\tilde{h}_n(x-y)\mu(dy) \le \epsilon$$

whenever $n+1 \ge \frac{1}{\delta}$. As ϵ is arbitrary, $f(x) = \gamma = \lim_{n\to\infty}(f * \tilde{h}_n)(x)$; and this is true for μ-almost every x.

(f) Apply 444T to the indefinite-integral measure $\tilde{h}_n\mu$ over μ defined by $\tilde{h}_n$; use 444Pa for the identification of $(\tilde{h}_n\mu) * f$ with $\tilde{h}_n * f = f * \tilde{h}_n$.

473F Lemma For any measure space (X,Σ,λ) and any non-negative $f_1,\dots,f_k \in \mathcal{L}^0(\lambda)$,

$$\int \prod_{i=1}^k f_i^{1/k} d\lambda \le \prod_{i=1}^k \bigl(\int f_i d\lambda\bigr)^{1/k}.$$

proof Induce on k. Note that we can suppose that every f_i is integrable; for if any $\int f_i$ is zero, then $f_i = 0$ a.e. and the result is trivial; and if all the $\int f_i$ are greater than zero and any of them is infinite, the result is again trivial.

The induction starts with the trivial case $k = 1$. For the inductive step to $k \ge 2$, we have

$$\int \prod_{i=1}^k f_i^{1/k} d\lambda \le \| \prod_{i=1}^{k-1} f_i^{1/k} \|_{k/(k-1)} \| f_k^{1/k} \|_k$$

(by Hölder's inequality, 244E)

$$= \bigl(\int \prod_{i=1}^{k-1} f_i^{1/(k-1)} d\lambda\bigr)^{(k-1)/k} \bigl(\int f_k d\lambda\bigr)^{1/k}$$

$$\le \bigl(\prod_{i=1}^{k-1} (\int f_i d\lambda)^{1/(k-1)}\bigr)^{(k-1)/k} \bigl(\int f_k d\lambda\bigr)^{1/k}$$

(by the inductive hypothesis)

$$= \prod_{i=1}^k (\int f_i d\lambda)^{1/k},$$

as required.

473G Lemma Let (X,Σ,λ) be a σ-finite measure space and $k\ge 2$ an integer. Write λ_k for the product measure on X^k. For $x=(\xi_1,\dots,\xi_k)\in X^k$, $t\in X$ and $1\le i\le k$ set $S_i(x,t)=(\xi'_1,\dots,\xi'_k)$ where $\xi'_i=t$ and $\xi'_j=\xi_j$ for $j\ne i$. Then if $h\in\mathcal{L}^1(\lambda_k)$ is non-negative, and we set $h_i(x)=\int h(S_i(x,t))\lambda(dt)$ whenever this is defined in $\mathbb{R}$, we have

$$\int(\prod_{i=1}^k h_i)^{1/(k-1)}d\lambda_k\le(\int h\,d\lambda_k)^{k/(k-1)}.$$

proof Induce on k.

(a) If $k=2$, we have

$$\begin{aligned}\int h_1\times h_2 d\lambda_k&=\iint\bigl(\int h(\tau_1,\xi_2)d\tau_1\bigr)\bigl(\int h(\xi_1,\tau_2)d\tau_2\bigr)d\xi_1 d\xi_2\\&=\iiiint h(\tau_1,\xi_2)h(\xi_1,\tau_2)d\tau_1 d\tau_2 d\xi_1 d\xi_2\\&=\iint h(\tau_1,\xi_2)d\tau_1 d\xi_2\cdot\iint h(\xi_1,\tau_2)d\tau_2 d\xi_1=\bigl(\int h\,d\lambda_2\bigr)^2\end{aligned}$$

by Fubini's theorem (252B) used repeatedly, because (by 253D) $(\tau_1,\tau_2,\xi_1,\xi_2)\mapsto h(\xi_1,\tau_2)h(\tau_1,\xi_2)$ is λ_4-integrable. (See 251W for a sketch of the manipulations needed to apply 252B, as stated, to the integrals above.)

(b) For the inductive step to $k\ge 3$, argue as follows. For $y\in X^{k-1}$, set $g(y)=\int h(y,t)dt$ whenever this is defined in $\mathbb{R}$, identifying X^k with $X^{k-1}\times X$, so that $g(y)=h_k(y,t)$ whenever either is defined. If $1\le i<k$, we can consider $S_i(y,t)$ for $y\in X^{k-1}$ and $t\in X$, and we have

$$\int g(S_i(y,t))dt=\iint h(S_i(y,t),u)du dt=\int h_i(y,t)dt$$

for almost every $y\in X^{k-1}$. So

$$\begin{aligned}\int(\prod_{i=1}^k h_i)^{1/(k-1)}d\lambda_k&=\iint(\prod_{i=1}^{k-1}h_i(y,t))^{1/(k-1)}g(y)^{1/(k-1)}dt\,\lambda_{k-1}(dy)\\&=\int g(y)^{1/(k-1)}\int(\prod_{i=1}^{k-1}h_i(y,t))^{1/(k-1)}dt\,\lambda_{k-1}(dy)\\&\le\int g(y)^{1/(k-1)}\prod_{i=1}^{k-1}\bigl(\int h_i(y,t)dt\bigr)^{1/(k-1)}\lambda_{k-1}(dy)\end{aligned}$$

(473F)

$$=\int g(y)^{1/(k-1)}\prod_{i=1}^{k-1}g_i(y)^{1/(k-1)}\lambda_{k-1}(dy)$$

(where g_i is defined from g in the same way as h_i is defined from h)

$$\le\bigl(\int g(y)\lambda_{k-1}(dy)\bigr)^{1/(k-1)}\bigl(\int\prod_{i=1}^{k-1}g_i(y)^{1/(k-2)}\lambda_{k-1}(dy)\bigr)^{(k-2)/(k-1)}$$

(by Hölder's inequality again, this time with $\frac{1}{k-1}+\frac{k-2}{k-1}=1$)

$$\le\bigl(\int g(y)\lambda_{k-1}(dy)\bigr)^{1/(k-1)}\cdot\int g(y)\lambda_{k-1}(dy)$$

(by the inductive hypothesis)

$$=\bigl(\int g(y)\lambda_{k-1}(dy)\bigr)^{k/(k-1)}=\bigl(\int h(x)\lambda_k(dx)\bigr)^{k/(k-1)},$$

and the induction proceeds.

473H Gagliardo-Nirenberg-Sobolev inequality Suppose that $f:\mathbb{R}^r\to\mathbb{R}$ is a Lipschitz function with compact support. Then $\|f\|_{r/(r-1)}\le\int\|\operatorname{grad}f\|d\mu$.

proof By 473Cc, $\operatorname{grad}f$ is measurable and bounded, so $\|\operatorname{grad}f\|$ also is; since it must have compact support, it is integrable.

For $1\le i\le r$, $x=(\xi_1,\ldots,\xi_r)\in\mathbb{R}^r$ and $t\in\mathbb{R}$ write $S_i(x,t)=(\xi'_1,\ldots,\xi'_r)$ where $\xi'_i=t$ and $\xi'_j=\xi_j$ for $j\ne i$. Set $h_i(x)=\int_{-\infty}^{\infty}\|\operatorname{grad}f(S_i(x,t))\|dt$ when this is defined, which will be the case for almost every x. Now, whenever $h_i(x)$ is defined,

$$|f(x)|=|f(S_i(x,\xi_i))|=|\int_{-\infty}^{\xi_i}\tfrac{\partial}{\partial t}f(S_i(x,t))dt|\le h_i(x).$$

(Use 225E and the fact that a Lipschitz function on any bounded interval in $\mathbb{R}$ is absolutely continuous.) So $|f|\le_{\text{a.e.}}h_i$ for every $i\le r$. Accordingly

$$\int|f(x)|^{r/(r-1)}\mu(dx)\le\int\prod_{i=1}^r h_i(x)^{1/(r-1)}\mu(dx)\le\bigl(\int\|\operatorname{grad}f(x)\|\mu(dx)\bigr)^{r/(r-1)}$$

by 473G. Raising both sides to the power $(r-1)/r$ we have the result.

473I Lemma For any Lipschitz function $f:B(\mathbf{0},1)\to\mathbb{R}$,

$$\int_{B(\mathbf{0},1)}|f|^{r/(r-1)}d\mu\le\bigl(2^{r+4}\sqrt{r}\int_{B(\mathbf{0},1)}\|\operatorname{grad}f\|+|f|d\mu\bigr)^{r/(r-1)}.$$

proof (a) Set $g(x)=\max(0,2\|x\|^2-1)$ for $x\in B(\mathbf{0},1)$. Then $\operatorname{grad}g$ is defined at every point x such that $\|x\|<1$ and $\|x\|\ne\frac{1}{\sqrt{2}}$, and at all such points $\frac{\partial g}{\partial\xi_i}$ is either 0 or $4\xi_i$ for each i, so that $\|\operatorname{grad}g(x)\|\le4\|x\|\le4$. Hence (or otherwise) g is Lipschitz. So $f_1=f\times g$ is Lipschitz (473Ca).

By Rademacher's theorem again, $\operatorname{grad}f_1$ is defined almost everywhere in $B(\mathbf{0},1)$. Now

$$\int_{B(\mathbf{0},1)}\|\operatorname{grad}f_1\|d\mu=\int_{B(\mathbf{0},1)}\|f(x)\operatorname{grad}g(x)+g(x)\operatorname{grad}f(x)\|\mu(dx)$$

(473Bd)

$$\le\int_{B(\mathbf{0},1)}4|f|+\|\operatorname{grad}f\|d\mu.$$

(b) It will be convenient to have an elementary fact out in the open. Set $\phi(x)=\frac{x}{\|x\|^2}$ for $x\in\mathbb{R}^r\setminus\{\mathbf{0}\}$; note that $\phi^2(x)=x$. Then $\phi\restriction\{x:\|x\|\ge\delta\}$ is Lipschitz, for any $\delta>0$. **P** If $\|x\|=\alpha\ge\delta$, $\|y\|=\beta\ge\delta$, then we have

$$\|\phi(x)-\phi(y)\|^2=\frac{1}{\alpha^4}\|x\|^2-\frac{2}{\alpha^2\beta^2}x\,.\,y+\frac{1}{\beta^4}\|y\|^2$$

$$=\frac{1}{\alpha^2\beta^2}\bigl(\|y\|^2-2x\,.\,y+\|x\|^2\bigr)\le\frac{1}{\delta^4}\|x-y\|^2,$$

so $\frac{1}{\delta^2}$ is a Lipschitz constant for $\phi\restriction\mathbb{R}^r\setminus B(\mathbf{0},\delta)$. **Q**

(c) Set $f_2(x)=f(x)$ if $\|x\|\le1$, $f_1\phi(x)$ if $\|x\|\ge1$. Then f_2 is well-defined (because $f_1(x)=f(x)$ if $\|x\|=1$), is zero outside $B(\mathbf{0},\sqrt{2})$ (because $g(x)=0$ if $\|x\|\le\frac{1}{\sqrt{2}}$), and is Lipschitz. **P** By 473Cb, it will be enough to show that $f_2\restriction F$ is Lipschitz, where $F=\{x:\|x\|\ge1\}$. But (b) shows that $\phi\restriction F$ is 1-Lipschitz, so any Lipschitz constant for f_1 is also a Lipschitz constant for $f_2\restriction F$. **Q**

If $\|x\|>1$, then, for any $i\le r$,

$$\frac{\partial f_2}{\partial\xi_i}(x)=\sum_{j=1}^r\frac{\partial f_1}{\partial\xi_j}(\phi(x))\cdot\frac{\partial}{\partial\xi_i}\Bigl(\frac{\xi_j}{\|x\|^2}\Bigr)=\frac{\partial f_1}{\partial\xi_i}(\phi(x))\cdot\frac{1}{\|x\|^2}-2\sum_{j=1}^r\frac{\partial f_1}{\partial\xi_j}(\phi(x))\cdot\frac{\xi_i\xi_j}{\|x\|^4}$$

$$=\frac{\partial f_1}{\partial\xi_i}(\phi(x))\cdot\frac{1}{\|x\|^2}-\frac{2\xi_i}{\|x\|^4}x\,.\operatorname{grad}f(\phi(x))$$

wherever the right-hand side is defined, that is, wherever all the partial derivatives $\frac{\partial f_1}{\partial \xi_j}(\phi(x))$ are defined. But $H = B(\mathbf{0},1) \setminus \operatorname{dom}(\operatorname{grad} f_1)$ is negligible, and does not meet $\{x : \|x\| < \frac{1}{\sqrt{2}}\}$, so $\phi \restriction H$ is Lipschitz and $\phi[H] = \phi^{-1}[H]$ is negligible (262D); while $\operatorname{grad} f_1(\phi(x))$ is defined whenever $\|x\| > 1$ and $x \notin \phi^{-1}[H]$. So the formula here is valid for almost every $x \in F$, and

$$\begin{aligned}|\frac{\partial f_2}{\partial \xi_i}(x)| &\le \|\operatorname{grad} f_1(\phi(x))\| \cdot \frac{1}{\|x\|^2} + \frac{2|\xi_i|}{\|x\|^4}\|\operatorname{grad} f_1(\phi(x))\|\|x\| \\ &= \|\operatorname{grad} f_1(\phi(x))\|\frac{\|x\|+2|\xi_i|}{\|x\|^3} \le 3\|\operatorname{grad} f_1(\phi(x))\|\end{aligned}$$

for almost every $x \in F$. But (since we know that $\operatorname{grad} f_2$ is defined almost everywhere, by Rademacher's theorem, as usual) we have

$$\|\operatorname{grad} f_2(x)\| \le 3\sqrt{r}\|\operatorname{grad} f_1(\phi(x))\|$$

for almost every $x \in F$.

(d) We are now in a position to estimate

$$\int_F \|\operatorname{grad} f_2\| d\mu = \int_{B(\mathbf{0},\sqrt{2})} \|\operatorname{grad} f_2\| d\mu - \int_{B(\mathbf{0},1)} \|\operatorname{grad} f_2\| d\mu$$

(because $f_2(x) = 0$ if $\|x\| \ge \sqrt{2}$)

$$= \int_1^{\sqrt{2}} \int_{\partial B(\mathbf{0},t)} \|\operatorname{grad} f_2(x)\| \nu(dx) dt$$

(265G, as usual)

$$\le 3\sqrt{r} \int_1^{\sqrt{2}} \int_{\partial B(\mathbf{0},t)} \|\operatorname{grad} f_1(\tfrac{1}{t^2}x)\| \nu(dx) dt$$

(by (b) above)

$$\le 3\sqrt{r} \int_1^{\sqrt{2}} \int_{\partial B(\mathbf{0},1/t)} t^{2r-2} \|\operatorname{grad} f_1(y)\| \nu(dy) dt$$

substituting $x = t^2 y$ in the inner integral; the point being that as the function $y \mapsto t^2 y$ changes all distances by a scalar multiple t^2, it must transform Hausdorff $(r-1)$-dimensional measure by a multiple t^{2r-2}. But now, substituting $s = \frac{1}{t}$ in the outer integral, we have

$$\begin{aligned}\int_F \|\operatorname{grad} f_2\| d\mu &\le 3\sqrt{r} \int_{1/\sqrt{2}}^1 \frac{1}{s^{2r}} \int_{\partial B(\mathbf{0},s)} \|\operatorname{grad} f_1(y)\| \nu(dy) ds \\ &\le 2^r \cdot 3\sqrt{r} \int_{1/\sqrt{2}}^1 \int_{\partial B(\mathbf{0},s)} \|\operatorname{grad} f_1(y)\| \nu(dy) ds \\ &= 2^r \cdot 3\sqrt{r} \int_{B(\mathbf{0},1)} \|\operatorname{grad} f_1\| d\mu \\ &\le 2^{r+2}\sqrt{r} \int_{B(\mathbf{0},1)} 4|f| + \|\operatorname{grad} f\| d\mu\end{aligned}$$

by (a) above.

(e) Accordingly

$$\begin{aligned}\int_{\mathbb{R}^r} \|\operatorname{grad} f_2\| d\mu &= \int_{B(\mathbf{0},1)} \|\operatorname{grad} f\| d\mu + \int_F \|\operatorname{grad} f_2\| d\mu \\ &\le 2^{r+4}\sqrt{r} \int_{B(\mathbf{0},1)} |f| + \|\operatorname{grad} f\| d\mu.\end{aligned}$$

But now we can apply 473H to see that

$$\int_{B(\mathbf{0},1)} |f|^{r/(r-1)} d\mu \le \int |f_2|^{r/r-1} d\mu \le (\int \| \operatorname{grad} f_2 \| d\mu)^{r/(r-1)}$$
$$\le \bigl(2^{r+4}\sqrt{r}\int_{B(\mathbf{0},1)} |f| + \| \operatorname{grad} f \| d\mu\bigr)^{r/(r-1)},$$

as claimed.

473J Lemma Let $f : \mathbb{R}^r \to \mathbb{R}$ be a Lipschitz function. Then

$$\int_{B(y,\delta)} |f(x) - f(z)|\mu(dx) \le \frac{2^r}{r}\delta^r \int_{B(y,\delta)} \| \operatorname{grad} f(x)\| \|x - z\|^{1-r}\mu(dx)$$

whenever $y \in \mathbb{R}^r$, $\delta > 0$ and $z \in B(y, \delta)$.

proof (a) To begin with, suppose that f is smooth. In this case, for any x, $z \in B(y, \delta)$,

$$|f(x) - f(z)| = \bigl|\int_0^1 \frac{d}{dt} f(z + t(x - z))dt\bigr|$$
$$= \bigl|\int_0^1 (x - z)\,.\, \operatorname{grad} f(z + t(x - z))dt\bigr|$$
$$\le \|x - z\| \int_0^1 \| \operatorname{grad} f(z + t(x - z))\| dt.$$

So, for $\eta > 0$,

$$\int_{B(y,\delta)\cap\partial B(z,\eta)} |f(x) - f(z)|\nu(dx)$$
$$\le \eta \int_0^1 \int_{B(y,\delta)\cap\partial B(z,\eta)} \| \operatorname{grad} f(z + t(x - z))\|\nu(dx)dt$$

(grad f is continuous and bounded, and the subspace measure induced by ν on $\partial B(z, \eta)$ is a (quasi-)Radon measure (471E, 471Dh), so its product with Lebesgue measure also is (417T), and there is no difficulty with the change in order of integration)

$$\le \eta \int_0^1 \frac{1}{t^{r-1}} \int_{B(y,\delta)\cap\partial B(z,t\eta)} \| \operatorname{grad} f(w)\|\nu(dw)dt$$

(because if $\phi(x) = z + t(x - z)$, then $\nu\phi^{-1}[E] = \frac{1}{t^{r-1}}\nu E$ whenever ν measures E and $t > 0$, while $\phi(x) \in B(y, \delta)$ whenever $x \in B(y, \delta)$)

$$= \eta^r \int_0^1 \int_{B(y,\delta)\cap\partial B(z,t\eta)} \| \operatorname{grad} f(w)\| \|w - z\|^{1-r}\nu(dw)dt$$
$$= \eta^{r-1} \int_0^\eta \int_{B(y,\delta)\cap\partial B(z,s)} \| \operatorname{grad} f(w)\| \|w - z\|^{1-r}\nu(dw)ds$$

(substituting $s = t\eta$)

$$= \eta^{r-1} \int_{B(y,\delta)\cap B(z,\eta)} \| \operatorname{grad} f(w)\| \|w - z\|^{1-r}\mu(dw).$$

So

$$\begin{aligned}\int_{B(y,\delta)}|f(x)-f(z)|\mu(dx)&=\int_0^{2\delta}\int_{B(y,\delta)\cap\partial B(z,\eta)}|f(x)-f(z)|\nu(dx)d\eta\\&\le\int_0^{2\delta}\eta^{r-1}\int_{B(y,\delta)\cap B(z,\eta)}\|\operatorname{grad}f(w)\|\|w-z\|^{1-r}\mu(dw)d\eta\\&\le\int_0^{2\delta}\eta^{r-1}\int_{B(y,\delta)}\|\operatorname{grad}f(w)\|\|w-z\|^{1-r}\mu(dw)d\eta\\&=\frac{2^r}{r}\delta^r\int_{B(y,\delta)}\|\operatorname{grad}f(w)\|\|w-z\|^{1-r}\mu(dw).\end{aligned}$$

(b) Now turn to the general case in which f is not necessarily differentiable everywhere, but is known to be Lipschitz and bounded. We need to know that $\int_{B(y,\delta)}\|x-z\|^{1-r}\mu(dx)$ is finite; this is because

$$\begin{aligned}\int_{B(y,\delta)}\|x-z\|^{1-r}\mu(dx)\le\int_{B(z,2\delta)}\|x-z\|^{1-r}\mu(dx)&=\int_0^{2\delta}t^{1-r}\nu(\partial B(z,t))dt\\&=\int_0^{2\delta}t^{1-r}r\beta_r t^{r-1}dt=2\delta r\beta_r.\end{aligned}$$

Take the sequence $\langle\tilde h_n\rangle_{n\in\mathbb{N}}$ from 473E. Then $\langle f*\tilde h_n\rangle_{n\in\mathbb{N}}$ converges uniformly to f (473Ed), while $\langle\operatorname{grad}(f*\tilde h_n)\rangle_{n\in\mathbb{N}}=\langle\tilde h_n*\operatorname{grad}f\rangle_{n\in\mathbb{N}}$ (473Dd) is uniformly bounded (473Cc, 473Dg) and converges almost everywhere to $\operatorname{grad}f$ (473Ee). But this means that, setting $f_n=f*\tilde h_n$,

$$\begin{aligned}\int_{B(y,\delta)}|f(x)-f(z)|\mu(dx)&=\lim_{n\to\infty}\int_{B(y,\delta)}|f_n(x)-f_n(z)|\mu(dx)\\&\le\lim_{n\to\infty}\frac{2^r}{r}\delta^r\int_{B(y,\delta)}\|\operatorname{grad}f_n(x)\|\|x-z\|^{1-r}\mu(dx)\end{aligned}$$

(because every f_n is smooth, by 473De)

$$=\frac{2^r}{r}\delta^r\int_{B(y,\delta)}\|\operatorname{grad}f(x)\|\|x-z\|^{1-r}\mu(dx)$$

by Lebesgue's Dominated Convergence Theorem.

(c) Finally, if f is not bounded on the whole of $\mathbb{R}^r$, it is surely bounded on $B(y,\delta)$, so we can apply (b) to the function $x\mapsto\operatorname{med}(-M,f(x),M)$ for a suitable $M\ge0$ to get the result as stated.

473K Poincaré's inequality for balls Let $B\subseteq\mathbb{R}^r$ be a non-trivial closed ball, and $f:B\to\mathbb{R}$ a Lipschitz function. Set $\gamma=\frac{1}{\mu B}\int_B fd\mu$. Then

$$\Bigl(\int_B|f-\gamma|^{r/(r-1)}d\mu\Bigr)^{(r-1)/r}\le c\int_B\|\operatorname{grad}f\|d\mu,$$

where $c=2^{r+4}\sqrt{r}(1+2^{r+1})$.

proof (a) To begin with (down to the end of (b)) suppose that B is the unit ball $B(\mathbf{0},1)$. Then, for any $x\in B$,

$$\begin{aligned}|f(x)-\gamma|&=\frac{1}{\mu B}\Bigl|\int_B f(x)-f(z)\mu(dz)\Bigr|\\&\le\frac{1}{\mu B}\int_B|f(x)-f(z)|\mu(dz)\\&\le\frac{2^r}{r}\cdot\frac{1}{\mu B}\int_B\|\operatorname{grad}f(z)\|\|x-z\|^{1-r}\mu(dz),\end{aligned}$$

by 473J. Also, for any $z \in B$,

$$\begin{aligned}\int_{B(z,2)} \|x-z\|^{1-r}\mu(dx) &= \int_0^2 \int_{\partial B(z,t)} \|x-z\|^{1-r}\nu(dx)dt \\ &= \int_0^2 t^{1-r}\nu(\partial B(z,t))dt = \int_0^2 t^{1-r}t^{r-1}\nu S_{r-1}dt = 2r\beta_r.\end{aligned}$$

So

$$\begin{aligned}\int_B |f(x)-\gamma|\mu(dx) &\le \frac{2^r}{r}\cdot\frac{1}{\mu B}\int_B\int_B \|\operatorname{grad} f(z)\|\|x-z\|^{1-r}\mu(dz)\mu(dx) \\ &= \frac{2^r}{r\beta_r}\int_B\int_B \|\operatorname{grad} f(z)\|\|x-z\|^{1-r}\mu(dx)\mu(dz) \\ &\le \frac{2^r}{r\beta_r}\int_B \|\operatorname{grad} f(z)\|\int_{B(z,2)} \|x-z\|^{1-r}\mu(dx)\mu(dz) \\ &\le 2^{r+1}\int_B \|\operatorname{grad} f(z)\|\mu(dz).\end{aligned}$$

(b) Now apply 473I to $g = f(x)-\gamma$. We have

$$\begin{aligned}\int_B |f-\gamma|^{r/(r-1)}d\mu &\le \bigl(2^{r+4}\sqrt{r}\int_B \|\operatorname{grad} f\| + |g|d\mu\bigr)^{r/(r-1)} \\ &\le \bigl(2^{r+4}\sqrt{r}(1+2^{r+1})\int_B \|\operatorname{grad} f\|d\mu\bigr)^{r/(r-1)}\end{aligned}$$

(by (a))

$$= \bigl(c\int_B \|\operatorname{grad} f\|d\mu\bigr)^{r/(r-1)}.$$

(c) For the general case, express B as $B(y,\delta)$, and set $h(x) = f(y+\delta x)$ for $x \in B(\mathbf{0},1)$. Then $\operatorname{grad} h(x) = \delta \operatorname{grad} f(y+\delta x)$ for almost every $x \in B(\mathbf{0},1)$. Now

$$\int_{B(\mathbf{0},1)} h\,d\mu = \frac{1}{\delta^r}\int_{B(y,\delta)} f d\mu,$$

so

$$\frac{1}{\mu B(\mathbf{0},1)}\int_{B(\mathbf{0},1)} h\,d\mu = \frac{1}{\mu B(y,\delta)}\int_{B(y,\delta)} f d\mu = \gamma.$$

We therefore have

$$\begin{aligned}\int_{B(y,\delta)} |f-\gamma|^{r/(r-1)}d\mu &= \delta^r\int_{B(\mathbf{0},1)} |h-\gamma|^{r/(r-1)}d\mu \\ &\le \delta^r\bigl(c\int_{B(\mathbf{0},1)} \|\operatorname{grad} h\|d\mu\bigr)^{r/(r-1)}\end{aligned}$$

(by (a)-(b) above)

$$\begin{aligned}&= \delta^r\bigl(\frac{\delta c}{\delta^r}\int_{B(y,\delta)} \|\operatorname{grad} f\|d\mu\bigr)^{r/(r-1)} \\ &= \bigl(c\int_{B(y,\delta)} \|\operatorname{grad} f\|d\mu\bigr)^{r/(r-1)}.\end{aligned}$$

Raising both sides to the power $(r-1)/r$ we have the result as stated.

Remark As will be plain from the way in which the proof here is constructed, there is no suggestion that the formula offered for c gives anything near the best possible value.

473L Corollary Let $B\subseteq\mathbb{R}^r$ be a non-trivial closed ball, and $f:B\to[0,1]$ a Lipschitz function. Set

$$F_0=\{x:x\in B,\,f(x)\le\tfrac{1}{4}\},\quad F_1=\{x:x\in B,\,f(x)\ge\tfrac{3}{4}\}.$$

Then

$$\bigl(\min(\mu F_0,\mu F_1)\bigr)^{(r-1)/r}\le 4c\int_B\|\operatorname{grad}f\|d\mu,$$

where $c=2^{r+4}\sqrt{r}(1+2^{r+1})$.

proof Setting $\gamma=\frac{1}{\mu B}\int_B f d\mu$,

$$\begin{aligned}\int_B|f-\gamma|^{r/(r-1)}d\mu&\ge\frac{1}{4^{r/(r-1)}}\mu F_0\text{ if }\gamma\ge\frac{1}{2},\\&\ge\frac{1}{4^{r/(r-1)}}\mu F_1\text{ if }\gamma\le\frac{1}{2}.\end{aligned}$$

So 473K tells us that

$$\frac{1}{4}\bigl(\min(\mu F_0,\mu F_1)\bigr)^{(r-1)/r}\le c\int_B\|\operatorname{grad}f\|d\mu,$$

as required.

473M The case $r=1$ The general rubric for this section declares that r is taken to be at least 2, which is clearly necessary for the formula in 473K to be appropriate. For the sake of an application in the next section, however, I mention the elementary corresponding result when $r=1$. In this case, B is just a closed interval, and $\operatorname{grad}f$ is the ordinary derivative of f; interpreting $(\int_B|f-\gamma|^{r/(r-1)})^{(r-1)/r}$ as $\|f\times\chi B-\gamma\chi B\|_{r/(r-1)}$, it is natural to look at

$$\|f\times\chi B-\gamma\chi B\|_\infty=\sup_{x\in B}|f(x)-\gamma|\le\sup_{x,y\in B}|f(x)-f(y)|\le\int_B|f'|d\mu,$$

giving a version of 473K for $r=1$. We see that the formula for c remains valid in the case $r=1$, with a good deal to spare. As for 473L, if $\int_B|f'|<\frac{1}{2}$ then at least one of F_0, F_1 must be empty.

473X Basic exercises (a) Set $f(x)=\max(0,-\ln\|x\|)$, $f_k(x)=\min(k,f(x))$ for $x\in\mathbb{R}^2\setminus\{0\}$, $k\in\mathbb{N}$. Show that $\lim_{k\to\infty}\|f-f_k\|_2=\lim_{k\to\infty}\|\operatorname{grad}f-\operatorname{grad}f_k\|_1=0$, so that all the inequalities 473H-473L are valid for f.

(b) Let $k\in[1,r]$ be an integer, and set $m=\frac{(r-1)!}{(k-1)!(r-k)!}$. Let $e_1,\dots,e_r$ be the standard orthonormal basis of $\mathbb{R}^r$ and $\mathcal{J}$ the family of subsets of $\{1,\dots,r\}$ with k members. For $J\in\mathcal{J}$ let V_J be the linear span of $\{e_i:i\in J\}$, $\pi_J:\mathbb{R}^r\to V_J$ the orthogonal projection and ν_J the normalized k-dimensional Hausdorff measure on V_J. Show that if $A\subseteq\mathbb{R}^r$ then $(\mu^*A)^m\le\prod_{J\in\mathcal{J}}\nu_J^*\pi_J[A]$. (*Hint*: start with $A\subseteq[0,1]^r$ and note that $([0,1]^r)^m$ can be identified with $\prod_{J\in\mathcal{J}}[0,1]^J$.)

473Y Further exercises (a) Let $D\subseteq\mathbb{R}^r$ be any set and $\phi:D\to\mathbb{R}^s$ any function. Show that $D_0=\{x:x\in D,\,\phi$ is differentiable at $x\}$ is a Borel subset of $\mathbb{R}^r$, and that the derivative of ϕ is a Borel measurable function. (Compare 225J.)

473 Notes and comments The point of all the inequalities 473H-473L is that they bound some measure of variance of a function f by the integral of $\|\operatorname{grad}f\|$. If $r=2$, indeed, we are looking at $\|f\|_2$ (473H) or $\int_B|f|^2$ (473I) or something essentially equal to the variance of probability theory (473K). In higher dimensions we need to look at $\|\ \|_{r/(r-1)}$ in place of $\|\ \|_2$, and when $r=1$ we can interpret the inequalities in terms of the supremum norm $\|\ \|_\infty$ (473M). In all cases we want to develop inequalities which will enable us to discuss a function in terms of its first derivative. In one dimension, this is the familiar Fundamental Theorem of Calculus (Chapter 22). We find there a straightforward criterion ('absolute continuity') to determine whether a given function of one variable is an indefinite integral, and that if so it is the indefinite integral of its own derivative. Even in two dimensions, this simplicity disappears. The essential problem is that a function can be the indefinite integral of an integrable gradient function without being bounded (473Xa). The principal results of this section are stated for Lipschitz functions, but in fact they apply much more widely. The argument suggested in 473Xa involves approximating the unbounded function f by Lipschitz functions f_k in a sharp enough sense to make it possible to read off all the inequalities for f from the corresponding inequalities for the f_k. This idea leads naturally to the concept of 'Sobolev space', which I leave on one side for the moment; see Evans & Gariepy 92, chap. 4, for details.

Version of 17.11.12

474 The distributional perimeter

The next step is a dramatic excursion, defining (for appropriate sets E) a perimeter *measure* for which a version of the Divergence Theorem is true (474E). I begin the section with elementary notes on the divergence of a vector field (474B-474C). I then define 'locally finite perimeter' (474D), 'perimeter measure' and 'outward normal' (474F) and 'reduced boundary' (474G). The definitions rely on the Riesz representation theorem, and we have to work very hard to relate them to any geometrically natural idea of 'boundary'. Even half-spaces (474I) demand some attention. From Poincaré's inequality (473K) we can prove isoperimetric inequalities for perimeter measures (474L). With some effort we can locate the reduced boundary as a subset of the topological boundary (474Xc), and obtain asymptotic inequalities on the perimeter measures of small balls (474N). With much more effort we can find a geometric description of outward normal functions in terms of 'Federer exterior normals' (474R), and get a tight asymptotic description of the perimeter measures of small balls (474S). I end with the Compactness Theorem for sets of bounded perimeter (474T).

474A Notation I had better repeat some of the notation from §473. $r \geq 2$ is a fixed integer. μ is Lebesgue measure on $\mathbb{R}^r$, and $\beta_r = \mu B(\mathbf{0}, 1)$ is the volume of the unit ball. $S_{r-1} = \partial B(\mathbf{0}, 1)$ is the unit sphere. ν is normalized $(r-1)$-dimensional Hausdorff measure on $\mathbb{R}^r$. We shall sometimes need to look at Lebesgue measure on $\mathbb{R}^{r-1}$, which I will denote μ_{r-1}. As in §473, I will use Greek letters to represent coordinates, so that $x = (\xi_1, \dots, \xi_r)$ for $x \in \mathbb{R}^r$, etc., and β_r will be the r-dimensional volume of the unit ball in $\mathbb{R}^r$.

$\mathcal{D}$ is the set of smooth functions $f : \mathbb{R}^r \to \mathbb{R}$ with compact support; $\mathcal{D}_r$ the set of smooth functions $\phi : \mathbb{R}^r \to \mathbb{R}^r$ with compact support, that is, the set of functions $\phi = (\phi_1, \dots, \phi_r) : \mathbb{R}^r \to \mathbb{R}^r$ such that $\phi_i \in \mathcal{D}$ for every i.

I continue to use the sequence $\langle \tilde{h}_n \rangle_{n\in\mathbb{N}}$ from 473E; these functions all belong to $\mathcal{D}$, are non-negative everywhere and zero outside $B(\mathbf{0}, \frac{1}{n+1})$, are even, and have integral 1.

474B The divergence of a vector field (a) For a function ϕ from a subset of $\mathbb{R}^r$ to $\mathbb{R}^r$, write $\operatorname{div} \phi = \sum_{i=1}^r \frac{\partial \phi_i}{\partial \xi_i}$, where $\phi = (\phi_1, \dots, \phi_r)$; for definiteness, let us take the domain of $\operatorname{div} \phi$ to be the set of points at which ϕ is differentiable (in the strict sense of 262Fa). Note that $\operatorname{div} \phi \in \mathcal{D}$ for every $\phi \in \mathcal{D}_r$. We need the following elementary facts.

(b) If $f : \mathbb{R}^r \to \mathbb{R}$ and $\phi : \mathbb{R}^r \to \mathbb{R}^r$ are functions, then $\operatorname{div}(f \times \phi) = \phi \,\boldsymbol{.}\, \operatorname{grad} f + f \times \operatorname{div} \phi$ at any point at which f and ϕ are both differentiable. (Use 473Bc; compare 473Bd.)

(c) If $\phi : \mathbb{R}^r \to \mathbb{R}^r$ is a Lipschitz function with compact support, then $\int \operatorname{div} \phi \, d\mu = 0$. **P** $\operatorname{div} \phi$ is defined almost everywhere (by Rademacher's theorem, 262Q), measurable (473Be), bounded (473Cc) and with compact support, so

$$\int \operatorname{div} \phi \, d\mu = \sum_{i=1}^r \int \frac{\partial \phi_i}{\partial \xi_i} d\mu$$

is defined in $\mathbb{R}$. For each $i \leq r$, Fubini's theorem tells us that we can replace integration with respect to μ by a repeated integral, in which the inner integral is

$$\int_{-\infty}^{\infty} \frac{\partial \phi_i}{\partial \xi_i}(\xi_1, \dots, \xi_r) d\xi_i = 0$$

because $\phi_i(\xi_1,\dots,\xi_r)=0$ whenever $|\xi_i|$ is large enough. So $\int\frac{\partial\phi_i}{\partial\xi_i}d\mu$ also is zero. Summing over i, we have the result. **Q**

(d) If $\phi:\mathbb{R}^r\to\mathbb{R}^r$ and $f:\mathbb{R}^r\to\mathbb{R}$ are Lipschitz functions, one of which has compact support, then $f\times\phi$ is Lipschitz. **P** Take $n\in\mathbb{N}$ such that $f(x)\phi(x)=\mathbf{0}$ for $\|x\|>n$, and $\gamma\ge 0$ such that $|f(x)-f(y)|\le\gamma\|x-y\|$ and $\|\phi(x)-\phi(y)\|\le\gamma\|x-y\|$ for all x, $y\in\mathbb{R}^r$, while also $|f(x)|\le\gamma$ whenever $\|x\|\le n+1$ and $\|\phi(x)\|\le\gamma$ whenever $\|x\|\le n+1$. If x, $y\in\mathbb{R}^r$ then

— if $\|x\|\le n+1$ and $\|y\|\le n+1$,

$$\|f(x)\phi(x)-f(y)\phi(y)\|\le|f(x)|\|\phi(x)-\phi(y)\|+\|\phi(y)\||f(x)-f(y)|\le 2\gamma^2\|x-y\|;$$

— if $\|x\|\le n$ and $\|y\|>n+1$,

$$|f(x)\phi(x)-f(y)\phi(y)\|=|f(x)|\|\phi(x)\|\le\gamma^2\le\gamma^2\|x-y\|;$$

— if $\|x\|>n$ and $\|y\|>n$, $|f(x)\phi(x)-f(y)\phi(y)\|=0$.

So $2\gamma^2$ is a Lipschitz constant for $f\times\phi$. **Q**

It follows that

$$\int\phi\,\boldsymbol{.}\,\operatorname{grad}f\,d\mu+\int f\times\operatorname{div}\phi\,d\mu=0.$$

P f and ϕ and $f\times\phi$ are all differentiable almost everywhere. So

$$\int\phi\,\boldsymbol{.}\,\operatorname{grad}f\,d\mu+\int f\times\operatorname{div}\phi\,d\mu=\int\operatorname{div}(f\times\phi)d\mu=0$$

by (b) and (c) above. **Q**

(e) If $f\in\mathcal{L}^\infty(\mu)$, $g\in\mathcal{L}^1(\mu)$ is even (that is, $g(-x)$ is defined and equal to $g(x)$ for every $x\in\operatorname{dom}g$), and $\phi:\mathbb{R}^r\to\mathbb{R}^r$ is a Lipschitz function with compact support, then $\int(f*g)\times\operatorname{div}\phi=\int f\times\operatorname{div}(g*\phi)$, where $g*\phi=(g*\phi_1,\dots,g*\phi_r)$. **P** For each i,

$$\int(f*g)\times\frac{\partial\phi_i}{\partial\xi_i}d\mu=\iint f(x)g(y)\frac{\partial\phi_i}{\partial\xi_i}(x+y)\mu(dy)\mu(dx)$$

(255G/444Od)

$$=\iint f(x)g(-y)\frac{\partial\phi_i}{\partial\xi_i}(x+y)\mu(dy)\mu(dx)$$

(because g is even)

$$=\int f\times(g*\frac{\partial\phi_i}{\partial\xi_i})d\mu=\int f\times\frac{\partial}{\partial\xi_i}(g*\phi_i)d\mu$$

as in 473Dd. Now take the sum over i of both sides. **Q**

474C Invariance under isometries: Proposition Suppose that $T:\mathbb{R}^r\to\mathbb{R}^r$ is an isometry, and that ϕ is a function from a subset of $\mathbb{R}^r$ to $\mathbb{R}^r$. Then

$$\operatorname{div}(T^{-1}\phi T)=(\operatorname{div}\phi)T.$$

proof Set $z=T(\mathbf{0})$. By 4A4Jb, the isometry $x\mapsto T(x)-z$ is linear and preserves inner products, so there is an orthogonal matrix S such that $T(x)=z+S(x)$ for every $x\in\mathbb{R}^r$. Now suppose that $x\in\mathbb{R}^r$ is such that $(\operatorname{div}\phi)(T(x))$ is defined. Then $T(y)-T(x)-S(y-x)=0$ for every y, so T is differentiable at x, with derivative S, and ϕT is differentiable at x, with derivative DS, where D is the derivative of ϕ at $T(x)$, by 473Bc. Also $T^{-1}(y)=S^{-1}(y-z)$ for every y, so T^{-1} is differentiable at $\phi(T(x))$ with derivative S^{-1}, and $T^{-1}\phi T$ is differentiable at x, with derivative $S^{-1}DS$. Now if D is $\langle\delta_{ij}\rangle_{1\le i,j\le r}$ and S is $\langle\sigma_{ij}\rangle_{1\le i,j\le r}$ and $S^{-1}DS$ is $\langle\tau_{ij}\rangle_{1\le i,j\le r}$, then S^{-1} is the transpose $\langle\sigma_{ji}\rangle_{1\le i,j\le r}$ of S, because S is orthogonal, so

$$\operatorname{div}(T^{-1}\phi T)(x)=\sum_{i=1}^r\tau_{ii}=\sum_{i=1}^r\sum_{j=1}^r\sigma_{ji}\sum_{k=1}^r\delta_{jk}\sigma_{ki}$$
$$=\sum_{j=1}^r\sum_{k=1}^r\delta_{jk}\sum_{i=1}^r\sigma_{ji}\sigma_{ki}=\sum_{j=1}^r\delta_{jj}=\operatorname{div}\phi(T(x))$$

because $\sum_{i=1}^{r}\sigma_{ji}\sigma_{jk}=1$ if $j=k$ and 0 otherwise. If $\operatorname{div}(T^{-1}\phi T)(x)$ is defined, then (because T^{-1} also is an isometry)

$$(\operatorname{div}\phi)(T(x))=\operatorname{div}(TT^{-1}\phi TT^{-1})(T(x))=\operatorname{div}(T^{-1}\phi T)(T^{-1}T(x))=\operatorname{div}(T^{-1}\phi T)(x).$$

So the functions $\operatorname{div}(T^{-1}\phi T)$ and $(\operatorname{div}\phi)T$ are identical.

474D Locally finite perimeter: Definition Let $E\subseteq\mathbb{R}^r$ be a Lebesgue measurable set. Its **perimeter** $\operatorname{per}E$ is

$$\sup\{|\int_E\operatorname{div}\phi\,d\mu|:\phi:\mathbb{R}^r\to B(\mathbf{0},1)\text{ is a Lipschitz function with compact support}\}$$

(allowing ∞). A set $E\subseteq\mathbb{R}^r$ has **locally finite perimeter** if it is Lebesgue measurable and

$$\sup\{|\int_E\operatorname{div}\phi\,d\mu|:\phi:\mathbb{R}^r\to\mathbb{R}^r\text{ is a Lipschitz function, }\|\phi\|\le\chi B(\mathbf{0},n)\}$$

is finite for every $n\in\mathbb{N}$. Of course a Lebesgue measurable set with finite perimeter also has locally finite perimiter.

474E Theorem Suppose that $E\subseteq\mathbb{R}^r$ has locally finite perimeter.

(i) There are a Radon measure λ_E^∂ on $\mathbb{R}^r$ and a Borel measurable function $\psi:\mathbb{R}^r\to S_{r-1}$ such that

$$\int_E\operatorname{div}\phi\,d\mu=\int\phi\,\boldsymbol{.}\,\psi\,d\lambda_E^\partial$$

for every Lipschitz function $\phi:\mathbb{R}^r\to\mathbb{R}^r$ with compact support.

(ii) This formula uniquely determines λ_E^∂, which can also be defined by saying that

$$\lambda_E^\partial(G)=\sup\{|\int_E\operatorname{div}\phi\,d\mu|:\phi:\mathbb{R}^r\to\mathbb{R}^r\text{ is Lipschitz, }\|\phi\|\le\chi G\}$$

whenever $G\subseteq\mathbb{R}^r$ is open.

(iii) If $\hat\psi$ is another function defined λ_E^∂-a.e. and satisfying the formula in (i), then $\hat\psi$ and ψ are equal λ_E^∂-almost everywhere.

proof (a)(i) For each $l\in\mathbb{N}$, set

$$\gamma_l=\sup\{|\int_E\operatorname{div}\phi\,d\mu|:\phi:\mathbb{R}^r\to\mathbb{R}^r\text{ is Lipschitz, }\|\phi\|\le\chi B(\mathbf{0},l)\}.$$

If $f:\mathbb{R}^r\to\mathbb{R}$ is a Lipschitz function and $f(x)=0$ for $\|x\|\ge l$, then $|\int_E\frac{\partial f}{\partial\xi_i}d\mu|\le\gamma_l\|f\|_\infty$ for every $i\le r$. **P** It is enough to consider the case $\|f\|_\infty=1$, since the result is trivial if $\|f\|_\infty=0$, and otherwise we can look at an appropriate scalar multiple of f. In this case, set $\phi(x)=f(x)e_i$ for every x, where e_i is the unit vector $(0,\dots,0,1,0,\dots,0)$ with a 1 in the ith place. Then ϕ is Lipschitz and $\|\phi\|=|f|\le\chi B(\mathbf{0},l)$, so

$$|\int_E\frac{\partial f}{\partial\xi_i}d\mu|=|\int_E\operatorname{div}\phi\,d\mu|\le\gamma_l.\ \mathbf{Q}$$

(ii) Write C_k for the space of continuous functions with compact support from $\mathbb{R}^r$ to $\mathbb{R}$. By 473Dc and 473De, $f*\tilde h_n\in\mathcal{D}$ for every $f\in C_k$ and $n\in\mathbb{N}$. Now the point is that

$$L_i(f)=\lim_{n\to\infty}\int_E\frac{\partial}{\partial\xi_i}(f*\tilde h_n)d\mu$$

is defined whenever $f\in C_k$ and $i\le r$. **P** Applying 473Ed, we see that $\|f-f*\tilde h_n\|_\infty\to0$ as $n\to\infty$. Let l be such that $f(x)=0$ for $\|x\|\ge l$. Then $\|(f*\tilde h_m)-(f*\tilde h_n)\|_\infty\to0$ as $m,n\to\infty$, while all the $f*\tilde h_m$ are zero outside $B(\mathbf{0},l+1)$ (473Dc), so that

$$\Bigl|\int_E\frac{\partial}{\partial\xi_i}(f*\tilde h_m)d\mu-\int_E\frac{\partial}{\partial\xi_i}(f*\tilde h_n)d\mu\Bigr|\le\gamma_{l+1}\|(f*\tilde h_m)-(f*\tilde h_n)\|_\infty\to0$$

as $m,n\to\infty$. Thus $\langle\int_E\frac{\partial}{\partial\xi_i}(f*\tilde h_n)d\mu\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence and must have a limit. **Q**

(iii) If $f\in C_k$ is Lipschitz and zero outside $B(\mathbf{0},l)$, then

$$|\int_E\frac{\partial f}{\partial\xi_i}d\mu-\int_E\frac{\partial}{\partial\xi_i}(f*\tilde h_n)d\mu|\le\gamma_{l+1}\|f-f*\tilde h_n\|_\infty\to0$$

as $n \to \infty$, and $L_i(f) = \int_E \frac{\partial f}{\partial \xi_i} d\mu$. Consequently $|L_i(f)| \le \gamma_l \|f\|_\infty$.

(b) Because all the functionals $f \mapsto \int_E \frac{\partial}{\partial \xi_i}(f * \tilde{h}_n) d\mu$ are linear, L_i is linear. Moreover, by the last remark in (a-iii), it is order-bounded when regarded as a linear functional on the Riesz space C_k, so is expressible as a difference $L_i^+ - L_i^-$ of positive linear functionals (356B).

By the Riesz Representation Theorem (436J), we have Radon measures λ_i^+, λ_i^- on $\mathbb{R}^r$ such that $L_i^+(f) = \int f d\lambda_i^+$, $L_i^-(f) = \int f d\lambda_i^-$ for every $f \in C_k$. Let $\hat{\lambda}$ be the sum $\sum_{i=1}^r \lambda_i^+ + \lambda_i^-$, so that $\hat{\lambda}$ is a Radon measure (416De) and every λ_i^+, λ_i^- is an indefinite-integral measure over $\hat{\lambda}$ (416Sb).

For each $i \le r$, let g_i^+, g_i^- be Radon-Nikodým derivatives of λ_i^+, λ_i^- with respect to $\hat{\lambda}$. Adjusting them on a $\hat{\lambda}$-negligible set if necessary, we may suppose that they are all bounded non-negative Borel measurable functions from $\mathbb{R}^r$ to $\mathbb{R}$. (Recall from 256C that $\hat{\lambda}$ must be the completion of its restriction to the Borel σ-algebra.) Set $g_i = g_i^+ - g_i^-$ for each i. Then

$$\begin{aligned}\int_E \frac{\partial f}{\partial \xi_i} d\mu = L_i^+(f) - L_i^-(f) &= \int f d\lambda_i^+ - \int f d\lambda_i^- \\ = \int f \times g_i^+ d\hat{\lambda} &- \int f \times g_i^- d\hat{\lambda} = \int f \times g_i d\hat{\lambda}\end{aligned}$$

for every Lipschitz function f with compact support (235K). Set $g = \sqrt{\sum_{i=1}^r g_i^2}$. For $i \le r$, set $\psi_i(x) = \frac{g_i(x)}{g(x)}$ when $g(x) \ne 0$, $\frac{1}{\sqrt{r}}$ when $g(x) = 0$, so that $\psi = (\psi_1, \dots, \psi_r) : \mathbb{R}^r \to S_{r-1}$ is Borel measurable. Let λ_E^∂ be the indefinite-integral measure over $\hat{\lambda}$ defined by g; then λ_E^∂ is a Radon measure on $\mathbb{R}^r$ (256E/416Sa).

(c) Now take any Lipschitz function $\phi : \mathbb{R}^r \to \mathbb{R}^r$ with compact support. Express it as $(\phi_1, \dots, \phi_r)$ where $\phi_i : \mathbb{R}^r \to \mathbb{R}$ is a Lipschitz function with compact support for each i. Then

$$\begin{aligned}\int_E \operatorname{div} \phi \, d\mu &= \sum_{i=1}^r \int_E \frac{\partial \phi_i}{\partial \xi_i} d\mu = \sum_{i=1}^r L_i(\phi_i) \\ &= \sum_{i=1}^r L_i^+(\phi_i) - \sum_{i=1}^r L_i^-(\phi_i) = \sum_{i=1}^r \int \phi_i d\hat{\lambda}_i^+ - \sum_{i=1}^r \int \phi_i d\hat{\lambda}_i^- \\ &= \sum_{i=1}^r \int \phi_i \times g_i^+ d\hat{\lambda} - \sum_{i=1}^r \int \phi_i \times g_i^- d\hat{\lambda}\end{aligned}$$

(by 235K again)

$$= \sum_{i=1}^r \int \phi_i \times g_i d\hat{\lambda} = \sum_{i=1}^r \int \phi_i \times \psi_i d\lambda_E^\partial$$

(235K once more, because $\psi_i \times g = g_i$)

$$= \int \phi \cdot \psi \, d\lambda_E^\partial.$$

So we have λ_E^∂ and ψ satisfying (i).

(d) Now suppose that $G \subseteq \mathbb{R}^r$ is open. If $\phi : \mathbb{R}^r \to \mathbb{R}^r$ is a Lipschitz function with compact support and $\|\phi\| \le \chi G$, then

$$|\int_E \operatorname{div} \phi \, d\mu| = |\int \phi \cdot \psi \, d\lambda_E^\partial| \le \int \|\phi\| d\lambda_E^\partial \le \lambda_E^\partial(G).$$

On the other hand, if $\gamma < \lambda_E^\partial(G)$, let $G_0 \subseteq G$ be a bounded open set such that $\gamma < \lambda_E^\partial(G_0)$, and set $\epsilon = \frac{1}{3}(\lambda_E^\partial(G_0) - \gamma)$. Let $K \subseteq G_0$ be a compact set such that $\lambda_E^\partial(G_0 \setminus K) \le \epsilon$. Let $\delta > 0$ be such that $\|x - y\| \ge 2\delta$ whenever $y \in K$ and $x \in \mathbb{R}^r \setminus G_0$, and set $H = \{x : \inf_{y \in K} \|x - y\| < \delta\}$. Now there are $f_1, \dots, f_r \in C_k$ such that

$$\sum_{i=1}^r f_i^2 \le \chi H, \quad \int \sum_{i=1}^r f_i \times \psi_i d\lambda_E^\partial \ge \gamma.$$

P For each $i \le r$, we can find a sequence $\langle g_{mi} \rangle_{m\in\mathbb{N}}$ in C_k such that $\int |g_{mi} - (\psi_i \times \chi K)| d\lambda_E^\partial \le 2^{-m}$ for every $m \in \mathbb{N}$ (416I); multiplying the g_{mi} by a function which takes the value 1 on K and 0 outside H if necessary, we can suppose that $g_{mi}(x) = 0$ for $x \notin H$. Set

$$f_{mi} = \frac{g_{mi}}{\max(1, \sqrt{\sum_{j=1}^r g_{mj}^2})} \in C_k$$

for each m and i. Now $\lim_{m\to\infty} f_{mi}(x) = \psi_i(x)$ for every $i \le r$ whenever $\lim_{m\to\infty} g_{mi}(x) = \psi_i(x)$ for every $i \le r$, which is the case for λ_E^∂-almost every $x \in K$. Also $\sum_{i=1}^r f_{mi}^2 \le \chi H$ for every m, so $|\sum_{i=1}^r f_{mi} \times \psi_i| \le \chi H$ for every m, while

$$\lim_{m\to\infty} \sum_{i=1}^r \int_K f_{mi} \times \psi_i d\lambda_E^\partial = \sum_{i=1}^r \int_K \psi_i^2 d\lambda_E^\partial = \lambda_E^\partial(K).$$

At the same time,

$$|\sum_{i=1}^r \int_{\mathbb{R}^r \setminus K} f_{mi} \times \psi_i d\lambda_E^\partial| \le \lambda_E^\partial(H \setminus K) \le \epsilon$$

for every m, so

$$\sum_{i=1}^r \int f_{mi} \times \psi_i d\lambda_E^\partial \ge \lambda_E^\partial(G_0) - 3\epsilon = \gamma$$

for all m large enough, and we may take $f_i = f_{mi}$ for such an m. **Q**

Now, for $n \in \mathbb{N}$, set

$$\phi_n = (f_1 * \tilde{h}_n, \dots, f_r * \tilde{h}_n) \in \mathcal{D}_r.$$

For all n large enough, we shall have $\|x - y\| \ge \frac{1}{n+1}$ for every $x \in \mathbb{R}^r \setminus G_0$ and $y \in H$, so that $\phi_n(x) = 0$ if $x \notin G_0$. By 473Dg,

$$\|\phi_n(x)\| \le \sup_{y\in\mathbb{R}} \sqrt{\sum_{i=1}^r f_i(y)^2} \le 1$$

for every x and n, so that $\|\phi_n\| \le \chi G_0$ for all n large enough. Next, $\lim_{n\to\infty} \phi_n(x) = (f_1(x), \dots, f_r(x))$ for every $x \in \mathbb{R}^r$ (473Ed), so

$$\int_E \operatorname{div} \phi_n d\mu = \int \phi_n \cdot \psi \, d\lambda_E^\partial \to \int \sum_{i=1}^r f_i \times \psi_i d\lambda_E^\partial \ge \gamma$$

as $n \to \infty$, by Lebesgue's Dominated Convergence Theorem. As γ is arbitrary,

$$\begin{aligned}\lambda_E^\partial(G) &\le \sup\{\int_E \operatorname{div} \phi \, d\mu : \phi \in \mathcal{D}_r, \|\phi\| \le \chi G\} \\ &\le \sup\{\int_E \operatorname{div} \phi \, d\mu : \phi \text{ is Lipschitz}, \|\phi\| \le \chi G\}\end{aligned}$$

and we have equality.

(e) Thus λ_E^∂ must satisfy (ii). By 416Eb, it is uniquely defined. Now suppose that $\hat{\psi}$ is another function from a λ_E^∂-conegligible set to $\mathbb{R}^r$ and satisfies (i). Then

$$\int \phi \cdot \psi \, d\lambda_E^\partial = \int \phi \cdot \hat{\psi} \, d\lambda_E^\partial$$

for every Lipschitz function $\phi : \mathbb{R}^r \to \mathbb{R}^r$ with compact support. Take any $i \le r$ and any compact set $K \subseteq \mathbb{R}^r$. For $m \in \mathbb{N}$, set $f_m(x) = \max(0, 1 - 2^m \inf_{y\in K} \|y - x\|)$ for $x \in \mathbb{R}^r$, so that $\langle f_m \rangle_{m\in\mathbb{N}}$ is a sequence of Lipschitz functions with compact support and $\lim_{m\to\infty} f_m = \chi K$. Set

$$\phi_m = (0, \dots, f_m, \dots, 0),$$

where the non-zero term is in the ith position. Then

$$\begin{aligned}\int_K \psi_i d\lambda_E^\partial &= \lim_{m\to\infty} \int f_m \times \psi_i d\lambda_E^\partial = \lim_{m\to\infty} \int \phi_m \cdot \psi \, d\lambda_E^\partial \\ &= \lim_{m\to\infty} \int \phi_m \cdot \hat{\psi} \, d\lambda_E^\partial = \int_K \hat{\psi}_i d\lambda_E^\partial.\end{aligned}$$

By the Monotone Class Theorem (136C), or otherwise, $\int_F \hat{\psi}_i d\lambda^\partial_E = \int_F \psi_i d\lambda^\partial_E$ for every bounded Borel set F, so that $\hat{\psi}_i = \psi_i$ λ^∂_E-a.e.; as i is arbitrary, $\psi = \hat{\psi}$ λ^∂_E-a.e. This completes the proof.

474F Definitions In the context of 474E, I will call λ^∂_E the **perimeter measure** of E, and if ψ is a function from a λ^∂_E-conegligible subset of $\mathbb{R}^r$ to S_{r-1} which has the property in (i) of the theorem, I will call it an **outward-normal** function for E.

The words 'perimeter' and 'outward normal' are intended to suggest geometric interpretations; much of this section and the next will be devoted to validating this suggestion.

Observe that if E has locally finite perimeter, then $\operatorname{per} E = \lambda^\partial_E(\mathbb{R}^r)$. The definitions in 474D-474E also make it clear that if E, $F \subseteq \mathbb{R}$ are Lebesgue measurable and $\mu(E \triangle F) = 0$, then $\operatorname{per} E = \operatorname{per} F$ and E has locally finite perimeter iff F has; and in this case $\lambda^\partial_E = \lambda^\partial_F$ and an outward-normal function for E is an outward-normal function for F.

474G The reduced boundary Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter; let λ^∂_E be its perimeter measure and ψ an outward-normal function for E. The **reduced boundary** $\partial^\$ E$ is the set of those $y \in \mathbb{R}^r$ such that, for some $z \in S_{r-1}$,

$$\lim_{\delta\downarrow 0} \frac{1}{\lambda^\partial_E B(y,\delta)} \int_{B(y,\delta)} \|\psi(x) - z\| \, \lambda^\partial_E(dx) = 0.$$

(When requiring that the limit be defined, I mean to insist that $\lambda^\partial_E B(y,\delta)$ should be non-zero for every $\delta > 0$, that is, that y belongs to the support of λ^∂_E. **Warning!** Some authors use the phrase 'reduced boundary' for a slightly larger set.) Note that, writing $\psi = (\psi_1, \dots, \psi_r)$ and $z = (\zeta_1, \dots, \zeta_r)$, we must have

$$\zeta_i = \lim_{\delta\downarrow 0} \frac{1}{\lambda^\partial_E B(y,\delta)} \int_{B(y,\delta)} \psi_i d\lambda^\partial_E,$$

so that z is uniquely defined; call it $\psi_E(y)$. Of course $\partial^\$ E$ and ψ_E are determined entirely by the set E, because λ^∂_E is uniquely determined and ψ is determined up to a λ^∂_E-negligible set (474E).

By Besicovitch's Density Theorem (472Db),

$$\lim_{\delta\downarrow 0} \frac{1}{\lambda^\partial_E B(x,\delta)} \int_{B(x,\delta)} |\psi_i(x) - \psi_i(y)| \lambda^\partial_E(dx) = 0$$

for every $i \le r$, for λ^∂_E-almost every $y \in \mathbb{R}^r$; and for any such y, $\psi_E(y)$ is defined and equal to $\psi(y)$. Thus $\partial^\$ E$ is λ^∂_E-conegligible and ψ_E is an outward-normal function for E. I will call $\psi_E : \partial^\$ E \to S_{r-1}$ the **canonical outward-normal function** of E.

Once again, we see that if E, $F \subseteq \mathbb{R}^r$ are sets with locally finite perimeter and $E \triangle F$ is Lebesgue negligible, then they have the same reduced boundary and the same canonical outward-normal function.

474H Invariance under isometries: Proposition Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter. Let λ^∂_E be its perimeter measure, and ψ_E its canonical outward-normal function. If $T : \mathbb{R}^r \to \mathbb{R}^r$ is any isometry, then $T[E]$ has locally finite perimeter, $\lambda^\partial_{T[E]}$ is the image measure $\lambda^\partial_E T^{-1}$, the reduced boundary $\partial^\$ T[E]$ is $T[\partial^\$ E]$, and the canonical outward-normal function of $T[E]$ is $S\psi_E T^{-1}$, where S is the derivative of T.

proof (a) As noted in 474C, the derivative of T is constant, and is an orthogonal matrix. Suppose that $n \in \mathbb{N}$. Let $\phi : \mathbb{R}^r \to \mathbb{R}^r$ be a Lipschitz function such that $\|\phi\| \le \chi B(\mathbf{0}, n)$. Then

$$|\int_{T[E]} \operatorname{div} \phi \, d\mu| = |\int_E (\operatorname{div} \phi) T \, d\mu|$$

(263D, because $|\det S| = 1$)

$$= |\int_E \operatorname{div}(T^{-1}\phi T) d\mu|$$

(474C)

$$= |\int_E \operatorname{div}(S^{-1}\phi T)d\mu|$$

(because $S^{-1}\phi T$ and $T^{-1}\phi T$ differ by a constant, and must have the same derivative)

$$\leq \lambda_E^\partial(T^{-1}[B(\mathbf{0},n)])$$

because $S^{-1}\phi T$ is a Lipschitz function and

$$\|S^{-1}\phi T\| = \|\phi T\| \leq \chi T^{-1}[B(\mathbf{0},n)].$$

Since $T^{-1}[B(\mathbf{0},n)]$ is bounded, $\lambda_E^\partial(T^{-1}[B(\mathbf{0},n)])$ is finite for every n, and $T[E]$ has locally finite perimeter.

(b) We can therefore speak of its perimeter measure $\lambda_{T[E]}^\partial$. Let $G \subseteq \mathbb{R}^r$ be an open set. If $\phi : \mathbb{R}^r \to \mathbb{R}^r$ is a Lipschitz function and $\|\phi\| \leq \chi T[G]$, then

$$|\int_{T[E]} \operatorname{div} \phi \, d\mu| = |\int_E \operatorname{div}(S^{-1}\phi T)d\mu| \leq \lambda_E^\partial(G)$$

because $S^{-1}\phi T$ is a Lipschitz function dominated by χG. As ϕ is arbitrary, $\lambda_{T[E]}^\partial(T[G]) \leq \lambda_E^\partial(G)$. Applying the same argument in reverse, with T^{-1} in the place of T, we see that $\lambda_E^\partial(G) \leq \lambda_{T[E]}^\partial(T[G])$, so the two are equal. This means that the Radon measures $\lambda_{T[E]}^\partial$ and $\lambda_E^\partial T^{-1}$ (418I) agree on open sets, and must be identical (416Eb again).

(c) Now consider $S\psi_E T^{-1}$. Since ψ_E is defined λ_E^∂-almost everywhere and takes values in S_{r-1}, $\psi_E T^{-1}$ and $S\psi_E T^{-1}$ are defined $\lambda_{T[E]}^\partial$-almost everywhere and take values in S_{r-1}. If $\phi : \mathbb{R}^r \to \mathbb{R}^r$ is a Lipschitz function with compact support,

$$\int_{T[E]} \operatorname{div} \phi \, d\mu = \int_E (\operatorname{div} \phi)T \, d\mu = \int_E \operatorname{div}(T^{-1}\phi T)d\mu$$

$$= \int_E \operatorname{div}(S^{-1}\phi T)d\mu = \int (S^{-1}\phi T) \cdot \psi_E \, d\lambda_E^\partial$$

$$= \int (\phi T) \cdot (S\psi_E) d\lambda_E^\partial$$

(because S is orthogonal)

$$= \int \phi \cdot (S\psi_E T^{-1}) d(\lambda_E^\partial T^{-1})$$

(235G)

$$= \int \phi \cdot (S\psi_E T^{-1}) d\lambda_{T[E]}^\partial.$$

Accordingly $S\psi_E T^{-1}$ is an outward-normal function for $T[E]$. Write $\psi_{T[E]}$ for the canonical outward-normal function of $T[E]$.

(d) Take $y \in \mathbb{R}$ and consider

$$\frac{1}{\lambda_{T[E]}^\partial B(y,\delta)} \int_{B(y,\delta)} \|S\psi_E T^{-1}(x) - S\psi_E T^{-1}(y)\| \lambda_{T[E]}^\partial(dx)$$

$$= \frac{1}{\lambda_E^\partial B(T^{-1}(y),\delta)} \int_{B(T^{-1}(y),\delta)} \|S\psi_E(x) - S\psi_E T^{-1}(y)\| \lambda_E^\partial(dx)$$

$$= \frac{1}{\lambda_E^\partial B(T^{-1}(y),\delta)} \int_{B(T^{-1}(y),\delta)} \|\psi_E(x) - \psi_E T^{-1}(y)\| \lambda_E^\partial(dx)$$

for any $\delta > 0$ for which

$$\lambda_{T[E]}^\partial B(y,\delta) = \lambda_E^\partial T^{-1}[B(y,\delta)] = \lambda_E^\partial B(T^{-1}(y),\delta)$$

is non-zero. We see that

$$\lim_{\delta\downarrow 0}\frac{1}{\lambda^{\partial}_{T[E]}B(y,\delta)}\int_{B(y,\delta)}\|S\psi_E T^{-1}(x)-S\psi_E T^{-1}(y)\|\lambda^{\partial}_{T[E]}(dx)$$

is defined and equal to 0 whenever

$$\frac{1}{\lambda^{\partial}_{E}B(T^{-1}(y),\delta)}\int_{B(T^{-1}(y),\delta)}\|\psi_E(x)-\psi_E T^{-1}(y)\|\lambda^{\partial}_{E}(dx)$$

is defined and equal to 0, that is, $T^{-1}(y)\in\partial^{\$}E$. In this case, $y\in\partial^{\$}T[E]$ and $S\psi_E T^{-1}(y)=\psi_{T[E]}(y)$. So $\partial^{\$}T[E]\supseteq T[\partial^{\$}E]$ and $S\psi_E T^{-1}$ extends $\psi_{T[E]}$.

Applying the argument to T^{-1}, we see that $S^{-1}\psi_{T[E]}T$ extends ψ_E, that is, $\psi_{T[E]}$ extends $S\psi_E T^{-1}$. So $S\psi_E T^{-1}$ is exactly the canonical outward-normal function of $T[E]$, and its domain $T[\partial^{\$}E]$ is $\partial^{\$}T[E]$.

474I Half-spaces It will be useful, and perhaps instructive, to check the most elementary special case.

Proposition Let $H\subseteq\mathbb{R}^r$ be a half-space $\{x:x\,.\,v\le\alpha\}$, where $v\in S^{r-1}$. Then H has locally finite perimeter; its perimeter measure λ^{∂}_H is defined by saying

$$\lambda^{\partial}_H(F)=\nu(F\cap\partial H)$$

whenever $F\subseteq\mathbb{R}^r$ is such that ν measures $F\cap\partial H$, and the constant function with value v is an outward-normal function for H.

proof (a) Suppose, to begin with, that v is the unit vector $(0,\dots,0,1)$ and that $\alpha=0$, so that $H=\{x:\xi_r\le 0\}$. Let $\phi:\mathbb{R}^r\to\mathbb{R}^r$ be a Lipschitz function with compact support. Then for any $i<r$

$$\int_H\frac{\partial\phi_i}{\partial\xi_i}\mu(dx)=0$$

because we can regard this as a multiple integral in which the inner integral is with respect to ξ_i and is therefore always zero. On the other hand, integrating with respect to the rth coordinate first,

$$\begin{aligned}\int_H\frac{\partial\phi_r}{\partial\xi_r}\mu(dx)&=\int_{\mathbb{R}^{r-1}}\int_{-\infty}^{0}\frac{\partial\phi_r}{\partial\xi_r}(z,t)dt\,\mu_{r-1}(dz)\\&=\int_{\mathbb{R}^{r-1}}\phi_r(z,0)\mu_{r-1}(dz)=\int_{\partial H}\phi_r(x)\nu(dx)\end{aligned}$$

(identifying ν on $\mathbb{R}^{r-1}\times\{0\}$ with μ_{r-1} on $\mathbb{R}^{r-1}$)

$$=\int_{\partial H}\phi\,.\,v\,d\nu=\int\phi\,.\,v\,d\lambda$$

where λ is the indefinite-integral measure over ν defined by the function $\chi(\partial H)$. Note that (by 234La) λ can also be regarded as $\nu_{\partial H}\iota^{-1}$, where $\nu_{\partial H}$ is the subspace measure on ∂H and $\iota:\partial H\to\mathbb{R}^r$ is the identity map. Now $\nu_{\partial H}$ can be identified with Lebesgue measure on $\mathbb{R}^{r-1}$, by 265B or otherwise, so in particular is a Radon measure, and λ also is a Radon measure, by 418I again or otherwise.

This means that λ and the constant function with value v satisfy the conditions of 474E, and must be the perimeter measure of H and an outward-normal function.

(b) For the general case, let S be an orthogonal matrix such that $S(0,\dots,0,1)=v$, and set $T(x)=S(x)+\alpha v$ for every x, so that $H=T[\{x:\xi_r\le 0\}]$. By 474H, the perimeter measure of H is λT^{-1} and the constant function with value v is an outward-normal function for H. Now the Radon measure $\lambda^{\partial}_H=\lambda T^{-1}$ is defined by saying that

$$\lambda^{\partial}_H F=\lambda T^{-1}[F]=\nu(T^{-1}[F]\cap\{x:\xi_r=0\})=\nu(F\cap T[\{x:\xi_r=0\}])=\nu(F\cap\partial H)$$

whenever $\nu(F\cap\partial H)$ is defined, because ν (being a scalar multiple of a Hausdorff measure) must be invariant under the isometry T.

474J Lemma Let $E\subseteq\mathbb{R}^r$ be a set with locally finite perimeter. Let λ^{∂}_E be the perimeter measure of E, and ψ_E its canonical outward-normal function. Then $\mathbb{R}^r\setminus E$ also has locally finite perimeter; its perimeter measure is λ^{∂}_E, its reduced boundary is $\partial^{\$}E$, and its canonical outward-normal function is $-\psi_E$.

proof Of course $\mathbb{R}^r \setminus E$ is Lebesgue measurable. By 474Bc,

$$\int_{\mathbb{R}^r\setminus E} \operatorname{div}\phi\, d\mu = -\int_E \operatorname{div}\phi\, d\mu = \int \phi \centerdot (-\psi_E)\, d\lambda^\partial_E$$

for every Lipschitz function $\phi : \mathbb{R}^r \to \mathbb{R}^r$ with compact support. The uniqueness assertions in 474E tell us that $\mathbb{R}^r \setminus E$ has locally finite perimeter, that its perimeter measure is λ^∂_E, and that $-\psi_E$ is an outward-normal function for $\mathbb{R}^r \setminus E$. Referring to the definition of 'reduced boundary' in 474G, we see at once that $\partial^\$(\mathbb{R}^r \setminus E) = \partial^\$ E$ and that $\psi_{\mathbb{R}^r\setminus E} = -\psi_E$.

474K Lemma Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter; let λ^∂_E be its perimeter measure, and ψ an outward-normal function for E. Let $\phi : \mathbb{R}^r \to \mathbb{R}^r$ be a Lipschitz function with compact support, and $g \in \mathcal{D}$ an even function. Then

$$\int \phi \centerdot \operatorname{grad}(g * \chi E) d\mu + \int (g * \phi) \centerdot \psi\, d\lambda^\partial_E = 0.$$

proof

$$\int \phi \centerdot \operatorname{grad}(g * \chi E) d\mu = -\int (g * \chi E) \times \operatorname{div}\phi\, d\mu$$

(474Bd, using 473Dd to see that $g * \chi E$ is Lipschitz)

$$= -\int \chi E \times \operatorname{div}(g * \phi) d\mu$$

(474Be)

$$= -\int (g * \phi) \centerdot \psi\, d\lambda^\partial_E$$

(because $g * \phi$ is smooth and has compact support, so is Lipschitz), as required.

474L Two isoperimetric inequalities: Theorem Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter, and λ^∂_E its perimeter measure.

(a) If E is bounded, then $(\mu E)^{(r-1)/r} \le \operatorname{per} E$.

(b) If $B \subseteq \mathbb{R}^r$ is a closed ball, then

$$\min(\mu(B \cap E), \mu(B \setminus E))^{(r-1)/r} \le 2c\lambda^\partial_E(\operatorname{int} B),$$

where $c = 2^{r+4}\sqrt{r}(1 + 2^{r+1})$.

proof (a) Let $\epsilon > 0$. By 473Ef, there is an $n \in \mathbb{N}$ such that $\|f - \chi E\|_{r/(r-1)} \le \epsilon$, where $f = \chi E * \tilde{h}_n$. Note that f is smooth (473De again) and has compact support, because E is bounded. Let $\eta > 0$ be such that

$$\int \|\operatorname{grad} f\| d\mu \le \int \frac{\|\operatorname{grad} f\|^2}{\sqrt{\eta + \|\operatorname{grad} f\|^2}} d\mu + \epsilon,$$

and set $\phi = \frac{\operatorname{grad} f}{\sqrt{\eta + \|\operatorname{grad} f\|^2}}$. Then $\phi \in \mathcal{D}_r$ and $\|\phi(x)\| \le 1$ for every $x \in \mathbb{R}^r$. Now we can estimate

$$\int \|\operatorname{grad} f\| d\mu \le \int \phi \centerdot \operatorname{grad} f\, d\mu + \epsilon$$

$$= -\int (\tilde{h}_n * \phi) \centerdot \psi\, d\lambda^\partial_E + \epsilon$$

(where ψ is an outward-normal function for E, by 474K)

$$\le \int \|\tilde{h}_n * \phi\| d\lambda^\partial_E + \epsilon \le \operatorname{per} E + \epsilon$$

because $\|(\tilde{h}_n * \phi)(x)\| \le 1$ for every $x \in \mathbb{R}^r$, by 473Dg. Accordingly

$$(\mu E)^{(r-1)/r} = \|\chi E\|_{r/(r-1)} \le \|f\|_{r/(r-1)} + \epsilon \le \int \|\operatorname{grad} f\| d\mu + \epsilon$$

(473H)

$$\le \operatorname{per} E + 2\epsilon.$$

As ϵ is arbitrary, we have the result.

(b)(i) Set $\alpha = \min(\mu(B \cap E), \mu(B \setminus E))$. If $\alpha = 0$, the result is trivial; so suppose that $\alpha > 0$. Take any $\epsilon \in]0, \alpha]$. Let B_1 be a closed ball, with the same centre as B and strictly smaller non-zero radius, such that $\mu(B \setminus B_1) \le \epsilon$; then $\alpha - \epsilon \le \min(\mu(B_1 \cap E), \mu(B_1 \setminus E))$. For $f \in \mathcal{L}^{r/(r-1)}(\mu)$ set

$$\gamma_0(f) = \frac{1}{\mu B_1} \int_{B_1} f d\mu, \quad \gamma_1(f) = \|(f \times \chi B_1) - \gamma_0(f)\chi B_1\|_{r/(r-1)};$$

then both γ_0 and γ_1 are continuous functions on $\mathcal{L}^{r/(r-1)}(\mu)$ if we give it its usual pseudometric $(f, g) \mapsto \|f - g\|_{r/(r-1)}$. Now $\gamma_1(\chi(E \cap B)) \ge \frac{1}{2}(\alpha - \epsilon)^{(r-1)/r}$. **P** We have

$$\gamma_0(\chi(E \cap B)) = \frac{\mu(B_1 \cap E)}{\mu B_1},$$

$$\begin{aligned}\gamma_1(\chi(E \cap B))^{r/(r-1)} &= \int_{B_1} |\chi(E \cap B) - \gamma_0(\chi(E \cap B))|^{r/(r-1)} \\ &= \mu(B_1 \cap E)\big(1 - \frac{\mu(B_1 \cap E)}{\mu B_1}\big)^{r/(r-1)} + \mu(B_1 \setminus E)\big(\frac{\mu(B_1 \cap E)}{\mu B_1}\big)^{r/(r-1)} \\ &= \mu(B_1 \cap E)\big(\frac{\mu(B_1 \setminus E)}{\mu B_1}\big)^{r/(r-1)} + \mu(B_1 \setminus E)\big(\frac{\mu(B_1 \cap E)}{\mu B_1}\big)^{r/(r-1)}.\end{aligned}$$

Either $\mu(B_1 \cap E) \ge \frac{1}{2}\mu B_1$ or $\mu(B_1 \setminus E) \ge \frac{1}{2}\mu B_1$; suppose the former. Then

$$\gamma_1(\chi(E \cap B))^{r/(r-1)} \ge \frac{1}{2^{r/(r-1)}} \mu(B_1 \setminus E) \ge \frac{1}{2^{r/(r-1)}}(\alpha - \epsilon)$$

and $\gamma_1(\chi(E \cap B)) \ge \dfrac{1}{2}(\alpha - \epsilon)^{(r-1)/r}$. Exchanging $B_1 \cap E$ and $B_1 \setminus E$ we have the same result if $\mu(B_1 \cap E) \ge \frac{1}{2}\mu B_1$. **Q**

(ii) Express B as $B(y, \delta)$ and B_1 as $B(y, \delta_1)$. Take $n_0 \ge \frac{2}{\delta - \delta_1}$. Because γ_1 is $\|\;\|_{r/(r-1)}$-continuous, there is an $n \ge n_0$ such that $\gamma_1(f) \ge \frac{1}{2}(\alpha - \epsilon)^{(r-1)/r} - \epsilon$, where $f = \tilde{h}_n * \chi(E \cap B)$ (473Ef); as in part (a) of the proof, $f \in \mathcal{D}$. Let $\eta > 0$ be such that

$$\int_{B_1} \frac{\|\operatorname{grad} f\|^2}{\sqrt{\eta + \|\operatorname{grad} f\|^2}} d\mu \ge \int_{B_1} \|\operatorname{grad} f\| d\mu - \epsilon.$$

Let $m \ge n_0$ be such that $\int \phi \centerdot \operatorname{grad} f \, d\mu \ge \int_{B_1} \|\operatorname{grad} f\| d\mu - 2\epsilon$, where

$$\phi = \tilde{h}_m * \big(\frac{\operatorname{grad} f}{\sqrt{\eta + \|\operatorname{grad} f\|^2}} \times \chi B_1\big).$$

Note that $\phi(x) = 0$ if $\|x - y\| \ge \frac{1}{2}(\delta + \delta_1)$, so that $(\tilde{h}_n * \phi)(x) = 0$ if $x \notin \operatorname{int} B$. By 473Dg, $\|\phi(x)\| \le 1$ for every x and $\|(\tilde{h}_n * \phi)(x)\| \le 1$ for every x, so $\|\tilde{h}_n * \phi\| \le \chi(\operatorname{int} B)$.

Now we have

$$\begin{aligned}\int \phi \centerdot \operatorname{grad} f \, d\mu &= \int \phi \centerdot \operatorname{grad}(\tilde{h}_n * \chi(E \cap B)) \, d\mu \\ &= -\int (\tilde{h}_n * \chi(E \cap B)) \times \operatorname{div} \phi \, d\mu\end{aligned}$$

(474Bd)

$$= -\int_{E\cap B} \operatorname{div}(\tilde{h}_n * \phi)d\mu$$

(474Be)

$$= -\int_{E} \operatorname{div}(\tilde{h}_n * \phi)d\mu \le \lambda_E^\partial(\operatorname{int} B)$$

(474E).

(iii) Accordingly

$$\frac{1}{2}(\alpha-\epsilon)^{(r-1)/r} - \epsilon \le \gamma_1(f) \le c\int_{B_1} \|\operatorname{grad} f\| d\mu$$

(473K)

$$\le c\Bigl(\int \phi \centerdot \operatorname{grad} f\, d\mu + 2\epsilon\Bigr) \le c(\lambda_E^\partial(\operatorname{int} B) + 2\epsilon).$$

As ϵ is arbitrary, $\alpha^{(r-1)/r} \le 2c\lambda_E^\partial(\operatorname{int} B)$, as claimed.

474M Lemma Suppose that $E \subseteq \mathbb{R}^r$ has locally finite perimeter, with perimeter measure λ_E^∂ and an outward-normal function ψ. Then for any $y \in \mathbb{R}^r$ and any Lipschitz function $\phi : \mathbb{R}^r \to \mathbb{R}^r$,

$$\int_{E\cap B(y,\delta)} \operatorname{div}\phi\, d\mu = \int_{B(y,\delta)} \phi \centerdot \psi\, d\lambda_E^\partial + \int_{E\cap\partial B(y,\delta)} \phi(x) \centerdot \tfrac{1}{\delta}(x-y)\,\nu(dx)$$

for almost every $\delta > 0$.

proof (a) For $t > 0$, set

$$w(t) = \int_{E\cap\partial B(y,t)} \phi(x) \centerdot \tfrac{1}{t}(x-y)\,\nu(dx)$$

when this is defined. By 265G, applied to functions of the form

$$x \mapsto \begin{cases} \phi(x) \centerdot \frac{x-y}{\|x-y\|} & \text{if } x \in E \text{ and } 0 < \|x-y\| \le \alpha \\ 0 & \text{otherwise,} \end{cases}$$

w is defined almost everywhere in $]0,\infty[$ and is measurable (for Lebesgue measure on $\mathbb{R}$).

Let $\delta > 0$ be any point in the Lebesgue set of w (223D). Then

$$\lim_{t\downarrow 0} \frac{1}{t}\int_\delta^{\delta+t} |w(s) - w(\delta)|ds \le 2\lim_{t\downarrow 0} \frac{1}{2t}\int_{\delta-t}^{\delta+t} |w(s)-w(\delta)| = 0.$$

Let $\epsilon > 0$. Then there is an $\eta > 0$ such that

$$\frac{1}{\eta}\int_\delta^{\delta+\eta} |w(s)-w(\delta)|ds \le \epsilon, \quad \int_{B(y,\delta+\eta)\setminus B(y,\delta)} \|\phi\| d\lambda_E^\partial \le \epsilon,$$

$$\int_{B(y,\delta+\eta)\setminus B(y,\delta)} \|\operatorname{div}\phi\| d\mu \le \epsilon.$$

(b) Set

$$\begin{aligned} g(x) &= 1 \text{ if } \|x-y\| \le \delta, \\ &= 1 - \frac{1}{\eta}(\|x-y\| - \delta) \text{ if } \delta \le \|x-y\| \le \delta+\eta, \\ &= 0 \text{ if } \|x-y\| \ge \delta+\eta. \end{aligned}$$

Then g is continuous, and $\operatorname{grad} g(x) = \mathbf{0}$ if $\|x-y\| < \delta$ or $\|x-y\| > \delta+\eta$; while if $\delta < \|x-y\| < \delta+\eta$, $\operatorname{grad} g(x) = -\frac{x-y}{\eta\|x-y\|}$. This means that

$$\int_E \phi\,.\,\operatorname{grad} g\,d\mu = -\frac{1}{\eta}\int_\delta^{\delta+\eta}\int_{E\cap\partial B(y,t)}\frac{1}{t}(x-y)\,.\,\phi(x)\,\nu(dx)dt$$
$$= -\frac{1}{\eta}\int_\delta^{\delta+\eta} w(t)dt.$$

By the choice of η,

$$|\int_E \phi\,.\,\operatorname{grad} g\,d\mu + w(\delta)| \le \epsilon.$$

(c) By 474E and 474Bb we have

$$\int (g\times\phi)\,.\,\psi\,d\lambda_E^\partial = \int_E \operatorname{div}(g\times\phi)d\mu$$

(of course $g\times\phi$ is Lipschitz, by 473Ca and 262Ba)

$$= \int_E \phi\,.\,\operatorname{grad} g\,d\mu + \int_E g\times\operatorname{div}\phi\,d\mu.$$

Next, by the choice of η,

$$|\int((g\times\phi)\,.\,\psi\,d\lambda_E^\partial - \int_{B(y,\delta)}\phi\,.\,\psi\,d\lambda_E^\partial| \le \int_{B(y,\delta+\eta)\setminus B(y,\delta)}\|\phi\|d\lambda_E^\partial \le \epsilon,$$

while

$$|\int_E \phi\,.\,\operatorname{grad} g\,d\mu + \int_{E\cap\partial B(y,\delta)}\phi(x)\,.\,\frac{1}{\delta}(x-y)\nu(dx)| = |\int_E \phi\,.\,\operatorname{grad} g\,d\mu + w(\delta)| \le \epsilon$$

and

$$\Bigl|\int_E g\times\operatorname{div}\phi\,d\mu - \int_{E\cap B(y,\delta)}\operatorname{div}\phi\,d\mu\Bigr|$$
$$\le \int_{B(y,\delta+\eta)\setminus B(y,\delta)}\|\operatorname{div}\phi\|d\mu \le \epsilon.$$

Putting these together, we have

$$|\int_{E\cap B(y,\delta)}\operatorname{div}\phi\,d\mu - \int_{B(y,\delta)}\phi\,.\,\psi\,d\lambda_E^\partial - \int_{E\cap\partial B(y,\delta)}\phi(x)\,.\,\frac{1}{\delta}(x-y)\nu(dx)| \le 3\epsilon.$$

As ϵ is arbitrary, this gives the result.

474N Lemma Let $E\subseteq\mathbb{R}^r$ be a set with locally finite perimeter, and λ_E^∂ its perimeter measure. Then, for any $y\in\partial^\$E$,

(i) $\liminf_{\delta\downarrow 0}\frac{\mu(B(y,\delta)\cap E)}{\delta^r} \ge \frac{1}{(3r)^r}$;

(ii) $\liminf_{\delta\downarrow 0}\frac{\mu(B(y,\delta)\setminus E)}{\delta^r} \ge \frac{1}{(3r)^r}$;

(iii) $\liminf_{\delta\downarrow 0}\frac{\lambda_E^\partial B(y,\delta)}{\delta^{r-1}} \ge \frac{1}{2c(3r)^{r-1}}$,

where $c = 2^{r+4}\sqrt{r}(1+2^{r+1})$;

(iv) $\limsup_{\delta\downarrow 0}\frac{\lambda_E^\partial B(y,\delta)}{\delta^{r-1}} \le 4\pi\beta_{r-2}$.

proof (a) Let ψ_E be the canonical outward-normal function of E (474G). Take $y\in\partial^\$E$. Set

$$\Phi = \{\phi : \phi \text{ is a Lipschitz function from } \mathbb{R}^r \text{ to } B(\mathbf{0},1)\}.$$

Because the space $L^1(\mu)$ is separable in its usual (norm) topology (244I), so is $\{(\operatorname{div}\phi\times\chi B(y,1))^\bullet : \phi\in\Phi\}$ (4A2P(a-iv)), and there must be a countable set $\Phi_0\subseteq\Phi$ such that

whenever $\phi\in\Phi$ and $m\in\mathbb{N}$ there is a $\hat\phi\in\Phi_0$ such that $\int_{B(y,1)}|\operatorname{div}\phi - \operatorname{div}\hat\phi|d\mu \le 2^{-m}$.

Now, for each $\phi \in \Phi_0$,

$$|\int_{E\cap B(y,\delta)} \operatorname{div}\phi\,d\mu| = |\int_{B(y,\delta)} \phi\,.\,\psi_E\,d\lambda_E^\partial + \int_{E\cap\partial B(y,\delta)} \phi(x)\,.\,\frac{1}{\delta}(x-y)\nu(dx)|$$
$$\le \lambda_E^\partial B(y,\delta) + \nu(E\cap\partial B(y,\delta))$$

for almost every $\delta > 0$, by 474M. But this means that, for almost every $\delta \in]0,1]$,

$$\operatorname{per}(E\cap B(y,\delta)) = \sup_{\phi\in\Phi} |\int_{E\cap B(y,\delta)} \operatorname{div}\phi\,d\mu|$$
$$= \sup_{\phi\in\Phi_0} |\int_{E\cap B(y,\delta)} \operatorname{div}\phi\,d\mu| \le \lambda_E^\partial B(y,\delta) + \nu(E\cap\partial B(y,\delta)).$$

(b) It follows that, for some $\delta_0 > 0$,

$$\operatorname{per}(E\cap B(y,\delta)) \le 3\nu(E\cap\partial B(y,\delta))$$

for almost every $\delta \in]0,\delta_0]$. **P** Applying 474M with $\phi(x) = \psi_E(y)$ for every x, we have

$$0 = \int_{B(y,\delta)} \psi_E(y)\,.\,\psi_E(x)\lambda_E^\partial(dx) + \int_{E\cap\partial B(y,\delta)} \psi_E(y)\,.\,\frac{1}{\delta}(x-y)\nu(dx)$$

for almost every $\delta \in [0,1]$. But by the definition of $\psi_E(y)$,

$$\lim_{\delta\downarrow0} \frac{1}{\lambda_E^\partial B(y,\delta)}\int_{B(y,\delta)} \psi_E(y)\,.\,\psi_E\,d\lambda_E^\partial = \lim_{\delta\downarrow0} \frac{1}{\lambda_E^\partial B(y,\delta)}\int_{B(y,\delta)} \psi_E(y)\,.\,\psi_E(y)\,d\lambda_E^\partial = 1.$$

So there is some $\delta_0 > 0$ such that, for almost every $\delta \in]0,\delta_0]$,

$$\lambda_E^\partial B(y,\delta) \le 2\int_{B(y,\delta)} \psi_E(y)\,.\,\psi_E\,d\lambda_E^\partial$$
$$= -2\int_{E\cap\partial B(y,\delta)} \psi_E(y)\,.\,\frac{1}{\delta}(x-y)\nu(dx) \le 2\nu(E\cap\partial B(y,\delta)). \qquad (\dagger)$$

But this means that, for almost every such δ,

$$\operatorname{per}(E\cap B(y,\delta)) \le \lambda_E^\partial B(y,\delta) + \nu(E\cap\partial B(y,\delta)) \le 3\nu(E\cap\partial B(y,\delta)). \text{ **Q**}$$

(c) Set $g(t) = \mu(E\cap B(y,t))$ for $t \ge 0$. By 265G, $g(t) = \int_0^t \nu(E\cap\partial B(y,s))ds$ for every t, so g is absolutely continuous on $[0,1]$ and $g'(t) = \nu(E\cap\partial B(y,t))$ for almost every t. Now we can estimate

$$g(t)^{(r-1)/r} = \mu(E\cap B(y,t))^{(r-1)/r} \le \operatorname{per}(E\cap B(y,t))$$

(474La)

$$\le 3\nu(E\cap\partial B(y,t)) = 3g'(t)$$

for almost every $t \in [0,\delta_0]$. So

$$\frac{d}{dt}\big(g(t)^{1/r}\big) = \frac{1}{r}g(t)^{(1-r)/r}g'(t) \ge \frac{1}{3r}$$

for almost every $t \in [0,\delta_0]$; since $t \mapsto g(t)^{1/r}$ is non-decreasing, $g(t)^{1/r} \ge \frac{t}{3r}$ (222C) and $g(t) \ge (3r)^{-r}t^r$ for every $t \in [0,\delta_0]$.

(d) Accordingly

$$\liminf_{\delta\downarrow0} \frac{\mu(B(y,\delta)\cap E)}{\delta^r} \ge \inf_{0<\delta\le\delta_0} \frac{\mu(B(y,\delta)\cap E)}{\delta^r} \ge \frac{1}{(3r)^r}.$$

This proves (i).

(e) Because $\lambda^{\partial}_{\mathbb{R}^r\setminus E}=\lambda^{\partial}_E$ and $-\psi_E$ is the canonical outward-normal function of $\mathbb{R}^r\setminus E$ (474J), y also belongs to $\partial^{\$}(\mathbb{R}^r\setminus E)$, so the second formula of this lemma follows from the first.

(f) By 474Lb,

$$\lambda^{\partial}_E B(y,\delta)\ge\frac{1}{2c}\min\bigl(\mu(B(y,\delta)\cap E),\mu(B(y,\delta)\setminus E)\bigr)^{(r-1)/r}$$

for every $\delta\ge 0$. So

$$\begin{aligned}\liminf_{\delta\downarrow 0}\frac{\lambda^{\partial}_E B(y,\delta)}{\delta^{r-1}}&\ge\frac{1}{2c}\liminf_{\delta\downarrow 0}\min\bigl(\frac{\mu(B(y,\delta)\cap E)}{\delta^r},\frac{\mu(B(y,\delta)\setminus E)}{\delta^r}\bigr)^{(r-1)/r}\\&\ge\frac{1}{2c}\min\bigl(\liminf_{\delta\downarrow 0}\frac{\mu(B(y,\delta)\cap E)}{\delta^r},\liminf_{\delta\downarrow 0}\frac{\mu(B(y,\delta)\setminus E)}{\delta^r}\bigr)^{(r-1)/r}\\&\ge\frac{1}{2c}\bigl(\frac{1}{(3r)^r}\bigr)^{(r-1)/r}=\frac{1}{2c(3r)^{r-1}}.\end{aligned}$$

Thus (iii) is true.

(g) Returning to the inequality (†) in the proof of (b) above, we have a $\delta_0>0$ such that

$$\lambda^{\partial}_E B(y,\delta)\le 2\nu(E\cap\partial B(y,\delta))\le 2\nu(\partial B(y,\delta))=4\pi\beta_{r-2}\delta^{r-1}$$

(265F) for almost every $\delta\in\,]0,\delta_0]$. But this means that, for any $\delta\in[0,\delta_0[$,

$$\lambda^{\partial}_E B(y,\delta)\le\inf_{t>\delta}\lambda^{\partial}_E B(y,t)\le\inf_{t>\delta}4\pi\beta_{r-2}t^{r-1}=4\pi\beta_{r-2}\delta^{r-1},$$

and (iv) is true.

474O Definition Let $A\subseteq\mathbb{R}^r$ be any set, and $y\in\mathbb{R}^r$. A **Federer exterior normal to** A **at** y is a $v\in S_{r-1}$ such that,

$$\lim_{\delta\downarrow 0}\tfrac{\mu^*((H\triangle A)\cap B(y,\delta))}{\mu B(y,\delta)}=0,$$

where H is the half-space $\{x:(x-y)\,\boldsymbol{.}\,v\le 0\}$.

474P Lemma If $A\subseteq\mathbb{R}^r$ and $y\in\mathbb{R}^r$, there can be at most one Federer exterior normal to A at y.

proof Suppose that v, $v'\in S_{r-1}$ are two Federer exterior normals to E at y. Set

$$H=\{x:(x-y)\,\boldsymbol{.}\,v\le 0\},\quad H'=\{x:(x-y)\,\boldsymbol{.}\,v'\le 0\}.$$

Then

$$\lim_{\delta\downarrow 0}\tfrac{\mu((H\triangle H')\cap B(y,\delta))}{\mu B(y,\delta)}\le\lim_{\delta\downarrow 0}\tfrac{\mu^*((H\triangle A)\cap B(y,\delta))}{\mu B(y,\delta)}+\lim_{\delta\downarrow 0}\tfrac{\mu^*((H'\triangle A)\cap B(y,\delta))}{\mu B(y,\delta)}=0.$$

But for any $\delta>0$,

$$(H\triangle H')\cap B(y,\delta)=y+\delta((H_0\triangle H_0')\cap B(\mathbf{0},1)),$$

where

$$H_0=\{x:x\,\boldsymbol{.}\,v\le 0\},\quad H_0'=\{x:x\,\boldsymbol{.}\,v'\le 0\}.$$

So

$$0=\lim_{\delta\downarrow 0}\tfrac{\mu((H\triangle H')\cap B(y,\delta))}{\mu B(y,\delta)}=\tfrac{\mu((H_0\triangle H_0')\cap B(\mathbf{0},1))}{\mu B(\mathbf{0},1)}=\tfrac{\mu((H_0\triangle H_0')\cap B(\mathbf{0},n))}{\mu B(\mathbf{0},n)}$$

for every $n\ge 1$, and $\mu(H_0\triangle H_0')=0$. Since μ is strictly positive, and H_0 and H_0' are both the closures of their interiors, they must be identical; and it follows that $v=v'$.

474Q Lemma Set $c'=2^{r+3}\sqrt{r-1}(1+2^r)$. Suppose that c^*, ϵ and δ are such that

$$c^*\ge 0,\quad\delta>0,\quad 0<\epsilon<\frac{1}{\sqrt{2}},\quad c^*\epsilon^3<\tfrac14\beta_{r-1},\quad 4c'\epsilon\le\tfrac18\beta_{r-1}.$$

Set $V_\delta=\{z:z\in\mathbb{R}^{r-1},\ \|z\|\le\delta\}$ and $C_\delta=V_\delta\times[-\delta,\delta]$, regarded as a cylinder in $\mathbb{R}^r$. Let $f\in\mathfrak{D}$ be such that

$$\int_{C_\delta}\|\operatorname{grad}_{r-1}f\|+\max(\frac{\partial f}{\partial\xi_r},0)d\mu\le c^*\epsilon^3\delta^{r-1},$$

where $\operatorname{grad}_{r-1}f=(\frac{\partial f}{\partial\xi_1},\dots,\frac{\partial f}{\partial\xi_{r-1}},0)$. Set

$$F=\{x:x\in C_\delta,\ f(x)\ge\frac{3}{4}\},\quad F'=\{x:x\in C_\delta,\ f(x)\le\frac{1}{4}\}.$$

and for $\gamma\in\mathbb{R}$ set $H_\gamma=\{x:x\in\mathbb{R}^r,\ \xi_r\le\gamma\}$. Then there is a $\gamma\in\mathbb{R}$ such that

$$\mu(F\triangle(H_\gamma\cap C_\delta))\le 9\mu(C_\delta\setminus(F\cup F'))+(c^*\beta_{r-1}+16c')\epsilon\delta^r.$$

proof (a) For $t\in[-\delta,\delta]$ set

$$f_t(z)=f(z,t)\text{ for }z\in\mathbb{R}^{r-1},$$

$$F_t=\{z:z\in V_\delta,\ f_t(z)\ge\frac{3}{4}\},\quad F'_t=\{z:z\in V_\delta,\ f_t(z)\le\frac{1}{4}\};$$

set

$$\gamma=\sup\bigl(\{-\delta\}\cup\{t:t\in[-\delta,\delta],\ \mu_{r-1}F_t\ge\frac{3}{4}\mu_{r-1}V_\delta\}\bigr),$$

$$G=\{t:t\in[-\delta,\delta],\ \int_{V_\delta}\|\operatorname{grad}f_t\|d\mu_{r-1}\ge\epsilon^2\delta^{r-2}\}.$$

Note that $(\operatorname{grad}_{r-1}f)(z,t)=((\operatorname{grad}f_t)(z),0)$, so we have

$$\int_{-\delta}^{\delta}\int_{V_\delta}\|\operatorname{grad}f_t\|d\mu_{r-1}dt=\int_{C_\delta}\|\operatorname{grad}_{r-1}f\|d\mu\le c^*\epsilon^3\delta^{r-1}$$

and $\mu_1G\le c^*\epsilon\delta$, where μ_1 is Lebesgue measure on $\mathbb{R}$.

(b) If $t\in[-\delta,\delta]\setminus G$, then

$$\min(\mu_{r-1}F'_t,\mu_{r-1}F_t)\le 4c'\epsilon\delta^{r-1}.$$

P If $r>2$,

$$\begin{aligned}\min\bigl(\mu_{r-1}F'_t,\mu_{r-1}F_t)\bigr)^{(r-2)/(r-1)}&\le 4c'\int_{V_\delta}\|\operatorname{grad}f_t\|d\mu_{r-1}\\ &\le 4c'\epsilon^2\delta^{r-2}\end{aligned}$$

(473L)

because $t\notin G$, so that

$$\begin{aligned}\min(\mu_{r-1}F't,\mu_{r-1}F_t)&\le(4c'\epsilon^2)^{(r-1)/(r-2)}\delta^{r-1}\\ &\le 4c'\epsilon\delta^{r-1}\end{aligned}$$

because $4c'\ge1$ and $\frac{2(r-1)}{r-2}\ge1$ and $\epsilon\le1$. If $r=2$, then

$$\int_{V_\delta}\|\operatorname{grad}f_t\|d\mu_{r-1}\le\epsilon^2<\frac{1}{2},$$

so at least one of F'_t, F_t is empty, as noted in 473M, and $\min(\mu_{r-1}F'_t,\mu_{r-1}F_t)=0$. **Q**

(c) If $-\delta\le s<t\le\delta$, then

$$\int_{-\delta}^{\delta}\max(\frac{\partial f}{\partial\xi_r}(z,\xi),0)d\xi\ge\int_s^t\frac{\partial f}{\partial\xi_r}(z,\xi)d\xi=f(z,t)-f(z,s)\ge\frac{1}{2}$$

for every $z\in F'_s\cap F_t$. Accordingly

$$\frac{1}{2}\mu_{r-1}(F'_s\cap F_t)\le\int_{C_\delta}\max(\frac{\partial f}{\partial\xi_r},0)d\mu\le c^*\epsilon^3\delta^{r-1}<\frac{1}{4}\beta_{r-1}\delta^{r-1}=\frac{1}{4}\mu_{r-1}V_\delta$$

and $\mu_{r-1}(F'_s\cap F_t)<\frac{1}{2}\mu_{r-1}V_\delta$. It follows that if $-\delta\le s<\gamma$, so that there is a $t>s$ such that $\mu_{r-1}F_t\ge\frac{3}{4}\mu_{r-1}V_\delta$, then $\mu_{r-1}F'_s<\frac{3}{4}\mu_{r-1}V_\delta$.

(d) Now

$$\mu((F\triangle(H_\gamma\cap C_\delta))\le 9\mu(C_\delta\setminus(F\cup F'))+\epsilon\delta^r(c^*\beta_{r-1}+16c').$$

P Set

$$\tilde G=\{t:-\delta\le t\le\delta,\ \mu_{r-1}(F_t\cup F'_t)\le\frac{7}{8}\mu_{r-1}V_\delta\},$$

$$\hat G=\{t:-\delta\le t\le\delta,\ \mu_{r-1}F_t\le 4c'\epsilon\delta^{r-1}\},$$

$$\hat G'=\{t:-\delta\le t\le\delta,\ \mu_{r-1}F'_t\le 4c'\epsilon\delta^{r-1}\}.$$

Then

$$\frac{1}{8}\mu_{r-1}V_\delta\cdot\mu_1\tilde G\le\mu(C_\delta\setminus(F\cup F')),$$

$$\mu(F\cap(V_\delta\times\hat G))\le 8c'\epsilon\delta^r,$$

$$\mu(F'\cap(V_\delta\times\hat G'))\le 8c'\epsilon\delta^r.$$

So if we set

$$\begin{aligned}W=(C_\delta\setminus(F\cup F'))&\cup(V_\delta\times(\tilde G\cup G\cup\{\gamma\}))\\&\cup(F\cap(V_\delta\times\hat G))\cup(F'\cap(V_\delta\times\hat G')),\end{aligned}$$

we shall have

$$\begin{aligned}\mu W&\le\mu(C_\delta\setminus(F\cup F'))+\mu_1\tilde G\cdot\mu_{r-1}V_\delta+\mu_1G\cdot\mu_{r-1}V_\delta+16c'\epsilon\delta^r\\&\le 9\mu(C_\delta\setminus(F\cup F'))+c^*\epsilon\delta\mu_{r-1}V_\delta+16c'\epsilon\delta^r\\&=9\mu(C_\delta\setminus(F\cup F'))+\epsilon\delta^r(c^*\beta_{r-1}+16c')\end{aligned}$$

(using the estimate of μ_1G in (a)).

? Suppose, if possible, that there is a point $(z,t)\in(F\triangle(H_\gamma\cap C_\delta))\setminus W$. Since $t\notin G$, (b) tells us that

$$\min(\mu_{r-1}F'_t,\mu_{r-1}F_t)\le 4c'\epsilon\delta^{r-1}\le\frac{1}{8}\mu_{r-1}V_\delta.$$

So $t\in\hat G\cup\hat G'$. Also, since $t\notin\tilde G$, $\mu_{r-1}F_t+\mu_{r-1}F'_t\ge\frac{7}{8}\mu_{r-1}V_\delta$; so (since $t\ne\gamma$) either $\mu_{r-1}F_t\ge\frac{3}{4}\mu_{r-1}V_\delta$ and $t<\gamma$, or $\mu_{r-1}F'_t\ge\frac{3}{4}\mu_{r-1}V_\delta$ and $t>\gamma$ (by (c)). Now

$$\begin{aligned}t<\gamma&\Longrightarrow\mu_{r-1}F_t\ge\frac{3}{4}\mu_{r-1}V_\delta\\&\Longrightarrow\mu_{r-1}F'_t\le 4c'\epsilon\delta^{r-1}\\&\Longrightarrow t\in\hat G'\\&\Longrightarrow(z,t)\notin F'\end{aligned}$$

(because $(z,t)\notin F'\cap(V_\delta\times\hat G')$)

$$\Longrightarrow(z,t)\in F$$

(because $(z,t)\notin C_\delta\setminus(F\cup F')$)

$$\Longrightarrow(z,t)\in F\cap H_\gamma,$$

which is impossible. And similarly

$$\begin{aligned} t > \gamma &\Longrightarrow \mu_{r-1}F' \geq \frac{3}{4}\mu_{r-1}V_\delta \\ &\Longrightarrow \mu_{r-1}F_t \leq 4c'\epsilon\delta^{r-1} \\ &\Longrightarrow t \in \hat{G} \\ &\Longrightarrow (z,t) \notin F \\ &\Longrightarrow (z,t) \notin F \cup H_\gamma, \end{aligned}$$

which is equally impossible. **X**

Thus $F\triangle(H_\gamma \cap C_\delta) \subseteq W$ has measure at most

$$9\mu(C_\delta \setminus (F \cup F')) + \epsilon\delta^r(c^*\beta_{r-1} + 16c'),$$

as claimed. **Q**

474R Theorem Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter, ψ_E its canonical outward-normal function, and y any point of its reduced boundary $\partial^{\$}E$. Then $\psi_E(y)$ is the Federer exterior normal to E at y.

proof Write λ_E^∂ for the perimeter measure of E, as usual.

(a) To begin with (down to the end of (c-ii) below) suppose that $y = \mathbf{0}$ and that $\psi_E(y) = (0, \dots, 0, 1) = v$ say. Set

$$c = 2^{r+4}\sqrt{r}(1 + 2^{r+1}), \quad c' = 2^{r+3}\sqrt{r-1}(1 + 2^r),$$

$$c_1 = 1 + \max\bigl(4\pi\beta_{r-2}, 2c(3r)^{r-1}\bigr),$$

(counting β_0 as 1, if $r = 2$),

$$c^* = \sqrt{2}(2\sqrt{2})^{r-1}c_1, \quad c_1^* = 10 + \frac{1}{2}(c^* + \frac{16c'}{\beta_{r-1}}).$$

As in 474Q, set

$$V_\delta = \{z : z \in \mathbb{R}^{r-1}, \ \|z\| \leq \delta\}, \quad C_\delta = V_\delta \times [-\delta, \delta]. \quad H_\gamma = \{x : \xi_r \leq \gamma\}$$

for $\delta > 0$ and $\gamma \in \mathbb{R}$, and $\operatorname{grad}_{r-1} f = (\frac{\partial f}{\partial \xi_1}, \dots, \frac{\partial f}{\partial \xi_{r-1}}, 0)$ for $f \in \mathcal{D}$.

(b)(i) Take any $\epsilon > 0$ such that

$$\epsilon < \frac{1}{\sqrt{2}}, \quad c^*\epsilon^3 < \frac{1}{4}\beta_{r-1}, \quad 2^{r+1}c'\epsilon < \frac{1}{8}\beta_{r-1}.$$

Then there is a $\delta_0 \in \mathopen]0, 1]$ such that

$$\frac{1}{\lambda_E^\partial B(\mathbf{0},\delta)}\int_{B(\mathbf{0},\delta)} \|\psi_E(x) - v\|\lambda_E^\partial(dx) \leq \epsilon^3,$$

$$\frac{1}{c_1}\delta^{r-1} \leq \lambda_E^\partial B(\mathbf{0}, \delta) \leq c_1\delta^{r-1}$$

for every $\delta \in \mathopen]0, 2\delta_0\sqrt{2}]$ (using 474N(iii) and 474N(iv) for the inequalities bounding $\lambda_E^\partial B(\mathbf{0}, \delta)$).

(ii) Suppose that $0 < \delta \leq \delta_0$. Note first that

$$\begin{aligned} \int_{C_{2\delta}} \|v - \psi_E\| d\lambda_E^\partial &\leq \int_{B(\mathbf{0},2\delta\sqrt{2})} \|v - \psi_E\| d\lambda_E^\partial \leq \epsilon^3\lambda_E^\partial B(\mathbf{0}, 2\delta\sqrt{2}) \\ &\leq c_1\epsilon^3(2\delta\sqrt{2})^{r-1} = \frac{c^*}{\sqrt{2}}\epsilon^3\delta^{r-1}. \end{aligned}$$

(iii) $\lim_{n\to\infty}\tilde h_n*\chi E=_{\text{a.e.}}\chi E$ (473Ee), so there is an $n\ge\frac{1}{\delta}$ such that $\int_{C_\delta}|\tilde h_n*\chi E-\chi E|d\mu\le\frac14\epsilon\mu C_\delta$. Setting

$$f=\tilde h_n*\chi E,\quad F=\{x:x\in C_\delta,\,f(x)\ge\tfrac34\},\quad F'=\{x:x\in C_\delta,\,f(x)\le\tfrac14\},$$

we have $f\in\mathcal{D}$ (473De once more) and

$$\mu(C_\delta\setminus(F\cup F'))\le\epsilon\mu C_\delta,\quad \mu(F\triangle(E\cap C_\delta))\le\epsilon\mu C_\delta.$$

(iv)

$$\int_{C_\delta}\|\operatorname{grad}_{r-1}f\|+\max(\frac{\partial f}{\partial\xi_r},0)d\mu\le c^*\epsilon^3\delta^{r-1}.$$

P? Suppose, if possible, otherwise. Note that because $\mu C_\delta=2\beta_{r-1}\delta^r$, $\lim_{\delta'\uparrow\delta}\mu C_{\delta'}=\mu C_\delta$, so there is some $\delta'<\delta$ such that

$$\int_{C_{\delta'}}\|\operatorname{grad}_{r-1}f\|+\max(\frac{\partial f}{\partial\xi_r},0)d\mu>c^*\epsilon^3\delta^{r-1}.$$

For $1\le i\le r$ and $x\in\mathbb{R}^r$, set

$$\begin{aligned}\theta_i(x)&=\frac{\frac{\partial f}{\partial\xi_i}(x)}{\|\operatorname{grad}_{r-1}f(x)\|}\text{ if }i<r,\ x\in C_{\delta'}\text{ and }\operatorname{grad}_{r-1}(x)\ne0,\\&=1\text{ if }i=r,\ x\in C_{\delta'}\text{ and }\frac{\partial f}{\partial\xi_r}(x)\ge0,\\&=0\text{ otherwise.}\end{aligned}$$

Then all the θ_i are μ-integrable. Setting $\theta=(\theta_1,\dots,\theta_r)$,

$$\int\theta\,.\operatorname{grad}f d\mu=\int_{C_{\delta'}}\|\operatorname{grad}_{r-1}f\|+\max(\frac{\partial f}{\partial\xi_r},0)d\mu>c^*\epsilon^3\delta^{r-1}.$$

By 473Ef, $\langle\|\theta_i-\theta_i*\tilde h_k\|_1\rangle_{k\in\mathbb{N}}\to0$ for each i; since $\operatorname{grad}f$ is bounded,

$$\int(\tilde h_k*\theta)\,.\operatorname{grad}f\,d\mu>c^*\epsilon^3\delta^{r-1}$$

for any k large enough. If we ensure also that $\frac{1}{k+1}\le\delta-\delta'$, and set $\phi=\tilde h_k*\theta$, we shall get a function $\phi\in\mathcal{D}$, with $\|\phi(x)\|\le\sqrt2\chi C_\delta$ for every x (by 473Dc and 473Dg), such that

$$\int\phi\,.\operatorname{grad}f d\mu>c^*\epsilon^3\delta^{r-1}.$$

Moreover, referring to the definition of $*$ in 473Dd and 473Dg,

$$(\tilde h_n*\phi)(x)\,.v=(\tilde h_n*(\tilde h_k*\theta_r))(x)\ge0$$

for every x, because $\tilde h_n$, $\tilde h_k$ and θ_r are all non-negative.

Now

$$\begin{aligned}c^*\epsilon^3\delta^{r-1}&<\int\phi\,.\operatorname{grad}f d\mu=\int\phi\,.\operatorname{grad}(\tilde h_n*\chi E)d\mu\\&=-\int(\tilde h_n*\phi)\,.\psi_E\,d\lambda_E^\partial\end{aligned}$$

(474K)

$$\le\int(\tilde h_n*\phi)\,.(v-\psi_E)\,d\lambda_E^\partial\le\sqrt2\int_{C_{2\delta}}\|v-\psi_E\|\,d\lambda_E^\partial$$

(because $\|(\tilde h_n*\phi)(x)\|\le\sqrt2$ for every x, by 473Dg again, and $(\tilde h_n*\phi)(x)=0$ if $x\notin C_\delta+C_{1/(n+1)}\subseteq C_{2\delta}$)

$$\le c^*\epsilon^3\delta^{r-1};$$

which is absurd. **XQ**

(v) By 474Q, there is a $\gamma \in \mathbb{R}$ such that

$$\mu(F\triangle(H_\gamma\cap C_\delta)) \le 9\mu(C_\delta\setminus(F\cup F')) + (c^*\beta_{r-1}+16c')\epsilon\delta^r$$
$$\le 9\epsilon\mu C_\delta + \frac{1}{2\beta_{r-1}}(c^*\beta_{r-1}+16c')\epsilon\mu C_\delta = (c_1^*-1)\epsilon\mu C_\delta,$$

and

$$\mu((E\triangle H_\gamma)\cap C_\delta) \le \mu(F\triangle(E\cap C_\delta)) + \mu(F\triangle(H_\gamma\cap C_\delta)) \le c_1^*\epsilon\mu C_\delta.$$

(vi) As ϵ is arbitrary, we see that

$$\lim_{\delta\downarrow 0}\inf_{\gamma\in\mathbb{R}}\frac{1}{\mu C_\delta}\mu((E\triangle H_\gamma)\cap C_\delta) = 0.$$

(c) Again take $\epsilon \in]0,1]$.

(i) By (b) above and 474N(i)-(ii) there is a $\delta_1 > 0$ such that whenever $0 < \delta \le \delta_1$ then

$$\mu(B(\mathbf{0},\delta)\cap E) \ge \frac{1}{2(3r)^r\beta_r}\mu B(\mathbf{0},\delta),$$

$$\mu(B(\mathbf{0},\delta)\setminus E) \ge \frac{1}{2(3r)^r\beta_r}\mu B(\mathbf{0},\delta)$$

and there is a $\gamma \in \mathbb{R}$ such that

$$\mu((E\triangle H_\gamma)\cap C_\delta) < \min(\epsilon, \frac{\epsilon^r}{4\beta_{r-1}(3r)^r})\mu C_\delta.$$

In this case, $|\gamma| \le \epsilon\delta$. **P?** Suppose, if possible, that $\gamma < -\epsilon\delta$. Then

$$\mu(B(\mathbf{0},\epsilon\delta)\cap E) \le \mu(E\cap C_\delta\setminus H_\gamma)$$
$$< \frac{\epsilon^r}{4\beta_{r-1}(3r)^r}\mu C_\delta = \frac{1}{2\beta_r(3r)^r}\mu B(\mathbf{0},\epsilon\delta)$$

which is impossible. **X** In the same way, **?** if $\gamma > \epsilon\delta$,

$$\mu(B(\mathbf{0},\epsilon\delta)\setminus E) \le \mu((C_\delta\setminus H_\gamma)\setminus E)$$
$$< \frac{\epsilon^r}{4\beta_{r-1}(3r)^r}\mu C_\delta = \frac{1}{2\beta_r(3r)^r}\mu B(\mathbf{0},\epsilon\delta).\text{ **XQ**}$$

(ii) It follows that

$$\mu((E\triangle H_0)\cap C_\delta) \le \mu((E\triangle H_\gamma)\cap C_\delta) + \mu((H_\gamma\triangle H_0)\cap C_\delta)$$
$$\le \epsilon\mu C_\delta + \epsilon\delta\mu_{r-1}V_\delta = \frac{3}{2}\epsilon\mu C_\delta.$$

As ϵ is arbitrary,

$$\lim_{\delta\downarrow 0}\frac{\mu((E\triangle H_0)\cap C_\delta)}{\mu C_\delta} = 0,$$

and

$$\lim_{\delta\downarrow 0}\frac{\mu((E\triangle H_0)\cap B_\delta)}{\mu B_\delta} \le \frac{2\beta_{r-1}}{\beta_r}\lim_{\delta\downarrow 0}\frac{\mu((E\triangle H_0)\cap C_\delta)}{\mu C_\delta} = 0.$$

(d) Thus v is a Federer exterior normal to E at $\mathbf{0}$ if $\psi_E(\mathbf{0}) = v$. For the general case, let S be an orthogonal matrix such that $S\psi_E(y) = v$, and set $T(x) = S(x-y)$ for every x. The point is of course that

$$\mathbf{0} = T(y) \in T[\partial^\$ E] = \partial^\$ T[E], \quad v = S\psi_E T^{-1}(\mathbf{0}) = \psi_{T[E]}(\mathbf{0})$$

(474H). So if we set

$$H = \{x : (x-y)\cdot\psi_E(y) \le 0\} = \{x : T(x)\cdot v \le 0\} = T^{-1}[H_0],$$

then

$$\frac{\mu((H\triangle E)\cap B(y,\delta))}{\mu B(y,\delta)} = \frac{\mu((H_0\triangle T[E])\cap B(\mathbf{0},\delta))}{\mu B(\mathbf{0},\delta)} \to 0$$

as $\delta \downarrow 0$, and $\psi_E(y)$ is a Federer exterior normal to E at y, as required.

474S Corollary Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter, and λ_E^∂ its perimeter measure. Let y be any point of the reduced boundary of E. Then

$$\lim_{\delta\downarrow 0} \frac{\lambda_E^\partial B(y,\delta)}{\beta_{r-1}\delta^{r-1}} = 1.$$

proof (a) Set $v = \psi_E(y)$ and $H = \{x : (x-y)\,\boldsymbol{.}\,v \le 0\}$, as in 474R. Now

$$\int_{H\cap\partial B(y,\delta)} v\,\boldsymbol{.}\,\frac{1}{\delta}(x-y)\nu(dx) = -\beta_{r-1}\delta^{r-1}$$

for almost every $\delta > 0$. **P** Set $\phi(x) = v$ for every $x \in \mathbb{R}^r$. By 474I, ϕ is an outward-normal function for H, so 474M tells us that, for almost every $\delta > 0$,

$$\begin{aligned}\int_{H\cap\partial B(y,\delta)} v\,\boldsymbol{.}\,\frac{1}{\delta}(x-y)\,\nu(dx) &= \int_{H\cap B(y,\delta)} \operatorname{div}\phi\,d\mu - \int_{B(y,\delta)} v\,\boldsymbol{.}\,v\,d\lambda_H^\partial \\ &= -\lambda_H^\partial B(y,\delta) = -\nu(B(y,\delta)\cap\partial H)\end{aligned}$$

(using the identification of λ_H^∂ in 474I)

$$= -\beta_{r-1}\delta^{r-1}$$

(identifying ν on the hyperplane ∂H with Lebesgue measure on $\mathbb{R}^{r-1}$, as usual). **Q**

(b) Now, given $\epsilon > 0$, there is a $\delta_0 > 0$ such that whenever $0 < \delta \le \delta_0$ there is an η such that $\delta \le \eta \le \delta(1+\epsilon)$ and $|\lambda_E^\partial B(y,\eta) - \beta_{r-1}\eta^{r-1}| \le \epsilon\eta^{r-1}$. **P** Let $\zeta > 0$ be such that

$$\zeta(1 + \frac{5\pi}{r}\beta_{r-2})(1+\epsilon)^r \le \epsilon^2.$$

By 474N(iv) and 474R and the definition of ψ_E, there is a $\delta_0 > 0$ such that

$$\lambda_E^\partial B(y,\delta) \le 5\pi\beta_{r-2}\delta^{r-1},$$

$$\mu((E\triangle H)\cap B(y,\delta)) \le \zeta\delta^r,$$

$$\int_{B(y,\delta)} \|\psi_E(x) - v\|\lambda_E^\partial(dx) \le \zeta\lambda_E^\partial B(y,\delta)$$

whenever $0 < \delta \le (1+\epsilon)\delta_0$. Take $0 < \delta \le \delta_0$. Then, for almost every $\eta > 0$, we have

$$\int_{B(y,\eta)} v\,\boldsymbol{.}\,\psi_E(x)\,\lambda_E^\partial(dx) + \int_{E\cap\partial B(y,\eta)} v\,\boldsymbol{.}\,\frac{1}{\eta}(x-y)\,\nu(dx) = 0$$

by 474M, applied with ϕ the constant function with value v. Putting this together with (a), we see that, for almost every $\eta \in \,]0,(1+\epsilon)\delta_0]$,

$$\begin{aligned}|\lambda_E^\partial B(y,\eta) - \beta_{r-1}\eta^{r-1}| &= |\int_{B(y,\eta)} v\,\boldsymbol{.}\,v\,d\lambda_E^\partial - \beta_{r-1}\eta^{r-1}| \\ &\le |\int_{B(y,\eta)} v\,\boldsymbol{.}\,(v-\psi_E)\,d\lambda_E^\partial| + |\int_{B(y,\eta)} v\,\boldsymbol{.}\,\psi_E\,d\lambda_E^\partial - \beta_{r-1}\eta^{r-1}| \\ &\le \int_{B(y,\eta)} \|\psi_E - v\| d\lambda_E^\partial \\ &\qquad + |\int_{B(y,\eta)} v\,\boldsymbol{.}\,\psi_E(x)\,\lambda_E^\partial(dx) + \int_{H\cap\partial B(y,\eta)} v\,\boldsymbol{.}\,\frac{1}{\eta}(x-y)\,\nu(dx)|\end{aligned}$$

(using (a) above)

$$\leq \zeta\lambda^\partial_E B(y,\eta)$$
$$+ |\int_{H\cap\partial B(y,\eta)} v\mathbin{.}\frac{1}{\eta}(x-y)\,\nu(dx) - \int_{E\cap\partial B(y,\eta)} v\mathbin{.}\frac{1}{\eta}(x-y)\,\nu(dx)|$$
$$\leq 5\pi\beta_{r-2}\zeta\eta^{r-1} + \nu((E\triangle H)\cap\partial B(y,\eta)).$$

Integrating with respect to η, we have

$$\int_0^{\delta(1+\epsilon)} |\lambda^\partial_E B(y,\eta) - \beta_{r-1}\eta^{r-1}|d\eta \leq \frac{5\pi}{r}\beta_{r-2}\zeta\delta^r(1+\epsilon)^r + \mu((E\triangle H)\cap B(y,\delta(1+\epsilon)))$$

(using 265G, as usual)

$$\leq \frac{5\pi}{r}\beta_{r-2}\zeta\delta^r(1+\epsilon)^r + \zeta\delta^r(1+\epsilon)^r \leq \epsilon^2\delta^r$$

by the choice of ζ. But this means that there must be some $\eta \in [\delta,\delta(1+\epsilon)]$ such that

$$|\lambda^\partial_E B(y,\eta) - \beta_{r-1}\eta^{r-1}| \leq \epsilon\delta^{r-1} \leq \epsilon\eta^{r-1}. \textbf{ Q}$$

(c) Now we see that

$$\lambda^\partial_E B(y,\delta) \leq \lambda^\partial_E B(y,\eta) \leq (\beta_{r-1}+\epsilon)\eta^{r-1} \leq (\beta_{r-1}+\epsilon)(1+\epsilon)^{r-1}\delta^{r-1}.$$

But by the same argument we have an $\hat\eta \in [\frac{\delta}{1+\epsilon},\delta]$ such that $|\lambda^\partial_E B(y,\hat\eta) - \beta_{r-1}\hat\eta^{r-1}| \leq \epsilon\hat\eta^{r-1}$, so that

$$\lambda^\partial_E B(y,\delta) \geq \lambda^\partial_E B(y,\hat\eta) \geq (\beta_{r-1}-\epsilon)\hat\eta^{r-1} \geq (\beta_{r-1}-\epsilon)(1+\epsilon)^{1-r}\delta^{r-1}.$$

Thus, for every $\delta \in]0,\delta_0]$,

$$(\beta_{r-1}-\epsilon)(1+\epsilon)^{1-r}\delta^{r-1} \leq \lambda^\partial_E B(y,\delta) \leq (\beta_{r-1}+\epsilon)(1+\epsilon)^{r-1}\delta^{r-1}.$$

As ϵ is arbitrary,

$$\lim_{\delta\downarrow 0}\frac{\lambda^\partial_E B(y,\delta)}{\delta^{r-1}} = \beta_{r-1},$$

as claimed.

474T The Compactness Theorem Let Σ be the algebra of Lebesgue measurable subsets of $\mathbb{R}^r$, and give it the topology $\mathfrak{T}_m$ of convergence in measure defined by the pseudometrics $\rho_H(E,F) = \mu((E\triangle F)\cap H)$ for measurable sets H of finite measure (cf. §§245 and 323). Then

(a) $\operatorname{per}:\Sigma\to[0,\infty]$ is lower semi-continuous;

(b) for any $\gamma<\infty$, $\{E : E\in\Sigma, \operatorname{per}E\leq\gamma\}$ is compact.

proof (a) Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be any $\mathfrak{T}_m$-convergent sequence in Σ with limit $E\in\Sigma$. If $\phi:\mathbb{R}^r\to B(\mathbf{0},1)$ is a Lipschitz function with compact support, then $\operatorname{div}\phi$ is integrable, so $F\mapsto\int_F\operatorname{div}\phi\,d\mu$ is truly continuous (225A), and

$$|\int_E\operatorname{div}\phi\,d\mu| = \lim_{n\to\infty}|\int_{E_n}\operatorname{div}\phi\,d\mu| \leq \sup_{n\in\mathbb{N}}\operatorname{per}E_n.$$

As ϕ is arbitrary, $\operatorname{per}E \leq \sup_{n\in\mathbb{N}}\operatorname{per}E_n$. This means that $\{E:\operatorname{per}E\leq\gamma\}$ is sequentially closed, therefore closed (4A2Ld), for any γ, and per is lower semi-continuous.

(b) Let us say that a 'dyadic cube' is a set expressible in the form $\prod_{1\leq i\leq r}[2^{-n}k_i, 2^{-n}(k_i+1)[$ where n, $k_1,\dots,k_r\in\mathbb{Z}$. Set $\mathcal{A} = \{E:\operatorname{per}E\leq\gamma\}$.

(i) For $E\in\mathcal{A}$, $n\in\mathbb{N}$ and $\epsilon\in]0,1]$ let $G(E,n,\epsilon)$ be the union of all the dyadic cubes D with side length 2^{-n} such that $\epsilon\mu D\leq\mu(E\cap D)\leq(1-\epsilon)\mu D$. Then $\mu G(E,n,\epsilon)\leq\frac{c_1}{2^n\epsilon}\gamma$, where $c_1 = 2^{r+5}(1+2^{r+1})(1+\sqrt{r})^{r+1}$.

P Express $G(E,n,\epsilon)$ as a disjoint union $\bigcup_{i\in I}D_i$ where each D_i is a dyadic cube of side length 2^{-n} and $\min(\mu(D_i\cap E),\mu(D_i\setminus E))\geq\epsilon\mu D_i$. Let x_i be the centre of D_i and B_i the ball $B(x_i,2^{-n-1}\sqrt{r})$, so that $D_i\subseteq B_i$ and $\mu B_i = \beta_r(\frac{1}{2}\sqrt{r})^r\mu D_i$. For any $x\in\mathbb{R}^r$, the ball $B(x,2^{-n-1}\sqrt{r})$ is included in a closed cube with side length $2^{-n}\sqrt{r}$, so can contain at the very most $(1+\sqrt{r})^r$ different x_i, because different x_i differ by at least 2^{-n} in some coordinate. Turning this round, $\sum_{i\in I}\chi B_i \le (1+\sqrt{r})^r\chi(\mathbb{R}^r)$.

Set $c = 2^{r+4}\sqrt{r}(1+2^{r+1})$. Then, for each $i \in I$,

$$2c\lambda^\partial_E B_i \ge \min(\mu(B_i\cap E),\mu(B_i\setminus E))^{(r-1)/r}$$

(474Lb)

$$\ge \min(\mu(D_i\cap E),\mu(D_i\setminus E))^{(r-1)/r} \ge (\epsilon\mu D_i)^{(r-1)/r} \ge 2^{-n(r-1)}\epsilon.$$

So

$$\mu G(E,n,\epsilon) = 2^{-nr}\#(I) \le \frac{2c}{2^n\epsilon}\sum_{i\in I}\lambda^\partial_E B_i$$

$$\le \frac{2c(1+\sqrt{r})^r}{2^n\epsilon}\lambda^\partial_E(\mathbb{R}^r) \le \frac{c_1}{2^n\epsilon}\gamma. \text{ **Q**}$$

(ii) Now let $\langle E_n\rangle_{n\in\mathbb{N}}$ be any sequence in $\mathcal{A}$. Then we can find a subsequence $\langle E'_n\rangle_{n\in\mathbb{N}}$ such that whenever $n\in\mathbb{N}$, D is a dyadic cube of side length 2^{-n} meeting $B(\mathbf{0},n)$, and i, $j\ge n$, then $|\mu(D\cap E'_i)-\mu(D\cap E'_j)| \le \frac{1}{(n+1)^{r+2}}\mu D$. Now

$$\mu((E'_n\triangle E'_{n+1})\cap B(\mathbf{0},n)) \le \frac{3\beta_r(n+\sqrt{r})^r}{(n+1)^{r+2}} + 2^{-n}(n+1)^{r+1}c_1\gamma$$

whenever $n\ge 1$. **P** Let $\mathcal{E}$ be the set of dyadic cubes of side length 2^{-n} meeting $B(\mathbf{0},n)$; then every member of $\mathcal{E}$ is included in $B(\mathbf{0},n+2^{-n}\sqrt{r})$, so $\mu(\bigcup\mathcal{E}) \le \beta_r(n+\sqrt{r})^r$. Let $\mathcal{E}_1$ be the collection of those dyadic cubes of side length 2^{-n} included in $G(E'_n,n,\frac{1}{(n+1)^{r+2}})$. If $D\in\mathcal{E}\setminus\mathcal{E}_1$, either $\mu(E'_n\cap D) \le \frac{1}{(n+1)^{r+2}}\mu D$ and $\mu(E'_{n+1}\cap D) \le \frac{2}{(n+1)^{r+2}}\mu D$ and $\mu((E'_n\triangle E'_{n+1})\cap D) \le \frac{3}{(n+1)^{r+2}}\mu D$, or $\mu(D\setminus E'_n) \le \frac{1}{(n+1)^{r+2}}\mu D$ and $\mu(D\setminus E'_{n+1}) \le \frac{2}{(n+1)^{r+2}}\mu D$ and $\mu((E'_n\triangle E'_{n+1})\cap D) \le \frac{3}{(n+1)^{r+2}}\mu D$. So

$$\mu((E'_n\triangle E'_{n+1})\cap B(\mathbf{0},n)) \le \sum_{D\in\mathcal{E}}\mu((E'_n\triangle E'_{n+1})\cap D)$$

$$\le \sum_{D\in\mathcal{E}\setminus\mathcal{E}_1}\mu((E'_n\triangle E'_{n+1})\cap D) + \mu(\bigcup\mathcal{E}_1)$$

$$\le \frac{3}{(n+1)^{r+2}}\mu(\bigcup\mathcal{E}) + \mu G(E'_n,n,\frac{1}{(n+1)^{r+2}})$$

$$\le \frac{3\beta_r(n+\sqrt{r})^r}{(n+1)^{r+2}} + 2^{-n}(n+1)^{r+2}c_1\gamma,$$

as claimed. **Q**

(iii) This means that $\sum_{i=0}^\infty\mu((E'_i\triangle E'_{i+1})\cap B(\mathbf{0},n))$ is finite for each $n\in\mathbb{N}$, so that if we set $E = \bigcup_{i\in\mathbb{N}}\bigcap_{j\ge i}E'_j$, then

$$\mu((E\triangle E'_i)\cap B(\mathbf{0},n)) \le \sum_{j=i}^\infty\mu((E'_j\triangle E'_{j+1})\cap B(\mathbf{0},n)) \to 0$$

as $i\to\infty$ for every $n\in\mathbb{N}$. It follows that $\lim_{i\to\infty}\rho_H(E,E'_i)=0$ whenever $\mu H<\infty$ (see the proofs of 245Eb and 323Gb). Thus we have a subsequence $\langle E'_i\rangle_{i\in\mathbb{N}}$ of the original sequence $\langle E_i\rangle_{i\in\mathbb{N}}$ which is convergent for the topology $\mathfrak{T}_m$ of convergence in measure. By (a), its limit belongs to $\mathcal{A}$. But since $\mathfrak{T}_m$ is pseudometrizable (245Eb/323Gb), this is enough to show that A is compact for $\mathfrak{T}_m$ (4A2Lf).

474X Basic exercises (a) Show that for any $E\subseteq\mathbb{R}^r$ with locally finite perimeter, its reduced boundary is a Borel set and its canonical outward-normal function is Borel measurable.

>(b) Show that if $E \subseteq \mathbb{R}^r$ has finite perimeter then either E or its complement has finite measure.

(c)(i) Show that if $E \subseteq \mathbb{R}^r$ has locally finite perimeter, then $\partial^{\$}E \subseteq \partial E$. (*Hint*: 474N(i)-(ii).) (ii) Show that if $H \subseteq \mathbb{R}^r$ is a half-space, as in 474I, then $\partial^{\$}E = \partial E$.

(d) Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter, and $y \in \partial^{\$}E$. Show that $\lim_{\delta\downarrow 0}\frac{\mu(E\cap B(y,\delta))}{\mu B(y,\delta)} = \frac{1}{2}$.

(e) In the proof of 474S, use 265E to show that

$$\int_{H\cap\partial B(y,\delta)} v\centerdot\frac{1}{\delta}(x-y)\nu(dx) = -\beta_{r-1}\delta^{r-1}$$

for *every* $\delta > 0$.

474Y Further exercises (a) In 474E, explain how to interpret the pair $(\psi, \lambda_E^\partial)$ as a vector measure (definition: 394O[1]) $\theta_E : \mathcal{B} \to \mathbb{R}^r$, where $\mathcal{B}$ is the Borel σ-algebra of $\mathbb{R}^r$, in such a way that we have $\int_E \operatorname{div}\phi\,d\mu = \int \phi\centerdot d\theta_E$ for Lipschitz functions ϕ with compact support.

474 Notes and comments When we come to the Divergence Theorem itself in the next section, it will be nothing but a repetition of Theorem 474E with the perimeter measure and the outward-normal function properly identified. The idea of the indirect approach here is to start by defining the pair $(\psi_E, \lambda_E^\partial)$ as a kind of 'distributional derivative' of the set E. I take the space to match the details with the language of the rest of this treatise, but really 474E amounts to nothing more than the Riesz representation theorem; since the functional $\phi \mapsto \int_E \operatorname{div}\phi\,d\mu$ is linear, and we restrict attention to sets E for which it is continuous in an appropriate sense (and can therefore be extended to arbitrary continuous functions ϕ with compact support), it must be representable by a (vector) measure, as in 474Ya. For the process to be interesting, we have to be able to identify at least some of the appropriate sets E with their perimeter measures and outward-normal functions. Half-spaces are straightforward enough (474I), and 474R tells us what the outward-normal functions have to be; but for a proper description of the family of sets with locally finite perimeter we must wait until the next section. I see no quick way to show from the results here that (for instance) the union of two sets with finite perimeter again has finite perimeter. And I notice that I have not even shown that balls have finite perimeters. After 475M things should be much clearer.

I have tried to find the shortest path to the Divergence Theorem itself, and have not attempted to give 'best' results in the intermediate material. In particular, in the isoperimetric inequality 474La, I show only that the measure of a set E is controlled by the magnitude of its perimeter measure. Simple scaling arguments show that if there is any such control, then it must be of the form $\gamma(\mu E)^{(r-1)/r} \le \operatorname{per} E$; the identification of the best constant γ as $r\beta_r^{1/r}$, giving equality for balls, is the real prize, to which I shall come in 476H. Similarly, there is a dramatic jump from the crude estimates in 474N to the exact limits in 474Xd and 474S. When we say that a set E has a Federer exterior normal at a point y, we are clearly saying that there is an 'approximate' tangent plane at that point, as measured by ordinary volume μ. 474S strengthens this by saying that, when measured by the perimeter measure, the boundary of E looks like a hyperplane through y with normalised $(r-1)$-dimensional measure. In 475G below we shall come to a partial explanation of this.

The laborious arguments of 474C and 474H are doing no more than establish the geometric invariance of the concepts here, which ought, one would think, to be obvious. The trouble is that I have given definitions of inner product and divergence and Lebesgue measure in terms of the standard coordinate system of $\mathbb{R}^r$. If these were not invariant under isometries they would be far less interesting. But even if we are confident that there must be a result corresponding to 474H, I think a little thought is required to identify the exact formulae involved in the transformation.

I leave the Compactness Theorem (474T) to the end of the section because it is off the line I have chosen to the Divergence Theorem (though it can be used to make the proof of 474R more transparent; see Evans & Gariepy 92, 5.7.2). I have expressed 474T in terms of the topology of convergence in measure on the algebra of Lebesgue measurable sets. But since the perimeter of a measurable set E is not altered if we

[1]Formerly 393O.

change E by a negligible set (474F), 'perimeter' can equally well be regarded as a function defined on the measure algebra, in which case 474T becomes a theorem about the usual topology of the measure algebra of Lebesgue measure, as described in §323.

Version of 24.1.13

475 The essential boundary

The principal aim of this section is to translate Theorem 474E into geometric terms. We have already identified the Federer exterior normal as an outward-normal function (474R), so we need to find a description of perimeter measures. Most remarkably, these turn out, in every case considered in 474E, to be just normalized Hausdorff measures (475G). This description needs the concept of 'essential boundary' (475B). In order to complete the programme, we need to be able to determine which sets have 'locally finite perimeter'; there is a natural criterion in the same language (475L). We now have all the machinery for a direct statement of the Divergence Theorem (for Lipschitz functions) which depends on nothing more advanced than the definition of Hausdorff measure (475N). (The definitions, at least, of 'Federer exterior normal' and 'essential boundary' are elementary.)

This concludes the main work of the first part of this chapter. But since we are now within reach of a reasonably direct proof of a fundamental fact about the $(r-1)$-dimensional Hausdorff measure of the boundaries of subsets of $\mathbb{R}^r$ (475Q), I continue up to Cauchy's Perimeter Theorem and the Isoperimetric Theorem for convex sets (475S, 475T).

475A Notation As in the last two sections, r will be an integer (greater than or equal to 2, unless explicitly permitted to take the value 1). μ will be Lebesgue measure on $\mathbb{R}^r$; I will sometimes write μ_{r-1} for Lebesgue measure on $\mathbb{R}^{r-1}$ and μ_1 for Lebesgue measure on $\mathbb{R}$. $\beta_r = \mu B(\mathbf{0},1)$ will be the measure of the unit ball in $\mathbb{R}^r$, and $S_{r-1} = \partial B(\mathbf{0},1)$ will be the unit sphere. ν will be normalized $(r-1)$-dimensional Hausdorff measure on $\mathbb{R}^r$ (265A), that is, $\nu = 2^{-r+1}\beta_{r-1}\mu_{H,r-1}$, where $\mu_{H,r-1}$ is $(r-1)$-dimensional Hausdorff measure on $\mathbb{R}^r$. Recall that $\nu S_{r-1} = r\beta_r$ (265H). I will take it for granted that $x \in \mathbb{R}^r$ has coordinates $(\xi_1, \dots, \xi_r)$.

If $E \subseteq \mathbb{R}^r$ has locally finite perimeter (474D), λ_E^∂ will be its perimeter measure (474F), $\partial^\$ E$ its reduced boundary (474G) and ψ_E its canonical outward-normal function (474G).

475B The essential boundary (In this paragraph I allow $r = 1$.) Let $A \subseteq \mathbb{R}^r$ be any set. The **essential closure** of A is the set

$$\mathrm{cl}^*A = \{x : \limsup_{\delta\downarrow 0} \tfrac{\mu^*(B(x,\delta)\cap A)}{\mu B(x,\delta)} > 0\}$$

(see 266B). Similarly, the **essential interior** of A is the set

$$\mathrm{int}^*A = \{x : \liminf_{\delta\downarrow 0} \tfrac{\mu_*(B(x,\delta)\cap A)}{\mu B(x,\delta)} = 1\}.$$

(If A is Lebesgue measurable, this is the lower Lebesgue density of A, as defined in 341E; see also 223Yf.) Finally, the **essential boundary** ∂^*A of A is the difference $\mathrm{cl}^*A \setminus \mathrm{int}^*A$.

Note that if $E \subseteq \mathbb{R}^r$ is Lebesgue measurable then $\mathbb{R}^r \setminus \partial^*E$ is the Lebesgue set of the function χE, as defined in 261E.

475C Lemma (In this lemma I allow $r = 1$.) Let A, $A' \subseteq \mathbb{R}^r$.

(a)

$$\mathrm{int}\,A \subseteq \mathrm{int}^*A \subseteq \mathrm{cl}^*A \subseteq \overline{A}, \quad \partial^*A \subseteq \partial A,$$

$$\mathrm{cl}^*A = \mathbb{R}^r \setminus \mathrm{int}^*(\mathbb{R}^r \setminus A), \quad \partial^*(\mathbb{R}^r \setminus A) = \partial^*A.$$

(b) If $A \setminus A'$ is negligible, then $\mathrm{cl}^*A \subseteq \mathrm{cl}^*A'$ and $\mathrm{int}^*A \subseteq \mathrm{int}^*A'$; in particular, if A itself is negligible, cl^*A, int^*A and ∂^*A are all empty.

(c) int^*A, cl^*A and ∂^*A are Borel sets.

(d) $\mathrm{cl}^*(A\cup A') = \mathrm{cl}^*A\cup\mathrm{cl}^*A'$ and $\mathrm{int}^*(A\cap A') = \mathrm{int}^*A\cap\mathrm{int}^*A'$, so $\partial^*(A\cup A')$, $\partial^*(A\cap A')$ and $\partial^*(A\triangle A')$ are all included in $\partial^*A\cup\partial^*A'$.

(e) $\mathrm{cl}^*A\cap\mathrm{int}^*A'\subseteq\mathrm{cl}^*(A\cap A')$, $\partial^*A\cap\mathrm{int}^*A'\subseteq\partial^*(A\cap A')$ and $\partial^*A\setminus\mathrm{cl}^*A'\subseteq\partial^*(A\cup A')$.

(f) $\partial^*(A\cap A')\subseteq(\mathrm{cl}^*A'\cap\partial A)\cup(\partial^*A'\cap\mathrm{int}\,A)$.

(g) If $E\subseteq\mathbb{R}^r$ is Lebesgue measurable, then $E\triangle\mathrm{int}^*E$, $E\triangle\mathrm{cl}^*E$ and ∂^*E are Lebesgue negligible.

(h) A is Lebesgue measurable iff ∂^*A is Lebesgue negligible.

proof (a) It is obvious that

$$\mathrm{int}\,A\subseteq\mathrm{int}^*A\subseteq\mathrm{cl}^*A\subseteq\overline{A},$$

so that $\partial^*A\subseteq\partial A$. Since

$$\tfrac{\mu^*(B(x,\delta)\cap A)}{\mu B(x,\delta)}+\tfrac{\mu_*(B(x,\delta)\setminus A)}{\mu B(x,\delta)}=1$$

for every $x\in\mathbb{R}^r$ and every $\delta>0$ (413Ec), $\mathbb{R}^r\setminus\mathrm{int}^*A=\mathrm{cl}^*(\mathbb{R}^r\setminus A)$. It follows that

$$\begin{aligned}\partial^*(\mathbb{R}^r\setminus A)&=\mathrm{cl}^*(\mathbb{R}^r\setminus A)\triangle\mathrm{int}^*(\mathbb{R}^r\setminus A)\\&=(\mathbb{R}^r\setminus\mathrm{int}^*A)\triangle(\mathbb{R}^r\setminus\mathrm{cl}^*A)=\mathrm{int}^*A\triangle\mathrm{cl}^*A=\partial^*A.\end{aligned}$$

(b) If $A\setminus A'$ is negligible, then

$$\mu_*(B(x,\delta)\cap A)\le\mu_*(B(x,\delta)\cap A'),\quad \mu^*(B(x,\delta)\cap A)\le\mu^*(B(x,\delta)\cap A')$$

for all x and δ, so $\mathrm{int}^*A\subseteq\mathrm{int}^*A'$ and $\mathrm{cl}^*A\subseteq\mathrm{cl}^*A'$.

(c) The point is just that $(x,\delta)\mapsto\mu^*(A\cap B(x,\delta))$ is continuous. **P** For any x, $y\in\mathbb{R}^r$ and δ, $\eta>0$ we have

$$\begin{aligned}|\mu^*(A\cap B(y,\eta))-\mu^*(A\cap B(x,\delta))|&\le\mu(B(y,\eta)\triangle B(x,\delta))\\&=2\mu(B(x,\delta)\cup B(y,\eta))-\mu B(x,\delta)-\mu B(y,\eta)\\&\le\beta_r\bigl(2(\max(\delta,\eta)+\|x-y\|)^r-\delta^r-\eta^r\bigr)\to 0\end{aligned}$$

as $(y,\eta)\to(x,\delta)$. **Q** So

$$x\mapsto\limsup_{\delta\downarrow 0}\tfrac{\mu^*(A\cap B(x,\delta))}{\mu B(x,\delta)}=\inf_{\alpha\in\mathbb{Q},\alpha>0}\sup_{\gamma\in\mathbb{Q},0<\gamma\le\alpha}\tfrac{1}{\beta_r\gamma^r}\mu^*(A\cap B(x,\gamma))$$

is Borel measurable, and

$$\mathrm{cl}^*A=\{x:\limsup_{\delta\downarrow 0}\tfrac{\mu^*(A\cap B(x,\delta))}{\mu B(x,\delta)}>0\}$$

is a Borel set.

Accordingly $\mathrm{int}^*A=\mathbb{R}^r\setminus\mathrm{cl}^*(\mathbb{R}^r\setminus A)$ and ∂^*A are also Borel sets.

(d) For any $x\in\mathbb{R}^r$,

$$\begin{aligned}\limsup_{\delta\downarrow 0}\tfrac{\mu^*((A\cup A')\cap B(x,\delta))}{\mu B(x,\delta)}&\le\limsup_{\delta\downarrow 0}\tfrac{\mu^*(A\cap B(x,\delta))}{\mu B(x,\delta)}+\tfrac{\mu^*(A'\cap B(x,\delta))}{\mu B(x,\delta)}\\&\le\limsup_{\delta\downarrow 0}\tfrac{\mu(A\cap B(x,\delta))}{\mu B(x,\delta)}+\limsup_{\delta\downarrow 0}\tfrac{\mu(A'\cap B(x,\delta))}{\mu B(x,\delta)},\end{aligned}$$

so $\mathrm{cl}^*(A\cup A')\subseteq\mathrm{cl}^*A\cup\mathrm{cl}^*A'$. By (b), $\mathrm{cl}^*A\cup\mathrm{cl}^*A'\subseteq\mathrm{cl}^*(A\cup A')$, so we have equality. Accordingly

$$\mathrm{int}^*(A\cap A')=\mathbb{R}^r\setminus\mathrm{cl}^*((\mathbb{R}^r\setminus A)\cup(\mathbb{R}^r\setminus A'))=\mathrm{int}^*A\cap\mathrm{int}^*A'.$$

Since $\mathrm{int}^*(A\cup A')\supseteq\mathrm{int}^*A\cup\mathrm{int}^*A'$, $\partial^*(A\cup A')\subseteq\partial^*A\cup\partial^*A'$. Now

$$\partial^*(A\cap A')=\partial^*(\mathbb{R}^r\setminus(A\cap A'))\subseteq\partial^*(\mathbb{R}^r\setminus A)\cup\partial^*(\mathbb{R}^r\setminus A'))=\partial^*A\cup\partial^*A'$$

and

$$\partial^*(A\triangle A')\subseteq\partial^*(A\cap(\mathbb{R}^r\setminus A'))\cup\partial^*(A'\cap(\mathbb{R}^r\setminus A))\subseteq\partial^*A\cup\partial^*A'.$$

(e) If $x \in \mathrm{cl}^*A \cap \mathrm{int}^*A'$, then $x \notin \mathrm{cl}^*(\mathbb{R}^r \setminus A')$ so $x \notin \mathrm{cl}^*(A \setminus A')$. But from (d) we know that $x \in \mathrm{cl}^*(A \cap A') \cup \mathrm{cl}^*(A \setminus A')$, so $x \in \mathrm{cl}^*(A \cap A')$.

Now

$$\begin{aligned}\partial^*A \cap \mathrm{int}^*A' &= (\mathrm{cl}^*A \cap \mathrm{int}^*A') \setminus \mathrm{int}^*A \\ &\subseteq \mathrm{cl}^*(A \cap A') \setminus \mathrm{int}^*(A \cap A') = \partial^*(A \cap A'),\end{aligned}$$

$$\begin{aligned}\partial^*A \setminus \mathrm{cl}^*A' &= \partial^*(\mathbb{R}^r \setminus A) \cap \mathrm{int}^*(\mathbb{R}^r \setminus A') \\ &\subseteq \partial^*((\mathbb{R}^r \setminus A) \cap (\mathbb{R}^r \setminus A')) \\ &= \partial^*(\mathbb{R}^r \setminus (A \cup A')) = \partial^*(A \cup A').\end{aligned}$$

(f) If $x \in \partial^*(A \cap A') \cap \partial A$, then of course $x \in \mathrm{cl}^*(A \cap A') \subseteq \mathrm{cl}^*A$, so $x \in \mathrm{cl}^*A \cap \partial A$. If $x \in \partial^*(A \cap A') \setminus \partial A$, then surely $x \in \overline{A}$, so $x \in \mathrm{int}\, A$. But this means that

$$\mu_*(B(x,\delta) \cap A') = \mu_*(B(x,\delta) \cap A \cap A'), \quad \mu^*(B(x,\delta) \cap A') = \mu^*(B(x,\delta) \cap A \cap A')$$

for all δ small enough, so $x \in \partial^*A'$ and $x \in \partial^*A' \cap \mathrm{int}\, A$.

(g) Applying 472Da to μ and χE, we see that

$$\begin{aligned}\partial^*E &\subseteq (E \triangle \mathrm{cl}^*E) \cup (E \triangle \mathrm{int}^*E) \\ &\subseteq \{x : \chi E(x) \ne \lim_{\delta\downarrow 0} \tfrac{\mu(E \cap B(x,\delta))}{\mu B(x,\delta)}\}\end{aligned}$$

are all μ-negligible.

(h) If A is Lebesgue measurable then (g) tells us that ∂^*A is negligible. If ∂^*A is negligible, let E be a measurable envelope of A. Then $\mu(E \cap B(x,\delta)) = \mu^*(A \cap B(x,\delta))$ for all x and δ, so $\mathrm{cl}^*E = \mathrm{cl}^*A$. Similarly, if F is a measurable envelope of $\mathbb{R}^r \setminus A$, then $\mathrm{cl}^*F = \mathrm{cl}^*(\mathbb{R}^r \setminus A) = \mathbb{R}^r \setminus \mathrm{int}^*A$ (using (a)). Now (g) tells us that

$$\mu(E \cap F) = \mu(\mathrm{cl}^*E \cap \mathrm{cl}^*F) = \mu(\mathrm{cl}^*A \setminus \mathrm{int}^*A) = 0.$$

But now $A \setminus E$ and $E \setminus A \subseteq (E \cap F) \cup ((\mathbb{R}^r \setminus A) \setminus F)$ are Lebesgue negligible, so A is Lebesgue measurable.

475D Lemma Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter, and $\partial^\$E$ its reduced boundary. Then $\partial^\$E \subseteq \partial^*E$ and $\nu(\partial^*E \setminus \partial^\$E) = 0$.

proof (a) By 474N(i), $\partial^\$E \subseteq \mathrm{cl}^*E$; by 474N(ii), $\partial^\$E \cap \mathrm{int}^*E = \emptyset$; so $\partial^\$E \subseteq \partial^*E$.

(b) For any $y \in \partial^*E$,

$$\limsup_{\delta\downarrow 0} \tfrac{1}{\delta^{r-1}} \lambda^\partial_E B(y,\delta) > 0.$$

P We have an $\epsilon \in \left]0, \tfrac{1}{2}\right]$ such that

$$\liminf_{\delta\downarrow 0} \tfrac{\mu(E \cap B(y,\delta))}{\mu B(y,\delta)} < 1 - \epsilon, \quad \limsup_{\delta\downarrow 0} \tfrac{\mu(E \cap B(y,\delta))}{\mu B(y,\delta)} > \epsilon.$$

Since the function $\delta \mapsto \frac{\mu(E \cap B(y,\delta))}{\mu B(y,\delta)}$ is continuous, there is a sequence $\langle \delta_n \rangle_{n\in\mathbb{N}}$ in $]0, \infty[$ such that $\lim_{n\to\infty} \delta_n = 0$ and

$$\epsilon\mu B(y,\delta_n) \le \mu(E \cap B(y,\delta_n)) \le (1-\epsilon)\mu B(y,\delta_n)$$

for every n. Now from 474Lb we have

$$\begin{aligned}(\epsilon\beta_r)^{(r-1)/r}\delta_n^{r-1} &= (\epsilon\mu B(y,\delta_n))^{(r-1)/r} \\ &\le \min(\mu(B(y,\delta_n) \cap E), \mu(B(y,\delta_n) \setminus E))^{(r-1)/r} \le 2c\lambda^\partial_E B(y,\delta_n)\end{aligned}$$

for every n, where $c > 0$ is the constant there. But this means that

$$\limsup_{\delta\downarrow 0}\frac{1}{\delta^{r-1}}\lambda_E^{\partial}B(y,\delta)\ge\limsup_{n\to\infty}\frac{1}{\delta_n^{r-1}}\lambda_E^{\partial}B(y,\delta_n)\ge\frac{1}{2c}(\epsilon\beta_r)^{(r-1)/r}>0.\ \mathbf{Q}$$

(c) Let $\epsilon>0$. Set

$$F_\epsilon=\{y:y\in\mathbb{R}^r\setminus\partial^{\$}E,\ \limsup_{\delta\downarrow 0}\frac{1}{\delta^{r-1}}\lambda_E^{\partial}B(y,\delta)>\epsilon\}.$$

Because $\partial^{\$}E$ is λ_E^{∂}-conegligible (474G), $\lambda_E^{\partial}F_\epsilon=0$. So there is an open set $G\supseteq F_\epsilon$ such that $\lambda_E^{\partial}G\le\epsilon^2$ (256Bb/412Wb). Let $\delta>0$. Let $\mathcal{I}$ be the family of all those non-singleton closed balls $B\subseteq G$ such that $\operatorname{diam}B\le\delta$ and $\lambda_E^{\partial}B\ge 2^{-r+1}\epsilon(\operatorname{diam}B)^{r-1}$. Then every point of F_ϵ is the centre of arbitrarily small members of $\mathcal{I}$. By Besicovitch's Covering Lemma (472B), there is a family $\langle\mathcal{I}_k\rangle_{k<5^r}$ of disjoint countable subsets of $\mathcal{I}$ such that $\mathcal{I}^*=\bigcup_{k<5^r}\bigcup\mathcal{I}_k$ covers F_ϵ. Now

$$\sum_{B\in\mathcal{I}^*}(\operatorname{diam}B)^{r-1}\le\sum_{k<5^r}\sum_{B\in\mathcal{I}_k}\frac{2^{r-1}}{\epsilon}\lambda_E^{\partial}B\le\frac{5^r2^{r-1}}{\epsilon}\lambda_E^{\partial}G\le 5^r2^{r-1}\epsilon.$$

As δ is arbitrary, $\mu_{H,r-1}^*F_\epsilon$ is at most $5^r2^{r-1}\epsilon$ (264Fb/471Dc) and $\nu^*F_\epsilon\le 5^r\beta_{r-1}\epsilon$. As ϵ is arbitrary,

$$\partial^*E\setminus\partial^{\$}E\subseteq\{y:y\in\mathbb{R}^r\setminus\partial^{\$}E,\ \limsup_{\delta\downarrow 0}\frac{1}{\delta^{r-1}}\lambda_E^{\partial}B(y,\delta)>0\}$$

is ν-negligible, as claimed.

475E Lemma Let $E\subseteq\mathbb{R}^r$ be a set with locally finite perimeter.
(a) If $A\subseteq\partial^{\$}E$, then $\nu^*A\le(\lambda_E^{\partial})^*A$.
(b) If $A\subseteq\mathbb{R}^r$ and $\nu A=0$, then $\lambda_E^{\partial}A=0$.

proof (a) Given ϵ, $\delta>0$ let $\mathcal{I}$ be the family of non-trivial closed balls $B\subseteq\mathbb{R}^r$ of diameter at most δ such that $\beta_{r-1}(\frac{1}{2}\operatorname{diam}B)^{r-1}\le(1+\epsilon)\lambda_E^{\partial}B$. By 474S, every point of A is the centre of arbitrarily small members of $\mathcal{I}$. By 472Cb, there is a countable family $\mathcal{I}_1\subseteq I$ such that $A\subseteq\bigcup\mathcal{I}_1$ and $\sum_{B\in\mathcal{I}_1}\lambda_E^{\partial}B\le(\lambda_E^{\partial})^*A+\epsilon$. But this means that

$$\sum_{B\in\mathcal{I}_1}(\operatorname{diam}B)^{r-1}\le(1+\epsilon)\frac{2^{r-1}}{\beta_{r-1}}((\lambda_E^{\partial})^*A+\epsilon).$$

As δ is arbitrary,

$$\nu^*A=\frac{\beta_{r-1}}{2^{r-1}}\mu_{H,r-1}^*A\le(1+\epsilon)((\lambda_E^{\partial})^*A+\epsilon).$$

As ϵ is arbitrary, we have the result.

(b) For $n\in\mathbb{N}$, set

$$A_n=\{x:x\in A,\ \lambda_E^{\partial}B(x,\delta)\le 2\beta_{r-1}\delta^{r-1}\text{ whenever }0<\delta\le 2^{-n}\}.$$

Now, given $\epsilon>0$, there is a sequence $\langle D_i\rangle_{i\in\mathbb{N}}$ of sets covering A_n such that $\operatorname{diam}D_i\le 2^{-n}$ for every i and $\sum_{i=0}^{\infty}(\operatorname{diam}D_i)^{r-1}\le\epsilon$. Passing over the trivial case $A_n=\emptyset$, we may suppose that for each $i\in\mathbb{N}$ there is an $x_i\in A_n\cap D_i$, so that $D_i\subseteq B(x_i,\operatorname{diam}D_i)$ and

$$\begin{aligned}(\lambda_E^{\partial})^*A_n&\le\sum_{i=0}^{\infty}(\lambda_E^{\partial})^*D_i\le\sum_{i=0}^{\infty}\lambda_E^{\partial}B(x_i,\operatorname{diam}D_i)\\&\le\sum_{i=0}^{\infty}2\beta_{r-1}(\operatorname{diam}D_i)^{r-1}\le 2\beta_{r-1}\epsilon.\end{aligned}$$

As ϵ is arbitrary, $\lambda_E^{\partial}A_n=0$. And this is true for every n. As $\bigcup_{n\in\mathbb{N}}A_n\supseteq A\cap\partial^{\$}E$ (474S again), $A\setminus\bigcup_{n\in\mathbb{N}}A_n$ is λ_E^{∂}-negligible (474G), and so is A.

475F Lemma Let $E\subseteq\mathbb{R}^r$ be a set with locally finite perimeter, and $\epsilon>0$. Then λ_E^{∂} is inner regular with respect to the family $\mathcal{E}=\{F:F\subseteq\mathbb{R}^r\text{ is Borel},\ \lambda_E^{\partial}F\le(1+\epsilon)\nu F\}$.

proof (a) We need some elementary bits of geometry.

(i) If $x \in \mathbb{R}^r$, $\delta > 0$, $\alpha \ge 0$ and $v \in S_{r-1}$, then

$$\mu\{z : z \in B(x,\delta),\, |(z-x)\mathbin{\bullet} v| \le \alpha\} \le 2\alpha\beta_{r-1}\delta^{r-1}.$$

P Translating and rotating, if necessary, we can reduce to the case $x = \mathbf{0}$, $v = (0,\dots,1)$. In this case we are looking at

$$\{z : \|z\| \le \delta,\, |\zeta_r| \le \alpha\} \subseteq \{u : u \in \mathbb{R}^{r-1},\, \|u\| \le \delta\} \times [-\alpha,\alpha]$$

which has measure $2\alpha\beta_{r-1}\delta^{r-1}$. **Q**

(ii) If $x \in \mathbb{R}^r$, $\delta > 0$, $0 < \eta \le \frac{1}{2}$, $v \in S_{r-1}$, $H = \{z : z\mathbin{\bullet} v \le \alpha\}$ and

$$|\mu(H \cap B(x,\delta)) - \frac{1}{2}\mu B(x,\delta)| < 2^{-r+1}\beta_{r-1}\eta\delta^r,$$

then $|x\mathbin{\bullet} v - \alpha| \le \eta\delta$. **P** Again translating and rotating if necessary, we may suppose that $x = \mathbf{0}$ and $v = (0,\dots,1)$. Set $H_0 = \{z : \zeta_r \le 0\}$. **?** If $\alpha > \eta\delta$, then $H \cap B(\mathbf{0},\delta)$ includes

$$(H_0 \cap B(\mathbf{0},\delta)) \cup (\{u : u \in \mathbb{R}^{r-1},\, \|u\| \le \frac{1}{2}\delta\} \times [0,\alpha'])$$

where $\alpha' = \min(|\alpha|, \frac{\sqrt{3}}{2}\delta) > \eta\delta)$, so

$$\begin{aligned}\mu(H \cap B(\mathbf{0},\delta)) - \frac{1}{2}\mu B(\mathbf{0},\delta) &= \mu(H \cap B(\mathbf{0},\delta)) - \mu(H_0 \cap B(\mathbf{0},\delta))\\ &\ge 2^{-r+1}\beta_{r-1}\delta^{r-1}\alpha' > 2^{-r+1}\beta_{r-1}\delta^r\eta,\end{aligned}$$

contrary to hypothesis. **X** Similarly, **?** if $\alpha < -\eta\delta$, then $H \cap B(\mathbf{0},\delta)$ is included in

$$H_0 \cap B(\mathbf{0},\delta) \setminus (\{u : u \in \mathbb{R}^{r-1},\, \|u\| \le \frac{1}{2}\delta\} \times \,]\alpha, 0]),$$

so

$$\begin{aligned}\frac{1}{2}\mu B(\mathbf{0},\delta) - \mu(H \cap B(\mathbf{0},\delta)) &= \mu(H_0 \cap B(\mathbf{0},\delta)) - \mu(H \cap B(\mathbf{0},\delta))\\ &\ge 2^{-r+1}\beta_{r-1}\delta^{r-1}\alpha' > 2^{-r+1}\beta_{r-1}\delta^r\eta,\end{aligned}$$

which is equally impossible. **X** So $|\alpha| \le \eta\delta$. **Q**

(b) Let F be such that $\lambda^{\partial}_E F > 0$. Let η, $\zeta > 0$ be such that

$$\eta < 1, \quad \frac{(1+\eta)^2}{(1-\eta)^{r-1}} \le 1 + \epsilon, \quad 2(1+2^r)\beta_r\zeta < 2^{-r}\beta_{r-1}\eta.$$

Because $\partial^{\$}E$ is λ^{∂}_E-conegligible (474G again), $\lambda^{\partial}_E(F \cap \partial^{\$}E) > 0$. Because λ^{∂}_E is a Radon measure (474E) and $\psi_E : \partial^{\$}E \to S_{r-1}$ is $\mathrm{dom}(\lambda^{\partial}_E)$-measurable (474E(i), 474G), there is a compact set $K_1 \subseteq F \cap \partial^{\$}E$ such that $\lambda^{\partial}_E K_1 > 0$ and $\psi_E \restriction K_1$ is continuous, by Lusin's theorem (418J). For $x \in \partial^{\$}E$, set $H_x = \{z : (z-x)\mathbin{\bullet}\psi_E(x) \le 0\}$. The function

$$(x,\delta) \mapsto \mu((E \triangle H_x) \cap B(x,\delta)) : K_1 \times \,]0,\infty[\, \to \mathbb{R}$$

is Borel measurable. **P** Take a Borel set E' such that $\mu(E \triangle E') = 0$. Then

$$\{(x,\delta,z) : x \in K_1,\, z \in (E' \triangle H_x) \cap B(x,\delta)\}$$

is a Borel set in $\mathbb{R}^r \times \,]0,\infty[\, \times \mathbb{R}^r$, so its sectional measure is a Borel measurable function, by 252P. **Q**

For each $x \in K_1$,

$$\lim_{n\to\infty} \frac{\mu((E \triangle H_x) \cap B(x,2^{-n}))}{\mu B(x,2^{-n})} = 0$$

(474R). So there is an $n_0 \in \mathbb{N}$ such that $\lambda^{\partial}_E F_1 > 0$, where F_1 is the Borel set

$$\{x : x \in K_1,\, \mu((E \triangle H_x) \cap B(x,2^{-n})) \le \zeta\mu B(x,2^{-n}) \text{ for every } n \ge n_0\}.$$

Let $K_2 \subseteq F_1$ be a compact set such that $\lambda^\partial_E K_2 > 0$.

For each $n \in \mathbb{N}$, the function

$$x \mapsto \lambda^\partial_E B(x, 2^{-n})) = \lambda^\partial_E (x + B(\mathbf{0}, 2^{-n}))$$

is Borel measurable (444Fe). Let $y \in K_2$ be such that $\lambda^\partial_E(K_2 \cap B(y,\delta)) > 0$ for every $\delta > 0$ (cf. 411Nd). Set $v = \psi_E(y)$. Let $n > n_0$ be so large that $2\beta_{r-1}\|\psi_E(x) - v\| \le \beta_r\zeta$ whenever $x \in K_1$ and $\|x - y\| \le 2^{-n}$. Set $K_3 = K_2 \cap B(y, 2^{-n-1})$, so that $\lambda^\partial_E K_3 > 0$.

(c) We have $|(x - z)\mathbin{.}v| \le \eta\|x - z\|$ whenever x, $z \in K_3$. **P** If $x = z$ this is trivial. Otherwise, let $k \ge n$ be such that $2^{-k-1} \le \|x - z\| \le 2^{-k}$, and set $\delta = 2^{-k}$. Set

$$H'_x = \{w : (w - x)\mathbin{.}v \le 0\}, \quad H'_z = \{w : (w - z)\mathbin{.}v \le 0\}.$$

Since $|(w - x)\mathbin{.}v - (w - x)\mathbin{.}\psi_E(x)| \le 2\delta\|\psi_E(x) - v\|$ whenever $w \in B(x, 2\delta)$,

$$(H_x \triangle H'_x) \cap B(x, 2\delta) \subseteq \{w : w \in B(x, 2\delta),\ |(w - x)\mathbin{.}v| \le 2\delta\|\psi_E(x) - v\|\}$$

has measure at most

$$4\delta\|\psi_E(x) - v\|\beta_{r-1}(2\delta)^{r-1} \le 2\delta\beta_r\zeta(2\delta)^{r-1} = \zeta\mu B(x, 2\delta),$$

using (a-i) for the first inequality. So

$$\begin{aligned}\mu((E \triangle H'_x) \cap B(x, 2\delta)) &\le \mu((E \triangle H_x) \cap B(x, 2\delta)) + \mu((H_x \triangle H'_x) \cap B(x, 2\delta))\\ &\le 2\zeta\mu B(x, 2\delta)\end{aligned}$$

because $k > n_0$ and $x \in F_1$. Similarly, $\mu((E \triangle H'_z) \cap B(z, \delta)) \le 2\zeta\mu B(z, \delta)$. Now observe that because $\|x - z\| \le \delta$, $B(z, \delta) \subseteq B(x, 2\delta)$,

$$\mu((E \triangle H'_x) \cap B(z, \delta)) \le 2\zeta\mu B(x, 2\delta) = 2^{r+1}\zeta\mu B(z, \delta),$$

and

$$\begin{aligned}\mu((H'_x \triangle H'_z) \cap B(z, \delta)) &\le \mu((E \triangle H'_x) \cap B(z, \delta)) + \mu((E \triangle H'_z) \cap B(z, \delta))\\ &\le (2 + 2^{r+1})\zeta\mu B(z, \delta).\end{aligned}$$

Since $\mu(H'_z \cap B(z, \delta)) = \frac{1}{2}\mu B(z, \delta)$,

$$|\mu(H'_x \cap B(z, \delta)) - \frac{1}{2}\mu B(z, \delta)| \le 2(1 + 2^r)\zeta\mu B(z, \delta) < 2^{-r}\beta_{r-1}\eta\delta^r,$$

and (using (a-ii) above)

$$|(x - z)\mathbin{.}v| \le \frac{1}{2}\eta\delta \le \eta\|x - v\|. \textbf{ Q}$$

(d) Let V be the hyperplane $\{w : w\mathbin{.}v = 0\}$, and let $T : K_3 \to V$ be the orthogonal projection, that is, $Tx = x - (x\mathbin{.}v)v$ for every $x \in K_3$. Then (c) tells us that if x, $z \in K_3$,

$$\|Tx - Tz\| \ge \|x - z\| - |(x - z)\mathbin{.}v| \ge (1 - \eta)\|x - z\|.$$

Because $\eta < 1$, T is injective. Consider the compact set $T[K_3]$. The inverse T^{-1} of T is $\frac{1}{1-\eta}$-Lipschitz, and $\nu K_3 > 0$ (by 475Eb), so $\nu(T[K_3]) \ge (1 - \eta)^{r-1}\nu K_3 > 0$ (264G/471J). Let $G \supseteq T[K_3]$ be an open set such that $\nu(G \cap V) \le (1 + \eta)\nu(T[K_3])$. (I am using the fact that the subspace measure ν_V induced by ν on V is a copy of Lebesgue measure on $\mathbb{R}^{r-1}$, so is a Radon measure.) Let $\mathcal{I}$ be the family of non-trivial closed balls $B \subseteq G$ such that $(1 - \eta)^{r-1}\lambda^\partial_E T^{-1}[B] \le (1 + \eta)\nu(B \cap V)$. Then every point of $T[K_3]$ is the centre of arbitrarily small members of $\mathcal{I}$. **P** If $x \in K_3$ and $\delta_0 > 0$, there is a $\delta \in \,]0, \delta_0]$ such that $B(Tx, \delta) \subseteq G$ and $\lambda^\partial_E B(x, \delta) \le (1+\eta)\beta_{r-1}\delta^{r-1}$ (474S once more). Now consider $B = B(Tx, (1-\eta)\delta)$. Then $T^{-1}[B] \subseteq B(x, \delta)$, so

$$\lambda^\partial_E T^{-1}[B] \le \lambda^\partial_E B(x, \delta) \le (1 + \eta)\beta_{r-1}\delta^{r-1} = \frac{1+\eta}{(1-\eta)^{r-1}}\nu(B \cap V). \textbf{ Q}$$

By 261B/472Ca, applied in $V\cong\mathbb{R}^{r-1}$, there is a countable disjoint family $\mathcal{I}_0\subseteq\mathcal{I}$ such that $\nu(T[K_3]\setminus\bigcup\mathcal{I}_0)=0$.

Now $\nu(K_3\setminus\bigcup_{B\in\mathcal{I}_0}T^{-1}[B])=0$, because T^{-1} is Lipschitz, so $\lambda^\partial_E(K_3\setminus\bigcup_{B\in\mathcal{I}_0}T^{-1}[B])=0$ (475Eb again), and

$$\begin{aligned}\lambda^\partial_E K_3&\le\sum_{B\in\mathcal{I}_0}\lambda^\partial_E T^{-1}[B]\le\tfrac{1+\eta}{(1-\eta)^{r-1}}\sum_{B\in\mathcal{I}_0}\nu(B\cap V)\\&\le\tfrac{1+\eta}{(1-\eta)^{r-1}}\nu(G\cap V)\le\tfrac{(1+\eta)^2}{(1-\eta)^{r-1}}\nu(T[K_3])\le\tfrac{(1+\eta)^2}{(1-\eta)^{r-1}}\nu K_3\end{aligned}$$

(because T is 1-Lipschitz)

$$\le(1+\epsilon)\nu K_3$$

by the choice of η. Thus $K_3\in\mathcal{E}$.

(e) This shows that every λ^∂_E-non-negligible set measured by λ^∂_E includes a λ^∂_E-non-negligible member of $\mathcal{E}$. As $\mathcal{E}$ is closed under disjoint unions, λ^∂_E is inner regular with respect to $\mathcal{E}$ (412Aa).

475G Theorem Let $E\subseteq\mathbb{R}^r$ be a set with locally finite perimeter. Then $\lambda^\partial_E=\nu\llcorner\partial^*E$, that is, for $F\subseteq\mathbb{R}^r$, $\lambda^\partial_E F=\nu(F\cap\partial^*E)$ whenever either is defined.

proof (a) Suppose first that F is a Borel set included in the reduced boundary $\partial^\$E$ of E. Then $\nu F\le\lambda^\partial_E F$, by 475Ea. On the other hand, for any $\epsilon>0$ and $\gamma<\lambda^\partial_E F$, there is an $F_1\subseteq F$ such that

$$\gamma\le\lambda^\partial_E F_1\le(1+\epsilon)\nu F_1\le(1+\epsilon)\nu F,$$

by 475F; so we must have $\lambda^\partial_E F=\nu F$.

(b) Now suppose that F is measured by λ^∂_E. Because $\partial^\$E$ is λ^∂_E-conegligible, and λ^∂_E is a σ-finite Radon measure, there is a Borel set $F'\subseteq F\cap\partial^\$E$ such that $\lambda^\partial_E(F\setminus F')=0$. Now $\nu F'=\lambda^\partial_E F'$, by (a), and $\nu(F\cap\partial^\$E\setminus F')=0$, by 475Ea, and $\nu(\partial^*E\setminus\partial^\$E)=0$, by 475D; so $\nu(F\cap\partial^*E)$ is defined and equal to $\lambda^\partial_E F'=\lambda^\partial_E F$.

(c) Let $\langle K_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of compact subsets of $\partial^\$E$ such that $\bigcup_{n\in\mathbb{N}}K_n$ is λ^∂_E-conegligible. By (b), $\nu(\partial^*E\setminus\bigcup_{n\in\mathbb{N}}K_n)=0$, while $\nu K_n=\lambda^\partial_E K_n$ is finite for each n. For each n, the subspace measure ν_{K_n} on K_n is a multiple of Hausdorff $(r-1)$-dimensional measure on K_n (471E), so is a Radon measure (471Dh, 471F), as is $(\lambda^\partial_E)_{K_n}$; since, by (b), ν_{K_n} and $(\lambda^\partial_E)_{K_n}$ agree on the Borel subsets of K_n, they are actually identical. So if $F\subseteq\mathbb{R}^r$ is such that ν measures $F\cap\partial^*E$, λ^∂_E will measure $F\cap K_n$ for every n, and therefore will measure F; so that in this case also $\lambda^\partial_E F=\nu(F\cap\partial^*E)$.

475H Proposition Let $V\subseteq\mathbb{R}^r$ be a hyperplane, and $T:\mathbb{R}^r\to V$ the orthogonal projection. Suppose that $A\subseteq\mathbb{R}^r$ is such that νA is defined and finite, and for $u\in V$ set

$$\begin{aligned}f(u)&=\#(A\cap T^{-1}[\{u\}])\text{ if this is finite,}\\&=\infty\text{ otherwise.}\end{aligned}$$

Then $\int_V f(u)\nu(du)$ is defined and at most νA.

proof (a) Because ν is invariant under isometries, we can suppose that $V=\{x:\xi_r=0\}$, so that $Tx=(\xi_1,\dots,\xi_{r-1},0)$ for $x=(\xi_1,\dots,\xi_r)$. For m, $n\in\mathbb{N}$ and $u\in V$ set

$$f_{mn}(u)=\#(\{k:k\in\mathbb{Z},\,|k|\le4^m,\,A\cap(\{u\}\times[2^{-m}k,2^{-m}(k+1-2^{-n})[)\ne\emptyset\});$$

so that $f(u)=\lim_{m\to\infty}\lim_{n\to\infty}f_{mn}(u)$ for every $u\in V$.

(b) Suppose for the moment that A is actually a Borel set. Then $T[A\cap(\mathbb{R}^r\times[\alpha,\beta[)]$ is always analytic (423Eb, 423Bb), therefore measured by ν (432A), and every f_{mn} is measurable. Next, given $\epsilon>0$ and m, $n\in\mathbb{N}$, there is a sequence $\langle F_i\rangle_{i\in\mathbb{N}}$ of closed sets of diameter at most 2^{-m-n}, covering A, such that $\sum_{i=0}^\infty2^{-r+1}\beta_{r-1}(\operatorname{diam}F_i)^{r-1}\le\nu A+\epsilon$. Now each $T[F_i]$ is a compact set of diameter at most $\operatorname{diam}F_i$, so

$\nu(T[F_i]) \le 2^{-r+1}\beta_{r-1}(\operatorname{diam} F_i)^{r-1}$ (264H); and if we set $g = \sum_{i=0}^{\infty} \chi T[F_i]$, $f_{mn} \le g$ everywhere on V, so $\int f_{mn} d\nu \le \int g\, d\nu \le \nu A + \epsilon$. As ϵ is arbitary, $\int f_{mn} d\nu \le \nu A$. Accordingly

$$\int f d\nu = \lim_{m\to\infty} \lim_{n\to\infty} \int f_{mn} d\nu$$

(because the limits are monotonic)

$$\le \nu A + \epsilon.$$

As ϵ is arbitrary, $\int f d\nu \le \nu A$.

(c) In general, there are Borel sets E, F such that $E \setminus F \subseteq A \subseteq E$ and $\nu F = 0$, by 264Fc/471Db. By (b),

$$\int \#(E \cap T^{-1}[\{u\}])\nu(du) \le \nu E, \quad \int \#(F \cap T^{-1}[\{u\}])\nu(du) \le \nu F = 0,$$

so $\int \#(A \cap T^{-1}[\{u\}])\nu(du)$ is defined and equal to $\int \#(E \cap T^{-1}[\{u\}])\nu(du) \le \nu A$.

475I Lemma (In this lemma I allow $r = 1$.) Let $\mathcal{K}$ be the family of compact sets $K \subseteq \mathbb{R}^r$ such that $K = \mathrm{cl}^* K$. Then μ is inner regular with respect to $\mathcal{K}$.

proof (a) Write $\mathcal{D}$ for the set of dyadic (half-open) cubes in $\mathbb{R}^r$, that is, sets expressible in the form $\prod_{i<r}[2^{-m}k_i, 2^{-m}(k_i+1)[$ where $m, k_0, \dots, k_{r-1} \in \mathbb{Z}$. For $m \in \mathbb{N}$ and $x \in \mathbb{R}^r$ write $C(x,m)$ for the dyadic cube with side of length 2^{-m} which contains x. Then, for any $A \subseteq \mathbb{R}^r$,

$$\lim_{m\to\infty} 2^{mr}\mu_*(A \cap C(x,m)) = 1$$

for every $x \in \mathrm{int}^* A$. **P** $C(x,m) \subseteq B(x, 2^{-m}\sqrt{r})$ for each m, so

$$2^{mr}\mu^*(C(x,m) \setminus A) \le \beta_r r^{r/2} \frac{\mu^*(B(x,2^{-m}\sqrt{r}) \setminus A)}{\mu B(x,2^{-m}\sqrt{r})} \to 0$$

as $m \to \infty$. **Q**

(b) Now, given a Lebesgue measurable set $E \subseteq \mathbb{R}^r$ and $\gamma < \mu E$, choose $\langle E_n\rangle_{n\in\mathbb{N}}$, $\langle \gamma_n\rangle_{n\in\mathbb{N}}$, $\langle m_n\rangle_{n\in\mathbb{N}}$ and $\langle K_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. Start with $E_0 = E$ and $\gamma_0 = \gamma$. Given that $\gamma_n < \mu E_n$, let $K_n \subseteq E_n$ be a compact set of measure greater than γ_n. Now, by (a), there is an $m_n \ge n$ such that $\mu^* E_{n+1} > \gamma_n$, where

$$E_{n+1} = \{x : x \in K_n,\ \mu(K_n \cap C(x,m_n)) \ge \frac{2}{3}\mu C(x,m_n)\};$$

note that E_{n+1} is of the form $K_n \cap \bigcup \mathcal{D}_0$ for some set $\mathcal{D}_0$ of half-open cubes of side 2^{-m_n}, so that E_{n+1} is measurable and

$$\mu(E_{n+1} \cap C(x,m_n)) = \mu(K_n \cap C(x,m_n)) \ge \frac{2}{3}\mu C(x,m_n)$$

for every $x \in E_{n+1}$. Now set

$$\gamma_{n+1} = \max(\gamma_n, \mu E_{n+1} - \frac{1}{3} \cdot 2^{-m_n r}),$$

and continue.

At the end of the induction, set $K = \bigcap_{n\in\mathbb{N}} K_n = \bigcap_{n\in\mathbb{N}} E_n$. Then K is compact, $K \subseteq E$ and

$$\mu K = \lim_{n\to\infty} \mu K_n = \lim_{n\to\infty} \gamma_n \ge \gamma.$$

If $x \in K$ and $n \in \mathbb{N}$, then

$$\begin{aligned}\mu(K \cap B(x, 2^{-m_n}\sqrt{r})) &\ge \mu(K \cap C(x,m_n)) \ge \mu(E_{n+1} \cap C(x,m_n)) - \mu E_{n+1} + \mu K \\ &\ge \frac{2}{3}\mu C(x,m_n) - \mu E_{n+1} + \gamma_{n+1} \\ &\ge \frac{1}{3} \cdot 2^{-m_n r} = \frac{1}{3\beta_r r^{r/2}}\mu B(x, 2^{-m_n}\sqrt{r}),\end{aligned}$$

so

$$\limsup_{\delta\downarrow 0}\frac{\mu(K\cap B(x,\delta))}{\mu B(x,\delta)}\ge\frac{1}{3\beta_r r^{r/2}}>0$$

and $x\in\mathrm{cl}^*K$.

(c) Thus $K\subseteq\mathrm{cl}^*K$. Since certainly $\mathrm{cl}^*K\subseteq\overline{K}=K$, we have $K=\mathrm{cl}^*K$.

475J Lemma Let E be a Lebesgue measurable subset of $\mathbb{R}^r$, identified with $\mathbb{R}^{r-1}\times\mathbb{R}$. For $u\in\mathbb{R}^{r-1}$, set $G_u=\{t:(u,t)\in\mathrm{int}^*E\}$, $H_u=\{t:(u,t)\in\mathrm{int}^*(\mathbb{R}^r\setminus E)\}$ and $D_u=\{t:(u,t)\in\partial^*E\}$, so that G_u, H_u and D_u are disjoint and cover $\mathbb{R}$.

(a) There is a μ_{r-1}-conegligible set $Z\subseteq\mathbb{R}^{r-1}$ such that whenever $u\in Z$, $t<t'$ in $\mathbb{R}$, $t\in G_u$ and $t'\in H_u$, there is an $s\in D_u\cap\,]t,t'[$.

(b) There is a μ_{r-1}-conegligible set $Z_1\subseteq\mathbb{R}^{r-1}$ such that whenever $u\in Z_1$, t, $t'\in\mathbb{R}$, $t\in G_u$ and $t'\in H_u$, there is a member of D_u between t and t'.

(c) If E has locally finite perimeter, there is a conegligible set $Z_2\subseteq Z_1$ such that, for every $u\in Z_2$, $D_u\cap[-n,n]$ is finite for every $n\in\mathbb{N}$, G_u and H_u are open, and $D_u=\partial G_u=\partial H_u$, so that the constituent intervals of $\mathbb{R}\setminus D_u$ lie alternately in G_u and H_u.

proof (a)(i) For $q\in\mathbb{Q}$, set $f_q(u)=\sup(G_u\cap\,]-\infty,q[)$, taking $-\infty$ for $\sup\emptyset$. Then $f_q:\mathbb{R}^{r-1}\to[-\infty,q]$ is Lebesgue measurable. **P** For $\alpha<q$,

$$\{u:f_q(u)>\alpha\}=\{u:\text{there is some }t\in\,]\alpha,q[\text{ such that }(u,t)\in\mathrm{int}^*E\}.$$

Because int^*E is a Borel set (475Cc), $\{u:f_q(u)>\alpha\}$ is analytic (423Eb, 423Bc), therefore measurable (432A again). **Q**

(ii) For any $q\in\mathbb{Q}$, $W_q=\{u:f_q(u)<q,\ f_q(u)\in G_u\}$ is negligible. **P?** Suppose, if possible, otherwise. Let $n\in\mathbb{N}$ be such that $\{u:u\in W_q,\ f_q(u)>-n\}$ is not negligible. If we think of $[-\infty,q]$ as a compact metrizable space, 418J again tells us that there is a Borel set $F\subseteq\mathbb{R}^{r-1}$ such that $f_q\restriction F$ is continuous and $F_1=\{u:u\in F\cap W_q,\ f_q(u)>-n\}$ is not negligible. Note that F_1 is measurable, being the projection of the Borel set $\{(u,f_q(u)):u\in F,\ -n<f_q(u)<q\}\cap\mathrm{int}^*E$. By 475I, there is a non-negligible compact set $K\subseteq F_1$ such that $K\subseteq\mathrm{cl}^*K$, interpreting cl^*K in $\mathbb{R}^{r-1}$. Because $g\restriction K$ is continuous, it attains its maximum at $u\in K$ say.

Set $x=(u,f_q(u))$. Then, whenever $0<\delta\le\min(n+f_q(u),q-f_q(u))$,

$$B(x,2\delta)\setminus\mathrm{int}^*E\supseteq\{(w,t):w\in K\cap V(u,\delta),\ |t-f_q(u)|\le\delta,\ f_q(w)<t<q\}$$

(writing $V(u,\delta)$ for $\{w:w\in\mathbb{R}^{r-1},\ \|w-u\|\le\delta\}$)

$$\supseteq\{(w,t):w\in K\cap V(u,\delta),\ f_q(u)<t<f_q(u)+\delta\}$$

because $f_q(w)\le f_q(u)$ for every $w\in K$. So, for such δ,

$$\begin{aligned}\mu(B(x,2\delta)\setminus\mathrm{int}^*E)&\ge\delta\mu_{r-1}(K\cap V(u,\delta))\\&=\frac{\beta_{r-1}}{2^r\beta_r}\cdot\frac{\mu_{r-1}(K\cap V(u,\delta))}{\mu_{r-1}V(u,\delta)}\cdot\mu B(x,2\delta),\end{aligned}$$

and

$$\begin{aligned}\limsup_{\delta\downarrow 0}\frac{\mu(B(x,\delta)\setminus E)}{\mu B(x,\delta)}&=\limsup_{\delta\downarrow 0}\frac{\mu(B(x,2\delta)\setminus\mathrm{int}^*E)}{\mu B(x,2\delta)}\\&\ge\frac{\beta_{r-1}}{2^r\beta_r}\limsup_{\delta\downarrow 0}\frac{\mu_{r-1}(K\cap V(u,\delta))}{\mu_{r-1}V(u,\delta)}>0\end{aligned}$$

because $u\in\mathrm{cl}^*K$; but $x=(u,f_q(u))$ is supposed to belong to int^*E. **XQ**

(iii) Similarly, setting

$$f'_q(u)=\inf(H_u\cap\,]u,\infty[),\qquad W'_q=\{u:q<f'_q(u)<\infty,\ f'_q(u)\in H_u\},$$

every W'_q is negligible. So $Z = \mathbb{R}^{r-1} \setminus \bigcup_{q\in\mathbb{Q}}(W_q \cup W'_q)$ is μ_{r-1}-conegligible.

Now if $u \in Z$, $t \in G_u$ and $t' \in H_u$, where $t < t'$, there is some $s \in \,]t, t'[\, \cap D_u$. **P?** Suppose, if possible, otherwise. Since, by hypothesis, neither t nor t' belongs to D_u, $D_u \cap [t, t'] = \emptyset$. Note that, because $u \in Z$,

$$s = \inf(G_u \cap \,]s, \infty[) \text{ for every } s \in G_u, \quad s = \sup(H_u \cap \,]-\infty, s[) \text{ for every } s \in H_u.$$

Choose $\langle s_n\rangle_{n\in\mathbb{N}}$, $\langle s'_n\rangle_{n\in\mathbb{N}}$ and $\langle \delta_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. Set $s_0 = t$, $s'_0 = t'$. Given that $t \le s_n < s'_n \le t'$ and $s_n \in G_u$ and $s'_n \in H_u$, where n is even, set $s'_{n+1} = \sup(G_u \cap [s_n, s'_n])$. Then either $s'_{n+1} = s'_n$, so $s'_{n+1} \in H_u$, or $s'_{n+1} < s'_n$ and $]s'_{n+1}, s'_n] \cap G_u = \emptyset$, so $s'_{n+1} \notin G_u$ and again $s'_{n+1} \in H_u$. Let $\delta_n > 0$ be such that $\delta_n \le 2^{-n}$ and $\mu(E \cap B((u, s'_{n+1}), \delta_n)) < \frac{1}{2}\beta_r\delta_n^r$. Because the function $s \mapsto \mu(E \cap B((u, s), \delta_n))$ is continuous (443C, or otherwise), there is an $s_{n+1} \in G_u \cap [s_n, s'_{n+1}[$ such that $\mu(E \cap B((u, s), \delta_n)) \le \frac{1}{2}\beta_r\delta_n^r$ whenever $s \in [s_{n+1}, s'_{n+1}]$.

This is the inductive step from even n. If n is odd and $s_n \in G_u$, $s'_n \in H_u$ and $s_n < s'_n$, set $s_{n+1} = \inf(H_u \cap [s_n, s'_n])$. This time we find that s_{n+1} must belong to G_u. Let $\delta_n \in \,]0, 2^{-n}]$ be such that $\mu(E \cap B((u, s_{n+1}), \delta_n)) > \frac{1}{2}\beta_r\delta_n^r$, and let $s'_{n+1} \in H_u \cap]s_{n+1}, s'_n]$ be such that $\mu(E \cap B((u, s), \delta_n)) \ge \frac{1}{2}\beta_r\delta_n^r$ whenever $s \in [s_{n+1}, s'_{n+1}]$.

The construction provides us with a non-increasing sequence $\langle [s_n, s'_n]\rangle_{n\in\mathbb{N}}$ of closed intervals, so there must be some s in their intersection. In this case

$$\mu(E \cap B((u, s), \delta_n) \le \tfrac{1}{2}\beta_r\delta_n^r \text{ if } n \text{ is even,}$$

$$\mu(E \cap B((u, s), \delta_n) \ge \tfrac{1}{2}\beta_r\delta_n^r \text{ if } n \text{ is odd.}$$

Since $\lim_{n\to\infty} \delta_n = 0$,

$$\liminf_{\delta\downarrow 0} \frac{\mu(E\cap B((u,s),\delta))}{\mu B((u,s),\delta)} \le \frac{1}{2}, \quad \limsup_{\delta\downarrow 0} \frac{\mu(E\cap B((u,s),\delta))}{\mu B((u,s),\delta)} \ge \frac{1}{2},$$

and $s \in D_u$, while of course $t \le s \le t'$. **XQ**

(iv) Thus the conegligible set Z has the property required by (a).

(b) Applying (a) to $\mathbb{R}^r \setminus E$, there is a conegligible set $Z' \subseteq \mathbb{R}^{r-1}$ such that if $u \in Z'$, $t \in H_u$, $t' \in G_u$ and $t < t'$, then D_u meets $]t, t'[$. So we can use $Z \cap Z'$.

(c) Now suppose that E has locally finite perimeter. We know that $\nu(\partial^*E \cap B(\mathbf{0}, n)) = \lambda_E^\partial B(\mathbf{0},n)$ is finite for every $n \in \mathbb{N}$ (475G). By 475H,

$$\int_{\|u\|\le n} \#(D_u \cap [-n, n])\mu_{r-1}(du) \le \nu(\partial^*E \cap B(\mathbf{0}, 2n)) < \infty$$

for every $n \in \mathbb{N}$; but this means that $D_u \cap [-n, n]$ must be finite for almost every u such that $\|u\| \le n$, for every n, and therefore that

$$Z'_1 = \{u : u \in Z_1,\ D_u \cap [-n, n] \text{ is finite for every } n\}$$

is conegligible. For $u \in Z'_1$, $\mathbb{R} \setminus D_u$ is an open set, so is made up of a disjoint sequence of intervals with endpoints in $D_u \cup \{-\infty, \infty\}$ (see 2A2I); and because $u \in Z_1$, each of these intervals is included in either G_u or H_u. Now

$$A = \{u : \text{there are successive constituent intervals of } \mathbb{R} \setminus D_u \text{ both included in } G_u\}$$

is negligible. **P?** Otherwise, there are rationals $q < q'$ such that

$$F = \{u :]q, q'[\setminus G_u \text{ contains exactly one point}\}$$

is not negligible. Note that, writing $T : \mathbb{R}^r \to \mathbb{R}^{r-1}$ for the orthogonal projection,

$$F = T[(\mathbb{R}^{r-1} \times]q, q'[) \setminus \mathrm{int}^*E] \setminus \bigcup_{q''\in\mathbb{Q}, q<q''<q'} T[(\mathbb{R}^{r-1} \times]q, q''[) \setminus \mathrm{int}^*E] \cap T[(\mathbb{R}^{r-1} \times]q'', q'[) \setminus \mathrm{int}^*E]$$

is measurable. Take any $u \in F \cap \mathrm{int}^*F$, and let t be the unique point in $]q, q'[\setminus G_u$. Then whenever $0 < \delta \le \min(t - q, q' - t)$ we shall have

$$\mu(B((u,t),\delta)\setminus E)=\mu(B((u,t),\delta)\setminus\mathrm{int}^*E)\le 2\delta\mu_{r-1}(V(u,\delta)\setminus F),$$

because if $w\in F$ then $(w,s)\in\mathrm{int}^*E$ for almost every $s\in[t-\delta,t+\delta]$. So

$$\limsup_{\delta\downarrow 0}\frac{\mu(B((u,t),\delta)\setminus E)}{\mu B((u,t),\delta)}\le\frac{2\beta_{r-1}}{\beta_r}\limsup_{\delta\downarrow 0}\frac{\mu_{r-1}(V(u,\delta)\setminus F)}{\mu_{r-1}V(u,\delta)}=0,$$

and $(u,t)\in\mathrm{int}^*E$. **XQ**

Similarly,

$$A'=\{u:\text{there are successive constituent intervals of }\mathbb{R}\setminus D_u\text{ both included in }H_u\}$$

is negligible. So $Z_2=Z_1'\setminus(A\cup A')$ is a conegligible set of the kind we need.

475K Lemma Suppose that $h:\mathbb{R}^r\to[-1,1]$ is a Lipschitz function with compact support; let $n\in\mathbb{N}$ be such that $h(x)=0$ for $\|x\|\ge n$. Suppose that $E\subseteq\mathbb{R}^r$ is a Lebesgue measurable set. Then

$$|\int_E\frac{\partial h}{\partial\xi_r}d\mu|\le 2\Bigl(\beta_{r-1}n^{r-1}+\nu(\partial^*E\cap B(\mathbf{0},n))\Bigr).$$

proof By Rademacher's theorem (262Q), $\dfrac{\partial h}{\partial\xi_r}$ is defined almost everywhere; as it is measurable and bounded, and is zero outside $B(\mathbf{0},n)$, the integral is well-defined. If $\nu(\partial^*E\cap B(\mathbf{0},n))$ is infinite, the result is trivial; so henceforth let us suppose that $\nu(\partial^*E\cap B(\mathbf{0},n))<\infty$. Identify $\mathbb{R}^r$ with $\mathbb{R}^{r-1}\times\mathbb{R}$. For $u\in\mathbb{R}^{r-1}$, set

$$\begin{aligned}f(u)&=\#(\{t:(u,t)\in\partial^*E\cap B(\mathbf{0},n)\})\text{ if this is finite,}\\&=\infty\text{ otherwise.}\end{aligned}$$

By 475H, $\int fd\mu_{r-1}\le\nu(\partial^*E\cap B(\mathbf{0},n))$. By 475Jb, there is a μ_{r-1}-conegligible set $Z\subseteq\mathbb{R}^{r-1}$ such that $f(u)$ is finite for every $u\in Z$ and

whenever $u\in Z$ and $(u,t)\in\mathrm{int}^*E$ and $(u,t')\in\mathbb{R}^r\setminus\mathrm{cl}^*E$, there is an s lying between t and t' such that $(u,s)\in\partial^*E$.

For $u\in Z$, set $D_u'=\{t:(u,t)\in B(\mathbf{0},n)\setminus\partial^*E\}$, and define $g_u:D_u'\to\{0,1\}$ by setting

$$\begin{aligned}g_u(t)&=1\text{ if }(u,t)\in B(\mathbf{0},n)\cap\mathrm{int}^*E,\\&=0\text{ if }(u,t)\in B(\mathbf{0},n)\setminus\mathrm{cl}^*E.\end{aligned}$$

Now if t, $t'\in D_u'$ and $g(t)\ne g(t')$, there is a point s between t and t' such that $(u,s)\in\partial^*E$; so the variation $\mathrm{Var}_{D_u'}\,g_u$ of g_u (224A) is at most $f(u)$. Setting $h_u(t)=h(u,t)$ for $u\in\mathbb{R}^{r-1}$ and $t\in\mathbb{R}$, we now have

$$\int_{-\infty}^{\infty}\frac{\partial h}{\partial\xi_r}(u,t)\chi(B(\mathbf{0},n)\cap\mathrm{int}^*E)(u,t)dt=\int_{D_u'}h_u'(t)g_u(t)dt$$

(because $h_u'(t)=0$ if $(u,t)\notin B(\mathbf{0},n)$)

$$\le(1+\mathrm{Var}_{D_u'}\,g_u)\sup_{a\le b}|\int_a^b h_u'(t)|dt$$

(by 224J, recalling that D_u' is either empty or a bounded interval with finitely many points deleted)

$$\le(1+f(u))\sup_{a\le b}|h_u(b)-h_u(a)|$$

(because h_u is Lipschitz, therefore absolutely continuous on any bounded interval)

$$\le 2(1+f(u)).$$

Integrating over u, we now have

$$|\int_E\frac{\partial h}{\partial\xi_r}d\mu|=|\int_{B(\mathbf{0},n)\cap\mathrm{int}^*E}\frac{\partial h}{\partial\xi_r}d\mu|$$

(475Cg)

$$= \int_{V(\mathbf{0},n)} \int_{-\infty}^{\infty} \frac{\partial h}{\partial \xi_r}(u,t)\chi(B(\mathbf{0},n)\cap \operatorname{int}^*E)(u,t)dt\,\mu_{r-1}(du)$$

(where $V(\mathbf{0},n) = \{u : u \in \mathbb{R}^{r-1}, \|u\| \le n\}$)

$$\le \int_{V(\mathbf{0},n)} 2(1+f(u))\mu_{r-1}(du)$$

$$\le 2\bigl(\beta_{r-1}n^{r-1} + \nu(\partial^*E \cap B(\mathbf{0},n))\bigr).$$

475L Theorem Suppose that $E \subseteq \mathbb{R}^r$. Then E has locally finite perimeter iff $\nu(\partial^*E\cap B(\mathbf{0},n))$ is finite for every $n \in \mathbb{N}$.

proof If E has locally finite perimeter, then $\nu(\partial^*E\cap B(\mathbf{0},n)) = \lambda_E^\partial B(\mathbf{0},n)$ is finite for every n, by 475G. So let us suppose that $\nu(\partial^*E \cap B(\mathbf{0},n))$ is finite for every $n \in \mathbb{N}$. Then $\mu(\partial^*E\cap B(\mathbf{0},n)) = 0$ for every n (471L), $\mu(\partial^*E) = 0$ and E is Lebesgue measurable (475Ch).

If $n \in \mathbb{N}$, then

$$\sup\{|\int_E \operatorname{div}\phi\,d\mu| : \phi : \mathbb{R}^r \to \mathbb{R}^r \text{ is Lipschitz, } \|\phi\| \le \chi B(\mathbf{0},n)\}$$
$$\le 2r\bigl(\beta_{r-1}n^{r-1} + \nu(\partial^*E\cap B(\mathbf{0},n))\bigr)$$

is finite. **P** If $\phi = (\phi_1,\dots,\phi_r) : \mathbb{R}^r \to \mathbb{R}^r$ is a Lipschitz function and $\|\phi\| \le \chi B(\mathbf{0},n)$, then 475K tells us that

$$|\int_E \frac{\partial \phi_r}{\partial \xi_r}d\mu| \le 2\Bigl(\beta_{r-1}n^{r-1} + \nu(\partial^*E\cap B(\mathbf{0},n))\Bigr).$$

But of course it is equally true that

$$|\int_E \frac{\partial \phi_j}{\partial \xi_j}d\mu| \le 2\Bigl(\beta_{r-1}n^{r-1} + \nu(\partial^*E\cap B(\mathbf{0},n))\Bigr)$$

for every other $j \le r$; adding, we have the result. **Q**

Since n is arbitrary, E has locally finite perimeter.

475M Corollary (a) The family of sets with locally finite perimeter is a subalgebra of the algebra of Lebesgue measurable subsets of $\mathbb{R}^r$.

(b) A set $E \subseteq \mathbb{R}^r$ is Lebesgue measurable and has finite perimeter iff $\nu(\partial^*E) < \infty$, and in this case $\nu(\partial^*E)$ is the perimeter of E.

(c) If $E \subseteq \mathbb{R}^r$ has finite measure, then $\operatorname{per} E = \liminf_{\alpha\to\infty} \operatorname{per}(E\cap B(\mathbf{0},\alpha))$.

proof (a) Recall that the definition in 474D insists that sets with locally finite perimeter should be Lebesgue measurable. If $E \subseteq \mathbb{R}^r$ has locally finite perimeter, then so has $\mathbb{R}^r\setminus E$, by 474J. If E, $F \subseteq \mathbb{R}^r$ have locally finite perimeter, then

$$\nu(\partial^*(E\cup F)\cap B(\mathbf{0},n)) \le \nu(\partial^*E\cap B(\mathbf{0},n)) + \nu(\partial^*F\cap B(\mathbf{0},n))$$

is finite for every $n \in \mathbb{N}$, by 475L and 475Cd; by 475L in the other direction, $E \cup F$ has locally finite perimeter.

(b) If E is Lebesgue measurable and has finite perimeter (on the definition in 474D), then $\nu(\partial^*E) = \lambda_E^\partial\mathbb{R}^r$ is the perimeter of E, by 475G. If $\nu(\partial^*E) < \infty$, then $\mu(\partial^*E) = 0$ and E is measurable (471L and 475Ch); now E has locally finite perimeter, by 475L, and 475G again tells us that $\nu(\partial^*E) = \lambda_E^\partial\mathbb{R}^r$ is the perimeter of E.

(c) Now suppose that $\mu E < \infty$. For any $\alpha \ge 0$,

$$\partial^*(E\cap B(\mathbf{0},\alpha)) \subseteq (\partial^*E\cap B(\mathbf{0},\alpha)) \cup (\operatorname{cl}^*E\cap \partial B(\mathbf{0},\alpha)) \subseteq \partial^*E \cup (\operatorname{cl}^*E\cap\partial B(\mathbf{0},\alpha))$$

by 475Cf. Now we know also that

$$\int_0^\infty \nu(\operatorname{cl}^*E\cap\partial B(\mathbf{0},\alpha))d\alpha = \mu(\operatorname{cl}^*E) = \mu E < \infty$$

(265G), so $\liminf_{\alpha\to\infty}\nu(\mathrm{cl}^*E\cap\partial B(\mathbf{0},\alpha))=0$. This means that

$$\begin{aligned}\liminf_{\alpha\to\infty}\mathrm{per}(E\cap B(\mathbf{0},\alpha))&=\liminf_{\alpha\to\infty}\nu(\partial^*(E\cap B(\mathbf{0},\alpha)))\\&\le\liminf_{\alpha\to\infty}\nu(\partial^*E)+\nu(\mathrm{cl}^*E\cap\partial B(\mathbf{0},\alpha))=\nu(\partial^*E).\end{aligned}$$

In the other direction,

$$\partial^*E\cap\mathrm{int}\,B(\mathbf{0},\alpha)=\partial^*E\cap\mathrm{int}^*B(\mathbf{0},\alpha)\subseteq\partial^*(E\cap B(\mathbf{0},\alpha))$$

for every α, by 475Ce, so

$$\begin{aligned}\mathrm{per}\,E=\nu(\partial^*E)&=\lim_{\alpha\to\infty}\nu(\partial^*E\cap\mathrm{int}\,B(\mathbf{0},\alpha))\\&\le\liminf_{\alpha\to\infty}\nu(\partial^*(E\cap B(\mathbf{0},\alpha)))=\liminf_{\alpha\to\infty}\mathrm{per}(E\cap B(\mathbf{0},\alpha))\end{aligned}$$

and we must have equality.

Remark See 475Xk.

475N The Divergence Theorem Let $E\subseteq\mathbb{R}^r$ be such that $\nu(\partial^*E\cap B(\mathbf{0},n))$ is finite for every $n\in\mathbb{N}$.

(a) E is Lebesgue measurable.

(b) For ν-almost every $x\in\partial^*E$, there is a Federer exterior normal v_x of E at x.

(c) For every Lipschitz function $\phi:\mathbb{R}^r\to\mathbb{R}^r$ with compact support,

$$\int_E\mathrm{div}\,\phi\,d\mu=\int_{\partial^*E}\phi(x)\,\boldsymbol{.}\,v_x\,\nu(dx).$$

proof By 475L, E has locally finite perimeter, and in particular is Lebesgue measurable. By 474R, there is a Federer exterior normal $v_x=\psi_E(x)$ of E at x for every $x\in\partial^{\$}E$; by 475D, ν-almost every point in ∂^*E is of this kind. By 474E-474F,

$$\int_E\mathrm{div}\,\phi\,d\mu=\int\phi\,\boldsymbol{.}\,\psi_E\,d\lambda_E^{\partial}=\int_{\partial^{\$}E}\phi\,\boldsymbol{.}\,\psi_E\,d\lambda_E^{\partial},$$

and this is also equal to $\int_{\partial^*E}\phi\,\boldsymbol{.}\,\psi_E\,d\lambda_E^{\partial}$, because $\partial^{\$}E\subseteq\partial^*E$ and $\partial^{\$}E$ is λ_E^{∂}-conegligible. But λ_E^{∂} and ν induce the same subspace measures on ∂^*E, by 475G, so

$$\int_E\mathrm{div}\,\phi\,d\mu=\int_{\partial^*E}\phi\,\boldsymbol{.}\,\psi_E\,d\nu=\int_{\partial^*E}\phi(x)\,\boldsymbol{.}\,v_x\,\nu(dx),$$

as claimed.

475O At the price of some technicalities which are themselves instructive, we can now proceed to some basic properties of the essential boundary.

Lemma Let $E\subseteq\mathbb{R}^r$ be a set with locally finite perimeter, and ψ_E its canonical outward-normal function. Let v be the unit vector $(0,\dots,0,1)$. Identify $\mathbb{R}^r$ with $\mathbb{R}^{r-1}\times\mathbb{R}$. Then we have sequences $\langle F_n\rangle_{n\in\mathbb{N}}$, $\langle g_n\rangle_{n\in\mathbb{N}}$ and $\langle g_n'\rangle_{n\in\mathbb{N}}$ such that

(i) for each $n\in\mathbb{N}$, F_n is a Lebesgue measurable subset of $\mathbb{R}^{r-1}$, and g_n, $g_n':F_n\to[-\infty,\infty]$ are Lebesgue measurable functions such that $g_n(u)<g_n'(u)$ for every $u\in F_n$;

(ii) if m, $n\in\mathbb{N}$ are distinct and $u\in F_m\cap F_n$, then $[g_m(u),g_m'(u)]\cap[g_n(u),g_n'(u)]=\emptyset$;

(iii) $\sum_{n=0}^{\infty}\int_{F_n}g_n'-g_n d\mu_{r-1}=\mu E$;

(iv) for any continuous function $h:\mathbb{R}^r\to\mathbb{R}$ with compact support,

$$\int_{\partial^*E}h(x)v\,\boldsymbol{.}\,\psi_E(x)\,\nu(dx)=\sum_{n=0}^{\infty}\int_{F_n}h(u,g_n'(u))-h(u,g_n(u))\mu_{r-1}(du),$$

where we interpret $h(u,\infty)$ and $h(u,-\infty)$ as 0 if necessary;

(v) for μ_{r-1}-almost every $u\in\mathbb{R}^{r-1}$,

$$\begin{aligned}\{t:(u,t)\in\partial^*E\}=\{g_n(u):\;&n\in\mathbb{N},\,u\in F_n,\,g_n(u)\ne-\infty\}\\&\cup\{g_n'(u):n\in\mathbb{N},\,u\in F_n,\,g_n'(u)\ne\infty\}.\end{aligned}$$

proof (a) Take a conegligible set $Z_2\subseteq\mathbb{R}^{r-1}$ as in 475Jc. Let $Z\subseteq Z_2$ be a conegligible Borel set. For $u\in Z$ set $D_u=\{t:(u,t)\in\partial^*E\}$.

(b) For each $q \in \mathbb{Q}$, set $F'_q = \{u : u \in Z,\ (u,q) \in \operatorname{int}^*E\}$, so that F'_q is a Borel set, and for $u \in F'_q$ set

$$f_q(u) = \sup(D_u \cap \,]{-\infty}, q[), \qquad f'_q(u) = \inf(D_u \cap \,]{-\infty}, q[),$$

allowing $-\infty$ as $\sup\emptyset$ and ∞ as $\inf\emptyset$. Observe that (because $D_u \cap [q-1, q+1]$ is finite) $f_q(u) < q < f'_q(u)$. Now f_q and f'_q are measurable, by the argument of part (a-i) of the proof of 475J. Enumerate $\mathbb{Q}$ as $\langle q_n\rangle_{n\in\mathbb{N}}$, and set

$$F_n = F'_{q_n} \setminus \bigcup_{m<n}\{u : u \in F'_{q_m},\ f_{q_m}(u) < q_n < f'_{q_m}(u)\}$$

for $n \in \mathbb{N}$. Set $g_n(u) = f_{q_n}(u)$, $g'_n(u) = f'_{q_n}(u)$ for $u \in F_n$.

The effect of this construction is to ensure that, for any $u \in Z$, $u \in F_n$ iff q_n is the first rational lying in one of those constituent intervals I of $\mathbb{R} \setminus D_u$ such that $\{u\} \times I \subseteq \operatorname{int}^*E$, and that now $g_n(u)$ and $g'_n(u)$ are the endpoints of that interval, allowing $\pm\infty$ as endpoints.

(c) Now let us look at the items (i)-(v) of the statement of this lemma. We have already achieved (i). If $u \in F_m \cap F_n$, then, in the language of 475J, $]g_m(u), g'_m(u)[$ is one of the constituent intervals of G_u, and $]g_n(u), g'_n(u)[$ is another; since these must be separated by one of the constituent intervals of H_u, their closures are disjoint. Thus (ii) is true. For any $u \in Z$,

$$\sum_{n\in\mathbb{N},u\in F_n} g'_n(u) - g_n(u) = \mu_1\{t : (u,t) \in \operatorname{int}^*E\},$$

so

$$\sum_{n=0}^{\infty}\int_{F_n} g'_n - g_n d\mu_{r-1} = \int_Z \mu_1\{t : (u,t) \in \operatorname{int}^*E\}\mu_{r-1}(du) = \mu(\operatorname{int}^*E) = \mu E.$$

So (iii) is true. Also, for $u \in Z$,

$$\begin{aligned}\{t : (u,t) \in \partial^*E\} &= D_u \\ &= \{g_n(u) : n \in \mathbb{N},\ u \in F_n,\ g_n(u) \ne -\infty\} \\ &\qquad \cup \{g'_n(u) : n \in \mathbb{N},\ u \in F_n,\ g'_n(u) \ne \infty\},\end{aligned}$$

so (v) is true.

(d) As for (iv), suppose first that $h : \mathbb{R}^r \to \mathbb{R}$ is a Lipschitz function with compact support. Set $\phi(x) = (0, \dots, h(x))$ for $x \in \mathbb{R}^r$. Then

$$\int_{\partial^*E} h(x)v \mathbin{.} \psi_E(x)\,\nu(dx) = \int h(x)v \mathbin{.} \psi_E(x)\,\lambda_E^{\partial}(dx)$$

(475G)

$$= \int \phi \mathbin{.} \psi_E\,d\lambda_E^{\partial} = \int_E \operatorname{div}\phi\,d\mu$$

(474E)

$$= \int_E \frac{\partial h}{\partial \xi_r}\,d\mu = \int_{\operatorname{int}^*E} \frac{\partial h}{\partial \xi_r}\,d\mu$$

$$= \int \sum_{n\in\mathbb{N},u\in F_n} \int_{g_n(u)}^{g'_n(u)} \frac{\partial h}{\partial \xi_r}(u,t)dt\,\mu_{r-1}(du)$$

$$= \int \sum_{n\in\mathbb{N},u\in F_n} h(u, g'_n(u)) - h(u, g_n(u))\mu_{r-1}(du)$$

(with the convention that $h(u, \pm\infty) = \lim_{t\to\pm\infty} h(u,t) = 0$)

$$= \sum_{n=0}^{\infty}\int_{F_n} h(u, g'_n(u)) - h(u, g_n(u))\mu_{r-1}(du)$$

because if $|h| \le m\chi B(\mathbf{0}, m)$ then

$$\int \sum_{n\in\mathbb{N},u\in F_n} |h(u,g_n'(u)) - h(u,g_n(u))|\mu_{r-1}(du)$$
$$\le 2m\int \#(\{t:(u,t)\in B(\mathbf{0},m)\cap\partial^*E\})\mu_{r-1}(du)$$
$$\le 2m\nu(B(\mathbf{0},m)\cap\partial^*E)$$

(475H) is finite.

For a general continuous function h of compact support, consider the convolutions $h_k = h*\tilde h_k$ for large k, where $\tilde h_k$ is defined in 473E. If $|h|\le m\chi B(\mathbf{0},m)$ then $|h_k|\le m\chi B(\mathbf{0},m+1)$ for every k, so that

$$\textstyle\sum_{n\in\mathbb{N},u\in F_n}|h_k(g_n'(u)) - h_k(g_n(u))| \le 2m\#(\{t:(u,t)\in B(\mathbf{0},m+1)\cap\partial^*E\})$$

for every $u\in\mathbb{R}^r$, $k\in\mathbb{N}$. Since $h_k\to h$ uniformly (473Ed),

$$\begin{aligned}\int_{\partial^*E} h(x)v\,.\,\psi_E(x)\,\nu(dx) &= \lim_{k\to\infty}\int_{\partial^*E} h_k(x)v\,.\,\psi_E(x)\,\nu(dx)\\ &= \lim_{k\to\infty}\int \sum_{n\in\mathbb{N},u\in F_n} h_k(u,g_n'(u)) - h_k(u,g_n(u))\mu_{r-1}(du)\\ &= \int \sum_{n\in\mathbb{N},u\in F_n} h(u,g_n'(u)) - h(u,g_n(u))\mu_{r-1}(du)\\ &= \sum_{n=0}^\infty \int_{F_n} h(u,g_n'(u)) - h(u,g_n(u))\mu_{r-1}(du),\end{aligned}$$

as required.

475P Lemma Let $v\in S_{r-1}$ be any unit vector, and $V\subseteq\mathbb{R}^r$ the hyperplane $\{x:x\,.\,v=0\}$. Let $T:\mathbb{R}^r\to V$ be the orthogonal projection. If $E\subseteq\mathbb{R}^r$ is any set with locally finite perimeter and canonical outward-normal function ψ_E, then

$$\textstyle\int_{\partial^*E}|v\,.\,\psi_E|d\nu = \int_V \#(\partial^*E\cap T^{-1}[\{u\}])\nu(du),$$

interpreting $\#(\partial^*E\cap T^{-1}[\{u\}])$ as ∞ if $\partial^*E\cap T^{-1}[\{u\}]$ is infinite.

proof As usual, we may suppose that the structure (E,v) is rotated until $v=(0,\dots,1)$, so that we can identify $T(\xi_1,\dots,\xi_r)$ with $(\xi_1,\dots,\xi_{r-1})\in\mathbb{R}^{r-1}$, and $\int_V\#(\partial^*E\cap T^{-1}[\{u\}])\nu(du)$ with $\int_{\mathbb{R}^{r-1}}\#(D_u)\mu_{r-1}(du)$, where $D_u=\{t:(u,t)\in\partial^*E\}$. For each $m\in\mathbb{N}$ and $u\in\mathbb{R}^{r-1}$, set $D_u^{(m)}=\{t:(u,t)\in\partial^*E,\ \|(u,t)\|<m\}$; note that $\int\#(D_u^{(m)})\mu_{r-1}(du)\le\nu(\partial^*E\cap B(\mathbf{0},m))$ is defined and finite (475H). It follows that the integral $\int_V\#(\partial^*E\cap T^{-1}[\{u\}])\nu(du)$ is defined in $[0,\infty]$.

(a) Write Φ for the set of continuous functions $h:\mathbb{R}^r\to[-1,1]$ with compact support. Then

$$\textstyle\int_{\partial^*E}|v\,.\,\psi_E|d\nu = \sup_{h\in\Phi}\int_{\partial^*E}h(x)v\,.\,\psi_E(x)\,\nu(dx).$$

P Of course

$$\textstyle\int_{\partial^*E}|v\,.\,\psi_E|d\nu \ge \int_{\partial^*E}h(x)v\,.\,\psi_E(x)\,\nu(dx)$$

for any $h\in\Phi$. On the other hand, if

$$\textstyle\gamma < \int_{\partial^*E}|v\,.\,\psi_E|d\nu = \int|v\,.\,\psi_E|d\lambda_E^\partial$$

(475G), then, because λ_E^∂ is a Radon measure, there is an $n\in\mathbb{N}$ such that

$$\textstyle\gamma < \int_{B(\mathbf{0},n)}|v\,.\,\psi_E|d\lambda_E^\partial = \int h_0(x)v\,.\,\psi_E(x)\,\lambda_E^\partial(dx)$$

where

$$h_0(x) = \frac{v\boldsymbol{.}\psi_E(x)}{|v\boldsymbol{.}\psi_E(x)|} \text{ if } x \in B(\mathbf{0},n) \text{ and } v\boldsymbol{.}\psi_E(x) \ne 0,$$
$$= 0 \text{ otherwise.}$$

Again because λ_E^∂ is a Radon measure, there is a continuous function $h_1 : \mathbb{R}^r \to \mathbb{R}$ with compact support such that

$$\int |h_1 - h_0| d\lambda_E^\partial \le \int_{B(\mathbf{0},n)} |v\boldsymbol{.}\psi_E| d\lambda_E^\partial - \gamma$$

(416I); of course we may suppose that h_1, like h, takes values in $[-1,1]$, so that $h_1 \in \Phi$. Now

$$\int_{\partial^* E} h_1(x) v\boldsymbol{.}\psi_E(x)\nu(dx) = \int h_1(x) v\boldsymbol{.}\psi_E(x)\, \lambda_E^\partial(dx) \ge \gamma.$$

As γ is arbitrary,

$$\sup_{h\in\Phi} \int h(x) v\boldsymbol{.}\psi_E(x)\, \nu(dx) = \int |v\boldsymbol{.}\psi_E| d\nu$$

as required. **Q**

(b) Now take $\langle F_n\rangle_{n\in\mathbb{N}}$, $\langle g_n\rangle_{n\in\mathbb{N}}$ and $\langle g'_n\rangle_{n\in\mathbb{N}}$ as in 475O. Then

$$\int \#(D_u)\mu_{r-1}(du) = \sup_{h\in\Phi} \sum_{n=0}^\infty \int_{F_n} h(u, g'_n(u)) - h(u, g_n(u))\mu_{r-1}(du).$$

(As in 475O(iv), interpret $h(u, \pm\infty)$ as 0 if necessary.) **P** By 475O(ii), $g'_m(u) \ne g_n(u)$ whenever m, $n \in \mathbb{N}$ and $u \in F_m \cap F_n$, while for almost all $u \in \mathbb{R}^{r-1}$

$$D_u = (\{g_n(u) : n \in \mathbb{N},\, u \in F_n\} \setminus \{-\infty\}) \cup (\{g'_n(u) : n \in \mathbb{N},\, u \in F_n\} \setminus \{\infty\}).$$

So if $h \in \Phi$,

$$\begin{aligned}\sum_{n\in\mathbb{N}, u\in F_n} h(u, g'_n(u)) - h(u, g_n(u)) &\le \#(\{g_n(u) : u \in F_n,\, g_n(u) \ne -\infty\}) \\ &\qquad + \#(\{g'_n(u) : u \in F_n,\, g'_n(u) \ne \infty\}) \\ &= \#(D_u)\end{aligned}$$

for almost every $u \in \mathbb{R}^{r-1}$, and

$$\sum_{n=0}^\infty \int_{F_n} h(u, g'_n(u)) - h(u, g_n(u))\mu_{r-1}(du) \le \int \#(D_u)\mu_{r-1}(du).$$

In the other direction, given $\gamma < \int \#(D_u)\mu_{r-1}(du)$, there is an $m \in \mathbb{N}$ such that $\gamma < \int \#(D_u^{(m)})\mu_{r-1}(du)$. Setting

$$H_n = \{u : u \in F_n,\, g_n(u) \in D_u^{(m)}\}, \quad H'_n = \{u : u \in F_n,\, g'_n(u) \in D_u^{(m)}\}$$

for $n \in \mathbb{N}$, we have

$$\gamma < \sum_{n=0}^\infty \mu_{r-1} H_n + \sum_{n=0}^\infty \mu_{r-1} H'_n \le \int \#(D_u^{(m)}\mu_{r-1}(du) < \infty.$$

By 418J once more, we can find $n \in \mathbb{N}$ and compact sets $K_i \subseteq H_i$, $K'_i \subseteq H'_i$ such that $g_i{\restriction}K_i$ and $g'_i{\restriction}K'_i$ are continuous for every i and

$$\begin{aligned}\gamma \le \sum_{i=0}^n (\mu_{r-1}K_i + \mu_{r-1}K'_i) &- \sum_{i=n+1}^\infty (\mu_{r-1}H_i + \mu_{r-1}H'_i) \\ &- \sum_{i=0}^n (\mu_{r-1}(H_i \setminus K_i) + \mu_{r-1}(H'_i \setminus K'_i)).\end{aligned}$$

Set

$$K = \bigcup_{i\le n} \{(u, g_i(u)) : u \in K_i\}, \quad K' = \bigcup_{i\le n} \{(u, g'_i(u)) : u \in K'_i\},$$

so that K and K' are disjoint compact subsets of $\operatorname{int} B(\mathbf{0}, m)$. Let $h : \mathbb{R}^r \to \mathbb{R}$ be a continuous function such that

$$\begin{aligned} h(x) &= 1 \text{ for } x \in K', \\ &= -1 \text{ for } x \in K, \\ &= 0 \text{ for } x \in \mathbb{R}^r \setminus \operatorname{int} B(\mathbf{0}, m) \end{aligned}$$

(4A2F(d-ix)); we can suppose that $-1 \le h(x) \le 1$ for every x. Then $h \in \Phi$, and

$$\sum_{i=0}^{\infty} \int_{F_i} h(u, g_i'(u)) - h(u, g_i(u))\mu_{r-1}(du)$$

$$= \sum_{i=0}^{\infty} \int_{H_i'} h(u, g_i'(u))\mu_{r-1}(du) - \sum_{i=0}^{\infty} \int_{H_i} h(u, g_i(u))\mu_{r-1}(du)$$

(because h is zero outside $\operatorname{int} B(\mathbf{0}, m)$)

$$\begin{aligned} &\ge \sum_{i=0}^{n} \int_{K_i'} h(u, g_i'(u))\mu_{r-1}(du) - \sum_{i=0}^{n} \int_{K_i} h(u, g_i(u))\mu_{r-1}(du) \\ &\qquad - \sum_{i=0}^{n}(\mu(H_i' \setminus K_i') + \mu(H_i \setminus K_i)) - \sum_{i=n+1}^{\infty} (\mu H_i' + \mu H_i) \\ &\ge \gamma. \end{aligned}$$

This shows that

$$\int \#(D_u)\mu_{r-1}(du) \le \sup_{h\in\Phi} \textstyle\sum_{n=0}^{\infty} \int_{F_n} h(u, g_n'(u)) - h(u, g_n(u))\mu_{r-1}(du),$$

and we have equality. **Q**

(c) Putting (a) and (b) together with 475O(iv), we have the result.

475Q Theorem (a) Let $E \subseteq \mathbb{R}^r$ be a set with finite perimeter. For $v \in S_{r-1}$ write V_v for $\{x : x \centerdot v = 0\}$, and let $T_v : \mathbb{R}^r \to V_v$ be the orthogonal projection. Then

$$\begin{aligned} \operatorname{per} E = \nu(\partial^* E) &= \frac{1}{2\beta_{r-1}} \int_{S_{r-1}} \int_{V_v} \#(\partial^* E \cap T_v^{-1}[\{u\}])\nu(du)\nu(dv) \\ &= \lim_{\delta\downarrow 0} \frac{1}{2\beta_{r-1}\delta} \int_{S_{r-1}} \mu(E \triangle (E + \delta v))\nu(dv). \end{aligned}$$

(b) Suppose that $E \subseteq \mathbb{R}^r$ is Lebesgue measurable. Set

$$\gamma = \sup_{x\in\mathbb{R}^r\setminus\{0\}} \frac{1}{\|x\|}\mu(E \triangle (E + x)).$$

Then $\gamma \le \operatorname{per} E \le \frac{r\beta_r\gamma}{2\beta_{r-1}}$.

proof (a)(i) I start with an elementary fact: there is a constant c such that $\int |w \centerdot v| \lambda^\partial_{S_{r-1}}(dv) = c$ for every $w \in S_{r-1}$; this is because whenever w, $w' \in S_{r-1}$ there is an orthogonal linear transformation taking w to w', and this transformation is an automorphism of the structure $(\mathbb{R}^r, \nu, S_{r-1}, \lambda^\partial_{S_{r-1}})$ (474H). (In (iii) below I will come to the calculation of c.)

(ii) Now, writing ψ_E for the canonical outward-normal function of E, we have

$$\int_{S_{r-1}} \int_{V_v} \#(\partial^* E \cap T_v^{-1}[\{u\}])\nu(du)\nu(dv) = \int_{S_{r-1}} \int_{\partial^* E} |\psi_E(x) \centerdot v|\nu(dx)\nu(dv)$$

(475P)

$$= \iint |\psi_E(x) \mathbin{.} v| \lambda^\partial_E(dx) \lambda^\partial_{S_{r-1}}(dv)$$

(by 235K, recalling that $\lambda^\partial_{S_{r-1}}$ and λ^∂_E are just indefinite-integral measures over ν, while S_{r-1} and $\partial^* E$ are Borel sets)

$$= \iint |\psi_E(x) \mathbin{.} v| \lambda^\partial_{S_{r-1}}(dv) \lambda^\partial_E(dx)$$

(because $\mathbin{.}$ is continuous, so $(x, v) \mapsto \psi_E(x) \mathbin{.} v$ is measurable, while $\lambda^\partial_{S_{r-1}}$ and λ^∂_E are totally finite, so we can use Fubini's theorem)

$$= c \operatorname{per} E$$

(by (i) above)

$$= c\nu(\partial^* E).$$

(iii) We have still to identify the constant c. But observe that the argument above applies whenever $\nu(\partial^* E)$ is finite, and in particular applies to $E = B(\mathbf{0}, 1)$. In this case, $\partial^* B(\mathbf{0}, 1) = S_{r-1}$, and for any $v \in S_{r-1}$, $u \in V_v$ we have

$$\begin{aligned} \#(\partial^* B(\mathbf{0}, 1) \cap T^{-1}[\{u\}]) &= 2 \text{ if } \|u\| < 1, \\ &= 1 \text{ if } \|u\| = 1, \\ &= 0 \text{ if } \|u\| > 1. \end{aligned}$$

Since we can identify ν on V_v as a copy of Lebesgue measure μ_{r-1},

$$\int_{V_v} \#(\partial^* B(\mathbf{0}, 1) \cap T^{-1}[\{u\}]) \nu(du) = 2\nu\{u : u \in V_v, \|u\| < 1\} = 2\beta_{r-1}.$$

This is true for every $v \in V_u$, so from (ii) we get

$$2\beta_{r-1}\nu S_{r-1} = \int_{S_{r-1}} \int_{V_v} \#(\partial^* E \cap T_v^{-1}[\{u\}]) \nu(du) \nu(dv) = c\nu S_{r-1},$$

and $c = 2\beta_{r-1}$. Substituting this into the result of (ii), we get

$$\operatorname{per} E = \nu(\partial^* E) = \frac{1}{2\beta_{r-1}} \int_{S_{r-1}} \int_{V_v} \#(\partial^* E \cap T_v^{-1}[\{u\}]) \nu(du) \nu(dv).$$

(iv) Before continuing with the main argument, it will help to set out another elementary fact, this time about translates of certain simple subsets of $\mathbb{R}$. Suppose that (G, H, D) is a partition of $\mathbb{R}$ such that G and H are open, $D = \partial G = \partial H$ is the common boundary of G and H, and D is locally finite, that is, $D \cap [-n, n]$ is finite for every $n \in \mathbb{N}$. Then

$$\sup_{\delta > 0} \frac{1}{\delta} \mu_1(G \triangle (G + \delta)) = \lim_{\delta \downarrow 0} \frac{1}{\delta} \mu_1(G \triangle (G + \delta)) = \#(D)$$

if you will allow me to identify '∞' with the cardinal ω. **P** If we look at the components of G, these are intervals with endpoints in D; and because $\partial G = \partial H$, distinct components of G have disjoint closures. Set $f(\delta) = \mu_1(G \triangle (G + \delta))$ for $\delta > 0$. If D is infinite, then for any $n \in \mathbb{N}$ we have disjoint bounded components $I_0, \ldots, I_n$ of G; for any δ small enough, $G \cap (I_j + \delta) \subseteq I_j$ for every $j \le n$ (because D is locally finite); so that

$$f(\delta) \ge \sum_{j=0}^n \mu(I_j \triangle (I_j + \delta)) = 2(n+1)\delta,$$

and $\liminf_{\delta \downarrow 0} \frac{1}{\delta} f(\delta) \ge 2(n+1)$. As n is arbitrary,

$$\sup_{\delta > 0} \frac{1}{\delta} f(\delta) = \lim_{\delta \downarrow 0} \frac{1}{\delta} f(\delta) = \infty.$$

If D is empty, then G is either empty or $\mathbb{R}$, and

$$\sup_{\delta > 0} \frac{1}{\delta} f(\delta) = \lim_{\delta \downarrow 0} \frac{1}{\delta} f(\delta) = 0.$$

If D is finite and not empty, let $I_0,\dots,I_n$ be the components of G. Then, for all small $\delta>0$, we have

$$f(\delta)=\sum_{j=0}^{n}\mu(I_j\triangle(I_j+\delta))=2(n+1)\delta=\delta\#(D),$$

so again

$$\sup_{\delta>0}\frac{1}{\delta}f(\delta)=\lim_{\delta\downarrow0}\frac{1}{\delta}f(\delta)=\#(D).\ \mathbf{Q}$$

(v) Returning to the proof in hand, we find that if $v\in S_{r-1}$ is such that $\int_{V_v}\#(\partial^*E\cap T_v^{-1}[\{u\}])\nu(du)$ is finite, then the integral is equal to

$$\lim_{\delta\downarrow0}\frac{1}{\delta}\mu(E\triangle(E+\delta v))=\sup_{\delta>0}\frac{1}{\delta}\mu(E\triangle(E+\delta v)).$$

P It is enough to consider the case in which v is the unit vector $(0,\dots,0,1)$, so that we can identify V_v with $\mathbb{R}^{r-1}$ and $\mathbb{R}^r$ with $\mathbb{R}^{r-1}\times\mathbb{R}$, as in 475J; in this case, $E\cap T_v^{-1}[\{u\}]$ turns into $E[\{u\}]$. Let $Z_2\subseteq\mathbb{R}^{r-1}$ and G_u, H_u and D_u, for $u\in\mathbb{R}^{r-1}$, be as in 475Jc. In this case, for any $\delta>0$,

$$\begin{aligned}\mu(E\triangle(E+\delta v))&=\mu(\operatorname{int}^*E\triangle(\operatorname{int}^*E+\delta v))\\&=\int_{\mathbb{R}^{r-1}}\mu_1((\operatorname{int}^*E\triangle(\operatorname{int}^*E+\delta v))[\{u\}])\mu_{r-1}(du)\\&=\int_{\mathbb{R}^{r-1}}\mu_1(G_u\triangle(G_u+\delta))\mu_{r-1}(du)\\&=\int_{Z_2}\mu_1(G_u\triangle(G_u+\delta))\mu_{r-1}(du)\end{aligned}$$

because Z_2 is conegligible. By (iv),

$$\lim_{\delta\downarrow0}\frac{1}{\delta}\mu_1(G_u\triangle(G_u+\delta))=\sup_{\delta>0}\frac{1}{\delta}\mu_1(G_u\triangle(G_u+\delta))=\#(D_u)$$

for any $u\in Z_2$. Applying Lebesgue's Dominated Convergence Theorem to arbitrary sequences $\langle\delta_n\rangle_{n\in\mathbb{N}}\downarrow0$, we see that

$$\lim_{\delta\downarrow0}\frac{1}{\delta}\mu(E\triangle(E+\delta v))=\int_{Z_2}\#(D_u)\nu(du)=\int_{\mathbb{R}^{r-1}}\#(\partial^*E\cap T_v^{-1}[\{u\}])\nu(du),$$

as required. To see that

$$\lim_{\delta\downarrow0}\frac{1}{\delta}\mu(E\triangle(E+\delta v))=\sup_{\delta>0}\frac{1}{\delta}\mu(E\triangle(E+\delta v)),$$

set $g(\delta)=\mu(E\triangle(E+\delta v))$ for $\delta>0$. Then for δ, $\delta'>0$ we have

$$\begin{aligned}g(\delta+\delta')&=\mu(E\triangle(E+(\delta+\delta')v))\le\mu(E\triangle(E+\delta v))+\mu((E+\delta v)\triangle(E+(\delta+\delta'v)))\\&=g(\delta)+\mu(\delta v+(E\triangle(E+\delta'v)))=g(\delta)+g(\delta').\end{aligned}$$

Consequently $g(\delta)\le ng(\frac{1}{n}\delta)$ whenever $\delta>0$ and $n\ge1$, so

$$\frac{1}{\delta}g(\delta)\le\liminf_{n\to\infty}\frac{n}{\delta}g(\tfrac{1}{n}\delta)=\lim_{\delta'\downarrow0}\frac{1}{\delta'}g(\delta')$$

for every $\delta>0$. **Q**

(vi) Putting (v) together with (i)-(iii) above,

$$\operatorname{per}E=\frac{1}{2\beta_{r-1}}\int_{S_{r-1}}\lim_{\delta\downarrow0}\frac{1}{\delta}\mu(E\triangle(E+\delta v))\nu(dv).$$

To see that we can exchange the limit and the integral, observe that we can again use the dominated convergence theorem, because

$$\int_{S_{r-1}}\sup_{\delta>0}\frac{1}{\delta}\mu(E\triangle(E+\delta v))=2\beta_{r-1}\operatorname{per}E$$

is finite. So

$$\operatorname{per} E = \lim_{\delta\downarrow 0} \frac{1}{2\beta_{r-1}\delta}\int_{S_{r-1}} \mu(E\triangle(E+\delta v))\nu(dv).$$

(b)(i) $\gamma \le \operatorname{per} E$. **P** We can suppose that E has finite perimeter.

(α) To begin with, suppose that $x = (0,\dots,0,\delta)$ where $\delta > 0$. As in part (a-v) of this proof, set $v = (0,\dots,0,1)$, identify $\mathbb{R}^r$ with $\mathbb{R}^{r-1}\times\mathbb{R}$, and define G_u, $D_u \subseteq \mathbb{R}$, for $u \in \mathbb{R}^{r-1}$, and $Z_2 \subseteq \mathbb{R}^{r-1}$ as in 475Jc. Suppose that $u \in Z_2$. Then $G_u = (\operatorname{int}^*E)[\{u\}]$ is an open set and its constituent intervals have endpoints in $D_u = (\partial^*E)[\{u\}]$. It follows that for any t in

$$G_u\triangle(G_u+\delta) = (\operatorname{int}^*E\triangle(\operatorname{int}^*E+\delta v))[\{u\}],$$

there must be an $s \in D_u \cap [t-\delta,\delta]$, and $t \in D_u + [0,\delta]$. Accordingly

$$(\operatorname{int}^*E\triangle(\operatorname{int}^*E+\delta v)) \cap (Z_2\times\mathbb{R}) \subseteq \partial^*E + [0,\delta v],$$

writing $[0,\delta v]$ for $\{tv : t \in [0,\delta]\}$. So

$$\mu(E\triangle(E+\delta v)) = \mu(\operatorname{int}^*E\triangle(\operatorname{int}^*E+\delta v)) \le \mu^*(\partial^*E+[0,\delta v]).$$

Take any $\epsilon > 0$. We have a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of sets, all of diameter at most ϵ, covering ∂^*E, and such that $2^{-r+1}\beta_{r-1}\sum_{n=0}^{\infty}(\operatorname{diam} A_n)^{r-1} \le \epsilon + \nu(\partial^*E)$; we can suppose that every A_n is closed. Taking $T : \mathbb{R}^r \to \mathbb{R}^{r-1}$ to be the natural projection, $T[A_n]$ is compact and has diameter at most $\operatorname{diam} A_n$, so that $\mu_{r-1}T[A_n] \le 2^{-r+1}\beta_{r-1}(\operatorname{diam} A_n)^{r-1}$ (264H again). For each $n \in \mathbb{N}$, the vertical sections of $A_n + [0,\delta v]$ have diameter at most $\epsilon+\delta$. So

$$\mu(A_n+[0,\delta v]) \le 2^{-r+1}\beta_{r-1}(\operatorname{diam} A_n)^{r-1}(\epsilon+\delta).$$

Consequently,

$$\begin{aligned}\mu(E\triangle(E+\delta v)) &\le \mu^*(\partial^*E+[0,\delta v]) \le \sum_{n=0}^{\infty}\mu(A_n+[0,\delta v])\\ &\le \sum_{n=0}^{\infty} 2^{-r+1}\beta_{r-1}(\operatorname{diam} A_n)^{r-1}(\epsilon+\delta) \le (\epsilon+\delta)(\epsilon+\nu(\partial^*E)).\end{aligned}$$

As ϵ is arbitrary,

$$\mu(E\triangle(E+x)) = \mu(E\triangle(E+\delta v)) \le \delta\nu(\partial^*E) = \|x\|\nu(\partial^*E).$$

(β) Of course the same must be true for all other non-zero $x \in \mathbb{R}^r$, so $\gamma \le \nu(\partial^*E) = \operatorname{per} E$. **Q**

(ii) For the other inequality, we need look only at the case in which γ is finite.

(α) In this case, E has finite perimeter. **P** Let $\phi : \mathbb{R}^r \to B(\mathbf{0},1)$ be a Lipschitz function with compact support. Take i such that $1 \le i \le r$, and consider

$$\Bigl|\int_E \frac{\partial\phi_i}{\partial\xi_i}d\mu\Bigr| = \Bigl|\int_E \lim_{n\to\infty} 2^n(\phi_i(x+2^{-n}e_i)-\phi_i(x))\mu(dx)\Bigr|$$

(where $\phi = (\phi_1,\dots,\phi_r)$)

$$= \Bigl|\lim_{n\to\infty}\int_E 2^n(\phi_i(x+2^{-n}e_i)-\phi_i(x))\mu(dx)\Bigr|$$

(by Lebesgue's Dominated Convergence Theorem, because ϕ_i is Lipschitz and has bounded support)

$$\begin{aligned}&= \lim_{n\to\infty} 2^n\Bigl|\int_E(\phi_i(x+2^{-n}e_i)-\phi_i(x))\mu(dx)\Bigr|\\ &= \lim_{n\to\infty} 2^n\Bigl|\int_{E+2^{-n}e_i}\phi_i d\mu - \int_E \phi_i d\mu\Bigr|\\ &\le \limsup_{n\to\infty} 2^n\mu((E+2^{-n}e_i)\triangle E)\end{aligned}$$

(because $\|\phi_i\|_\infty \le 1$)

$$\leq \gamma.$$

Summing over i, $|\int_E \operatorname{div}\phi\,d\mu| \leq r\gamma$. As ϕ is arbitrary, $\operatorname{per} E \leq r\gamma$ is finite. **Q**

(β) By (a), we have

$$\operatorname{per} E = \lim_{\delta\downarrow 0} \frac{1}{2\beta_{r-1}\delta} \int_{S_{r-1}} \mu(E\triangle(E+\delta v))\nu(dv)$$
$$\leq \frac{1}{2\beta_{r-1}}\gamma\nu S_{r-1} = \frac{r\beta_r\gamma}{2\beta_{r-1}},$$

as required.

475R Convex sets in $\mathbb{R}^r$ For the next result it will help to have some elementary facts about convex sets in finite-dimensional spaces out in the open.

Lemma (In this lemma I allow $r = 1$.) Let $C \subseteq \mathbb{R}^r$ be a convex set.

(a) If $x \in C$ and $y \in \operatorname{int} C$, then $ty + (1-t)x \in \operatorname{int} C$ for every $t \in \,]0,1]$.

(b) $\overline{C}$ and $\operatorname{int} C$ are convex.

(c) If $\operatorname{int} C \neq \emptyset$ then $\overline{C} = \overline{\operatorname{int} C}$.

(d) If $\operatorname{int} C = \emptyset$ then C lies within some hyperplane.

(e) $\operatorname{int}\overline{C} = \operatorname{int} C$.

proof (a) Setting $\phi(z) = x + t(z-x)$ for $z \in \mathbb{R}^r$, $\phi : \mathbb{R}^r \to \mathbb{R}^r$ is a homeomorphism and $\phi[C] \subseteq C$, so

$$\phi(y) \in \phi[\operatorname{int} C] = \operatorname{int}\phi[C] \subseteq \operatorname{int} C.$$

(b) It follows at once from (a) that $\operatorname{int} C$ is convex; $\overline{C}$ is convex because $(x,y) \mapsto tx+(1-t)y$ is continuous for every $t \in [0,1]$.

(c) From (a) we see also that if $\operatorname{int} C \neq \emptyset$ then $C \subseteq \overline{\operatorname{int} C}$, so that $\overline{C} \subseteq \overline{\operatorname{int} C}$ and $\overline{C} = \overline{\operatorname{int} C}$.

(d) It is enough to consider the case in which $\mathbf{0} \in C$, since if $C = \emptyset$ the result is trivial. **?** If $x_1, \dots, x_r$ are linearly independent elements of C, set $x = \frac{1}{r+1}\sum_{i=1}^r x_i$; then

$$x + \sum_{i=1}^r \alpha_i x_i = \sum_{i=1}^r (\alpha_i + \frac{1}{r+1})x_i \in C$$

whenever $\sum_{i=1}^r |\alpha_i| \leq \frac{1}{r+1}$. Also, writing $e_1, \dots, e_r$ for the standard orthonormal basis of $\mathbb{R}^r$, we can express e_i as $\sum_{j=1}^r \alpha_{ij}x_j$ for each j; setting $M = (r+1)\max_{i\leq r}\sum_{j=1}^r |\alpha_{ij}|$, we have $x \pm \frac{1}{M}e_i \in C$ for every $i \leq r$, so that $x + y \in C$ whenever $\|y\| \leq \frac{1}{M\sqrt{r}}$, and $x \in \operatorname{int} C$. **X**

So the linear subspace of $\mathbb{R}^r$ spanned by C has dimension at most $r-1$.

(e) If $\operatorname{int} C = \mathbb{R}^r$ the result is trivial. If $\operatorname{int} C$ is empty, then (d) shows that C is included in a hyperplane, so that $\operatorname{int}\overline{C}$ is empty. Otherwise, if $x \in \mathbb{R}^r \setminus \operatorname{int} C$, there is a non-zero $e \in \mathbb{R}^r$ such that $e\,.\,y \leq e\,.\,x$ for every $y \in \operatorname{int} C$ (4A4Db, or otherwise). Now, by (c), $e\,.\,y \leq e\,.\,x$ for every $y \in \overline{C}$, so $x \notin \operatorname{int}\overline{C}$. This shows that $\operatorname{int}\overline{C} \subseteq \operatorname{int} C$, so that the two are equal.

475S Corollary: Cauchy's Perimeter Theorem Let $C \subseteq \mathbb{R}^r$ be a bounded convex set with non-empty interior. For $v \in S_{r-1}$ write V_v for $\{x : x\,.\,v = 0\}$, and let $T_v : \mathbb{R}^r \to V_v$ be the orthogonal projection. Then

$$\nu(\partial C) = \frac{1}{\beta_{r-1}}\int_{S_{r-1}} \nu(T_v[C])\nu(dv).$$

proof (a) The first thing to note is that $\partial^* C = \partial C$. **P** Of course $\partial^* C \subseteq \partial C$ (475Ca). If $x \in \partial C$, there is a half-space V containing x and disjoint from $\operatorname{int} C$ (4A4Db again, because $\operatorname{int} C$ is convex), so that

$$\limsup_{\delta\downarrow 0}\frac{\mu(B(x,\delta)\setminus C)}{\mu B(x,\delta)}\ge\limsup_{\delta\downarrow 0}\frac{\mu(B(x,\delta)\setminus\operatorname{int}V)}{\mu B(x,\delta)}=\frac{1}{2},$$

and $x\notin\operatorname{int}^*C$. On the other hand, if $x_0\in\operatorname{int}C$ and $\eta>0$ are such that $B(x_0,\eta)\subseteq\operatorname{int}C$, then for any $\delta\in\,]0,1]$ we can set $t=\frac{\delta}{\eta+\|x_0-x\|}$, and then

$$B(x+t(x_0-x),t\eta)\subseteq B(x,\delta)\cap((1-t)x+tB(x_0,\eta))\subseteq B(x,\delta)\cap C,$$

so that

$$\mu(B(x,\delta)\cap C)\ge\beta_r t^r\eta^r=\big(\tfrac{\eta}{\|x_0-x\|+\eta}\big)^r\mu B(x,\delta);$$

as δ is arbitrary, $x\in\operatorname{cl}^*C$. This shows that $\partial C\subseteq\partial^*C$ so that $\partial^*C=\partial C$. **Q**

(b) We have a function $\phi:\mathbb{R}^r\to\overline{C}$ defined by taking $\phi(x)$ to be the unique point of C closest to x, for every $x\in\mathbb{R}^r$ (3A5Md). This function is 1-Lipschitz. **P** Take any x, $y\in\mathbb{R}^r$ and set $e=\phi(x)-\phi(y)$. We know that $\phi(x)-\epsilon e\in\overline{C}$, so that $\|x-\phi(x)-\epsilon e\|\ge\|x+\phi(x)\|$, for $0\le\epsilon\le 1$; it follows that $(x-\phi(x))\textbf{.}e\ge 0$. Similarly, $(y-\phi(y))\textbf{.}(-e)\ge 0$. Accordingly $(x-y)\textbf{.}e\ge e\textbf{.}e$ and $\|x-y\|\ge\|e\|$. As x and y are arbitrary, ϕ is 1-Lipschitz. **Q**

Now suppose that $C'\supseteq C$ is a closed bounded convex set. Then $\nu(\partial C')\ge\nu(\partial C)$. **P** Let ϕ be the function defined just above. By 264G/471J again, $\nu^*(\phi[\partial C'])\le\nu(\partial C')$. But if $x\in\partial C$, there is an $e\in\mathbb{R}^r\setminus\{0\}$ such that $x\textbf{.}e\ge y\textbf{.}e$ for every $y\in C$. Then $\phi(x+\alpha e)=x$ for every $\alpha\ge 0$. Because C' is closed and bounded, and $x\in C\subseteq C'$, there is a greatest $\alpha\ge 0$ such that $x+\alpha e\in C'$, and in this case $x+\alpha e\in\partial C'$; since $\phi(x+\alpha e)=x$, $x\in\phi[\partial C']$. As x is arbitrary, $\partial C\subseteq\phi[\partial C']$, and

$$\nu(\partial C)\le\nu^*(\phi[\partial C'])\le\nu(\partial C'). \textbf{ Q}$$

Since we can certainly find a closed convex set $C'\supseteq C$ such that $\nu(\partial C')$ is finite (e.g., any sufficiently large ball or cube), $\nu(\partial C)<\infty$. It follows at once that $\mu(\partial C)=0$ (471L once more).

(c) The argument so far applies, of course, to every $r\ge 1$ and every bounded convex set with non-empty interior in $\mathbb{R}^r$. Moving to the intended case $r\ge 2$, and fixing $v\in S_{r-1}$ for the moment, we see that, because T_v is an open map (if we give V_v its subspace topology), $T_v[C]$ is again a bounded convex set with non-empty (relative) interior. Since the subspace measure induced by ν on V_v is just a copy of Lebesgue measure, (b) tells us that $\nu T_v[C]=\nu(\operatorname{int}_{V_v}T_v[C])$, where here I write $\operatorname{int}_{V_v}T_v[C]$ for the interior of $T_v[C]$ in the subspace topology of V_v. Now the point is that $\operatorname{int}_{V_v}T_v[C]\subseteq T_v[\operatorname{int}C]$. **P** $\operatorname{int}C$ (taken in $\mathbb{R}^r$) is dense in C (475Rc), so $W=T_v[\operatorname{int}C]$ is a relatively open convex set which is dense in $T_v[C]$; now $W=\operatorname{int}_{V_v}\overline{W}$ (475Re, applied in $V_v\cong\mathbb{R}^{r-1}$), so $W\supseteq\operatorname{int}_{V_v}T_v[C]$. **Q**

It follows that $\#(\partial C\cap T_v^{-1}[\{u\}])=2$ for every $u\in\operatorname{int}_{V_v}T_v[C]$. **P** $T_v^{-1}[\{u\}]$ is a straight line meeting $\operatorname{int}C$ in y_0 say. Because $\overline{C}$ is a bounded convex set, it meets $T_v^{-1}[\{u\}]$ in a bounded convex set, which must be a non-trivial closed line segment with endpoints y_1, y_2 say. Now certainly neither y_1 nor y_2 can be in the interior of C. Moreover, the open line segments between y_1 and y_0, and between y_2 and y_0, are covered by $\operatorname{int}C$, by 475Ra; so $T_v^{-1}[\{u\}]\cap\partial C=\{y_1,y_2\}$ has just two members. **Q**

(d) This is true for every $v\in S_{r-1}$. But this means that we can apply 475Q to see that

$$\begin{aligned}\nu(\partial C)=\nu(\partial^*C)&=\frac{1}{2\beta_{r-1}}\int_{S_{r-1}}\int_{V_v}\#(\partial^*C\cap T_v^{-1}[\{u\}])\nu(du)\nu(dv)\\&=\frac{1}{2\beta_{r-1}}\int_{S_{r-1}}\int_{T_v[C]}\#(\partial^*C\cap T_v^{-1}[\{u\}])\nu(du)\nu(dv)\\&=\frac{1}{2\beta_{r-1}}\int_{S_{r-1}}\int_{\operatorname{int}_{V_v}T_v[C]}\#(\partial^*C\cap T_v^{-1}[\{u\}])\nu(du)\nu(dv)\\&=\frac{1}{\beta_{r-1}}\int_{S_{r-1}}\nu(\operatorname*{int}_{V_v}T_v[C])\nu(dv)=\frac{1}{\beta_{r-1}}\int_{S_{r-1}}\nu(T_v[C])\nu(dv),\end{aligned}$$

as required.

475T Corollary: the Convex Isoperimetric Theorem If $C\subseteq\mathbb{R}^r$ is a bounded convex set, then $\nu(\partial C)\le r\beta_r(\frac{1}{2}\operatorname{diam}C)^{r-1}$.

proof (a) If C is included in some $(r-1)$-dimensional affine subspace, then

$$\nu(\partial C) = \nu\overline{C} \le \beta_{r-1}(\frac{1}{2}\operatorname{diam} C)^{r-1}$$

by 264H once more. For completeness, I should check that $\beta_{r-1} \le r\beta_r$. **P** Comparing 265F with 265H, or working from the formulae in 252Q, we have $r\beta_r = 2\pi\beta_{r-2}$. On the other hand, by the argument of 252Q,

$$\beta_{r-1} = \beta_{r-2}\int_{-\pi/2}^{\pi/2} \cos^{r-1} t\,dt \le \pi\beta_{r-2},$$

so (not coincidentally) we have a factor of two to spare. **Q**

(b) Otherwise, C has non-empty interior (475Rd), and for any orthogonal projection T of $\mathbb{R}^r$ onto an $(r-1)$-dimensional linear subspace, $\operatorname{diam} T[C] \le \operatorname{diam} C$, so $\nu(T[C]) \le \beta_{r-1}(\frac{1}{2}\operatorname{diam} C)^{r-1}$. Now 475S tells us that

$$\nu(\partial C) \le (\frac{1}{2}\operatorname{diam} C)^{r-1}\nu S_{r-1} = r\beta_r(\frac{1}{2}\operatorname{diam} C)^{r-1}.$$

Remark Compare 476H below.

475X Basic exercises (a) Show that if $C \subseteq \mathbb{R}^r$ is convex, then either $\mu C = 0$ and $\partial^* C = \emptyset$, or $\partial^* C = \partial C$.

(b) Let A, $A' \subseteq \mathbb{R}^r$ be any sets. Show that

$$(\partial^* A \cap \operatorname{int}^* A') \cup (\partial^* A' \cap \operatorname{int}^* A) \subseteq \partial^*(A \cap A') \subseteq (\partial^* A \cap \operatorname{cl}^* A') \cup (\partial^* A' \cap \operatorname{cl}^* A).$$

(c) Let $A \subseteq \mathbb{R}^r$ be any set, and B a non-trivial closed ball. Show that

$$\partial^*(A \cap B) \triangle ((B \cap \partial^* A) \cup (A \cap \partial B)) \subseteq A \cap \partial B \setminus \partial^* A.$$

>(d) Let E, $F \subseteq \mathbb{R}^r$ be measurable sets, and v the Federer exterior normal to E at $x \in \operatorname{int}^* F$. Show that v is the Federer exterior normal to $E \cap F$ at x.

(e) Let $\mathfrak{T}$ be the density topology on $\mathbb{R}^r$ (414P) defined from lower Lebesgue density (341E). Show that, for any $A \subseteq \mathbb{R}^r$, $A \cup \operatorname{cl}^* A$ is the $\mathfrak{T}$-closure of A and $\operatorname{int}^* A$ is the $\mathfrak{T}$-interior of the $\mathfrak{T}$-closure of A.

(f) Let $A \subseteq \mathbb{R}^r$ be any set. Show that $A \setminus \operatorname{cl}^* A$ and $\operatorname{int}^* A \setminus A$ are Lebesgue negligible.

>(g) Let $E \subseteq \mathbb{R}^r$ be such that $\nu(\partial^* E)$ and μE are both finite. Show that, taking v_x to be the Federer exterior normal to E at any point x where this is defined,

$$\int_E \operatorname{div} \phi\, d\mu = \int_{\partial^* E} \phi(x) \,.\, v_x\, \nu(dx)$$

for every bounded Lipschitz function $\phi : \mathbb{R}^r \to \mathbb{R}^r$.

>(h) Let $\langle E_n \rangle_{n \in \mathbb{N}}$ be a sequence of measurable subsets of $\mathbb{R}^r$ such that (i) there is a measurable set E such that $\lim_{n\to\infty} \mu((E_n \triangle E) \cap B(\mathbf{0}, m)) = 0$ for every $m \in \mathbb{N}$ (ii) $\sup_{n\in\mathbb{N}} \nu(\partial^* E_n \cap B(\mathbf{0}, m))$ is finite for every $m \in \mathbb{N}$. Show that E has locally finite perimeter. (*Hint*: $\int_E \operatorname{div} \phi\, d\mu = \lim_{n\to\infty} \int_{E_n} \operatorname{div} \phi\, d\mu$ for every Lipschitz function ϕ with compact support.)

(i) Give an example of bounded convex sets E and F such that $\partial^\$(E \cup F) \not\subseteq \partial^\$ E \cup \partial^\$ F$.

(j)(i) Show that if A, $B \subseteq \mathbb{R}^r$ then $\partial^*(A \cap B) \cap \partial^*(A \cup B) \subseteq \partial^* A \cap \partial^* B$. (ii) Show that if E, $F \subseteq \mathbb{R}$ are Lebesgue measurable, then $\operatorname{per}(E \cap F) + \operatorname{per}(E \cup F) \le \operatorname{per} E + \operatorname{per} F$.

>(k) Let $E \subseteq \mathbb{R}^r$ be a set with finite Lebesgue measure and finite perimeter. (i) Show that if $H \subseteq \mathbb{R}^r$ is a half-space, then $\operatorname{per}(E \cap H) \le \operatorname{per} E$. (*Hint*: 475Ja.) (ii) Show that if $C \subseteq \mathbb{R}^r$ is convex, then $\operatorname{per}(E \cap C) \le \operatorname{per} E$. (*Hint*: by the Hahn-Banach theorem, C is a limit of polytopes; use 474Ta.) (iii) Show that in 475Mc we have $\operatorname{per} E = \lim_{\alpha\to\infty} \operatorname{per}(E \cap B(\mathbf{0}, \alpha))$.

(l) Let $E\subseteq\mathbb{R}^r$ be a set with finite measure and finite perimeter, and $f:\mathbb{R}^r\to\mathbb{R}$ a Lipschitz function. Show that for any unit vector $v\in\mathbb{R}^r$, $|\int_E v\,\boldsymbol{.}\,\operatorname{grad} f\,d\mu|\le\|f\|_\infty\operatorname{per}E$.

(m) For measurable $E\subseteq\mathbb{R}^r$ set $p(E)=\sup_{x\in\mathbb{R}^r\setminus\{0\}}\frac{1}{\|x\|}\mu(E\triangle(E+x))$. (i) Show that for any measurable E, $p(E)=\limsup_{x\to\mathbf{0}}\frac{1}{\|x\|}\mu(E\triangle(E+x))$. (ii) Show that for every $\epsilon>0$ there is an $E\subseteq\mathbb{R}^r$ such that $\operatorname{per}E=1$ and $p(E)\ge 1-\epsilon$. (iii) Show that if $E\subseteq\mathbb{R}^r$ is a non-trivial ball then $\operatorname{per}E=\frac{r\beta_r}{2\beta_{r-1}}p(E)$. (iv) Show that if $E\subseteq\mathbb{R}^r$ is a cube then $\operatorname{per}E=\sqrt{r}p(E)$. (*Hint*: 475Yf.)

(n) Suppose that $E\subseteq\mathbb{R}^r$ is a bounded set with finite perimeter, and ϕ, $\psi:\mathbb{R}^r\to\mathbb{R}$ two differentiable functions such that $\operatorname{grad}\phi$ and $\operatorname{grad}\psi$ are Lipschitz. Show that

$$\int_E \phi\times\nabla^2\psi-\psi\times\nabla^2\phi\,d\mu=\int_{\partial^*E}(\phi\times\operatorname{grad}\psi-\psi\times\operatorname{grad}\phi)\,\boldsymbol{.}\,v_x\,\nu(dx)$$

where v_x is the Federer exterior normal to E at x when this is defined. (This is **Green's second identity**.)

475Y Further exercises (a) Show that if $A\subseteq\mathbb{R}^r$ is Lebesgue negligible, then there is a Borel set $E\subseteq\mathbb{R}^r$ such that $A\subseteq\partial^*E$.

(b) Let (X,ρ) be a metric space and μ a strictly positive locally finite topological measure on X. Show that we can define operations cl*, int* and ∂^* on $\mathcal{P}X$ for which parts (a)-(f) of 475C will be true.

(c) Let B be a ball in $\mathbb{R}^r$ with centre y, and v, v' two unit vectors in $\mathbb{R}^r$. Set

$$H=\{x:x\in\mathbb{R}^r,\,(x-y)\,\boldsymbol{.}\,v\le 0\},\quad H'=\{x:x\in\mathbb{R}^r,\,(x-y)\,\boldsymbol{.}\,v'\le 0\}.$$

Show that $\mu((H\triangle H')\cap B)=\frac{1}{\pi}\arccos(v\,\boldsymbol{.}\,v')\mu B$.

(d) Show that μ is inner regular with respect to the family of compact sets $K\subseteq\mathbb{R}^r$ such that $\limsup_{\delta\downarrow 0}\frac{\mu(B(x,\delta)\cap K)}{\mu B(x,\delta)}\ge\frac{1}{2}$ for every $x\in K$.

(e) Let $\langle f_n\rangle_{n\in\mathbb{N}}$ be a sequence of functions from $\mathbb{R}^r$ to $\mathbb{R}$ which is uniformly bounded and **uniformly Lipschitz** in the sense that there is some $\gamma\ge 0$ such that every f_n is γ-Lipschitz. Suppose that $f=\lim_{n\to\infty}f_n$ is defined everywhere in $\mathbb{R}^r$. (i) Show that if $E\subseteq\mathbb{R}^r$ has finite measure, then $\int_E z\,\boldsymbol{.}\,\operatorname{grad} f d\mu=\lim_{n\to\infty}\int_E z\,\boldsymbol{.}\,\operatorname{grad} f_n d\mu$ for every $z\in\mathbb{R}^r$. (*Hint*: look at E of finite perimeter first.) (ii) Show that for any convex function $\phi:\mathbb{R}^r\to[0,\infty[$, $\int\phi(\operatorname{grad} f)d\mu\le\liminf_{n\to\infty}\int\phi(\operatorname{grad} f_n)d\mu$.

(f) Let $E\subseteq\mathbb{R}^r$ be a set with locally finite perimeter. Show that

$$\sup_{x\in\mathbb{R}^r\setminus\{0\}}\frac{1}{\|x\|}\mu(E\triangle(E+x))=\sup_{\|v\|=1}\int_{\partial^*E}|v\,\boldsymbol{.}\,v_x|\nu(dx),$$

where v_x is the Federer exterior normal of E at x when this is defined.

(g) Let $E\subseteq\mathbb{R}^r$ be Lebesgue measurable. (i) Show that int*E is an $\mathrm{F}_{\sigma\delta}$ $(=\mathbf{\Pi}^0_3)$ set, that is, is expressible as the intersection of a sequence of F_σ sets. (ii)(cf. Andretta & Camerlo 13) Show that if E is not negligible and cl*E has empty interior, then int*E is not $\mathrm{G}_{\delta\sigma}$ $(=\mathbf{\Sigma}^0_3)$, that is, cannot be expressed as the union of sequence of G_δ sets.

475 Notes and comments The successful identification of the distributionally-defined notion of 'perimeter', as described in §474, with the geometrically accessible concept of Hausdorff measure of an appropriate boundary, is of course the key to any proper understanding of the results of the last section as well as this one. The very word 'perimeter' would be unfair if the perimeter of $E\cup F$ were unrelated to the perimeters of E and F; and from this point of view the reduced boundary is less suitable than the essential boundary (475Cd, 475Xi). If we re-examine 474M, we see that it is saying, in effect, that for many balls B the boundary $\partial^*(E\cap B)$ is nearly $(B\cap\partial^*E)\cup(E\cap\partial B)$, and that an outward-normal function for $E\cap B$ can be assembled from outward-normal functions for E and B. But looking at 475Xc-475Xd we see that this is entirely natural; we need only ensure that $\nu(F \cap \partial B) = 0$ for a μ-negligible set F defined from E; and the 'almost every δ' in the statement of 474M is fully enough to arrange this. On the other hand, 475Xg seems to be very hard to prove without using the identification between $\nu(\partial^* E)$ and $\operatorname{per} E$.

Concerning 475Q, I ought to emphasize that it is *not* generally true that

$$\nu F = \frac{1}{2\beta_{r-1}} \int_{S_{r-1}} \int_{V_v} \#(F \cap T_v^{-1}[\{u\}])\nu(du)\nu(dv)$$

even for $r = 2$ and compact sets F with $\nu F < \infty$. We are here approaching one of the many fundamental concepts of geometric measure theory which I am ignoring. The key word is 'rectifiability'; for 'rectifiable' sets a wide variety of concepts of k-dimensional measure coincide, including the integral-geometric form above, and $\partial^* E$ is rectifiable whenever E has locally finite perimeter (Evans & Gariepy 92, 5.7.3). For the general theory of rectifiable sets, see the last quarter of Mattila 95, or Chapter 3 of Federer 69.

I have already noted that the largest volumes for sets of given diameter or perimeter are provided by balls (see 264H and the notes to §474). The isoperimetric theorem for convex sets (475T) is of the same form: once again, the best constant (here, the largest perimeter for a convex set of given diameter, or the smallest diameter for a convex set of given perimeter) is provided by balls.

475Qb gives an alternative characterization of 'set of finite perimeter', with bounds on the perimeter which are sometimes useful.

Version of 29.7.21

476 Concentration of measure

Among the myriad special properties of Lebesgue measure, a particularly interesting one is 'concentration of measure'. For a set of given measure in the plane, it is natural to feel that it is most 'concentrated' if it is a disk. There are many ways of defining 'concentration', and I examine three of them in this section (476F, 476G and 476H); all lead us to Euclidean balls as the 'most concentrated' shapes. On the sphere the same criteria lead us to caps (476K, 476Xe).

All the main theorems of this section will be based on the fact that semi-continuous functions on compact spaces attain their bounds. The compact spaces in question will be spaces of subsets, and I begin with some general facts concerning the topologies introduced in 4A2T (476A-476B). The particular geometric properties of Euclidean space which make all these results possible are described in 476D-476E, where I describe concentrating operators based on reflections. The actual theorems 476F-476H and 476K can now almost be mass-produced.

476A Proposition Let X be a topological space, $\mathcal{C}$ the family of closed subsets of X, $\mathcal{K} \subseteq \mathcal{C}$ the family of closed compact sets and μ a topological measure on X.

(a)(i) If μ is outer regular with respect to the open sets then $\mu\restriction\mathcal{C} : \mathcal{C} \to [0, \infty[$ is upper semi-continuous with respect to the Vietoris topology on $\mathcal{C}$.

(ii) If μ is locally finite and inner regular with respect to the closed sets then $\mu\restriction\mathcal{K}$ is upper semi-continuous with respect to the Vietoris topology.

(iii) If μ is inner regular with respect to the closed sets and f is a non-negative μ-integrable real-valued function then $F \mapsto \int_F f d\mu : \mathcal{C} \to \mathbb{R}$ is upper semi-continuous with respect to the Vietoris topology.

(b) Suppose that μ is tight.

(i) If μ is totally finite then $\mu\restriction\mathcal{C}$ is upper semi-continuous with respect to the Fell topology on $\mathcal{C}$.

(ii) If f is a non-negative μ-integrable real-valued function then $F \mapsto \int_F f d\mu : \mathcal{C} \to \mathbb{R}$ is upper semi-continuous with respect to the Fell topology.

(c) Suppose that X is metrizable, and that ρ is a metric on X defining its topology; let $\tilde{\rho}$ be the Hausdorff metric on $\mathcal{C} \setminus \{\emptyset\}$.

(i) If μ is totally finite, then $\mu\restriction\mathcal{C} \setminus \{\emptyset\}$ is upper semi-continuous with respect to $\tilde{\rho}$.

(ii) If μ is locally finite, then $\mu\restriction\mathcal{K} \setminus \{\emptyset\}$ is upper semi-continuous with respect to $\tilde{\rho}$.

(iii) If f is a non-negative μ-integrable real-valued function, then $F \mapsto \int_F f d\mu : \mathcal{C} \setminus \{\emptyset\} \to \mathbb{R}$ is upper semi-continuous with respect to $\tilde{\rho}$.

proof (a)(i) Suppose that $F \in \mathcal{C}$ and that $\mu F < \alpha$. Because μ is outer regular with respect to the open sets, there is an open set $G \supseteq F$ such that $\mu G < \alpha$. Now $\mathcal{V} = \{E : E \in \mathcal{C}, E \subseteq G\}$ is an open set for the Vietoris topology containing F, and $\mu E < \alpha$ for every $E \in \mathcal{V}$. As F and α are arbitrary, $\mu \restriction \mathcal{C}$ is upper semi-continuous for the Vietoris topology.

(ii) Given that $K \in \mathcal{K}$ and $\mu K < \alpha$, then, because μ is locally finite, there is an open set G of finite measure including K (cf. 411Ga). Now there is a closed set $F \subseteq G \setminus K$ such that $\mu F > \mu G - \alpha$, so that $\mathcal{V} = \{L : L \in \mathcal{K}, L \subseteq G \setminus F\}$ is a relatively open subset of $\mathcal{K}$ for the Vietoris topology containing K, and $\mu L < \alpha$ for every $L \in \mathcal{V}$.

(iii) Apply (i) to the indefinite-integral measure over μ defined by f; by 412Q this is still inner regular with respect to the closed sets.

(b) If $F \in \mathcal{C}$ and $\mu F < \alpha$, let $K \subseteq X \setminus F$ be a compact set such that $\mu K > \mu X - \alpha$. Then $\mathcal{V} = \{E : E \in \mathcal{C}, E \cap K = \emptyset\}$ is a neighbourhood of F and $\mu E < \alpha$ for every $E \in \mathcal{V}$. This proves (i). Now (ii) follows as in (a-iii) above.

(c)(i) If $F \in \mathcal{C} \setminus \{\emptyset\}$ and $\mu F < \alpha$, then for each $n \in \mathbb{N}$ set $F_n = \{x : \rho(x, F) \le 2^{-n}\}$. Since $\langle F_n \rangle_{n \in \mathbb{N}}$ is a non-increasing sequence of closed sets with intersection F, and μ is totally finite, there is an n such that $\mu F_n < \alpha$. If now we take $E \in \mathcal{C} \setminus \{\emptyset\}$ such that $\tilde{\rho}(E, F) \le 2^{-n}$, then $E \subseteq F_n$ so $\mu E < \alpha$. As F and α are arbitrary, $\mu \restriction \mathcal{C} \setminus \{\emptyset\}$ is upper semi-continuous.

(ii) If $K \in \mathcal{K} \setminus \{\emptyset\}$ and $\mu K < \alpha$, let $G \supseteq K$ be an open set of finite measure, as in (a-ii) above. The function $x \mapsto \rho(x, X \setminus G)$ is continuous and strictly positive on K, so has a non-zero lower bound on K, and there is some $m \in \mathbb{N}$ such that $\rho(x, X \setminus G) > 2^{-m}$ for every $x \in K$. If, as in (i) just above, we set $F_n = \{x : \rho(x, K) \le 2^{-n}\}$ for each n, $F_m \subseteq G$ has finite measure. So, as in (i), we have an $n \ge m$ such that $\mu F_n < \alpha$, and we can continue as before.

(iii) Once again this follows at once from (i).

476B Lemma Let (X, ρ) be a metric space, and $\mathcal{C}$ the family of closed subsets of X, with its Fell topology. For $\epsilon > 0$, set $U(A, \epsilon) = \{x : x \in X, \rho(x, A) < \epsilon\}$ if $A \subseteq X$ is not empty; set $U(\emptyset, \epsilon) = \emptyset$. Then for any τ-additive topological measure μ on X, the function

$$(F, \epsilon) \mapsto \mu U(F, \epsilon) : \mathcal{C} \times]0, \infty[\to [0, \infty]$$

is lower semi-continuous.

proof Set $Q = \{(F, \epsilon) : F \in \mathcal{C}, \epsilon > 0, \mu U(F, \epsilon) > \gamma\}$, where $\gamma \in \mathbb{R}$. Take any $(F_0, \epsilon_0) \in Q$. Note first that $\mu U(F_0, \epsilon_0) = \sup_{\epsilon < \epsilon_0} \mu U(F, \epsilon)$, so there is a $\delta \in \left]0, \frac{1}{2}\epsilon_0\right[$ such that $\mu U(F_0, \epsilon_0 - 2\delta) > \gamma$. Next, $\{U(x, \epsilon_0 - 2\delta) : x \in F_0\}$ is an open cover of $U(F_0, \epsilon_0 - 2\delta)$; because μ is τ-additive, there is a finite set $I \subseteq F_0$ such that $\mu(\bigcup_{x \in I} U(x, \epsilon_0 - 2\delta)) > \gamma$. Consider

$$\mathcal{V} = \{F : F \in \mathcal{C}, F \cap U(x, \delta) \ne \emptyset \text{ for every } x \in I\}.$$

By the definition of the Fell topology, $\mathcal{V}$ is open. So $\mathcal{V} \times]\epsilon_0 - \delta, \infty[$ is an open neighbourhood of (F_0, ϵ) in $\mathcal{C} \times \mathbb{R}$. If $F \in \mathcal{V}$ and $\epsilon > \epsilon_0 - \delta$, then

$$U(F, \epsilon) \supseteq \bigcup_{x \in I} U(x, \epsilon - \delta) \supseteq \bigcup_{x \in I} U(x, \epsilon_0 - 2\delta)$$

has measure greater than γ and $(F, \epsilon) \in Q$. As (F_0, ϵ_0) is arbitrary, Q is open; as γ is arbitrary, $(F, \epsilon) \mapsto \mu U(F, \epsilon)$ is lower semi-continuous.

Remark Recall that all 'ordinary' topological measures on metric spaces are τ-additive; see 438J.

476C Proposition Let (X, ρ) be a non-empty compact metric space, and suppose that its isometry group G acts transitively on X. Then X has a unique G-invariant Radon probability measure μ, which is strictly positive.

proof By 441G, G, with its topology of pointwise convergence, is a compact topological group, and the action of G on X is continuous. So 443Ud gives the result.

476D Concentration by partial reflection The following construction will be used repeatedly in the rest of the section.

(a) Let X be an inner product space. (In this section, X will be usually be $\mathbb{R}^r$, but in 493G below it will be helpful to be able to speak of abstract Hilbert spaces.) For any unit vector $e \in X$ and any $\alpha \in \mathbb{R}$, write $R = R_{e\alpha} : X \to X$ for the reflection in the hyperplane $V = V_{e\alpha} = \{x : x \in X, (x|e) = \alpha\}$, so that $R(x) = x + 2(\alpha - (x|e))e$ for every $x \in X$. Next, for any $A \subseteq X$, we can define a set $\psi(A) = \psi_{e\alpha}(A)$ by setting

$$\begin{aligned}\psi(A) = \{x : x \in A, (x|e) \ge \alpha\} &\cup \{x : x \in A, (x|e) < \alpha, R(x) \in A\} \\ &\cup \{x : x \in \mathbb{R}^r \setminus A, (x|e) \ge \alpha, R(x) \in A\} \\ = (W \cap (A \cup R[A])) &\cup (A \cap R[A]),\end{aligned}$$

where $W = W_{e\alpha}$ is the half-space $\{x : (x|e) \ge \alpha\}$. Geometrically, we construct $\psi(A)$ by moving those points of A on the 'wrong' side of the hyperplane V to their reflections, provided those points are not already occupied. We have the following facts.

(b)(i) If $A \subseteq B \subseteq X$, $\psi(A) \subseteq \psi(B)$. (ii) For any $A \subseteq X$, $\psi(R[A]) = \psi(A)$. (iii) If $F \subseteq X$ is closed, then $\psi(F)$ is closed. **P** Use the second formula in (a) for $\psi(F)$. **Q**

(c) We need a fragment of elementary geometry. If $x \in X \setminus W$ and $y \in W$ then $\|x - R(y)\| \le \|x - y\|$. **P** Write Y for the linear subspace of X generated by e, $Y^\perp$ for its orthogonal complement and P for the orthogonal projection of X onto $Y^\perp$. Then $P(e) = 0$, $PR(y) = P(y)$,

$$(x|e) < \alpha \le (y|e), \quad (R(y)|e) = (y|e) + 2(\alpha - (y|e)) = 2\alpha - (y|e),$$

$$|(x - y|e)| = |(x|e) - \alpha| + |(y|e) - \alpha| \ge |(x|e) - 2\alpha + (y|e)| = |(x - R(y)|e)|,$$

and

$$\|x - y\|^2 = (x - y|e)^2 + \|P(x - y)\|^2$$

(4A4Jf)

$$\ge (x - R(y)|e)^2 + \|P(x - R(y))\|^2 = \|x - R(y)\|^2. \text{ Q}$$

(d) For non-empty $A \subseteq X$ and $\epsilon > 0$, set $U(A, \epsilon) = \{x : \rho(x, A) < \epsilon\}$, where ρ is the standard metric on X. Now $U(\psi(A), \epsilon) \subseteq \psi(U(A, \epsilon))$. **P**

(i) If $x \in U(W \cap A, \epsilon) \setminus W$, there is a $y \in W \cap A$ such that $\|x - y\| < \epsilon$; by (c), $\|x - R(y)\| < \epsilon$ so $x \in U(R[A], \epsilon) = R[U(A, \epsilon)]$ (because $R : X \to X$ is an isometry) and

$$x \in U(A, \epsilon) \cap R[U(A, \epsilon)] \subseteq \psi(U(A, \epsilon));$$

thus $U(W \cap A, \epsilon) \setminus W \subseteq \psi(U(A, \epsilon))$; as also

$$U(W \cap A, \epsilon) \cap W \subseteq U(A, \epsilon) \cap W \subseteq \psi(U(A, \epsilon)),$$

we have $U(W \cap A, \epsilon) \subseteq \psi(U(A, \epsilon))$.

(ii) Now

$$\begin{aligned}U(\psi(A), \epsilon) &= U(A \cap R[A], \epsilon) \cup U(A \cap W, \epsilon) \cup U(R[A] \cap W, \epsilon) \\ &\subseteq (U(A, \epsilon) \cap U(R[A], \epsilon)) \cup \psi(U(A, \epsilon)) \cup \psi(U(R[A], \epsilon)) \\ &= (U(A, \epsilon) \cap R[U(A, \epsilon)]) \cup \psi(U(A, \epsilon)) \cup \psi(R[U(A, \epsilon)]) \\ &= (U(A, \epsilon) \cap R[U(A, \epsilon)]) \cup \psi(U(A, \epsilon))\end{aligned}$$

(using (b-i))

$$= \psi(U(A, \epsilon)). \text{ Q}$$

476E Lemma Let X be an inner product space, $e \in X$ a unit vector and $\alpha \in \mathbb{R}$. Let $R = R_{e\alpha} : X \to X$ be the reflection operator, and $\psi = \psi_{e\alpha} : \mathcal{P}X \to \mathcal{P}X$ the associated transformation, as described in 476D. For $x \in A \subseteq X$, define

$$\begin{aligned}\phi_A(x) &= x \text{ if } (x|e) \geq \alpha,\\ &= x \text{ if } (x|e) < \alpha \text{ and } R(x) \in A,\\ &= R(x) \text{ if } (x|e) < \alpha \text{ and } R(x) \notin A.\end{aligned}$$

Let ν be a topological measure on X which is R-invariant, that is, ν coincides with the image measure νR^{-1}.

(a) For any $A \subseteq X$, $\phi_A : A \to \psi(A)$ is a bijection. If $\alpha < 0$, then $\|\phi_A(x)\| \leq \|x\|$ for every $x \in A$, with $\|\phi_A(x)\| < \|x\|$ iff $(x|e) < \alpha$ and $R(x) \notin A$.

(b)(i) If $E \subseteq X$ is measured by ν, then $\psi(E)$ is measured by ν, $\nu\psi(E) = \nu E$ and ϕ_E is a measure space isomorphism for the subspace measures on E and $\psi(E)$ induced by ν.

(ii) For any $A \subseteq X$, $\nu^*\psi(A) \leq \nu^* A \leq 2\nu^*\psi(A)$.

(c) If $\alpha < 0$ and $E \subseteq X$ is measured by ν, then $\int_E \|x\|\nu(dx) \geq \int_{\psi(E)} \|x\|\nu(dx)$, with equality iff $\{x : x \in E,\ (x|e) < \alpha,\ R(x) \notin E\}$ is negligible.

(d) Suppose that X is separable. Let λ be the c.l.d. product measure of ν with itself on $X \times X$. If $E \subseteq X$ is measured by ν, then

$$\int_{E\times E} \|x - y\|\lambda(d(x, y)) \geq \int_{\psi(E)\times\psi(E)} \|x - y\|\lambda(d(x, y)).$$

(e) Now suppose that $X = \mathbb{R}^r$. Then $\nu(\partial^*\psi(A)) \leq \nu(\partial^* A)$ for every $A \subseteq \mathbb{R}^r$, where $\partial^* A$ is the essential boundary of A (definition: 475B).

proof (a) That $\phi_A : A \to \psi(A)$ is a bijection is immediate from the definitions of ψ and ϕ_A. If $\alpha < 0$, then for any $x \in A$ either $\phi_A(x) = x$ or $(x|e) < \alpha$ and $R(x) \notin A$. In the latter case

$$\|\phi_A(x)\|^2 = \|R(x)\|^2 = \|x + 2\gamma e\|^2$$

(where $\gamma = \alpha - (x|e) > 0$)

$$= \|x\|^2 + 4\gamma(x|e) + 4\gamma^2 = \|x\|^2 + 4\gamma\alpha < \|x\|^2,$$

so $\|\phi_A(x)\| < \|x\|$.

(b)(i) If we set

$$E_1 = \{x : x \in E,\ (x|e) \geq \alpha\},$$

$$E_2 = \{x : x \in E, (x|e) < \alpha,\ R(x) \in E\},$$

$$E_3 = \{x : x \in E,\ (x|e) < \alpha,\ R(x) \notin E\},$$

$$E_4 = \{x : x \in \mathbb{R}^r \setminus E,\ (x|e) > \alpha,\ R(x) \in E\},$$

then E_1, E_2, E_3 and E_4 are disjoint and measured by ν, $E = E_1 \cup E_2 \cup E_3$, $\psi(E) = E_1 \cup E_2 \cup E_4$ and $\phi_E{\restriction}E_3 = R{\restriction}E_3$ is a measure space isomorphism for the subspace measures on E_3 and E_4.

(ii) There is an $E \supseteq A$ such that ν measures E and $\nu^* A = \nu E$. Now $\psi(A) \subseteq \psi(E)$, so

$$\nu^*\psi(A) \leq \nu\psi(E) = \nu E = \nu^* A.$$

On the other hand, if $x \in A$ then at least one of x, $R(x)$ belongs to $\psi(A)$, so $A \subseteq \psi(A) \cup R[\psi(A)]$ and

$$\nu^* A \leq \nu^*\psi(A) + \nu^* R[\psi(A)] = 2\nu^*\psi(A).$$

(c) By (a),

$$\int_E \|x\|\nu(dx) \geq \int_E \|\phi_E(x)\|\nu(dx) = \int_{\psi(E)} \|x\|\nu(dx)$$

by 235Gc, because $\phi_E : E \to \psi(E)$ is inverse-measure-preserving, with equality only when

$$\{x : \|x\| > \|\phi_E(x)\|\} = \{x : x \in E,\ (x|e) < \alpha,\ R(x) \notin E\}$$

is negligible.

(d) Note first that if Λ is the domain of λ then Λ includes the Borel algebra of $X \times X$ (because X is second-countable, so this is just the σ-algebra generated by products of Borel sets, by 4A3Ga); so that $(x, y) \mapsto \|x - y\|$ is Λ-measurable, and the integrals are defined in $[0, \infty]$. Now consider the sets

$$W_1 = \{(x, y) : x \in E,\ y \in E,\ R(x) \notin E,\ R(y) \in E,\ (x|e) < \alpha,\ (y|e) < \alpha\},$$

$$W_1' = \{(x, y) : x \in E,\ y \in E,\ R(x) \notin E,\ R(y) \in E,\ (x|e) < \alpha,\ (y|e) > \alpha\},$$

$$W_2 = \{(x, y) : x \in E,\ y \in E,\ R(x) \in E,\ R(y) \notin E,\ (x|e) < \alpha,\ (y|e) < \alpha\},$$

$$W_2' = \{(x, y) : x \in E,\ y \in E,\ R(x) \in E,\ R(y) \notin E,\ (x|e) > \alpha,\ (y|e) < \alpha\}.$$

Then $(x, y) \mapsto (x, R(y)) : W_1' \to W_1$ is a measure space isomorphism for the subspace measures induced on W_1 and W_1' by λ, so

$$\begin{aligned}\int_{W_1} \|\phi_E(x) - \phi_E(y)\|\lambda(d(x, y)) &= \int_{W_1} \|R(x) - y\|\lambda(d(x, y))\\ &= \int_{W_1'} \|R(x) - R(y)\|\lambda(d(x, y))\\ &= \int_{W_1'} \|x - y\|\lambda(d(x, y)).\end{aligned}$$

Similarly,

$$\begin{aligned}\int_{W_1'} \|\phi_E(x) - \phi_E(y)\|\lambda(d(x, y)) &= \int_{W_1'} \|R(x) - y\|\lambda(d(x, y))\\ &= \int_{W_1} \|R(x) - R(y)\|\lambda(d(x, y))\\ &= \int_{W_1} \|x - y\|\lambda(d(x, y)).\end{aligned}$$

So we get

$$\int_{W_1 \cup W_1'} \|\phi_E(x) - \phi_E(y)\|\lambda(d(x, y)) = \int_{W_1 \cup W_1'} \|x - y\|\lambda(d(x, y)).$$

In the same way, $(x, y) \mapsto (R(x), y)$ is an isomorphism of the subspace measures on W_2 and W_2', and we have

$$\int_{W_2 \cup W_2'} \|\phi_E(x) - \phi_E(y)\|\lambda(d(x, y)) = \int_{W_2 \cup W_2'} \|x - y\|\lambda(d(x, y)).$$

On the other hand, for all $(x, y) \in (E \times E) \setminus (W_1 \cup W_1' \cup W_2 \cup W_2')$, we have $\|\phi_E(x) - \phi_E(y)\| \le \|x - y\|$. (Either x and y are both left fixed by ϕ_E, or both are moved, or one is on the reflecting hyperplane, or one is moved to the same side of the reflecting hyperplane as the other.) So we get

$$\begin{aligned}\int_{E \times E} \|x - y\|\lambda(d(x, y)) &\ge \int_{E \times E} \|\phi_E(x) - \phi_E(y)\|\lambda(d(x, y))\\ &= \int_{\psi(E) \times \psi(E)} \|x - y\|\lambda(d(x, y))\end{aligned}$$

because $(x, y) \mapsto (\phi_E(x), \phi_E(y))$ is an inverse-measure-preserving transformation for the subspace measures on $E \times E$ and $\psi(E) \times \psi(E)$.

(e)(i) Because R is both an isometry and a measure space automorphism for Lebesgue measure μ on $\mathbb{R}^r$, $\mathrm{cl}^*R[A] = R[\mathrm{cl}^*A]$ and $\mathrm{int}^*R[A] = R[\mathrm{int}^*A]$, where cl^*A and int^*A are the essential closure and the essential interior of A, as in 475B. Recall that cl^*A, int^*A and ∂^*A are all Borel sets (475Cc), so that ∂^*A and $\partial^*\psi(A)$ are measured by ν.

(ii) Suppose that $x\mathbin{.}e=\alpha$. Then $x\in\partial^*\psi(A)$ iff $x\in\partial^*A$. **P** For any $\delta>0$, it is easy to check that $B(x,\delta)\cap\psi(A)=\psi(B(x,\delta)\cap A)$, while $\mu^*(B(x,\delta)\cap A)$ lies between $\mu^*\psi(B(x,\delta)\cap A)$ and $2\mu^*\psi(B(x,\delta)\cap A)$ by (b-ii) above; so

$$\mu^*(B(x,\delta)\cap\psi(A))\le\mu^*(B(x,\delta)\cap A)\le 2\mu^*(B(x,\delta)\cap\psi(A))$$

for every $\delta>0$ and $x\in\mathrm{cl}^*A$ iff $x\in\mathrm{cl}^*\psi(A)$. If $x=R(x)\in\mathrm{int}^*A$, then $x\in\mathrm{int}^*R[A]$ so $x\in\mathrm{int}^*(A\cap R[A])$ (475Cd) and $x\in\mathrm{int}^*\psi(A)$. If $x\in\mathrm{int}^*\psi(A)$ then $x\in\mathrm{int}^*(R[\psi(A)]\cap\psi(A))\subseteq\mathrm{int}^*A$. **Q**

(iii) If $x\in\partial^*\psi(A)\setminus\partial^*A$ then $R(x)\in\partial^*A\setminus\partial^*\psi(A)$. **P** By (ii), $x\mathbin{.}e\ne\alpha$.

case 1 Suppose that $x\mathbin{.}e>\alpha$. Setting $\delta=x\mathbin{.}e-\alpha$, we see that $U(x,\delta)\cap\psi(A)=U(x,\delta)\cap(A\cup R[A])$, while $U(R(x),\delta)\cap\psi(A)=U(R(x),\delta)\cap A\cap R[A]$. Since $x\notin\mathrm{int}^*\psi(A)$, $x\notin\mathrm{int}^*(A\cup R[A])$ and $x\notin\mathrm{int}^*A$; since x also does not belong to ∂^*A, $x\notin\mathrm{cl}^*A$. However, $x\in\mathrm{cl}^*(A\cup R[A])=\mathrm{cl}^*A\cup\mathrm{cl}^*R[A]$ (475Cd), so $x\in\mathrm{cl}^*R[A]$ and $R(x)\in\mathrm{cl}^*A$. Next, $x\notin\mathrm{int}^*R[A]$, so $R(x)\notin\mathrm{int}^*A$ and $R(x)\in\partial^*A$. Since $x\notin\mathrm{cl}^*A$, $R(x)\notin\mathrm{cl}^*R[A]$, while $R(x)\mathbin{.}e<\alpha$; so $R(x)\notin\mathrm{cl}^*\psi(A)$ and $R(x)\in\partial^*A\setminus\partial^*\psi(A)$.

case 2 Suppose that $x\mathbin{.}e<\alpha$. This time, set $\delta=\alpha-x\mathbin{.}e$, so that $U(x,\delta)\cap\psi(A)=U(x,\delta)\cap A\cap R[A]$ and $U(R(x),\delta)\cap\psi(A)=U(R(x),\delta)\cap(A\cup R[A])$. As $x\in\mathrm{cl}^*\psi(A)$, $x\in\mathrm{cl}^*(A\cap R[A])$ and $R(x)\in\mathrm{cl}^*A$. Also $x\in\mathrm{cl}^*A$; as $x\notin\partial^*A$, $x\in\mathrm{int}^*A$, $R(x)\in\mathrm{int}^*R[A]$ and $R(x)\in\mathrm{int}^*\psi(A)$, so that $R(x)\notin\partial^*\psi(A)$. Finally, we know that $x\in\mathrm{int}^*A$ but $x\notin\mathrm{int}^*(A\cap R[A])$ (because $x\notin\mathrm{int}^*\psi(A)$); it follows that $x\notin\mathrm{int}^*R[A]$ so $R(x)\notin\mathrm{int}^*A$ and $R(x)\in\partial^*A\setminus\partial^*\psi(A)$.

Thus all possibilities are covered and we have the result. **Q**

(iv) What this means is that if we set $E=\partial^*\psi(A)\setminus\partial^*A$ then $R[E]\subseteq\partial^*A\setminus\partial^*\psi(A)$. So

$$\nu\partial^*\psi(A)=\nu E+\nu(\partial^*\psi(A)\cap\partial^*A)=\nu R[E]+\nu(\partial^*\psi(A)\cap\partial^*A)\le\nu\partial^*A,$$

as required.

476F Theorem Let $r\ge 1$ be an integer, and let μ be Lebesgue measure on $\mathbb{R}^r$. For non-empty $A\subseteq\mathbb{R}^r$ and $\epsilon>0$, write $U(A,\epsilon)$ for $\{x:\rho(x,A)<\epsilon\}$, where ρ is the Euclidean metric on $\mathbb{R}^r$. If μ^*A is finite, then $\mu U(A,\epsilon)\ge\mu U(B_A,\epsilon)$, where B_A is the closed ball with centre $\mathbf{0}$ and measure μ^*A.

proof (a) To begin with, suppose that A is bounded. Set $\gamma=\mu^*A$ and $\beta=\mu U(A,\epsilon)$. If $\gamma=0$ then (because $A\ne\emptyset$)

$$\mu U(A,\epsilon)\ge\mu U(\{\mathbf{0}\},\epsilon)=\mu U(B_A,\epsilon),$$

and we can stop. So let us suppose henceforth that $\gamma>0$. Let $M\ge 0$ be such that $A\subseteq B(\mathbf{0},M)$, and consider the family

$$\mathcal{F}=\{F:F\in\mathcal{C},\ F\subseteq B(\mathbf{0},M),\ \mu F\ge\gamma,\ \mu U(F,\epsilon)\le\beta\},$$

where $\mathcal{C}$ is the family of closed subsets of $\mathbb{R}^r$ with its Fell topology. Because $U(\overline{A},\epsilon)=U(A,\epsilon)$, $\overline{A}\in\mathcal{F}$ and $\mathcal{F}$ is non-empty. By the definition of the Fell topology, $\{F:F\subseteq B(\mathbf{0},M)\}$ is closed; by 476A(b-ii) (applied to the functional $F\mapsto\int_F\chi B(\mathbf{0},M)d\mu$) and 476B, $\mathcal{F}$ is closed in $\mathcal{C}$, therefore compact, by 4A2T(b-iii). Next, the function

$$F\mapsto\int_F\max(0,M-\|x\|)\mu(dx):\mathcal{C}\to[0,\infty[$$

is upper semi-continuous, by 476A(b-ii) again. It therefore attains its supremum on $\mathcal{F}$ at some $F_0\in\mathcal{F}$ (4A2Gl). Let $F_1\subseteq F_0$ be a closed self-supporting set of the same measure as F_0; then $U(F_1,\epsilon)\subseteq U(F_0,\epsilon)$ and $\mu F_1=\mu F_0$, so $F_1\in\mathcal{F}$; also

$$\int_{F_1}M-\|x\|\mu(dx)=\int_{F_0}M-\|x\|\mu(dx)\ge\int_F M-\|x\|\mu(dx)$$

for every $F\in\mathcal{F}$.

(b) Now F_1 is a ball with centre $\mathbf{0}$. **P?** Suppose, if possible, otherwise. Then there are $x_1\in F_1$ and $x_0\in\mathbb{R}^r\setminus F_1$ such that $\|x_0\|<\|x_1\|$. Set $e=\frac{1}{\|x_0-x_1\|}(x_0-x_1)$, so that e is a unit vector, and $\alpha=\frac{1}{2}e\mathbin{.}(x_0+x_1)$; then

$$\alpha=\frac{1}{2\|x_0-x_1\|}(x_0-x_1)\mathbin{.}(x_0+x_1)=\frac{1}{2\|x_0-x_1\|}(\|x_0\|^2-\|x_1\|^2)<0,$$

$$e\centerdot(x_0 - x_1) > 0, \quad e\centerdot x_0 > e\centerdot x_1, \quad e\centerdot x_0 > \alpha > e\centerdot x_1.$$

Define $R = R_{e\alpha} : \mathbb{R}^r \to \mathbb{R}^r$ and $\psi = \psi_{e\alpha}$ as in 476D; note that $R(x_1) = x_0$. Set $F = \psi(F_1)$. Then F is closed (476Db) and $\mu F = \mu F_1 \ge \mu^* A$ (476Eb). Also $U(F,\epsilon) \subseteq \psi(U(F_1,\epsilon))$ (476Dd), so

$$\mu U(F,\epsilon) \le \mu(\psi(U(F_1,\epsilon)) = \mu U(F_1,\epsilon) \le \beta$$

and $F \in \mathcal{F}$. It follows that $\int_F M - \|x\|\mu(dx) \le \int_{F_1} M - \|x\|\mu(dx)$; as $\mu F = \mu F_1$, $\int_F \|x\|\mu(dx) \ge \int_{F_1} \|x\|\mu(dx)$. By 476Ec, $G = \{x : x \in F_1, x\centerdot e < \alpha, R(x) \notin F_1\}$ is negligible. But G contains x_1 and is relatively open in F_1, and F_1 is supposed to be self-supporting; so this is impossible. **XQ**

(c) Since $\mu F_1 \ge \gamma$, $F_1 \supseteq B_A$, and

$$\mu U(B_A,\epsilon) \le \mu U(F_1,\epsilon) \le \beta = \mu U(A,\epsilon).$$

So we have the required result for bounded A. In general, given an unbounded set A of finite measure, let δ be the radius of B_A; then

$$\begin{aligned}\mu U(B_A,\epsilon) = \mu B(\mathbf{0},\delta+\epsilon) &= \sup_{\alpha<\delta} \mu B(\mathbf{0},\alpha+\epsilon) \\ &\le \sup_{A' \subseteq A \text{ is bounded}} \mu U(B_{A'},\epsilon) \le \sup_{A' \subseteq A \text{ is bounded}} \mu U(A',\epsilon) = \mu U(A,\epsilon)\end{aligned}$$

because $\{U(A',\epsilon) : A' \subseteq A \text{ is bounded}\}$ is an upwards-directed family of open sets with union $U(A,\epsilon)$, and μ is τ-additive. So the theorem is true for unbounded A as well.

476G Theorem Let $r \ge 1$ be an integer, and let μ be Lebesgue measure on $\mathbb{R}^r$; write λ for the product measure on $\mathbb{R}^r \times \mathbb{R}^r$. For any measurable set $E \subseteq \mathbb{R}^r$ of finite measure, write B_E for the closed ball with centre $\mathbf{0}$ and the same measure as E. Then

$$\int_{E\times E} \|x-y\|\lambda(d(x,y)) \ge \int_{B_E\times B_E} \|x-y\|\lambda(d(x,y)).$$

proof (a) Suppose for the time being (down to the end of (c) below) that E is compact and not empty, and that $\epsilon > 0$. Let $M \ge 0$ be such that $\|x\| \le M$ for every $x \in E$. For a non-empty set $A \subseteq \mathbb{R}^r$ set $U(A,\epsilon) = \{x : \rho(x,A) < \epsilon\}$, where ρ is Euclidean distance on $\mathbb{R}^r$. Set $\beta = \int_{U(E,\epsilon)\times U(E,\epsilon)} \|x-y\|\lambda(d(x,y))$. Let $\mathcal{F}$ be the family of non-empty closed subsets F of the ball $B(\mathbf{0},M) = \{x : \|x\| \le M\}$ such that $\mu F \ge \mu E$ and $\int_{U(F,\epsilon)\times U(F,\epsilon)} \|x-y\|\lambda(d(x,y)) \le \beta$. Then $\mathcal{F}$ is compact for the Fell topology on the family $\mathcal{C}$ of closed subsets of $\mathbb{R}^r$. **P** We know from 4A2T(b-iii) that $\mathcal{C}$ is compact, and from 476A(b-ii) that $\{F : \int_F \chi B(\mathbf{0},M) d\mu \ge \mu E\}$ is closed; also $\{F : F \subseteq B(\mathbf{0},M)\}$ is closed. Let σ be the metric on $\mathbb{R}^r \times \mathbb{R}^r$ defined by setting $\sigma((x,y),(x',y')) = \max(\|x-x'\|,\|y-y'\|)$, and ν the indefinite-integral measure over λ defined by the function $(x,y) \mapsto \|x-y\|$. Then

$$U(F,\epsilon) \times U(F,\epsilon) = \{(x,y) : \sigma((x,y),F\times F) < \epsilon\} = U(F\times F,\epsilon;\sigma)$$

for $F \in \mathcal{C}$ and $\epsilon > 0$. Now, writing $\mathcal{C}_2$ for the family of closed sets in $\mathbb{R}^r \times \mathbb{R}^r$ with its Fell topology, we know that

$F \mapsto F \times F : \mathcal{C} \to \mathcal{C}_2$ is continuous, by 4A2Td,

$E \mapsto \nu U(E,\epsilon;\sigma) : \mathcal{C}_2 \to \mathbb{R}$ is lower semi-continuous, by 476B;

so $F \mapsto \nu(U(F,\epsilon) \times U(F,\epsilon))$ is lower semi-continuous, and $\{F : \int_{U(F,\epsilon)\times U(F,\epsilon)} \|x-y\|\lambda(d(x,y)) \le \beta\}$ is closed. Putting these together, $\mathcal{F}$ is a closed subset of $\mathcal{C}$ and is compact. **Q**

(b) Since $E \in \mathcal{F}$, $\mathcal{F}$ is not empty. By 476A(b-ii), there is an $F_0 \in \mathcal{F}$ such that $\int_{F_0} M - \|x\|\mu(dx) \ge \int_F M - \|x\|\mu(dx)$ for every $F \in \mathcal{F}$. Let $F_1 \subseteq F_0$ be a closed self-supporting set of the same measure; then $U(F_1,\epsilon) \subseteq U(F_0,\epsilon)$, so $\int_{U(F_1,\epsilon)\times U(F_1,\epsilon)} \|x-y\|\lambda(d(x,y)) \le \beta$ and $F_1 \in \mathcal{F}$; also

$$\int_{F_1} M - \|x\|\mu(dx) = \int_{F_0} M - \|x\|\mu(dx) \ge \int_F M - \|x\|\mu(dx)$$

for every $F \in \mathcal{F}$.

Now F_1 is a ball with centre $\mathbf{0}$. **P?** Suppose, if possible, otherwise. Then (just as in the proof of 476F) there are $x_1 \in F_1$ and $x_0 \in \mathbb{R}^r \setminus F_1$ such that $\|x_1\| > \|x_0\|$. Once again, set $e = \frac{1}{\|x_0-x_1\|}(x_0 - x_1)$ and

$\alpha = \frac{1}{2}e\centerdot(x_0 + x_1) < 0$. Define $R = R_{e\alpha} : \mathbb{R}^r \to \mathbb{R}^r$ and $\psi = \psi_{e\alpha}$ as in 476D. Set $F = \psi(F_1)$. Then F is closed and $\mu F = \mu F_1 \ge \mu E$ and $U(F, \epsilon) \subseteq \psi(U(F_1, \epsilon))$. So

$$\begin{aligned}\int_{U(F,\epsilon)\times U(F,\epsilon)} \|x-y\|\lambda(d(x,y)) &\le \int_{\psi(U(F_1,\epsilon))\times\psi(U(F_1,\epsilon))} \|x-y\|\lambda(d(x,y))\\ &\le \int_{U(F_1,\epsilon)\times U(F_1,\epsilon)} \|x-y\|\lambda(d(x,y))\\ &\le \beta. \qquad \text{(476Ed)}\end{aligned}$$

This means that $F \in \mathcal{F}$. Accordingly $\int_{F_1} M - \|x\|\mu(dx) \ge \int_F M - \|x\|\mu(dx)$; since $\mu F = \mu F_1$, $\int_{F_1} \|x\|\mu(dx) \le \int_F \|x\|\mu(dx)$. By 476Ec, $G = \{x : x \in F_1,\ x\centerdot e < \alpha,\ R(x) \notin F_1\}$ must be negligible. But G contains x_1 and is relatively open in F_1, and F_1 is supposed to be self-supporting; so this is impossible. **XQ**

(c) Since $\mu F_1 \ge \mu E$, $F_1 \supseteq B_E$, and

$$\begin{aligned}\int_{B_E\times B_E} \|x-y\|\lambda(d(x,y)) &\le \int_{U(F_1,\epsilon)\times U(F_1,\epsilon)} \|x-y\|\lambda(d(x,y))\\ &\le \beta = \int_{U(E,\epsilon)\times U(E,\epsilon)} \|x-y\|\lambda(d(x,y)).\end{aligned}$$

At this point, recall that ϵ was arbitrary. Since E is compact,

$$\begin{aligned}\int_{E\times E} \|x-y\|\lambda(d(x,y)) &= \inf_{\epsilon>0} \int_{U(E,\epsilon)\times U(E,\epsilon)} \|x-y\|\lambda(d(x,y))\\ &\ge \int_{B_E\times B_E} \|x-y\|\lambda(d(x,y)).\end{aligned}$$

(d) Thus the result is proved for non-empty compact sets E. In general, given a measurable set E of finite measure, then if E is negligible the result is trivial; and otherwise, writing δ for the radius of B_E,

$$\begin{aligned}\int_{B_E\times B_E} \|x-y\|\lambda(d(x,y)) &= \sup_{\alpha<\delta} \int_{B(\mathbf{0},\alpha)\times B(\mathbf{0},\alpha)} \|x-y\|\lambda(d(x,y))\\ &\le \sup_{K\subseteq E \text{ is compact}} \int_{B_K\times B_K} \|x-y\|\lambda(d(x,y))\\ &\le \sup_{K\subseteq E \text{ is compact}} \int_{K\times K} \|x-y\|\lambda(d(x,y))\\ &= \int_{E\times E} \|x-y\|\lambda(d(x,y)),\end{aligned}$$

so the proof is complete.

476H The Isoperimetric Theorem Let $r \ge 1$ be an integer, and let μ be Lebesgue measure on $\mathbb{R}^r$. If $E \subseteq \mathbb{R}^r$ is a measurable set of finite measure, then $\operatorname{per} E \ge \operatorname{per} B_E$, where B_E is the closed ball with centre $\mathbf{0}$ and the same measure as E, while $\operatorname{per} E$ is the perimeter of E as defined in 474D.

proof (a) Suppose to begin with that $E \subseteq B(\mathbf{0}, M)$, where $M \ge 0$, and that $\operatorname{per} E < \infty$. Let $\mathcal{F}$ be the family of measurable sets $F \subseteq \mathbb{R}^r$ such that $F \setminus B(\mathbf{0}, M)$ is negligible, $\mu F \ge \mu E$ and $\operatorname{per} F \le \operatorname{per} E$, with the topology of convergence in measure (474T). Then

$$\begin{aligned}\mathcal{F} = \{F : \operatorname{per} F \le \operatorname{per} E,\ &\mu(F \cap B(\mathbf{0}, M)) \ge \mu E,\\ &\mu(F \cap B(\mathbf{0}, \alpha)) \le \mu(F \cap B(\mathbf{0}, M)) \text{ for every } \alpha \ge 0\}\end{aligned}$$

is a closed subset of $\{F : \operatorname{per} F \le \operatorname{per} E\}$, which is compact (474Tb), so $\mathcal{F}$ is compact. For $F \in \mathcal{F}$ set $h(F) = \int_F \|x\|\mu(dx)$; then $|h(F) - h(F')| \le M\mu((F \triangle F') \cap B(\mathbf{0}, M))$ for all F, $F' \in \mathcal{F}$, so h is continuous. There is therefore an $F_0 \in \mathcal{F}$ such that $h(F_0) \le h(F)$ for every $F \in \mathcal{F}$. Set $F_1 = \mathrm{cl}^* F_0$, so that $F_1 \triangle F_0$ is negligible (475Cg), $\operatorname{per} F_1 = \operatorname{per} F_0$ (474F), $F_1 \in \mathcal{F}$, $F_1 \subseteq B(\mathbf{0}, M)$ and $h(F_1) = h(F_0)$.

(b) Writing $\delta = \sup_{x \in F_1} \|x\|$, we have $U(\mathbf{0}, \delta) \subseteq F_1$. **P?** Otherwise, there are $x_0 \in \mathbb{R}^r \setminus F_1$ and $x_1 \in F_1$ such that $\|x_0\| < \|x_1\|$. Set $e = \frac{1}{\|x_0 - x_1\|}(x_0 - x_1)$, $\alpha = \frac{1}{2} e \,.\, (x_0 + x_1) < 0$, $R = R_{e\alpha} : \mathbb{R}^r \to \mathbb{R}^r$ and $\psi = \psi_{e\alpha}$. As before, $e \,.\, x_1 < \alpha < e \,.\, x_0$, and $R(x_0) = x_1$ so $R[B(x_0, \eta)] = B(x_1, \eta)$ for every $\eta > 0$. Set $F = \psi(F_1)$ and let $\phi = \phi_{F_1} : F_1 \to F$ be the function described in 476E. Then $\|\phi(x)\| \le \|x\|$ for every $x \in F_1$ (476Ea). In particular, $F = \phi[F_1] \subseteq B(\mathbf{0}, M)$. Now F is measurable and $\mu F = \mu F_1 \ge \mu E$, by 476Eb. Also, writing ν for normalized $(r-1)$-dimensional Hausdorff measure on $\mathbb{R}^r$,

$$\operatorname{per} F = \nu(\partial^* F) \le \nu(\partial^* F_1) = \operatorname{per} F_1 \le \operatorname{per} E,$$

by 475Mb and 476Ee. So $F \in \mathcal{F}$, and

$$\int_F \|x\|\mu(dx) \ge \int_{F_0} \|x\|\mu(dx) = \int_{F_1} \|x\|\mu(dx).$$

By 476Ec, $G = \{x : x \in F_1,\, x \,.\, e < \alpha,\, R(x) \notin F_1\}$ is negligible. But now consider $G \cap U(x_1, \eta)$ for small $\eta > 0$. Since x_1 belongs to $F_1 = \mathrm{cl}^* F_0 = \mathrm{cl}^* F_1$, but x_0 does not,

$$\limsup_{\eta \downarrow 0} \frac{\mu(F_1 \cap B(x_1, \eta))}{\mu B(x_1, \eta)} > 0 = \limsup_{\eta \downarrow 0} \frac{\mu(F_1 \cap B(x_0, \eta))}{\mu B(x_0, \eta)}.$$

There must therefore be some $\eta > 0$ such that $\eta < \frac{1}{2}\|x_1 - x_0\|$ and $\mu(F_1 \cap B(x_0, \eta)) < \mu(F_1 \cap B(x_1, \eta))$. In this case, however, $G \supseteq F_1 \cap B(x_1, \eta) \setminus R[F_1 \cap B(x_0, \eta)]$ has measure at least $\mu(F_1 \cap B(x_1, \eta)) - \mu(F_1 \cap B(x_0, \eta)) > 0$, which is impossible. **XQ**

(c) Thus $U(\mathbf{0}, \delta) \subseteq F_1 \subseteq B(\mathbf{0}, \delta)$ and $\operatorname{per} F_1 = \operatorname{per} B(\mathbf{0}, \delta)$. Since $\mu F_1 \ge \mu E$, the radius of B_E is at most δ and

$$\operatorname{per} B_E \le \operatorname{per} B(\mathbf{0}, \delta) = \operatorname{per} F_1 \le \operatorname{per} E.$$

(d) Thus the result is proved when E is bounded and has finite perimeter. Of course it is trivial when E has infinite perimeter. Now suppose that E is any measurable set with finite measure and finite perimeter. Set $E_\alpha = E \cap B(\mathbf{0}, \alpha)$ for $\alpha \ge 0$; then $\operatorname{per} E = \liminf_{\alpha \to \infty} \operatorname{per} E_\alpha$ (475Mc, 475Xk). By (a)-(c), $\operatorname{per} E_\alpha \ge \operatorname{per} B_{E_\alpha}$ for every $\alpha \ge 0$; since $\operatorname{per} B_{E_\alpha} \to \operatorname{per} B_E$ as $\alpha \to \infty$, $\operatorname{per} E \ge \operatorname{per} B_E$ in this case also.

476I Spheres in inner product spaces For the rest of the section I will use the following notation. Let X be a (real) inner product space. Then S_X will be the unit sphere $\{x : x \in X, \|x\| = 1\}$. Let H_X be the isometry group of S_X with its topology of pointwise convergence (441G).

A **cap** in S_X will be a set of the form $\{x : x \in S_X,\, (x|e) \ge \alpha\}$ where $e \in S_X$ and $-1 \le \alpha \le 1$.

When X is finite-dimensional, it is isomorphic, as inner product space, to $\mathbb{R}^r$, where $r = \dim X$ (4A4Je). If $r \ge 1$, S_X is non-empty and compact, so has a unique H_X-invariant Radon probability measure ν_X, which is strictly positive (476C). If $r \ge 1$ is an integer, we know that the $(r-1)$-dimensional Hausdorff measure of the sphere $S_{\mathbb{R}^r}$ is finite and non-zero (265F). Since Hausdorff measures are invariant under isometries (471J), and are quasi-Radon measures when totally finite (471Dh), $(r-1)$-dimensional Hausdorff measure on $S_{\mathbb{R}^r}$ is a multiple of the normalized invariant measure $\nu_{\mathbb{R}^r}$, by 476C. The same is therefore true in any r-dimensional inner product space.

476J Lemma Let X be a real inner product space and $f \in H_X$. Then $(f(x)|f(y)) = (x|y)$ for all x, $y \in S_X$. Consequently $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ whenever x, $y \in S_X$ and α, $\beta \in \mathbb{R}$ are such that $\alpha x + \beta y \in S_X$.

proof (a) We have

$$\rho(x, y)^2 = (x - y | x - y) = (x|x) - 2(x|y) + (y|y) = 2 - 2(x|y),$$

so

$$(x|y) = 1 - \frac{1}{2}\rho(x,y)^2 = 1 - \frac{1}{2}\rho(f(x),f(y))^2 = (f(x)|f(y)).$$

(b)

$$\begin{aligned}
&\|f(\alpha x+\beta y)-\alpha f(x)-\beta f(y)\|^2 \\
&\quad= (f(\alpha x+\beta y)-\alpha f(x)-\beta f(y)|f(\alpha x+\beta y)-\alpha f(x)-\beta f(y)) \\
&\quad= 1+\alpha^2+\beta^2-2\alpha(f(\alpha x+\beta y)|f(x)) \\
&\qquad\qquad -2\beta(f(\alpha x+\beta y)|f(y))+2\alpha\beta(f(x)|f(y)) \\
&\quad= 1+\alpha^2+\beta^2-2\alpha(\alpha x+\beta y|x)-2\beta(\alpha x+\beta y|y)+2\alpha\beta(x|y) \\
&\quad= \|(\alpha x+\beta y)-\alpha x-\beta y\|^2 = 0.
\end{aligned}$$

476K I give a theorem on concentration of measure on the sphere corresponding to 476F.

Theorem Let X be a finite-dimensional inner product space of dimension at least 2, S_X its unit sphere and ν_X the invariant Radon probability measure on S_X. For a non-empty set $A \subseteq S_X$ and $\epsilon > 0$, write $U(A,\epsilon) = \{x : \rho(x,A) < \epsilon\}$, where ρ is the usual metric of X. Then there is a cap $C \subseteq S_X$ such that $\nu_X C = \nu_X^* A$, and $\nu_X(S_X \cap U(A,\epsilon)) \ge \nu_X(S_X \cap U(C,\epsilon))$ for any such C and every $\epsilon > 0$.

proof In order to apply the results of 476D-476E directly, and simplify some of the formulae slightly, it will be helpful to write ν for the Radon measure on X defined by setting $\nu E = \nu_X(E \cap S_X)$ whenever this is defined. By 214Cd, ν^* agrees with ν_X^* on $\mathcal{P}S_X$.

(a) The first step is to check that there is a cap C of S_X such that $\nu C = \nu^* A$. **P** Take any $e_0 \in S_X$, and set $C_\alpha = \{x : x \in S_X, (x|e_0) \ge \alpha\}$ for $\alpha \in [-1,1]$. νC_α is defined for every $\alpha \in \mathbb{R}$ because every C_α is closed and ν is a topological measure. Now examine the formulae of 265F. We can identify X with $\mathbb{R}^{r+1}$ where $r+1 = \dim X$; do this in such a way that e_0 corresponds to the unit vector $(0,\dots,0,1)$. We have a parametrization $\phi_r : D_r \to S_X$, where D_r is a Borel subset of $\mathbb{R}^r$ with interior $]-\pi,\pi[\times]0,\pi[^{r-1}$ and ϕ_r is differentiable with continuous derivative. Moreover, if $x = (\xi_1,\dots,\xi_r) \in D_r$, then $\phi_r(x)\,.\,e_0 = \cos\xi_r$, and the Jacobian J_r of ϕ_r is bounded by 1 and never zero on $\operatorname{int} D_r$. Finally, the boundary ∂D_r is negligible. What this means is that $\nu_r C_\alpha = \int_{E_\alpha} J_r d\mu_r$, where μ_r is Lebesgue measure on $\mathbb{R}^r$, ν_r is normalized Hausdorff r-dimensional measure on $\mathbb{R}^{r+1}$, and $E_\alpha = \{x : x \in D_r, \cos\xi_r \ge \alpha\}$. So if $-1 \le \alpha \le \beta \le 1$ then

$$\nu_r C_\alpha - \nu_r C_\beta \le \mu_r(E_\alpha \setminus E_\beta) \le 2\pi^{r-1}(\arccos\alpha - \arccos\beta);$$

because arccos is continuous, so is $\alpha \mapsto \nu_r C_\alpha$. Also, if $\alpha < \beta$, then $E_\alpha \setminus E_\beta$ is non-negligible, so $\int_{E_\alpha\setminus E_\beta} J_r d\mu_r \ne 0$ and $\nu_r C_\alpha > \nu_r C_\beta$.

This shows that $\alpha \mapsto \nu_r C_\alpha$ is continuous and strictly decreasing; since ν_r is just a multiple of ν on S_X, the same is true of $\alpha \mapsto \nu C_\alpha$.

Since $\nu C_{-1} = \nu S_X = 1$ and $\nu C_1 = \nu\{e_0\} = 0$, the Intermediate Value Theorem tells us that there is a unique α such that $\nu C_\alpha = \nu^* A$, and we can set $C = C_\alpha$. **Q**

(b) Now take any non-empty set $A \subseteq S_X$ and any $\epsilon > 0$, and set $\gamma = \nu^* A$, $\beta = \nu U(A,\epsilon)$. Let C be a cap such that $\nu^* A = \nu C$; let e_0 be the centre of C. Consider the family

$$\mathcal{F} = \{F : F \in \mathcal{C},\, F \subseteq S_X,\, \nu F \ge \gamma,\, \nu U(F,\epsilon) \le \beta\},$$

where $\mathcal{C}$ is the family of closed subsets of X with its Fell topology. Because $U(\overline{A},\epsilon) = U(A,\epsilon)$, $\overline{A} \in \mathcal{F}$ and $\mathcal{F}$ is non-empty. By 476A(b-i) and 476B, $\mathcal{F}$ is closed in $\mathcal{C}$, therefore compact, by 4A2T(b-iii) once more. Next, the function

$$F \mapsto \int_F \max(0, 1+(x|e_0))\nu(dx) : \mathcal{C} \to [0,\infty[$$

is upper semi-continuous, by 476A(b-ii). It therefore attains its supremum on $\mathcal{F}$ at some $F_0 \in \mathcal{F}$. Let $F_1 \subseteq F_0$ be a self-supporting closed set with the same measure as F_0; then $F_1 \in \mathcal{F}$ and $\int_F (1+(x|e_0))\nu(dx) \le \int_{F_1}(1+(x|e_0))\nu(dx)$ for every $F \in \mathcal{F}$.

(c) F_1 is a cap with centre e_0. **P?** Otherwise, there are $x_0 \in S_X \setminus F_1$ and $x_1 \in F_1$ such that $(x_0|e_0) > (x_1|e_0)$. Set $e = \frac{1}{\|x_0-x_1\|}(x_0 - x_1)$. Then $e \in S_X$ and $(e|e_0) > 0$. Set $R = R_{e0}$ and $\psi = \psi_{e0}$ as defined in 476D; write F for $\psi(F_1)$. Note that $(x_0 + x_1|x_0 - x_1) = \|x_0\|^2 - \|x_1\|^2 = 0$, so $(x_0 + x_1|e) = 0$ and $R(x_1) = x_0$. Also $R[S_X] = S_X$, so ν is R-invariant, because ν is a multiple of Hausdorff $(r-1)$-dimensional measure on S_X and must be invariant under isometries of S_X.

We have $\nu F = \nu F_1 \ge \gamma$, by 476Eb, and

$$\nu U(F, \epsilon) \le \nu\psi(U(F_1, \epsilon)) = \nu U(F_1, \epsilon) \le \nu U(F_0, \epsilon) \le \beta$$

by 476Dd. So $F \in \mathcal{F}$. But consider the standard bijection $\phi = \phi_{F_1} : F_1 \to F$ as defined in 476E. We have

$$\int_{F_1}(1 + (\phi(x)|e_0))\nu(dx) = \int_F(1 + (x|e_0))\nu(dx) \le \int_{F_1}(1 + (x|e_0))\nu(dx).$$

If we examine the definition of ϕ, we see that $\phi(x) \ne x$ only when $(x|e) < 0$ and $\phi(x) = R(x)$, so that in this case $\phi(x) - x$ is a positive multiple of e and $(\phi(x)|e_0) > (x|e_0)$. So $G = \{x : x \in F_1, (x|e) < 0, R(x) \notin F_1\}$ must be ν-negligible. But G includes a relative neighbourhood of x_1 in F_1 and F_1 is supposed to be self-supporting for ν, so this is impossible. **XQ**

(d) Now $\nu^*A = \gamma \le \nu F_1$, so $C \subseteq F_1$ and

$$\nu U(A, \epsilon) = \beta \ge \nu U(F_1, \epsilon) \ge \nu U(C, \epsilon),$$

as claimed.

476L Corollary For any $\epsilon > 0$, there is an $r_0 \ge 1$ such that whenever X is a finite-dimensional inner product space of dimension at least r_0, A_1, $A_2 \subseteq S_X$ and $\min(\nu_X^*A_1, \nu_X^*A_2) \ge \epsilon$, then there are $x \in A_1$, $y \in A_2$ such that $\|x - y\| \le \epsilon$.

proof Take $r_0 \ge 2$ such that $r_0\epsilon^3 > 2$. Suppose that $\dim X = r \ge r_0$. Fix $e_0 \in S_X$. We need an estimate of $\nu_X C_{\epsilon/2}$, where $C_{\epsilon/2} = \{x : x \in S_X, (x|e_0) \ge \epsilon/2\}$ as in 476K. To get this, let $e_1, \dots, e_{r-1}$ be such that $e_0, \dots, e_{r-1}$ is an orthonormal basis of X (4A4Kc). For each $i < r$, there is an $f \in H_X$ such that $f(e_i) = e_0$, so that $(x|e_i) = (f(x)|e_0)$ for every x (476J), and

$$\int(x|e_i)^2\nu_X(dx) = \int(f(x)|e_0)^2\nu_X(dx) = \int(x|e_0)^2\nu_X(dx),$$

because $f : S_X \to S_X$ is inverse-measure-preserving for ν_X.

Accordingly

$$\begin{aligned}\nu_X C_{\epsilon/2} &= \frac{1}{2}\nu_X\{x : x \in S_X, |(x|e_0)| \ge \epsilon/2\} \le \frac{2}{\epsilon^2}\int_{S_X}(x|e_0)^2\nu_X(dx)\\ &< r\epsilon\int_{S_X}(x|e_0)^2\nu_X(dx) = \epsilon\sum_{i=0}^{r-1}\int_{S_X}(x|e_i)^2\nu_X(dx)\\ &= \epsilon\int_{S_X}\sum_{i=0}^{r-1}(x|e_i)^2\nu_X(dx) = \epsilon \le \nu_X^*A_1.\end{aligned}$$

So, taking C to be the cap of S_X with centre e_0 and measure $\nu_X^*A_1$, $C = C_\alpha$ where $\alpha < \frac{1}{2}\epsilon$, and

$$\nu_X(S_X \cap U(A_1, \tfrac{1}{2}\epsilon)) \ge \nu_X(S_X \cap U(C_\alpha, \tfrac{1}{2}\epsilon)) \ge \nu_X C_{\alpha-\epsilon/2} > \frac{1}{2}$$

by 476K. Similarly, $\nu_X(S_X \cap U(A_2, \frac{1}{2}\epsilon)) > \frac{1}{2}$ and there must be some $z \in S_X \cap U(A_1, \frac{1}{2}\epsilon) \cap U(A_2, \frac{1}{2}\epsilon)$. Take $x \in A_1$ and $y \in A_2$ such that $\|x - z\| < \frac{1}{2}\epsilon$ and $\|y - z\| < \frac{1}{2}\epsilon$; then $\|x - y\| \le \epsilon$, as required.

476X Basic exercises (a) Let X be a topological space, $\mathcal{C}$ the set of closed subsets of X, μ a topological measure on X and f a μ-integrable real-valued function; set $\phi(F) = \int_F f d\mu$ for $F \in \mathcal{C}$. (i) Show that if *either* μ is inner regular with respect to the closed sets and $\mathcal{C}$ is given its Vietoris topology *or* μ is tight and $\mathcal{C}$ is given its Fell topology, then ϕ is Borel measurable. (ii) Show that if X is metrizable and $\mathcal{C} \setminus \{\emptyset\}$ is given an appropriate Hausdorff metric, then $\phi\restriction\mathcal{C} \setminus \{\emptyset\}$ is Borel measurable.

(b) In the context of 476D, show that $\operatorname{diam}\psi_{e\alpha}(A) \le \operatorname{diam} A$ for all A, e and α.

>(c) Find an argument along the lines of those in 476F and 476G to prove 264H. (*Hint*: 476Xb.)

>(d) Let X be an inner product space and S_X its unit sphere. Show that every isometry $f : S_X \to S_X$ extends uniquely to an isometry $T_f : X \to X$ which is a linear operator. (*Hint*: first check the cases in which $\dim X \le 2$.) Show that f is surjective iff T_f is, so that we have a natural isomorphism between the isometry group of S_X and the group of invertible isometric linear operators. Show that this isomorphism is a homeomorphism for the topologies of pointwise convergence.

(e) Let X be a finite-dimensional inner product space, ν_X the invariant Radon probability measure on the sphere S_X, and $E \in \operatorname{dom}\nu_X$; let $C \subseteq S_X$ be a cap with the same measure as E, and let λ be the product measure of ν_X with itself on $S_X \times S_X$. Show that $\int_{C\times C} \|x-y\|\lambda(d(x,y)) \le \int_{E\times E} \|x-y\|\lambda(d(x,y))$.

(f) Let X be a finite-dimensional inner product space and ν_X the invariant Radon probability measure on the sphere S_X. (i) Without appealing to the formulae in §265, show that $\nu_X(S_X \cap H) = 0$ whenever $H \subseteq X$ is a proper affine subspace. (*Hint*: induce on $\dim H$.) (ii) Use this to prove that if $e \in S_X$ then $\alpha \mapsto \nu_X\{x : (x|e) \ge \alpha\}$ is continuous.

476Y Further exercises (a) Let X be a compact metric space and G its isometry group. Suppose that $H \subseteq G$ is a subgroup such that the action of H on X is transitive. Show that X has a unique H-invariant Radon probability measure which is also G-invariant.

(b) Let $r \ge 1$ be an integer, and $g \in C_0(\mathbb{R}^r)$ a non-negative γ-Lipschitz function, where $\gamma \ge 0$. Let F be the set of non-negative γ-Lipschitz functions $f \in C(\mathbb{R}^r)$ such that f has the same decreasing rearrangement as g with respect to Lebesgue measure μ on $\mathbb{R}^r$ (§373) and $\int \phi(\operatorname{grad} f)d\mu \le \int \phi(\operatorname{grad} g)d\mu$ for every convex function $\phi : \mathbb{R}^r \to [0,\infty[$. (i) Show that F is compact for the topology of pointwise convergence. (ii) Show that there is an $f \in F$ such that $f(x) \ge f(y)$ whenever $\|x\| \le \|y\|$. (*Hint*: parts (a) and (b-i) of the proof of 479V.)

476 Notes and comments The main theorems here (476F-476H, 476K), like 264H, are all 'classical'; they go back to the roots of geometric measure theory, and the contribution of the twentieth century was to extend the classes of sets for which balls or caps provide the bounding examples. It is very striking that they can all be proved with the same tools (see 476Xc). Of course I should remark that the Compactness Theorem (474T) lies at a much deeper level than the rest of the ideas here. (The proof of 474T relies on the distributional definition of 'perimeter' in 474D, while the arguments of 476Ee and 476H work with the Hausdorff measures of essential boundaries; so that we can join these ideas together only after proving all the principal theorems of §§472-475.) So while 'Steiner symmetrization' (264H) and 'concentration by partial reflection' (476D) are natural companions, 476H is essentially harder than the other results.

In all the theorems here, as in 264H, I have been content to show that a ball or a cap is an optimum for whatever inequality is being considered. I have not examined the question of whether, and in what sense, the optimum is unique. It seems that this requires deeper analysis.

Version of 4.1.08/2.1.10

477 Brownian motion

I presented §455 with an extraordinary omission: the leading example of a Lévy process, and the inspiration for the whole project, was relegated to an anonymous example (455Xg). In this section I will take the subject up again. The theorem that the sum of independent normally distributed random variables is again normally distributed (274B), when translated into the language of this volume, tells us that we have a family $\langle\lambda_t\rangle_{t>0}$ of centered normal distributions such that $\lambda_{s+t} = \lambda_s * \lambda_t$ for all s, $t > 0$. Consequently we have a corresponding example of a Lévy process on $\mathbb{R}$, and this is the process which we call 'Brownian motion' (477A). This is special in innumerable ways, but one of them is central: we can represent it in such a way that sample paths are continuous (477B), that is, as a Radon measure on the space of continuous paths starting at 0. In this form, it also appears as a limit, for the narrow topology, of interpolations of random walks (477C).

For the geometric ideas of §479, we need Brownian motion in three dimensions; the r-dimensional theory of 477D-477G gives no new difficulties. The simplest expression of Brownian motion in $\mathbb{R}^r$ is just to take a product measure (477Da), but in order to apply the results of §455, and match the construction with the ideas of §456, a fair bit of explanation is necessary. The geometric properties of Brownian motion begin with the invariant transformations of 477E. As for all Lévy processes, we have a strong Markov property, and Theorem 455U translates easily into the new formulation (477G), as does the theory of hitting times (477I). I conclude with a classic result on maximal values which will be useful later (477J), and with proofs that almost all Brownian paths are nowhere differentiable (477K) and have zero two-dimensional Hausdorff measure (477L).

477A Brownian motion: Theorem There are a probability space (Ω,Σ,μ) and a family $\langle X_t\rangle_{t\ge 0}$ of real-valued random variables on Ω such that

(i) $X_0=0$ almost everywhere;

(ii) whenever $0\le s<t$ then X_t-X_s is normally distributed with expectation 0 and variance $t-s$;

(iii) $\langle X_t\rangle_{t\ge 0}$ has independent increments.

First proof In 455P, take $U=\mathbb{R}$ and λ_t, for $t>0$, to be the distribution of a normal random variable with expectation 0 and variance t; that is, the distribution with probability density function $x\mapsto\frac{1}{\sqrt{2\pi t}}e^{-x^2/2t}$. By 272T[2], $\lambda_{s+t}=\lambda_s*\lambda_t$ for all s, $t>0$. If $\epsilon>0$, then

$$\lim_{t\downarrow 0}\lambda_t\,]-\epsilon,\epsilon[\,=\lim_{t\downarrow 0}\lambda_1\,]-\frac{\epsilon}{\sqrt{t}},\frac{\epsilon}{\sqrt{t}}[\,=1,$$

so $\langle\lambda_t\rangle_{t>0}$ satisfies the conditions of 455P. Accordingly we have a probability measure $\hat\mu$ on $\Omega=\mathbb{R}^{[0,\infty[}$ for which, setting $X_t(\omega)=\omega(t)$, $\langle X_t\rangle_{t\ge 0}$ has the required properties, as noted in 455Q-455R.

Second proof Let μ_L be Lebesgue measure on $\mathbb{R}$, and for $t\ge 0$ set $u_t=\chi[0,t]^\bullet$ in $L^2(\mu_L)$, so that $(u_s|u_t)=\min(s,t)$ for s, $t\ge 0$. By 456C, there is a centered Gaussian distribution μ on $\mathbb{R}^{[0,\infty[}$ with covariance matrix $\langle\min(s,t)\rangle_{s,t\ge 0}$. Set $X_t(x)=x(t)$ for $x\in\mathbb{R}^{[0,\infty[}$. Then X_0 has expectation and variance both 0, that is, $X_0=0$ a.e. If $0\le s<t$, then X_t-X_s is a linear combination of X_s and X_t, so is normally distributed with expectation 0, and its variance is

$$\mathbb{E}(X_t-X_s)^2=\mathbb{E}(X_t)^2-2\mathbb{E}(X_t\times X_s)+\mathbb{E}(X_s)^2=t-2s+s=t-s.$$

If $0\le t_0<\ldots<t_n$ and $Y_i=X_{t_{i+1}}-X_{t_i}$ for $i<n$, then $\langle Y_i\rangle_{i<n}$ has a centered Gaussian distribution, by 456Ba. Also, if $i<j<n$, then

$$\begin{aligned}\mathbb{E}(Y_i\times Y_j)&=\mathbb{E}(X_{t_{i+1}}\times X_{t_{j+1}})-\mathbb{E}(X_{t_{i+1}}\times X_{t_j})-\mathbb{E}(X_{t_i}\times X_{t_{j+1}})+\mathbb{E}(X_{t_i}\times X_{t_j})\\&=t_{i+1}-t_{i+1}-t_i+t_i=0.\end{aligned}$$

So 456E assures us that $\langle Y_i\rangle_{i<n}$ is independent.

Thus $\langle X_t\rangle_{t\ge 0}$ satisfies the conditions required.

477B These constructions of Brownian motion are sufficient to show that there is a process, satisfying the defining conditions (i)-(iii), which can be studied with the tools of measure theory. From 455H we see that we have a Radon measure on the space of càllàl functions representing the process, and from 455P that we have the option of moving to the càdlàg functions, with a corresponding description of the strong Markov property in terms of inverse-measure-preserving functions, as in 455U. But there is no hint yet of the most important property of Brownian motion, that 'sample paths are continuous'. With some simple inequalities from Chapter 27 and the ideas of 454Q-454S, we can find a proof of this, as follows.

[2]Formerly 272S.

Theorem Let $\langle X_t\rangle_{t\ge 0}$ be as in 477A, and $\hat\mu$ the distribution of the process $\langle X_t\rangle_{t\ge 0}$. Let $C([0,\infty[)_0$ be the set of continuous functions $\omega:[0,\infty[\,\to\mathbb{R}$ such that $\omega(0)=0$. Then $C([0,\infty[)_0$ has full outer measure for $\hat\mu$, and the subspace measure μ_W on $C([0,\infty[)_0$ induced by $\hat\mu$ is a Radon measure when $C([0,\infty[)_0$ is given the topology $\mathfrak{T}_c$ of uniform convergence on compact sets.

proof (a) The main part of the argument here (down to the end of (e)) is devoted to showing that $\hat\mu^*C([0,\infty[)=1$; the result will then follow easily from 454Sb.

(b) ? Suppose, if possible, that $\hat\mu^*C([0,\infty[)<1$. Then there is a non-negligible Baire set $H\subseteq\mathbb{R}^{[0,\infty[}\setminus C([0,\infty[)$. There is a countable set $D\subseteq[0,\infty[$ such that H is determined by coordinates in D (4A3Nb); we may suppose that D includes $\mathbb{Q}\cap[0,\infty[$.

(c) (The key.) Let q, q' be rational numbers such that $0\le q<q'$, and $\epsilon>0$. Then

$$\Pr(\sup_{t\in D\cap[q,q']}|X_t-X_q|>\epsilon)\le\frac{18\sqrt{q'-q}}{\epsilon\sqrt{2\pi}}e^{-\epsilon^2/18(q'-q)}.$$

P If $q=t_0<t_1<\ldots<t_n=q'$, set $Y_i=X_{t_i}-X_{t_{i-1}}$ for $1\le i\le n$, so that $X_{t_m}-X_q=\sum_{i=1}^m Y_i$ for $1\le m\le n$, and $Y_1,\ldots,Y_n$ are independent. By Etemadi's lemma (272V[3]),

$$\begin{aligned}\Pr(\sup_{i\le n}|X_{t_i}-X_q|>\epsilon)&\le 3\max_{i\le n}\Pr(|X_{t_i}-X_q|>\tfrac{1}{3}\epsilon)\\&=3\max_{1\le i\le n}\Pr(\frac{1}{\sqrt{t_i-q}}|X_{t_i}-X_q|>\frac{\epsilon}{3\sqrt{t_i-q}})\\&=6\max_{1\le i\le n}\frac{1}{\sqrt{2\pi}}\int_{\epsilon/3\sqrt{t_i-q}}^\infty e^{-x^2/2}dx\end{aligned}$$

(because $\dfrac{1}{\sqrt{t_i-q}}(X_{t_i}-X_q)$ is standard normal)

$$\begin{aligned}&=\frac{6}{\sqrt{2\pi}}\int_{\epsilon/3\sqrt{q'-q}}^\infty e^{-x^2/2}dx\\&\le\frac{18\sqrt{q'-q}}{\epsilon\sqrt{2\pi}}e^{-\epsilon^2/18(q'-q)}\end{aligned}$$

by 274Ma. Thus if $I\subseteq[q,q']$ is any finite set containing q and q',

$$\Pr(\sup_{t\in I}|X_t-X_q|>\epsilon)\le\frac{18\sqrt{q'-q}}{\epsilon\sqrt{2\pi}}e^{-\epsilon^2/18(q'-q)}.$$

Taking $\langle I_n\rangle_{n\in\mathbb{N}}$ to be a non-decreasing sequence of finite sets with union $D\cap[q,q']$, starting from $I_0=\{q,q'\}$, we get

$$\begin{aligned}\Pr(\sup_{t\in D\cap[q,q']}|X_t-X_q|>\epsilon)&=\lim_{n\to\infty}\Pr(\sup_{t\in I_n}|X_t-X_q|>\epsilon)\\&\le\frac{18\sqrt{q'-q}}{\epsilon\sqrt{2\pi}}e^{-\epsilon^2/18(q'-q)},\end{aligned}$$

as required. **Q**

(d) If $\epsilon>0$ and $n\ge 1$, then

$$\begin{aligned}\Pr(\text{there are } t,\,u\in D\cap[0,n] \text{ such that } |t-u|\le\frac{1}{n^2} \text{ and } |X_t-X_u|>3\epsilon)\\\le\frac{18n^2}{\epsilon\sqrt{2\pi}}e^{-n^2\epsilon^2/18}.\end{aligned}$$

P Divide $[0,n]$ into n^3 intervals $[q_i,q_{i+1}]$ of length $1/n^2$. For each of these,

[3] Formerly 272U.

$$\Pr(\sup_{t\in D\cap[q_i,q_{i+1}]}|X_t-X_{q_i}|>\epsilon)\le\frac{18}{n\epsilon\sqrt{2\pi}}e^{-n^2\epsilon^2/18}.$$

So

$$\Pr(\text{there are } i<n^3,\ t\in D\cap[q_i,q_{i+1}] \text{ such that } |X_t-X_{q_i}|>\epsilon)$$

is at most $\frac{18n^2}{\epsilon\sqrt{2\pi}}e^{-n^2\epsilon^2/18}$.

But if t, $u\in[0,n]$ and $|t-u|\le 1/n^2$ and $|X_t-X_u|>3\epsilon$, there must be an $i<n^3$ such that both t and u belong to $[q_i,q_{i+2}]$, so that either there is a $t'\in D\cap[q_i,q_{i+1}]$ such that $|X_{t'}-X_{q_i}|>\epsilon$ or there is a $t'\in D\cap[q_{i+1},q_{i+2}]$ such that $|X_{t'}-X_{q_{i+1}}|>\epsilon$. So

$$\begin{aligned}\Pr(&\text{there are } t,\ u\in D\cap[0,n] \text{ such that } |t-u|\le\frac{1}{n^2} \text{ and } |X_t-X_u|>3\epsilon)\\&\le\Pr(\text{there are } i<n^3,\ t\in D\cap[q_i,q_{i+1}] \text{ such that and } |X_t-X_{q_i}|>\epsilon)\\&\le\frac{18n^2}{\epsilon\sqrt{2\pi}}e^{-n^2\epsilon^2/18},\end{aligned}$$

as required. **Q**

(e) So if we take $G_{\epsilon n}$ to be the Baire set

$$\{\omega:\omega\in\mathbb{R}^{[0,\infty[},\ \text{there are } t,\ u\in D\cap[0,n] \text{ such that } |t-u|\le\frac{1}{n^2}\\ \text{and } |\omega(t)-\omega(u)|>3\epsilon\},$$

we have

$$\hat\mu G_{\epsilon n}\le\frac{18n^2}{\epsilon\sqrt{2\pi}}e^{-n^2\epsilon^2/18},$$

and $\lim_{n\to\infty}\hat\mu G_{\epsilon n}=0$. We can therefore find a strictly increasing sequence $\langle n_k\rangle_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that $\sum_{k=1}^{\infty}\hat\mu(G_{1/k,n_k})<\hat\mu H$, so that there is an $\omega\in H\setminus\bigcup_{k\ge1}G_{1/k,n_k}$.

What this means is that if $k\ge1$ and t, $u\in D\cap[0,n_k]$ are such that $|t-u|\le\frac{1}{n_k^2}$, then $|\omega(t)-\omega(u)|\le\frac{3}{k}$. Since $n_k\to\infty$ as $k\to\infty$, there is a continuous function $\omega':[0,\infty[\,\to\mathbb{R}$ such that $\omega'\restriction D=\omega\restriction D$. But H is determined by coordinates in D, so ω' belongs to $H\cap C([0,\infty[)$, which is supposed to be empty. **X**

(f) Thus $\hat\mu^*C([0,\infty[)=1$. Since $\hat\mu\{\omega:\omega(0)=0\}=1$, $C([0,\infty[)\setminus C([0,\infty[)_0$ is $\hat\mu$-negligible and $C([0,\infty[)_0$ has full outer measure for $\hat\mu$. By 454Sb, the subspace measure $\hat\mu_C$ on $C([0,\infty[)$ induced by $\hat\mu$ is a Radon measure for $\mathfrak{T}_c$; now $C([0,\infty[)_0$ is $\hat\mu_C$-conegligible. The subspace measure μ_W on $C([0,\infty[)_0$ induced by $\hat\mu$ is also the subspace measure induced by $\hat\mu_C$, so is a Radon measure for the topology on $C([0,\infty[)_0$ induced by $\mathfrak{T}_c$.

Remark We can put this together with the ideas of 455H. Following the First Proof of 477A, and using 455Pc, we see that there is a unique Radon measure $\tilde\mu$ on $\mathbb{R}^{[0,\infty[}$ (for the topology $\mathfrak{T}_p$ of pointwise convergence) extending $\hat\mu$. The identity map $\iota:C([0,\infty[)_0\to\mathbb{R}^{[0,\infty[}$ is continuous for $\mathfrak{T}_c$ and $\mathfrak{T}_p$, so the image measure $\mu_W\iota^{-1}$ is a Radon measure on $\mathbb{R}^{[0,\infty[}$ (418I). If $E\subseteq\mathbb{R}^{[0,\infty[}$ is a Baire set, then

$$\mu_W\iota^{-1}[E]=\mu_W(E\cap C([0,\infty[)_0)=\hat\mu E,$$

so $\mu_W\iota^{-1}$ agrees with $\tilde\mu$ on Baire sets, and the two must be equal. Now $C([0,\infty[)_0$ is $\tilde\mu$-conegligible, just because its complement has empty inverse image under ι. So μ_W is also the subspace measure on $C([0,\infty[)_0$ induced by $\tilde\mu$.

Equally, since of course $C([0,\infty[)_0$ is a subspace of the set C_{dlg} of càdlàg functions from $[0,\infty[$ to $\mathbb{R}$, μ_W is the subspace measure induced by the measure $\ddot\mu$ of Theorem 455O.

***477C** I star the next theorem because it is very hard work and will not be relied on later. Nevertheless I think the statement, at least, should be part of your general picture of Brownian motion.

Theorem For $\alpha > 0$, define $f_\alpha : \mathbb{R}^{\mathbb{N}} \to \Omega = C([0,\infty[)_0$ by setting $f_\alpha(z)(t) = \sqrt{\alpha}(\sum_{i<n} z(i) + \frac{1}{\alpha}(t-n\alpha)z(n))$ when $z \in \mathbb{R}^{\mathbb{N}}$, $n \in \mathbb{N}$ and $n\alpha \le t \le (n+1)\alpha$. Give Ω its topology $\mathfrak{T}_c$ of uniform convergence on compact sets, and $\mathbb{R}^{\mathbb{N}}$ its product topology; then f_α is continuous. For a Radon probability measure ν on $\mathbb{R}$, let $\mu_{\nu\alpha}$ be the image Radon measure $\nu^{\mathbb{N}} f_\alpha^{-1}$ on Ω, where $\nu^{\mathbb{N}}$ is the product measure on $\mathbb{R}^{\mathbb{N}}$. Let μ_W be the Radon measure of 477B, and U a neighbourhood of μ_W in the space $P_{\mathrm{R}}(\Omega)$ of Radon probability measures on Ω for the narrow topology (437Jd). Then there is a $\delta > 0$ such that $\mu_{\nu\alpha} \in U$ whenever $\alpha \in]0,\delta]$ and ν is a Radon probability measure on $\mathbb{R}$ with mean $0 = \int x\,\nu(dx)$ and variance $1 = \int x^2\nu(dx)$ and

$$\int_{\{x:|x|\ge\delta/\sqrt{\alpha}\}} x^2\nu(dx) \le \delta. \qquad (\dagger)$$

Remark The idea is that, for a given α and ν, we consider a random walk with independent identically distributed steps, with expectation 0 and variance α, at time intervals of α, and then interpolate to get a continuous function on $[0,\infty[$; and that if the step-lengths are small the result should look like Brownian motion. Moreover, this ought not to depend on the distribution ν; but in order to apply the Central Limit Theorem in a sufficiently uniform way, we need the extra regularity condition (†). On first reading you may well prefer to fix on a particular distribution ν with mean 0 and expectation 1 (e.g., the distribution which gives measure $\frac{1}{2}$ to each of $\{1\}$ and $\{-1\}$), so that (†) is satisfied whenever α is small enough compared with δ.

proof For $\delta > 0$ I will write $Q(\delta)$ for the set of pairs (ν,α) such that ν is a Radon probability measure on $\mathbb{R}$ with mean 0 and variance 1, $0 < \alpha \le \delta$ and $\int_{\{x:|x|\ge\delta/\sqrt{\alpha}\}} x^2\nu(dx) \le \delta$. Note that $Q(\delta') \subseteq Q(\delta)$ when $\delta' \le \delta$.

(a)(i) If γ, $\epsilon > 0$ there is a $\delta > 0$ such that whenever $(\nu,\alpha) \in Q(\delta)$, s, $t \ge 0$ are multiples of α such that $t - s \ge \gamma$, and $I \subseteq \mathbb{R}$ is an interval (open, closed or half-open), then

$$|\mu_{\nu\alpha}\{\omega : \omega \in \Omega,\, \omega(t) - \omega(s) \in I\} - \frac{1}{\sqrt{2\pi(t-s)}}\int_I e^{-x^2/2(t-s)}dx| \le \epsilon.$$

P For $\delta > 0$, $x \in \mathbb{R}$ set $\psi_\delta(x) = x^2$ if $|x| > \delta$, 0 if $|x| \le \delta$. Let $\eta > 0$ be such that whenever $Y_1, \dots, Y_k$ are independent random variables with finite variance and zero expectation, $\sum_{i=1}^k \mathrm{Var}(Y_i) = 1$ and $\sum_{i=1}^k \mathbb{E}(\psi_\eta(Y_i)) \le \eta$, then

$$|\Pr(\sum_{i=1}^k Y_i \le \beta) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\beta} e^{-x^2/2}dx| \le \frac{\epsilon}{2}$$

for every $\beta \in \mathbb{R}$ (274F); observe that in this case

$$|\Pr(\sum_{i=1}^k Y_i < \beta) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\beta} e^{-x^2/2}dx|$$
$$= \lim_{\beta'\uparrow\beta} |\Pr(\sum_{i=1}^k Y_i \le \beta') - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\beta'} e^{-x^2/2}dx| \le \frac{\epsilon}{2}$$

for every $\beta \in \mathbb{R}$, so that

$$|\Pr(\sum_{i=1}^k Y_i \in J) - \frac{1}{\sqrt{2\pi}}\int_J e^{-x^2/2}dx| \le \epsilon$$

for every interval $J \subseteq \mathbb{R}$.

Set $\delta = \min(\eta, \eta\sqrt{\gamma})$. If $I \subseteq \mathbb{R}$ is an interval, $(\nu,\alpha) \in Q(\delta)$ and s, t are multiples of α such that $t - s \ge \gamma$, set $j = \frac{s}{\alpha}$, $k = \frac{t-s}{\alpha}$ and $J = \sqrt{t-s}I = \sqrt{\frac{k}{\alpha}}I$. Then

$$\mu_{\nu\alpha}\{\omega : \omega(t) - \omega(s) \in I\} = \nu^{\mathbb{N}}\{z : f_\alpha(z)(t) - f_\alpha(z)(s) \in I\}$$
$$= \nu^{\mathbb{N}}\{z : \sqrt{\alpha}\sum_{i=j}^{j+k-1} z(i) \in I\} = \Pr(\sum_{i=0}^{k-1} Y_i \in J)$$

where $Y_i(z) = \frac{1}{\sqrt{k}}z(j+i)$. For each i, the mean and variance of Y_i are 0 and $\frac{1}{k}$, because the mean and expectation of ν are 0 and 1. Next,

$$\sum_{i=0}^{k-1}\mathbb{E}(\psi_\eta(Y_i)) = k\int_{\{x:|x|>\eta\sqrt{k}\}}\frac{1}{k}x^2\nu(dx) \le \int_{\{x:|x|>\eta\sqrt{\gamma/\alpha}\}}x^2\nu(dx)$$
$$\le \int_{\{x:|x|>\delta/\sqrt{\alpha}\}}x^2\nu(dx) \le \delta \le \eta,$$

so by the choice of η,

$$|\mu_{\nu\alpha}\{\omega : \omega\in\Omega,\, \omega(t)-\omega(s)\in I\} - \frac{1}{\sqrt{2\pi(t-s)}}\int_I e^{-x^2/2(t-s)}dx|$$
$$= |\Pr(\sum_{i=0}^{k-1}Y_i \in J) - \frac{1}{\sqrt{2\pi}}\int_J e^{-x^2/2}dx| \le \epsilon. \textbf{ Q}$$

(ii) If γ, $\epsilon>0$, there is a $\delta>0$ such that

$$\mu_{\nu\alpha}\{\omega : \operatorname{diam}(\omega[\,[\beta,\beta+\gamma]\,]) > 12\epsilon\} \le \frac{3\sqrt{\gamma}}{\epsilon}e^{-\epsilon^2/2\gamma}$$

whenever $(\nu,\alpha)\in Q(\delta)$ and $\beta\ge 0$. **P** Let $\eta>0$ be such that

$$6(\eta + \frac{\sqrt{\gamma+\eta}}{\epsilon\sqrt{2\pi}}e^{-\epsilon^2/2(\gamma+\eta)}) \le \frac{3\sqrt{\gamma}}{\epsilon}e^{-\epsilon^2/2\gamma},$$

and let $\delta_0>0$ be such that

$$|\mu_{\nu\alpha}\{\omega : \omega\in\Omega,\, \omega(t)-\omega(s)\in I\} - \frac{1}{\sqrt{2\pi(t-s)}}\int_I e^{-x^2/2(t-s)}dx| \le \eta$$

whenever $I\subseteq\mathbb{R}$ is an interval, $(\nu,\alpha)\in Q(\delta_0)$ and s and t are multiples of α such that $t-s\ge\frac{1}{4}\gamma$. Set $\delta=\min(\frac{1}{4}\gamma,\frac{1}{2}\eta,\delta_0)$.

Fix $(\nu,\alpha)\in Q(\delta)$. Applying the last formula with $I=[-\epsilon,\epsilon]$ and then taking complements,

$$\mu_{\nu\alpha}\{\omega : |\omega(t)-\omega(s)|>\epsilon\} \le \eta + \frac{2}{\sqrt{2\pi(t-s)}}\int_\epsilon^\infty e^{-x^2/2(t-s)}dx$$
$$= \eta + \frac{2}{\sqrt{2\pi}}\int_{\epsilon/\sqrt{t-s}}^\infty e^{-x^2/2}dx$$
$$\le \eta + \frac{2}{\sqrt{2\pi}}\int_{\epsilon/\sqrt{\gamma+\eta}}^\infty e^{-x^2/2}dx \le \eta + \frac{\sqrt{\gamma+\eta}}{\epsilon}e^{-\epsilon^2/2(\gamma+\eta)}$$

whenever s, t are multiples of α such that $\frac{1}{4}\gamma\le t-s\le\gamma+\eta$, using 274Ma for the last step, as in part (c) of the proof of 477B. Now if s, t are multiples of α such that $s\le t\le\gamma+\eta$, either $t-s\ge\frac{1}{4}\gamma$ and

$$\mu_{\nu\alpha}\{\omega : |\omega(t)-\omega(s)|>2\epsilon\} \le \eta + \frac{\sqrt{\gamma+\eta}}{\epsilon}e^{-\epsilon^2/2(\gamma+\eta)},$$

or $t\le s+\frac{1}{4}\gamma$ and there is a multiple u of α such that $t+\frac{1}{4}\gamma\le u\le t+\frac{1}{2}\gamma$, in which case

$$\mu_{\nu\alpha}\{\omega : |\omega(t)-\omega(s)|>2\epsilon\} \le \mu_{\nu\alpha}\{\omega : |\omega(u)-\omega(s)|>\epsilon\} + \mu_{\nu\alpha}\{\omega : |\omega(u)-\omega(t)|>\epsilon\}$$
$$\le 2(\eta + \frac{\sqrt{\gamma+\eta}}{\epsilon}e^{-\epsilon^2/2(\gamma+\eta)}).$$

Let j, k be such that $\beta-\frac{1}{2}\eta<j\alpha\le\beta$ and $\beta+\gamma\le k\alpha<\beta+\gamma+\frac{1}{2}\eta$. We have

$$\begin{aligned}\mu_{\nu\alpha}\{\omega : &\operatorname{diam}(\omega[\,[\beta,\beta+\gamma]\,]) > 12\epsilon\}\\ &= \nu^{\mathbb{N}}\{z : \operatorname{diam}(f_\alpha(z)[\,[\beta,\beta+\gamma]\,]) > 12\epsilon\}\\ &\le \nu^{\mathbb{N}}\{z : \sup_{t\in[\beta,\beta+\gamma]} |f_\alpha(z)(t) - f_\alpha(z)(j\alpha)| > 6\epsilon\}\\ &\le \nu^{\mathbb{N}}\{z : \text{there is an } l \text{ such that } j < l \le k \text{ and } |f_\alpha(z)(l\alpha) - f_\alpha(z)(j\alpha)| > 6\epsilon\}\end{aligned}$$

(because $f_\alpha(z)$ is linear between its determining values at multiples of α)

$$\begin{aligned}&= \nu^{\mathbb{N}}\{z : \text{there is an } l \text{ such that } j < l \le k \text{ and } |\sum_{i=j}^{l-1} z(i)| > \frac{6\epsilon}{\sqrt{\alpha}}\}\\ &\le 3 \sup_{j<l\le k} \nu^{\mathbb{N}}\{z : |\sum_{i=j}^{l-1} z(i)| > \frac{2\epsilon}{\sqrt{\alpha}}\}\end{aligned}$$

(Etemadi's lemma, 272V)

$$\begin{aligned}&= 3 \sup_{j<l\le k} \mu_{\nu\alpha}\{\omega : |\omega(l\alpha) - \omega(j\alpha)| > 2\epsilon\}\\ &\le 6(\eta + \tfrac{\sqrt{\gamma+\eta}}{\epsilon} e^{-\epsilon^2/2(\gamma+\eta)}) \le \tfrac{3\sqrt{\gamma}}{\epsilon} e^{-\epsilon^2/2\gamma},\end{aligned}$$

as required. **Q**

(iii) If γ, $\epsilon > 0$ there is a $\delta > 0$ such that

$$\mu_{\nu\alpha}\{\omega : \text{there are } s,\, t \in [0,\gamma] \text{ such that } |t-s| \le \delta \text{ and } |\omega(t) - \omega(s)| > \epsilon\} \le \epsilon$$

whenever $(\nu,\alpha) \in Q(\delta)$. **P** Set $\eta = \epsilon/12$, and let $k \ge 1$ be such that $\frac{6\gamma k}{\eta} e^{-k^2\eta^2/2} \le \epsilon$; set $m = \lfloor 2k^2\gamma \rfloor$. By (ii), there is a $\delta \in \left]0, \frac{1}{2k^2}\right]$ such that

$$\mu_{\nu\alpha}\{\omega : \operatorname{diam}(\omega[\,[\beta,\beta+\tfrac{1}{k^2}]\,]) > 12\eta\} \le \frac{3}{k\eta} e^{-k^2\eta^2/2}$$

whenever $(\nu,\alpha) \in Q(\delta)$ and $\beta \ge 0$. Now, for such ν and α,

$$\begin{aligned}\mu_{\nu\alpha}\{\omega : &\text{there are } s,\, t \in [0,\gamma] \text{ such that } |t-s| \le \delta \text{ and } |\omega(t) - \omega(s)| > \epsilon\}\\ &\le \mu_{\nu\alpha}(\bigcup_{i<m} \{\omega : \operatorname{diam}(\omega[\,[\tfrac{i}{2k^2}, \tfrac{i+2}{2k^2}]\,] > 12\eta\})\\ &\le \frac{3m}{k\eta} e^{-k^2\eta^2/2} \le \frac{6\gamma k}{\eta} e^{-k^2\eta^2/2} \le \epsilon,\end{aligned}$$

as required. **Q**

(b) Suppose that $0 = t_0 < t_1 < \ldots < t_n$ and that $E_0, \ldots, E_{n-1}$ are intervals in $\mathbb{R}$; set $E = \{\omega : \omega \in \Omega,\ \omega(t_{i+1}) - \omega(t_i) \in E_i \text{ for } i < n\}$. Then for every $\epsilon > 0$ there is a $\delta > 0$ such that $\mu_W E \le 3\epsilon + \mu_{\nu\alpha} E$ whenever $(\nu,\alpha) \in Q(\delta)$. **P** Of course

$$\mu_W E = \prod_{i<n} \frac{1}{\sqrt{2\pi(t_{i+1}-t_i)}} \int_{E_i} e^{-x^2/2(t_{i+1}-t_i)} dx.$$

For $\eta > 0$ and $i < n$, let $F_{i\eta}$ be the interval $\{x : [x-2\eta, x+2\eta] \subseteq E_i\}$. Set $\gamma = \frac{1}{2}\min_{i<n}(t_{i+1} - t_i)$; let $\eta \in \left]0,\gamma\right]$ be such that

$$\prod_{i<n} \frac{1}{\sqrt{2\pi\gamma_i}} \int_{F_{i\eta}} e^{-x^2/2\gamma_i} dx \ge \mu_W E - \epsilon$$

whenever $|\gamma_i - (t_{i+1} - t_i)| \le \eta$ for every $i < n$. Next, by (a-i) and (a-iii), there is a $\delta \in \left]0, \frac{1}{2}\eta\right]$ such that

$$\prod_{i<n}\frac{1}{\sqrt{2\pi(s_{i+1}-s_i)}}\int_{F_{i\eta}}e^{-x^2/2(s_{i+1}-s_i)}dx$$
$$\le \epsilon+\prod_{i<n}\mu_{\nu\alpha}\{\omega:\omega(s_{i+1})-\omega(s_i)\in F_{i\eta}\},$$

$$\mu_{\nu\alpha}\{\omega:\text{there are }s,\,t\in[0,t_n+\eta]$$
$$\text{such that }|s-t|\le\delta\text{ and }|\omega(s)-\omega(t)|>\eta\}\le\epsilon$$

whenever $(\nu,\alpha)\in Q(\delta)$ and $s_0,\dots,s_n$ are multiples of α such that $s_{i+1}-s_i\ge\gamma$ for every $i\le n$. Take any $(\nu,\alpha)\in Q(\delta)$, and for each $i\le n$ let s_i be a multiple of α such that $t_i\le s_i\le t_i+\alpha$. Then

$$\{\omega:\omega(s_{i+1})-\omega(s_i)\in F_{i\eta}\text{ for every }i<n\}\setminus E$$
$$=\bigcup_{i<n}\{\omega:\omega(s_{i+1})-\omega(s_i)\in F_{i\eta},\,\omega(t_{i+1})-\omega(t_i)\notin E_i\}$$
$$\subseteq\bigcup_{i<n}\{\omega:|(\omega(s_{i+1})-\omega(s_i))-(\omega(t_{i+1}-\omega(t_i))|>2\eta\}$$
$$\subseteq\bigcup_{i\le n}\{\omega:|\omega(s_i)-\omega(t_i)|>\eta\}$$
$$\subseteq\{\omega:\text{there are }s,\,t\in[0,t_n+\eta]$$
$$\text{such that }|s-t|\le\delta\text{ and }|\omega(s_i)-\omega(t_i)|>\eta\},$$

so

$$\mu_{\nu\alpha}\{\omega:\omega(s_{i+1})-\omega(s_i)\in F_{i\eta}\text{ for every }i<n\}\le\epsilon+\mu_{\nu\alpha}E.$$

Next, if $s_i=k_i\alpha$ for each i,

$$\mu_{\nu\alpha}\{\omega:\omega(s_{i+1})-\omega(s_i)\in F_{i\eta}\text{ for every }i<n\}$$
$$=\nu^{\mathbb{N}}\{z:f_\alpha(z)(s_{i+1})-f_\alpha(z)(s_i)\in F_{i\eta}\text{ for every }i<n\}$$
$$=\nu^{\mathbb{N}}\{z:\sqrt{\alpha}\sum_{j=k_i}^{k_{i+1}-1}z(j)\in F_{i\eta}\text{ for every }i<n\}$$
$$=\prod_{i<n}\nu^{\mathbb{N}}\{z:\sqrt{\alpha}\sum_{j=k_i}^{k_{i-1}}z(j)\in F_{i\eta}\}$$
$$=\prod_{i<n}\mu_{\nu\alpha}\{\omega:\omega(s_{i+1})-\omega(s_i)\in F_{i\eta}\}.$$

So

$$\mu_W E\le\epsilon+\prod_{i<n}\frac{1}{\sqrt{2\pi(s_{i+1}-s_i)}}\int_{F_{i\eta}}e^{-x^2/2(s_{i+1}-s_i)}dx$$

(because $|(s_{i+1}-s_i)-(t_{i+1}-t_i)|\le\alpha\le\eta$ for $i<n$)

$$\le 2\epsilon+\prod_{i<n}\mu_{\nu\alpha}\{\omega:\omega(s_{i+1})-\omega(s_i)\in F_{i\eta}\}$$

(because $s_{i+1}-s_i\ge\gamma$ for every $i<n$)

$$=2\epsilon+\mu_{\nu\alpha}\{\omega:\omega(s_{i+1})-\omega(s_i)\in F_{i\eta}\text{ for every }i<n\}$$
$$\le 3\epsilon+\mu_{\nu\alpha}E,$$

as required. **Q**

(c)(i) For $k \in \mathbb{N}$ let $\delta_k > 0$ be such that $\mu_{\nu\alpha} G_k \le 2^{-k}$ whenever $(\nu, \alpha) \in Q(\delta_k)$, where

$$G_k = \{\omega : \text{there are } s,\, t \in [0, k] \text{ such that } |t - s| \le \delta_k \text{ and } |\omega(t) - \omega(s)| > 2^{-k}\};$$

such exists by (a-iii) above. For k, $n \in \mathbb{N}$ set $H_{kn} = \bigcup_{i \le n} G_{k+i}$. If $k \in \mathbb{N}$ and $\langle \omega'_n \rangle_{n\in\mathbb{N}}$ is a sequence such that $\omega'_n \in \Omega \setminus H_{kn}$ for every $n \in \mathbb{N}$, $\{\omega'_n : n \in \mathbb{N}\}$ is relatively compact in Ω. **P** If $\gamma \ge 0$ and $\epsilon > 0$, there is an $n \in \mathbb{N}$ such that $2^{-k-n} \le \epsilon$ and $k + n \ge \gamma$; now for $m \ge n$, $\omega'_m \notin G_{k+n}$ so $|\omega'_m(t) - \omega'_m(s)| \le \epsilon$ whenever s, $t \in [0, \gamma]$ and $|s - t| \le \delta_{k+n}$. Of course there is a $\delta \in \,]0, \delta_{k+n}]$ such that $|\omega'_m(s) - \omega'_m(t)| \le \epsilon$ whenever $m < k+n$ and s, $t \in [0, \gamma]$ are such that $|s-t| \le \delta$. Since $\omega'_n(0) = 0$ for every n, the conditions of 4A2U(e-ii) are satisfied, and $\{\omega'_n : n \in \mathbb{N}\}$ is relatively compact in $C([0, \infty[)$, therefore in its closed subset Ω. **Q**

Now if we have a compact set $K \subseteq \Omega$, an open set $G \subseteq \Omega$ including K, and $k \in \mathbb{N}$, there are an $n \in \mathbb{N}$ and a finite set $I \subseteq [0, \infty[$ such that $\omega' \in G \cup H_{kn}$ whenever $\omega \in K$, $\omega' \in \Omega$ and $|\omega'(s) - \omega(s)| \le 2^{-n}$ for every $s \in I$. **P?** Otherwise, let $\langle q_i \rangle_{i\in\mathbb{N}}$ enumerate $\mathbb{Q} \cap [0, \infty[$. For each $n \in \mathbb{N}$ we have $\omega_n \in K$ and $\omega'_n \in \Omega \setminus (G \cup H_{kn})$ such that $|\omega'_n(q_i) - \omega_n(q_i)| \le 2^{-n}$ for every $i \le n$. Since the topology $\mathfrak{T}_c$ on Ω is metrizable (4A2U(e-i)), and both $\{\omega_n : n \in \mathbb{N}\}$ and $\{\omega'_n : n \in \mathbb{N}\}$ are relatively compact, there is a strictly increasing sequence $\langle n_i \rangle_{i\in\mathbb{N}}$ such that $\omega = \lim_{i\to\infty} \omega_{n_i}$ and $\omega' = \lim_{i\to\infty} \omega'_{n_i}$ are both defined (use 4A2Lf twice). Since $|\omega'(q) - \omega(q)| = \lim_{i\to\infty} |\omega'_{n_i}(q) - \omega_{n_i}(q)|$ is zero for every $q \in \mathbb{Q} \cap [0, \infty[$, $\omega = \omega'$; but $\omega \in K$ and $\omega' \notin G$, so this is impossible. **XQ**

(ii) Suppose that $G \subseteq \Omega$ is open and $\gamma < \mu_W E$. Then there is a $\delta > 0$ such that $\mu_{\nu\alpha} G > \gamma$ whenever $(\nu, \alpha) \in Q_\delta$. **P** Let $K \subseteq G$ be a compact set such that $\mu_W K > \gamma$. Let $k \in \mathbb{N}$, $\epsilon > 0$ be such that $\mu_W K \ge \gamma + \epsilon + 2^{-k+1}$. By (i), there are an $n \in \mathbb{N}$ and a finite set $I \subseteq [0, \infty[$ such that $\omega' \in G \cup H_{kn}$ whenever $\omega' \in \Omega$, $\omega \in K$ and $|\omega'(t) - \omega(t)| \le 2^{-n}$ for every $t \in I$; of course we can suppose that $0 \in I$ and that $\#(I) \ge 2$. Enumerate I in increasing order as $\langle t_i \rangle_{i \le m}$. For $z \in \mathbb{Z}^m$, set

$$E_z = \{\omega : \omega \in \Omega,\ \lfloor 2^n m(\omega(t_{i+1} - \omega(t_i)) \rfloor = z(i) \text{ for every } i < m\};$$

set $D = \{z : z \in \mathbb{Z}^m,\ E_z \cap K \ne \emptyset\}$ and $F = \bigcup_{z\in D} E_z$. If $z \in D$ and $\omega' \in E_z$, there is an $\omega \in K \cap E_z$, in which case

$$|(\omega'(t_{i+1}) - \omega'(t_i)) - (\omega(t_{i+1} - \omega(t_i))| \le \frac{2^{-n}}{m} \text{ for every } i < m,$$

$$|\omega'(t_i) - \omega(t_i)| \le 2^{-n} \text{ for every } i \le m$$

and $\omega' \in G \cup H_{kn}$. Thus $F \subseteq G \cup H_{kn}$. As K is compact, $\{\omega(t_i) : \omega \in K\}$ is bounded for every i and D is finite. By (b) there is a $\delta > 0$ such that $\delta \le \delta_{k+i}$ for every $i \le n$ and

$$\mu_W E_z \le \frac{\epsilon}{1+\#(D)} + \mu_{\nu\alpha} E_z$$

whenever $z \in D$ and $(\nu, \alpha) \in Q(\delta)$. Now, for such ν and α,

$$\begin{aligned}
\epsilon + 2^{-k+1} + \gamma \le \mu_W K \le \mu_W F = \sum_{z\in D} \mu_W E_z &\le \epsilon + \sum_{z\in D} \mu_{\nu\alpha} E_z \\
&= \epsilon + \mu_{\nu\alpha} F \le \epsilon + \mu_{\nu\alpha} G + \sum_{i=0}^{n} \mu_{\nu\alpha} G_{k+i} \\
&\le \epsilon + \mu_{\nu\alpha} G + \sum_{i=0}^{n} 2^{-k-i} < \epsilon + 2^{-k+1} + \mu_{\nu\alpha} G
\end{aligned}$$

and $\mu_{\nu\alpha} G > \gamma$, as required. **Q**

(iii) So if U is a neighbourhood of μ_W for the narrow topology on $P_{\mathrm{R}}(\Omega)$, there is a $\delta > 0$ such that $\mu_{\nu\alpha} \in U$ whenever $(\nu, \alpha) \in Q(\delta)$. **P** There are open sets $G_0, \dots, G_n$ and $\gamma_0, \dots, \gamma_n$ such that $\gamma_i < \mu_W G_i$ for each $i < n$ and U includes $\{\mu : \mu \in P_{\mathrm{R}}(\Omega),\ \mu G_i > \gamma_i \text{ for every } i < n\}$. But from (ii) we see that for each $i \le n$ there will be a $\delta'_i > 0$ such that $\mu_{\nu\alpha} G_i > \gamma_i$ for every i whenever $(\nu, \alpha) \in Q(\delta'_i)$; so setting $\delta = \min_{i\le n} \delta'_i$ we get the result. **Q**

And this is just the conclusion declared in the statement of the theorem, rephrased in the language developed in the course of the proof.

477D Multidimensional Brownian motion In §§478-479 we shall need the theory of Brownian motion in r-dimensional space. I sketch the relevant details. Fix an integer $r \geq 1$.

(a) Let μ_{W1} be the Radon probability measure on $\Omega_1 = C([0,\infty[)_0$ described in 477B; I will call it **one-dimensional Wiener measure**. We can identify the power Ω_1^r with $\Omega = C([0,\infty[\,;\mathbb{R}^r)_0$, the space of continuous functions $\omega : [0,\infty[\, \to \mathbb{R}^r$ such that $\omega(0) = 0$, with the topology of uniform convergence on compact sets; note that Ω_1 is Polish (4A2U(e-i)), so Ω_1^r also is. Because Ω_1 is separable and metrizable, the c.l.d. product measure μ_{W1}^r measures every Borel set (4A3Dc, 4A3E), while it is inner regular with respect to the compact sets (412Sb), so it is a Radon measure. I will say that $\mu_W = \mu_{W1}^r$, interpreted as a measure on $C([0,\infty[\,;\mathbb{R}^r)_0$, is **$r$-dimensional Wiener measure**.

As observed in 477B, μ_{W1} is the subspace measure on Ω_1 induced by the distribution $\hat{\mu}$ of the process $\langle X_t\rangle_{t\geq 0}$ in 477A. So μ_W here, regarded as a measure on $C([0,\infty[)_0^r$, is the subspace measure induced by the measure $\hat{\mu}^r$ on $(\mathbb{R}^{[0,\infty[})^r \cong \mathbb{R}^{[0,\infty[\times r}$ (254La).

(b) For $\omega \in \Omega$, $t \geq 0$ and $i < r$, set $X_t^{(i)}(\omega) = \omega(t)(i)$. Then $\langle X_t^{(i)}\rangle_{t\geq 0, i<r}$ is a centered Gaussian process, with covariance matrix

$$\begin{aligned} \mathbb{E}(X_s^{(i)} \times X_t^{(j)}) &= 0 \text{ if } i \neq j, \\ &= \min(s,t) \text{ if } i = j. \end{aligned}$$

P Taking μ, $\hat{\mu}$ and $\hat{\mu}^r$ as in (a), $\hat{\mu}^r$, like $\hat{\mu}$, is a centered Gaussian distribution (456Be); but it is easy to check from the formula in 454J(i) that $\hat{\mu}^r$ can be identified with the distribution of the family $\langle X_t^{(i)}\rangle_{t\geq 0, i<r}$. So $\langle X_t^{(i)}\rangle_{t\geq 0, i<r}$ is a centered Gaussian process. As for the covariance matrix, if $i \neq j$ then $X_s^{(i)}$ and $X_t^{(j)}$ are determined by different factors in the product $\Omega = \Omega_1^r$, so must be independent; while if $i = j$ then $(X_s^{(i)}, X_t^{(i)})$ have the same joint distribution as (X_s, X_t) in 477A. **Q**

(c) We shall need a variety of characterizations of the Radon measure μ_W.

(i) μ_W is the only Radon probability measure on Ω such that the process $\langle X_t^{(i)}\rangle_{t\geq 0, i<r}$ described in (b) is a Gaussian process with the covariance matrix there. **P** Suppose ν is another measure with these properties. The distribution of $\langle X_t^{(i)}\rangle_{t\geq 0, i<r}$ (with respect to ν) must be a centered Gaussian process on $\mathbb{R}^{r\times[0,\infty[} \cong (\mathbb{R}^{[0,\infty[})^r$, and because it has the same covariance matrix it must be equal to $\hat{\mu}^r$, by 456Bb. But this says just that $\langle X_t^{(i)}\rangle_{t\geq 0, i<r}$ has the same joint distribution with respect to μ_W and ν. By 454N, $\nu = \mu_W$. **Q**

(ii) Another way of looking at the family $\langle X_t^{(i)}\rangle_{i<r, t\geq 0}$ is to write $X_t(\omega) = \omega(t)$ for $t \geq 0$, so that $\langle X_t\rangle_{t\geq 0}$ is now a family of $\mathbb{R}^r$-valued random variables defined on Ω. We can describe its distribution in terms matching those of 455Q and 477A, which become

(i) $X_0 = 0$ everywhere (on Ω, that is);

(ii) whenever $0 \leq s < t$ then $\frac{1}{\sqrt{t-s}}(X_t - X_s)$ has the standard Gaussian distribution μ_G^r (that is, $\omega \mapsto \frac{1}{\sqrt{t-s}}(\omega(t) - \omega(s))$ is inverse-measure-preserving for μ_W and μ_G^r);

(iii) whenever $0 \leq t_1 < \ldots < t_n$, then $X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent (that is, taking T_i to be the σ-algebra $\{\{\omega : \omega(t_{i+1}) - \omega(t_i) \in E\} : E \subseteq \mathbb{R}^r$ is a Borel set$\}$, $\mathrm{T}_1, \ldots, \mathrm{T}_{n-1}$ are independent).

Note that these properties also determine the Radon measure μ_W. **P** Once again, suppose ν is a Radon probability measure on Ω for which (ii) and (iii) are true. We wish to show that μ_W and ν give the same distribution to $\langle X_t^{(i)}\rangle_{i<r, t\geq 0}$. If $0 = t_0 < t_1 < \ldots < t_n$, we know that μ_W and ν give the same distribution to each of the differences $Y_j = X_{t_{j+1}} - X_{t_j}$ (or, if you prefer, to each of the families $\langle Y_j^{(i)}\rangle_{i<r}$, where $Y_j^{(i)} = X_{t_{j+1}}^{(i)} - X_{t_j}^{(i)}$); moreover, if Σ_j is the σ-algebra generated by $\{Y_j^{(i)} : i < r\}$ for each j, then μ_W and ν agree that $\langle \Sigma_j\rangle_{j<n}$ is independent. So $\mu_W E = \nu E$ whenever E is of the form $\bigcap_{j<n} E_j$ where $E_j \in \Sigma_j$ for each $j < n$. By the Monotone Class Theorem, μ_W and ν agree on the σ-algebra Σ generated by sets of this type, which is the σ-algebra generated by $\{Y_j^{(i)} : i < r, j < n\}$. But as $X_{t_j} = \sum_{i<j} Y_i$ for every $j \leq n$, every

$X^{(i)}_{t_j}$ is Σ-measurable, and μ_W and ν give the same distribution to $\langle X^{(i)}_{t_j}\rangle_{i<r,j\le n}$. As this is true whenever $0=t_0<\ldots<t_n$, μ_W and ν give the same distribution to the whole family $\langle X^{(i)}_t\rangle_{i<r,t\ge 0}$, and must be equal. **Q**

(d) In order to apply Theorem 455U, we need to go a little deeper, in order to relate the product-measure definition of μ_W to the construction in 455P. I will use the ideas of part (b) of the proof of 455R. Consider the process $\langle X^{(i)}_t\rangle_{t\ge 0,i<r}$ and the associated distribution $\hat\mu^r$ on $(\mathbb{R}^{[0,\infty[})^r\cong(\mathbb{R}^r)^{[0,\infty[}$. Setting $X_t=\langle X^{(i)}_t\rangle_{i<r}$, $\langle X_t\rangle_{t\ge 0}$ is an $\mathbb{R}^r$-valued process satisfying the conditions of 455Q with $U=\mathbb{R}^r$. **P** $X_0=0$ a.e. because every $X^{(i)}_0$ is zero a.e. If $0\le s<t$ then $X_t-X_s=\langle X^{(i)}_t-X^{(i)}_s\rangle_{i<r}$ has the same distribution as X_{t-s} because $X^{(i)}_t-X^{(i)}_s$ has the same distribution as $X^{(i)}_{t-s}$ for each i and $\langle X^{(i)}_t-X^{(i)}_s\rangle_{i<r}$, $\langle X^{(i)}_{t-s}\rangle_{i<r}$ are both independent. If $0\le t_0<t_1<\ldots<t_n$ then $\langle X^{(i)}_{t_{j+1}}-X^{(i)}_{t_j}\rangle_{i<r,j<n}$ is independent so $\langle X_{t_{j+1}}-X_{t_j}\rangle_{j<n}$ is independent (using 272K, or otherwise). Finally, when $t\downarrow 0$, $X_t\to 0$ in measure because $X^{(i)}_t\to 0$ in measure for each i. **Q**

For $t>0$, let λ_t be the distribution of X_t. Then λ_t is the centered Gaussian distribution on $\mathbb{R}^r$ with covariance matrix $\langle\sigma_{ij}\rangle_{i,j<r}$ where $\sigma_{ij}=t$ if $i=j$ and 0 if $i\ne j$ (456Ba, with $T(\omega)=\omega(t)$ for $\omega\in\mathbb{R}^{[0,\infty[\times r}\cong(\mathbb{R}^r)^{[0,\infty[}$). By 455R, the process of 455P can be applied to $\langle\lambda_t\rangle_{t>0}$ to give us a measure $\hat\nu$ on $(\mathbb{R}^r)^{[0,\infty[}$, the completion of a Baire measure, such that

$$\begin{aligned}\hat\nu\{\omega:\omega(t_i)\in F_i\text{ for every }i\le n\}&=\Pr(X_{t_i}\in F_i\text{ for every }i\le n)\\&=\hat\mu^r\{\omega:\omega(t_i)\in F_i\text{ for every }i\le n\}\end{aligned}$$

whenever $F_0,\ldots,F_n\subseteq\mathbb{R}^r$ are Borel sets and $t_0,\ldots,t_n\in[0,\infty[$. Since sets of this kind generate the Baire σ-algebra of $(\mathbb{R}^r)^{[0,\infty[}$, $\hat\nu$ must be equal to $\hat\mu^r$, that is, $\hat\mu^r$ is the result of applying 455P to $\langle\lambda_t\rangle_{t>0}$.

By 455H, $\hat\nu$ has a unique extension to a measure $\tilde\nu$ on $(\mathbb{R}^r)^{[0,\infty[}$ which is a Radon measure for the product topology. But if we write $\iota:C([0,\infty[;\mathbb{R}^r)_0\to(\mathbb{R}^r)^{[0,\infty[}$ for the identity map, the image measure $\mu_W\iota^{-1}$ is a Radon measure on $(\mathbb{R}^r)^{[0,\infty[}$ for the product topology and extends $\hat\mu^r$, so must be equal to $\tilde\nu$. Thus $\tilde\nu C([0,\infty[;\mathbb{R}^r)_0=1$. Of course $C([0,\infty[;\mathbb{R}^r)_0$ is included in the space of càllàl functions from $[0,\infty[$ to $\mathbb{R}^r$, so that we have a strengthening of the results in §455. Similarly, writing $\ddot\nu$ for the subspace measure induced by $\hat\nu$ or $\hat\mu^r$ on the space $C_{\rm dlg}$ of càdlàg functions from $[0,\infty[$ to $\mathbb{R}^r$, μ_W is the subspace measure on $C([0,\infty[;\mathbb{R}^r)_0$ induced by $\ddot\nu$.

By 4A3Qa, every Baire subset of $C_{\rm dlg}$ is the intersection of $C_{\rm dlg}$ with a Baire subset of $(\mathbb{R}^r)^{[0,\infty[}$, and is therefore measured by $\ddot\nu$. In particular, $C([0,\infty[;\mathbb{R}^r)$ and $C([0,\infty[;\mathbb{R}^r)_0$ are measured by $\ddot\nu$ (4A3Qd).

477E Invariant transformations of Wiener measure: Proposition Let $r\ge 1$ be an integer, and μ_W Wiener measure on $\Omega=C([0,\infty[;\mathbb{R}^r)_0$. Let $\hat\mu^r$ be the product measure on $(\mathbb{R}^{[0,\infty[})^r$ as described in 477D.

(a) Suppose that $f:(\mathbb{R}^{[0,\infty[})^r\to(\mathbb{R}^{[0,\infty[})^r$ is inverse-measure-preserving for $\hat\mu^r$, and that $\Omega_0\subseteq\Omega$ is a μ_W-conegligible set such that $f[\Omega_0]\subseteq\Omega_0$. Then $f{\restriction}\Omega_0$ is inverse-measure-preserving for the subspace measure induced by μ_W on Ω_0.

(b) Suppose that $\hat T:\mathbb{R}^{r\times[0,\infty[}\to\mathbb{R}^{r\times[0,\infty[}$ is a linear operator such that, for i, $j<r$ and s, $t\ge 0$,

$$\begin{aligned}\int(\hat T\omega)(i,s)(\hat T\omega)(j,t)\hat\mu^r(d\omega)&=\min(s,t)\text{ if }i=j,\\&=0\text{ if }i\ne j.\end{aligned}$$

Then, identifying $\mathbb{R}^{r\times[0,\infty[}$ with $(\mathbb{R}^{[0,\infty[})^r$, $\hat T$ is inverse-measure-preserving for $\hat\mu^r$.

(c) Suppose that $t\ge 0$. Define $S_t:\Omega\to\Omega$ by setting $(S_t\omega)(s)=\omega(s+t)-\omega(s)$ for $s\ge 0$ and $\omega\in\Omega$. Then S_t is inverse-measure-preserving for μ_W.

(d) Let $T:\mathbb{R}^r\to\mathbb{R}^r$ be an orthogonal transformation. Define $\tilde T:\Omega\to\Omega$ by setting $(\tilde T\omega)(t)=T(\omega(t))$ for $t\ge 0$ and $\omega\in\Omega$. Then $\tilde T$ is an automorphism of (Ω,μ_W).

(e) Suppose that $\alpha>0$. Define $U_\alpha:\Omega\to\Omega$ by setting $U_\alpha(\omega)(t)=\frac{1}{\sqrt{\alpha}}\omega(\alpha t)$ for $t\ge 0$ and $\omega\in\Omega$. Then U_α is an automorphism of (Ω,μ_W).

(f) Set

$$\Omega_0 = \{\omega : \omega \in \Omega,\ \lim_{t\to\infty} \frac{1}{t}\omega(t) = 0\},$$

and define $R : \Omega_0 \to \Omega_0$ by setting

$$\begin{aligned}(R\omega)(t) &= t\omega(\tfrac{1}{t}) \text{ if } t > 0,\\ &= 0 \text{ if } t = 0.\end{aligned}$$

Then Ω_0 is μ_W-conegligible and R is an automorphism of Ω_0 with its subspace measure.

(g) Suppose that $1 \le r' \le r$, and that $\mu_W^{(r')}$ is Wiener measure on $C([0,\infty[\,;\mathbb{R}^{r'})_0$. Define $P : \Omega \to C([0,\infty[\,;\mathbb{R}^{r'})_0$ by setting $(P\omega)(t)(i) = \omega(t)(i)$ for $t \ge 0$, $i < r'$ and $\omega \in \Omega$. Then $\mu_W^{(r')}$ is the image measure $\mu_W P^{-1}$.

proof The following arguments will unscrupulously identify $C([0,\infty[\,;\mathbb{R}^r)_0$ with $C([0,\infty[)_0^r$, and $\mathbb{R}^{r\times[0,\infty[}$ with $(\mathbb{R}^r)^{[0,\infty[}$ and $(\mathbb{R}^{[0,\infty[})^r$.

(a) Because μ_W is the subspace measure on Ω induced by $\hat\mu^r$ (477Da), the subspace measure ν on Ω_0 induced by μ_W is also the subspace measure on Ω_0 induced by $\hat\mu^r$ (214Ce). If $E \subseteq \Omega_0$ is measured by ν, there is an $F \in \operatorname{dom}\hat\mu^r$ such that $E = F \cap \Omega_0$, and now

$$\nu E = \hat\mu^r F = \hat\mu^r f^{-1}[F] = \nu(\Omega_0 \cap f^{-1}[F]) = \nu(f\restriction\Omega_0)^{-1}[E].$$

As E is arbitrary, $f\restriction\Omega_0$ is inverse-measure-preserving for ν.

(b) By 456Ba, $\hat\mu^r\hat T^{-1}$ is a centered Gaussian distribution on $\mathbb{R}^{r\times[0,\infty[}$. The hypothesis asserts that its covariance matrix is the same as that of $\hat\mu^r$ (477Db), so that $\hat\mu^r = \hat\mu^r\hat T^{-1}$ (456Bb), that is, $\hat T$ is inverse-measure-preserving for $\hat\mu^r$.

(c) Define $\hat S_t : \mathbb{R}^{r\times[0,\infty[} \to \mathbb{R}^{r\times[0,\infty[}$ by setting $(\hat S_t\omega)(i,s) = \omega(i,s+t) - \omega(i,s)$, this time for $\omega \in \mathbb{R}^{r\times[0,\infty[}$, $i < r$ and $s \ge 0$. Then $\hat S_t$ is linear, and for s, $u \in [0,\infty[$, i, $j < r$

$$\begin{aligned}&\int (\hat S_t\omega)(i,s)(\hat S_t\omega)(j,u)\mu^r(d\omega)\\ &\quad= \int (\omega(i,s+t) - \omega(t))(\omega(j,u+t) - \omega(j,t))\hat\mu^r(d\omega)\\ &\quad= 0 \text{ if } i \ne j,\\ &\quad= \min(s+t,u+t) - \min(s+t,t)\\ &\qquad\qquad - \min(t,u+t) + \min(t,t) = \min(s,u) \text{ if } i = j.\end{aligned}$$

By (b), $\hat S_t$ is inverse-measure-preserving for $\hat\mu^r$. Now $\hat S_t[\Omega] \subseteq \Omega$, so $S_t = \hat S_t\restriction\Omega$ is inverse-measure-preserving for μW, by (a).

(d) If we define $\hat T : (\mathbb{R}^r)^{[0,\infty[} \to (\mathbb{R}^r)^{[0,\infty[}$ by setting $(\hat T)(\omega)(t) = T(\omega(t))$ for $x \in (\mathbb{R}^r)^{[0,\infty[}$ and $t \ge 0$, then $\hat T$ is linear. Suppose that T is defined by the matrix $\langle\alpha_{ij}\rangle_{i,j<r}$. For $\omega \in \mathbb{R}^{r\times[0,\infty[}$, $t \in [0,\infty[$ and $i < r$,

$$(\hat T\omega)(i,t) = \sum_{k=0}^{r-1} \alpha_{ik}\omega(k,t).$$

So, for i, $j < r$ and s, $t \ge 0$,

$$\begin{aligned}\int (\hat T\omega)(i,s)(\hat T\omega)(j,t)\hat\mu^r(d\omega) &= \sum_{k=0}^{r-1}\sum_{l=0}^{r-1} \alpha_{ik}\alpha_{jl} \int \omega(k,s)\omega(l,t)\hat\mu^r(d\omega)\\ &= \sum_{k=0}^{r-1} \alpha_{ik}\alpha_{jk} \min(s,t)\\ &= \min(s,t) \text{ if } i = j,\\ &= 0 \text{ if } i \ne j.\end{aligned}$$

So $\hat{T}$ is $\hat{\mu}^r$-inverse-measure-preserving. If we think of $\hat{T}$ as operating on $(\mathbb{R}^r)^{[0,\infty[}$, then $\hat{T}(\omega) = T\omega$ is continuous for every $\omega \in C([0,\infty[\,;\mathbb{R}^r)$, so $\hat{T}[\Omega] \subseteq \Omega$ and $\tilde{T} = \hat{T}\restriction\Omega$ is μ_W-inverse-measure-preserving.

Now the same argument applies to T^{-1}, so $\tilde{T}^{-1} = (T^{-1})^{\sim}$ also is inverse-measure-preserving, and $\tilde{T}$ is an automorphism of (Ω, μ_W).

(e) This time, we have $\hat{U}_\alpha : \mathbb{R}^{r\times[0,\infty[} \to \mathbb{R}^{r\times[0,\infty[}$ defined by the formula $(\hat{U}_\alpha\omega)(i,t) = \frac{1}{\sqrt{\alpha}}\omega(i,\alpha t)$ for $i < r$, $t \ge 0$ and $\omega \in \mathbb{R}^{r\times[0,\infty[}$. Once again, $\hat{U}_\alpha$ is linear. This time,

$$\begin{aligned}\int(\hat{U}_\alpha\omega)(i,s)(\hat{U}_\alpha\omega)(j,t)\hat{\mu}^r(d\omega) &= \frac{1}{\alpha}\int\omega(i,\alpha s)\omega(j,\alpha t)\hat{\mu}^r(d\omega)\\ &= 0 \text{ if } i \ne j,\\ &= \frac{1}{\alpha}\min(\alpha s,\alpha t) = \min(s,t) \text{ if } i = j.\end{aligned}$$

As before, it follows that $\hat{U}_\alpha$ is $\hat{\mu}^r$-inverse-measure-preserving, so that $U_\alpha = \hat{U}_\alpha\restriction\Omega$ is μ_W-inverse-measure-preserving. In the same way as in (d), $U_\alpha^{-1} = U_{1/\alpha}$ is μ_W-inverse-measure-preserving, so U_α is an automorphism of (Ω, μ_W).

(f) Define $\hat{R} : \mathbb{R}^{r\times[0,\infty[} \to \mathbb{R}^{r\times[0,\infty[}$ by setting

$$\begin{aligned}\hat{R}(\omega)(i,t) &= t\omega(i,\tfrac{1}{t}) \text{ if } i < r \text{ and } t > 0,\\ &= \omega(i,0) \text{ if } i < r \text{ and } t = 0.\end{aligned}$$

Then, if i, $j < r$ and s, $t > 0$,

$$\begin{aligned}\int(\hat{R}\omega)(i,s)(\hat{R}\omega)(j,t)\hat{\mu}^r(d\omega) &= st\int\omega(i,\tfrac{1}{s})\omega(j,\tfrac{1}{t})\hat{\mu}^r(d\omega)\\ &= 0 \text{ if } i \ne j,\\ &= st\min(\tfrac{1}{s},\tfrac{1}{t}) = \min(s,t) \text{ if } i = j.\end{aligned}$$

If $s = 0$, then $(\hat{R}\omega)(i,s) = \omega(i,s) = 0$ for almost every ω, so that $\int(\hat{R}\omega)(i,s)(\hat{R}\omega)(j,t)\hat{\mu}^r(d\omega) = 0$; and similarly if $t = 0$. So $\hat{R}$ is $\hat{\mu}^r$-inverse-measure-preserving.

At this point I think we need a new argument. Consider the set

$$E = \{\omega : \omega \in (\mathbb{R}^r)^{[0,\infty[},\ \lim_{q\in\mathbb{Q},q\downarrow 0}\omega(q) = 0\}.$$

Then E is a Baire set in $(\mathbb{R}^r)^{[0,\infty[} \cong \mathbb{R}^{r\times[0,\infty[}$. Since $E \supseteq \Omega$, $\hat{\mu}^r E = 1$. Consequently $\hat{\mu}^r\hat{R}^{-1}[E] = 1$. But, for $\omega \in (\mathbb{R}^r)^{[0,\infty[}$,

$$\omega \in \hat{R}^{-1}[E] \iff 0 = \lim_{q\in\mathbb{Q},q\downarrow 0}(\hat{R}\omega)(q) = \lim_{q\in\mathbb{Q},q\downarrow 0} q\omega(\tfrac{1}{q}) = \lim_{q\in\mathbb{Q},q\to\infty}\frac{1}{q}\omega(q).$$

So

$$\Omega_0 = \{\omega : \omega \in \Omega,\ \lim_{t\to\infty}\frac{1}{t}\omega(t) = 0\} = \{\omega : \omega \in \Omega,\ \lim_{q\in\mathbb{Q},q\to\infty}\frac{1}{q}\omega(q) = 0\}$$

(because every member of Ω is continuous)

$$= \Omega \cap \hat{R}^{-1}[E]$$

is μ_W-conegligible. Next, for $\omega \in \Omega_0$, $\hat{R}\omega$ is continuous on $]0,\infty[$ and

$$\begin{aligned}0 = \omega(0) &= \lim_{t\to\infty}\frac{1}{t}\omega(t) = \lim_{t\downarrow 0}\omega(t)\\ &= (\hat{R}\omega)(0) = \lim_{t\to\infty}(\hat{R}\omega)(\tfrac{1}{t}) = \lim_{t\downarrow 0}t(\hat{R}\omega)(\tfrac{1}{t})\\ &= \lim_{t\downarrow 0}(\hat{R}\omega)(t) = \lim_{t\to\infty}\frac{1}{t}(\hat{R}\omega)(t),\end{aligned}$$

so $\hat{R}\omega \in \Omega_0$. By (a), $R = \hat{R}\restriction\Omega_0$ is inverse-measure-preserving for the subspace measure $\nu = (\mu_W)_{\Omega_0}$; being an involution, it is an automorphism of (Ω_0, ν).

(g) If we identify μ_W and $\mu_W^{(r')}$ with μ_{W1}^r and $\mu_{W1}^{r'}$, as in 477Da, this is elementary.

477F Proposition Let $r \ge 1$ be an integer. Then Wiener measure on $\Omega = C([0,\infty[\,;\mathbb{R}^r)_0$ is strictly positive for the topology $\mathfrak{T}_c$ of uniform convergence on compact sets.

proof (a) Let $\mathcal{I}$ be a partition of $[0,\infty[$ into bounded intervals (open, closed or half-open). As usual, set $X_t(\omega) = \omega(t)$ for $t \in [0,\infty[$ and $\omega \in (\mathbb{R}^r)^{[0,\infty[}$. Define $\langle Y_t\rangle_{t\ge 0}$, $\langle Z_t\rangle_{t\ge 0}$ as follows. If $t \in I \in \mathcal{I}$, $a = \inf I$ and $b = \sup I$, then

$$\begin{aligned} Y_t &= X_a + \tfrac{t-a}{b-a}(X_b - X_a) \text{ if } a < b,\\ &= X_a = X_t = X_b \text{ if } a = b,\\ Z_t &= X_t - Y_t. \end{aligned}$$

(i) With respect to the centered Gaussian distribution $\hat{\mu}^r$ on $(\mathbb{R}^{[0,\infty[})^r \cong (\mathbb{R}^r)^{[0,\infty[}$, $(\langle Y_t\rangle_{t\ge 0}, \langle Z_t\rangle_{t\ge 0})$ is a centered Gaussian process. **P** The map $\omega \mapsto (\langle Y_t(\omega)\rangle_{t\ge 0}, \langle Z_t(\omega)\rangle_{t\ge 0})$ is linear and continuous, so we can apply 456Ba (strictly speaking, we apply this to the family $(\langle Y_t^{(i)}(\omega)\rangle_{i<r,t\ge 0}, \langle Z_t^{(i)}(\omega)\rangle_{i<r,t\ge 0})$ regarded as linear operators from $\mathbb{R}^{r\times[0,\infty[}$ to its square). **Q**

(ii) If s, $t \in [0,\infty[$ then $\mathbb{E}(Y_s \times Y_t) = \mathbb{E}(X_s \times Y_t)$. **P** Let I, J be the members of $\mathcal{I}$ containing s, t respectively; set $a = \min I$, $b = \max I$, $c = \min J$ and $d = \max J$.

case 1 If $a = b$ then $Y_s = X_s$ and we can stop.

case 2 If $a < b \le c$, then

$$\mathbb{E}(X_a \times X_c) = \mathbb{E}(X_a \times X_d) = a, \quad \mathbb{E}(X_a \times Y_t) = a,$$

$$\mathbb{E}(X_b \times X_c) = \mathbb{E}(X_b \times X_d) = b, \quad \mathbb{E}(X_b \times Y_t) = b,$$

$$\mathbb{E}(X_s \times X_c) = \mathbb{E}(X_s \times X_d) = s, \quad \mathbb{E}(X_s \times Y_t) = s,$$

$$\mathbb{E}(Y_s \times Y_t) = a + \tfrac{s-a}{b-a}(b-a) = s = \mathbb{E}(X_s \times Y_t).$$

case 3 If $d \le a < b$, then

$$c = \mathbb{E}(X_s \times X_c) = \mathbb{E}(X_a \times X_c) = \mathbb{E}(X_b \times X_c) = \mathbb{E}(Y_s \times X_c),$$

$$d = \mathbb{E}(X_s \times X_d) = \mathbb{E}(X_a \times X_d) = \mathbb{E}(X_b \times X_d) = \mathbb{E}(Y_s \times X_d);$$

since Y_t is a convex combination of X_c and X_d, $\mathbb{E}(Y_s \times Y_t) = \mathbb{E}(X_s \times Y_t)$.

case 4 If $a = c < b = d$, then

$$\mathbb{E}(X_a \times Y_t) = \mathbb{E}(X_a \times X_a) + \tfrac{t-a}{b-a}\,\mathbb{E}(X_a \times (X_b - X_a)) = a,$$

$$\mathbb{E}(X_b \times Y_t) = \mathbb{E}(X_b \times X_a) + \tfrac{t-a}{b-a}\,\mathbb{E}(X_b \times (X_b - X_a)) = a + \tfrac{t-a}{b-a}(b-a) = t,$$

$$\mathbb{E}(X_s \times Y_t) = \mathbb{E}(X_s \times X_a) + \tfrac{t-a}{b-a}\,\mathbb{E}(X_s \times (X_b - X_a)) = a + \tfrac{t-a}{b-a}(s-a),$$

$$\begin{aligned} \mathbb{E}(Y_s \times Y_t) &= \mathbb{E}(X_a \times Y_t) + \tfrac{s-a}{b-a}\,\mathbb{E}((X_b - X_a) \times Y_t)\\ &= a + \tfrac{s-a}{b-a}(t-a) = \mathbb{E}(X_s \times Y_t). \text{ **Q** } \end{aligned}$$

(iii) Accordingly $\mathbb{E}(Z_s \times Y_t) = 0$ for all s, $t \geq 0$. It follows that if Σ_1, Σ_2 are the σ-algebras of subsets of $(\mathbb{R}^r)^{[0,\infty[}$ defined by $\{Y_t : t \geq 0\}$ and $\{Z_t : t \geq 0\}$ respectively, Σ_1 and Σ_2 are $\hat{\mu}^r$-independent (456Eb).

(b) Observe next that if $x_1, \ldots, x_n \in \mathbb{R}^r$, $0 < t_1 \leq \ldots \leq t_n$ and $\delta > 0$, then $\hat{\mu}^r\{\omega : \|\omega(t_i) - x_i\| \leq \delta$ for $1 \leq i \leq n\}$ is greater than 0. **P** Set $x_0 = 0$ in $\mathbb{R}^r$ and $t_0 = 0$. For each $i < n$, the distribution of $\frac{1}{\sqrt{t_{i+1}-t_i}}(X_{t_{i+1}} - X_{t_i})$ is the standard Gaussian distribution μ_G^r on $\mathbb{R}^r$, which has strictly positive probability density function with respect to Lebesgue measure. So

$$\begin{aligned}\Pr(\|(X_{t_{i+1}} - X_{t_i}) - (x_{i+1} - x_i)\| \leq \frac{1}{n}\delta) &= \mu_G^r\{x : \|\sqrt{t_{i+1} - t_i}x - x_{i+1} + x_i\| \leq \frac{1}{n}\delta\} \\ &> 0.\end{aligned}$$

Next, $X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent, so

$$\begin{aligned}0 &< \prod_{i<n} \Pr(\|(X_{t_{i+1}} - X_{t_i}) - (x_{i+1} - x_i)\| \leq \frac{1}{n}\delta) \\ &= \Pr(\|(X_{t_{i+1}} - X_{t_i}) - (x_{i+1} - x_i)\| \leq \frac{1}{n}\delta \text{ for every } i < n) \\ &\leq \Pr(\|X_{t_i} - x_i\| \leq \delta \text{ for every } i \leq n). \textbf{ Q}\end{aligned}$$

(c) Let $G \subseteq \Omega$ be a non-empty $\mathfrak{T}_c$-open set. Then there are $\omega_0 \in \Omega$, $m \in \mathbb{N}$ and $\delta > 0$ such that G includes

$$V = \{\omega : \omega \in \Omega,\ \|\omega(t) - \omega_0(t)\| \leq 6\delta \text{ for every } t \in [0, m]\}.$$

For $n \in \mathbb{N}$, let F_n be

$$\begin{aligned}\{\omega : \omega \in (\mathbb{R}^r)^{[0,\infty[}, \|\omega(q) - \omega(q')\| \leq \delta \\ \text{whenever } q,\ q' \in \mathbb{Q} \cap [0, m] \text{ and } |q - q'| \leq 2^{-n}\}.\end{aligned}$$

Then $\Omega \subseteq \bigcup_{n\in\mathbb{N}} F_n$ so there is an $n \geq 0$ such that $\omega_0 \in F_n$ and $\hat{\mu}^r F_n > 0$. Let $\mathcal{I}$ be $\{[2^{-n}k, 2^{-n}(k+1)[: k \in \mathbb{N}\}$, and let $\langle Y_t \rangle_{t\geq 0}$, $\langle Z_t \rangle_{t\geq 0}$ be the families of random variables defined from $\mathcal{I}$ by the method of (a) above, with corresponding independent σ-algebras Σ_1, Σ_2. If $\omega \in F_n$, then $\|Z_t(\omega)\| \leq \delta$ for every $t \in \mathbb{Q} \cap [0, m]$. **P** If $t \in [a, b[\in \mathcal{I}$, then $b - a = 2^{-n}$ so $\|X_t(\omega) - X_a(\omega)\|$, $\|X_t(\omega) - X_b(\omega)\|$ and therefore $\|Z_t(\omega)\| = \|X_t(\omega) - Y_t(\omega)\|$ are all at most δ. **Q** Set

$$F = \{\omega : \omega \in (\mathbb{R}^r)^{[0,\infty[},\ \|Z_t(\omega)\| \leq \delta \text{ for every } t \in \mathbb{Q} \cap [0, m]\};$$

then $F \in \Sigma_2$ and $\hat{\mu}^r F > 0$.

Next, set

$$E = \{\omega : \omega \in (\mathbb{R}^r)^{[0,\infty[},\ \|\omega(2^{-n}k) - \omega_0(2^{-n}k)\| \leq \delta \text{ for } 1 \leq k \leq 2^n m\}.$$

By (b), $\hat{\mu}^r E > 0$. But since $Y_{2^{-n}k}(\omega) = X_{2^{-n}k}(\omega) = \omega(2^{-n}k)$ whenever $k \leq 2^n m$, $E \in \Sigma_1$. Accordingly $\hat{\mu}^r(E \cap F) = \hat{\mu}^r E \cdot \hat{\mu}^r F > 0$. But $E \cap F \cap \Omega \subseteq V$. **P** If $\omega \in E \cap F \cap \Omega$, then $t \mapsto X_t(\omega)$, $t \mapsto Y_t(\omega)$ and $t \mapsto Z_t(\omega)$ are all continuous, so $\|Z_t(\omega)\| \leq \delta$ for every $t \in [0, m]$. If $t \in [0, m]$, let $k < 2^n m$ be such that $2^{-n}k \leq t \leq 2^{-n}(k + 1)$. Then

$$\begin{aligned}\|\omega(t) - \omega_0(t)\| &\leq \|\omega(t) - \omega(2^{-n}k)\| + \|\omega(2^{-n}k) - \omega_0(2^{-n}k)\| + \|\omega_0(2^{-n}k) - \omega_0(t)\| \\ &\leq \|Z_t(\omega)\| + \|Y_t(\omega) - Y_{2^{-n}k}(\omega)\| + 2\delta \\ &\leq \delta + \|\omega(2^{-n}(k + 1)) - \omega(2^{-n}k)\| + 2\delta \\ &\leq \|\omega(2^{-n}(k + 1)) - \omega_0(2^{-n}(k + 1))\| + \|\omega_0(2^{-n}(k + 1)) - \omega_0(2^{-n}k)\| \\ &\qquad + \|\omega_0(2^{-n}k) - \omega(2^{-n}k)\| + 3\delta \\ &\leq 6\delta.\end{aligned}$$

As t is arbitrary, $\omega \in V$. **Q**

Accordingly

$$\mu_W G \geq \mu_W(E \cap F \cap \Omega) = \hat{\mu}^r(E \cap F) > 0.$$

As G is arbitrary, μ_W is strictly positive.

477G The strong Markov property With the identification in 477Dd, we are ready for one of the most important properties of Brownian motion.

Theorem Suppose that $r \geq 1$, μ_W is Wiener measure on $\Omega = C([0,\infty[\,;\mathbb{R}^r)_0$ and Σ is its domain. For $t \geq 0$ let Σ_t be

$$\{F : F \in \Sigma,\ \omega' \in F \text{ whenever } \omega \in F,\ \omega' \in \Omega \text{ and } \omega'\restriction[0,t] = \omega\restriction[0,t]\},$$

$$\Sigma_t^+ = \bigcap_{s>t} \Sigma_s,$$

and let $\tau : \Omega \to [0,\infty]$ be a stopping time adapted to the family $\langle \Sigma_t^+ \rangle_{t\geq 0}$. Define $\phi_\tau : \Omega \times \Omega \to \Omega$ by saying that

$$\begin{aligned} \phi_\tau(\omega,\omega')(t) &= \omega(t) \text{ if } t \leq \tau(\omega), \\ &= \omega(\tau(\omega)) + \omega'(t - \tau(\omega)) \text{ if } t \geq \tau(\omega). \end{aligned}$$

Then ϕ_τ is inverse-measure-preserving for $\mu_W \times \mu_W$ and μ_W.

proof (a) At this point I apply the general theory of §455 in something like its full strength. As in 477Dd, let $\langle \lambda_t \rangle_{t>0}$ be the standard family of Gaussian distributions on $\mathbb{R}^r$, $\ddot{\nu}$ the corresponding measure on the space C_{dlg} of càdlàg functions from $[0,\infty[$ to $\mathbb{R}^r$, and $\ddot{\Sigma}$ its domain; then $\ddot{\nu}\Omega = 1$ and μ_W is the subspace measure on Ω induced by $\ddot{\nu}$. As in 455U, let $\ddot{\Sigma}_t$ be

$$\{F : F \in \ddot{\Sigma},\ \omega' \in F \text{ whenever } \omega \in F,\ \omega' \in C_{\mathrm{dlg}} \text{ and } \omega'\restriction[0,t] = \omega\restriction[0,t]\},$$

and $\ddot{\Sigma}_t^+ = \bigcap_{s>t} \ddot{\Sigma}_s$, for $t > 0$.

(b) For $\omega \in C_{\mathrm{dlg}}$ set

$$\ddot{\tau}(\omega) = \inf\{t : \text{there is an } \omega' \in \Omega \text{ such that } \omega'\restriction[0,t] = \omega\restriction[0,t] \text{ and } \tau(\omega') \leq t\},$$

counting $\inf\emptyset$ as ∞. Then $\ddot{\tau}$ is a stopping time adapted to $\langle \ddot{\Sigma}_t^+ \rangle_{t\geq 0}$. **P** For $t \geq 0$, set

$$F_t = \{\omega : \omega \in \Omega,\ \tau(\omega) < t\} \in \Sigma_t$$

(455Lb), and

$$F'_t = \{\omega : \omega \in C_{\mathrm{dlg}}, \text{ there is an } \omega' \in F_t \text{ such that } \omega\restriction[0,t] = \omega'\restriction[0,t]\}.$$

Since $F_t \in \Sigma_t$, $F'_t \cap \Omega = F_t$; as Ω is $\ddot{\nu}$-conegligible and $\ddot{\nu}$ is complete, $F'_t \in \ddot{\Sigma}$; now of course $F'_t \in \ddot{\Sigma}_t$. If $\omega \in F'_t$, let $\omega' \in F_t$ be such that $\omega\restriction[0,t] = \omega'\restriction[0,t]$; then ω' witnesses that $\ddot{\tau}(\omega) \leq \tau(\omega') < t$. If $\omega \in C_{\mathrm{dlg}}$ and $\ddot{\tau}(\omega) < t$, let $q < t$ and $\omega' \in \Omega$ be such that q is rational, $\omega\restriction[0,q] = \omega'\restriction[0,q]$ and $\tau(\omega') < q$; then

$$\omega \in F'_q \in \ddot{\Sigma}_q \subseteq \ddot{\Sigma}_t.$$

This shows that

$$\{\omega : \omega \in C_{\mathrm{dlg}},\ \ddot{\tau}(\omega) < t\} = \bigcup_{q\in[0,t]\cap\mathbb{Q}} F'_q \in \ddot{\Sigma}_t.$$

By 455Lb in the other direction, $\ddot{\tau}$ is a stopping time adapted to $\langle \ddot{\Sigma}_t^+ \rangle_{t\geq 0}$. **Q**

(c) By 455U, $\ddot{\phi} : C_{\mathrm{dlg}} \times C_{\mathrm{dlg}} \to C_{\mathrm{dlg}}$ is inverse-measure-preserving for $\ddot{\nu} \times \ddot{\nu}$ and $\ddot{\nu}$, where

$$\begin{aligned} \ddot{\phi}(\omega,\omega')(t) &= \omega(t) \text{ if } t < \ddot{\tau}(\omega), \\ &= \omega(\ddot{\tau}(\omega)) + \omega'(t - \ddot{\tau}(\omega)) \text{ if } t \geq \ddot{\tau}(\omega). \end{aligned}$$

Now $\phi_\tau = \ddot{\phi}\restriction\Omega \times \Omega$ and $\mu_W \times \mu_W$ is the subspace measure on $\Omega \times \Omega$ induced by $\ddot{\nu} \times \ddot{\nu}$ (251Q), so ϕ_τ also is inverse-measure-preserving.

477H Some families of σ-algebras The σ-algebras considered in Theorem 477G can be looked at in other ways which are sometimes useful.

Proposition Let $r \geq 1$ be an integer, μ_W r-dimensional Wiener measure on $\Omega = C([0,\infty[\,;\mathbb{R}^r)_0$ and Σ its domain. Set $X_t^{(i)}(\omega) = \omega(t)(i)$ for $t \geq 0$ and $i < r$. For $I \subseteq [0,\infty[$, let T_I be the σ-algebra of subsets of Ω generated by $\{X_s^{(i)} - X_t^{(i)} : s,\, t \in I,\, i < r\}$, and $\hat{\mathrm{T}}_I$ the σ-algebra $\{E \triangle F : E \in \mathrm{T}_I,\, \mu_W F = 0\}$.

(a) $\mathrm{T}_{[0,\infty[}$ is the Borel σ-algebra of Ω either for the topology of pointwise convergence inherited from $(\mathbb{R}^r)^{[0,\infty[}$ or $\mathbb{R}^{r\times[0,\infty[}$, or for the topology of uniform convergence on compact sets.

(b) If $\mathcal{I}$ is a family of subsets of $[0,\infty[$ such that for all distinct I, $J \in \mathcal{I}$ either $\sup I \leq \inf J$ or $\sup J \leq \inf I$ (counting $\inf\emptyset$ as ∞ and $\sup\emptyset$ as 0), then $\langle\hat{\mathrm{T}}_I\rangle_{I\in\mathcal{I}}$ is an independent family of σ-algebras.

(c) For $t \geq 0$, let Σ_t be the σ-algebra of sets $F \in \Sigma$ such that $\omega' \in F$ whenever $\omega \in F$, $\omega' \in \Omega$ and $\omega'\restriction[0,t] = \omega\restriction[0,t]$, and $\Sigma_t^+ = \bigcap_{s>t}\Sigma_s$. Write $\hat{\mathrm{T}}_{[0,t]}^+$ for $\bigcap_{s>t}\hat{\mathrm{T}}_{[0,s]}$. Then, for any $t \geq 0$,

$$\mathrm{T}_{[0,t]} \subseteq \Sigma_t \subseteq \Sigma_t^+ \subseteq \hat{\mathrm{T}}_{[0,t]}^+ = \hat{\mathrm{T}}_{[0,t]} = \hat{\mathrm{T}}_{[0,t[}.$$

(d) On the tail σ-algebra $\bigcap_{t\geq0}\hat{\mathrm{T}}_{[t,\infty[}$, μ_W takes only the values 0 and 1.

proof (a) Write $\mathcal{B}(\Omega,\mathfrak{T}_p)$, $\mathcal{B}(\Omega,\mathfrak{T}_c)$ for the Borel algebras under the topologies $\mathfrak{T}_p$, $\mathfrak{T}_c$ of pointwise convergence and uniform convergence on compact sets. Then $\mathrm{T}_{[0,\infty[} \subseteq \mathcal{B}(\Omega,\mathfrak{T}_p)$ because the functionals $X_t^{(i)}$ are all $\mathfrak{T}_p$-continuous, and $\mathcal{B}(\Omega,\mathfrak{T}_p) \subseteq \mathcal{B}(\Omega,\mathfrak{T}_c)$ because $\mathfrak{T}_p \subseteq \mathfrak{T}_c$.

Now $\mathrm{T}_{[0,\infty[}$ includes a base for $\mathfrak{T}_c$. **P** Suppose that $\omega \in \Omega$, $n \in \mathbb{N}$ and $V = \{\omega' : \omega' \in \Omega,\, \sup_{t\in[0,n]}\|\omega'(t) - \omega(t)\| < 2^{-n}\}$. Then (because every member of Ω is continuous)

$$V = \bigcup_{m\in\mathbb{N}}\bigcap_{q\in\mathbb{Q}\cap[0,n]}\{\omega' : \sum_{i=0}^{r-1}|X_q^{(i)}(\omega') - X_q^{(i)}(\omega)|^2 \leq 2^{-2n} - 2^{-m}\}$$

belongs to $\mathrm{T}_{[0,\infty[}$; but such sets form a base for $\mathfrak{T}_c$. **Q**

Since $\mathfrak{T}_c$ is separable and metrizable, $\mathcal{B}(\Omega,\mathfrak{T}_c) \subseteq \mathrm{T}_{[0,\infty[}$ (4A3Da).

(b)(i) Suppose to begin with that $\mathcal{I}$ is finite and every member of $\mathcal{I}$ is finite. If we enumerate $\{0\}\cup\bigcup\mathcal{I}$ in ascending order as $\langle t_j\rangle_{j\leq n}$, $\langle X_{t_{j+1}}^{(i)} - X_{t_j}^{(i)}\rangle_{j<n,i<r}$ is an independent family of real-valued random variables. Taking $J_I = \{t_j : j < n,\, t_j \in I,\, t_{j+1} \in I\}$ for $I \in \mathcal{I}$, $\{X_{t_{j+1}}^{(i)} - X_{t_j}^{(i)} : j \in J_I\}$ generates T_I for each $I \in \mathcal{I}$, because of the separation property of the members of $\mathcal{I}$, and $\langle J_I\rangle_{I\in\mathcal{I}}$ is disjoint. By 272K, $\langle\mathrm{T}_I\rangle_{I\in\mathcal{I}}$ is independent.

(ii) Now suppose only that $\mathcal{I}$ is finite and not empty. For $I \in \mathcal{I}$, set $\mathrm{T}'_I = \bigcup_{J\subseteq I \text{ is finite}}\mathrm{T}_J$ for $I \in \mathcal{I}$; then T'_I is an algebra of sets, and T_I is the σ-algebra generated by T'_I. If $E_I \in \mathrm{T}'_I$ for $I \in \mathcal{I}$, there are $J_I \in [I]^{<\omega}$ such that $E_I \in \mathrm{T}_{J_I}$ for $I \in \mathcal{I}$, so $\mu_W(\bigcap_{I\in\mathcal{I}}E_I) = \prod_{I\in\mathcal{I}}\mu_W(E_I)$, by ($\alpha$). Inducing on n, and using the Monotone Class Theorem for the inductive step, we see that $\mu_W(\bigcap_{I\in\mathcal{I}}E_I) = \prod_{I\in\mathcal{I}}\mu_W(E_I)$ whenever $E_I \in \mathrm{T}_I$ for every $I \in \mathcal{I}$ and $\#(\{I : E_I \notin \mathrm{T}'_I\}) \leq n$. At the end of the induction, with $n = \#(\mathcal{I})$, we have $\mu_W(\bigcap_{I\in\mathcal{I}}E_I) = \prod_{I\in\mathcal{I}}\mu_W(E_I)$ whenever $E_I \in \mathrm{T}_I$ for every $I \in \mathcal{I}$; that is, $\langle\mathrm{T}_I\rangle_{I\in\mathcal{I}}$ is independent.

(iii) Thus $\langle\mathrm{T}_I\rangle_{I\in\mathcal{J}}$ is independent for every non-empty finite $\mathcal{J} \subseteq \mathcal{I}$, and $\langle\mathrm{T}_I\rangle_{I\in\mathcal{I}}$ is independent (272Bb).

(c)(i) If s, $s' \leq t$ and $i < r$ then $X_s^{(i)}$, $X_{s'}^{(i)}$ and $X_s^{(i)} - X_{s'}^{(i)}$ are Σ_t-measurable, so $\mathrm{T}_{[0,t]} \subseteq \Sigma_t$. Of course $\Sigma_t \subseteq \Sigma_t^+$.

(ii) $\Sigma_t \subseteq \hat{\mathrm{T}}_{[0,t]}$. **P** Suppose that $F \in \Sigma_t$. Set $D = [0,t]\cap(\mathbb{Q}\cup\{t\})$, and set $g(\omega) = \omega\restriction D$ for $\omega \in \Omega$; then $g : \Omega \to (\mathbb{R}^r)^D$ is continuous (when Ω is given the topology of pointwise convergence inherited from $(\mathbb{R}^r)^{[0,\infty[}$, for definiteness), and $F = g^{-1}[g[F]]$. Now the Borel σ-algebra of $(\mathbb{R}^r)^D \cong \mathbb{R}^{r\times D}$ is the σ-algebra generated by the functionals $\omega \mapsto \omega(t)(i) : (\mathbb{R}^r)^D \to \mathbb{R}$ for $t \in D$ and $i < r$ (4A3D(c-i)), and for such t and i, $\omega \mapsto g(\omega)(t)(i) = X_t^{(i)}(\omega)$ is $\mathrm{T}_{[0,t]}$-measurable; so g is $\mathrm{T}_{[0,t]}$-measurable. Now there is a sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ of compact subsets of F such that $\sup_{n\in\mathbb{N}}\mu_W K_n = \mu_W F$. In this case, $g[K_n] \subseteq (\mathbb{R}^r)^D$ is compact and $K'_n = g^{-1}[g[K_n]]$ belongs to $\mathrm{T}_{[0,t]}$, for each n. So $F' = \bigcup_{n\in\mathbb{N}}K'_n$ belongs to $\mathrm{T}_{[0,t]}$, and $\mu_W(F\setminus F') = 0$.

Similarly, applying the same argument to $\Omega\setminus F$, we have an $F'' \in \mathrm{T}_{[0,t]}$ such that $F'' \supseteq F$ and $\mu_W(F''\setminus F) = 0$. So $F \in \hat{\mathrm{T}}_{[0,t]}$. **Q**

Consequently $\Sigma_t^+ \subseteq \hat{\mathrm{T}}_{[0,t]}^+$.

(iii)(α) Let $\mathcal{A}$ be the family of those sets $G \in \Sigma$ such that χG has a conditional expectation on $\hat{\mathrm{T}}^+_{[0,t]}$ which is $\mathrm{T}_{[0,t]}$-measurable. Then $\mathcal{A}$ is a Dynkin class (definition: 136A). If $E \in \mathrm{T}_{[0,t]}$ and $F \in \mathrm{T}_{[s,\infty[}$ where $s > t$, then $(\mu_W F)\chi\Omega$ is a conditional expectation of χF on $\hat{\mathrm{T}}^+_{[0,t]}$, because $\hat{\mathrm{T}}^+_{[0,t]} \subseteq \hat{\mathrm{T}}_{[0,s]}$ and $\mathrm{T}_{[s,\infty[}$ are independent. As $E \in \hat{\mathrm{T}}^+_{[0,t]}$, $(\mu_W F)\chi E$ is a conditional expectation of $\chi(E \cap F)$ on $\hat{\mathrm{T}}^+_{[0,t]}$ (233Eg), and $E \cap F \in \mathcal{A}$. Since $\mathcal{E} = \{E \cap F : E \in \Sigma_t,\, F \in \bigcup_{s>t} \mathrm{T}_{[s,\infty[}\}$ is closed under finite intersections, the Monotone Class Theorem, in the form 136B, shows that $\mathcal{A}$ includes the σ-algebra T generated by $\mathcal{E}$; note that T includes $\mathrm{T}_{[0,t]} \cup \mathrm{T}_{[s,\infty[}$ whenever $s > t$. Now $X^{(i)}_u - X^{(i)}_s$ is T-measurable whenever $0 \le s \le u$. **P** If $u \le t$, $X^{(i)}_u - X^{(i)}_s$ is $\mathrm{T}_{[0,t]}$-measurable, therefore T-measurable; if $t < s$, $X^{(i)}_u - X^{(i)}_s$ is $\mathrm{T}_{[s,\infty[}$-measurable, therefore T-measurable. If $s \le t < u$, let $\langle t_n \rangle_{n\in\mathbb{N}}$ be a sequence in $]t, u]$ with limit t. Then

$$X^{(i)}_u - X^{(i)}_s = \lim_{n\to\infty}(X^{(i)}_u - X^{(i)}_{t_n}) + (X^{(i)}_t - X^{(i)}_s)$$

is T-measurable. **Q**

(β) This means that T includes $\mathrm{T}_{[0,\infty[}$. It follows that $\mathcal{A} = \Sigma$, because for any $G \in \Sigma$ there is a $G' \in \mathrm{T}_{[0,\infty[}$ such that $G \triangle G'$ is negligible, and now χG and $\chi G'$ have the same conditional expectations. So $\hat{\mathrm{T}}^+_{[0,t]} = \hat{\mathrm{T}}_{[0,t]}$. **P** Of course $\hat{\mathrm{T}}^+_{[0,t]} \supseteq \hat{\mathrm{T}}_{[0,t]}$. If $H \in \hat{\mathrm{T}}^+_{[0,t]}$ there is a $\mathrm{T}_{[0,t]}$-measurable function g which is a conditional expectation of χH on $\hat{\mathrm{T}}^+_{[0,t]}$. But in this case $g =_{\text{a.e.}} \chi H$, so, setting $E = \{\omega : g(\omega) = 1\} \in \mathrm{T}_{[0,t]}$, $E \triangle H$ is negligible and $H \in \hat{\mathrm{T}}_{[0,t]}$. Thus $\hat{\mathrm{T}}^+_{[0,t]} \subseteq \hat{\mathrm{T}}_{[0,t]}$. **Q**

(γ) Observe next that $X^{(i)}_t$ is $\mathrm{T}_{[0,t[}$-measurable for $i < r$. **P** If $t = 0$ then $X^{(i)}_t$ is the constant function with value 0. Otherwise, there is a strictly increasing sequence $\langle s_n \rangle_{n\in\mathbb{N}}$ in $[0, t[$ with limit t, so that $X^{(i)}_t = \lim_{n\to\infty} X^{(i)}_{s_n}$ is the limit of a sequence of $\mathrm{T}_{[0,t[}$-measurable functions and is itself $\mathrm{T}_{[0,t[}$-measurable. **Q** But this means that $\mathrm{T}_{[0,t]} = \mathrm{T}_{[0,t[}$, so $\hat{\mathrm{T}}^+_{[0,t]} \subseteq \hat{\mathrm{T}}_{[0,t]} = \hat{\mathrm{T}}_{[0,t[}$. In the other direction, of course $\mathrm{T}_{[0,t[} \subseteq \mathrm{T}^+_{[0,t]}$ and $\hat{\mathrm{T}}_{[0,t[} \subseteq \hat{\mathrm{T}}^+_{[0,t]}$, so we have equality.

(d) Set $\mathrm{T}' = \bigcup_{t\ge 0} \mathrm{T}_{[0,t]}$. If $E \in \bigcap_{t\ge 0} \hat{\mathrm{T}}_{[t,\infty[}$, then $\mu_W(E \cap F) = \mu_W E \cdot \mu_W F$ for every $F \in \mathrm{T}'$. By the Monotone Class Theorem again, $\mu_W(E \cap F) = \mu_W E \cdot \mu_W F$ for every F in the σ-algebra generated by T', which is $\mathcal{B}(\Omega)$, by (a). Now $\mu_W \llcorner E$ (definition: 234M[4]) and $(\mu_W E)\mu_W$ are Radon measures on Ω (416Sa) which agree on $\mathcal{B}(\Omega)$, so must be identical. In particular,

$$\mu_W E = (\mu_W \llcorner E)(E) = (\mu_W E)^2$$

and $\mu_W E$ must be either 0 or 1.

477I Hitting times In 455M I introduced 'hitting times'. I give a paragraph now to these in the special case of Brownian motion; such stopping times will dominate the applications of the theory in §§478-479. Take $r \ge 1$, and let μ_W be Wiener measure on $\Omega = C([0,\infty[;\mathbb{R}^r)_0$ and Σ its domain; for $t \ge 0$ define Σ^+_t and $\mathrm{T}_{[0,t]}$ as in 477G and 477H. Give Ω its topology of uniform convergence on compact sets.

(a) Suppose that $A \subseteq \mathbb{R}^r$. For $\omega \in \Omega$ set $\tau(\omega) = \inf\{t : t \in [0,\infty[,\, \omega(t) \in A\}$, counting $\inf\emptyset$ as ∞. I will call τ the **Brownian hitting time** to A, or the **Brownian exit time** from $\mathbb{R}^r \setminus A$. I will say that the **Brownian hitting probability** of A, or the **Brownian exit probability** of $\mathbb{R}^r \setminus A$, is $\mathrm{hp}(A) = \mu_W\{\omega : \tau(\omega) < \infty\}$ if this is defined. More generally, I will write

$$\mathrm{hp}^*(A) = \mu^*_W\{\omega : \tau(\omega) < \infty\} = \mu^*_W\{\omega : \omega^{-1}[A] \ne \emptyset\},$$

the **outer Brownian hitting probability**, for any $A \subseteq \mathbb{R}^r$.

(b) If $A \subseteq \mathbb{R}^r$ is analytic, the Brownian hitting time to A is a stopping time adapted to the family $\langle \Sigma^+_t \rangle_{t\ge 0}$. **P** Let C_{dlg} be the space of càdlàg functions from $[0,\infty[$ to $\mathbb{R}^r$, and define $\ddot{\Sigma}$ as in the proof of 477G; let $\ddot{\tau}$ be the hitting time on C_{dlg} defined by A. By 455Ma, $\ddot{\tau}$ is $\ddot{\Sigma}$-measurable, so $\tau = \ddot{\tau}\restriction\Omega$ is Σ-measurable. Now (as in 455Mb) $\{\omega : \tau(\omega) < t\} \in \Sigma_t$ for every t, so τ is adapted to $\langle \Sigma^+_t \rangle_{t\ge 0}$. **Q**

In particular, there is a well-defined Brownian hitting probability of A.

[4]Formerly 234E.

(c) Let $F \subseteq \mathbb{R}^r$ be a closed set, and τ the Brownian hitting time to F.

(i) If $\tau(\omega) < \infty$, then

$$\tau(\omega) = \inf \omega^{-1}[F] = \min \omega^{-1}[F]$$

because ω is continuous. If $0 \notin F$ and $\tau(\omega) < \infty$, then $\omega(\tau(\omega)) \in \partial F$.

(ii) τ is lower semi-continuous. **P** For any $t \in [0, \infty[$,

$$\{\omega : \tau(\omega) > t\} = \{\omega : \omega(s) \notin F \text{ for every } s \le t\}$$

is open in Ω. **Q**

(iii) τ is adapted to $\langle \mathrm{T}_{[0,t]} \rangle_{t \ge 0}$. **P** Let $\langle G_n \rangle_{n \in \mathbb{N}}$ be a non-increasing sequence of open sets including F such that $F = \bigcap_{n \in \mathbb{N}} \overline{G}_n$. Then, for $\omega \in \Omega$ and $t > 0$,

$$\tau(\omega) \le t \iff \omega[\,[0,t]\,] \cap F \ne \emptyset$$

(because ω is continuous)

$$\iff \omega[\,[0,t]\,] \cap G_n \ne \emptyset \text{ for every } n \in \mathbb{N}$$

(because $\omega[\,[0,t]\,]$ is compact)

$$\iff \quad \text{for every } n \in \mathbb{N} \text{ there is a rational } q \le t \text{ such that } \omega(q) \in G_n.$$

So

$$\{\omega : \tau(\omega) \le t\} = \bigcap_{n \in \mathbb{N}} \bigcup_{q \in \mathbb{Q} \cap [0,t]} \{\omega : \omega(q) \in G_n\} \in \mathrm{T}_{[0,t]}.$$

Of course $\{\omega : \tau(\omega) = 0\}$ is either Ω (if $0 \in F$) or $\emptyset$ (if $0 \notin F$), so belongs to $\mathrm{T}_{[0,0]}$. **Q**

In the language of 477G, we have $\mathrm{T}_{[0,t]} \subseteq \Sigma_t$ for every $t \ge 0$ (477Hc), so τ must also be adapted to $\langle \Sigma_t \rangle_{t \ge 0}$.

(d) If $A \subseteq \mathbb{R}^r$ is any set, then

$$\mathrm{hp}^*(A) = \min\{\mathrm{hp}(B) : B \supseteq A \text{ is an analytic set}\} = \min\{\mathrm{hp}(E) : E \supseteq A \text{ is a } \mathrm{G}_\delta \text{ set}\}.$$

P Of course

$$\begin{aligned} \mathrm{hp}^*(A) &\le \inf\{\mathrm{hp}(B) : B \supseteq A \text{ is an analytic set}\} \\ &= \min\{\mathrm{hp}(B) : B \supseteq A \text{ is an analytic set}\} \\ &\le \inf\{\mathrm{hp}(E) : E \supseteq A \text{ is a } \mathrm{G}_\delta \text{ set}\} = \min\{\mathrm{hp}(E) : E \supseteq A \text{ is a } \mathrm{G}_\delta \text{ set}\} \end{aligned}$$

just because hp^* is an order-preserving function. If $\gamma > \mathrm{hp}^*(A)$, there is a compact $K \subseteq \Omega$ such that $\omega^{-1}[A] = \emptyset$ for every $\omega \in K$ and $\mu_W K \ge 1 - \gamma$. Now $F = \{\omega(t) : \omega \in K, \, t \in [0, \infty[\}$ is a K_σ set not meeting A, so $E = \mathbb{R}^r \setminus F$ is a G_δ set including A. Since $\omega^{-1}[E]$ is empty for every $\omega \in K$, $\mathrm{hp}(E) \le \mu_W(\Omega \setminus K) \le \gamma$. As γ is arbitrary,

$$\inf\{\mathrm{hp}(E) : E \supseteq A \text{ is a } \mathrm{G}_\delta \text{ set}\} \le \mathrm{hp}^*(A)$$

and we have equality throughout. **Q**

(e) If $A \subseteq \mathbb{R}^r$ is analytic, then $\mathrm{hp}(A) = \sup\{\mathrm{hp}(K) : K \subseteq A \text{ is compact}\}$. **P** Suppose that $\gamma < \mathrm{hp}(A)$. Set $E = \{(\omega, t) : \omega \in \Omega, \, t \ge 0, \, \omega(t) \in A\}$. Then E is analytic and $\mathrm{hp}(A) = \mu_W \pi_1[E]$, where $\pi_1(\omega, t) = \omega$ for $(\omega, t) \in E$. Let λ be the subspace measure $(\mu_W)_{\pi_1[E]}$. By 433D, there is a Radon measure λ' on E such that $\lambda = \lambda' \pi_1^{-1}$. Then $\lambda' E = \mathrm{hp}(A) > \gamma$, so there is a compact set $L \subseteq E$ such that $\lambda' L \ge \gamma$. Set $K = \{\omega(t) : (\omega, t) \in L\}$; then $K \subseteq \mathbb{R}^r$ is compact, and

$$\mathrm{hp}(K) = \mu_W\{\omega : \omega(t) \in K \text{ for some } t \ge 0\} \ge \mu_W \pi_1[L] \ge \lambda' L \ge \gamma.$$

As γ is arbitrary, $\mathrm{hp}(A) \le \sup\{\mathrm{hp}(K) : K \subseteq A \text{ is compact}\}$; the reverse inequality is trivial. **Q**

Remark 477Id-477Ie are characteristic of Choquet capacities (432J-432L); see 478Xe below.

477J As an example of the use of 477G, I give a classical result on one-dimensional Brownian motion.

Proposition Let μ_W be Wiener measure on $\Omega = C([0,\infty[)_0$. Set $X_t(\omega) = \omega(t)$ for $\omega \in \Omega$. Then

$$\Pr(\max_{s\le t} X_s \ge \alpha) = 2\Pr(X_t \ge \alpha) = \frac{2}{\sqrt{2\pi}}\int_{\alpha/\sqrt{t}}^{\infty} e^{-u^2/2}du$$

whenever $t > 0$ and $\alpha \ge 0$.

proof Let τ be the Brownian hitting time to $F = \{x : x \in \mathbb{R},\, x \ge \alpha\}$; because F is closed, τ is a stopping time adapted to $\langle\Sigma_t\rangle_{t\ge0}$, as in 477Ic. Let $\phi_\tau : \Omega\times\Omega \to \Omega$ be the corresponding inverse-measure-preserving function as in 477G, and set $E = \{\omega : \tau(\omega) < t\}$. Note that as $\omega(\tau(\omega)) = \alpha$ whenever $\tau(\omega)$ is finite, $\Pr(\tau = t) \le \Pr(X_t = \alpha) = 0$, and

$$\mu_W E = \Pr(\tau \le t) = \Pr(\max_{s\le t} X_s \ge \alpha).$$

Now

$$\Pr(X_t \ge \alpha) = \mu_W\{\omega : \omega(t) \ge \alpha\} = \mu_W^2\{(\omega,\omega') : \phi_\tau(\omega,\omega')(t) \ge \alpha\}$$
$$= \mu_W^2\{(\omega,\omega') : \tau(\omega) \le t,\, \phi_\tau(\omega,\omega')(t) \ge \alpha\}$$

(because if $\tau(\omega) > t$ then $\phi_\tau(\omega,\omega')(t) = \omega(t) < \alpha$)

$$= \mu_W^2\{(\omega,\omega') : \tau(\omega) < t,\, \phi_\tau(\omega,\omega')(t) \ge \alpha\}$$

(because $\{\omega : \tau(\omega) = t\}$ is negligible)

$$= \mu_W^2\{(\omega,\omega') : \tau(\omega) < t,\, \omega(\tau(\omega)) + \omega'(t-\tau(\omega)) \ge \alpha\}$$
$$= \mu_W^2\{(\omega,\omega') : \tau(\omega) < t,\, \omega'(t-\tau(\omega)) \ge 0\}$$
$$= \int_E \mu_W\{\omega' : \omega'(t-\tau(\omega)) \ge 0\}\mu_W(d\omega)$$
$$= \frac{1}{2}\mu_W E = \frac{1}{2}\Pr(\max_{s\le t} X_s \ge \alpha).$$

To compute the value, observe that X_t has the same distribution as $\sqrt{t}Z$ where Z is a standard normal random variable, so that

$$\Pr(X_t \ge \alpha) = \Pr(Z \ge \tfrac{\alpha}{\sqrt{t}}) = \frac{1}{\sqrt{2\pi}}\int_{\alpha/\sqrt{t}}^{\infty} e^{-u^2/2}du.$$

477K Typical Brownian paths A vast amount is known concerning the nature of 'typical' members of Ω; that is to say, a great many interesting μ_W-conegligible sets have been found. Here I will give only a couple of basic results; the first because it is essential to any picture of Brownian motion, and the second because it is relevant to a question in §479. Others are in 478M, 478Yi and 479R.

Proposition Let μ_W be one-dimensional Wiener measure on $\Omega = C([0,\infty[)_0$. Then μ_W-almost every element of Ω is nowhere differentiable.

proof Note first that if $\eta > 0$ and Z is a standard normal random variable, then $\Pr(|Z| \le \eta) \le \eta$, because the maximum value of the probability density function of Z is $\frac{1}{\sqrt{2\pi}} \le \frac{1}{2}$. For m, n, $k \in \mathbb{N}$, set

$$F_m = \{\omega : \omega \in \Omega \text{ and there is a } t \in [0,m[\text{ such that } \limsup_{s\downarrow0} \tfrac{|\omega(s)-\omega(t)|}{s-t} < m\},$$

$$E_{mnk} = \{\omega : \omega \in \Omega,\, |\omega(2^{-n}(k+2)) - \omega(2^{-n}(k+1))| \le 3\cdot2^{-n}m,$$
$$|\omega(2^{-n}(k+3)) - \omega(2^{-n}(k+2))| \le 5\cdot2^{-n}m,$$
$$|\omega(2^{-n}(k+4)) - \omega(2^{-n}(k+3))| \le 7\cdot2^{-n}m\},$$

$$E_{mn} = \bigcup_{k<2^n m} E_{mnk}.$$

Now we can estimate the measure of E_{mnk}, because for any α, $t \ge 0$, $2^{n/2}(X_{t+2^{-n}} - X_t)$ has a standard normal distribution (taking $X_t(\omega) = \omega(t)$, as usual), so

$$\Pr(|X_{t+2^{-n}} - X_t| \le \alpha) \le 2^{n/2}\alpha;$$

since E_{mnk} is the intersection of three independent sets of this type,

$$\mu_W E_{mnk} \le 2^{n/2}\cdot 3\cdot 2^{-n}m\cdot 2^{n/2}\cdot 5\cdot 2^{-n}m\cdot 2^{n/2}\cdot 7\cdot 2^{-n}m = 105\,m^3 2^{-3n/2}.$$

Accordingly

$$\mu_W E_{mn} \le \sum_{k<2^n m}\mu_W E_{mnk} \le 105\,m^4 2^{-n/2}.$$

Next, observe that $F_m \subseteq \bigcup_{l\in\mathbb{N}}\bigcap_{n\ge l} E_{mn}$. **P** If $\omega \in F_m$, let $t \in [0,m[$ be such that $\limsup_{s\downarrow 0}\frac{|\omega(s)-\omega(t)|}{s-t} < m$, and $l \in \mathbb{N}$ such that $|\omega(s)-\omega(t)| \le m(s-t)$ whenever $t < s \le t + 4\cdot 2^{-l}$. Take any $n \ge l$. Then there is a $k < 2^n m$ such that $2^{-n}k \le t < 2^{-n}(k+1)$. In this case,

$$|\omega(2^{-n}(k+j)) - \omega(t)| \le 2^{-n}jm$$

for $1 \le j \le 4$,

$$|\omega(2^{-n}(k+j+1)) - \omega(2^{-n}(k+j))| \le (2j+1)2^{-n}m$$

for $1 \le j \le 3$, and $\omega \in E_{mnk} \subseteq E_{mn}$. **Q**

Since

$$\mu_W(\bigcup_{l\in\mathbb{N}}\bigcap_{n\ge l} E_{mn}) \le \liminf_{n\to\infty}\mu_W E_{mn} = 0,$$

F_m is negligible. So $F = \bigcup_{m\in\mathbb{N}} F_m$ is negligible. But F includes any member of ω which is differentiable at any point of $]0,\infty[$, and more. So almost every path is nowhere differentiable.

477L Theorem Let $r \ge 1$ be an integer, and μ_W Wiener measure on $\Omega = C([0,\infty[;\mathbb{R}^r)_0$; for $s > 0$ let μ_{Hs} be s-dimensional Hausdorff measure on $\mathbb{R}^r$.

(a) (Taylor 53) $\{\omega(t) : t \in [0,\infty[\}$ is μ_{H2}-negligible for μ_W-almost every ω.

(b) Now suppose that $r \ge 2$. For $\omega \in \Omega$, let F_ω be the compact set $\{\omega(t) : t \in [0,1]\}$. Then for μ_W-almost every $\omega \in \Omega$, $\mu_{Hs}F_\omega = \infty$ for every $s \in]0,2[$.

proof (a)(i) For $0 \le s \le t$ and $\omega \in \Omega$ set $K_{st}(\omega) = \{\omega(u) : s \le u \le t\}$ and $d_{st}(\omega) = \operatorname{diam} K_{st}(\omega)$. Note that $d_{st} : \Omega \to [0,\infty[$ is continuous (for the topology of uniform convergence on compact sets, of course).

($\boldsymbol{\alpha}$) If $0 \le s \le t$ then $\mathbb{E}(d_{st}^2) \le 8r(t-s)$. **P** As $\langle X_{u+s} - X_s\rangle_{u\ge 0}$ and $\langle X_u\rangle_{u\ge 0}$ have the same distribution, d_{st} has the same distribution as $d_{0,t-s}$, and we may suppose that $s = 0$. [If you prefer: if $S_s : \Omega \to \Omega$ is the shift operator of 477Ec, $K_{st}(\omega) = \omega(s) + K_{0,t-s}(S_s\omega)$, so $d_{st}(\omega) = d_{0,t-s}(S_s\omega)$, while S_s is inverse-measure-preserving.] In this case,

$$d_{0t}(\omega)^2 \le 4\max_{s\in[0,t]}\|\omega(s)\|^2 \le 4\sum_{j=0}^{r-1}\max_{s\in[0,t]}\omega(s)(j)^2.$$

For each $j < r$,

$$\begin{aligned}\int \max_{s\in[0,t]}\omega(s)(j)^2\mu_W(d\omega) &= \int_0^\infty \mu_W\{\omega : \max_{s\in[0,t]}\omega(s)(j)^2 \ge \beta\}d\beta\\ &\le \int_0^\infty \mu_W\{\omega : \max_{s\in[0,t]}\omega(s)(j) \ge \sqrt{\beta}\}\\ &\qquad + \mu_W\{\omega : \min_{s\in[0,t]}\omega(s)(j) \le -\sqrt{\beta}\}d\beta\\ &= 2\int_0^\infty \mu_W\{\omega : \max_{s\in[0,t]}\omega(s)(j) \ge \sqrt{\beta}\}d\beta\end{aligned}$$

(because μ_W is invariant under reflections in $\mathbb{R}^r$, see 477Ed)

$$= 4\int_0^\infty \mu_W\{\omega : \omega(t)(j) \ge \sqrt{\beta}\}d\beta$$

(by 477J, applied to the jth coordinate projection of Ω onto $C([0,\infty[)_0$, which is inverse-measure-preserving, by 477Da or 477Ed and 477Eg)

$$= 2\int_0^\infty \mu_W\{\omega : \omega(t)(j)^2 \ge \beta\}d\beta$$

(again because μ_W is symmetric)

$$= 2\int_\Omega \omega(t)(j)^2\mu_W(d\omega) = 2\mathbb{E}(tZ^2)$$

(where Z is a standard normal random variable)

$$= 2t.$$

Summing,

$$\mathbb{E}(d_{0t}^2) \le 4\sum_{j=0}^{r-1}\int_\Omega \max_{s\in[0,t]}\omega(s)(j)^2\mu_W(d\omega) \le 8rt. \quad \mathbf{Q}$$

(β) For any $\epsilon > 0$, $\Pr(d_{01} \le \epsilon) > 0$. **P** $\{\omega : d_{01}(\omega) \le \epsilon\}$ is a neighbourhood of 0 for the topology of uniform convergence on compact sets, so has non-zero measure, by 477F. **Q**

(ii) For a non-empty finite set $I \subseteq [0,\infty[$ and $\omega \in \Omega$ set

$$g_I(\omega) = \sum_{j=0}^{n-1} d_{t_{j-1},t_j}(\omega)^2$$

where $\langle t_j\rangle_{j\le n}$ enumerates I in increasing order. For $0 \le s \le t$ and $\omega \in \Omega$ set

$$h_{st}(\omega) = \inf_{\{s,t\}\subseteq I\subseteq[s,t] \text{ is finite}} g_I(\omega);$$

then h_{st} is $\mathrm{T}_{[s,t]}$-measurable, in the language of 477H. **P** The point is that $I \mapsto g_I(\omega)$ is a continuous function of the members of I, at least if we restrict attention to sets I of a fixed size. So if D is any countable dense subset of $[s,t]$ containing s and t,

$$h_{st} = \inf_{\{s,t\}\subseteq I\subseteq D \text{ is finite}} g_I.$$

On the other hand, if $I \subseteq D$ is enumerated as $\langle t_i\rangle_{i\le n}$,

$$g_I(\omega) = \sum_{i=0}^{n-1} \max_{u,u'\in D\cap[t_i,t_{i+1}]} \|\omega(u) - \omega(u')\|^2,$$

so g_I is $\mathrm{T}_{[s,t]}$-measurable. **Q**

(iii) We need the following facts about the h_{st}.

(α) If $0 \le s \le t$, then the distribution of h_{st} is the same as the distribution of $h_{0,t-s}$, again because $\langle X_{s+u} - X_s\rangle_{u\ge0}$ has the same distribution as $\langle X_u\rangle_{u\ge0}$. [In the language suggested in the proof of (i-α), we have $g_{s+I}(\omega) = g_I(S_s(\omega))$ for any $\omega \in \Omega$ and non-empty finite $I \subseteq [0,\infty[$, so $h_{st}(\omega) = h_{0,t-s}(S_s\omega)$.]

(β) h_{0t} has finite expectation. **P** $h_{0t} \le g_{\{0,t\}} = d_{0t}^2$, so we can use (i). **Q**

(γ) $h_{su} \le h_{st} + h_{tu}$ if $s \le t \le u$. **P** If $\{s,t\} \subseteq I \subseteq [s,t]$ and $\{t,u\} \subseteq J \subseteq [t,u]$ then $\{s,u\} \subseteq I \cup J \subseteq [s,u]$ and $g_{I\cup J} = g_I + g_J$. **Q**

(δ) If $s \le t \le u$ then h_{st} and h_{tu} are independent, because $\mathrm{T}_{[s,t]}$ and $\mathrm{T}_{[t,u]}$ are independent (477H(b-i)).

(ϵ) The distribution of h_{0t} is the same as the distribution of th_{01} whenever $t \ge 0$. **P** The case $t = 0$ is trivial. For $t > 0$, define $U_t : \Omega \to \Omega$ by saying that $U_t(\omega)(s) = \frac{1}{\sqrt{t}}\omega(ts)$, as in 477Ee. Then

$$K_{su}(U_t(\omega)) = \frac{1}{\sqrt{t}}K_{ts,tu}(\omega), \quad d_{su}(U_t(\omega)) = \frac{1}{\sqrt{t}}d_{ts,tu}(\omega),$$

$$g_I(U_t(\omega)) = \frac{1}{t}g_{tI}(\omega), \quad h_{su}(U_t(\omega)) = \frac{1}{t}h_{ts,tu}(\omega),$$

whenever $s \le u$, $\{s,u\} \subseteq I \subseteq [s,u]$ and $\omega \in \Omega$, and

$$\mu_W\{\omega : th_{01}(\omega) \ge \alpha\} = \mu_W\{\omega : th_{01}(U_t(\omega)) \ge \alpha\}$$

(because U_t is an automorphism of (Ω, μ_W))

$$= \mu_W\{\omega : h_{0t}(\omega) \ge \alpha\}$$

for every $\alpha \in \mathbb{R}$. **Q**

(ζ) Consequently

$$\mathbb{E}(h_{st}) = \mathbb{E}(h_{0,t-s}) = (t-s)\mathbb{E}(h_{01})$$

whenever $s \le t$, and

$$\mathbb{E}(h_{st}) + \mathbb{E}(h_{tu}) = \mathbb{E}(h_{su})$$

whenever $s \le t \le u$. Since $h_{st} + h_{tu} \ge h_{su}$, by (iii), we must have $h_{st} + h_{tu} =_{\text{a.e.}} h_{su}$.

(η) For any $\eta > 0$, $\Pr(h_{01} \le 4\eta^2) > 0$. **P** By (i-$\beta$), $\Pr(d_{01} \le 2\eta) > 0$, and $h_{01} \le g_{\{0,1\}} = d_{01}^2$. **Q**

(iv) For $t \ge 0$ let φ_t be the characteristic function of h_{0t}, that is, $\varphi_t(\alpha) = \mathbb{E}(\exp(i\alpha h_{0t}))$ for $\alpha \in \mathbb{R}$ (285Ab). Working through the facts listed above, we see that

$$\varphi_t(1) = \mathbb{E}(\exp(ih_{0t})) = \mathbb{E}(\exp(ith_{01}))$$

(by (iii-ϵ))

$$= \varphi_1(t),$$

$$\varphi_1(s)\varphi_1(t) = \varphi_s(1)\varphi_t(1) = \mathbb{E}(\exp(ih_{0s}))\mathbb{E}(\exp(ih_{0t}))$$

$$= \mathbb{E}(\exp(ih_{0s}))\mathbb{E}(\exp(ih_{s,s+t}))$$

(by (iii-α))

$$= \mathbb{E}(\exp(ih_{0s})\exp(ih_{s,s+t}))$$

(because h_{0s} and $h_{s,s+t}$ are independent, by (c-iv))

$$= \mathbb{E}(\exp(i(h_{0s} + h_{s,s+t}))) = \mathbb{E}(ih_{0,s+t})$$

(by (iii-ζ))

$$= \varphi_1(s+t),$$

for all s, $t \ge 0$; while φ_1 is differentiable, because h_{01} has finite expectation ((iii-α) above and 285Fd). It follows that there is a $\gamma \in \mathbb{R}$ such that $\varphi_1(t) = e^{i\gamma t}$ for every $t \in \mathbb{R}$. **P** Set $\gamma = \frac{1}{i}\varphi_1'(0) = \mathbb{E}(h_{01})$ (285Fd) and $\psi(t) = e^{-i\gamma t}\varphi_1(t)$ for $t \in \mathbb{R}$. If $t > 0$, then

$$\varphi_1'(t) = \lim_{s\downarrow 0} \frac{1}{s}(\varphi_1(t+s) - \varphi_1(t)) = \varphi_1(t)\lim_{s\downarrow 0}\frac{1}{s}(\varphi_1(s) - 1) = i\varphi_1(t)\gamma$$

and $\psi'(t) = 0$. Since ψ is continuous on $[0,\infty[$ (285Fb), it must be constant, and $\varphi_1(t) = e^{i\gamma t}\psi(0) = e^{i\gamma t}$ for every $t \ge 0$. As for negative t, we have

$$\varphi_1(t) = \overline{\varphi_1(-t)} = \overline{e^{-i\gamma t}} = e^{i\gamma t}$$

for $t \le 0$, by 285Fc. **Q**

(v) Thus we see that h_{01} has the same characteristic function as the Dirac measure concentrated at γ, and this must therefore be the distribution of h_{01} (285M); that is, $h_{01} =_{\text{a.e.}} \gamma$. Now (iii-$\eta$) tells us that $\gamma = 0$.

Since h_{0t} has the same distribution as th_{01}, $h_{0t} =_{\text{a.e.}} 0$ for every $t \ge 0$. But now observe that if $t \ge 0$, $\omega \in \Omega$ and $h_{0t}(\omega) = 0$, then for any $\eta > 0$ there is a finite $I \subseteq [0,t]$, containing 0 and t, such that $g_I(\omega) \le \eta^2$. This means that $K_{0t}(\omega)$ can be covered by finitely many sets $K_{t_j,t_{j+1}}(\omega)$ with $\sum_{j=0}^{n-1} \operatorname{diam} K_{t_j,t_{j+1}}(\omega)^2 \le \eta^2$. All the diameters here must of course be less than or equal to η. As η is arbitrary, $\mu_{H2}K_{0t}(\omega) = 0$.

For each $t \geq 0$, this is true for almost every ω. But this means that, for almost every ω, $\mu_{H2}K_{0n}(\omega) = 0$ for every n, and $\mu_{H2}\{\omega(t) : t \geq 0\} = 0$, as claimed.

(b)(i) To begin with, take a fixed $s \in]0, 2[$. Let μ_{L1} be Lebesgue measure on $[0, 1]$. For each $\omega \in \Omega$, let ζ_ω be the image measure $\mu_{L1}(\omega\restriction[0, 1])^{-1}$ on F_ω. Then

$$\int_\Omega \int_{F_\omega} \int_{F_\omega} \frac{1}{\|x-y\|^s}\zeta_\omega(dx)\zeta_\omega(dy)\mu_W(d\omega)$$
$$= \int_\Omega \int_0^1 \int_0^1 \frac{1}{\|\omega(t)-\omega(u)\|^s} dt\,du\,\mu_W(d\omega)$$
$$= \int_0^1 \int_0^1 \int_\Omega \frac{1}{\|\omega(t)-\omega(u)\|^s}\mu_W(d\omega)\,dt\,du$$

(of course $(\omega, t, u) \mapsto \frac{1}{\|\omega(t)-\omega(u)\|^s}$ is continuous and non-negative, so there is no difficulty with the change in order of integration)

$$= 2\int_0^1 \int_u^1 \int_\Omega \frac{1}{\|\omega(t)-\omega(u)\|^s}\mu_W(d\omega)\,dt\,du$$
$$= 2\int_0^1 \int_u^1 \int_\Omega \frac{1}{\|\omega(t-u)\|^s}\mu_W(d\omega)\,dt\,du$$

(because $X_t - X_u$ has the same distribution as X_{t-u}, as in (a-i-α))

$$= 2\int_0^1 \int_0^{1-u} \int_\Omega \frac{1}{\|\omega(t)\|^s}\mu_W(d\omega)\,dt\,du \leq 2\int_0^1 \int_\Omega \frac{1}{\|\omega(t)\|^s}\mu_W(d\omega)\,dt$$
$$= 2\int_0^1 \int_{\mathbb{R}^r} \frac{1}{(\sqrt{2\pi t})^r}\frac{1}{\|x\|^s}e^{-\|x\|^2/2t}\mu(dx)\,dt$$

(here μ is Lebesgue measure on $\mathbb{R}^r$)

$$= \frac{2}{(\sqrt{2\pi})^r}\int_0^1 \frac{1}{t^{r/2}}\int_0^\infty \frac{1}{\alpha^s} \cdot r\beta_r\alpha^{r-1}e^{-\alpha^2/2t}d\alpha\,dt$$
$$= \frac{2r\beta_r}{(\sqrt{2\pi})^r}\int_0^1 \frac{1}{t^{r/2}}\int_0^\infty \frac{(\beta\sqrt{t})^{r-1}}{(\beta\sqrt{t})^s}e^{-\beta^2/2}\sqrt{t}d\beta\,dt$$
$$= \frac{2r\beta_r}{(\sqrt{2\pi})^r}\int_0^1 \frac{1}{t^{s/2}}dt\int_0^\infty \beta^{r-s-1}e^{-\beta^2/2}d\beta < \infty$$

because $\frac{s}{2} < 1$ and $r - s > 0$. So $\int_{F_\omega}\int_{F_\omega} \frac{1}{\|x-y\|^s}\zeta_\omega(dx)\zeta_\omega(dy)$ is finite for almost every ω. Since ζ_ω is always a probability measure with support included in F_ω, $\mu_{Hs}F_\omega = \infty$ for all such ω (471Tb).

(ii) Setting $s_n = 2 - 2^{-n}$ for each n, we see that, for almost every $\omega \in \Omega$, $\mu_{Hs_n}F_\omega = \infty$ for every n. But for any such ω, $\mu_{Hs}F_\omega = \infty$ for every $s \in]0, 2[$, by 471L.

477X Basic exercises (a) Use 272Yc[5] to simplify the formulae in the proof of 477B.

(b) Let μ_W be Wiener measure on $\Omega = C([0, \infty[)_0$, and set $X_t(\omega) = \omega(t)$ for $t \geq 0$ and $\omega \in \Omega$. Let f be a real-valued tempered function on $\mathbb{R}$ (definition: 284D). For $x \in \mathbb{R}$ and $0 < t < b$, let $\nu_x^{(t,b)}$ be the distribution of a normally distributed random variable with mean x and variance $b - t$, so that $g(x, t) = \int f(y)\nu_x^{(t,b)}(dy)$ can be regarded as the expectation of $f(X_b)$ given that $X_t = x$. (i) Show that g satisfies the **backwards heat equation** $2\frac{\partial g}{\partial t} + \frac{\partial^2 g}{\partial x^2} = 0$. (ii) Interpret this in terms of the disintegration $\nu_x^{(t,b)} = \int \nu_z^{(u,b)}\nu_x^{(t,u)}(dz)$ as $u \downarrow t$.

[5]Formerly 272Ye.

(c)(i) Show that the measure $\hat{\mu}^r$ of 477Da can be constructed directly by applying 455A with $(X_t, \mathcal{B}_t) = (\mathbb{R}^r, \mathcal{B}(\mathbb{R}^r))$ for every $t \ge 0$ and suitable Gaussian distributions $\nu_x^{(s,t)}$ on $\mathbb{R}^r$. (ii) Show that the measure $\hat{\mu}^r$ can be constructed by applying 455A to $T = r \times [0, \infty[$ with its lexicographic ordering and suitable Gaussian distributions $\nu_x^{(s,t)}$ on $\mathbb{R}$.

(d) Let $r \ge 1$ be an integer. (i) Show that there is a centered Gaussian process $\langle Y_t \rangle_{t\in[0,1]} = \langle Y_t^{(i)} \rangle_{t\in[0,1],i<r}$ such that $\mathbb{E}(Y_s^{(i)} \times Y_t^{(j)}) = 0$ if $i \ne j$, $\min(s,t) - st$ otherwise. (ii) Show that if $\langle X_t \rangle_{t\ge 0}$ is ordinary r-dimensional Brownian motion, then $\langle Y_t \rangle_{t\in[0,1]}$ has the same distribution as $\langle X_t - tX_1 \rangle_{t\in[0,1]}$. (iii) Show that the process $\langle Y_t \rangle_{t\in[0,1]}$ (the **Brownian bridge**) can be represented by a Radon probability measure μ_{bridge} on the space $C([0,1];\mathbb{R}^r)_{00}$ of continuous functions from $[0,1]$ to $\mathbb{R}^r$ taking the value 0 at both ends of the interval. (iv) For $\omega \in C([0,1];\mathbb{R}^r)_{00}$ define $\overleftrightarrow{\omega} \in C([0,1];\mathbb{R}^r)_{00}$ by setting $\overleftrightarrow{\omega}(t) = \omega(1-t)$ for $t \in [0,1]$. Show that $\omega \mapsto \overleftrightarrow{\omega}$ is an automorphism of $(C([0,1];\mathbb{R}^r)_{00}, \mu_{\mathrm{bridge}})$.

(e) Let (Ω, Σ, μ) be a complete probability space and $\langle X_t \rangle_{t\ge 0}$ a family of real-valued random variables on Ω with independent increments. For $I \subseteq [0,\infty[$ let T_I be the σ-algebra generated by $\{X_s - X_t : s, t \in I\}$. Let $\mathcal{I}$ be a family of subsets of $[0,\infty[$ such that for all distinct I, $J \in \mathcal{I}$ either $\sup I \le \inf J$ or $\sup J \le \inf I$. Show that $\langle \mathrm{T}_I \rangle_{I\in\mathcal{I}}$ is an independent family of σ-algebras.

(f) Suppose that $H \subseteq \mathbb{R}^r$ is an F_σ set and that $\tau : \Omega \to [0,\infty]$ is the Brownian hitting time to H, as defined in 477I. Show that τ is Borel measurable.

(g) Let μ_W be one-dimensional Wiener measure, and τ the hitting time to $\{1\}$. Show that the distribution of τ has probability density function $x \mapsto \frac{1}{x\sqrt{2\pi x}} e^{-1/2x}$ for $x > 0$.

(h) Let μ_W be one-dimensional Wiener measure on $\Omega = C([0,\infty[)_0$. Show that, for μ_W-almost every $\omega \in \Omega$, the total variation $\mathrm{Var}_{[s,t]}(\omega)$ is infinite whenever $0 \le s < t$.

477Y Further exercises (a) Write D_n for $\{2^{-n}i : i \in \mathbb{N}\}$ and $D = \bigcup_{n\in\mathbb{N}} D_n$, the set of dyadic rationals in $[0,\infty[$. For $d \in D$ define $f_d \in C([0,\infty[)$ as follows. If $n \in \mathbb{N}$, $f_n(t) = 0$ if $t \le n$, $t - n$ if $n \le t \le n+1$, 1 if $t \ge n+1$. If $d = 2^{-n}k$ where $n \ge 1$ and $k \in \mathbb{N}$ is odd, $f_d(t) = \max(0, \frac{1}{\sqrt{2^{n+1}}}(1 - 2^n|t-d|))$ for $t \ge 0$. Now let $\langle Z_d \rangle_{d\in D}$ be an independent family of standard normal distributions, and set $\omega_n(t) = \sum_{d\in D_n} f_d(t)Z_d$ for $t \ge 0$, so that each ω_n is a random continuous function on $[0,\infty[$. Show that for any $n \in \mathbb{N}$ and $\epsilon > 0$,

$$\Pr(\sup_{t\in[0,n]} |\omega_{n+1}(t) - \omega_n(t)| \ge \epsilon) \le 2^n n \cdot \frac{2}{\sqrt{2\pi}} \int_{2\epsilon\sqrt{2^n}}^\infty e^{-x^2/2} dx,$$

and hence that $\langle \omega_n \rangle_{n\in\mathbb{N}}$ converges almost surely to a continuous function. Explain how to interpret this as a construction of Wiener measure on $\Omega = C([0,\infty[)_0$, as the image measure $\mu_G^D g^{-1}$ where $g : \mathbb{R}^D \to \Omega$ is almost continuous (for the topology $\mathfrak{T}_c$ on Ω) and μ_G^D is the product of copies of the standard normal distribution μ_G.

(b) Fix $p \in\,]0,1[\setminus \{\frac{1}{2}\}$. (i) For $\alpha \in \mathbb{R}$ set $h_\alpha(t) = |t-\alpha|^{p-\frac{1}{2}} - |t|^{p-\frac{1}{2}}$ when this is defined. Show that $h_\alpha \in \mathcal{L}^2(\mu_L)$, where μ_L is Lebesgue measure, and that $\|h_\alpha\|_2^2 = |\alpha|^{2p}\|h_1\|_2^2$ and $\|h_\alpha - h_\beta\|_2 = \|h_{\alpha-\beta}\|_2$ for all α, $\beta \in \mathbb{R}$. (ii) Show that there is a centered Gaussian process $\langle X_\alpha \rangle_{\alpha\in\mathbb{R}}$ such that $\mathbb{E}(X_\alpha \times X_\beta) = |\alpha|^{2p} + |\beta|^{2p} - |\alpha - \beta|^{2p}$ for all α, $\beta \in \mathbb{R}$. (iii) Show that such a process can be represented by a Radon measure on $C(\mathbb{R})$. (*Hint*: 477Ya.) (This is **fractional Brownian motion**.)

(c) Let μ_W be Wiener measure on $\Omega = C([0,\infty[;\mathbb{R}^r)_0$, where $r \ge 1$, and set $X_t(\omega) = \omega(t)$ for $t \ge 0$ and $\omega \in \Omega$. Let f be a real-valued tempered function on $\mathbb{R}^r$ (definition: 284Wa). For $x \in \mathbb{R}^r$ and $0 < t < b$, let $\nu_x^{(t,b)}$ be the distribution of $x + X_{b-t}$, so that $g(x,t) = \int f(y)\nu_x^{(t,b)}(dy)$ can be regarded as the expectation of $f(X_b)$ given that $X_t = x$. Show that g satisfies the backwards heat equation $2\frac{\partial g}{\partial t} + \sum_{i=0}^{r-1} \frac{\partial^2 g}{\partial \xi_i^2} = 0$.

(d) Let $r \ge 1$ be an integer, and ν a Radon probability measure on $\mathbb{R}^r$ such that $x \mapsto a \centerdot x$ has expectation 0 and variance $\|a\|^2$ for every $a \in \mathbb{R}^r$. Let Ω be $C([0,\infty[;\mathbb{R}^r)_0$, and for $\alpha > 0$ define $f_\alpha : (\mathbb{R}^r)^{\mathbb{N}} \to \Omega$ by setting $f_\alpha(x)(t) = \sqrt{\alpha}(\sum_{i<n} z(i) + \frac{1}{\alpha}(t - n\alpha)z(n))$ when $z \in (\mathbb{R}^r)^{\mathbb{N}}$, $n \in \mathbb{N}$ and $n\alpha \le t \le (n+1)\alpha$; let μ_α be the image Radon measure $\nu^{\mathbb{N}} f_\alpha^{-1}$ on Ω. Show that Wiener measure μ_W is the limit $\lim_{\alpha\downarrow 0} \mu_\alpha$ for the narrow topology.

(e) Let μ_W be Wiener measure on $C([0,\infty[)_0$, and $\gamma > \frac{1}{2}$. Show that $\lim_{t\to\infty} \frac{1}{t^\gamma}\omega(t) = \lim_{t\downarrow 0} \frac{1}{t^{1-\gamma}}\omega(t) = 0$ for μ_W-almost every ω.

(f) Write out a proof of 477G which works directly from the Gaussian-distribution characterization of Wiener measure, without appealing to results from §455 other than 455L. (I think you will need to start with stopping times taking finitely and countably many values, as in 455C; but you will find great simplifications.)

(g) Let $\hat{\mu}$ be the Gaussian distribution on $\mathbb{R}^{[0,\infty[}$ corresponding to Brownian motion, as in 477A. For $t \ge 0$ let Σ_t be the family of Baire subsets of $\mathbb{R}^{[0,\infty[}$ determined by coordinates in $[0,t]$, and $\Sigma_t^+ = \bigcap_{s>t} \Sigma_s$. For $\omega \in \mathbb{R}^{[0,\infty[}$ set $\tau(\omega) = \inf\{q : q \in \mathbb{Q},\ \omega(q) \ge 1\}$, counting $\inf \emptyset$ as ∞. Show that τ is a stopping time adapted to $\langle \Sigma_t^+ \rangle_{t\ge 0}$. For ω, $\omega' \in \mathbb{R}^{[0,\infty[}$ define $\phi_\tau(\omega,\omega') \in \mathbb{R}^{[0,\infty[}$ by setting $\phi_\tau(\omega,\omega')(t) = \omega(t)$ if $t \le \tau(\omega)$, $\omega(\tau(\omega)) + \omega'(t - \tau(\omega))$ if $t > \tau(\omega)$. Show that ϕ_τ is not inverse-measure-preserving for the product measure $\hat{\mu}^2$ and $\hat{\mu}$. (*Hint*: show that $\{\omega : \tau(\omega) \in D\}$ is negligible for every countable set D.)

(h) Let μ_W be one-dimensional Wiener measure on $\Omega = C([0,\infty[)_0$. (i) Show by induction on k that $\Pr(\text{there are } t_0 < t_1 < \ldots < t_k \le t \text{ such that } X_{t_j} = (-1)^j \text{ for every } j \le k) = \Pr(\text{there is an } s \le t \text{ such that } X_s = 2k+1)$ for any $t \ge 0$. (*Hint*: 477J.) (ii) For $\omega \in \Omega$, $k \in \mathbb{N}$ define $\tau_k(\omega)$ by saying that $\tau_0(\omega) = \inf\{t : |\omega(t)| = 1\}$, $\tau_{k+1}(\omega) = \inf\{t : t > \tau_k(\omega),\ \omega(t) = -\omega(\tau_k(\omega))\}$. Show that τ_k is a stopping time adapted to the family $\langle \Sigma_t \rangle_{t\ge 0}$ of 477H, and is finite μ_W-a.e. (iii) Set $E_k = \{\omega : \tau_k(\omega) \le 1 < \tau_{k+1}(\omega)\}$, $p_k = \mu_W E_k$, $F_k = \{\omega : \text{there is an } s \le 1 \text{ such that } \omega(s) = 2k+1\}$, $q_k = \mu_W F_k$. Show that

$$q_k = \frac{1}{2}p_k + \sum_{j=k+1}^\infty p_j = \frac{2}{\sqrt{2\pi}}\int_{2k+1}^\infty e^{-x^2/2}dx.$$

(iv) Show that $\Pr(\tau_0 \le 1) = \sum_{k=0}^\infty p_k = 2\sum_{k=0}^\infty (-1)^k q_k$.

(i) Let μ_W be Wiener measure on $\Omega = C([0,\infty[\,;\mathbb{R}^r)_0$, where $r \ge 1$. Show that, for μ_W-almost every $\omega \in \Omega$, $\{\frac{\omega(t)}{\|\omega(t)\|} : t \ge t_0,\ \omega(t) \ne 0\}$ is dense in $\partial B(\mathbf{0},1)$ for every $t_0 \ge 0$.

(j)(i) Show that, for any $r \ge 1$, the topology $\mathfrak{T}_c$ of uniform convergence on compact sets is a complete linear space topology on $\Omega = C([0,\infty[\,;\mathbb{R}^r)_0$. (ii) Show that Wiener measure on Ω is a centered Gaussian measure in the sense of 466N. (*Hint*: 466Ye.)

477 Notes and comments The 'first proof' of 477A calls on a result which is twenty-five pages into §455, and you will probably be glad to be assured that all it really needs is 455A, fragments from the proof of 455P, and part (a) of the proof of 455R. So it is not enormously harder than the 'second proof', based on the elementary theory of Gaussian processes.

With this theorem, we have two routes to the first target: setting up a measure space with a family of random variables representing Brownian motion. I repeat that this is a secondary issue. Brownian motion begins with the family of joint distributions of finite indexed sets $(X_{t_0},\ldots,X_{t_n})$ satisfying the properties listed in 477A. It is one of the triumphs of Kolmogorov's theory of probability that these distributions can be represented by a family of real-valued functions on a set with a countably additive measure; but they would still be of the highest importance if they could not. In order to show that it can be done, we can use either the time-dependent approach based on conditional expectations, as in 455A, 455P and the 'first proof' of 477A, or the timeless Gaussian-distribution approach through 456C, as in the 'second proof' of 477A. Both, of course, depend on Kolmogorov's theorem 454D. They have different advantages, and it will be very useful to be able to call on the intuitions of both. The 'first proof' leads us naturally into the theory of Lévy processes, in which other families of distributions replace normal distributions.

To get to the continuity of sample paths, we need to do quite a bit more, and the proof of 477B is one way of filling the gap. At this point it becomes tempting to abandon both proofs of 477A and start again with the method of 477Ya, the 'Lévy-Ciesielski' construction, *not* using Kolmogorov's theorem. But if we do this, we shall have to devise a new argument to prove the strong Markov property 477G, rather than quoting 455U. Of course the special properties of Gaussian processes mean that a direct proof of 477G is still quite a bit easier than the general results of §455 (477Yf). I make no claim that one approach is 'better' than another; they all throw light on the result.

What I here call 'Wiener measure' (477B) is a particular realization of Brownian motion. It is so convincing that it is tempting to regard it as 'the' real basis of Brownian motion. I do not mean to assert this in any way which will bind me in future. But (as a measure theorist, rather than a probabilist) I think that the specific measures of 477B and 477D are worth as much attention as any. One reason for not insisting that the space $C([0,\infty[)_0$ is the only right place to start is that we may at any moment wish to move to something smaller, as in 477Ef. The approach here gives a very direct language in which to express theorems of the form 'almost every Brownian path is ... ' (477K, 477L), and every such theorem carries an implicit suggestion that we could move to a conegligible set and a subspace measure.

In 477C I sketch an alternative characterization of one-dimensional Wiener measure. Five pages seem to be rather a lot for a proof of something which surely has to be true, if we can get the hypotheses right; but I do not see a genuinely shorter route, and I think in fact that the indigestibility of the argument as presented is due to compression more than to pedantry. At least I have tried to put the key step into part (a-i) of the proof. We have to use the Central Limit Theorem; we have to use a finite-approximation version of it, rather than a limit version; the ideas of this proof do not demand Lindeberg's formulation, but this is what we have to hand in Volume 2; and if we are going to consider interpolations for general random walks, we need something to force a sufficient degree of equicontinuity, and (†) is what comes naturally from the result in 274F. It is surely obvious that I have been half-hearted in the generality of the theorem as given. There can be no reason for insisting on steps being at uniform time intervals, or on stationary processes, or even on variances being exactly correct, provided that everything averages out nicely in the limit. The idea does require that steps be independent, but after that we just need hypotheses adequate for the application of the Central Limit Theorem.

Clearly we can also look for r-dimensional versions of the theorem. I have not done so because they would inevitably demand vector-valued versions of the Central Limit Theorem, and while a combination of the ideas of §§274 and 456 would take us a long way, it would not belong to this section. However I give 477Yd as an example which can be dealt with without much general theory.

Already in the elementary results 477Eb-477Eg we see that Wiener measure is a remarkable construction. It is a general principle that the more symmetries an object has, the more important it is; this one has a surprising symmetry (477Ef), which is even better. I take it as confirmation that we have a good representation, that all these symmetries can be represented by actual inverse-measure-preserving functions, rather than leaving them as manipulations of distributions.

477F is a natural result, and a further confirmation that in $C([0,\infty[;\mathbb{R}^r)_0$ we have got hold of an appropriate space of functions. The proof I give depends on an aspect of the structure developed in 477Ya.

The next really important result is the 'strong Markov property' (477G). This is clearly a central property of Brownian motion. It may not be quite so clear what the formulation here is trying to say. As in 477E, I am expressing the result in terms of an inverse-measure-preserving function. This makes no sense unless we have a probability space Ω in which we can put two elements ω, ω' together to form a third; so we are more or less forced to look at a space of paths. But not all spaces of paths will do. For an indication of what can happen if we work with the wrong realization, see 477Yg.

In 477H we have two kinds of zero-one law. One, 477Hd, is explicit; the tail σ-algebra $\bigcap_{t\ge 0}\hat{\mathrm{T}}_{[t,\infty[}$ behaves like the tail of an independent sequence of σ-algebras (272O). But the formula '$\bigcap_{s>t}\Sigma_s\subseteq\hat{\mathrm{T}}_{[0,t]}$' in 477Hc can be thought of as a relative zero-one law. There are many events (e.g., $\{\omega:\liminf_{s\downarrow t}\frac{1}{s-t}\omega(s)\ge 0\}$) which belong to $\bigcap_{s>t}\Sigma_s$ and not to $\mathrm{T}_{[0,t]}$ or Σ_t, but all collapse to events in $\mathrm{T}_{[0,t]}$ if we rearrange them on appropriate negligible sets. This is really a special case of 455T.

The formulae in the first application of the strong Markov property (477J) demand a bit of concentrated attention, but I think that the key step at the end of the proof (moving from μ_W^2 to $\int\ldots d\mu_W$) faithfully represents the intuition: once we have reached the level α, we have an even chance of rising farther. For the discrete case, 272Yc is a version of the same idea. From the distribution of the hitting time to $\{\alpha\}$ we can deduce the distribution of the hitting time to $\{-1,1\}$ (477Yh); but I do not know of a corresponding exact result for the hitting time to the unit circle for two-dimensional Brownian motion.

I expect you have been shown a continuous function which is nowhere differentiable. In 477K we see that 'almost every' function is of this type. What a hundred and fifty years ago seemed to be an exotic counter-example now presents itself as a representative of the typical case. The very crude estimates in the proof of 477K are supposed to furnish a straightforward proof of the result, without asking for anything which might lead to refinements. Of course there is much more to be said, starting with 477Xh. In 477L we have an interesting result which will be useful in §479, when I return to geometric properties of Brownian paths.

Version of 4.6.09/7.8.20

478 Harmonic functions

In this section and the next I will attempt an introduction to potential theory. This is an enormous subject and my choice of results is necessarily somewhat arbitrary. My principal aim is to give the most elementary properties of Newtonian capacity, which will appear in §479. It seems that this necessarily involves a basic understanding of harmonic and superharmonic functions. I approach these by the 'probabilistic' route, using Brownian motion as described in §477.

The first few paragraphs, down to 478J, do not in fact involve Brownian motion; they rely on multidimensional advanced calculus and on the Divergence Theorem. (The latter is applied only to continuously differentiable functions and domains of very simple types, so we need far less than the quoted result in 475N.) Defining 'harmonic function' in terms of average values over concentric spherical shells (478B), the first step is to identify this with the definition in terms of the Laplacian differential operator (478E). An essential result is a formula for a harmonic function inside a sphere in terms of its values on the boundary and the 'Poisson kernel' (478Ib), and we also need to understand the effects of smoothing by convolution with appropriate functions (478J, following 473D-473E). I turn to Brownian motion with Dynkin's formula (478K), relating the expected value of $f(X_\tau)$ for a stopped Brownian process X_τ to an integral in terms of $\nabla^2 f$. This is already enough to deal with the asymptotic behaviour of Brownian paths, which depends in a striking way on the dimension of the space (478M).

We can now approach Dirichlet's problem. If we have a bounded open set $G \subseteq \mathbb{R}^r$, there is a family $\langle \mu_x \rangle_{x \in G}$ of probability measures such that whenever $f : \overline{G} \to \mathbb{R}$ is continuous and $f \restriction G$ is harmonic, then $f(x) = \int f d\mu_x$ for every $x \in G$ (478Pc). So this family of 'harmonic measures' gives a formula continuously extending a function on ∂G to a harmonic function on G, whenever such an extension exists (478S). The method used expresses μ_x in terms of the distribution of points at which Brownian paths starting at x strike ∂G, and relies on Dynkin's formula through Theorem 478O. The strong Markov property of Brownian motion now enables us to relate harmonic measures associated with different sets (478R).

478A Notation $r \geq 1$ will be an integer; if you find it easier to focus on one dimensionality at a time, you should start with $r = 3$, because $r = 1$ is too easy and $r = 2$ is exceptional. μ will be Lebesgue measure on $\mathbb{R}^r$, and $\| \, \|$ the Euclidean norm on $\mathbb{R}^r$; ν will be normalized $(r-1)$-dimensional Hausdorff measure on $\mathbb{R}^r$. In the elementary case $r = 1$, ν will be counting measure on $\mathbb{R}$.

β_r will be the volume of the unit ball in $\mathbb{R}^r$, that is,

$$\begin{aligned}\beta_r &= \tfrac{1}{k!}\pi^k \text{ if } r = 2k \text{ is even,}\\ &= \tfrac{2^{2k+1}k!}{(2k+1)!}\pi^k \text{ if } r = 2k+1 \text{ is odd.}\end{aligned}$$

Recall that

$$\begin{aligned}\nu(\partial B(\mathbf{0},1)) = r\beta_r &= \tfrac{2}{(k-1)!}\pi^k \text{ if } r = 2k \text{ is even,}\\ &= \tfrac{2^{2k+1}k!}{(2k)!}\pi^k \text{ if } r = 2k+1 \text{ is odd}\end{aligned}$$

(265F/265H).

In the formulae below, there are repeated expressions of the form $\frac{1}{\|x-y\|^{r-1}}$, $\frac{1}{\|x-y\|^{r-2}}$; in these, it will often be convenient to interpret $\frac{1}{0}$ as ∞, so that we have $[0,\infty]$-valued functions defined everywhere.

It will be convenient to do some calculations in the one-point compactification $\mathbb{R}^r \cup \{\infty\}$ of $\mathbb{R}^r$ (3A3O). For a set $A \subseteq \mathbb{R}^r$, write $\overline{A}^\infty$ and $\partial^\infty A$ for its closure and boundary taken in $\mathbb{R}^r \cup \{\infty\}$; that is,

$$\overline{A}^\infty = \overline{A}, \quad \partial^\infty A = \partial A$$

if A is bounded, and

$$\overline{A}^\infty = \overline{A} \cup \{\infty\}, \quad \partial^\infty A = \partial A \cup \{\infty\}$$

if A is unbounded. Note that $\overline{A}^\infty$ and $\partial^\infty A$ are always compact. In this context I will take $x + \infty = \infty$ for every $x \in \mathbb{R}^r$.

μ_W will be r-dimensional Wiener measure on $\Omega = C([0,\infty[\,;\mathbb{R}^r)_0$, the space of continuous functions ω from $[0,\infty[$ to $\mathbb{R}^r$ such that $\omega(0) = 0$ (477D), endowed with the topology of uniform convergence on compact sets; Σ will be the domain of μ_W. The probabilistic notations $\mathbb{E}$ and $\Pr$ will always be with respect to μ_W or some directly associated probability. μ_W^2 will be the product measure on $\Omega \times \Omega$. I will write $X_t(\omega) = \omega(t)$ for $t \in [0,\infty[$ and $\omega \in \Omega$, and if $\tau : \Omega \to [0,\infty]$ is a function, I will write $X_\tau(\omega) = \omega(\tau(\omega))$ whenever $\omega \in \Omega$ and $\tau(\omega)$ is finite.

As in 477Hc, I will write Σ_t for the σ-algebra of sets $F \in \Sigma$ such that $\omega' \in F$ whenever $\omega \in F$, $\omega' \in \Omega$ and $\omega'\restriction[0,t] = \omega\restriction[0,t]$, and $\Sigma_t^+ = \bigcap_{s>t} \Sigma_s$. $\mathrm{T}_{[0,t]}$ will be the σ-algebra of subsets of Ω generated by $\{X_s : s \le t\}$.

478B Harmonic and superharmonic functions Let $G \subseteq \mathbb{R}^r$ be an open set and $f : G \to [-\infty,\infty]$ a function.

(a) f is **superharmonic** if $\frac{1}{\nu(\partial B(x,\delta))}\int_{\partial B(x,\delta)} f d\nu$ is defined in $[-\infty,\infty]$ and less than or equal to $f(x)$ whenever $x \in G$, $\delta > 0$ and $B(x,\delta) \subseteq G$.

(b) f is **subharmonic** if $-f$ is superharmonic, that is, $\frac{1}{\nu(\partial B(x,\delta))}\int_{\partial B(x,\delta)} f d\nu$ is defined in $[-\infty,\infty]$ and greater than or equal to $f(x)$ whenever $x \in G$, $\delta > 0$ and $B(x,\delta) \subseteq G$.

(c) f is **harmonic** if it is both superharmonic and subharmonic, that is, $\frac{1}{\nu(\partial B(x,\delta))}\int_{\partial B(x,\delta)} f d\nu$ is defined and equal to $f(x)$ whenever $x \in G$, $\delta > 0$ and $B(x,\delta) \subseteq G$.

478C Elementary facts Let $G \subseteq \mathbb{R}^r$ be an open set.

(a) If $f : G \to [-\infty,\infty]$ is a function, then f is superharmonic iff $-f$ is subharmonic.

(b) If f, $g : G \to [-\infty,\infty[$ are superharmonic functions, then $f + g$ is superharmonic. **P** If $x \in G$, $\delta > 0$ and $B(x,\delta) \subseteq G$, then $\int_{\partial B(x,\delta)} f d\nu$ and $\int_{\partial B(x,\delta)} g\,d\nu$ are defined in $[-\infty,\infty[$, so $\int_{\partial B(x,\delta)} f + g\,d\nu$ is defined and is

$$\int_{\partial B(x,\delta)} f d\nu + \int_{\partial B(x,\delta)} g\,d\nu \le \nu(\partial B(x,\delta))\,(f(x) + g(x)). \text{ **Q**}$$

(c) If f, $g : G \to [-\infty,\infty]$ are superharmonic functions, then $f \wedge g$ is superharmonic. **P** If $x \in G$, $\delta > 0$ and $B(x,\delta) \subseteq G$, then $\int_{\partial B(x,\delta)} f d\nu$ and $\int_{\partial B(x,\delta)} g\,d\nu$ are defined in $[-\infty,\infty]$, so $\int_{\partial B(x,\delta)} f \wedge g\,d\nu$ is defined and is at most

$$\min\Bigl(\int_{\partial B(x,\delta)} f d\nu, \int_{\partial B(x,\delta)} g\,d\nu\Bigr) \le \nu(\partial B(x,\delta))\,\min(f(x), g(x)). \text{ **Q**}$$

(d) Let $f : G \to \mathbb{R}$ be a harmonic function which is locally integrable with respect to Lebesgue measure on G (that is, every point of G has a neighbourhood V such that $\int_V f d\mu$ is defined and finite). Then

$$f(x) = \frac{1}{\mu B(x,\delta)}\int_{B(x,\delta)} f d\mu$$

whenever $x \in G$, $\delta > 0$ and $B(x,\delta) \subseteq G$. **P** By 265G,

$$\int_{B(x,\delta)} f d\mu = \int_0^t \int_{\partial B(x,t)} f d\nu\, dt = \int_0^t \nu(\partial B(x,\delta)) f(x) dt = \beta_r \delta^r f(x). \quad \mathbf{Q}$$

So f is continuous. **P** If $x \in G$, take $\delta > 0$ such that $B(x,2\delta) \subseteq G$, and set $f_1(y) = f(y)$ for $y \in B(x,2\delta)$, 0 for $y \in \mathbb{R}^r \setminus B(x,2\delta)$. Set $g = \frac{1}{\mu B(\mathbf{0},\delta)} \chi B(\mathbf{0},\delta)$. Then f_1 is integrable, so the convolution $f_1 * g$ is continuous (444Rc). Also, for any $y \in B(x,\delta)$,

$$(f_1 * g)(y) = \int f_1(z) g(y-z) \mu(dz) = \int_{B(y,\delta)} \frac{f_1(z)}{\mu B(y,\delta)} \mu(dz)$$
$$= \frac{1}{\mu B(y,\delta)} \int_{B(y,\delta)} f(z) \mu(dz) = f(y),$$

so f is continuous at x. **Q**

478D Maximal principle One of the fundamental properties of harmonic functions will hardly be used in the exposition here, but I had better give it a suitably prominent place.

Proposition Let $G \subseteq \mathbb{R}^r$ be a non-empty open set. Suppose that $g : \overline{G}^\infty \to \left]-\infty, \infty\right]$ is lower semi-continuous, $g(y) \ge 0$ for every $y \in \partial^\infty G$, and $g \restriction G$ is superharmonic. Then $g(x) \ge 0$ for every $x \in G$.

proof ? Otherwise, set $\gamma = \inf_{x \in G} g(x) = \inf\{g(y) : y \in \overline{G}^\infty\}$. Because $\overline{G}^\infty$ is compact and g is lower semi-continuous, $K = \{x : x \in G,\ g(x) = \gamma\}$ is non-empty and compact (4A2B(d-viii)). Let $x \in K$ be such that $\|x\|$ is maximal, and $\delta > 0$ such that $B(x,\delta) \subseteq G$. Then $\frac{1}{\nu(\partial B(x,\delta))} \int_{\partial B(x,\delta)} g\, d\nu \le g(x)$. But $g(y) \ge g(x)$ for every $y \in \partial B(x,\delta)$ and

$$\{y : y \in \partial B(x,\delta),\ g(y) > g(x)\} \supseteq \{y : y \in \partial B(x,\delta),\ (y-x) \textbf{.} x \ge 0\}$$

is not ν-negligible, so this is impossible. **X**

478E Theorem Let $G \subseteq \mathbb{R}^r$ be an open set and $f : G \to \mathbb{R}$ a function with continuous second derivative. Write $\nabla^2 f$ for its Laplacian $\operatorname{div} \operatorname{grad} f = \sum_{i=1}^r \frac{\partial^2 f}{\partial \xi_i^2}$.

(a) f is superharmonic iff $\nabla^2 f \le 0$ everywhere in G.

(b) f is subharmonic iff $\nabla^2 f \ge 0$ everywhere in G.

(c) f is harmonic iff $\nabla^2 f = 0$ everywhere in G.

proof (a)(i) For $x \in G$ set

$$R_x = \rho(x, \mathbb{R}^r \setminus G) = \inf_{y \in \mathbb{R}^r \setminus G} \|x - y\|,$$

counting $\inf \emptyset$ as ∞; for $0 < \gamma < R_x$ set

$$g_x(\gamma) = \frac{1}{\gamma^{r-1}} \int_{\partial B(x,\gamma)} f(y) \nu(dy) = \int_{\partial B(\mathbf{0},1)} f(x + \gamma z) \nu(dz).$$

Because f is continuously differentiable, $g_x'(\gamma)$ is defined and equal to $\int_{\partial B(\mathbf{0},1)} \frac{\partial}{\partial \gamma} f(x+\gamma z) \nu(dz)$ for $\gamma \in \left]0, R_x\right[$.

Set $\phi = \operatorname{grad} f$, so that $\nabla^2 f = \operatorname{div} \phi$. Each ball $B(x,\gamma)$ has finite perimeter; its essential boundary is its ordinary boundary; the Federer exterior normal v_y at y is $\frac{1}{\gamma}(y-x)$; and if $y = x + \gamma z$, where $\|z\| = 1$, then $\phi(y) \textbf{.} v_y$ is $\frac{\partial}{\partial \gamma} f(x + \gamma z)$. So the Divergence Theorem (475N) tells us that

$$\int_{B(x,\gamma)} \nabla^2 f d\mu = \int_{\partial B(x,\gamma)} \phi(y) \textbf{.} v_y \nu(dy)$$
$$= \gamma^{r-1} \int_{\partial B(\mathbf{0},1)} \frac{\partial}{\partial \gamma} f(x + \gamma z) \nu(dz) = \gamma^{r-1} g_x'(\gamma).$$

(ii) If $\nabla^2 f \le 0$ everywhere in G, and $B(x,\gamma) \subseteq G$, then $g'_x(t) \le 0$ for $0 < t \le \gamma$, so

$$g_x(\gamma) \le \lim_{t\downarrow 0} g_x(t) = r\beta_r f(x);$$

as x and γ are arbitrary, f is superharmonic.

(iii) If f is superharmonic, and $x \in G$, then

$$g_x(\gamma) \le r\beta_r f(x) = \lim_{t\downarrow 0} g_x(t)$$

for every $\gamma \in \,]0, R_x[$. So there must be arbitrarily small $\gamma > 0$ such that $g'_x(\gamma) \le 0$ and $\int_{B(x,\gamma)} \nabla^2 f d\mu \le 0$; as $\nabla^2 f$ is continuous, $(\nabla^2 f)(x) \le 0$.

(b)-(c) are now immediate.

478F Basic examples (a) For any y, $z \in \mathbb{R}^r$,

$$x \mapsto \frac{1}{\|x-z\|^{r-2}}, \quad x \mapsto \frac{(x-z)\,.\,y}{\|x-z\|^r},$$

$$x \mapsto \frac{\|y-z\|^2 - \|x-y\|^2}{\|x-z\|^r} = 2\frac{(x-z)\,.\,(y-z)}{\|x-z\|^r} - \frac{1}{\|x-z\|^{r-2}}$$

are harmonic on $\mathbb{R}^r \setminus \{z\}$.

(b) For any $z \in \mathbb{R}^2$,

$$x \mapsto \ln \|x - z\|$$

is harmonic on $\mathbb{R}^2 \setminus \{z\}$.

proof The Laplacians are easiest to calculate when $z = 0$, of course, but in any case you only have to get the algebra right to apply 478Ec.

Remark The function $x \mapsto \frac{\|y-z\|^2 - \|x-y\|^2}{\|x-z\|^r}$ is the **Poisson kernel**; see 478I below.

478G We shall need a pair of exact integrals involving the functions here, with an easy corollary.

Lemma (a) $\frac{1}{\nu(\partial B(\mathbf{0},\delta))}\int_{\partial B(\mathbf{0},\delta)} \frac{1}{\|x-z\|^{r-2}}\nu(dz) = \frac{1}{\max(\delta,\|x\|)^{r-2}}$ whenever $x \in \mathbb{R}^r$ and $\delta > 0$.

(b) $\frac{1}{\nu(\partial B(\mathbf{0},\delta))}\int_{\partial B(\mathbf{0},\delta)} \frac{|\delta^2 - \|x\|^2|}{\|x-z\|^r}\nu(dz) = \frac{1}{\max(\delta,\|x\|)^{r-2}}$ whenever $x \in \mathbb{R}^r$, $\delta > 0$ and $\|x\| \ne \delta$.

(c) $\int_{B(\mathbf{0},\delta)} \frac{1}{\|x-z\|^{r-2}}\mu(dz) \le \frac{1}{2}r\beta_r\delta^2$ whenever $x \in \mathbb{R}^r$ and $\delta > 0$.

proof (a)(i) The first thing to note is that there is a function $g : [0,\infty[\setminus \{\delta\} \to [0,\infty[$ such that $g(\|x\|) = \frac{1}{\nu(\partial B(\mathbf{0},\delta))}\int_{\partial B(\mathbf{0},\delta)} \frac{1}{\|x-z\|^{r-2}}\nu(dz)$ whenever $\|x\| \ne \delta$. **P** If $\|x\| = \|y\|$, then there is an orthogonal transformation $T : \mathbb{R}^r \to \mathbb{R}^r$ such that $Tx = y$, so that

$$\int_{\partial B(\mathbf{0},\delta)} \frac{1}{\|y-z\|^{r-2}}\nu(dz) = \int_{\partial B(\mathbf{0},\delta)} \frac{1}{\|y-Tz\|^{r-2}}\nu(dz)$$

(because T is an automorphism of $(\mathbb{R}^r, B(\mathbf{0},\delta), \nu)$)

$$= \int_{\partial B(\mathbf{0},\delta)} \frac{1}{\|Tx-Tz\|^{r-2}}\nu(dz) = \int_{\partial B(\mathbf{0},\delta)} \frac{1}{\|x-z\|^{r-2}}\nu(dz). \textbf{ Q}$$

(ii) Now suppose that $0 < \gamma < \delta$. Then

$$g(\gamma) = \frac{1}{\nu(\partial B(\mathbf{0},\gamma))} \int_{\partial B(\mathbf{0},\gamma)} g(\gamma)\nu(dx)$$

$$= \frac{1}{\nu(\partial B(\mathbf{0},\gamma))} \int_{\partial B(\mathbf{0},\gamma)} \frac{1}{\nu(\partial B(\mathbf{0},\delta))} \int_{\partial B(\mathbf{0},\delta)} \frac{1}{\|x-z\|^{r-2}}\nu(dz)\nu(dx)$$

$$= \frac{1}{\nu(\partial B(\mathbf{0},\delta))} \int_{\partial B(\mathbf{0},\delta)} \frac{1}{\nu(\partial B(\mathbf{0},\gamma))} \int_{\partial B(\mathbf{0},\gamma)} \frac{1}{\|x-z\|^{r-2}}\nu(dx)\nu(dz)$$

$$= \frac{1}{\nu(\partial B(\mathbf{0},\delta))} \int_{\partial B(\mathbf{0},\delta)} \frac{1}{\|z\|^{r-2}}\nu(dz)$$

(because the function $x \mapsto \frac{1}{\|x-z\|^2}$ is harmonic in $\mathbb{R}^r \setminus \{z\}$, by 478Fa)

$$= \frac{1}{\delta^{r-2}}.$$

(iii) Next, if $\gamma > \delta$,

$$g(\gamma) = \frac{1}{\nu(\partial B(\mathbf{0},\delta))} \int_{\partial B(\mathbf{0},\delta)} \frac{1}{\nu(\partial B(\mathbf{0},\gamma))} \int_{\partial B(\mathbf{0},\gamma)} \frac{1}{\|x-z\|^{r-2}}\nu(dx)\nu(dz)$$

(as in (ii))

$$= \frac{1}{\nu(\partial B(\mathbf{0},\delta))} \int_{\partial B(\mathbf{0},\delta)} \frac{1}{\gamma^{r-2}}\nu(dz)$$

(by (ii), with γ and δ interchanged)

$$= \frac{1}{\gamma^{r-2}}.$$

(iv) So we have the result if $\|x\| \neq \delta$. If $\|x\| = \delta$ and $r \geq 2$, set $x_n = (1+2^{-n})x$ for each $n \in \mathbb{N}$. If $z \in \partial B(\mathbf{0},\delta)$, $\langle \|x_n - z\| \rangle_{n\in\mathbb{N}}$ is a decreasing sequence with limit $\|x-z\|$, so $\langle \frac{1}{\|x_n-z\|^{r-2}} \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with limit $\frac{1}{\|x-z\|^{r-2}}$. By B.Levi's theorem,

$$\frac{1}{\nu(\partial B(\mathbf{0},\delta))} \int_{\partial B(\mathbf{0},\delta)} \frac{1}{\|x-z\|^{r-2}}\nu(dz) = \lim_{n\to\infty} \frac{1}{\nu(\partial B(\mathbf{0},\delta))} \int_{\partial B(\mathbf{0},\delta)} \frac{1}{\|x_n-z\|^{r-2}}\nu(dz)$$

$$= \lim_{n\to\infty} \frac{1}{\|x_n\|^{r-2}} = \frac{1}{\|x\|^{r-2}} = \frac{1}{\delta^{r-2}}.$$

Finally, if $r = 1$ and $\|x\| = |x| = \delta$, we are just trying to take the average of $|x-\delta|$ and $|x-(-\delta)|$, which will be $\delta = \frac{1}{\delta^{r-2}}$.

(b) We can follow the same general line.

(i) Define $f : \mathbb{R}^r \setminus \partial B(\mathbf{0},\delta) \to \mathbb{R}$ by setting $f(x) = \frac{1}{\nu(\partial B(\mathbf{0},\delta))} \int_{\partial B(\mathbf{0},\delta)} \frac{\|x\|^2-\delta^2}{\|x-z\|^r}\nu(dz)$ when $\|x\| \neq \delta$. Then f is harmonic. **P** If x and $\gamma > 0$ are such that $B(x,\gamma) \subseteq \mathbb{R}^r \setminus \partial B(\mathbf{0},\delta)$, then

$$\frac{1}{\nu(\partial B(x,\gamma))}\int_{\partial B(x,\gamma)} f(y)\nu(dy)$$
$$= \frac{1}{\nu(\partial B(x,\gamma))}\int_{\partial B(x,\gamma)} \frac{1}{\nu(\partial B(\mathbf{0},\delta))}\int_{\partial B(\mathbf{0},\delta)} \frac{\|y\|^2-\delta^2}{\|y-z\|^r}\nu(dz)\nu(dy)$$
$$= \frac{1}{\nu(\partial B(\mathbf{0},\delta))}\int_{\partial B(\mathbf{0},\delta)} \frac{1}{\nu(\partial B(x,\gamma))}\int_{\partial B(x,\gamma)} \frac{\|y\|^2-\delta^2}{\|y-z\|^r}\nu(dy)\nu(dz)$$
$$= \frac{1}{\nu(\partial B(\mathbf{0},\delta))}\int_{\partial B(\mathbf{0},\delta)} \frac{\|x\|^2-\delta^2}{\|x-z\|^r}\nu(dz)$$

(because the functions $y \mapsto \frac{\|y\|^2-\delta^2}{\|y-z\|^r}$ are harmonic when $\|z\| = \delta$, by 478Fa)

$$= f(x).\mathbf{Q}$$

Since f is smooth, $\nabla^2 f = 0$ everywhere off $\partial B(\mathbf{0},\delta)$, by 478Ec.

(ii) As before, we have a function $h : [0,\infty[\setminus\{\delta\} \to [0,\infty[$ such that $f(x) = h(\|x\|)$ whenever $\|x\| \ne \delta$. If $0 < \gamma < \delta$ then

$$h(\gamma) = \frac{1}{\nu(\partial B(\mathbf{0},\gamma))}\int_{\partial B(\mathbf{0},\gamma)} h(\gamma)\nu(dy) = \frac{1}{\nu(\partial B(\mathbf{0},\gamma))}\int_{\partial B(\mathbf{0},\gamma)} f(y)\nu(dy)$$
$$= f(0) = \frac{1}{\nu(\partial B(\mathbf{0},\delta))}\int_{\partial B(\mathbf{0},\delta)} \frac{-\delta^2}{\|z\|^r}\nu(dz) = -\frac{1}{\delta^{r-2}},$$

and

$$\frac{1}{\nu(\partial B(\mathbf{0},\delta))}\int_{\partial B(\mathbf{0},\delta)} \frac{|\delta^2-\|x\|^2|}{\|x-z\|^r}\nu(dz) = -h(\gamma) = \frac{1}{\delta^{r-2}}$$

if $\|x\| = \gamma < \delta$.

(iii) For $\gamma > \delta$ I start with an elementary estimate. If $\|x\| = \gamma > \delta$ then $\frac{\|x\|^2-\delta^2}{\|x-z\|^r}$ lies between $\frac{\gamma^2-\delta^2}{(\gamma+\delta)^r}$ and $\frac{\gamma^2-\delta^2}{(\gamma-\delta)^r}$ for every $z \in \partial B(x,\delta)$, so that $\gamma^{r-2}f(x)$ lies between $\frac{1-(\delta/\gamma)^2}{(1+(\delta/\gamma))^r}$ and $\frac{1-(\delta/\gamma)^2}{(1-(\delta/\gamma))^r}$, and is approximately 1 if γ is large.

(iv) Now we can use the Divergence Theorem again, as follows. If $\delta < \gamma < \beta$ consider the region $E = B(\mathbf{0},\beta)\setminus B(\mathbf{0},\gamma)$ and the function $\phi = \operatorname{grad} f$. As f is smooth, ϕ is defined everywhere off $\partial B(\mathbf{0},\delta)$, and $\phi(x) = \frac{h'(\|x\|)}{\|x\|}x$ at every $x \in \mathbb{R}^r \setminus \partial B(\mathbf{0},\delta)$. The essential boundary of E is $\partial E = \partial B(\mathbf{0},\gamma)\cup\partial B(\mathbf{0},\beta)$; the Federer exterior normal at $x \in \partial B(\mathbf{0},\gamma)$ is $v_x = -\frac{1}{\gamma}x$ and at $x \in \partial B(\mathbf{0},\beta)$ it is $v_x = \frac{1}{\beta}x$; and $\operatorname{div}\phi = \nabla^2 f$ is zero everywhere on E. So 475N tells us that

$$0 = \int_{\partial B(\mathbf{0},\beta)} \phi(x)\,.\,v_x\nu(dx) + \int_{\partial B(\mathbf{0},\gamma)} \phi(x)\,.\,v_x\nu(dx)$$
$$= \int_{\partial B(\mathbf{0},\beta)} h'(\beta)\nu(dx) - \int_{\partial B(\mathbf{0},\gamma)} h'(\gamma)\nu(dx)$$
$$= r\beta_r\beta^{r-1}h'(\beta) - r\beta_r\gamma^{r-1}h'(\gamma).$$

This shows that $h'(\gamma)$ is inversely proportional to γ^{r-1}.

(v) If $r \ge 3$, there are α, $\beta \in \mathbb{R}$ such that $h(\gamma) = \alpha + \frac{\beta}{\gamma^{r-2}}$ for every $\gamma > \delta$. But since (iii) shows us that $\lim_{\gamma\to\infty}\gamma^{r-2}h(\gamma) = 1$, we must have $h(\gamma) = \frac{1}{\gamma^{r-2}}$ for $\gamma > \delta$, as declared. If $r = 2$, then we can express h in the form $h(\gamma) = \alpha + \beta\ln\gamma$; this time, $\lim_{\gamma\to\infty} h(\gamma) = 1$, so once more $h(\gamma) = 1 = \frac{1}{\gamma^{r-2}}$ for every γ.

(vi) Finally, if $r=1$ and $|x|>\delta$, then, as in (a-iv) above, $\frac{1}{\nu(\partial B(\mathbf{0},\delta))}\int_{\partial B(\mathbf{0},\delta)}\frac{|\delta^2-\|x\|^2|}{\|x-z\|^r}\nu(dz)$ is the average of $\frac{x^2-\delta^2}{|x-\delta|}=|x+\delta|$ and $\frac{x^2-\delta^2}{|x+\delta|}=|x-\delta|$, so is $|x|=\frac{1}{|x|^{r-2}}$.

(c) This follows easily from (a);

$$\begin{aligned}\int_{B(\mathbf{0},\delta)}\frac{1}{\|x-z\|^{r-2}}\mu(dz)&=\int_0^\delta\int_{\partial B(\mathbf{0},t)}\frac{1}{\|x-z\|^{r-2}}\nu(dz)dt\\&=\int_0^\delta\frac{r\beta_r t^{r-1}}{\max(t,\|x\|)^{r-2}}dt\le\int_0^\delta r\beta_r t\,dt=\frac{1}{2}r\beta_r\delta^2.\end{aligned}$$

478H Corollary If $r\ge 2$, then $x\mapsto\frac{1}{\|x-z\|^{r-2}}:\mathbb{R}^r\to[0,\infty]$ is superharmonic for any $z\in\mathbb{R}^r$.

proof If $\delta>0$ and $x\in\mathbb{R}^r$,

$$\begin{aligned}\frac{1}{\nu\partial B(x,\delta)}\int_{\partial B(x,\delta)}\frac{1}{\|y-z\|^{r-2}}\nu(dy)&=\frac{1}{\nu\partial B(\mathbf{0},\delta)}\int_{\partial B(\mathbf{0},\delta)}\frac{1}{\|y+x-z\|^{r-2}}\nu(dy)\\&=\frac{1}{\max(\delta,\|x-z\|)^{r-2}}\le\frac{1}{\|x-z\|^{r-2}}.\end{aligned}$$

478I The Poisson kernel gives a basic method of building and describing harmonic functions.

Theorem Suppose that $y\in\mathbb{R}^r$ and $\delta>0$; let $S=\partial B(y,\delta)$ be the sphere with centre y and radius δ.

(a) Let ζ be a totally finite Radon measure on S, and define f on $\mathbb{R}^r\setminus S$ by setting

$$f(x)=\frac{1}{r\beta_r\delta}\int_S\frac{|\delta^2-\|x-y\|^2|}{\|x-z\|^r}\zeta(dz)$$

for $x\in\mathbb{R}^r\setminus S$. Then f is continuous and harmonic.

(b) Let $g:S\to\mathbb{R}$ be a ν_S-integrable function, where ν_S is the subspace measure on S induced by ν, and define $f:\mathbb{R}^r\to\mathbb{R}$ by setting

$$\begin{aligned}f(x)&=\frac{1}{r\beta_r\delta}\int_S g(z)\frac{|\delta^2-\|x-y\|^2|}{\|x-z\|^r}\nu(dz)\text{ if }x\in\mathbb{R}^r\setminus S,\\&=g(x)\text{ if }x\in S.\end{aligned}$$

(i) f is continuous and harmonic in $\mathbb{R}^r\setminus S$.

(ii) If $r\ge 2$, then

$$\liminf_{z\in S,z\to z_0}g(x)=\liminf_{x\to z_0}f(x),\quad\limsup_{x\to z_0}f(x)=\limsup_{z\in S,z\to z_0}g(x)$$

for every $z_0\in S$.

(iii) f is continuous at any point of S where g is continuous, and if g is lower semi-continuous then f also is.

(iv) $\sup_{x\in\mathbb{R}^r}f(x)=\sup_{z\in S}g(z)$ and $\inf_{x\in\mathbb{R}^r}f(x)=\inf_{z\in S}g(z)$.

proof (a) f is continuous just because $x\mapsto\frac{\delta^2-\|x-y\|^2}{\|x-z\|^r}$ is continuous for each $z\in S$ and uniformly bounded for $z\in S$ and x running over any compact set not meeting S.

Suppose that $\|x-y\|<\delta$ and $\eta>0$ is such that $B(x,\eta)\cap S=\emptyset$. Then

$$\frac{1}{\nu(\partial B(x,\eta))}\int_{\partial B(x,\eta)} f d\nu = \frac{1}{\nu(\partial B(x,\eta))}\int_{\partial B(x,\eta)} \frac{1}{r\beta_r\delta}\int_S \frac{\delta^2-\|w-y\|^2}{\|w-z\|^r}\zeta(dz)\nu(dw)$$
$$= \frac{1}{r\beta_r\delta}\int_S \frac{1}{\nu(\partial B(x,\eta))}\int_{\partial B(x,\eta)} \frac{\delta^2-\|w-y\|^2}{\|w-z\|^r}\nu(dw)\zeta(dz)$$
$$= \frac{1}{r\beta_r\delta}\int_S \frac{\delta^2-\|x-y\|^2}{\|x-z\|^r}\zeta(dz)$$

(because $w \mapsto \frac{\delta^2-\|w-y\|^2}{\|w-z\|^r}$ is harmonic on $\mathbb{R}^r \setminus \{z\}$ whenever $z \in S$, by 478Fa)

$$= f(x).$$

As x and η are arbitrary, f is harmonic on $\operatorname{int} B(y,\delta)$. Similarly, it is harmonic on $\mathbb{R}^r \setminus B(y,\delta)$.

(b)(i) Applying (a) to the indefinite-integral measures over the subspace measure ν_S defined by the positive and negative parts of g, we see that f is continuous and harmonic in $\mathbb{R}^r \setminus S$.

(ii)(α) If $x \notin S$,

$$\int_S \frac{|\delta^2-\|x-y\|^2|}{\|x-z\|^r}\nu(dz) = \int_{\partial B(\mathbf{0},\delta)} \frac{|\delta^2-\|x-y\|^2|}{\|x-y-z\|^r}\nu(dz)$$
$$= \frac{\nu(\partial B(\mathbf{0},\delta))}{\max(\delta,\|x-y\|)^{r-2}}$$

by 478Gb. In particular, if x is close to, but not on, the sphere S, $\int_S \frac{|\delta^2-\|x-y\|^2|}{\|x-z\|^r}\nu(dz)$ is approximately $\frac{\nu(\partial B(\mathbf{0},\delta)}{\delta^{r-2}} = r\beta_r\delta$.

(β) Set $M = \int_S |g|d\nu$, and take $z_0 \in S$; set $\gamma = \limsup_{x\in S,x\to z_0} g(x)$. If $\gamma = \infty$ then certainly $\limsup_{x\to z_0} f(x) \le \gamma$. Otherwise, take $\eta > 0$. Let $\alpha_0 \in]0,\delta[$ be such that

$$|\frac{1}{r\beta_r\delta}\int_S \frac{|\delta^2-\|x-y\|^2|}{\|x-z\|^r}\nu(dz) - 1| \le \eta$$

whenever $0 < |\delta - \|x-y\|| \le \alpha_0$, and $g(z) \le \gamma + \eta$ whenever $z \in S$ and $0 < \|z - z_0\| \le 2\alpha_0$. Let $\alpha \in]0,\alpha_0]$ be such that $(2\delta + \alpha_0)(M + |\gamma|)\alpha\nu S \le 2^r r\beta_r\delta\alpha_0^r\eta$.

If $\|x-z_0\| \le \alpha$ and $\|x-y\| \ne \delta$, then $|\delta - \|x-y\|| \le \|x-z_0\| \le \alpha_0$ and $|\delta^2 - \|x-y\|^2| \le \|x-z_0\|(2\delta+\alpha_0)$, so

$$f(x) - \gamma \le \eta|\gamma| + \frac{1}{r\beta_r\delta}\int_S (g(z) - \gamma)\frac{|\delta^2-\|x-y\|^2|}{\|x-z\|^r}\nu(dz)$$
$$\le \eta|\gamma| + \frac{1}{r\beta_r\delta}\int_{S\cap B(z_0,2\alpha_0)} (g(z) - \gamma)\frac{|\delta^2-\|x-y\|^2|}{\|x-z\|^r}\nu(dz)$$
$$+ \frac{1}{r\beta_r\delta}\int_{S\setminus B(z_0,2\alpha_0)} (|g(z)| + |\gamma|)\frac{|\delta^2-\|x-y\|^2|}{\|x-z\|^r}\nu(dz)$$
$$\le \eta|\gamma| + \frac{\eta}{r\beta_r\delta}\int_S \frac{|\delta^2-\|x-y\|^2|}{\|x-z\|^r}\nu(dz)$$
$$+ \frac{1}{r\beta_r\delta}\int_{S\setminus B(z_0,2\alpha_0)} (|g(z)| + |\gamma|)\frac{\|x-z_0\|(2\delta+\alpha_0)}{2^r\alpha_0^r}\nu(dz)$$

(because $r \ge 2$, so $\nu\{z_0\} = 0$)

$$\le \eta|\gamma| + \eta(1+\eta) + (M + |\gamma|)\frac{\alpha(2\delta+\alpha_0)}{2^r r\beta_r\delta\alpha_0^r}\nu S$$
$$\le (|\gamma| + 1 + \eta + 1)\eta.$$

Also, of course, $f(x)-\gamma=g(x)-\gamma\le\eta$ if $0<\|x-z_0\|\le\alpha$ and $\|x-y\|=\delta$. As η is arbitrary, $\limsup_{x\to z_0}f(x)\le\gamma$. In the other direction, $\limsup_{x\to z_0}f(x)\ge\gamma$ just because f extends g.

(γ) Similarly, or applying (β) to $-g$, $\liminf_{x\to z_0}f(x)=\liminf_{x\in S,x\to z_0}g(x)$.

(iii)(α) If $r\ge 2$, it follows at once from (ii) that if g is continuous at $z_0\in S$, so is f, and that if g is lower semi-continuous (so that $g(z_0)\le\liminf_{x\in S,x\to z_0}g(x)$ for every $z_0\in S$) then f also is lower semi-continuous.

(β) If $r=1$, then $S=\{y-\delta,y+\delta\}$ and $\nu\{y-\delta\}=\nu\{y+\delta\}=1$, so

$$\begin{aligned}f(x)&=\tfrac{1}{2\delta}\bigl(\tfrac{|\delta^2-(x-y)^2|}{|x-(y-\delta)|}g(y-\delta)+\tfrac{|\delta^2-(x-y)^2|}{|x-(y+\delta)|}g(y+\delta)\bigr)\\&=\frac{1}{2\delta}\bigl(|x-(y+\delta)|g(y-\delta)+|x-(y-\delta)|g(y+\delta)\bigr)\end{aligned}$$

for $x\in\mathbb{R}\setminus S$, and $\lim_{x\mapsto y\pm\delta}f(x)=g(y\pm\delta)$, so f is continuous.

(iv) If $g(z)\le\alpha<\infty$ for every $z\in S$, then

$$\begin{aligned}f(x)&=\frac{1}{r\beta_r\delta}\int_S g(z)\tfrac{|\delta^2-\|x-y\|^2|}{\|x-z\|^r}\nu(dz)\le\frac{\alpha}{r\beta_r\delta}\int_{\partial B(\mathbf{0},\delta)}\tfrac{|\delta^2-\|x-y\|^2|}{\|x-y-z\|^r}\nu(dz)\\&=\tfrac{\alpha}{r\beta_r\delta}\cdot\tfrac{\nu(\partial B(\mathbf{0},\delta))}{\max(\delta,\|x-y\|)^{r-2}}=\tfrac{\alpha\delta^{r-2}}{\max(\delta,\|x-y\|)^{r-2}}\le\alpha\end{aligned}$$

for every $x\in\mathbb{R}^r\setminus S$. So $\sup_{x\in\mathbb{R}^r}f(x)=\sup_{z\in S}g(z)$; similarly, $\inf_{x\in\mathbb{R}^r}f(x)=\inf_{z\in S}g(z)$.

478J Convolutions and smoothing: Proposition (a) Suppose that $f:\mathbb{R}^r\to[0,\infty]$ is Lebesgue measurable, and $G\subseteq\mathbb{R}^r$ an open set such that $f\restriction G$ is superharmonic. Let $h:\mathbb{R}^r\to[0,\infty]$ be a Lebesgue integrable function, and $f*h$ the convolution of f and h. If $H\subseteq G$ is an open set such that $H-\{z:h(z)\ne 0\}\subseteq G$, then $(f*h)\restriction H$ is superharmonic.

(b) Suppose, in (a), that $h(y)=h(z)$ whenever $\|y\|=\|z\|$ and that $\int_{\mathbb{R}^r}h\,d\mu\le 1$. If $x\in G$ and $\gamma>0$ are such that $B(x,\gamma)\subseteq G$ and $h(y)=0$ whenever $\|y\|\ge\gamma$, then $(f*h)(x)\le f(x)$.

(c) Let $f:\mathbb{R}^r\to[0,\infty]$ be a lower semi-continuous function, and $\langle\tilde h_n\rangle_{n\in\mathbb{N}}$ the sequence of 473E. If $G\subseteq\mathbb{R}^r$ is an open set such that $f\restriction G$ is superharmonic, then $f(x)=\lim_{n\to\infty}(f*\tilde h_n)(x)$ for every $x\in G$.

(d) Let $G\subseteq\mathbb{R}^r$ be an open set, $K\subseteq G$ a compact set and $g:G\to\mathbb{R}$ a smooth function. Then there is a smooth function $f:G\to\mathbb{R}$ with compact support included in G such that f agrees with g on an open set including K.

proof (a) If $x\in H$ and $\delta>0$ are such that $B(x,\delta)\subseteq H$, then

$$\begin{aligned}\int_{\partial B(x,\delta)}(f*h)(y)\nu(dy)&=\int_{\partial B(x,\delta)}\int_{\mathbb{R}^r}f(y-z)h(z)\mu(dz)\nu(dy)\\&=\int_{\mathbb{R}^r}h(z)\int_{\partial B(x,\delta)}f(y-z)\nu(dy)\mu(dz)\\&=\int_{\mathbb{R}^r}h(z)\int_{\partial B(x-z,\delta)}f(y)\nu(dy)\mu(dz)\\&\le r\beta_r\delta^{r-1}\int_{\mathbb{R}^r}h(z)f(x-z)\mu(dz)\end{aligned}$$

(because if $h(z)\ne 0$ then $B(x-z,\delta)=B(x,\delta)-z$ is included in G)

$$=\nu(\partial B(x,\delta))\cdot(f*h)(x).$$

(b) Let $g:[0,\infty[\,\to[0,\infty]$ be such that $h(y)=g(\|y\|)$ for every y. Then

$$(f*h)(x) = \int_{\mathbb{R}^r} f(y)h(x-y)\mu(dy) = \int_0^\gamma \int_{\partial B(x,t)} f(y)g(t)\nu(dy)dt$$
$$\le \int_0^\gamma r\beta_r t^{r-1} f(x)g(t)dt = f(x)\int_{\mathbb{R}^r} h\,d\mu \le f(x).$$

(c) By (b), $(f*\tilde{h}_n)(x) \le f(x)$ for every sufficiently large n, so $\limsup_{n\to\infty}(f*\tilde{h}_n)(x) \le f(x)$. In the other direction, if $x \in G$ and $\alpha < f(x)$, there is a $\delta > 0$ such that $B(x,\delta) \subseteq G$ and $f(y) \ge \alpha$ for every $y \in B(x,\delta)$. Now there is an $m \in \mathbb{N}$ such that $\tilde{h}_n(y) = 0$ whenever $n \ge m$ and $\|y\| \ge \delta$; so that

$$(f*\tilde{h}_n)(x) = \int f(y)\tilde{h}_n(x-y)\mu(dy) = \int_{B(x,\delta)} f(y)\tilde{h}_n(x-y)\mu(dy)$$
$$\ge \alpha \int_{B(x,\delta)} \tilde{h}_n(x-y)\mu(dy) = \alpha$$

whenever $n \ge m$. As α is arbitrary, $f(x) = \lim_{n\to\infty}(f*\tilde{h}_n)(x)$.

(d) If K is empty we can take f to be the constant function with value 0, so suppose otherwise. Let $\delta > 0$ be such that $B(x,4\delta) \subseteq G$ for every $x \in K$, and set $H_k = \{x : \rho(x,K) < k\delta\}$ for $k \in \mathbb{N}$. Let n be such that $\frac{1}{n+1} \le \delta$, and set $f = g \times (\chi H_2 * \tilde{h}_n)$. By 473De, $\chi H_2 * \tilde{h}_n$ and therefore f are smooth. If $x \in H_1$ then $(\chi H_2 * \tilde{h}_n)(x) = 1$ so $f(x) = g(x)$, while if $x \in G \setminus H_3$ then $f(x) = (\chi H_2 * \tilde{h}_n)(x) = 0$ so the support of f is included in the compact set $\overline{H}_3 \subseteq G$.

478K Dynkin's formula: Lemma Let μ_W be r-dimensional Wiener measure on $\Omega = C([0,\infty[\,;\mathbb{R}^r)_0$; set $X_t(\omega) = \omega(t)$ for $\omega \in \Omega$ and $t \ge 0$. Let $f : \mathbb{R}^r \to \mathbb{R}$ be a three-times-differentiable function such that f and its first three derivatives are continuous and bounded.

(a) $\mathbb{E}(f(X_t)) = f(0) + \frac{1}{2}\mathbb{E}(\int_0^t (\nabla^2 f)(X_s)ds)$ for every $t \ge 0$.

(b) If $\tau : \Omega \to [0,\infty[$ is a stopping time adapted to $\langle \Sigma_t^+\rangle_{t\ge 0}$ and $\mathbb{E}(\tau)$ is finite, then

$$\mathbb{E}(f(X_\tau)) = f(0) + \frac{1}{2}\mathbb{E}(\int_0^\tau (\nabla^2 f)(X_s)ds).$$

proof (a)(i) We need a special case of the multidimensional Taylor's theorem. If $f : \mathbb{R}^r \to \mathbb{R}$ is three times differentiable and $x = (\xi_1,\dots,\xi_r)$, $y = (\eta_1,\dots,\eta_r) \in \mathbb{R}^r$, then there is a z in the line segment $[x,y]$ such that

$$f(y) = f(x) + \sum_{i=1}^r (\eta_i - \xi_i)\frac{\partial f}{\partial \xi_i}(x) + \frac{1}{2}\sum_{i=1}^r\sum_{j=1}^r (\eta_i-\xi_i)(\eta_j-\xi_j)\frac{\partial^2 f}{\partial\xi_i\partial\xi_j}(x)$$
$$+ \frac{1}{6}\sum_{i=1}^r\sum_{j=1}^r\sum_{k=1}^r (\eta_i-\xi_i)(\eta_j-\xi_j)(\eta_k-\xi_k)\frac{\partial^3 f}{\partial\xi_i\partial\xi_j\partial\xi_k}(z).$$

P Set $g(\beta) = f(\beta y + (1-\beta)x)$ for $\beta \in \mathbb{R}$. Then g is three times differentiable, with

$$g'(\beta) = \sum_{k=1}^r (\eta_k - \xi_k)\frac{\partial f}{\partial\xi_k}(\beta y + (1-\beta)x),$$
$$g''(\beta) = \sum_{j=1}^r\sum_{k=1}^r (\eta_j-\xi_j)(\eta_k-\xi_k)\frac{\partial^2 f}{\partial\xi_j\partial\xi_k}(\beta y + (1-\beta)x),$$
$$g'''(\beta) = \sum_{i=1}^r\sum_{j=1}^r\sum_{k=1}^r (\eta_i-\xi_i)(\eta_j-\xi_j)(\eta_k-\xi_k)\frac{\partial^3 f}{\partial\xi_i\partial\xi_j\partial\xi_k}(\beta y + (1-\beta)x).$$

Now by Taylor's theorem with remainder, in one dimension, there is a $\beta \in\,]0,1[$ such that

$$g(1) = g(0) + g'(0) + \frac{1}{2}g''(0) + \frac{1}{6}g'''(\beta)$$

and all we have to do is to set $z = \beta y + (1-\beta)x$ and substitute in the values for $g(1), \dots, g'''(\beta)$. **Q**

(ii) Let $M \ge 0$ be such that $\|\frac{\partial^3 f}{\partial\xi_i\partial\xi_j\partial\xi_k}\|_\infty \le M$ whenever $1 \le i,\ j,\ k \le r$. Let K be $\mathbb{E}((\sum_{i=1}^r |Z_i|)^3)$ when $Z_1, \dots, Z_r$ are independent real-valued random variables with standard normal distribution. (To see that this is finite, observe that

$$\mathbb{E}((\sum_{i=1}^r |Z_i|)^3) \le \mathbb{E}(r^3 \max_{i\le r} |Z_i|^3) \le r^3\,\mathbb{E}(\sum_{i=1}^r |Z_i|^3) = r^4\,\mathbb{E}(|Z|^3)$$

(where Z is a random variable with standard normal distribution)

$$= \frac{2r^4}{\sqrt{2\pi}}\int_0^\infty t^3 e^{-t^2/2} dt < \infty.)$$

For any $x,\ y \in \mathbb{R}^r$ we have

$$\bigl|f(y) - f(x) - \sum_{i=1}^r (\eta_i - \xi_i)\frac{\partial f}{\partial\xi_i}(x) + \frac{1}{2}\sum_{i=1}^r\sum_{j=1}^r (\eta_i - \xi_i)(\eta_j - \xi_j)\frac{\partial^2 f}{\partial\xi_i\partial\xi_j}(x)\bigr|$$
$$\le \frac{M}{6}(\sum_{i=1}^r |\eta_i - \xi_i|)^3.$$

If $0 \le s \le t$ and $\omega \in \Omega$, then

$$\bigl|f(\omega(t)) - f(\omega(s)) - \sum_{i=1}^r \frac{\partial f}{\partial\xi_i}(\omega(s))(\omega_i(t) - \omega_i(s))$$
$$- \frac{1}{2}\sum_{i=1}^r\sum_{j=1}^r \frac{\partial^2 f}{\partial\xi_i\partial\xi_j}(\omega(s))(\omega_i(t) - \omega_i(s))(\omega_j(t) - \omega_j(s))\bigr|$$
$$\le \frac{M}{6}(\sum_{i=1}^r |\omega_i(t) - \omega_i(s)|)^3,$$

writing $\omega_1, \dots, \omega_r \in C([0,\infty[)_0$ for the coordinates of $\omega \in \Omega$. Integrating with respect to ω, we have

$$|\mathbb{E}(f(X_t) - f(X_s) - \frac{1}{2}(t-s)(\nabla^2 f)(X_s))|$$
$$= \bigl|\mathbb{E}(f(X_t) - f(X_s)) - \sum_{i=1}^r \mathbb{E}(\frac{\partial f}{\partial\xi_i}(X_s))\mathbb{E}(X_t^{(i)} - X_s^{(i)})$$
$$- \frac{1}{2}\sum_{i=1}^r \mathbb{E}(\frac{\partial^2 f}{\partial\xi_i^2}(X_s))\mathbb{E}(X_t^{(i)} - X_s^{(i)})^2$$
$$- \frac{1}{2}\sum_{i=1}^r\sum_{j\ne i} \mathbb{E}(\frac{\partial^2 f}{\partial\xi_i\partial\xi_j})(X_s)\mathbb{E}(X_t^{(i)} - X_s^{(i)})\mathbb{E}(X_t^{(j)} - X_s^{(j)})\bigr|$$

(writing $X_t^{(i)}(\omega) = \omega_i(t)$ for $1 \le i \le r$, and recalling that $\mathbb{E}(X_t^{(i)} - X_s^{(i)}) = 0$ for every i, while $\mathbb{E}(X_t^{(i)} - X_s^{(i)})^2 = t - s$)

$$= \big|\mathbb{E}(f(X_t)-f(X_s)) - \sum_{i=1}^{r}\mathbb{E}((X_t^{(i)}-X_s^{(i)})\frac{\partial f}{\partial \xi_i}(X_s))$$
$$-\frac{1}{2}\sum_{i=1}^{r}\mathbb{E}((X_t^{(i)}-X_s^{(i)})^2\frac{\partial^2 f}{\partial \xi_i^2}(X_s))$$
$$-\frac{1}{2}\sum_{i=1}^{r}\sum_{j\ne i}\mathbb{E}((X_t^{(i)}-X_s^{(i)})(X_t^{(j)}-X_s^{(j)})\frac{\partial^2 f}{\partial\xi_i\partial\xi_j}(X_s))\big|$$

(because for any $i \le r$ the random variables $\frac{\partial f}{\partial \xi_i}(X_s)$ and $X_t^{(i)}-X_s^{(i)}$ are independent, while for any distinct i, $j \le r$ the random variables $\frac{\partial^2 f}{\partial\xi_i\partial\xi_j}(X_s)$, $X_t^{(i)}-X_s^{(i)}$ and $X_t^{(j)}-X_s^{(j)}$ are independent)

$$= \big|\mathbb{E}\big(f(X_t)-f(X_s)-\sum_{i=1}^{r}(X_t^{(i)}-X_s^{(i)})\frac{\partial f}{\partial\xi_i}(X_s)$$
$$-\frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{r}(X_t^{(i)}-X_s^{(i)})(X_t^{(j)}-X_s^{(j)})\frac{\partial^2 f}{\partial\xi_i\partial\xi_j}(X_s)\big)\big|$$
$$\le \frac{M}{6}\mathbb{E}((\sum_{i=1}^{r}|X_t^{(i)}-X_s^{(i)}|)^3) \le \frac{MK}{6}(t-s)^{3/2}$$

because the $X_t^{(i)}-X_s^{(i)}$ are independent random variables all with the same distribution as $\sqrt{t-s}\,Z$ where Z is standard normal.

(iii) Now fix $t \ge 0$ and $n \ge 1$; set $s_k = \frac{k}{n}t$ for $k \le n$. Set

$$g_n(\omega) = \sum_{k=0}^{n-1}(s_{k+1}-s_k)(\nabla^2 f)(\omega(s_k))$$

for $\omega \in \Omega$. Then

$$\big|\int\big(f(\omega(t))-f(0)-\tfrac{1}{2}g_n(\omega)\big)\mu_W(d\omega)\big|$$
$$= \big|\sum_{k=0}^{n-1}\mathbb{E}(f(X_{s_{k+1}})-f(X_{s_k})-\frac{1}{2}(s_{k+1}-s_k)(\nabla^2 f)(X_{s_k}))\big|$$
$$\le \sum_{k=0}^{n-1}\frac{MK}{6}(\frac{t}{n})^{3/2} = \frac{MKt\sqrt{t}}{6\sqrt{n}}.$$

On the other hand,

$$\lim_{n\to\infty} g_n(\omega) = \int_0^t(\nabla^2 f)(\omega(s))ds$$

for every ω (the Riemann integral $\int_0^t(\nabla^2 f)(\omega(s))ds$ is defined because $s \mapsto (\nabla^2 f)(\omega(s))$ is continuous), and $|g_n(\omega)| \le t\|\nabla^2 f\|_\infty < \infty$ for every ω, so by Lebesgue's Dominated Convergence Theorem

$$\mathbb{E}(f(X_t)-f(0)) = \frac{1}{2}\lim_{n\to\infty}\int g_n(\omega)\mu_W(d\omega) = \frac{1}{2}\int\lim_{n\to\infty}g_n(\omega)\mu_W(d\omega)$$
$$= \frac{1}{2}\int\int_0^t(\nabla^2 f)(\omega(s))ds\,\mu_W(d\omega) = \frac{1}{2}\mathbb{E}\big(\int_0^t(\nabla^2 f)(X_s)ds\big)$$

as claimed.

(b)(i) Consider first the case in which τ takes values in a finite set $I \subseteq [0,\infty[$. In this case we can induce on $\#(I)$. If $I = \{t_0\}$ then

$$\mathbb{E}(f(X_\tau)) = \mathbb{E}(f(X_{t_0})) = f(0)+\tfrac{1}{2}\mathbb{E}(\int_0^{t_0}(\nabla^2 f)(X_s)ds) = f(0)+\tfrac{1}{2}\mathbb{E}(\int_0^{\tau}(\nabla^2 f)(X_s)ds)$$

by (a). For the inductive step to $\#(I) > 1$, set $t_0 = \min I$, $E = \{\omega : \tau(\omega) = t_0\}$ and

$$\begin{aligned}\phi(\omega, \omega')(t) &= \omega(t) \text{ if } t \le t_0, \\ &= \omega(t_0) + \omega'(t - t_0) \text{ if } t \ge t_0,\end{aligned}$$

for ω, $\omega' \in \Omega$, so that ϕ is inverse-measure-preserving (477G). Set

$$\sigma_\omega(\omega') = \tau(\phi(\omega, \omega')) - t_0$$

for ω, $\omega' \in \Omega$. If $\omega \in \Omega \setminus E$, σ_ω is a stopping time adapted to $\langle \Sigma_t^+ \rangle_{t \ge 0}$, taking fewer than $\#(I)$ values. **P** Suppose that $t > 0$ and $F = \{\omega' : \sigma_\omega(\omega') < t\}$. If $\omega' \in F$, $\tilde{\omega}' \in \Omega$ and $\tilde{\omega}' \restriction [0, t] = \omega' \restriction [0, t]$, then $\tau(\phi(\omega, \omega')) < t + t_0$, while $\phi(\omega, \tilde{\omega}')(s) = \phi(\omega, \omega')(s)$ whenever $s \le t + t_0$; so that

$$\sigma_\omega(\tilde{\omega}') + t_0 = \tau(\phi(\omega, \tilde{\omega}')) < t + t_0$$

and $\tilde{\omega}' \in F$. Thus $F \in \Sigma_t$; as t is arbitrary, σ_ω is adapted to $\langle \Sigma_t^+ \rangle_{t \ge 0}$ (455Lb). Also every value of σ_ω belongs to $\{t - t_0 : t \in I, t > t_0\}$ which is smaller than I. **Q**

Writing $\int \ldots d\omega$ and $\int \ldots d\omega'$ for integration with respect to μ_W,

$$\begin{aligned}
&\mathbb{E}\Bigl(\int_0^\tau (\nabla^2 f)(X_s) ds\Bigr) \\
&\quad = \int_\Omega \int_0^{\tau(\omega)} (\nabla^2 f)(\omega(s)) ds\, d\omega \\
&\quad = \int_\Omega \int_\Omega \int_0^{\tau(\phi(\omega,\omega'))} (\nabla^2 f)(\phi(\omega, \omega')(s)) ds\, d\omega' d\omega \\
&\quad = \int_E \int_\Omega \int_0^{t_0} (\nabla^2 f)(\omega(s)) ds\, d\omega' d\omega \\
&\qquad\qquad + \int_{\Omega \setminus E} \int_\Omega \int_0^{t_0 + \sigma_\omega(\omega')} (\nabla^2 f)(\phi(\omega, \omega')(s)) ds\, d\omega' d\omega \\
&\quad = \int_E \int_0^{t_0} (\nabla^2 f)(\omega(s)) ds\, d\omega + \int_{\Omega \setminus E} \int_\Omega \int_0^{t_0} (\nabla^2 f)(\phi(\omega, \omega')(s)) ds\, d\omega' d\omega \\
&\qquad\qquad + \int_{\Omega \setminus E} \int_\Omega \int_{t_0}^{t_0 + \sigma_\omega(\omega')} (\nabla^2 f)(\phi(\omega, \omega')(s)) ds\, d\omega' d\omega \\
&\quad = \int_\Omega \int_0^{t_0} (\nabla^2 f)(\omega(s)) ds\, d\omega \\
&\qquad\qquad + \int_{\Omega \setminus E} \int_\Omega \int_{t_0}^{t_0 + \sigma_\omega(\omega')} (\nabla^2 f)(\omega(t_0) + \omega'(s - t_0)) ds\, d\omega' d\omega \\
&\quad = \int_\Omega \int_0^{t_0} (\nabla^2 f)(\omega(s)) ds\, d\omega + \int_{\Omega \setminus E} \int_\Omega \int_0^{\sigma_\omega(\omega')} (\nabla^2 f)(\omega(t_0) + \omega'(s)) ds\, d\omega' d\omega \\
&\quad = 2 \int_\Omega f(\omega(t_0)) - f(0) d\omega \\
&\qquad\qquad + 2 \int_{\Omega \setminus E} \int_\Omega f(\omega(t_0) + \omega'(\sigma_\omega(\omega'))) - f(\omega(t_0)) d\omega' d\omega
\end{aligned}$$

(applying the inductive hypothesis to the function $x \mapsto f(\omega(t_0) + x)$ and the stopping time σ_ω)

$$= 2 \int_E f(\omega(t_0)) - f(0) d\omega + 2 \int_{\Omega \setminus E} \int_\Omega f(\omega(t_0) + \omega'(\sigma_\omega(\omega'))) - f(0) d\omega' d\omega$$

$$= 2\int_E\int_\Omega f\bigl(\phi(\omega,\omega')(\tau(\phi(\omega,\omega')))\bigr) - f(0)d\omega' d\omega$$

$$+ 2\int_{\Omega\setminus E}\int_\Omega f\bigl(\phi(\omega,\omega')(\tau(\phi(\omega,\omega')))\bigr) - f(0)d\omega' d\omega$$

$$= 2\int_\Omega\int_\Omega f\bigl(\phi(\omega,\omega')(\tau(\phi(\omega,\omega')))\bigr) - f(0)d\omega' d\omega$$

$$= 2\int_\Omega f(\omega(\tau(\omega))) - f(0)d\omega = 2\bigl(\mathbb{E}(f(X_\tau)) - f(0)\bigr).$$

Turning this around, we have the formula we want, so the induction proceeds.

(ii) Now suppose that every value of τ belongs to an infinite set of the form $\{t_n : n \in \mathbb{N}\} \cup \{\infty\}$ where $\langle t_n\rangle_{n\in\mathbb{N}}$ is a strictly increasing sequence in $[0,\infty[$. In this case, for $n \in \mathbb{N}$, define

$$\tau_n(\omega) = \min(\tau(\omega), t_n)$$

for $\omega \in \Omega$, so that τ_n takes values in the finite set $\{t_0, \dots , t_n\}$, and

$$\{\omega : \tau_n(\omega) < t\} = \{\omega : \tau(\omega) < t\} \in \Sigma_t \text{ if } t \le t_n,$$
$$= \Omega \in \Sigma_t \text{ if } t > t_n.$$

Now τ is finite a.e., so $\tau =_{\text{a.e.}} \lim_{n\to\infty} \tau_n$; it follows that

$$f(X_\tau(\omega)) = f(\omega(\tau(\omega))) = \lim_{n\to\infty} f(\omega(\tau_n(\omega))) = \lim_{n\to\infty} f(X_{\tau_n}(\omega))$$

for almost every ω; because f is bounded,

$$\mathbb{E}(f(X_\tau)) = \lim_{n\to\infty} \mathbb{E}(f(X_{\tau_n})).$$

On the other side,

$$\int_0^{\tau(\omega)}(\nabla^2 f)(\omega(s))ds = \lim_{n\to\infty}\int_0^{\tau_n(\omega)}(\nabla^2 f)(\omega(s))ds$$

for almost every ω. At this point, recall that we are supposing that τ has finite expectation and that $\nabla^2 f$ is bounded. So

$$|\int_0^{\tau_n(\omega)}(\nabla^2 f)(\omega(s))ds| \le \int_0^{\tau(\omega)}|(\nabla^2 f)(\omega(s))ds| \le \|\nabla^2 f\|_\infty \tau(\omega)$$

for every ω, and the dominated convergence theorem assures us that

$$\mathbb{E}\bigl(\int_0^\tau(\nabla^2 f)(X_s)ds\bigr) = \int_\Omega\int_0^{\tau(\omega)}(\nabla^2 f)(\omega(s))ds\,d\omega$$

$$= \lim_{n\to\infty}\int_\Omega\int_0^{\tau_n(\omega)}(\nabla^2 f)(\omega(s))ds\,d\omega$$

$$= \lim_{n\to\infty}\mathbb{E}\bigl(\int_0^{\tau_n}(\nabla^2 f)(X_s)ds\bigr).$$

Accordingly

$$\mathbb{E}(f(X_\tau)) = \lim_{n\to\infty}\mathbb{E}(f(X_{\tau_n})) = \lim_{n\to\infty} f(0) + \tfrac{1}{2}\mathbb{E}\bigl(\int_0^{\tau_n}(\nabla^2 f)(X_s)ds\bigr)$$

(by (i))

$$= f(0) + \tfrac{1}{2}\mathbb{E}\bigl(\int_0^\tau(\nabla^2 f)(X_s)ds\bigr),$$

as required.

(iii) Suppose just that τ has finite expectation. This time, for $n \in \mathbb{N}$, define a stopping time τ_n by saying that

$$\tau_n(\omega) = 2^{-n}k \text{ if } k \ge 1 \text{ and } 2^{-n}(k-1) \le \tau(\omega) < 2^{-n}k,$$
$$= \infty \text{ if } \tau(\omega) = \infty.$$

If $t > 0$, set $t' = 2^{-n}k$ where $2^{-n}k < t \le 2^{-n}(k+1)$; then

$$\{\omega : \tau_n(\omega) < t\} = \{\omega : \tau(\omega) < t'\} \in \Sigma_{t'} \subseteq \Sigma_t.$$

So τ_n is a stopping time adapted to $\langle \Sigma_t^+ \rangle_{t \ge 0}$; as $\tau_n \le 2^{-n} + \tau$, $\mathbb{E}(\tau_n) < \infty$. Again we have $\tau(\omega) = \lim_{n\to\infty} \tau_n(\omega)$ for every ω. The arguments of (ii) now tell us that, as before,

$$f(X_\tau(\omega)) = \lim_{n\to\infty} f(X_{\tau_n}(\omega))$$

(because f is continuous),

$$\int_0^{\tau(\omega)} (\nabla^2 f)(\omega(s))ds = \lim_{n\to\infty} \int_0^{\tau_n(\omega)} (\nabla^2 f)(\omega(s))ds$$

for almost every ω, so that

$$\mathbb{E}(f(X_\tau)) = \lim_{n\to\infty} \mathbb{E}(f(X_{\tau_n})),$$

$$\mathbb{E}(\int_0^\tau (\nabla^2 f)(X_s)ds) = \lim_{n\to\infty} \mathbb{E}(\int_0^{\tau_n} (\nabla^2 f)(X_s)ds).$$

(This time, of course, we need to check that

$$|\int_0^{\tau_n(\omega)} (\nabla^2 f)(\omega(s))ds| \le \|\nabla^2 f\|_\infty \tau_0(\omega)$$

for almost every ω, to confirm that we have dominated convergence.) So once again the desired formula can be got by taking the limit of a sequence of equalities we already know.

478L Theorem Let μ_W be r-dimensional Wiener measure on $\Omega = C([0,\infty[\,;\mathbb{R}^r)_0$, $f : \mathbb{R}^r \to [0,\infty]$ a lower semi-continuous superharmonic function, and $\tau : \Omega \to [0,\infty]$ a stopping time adapted to $\langle \Sigma_t^+ \rangle_{t\ge0}$. Set $H = \{\omega : \omega \in \Omega,\ \tau(\omega) < \infty\}$. Then

$$f(x) \ge \int_H f(x + \omega(\tau(\omega)))\mu_W(d\omega)$$

for every $x \in \mathbb{R}^r$.

proof (a) To begin with, suppose that f is real-valued and bounded. Let $\langle \tilde{h}_m \rangle_{m\in\mathbb{N}}$ be the sequence of 473E/478J, and for $m \in \mathbb{N}$ set $f_m = f * \tilde{h}_m$. Then each f_m is non-negative, smooth with bounded derivatives of all orders (473De) and superharmonic (478Ja), so $\nabla^2 f_m \le 0$ (478Ea). Set $\tau_n(\omega) = \min(n, \tau(\omega))$ for $n \in \mathbb{N}$ and $\omega \in \Omega$; then each τ_n is a stopping time, adapted to $\langle \Sigma_t^+ \rangle_{t\ge0}$, with finite expectation. In the language of 478K,

$$\mathbb{E}(f_m(x + X_{\tau_n})) = f_m(x) + \frac{1}{2}\mathbb{E}\Big(\int_0^{\tau_n} (\nabla^2 f_m)(x + X_s)ds\Big) \le f_m(x)$$

whenever $m, n \in \mathbb{N}$. Letting $n \to \infty$,

$$\int_H f_m(x + \omega(\tau(\omega)))\mu_W(d\omega) = \lim_{n\to\infty} \int_H f_m(x + \omega(\tau_n(\omega)))\mu_W(\omega)$$

(because f_m, and every $\omega \in \Omega$, are continuous, and $\tau(\omega) = \lim_{n\to\infty} \tau_n(\omega)$ for every ω)

$$\le \liminf_{n\to\infty} \int_{\mathbb{R}^r} f_m(x + \omega(\tau_n(\omega)))\mu_W(d\omega)$$

(because f_m is non-negative)

$$\le f_m(x)$$

by Fatou's Lemma. Now $f = \lim_{m\to\infty} f_m$ (478Jc), so

$$\int_H f(x + \omega(\tau(\omega)))\mu_W(\omega) \le \liminf_{m\to\infty} \int_H f_m(x + \omega(\tau(\omega)))\mu_W(\omega)$$
$$\le \liminf_{m\to\infty} f_m(x) = f(x),$$

which is what we need to know.

(b) For the general case, set $g_k = f \wedge k\chi\mathbb{R}^r$ for each $k \in \mathbb{N}$. Then g_k is non-negative, lower semi-continuous, superharmonic (478Cc) and bounded. So

$$\begin{aligned}\int_H f(x+\omega(\tau(\omega)))\mu_W(d\omega) &= \lim_{k\to\infty}\int_H g_k(x+\omega(\tau(\omega)))\mu_W(d\omega)\\ &\le \lim_{k\to\infty} g_k(x) = f(x).\end{aligned}$$

478M Proposition (a) If $r = 1$, then $\{\omega(t) : t \ge 0\} = \mathbb{R}$ for almost every $\omega \in \Omega$.
(b) If $r \le 2$, then $\{\omega(t) : t \ge 0\}$ is dense in $\mathbb{R}^2$ for almost every $\omega \in \Omega$.
(c) If $r \ge 2$, then for every $z \in \mathbb{R}^2$, $z \notin \{\omega(t) : t > 0\}$ for almost every $\omega \in \Omega$.
(d) If $r \ge 3$, then $\lim_{t\to\infty} \|\omega(t)\| = \infty$ for almost every $\omega \in \Omega$.

proof (a) Suppose that α, $\beta > 0$ and that τ is the Brownian exit time from $]-\alpha,\beta[$; then τ is a stopping time adapted to $\langle\Sigma_t\rangle_{t\ge0}$ (477Ic). Now τ is almost everywhere finite and $\Pr(X_\tau = \beta) = \frac{\alpha}{\alpha+\beta}$. **P** Since $\Pr(|X_t| \le \max(\alpha,\beta)) \to 0$ as $t \to \infty$, τ is finite a.e., and $\Pr(X_\tau = \beta) + \Pr(X_\tau = -\alpha) = 1$. Set $\tau_n(\omega) = \min(n,\tau(\omega))$ for each n, and let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function with compact support such that $f(x) = x$ for $x \in [-\alpha-1, \beta+1]$ (478Jd). Now 478K tells us that

$$\mathbb{E}(X_{\tau_n}) = \mathbb{E}(f(X_{\tau_n})) = f(0) + \frac{1}{2}\mathbb{E}(\int_0^{\tau_n}(\nabla^2 f)(X_s)ds) = f(0) = 0.$$

Since $\langle X_{\tau_n}\rangle_{n\in\mathbb{N}}$ is a uniformly bounded sequence converging almost everywhere to X_τ,

$$\beta\Pr(X_\tau = \beta) - \alpha\Pr(X_\tau = -\alpha) = \mathbb{E}(X_\tau) = \lim_{n\to\infty}\mathbb{E}(X_{\tau_n}) = 0,$$

and $\Pr(X_\tau = \beta) = \frac{\alpha}{\alpha+\beta}$. **Q**

Letting $\alpha \to \infty$, we see that $\Pr(\exists\, t \ge 0,\ X_t = \beta) = 1$. Similarly, $-\alpha$ lies on almost every sample path.

Thus almost every sample path must pass through every point of $\mathbb{Z}$; since sample paths are continuous, they almost all cover $\mathbb{R}$.

(b) For $r = 1$ this is covered by (a); take $r = 2$. Suppose that $z \in \mathbb{R}^2$ and that $\delta > 0$. Then almost every sample path meets $B(z,\delta)$. **P** If $\delta \ge \|z\|$ this is trivial. Otherwise, take $R > \|z\|$ and let τ be the Brownian exit time from $G = \operatorname{int} B(z,R) \setminus B(z,\delta)$. We have $\Pr(\|X_t\| \le R + \|z\|) \to 0$ as $t \to \infty$ (because $\Pr(\|X_t\| \le \alpha) \le \Pr(|Z| \le \frac{\alpha}{\sqrt{t}})$ where Z is a standard normal random variable), so τ is finite a.e. Once again, set $\tau_n(\omega) = \min(n,\tau(\omega))$ for $n \in \mathbb{N}$ and $\omega \in \Omega$; this time, take a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ with compact support such that $f(x) = \ln\|x - z\|$ for $x \in B(z,2R) \setminus B(z,\frac{1}{2}\delta)$. Then

$$\mathbb{E}(f(X_{\tau_n})) = f(0) = \ln\|z\|$$

(use 478Fb), so

$$\begin{aligned}\ln R\cdot\Pr(X_\tau \in \partial B(z,R)) &+ \ln\delta\cdot\Pr(X_\tau \in \partial B(z,\delta))\\ &= \mathbb{E}(f(X_\tau)) = \lim_{n\to\infty}\mathbb{E}(f(X_{\tau_n})) = \ln\|z\|\end{aligned}$$

and

$$\Pr(X_\tau \in \partial B(z,\delta)) = \frac{\ln R - \ln\|z\|}{\ln R - \ln\delta}.$$

Letting $R \to \infty$, we see that $\Pr(\exists\ t \ge 0, \omega(t) \in B(z,\delta)) = 1$; that is, almost every sample path meets $B(z,\delta)$. **Q**

Letting $B(z,\delta)$ run over a sequence of balls constituting a network for the topology of $\mathbb{R}^2$, we see that almost every path meets every non-empty open set and is dense in $\mathbb{R}^2$.

(c)(i) Consider first the case $r = 2$.

(α) Suppose that $z \neq 0$. In this case, take δ, R such that $0 < \delta < \|z\| < R$ and let τ be the Brownian exit time from $G = \operatorname{int} B(z, R) \setminus B(z, \delta)$, as in the proof of (b). As before, we have

$$\Pr(X_\tau \in \partial B(z, \delta)) = \frac{\ln R - \ln \|z\|}{\ln R - \ln \delta}.$$

This time, looking at the limit as $\delta \downarrow 0$, we see that

$$\{\omega : \text{there is a } t \geq 0 \text{ such that } \omega(t) = z \text{ but } \|\omega(s) - z\| < R \text{ for every } s \leq t\}$$

is negligible. Taking the union of these sets over large integer R, we see that

$$\{\omega : \text{there is a } t \geq 0 \text{ such that } \omega(t) = z\}$$

is negligible, as required.

(β) As for $z = 0$, take any $\epsilon > 0$. Then

$$\begin{aligned}
\mu_W\{\omega &: \text{there is some } t \geq \epsilon \text{ such that } \omega(t) = 0\} \\
&= \mu_W^2\{(\omega, \omega') : \text{ there is some } t \geq 0 \text{ such that } \omega'(t) = -\omega(\epsilon)\} \\
&= \mu_W^2\{(\omega, \omega') : \omega(\epsilon) \neq 0 \text{ and there is some } t \geq 0 \text{ such that } \omega'(t) = -\omega(\epsilon)\}
\end{aligned}$$

(because the distribution of X_ϵ is atomless, so $\{\omega : \omega(\epsilon) = 0\}$ is negligible)

$$= 0$$

by (α). Taking the union over rational $\epsilon > 0$, $\{\omega :$ there is some $t > 0$ such that $\omega(t) = 0\}$ is negligible.

(ii) If $r > 2$, set $Tx = (\xi_1, \xi_2)$ for $x = (\xi_1, \dots, \xi_r) \in \mathbb{R}^r$. Then $\omega \mapsto T\omega : C([0, \infty[\,; \mathbb{R}^r)_0 \to C([0, \infty[\,; \mathbb{R}^2)_0$ is inverse-measure-preserving for r-dimensional Wiener measure μ_{W^r} on $C([0, \infty[\,; \mathbb{R}^r)_0$ and two-dimensional Wiener measure μ_{W^2} on $C([0, \infty[\,; \mathbb{R}^2)_0$, by 477D(c-i) or otherwise. So

$$\{\omega : z \in \omega[\,]0, \infty[\,]\} \subseteq \{\omega : Tz \in (T\omega)[\,]0, \infty[\,]\}$$

is negligible.

(d)(i) Fix $\gamma \in [0, \infty[$ and $\epsilon > 0$ for the moment. Set $g(x) = \int \frac{1}{\|y\|^{r-2}} \tilde{h}_0(x - y) \mu(dy)$ for $x \in \mathbb{R}^r$, where $\tilde{h}_0$ is the function of 473E; then g is smooth (473De), strictly positive and superharmonic (478Ja). In addition, we have the following.

(α) All the derivatives of g are bounded. **P** As shown in the proof of 473De, $\frac{\partial g}{\partial \xi_i}(x) = \int \frac{1}{\|y\|^{r-2}} \frac{\partial}{\partial \xi_i} \tilde{h}_0(x - y) \mu(dy)$ for $1 \leq i \leq r$ and $x \in \mathbb{R}^r$. Inducing on the order of D, and using 478Gc at the last step, we see that

$$\begin{aligned}
(Dg)(x) &= \int \frac{1}{\|y\|^{r-2}} (D\tilde{h}_0)(x - y) \mu(dy) = \int \frac{1}{\|x-y\|^{r-2}} (D\tilde{h}_0)(y) \mu(dy) \\
&= \int_{B(\mathbf{0},1)} \frac{1}{\|x-y\|^{r-2}} (D\tilde{h}_0)(y) \mu(dy) \\
&\leq \|D\tilde{h}_0\|_\infty \int_{B(\mathbf{0},1)} \frac{1}{\|x-y\|^{r-2}} \mu(dy) \leq \frac{1}{2} r \beta_r \|D\tilde{h}_0\|_\infty
\end{aligned}$$

for any partial differential operator D and any $x \in \mathbb{R}^r$. **Q**

(β) $\lim_{\|x\| \to \infty} g(x) = 0$, because

$$g(x) \leq \|\tilde{h}_0\|_\infty \int_{B(\mathbf{0},1)} \frac{1}{\|x-y\|^{r-2}} \mu(dy) \leq \|\tilde{h}_0\|_\infty \frac{\beta_r}{(\|x\|-1)^{r-2}}$$

whenever $\|x\| > 1$.

(γ) $g(x) = g(y)$ whenever $\|x\| = \|y\|$. **P** Let $T : \mathbb{R}^r \to \mathbb{R}^r$ be an orthogonal transformation such that $Tx = y$. Then

$$g(x) = \int \frac{1}{\|x-z\|^{r-2}}\tilde{h}_0(z)\mu(dz) = \int \frac{1}{\|T(x-z)\|^{r-2}}\tilde{h}_0(Tz)\mu(dz)$$

(because $\tilde{h}_0 T = \tilde{h}_0$)

$$= \int \frac{1}{\|y-Tz\|^{r-2}}\tilde{h}_0(Tz)\mu(dz) = \int \frac{1}{\|y-z\|^{r-2}}\tilde{h}_0(z)\mu(dz)$$

(because T is an automorphism of $(\mathbb{R}^r, \mu)$)

$$= g(y). \text{ } \mathbf{Q}$$

(ii) Let $\beta > 0$ be the common value of $g(y)$ for $\|y\| = \gamma$. Take $x \in \mathbb{R}^r$ such that $\|x\| > \gamma$, and $n \in \mathbb{N}$. Define

$$\tau(\omega) = \min(\{n\} \cup \{t : \|x + \omega(t)\| \le \gamma\})$$

for $\omega \in \Omega$. Then τ is a bounded stopping time adapted to $\langle \Sigma_t \rangle_{t \ge 0}$, so

$$\begin{aligned}\beta \Pr(\tau < n) &\le \mathbb{E}(g(x + X_\tau)) \\ &= g(x) + \frac{1}{2}\mathbb{E}(\int_0^\tau (\nabla^2 g)(x + X_s)ds) \le g(x).\end{aligned}$$

Letting $n \to \infty$, we see that

$$\mu_W\{\omega : \|x + \omega(t)\| \le \gamma \text{ for some } t \ge 0\} \le \frac{1}{\beta}g(x).$$

(iii) Now let $n > \gamma$ be an integer such that $\frac{1}{\beta}g(x) \le \epsilon$ whenever $\|x\| \ge n$. As in (a) and (b-i) above, $\lim_{t\to\infty} \Pr(\|X_t\| \le n) = 0$; take $m \in \mathbb{N}$ such that $\Pr(\|X_m\| \le n) \le \epsilon$. Let σ be the stopping time with constant value m, with $\phi_\sigma : \Omega \times \Omega \to \Omega$ the corresponding inverse-measure-preserving function (477G). Set $F = \{\omega : \|\omega(m)\| > n\}$. Now

$$\begin{aligned}&\Pr(\|X_t\| \le \gamma \text{ for some } t \ge m) \\ &\qquad = \mu_W^2\{(\omega, \omega') : \|\phi_\sigma(\omega, \omega')(t)\| \le \gamma \text{ for some } t \ge m\}\end{aligned}$$

(where μ_W^2 is the product measure on $\Omega \times \Omega$)

$$\begin{aligned}&= \mu_W^2\{(\omega, \omega') : \|\omega(m) + \omega'(t - m)\| \le \gamma \text{ for some } t \ge m\} \\ &\le \mu_W^2\{(\omega, \omega') : \|\omega(m)\| \le n \text{ or } \|\omega(m)\| \ge n \\ &\qquad\qquad \text{and } \|\omega(m) + \omega'(t)\| \le \gamma \text{ for some } t \ge 0\} \\ &\le \mu\{\omega : \|\omega(m)\| \le n\} \\ &\qquad + \int_F \mu_W\{\omega' : \|\omega(m) + \omega'(t)\| \le \gamma \text{ for some } t \ge 0\}\mu_W(d\omega) \\ &\le \epsilon + \int_F \frac{1}{\beta}g(\omega(m))\mu_W(d\omega) \le \epsilon + \epsilon\mu_W F \le 2\epsilon.\end{aligned}$$

As ϵ is arbitrary, $\Pr(\liminf_{t\to\infty} \|X_t\| < \gamma) = 0$; as γ is arbitrary, $\Pr(\lim_{t\to\infty} \|X_t\| = \infty) = 1$.

Remark In 479R I will show that there is a surprising difference between the cases $r = 3$ and $r \ge 4$.

478N Wandering paths Let $G \subseteq \mathbb{R}^r$ be an open set, and for $x \in G$ set

$$F_x(G) = \{\omega : \text{either } \tau_x(\omega) < \infty \text{ or } \lim_{t\to\infty} \|\omega(t)\| = \infty\}$$

where τ_x is the Brownian exit time from $G - x$. I will say that G has **few wandering paths** if $F_x(G)$ is conegligible for every $x \in G$. In this case we can be sure that, if $x \in G$, then for almost every ω *either* $\lim_{t\to\infty} \|\omega(t)\| = \infty$ *or* $\omega(t) \notin G - x$ for some t. So we can speak of $X_{\tau_x}(\omega) = \omega(\tau_x(\omega))$, taking this to be

∞ if $\omega \in F_x(G)$ and $\tau_x(\omega) = \infty$; and ω will be continuous on $[0, \tau_x(\omega)]$ for every $\omega \in F_x(G)$. We find that $X_{\tau_x} : \Omega \to \partial^\infty(G - x)$ is Borel measurable. **P** τ_x is the Brownian hitting time to the closed set $\mathbb{R}^r \setminus (G - x)$, so is a stopping time adapted to $\langle \mathrm{T}_{[0,t]} \rangle_{t \ge 0}$ (477Ic). Let $\mathcal{B}(\Omega)$ be the Borel σ-algebra of Ω for the topology of uniform convergence on compact sets; then $\mathrm{T}_{[0,t]} \subseteq \mathcal{B}(\Omega)$ for every $t \ge 0$. The function

$$(t, \omega) \mapsto X_t(\omega) : [0, \infty[\times \Omega \to \mathbb{R}^r$$

is continuous, therefore $\mathcal{B}([0,\infty[) \widehat{\otimes} \mathcal{B}(\Omega)$-measurable (4A3D(c-i)); so X_{τ_x} is $\mathcal{B}(\Omega)$-measurable (455Ld). **Q**

From 478M, we see that if $r \ge 3$ then any open set in $\mathbb{R}^r$ will have few wandering paths, while if $r \le 2$ then G will have few wandering paths whenever it is not dense in $\mathbb{R}^r$. Note that if $G \subseteq \mathbb{R}^r$ is open, H is a component of G, and $x \in H$, then the exit times from $H - x$ and $G - x$ are the same, just because sample paths are continuous, and $F_x(G) = F_x(H)$. It follows at once that if G has more than one component then it has few wandering paths.

478O Theorem Let $G \subseteq \mathbb{R}^r$ be an open set with few wandering paths and $f : \overline{G}^\infty \to \mathbb{R}$ a bounded lower semi-continuous function such that $f \restriction G$ is superharmonic. Take $x \in G$ and let $\tau : \Omega \to [0, \infty]$ be the Brownian exit time from $G - x$ (477Ia). Then $f(x) \ge \mathbb{E}(f(x + X_\tau))$.

proof It will be enough to deal with the case $f \ge 0$.

(a) Extend f to a function $\tilde f : \mathbb{R}^r \cup \{\infty\} \to \mathbb{R}$ by setting $\tilde f(x) = 0$ for $x \notin \overline{G}^\infty$. Since f is bounded, so is $\tilde f$. Let $\langle \tilde h_n \rangle_{n \in \mathbb{N}}$ be the sequence of 473E/478Jc, and for $n \in \mathbb{N}$ set $f_n = (\tilde f \restriction \mathbb{R}^r) * \tilde h_n$. Also, for $n \in \mathbb{N}$, set

$$G_n = \{y : y \in G,\ \|y\| < n,\ \rho(y, \mathbb{R}^r \setminus G) > \tfrac{1}{n+1}\}$$

(interpreting $\rho(y, \emptyset)$ as ∞ if $G = \mathbb{R}^r$), and let τ_n be the Brownian exit time from $G_n - x$.

(b) For $y \in G$, $f_n(y) \le f(y)$ for all sufficiently large n and $f(y) = \lim_{n\to\infty} f_n(y)$ (478Jb). Also $f_n \restriction G_n$ is superharmonic (478Ja). Each f_n is smooth with bounded derivatives of all orders (473De), and $(\nabla^2 f_n)(y) \le 0$ for $y \in G_n$ (478Ea).

If $m \ge n$,

$$\mathbb{E}(f_m(x + X_{\tau_n})) = f_m(x) + \tfrac{1}{2}\mathbb{E}(\int_0^{\tau_n} (\nabla^2 f_m)(x + X_s) ds) \le f_m(x)$$

(478K). Consequently

$$\begin{aligned} \mathbb{E}(f(x + X_{\tau_n})) &\le \liminf_{m\to\infty} \mathbb{E}(f_m(x + X_{\tau_n})) \\ &\le \liminf_{m\to\infty} f_m(x) = f(x). \end{aligned}$$

(c) For every $\omega \in \Omega$, $\langle \tau_n(\omega) \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence with limit $\tau(\omega)$. **P** Since $\tau_n(\omega) \le \tau_{n+1}(\omega) \le \tau(\omega)$ for every n, $t = \lim_{n \in \mathbb{N}} \tau_n(\omega)$ is defined in $[0, \infty]$. If $t = \infty$ then surely $t = \tau(\omega)$. Otherwise, $\omega(t) = \lim_{n\to\infty} \omega(\tau_n(\omega)) \notin G - x$, so again $t = \tau(\omega)$. **Q**

Consequently

$$f(x + \omega(\tau(\omega))) \le \liminf_{n\to\infty} f(x + \omega(\tau_n(\omega)))$$

for almost every ω. **P** In the language of 478N, we can suppose that $\omega \in F_x(G)$, so that $\omega(\tau(\omega)) = \lim_{n\to\infty} \omega(\tau_n(\omega))$ in $\mathbb{R}^r \cup \{\infty\}$, and we can use the fact that f is lower semi-continuous. **Q** So

$$\mathbb{E}(f(x + X_\tau)) \le \liminf_{n\to\infty} \mathbb{E}(f(x + X_{\tau_n})) \le f(x)$$

as required.

478P Harmonic measures (a) Let $A \subseteq \mathbb{R}^r$ be an analytic set and $x \in \mathbb{R}^r$. Let $\tau : \Omega \to [0, \infty]$ be the Brownian hitting time to $A - x$ (477I). Then τ is Σ-measurable, where Σ is the domain of μ_W (455Ma). Setting $H = \{\omega : \tau(\omega) < \infty\}$, $X_\tau : H \to \mathbb{R}^r$ is Σ-measurable. **P** By 4A3Qc, $(\omega, t) \mapsto \omega(t)$ is $\Sigma \widehat{\otimes} \mathcal{B}([0,\infty[)$-measurable, while $\omega \mapsto (\omega, \tau(\omega))$ is $(\Sigma, \Sigma \widehat{\otimes} \mathcal{B}([0,\infty[))$-measurable. So $\omega \mapsto \omega(\tau(\omega))$ is Σ-measurable on H. **Q**

Consider the function $\omega \mapsto x + \omega(\tau(\omega)) : H \to \mathbb{R}^r$. This induces a Radon image measure μ_x on $\mathbb{R}^r$ defined by saying that

$$\mu_x F = \mu_W\{\omega : \omega \in H,\, x+\omega(\tau(\omega)) \in F\} = \Pr(x + X_\tau \in F)$$

whenever this is defined. Because every $\omega \in \Omega$ is continuous, $X_\tau(\omega) \in \partial(A-x)$ for every $\omega \in H$, and ∂A is conegligible for μ_x. I will call μ_x the **harmonic measure for arrivals in** A **from** x. Of course $\mu_x\mathbb{R}^r$ is the Brownian hitting probability of A.

Note that if $F \subseteq \mathbb{R}^r$ is closed and $x \in \mathbb{R}^r \setminus F$, then the Brownian hitting time to $F-x$ is the same as the Brownian hitting time to $\partial F - x$, because all paths are continuous, so that the harmonic measure for arrivals in F from x coincides with the harmonic measure for arrivals in ∂F from x.

(b) We now have an easy corollary of 478L. Let $A \subseteq \mathbb{R}^r$ be an analytic set, $x \in \mathbb{R}^r$, and μ_x the harmonic measure for arrivals in A from x. If $f : \mathbb{R}^r \to [0,\infty]$ is a lower semi-continuous superharmonic function, $f(x) \ge \int f d\mu_x$. **P** Let τ be the Brownian hitting time to $A - x$, and $H = \{\omega : \tau(\omega) < \infty\}$. Then

$$\int f d\mu_x = \int_H f(x+\omega(\tau(\omega)))d\mu_W$$

(because μ_x is the image measure of the subspace measure $(\mu_W)_H$ under $\omega \mapsto x + \omega(\tau(\omega))$)

$$\le f(x)$$

by 478L. **Q**

(c) We can re-interpret 478O in this language. Let $G \subseteq \mathbb{R}^r$ be an open set with few wandering paths, and $x \in G$. Let μ_x be the harmonic measure for arrivals in $\mathbb{R}^r \setminus G$ from x. In this case, taking τ to be the Brownian exit time from $G - x$ and $H = \{\omega : \tau(\omega) < \infty\}$, we know that $\lim_{t\to\infty} \|\omega(t)\| = \infty$ for almost every $\omega \in \Omega \setminus H$. If $f : \partial^\infty G \to [-\infty,\infty]$ is a function, then

$$\mathbb{E}(f(x+X_\tau)) = \int_H f(x + X_\tau(\omega))\mu_W(d\omega) + f(\infty)\mu_W(\Omega \setminus H)$$

(counting $f(\infty)$ as zero if G is bounded)

$$= \int f d\mu_x + f(\infty)(1 - \mu_x\mathbb{R}^r)$$

if either integral is defined in $[-\infty,\infty]$ (235J[6]). In particular, if $f : \overline{G}^\infty \to \mathbb{R}$ is a bounded lower semi-continuous function and $f{\restriction}G$ is superharmonic, then $f(x) \ge \int f d\mu_x + f(\infty)(1-\mu_x\mathbb{R}^r)$, by 478O. Similarly, if $f : \overline{G}^\infty \to \mathbb{R}$ is continuous and $f{\restriction}G$ is harmonic, then $f(x) = \int f d\mu_x + f(\infty)(1-\mu_x\mathbb{R}^r)$ for every $x \in G$.

(d) Suppose that $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of analytic subsets of $\mathbb{R}^r$, with union A. For $x \in \mathbb{R}^r$, let $\mu_x^{(n)}$, μ_x be the harmonic measures for arrivals in A_n, A respectively from x. Then μ_x is the limit $\lim_{n\to\infty} \mu_x^{(n)}$ for the narrow topology on the space of totally finite Radon measures on $\mathbb{R}^r$ (437Jd). **P** Let τ_n, τ be the Brownian hitting times for $A_n - x$, $A - x$ respectively. Then $\langle \tau_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence with limit τ. Since every $\omega \in \Omega$ is continuous, $X_\tau(\omega) = \lim_{n\to\infty} X_{\tau_n}(\omega)$ whenever $\tau(\omega) < \infty$. Set

$$H_n = \{\omega : \tau_n(\omega) < \infty\} \text{ for each } n,\ H = \{\omega : \tau(\omega) < \infty\} = \textstyle\bigcup_{n\in\mathbb{N}} H_n.$$

If $f \in C_b(\mathbb{R}^r)$, then $f(x + X_\tau(\omega)) = \lim_{n\to\infty} f(x + X_{\tau_n}(\omega))$ for every $\omega \in H$, so

$$\textstyle\int f d\mu_x = \int_H f(x + X_\tau) d\mu_W = \lim_{n\to\infty} \int_{H_n} f(x + X_{\tau_n}) d\mu_W = \lim_{n\to\infty} \int f d\mu_x^{(n)}.$$

As f is arbitrary, $\mu_x = \lim_{n\to\infty} \mu_x^{(n)}$ (437Kc). **Q**

478Q It is generally difficult to find formulae describing harmonic measures. Theorem 478I, however, gives us a technique for an important special case.

[6] Formerly 235L.

Proposition Let S be the sphere $\partial B(y,\delta)$, where $y \in \mathbb{R}^r$ and $\delta > 0$. For $x \in \mathbb{R}^r \setminus S$, let ζ_x be the indefinite-integral measure over ν defined by the function

$$\begin{aligned} z &\mapsto \tfrac{|\delta^2-\|x-y\|^2|}{r\beta_r\delta\|x-z\|^r} \text{ if } z \in S, \\ &\mapsto 0 \text{ if } z \in \mathbb{R}^r \setminus S. \end{aligned}$$

(a) If $x \in \operatorname{int} B(y,\delta)$, then the harmonic measure μ_x for arrivals in S from x is ζ_x.

(b) In particular, the harmonic measure μ_y for arrivals in S from y is $\frac{1}{\nu S}\nu\llcorner S$.

(c) Suppose that $r \ge 2$. If $x \in \mathbb{R}^r \setminus B(y,\delta)$, then the harmonic measure μ_x for arrivals in S from x is ζ_x. In particular, $\mu_x\mathbb{R}^r = \frac{\delta^{r-2}}{\|x-y\|^{r-2}}$.

proof (a) If $g \in C_b(\mathbb{R}^r)$, then 478Ib tells us that we have a continuous function f_g which extends g, is harmonic on $\mathbb{R}^r \setminus S$ and is such that $f_g(x) = \int g\,d\zeta_x$. Now $G = \operatorname{int} B(y,\delta)$ is bounded, so it has few wandering paths (478N) and the harmonic measure μ_x is defined, with $f_g(x) = \int f_g\,d\mu_x$, by 478Pc. But this means that

$$\int g\,d\mu_x = \int f_g d\mu_x = f_g(x) = \int g\,d\zeta_x.$$

As g is arbitrary, $\mu_x = \zeta_x$ (415I), as claimed.

(b) If $x = y$ then

$$\frac{\delta^2-\|x-y\|^2}{r\beta_r\delta\|x-z\|^r} = \frac{\delta^2}{r\beta_r\delta^{r+1}} = \frac{1}{\nu S}$$

if $z \in S$. Since $\nu\llcorner S$ is the indefinite-integral measure over ν defined by χS (234M[7]), we have the result.

(c)(i) To see that ζ_x is the harmonic measure, we can use the same argument as in (a), with decorations. If $g \in C_b(\mathbb{R}^r)$, then 478Ib gives us a bounded continuous function f_g, harmonic on $H = \mathbb{R}^r \setminus B(y,\delta)$, such that f_g agrees with g on S, and $f_g(x) = \int g\,d\zeta_x$. S is conegligible for both μ_x and ζ_x.

(α) If $r \ge 3$, then

$$\limsup_{\|x\|\to\infty} |f_g(x)| \le \frac{\nu S}{r\beta_r\delta} \limsup_{\|x\|\to\infty} \frac{\|x-y\|^2-\delta^2}{(\|x\|-\delta-\|y\|)^r} = 0.$$

So setting $f_g(\infty) = 0$, $f_g : \overline{H}^\infty \to \mathbb{R}$ is continuous and bounded, and harmonic on H; so that

$$\int g\,d\zeta_x = f_g(x) = \int f_g d\mu_x + f_g(\infty)(1-\mu_x\mathbb{R}^r) = \int g\,d\mu_x.$$

As in (a), we conclude that $\zeta_x = \mu_x$.

(β) If $r = 2$, then by 478Mb we see that almost every $\omega \in \Omega$ takes values in $B(y,\delta)-x$; so $\mu_x\mathbb{R}^r = 1$. Set $\underline{f}_g(x) = f_g(x)$ for $x \in \mathbb{R}^2$, $\underline{f}_g(\infty) = \liminf_{\|x\|\to\infty} f_g(x)$. Then $\underline{f}_g$ is lower semi-continuous on $\overline{H}^\infty$ and harmonic on H, so

$$\int g\,d\zeta_x = f_g(x) = \underline{f}_g(x) \ge \int \underline{f}_g d\mu_x + \underline{f}_g(\infty)(1-\mu_x\mathbb{R}^r) = \int g\,d\mu_x.$$

Applying the same argument to $-g$, we see that $\int g\,d\zeta_x \le \int g\,d\mu_x$, so in fact the integrals are equal, and we have the result in this case also.

(ii) Now

$$\begin{aligned} \mu_x\mathbb{R}^r = \zeta_x S &= \int_S \frac{\|x-y\|^2-\delta^2}{r\beta_r\delta\|x-z\|^r}\nu(dz) \\ &= \int_{\partial B(\mathbf{0},\delta)} \frac{\|x-y\|^2-\delta^2}{r\beta_r\delta\|x-y-z\|^r}\nu(dz) = \frac{\nu(\partial B(\mathbf{0},\delta))}{r\beta_r\delta\|x-y\|^{r-2}} \end{aligned}$$

(478Gb)

[7]Formerly 234E.

$$= \frac{\delta^{r-2}}{\|x-y\|^{r-2}}.$$

478R Theorem Let A, $B \subseteq \mathbb{R}^r$ be analytic sets with $A \subseteq B$. For $x \in \mathbb{R}^r$, let $\mu_x^{(A)}$, $\mu_x^{(B)}$ be the harmonic measures for arrivals in A, B respectively from x. Then, for any $x \in \mathbb{R}^r$, $\langle \mu_y^{(A)} \rangle_{y \in \mathbb{R}^r}$ is a disintegration of $\mu_x^{(A)}$ over $\mu_x^{(B)}$.

proof (a) Let τ be the Brownian hitting time to $B - x$, and τ' the hitting time to $A - x$; then $\tau(\omega) \le \tau'(\omega)$ for every ω. If $\tau(\omega) < \infty$, set $f(\omega) = x + \omega(\tau(\omega))$, so that $\mu_x^{(A)}$ is the image measure $(\mu_W)_H f^{-1}$, where $H = \{\omega : \tau(\omega) < \infty\}$. Define $\phi_\tau : \Omega \times \Omega \to \Omega$ as in 477G, so that

$$\begin{aligned} \phi_\tau(\omega, \omega')(t) &= \omega(t) \text{ if } t \le \tau(\omega), \\ &= \omega(\tau(\omega)) + \omega'(t - \tau(\omega)) \text{ if } t \ge \tau(\omega). \end{aligned}$$

Then we have $x + \phi_\tau(\omega, \omega')(t) \in A$ iff $t \ge \tau(\omega)$ and $f(\omega) + \omega'(t - \tau(\omega)) \in A$; so if we write σ_ω for the Brownian hitting time to $A - f(\omega)$ when $\omega \in H$, $\tau'(\phi_\tau(\omega, \omega')) = \tau(\omega) + \sigma_\omega(\omega')$.

(b) Now suppose that $E \subseteq \mathbb{R}^r$ is a Borel set. Then

$$\begin{aligned} \mu_x^{(A)}(E) &= \mu_W\{\omega : \tau'(\omega) < \infty,\ x + \omega(\tau'(\omega)) \in E\} \\ &= \mu_W^2\{(\omega, \omega') : \tau'(\phi_\tau(\omega, \omega')) < \infty,\ x + \phi_\tau(\omega, \omega')(\tau'(\phi_\tau(\omega, \omega'))) \in E\} \\ &= \mu_W^2\{(\omega, \omega') : \tau(\omega) < \infty,\ \sigma_\omega(\omega') < \infty,\ f(\omega) + \omega'(\sigma_\omega(\omega')) \in E\} \\ &= \int_H \mu_W\{\omega' : \sigma_\omega(\omega') < \infty,\ f(\omega) + \omega'(\sigma_\omega(\omega')) \in E\}\mu_W(d\omega) \\ &= \int_H \mu_{f(\omega)}^{(A)}(E)\mu_W(d\omega) = \int \mu_y^{(A)}(E)\mu_x^{(B)}(dy). \end{aligned}$$

The definition in 452E demands that this formula should be valid whenever E is measured by $\mu_x^{(A)}$; but in general there will be Borel sets E', E'' such that $E' \subseteq E \subseteq E''$ and $\mu_x^{(A)}(E') = \mu_x^{(A)}(E) = \mu_x^{(A)}(E'')$, in which case we must have $\mu_y^{(A)}(E') = \mu_y^{(A)}(E) = \mu_y^{(A)}(E'')$ for $\mu_x^{(B)}$-almost every y, and again $\mu_x^{(A)}(E) = \int \mu_y^{(A)}(E)\mu_x^{(B)}(dy)$.

478S Corollary Let $A \subseteq \mathbb{R}^r$ be an analytic set, and $f : \partial A \to \mathbb{R}$ a bounded universally measurable function. For $x \in \mathbb{R}^r \setminus \overline{A}$ set $g(x) = \int f d\mu_x$, where μ_x is the harmonic measure for arrivals in A from x. Then g is harmonic.

proof Suppose that $\delta > 0$ is such that $B(x, \delta) \cap \overline{A} = \emptyset$, and set $S = \partial B(x, \delta) = \partial(\mathbb{R}^r \setminus \operatorname{int} B(x, \delta))$. Then the harmonic measure for arrivals in $\mathbb{R}^r \setminus \operatorname{int} B(x, \delta)$ from x is $\frac{1}{\nu S}\nu \llcorner S$ (478Qb). So

$$g(x) = \int f d\mu_x = \int_S \frac{1}{\nu S} \int f d\mu_y \nu(dy)$$

(478R, 452F)

$$= \frac{1}{\nu S} \int_S g(y)\nu(dy).$$

As x and δ are arbitrary, g is harmonic.

478T Corollary Let $A \subseteq \mathbb{R}^r$ be an analytic set, and for $x \in \mathbb{R}^r$ let μ_x be the harmonic measure for arrivals in A from x. Then $x \mapsto \mu_x$ is continuous on $\mathbb{R}^r \setminus \overline{A}$ for the total variation metric on the set of totally finite Radon measures on $\mathbb{R}^r$ (definition: 437Qa).

proof Take any $y \in \mathbb{R}^r \setminus \overline{A}$. Let $\delta > 0$ be such that $B(y,\delta) \cap A = \emptyset$, and set $S = \partial B(y,\delta)$. For $x \in \operatorname{int} B(y,\delta)$, let ζ_x be the harmonic measure for arrivals in $\mathbb{R}^r \setminus B(y,\delta)$ from x, so that ζ_x is the indefinite-integral measure over ν defined by the function

$$\begin{aligned} z &\mapsto \frac{\delta^2 - \|x-y\|^2}{r\beta_r\delta\|x-z\|^r} \text{ if } z \in S, \\ &\mapsto 0 \text{ if } z \in \mathbb{R}^r \setminus S \end{aligned}$$

(478Qa). Then, for any $x \in \operatorname{int} B(y,\delta)$, $\langle \mu_z \rangle_{z \in \mathbb{R}^r}$ is a disintegration of μ_x over ζ_x. So if $E \subseteq \mathbb{R}^r$ is a Borel set,

$$\mu_x E = \int \mu_z(E)\,\zeta_x(dz) = \frac{1}{r\beta_r\delta}\int_S \mu_z(E)\,\frac{\delta^2 - \|x-y\|^2}{\|x-z\|^r}\nu(dz).$$

But this means that

$$|\mu_x(E) - \mu_y(E)| \le \frac{\nu S}{r\beta_r\delta}\sup_{z \in S}\Bigl|\frac{\delta^2 - \|x-y\|^2}{\|x-z\|^r} - \frac{\delta^2}{\|y-z\|^r}\Bigr|;$$

as E is arbitrary, the distance from μ_x to μ_y is at most

$$\delta^{r-2}\sup_{z \in S}\Bigl|\frac{\delta^2 - \|x-y\|^2}{\|x-z\|^r} - \frac{\delta^2}{\|y-z\|^r}\Bigr|,$$

which is small if x is close to y.

478U A variation on the technique of 478R enables us to say something about Brownian paths starting from a point in the essential closure of a set.

Proposition Suppose that $A \subseteq \mathbb{R}^r$ and that 0 belongs to the essential closure cl^*A of A as defined in 475B. Then the outer Brownian hitting probability $\mathrm{hp}^*(A)$ of A (477Ia) is 1.

proof (a) Take that $\alpha \in \,]0,1[$ such that $\frac{1-\alpha^2}{(1+\alpha)^r} = \frac{1}{2}$. Suppose that $E \subseteq \mathbb{R}^r$ is analytic, and that $0 < \delta_0 < \ldots < \delta_n$ are such that $\delta_i \le \alpha\delta_{i+1}$ for $i < n$. For $i \le n$, let τ_i be the Brownian hitting time to $S_i = \partial B(\mathbf{0},\delta_i)$. Then

$$\mu_W\{\omega : \omega(\tau_i(\omega)) \notin E \text{ for every } i \le n\} \le \prod_{i=0}^n\Bigl(1 - \frac{\nu(E \cap S_i)}{2\nu S_i}\Bigr).$$

$\mathbf{P}$ Induce on n. If $n = 0$, then

$$\mu_W\{\omega : \omega(\tau_0(\omega)) \notin E\} = 1 - \mu_0^{(S_i)}(E)$$

(where $\mu_0^{(S_i)}$ is the harmonic measure for arrivals in S_i from 0)

$$= 1 - \frac{\nu(E \cap S_i)}{\nu S_i} \le 1 - \frac{\nu(E \cap S_i)}{2\nu S_i}$$

(478Qb). For the inductive step to $n+1 \ge 1$, let $\phi : \Omega \times \Omega \to \Omega$ be the inverse-measure-preserving function corresponding to the stopping time τ_n as in 477G; when $\tau_n(\omega)$ is finite, set $y(\omega) = \omega(\tau_n(\omega)) \in S_n$. Since $\tau_i(\omega) < \tau_{i+1}(\omega)$ whenever $i \le n$ and $\tau_i(\omega)$ is finite,

$$\tau_i(\phi(\omega,\omega')) = \tau_i(\omega)$$

for $i \le n$ and ω, $\omega' \in \Omega$. As for $\tau_{n+1}(\phi(\omega,\omega'))$, this is infinite if $\tau_n(\omega) = \infty$, and otherwise is $\sigma_{y(\omega)}(\omega')$, where σ_y is the Brownian hitting time of $S_{n+1} - y$. Now if $y \in S_n$, then

$$\mu_y^{(S_{n+1})}(E) = \int_{E \cap S_{n+1}} \frac{\delta_{n+1}^2 - \delta_n^2}{r\beta_r\delta_{n+1}\|x-y\|^r}\nu(dx)$$

(478Qa)

$$\geq \frac{\delta_{n+1}^2-\delta_n^2}{r\beta_r\delta_{n+1}(\delta_{n+1}+\delta_n)^r}\nu(E\cap S_{n+1})$$
$$\geq \frac{1-\alpha^2}{r\beta_r\delta_{n+1}^{r-1}(1+\alpha)^r}\nu(E\cap S_{n+1}) = \frac{\nu(E\cap S_{n+1})}{2\nu S_{n+1}}.$$

Consequently

$$\begin{aligned}\mu_W\{\omega : \omega(\tau_i(\omega)) \notin E \text{ for every } i \leq n+1\} \\ = (\mu_W \times \mu_W)\{(\omega,\omega') : \phi(\omega,\omega')(\tau_i(\phi(\omega,\omega'))) \notin E \text{ for every } i \leq n+1\} \\ = (\mu_W \times \mu_W)\{(\omega,\omega') : \omega(\tau_i(\omega)) \notin E \text{ for every } i \leq n,\ \omega'(\sigma_{y(\omega)}(\omega')) \notin E\} \\ = \int_V \mu_W\{\omega' : \omega'(\sigma_{y(\omega)}(\omega')) \notin E\}\mu_W(d\omega)\end{aligned}$$

(setting $V = \{\omega : \omega(\tau_i(\omega)) \notin E$ for every $i \leq n\}$)

$$\leq \mu_W V \cdot \sup_{y\in S_n}(1-\mu_y^{(S_{n+1})}E)$$
$$\leq \mu_W V \cdot (1-\frac{\nu(E\cap S_{n+1})}{2\nu S_{n+1}}) \leq \prod_{i=0}^{n+1}(1-\frac{\nu(E\cap S_i)}{2\nu S_i})$$

by the inductive hypothesis. So the induction continues. **Q**

(b) In particular, under the conditions of (a), $\mathrm{hp}(E) \geq 1-\prod_{i=0}^n(1-\frac{\nu(E\cap S_i)}{2\nu S_i})$. Now suppose that $A \subseteq \mathbb{R}^r$ and that $0 \in \mathrm{cl}^*A$. Let $E \supseteq A$ be an analytic set such that $\mathrm{hp}(E) = \mathrm{hp}^*(A)$ (477Id). Then $0 \in \mathrm{cl}^*E$; set $\gamma = \frac{1}{3}\limsup_{\delta\downarrow 0}\frac{\mu(E\cap B(\mathbf{0},\delta))}{\mu B(\mathbf{0},\delta)} > 0$. For any $\delta > 0$, there is a $\delta' \in\ ]0,\delta]$ such that $\frac{\nu(E\cap\partial B(\mathbf{0},\delta'))}{\nu\partial B(\mathbf{0},\delta')} \geq 2\gamma$. **P** Let $\beta \in\ ]0,\delta]$ be such that $\mu(E\cap B(\mathbf{0},\beta)) \geq 2\gamma\mu B(\mathbf{0},\beta)$. Then

$$\int_0^\beta \nu(E\cap\partial B(\mathbf{0},t))dt \geq 2\gamma\int_0^\beta \nu\partial B(\mathbf{0},t)dt,$$

so there must be a $\delta' \in\ ]0,\beta]$ such that $\nu(E\cap\partial B(\mathbf{0},\delta')) \geq 2\gamma\nu\partial B(\mathbf{0},\delta')$. **Q**

We can therefore find, for any $n \in \mathbb{N}$, $0 < \delta_0 < \ldots < \delta_n$ such that $\delta_i \leq \alpha\delta_{i+1}$ for every $i < n$ (where α is chosen as in (a) above) and $\nu(E\cap\partial B(\mathbf{0},\delta_i)) \geq 2\gamma\nu\partial B(\mathbf{0},\delta_i)$ for every i. As noted at the beginning of this part of the proof, it follows that $\mathrm{hp}(E) \geq 1-(1-\gamma)^{n+1}$. As this is true for every $n \in \mathbb{N}$, $\mathrm{hp}(E) = 1$, so $\mathrm{hp}^*(A) = 1$, as claimed.

***478V Theorem** (a) Let $G \subseteq \mathbb{R}^r$ be an open set with few wandering paths and $f : \overline{G}^\infty \to \mathbb{R}$ a continuous function such that $f\restriction G$ is harmonic. For $x \in \mathbb{R}^r$ let $\tau_x : \Omega \to [0,\infty]$ be the Brownian exit time from $G-x$. Set

$$\begin{aligned}g_{\tau_x}(\omega) &= f(x+\omega(\tau_x(\omega))) \text{ if } \tau_x(\omega) < \infty, \\ &= f(\infty) \text{ if } \lim_{t\to\infty}\|\omega(t)\| = \infty \text{ and } \tau_x(\omega) = \infty.\end{aligned}$$

Then $f(x) = \mathbb{E}(g_{\tau_x})$.

(b) Now suppose that σ is a stopping time adapted to $\langle\Sigma_t\rangle_{t\geq 0}$ such that $\sigma(\omega) \leq \tau_x(\omega)$ for every ω. Set

$$\begin{aligned}g_\sigma(\omega) &= g_{\tau_x}(\omega) \text{ if } \sigma(\omega) = \tau_x(\omega) = \infty, \\ &= f(x+\omega(\sigma(\omega))) \text{ otherwise.}\end{aligned}$$

As in 455Lc, set $\Sigma_\sigma = \{E : E \in \operatorname{dom}\mu_W,\ E\cap\{\omega : \sigma(\omega) \leq t\} \in \Sigma_t$ for every $t \geq 0\}$. Then g_σ is a conditional expectation of g_{τ_x} on Σ_σ.

proof (a)(i) Of course if $x \notin G$ then $\tau_x(\omega) = 0$ and $g_{\tau_x}(\omega) = f(x)$ for every ω and the result is trivial. So we can suppose that $x \in G$. Note next that if there is any ω such that $\lim_{t\to\infty}\|\omega(t)\| = \infty$ and $\tau_x(\omega) = \infty$, then G must be unbounded, so $f(\infty)$ will be defined. Because G has few wandering paths, g_{τ_x} is defined almost everywhere.

(ii) Let $m \in \mathbb{N}$ be such that $\rho(x, \mathbb{R}^r \setminus G) > \frac{1}{m+1}$ and $\|x\| < m$; for $n \ge m$, set $G_n = \{y : \|y\| < n, \rho(y, \mathbb{R}^r \setminus G) > \frac{1}{n+1}\}$, let τ'_{xn} be the Brownian exit time from $G_n - x$ and set $\tau_{xn}(\omega) = \min(n, \tau'_{xn}(\omega))$ for every ω. Note that by 477I(c-i), $x + \omega(\tau_{xn}(\omega)) \in \overline{G}_n$ for every ω.

By 477I(c-iii) and 455L(c-v), τ_{xn} is a stopping time adapted to $\langle \Sigma_t \rangle_{t \ge 0}$. Let $\langle \tilde{h}_n \rangle_{n \in \mathbb{N}}$ be the sequence of 473E, and for $n \ge m$ set $f_n = \tilde{f} * \tilde{h}_n$, where $\tilde{f}$ is the extension of $f \restriction \overline{G}$ to $\mathbb{R}^r$ which takes the value 0 on $\mathbb{R}^r \setminus \overline{G}$. Then f_n and all its derivatives are smooth and bounded. By 478Jb (applied in turn to the functions $\tilde{f} + M\chi\mathbb{R}^r$ and $-\tilde{f} + M\chi\mathbb{R}^r$ where $M = \sup_{y \in G} |f(y)|$, which of course are both superharmonic on G), f_n agrees with f on G_n, so that $f_n \restriction G_n$ is harmonic and $\nabla^2 f_n$ is zero on G_n (478Ec). Also, because both f_n and f are continuous, they agree on $\overline{G}_n$ and $f(x + \omega(\tau_{xn}(\omega))) = f_n(x + \omega(\tau_{xn}(\omega)))$ for every ω.

If $n \ge m$, $\omega \in \Omega$ and $0 \le s < \tau_{xn}(\omega)$, then $x + \omega(s) \in G_n$ so $(\nabla^2 f_n)(x + \omega(s)) = 0$. Dynkin's formula (478K), applied to the function $y \mapsto f_n(x + y)$, therefore tells us that $f(x) = f_n(x) = \int f_n(x + \omega(\tau_{xn}(\omega)))\mu_W(d\omega)$.

(iii) If $\omega \in \Omega$ and $t < \tau_x(\omega)$, then the compact set $x + \omega[\,[0,t]\,]$ is included in the open set G and there is an $n \ge \max(m, t)$ such that it is included in G_n. So $\lim_{n\to\infty} \tau_{xn}(\omega) = \tau_x(\omega)$ and, because f is continuous on $\overline{G}^\infty$,

$$g_{\tau_x}(\omega) = \lim_{n\to\infty} f(x + \omega(\tau_{xn}(\omega))) = \lim_{n\to\infty} f_n(x + \omega(\tau_{xn}(\omega)))$$

for almost every ω. Since $\|f_n\|_\infty \le \|f\|_\infty < \infty$ for every n, Lebesgue's Dominated Convergence Theorem tells us that

$$\mathbb{E}(g_{\tau_x}) = \lim_{n\to\infty} \int f_n(x + \omega(\tau_{xn}(\omega)))\mu_W(d\omega) = f(x),$$

as required.

(b)(i) If ω_0, $\omega_1 \in \Omega$, $\sigma(\omega_0) = t$ and $\omega_1 \restriction [0,t] = \omega_0 \restriction [0,t]$, then $\sigma(\omega_1) = t$. **P** The set $\{\omega : \sigma(\omega) \le t\}$ belongs to Σ_t; as it contains ω_0, it contains ω_1, and $\sigma(\omega_1) \le t$. But now ω_0 agrees with ω_1 on $[0, \sigma(\omega_1)]$, so $\sigma(\omega_1) \ge \sigma(\omega) = t$. **Q**

If $H \in \Sigma_\sigma$ then $\omega_0 \in H$ iff $\omega_1 \in H$. **P** For every $t \ge 0$, $H \cap \{\omega : \sigma(\omega) \le t\}$ belongs to Σ_t, so contains ω_0 iff it contains ω_1. **Q**

(ii) Of course $E_\infty = \{\omega : \sigma(\omega) = \infty\}$ belongs to Σ_σ, because it has empty intersection with every set $\{\omega : \sigma(\omega) \le t\}$.

(iii) g_σ is Σ_σ-measurable. **P** For $n \in \mathbb{N}$, $\omega \in \Omega$ and $t \ge 0$, set

$$\begin{aligned} h_n(t, \omega) &= f(\omega(2^{-n}\lfloor 2^n t \rfloor)) \text{ if } \omega(2^{-n}\lfloor 2^n t \rfloor) \in G, \\ &= 0 \text{ otherwise.} \end{aligned}$$

Then h_n is $(\mathcal{B}([0,\infty[) \times \Sigma)$-measurable, so if we set $h(t,\omega) = \lim_{n\to\infty} h_n(t,\omega)$ when this is defined, h also will be $(\mathcal{B}([0,\infty[) \times \Sigma)$-measurable, and $\omega \mapsto h(\sigma(\omega), \omega)$ is Σ-measurable. Now, because $\sigma \le \tau$, $g_\sigma(\omega) = h(\sigma(\omega), \omega)$ for almost every $\omega \in \Omega \setminus E_\infty$; because μ_W is complete, g_σ is Σ-measurable. But now observe that if $t \ge 0$ and $\alpha \in \mathbb{R}$, $\{\omega : \sigma(\omega) \le t, \, g_\sigma(\omega) \ge \alpha\}$ belongs to Σ and is determined by coordinates less than or equal to t, so belongs to Σ_t. As t is arbitrary, $\{\omega : g_\sigma(\omega) \ge \alpha\} \in \Sigma_\sigma$; as α is arbitrary, g_σ is Σ_σ-measurable. **Q**

(iv) As in 477G, define $\phi_\sigma : \Omega \times \Omega \to \Omega$ by saying that

$$\begin{aligned} \phi_\sigma(\omega, \omega')(t) &= \omega(t) \text{ if } t \le \sigma(\omega), \\ &= \omega(\sigma(\omega)) + \omega'(t - \sigma(\omega)) \text{ if } t \ge \sigma(\omega). \end{aligned}$$

Then 477G tells us that ϕ_σ is inverse-measure-preserving.

(v) $\tau_x(\phi_\sigma(\omega, \omega')) = \sigma(\omega) + \tau_{x+\omega(\sigma(\omega))}(\omega')$ for all ω, $\omega' \in \Omega$. **P** If $\sigma(\omega) = \infty$ then $\phi_\sigma(\omega, \omega') = \omega$ and

$$\tau_x(\phi_\sigma(\omega, \omega')) = \tau_x(\omega) = \sigma(\omega).$$

If $\sigma(\omega) = \tau_x(\omega)$ is finite then $\omega(\sigma(\omega)) \notin G - x$ and $\phi_\sigma(\omega, \omega') \restriction [0, \tau_x(\omega)] = \omega \restriction [0, \tau_x(\omega)]$, so

$$\tau_x(\phi_\sigma(\omega, \omega')) = \tau_x(\omega) = \sigma(\omega) = \sigma(\omega) + \tau_{x+\omega(\sigma(\omega))}(\omega').$$

If $\sigma(\omega)<\tau_x(\omega)$ then $\omega(t)=\phi_\sigma(\omega,\omega')(t)$ belongs to $G-x$ for every $t\le\sigma(\omega))$ and

$$\begin{aligned}\tau_x(\phi_\sigma(\omega,\omega'))&=\inf\{t:t\ge\sigma(\omega),\,\omega(\sigma(\omega))+\omega'(t-\sigma(\omega))\notin G-x\}\\&=\sigma(\omega)+\inf\{t:\omega'(t)\notin G-x-\omega(\sigma(\omega))\}=\sigma(\omega)+\tau_{x+\omega(\sigma(\omega))}(\omega').\ \mathbf{Q}\end{aligned}$$

Consequently, if $\omega\in\Omega$, $\sigma(\omega)<\infty$ and $y=\omega(\sigma(\omega))$,

$$\phi_\sigma(\omega,\omega')(\tau_x(\phi_\sigma(\omega,\omega')))=\phi_\sigma(\omega,\omega')(\sigma(\omega)+\tau_{x+y}(\omega'))=y+\omega'(\tau_{x+y}(\omega'))$$

whenever either $\tau_x(\phi_\sigma(\omega,\omega'))$ or $\tau_{x+y}(\omega')$ is finite,

$$g_{\tau_x}(\phi_\sigma(\omega,\omega'))=f(x+\phi_\sigma(\omega,\omega')(\tau_x(\phi_\sigma(\omega,\omega'))))=f(x+y+\omega'(\tau_{x+y}(\omega')))$$

for almost every ω'.

(vi) If $H\in\Sigma_\sigma$ then $\phi_\sigma^{-1}[H]=H\times\Omega$. $\mathbf{P}$ If ω, $\omega'\in\Omega$ then $\phi_\sigma(\omega,\omega')\restriction[0,\sigma(\omega)]=\omega\restriction[0,\sigma(\omega)]$, so by (i) above $\phi_\sigma(\omega,\omega')\in H$ iff $\omega\in H$. $\mathbf{Q}$

If $H\cap E_\infty=\emptyset$, we now have

$$\begin{aligned}\int_H g_{\tau_x}&=\int_{\phi_\sigma^{-1}[H]}g_{\tau_x}(\phi_\sigma(\omega,\omega'))d(\omega,\omega')\\&=\int_H\int f(x+\omega(\sigma(\omega))+\omega'(\tau_{x+\omega(\sigma(\omega))}(\omega)))d\omega' d\omega\\&=\int_H f(x+\omega(\sigma(\omega)))d\omega\end{aligned}$$

(by (a) above)

$$=\int_H g_\sigma.$$

Of course we also have $\int_H g_{\tau_x}=\int_H g_\sigma$ if $H\subseteq E_\infty$. So $\int_H g_\sigma=\int_H g_{\tau_x}$ for every $H\in\Sigma_\sigma$, and g_σ is a conditional expectation of g_{τ_x} on Σ_σ.

478X Basic exercises (a) Let $G\subseteq\mathbb{R}^r$ be an open set, and $\langle f_n\rangle_{n\in\mathbb{N}}$ a sequence of superharmonic functions from G to $[0,\infty]$. Show that $\liminf_{n\to\infty}f_n$ is superharmonic.

(b) Let $G\subseteq\mathbb{R}^r$ be an open set, and $f:G\to\mathbb{R}$ a continuous harmonic function. Show that f is smooth. (*Hint*: put 478I and 478D together.)

(c) Let $G\subseteq\mathbb{R}^r$ be an open set, and $f:G\to[0,\infty]$ a lower semi-continuous superharmonic function. Show that there is are sequences $\langle G_n\rangle_{n\in\mathbb{N}}$, $\langle f_n\rangle_{n\in\mathbb{N}}$ such that (i) $\langle G_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of open sets with union G (ii) for each $n\in\mathbb{N}$, $f_n:G_n\to[0,\infty[$ is a bounded smooth superharmonic function and $f_n\le f\restriction G_n$ (iii) $f(x)=\lim_{n\to\infty}f_n(x)$ for every $x\in G$.

>(d) Let $\langle X_t\rangle_{t\ge0}$ be Brownian motion in $\mathbb{R}^r$, and $\delta>0$. Let τ be the Brownian hitting time to $\{x:\|x\|\ge\delta\}$. Show that $\mathbb{E}(\tau)=\frac{\delta^2}{r}$. (*Hint*: in 478K, take f extending $x\mapsto\|x\|^2:B(0,\delta)\to\mathbb{R}$.)

(e) Show that $\mathrm{hp}^*:\mathcal{P}\mathbb{R}^r\to[0,1]$ is an outer regular Choquet capacity (definition: 432J) iff $r\ge3$. (*Hint*: if $r\ge3$, μ_W is inner regular with respect to $\{K:K\subseteq\Omega,\ \lim_{t\to\infty}\inf_{\omega\in K}\|\omega(t)\|=\infty\}$.)

(f) Show that an open subset of $\mathbb{R}$ has few wandering paths iff it is not $\mathbb{R}$ itself.

(g) Suppose $r=2$. (i) Show that if $x\in\mathbb{R}^2\setminus\{0\}$, then the Brownian exit time from $\mathbb{R}^2\setminus\{x\}$ is infinite a.e. (*Hint*: use the method of part (b-ii) of the proof of 478M to show that if $R>\|x\|$ and $\delta>0$ is small enough then most sample paths meet $\partial B(x,R)$ before they meet $B(x,\delta)$.) (ii) Show that if $G\subseteq\mathbb{R}^2$ is an open set with countable complement then G does not have few wandering paths.

(h) Suppose that $r \ge 3$. Let $\langle A_n\rangle_{n\in\mathbb{N}}$ be a non-increasing sequence of analytic sets in $\mathbb{R}^r$ such that A_0 is bounded and $\bigcap_{n\in\mathbb{N}} \bar{A}_n = \bigcap_{n\in\mathbb{N}} A_n$, and $x \in \mathbb{R}^r$. Let $\mu_x^{(n)}$, μ_x be the harmonic measures for arrivals in A_n, $\bigcap_{m\in\mathbb{N}} A_m$ from x. Show that $\mu_x = \lim_{n\to\infty} \mu_x^{(n)}$ for the narrow topology on the set of totally finite Radon measures on $\mathbb{R}^r$. (*Hint*: 478Xe, 478Pd.)

(i) Let $A \subseteq \mathbb{R}$ be an analytic set, $x \in \mathbb{R}$ and μ_x the harmonic measure for arrivals in A from x. For $y \in \mathbb{R}$ let δ_y be the Dirac measure on $\mathbb{R}$ concentrated at y. Show that (i) if A is empty, then μ_x is the zero measure; (ii) if $A \ne \emptyset$ but $A \cap [x,\infty[\, = \emptyset$ then $\mu_x = \delta_{\sup A}$; (iii) if $A \ne \emptyset$ but $A \cap\,]-\infty, x] = \emptyset$ then $\mu_x = \delta_{\inf A}$; (iv) if A meets both $]-\infty, x]$ and $[x,\infty[$, and $y = \sup(A \cap\,]-\infty, x])$, $z = \inf(A \cap [x,\infty[)$, then $\mu_x = \delta_x$ if $y = z = x$, and otherwise $\mu_x = \frac{z-x}{z-y}\delta_y + \frac{x-y}{z-y}\delta_z$.

(j) Prove 478Qb by a symmetry argument not involving the calculations of 478I.

>(k) Let $G \subseteq \mathbb{R}^r$ be an open set, and for $x \in G$ let μ_x be the harmonic measure for arrivals in $\mathbb{R}^r \setminus G$ from x. Show that for any bounded universally measurable function $f : \partial G \to \mathbb{R}$, the function $x \mapsto \int f d\mu_x : G \to \mathbb{R}$ is continuous and harmonic.

(l) (i) Suppose that $r = 2$, and that x, y, $z \in \mathbb{R}^r$ are such that $\|x\| < 1 = \|z\|$ and $y = 0$. Identify $\mathbb{R}^2$ with $\mathbb{C}$, and express x, z as $\gamma e^{i\theta}$ and e^{it} respectively. Show that, in the language of 272Yg, $\frac{\|y-z\|^2-\|x-y\|^2}{\|x-z\|^r} = A_\gamma(\theta - t)$. (ii) Compare 478I(b-iii) with 272Yg(iii).

478Y Further exercises (a)(i) Show that there is a function $f : \mathbb{R} \to \mathbb{Q}$ which is 'harmonic' in the sense of 478B, but is not continuous. (*Hint*: take f to be a linear operator when $\mathbb{R}$ is regarded as a linear space over $\mathbb{Q}$.) (ii) Show that if the continuum hypothesis is true, there is a surjective function $f : \mathbb{R}^2 \to \{0, 1\}$ which is 'harmonic' in the sense of 478B.

(b) Let $G \subseteq \mathbb{R}^r$ be a connected open set, and $f : G \to [0,\infty]$ a superharmonic Lebesgue measurable function which is not everywhere infinite. Show that f is locally integrable.

(c) Let $G \subseteq \mathbb{R}^2$ be an open set, and $f : G \to \mathbb{C}$ a function which is analytic when regarded as a function of a complex variable. Show that $\mathcal{R}\mathrm{e}\, f$ is harmonic. (*Hint*: The non-trivial part is the theorem that f has continuous second partial derivatives.)

(d) Define $\psi : \mathbb{R}^r \setminus \{0\} \to \mathbb{R}^r$ by setting $\psi(x) = \frac{x}{\|x\|^2}$. For a $[-\infty,\infty]$-valued function f defined on a subset of $\mathbb{R}^r$, set $f^*(x) = \frac{1}{\|x\|^{r-2}} f(\psi(x))$ for $x \in \psi^{-1}[\mathrm{dom}\, f] \setminus \{0\}$. (This is the **Kelvin transform** of f relative to the sphere $\partial B(\mathbf{0}, 1)$.) (i) Show that if f is real-valued and twice continuously differentiable, then $(\nabla^2 f^*)(x) = \frac{1}{\|x\|^{r+2}}(\nabla^2 f)(\psi(x))$ for $x \in \mathrm{dom}\, f^*$. (ii) Show that if $\mathrm{dom}\, f$ is open and f is non-negative, lower semi-continuous and superharmonic, then f^* is superharmonic.

(e) Let $G \subseteq \mathbb{R}^2$ be an open set. Show that G has few wandering paths iff there is an $x \in \mathbb{R}^2$ such that $\mathrm{hp}((\mathbb{R}^2 \setminus (G \cup \{x\})) - x) > 0$.

(f) Show that 478K remains true if we replace 'three-times-differentiable function such that f and its first three derivatives are continuous and bounded' with 'twice-differentiable function such that f and its first two derivatives are continuous and bounded'.

(g) Let $f : \mathbb{R}^r \to \mathbb{R}$ be a twice-differentiable function such that f and its first two derivatives are continuous and bounded. Show that

$$\lim_{\delta\downarrow 0} \frac{1}{\delta^2}\Bigl(f(x) - \frac{1}{\mu B(x,\delta)}\int_{B(x,\delta)} f d\mu\Bigr) = -\frac{(\nabla^2 f)(x)}{2(r+2)}$$

for every $x \in \mathbb{R}^r$.

(h) Let $G \subseteq \mathbb{R}^r$ be an open set and $f : G \to \mathbb{R}$ a continuous function such that

$$\lim_{\delta\downarrow 0}\frac{1}{\delta^2}\bigl(f(x) - \frac{1}{\mu B(x,\delta)}\int_{B(x,\delta)} f d\mu\bigr) = 0$$

for every $x \in G$. Show that f is harmonic.

(i) Let $f : \mathbb{R}^r \to [0,\infty]$ be a lower semi-continuous superharmonic function. Show that $f\omega$ is continuous for μ_W-almost every $\omega \in \Omega$.

(j) Suppose that $A \subseteq \mathbb{R}^r$. Show that $x \mapsto \mathrm{hp}^*(A - x)$ is lower semi-continuous at every point of $\mathbb{R}^r \setminus A$, and continuous at every point of $\mathbb{R}^r \setminus \overline{A}$.

(k) Suppose that $A \subseteq \mathbb{R}^r$ is such that $\inf_{\delta>0}\mathrm{hp}^*(A \cap B(\mathbf{0},\delta)) > 0$. Show that $\mathrm{hp}^*(A) = 1$.

(l) Let μ_W be three-dimensional Wiener measure on $\Omega = C([0,\infty[\,;\mathbb{R}^3)_0$, and e a unit vector in $\mathbb{R}^3$. Set $Y_t(\omega) = \frac{1}{\|\omega(t)-e\|}$ for $\omega \in \Omega$ and $t \in [0,\infty[$ such that $\omega(t) \ne e$. (i) Show that if $R > 1$ then

$$\begin{aligned}\mathbb{E}(Y_t) &= \frac{1}{(\sqrt{2\pi t})^3}\int \frac{\exp(-\|x\|^2/2t)}{\|x-e\|}\mu(dx)\\ &\le \frac{1}{(\sqrt{2\pi t})^3}\int_{B(\mathbf{0},1)}\frac{1}{\|x\|}\mu(dx) + \frac{1}{(\sqrt{2\pi t})^3}\int_{B(e,R)\setminus B(e,1)} e^{-\|x\|^2/2t}\mu(dx) + \frac{1}{R} \to \frac{1}{R}\end{aligned}$$

as $t \to \infty$, so that $\lim_{t\to\infty}\mathbb{E}(Y_t) = 0$ and $\langle Y_t\rangle_{t\ge 0}$ is not a martingale. (ii) Show that if $n \ge 1$ and τ_n is the Brownian hitting time to $B(e,2^{-n})$, then $\langle Y_{t\wedge\tau_n}\rangle_{t\ge 0}$ is a martingale, where $t \wedge \tau_n$ is the stopping time $\omega \mapsto \min(t,\tau_n(\omega))$. (iii) Show that $\langle\tau_n(\omega)\rangle_{n\in\mathbb{N}}$ is a strictly increasing sequence with limit ∞ for almost every ω. ($\langle Y_t\rangle_{t\ge 0}$ is a 'local martingale'.)

(m) Taking $r = 2$, set $S = \{(\xi,1) : \xi \in \mathbb{R}\}$ and let ζ be the indefinite-integral measure over ν defined by the function

$$\begin{aligned}(\xi,\eta) &\mapsto \frac{1}{\pi(1+\xi^2)} \text{ if } \eta = 1,\\ &\mapsto 0 \text{ otherwise.}\end{aligned}$$

Show that ζ is the harmonic measure for arrivals in S from $(0,0)$.

478 Notes and comments I find that books are still being published on the subject of potential theory which ignore Brownian motion. To my eye, Newtonian potential, at least (and this is generally acknowledged to be the core of the subject), is an essentially geometric concept, and random walks are an indispensable tool for understanding it. So I am giving these priority, at the cost of myself ignoring Green's functions.

The definitions in 478B are already unconventional; most authors take it for granted that harmonic functions should be finite-valued and continuous (see 478Cd). All the work of this section refers to measurable functions. But there are things which can be said about non-measurable functions satisfying the definitions here (478Ya). Let me draw your attention to 478Fa and 478H. If we want to say that $x \mapsto \frac{1}{\|x\|^{r-2}}$ is harmonic, we have to be careful not to define it at 0. If (for $r \ge 3$) we allow $\frac{1}{0^{r-2}} = \infty$, we get a superharmonic function. If (for $r = 1$) we allow $\frac{1}{0^{-1}} = 0$, we get a subharmonic function. The slightly paradoxical phenomenon of 478Yl is another manifestation of this.

I hope that using the operations ${}^{-\infty}$ and ∂^∞ does not make things more difficult. The point is that by compactifying $\mathbb{R}^r$ we get an efficient way of talking about $\lim_{\|x\|\to\infty} f(x)$ when we need to. This is particularly effective for Brownian paths, since in three and more dimensions almost all paths go off to infinity (478Md). In two dimensions the situation is more complex (478Mb-478Mc), and we have to consider the possibility that a path ω in an open set may be 'wandering', in the sense that it neither strikes the boundary nor goes to infinity, and $\lim_{t\to\infty} f(\omega(t))$ may fail to exist even for the best-behaved functions

f. Of course this already happens in one dimension, but only when $G=\mathbb{R}$, and classical potential theory (though not, I think, Brownian potential theory) is nearly trivial in the one-dimensional case.

Many readers will also find that setting $r=3$ and $r\beta_r=4\pi$ will make the formulae easier to digest. I allow for variations in r partly in order to cover the case $r=2$ (in this section, though not in the next, many of the ideas translate directly into the one- and two-dimensional cases), and partly because it is not always easy to guess at a formula for $r\ge 4$ from the formula for $r=3$. There is little extra work to be done, given that §§472-475 cover the general case.

I call 478K a 'lemma' because I have made no attempt to look for weakest adequate hypotheses; of course we don't really need third derivatives (478Yf). The 'theorems' are 478L and 478O, where the hypotheses seem to mark natural boundaries of the arguments given. In 478O I use a language which is both unusual and slightly contorted, in order to do as much as possible without splitting the cases $r\le 2$ from the rest. Of course any result involving the notion of 'few wandering paths' really has two forms; one when $r\ge 3$, so that there is no restriction on the open set and we are genuinely making use of the one-point compactification of $\mathbb{R}^r$, and one when $r\le 2$, in which essentially all our paths are bounded.

Theorem 478O leads directly to a solution of Dirichlet's problem, in the sense that, for an open set G with few wandering paths, we have a family of measures enabling us to calculate the values within G of a continuous function on $\overline{G}^\infty$ which satisfies Laplace's equation inside G (478Pc). We do not get a satisfactory existence theorem; we can use harmonic measures to generate many harmonic functions on G (478S), but we do not get good information on their behaviour near ∂G, and are left guessing at which continuous functions on $\partial^\infty G$ will be extended continuously. The method does, however, make it clear that what matters is the geometry of the boundary; we need to know whether, starting from a point near the boundary, a random walk will hit the boundary soon. So at least we can see from 478M that (if $r\ge 2$) an isolated point of ∂G will be at worst an irrelevant distraction. For $r\ge 3$ the next section will give some useful information (479P *et seq.*), though I shall not have space for a proper analysis.

The idea of 478Vb is that we have a particularly dramatic kind of martingale. Writing S for the set of stopping times $\sigma\le\tau$, it is easy to see that the family $\langle g_\sigma\rangle_{\sigma\in S}$ is a martingale in the sense that $\Sigma_\sigma\subseteq\Sigma_{\sigma'}$ and g_σ is a conditional expectation of $g_{\sigma'}$ on Σ_σ whenever $\sigma\le\sigma'$ in S.

Version of 15.2.10/28.4.10

479 Newtonian capacity

I end the chapter with a sketch of fragments of the theory of Newtonian capacity. I introduce equilibrium measures as integrals of harmonic measures (479B); this gives a quick definition of capacity (479C), with a substantial number of basic properties (479D, 479E), including its extendability to a Choquet capacity (479Ed). I give sufficient fragments of the theory of Newtonian potentials (479F, 479J) and harmonic analysis (479G, 479I) to support the classical definitions of capacity and equilibrium measures in terms of potential and energy (479K, 479N). The method demands some Fourier analysis extending that of Chapter 28 (479H). 479P is a portmanteau theorem on generalized equilibrium measures and potentials with an exact description of the latter in terms of outer Brownian hitting probabilities. I continue with notes on capacity and Hausdorff measure (479Q), self-intersecting Brownian paths (479R) and an example of a discontinuous equilibrium potential (479S). Yet another definition of capacity, for compact sets, can be formulated in terms of gradients of potential functions (479U); this leads to a simple inequality relating capacity to Lebesgue measure (479V). The section ends with an alternative description of capacity in terms of a measure on the family of closed subsets of $\mathbb{R}^r$ (479W).

479A Notation In this section, unless otherwise stated, r will be a fixed integer greater than or equal to 3. As in §478, μ will be Lebesgue measure on $\mathbb{R}^r$, and β_r the measure of the unit ball $B(\mathbf{0},1)$; ν will be normalized $(r-1)$-dimensional Hausdorff measure on $\mathbb{R}^r$, so that $\nu(\partial B(\mathbf{0},1))=r\beta_r$.

Recall that if ζ is a measure on a space X, and $E\in\operatorname{dom}\zeta$, then $\zeta\llcorner E$ is defined by saying that $(\zeta\llcorner E)(F)=\zeta(E\cap F)$ whenever $F\subseteq X$ and ζ measures $E\cap F$ (234M[8]). If ζ is a Radon measure, so is $\zeta\llcorner E$ (416Sa).

As in §478, Ω will be $C([0,\infty[\,;\mathbb{R}^r)_0$, with the topology of uniform convergence on compact sets; μ_W will be Wiener measure on Ω. Recall that the Brownian hitting probability $\operatorname{hp}(D)$ of a set $D\subseteq\mathbb{R}^r$ is

[8] Formerly 234E.

$\mu_W\{\omega : \omega^{-1}[D] \neq \emptyset\}$ if this is defined, and that for any $D \subseteq \mathbb{R}^r$ the outer Brownian hitting probability is $\mathrm{hp}^*(D) = \mu_W^*\{\omega : \omega^{-1}[D] \neq \emptyset\}$ (477Ia).

If $x \in \mathbb{R}^r$ and $A \subseteq \mathbb{R}^r$ is an analytic set, $\mu_x^{(A)}$ will be the harmonic measure for arrivals in A from x (478P); note that $\mu_x^{(A)}(\mathbb{R}^r) = \mu_x^{(A)}(\partial A) = \mathrm{hp}(A - x)$.

I will write ρ_{tv} for the total variation metric on the space $M_{\mathrm{R}}^+(\mathbb{R}^r)$ of totally finite Radon measures on $\mathbb{R}^r$, so that

$$\rho_{\mathrm{tv}}(\lambda, \zeta) = \sup_{E,F\subseteq\mathbb{R}^r \text{ are Borel}} \lambda E - \zeta E - \lambda F + \zeta F$$

for λ, $\zeta \in M_{\mathrm{R}}^+(\mathbb{R}^r)$ (437Qa).

479B Theorem Let $A \subseteq \mathbb{R}^r$ be a bounded analytic set.

(i) There is a Radon measure λ_A on $\mathbb{R}^r$, with support included in ∂A, defined by saying that $\langle \frac{1}{r\beta_r\gamma}\mu_x^{(A)}\rangle_{x\in\partial B(\mathbf{0},\gamma)}$ is a disintegration of λ_A over the subspace measure $\nu_{\partial B(\mathbf{0},\gamma)}$ whenever $\gamma > 0$ and $\overline{A} \subseteq \operatorname{int} B(\mathbf{0}, \gamma)$.

(ii) λ_A is the limit $\lim_{\|x\|\to\infty} \|x\|^{r-2}\mu_x^{(A)}$ for the total variation metric on $M_{\mathrm{R}}^+(\mathbb{R}^r)$.

proof (a) Suppose that $\gamma > 0$ is such that $\overline{A} \subseteq \operatorname{int} B(\mathbf{0}, \gamma)$. By 478T, $x \mapsto \mu_x^{(A)}(E) : \partial B(\mathbf{0}, \gamma) \to [0, \infty[$ is continuous for every Borel set $E \subseteq \mathbb{R}^r$, and $\mu_x^{(A)}(\mathbb{R}^r \setminus \overline{A}) = 0$ for every x, so $\{\mu_x^{(A)} : x \in \partial B(\mathbf{0}, \gamma)\}$ is uniformly tight. By 452Da, we have a unique totally finite Radon measure ζ_γ such that $\langle \frac{1}{r\beta_r\gamma}\mu_x^{(A)}\rangle_{x\in\partial B(\mathbf{0},\gamma)}$ is a disintegration of ζ_γ over the subspace measure $\nu_{\partial B(\mathbf{0},\gamma)}$. Since $\mathbb{R}^r \setminus \partial A$ is $\mu_x^{(A)}$-negligible for every $x \in \partial B(\mathbf{0}, \gamma)$ (478Pa), it is also ζ_γ-negligible.

(b) Now suppose that $\overline{A} \subseteq \operatorname{int} B(\mathbf{0}, \gamma)$ and that $\|x\| = M\gamma$, where $M > 1$. Then 478R tells us that $\langle \mu_y^{(A)}\rangle_{y\in\mathbb{R}^r}$ is a disintegration of $\mu_x^{(A)}$ over $\mu_x^{(B(\mathbf{0},\gamma))}$. So, for any Borel set $E \subseteq \mathbb{R}^r$,

$$|\zeta_\gamma E - \|x\|^{r-2}\mu_x^{(A)}E| = \frac{1}{r\beta_r\gamma}\Bigl|\int_S \mu_y^{(A)}E\,\nu(dy) - \|x\|^{r-2}\int_{\mathbb{R}^r} \mu_y^{(A)}E\,\mu_x^{(B(\mathbf{0},\gamma))}(dy)\Bigr|$$

(where $S = \partial B(\mathbf{0}, \gamma)$)

$$= \frac{1}{r\beta_r\gamma}\Bigl|\int_S \mu_y^{(A)}E\,\nu(dy) - \|x\|^{r-2}\int_S \mu_y^{(A)}E\,\mu_x^{(S)}(dy)\Bigr|$$

(478Pa)

$$= \frac{1}{r\beta_r\gamma}\Bigl|\int_S \mu_y^{(A)}E\,\nu(dy) - \|x\|^{r-2}\int_S \frac{\|x\|^2-\gamma^2}{r\beta_r\gamma\|x-y\|^r}\mu_y^{(A)}E\,\nu(dy)\Bigr|$$

(478Q)

$$\le \frac{1}{r\beta_r\gamma}\int_S |1 - \|x\|^{r-2}\frac{\|x\|^2-\gamma^2}{\|x-y\|^r}|\mu_y^{(A)}E\nu(dy)$$

$$\le \frac{\nu S}{r\beta_r\gamma}\sup_{y\in S}|1 - \|x\|^{r-2}\frac{\|x\|^2-\gamma^2}{\|x-y\|^r}|$$

$$\le \gamma^{r-2}\sup_{y\in S}(|1 - \frac{\|x\|^r}{\|x-y\|^r}| + \frac{\gamma^2\|x\|^{r-2}}{\|x-y\|^r})$$

$$= \gamma^{r-2}\bigl(|\frac{M^r\gamma^r}{(M\gamma-\gamma)^r} - 1| + \frac{\gamma^r M^{r-2}}{\gamma^r(M-1)^r}\bigr)$$

$$= \gamma^{r-2}\bigl(|\frac{M^{r-2}}{(M-1)^r} - 1| + \frac{M^r}{(M-1)^r}\bigr).$$

(c) Accordingly

$$\rho_{\mathrm{tv}}(\zeta_\gamma, \|x\|^{r-2}\mu_x^{(A)}) \le 2\gamma^{r-2}\bigl(|\frac{M^r}{(M-1)^r} - 1| + \frac{M^{r-2}}{(M-1)^r}\bigr)$$

whenever $\gamma > 0$, $\overline{A} \subseteq \operatorname{int} B(\mathbf{0}, \gamma)$ and $\|x\| = M\gamma > \gamma$; so that

$$\zeta_\gamma = \lim_{\|x\|\to\infty} \|x\|^{r-2}\mu_x^{(A)}$$

for the total variation metric whenever $\gamma > 0$ and $\overline{A} \subseteq \operatorname{int} B(\mathbf{0},\gamma)$. We can therefore write λ_A for the limit, and both (i) and (ii) will be true, since I have already checked that $\operatorname{supp}(\zeta_\gamma) \subseteq \partial A$ for all large γ.

479C Definitions (a)(i) In the context of 479B, I will call λ_A the **equilibrium measure** of the bounded analytic set A, and $\lambda_A\mathbb{R}^r = \lambda_A(\partial A)$ the **Newtonian capacity** $\operatorname{cap} A$ of A.

(ii) For any $D \subseteq \mathbb{R}^r$, its **Choquet-Newton capacity** will be

$$c(D) = \inf_{G\supseteq D \text{ is open}} \sup_{K\subseteq G \text{ is compact}} \operatorname{cap} K.$$

(I will confirm in 479Ed below that c is in fact a capacity as defined in §432.) Sets with zero Choquet-Newton capacity are called **polar**.

(b) If ζ is a Radon measure on $\mathbb{R}^r$, the **Newtonian potential** associated with ζ is the function $W_\zeta : \mathbb{R}^r \to [0,\infty]$ defined by the formula

$$W_\zeta(x) = \int_{\mathbb{R}^r} \frac{1}{\|y-x\|^{r-2}}\zeta(dy)$$

for $x \in \mathbb{R}^r$. The **energy** of ζ is now

$$\operatorname{energy}(\zeta) = \int W_\zeta d\zeta = \int_{\mathbb{R}^r}\int_{\mathbb{R}^r} \frac{1}{\|x-y\|^{r-2}}\zeta(dy)\zeta(dx).$$

If A is a bounded analytic subset of $\mathbb{R}^r$, the potential $\tilde{W}_A = W_{\lambda_A}$ is the **equilibrium potential** of A.

(In 479P below I will describe constructions of equilibrium measures and potentials for arbitrary subsets D of $\mathbb{R}^r$ such that $c(D)$ is finite.)

(c) If ζ is a Radon measure on $\mathbb{R}^r$, I will write U_ζ for the **$(r-1)$-potential** of ζ, defined by saying that $U_\zeta(x) = \int_{\mathbb{R}^r} \frac{1}{\|x-y\|^{r-1}}\zeta(dy) \in [0,\infty]$ for $x \in \mathbb{R}^r$.

479D The machinery in Theorem 479B gives an efficient method of approaching several fundamental properties of equilibrium measures. I start with some elementary calculations.

Proposition (a) For any $\gamma > 0$ and $z \in \mathbb{R}^r$, the Newtonian capacity of $B(z,\gamma)$ is γ^{r-2}, the equilibrium measure of $B(z,\gamma)$ is $\frac{1}{r\beta_r\gamma}\nu\llcorner\partial B(z,\gamma)$, and the equilibrium potential of $B(z,\gamma)$ is given by

$$\tilde{W}_{B(z,\gamma)}(x) = \min(1, \frac{\gamma^{r-2}}{\|x-z\|^{r-2}})$$

for every $x \in \mathbb{R}^r$.

(b) Let $A \subseteq \mathbb{R}^r$ be a bounded analytic set with equilibrium measure λ_A and equilibrium potential $\tilde{W}_A$.

(i) $\tilde{W}_A(x) \le 1$ for every $x \in \mathbb{R}^r$.

(ii) If $B \subseteq A$ is another analytic set, $\tilde{W}_B \le \tilde{W}_A$.

(iii) $\tilde{W}_A(x) = 1$ for every $x \in \operatorname{int} A$.

(c) Let A, $B \subseteq \mathbb{R}^r$ be bounded analytic sets.

(i) Defining $+$ and $\le$ as in 234G[9] and 234P, $\lambda_{A\cup B} \le \lambda_A + \lambda_B$.

(ii) $\lambda_A B \le \operatorname{cap} B$.

proof (a) For $x \in \mathbb{R}^r \setminus B(z,\gamma)$, $\mu_x^{(B(z,\gamma))}$ is the indefinite-integral measure over $\nu\llcorner\partial B(z,\gamma)$ defined by the function $y \mapsto \frac{\|x-z\|^2-\gamma^2}{r\beta_r\gamma\|x-y\|^r}$ (478Qc). So $\|x\|^{r-2}\mu_x^{(B(z,\gamma))}$ is the indefinite-integral measure defined by

$$y \mapsto f_x(y) = \frac{\|x\|^{r-2}(\|x-z\|^2-\gamma^2)}{r\beta_r\gamma\|x-y\|^r}.$$

[9] Formerly 112Xe.

As $\|x\| \to \infty$, $f_x(y) \to \frac{1}{r\beta_r\gamma}$ uniformly for $y \in \partial B(z,\gamma)$, so $\lambda_{B(\mathbf{0},\gamma)} = \lim_{\|x\|\to\infty} \|x\|^{r-2}\mu_x^{(B(z,\gamma))}$ is $\frac{1}{r\beta_r\gamma}\nu\llcorner\partial B(z,\gamma)$. Consequently the capacity of $B(z,\gamma)$ is $\frac{1}{r\beta_r\gamma}\nu(\partial B(z,\gamma)) = \gamma^{r-2}$, and the equilibrium potential is

$$\tilde W_{B(z,\gamma)}(x) = \frac{1}{r\beta_r\gamma}\int_{\partial B(z,\gamma)} \frac{1}{\|y-x\|^{r-2}}\nu(dy)$$
$$= \frac{1}{r\beta_r\gamma}\int_{\partial B(\mathbf{0},\gamma)} \frac{1}{\|y+z-x\|^{r-2}}\nu(dy) = \frac{1}{r\beta_r\gamma}\cdot\frac{\nu(\partial B(\mathbf{0},\gamma))}{\max(\gamma,\|x-z\|)^{r-2}}$$

(478Ga)

$$= \min(1, \frac{\gamma^{r-2}}{\|x-z\|^{r-2}}).$$

(b)(i) Let $\gamma > 0$ be such that $\overline{A} \subseteq \operatorname{int} B(\mathbf{0},\gamma)$. Then

$$\tilde W_A(x) = \int_{\mathbb{R}^r} \frac{1}{\|x-y\|^{r-2}}\lambda_A(dy) = \frac{1}{r\beta_r\gamma}\int_{\partial B(\mathbf{0},\gamma)}\int_{\mathbb{R}^r} \frac{1}{\|x-y\|^{r-2}}\mu_z^{(A)}(dy)\nu(dz)$$

(452F)

$$\le \frac{1}{r\beta_r\gamma}\int_{\partial B(\mathbf{0},\gamma)} \frac{1}{\|x-z\|^{r-2}}\nu(dz)$$

(478Pb, 478H)

$$= \frac{\nu(\partial B(\mathbf{0},\gamma))}{r\beta_r\gamma}\frac{1}{\max(\gamma,\|x\|)^{r-2}}$$

(478Ga)

$$\le 1.$$

(ii) Let $\gamma > 0$ be such that $\overline{A} \subseteq \operatorname{int} B(\mathbf{0},\gamma)$. Then

$$\tilde W_B(x) = \int_{\mathbb{R}^r} \frac{1}{\|x-y\|^{r-2}}\lambda_B(dy)$$
$$= \frac{1}{r\beta_r\gamma}\int_{\partial B(\mathbf{0},\gamma)}\int_{\mathbb{R}^r} \frac{1}{\|x-y\|^{r-2}}\mu_z^{(B)}(dy)\nu(dz)$$
$$= \frac{1}{r\beta_r\gamma}\int_{\partial B(\mathbf{0},\gamma)}\int_{\mathbb{R}^r}\int_{\mathbb{R}^r} \frac{1}{\|x-y\|^{r-2}}\mu_w^{(B)}(dy)\mu_z^{(A)}(dw)\nu(dz)$$

(because $\langle\mu_w^{(B)}\rangle_{w\in\mathbb{R}^r}$ is a disintegration of $\mu_z^{(B)}$ over $\mu_z^{(A)}$ for every z, by 478R)

$$\le \frac{1}{r\beta_r\gamma}\int_{\partial B(\mathbf{0},\gamma)}\int_{\mathbb{R}^r} \frac{1}{\|x-w\|^{r-2}}\mu_z^{(A)}(dw)\nu(dz)$$

(by 478Pb, because $y \mapsto \frac{1}{\|x-y\|^{r-2}}$ is continuous and superharmonic)

$$= \tilde W_A(x).$$

(iii) If $x \in \operatorname{int} A$, there is a $\gamma > 0$ such that $B(x,\gamma) \subseteq A$; now, putting (a) and (ii) above together,

$$\tilde W_A(x) \ge \tilde W_{B(x,\gamma)}(x) = 1.$$

Since we know from (i) that $\tilde W_A(x) \le 1$, we have equality.

(c)(i) Suppose that $K \subseteq \mathbb{R}^r$ is compact and that $x \in \mathbb{R}^r$. Let τ_A, τ_B and $\tau_{A\cup B}$ be the Brownian hitting times to $A - x$, $B - x$ and $(A \cup B) - x$ respectively. Then $\tau_{A\cup B} = \tau_A \wedge \tau_B$. Now

$$\begin{aligned}\mu_x^{(A\cup B)}(K) &= \mu_W\{\omega : \tau_{A\cup B}(\omega) < \infty,\, x+\omega(\tau_{A\cup B}(\omega)) \in K\}\\ &\le \mu_W\{\omega : \tau_{A\cup B}(\omega) = \tau_A(\omega) < \infty,\, x+\omega(\tau_A(\omega)) \in K\}\\ &\qquad\qquad + \mu_W\{\omega : \tau_{A\cup B}(\omega) = \tau_B(\omega) < \infty,\, x+\omega(\tau_B(\omega)) \in K\}\\ &\le \mu_x^{(A)}(K) + \mu_x^{(B)}(K).\end{aligned}$$

Multiplying by $\|x\|^{r-2}$ and letting $\|x\| \to \infty$,

$$\lambda_{A\cup B}(K) \le \lambda_A K + \lambda_B K = (\lambda_A + \lambda_B)(K)$$

for every K, which is the criterion of 416E(a-ii).

(ii) For any $x \in \mathbb{R}^r$,

$$\mu_x^{(A)}(B) = \mu_W\{\omega : \tau_A(\omega) < \infty,\, x+\omega(\tau_A(\omega)) \in B\}$$

(where $\tau_A(\omega)$ is the Brownian hitting time to $A-x$)

$$\le \mu_W\{\omega : \omega^{-1}[B-x] \ne \emptyset\} = \mu_x^{(B)}(\mathbb{R}^r).$$

Multiplying by $\|x\|^{r-2}$ and taking the limit as $\|x\| \to \infty$, $\lambda_A B \le \operatorname{cap} B$.

479E Theorem (a) Newtonian capacity cap is submodular (definition: 413Qb).

(b) Suppose that $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of analytic subsets of $\mathbb{R}^r$ with bounded union A.

(i) The equilibrium measure λ_A is the limit $\lim_{n\to\infty}\lambda_{A_n}$ for the narrow topology on the space $M_{\mathrm{R}}^+(\mathbb{R}^r)$ of totally finite Radon measures on $\mathbb{R}^r$.

(ii) $\operatorname{cap} A = \lim_{n\to\infty}\operatorname{cap} A_n$.

(iii) The equilibrium potential $\tilde W_A$ is $\lim_{n\to\infty}\tilde W_{A_n} = \sup_{n\in\mathbb{N}}\tilde W_{A_n}$.

(c) Suppose that $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of bounded analytic subsets of $\mathbb{R}^r$ such that $\bigcap_{n\in\mathbb{N}} A_n = \bigcap_{n\in\mathbb{N}}\overline{A_n} = A$ say.

(i) λ_A is the limit $\lim_{n\to\infty}\lambda_{A_n}$ for the narrow topology on $M_{\mathrm{R}}^+(\mathbb{R}^r)$.

(ii) $\operatorname{cap} A = \lim_{n\to\infty}\operatorname{cap} A_n$.

(d)(i) Choquet-Newton capacity $c : \mathcal{P}\mathbb{R}^r \to [0,\infty]$ is the unique outer regular Choquet capacity on $\mathbb{R}^r$ extending cap.

(ii) c is submodular.

(iii) $c(A) = \sup\{\operatorname{cap} K : K \subseteq A$ is compact$\}$ for every analytic set $A \subseteq \mathbb{R}^r$.

proof (a) Let A, $B \subseteq \mathbb{R}^r$ be bounded analytic sets. If $x \in \mathbb{R}^r$, then

$$\operatorname{hp}((A\cup B)-x) + \operatorname{hp}((A\cap B)-x) \le \operatorname{hp}(A-x) + \operatorname{hp}(B-x).$$

P For $C \subseteq \mathbb{R}^r$ set

$$H_C = \{\omega : \omega \in \Omega, \text{ there is some } t \ge 0 \text{ such that } x+\omega(t) \in C\},$$

so that if C is an analytic set, $\operatorname{hp}(C-x) = \mu_W H_C$. Then

$$H_{A\cup B} = H_A \cup H_B, \quad H_{A\cap B} \subseteq H_A \cap H_B,$$

so

$$\begin{aligned}\operatorname{hp}((A\cup B)-x) + \operatorname{hp}((A\cap B)-x) &= \mu_W H_{A\cup B} + \mu_W H_{A\cap B}\\ &\le \mu_W(H_A\cup H_B) + \mu_W(H_A\cap H_B)\\ &= \mu_W H_A + \mu_W H_B\\ &= \operatorname{hp}(A-x) + \operatorname{hp}(B-x).\ \mathbf{Q}\end{aligned}$$

Consequently

$$\begin{aligned}\operatorname{cap}(A\cup B)+\operatorname{cap}(A\cap B)&=\lambda_{A\cup B}(\mathbb{R}^r)+\lambda_{A\cap B}(\mathbb{R}^r)\\&=\lim_{\|x\|\to\infty}\|x\|^{r-2}(\operatorname{hp}((A\cup B)-x)+\operatorname{hp}((A\cap B)-x)\\&\le\lim_{\|x\|\to\infty}\|x\|^{r-2}(\operatorname{hp}(A-x)+\operatorname{hp}(B-x))\\&=\operatorname{cap}A+\operatorname{cap}B.\end{aligned}$$

As A and B are arbitrary, cap is submodular.

(b)(i) Let $f:\mathbb{R}^r\to\mathbb{R}$ be any bounded continuous function. For any $x\in\mathbb{R}^r$, $\int fd\mu_x^{(A)}=\lim_{n\to\infty}\int fd\mu_x^{(A_n)}$. **P** Let τ, τ_n be the Brownian hitting times to $A-x$, A_n-x respectively. Observe that $\langle\tau_n(\omega)\rangle_{n\in\mathbb{N}}$ is non-increasing and

$$\tau(\omega)=\inf\{t:x+\omega(t)\in\textstyle\bigcup_{n\in\mathbb{N}}A_n\}=\lim_{n\to\infty}\tau_n(\omega)$$

for every $\omega\in\Omega$. Set $H=\{\omega:\tau(\omega)<\infty\}$, $H_n=\{\omega:\tau_n(\omega)<\infty\}$. Then $\langle H_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with union H, and for $\omega\in H$

$$f(x+\tau(\omega))=\lim_{n\to\infty}f(x+\tau_n(\omega))$$

because f and ω are continuous. Accordingly

$$\begin{aligned}\int fd\mu_x^{(A)}&=\int_H f(x+\omega(\tau(\omega)))\\&=\lim_{n\to\infty}\int_{H_n}f(x+\omega(\tau_n(\omega)))=\lim_{n\to\infty}\int fd\mu_x^{(A_n)}.\ \mathbf{Q}\end{aligned}$$

Taking $\gamma>0$ such that $\overline{A}\subseteq\operatorname{int}B(\mathbf{0},\gamma)$,

$$\begin{aligned}\int fd\lambda_A&=\frac{1}{r\beta_r\gamma}\int_{\partial B(\mathbf{0},\gamma)}\int fd\mu_x^{(A)}\nu(dx)\\&\qquad\text{(452F)}\\&=\frac{1}{r\beta_r\gamma}\int_{\partial B(\mathbf{0},\gamma)}\lim_{n\to\infty}\int fd\mu_x^{(A_n)}\nu(dx)\\&=\lim_{n\to\infty}\frac{1}{r\beta_r\gamma}\int_{\partial B(\mathbf{0},\gamma)}\int fd\mu_x^{(A_n)}\nu(dx)=\lim_{n\to\infty}\int fd\lambda_{A_n}.\end{aligned}$$

As f is arbitrary, $\lambda_A=\lim_{n\to\infty}\lambda_{A_n}$ for the narrow topology (437Kc).

(ii) Taking $f=\chi\mathbb{R}^r$ in (i), we see that $\operatorname{cap}A=\lim_{n\to\infty}\operatorname{cap}A_n$.

(iii) For any $x\in\mathbb{R}^r$,

$$\tilde{W}_A(x)=\int\frac{1}{\|y-x\|^{r-2}}\lambda_A(dy)\le\liminf_{n\to\infty}\int\frac{1}{\|y-x\|^{r-2}}\lambda_{A_n}(dy)$$

because $y\mapsto\frac{1}{\|y-x\|^{r-2}}$ is non-negative and continuous (437Jg). As $\tilde{W}_{A_n}(x)\le\tilde{W}_A(x)$ for every n (479D(b-ii)), $\tilde{W}_A(x)=\lim_{n\to\infty}\tilde{W}_{A_n}(x)=\sup_{n\in\mathbb{N}}\tilde{W}_{A_n}(x)$.

(c) Most of the ideas of (b) still work. Again take $f\in C_b(\mathbb{R}^r)$. Then $\int fd\mu_x^{(A)}=\lim_{n\to\infty}\int fd\mu_x^{(A_n)}$ for any $x\in\mathbb{R}^r$. **P** As before, let τ, τ_n be the Brownian hitting times to $A-x$, A_n-x respectively. This time, $\langle\tau_n\rangle_{n\in\mathbb{N}}$ is non-decreasing. Let Ω' be the conegligible subset of Ω consisting of those functions ω such that $\lim_{t\to\infty}\|\omega(t)\|=\infty$. If $\omega\in\Omega'$ and $t=\lim_{n\to\infty}\tau_n(\omega)$ is finite, then for every $n\in\mathbb{N}$ there is a $t_n\le t+2^{-n}$ such that $x+\omega(t_n)\in A_n$. Let $s\in[0,t]$ be a cluster point of $\langle t_n\rangle_{n\in\mathbb{N}}$; then $x+\omega(s)$ is a cluster point of $\langle x+\omega(t_n)\rangle_{n\in\mathbb{N}}$, so belongs to $\bigcap_{n\in\mathbb{N}}\overline{A}_n=A$, and $\tau(\omega)\le s\le t$. Since $\tau(\omega)\ge\tau(\omega_n)$ for every n, we have $\tau(\omega)=\lim_{n\to\infty}\tau(\omega_n)$.

Setting $H=\{\omega:\tau(\omega)<\infty\}$ and $H_n=\{\omega:\tau_n(\omega)<\infty\}$, $\langle H_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence with intersection H, and for $\omega\in H$

$$f(x+\tau(\omega))=\lim_{n\to\infty}f(x+\tau_n(\omega)).$$

So once again

$$\begin{aligned}\int fd\mu_x^{(A)}&=\int_H f(x+\omega(\tau(\omega)))\mu_W(d\omega)\\&=\lim_{n\to\infty}\int_{H_n}f(x+\omega(\tau_n(\omega)))\mu_W(d\omega)=\lim_{n\to\infty}\int fd\mu_x^{(A_n)}.\ \mathbf{Q}\end{aligned}$$

The rest of the argument follows (b-i) and (b-ii) unchanged.

(d)(i) I seek to apply 432Lb.

(α) Let $\mathcal{K}$ be the family of compact subsets of $\mathbb{R}^r$ and set $c_1=\operatorname{cap}\restriction\mathcal{K}$. By (a), c_1 is submodular. If $G\subseteq\mathbb{R}^r$ is a bounded open set, then it is expressible as the union of a non-decreasing sequence of compact sets, so by (b-ii) we have $\operatorname{cap}G=\sup\{c_1(L):L\in\mathcal{K},\ L\subseteq G\}$; and if $K\in\mathcal{K}$, there is a non-increasing sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ of bounded open sets such that $K=\bigcap_{n\in\mathbb{N}}G_n=\bigcap_{n\in\mathbb{N}}\overline{G}_n$, and now $c_1(K)=\lim_{n\to\infty}\operatorname{cap}G_n$, by (c-ii). But this means that

$$c_1(K)\le\inf_{G\supseteq K\text{ is open}}\sup_{L\subseteq G\text{ is compact}}c_1(L)\le\inf_{n\in\mathbb{N}}\operatorname{cap}G_n=c_1(K).$$

So all the conditions of 432Lb are satisfied, and c, as defined in 479C(a-ii), is the unique extension of c_1 to an outer regular Choquet capacity on $\mathbb{R}^r$.

(β) Now $c(A)=\operatorname{cap}A$ for every bounded analytic set $A\subseteq\mathbb{R}^r$. **P**

$$\begin{aligned}c(A)&=\sup_{K\subseteq A\text{ is compact}}c(K)\qquad\text{(432K)}\\&=\sup_{K\subseteq A\text{ is compact}}\operatorname{cap}K\le\operatorname{cap}A\le\inf_{G\supseteq A\text{ is bounded and open}}\operatorname{cap}G\\&=\inf_{G\supseteq A\text{ is open}}\ \sup_{L\subseteq G\text{ is compact}}c(L)=c(A).\ \mathbf{Q}\end{aligned}$$

So c extends cap, as claimed, and must be the unique outer regular Choquet capacity doing so.

(ii)-(iii) By 432Lb, c is submodular; and (iii) is covered by the argument in (i-β).

479F I now wish to describe an entirely different characterization of the capacity and equilibrium measure of a compact set, which demands a substantial investment in harmonic analysis (down to 479I) and an excursion into Fourier analysis (479H). I begin with general remarks about Newtonian potentials.

Theorem Let ζ be a totally finite Radon measure on $\mathbb{R}^r$, and set $G=\mathbb{R}^r\setminus\operatorname{supp}\zeta$, where $\operatorname{supp}\zeta$ is the support of ζ (411Nd). Let W_ζ be the Newtonian potential associated with ζ.

(a) $W_\zeta:\mathbb{R}^r\to[0,\infty]$ is lower semi-continuous, and $W_\zeta\restriction G:G\to[0,\infty[$ is continuous.

(b) W_ζ is superharmonic, and $W_\zeta\restriction G$ is harmonic.

(c) W_ζ is locally μ-integrable; in particular, it is finite μ-a.e.

(d) If ζ has compact support, then $\zeta\mathbb{R}^r=\lim_{\|x\|\to\infty}\|x\|^{r-2}W_\zeta(x)$.

(e) If $W_\zeta\restriction\operatorname{supp}\zeta$ is continuous then W_ζ is continuous.

(f) If K is a compact set such that $W_\zeta\restriction K$ is continuous and finite-valued then $W_{\zeta\llcorner K}$ is continuous.

(g) If W_ζ is finite ζ-a.e. and $f:\mathbb{R}^r\to[0,\infty]$ is a lower semi-continuous superharmonic function such that $f\ge W_\zeta$ ζ-a.e., then $f\ge W_\zeta$.

(h) If ζ' is another Radon measure on $\mathbb{R}^r$ and $\zeta'\le\zeta$, then $W_{\zeta'}\le W_\zeta$ and $\operatorname{energy}(\zeta')\le\operatorname{energy}(\zeta)$.

proof (a) If $\langle x_n\rangle_{n\in\mathbb{N}}$ is a convergent sequence in $\mathbb{R}^r$ with limit x, then $\frac{1}{\|y-x\|^{r-2}} = \lim_{n\to\infty}\frac{1}{\|y-x_n\|^{r-2}}$ for every y (counting $\frac{1}{0}$ as ∞, as usual), so that $W_\zeta(x)\le\liminf_{n\to\infty}W_\zeta(x_n)$, by Fatou's Lemma. As x and $\langle x_n\rangle_{n\in\mathbb{N}}$ are arbitrary, W_ζ is lower semi-continuous.

If $x\in G$, then $\frac{1}{\|y-x_n\|^{r-2}}\le\frac{2}{\rho(x,\operatorname{supp}\zeta)^{r-2}}$ for all n large enough and all $y\in\operatorname{supp}\zeta$, so Lebesgue's Dominated Convergence Theorem tells us that $W_\zeta(x)=\lim_{n\to\infty}W_\zeta(x_n)$ and that W_ζ is continuous at x, as well as finite-valued there.

(b) If $x\in\mathbb{R}^r$ and $\delta>0$, then

$$\begin{aligned}\frac{1}{\nu(\partial B(x,\delta))}\int_{\partial B(x,\delta)}W_\zeta d\nu &= \frac{1}{\nu(\partial B(x,\delta))}\int_{\partial B(x,\delta)}\int_{\mathbb{R}^r}\frac{1}{\|z-y\|^{r-2}}\zeta(dz)\nu(dy)\\ &=\int_{\mathbb{R}^r}\frac{1}{\nu(\partial B(x,\delta))}\int_{\partial B(x,\delta)}\frac{1}{\|z-y\|^{r-2}}\nu(dy)\zeta(dz)\\ &=\int_{\mathbb{R}^r}\frac{1}{\nu(\partial B(\mathbf{0},\delta))}\int_{\partial B(\mathbf{0},\delta)}\frac{1}{\|z-x-y\|^{r-2}}\nu(dy)\zeta(dz)\\ &\ge\int_{\mathbb{R}^r}\frac{1}{\|z-x\|^{r-2}}\zeta(dz)=W_\zeta(x)\end{aligned}$$

(478Ga) with equality if $B(x,\delta)$ does not meet $\operatorname{supp}\zeta$, since then $z-x\notin B(\mathbf{0},\delta)$ and

$$\frac{1}{\nu(\partial B(x,\delta))}\int_{\partial B(x,\delta)}\frac{1}{\|z-x-y\|^{r-2}}\nu(dy)=\frac{1}{\|z-x\|^{r-2}}$$

for ζ-almost every z.

(c) For any $\gamma>0$ and $y\in\mathbb{R}^r$, $\int_{B(\mathbf{0},\gamma)}\frac{1}{\|y-x\|^{r-2}}\mu(dx)\le\frac{1}{2}r\beta_r\gamma^2$ (478Gc), so

$$\begin{aligned}\int_{B(\mathbf{0},\gamma)}W_\zeta d\mu &= \int_{B(\mathbf{0},\gamma)}\int_{\mathbb{R}^r}\frac{1}{\|y-x\|^{r-2}}\zeta(dy)\mu(dx)\\ &=\int_{\mathbb{R}^r}\int_{B(\mathbf{0},\gamma)}\frac{1}{\|y-x\|^{r-2}}\mu(dx)\zeta(dy)\le\frac{1}{2}r\beta_r\gamma^2\zeta\mathbb{R}^r\end{aligned}$$

is finite.

(d) If ζ has compact support, there is an $\gamma>0$ such that $\operatorname{supp}\zeta\subseteq B(\mathbf{0},\gamma)$. In this case, for $\|x\|>\gamma$, we have

$$\frac{\|x\|^{r-2}}{(\|x\|+\gamma)^{r-2}}\zeta\mathbb{R}^r\le\int_{\mathbb{R}^r}\frac{\|x\|^{r-2}}{\|x-y\|^{r-2}}\zeta(dy)=W_\zeta(x)\|x\|^{r-2}\le\frac{\|x\|^{r-2}}{(\|x\|-\gamma)^{r-2}}\zeta\mathbb{R}^r$$

so all the terms converge to $\zeta\mathbb{R}^r$ as $\|x\|\to\infty$.

(e) Since W_ζ is lower semi-continuous, it will be enough to show that $H=\{x:W_\zeta(x)<\gamma\}$ is open for every $\gamma\in\mathbb{R}$. Take $x_0\in H$. If $x_0\notin\operatorname{supp}\zeta$ then W_ζ is continuous at x_0, by 479Fa, and H is certainly a neighbourhood of x_0. If $x_0\in\operatorname{supp}\zeta$ take $\eta\in\,]0,2^{-r}(\gamma-W_\zeta(x_0))[$. Because $W_\zeta(x_0)$ is finite, $\zeta\{x_0\}=0$; because $W_\zeta\restriction\operatorname{supp}\zeta$ is continuous, there is a $\delta>0$ such that $\int_{B(x_0,\delta)}\frac{1}{\|x_0-y\|^{r-2}}\zeta(dy)\le\eta$ and $|W_\zeta(x)-W_\zeta(x_0)|\le\eta$ whenever $x\in B(x_0,\delta)\cap\operatorname{supp}\zeta$. Let $\delta'\in\,]0,\delta[$ be such that

$$\Bigl|\frac{1}{\|x-y\|^{r-2}}-\frac{1}{\|x_0-y\|^{r-2}}\Bigr|\le\frac{\eta}{\zeta\mathbb{R}^r}$$

whenever $\|x-x_0\|\le\delta'$ and $\|y-x_0\|\ge\delta$; then

$$\Bigl|\int_{\mathbb{R}^r\setminus B(x_0,\delta)}\frac{1}{\|x-y\|^{r-2}}\zeta(dy)-\int_{\mathbb{R}^r\setminus B(x_0,\delta)}\frac{1}{\|x_0-y\|^{r-2}}\zeta(dy)\Bigr|\le\eta$$

whenever $x\in B(x_0,\delta')$.

Take $x \in B(x_0, \frac{1}{2}\delta')$, and let $z \in B(x_0, \delta) \cap \operatorname{supp} \zeta$ be such that $\|x-z\| = \rho(x, B(x_0, \delta) \cap \operatorname{supp} \zeta)$. We have $\|x-z\| \le \|x-x_0\|$ so $\|z-x_0\| \le 2\|x-x_0\| \le \delta'$. If $y \in B(x_0, \delta) \cap \operatorname{supp} \zeta$, then $\|y-z\| \le \|x-y\| + \|x-z\| \le 2\|x-y\|$; so

$$\begin{aligned}\int_{B(x_0,\delta)} \frac{1}{\|x-y\|^{r-2}}\zeta(dy) &\le 2^{r-2}\int_{B(x_0,\delta)} \frac{1}{\|z-y\|^{r-2}}\zeta(dy)\\ &= 2^{r-2}(W_\zeta(z) - \int_{\mathbb{R}^r \setminus B(x_0,\delta)} \frac{1}{\|z-y\|^{r-2}}\zeta(dy))\\ &\le 2^{r-2}(2\eta + W_\zeta(x_0) - \int_{\mathbb{R}^r \setminus B(x_0,\delta)} \frac{1}{\|x_0-y\|^{r-2}}\zeta(dy))\\ &= 2^{r-1}\eta + 2\int_{B(x_0,\delta)} \frac{1}{\|x_0-y\|^{r-2}}\zeta(dy)\\ &\le 2^{r-1}\eta + 2\eta \le (2^r-1)\eta,\end{aligned}$$

$$\begin{aligned}W_\zeta(x) &\le (2^r-1)\eta + \int_{\mathbb{R}^r \setminus B(x_0,\delta)} \frac{1}{\|x-y\|^{r-2}}\zeta(dy)\\ &\le 2^r\eta + \int_{\mathbb{R}^r \setminus B(x_0,\delta)} \frac{1}{\|x_0-y\|^{r-2}}\zeta(dy) \le 2^r\eta + W_\zeta(x_0) < \gamma.\end{aligned}$$

Thus $B(x_0, \frac{1}{2}\delta') \subseteq H$ and again H is a neighbourhood of x_0. As x_0 is arbitrary, H is open; as γ is arbitrary, W_ζ is continuous.

(f) Setting $H = \mathbb{R}^r \setminus K$, $\zeta = \zeta \llcorner K + \zeta \llcorner H$, so $W_\zeta = W_{\zeta \llcorner K} + W_{\zeta \llcorner H}$ (234Hc[10]). Now $W_{\zeta \llcorner K}$ and $W_{\zeta \llcorner H}$ are both lower semi-continuous and non-negative, so if $W_\zeta \restriction K$ is continuous and finite-valued then $W_{\zeta \llcorner K} \restriction K$ is continuous (4A2B(d-ix)). Since $\operatorname{supp}(\zeta \llcorner K) \subseteq K$, (e) tells us that $W_{\zeta \llcorner K}$ is continuous.

(g) ? Suppose that $f(x_0) < W_\zeta(x_0)$. Since $\{x : f(x) \ge W_\zeta(x),\ W_\zeta(x) < \infty\}$ is ζ-conegligible, and W_ζ is measurable therefore ζ-almost continuous (418J), there is a compact set K such that $W_\zeta(x) < \infty$ and $W_\zeta(x) \le f(x)$ for every $x \in K$, $W_\zeta \restriction K$ is continuous and $\int_K \frac{1}{\|x_0-y\|^{r-2}}\zeta(dy) > f(x_0)$. Set $\zeta' = \zeta \llcorner K$. By (f), $W_{\zeta'}$ is continuous; $W_{\zeta'}(x_0) > f(x_0)$; and $f(x) \ge W_\zeta(x) \ge W_{\zeta'}(x)$ for every $x \in K \supseteq \operatorname{supp} \zeta'$.

Set $g = f - W_{\zeta'}$ and $\alpha = \inf_{x \in \mathbb{R}^r} g(x) < 0$. Because f is lower semi-continuous and $W_{\zeta'}$ is continuous, g is lower semi-continuous; because ζ' has compact support, $\lim_{\|x\| \to \infty} W_{\zeta'}(x) = 0$ ((d) above) and $\liminf_{\|x\| \to \infty} g(x) \ge 0$; so $L = \{x : g(x) = \alpha\}$ is non-empty and compact. Note that L is disjoint from K. Let $x_1 \in L$ be a point of maximum norm. Then $x_1 \notin K$, while $W_{\zeta'} \restriction \mathbb{R}^r \setminus K$ is harmonic ((b) above). Let $\delta > 0$ be such that $B(x_1, \delta) \cap K = \emptyset$. Then we have

$$\frac{1}{\nu(\partial B(x_1,\delta))}\int_{\partial B(x_1,\delta)} g\,d\nu > \alpha$$

because $g(x) \ge \alpha$ for every x and $g(x) > \alpha$ whenever $x \ne x_1$ and $(x - x_1) . x_1 \ge 0$. But we also have

$$\begin{aligned}\frac{1}{\nu(\partial B(x_1,\delta))}\int_{\partial B(x_1,\delta)} g\,d\nu &= \frac{1}{\nu(\partial B(x_1,\delta))}\int_{\partial B(x_1,\delta)} f\,d\nu - \frac{1}{\nu(\partial B(x_1,\delta))}\int_{\partial B(x_1,\delta)} W_{\zeta'}\,d\nu\\ &\le f(x_1) - W_{\zeta'}(x_1) = \alpha,\end{aligned}$$

which is impossible. **X**

(h) By 234Qc, $W_{\zeta'} \le W_\zeta$; so

$$\operatorname{energy}(\zeta') = \int W_{\zeta'} d\zeta' \le \int W_\zeta d\zeta' \le \int W_\zeta d\zeta = \operatorname{energy}(\zeta).$$

479G At this point I embark on an extended parenthesis, down to 479I, covering some essential material from harmonic analysis and Fourier analysis. The methods here apply equally to the cases $r = 1$ and $r = 2$.

[10] Formerly 212Xh.

Lemma (In this result, r may be any integer greater than or equal to 1.) For $\alpha \in \mathbb{R}$, set $k_\alpha(x) = \frac{1}{\|x\|^\alpha}$ for $x \in \mathbb{R}^r \setminus \{0\}$. If $\alpha < r$, $\beta < r$ and $\alpha + \beta > r$, then $k_{\alpha+\beta-r}$ is a constant multiple of the convolution $k_\alpha * k_\beta$ (definition: 255E, 444O).

proof (a) First note that

$$\int_{B(\mathbf{0},1)} k_\alpha(x)\mu(dx) = r\beta_r \int_0^1 \frac{t^{r-1}}{t^\alpha} dt = \frac{r\beta_r}{r-\alpha}$$

is finite. Consequently k_α is expressible as the sum of an integrable function and a bounded function, and in particular is locally integrable.

(b) For $x \in \mathbb{R}^r$, set $f(x) = \int_{\mathbb{R}^r} k_\alpha(x-y)k_\beta(x-y)\mu(dy) \in [0,\infty]$. If $e \in \mathbb{R}^r$ is a unit vector, then $f(e)$ is finite. **P** For any $y \in \mathbb{R}^r$, at least one of $\|e-y\|$, $\|e+y\|$ is greater than or equal to 1, so $k_\alpha(e-y)k_\beta(e+y) \le k_\alpha(e-y) + k_\beta(e+y)$. Consequently

$$\int_{B(\mathbf{0},2)} k_\alpha(e-y)k_\alpha(e+y)\mu(dy) \le \int_{B(\mathbf{0},2)} k_\alpha(e-y) + k_\alpha(e+y)\mu(dy)$$

is finite. On the other hand, if $\|y\| \ge 2$, $\|e-y\|$ and $\|e-y\|$ are both at least $\frac{1}{2}\|y\|$, so

$$\begin{aligned}\int_{\mathbb{R}^r \setminus B(\mathbf{0},2)} k_\alpha(e-y)k_\beta(e+y)\mu(dy) &\le \int_{\mathbb{R}^r \setminus B(\mathbf{0},2)} \frac{2^\alpha}{\|y\|^\alpha} \cdot \frac{2^\beta}{\|y\|^\beta}\mu(dy) \\ &= 2^{\alpha+\beta} r\beta_r \int_2^\infty \frac{t^{r-1}}{t^{\alpha+\beta}} dt = \frac{2^r r\beta_r}{\alpha+\beta-r}\end{aligned}$$

is finite. Putting these together, $f(e) = \int_{\mathbb{R}^r} k_\alpha(e-y)k_\beta(e-y)\mu(dy)$ is finite. **Q**

(c) If e, $e' \in \mathbb{R}^r$ are unit vectors, then $f(e) = f(e')$. **P** Let $T : \mathbb{R}^r \to \mathbb{R}^r$ be an orthogonal transformation such that $Te = e'$. Then

$$f(e') = \int_{\mathbb{R}^r} k_\alpha(Te-y)k_\beta(Te-y)\mu(dy) = \int_{\mathbb{R}^r} k_\alpha(Te-Ty)k_\beta(Te-Ty)\mu(dy)$$

(because T is an automorphism of $(\mathbb{R}^r, \mu)$)

$$= \int_{\mathbb{R}^r} k_\alpha(e-y)k_\beta(e-y)\mu(dy)$$

(because $k_\alpha(x)$, $k_\beta(x)$ are functions of $\|x\|$)

$$= f(e). \text{ } \mathbf{Q}$$

Let c be the constant value of $f(e)$ for $\|e\| = 1$.

(d) If $x \in \mathbb{R}^r \setminus \{0\}$, $f(x) = \frac{c}{\|x\|^{\alpha+\beta-r}}$. **P** Set $t = \|x\|$, $e = \frac{1}{t}x$. Then

$$f(x) = \int_{\mathbb{R}^r} k_\alpha(te-y)k_\beta(te-y)\mu(dy) = \int_{\mathbb{R}^r} t^r k_\alpha(te-tz)k_\beta(te-tz)\mu(dz)$$

(substituting $y = tz$)

$$= t^{r-\alpha-\beta} \int_{\mathbb{R}^r} k_\alpha(e-z)k_\beta(e-z)\mu(dz) = \frac{c}{\|x\|^{\alpha+\beta-r}}. \text{ } \mathbf{Q}$$

(e) If $x \in \mathbb{R}^r \setminus \{0\}$, $(k_\alpha * k_\beta)(x) = 2^{\alpha+\beta-r} c k_{\alpha+\beta-r}(x)$. **P** Set $z = \frac{1}{2}x$. Then

$$\begin{aligned}(k_\alpha * k_\beta)(x) &= \int_{\mathbb{R}^r} \frac{1}{\|x-y\|^\alpha \|y\|^\beta}\mu(dy) = \int_{\mathbb{R}^r} \frac{1}{\|x-y-z\|^\alpha \|y+z\|^\beta}\mu(dy) \\ &= \int_{\mathbb{R}^r} \frac{1}{\|z-y\|^\alpha \|z+y\|^\beta}\mu(dy) = \frac{c}{\|z\|^{\alpha+\beta-r}} = 2^{\alpha+\beta-r} c k_{\alpha+\beta-r}(x). \text{ } \mathbf{Q}\end{aligned}$$

(f) Of course it is of no importance what happens at 0, but for completeness: $k_{\alpha+\beta-r}$ is declared to be undefined there, and $\int_{\mathbb{R}^r} \frac{1}{\|y\|^\alpha \|y\|^\beta}\mu(dy)$ is infinite for any α and β, so $k_\alpha * k_\beta$ also is undefined at 0 on the convention of 255E or 444O. Thus we have $k_\alpha * k_\beta = 2^{\alpha+\beta-r} c k_{\alpha+\beta-r}$ in the strict sense.

Remark The functions k_α are called **Riesz kernels**. It will be helpful later to have a name for the constant arising here in a special case. If $r \ge 3$, I will take $c_r > 0$ to be the constant such that $c_r k_{r-2} = k_{r-1} * k_{r-1}$.

479H Now for some Fourier analysis which wasn't quite reached in Chapter 28. In the following, I will define the Fourier transform $\hat{f}$ and the inverse Fourier transform $\check{f}$, for μ-measurable complex-valued functions f defined μ-almost everywhere in $\mathbb{R}^r$, as in 283W and 284W, and $\hat{\zeta}$, for a totally finite Radon measure ζ on $\mathbb{R}^r$, by the formula offered in 285Ya for probability measures. The convolution $\zeta * f$ of a measure and a function will be defined as in 444H. Thus the basic formulae are

$$\hat{f}(y) = \frac{1}{(\sqrt{2\pi})^r}\int_{\mathbb{R}^r} e^{-iy\cdot x} f(x)\mu(dx), \quad \check{f}(y) = \frac{1}{(\sqrt{2\pi})^r}\int_{\mathbb{R}^r} e^{iy\cdot x} f(x)\mu(dx)$$

for μ-integrable f,

$$\hat{\zeta}(y) = \frac{1}{(\sqrt{2\pi})^r}\int_{\mathbb{R}^r} e^{-iy\cdot x}\zeta(dx), \quad (\zeta * f)(x) = \int_{\mathbb{R}^r} f(x-y)\zeta(dy).$$

Theorem (In this result, r may be any integer greater than or equal to 1.) Let ζ be a totally finite Radon measure on $\mathbb{R}^r$.

(a) If $f \in \mathcal{L}^1_{\mathbb{C}}(\mu)$, then $\zeta * f$ is μ-integrable and $(\zeta * f)^\wedge = (\sqrt{2\pi})^r \hat{\zeta} \times \hat{f}$.

(b) If ζ has compact support and $h : \mathbb{R}^r \to \mathbb{C}$ is a rapidly decreasing test function (284Wa), then $\zeta * h$ and $h \times \hat{\zeta}$ are rapidly decreasing test functions.

(c) Suppose that f is a tempered function on $\mathbb{R}^r$ (284Wa). If *either* ζ has compact support *or* f is expressible as the sum of a μ-integrable function and a bounded function, then $\zeta * f$ is defined μ-almost everywhere and is a tempered function.

(d) Suppose that f, g are tempered functions on $\mathbb{R}^r$ such that g represents the Fourier transform of f (284Wd). If *either* ζ has compact support *or* f is expressible as the sum of a bounded function and a μ-integrable function, then $(\sqrt{2\pi})^r \hat{\zeta} \times g$ represents the Fourier transform of $\zeta * f$.

proof (a)(i) To begin with, suppose that f is real-valued and non-negative. As in §444, I will write $f\mu$ for the indefinite-integral measure defined by f over μ. By 444K, $\zeta * f$ is μ-integrable and $(\zeta * f)\mu = \zeta * f\mu$.

As the formula used here for $\hat{\zeta}$ does not quite match that of 445C, whatever parametrization we use for the characters of the topological group $\mathbb{R}^r$, I had better not try to quote Chapter 44 when discussing Fourier transforms. Going back to first principles,

$$\begin{aligned}
(\zeta * f)^\wedge(y) &= \frac{1}{(\sqrt{2\pi})^r}\int e^{-iy\cdot x}(\zeta * f)(x)\mu(dx) = \frac{1}{(\sqrt{2\pi})^r}\int e^{-iy\cdot x}(\zeta * f)\mu(dx) \\
&= \frac{1}{(\sqrt{2\pi})^r}\int e^{-iy\cdot x}(\zeta * f\mu)(dx) = \frac{1}{(\sqrt{2\pi})^r}\iint e^{-iy\cdot(x+z)}\zeta(dz)(f\mu)(dx) \\
\text{(444C)} \qquad & \\
&= \frac{1}{(\sqrt{2\pi})^r}\int e^{-iy\cdot z}\zeta(dz)\int e^{-iy\cdot x}(f\mu)(dx) \\
&= \hat{\zeta}(y)\int e^{-iy\cdot x} f(x)\mu(dx) = (\sqrt{2\pi})^r \hat{\zeta}(y)\hat{f}(y)
\end{aligned}$$

for every $y \in \mathbb{R}^r$.

(ii) For general integrable complex-valued functions f, apply (i) to the positive and negative parts of the real and imaginary parts of f.

(b)(i) Because h is continuous, so is $\zeta * h$ (444Ib). If $j < r$, then, as in 123D,

$$(\frac{\partial}{\partial\xi_j}(\zeta * h))(x) = \frac{\partial}{\partial\xi_j}\int h(x-y)\zeta(dy) = \int \frac{\partial}{\partial\xi_j}h(x-y)\zeta(dy) = (\zeta * \frac{\partial h}{\partial\xi_j})(x)$$

because $\frac{\partial h}{\partial\xi_j}$ is bounded. Since $\frac{\partial h}{\partial\xi_j}$ is again a rapidly decreasing test function, we can repeat this process to see that $\zeta * h$ is smooth. Next, let $\gamma > 0$ be such that the support of ζ is included in $B(\mathbf{0},\gamma)$. If $k \in \mathbb{N}$, then $M = \sup_{x\in\mathbb{R}^r}(\gamma^k + \|x\|^k)|h(x)|$ is finite, and

$$\|x\|^k|h(y)| \le (\|y\| + \gamma)^k|h(y)| \le 2^k M$$

whenever $\|x - y\| \le \gamma$. So

$$\|x\|^k|(\zeta * h)(x)| \le \zeta\mathbb{R}^r \cdot \|x\|^k \sup_{\|y-x\|\le\gamma}|h(y)| \le 2^k M\zeta\mathbb{R}^r$$

for every $x \in \mathbb{R}^r$. Applying this to all the partial derivatives of h, we see that $\zeta * h$ is a rapidly decreasing test function.

(ii) Again suppose that $j < r$. Because ζ has compact support, $\int \|x\|\zeta(dx)$ is finite, so $\frac{\partial}{\partial\eta_j}\hat{\zeta}(y)$ is defined and equal to $-i\int \xi_j e^{-iy\cdot x}\zeta(dx)$ for every $y \in \mathbb{R}^r$ (cf. 285Fd). More generally, whenever $j_1, \dots, j_m < r$,

$$\frac{\partial^m}{\partial\eta_{j_1}\dots\partial\eta_{j_m}}\hat{\zeta}(y) = (-i)^m\int \xi_{j_1}\dots\xi_{j_m}e^{-iy\cdot x}\zeta(dx),$$

so all the partial derivatives of $\hat{\zeta}$ are defined everywhere and bounded. It follows that $\hat{\zeta}$ is smooth and $h \times \hat{\zeta}$ is a rapidly decreasing test function.

(c)(i) To begin with, suppose that f is real and non-negative, and that ζ has compact support. Set $f_n = f \times \chi B(\mathbf{0}, n)$ for each $n \in \mathbb{N}$. Then f_n is integrable, so $\zeta * f_n$ is defined μ-a.e.; also $\langle \zeta * f_n\rangle_{n\in\mathbb{N}}$ is non-decreasing, and $(\zeta * f)(x) = \sup_{n\in\mathbb{N}}(\zeta * f_n)(x)$ whenever the latter is defined and finite.

Let $\gamma \ge 1$ be such that the support of ζ is included in $B(\mathbf{0},\gamma)$, and let $k \in \mathbb{N}$ be such that $\int_{\mathbb{R}^r}\frac{1}{1+\|x\|^k}f(x)\mu(dx)$ is finite. If $y \in B(\mathbf{0},\gamma)$ and $x \in \mathbb{R}^r$, then $\|x\| \le 2\max(\gamma, \|x + y\|)$, so $\frac{1}{1+\|x+y\|^k} \le \frac{M}{1+\|x\|^k}$, where $M = 1 + 2^k\gamma^k$. It follows that

$$\begin{aligned}\int_{\mathbb{R}^r}\frac{1}{1+\|x\|^k}(\zeta * f_n)(x)\mu(dx) &= \int_{\mathbb{R}^r}\int_{B(\mathbf{0},\gamma)}\frac{1}{1+\|x\|^k}f_n(x-y)\zeta(dy)\mu(dx)\\ &= \int_{B(\mathbf{0},\gamma)}\int_{\mathbb{R}^r}\frac{1}{1+\|x\|^k}f_n(x-y)\mu(dx)\zeta(dy)\\ &= \int_{B(\mathbf{0},\gamma)}\int_{\mathbb{R}^r}\frac{1}{1+\|x+y\|^k}f_n(x)\mu(dx)\zeta(dy)\\ &\le M\int_{B(\mathbf{0},\gamma)}\int_{\mathbb{R}^r}\frac{1}{1+\|x\|^k}f(x)\mu(dx)\zeta(dy)\end{aligned}$$

for every $n \in \mathbb{N}$. Consequently $\int_{\mathbb{R}^r}\frac{1}{1+\|x\|^k}(\zeta * f)(x)\mu(dx)$ is defined and finite.

(ii) Now suppose that f is expressible as $f_1 + f_\infty$, where f_1 is μ-integrable, f_∞ is bounded and both are real-valued and non-negative. Adjusting f_1 and f_2 on a μ-negligible set if necessary, we can suppose that f_∞ is Borel measurable and defined everywhere on $\mathbb{R}^r$. By (a), $\zeta * f_1$ is defined and μ-integrable. Next, $\zeta * f_\infty$ is defined everywhere, is bounded, and is Borel measurable (444Ia). So $\zeta * f =_{\text{a.e.}} \zeta * f_1 + \zeta * f_\infty$ is the sum of a μ-integrable function and a bounded Borel measurable function, and is tempered.

(iii) These arguments deal with the case in which $f \ge 0$. For the general case, apply (i) or (ii) to the four parts of f, as in (a-ii).

(d)(i) Suppose to begin with that ζ has compact support. Let h be a rapidly decreasing test function. Set $\overset{\leftrightarrow}{h}(x) = h(-x)$ for every $x \in \mathbb{R}^r$. Then $\overset{\leftrightarrow}{h}$ is a rapidly decreasing test function, and

$$(\zeta * \overset{\leftrightarrow}{h})(-x) = \int \overset{\leftrightarrow}{h}(-x-y)\zeta(dy) = \int h(x+y)\zeta(dy)$$

for every $x \in \mathbb{R}^r$. Accordingly

$$\int (\zeta * f) \times h\,d\mu = \iint h(x)f(x-y)\zeta(dy)\mu(dx)$$
$$= \iint h(x)f(x-y)\mu(dx)\zeta(dy)$$

(because $\zeta * |f|$ is tempered, so $\iint |h(x)f(x-y)|\zeta(dy)\mu(dx) = \int |h| \times (\zeta * |f|)d\mu$ is finite)

$$= \iint h(x+y)f(x)\mu(dx)\zeta(dy)$$
$$= \iint h(x+y)f(x)\zeta(dy)\mu(dx)$$

(because $\iint |h(x+y)f(x)|\mu(dx)\zeta(dy) = \iint |h(x)f(x-y)|\mu(dx)\zeta(dy)$ is finite)

$$= \int f \times (\zeta * \overset{\leftrightarrow}{h})^{\leftrightarrow} d\mu = \int g \times ((\zeta * \overset{\leftrightarrow}{h})^{\leftrightarrow})^{\vee} d\mu$$

(because $\zeta * \overset{\leftrightarrow}{h}$ and $(\zeta * \overset{\leftrightarrow}{h})^{\leftrightarrow}$ are rapidly decreasing test functions, by (b))

$$= \int g \times (\zeta * \overset{\leftrightarrow}{h})^{\wedge} d\mu = (\sqrt{2\pi})^r \int g \times \hat{\zeta} \times (\overset{\leftrightarrow}{h})^{\wedge} d\mu$$

(by (a))

$$= (\sqrt{2\pi})^r \int g \times \hat{\zeta} \times \check{h}\,d\mu.$$

As h is arbitrary, $(\sqrt{2\pi})^r g \times \hat{\zeta}$ represents the Fourier transform of $\zeta * f$.

(ii) Now suppose that f is expressible as $f_1 + f_\infty$ where f_1 is μ-integrable and f_∞ is bounded. By (c), $\zeta * |f|$ is defined almost everywhere and is a tempered function. Set $\zeta_n = \zeta \llcorner B(\mathbf{0}, n)$ for each n. Then $(\sqrt{2\pi})^r g \times \hat{\zeta}_n$ represents the Fourier transform of $\zeta_n * f$, for each n. Now $\langle \hat{\zeta}_n \rangle_{n\in\mathbb{N}}$ converges uniformly to $\hat{\zeta}$, and $\langle \zeta_n * f \rangle_{n\in\mathbb{N}}$ converges to $\zeta * f$ at every point at which $\zeta * |f|$ is defined and finite, which is μ-almost everywhere. So if h is a rapidly decreasing test function,

$$\int h \times (\sqrt{2\pi})^r g \times \hat{\zeta} = \lim_{n\to\infty} \int h \times (\sqrt{2\pi})^r g \times \hat{\zeta}_n$$

(the convergence is dominated by the integrable function $(\sqrt{2\pi})^r \zeta\mathbb{R}^r \cdot |h \times g|$)

$$= \lim_{n\to\infty} \int \hat{h} \times (\zeta_n * f) = \int \hat{h} \times (\zeta * f)$$

(this convergence being dominated by the integrable function $|\hat{h}| \times (\zeta * |f|)$). As h is arbitrary, $(\sqrt{2\pi})^r g \times \hat{\zeta}$ represents the Fourier transform of $\zeta * f$.

479I Proposition (In this result, r may be any integer greater than or equal to 1.)

(a) Suppose that $0 < \alpha < r$.

(i) There is a tempered function representing the Fourier transform of k_α.

(ii) There is a measurable function g_0, defined almost everywhere on $[0, \infty[$, such that $y \mapsto g_0(\|y\|)$ represents the Fourier transform of k_α.

(iii) In (ii),

$$2^{\alpha/2}\Gamma(\tfrac{\alpha}{2})\int_0^\infty t^{r-1} g_0(t) e^{-\epsilon t^2} dt = 2^{(r-\alpha)/2}\Gamma(\tfrac{r-\alpha}{2})\int_0^\infty t^{\alpha-1} e^{-\epsilon t^2} dt$$

for every $\epsilon > 0$.

(iv) $2^{\alpha/2}\Gamma(\frac{\alpha}{2}) g_0(t) = 2^{(r-\alpha)/2}\Gamma(\frac{r-\alpha}{2}) t^{\alpha-r}$ for almost every $t > 0$.

(v) $2^{(r-\alpha)/2}\Gamma(\frac{r-\alpha}{2}) k_{r-\alpha}$ represents the Fourier transform of $2^{\alpha/2}\Gamma(\frac{\alpha}{2}) k_\alpha$.

(b) Suppose that ζ_1, ζ_2 are totally finite Radon measures on $\mathbb{R}^r$, and $0<\alpha<r$. If $\zeta_1*k_\alpha=\zeta_2*k_\alpha$ μ-a.e., then $\zeta_1=\zeta_2$.

proof (a)(i) Set $\beta=\frac{1}{2}(\alpha+r)$. Then k_β is expressible as f_1+f_2 where f_1 is integrable and f_2 is square-integrable. **P**

$$\int_{B(\mathbf{0},1)}k_\beta d\mu=r\beta_r\int_0^1\frac{t^{r-1}}{t^\beta}dt$$

is finite because $\beta<r$;

$$\int_{\mathbb{R}^r\setminus B(\mathbf{0},1)}k_\beta^2 d\mu=r\beta_r\int_1^\infty\frac{t^{r-1}}{t^{2\beta}}dt$$

is finite because $2\beta>r$. So we can take $f_1=k_\alpha\times\chi B(\mathbf{0},1)$ and $f_2=k_\alpha-f_1$. **Q**

479G tells us that there is a constant c such that

$$k_\alpha=ck_\beta*k_\beta=c(f_1*f_1+2f_1*f_2+f_2*f_2).$$

Now f_1*f_1 is integrable and f_1*f_2 is square-integrable (444Ra), so both have Fourier transforms represented by tempered functions; while the continuous function f_2*f_2 also has a Fourier transform represented by an integrable function (284Wi). Assembling these, k_α has a Fourier transform represented by a tempered function.

(ii) We can therefore represent the Fourier transform of k_α by the function g, where

$$g(y)=\lim_{n\to\infty}\frac{1}{(\sqrt{2\pi})^r}\int e^{-iy\,.\,x}e^{-\|x\|^2/n}k_\alpha(x)\mu(dx)$$

is defined μ-almost everywhere (284M/284Wg). Now suppose that $T:\mathbb{R}^r\to\mathbb{R}^r$ is any orthogonal transformation, and that $y\in\operatorname{dom}g$. Then

$$\begin{aligned}g(y)&=\lim_{n\to\infty}\frac{1}{(\sqrt{2\pi})^r}\int e^{-iy\,.\,x}e^{-\|x\|^2/n}k_\alpha(x)\mu(dx)\\&=\lim_{n\to\infty}\frac{1}{(\sqrt{2\pi})^r}\int e^{-iy\,.\,T^\top x}e^{-\|T^\top x\|^2/n}k_\alpha(T^\top x)\mu(dx)\end{aligned}$$

(because the transpose $T^\top$ of T acts as an automorphism of $(\mathbb{R}^r,\mu)$)

$$=\lim_{n\to\infty}\frac{1}{(\sqrt{2\pi})^r}\int e^{-iTy\,.\,x}e^{-\|x\|^2/n}k_\alpha(x)\mu(dx),$$

and $g(Ty)$ is defined and equal to $g(y)$. So we can set $g_0(t)=g(y)$ whenever $y\in\operatorname{dom}g$ and $\|y\|=t$, and we shall have $y\mapsto g_0(\|y\|)$ representing the Fourier transform of k_α.

(iii) If $\epsilon>0$, then $x\mapsto e^{-\epsilon\|x\|^2}$ is a rapidly decreasing test function, and its Fourier transform is the function $x\mapsto\frac{1}{(\sqrt{2\epsilon})^r}e^{-\|x\|^2/4\epsilon}$ (283N/283Wi[11]). We therefore have

$$\int g_0(\|y\|)e^{-\epsilon\|y\|^2}\mu(dy)=\frac{1}{(\sqrt{2\epsilon})^r}\int k_\alpha(x)e^{-\|x\|^2/4\epsilon}\mu(dx),$$

that is,

$$r\beta_r\int_0^\infty t^{r-1}g_0(t)e^{-\epsilon t^2}dt=\frac{r\beta_r}{(\sqrt{2\epsilon})^r}\int_0^\infty\frac{t^{r-1}}{t^\alpha}e^{-t^2/4\epsilon}dt;$$

simplifying,

$$\begin{aligned}\int_0^\infty t^{r-1}g_0(t)e^{-\epsilon t^2}dt&=\frac{1}{(\sqrt{2\epsilon})^r}\int_0^\infty t^{r-1-\alpha}e^{-t^2/4\epsilon}dt\\&=\frac{2^{r-\alpha}}{2\cdot 2^{r/2}\epsilon^{\alpha/2}}\int_0^\infty u^{(r-\alpha-2)/2}e^{-u}du\end{aligned}$$

(substituting $u=t^2/4\epsilon$)

[11] Formerly 283We.

$$= \frac{2^{r-\alpha}}{2\cdot 2^{r/2}\epsilon^{\alpha/2}}\Gamma(\frac{r-\alpha}{2}).$$

On the other hand,

$$\int_0^\infty t^{\alpha-1}e^{-\epsilon t^2}dt = \int_0^\infty \frac{(\sqrt{u})^{\alpha-2}}{2\epsilon^{\alpha/2}}e^{-u}du = \frac{1}{2\epsilon^{\alpha/2}}\Gamma(\frac{\alpha}{2}).$$

Putting these together,

$$2^{\alpha/2}\Gamma(\tfrac{\alpha}{2})\int_0^\infty t^{r-1}g_0(t)e^{-\epsilon t^2}dt = 2^{(r-\alpha)/2}\Gamma(\tfrac{r-\alpha}{2})\int_0^\infty t^{(r-1)-(r-\alpha)}e^{-\epsilon t^2}dt$$

for every $\epsilon > 0$.

(iv) Set

$$g_1(t) = t^{r-1}e^{-t^2}\big(2^{\alpha/2}\Gamma(\tfrac{\alpha}{2})g_0(t) - 2^{(r-\alpha)/2}\Gamma(\tfrac{r-\alpha}{2})t^{\alpha-r}\big)$$

for $t > 0$. Then g_1 is integrable and $\int_0^\infty g_1(t)e^{-\epsilon t^2}dt = 0$ for every $\epsilon \ge 0$. It follows that $g_1 = 0$ a.e. **P** Consider the linear span A of the functions $t \mapsto e^{-\epsilon t^2}$ for $\epsilon \ge 0$. This is a subalgebra of $C_b([0,\infty[)$ containing the constant functions and separating the points of $[0,\infty[$. It follows that for every $\gamma \ge 0$, $\delta > 0$ and $h \in C_b([0,\infty[)$, there is an $f \in A$ such that $|f(t) - h(t)| \le \delta$ for $t \in [0,\gamma]$ and $\|f\|_\infty \le \|h\|_\infty$ (281E). Since $\int_0^\infty g_1 \times f = 0$, we must have

$$|\int_0^\infty g_1 \times h| \le \delta\|g_1\|_1 + 2\|h\|_\infty\int_\gamma^\infty |g_1(t)|dt.$$

As δ and γ are arbitrary, $\int_0^\infty g_1 \times h = 0$; as h is arbitrary, $\int_0^a g_1 = 0$ for every $a \ge 0$, and g_1 must be zero almost everywhere (222D). **Q**

Accordingly $2^{\alpha/2}\Gamma(\frac{\alpha}{2})g_0(t) = 2^{(r-\alpha)/2}\Gamma(\frac{r-\alpha}{2})t^{\alpha-r}$ for almost every $t \ge 0$.

(v) Now

$$y \mapsto 2^{(r-\alpha)/2}\Gamma(\tfrac{r-\alpha}{2})k_{r-\alpha}(y) =_{\text{a.e.}} 2^{\alpha/2}\Gamma(\tfrac{\alpha}{2})g_0(\|y\|)$$

represents the Fourier transform of $2^{\alpha/2}\Gamma(\frac{\alpha}{2})k_\alpha$.

(b) By (a), the Fourier transform of k_α is represented by a tempered function g which is non-zero μ-a.e. As k_α is the sum of an integrable function and a bounded function, 479Hd tells us that the Fourier transform of $\zeta_1 * k_\alpha$ is represented by $(\sqrt{2\pi})^r\hat{\zeta}_1 \times g$; and similarly for ζ_2. As $\zeta_1 * k_\alpha =_{\text{a.e.}} \zeta_2 * k_\alpha$, $\hat{\zeta}_1 \times g =_{\text{a.e.}} \hat{\zeta}_2 \times g$ (284Ib) and $\hat{\zeta}_1 =_{\text{a.e.}} \hat{\zeta}_2$. Since $\hat{\zeta}_1$ and $\hat{\zeta}_2$ are both continuous (285Fb), they are equal everywhere; in particular,

$$\zeta_1\mathbb{R}^r = \hat{\zeta}_1(0) = \hat{\zeta}_2(0) = \zeta_2\mathbb{R}^r.$$

If $\zeta_1 = \zeta_2$ is the zero measure, we can stop. Otherwise, they can be expressed as $\gamma\zeta_1'$ and $\gamma\zeta_2'$ where ζ_1' and ζ_2' are probability measures and $\gamma > 0$. In this case, ζ_1' and ζ_2' have the same characteristic function (285D) and must be equal (285M); so $\zeta_1 = \zeta_2$, as claimed.

Remark The functions $\zeta * k_\alpha$ are called **Riesz potentials**.

479J Now I return to the study of Newtonian potential when $r \ge 3$.

Lemma (a) Let ζ be a totally finite Radon measure on $\mathbb{R}^r$. Let U_ζ be the $(r-1)$-potential of ζ and W_ζ the Newtonian potential of ζ; let k_{r-1} and k_{r-2} be the Riesz kernels. Then $U_\zeta =_{\text{a.e.}} \zeta * k_{r-1}$ and $W_\zeta =_{\text{a.e.}} \zeta * k_{r-2}$.

(b) Let ζ, ζ_1 and ζ_2 be totally finite Radon measures on $\mathbb{R}^r$.

(i) $\int_{\mathbb{R}^r} W_{\zeta_1}d\zeta_2 = \int_{\mathbb{R}^r} W_{\zeta_2}d\zeta_1 = \frac{1}{c_r}\int_{\mathbb{R}^r} U_{\zeta_1} \times U_{\zeta_2}d\mu$, where c_r is the constant of 479G.

(ii) The energy energy(ζ) of ζ is $\frac{1}{c_r}\|U_\zeta\|_2^2$, counting $\|U_\zeta\|_2$ as ∞ if $U_\zeta \notin \mathcal{L}^2(\mu)$.

(iii) If $\zeta = \zeta_1 + \zeta_2$ then $U_\zeta = U_{\zeta_1} + U_{\zeta_2}$ and $W_\zeta = W_{\zeta_1} + W_{\zeta_2}$; similarly, $U_{\alpha\zeta} = \alpha U_\zeta$ and $W_{\alpha\zeta} = \alpha W_\zeta$ for $\alpha \ge 0$.

(iv) If $U_{\zeta_1} = U_{\zeta_2}$ μ-a.e., then $\zeta_1 = \zeta_2$.

(v) If $W_{\zeta_1} = W_{\zeta_2}$ μ-a.e., then $\zeta_1 = \zeta_2$.

(vi) $\zeta\mathbb{R}^r = \lim_{\gamma\to\infty} \frac{1}{r\beta_r\gamma}\int_{\partial B(0,\gamma)} W_\zeta d\nu$.

(c) Let $M_{\mathrm{R}}^+(\mathbb{R}^r)$ be the set of totally finite Radon measures on $\mathbb{R}^r$, with its narrow topology. Then energy : $M_{\mathrm{R}}^+(\mathbb{R}^r) \to [0,\infty]$ is lower semi-continuous.

proof (a) As k_{r-1} and k_{r-2} are both expressible as sums of integrable functions and bounded functions, $\zeta * k_{r-1}$ and $\zeta * k_{r-2}$ are both defined a.e. (479Hc); and now we have only to read the definitions to see that U_ζ and W_ζ are these convolutions with the technical adjustment that they are permitted to take the value ∞.

(b)(i) For any x, $y \in \mathbb{R}^r$,

$$\begin{aligned}\frac{1}{\|x-y\|^{r-2}} &= k_{r-2}(x-y) = \frac{1}{c_r}(k_{r-1} * k_{r-1})(x-y)\\ &= \frac{1}{c_r}\int_{\mathbb{R}^r}\frac{1}{\|x-y-z\|^{r-1}\|z\|^{r-1}}\mu(dz)\\ &= \frac{1}{c_r}\int_{\mathbb{R}^r}\frac{1}{\|x-z\|^{r-1}\|z-y\|^{r-1}}\mu(dz) = \frac{1}{c_r}\int_{\mathbb{R}^r}\frac{1}{\|x-z\|^{r-1}\|y-z\|^{r-1}}\mu(dz).\end{aligned}$$

So

$$\begin{aligned}\int_{\mathbb{R}^r} W_{\zeta_1} d\zeta_2 &= \int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\frac{1}{\|x-y\|^{r-2}}\zeta_1(dx)\zeta_2(dy)\\ &= \frac{1}{c_r}\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\frac{1}{\|x-z\|^{r-1}\|y-z\|^{r-1}}\mu(dz)\zeta_1(dx)\zeta_2(dy)\\ &= \frac{1}{c_r}\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\frac{1}{\|x-z\|^{r-1}\|y-z\|^{r-1}}\zeta_1(dx)\zeta_2(dy)\mu(dz)\\ &= \frac{1}{c_r}\int_{\mathbb{R}^r} U_{\zeta_1}(z)U_{\zeta_2}(z)\mu(dz) = \int_{\mathbb{R}^r} U_{\zeta_1}\times U_{\zeta_2} d\mu.\end{aligned}$$

Hence (or otherwise)

$$\int_{\mathbb{R}^r} W_{\zeta_2} d\zeta_1 = \frac{1}{c_r}\int_{\mathbb{R}^r} U_{\zeta_2}\times U_{\zeta_1} d\mu = \int_{\mathbb{R}^r} W_{\zeta_1} d\zeta_2.$$

(ii) Take $\zeta_1 = \zeta_2 = \zeta$ in (i).

(iii) This is immediate from 234Hc.

(iv)-(v) Put (a) and 479Ib together.

(vi) For any $\gamma > 0$,

$$\frac{1}{r\beta_r\gamma}\int_{\partial B(0,\gamma)} W_\zeta d\nu = \int W_\zeta d\lambda_{B(0,\gamma)}$$

(479Da)

$$= \int \tilde W_{B(0,\gamma)} d\zeta$$

((i) above)

$$= \int \min(1, \frac{\gamma^{r-2}}{\|x-z\|^{r-2}})\zeta(dx)$$

(479Da again)

$$\to \zeta\mathbb{R}^r$$

as $\gamma \to \infty$.

(c) The map $\zeta \mapsto \zeta \times \zeta : M_{\mathrm{R}}^{+}(\mathbb{R}^r) \to M_{\mathrm{R}}^{+}(\mathbb{R}^r \times \mathbb{R}^r)$ is continuous, by 437Ma. Next, the function $\lambda \mapsto \int_{\mathbb{R}^r} \frac{1}{\|x-y\|^{r-2}} \lambda(d(x,y)) : M_{\mathrm{R}}^{+}(\mathbb{R}^r \times \mathbb{R}^r) \to [0,\infty]$ is lower semi-continuous, by 437Jg again. So energy is the composition of a lower semi-continuous function with a continuous function, and is lower semi-continuous (4A2B(d-ii)).

479K Lemma Let $K \subseteq \mathbb{R}^r$ be a compact set, with equilibrium measure λ_K. Then $\lambda_K K = \operatorname{cap} K = \operatorname{energy}(\lambda_K)$, and if ζ is any Radon measure on $\mathbb{R}^r$ such that $\zeta K \ge \operatorname{cap} K \ge \operatorname{energy}(\zeta)$, $\zeta = \lambda_K$.

proof (a) We know that $\lambda_K K = \lambda_K \mathbb{R}^r = \operatorname{cap} K$ (479C(a-i)). So if K has zero capacity then λ_K is the zero measure and $\operatorname{energy}(\lambda_K) = 0$; also the only Radon measure on $\mathbb{R}^r$ with zero energy is λ_K, and we can stop. So henceforth let us suppose that $\operatorname{cap} K > 0$.

Set

$$e = \inf\{\operatorname{energy}(\zeta) : \zeta \text{ is a Radon probability measure on } \mathbb{R}^r \text{ such that } \zeta K \ge \operatorname{cap} K\}.$$

Because $\tilde{W}_K(x) \le 1$ for every $x \in \mathbb{R}^r$ (479D(b-i)),

$$e \le \operatorname{energy}(\lambda_K) = \int \tilde{W}_K d\lambda_K \le \lambda_K \mathbb{R}^r = \operatorname{cap} K$$

is finite.

(b) Consider the set Q of Radon measures ζ on $\mathbb{R}^r$ such that $\zeta K = \zeta \mathbb{R}^r = \operatorname{cap} K$. With its narrow topology, Q is homeomorphic to the set of Radon measures on K of magnitude $\operatorname{cap} K$, which is compact (437R(f-ii)). Since energy : $Q \to [0,\infty]$ is lower semi-continuous (479Jc), there is a $\lambda \in Q$ with energy e (4A2B(d-viii)).

In fact there is exactly one such member of Q. **P** Suppose that ζ is any other member of Q with energy e. Write u_ζ for the equivalence class of U_ζ in L^2. Then $\frac{1}{2}(\zeta + \lambda)$ belongs to Q and $U_{\frac{1}{2}(\zeta+\lambda)} = \frac{1}{2}(U_\zeta + U_\lambda)$ (479J(b-iii)). So, defining c_r as in 479G,

$$\begin{aligned} e + \frac{1}{c_r}\|u_\zeta - u_\lambda\|_2^2 &\le \operatorname{energy}(\tfrac{1}{2}(\zeta+\lambda)) + \frac{1}{4c_r}(u_\zeta - u_\lambda | u_\zeta - u_\lambda) \\ &= \frac{1}{4c_r}(u_\zeta + u_\lambda | u_\zeta + u_\lambda) + \frac{1}{4c_r}(u_\zeta - u_\lambda | u_\zeta - u_\lambda) \end{aligned}$$

(479J(b-ii))

$$= \frac{1}{2c_r}(\|u_\zeta\|_2^2 + \|u_\lambda\|_2^2) = e.$$

It follows that $\|u_\zeta - u_\lambda\|_2 = 0$ and $U_\zeta =_{\text{a.e.}} U_\lambda$. Consequently $\zeta = \lambda$ (479J(b-iv)). **Q**

(c)(i) If ζ is any Radon measure on $\mathbb{R}^r$ with finite energy, then $\int W_\zeta d\lambda \ge \frac{e\,\zeta K}{\operatorname{cap} K}$. **P** If $\zeta K = 0$ this is trivial. Otherwise, set $\zeta' = \frac{\operatorname{cap} K}{\zeta K}\zeta \llcorner K$. Then ζ' has finite energy (479Fh) and belongs to Q, so for any $\alpha \in [0,1]$ we have $\alpha\zeta' + (1-\alpha)\lambda \in Q$, and

$$\begin{aligned} c_r e \le c_r \operatorname{energy}(\alpha\zeta' + (1-\alpha)\lambda) &= \|\alpha u_{\zeta'} + (1-\alpha)u_\lambda\|_2^2 \\ &= \alpha^2\|u_{\zeta'}\|_2^2 + 2\alpha(1-\alpha)(u_{\zeta'}|u_\lambda) + (1-\alpha)^2\|u_\lambda\|_2^2 \\ &= \alpha^2\|u_{\zeta'}\|_2^2 + 2\alpha(1-\alpha)(u_{\zeta'}|u_\lambda) + (1-\alpha)^2 c_r e \\ &= c_r e + 2\alpha((u_{\zeta'}|u_\lambda) - c_r e) + \alpha^2(\|u_{\zeta'}\|_2^2 - 2(u_{\zeta'}|u_\lambda) + c_r e). \end{aligned}$$

It follows that $(u_{\zeta'}|u_\lambda) - c_r e \ge 0$ and

$$\int W_\zeta d\lambda \ge \int W_{\zeta \llcorner K} d\lambda$$

(479Fh)

$$= \frac{\zeta K}{\operatorname{cap} K} \int W_{\zeta'} d\lambda = \frac{\zeta K}{c_r \operatorname{cap} K} \int U_{\zeta'} \times U_\lambda d\mu$$

(479J(b-i))

$$= \frac{\zeta K}{c_r \operatorname{cap} K} (u_{\zeta'} | u_\zeta) \ge \frac{\zeta K}{c_r \operatorname{cap} K} c_r e = \frac{e\zeta K}{\operatorname{cap} K},$$

as claimed. **Q**

(ii) If ζ is any Radon measure on $\mathbb{R}^r$ with finite energy, then $W_\lambda(x) \ge \frac{e}{\operatorname{cap} K}$ for ζ-almost every $x \in K$. **P?** Otherwise, set $E = \{x : x \in K, W_\lambda(x) < \frac{e}{\operatorname{cap} K}\}$, and consider $\zeta' = \zeta \llcorner E$. Then

$$\int W_{\zeta'} d\lambda = \int W_\lambda d\zeta'$$

(479J(b-i))

$$< \frac{e}{\operatorname{cap} K} \zeta' E \le \frac{e\zeta' K}{\operatorname{cap} K},$$

contradicting (i). **XQ**

(iii) $W_\lambda(x) = \frac{e}{\operatorname{cap} K}$ for λ-almost every $x \in K$. **P** Since λ has finite energy, (ii) tells us that $W_\lambda(x) \ge \frac{e}{\operatorname{cap} K}$ for λ-almost every $x \in K$. Since

$$\int_K W_\lambda d\lambda \le \int W_\lambda d\lambda = e = \frac{e}{\operatorname{cap} K} \lambda K,$$

we must have $W_\lambda(x) = \frac{e}{\operatorname{cap} K}$ for λ-almost every $x \in K$. **Q**

(iv) Since $\lambda K = \lambda \mathbb{R}^r$, 479Fg, with f the constant function with value $\frac{e}{\operatorname{cap} K}$, tells us that $W_\lambda(x) \le \frac{e}{\operatorname{cap} K}$ for every $x \in \mathbb{R}^r$.

(d) For $x \in \mathbb{R}^r \setminus K$, $W_\lambda(x) \le \frac{e}{\operatorname{cap} K} \operatorname{hp}(K - x)$. **P** Set $G = \mathbb{R}^r \setminus K$, and let τ be the Brownian exit time from $G - x$. Define $f : \overline{G}^\infty \to [0, 1]$ by setting

$$\begin{aligned} f(y) &= 0 \text{ if } y \in \partial G = \partial K, \\ &= \frac{e}{\operatorname{cap} K} - W_\lambda(y) \text{ if } y \in G, \\ &= \frac{e}{\operatorname{cap} K} \text{ if } y = \infty. \end{aligned}$$

Because $W_\lambda \restriction G$ is continuous and harmonic (479Fa), so is $f \restriction G$. Because λ has compact support, $\lim_{y\to\infty} W_\lambda(y) = 0$ (479Fd), so f is continuous at ∞; because $W_\lambda(y) \le \frac{e}{\operatorname{cap} K}$ for every y, f is lower semi-continuous. So

$$\frac{e}{\operatorname{cap} K} - W_\lambda(x) = f(x) \ge \mathbb{E}(f(x + X_\tau))$$

(478O, because $r \ge 3$ and $\mathbb{R}^r$ has few wandering paths)

$$= \frac{e}{\operatorname{cap} K} \Pr(\tau = \infty) = \frac{e}{\operatorname{cap} K} (1 - \Pr(\tau < \infty)).$$

Thus $W_\lambda(x)$ is at most $\frac{e}{\operatorname{cap} K} \Pr(\tau < \infty)$. But $\Pr(\tau < \infty)$ is just the Brownian hitting probability $\operatorname{hp}(K - x)$. **Q**

(e) $e = \text{energy}(\lambda_K)$. **P**

$$e \le \text{energy}(\lambda_K) \le \operatorname{cap} K$$

((a) above)

$$= \lambda K = \lambda \mathbb{R}^r = \lim_{\|x\| \to \infty} \|x\|^{r-2} W_\lambda(x)$$

(479Fd)

$$\le \frac{e}{\operatorname{cap} K} \liminf_{\|x\| \to \infty} \|x\|^{r-2} \operatorname{hp}(K - x)$$

(by (d) of this proof)

$$= \frac{e}{\operatorname{cap} K} \cdot \operatorname{cap} K$$

(479B(ii))

$$= e. \text{ **Q**}$$

(f) From this we see at once that $\lambda = \lambda_K$ and $\operatorname{cap} K = \text{energy}(\lambda_K)$. Now suppose that ζ is a Radon measure on $\mathbb{R}^r$ such that $\zeta K \ge \operatorname{cap} K \ge \text{energy}(\zeta)$. Set $\zeta' = \frac{\operatorname{cap} K}{\zeta K} \zeta \llcorner K$; then $\zeta' \in Q$, while $\zeta' \le \zeta$, so

$$e = \operatorname{cap} K \ge \text{energy}(\zeta) \ge \text{energy}(\zeta')$$

by 479Fh. It follows that $\zeta' = \lambda$ and $\lambda \le \zeta$. Accordingly $W_\lambda \le W_\zeta$ (479Fh again),

$$\text{energy}(\zeta) = \int W_\zeta d\zeta \ge \int W_\lambda d\zeta \ge \int_K W_\lambda d\zeta \ge \zeta K$$

((c-ii) above)

$$\ge \operatorname{cap} K \ge \text{energy}(\zeta),$$

and we have equality throughout. Since λ is non-zero and the kernel $(x, y) \mapsto \frac{1}{\|x-y\|^{r-2}}$ is strictly positive, W_λ is strictly positive. It follows that $\zeta(\mathbb{R}^r \setminus K) = 0$ and $\zeta \in Q$; consequently $\zeta = \lambda = \lambda_K$, as required.

479L I shall wish later to quote a couple of the facts which appeared in the course of the proof above, and I think it will be safer to list them now.

Corollary Let $K \subseteq \mathbb{R}^r$ be a compact set with equilibrium potential $\tilde{W}_K$.

(a) If ζ is any Radon measure on $\mathbb{R}^r$ with finite energy, then $\tilde{W}_K(x) = 1$ for ζ-almost every $x \in K$.

(b) If ζ is a Radon measure on $\mathbb{R}^r$ such that $W_\zeta \le 1$ everywhere on K, $\zeta K \le \operatorname{cap} K$.

(c) $\tilde{W}_K(x) \le \operatorname{hp}(K - x)$ for every $x \in \mathbb{R}^r \setminus K$.

proof (a)(i) Suppose first that $\operatorname{cap} K > 0$. Working through the proof of 479K, we discover, in parts (e)-(f) of the proof, that $e = \operatorname{cap} K$ and $\lambda = \lambda_K$, so we just have to put (c-ii) of the proof together with 479D(b-i).

(ii) If $\operatorname{cap} K = 0$, let B be a non-trivial closed ball disjoint from A, and consider $L = K \cup B$. Then $\operatorname{cap} B = \operatorname{cap} L$ (479Ea) and $\lambda_L K = 0$, by 479D(c-ii), so

$$\lambda_L B = \lambda_L L = \operatorname{cap} L = \text{energy}(\lambda_L) = \operatorname{cap} B$$

and $\lambda_L = \lambda_B$ (479K). Now $\tilde{W}_L = 1$ ζ-a.e. on L, while

$$\tilde{W}_L(x) = \tilde{W}_B(x) < 1$$

for every $x \in \mathbb{R}^r \setminus B$ (479Da), and in particular for every $x \in K$; so K must be ζ-negligible.

(b) Set $\zeta' = \zeta \llcorner K$; then $W_{\zeta'} \le W_\zeta$, so $\text{energy}(\zeta') = \int_K W_{\zeta'} d\zeta' \le \zeta' K$ is finite. By (a), $\tilde{W}_K \ge 1$ ζ'-a.e., so

$$\zeta K = \zeta' K \le \int \tilde{W}_K d\zeta' = \int W_{\zeta'} d\lambda_K$$

(479J(b-i))

$$\le \lambda_K \mathbb{R}^r = \operatorname{cap} K.$$

(c) If $\operatorname{cap} K = 0$ then λ_K is the zero measure and the result is trivial. Otherwise, again look at the proof of 479K; in part (d), we saw that $W_\lambda(x) \le \frac{e}{\operatorname{cap} K} \operatorname{hp}(K - x)$; but we now know that $e = \operatorname{cap} K$ and $\lambda = \lambda_K$, so we get $\tilde{W}_K(x) \le \operatorname{hp}(K - x)$, as claimed.

479M In 479Ed we saw that there is a natural extension of Newtonian capacity to a Choquet capacity defined on every subset of $\mathbb{R}^r$. However the importance of Newtonian capacity lies as much in the equilibrium measures and potentials as in the simple quantity of capacity itself, and the methods of 479B-479E do not seem to yield these by any direct method. With the new ideas of 479K-479L, we can now approach the problem of defining equilibrium measures for unbounded analytic sets of finite capacity.

Lemma Let $A \subseteq \mathbb{R}^r$ be an analytic set with finite Choquet-Newton capacity $c(A)$.

(a) $\lim_{\gamma\to\infty} c(A \setminus B(\mathbf{0}, \gamma)) = 0$.

(b) $\lambda_A = \lim_{\gamma\to\infty} \lambda_{A\cap B(\mathbf{0},\gamma)}$ is defined for the total variation metric on the space $M_{\mathrm{R}}^+(\mathbb{R}^r)$ of totally finite Radon measures on $\mathbb{R}^r$.

(c)(i) $\lambda_A \mathbb{R}^r = c(A)$.

(ii) $\operatorname{supp}(\lambda_A) \subseteq \partial A$.

(iii) If $B \subseteq \mathbb{R}^r$ is another analytic set such that $c(B) < \infty$, then $\lambda_{A\cup B} \le \lambda_A + \lambda_B$.

(d)(i) $\tilde{W}_A = W_{\lambda_A}$ is the limit $\lim_{\gamma\to\infty} \tilde{W}_{A\cap B(\mathbf{0},\gamma)} = \sup_{\gamma\ge 0} \tilde{W}_{A\cap B(\mathbf{0},\gamma)}$.

(ii) $\tilde{W}_A(x) \le 1$ for every $x \in \mathbb{R}^r$.

(iii) If ζ is any Radon measure on $\mathbb{R}^r$ with finite energy, $\tilde{W}_A(x) = 1$ for ζ-almost every $x \in A$.

(iv) $\operatorname{energy}(\lambda_A) = c(A)$.

proof (a) ? Otherwise, set

$$\alpha = \lim_{\gamma\to\infty} c(A \setminus B(\mathbf{0}, \gamma)) = \inf_{\gamma>0} c(A \setminus B(\mathbf{0}, \gamma)) > 0.$$

Set $\epsilon = \frac{1}{8}\alpha$ and $\delta = \frac{3}{2}\sqrt{\alpha}$. Let γ be such that $c(A \setminus B(\mathbf{0}, \gamma)) \le \alpha + \epsilon$, and let $K \subseteq A \setminus B(\mathbf{0}, \gamma)$ be a compact set such that $\operatorname{cap} K \ge \alpha - \epsilon$ (479E(d-iii)). Let γ' be such that $K \subseteq B(\mathbf{0}, \gamma')$, and let $L \subseteq A \setminus B(\mathbf{0}, \gamma' + \delta)$ be a compact set such that $\operatorname{cap} L \ge \alpha - \epsilon$.

Set $\zeta = \frac{2}{3}(\lambda_K + \lambda_L)$. Then $W_\zeta = \frac{2}{3}(\tilde{W}_K + \tilde{W}_L)$. If $x \in K$, then $\|x - y\| \ge \delta$ for every $y \in L$, so

$$\tilde{W}_L(x) \le \tfrac{1}{\delta^2} \lambda_L L \le \tfrac{\alpha+\epsilon}{\delta^2} = \tfrac{1}{2};$$

similarly, $\tilde{W}_K(x) \le \frac{1}{2}$ for every $x \in L$. So $W_\zeta(x) \le 1$ for every $x \in K \cup L$, and therefore for every $x \in \mathbb{R}^r$, by 479Fg. But this means that

$$c(A \setminus B(\mathbf{0}, \gamma)) \ge \operatorname{cap}(K \cup L) \ge \zeta(K \cup L)$$

(479Lb)

$$= \frac{2}{3}(\operatorname{cap} K + \operatorname{cap} L) \ge \frac{4}{3}(\alpha - \epsilon) > \alpha + \epsilon,$$

which is impossible. **X**

(b) For $\gamma \ge 0$ set $\alpha_\gamma = c(A \setminus B(\mathbf{0}, \gamma))$ and $\zeta_\gamma = \lambda_{A\cap B(\mathbf{0},\gamma)}$. If $0 \le \gamma \le \gamma'$ and $E \subseteq \mathbb{R}^r$ is Borel, then $|\zeta_\gamma E - \zeta_{\gamma'} E| \le \alpha_\gamma$. **P**

$$\zeta_{\gamma'}E \le \zeta_\gamma E + \lambda_{E\cap B(\mathbf{0},\gamma')\setminus B(\mathbf{0},\gamma)}(K)$$

(479D(c-i))

$$\le \zeta_\gamma E + c(A \cap B(\mathbf{0},\gamma') \setminus B(\mathbf{0},\gamma)) \le \zeta_\gamma E + c(A \setminus B(\mathbf{0},\gamma)) = \zeta_\gamma E + \alpha_\gamma.$$

On the other side we now have

$$\begin{aligned}\zeta_\gamma E &= c(A\cap B(\mathbf{0},\gamma)) - \zeta_\gamma(\mathbb{R}^r \setminus E)\\ &\le c(A\cap B(\mathbf{0},\gamma')) - \zeta_{\gamma'}(\mathbb{R}^r\setminus E) + \alpha_\gamma = \zeta_{\gamma'}E + \alpha_\gamma.\end{aligned}$$

So $|\zeta_\gamma E - \zeta_{\gamma'}E| \le \alpha_\gamma$. **Q** It follows at once that $\rho_{\mathrm{tv}}(\zeta_\gamma, \zeta_{\gamma'}) \le 2\alpha_\gamma$.

Since $\lim_{\gamma\to\infty}\alpha_\gamma = 0$, by (a), $\langle\zeta_n\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence for ρ_{tv}. As noted in 437Q(a-iii), $M_{\mathrm{R}}^+(\mathbb{R}^r)$ is complete, so $\lim_{\gamma\to\infty}\lambda_{A\cap B(\mathbf{0},\gamma)} = \lim_{n\to\infty}\zeta_n$ is defined, and we have our measure λ_A.

(c)(i) Now

$$\lambda_A\mathbb{R}^r = \lim_{n\to\infty}\lambda_{A\cap B(\mathbf{0},n)}(\mathbb{R}^r) = \lim_{n\to\infty} c(A\cap B(\mathbf{0},n)) = c(A).$$

(ii) For any $\gamma \ge 0$,

$$\lambda_{A\cap B(\mathbf{0},\gamma)}(\mathbb{R}^r\setminus\partial A) \le \lambda_{A\cap B(\mathbf{0},\gamma)}(\partial B(\mathbf{0},\gamma))$$

(because the support of $\lambda_{A\cap B(\mathbf{0},\gamma)}$ is included in $\partial(A\cap B(\mathbf{0},\gamma)) \subseteq \partial A \cup \partial B(\mathbf{0},\gamma)$)

$$\begin{aligned}&\le |\lambda_A(\partial B(\mathbf{0},\gamma)) - \lambda_{A\cap B(\mathbf{0},\gamma)}(\partial B(\mathbf{0},\gamma))| + \lambda_A(\partial B(\mathbf{0},\gamma))\\ &\le \rho_{\mathrm{tv}}(\lambda_A, \lambda_{A\cap B(\mathbf{0},\gamma)}) + \lambda_A(\partial B(\mathbf{0},\gamma)).\end{aligned}$$

So

$$\begin{aligned}\lambda_A(\mathbb{R}^r\setminus\partial A) &= \lim_{\gamma\to\infty}\lambda_{A\cap B(\mathbf{0},\gamma)}(\mathbb{R}^r\setminus\partial A)\\ &\le \lim_{\gamma\to\infty}\rho_{\mathrm{tv}}(\lambda_A,\lambda_{A\cap B(\mathbf{0},\gamma)}) + \lim_{\gamma\to\infty}\lambda_A(\partial B(\mathbf{0},\gamma)) = 0.\end{aligned}$$

(iii) For any compact set $K \subseteq \mathbb{R}^r$,

$$\lambda_{A\cup B}(K) = \lim_{\gamma\to\infty}\lambda_{(A\cup B)\cap B(\mathbf{0},\gamma)}(K) \le \lim_{\gamma\to\infty}\lambda_{A\cap B(\mathbf{0},\gamma)}(K) + \lambda_{B\cap B(\mathbf{0},\gamma)}(K)$$

(479D(c-i))

$$= \lambda_A(K) + \lambda_B(K) = (\lambda_A+\lambda_B)(K).$$

By 416Ea, $\lambda_{A\cup B} \le \lambda_A + \lambda_B$.

(d)(i) By 479D(b-ii), the supremum and the limit are the same. Suppose that $x\in\mathbb{R}^r$ and $\epsilon > 0$. Start with $\gamma > \|x\|$. Since $\tilde W_{A\cap B(\mathbf{0},\gamma)}(x)$ is finite, there is a $\delta \in \left]0,\gamma-\|x\|\right[$ such that $\int_{B(\mathbf{0},\delta)}\frac{1}{\|x-y\|^{r-2}}\lambda_{A\cap B(\mathbf{0},\gamma)}(dy) \le \epsilon$. If $\gamma' \ge \gamma \ge 0$, then

$$\lambda_{A\cap B(\mathbf{0},\gamma')} \le \lambda_{A\cap B(\mathbf{0},\gamma)} + \lambda_{A\cap B(\mathbf{0},\gamma')\setminus B(\mathbf{0},\gamma)},$$

so

$$\begin{aligned}\int_{B(\mathbf{0},\delta)}\frac{1}{\|x-y\|^{r-2}}\lambda_{A\cap B(\mathbf{0},\gamma')}(dy) &\le \int_{B(\mathbf{0},\delta)}\frac{1}{\|x-y\|^{r-2}}\lambda_{A\cap B(\mathbf{0},\gamma)}(dy)\\ &\qquad + \int_{B(\mathbf{0},\delta)}\frac{1}{\|x-y\|^{r-2}}\lambda_{A\cap B(\mathbf{0},\gamma')\setminus B(\mathbf{0},\gamma)}(dy)\end{aligned}$$

(234Hc, 234Qc)

$$= \int_{B(\mathbf{0},\delta)} \frac{1}{\|x-y\|^{r-2}} \lambda_{A\cap B(\mathbf{0},\gamma)}(dy)$$

(because $\operatorname{int} B(\mathbf{0},\gamma)$ is $\lambda_{A\cap B(\mathbf{0},\gamma')\setminus B(\mathbf{0},\gamma)}$-negligible)

$$\le \epsilon.$$

So, setting $M = \frac{1}{\delta^{r-2}}$,

$$|\tilde{W}_{A\cap B(\mathbf{0},\gamma')}(x) - \int \min(M, \frac{1}{\|x-y\|^{r-2}})\lambda_{A\cap B(\mathbf{0},\gamma')}(dy)| \le \epsilon.$$

Using (c-ii) and (c-iii) to apply the same argument with A in place of $A \cap B(\mathbf{0},\gamma')$, we get

$$|\tilde{W}_A(x) - \int \min(M, \frac{1}{\|x-y\|^{r-2}})\lambda_A(dy)| \le \epsilon.$$

On the other hand,

$$\int \min(M, \frac{1}{\|x-y\|^{r-2}})\lambda_A(dy) = \lim_{\gamma'\to\infty} \int \min(M, \frac{1}{\|x-y\|^{r-2}})\lambda_{A\cap B(\mathbf{0},\gamma')}(dy)$$

(437Q(a-ii)), so

$$\limsup_{\gamma'\to\infty} |\tilde{W}_{A\cap B(\mathbf{0},\gamma')}(x) - \tilde{W}_A(x)| \le 2\epsilon.$$

As ϵ is arbitrary, $W_A(x) = \lim_{\gamma'\to\infty} W_{A\cap B(\mathbf{0},\gamma')}(x)$, as claimed.

(ii) It follows at once that $\tilde{W}_A \le 1$ everywhere.

(iii) Write $E = \{x : x \in A, \tilde{W}_A(x) < 1\}$, and let ζ be a Radon measure on $\mathbb{R}^r$ of finite energy. **?** If $\zeta E > 0$, there is a compact set $K \subseteq E$ such that $\zeta K > 0$. Now there is a $\gamma > 0$ such that $K \subseteq B(\mathbf{0},\gamma)$, in which case

$$\tilde{W}_K(x) \le \tilde{W}_{A\cap B(\mathbf{0},\gamma)}(x) < 1$$

for every $x \in K$, and $\zeta K = 0$, by 479La. **X** So $\zeta E = 0$, as required.

(iv) By (ii) and (c-i),

$$\operatorname{energy}(\lambda_A) = \int \tilde{W}_A d\lambda_A \le \lambda_A \mathbb{R}^r = c(A).$$

In the other direction, for any $\gamma \ge 0$,

$$\operatorname{energy}(\lambda_A) = \int \tilde{W}_A d\lambda_A \ge \int \tilde{W}_{A\cap B(\mathbf{0},\gamma)} d\lambda_A = \int \tilde{W}_A d\lambda_{A\cap B(\mathbf{0},\gamma)}$$

(479J(b-i))

$$\ge \int \tilde{W}_{A\cap B(\mathbf{0},\gamma)} d\lambda_{A\cap B(\mathbf{0},\gamma)} = c(A \cap B(\mathbf{0},\gamma));$$

taking the limit as $\gamma \to \infty$, $\operatorname{energy}(\lambda_A) \ge c(A)$ and we have equality.

479N We are ready to match the definitions in 479C to some alternative definitions of capacity.

Theorem Let $A \subseteq \mathbb{R}^r$ be an analytic set with finite Choquet-Newton capacity $c(A)$.

(a) Writing W_ζ for the Newtonian potential of a Radon measure ζ on $\mathbb{R}^r$,

$$c(A) = \sup\{\zeta A : \zeta \text{ is a Radon measure on } \mathbb{R}^r,\ W_\zeta(x) \le 1 \text{ for every } x \in \mathbb{R}^r\};$$

if A is closed, the supremum is attained.

(b) $c(A) = \inf\{\operatorname{energy}(\zeta) : \zeta$ is a Radon measure on $\mathbb{R}^r$, $\zeta A \ge c(A)\}$; if A is closed, the infimum is attained.

(c) If $A \ne \emptyset$, $c(A) = \sup\{\frac{1}{\operatorname{energy}(\zeta)} : \zeta$ is a Radon measure on $\mathbb{R}^r$ such that $\zeta A = 1\}$, counting $\frac{1}{\infty}$ as zero; if A is closed, the supremum is attained.

proof Note first that if there is a Radon measure ζ on $\mathbb{R}^r$, with finite energy, such that $\zeta A > 0$, then $c(A) > 0$. **P** By 479M(d-iii), $\tilde{W}_A = 1$ ζ-a.e. on A. So $\tilde{W}_A$ cannot be identically 0, and $0 < \lambda_A\mathbb{R}^r = c(A)$, by 479M(c-i). **Q**

(a)(i) We know from 479E(d-i) and 479D(b-i) that

$$\begin{aligned} c(A) &= \sup\{\operatorname{cap} K : K \subseteq A \text{ is compact}\} = \sup\{\lambda_K K : K \subseteq A \text{ is compact}\} \\ &= \sup\{\lambda_K A : K \subseteq A \text{ is compact}\} \le \sup\{\zeta A : W_\zeta \le \chi\mathbb{R}^r\}. \end{aligned}$$

(ii) If ζ is a Radon measure on $\mathbb{R}^r$ and $W_\zeta \le \chi\mathbb{R}^r$, then

$$\zeta A = \sup_{K\subseteq A \text{ is compact}} \zeta K \le \sup_{K\subseteq A \text{ is compact}} \operatorname{cap} K$$

(479Lb)

$$= c(A).$$

Thus $\sup\{\zeta A : W_\zeta \le \chi\mathbb{R}^r\} \le c(A)$ and we have equality.

(iii) If A is closed, then by 479M(c-ii)

$$\lambda_A A = \lambda_A(\partial A) = \lambda_A\mathbb{R}^r = c(A)$$

so λ_A witnesses that the supremum is attained.

(b)(i) ? Suppose, if possible, that there is a Radon measure ζ on $\mathbb{R}^r$ such that $\zeta A \ge c(A) > \operatorname{energy}(\zeta)$. Let $\alpha \in \,]0,1[$ be such that $\alpha^4 c(A) \ge \operatorname{energy}(\zeta)$. Since

$$\zeta A = \sup\{\zeta K : K \subseteq A \text{ is compact}\}, \quad c(A) = \sup\{\operatorname{cap} K : K \subseteq A \text{ is compact}\},$$

there is a compact $K \subseteq A$ such that $\zeta K \ge \alpha\zeta A$ and $\operatorname{cap} K > \alpha^2 c(A)$. Set $\zeta' = \frac{\operatorname{cap} K}{\zeta K}\zeta$. Then

$$\begin{aligned} \operatorname{energy}(\zeta') &= \big(\frac{\operatorname{cap} K}{\zeta K}\big)^2 \operatorname{energy}(\zeta) \le \big(\frac{c(A)}{\alpha\zeta A}\big)^2 \alpha^4 c(A) \\ &\le \alpha^2 c(A) < \operatorname{cap} K = \zeta' K; \end{aligned}$$

which is impossible, by 479K. **X**

So $c(A) \le \inf\{\operatorname{energy}(\zeta) : \zeta A \ge c(A)\}$.

(ii) Take any $\epsilon > 0$. Then there is a compact set $K \subseteq A$ such that $(1+\epsilon)\operatorname{cap} K \ge c(A)$. Set $\zeta = (1+\epsilon)\lambda_K$; then

$$\zeta A \ge c(A), \quad \operatorname{energy}(\zeta) = (1+\epsilon)^2 \operatorname{energy}(\lambda_K) = (1+\epsilon)^2 \operatorname{cap} K \le (1+\epsilon)^2 c(A).$$

As ϵ is arbitrary, $c(A) \ge \inf\{\operatorname{energy}(\zeta) : \zeta A \ge c(A)\}$ and we have equality.

(iii) If A is closed, then

$$\lambda_A A = \lambda_A(\partial A) = \lambda_A\mathbb{R}^r = c(A)$$

by 479M(c-i) and (c-ii), while $\operatorname{energy}(\lambda_A) = c(A)$ by 479M(d-iv). So λ_A witnesses that $c(A) = \min\{\operatorname{energy}(\zeta) : \zeta A \ge c(A)\}$.

(c)(i) Suppose that ζ is a Radon measure on $\mathbb{R}^r$ such that $\zeta A = 1$. If $\operatorname{energy}(\zeta) = \infty$ then of course $\frac{1}{\operatorname{energy}(\zeta)} \le c(A)$. Otherwise, $c(A) > 0$, as remarked at the beginning of this part of the proof. Set $\zeta' = c(A)\zeta$. By (b),

$$c(A) \le \operatorname{energy}(\zeta') = c(A)^2 \operatorname{energy}(\zeta),$$

so $\frac{1}{\operatorname{energy}(\zeta)} \le c(A)$.

Thus $\sup\{\frac{1}{\operatorname{energy}(\zeta)} : \zeta A = \zeta\mathbb{R}^r = 1\} \le c(A)$.

(ii) If $c(A) = 0$ then the supremum is attained by any Radon measure ζ such that $\zeta A = 1$, so we can stop. If $c(A) > 0$, then for any $\alpha \in \,]0,1[$ there is a compact set $K \subseteq A$ such that $\operatorname{cap} K \geq \alpha c(A)$. Set $\zeta = \frac{1}{\operatorname{cap} K}\lambda_K$; then

$$\zeta K = \zeta \mathbb{R}^r = \zeta A = 1$$

and

$$\frac{1}{\operatorname{energy}(\zeta)} = \frac{(\operatorname{cap} K)^2}{\operatorname{energy}(\lambda_K)} = \operatorname{cap} K \geq \alpha c(A).$$

As α is arbitrary, $c(A) \leq \sup\{\frac{1}{\operatorname{energy}(\zeta)} : \zeta A = \zeta \mathbb{R}^r = 1\}$ and we have equality.

(iii) If A is closed and $c(A) > 0$, then $\zeta = \frac{1}{c(A)}\lambda_A$ witnesses that the supremum is attained, as in (b) above.

479O Polar sets To make the final step, to arbitrary sets with finite Choquet-Newton capacity, we seem to need an alternative description of polar sets.

Proposition For a set $D \subseteq \mathbb{R}^r$, the following are equiveridical:

(i) D is polar, that is, $c(D) = 0$;

(ii) there is a totally finite Radon measure ζ on $\mathbb{R}^r$ such that its Newtonian potential W_ζ is infinite at every point of D;

(iii) there is an analytic set $E \supseteq D$ such that $\zeta E = 0$ whenever ζ is a Radon measure on $\mathbb{R}^r$ with finite energy.

proof (i)⇒(ii) If (i) is true, then for each $n \in \mathbb{N}$ there is a bounded open set $G_n \supseteq D \cap B(\mathbf{0}, n)$ such that $c(G_n) \leq 2^{-n}$. Try $\zeta = \sum_{n=0}^\infty \lambda_{G_n}$, defining the sum as in 234G. Then $\zeta \mathbb{R}^r = \sum_{n=0}^\infty c(G_n)$ is finite, and $W_\zeta = \sum_{n=0}^\infty \tilde{W}_{G_n}$ (234Hc). If $x \in D \cap B(\mathbf{0}, n)$, then $\tilde{W}_{G_m}(x) = 1$ for every $m \geq n$ (479D(b-iii)), so $W_\zeta(x) = \infty$. Thus ζ witnesses that (ii) is true.

(ii)⇒(iii) Suppose that λ is a totally finite Radon measure such that $W_\lambda(x) = \infty$ for every $x \in D$. Set $E = \{x : W_\lambda(x) = \infty\}$; then E is a G_δ set, because W_λ is lower semi-continuous (479Fa). **?** If there is a Radon measure ζ on $\mathbb{R}^r$, with finite energy, such that $\zeta E > 0$, let $K \subseteq E$ be a compact set such that $\zeta K > 0$. Set $\zeta_1 = \frac{1}{\zeta K}\zeta \llcorner K$; then ζ_1 has finite energy and $\zeta_1 K = 1$, so $\operatorname{cap} K \geq \frac{1}{\operatorname{energy}(\zeta_1)} > 0$, by 479Nc.

Let $G \supseteq K$ be a bounded open set; set $\lambda_1 = \lambda \llcorner G$ and $\lambda_2 = \lambda \llcorner (\mathbb{R}^r \setminus G)$, so that $\lambda = \lambda_1 + \lambda_2$ and $W_\lambda = W_{\lambda_1} + W_{\lambda_2}$ (234Hc). Since $W_{\lambda_2}(x)$ is finite for $x \in G$ (479Fa), $W_{\lambda_1}(x) = \infty$ for every $x \in K$. Let $\epsilon > 0$ be such that $\epsilon \lambda_1 \mathbb{R}^r < \operatorname{cap} K$. Then ϵW_{λ_1} is a lower semi-continuous superharmonic function greater than or equal to $\tilde{W}_K$ on $K \supseteq \operatorname{supp}(\lambda_K)$, so $\epsilon W_{\lambda_1} \geq \tilde{W}_K$ everywhere (479Fg). But this means that

$$\epsilon \lambda_1 \mathbb{R}^r = \epsilon \lim_{\|x\| \to \infty} \|x\|^{r-2} W_{\lambda_1}(x)$$

(479Fd)

$$\geq \lim_{\|x\| \to \infty} \|x\|^{r-2} \tilde{W}_K(x) = \lambda_K \mathbb{R}^r = \operatorname{cap} K > \epsilon \lambda_1 \mathbb{R}^r,$$

which is absurd. **X**

So E witnesses that (iii) is true.

(iii)⇒(i) Suppose that $E \supseteq D$ is analytic and that $\zeta E = 0$ whenever $\operatorname{energy}(\zeta)$ is finite. If $K \subseteq E$ is compact and ζ is a Radon probability measure on $\mathbb{R}^r$ such that $\zeta K = 1$, then $\operatorname{energy}(\zeta)$ must be infinite; by 479Nc, $\operatorname{cap} K = 0$. As K is arbitrary, $c(E) = 0$ and $c(D) = 0$.

479P At last I come to my final extension of the notions of equilibrium measure and potential, together with a direct expression of the latter in terms of Brownian hitting probabilities.

Theorem Let $D \subseteq \mathbb{R}^r$ be a set with finite Choquet-Newton capacity $c(D)$.

(a) There is a totally finite Radon measure λ_D on $\mathbb{R}^r$ such that $\lambda_D = \lambda_A$, as defined in 479Mb, whenever $A \supseteq D$ is analytic and $c(A) = c(D)$.

(b) Write $\tilde{W}_D = W_{\lambda_D}$ for the equilibrium potential corresponding to the equilibrium measure λ_D. Then $\tilde{W}_D(x) = \text{hp}^*((D \setminus \{x\}) - x)$ for every $x \in \mathbb{R}^r$.

(c)(i)(α) $\lambda_D \mathbb{R}^r = c(D)$;

(β) if ζ is any Radon measure on $\mathbb{R}^r$ with finite energy, $\tilde{W}_D(x) = 1$ for ζ-almost every $x \in D$;

(γ) energy$(\lambda_D) = c(D)$;

(δ) if $D' \subseteq D$ and $c(D') = c(D)$, then $\lambda_{D'} = \lambda_D$.

(ii) $\text{supp}(\lambda_D) \subseteq \partial D$.

(iii) For any $D' \subseteq \mathbb{R}^r$ such that $c(D') < \infty$,

(α) $\lambda_D^*(D') \le c(D')$;

(β) $\lambda_{D \cup D'} \le \lambda_D + \lambda_{D'}$;

(γ) $\tilde{W}_{D \cap D'} + \tilde{W}_{D \cup D'} \le \tilde{W}_D + \tilde{W}_{D'}$;

(δ) $\rho_{\text{tv}}(\lambda_D, \lambda_{D'}) \le 2c(D \triangle D')$.

(iv) If $\langle D_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence of sets with union D, then

(α) $\tilde{W}_D = \lim_{n\to\infty} \tilde{W}_{D_n} = \sup_{n \in \mathbb{N}} \tilde{W}_{D_n}$;

(β) $\langle \lambda_{D_n} \rangle_{n \in \mathbb{N}} \to \lambda_D$ for the narrow topology on $M_{\text{R}}^+(\mathbb{R}^r)$.

(v) $c(D) = \inf\{\zeta \mathbb{R}^r : \zeta$ is a Radon measure on $\mathbb{R}^r$, $W_\zeta \ge \chi D\}$
$= \inf\{\text{energy}(\zeta) : \zeta$ is a Radon measure on $\mathbb{R}^r$, $W_\zeta \ge \chi D\}$.

(vi) Writing cl*D for the essential closure of D, $c(\text{cl}^*D) \le c(D)$ and $\tilde{W}_{\text{cl}^*D} \le \tilde{W}_D$.

(vii) Suppose that $f : D \to \mathbb{R}^r$ is γ-Lipschitz, where $\gamma \ge 0$. Then $c(f[D]) \le \gamma^{r-2} c(D)$.

proof (a)(i) If A, $B \subseteq \mathbb{R}^r$ are analytic sets, $c(B) < \infty$ and $A \subseteq B$, then

$$\tilde{W}_A = \sup_{n \in \mathbb{N}} \tilde{W}_{A \cap B(\mathbf{0},n)}$$

(479M(d-i))

$$\le \sup_{n \in \mathbb{N}} \tilde{W}_{B \cap B(\mathbf{0},n)}$$

(479D(b-ii))

$$= \tilde{W}_B.$$

If $c(A) = c(B)$, then $\lambda_A = \lambda_B$. **P**

$$c(A) = \text{energy}(\lambda_A)$$

(479M(d-iv))

$$= \int \tilde{W}_A d\lambda_A \le \int \tilde{W}_B d\lambda_A = \int \tilde{W}_A d\lambda_B$$

(479J(b-i))

$$\le \int \tilde{W}_B d\lambda_A = \text{energy}(\lambda_B) = c(B) = \lambda_B \mathbb{R}^r$$

(479M(c-i)). So we must have equality throughout, and $\tilde{W}_A = \tilde{W}_B$ λ_B-a.e. By 479Fg, $\tilde{W}_A \ge \tilde{W}_B$ everywhere and

$$W_{\lambda_B} = \tilde{W}_B = \tilde{W}_A = W_{\lambda_A}.$$

By 479J(b-v), $\lambda_B = \lambda_A$. **Q**

(ii) Now consider the given set D. By 479E(d-i), there is an analytic set $A \supseteq D$ such that $c(A) = c(D)$. If B is another such set, then $c(A \cap B) = c(A) = c(B)$, so $\lambda_{A \cap B} = \lambda_A = \lambda_B$. We therefore have a common measure which we can take to be λ_D. Of course this agrees with 479Mb if D itself is analytic, and with 479B if D is bounded and analytic.

(b) Write $h_D(x)$ for $\mathrm{hp}^*((D\setminus\{x\})-x)$.

(i) To begin with, suppose that $D=K$ is compact and that $x\notin K$, so that $h_D(x)=h_K(x)=\mathrm{hp}(K-x)$.

(α) $h_K(x)\ge\tilde W_K(x)$. **P** 479Lc. **Q**

(β) In fact $h_K(x)=\tilde W_K(x)$. **P** Let $\epsilon>0$. Set $E=\{y:y\in K,\ \tilde W_K(y)<1\}$. Because $\tilde W_K$ is lower semi-continuous, E is an F_σ set, therefore analytic; by 479M(d-iii), E satisfies condition (iii) of 479O, and is polar. By 479O(ii), there is a totally finite Radon measure ζ on $\mathbb{R}^r$ such that $W_\zeta(y)=\infty$ for every $y\in E$. Let H be a bounded open set, including K, such that $x\notin\overline{H}$; set $\zeta_1=\zeta\llcorner H$ and $\zeta_2=\zeta\llcorner(\mathbb{R}^r\setminus H)$. Then $\zeta=\zeta_1+\zeta_2$, so $W_\zeta=W_{\zeta_1}+W_{\zeta_2}$ (479J(b-iii)). Since H is open and ζ_2-negligible, $W_{\zeta_2}(y)$ is finite for every $y\in K$ (479Fa), and $W_{\zeta_1}(y)=\infty$ for every $y\in E$; while $W_{\zeta_1}(x)$ is finite because the support of ζ_1 is included in $\overline{H}$.

There is therefore an $\eta>0$ such that $\eta W_{\zeta_1}(x)\le\epsilon$. Consider $\lambda=\lambda_K+\eta\zeta_1$. We have $W_\lambda(y)\ge1$ for every $y\in K$, while W_λ is superharmonic and lower semi-continuous (479Fa, 479Fb); as the support of λ is included in the compact set $\overline{H}$, $\lim_{\|y\|\to\infty}W_\lambda(y)=0$ (479Fd). Consequently

$$h_K(x)=\mu_x^{(K)}(K)\le\int W_\lambda d\mu_x^{(K)}\le W_\lambda(x)$$

(478Pc, with $G=\mathbb{R}^r\setminus K$)

$$\le\tilde W_K(x)+\epsilon.$$

As ϵ is arbitrary, $h_K(x)\le\tilde W_K(x)$ and we have equality. **Q**

(ii) If $D=A$ is analytic, note that $\mathrm{cap}\{x\}=0$ (479Da, or otherwise), so $c(A\setminus\{x\})=c(A)$, because c is monotonic and submodular, therefore subadditive (479E(d-ii)). Now we know that

$$h_A(x)=\sup\{\mathrm{hp}(K-x):K\subseteq A\setminus\{x\}\text{ is compact}\}$$

(477Ie) and

$$c(A\setminus\{x\})=\sup\{\mathrm{cap}\,K:K\subseteq A\setminus\{x\}\text{ is compact}\}$$

(479E(d-iii)). So there is a non-decreasing sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ of compact subsets of $A\setminus\{x\}$ such that

$$h_A(x)=\sup_{n\in\mathbb{N}}\mathrm{hp}(K_n-x),\quad c(A\setminus\{x\})=\sup_{n\in\mathbb{N}}\mathrm{cap}\,K_n.$$

Set $E=\bigcup_{n\in\mathbb{N}}K_n$; then $E\subseteq A$ and $c(E)=c(A)$, so $\lambda_E=\lambda_A$ ((a-i) above) and $\tilde W_E=\tilde W_A$. Accordingly

$$h_A(x)=\sup_{n\in\mathbb{N}}\mathrm{hp}(K_n-x)=\sup_{n\in\mathbb{N}}\tilde W_{K_n}(x)$$

((a-i) above)

$$=\sup_{m,n\in\mathbb{N}}\tilde W_{K_n\cap B(\mathbf{0},m)}(x)=\sup_{m\in\mathbb{N}}\tilde W_{E\cap B(\mathbf{0},m)}(x)$$

(apply 479E(b-iii) twice)

$$=\tilde W_E(x)=\tilde W_A(x).$$

(iii) For the general case, note first that $h_D\le\tilde W_D$. **P** There is a G_δ set $E\supseteq D$ such that $c(E)=c(D)$, so $\lambda_E=\lambda_D$. Now, for any $x\in\mathbb{R}^r$,

$$h_D(x)\le h_E(x)=\tilde W_E(x)=\tilde W_D(x),$$

using (ii) for the central equality. **Q**

Equally, $h_D\ge\tilde W_D$. **P** If $x\in\mathbb{R}^r$, there is a G_δ set $H\supseteq(D\setminus\{x\})-x$ such that

$$h_D(x)=\mathrm{hp}^*((D\setminus\{x\})-x)=\mathrm{hp}\,H$$

(477Id). Set $A=(H+x)\cup\{x\}$; then $A\supseteq D$ and

$$h_D(x)=h_A(x)=\tilde W_A(x)\ge\tilde W_{A\cap E}(x)=\tilde W_D(x),$$

using (a-i) again for the inequality. **Q**

So $h_D = \tilde{W}_D$, as claimed.

(c) Fix an analytic set $A \supseteq D$ such that $c(A) = c(D)$; replacing A by $A \cap \overline{D}$ if necessary, we may suppose that $A \subseteq \overline{D}$. We have $\lambda_D = \lambda_A$ and $\tilde{W}_D = \tilde{W}_A$.

(i)(α)

$$\lambda_D \mathbb{R}^r = \lambda_A \mathbb{R}^r = c(A) = c(D)$$

by 479M(c-i).

(β)-(γ) 479M(d-iii) tells us that $\tilde{W}_D(x) = \tilde{W}_A(x) = 1$ for ζ-almost every $x \in A$, and therefore for ζ-almost every $x \in D$. At the same time,

$$\text{energy}(\lambda_D) = \text{energy}(\lambda_A) = c(A) = c(D)$$

by 479M(d-iv).

(δ) Of course $A \supseteq D'$ and $c(A) = c(D')$, so $\lambda_{D'} = \lambda_A = \lambda_D$.

(ii) $\overline{A} = \overline{D}$ and $\operatorname{int} A \supseteq \operatorname{int} D$, so $\partial A \subseteq \partial D$ and

$$\lambda_D(\mathbb{R}^r \setminus \partial D) = \lambda_A(\mathbb{R}^r \setminus \partial D) \le \lambda_A(\mathbb{R}^r \setminus \partial A) = 0$$

by 479M(c-ii). As ∂D is closed, it includes $\operatorname{supp}(\lambda_D)$.

(iii) Let $A' \supseteq D'$ be an analytic set such that $c(A') = c(D')$.

(α)

$$\lambda_D^*(D') \le \lambda_D(A') = \lambda_A(A') \le \sup_{m\in\mathbb{N}} \lambda_{A\cap B(\mathbf{0},m)}(A')$$

(479Mb)

$$= \sup_{m,n\in\mathbb{N}} \lambda_{A\cap B(\mathbf{0},m)}(A' \cap B(\mathbf{0},n)) \le \sup_{n\in\mathbb{N}} c(A' \cap B(\mathbf{0},n))$$

(479D(c-ii))

$$= c(A')$$

(because c is a capacity)

$$= c(D').$$

(β) Because c is subadditive, we know that $c(D \cup D')$ is finite. Let $B \supseteq D \cup D'$ be an analytic set such that $c(B) = c(D \cup D')$. Then

$$\lambda_{D\cup D'} = \lambda_{B\cap(A\cup A')} \le \lambda_{B\cap A} + \lambda_{B\cap A'}$$

(479M(c-iii))

$$= \lambda_D + \lambda_{D'}.$$

(γ) This is immediate from (b) and the general fact that $\zeta^*(U \cap V) + \zeta^*(U \cup V) \le \zeta^* U + \zeta^* V$ for any measure ζ and any sets U and V (132Xk).

(δ) As usual, it will be enough to show that $|\lambda_D E - \lambda_{D'} E| \le c(D \triangle D')$ for every Borel set $E \subseteq \mathbb{R}^r$; by symmetry, all we need to check is that $\lambda_{D'} E \le \lambda_D E + c(D \triangle D')$ for every Borel set E. **P**

case 1 Suppose that D and D' are both bounded Borel sets. Take $x \in \mathbb{R}^r$, and let τ, $\tau' : \Omega \to [0, \infty]$ be the Brownian arrival times to $D - x$, $D' - x$ respectively. Then

$$\begin{aligned}\mu_x^{(D')}E &= \mu_W\{\omega : \tau'(\omega) < \infty,\, x + \omega(\tau'(\omega)) \in E\} \\ &\le \mu_W\{\omega : \tau(\omega) < \infty,\, x + \omega(\tau(\omega)) \in E\} + \mu_W\{\omega : \tau(\omega) \ne \tau'(\omega)\} \\ &\le \mu_x^{(D)}E + \mu_W\{\omega : \text{there is some } t \ge 0 \text{ such that } x + \omega(t) \in D\triangle D'\} \\ &= \mu_x^{(D)}E + \mu_x^{(D\triangle D')}\mathbb{R}^r.\end{aligned}$$

So

$$\begin{aligned}\lambda_{D'}E &= \lim_{\|x\|\to\infty} \|x\|^{r-2}\mu_x^{(D')}E \\ &\le \lim_{\|x\|\to\infty} \|x\|^{r-2}\mu_x^{(D)}E + \lim_{\|x\|\to\infty} \|x\|^{r-2}\mu_x^{(D\triangle D')}\mathbb{R}^r = \lambda_D E + c(D\triangle D').\end{aligned}$$

case 2 Suppose that D, D' are Borel sets, not necessarity bounded. Set $D_n = D \cap B(\mathbf{0}, n)$, $D'_n = D' \cap B(\mathbf{0}, n)$. Then

$$\lambda_{D'}E = \lim_{n\to\infty} \lambda_{D'_n}E$$

(479Mb)

$$\le \lim_{n\to\infty} \lambda_{D_n}E + \lim_{n\to\infty} c(D_n \triangle D'_n)$$

(by case 1)

$$= \lambda_D E + \lim_{n\to\infty} c((D\triangle D') \cap B(\mathbf{0}, n)) = \lambda_D E + c(D\triangle D')$$

because c is a capacity.

case 3 In general, let $G \supseteq D$, $G' \supseteq D'$ and $H \supseteq D\triangle D'$ be G_δ sets such that $c(G) = c(D)$, $c(G') = c(D')$ and $c(H) = c(D\triangle D')$. Set

$$G_1 = G \cap (G' \cup H), \quad G'_1 = G' \cap (G \cup H);$$

these are Borel sets, while $D \subseteq G_1 \subseteq G$, $D' \subseteq G'_1 \subseteq G'$ and $G_1 \triangle G_2 \subseteq H$. So

$$\lambda_{D'}E = \lambda_{G'_1}E \le \lambda_{G_1}E + c(G\triangle G_1)$$

(by case 2)

$$\le \lambda_D E + c(H) = \lambda_D E + c(D\triangle D')$$

and we have the result in this case also. So we're done. **Q**

(iv)(α) This follows immediately from (b) above.

(β) Consider first the case in which every D_n is analytic. Returning to the proof of 479M, or putting 479Ma together with (iii-δ) here, we see that for any m, $n \in \mathbb{N}$ we shall have

$$\rho_{\mathrm{tv}}(\lambda_{D_n}, \lambda_{D_n \cap B(\mathbf{0},m)}) \le 2c(D_n \setminus B(\mathbf{0}, m)) \le 2c(D \setminus B(\mathbf{0}, m)) = 2\alpha_m$$

say, and that $\lim_{m\to\infty} \alpha_m = 0$. So if $G \subseteq \mathbb{R}^r$ is any open set,

$$\lambda_D G = \lim_{m\to\infty} \lambda_{D\cap B(\mathbf{0},m)}G \le \lim_{m\to\infty} \liminf_{n\to\infty} \lambda_{D_n \cap B(\mathbf{0},m)}G$$

(479E(c-i))

$$\le \lim_{m\to\infty} \liminf_{n\to\infty} \lambda_{D_n}G + 2\alpha_m = \liminf_{n\to\infty} \lambda_{D_n}G.$$

Since we know also that

$$\lambda_D \mathbb{R}^r = c(D) = \lim_{n\to\infty} c(D_n) = \lim_{n\to\infty} \lambda_{D_n}\mathbb{R}^r,$$

$\langle \lambda_{D_n} \rangle_{n\in\mathbb{N}} \to \lambda_D$ for the narrow topology.

For the general case, take analytic sets $A_n \supseteq D_n$, $A \supseteq D$ such that $c(A_n) = c(D_n)$ for every n and $c(A) = c(D)$. Set $A'_n = A \cap \bigcap_{m\ge n} A_m$ for each n, $A' = \bigcup_{n\in\mathbb{N}} A_n$; then

$$\langle \lambda_{D_n} \rangle_{n\in\mathbb{N}} = \langle \lambda_{A'_n} \rangle_{n\in\mathbb{N}} \to \lambda_{A'} = \lambda_D$$

for the narrow topology.

(v) Let Q be the set of Radon measures ζ on $\mathbb{R}^r$ such that $W_\zeta \ge \chi D$.

(α) I show first that $\inf_{\zeta\in Q} \zeta\mathbb{R}^r$ and $\inf_{\zeta\in Q} \operatorname{energy}(\zeta)\}$ are both less than or equal to $c(D)$. **P** Let $\epsilon > 0$. Because c is outer regular (479E(d-i)), there is an open set $G \supseteq D$ such that $c(G) \le c(D) + \epsilon$. Set $\zeta = \lambda_G$. Then

$$W_\zeta = \tilde{W}_G \ge \chi G \ge \chi D$$

(479D(b-iii)), so $\zeta \in Q$, while

$$\zeta\mathbb{R}^r = \operatorname{energy}(\zeta) = c(G) \le c(D) + \epsilon. \ \mathbf{Q}$$

(β) Now suppose that $\zeta \in Q$. Then $c(D) \le \min(\zeta\mathbb{R}^r, \operatorname{energy}(\zeta))$. **P** Take any $\gamma < c(D)$ and $\epsilon > 0$. Let $A \supseteq D$ be an analytic set such that $c(A) = c(D)$; replacing A by $\{x : x \in A,\ W_\zeta(x) \ge 1\}$ if necessary, we can suppose that $W_\zeta \ge \chi A$. For each $n \in \mathbb{N}$, let ζ_n be the totally finite measure $(1+\epsilon)\zeta \llcorner B(\mathbf{0}, n)$. Then $\langle W_{\zeta_n} \rangle_{n\in\mathbb{N}}$ is non-decreasing and has supremum $(1+\epsilon)W_\zeta$ (479M(d-i)), so $A = \bigcup_{n\in\mathbb{N}} A_n$, where $A_n = \{x : x \in A,\ W_{\zeta_n}(x) \ge 1\}$. There are an $n \in \mathbb{N}$ such that $c(A_n) > \gamma$ and a compact $K \subseteq A_n$ such that $\operatorname{cap} K \ge \gamma$ (432K). Now $W_{\zeta_n} \ge \tilde{W}_K$ λ_K-a.e., so $W_{\zeta_n} \ge \tilde{W}_K$ everywhere (479Fg) and

$$\gamma \le \operatorname{cap} K = \int \tilde{W}_K d\lambda_K \le \int W_{\zeta_n} d\lambda_K = \int \tilde{W}_K d\zeta_n$$

(479J(b-i))

$$\le \int W_{\zeta_n} d\zeta_n \le (1+\epsilon) \int W_\zeta d\zeta_n \le (1+\epsilon)^2 \int W_\zeta d\zeta = (1+\epsilon)^2 \operatorname{energy}(\zeta).$$

Moreover, 479J(c-vi), applied to ζ_n and λ_K, tells us that

$$\zeta\mathbb{R}^r \ge \zeta_n\mathbb{R}^r \ge \lambda_K\mathbb{R}^r = \operatorname{cap} K \ge \gamma.$$

As γ and ϵ are arbitrary, $c(D) \le \min(\operatorname{energy}(\zeta), \zeta\mathbb{R}^r)$, as claimed. **Q**

(γ) Putting these together, we see that $c(D) = \inf_{\zeta\in Q} \zeta\mathbb{R}^r = \inf_{\zeta\in Q} \operatorname{energy}(\zeta)$.

(vi) If $x \in \mathrm{cl}^*A$, then $0 \in \mathrm{cl}^*((A \setminus \{x\}) - x)$ and $\tilde{W}_A(x) = \mathrm{hp}^*((A \setminus \{x\}) - x) = 1$ for every $x \in \mathrm{cl}^*E$, by 478U and (b) above. Now

$$c(\mathrm{cl}^*D) \le c(\mathrm{cl}^*A) \le \operatorname{energy}(\lambda_A)$$

((v) above)

$$= c(A)$$

(479M(d-iv))

$$= c(D).$$

(vii)(α) Consider first the case $D = A$, so that $f[D] = f[A]$ is analytic. We can suppose that $c(f[A]) > 0$, in which case $A \ne \emptyset$ and $\gamma > 0$. Take any $\epsilon > 0$. By 479Nc there is a Radon measure ζ on $\mathbb{R}^r$ such that $\zeta f[A] = 1$ and $c(f[A]) \le \frac{1+\epsilon}{\operatorname{energy}(\zeta)}$. Applying 433D to the subspace measure $\zeta_{f[A]}$, we see that there is a Radon probability measure ζ' on A such that $\zeta_{f[A]}$ is the image measure $\zeta' f^{-1}$; let λ be the Radon probability measure on $\mathbb{R}^r$ extending ζ'. Then

$$\begin{aligned}\text{energy}(\zeta) &= \int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\frac{1}{\|x-y\|^{r-2}}\zeta(dx)\zeta(dy) \ge \int_{f[A]}\int_{f[A]}\frac{1}{\|x-y\|^{r-2}}\zeta(dx)\zeta(dy)\\ &= \int_A\int_{f[A]}\frac{1}{\|x-f(v)\|^{r-2}}\zeta(dx)\zeta'(dv) = \int_A\int_A\frac{1}{\|f(u)-f(v)\|^{r-2}}\zeta'(du)\zeta'(dv)\end{aligned}$$

(applying 235J[12] twice)

$$\begin{aligned}&\ge \int_A\int_A\frac{1}{\gamma^{r-2}\|u-v\|^{r-2}}\zeta'(du)\zeta'(dv)\\ &= \frac{1}{\gamma^{r-2}}\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\frac{1}{\|u-v\|^{r-2}}\lambda(du)\lambda(dv) = \frac{1}{\gamma^{r-2}}\,\text{energy}(\lambda).\end{aligned}$$

By 479Nc in the other direction,

$$c(A) \ge \frac{1}{\text{energy}(\lambda)} \ge \frac{1}{\gamma^{r-2}\,\text{energy}(\zeta)} \ge \frac{1}{(1+\epsilon)\gamma^{r-2}}c(f[A]).$$

As ϵ is arbitrary, $c(f[A]) \le \gamma^{r-2}c(A)$.

(β) In general, since $f : D \to \mathbb{R}^r$ is certainly uniformly continuous, it has a continuous extension $g : \overline{D} \to \mathbb{R}^r$ (3A4G), which is still γ-Lipschitz. Now (α) tells us that

$$c(f[D]) \le c(g[A]) \le \gamma^{r-2}c(A) = \gamma^{r-2}c(D),$$

as required.

479Q Hausdorff measure: Theorem For $s \in \,]0,\infty[$ let μ_{Hs} be Hausdorff s-dimensional measure on $\mathbb{R}^r$. Let D be any subset of $\mathbb{R}^r$.

(a) If the Choquet-Newton capacity $c(D)$ is non-zero, then $\mu^*_{H,r-2}D = \infty$.

(b) If $s > r-2$ and $\mu^*_{Hs}D > 0$, then $c(D) > 0$.

proof (a) Let $E \supseteq D$ be a G_δ set such that $\mu_{H,r-2}E = \mu^*_{H,r-2}D$ (471Db). Then $c(E) > 0$. Let $K \subseteq E$ be a compact set such that $\operatorname{cap} K > 0$. Then

$$\operatorname{cap} K = \int_K\int_K \frac{1}{\|x-y\|^{r-2}}\lambda_K(dx)\lambda_K(dy)$$

is finite and not 0; applying 471Tb to the subspace measure on K,

$$\infty = \mu_{H,r-2}K = \mu_{H,r-2}E = \mu^*_{H,r-2}D.$$

(b) Let $E \supseteq D$ be a G_δ set such that $c(E) = c(D)$ (479E(d-i)). Then $\mu_{Hs}E > 0$. By 471Ta, there is a non-zero Radon measure ζ_0 on E such that $\int_E\int_E \frac{1}{\|x-y\|^{r-2}}\zeta_0(dx)\zeta_0(dy)$ is finite. Let $K \subseteq E$ be a compact set such that $\zeta_0 K > 0$, and let ζ be the Radon measure on $\mathbb{R}^r$ such that $\zeta H = \zeta_0(K\cap H)$ for every Borel set $H \subseteq \mathbb{R}^r$; then

$$\text{energy}(\zeta) = \int_K\int_K \frac{1}{\|x-y\|^{r-2}}\zeta_0(dx)\zeta_0(dy)$$

is finite, while K is ζ-conegligible. By 479Nc (or, more directly, by the first remark in the proof of 479N), $\operatorname{cap} K > 0$, so that $c(D) = c(E) \ge \operatorname{cap} K > 0$.

479R I come to the promised difference between Brownian motion in $\mathbb{R}^3$ and in higher dimensions, following 478M.

Proposition (a) Suppose that $r = 3$. Then almost every $\omega \in \Omega$ is not injective.

(b) If $r \ge 4$, then almost every $\omega \in \Omega$ is injective.

proof In this proof, I will take $X_t(\omega) = \omega(t)$ for $\omega \in \Omega$ and $t \ge 0$.

[12] Formerly 235L.

(a)(i) For $\omega \in \Omega$ set $F_\omega = \{\omega(t) : t \in [0,1]\}$. Then $\operatorname{cap} F_\omega > 0$ for μ_W-almost every ω. **P** By 477Lb, $\mu_{H,3/2} F_\omega = \infty$ for almost every ω. For any such ω, there is a non-zero Radon measure ζ_0 on F_ω such that $\int_{F_\omega}\int_{F_\omega} \frac{1}{\|x-y\|}\zeta_0(dx)\zeta_0(dy)$ is finite (471Ta). Let ζ be the Radon measure on $\mathbb{R}^r$, extending ζ_0, for which F_ω is conegligible. Then $\zeta(F_\omega) > 0$ and $\operatorname{energy}(\zeta) < \infty$. (This is where we need to know that $r = 3$.) So $\operatorname{cap} F_\omega > 0$ (479Nc). **Q**

(ii) Consider $E_0 = \{\omega :$ there are $s \le 1$ and $t \ge 2$ such that $\omega(s) = \omega(t)\}$. (This is an F_σ set, so is measurable.) Take τ to be the stopping time with constant value 2 and $\phi_\tau : \Omega \times \Omega \to \Omega$ the corresponding inverse-measure-preserving function as in 477G; set $H = \{\omega : \omega \in \Omega,\ \omega(2) \notin F_\omega\}$. Then

$$\begin{aligned}\mu_W E_0 &= \int_\Omega \mu_W\{\omega' : \phi_\tau(\omega,\omega') \in E_0\}\mu_W(d\omega)\\ &= \int_\Omega \mu_W\{\omega' : \text{ there is some } t \ge 0 \text{ such that } \omega(2) + \omega'(t) \in F_\omega\}\mu_W(d\omega)\\ &= \mu_W(\Omega \setminus H) + \int_H \tilde W_{F_\omega}(\omega(2))\mu_W(d\omega)\\ &> 0 \end{aligned} \tag{479Pb}$$

because $\tilde W_{F_\omega}(\omega(2)) > 0$ whenever $\omega \in H$ and $\operatorname{cap} F_\omega > 0$, which is so for almost every $\omega \in H$.

(iii) Now, setting

$$E_n = \{\omega : \text{there are } s \in [n, n+1] \text{ and } t \ge n+2 \text{ such that } \omega(s) = \omega(t)\},$$

we have $\mu_W E_n = \mu_W E_0$ for every n, because $\langle X_{s+n} - X_n\rangle_{s\ge 0}$ has the same distribution as $\langle X_s\rangle_{s\ge0}$. So if $E = \bigcap_{n\in\mathbb{N}}\bigcup_{m\ge n} E_m$, $\mu_W E > 0$. But E belongs to the tail σ-algebra $\bigcap_{t\ge0} \mathrm{T}_{[t,\infty[}$, so has measure either 0 or 1 (477Hd), and must be conegligible. Since every $\omega \in E$ is self-intersecting, we see that almost every Brownian path is self-intersecting.

(b)(i) Suppose that q, $q' \in \mathbb{Q}$ are such that $0 \le q < q'$. This time, set $F_\omega = \{\omega(t) : t \in [0,q]\}$. For almost every ω, F_ω has zero two-dimensional Hausdorff measure (477La), so has zero $(r-2)$-dimensional Hausdorff measure (because $r \ge 4$), and therefore has zero capacity (479Qa). Also

$$\mu_W\{\omega : \omega(q') \in F_\omega\} = (\mu_W \times \mu_W)\{(\omega,\omega') : \omega'(q'-q) \in F_\omega - \omega(q)\} = 0$$

because the distribution of $X_{q'-q}$ is absolutely continuous with respect to Lebesgue measure and $\mu F_\omega = 0$ for μ_W-almost every ω. But this means that

$$\begin{aligned}&\mu_W\{\omega : \text{there is a } t \ge q' \text{ such that } \omega(t) \in F_\omega\}\\ &\qquad= (\mu_W \times \mu_W)\{(\omega,\omega') : \text{there is a } t \ge 0 \text{ such that } \omega'(t) \in F_\omega - \omega(q')\}\\ &\qquad= \int_\Omega \tilde W_{F_\omega}(\omega(q'))\mu(d\omega) = 0,\end{aligned}$$

that is,

$$\{\omega : \text{there are } s \le q,\ t \ge q' \text{ such that } \omega(s) = \omega(t)\}$$

is negligible. As q and q' are arbitrary, almost every sample path is injective.

479S A famous classical problem concerned, in effect, the continuity of potential functions, in particular the continuity of functions of the form $\tilde W_K$. I think that even with the modern theory as sketched above, this is not quite trivial, so I spell out an example.

Example Suppose that $e \in \mathbb{R}^r$ is a unit vector. Then there is a sequence $\langle\delta_n\rangle_{n\in\mathbb{N}}$ of strictly positive real numbers such that the equilibrium potential $\tilde W_K$ is discontinuous at e whenever $K \subseteq B(\mathbf{0},1)$ is compact, $e \in \overline{\operatorname{int} K}$ and $\|x - te\| \le \delta_n$ whenever $n \in \mathbb{N}$, $t \in [1-2^{-n}, 1]$, $x \in K$ and $\|x\| = t$.

proof For $n \in \mathbb{N}$, let K_n be the line segment $\{te : 1-2^{-n} \le t \le 1-2^{-n-1}\}$. Then the one-dimensional Hausdorff measure of K_n is finite, so $\operatorname{cap} K_n = 0$ (479Qa). By 479E(c-ii), $\lim_{\delta\downarrow 0} \operatorname{cap}(K_n + B(\mathbf{0},\delta)) = 0$; let $\delta_n \in \left]0, 2^{-n-2}\right[$ be such that $\operatorname{cap}(K_n + B(\mathbf{0},\delta_n)) \le 2^{-3n-6}$. Setting $L_n = K_n + B(\mathbf{0},\delta_n)$, the distance from e to L_n is at least 2^{-n-2}. By 479Pb,

$$\operatorname{hp}(L_n - e) = \tilde{W}_{L_n}(e) \le 4^{n+2}\lambda_{L_n}(\mathbb{R}^r) = 4^{n+2}\operatorname{cap} L_n \le 2^{-n-2}.$$

Suppose that $K \subseteq B(\mathbf{0},1)$ is compact, $e \in \overline{\operatorname{int} K}$ and $\|x - te\| \le \delta_n$ whenever $n \in \mathbb{N}$, $t \in [1-2^{-n}, 1]$, $x \in K$ and $\|x\| = t$. Then $K \subseteq \bigcup_{n\in\mathbb{N}} L_n \cup \{e\}$. Using the full strength of 479Pb,

$$\tilde{W}_K(e) = \operatorname{hp}((K \setminus \{e\}) - e) \le \operatorname{hp}(\bigcup_{n\in\mathbb{N}} L_n - e) \le \sum_{n=0}^{\infty} \operatorname{hp}(L_n - e) \le \frac{1}{2}.$$

On the other hand, $\tilde{W}_K(x) = 1$ for every $x \in \operatorname{int} K$ (479D(b-iii)), so $\tilde{W}_K$ is not continuous at e.

***479T** This concludes the main argument of the section, which you may feel is quite enough. However, there is an important alternative method of calculating the capacity of a compact set, based on gradients of potential functions (479U), and a couple of further results are reasonably accessible (479V-479W) which reflect other concerns of this volume.

Lemma (a) If $g : \mathbb{R}^r \to \mathbb{R}$ is a smooth function with compact support,

$$\int_{\mathbb{R}^r} \frac{1}{\|x-y\|^{r-2}} \nabla^2 g\, d\mu = -r(r-2)\beta_r g(x)$$

for every $x \in \mathbb{R}^r$.

(b) Let g, $h : \mathbb{R}^r \to \mathbb{R}$ be smooth functions with compact support. Then

$$\int_{\mathbb{R}^r} h \times \nabla^2 g\, d\mu = \int_{\mathbb{R}^r} g \times \nabla^2 h = -\int_{\mathbb{R}^r} \operatorname{grad} h \,\boldsymbol{.}\, \operatorname{grad} g\, d\mu.$$

(c) Let ζ be a totally finite Radon measure on $\mathbb{R}^r$, and $W_\zeta : \mathbb{R}^r \to [0,\infty]$ the associated Newtonian potential. Then $\int_{\mathbb{R}^r} W_\zeta \times \nabla^2 g\, d\mu = -r(r-2)\beta_r \int_{\mathbb{R}^r} g\, d\zeta$ for every smooth function $g : \mathbb{R}^r \to \mathbb{R}$ with compact support.

(d) Let ζ be a totally finite Radon measure on $\mathbb{R}^r$ such that W_ζ is finite-valued everywhere and Lipschitz. Then $\int_{\mathbb{R}^r} \operatorname{grad} f \,\boldsymbol{.}\, \operatorname{grad} W_\zeta d\mu = r(r-2)\beta_r \int_{\mathbb{R}^r} f\, d\zeta$ for every Lipschitz function $f : \mathbb{R}^r \to \mathbb{R}$ with compact support.

(e) Let $K \subseteq \mathbb{R}^r$ be a compact set, and $\epsilon > 0$. Then there is a Radon measure ζ on $\mathbb{R}^r$, with support included in $K + B(\mathbf{0},\epsilon)$, such that W_ζ is a smooth function with compact support, $W_\zeta \ge \chi K$, $\zeta\mathbb{R}^r \le \operatorname{cap} K + \epsilon$ and

$$\int_{\mathbb{R}^r} \|\operatorname{grad} W_\zeta\|^2 d\mu = r(r-2)\beta_r \operatorname{energy}(\zeta) \le r(r-2)\beta_r \zeta\mathbb{R}^r.$$

proof (a)(i) Consider first the case $x = 0$. Setting $f(y) = \frac{1}{\|y\|^{r-2}}$ for $y \ne 0$, we have $\operatorname{grad} f(y) = -\frac{r-2}{\|y\|^r} y$ and $(\nabla^2 f)(y) = 0$ for $y \ne 0$ (478Fa); also f is locally integrable, by 478Ga. So $\int_{\mathbb{R}^r} f \times \nabla^2 g\, d\mu$ is well-defined.

Let $R > 0$ be such that g is zero outside $B(\mathbf{0},R)$, and set $M = \|\operatorname{grad} g\|_\infty$; take $\epsilon \in]0, R[$. Then

$$\int_{\mathbb{R}^r \setminus B(\mathbf{0},\epsilon)} f \times \nabla^2 g\, d\mu = \int_{B(\mathbf{0},R)\setminus B(\mathbf{0},\epsilon)} f \times \nabla^2 g - g \times \nabla^2 f\, d\mu$$

$$= \int_{B(\mathbf{0},R)\setminus B(\mathbf{0},\epsilon)} \operatorname{div}(f \times \operatorname{grad} g - g \times \operatorname{grad} f) d\mu$$

(use 474Bb)

$$= \int_{\partial B(\mathbf{0},R)} \Bigl(\frac{1}{\|y\|^{r-2}} \operatorname{grad} g(y) + \frac{(r-2)g(y)}{\|y\|^r} y\Bigr) \boldsymbol{.} \frac{y}{\|y\|} \nu(dy)$$

$$- \int_{\partial B(\mathbf{0},\epsilon)} \Bigl(\frac{1}{\|y\|^{r-2}} \operatorname{grad} g(y) + \frac{(r-2)g(y)}{\|y\|^r} y\Bigr) \boldsymbol{.} \frac{y}{\|y\|} \nu(dy)$$

(475Nc)

$$= -\int_{\partial B(\mathbf{0},\epsilon)} (\frac{1}{\|y\|^{r-1}} y \,\boldsymbol{.}\, \operatorname{grad} g(y) + \frac{(r-2)g(y)}{\|y\|^{r-1}}) \nu(dy)$$

$$= -\frac{1}{\epsilon^{r-1}} \int_{\partial B(\mathbf{0},\epsilon)} (y \,\boldsymbol{.}\, \operatorname{grad} g(y) + (r-2)g(y)) \nu(dy).$$

Now we have

$$|\int_{\partial B(\mathbf{0},\epsilon)} y \,\boldsymbol{.}\, \operatorname{grad} g(y) \nu(dy)| \le \epsilon M \nu(\partial B(\mathbf{0},\epsilon)) \le r\beta_r \epsilon^r M,$$

so

$$|\int_{\mathbb{R}^r \setminus B(\mathbf{0},\epsilon)} f \times \nabla^2 g \, d\mu + r(r-2)\beta_r g(0)|$$

$$\le r\beta_r \epsilon M + \frac{1}{\epsilon^{r-1}} |r\beta_r \epsilon^{r-1}(r-2)g(0) - \int_{\partial B(\mathbf{0},\epsilon)} (r-2)g(y)\nu(dy)|$$

$$\le r\beta_r \epsilon M + \frac{r-2}{\epsilon^{r-1}} \int_{\partial B(\mathbf{0},\epsilon)} |g(0) - g(y)| \nu(dy)$$

$$\le r\beta_r \epsilon M + r(r-2)\beta_r \sup_{y \in \partial B(\mathbf{0},\epsilon)} |g(0) - g(y)| \to 0$$

as $\epsilon \downarrow 0$; that is,

$$\int_{\mathbb{R}^r} f \times \nabla^2 g \, d\mu = -r(r-2)\beta_r g(0).$$

(ii) For the general case, apply (i) to the function $y \mapsto g(x+y)$.

(b) Take $R > 0$ so large that both g and h are zero outside $B(\mathbf{0}, R)$, and $M \ge \max(\|\nabla^2 g\|_\infty, \|\nabla^2 h\|_\infty)$.

(i) We have

$$\int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \frac{1}{\|x-y\|^{r-2}} |(\nabla^2 g)(x)(\nabla^2 h)(y)| \mu(dx)\mu(dy)$$

$$\le M^2 \int_{B(\mathbf{0},R)} \int_{B(\mathbf{0},R)} \frac{1}{\|x-y\|^{r-2}} \mu(dx)\mu(dy) \le M^2 \int_{B(\mathbf{0},R)} \frac{1}{2} r\beta_r R^2 \mu(dy)$$

(478Gc)

$$< \infty.$$

So

$$-r(r-2)\beta_r \int_{\mathbb{R}^r} h \times \nabla^2 g \, d\mu = \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \frac{1}{\|x-y\|^{r-2}} (\nabla^2 h)(x)(\nabla^2 g)(y) \mu(dx)\mu(dy)$$

(by (a))

$$= \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \frac{1}{\|x-y\|^{r-2}} (\nabla^2 g)(y)(\nabla^2 h)(x) \mu(dy)\mu(dx)$$

$$= -r(r-2)\beta_r \int_{\mathbb{R}^r} g \times \nabla^2 h \, d\mu.$$

Thus $\int_{\mathbb{R}^r} g \times \nabla^2 h \, d\mu = \int_{\mathbb{R}^r} h \times \nabla^2 g \, d\mu$.

(ii) By 473Bd, $\operatorname{grad}(g \times h) = g \times \operatorname{grad} h + h \times \operatorname{grad} g$, so 474Bb tells us that $\nabla^2(g \times h) = 2 \operatorname{grad} g \,\boldsymbol{.}\, \operatorname{grad} h + g \times \nabla^2 h + h \times \nabla^2 g$, and

$$\int_{\mathbb{R}^r} \nabla^2(g \times h) d\mu = \int_{B(\mathbf{0},R)} \nabla^2(g \times h) d\mu$$

$$= \int_{\partial B(\mathbf{0},R)} \operatorname{grad}(g \times h) \,\boldsymbol{.}\, \frac{x}{\|x\|} \nu(dx) = 0,$$

so

$$\int_{\mathbb{R}^r}\operatorname{grad} g \mathbin{.} \operatorname{grad} h\,d\mu = -\frac{1}{2}\int_{\mathbb{R}^r} g\times\nabla^2 h + h\times\nabla^2 g\,d\mu = -\int_{\mathbb{R}^r} g\times\nabla^2 h\,d\mu.$$

(c) If $g(x)=0$ for $\|x\|\ge R$ and $|(\nabla^2 g)(x)|\le M$ for every x, then

$$\int_{\mathbb{R}^r}\frac{1}{\|x-y\|^{r-2}}|(\nabla^2 g)(x)|\mu(dx)\le M\int_{B(\mathbf{0},R)}\frac{1}{\|x-y\|^{r-2}}\mu(dx)\le\frac{1}{2}Mr\beta_r R^2$$

for every y (478Gc again). We can therefore apply (a) and integrate with respect to ζ to see that

$$\begin{aligned}-r(r-2)\beta_r\int_{\mathbb{R}^r} g\,d\zeta &= \int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\frac{1}{\|x-y\|^{r-2}}(\nabla^2 g)(x)\mu(dx)\zeta(dy)\\ &= \int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\frac{1}{\|x-y\|^{r-2}}(\nabla^2 g)(x)\zeta(dy)\mu(dx)\\ &= \int_{\mathbb{R}^r} W_\zeta(x)(\nabla^2 g)(x)\mu(dx),\end{aligned}$$

as required.

(d) Let $\langle\tilde h_n\rangle_{n\in\mathbb{N}}$ be the smoothing sequence of 473E.

(i) Suppose to begin with that f is smooth. For $n\in\mathbb{N}$ set $g_n=\tilde h_n * W_\zeta$. As W_ζ is continuous, $\lim_{n\to\infty} g_n = W_\zeta$ (473Ec); as $\|W_\zeta\|_\infty\le 1$, $\|g_n\|_\infty\le 1$ for every n (473Da). Because f has compact support,

$$\lim_{n\to\infty}\int_{\mathbb{R}^r} g_n\times\nabla^2 f\,d\mu = \int_{\mathbb{R}^r} W_\zeta\times\nabla^2 f\,d\mu$$

by the dominated convergence theorem. Next, $\operatorname{grad} g_n = \tilde h_n * \operatorname{grad} W_\zeta$ for each n (473Dd). As $\operatorname{grad} W_\zeta$ is essentially bounded (473Cc), all its coordinates are locally integrable, so $\operatorname{grad} W_\zeta =_{\text{a.e.}} \lim_{n\to\infty}\operatorname{grad} g_n$ (473Ee). We therefore have

$$\begin{aligned}\int_{\mathbb{R}^r}\operatorname{grad} f\mathbin{.}\operatorname{grad} W_\zeta d\mu &= \lim_{n\to\infty}\int_{\mathbb{R}^r}\operatorname{grad} f\mathbin{.}\operatorname{grad} g_n d\mu\\ &= -\lim_{n\to\infty}\int_{\mathbb{R}^r} g_n\times\nabla^2 f\,d\mu\end{aligned}$$

((b) above)

$$= -\int_{\mathbb{R}^r} W_\zeta\times\nabla^2 f\,d\mu = r(r-2)\beta_r\int_{\mathbb{R}^r} f\,d\zeta$$

by (c).

(ii) For the general case, smooth on the other side, setting $f_n=\tilde h_n * f$ for every n. This time, $f_n\to f$ uniformly (473Ed), so $\int_{\mathbb{R}^r} f d\zeta = \lim_{n\to\infty}\int_{\mathbb{R}^r} f_n d\zeta$. On the other hand, if f is M-Lipschitz, $\operatorname{grad} f_n = \tilde h_n * \operatorname{grad} f$ converges μ-a.e. to $\operatorname{grad} f$, and $\|\operatorname{grad} f_n\|_\infty$ is at most M for every n; also there is a bounded set outside which all the f_n and $\operatorname{grad} f_n$ are zero, and $\|\operatorname{grad} W_\zeta\|$ is bounded. So

$$\int_{\mathbb{R}^r}\operatorname{grad} f\mathbin{.}\operatorname{grad} W_\zeta d\mu = \lim_{n\to\infty}\int_{\mathbb{R}^r}\operatorname{grad} f_n\mathbin{.}\operatorname{grad} W_\zeta d\mu.$$

Applying (i) to each f_n and taking the limit, we get the equality we seek.

(e)(i) There is a compact set $L\subseteq K+B(\mathbf{0},\frac{\epsilon}{2})$ such that $K\subseteq\operatorname{int} L$ and $\operatorname{cap} L\le\operatorname{cap} K+\epsilon$ (479Ed). Let $n\in\mathbb{N}$ be such that $\frac{1}{n+1}\le\frac{\epsilon}{2}$ and $K+B(\mathbf{0},\frac{1}{n+1})\subseteq L$. Set $h=\lambda_L * \tilde h_n$, where $\tilde h_n$ is the function of 473E, as before; let $\zeta = h\mu$ be the corresponding indefinite-integral measure over μ. Because $\tilde h_n$ is zero outside $B(\mathbf{0},\frac{1}{n+1})$ and the support of λ_L is included in L, the support of ζ is included in $L+B(\mathbf{0},\frac{1}{n+1})\subseteq K+B(\mathbf{0},\epsilon)$.

(ii) By 444Pa, we have

$$W_\zeta = \zeta * k_{r-2} = (h\mu) * k_{r-2} = h * k_{r-2}$$

where k_{r-2} is the Riesz kernel (479G). Now $W_\zeta = \tilde{W}_L * \tilde{h}_n$. **P** For $m \in \mathbb{N}$, set $f_m = k_{r-2} \times \chi B(\mathbf{0}, m)$, so that f_m is μ-integrable. Observe that

$$\tilde{W}_L(x) = \int_{\mathbb{R}^r} k_{r-2}(x-y)\lambda_L(dy) = \int_{\mathbb{R}^r} \lim_{m\to\infty} f_m(x-y)\lambda_L(dy)$$
$$= \lim_{m\to\infty} \int_{\mathbb{R}^r} f_m(x-y)\lambda_L(dy) = \lim_{m\to\infty} (\lambda_L * f_m)(x)$$

for each x; moreover, because $\langle f_m \rangle_{m\in\mathbb{N}}$ is non-decreasing, so is $\langle \lambda_L * f_m \rangle_{m\in\mathbb{N}}$. For each m,

$$h * f_m = (h\mu) * f_m = (\lambda_L * \tilde{h}_n)\mu * f_m = (\lambda_L * \tilde{h}_n\mu) * f_m$$

(444K)

$$= \lambda_L * (\tilde{h}_n\mu * f_m)$$

(444Ic)

$$= \lambda_L * (\tilde{h}_n * f_m) = \lambda_L * (f_m * \tilde{h}_n) = \lambda_L * (f_m\mu * \tilde{h}_n)$$
$$= (\lambda_L * f_m\mu) * \tilde{h}_n = (\lambda_L * f_m)\mu * \tilde{h}_n = (\lambda_L * f_m) * \tilde{h}_n.$$

Now, for each x,

$$W_\zeta(x) = \int_{\mathbb{R}^r} h(x-y)k_{r-2}(y)\mu(dy) = \lim_{m\to\infty} \int_{\mathbb{R}^r} h(x-y)f_m(y)\mu(dy)$$
$$= \lim_{m\to\infty} (h * f_m)(x) = \lim_{m\to\infty} ((\lambda_L * f_m) * \tilde{h}_n)(x)$$
$$= \lim_{m\to\infty} \int_{\mathbb{R}^r} (\lambda_L * f_m)(y)\tilde{h}_n(x-y)\mu(dy)$$
$$= \int_{\mathbb{R}^r} \lim_{m\to\infty} (\lambda_L * f_m)(y)\tilde{h}_n(x-y)\mu(dy)$$
$$= \int_{\mathbb{R}^r} \tilde{W}_L(y)\tilde{h}_n(x-y)\mu(dy) = (\tilde{W}_L * \tilde{h}_n)(x). \textbf{ Q}$$

(iii) Since $\tilde{W}_L(x) = 1$ whenever $x \in \operatorname{int} L$ (479D(b-iii)), and $x + y \in \operatorname{int} L$ whenever $x \in K$ and $\tilde{h}_n(y) \ne 0$, $W_\zeta(x) = 1$ for every $x \in K$. Because both $\tilde{W}_L$ and $\tilde{h}_n$ have compact support, so does W_ζ; because $\tilde{h}_n$ is smooth, so is W_ζ (473De).

(iv) Now

$$\int_{\mathbb{R}^r} \|\operatorname{grad} W_\zeta\|^2 d\mu = -\int_{\mathbb{R}^r} W_\zeta \times \nabla^2 W_\zeta \, d\mu$$

((b) above)

$$= r(r-2)\beta_r \int_{\mathbb{R}^r} W_\zeta \, d\zeta$$

((c) above)

$$= r(r-2)\beta_r \operatorname{energy}(\zeta) \le r(r-2)\beta_r \zeta\mathbb{R}^r$$

because $\|W_\zeta\|_\infty \le \|\tilde{W}_L\|_\infty \|\tilde{h}_n\|_1 \le 1$.

(v) Finally,

$$\zeta\mathbb{R}^r = (h\mu)\mathbb{R}^r = (\lambda_L * \tilde{h}_n\mu)\mathbb{R}^r = \lambda_L\mathbb{R}^r \cdot (\tilde{h}_n\mu)\mathbb{R}^r$$
$$= \lambda_L\mathbb{R}^r = \operatorname{cap} L \le \epsilon + \operatorname{cap} K.$$

***479U Theorem** Let $K \subseteq \mathbb{R}^r$ be compact, and let Φ be the set of Lipschitz functions $g : \mathbb{R}^r \to \mathbb{R}$ such that $g(x) \ge 1$ for every $x \in K$ and $\lim_{\|x\|\to\infty} g(x) = 0$. Then

$$\begin{aligned} r(r-2)\beta_r \operatorname{cap} K &= \inf\{\int_{\mathbb{R}^r} \|\operatorname{grad} g\|^2 d\mu : g \in \Phi \text{ is smooth and has compact support}\} \\ &= \inf\{\int_{\mathbb{R}^r} \|\operatorname{grad} g\|^2 d\mu : g \in \Phi\}. \end{aligned}$$

proof (a) By 479Te,

$$\inf\{\textstyle\int_{\mathbb{R}^r} \|\operatorname{grad} g\|^2 d\mu : g \in \Phi \text{ is smooth and has compact support}\} \le r(r-2)\beta_r \operatorname{cap} K.$$

(b) Now suppose that $g \in \Phi$ is a smooth function with compact support. Then $r(r-2)\beta_r \operatorname{cap} K \le \int_{\mathbb{R}^r} \|\operatorname{grad} g\|^2 d\mu$. **P** Take any $\epsilon \in \,]0,1[$. Then there is a $\delta > 0$ such that $g(x) \ge 1-\epsilon$ for every $x \in K + B(\mathbf{0}, \delta)$. By 479Te, there is a Radon measure ζ on $\mathbb{R}^r$, with support included in $K + B(\mathbf{0},\delta)$, such that W_ζ is smooth and has compact support, $W_\zeta \ge \chi K$, $\zeta\mathbb{R}^r \le \operatorname{cap} K + \epsilon$ and

$$\textstyle\int_{\mathbb{R}^r} \|\operatorname{grad} W_\zeta\|^2 d\mu = r(r-2)\beta_r \operatorname{energy}(\zeta) \le r(r-2)\beta_r\zeta\mathbb{R}^r.$$

In this case,

$$\int_{\mathbb{R}^r} \operatorname{grad} g \,.\, \operatorname{grad} W_\zeta d\mu = r(r-2)\beta_r \int_{\mathbb{R}^r} g\, d\zeta$$

(479Td)

$$\ge (1-\epsilon) r(r-2)\beta_r \zeta\mathbb{R}^r \ge (1-\epsilon)\int_{\mathbb{R}^r} \|\operatorname{grad} W_\zeta\|^2.$$

Setting $v = (1-\epsilon)\operatorname{grad} W_\zeta$, we have

$$\textstyle\int_{\mathbb{R}^r} v \,.\, \operatorname{grad} g d\mu \ge \int_{\mathbb{R}^r} \|v\|^2 d\mu.$$

But this means that

$$\begin{aligned} \int_{\mathbb{R}^r} \|\operatorname{grad} g\|^2 d\mu &= 2\int_{\mathbb{R}^r} v \,.\, \operatorname{grad} g d\mu - \int_{\mathbb{R}^r} \|v\|^2 d\mu + \int_{\mathbb{R}^r} \|v - \operatorname{grad} g\|^2 d\mu \\ &\ge \int_{\mathbb{R}^r} \|v\|^2 d\mu \ge (1-\epsilon)^2 \int_{\mathbb{R}^r} \|\operatorname{grad} W_\zeta\|^2 d\mu \\ &= (1-\epsilon)^2 r(r-2)\beta_r \int_{\mathbb{R}^r} W_\zeta d\zeta \ge (1-\epsilon)^2 r(r-2)\beta_r \int_{\mathbb{R}^r} \tilde W_K d\zeta \end{aligned}$$

(because $W_\zeta \ge \tilde W_K$ on K, so $W_\zeta \ge \tilde W_K$ everywhere, by 479Fg)

$$= (1-\epsilon)^2 r(r-2)\beta_r \int_{\mathbb{R}^r} W_\zeta d\lambda_K$$

(479J(b-i))

$$\ge (1-\epsilon)^2 r(r-2)\beta_r \int_{\mathbb{R}^r} \tilde W_K d\lambda_K = (1-\epsilon)^2 r(r-2)\beta_r \operatorname{cap} K.$$

As ϵ is arbitrary, $r(r-2)\beta_r \operatorname{cap} K \le \int_{\mathbb{R}^r} \|\operatorname{grad} g\|^2 d\mu$. **Q**

(c) If $g \in \Phi$ has compact support, then $\int_{\mathbb{R}^r} \|\operatorname{grad} g\|^2 d\mu \ge r(r-2)\beta_r \operatorname{cap} K$. **P** Let $R > 0$ be such that g is zero outside $B(\mathbf{0}, R)$. Let $M \ge 0$ be such that g is M-Lipschitz; then $\|\operatorname{grad} g(x)\| \le M$ for every $x \in \operatorname{dom} \operatorname{grad} g$ (473Cc). Take any $\epsilon > 0$. As in 479T, let $\langle \tilde h_n\rangle_{n\in\mathbb{N}}$ be the smoothing sequence of 473E. For $n \in \mathbb{N}$, set $g_n = (1+\epsilon)\tilde h_n * g$. Then $\operatorname{grad} g_n = (1+\epsilon)\tilde h_n * \operatorname{grad} g$ (473Dd) and $\|\operatorname{grad} g_n\|_\infty \le M(1+\epsilon)$ (473Da). In the limit, $(1+\epsilon)\operatorname{grad} g =_{\text{a.e.}} \lim_{n\to\infty} \operatorname{grad} g_n$ (473Ee).

There is an $m \in \mathbb{N}$ such that $(1+\epsilon)g(x) \ge 1$ for every $x \in K + B(\mathbf{0}, \frac{1}{m+1})$; now if $n \ge m$,

$$g_n(x) \ge (1+\epsilon)\inf_{\|y\|\le 1/(n+1)} g(x-y) \ge 1$$

for every $x \in K$. So

$$(1+\epsilon)^2 \int_{\mathbb{R}^r} \|\operatorname{grad} g\|^2 d\mu = (1+\epsilon)^2 \int_{B(\mathbf{0},R+1)} \|\operatorname{grad} g\|^2 d\mu$$

$$= \lim_{n\to\infty} \int_{B(\mathbf{0},R+1)} \|\operatorname{grad} g_n\|^2 d\mu$$

(by the dominated convergence theorem)

$$= \lim_{n\to\infty} \int_{\mathbb{R}^r} \|\operatorname{grad} g_n\|^2 d\mu$$

(because every g_n is zero outside $B(\mathbf{0}, R+1)$)

$$\ge r(r-2)\beta_r \operatorname{cap} K$$

(applying (b) to g_n for $n \ge m$). As ϵ is arbitrary, $r(r-2)\beta_r \operatorname{cap} K \le \int_{\mathbb{R}^r} \|\operatorname{grad} g\|^2 d\mu$. **Q**

(d) If $g \in \Phi$, then $\int_{\mathbb{R}^r} \|\operatorname{grad} g\|^2 d\mu \ge r(r-2)\beta_r \operatorname{cap} K$. **P** Let $\epsilon > 0$. Set $g_1(x) = \max(0, (1+\epsilon)g(x) - \epsilon)$ for $x \in \mathbb{R}^r$. Then $g_1 \in \Phi$ has compact support, and $\|g_1(x) - g_1(y)\| \le (1+\epsilon)\|g(x) - g(y)\|$ for all x, $y \in \mathbb{R}^r$, so $\|\operatorname{grad} g_1(x)\| \le (1+\epsilon)\|\operatorname{grad} g(x)\|$ whenever both gradients are defined. Accordingly

$$(1+\epsilon)^2 \int_{\mathbb{R}^r} \|\operatorname{grad} g\|^2 d\mu \ge \int_{\mathbb{R}^r} \|\operatorname{grad} g_1\|^2 d\mu \ge r(r-2)\beta_r \operatorname{cap} K$$

by (c). As ϵ is arbitrary, we have the result. **Q**

(e) Putting (a) and (d) together, the theorem is proved.

***479V** We are ready for another theorem along the lines of 476H, this time relating capacity and Lebesgue measure.

Theorem Let $D \subseteq \mathbb{R}^r$ be a set of finite outer Lebesgue measure, and B_D the closed ball with centre 0 and the same outer measure as D. Then the Choquet-Newton capacity $c(D)$ of D is at least $\operatorname{cap} B_D$.

proof (a) We need an elementary fact about gradients. Suppose that f, $g : \mathbb{R}^r \to \mathbb{R}$ and $x \in \mathbb{R}^r$ are such that $\operatorname{grad} f$, $\operatorname{grad} g$, $\operatorname{grad}(f \vee g)$ and $\operatorname{grad}(f \wedge g)$ are all defined at x. Then $\{\operatorname{grad}(f\vee g)(x), \operatorname{grad}(f\wedge g)(x)\} = \{\operatorname{grad} f(x), \operatorname{grad} g(x)\}$. **P** (i) If $f(x) > g(x)$ then (because f and g are both continuous at x) we have $\operatorname{grad}(f\vee g)(x) = \operatorname{grad} f(x)$, $\operatorname{grad}(f\wedge g)(x) = \operatorname{grad} g(x)$ and the result is immediate. (ii) The same argument applies if $f(x) < g(x)$. (iii) If $f(x) = g(x)$, consider $h = |f-g| = (f\vee g) - (f\wedge g)$. Then $\operatorname{grad} h(x)$ is defined, and $h(x) = 0 \le h(y)$ for every y. So all the partial derivatives of h have to be zero at x, and $\operatorname{grad} h(x) = 0$, that is, $\lim_{y\to x} \frac{1}{\|y-x\|} h(y) = 0$. It follows at once that $\operatorname{grad} f(x) = \operatorname{grad} g(x)$, and therefore both are equal to $\operatorname{grad}(f\vee g)(x)$ and $\operatorname{grad}(f\wedge g)(x)$. So again we have the result. **Q**

(b) Now for a further clause to add to Lemma 476E. Suppose that $e \in S_{r-1} = \partial B(\mathbf{0}, 1)$ and $\alpha \in \mathbb{R}$; let $R = R_{e\alpha}$ be the reflection in the plane $\{x : x \centerdot e = \alpha\}$, and $\psi = \psi_{e\alpha} : \mathcal{P}\mathbb{R}^r \to \mathcal{P}\mathbb{R}^r$ the partial-reflection operator of 476D-476E, that is,

$$\psi(D) = (W \cap (D \cup R[D])) \cup (W' \cap D \cap R[D])$$

for $D \subseteq \mathbb{R}^r$, where $W = \{x : x\centerdot e \ge \alpha\}$ and $W' = \{x : x\centerdot e \le \alpha\}$. Then $c(\psi(D)) \le c(D)$ for every $D \subseteq \mathbb{R}^r$.

P(i) Suppose first that $D = K$ is compact. Take any $\gamma > r(r-2)\beta_r \operatorname{cap} K$. By 479U, there is a Lipschitz function $f : \mathbb{R}^r \to \mathbb{R}$ such that $f(x) \ge 1$ for every $x \in K$, $\lim_{\|x\|\to\infty} f(x) = 0$ and $\int_{\mathbb{R}^r} \|\operatorname{grad} f\|^2 d\mu \le \gamma$. Set $g = fR$. Of course g is Lipschitz and $\int_{\mathbb{R}^r} \|\operatorname{grad} g\|^2 d\mu = \int_{\mathbb{R}^r} \|\operatorname{grad} f\|^2 d\mu$. Now $f \vee g$ and $f \wedge g$ are also Lipschitz, so for almost every $x \in \mathbb{R}^r$ all the gradients $\operatorname{grad} f(x)$, $\operatorname{grad} g(x)$, $\operatorname{grad}(f\vee g)(x)$ and $\operatorname{grad}(f\wedge g)(x)$ are defined; by (a), $\|\operatorname{grad} f\|^2 + \|\operatorname{grad} g\|^2 =_{\text{a.e.}} \|\operatorname{grad}(f\vee g)\|^2 + \|\operatorname{grad}(f\wedge g)\|^2$.

Now consider the function h defined by saying that

$$h(x) = (f\vee g)(x) \text{ if } x \in W,$$
$$= (f\wedge g)(x) \text{ if } x \in W'.$$

h is Lipschitz and $\lim_{\|x\|\to\infty} h(x) = 0$; also $h(x) \ge 1$ for every $x \in \psi(K)$. For $x \in W \setminus W'$, $h(x) = (f \vee g)(x)$ and $h(Rx) = (f \wedge g)(x)$, so $\{\operatorname{grad} h(x), \operatorname{grad}(hR)(x)\} = \{\operatorname{grad}(f \vee g)(x), \operatorname{grad}(f \wedge g)(x)\}$ if the gradients are defined; for $x \in W' \setminus W$, $h(x) = (f \wedge g)(x)$ and $h(Rx) = (f \vee g)(x)$, so $\{\operatorname{grad} h(x), \operatorname{grad}(hR)(x)\} = \{\operatorname{grad}(f \vee g)(x), \operatorname{grad}(f \wedge g)(x)\}$ if the gradients are defined. Accordingly

$$\begin{aligned}\|\operatorname{grad} h\|^2 + \|\operatorname{grad}(hR)\|^2 &=_{\text{a.e.}} \|\operatorname{grad}(f \vee g)\|^2 + \|\operatorname{grad}(f \wedge g)\|^2 \\ &=_{\text{a.e.}} \|\operatorname{grad} f\|^2 + \|\operatorname{grad} g\|^2.\end{aligned}$$

By 479U again,

$$\begin{aligned}r(r-2)\beta_r \operatorname{cap}(\psi(K)) &\le \int_{\mathbb{R}^r} \|\operatorname{grad} h\|^2 d\mu = \frac{1}{2}\int_{\mathbb{R}^r} (\|\operatorname{grad} h\|^2 + \|\operatorname{grad}(hR)\|^2) d\mu \\ &= \frac{1}{2}\int_{\mathbb{R}^r} (\|\operatorname{grad} f\|^2 + \|\operatorname{grad} g\|^2) d\mu \le \gamma.\end{aligned}$$

As γ is arbitrary, $\operatorname{cap}(\psi(K)) \le \operatorname{cap} K$.

(ii) Now suppose that $D = G$ is open. Then there is a non-decreasing sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ of compact sets with union G, and $\langle \psi(K_n)\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with union $\psi(G)$. So

$$c(\psi(G)) = \sup_{n\in\mathbb{N}} \operatorname{cap}(\psi(K_n)) \le \sup_{n\in\mathbb{N}} \operatorname{cap} K_n = c(G).$$

(iii) Finally, for arbitrary $D \subseteq \mathbb{R}^r$, take any $\gamma > c(D)$. Then there is an open set G such that $D \subseteq G$ and $c(G) \le \gamma$ (because c is outer regular, see 479E(d-i)). In this case, $\psi(D) \subseteq \psi(G)$, so

$$c(\psi(D)) \le c(\psi(G)) \le c(G) \le \gamma.$$

As γ is arbitrary, $c(\psi(D)) \le c(D)$ and we are done. **Q**

(c) Now suppose that E is a bounded Lebesgue measurable subset of $\mathbb{R}^r$ with finite perimeter.

(i) Let $M \ge 0$ be such that $E \subseteq B(\mathbf{0}, M)$. Consider

$$\begin{aligned}\mathcal{E} = \{F : F \subseteq B(\mathbf{0}, M) &\text{ is Lebesgue measurable,} \\ &\mu F = \mu E,\ \operatorname{per} F \le \operatorname{per} E,\ c(F) \le c(E)\}.\end{aligned}$$

Then $\mathcal{E}$ is compact for the topology $\mathfrak{T}_m$ of convergence in measure as described in 474T. **P** By 474T,

$$\mathcal{E}_1 = \{F : F \text{ is Lebesgue measurable, } \operatorname{per} F \le \operatorname{per} E\}$$

is compact. So if $\langle F_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\mathcal{E}$, it has a subsequence $\langle F'_n\rangle_{n\in\mathbb{N}}$ which is $\mathfrak{T}_m$-convergent to $F \in \mathcal{E}_1$ say (4A2Le; recall that, as noted in the proof of 474T, $\mathfrak{T}_m$ is pseudometrizable). Taking a further subsequence if necessary, we can suppose that $\mu((F\triangle F'_n) \cap B(\mathbf{0}, M)) \le 2^{-n}$ for every $n \in \mathbb{N}$. Set $F' = \bigcup_{m\in\mathbb{N}} \bigcap_{n\ge m} F'_n$. Because every F'_n is included in $B(\mathbf{0}, M)$, F' is a $\mathfrak{T}_m$-limit of $\langle F'_n\rangle_{n\in\mathbb{N}}$. So $\mu F' = \lim_{n\to\infty} \mu F'_n = \mu E$, and

$$c(F') = \lim_{m\to\infty} c(\bigcap_{n\ge m} F'_n) \le c(E).$$

Finally, $F'\triangle F$ is negligible, so $\partial^* F' = \partial^* F$ and $\operatorname{per}(F') = \operatorname{per} F \le \operatorname{per} E$. Thus $F' \in \mathcal{E}$. As $\langle F_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathcal{E}$ is relatively compact, by 4A2Le in the opposite direction. **Q**

(ii) Because $B(\mathbf{0}, M)$ is bounded, the function $F \mapsto \int_F \|x\|\mu(dx) : \mathcal{E} \to [0, \infty[$ is continuous, and must attain its infimum at H say. Let B_E be the ball with centre 0 and the same measure as E. Then $B_E \subseteq \operatorname{cl}^* H$. **P** (Compare part (b) of the proof of 476H.) **?** Otherwise, take $z \in B_E \setminus \operatorname{cl}^* H$. Then

$$\lim_{\delta>0} \frac{\mu(B(z,\delta)\setminus H)}{\mu B(z,\delta)} = 1, \quad \lim_{\delta>0} \frac{\mu(B(z,\delta)\setminus B_E)}{\mu B(z,\delta)} \le \frac{1}{2},$$

so there is a $\delta > 0$ such that $\mu(B(z,\delta) \setminus B_E) < \mu(B(z,\delta) \setminus H)$ and $\mu(B_E \setminus H) > 0$. Because

$$\mu(\operatorname{cl}^* H) = \mu H = \mu E = \mu B_E,$$

$\operatorname{cl}^* H \setminus B_E$ is also non-negligible. Take $x_1 \in \operatorname{cl}^* H \setminus B_E$ and $x_0 \in B_E \setminus \operatorname{cl}^* H$. Then $\delta_0 = \|x_1\| - \|x_0\|$ is greater than 0. Since

$$\limsup_{\delta\downarrow 0}\frac{\mu(H\cap B(x_1,\delta))}{\mu B(x_1,\delta)}>0=\lim_{\delta\downarrow 0}\frac{\mu(H\cap B(x_0,\delta))}{\mu B(x_0,\delta)},$$

there is a $\delta\in\left]0,\frac{1}{2}\delta_0\right[$ such that $\mu(H\cap B(x_1,\delta))>\mu(H\cap B(x_0,\delta))$. Now let e be the unit vector $\frac{1}{\|x_0-x_1\|}(x_0-x_1)$, and set $\alpha=e\centerdot\frac{1}{2}(x_0+x_1)$. Consider the reflection $R=R_{e\alpha}$ and the operator $\psi=\psi_{e\alpha}$; set $H_1=\psi(H)$ and let $\phi=\phi_{H_1}:H\to H_1$ be the function of 476E. As $\alpha<0$, $\|\phi(x)\|\le\|x\|$ for every $x\in H$; moreover, $R[B(x_1,\delta)]=B(x_0,\delta)$, so

$$\{x:\|\phi(x)\|<\|x\|\}\supseteq\{x:x\in B(x_1,\delta)\cap H,\ Rx\notin H\}$$

is not negligible. So $\int_{H_1}\|x\|\mu(dx)<\int_H\|x\|\mu(dx)$. On the other hand, we surely have $H_1\subseteq B(\mathbf{0},M)$, $\mu H_1=\mu H=\mu E$ and $\operatorname{per}H_1\le\operatorname{per}H\le\operatorname{per}E$ (476Ee); and, finally, $c(H_1)\le c(H)\le c(E)$, by (b) of this proof. Thus $H_1\in\mathcal{E}$ and the functional $F\mapsto\int_F\|x\|\mu(dx)$ is not minimized at H. **XQ**

(iii) Accordingly

$$\operatorname{cap}B_E\le c(\operatorname{cl}^*H)\le c(H)$$

(479P(c-vi))

$$\le c(E).$$

(d) Thus $\operatorname{cap}B_E\le c(E)$ whenever $E\subseteq\mathbb{R}^r$ is Lebesgue measurable, bounded and has finite perimeter. Consequently $\operatorname{cap}B_K\le\operatorname{cap}K$ for every compact set $K\subseteq\mathbb{R}^r$. **P** If $\epsilon>0$, there is an open set $G\supseteq K$ such that $c(G)\le\operatorname{cap}K+\epsilon$. Now there is a set E, a finite union of balls, such that $K\subseteq E\subseteq G$. In this case, E has finite perimeter and is bounded, while of course $B_E\supseteq B_K$. So

$$\operatorname{cap}B_K\le\operatorname{cap}B_E\le c(E)\le c(G)\le\operatorname{cap}K+\epsilon.$$

As ϵ is arbitrary, $\operatorname{cap}B_K\le\operatorname{cap}K$. **Q**

It follows that $\operatorname{cap}B_E\le c(E)$ for every measurable set $E\subseteq\mathbb{R}^r$ of finite measure. **P** If $K\subseteq E$ is compact, then $\operatorname{cap}B_K\le\operatorname{cap}K\le c(E)$. But as $\mu E=\sup\{\mu K:K\subseteq E$ is compact$\}$, $\operatorname{diam}B_E=\sup\{\operatorname{diam}B_K:K\subseteq E$ is compact$\}$; because capacity is a continuous function of radius (479Da),

$$\begin{aligned}\operatorname{cap}B_E)&=\sup\{\operatorname{cap}B_K:K\subseteq E\text{ is compact}\}\\&\le\sup\{\operatorname{cap}K:K\subseteq E\text{ is compact }\}\le c(E).\ \mathbf{Q}\end{aligned}$$

Finally, if D is any set of finite outer measure, there is a G_δ set $E\supseteq D$ such that $c(E)=c(D)$ and $\mu E=\mu^*D$, so that

$$\operatorname{cap}B_D=\operatorname{cap}B_E\le c(E)=c(D),$$

and we have the general result claimed.

***479W** I conclude with an alternative representation of Choquet-Newton capacity c in terms of a measure on the space of closed subsets of $\mathbb{R}^r$.

Theorem Let $\mathcal{C}^+$ be the family of non-empty closed subsets of $\mathbb{R}^r$, with its Fell topology (4A2T). Then there is a unique Radon measure θ on $\mathcal{C}^+$ such that $\theta^*\{C:C\in\mathcal{C}^+,\ D\cap C\ne\emptyset\}$ is the Choquet-Newton capacity $c(D)$ of D for every $D\subseteq\mathbb{R}^r$.

proof (a) Recall that the Fell topology on $\mathcal{C}=\mathcal{C}^+\cup\{\emptyset\}$ is compact (4A2T(b-iii)) and metrizable (4A2Tf), so $\mathcal{C}^+$ is locally compact and Polish. For $D\subseteq\mathbb{R}^r$, set $\Psi D=\{C:C\in\mathcal{C}^+,\ C\cap D\ne\emptyset\}$. Of course $\Psi(\bigcup\mathcal{A})=\bigcup_{D\in\mathcal{A}}\Psi D$ for every family $\mathcal{A}$ of subsets of $\mathbb{R}^r$.

(b) Let Ω' be the set of those $\omega\in\Omega$ such that $\lim_{t\to\infty}\|\omega(t)\|=\infty$; because $r\ge 3$, Ω' is conegligible in Ω (478Md). If $\omega\in\Omega'$, then $\omega[[0,\infty[]$ is closed. For $x\in\mathbb{R}^r$ and $\omega\in\Omega'$, set $h_x(\omega)=x+\omega[[0,\infty[]\in\mathcal{C}^+$. Then $h_x:\Omega'\to\mathcal{C}^+$ is Borel measurable. **P** (α) If $G\subseteq\mathbb{R}^r$ is open, then

$$\{\omega:\omega\in\Omega',\ h_x(\omega)\cap G\ne\emptyset\}=\bigcup_{t\ge 0}\{\omega:x+\omega(t)\in G\}$$

is relatively open in Ω'. (β) If $K \subseteq \mathbb{R}^r$ is compact,

$$\{(\omega, t) : x + \omega(t) \in K\}$$

is closed in $\Omega \times [0, \infty[$, so its projection $\{\omega : x + \omega[\,[0, \infty[\,] \cap K \neq \emptyset\}$ is F_σ, and $\{\omega : \omega \in \Omega',\, h_x(\omega) \cap K = \emptyset\}$ is a G_δ set in Ω'. (γ) Because $\mathcal{C}^+$ is hereditarily Lindelöf, this is enough to prove that h_x is Borel measurable (4A3Db). **Q**

(c) Let T be the ring of subsets of $\mathcal{C}^+$ generated by sets of the form ΨE where $E \subseteq \mathbb{R}^r$ is bounded and is either compact or open. Then we have an additive functional $\phi : \mathrm{T} \to [0, \infty[$ such that $\phi(\Psi K) = \operatorname{cap} K$ for every compact set $K \subseteq \mathbb{R}^r$. **P** For $x \in \mathbb{R}^r$ let $h_x : \Omega' \to \mathcal{C}^+$ be as in (b). Then we have a corresponding scaled Radon image measure $\phi_x = \|x\|^{r-2}(\mu_W)_{\Omega'} h_x^{-1}$ on $\mathcal{C}^+$ (418I), defined by setting $\phi_x H = \|x\|^{r-2}\mu_W\{\omega : x + \omega[\,[0, \infty[\,] \in H\}$ whenever this is defined. If $E \subseteq \mathbb{R}^r$ is either compact or open, then

$$\{\omega : x + \omega[\,[0, \infty[\,] \cap E \neq \emptyset\}$$

is F_σ or open, respectively, so $\phi_x(\Psi E)$ is defined; accordingly $\phi_x H$ is defined for every $H \in \mathrm{T}$. If $\gamma > 0$, $E \subseteq B(\mathbf{0}, \gamma)$ and $\|x\| > \gamma$, then

$$\phi_x^*(\Psi E) \le \phi_x(\Psi(B(\mathbf{0}, \gamma))) = \|x\|^{r-2} \operatorname{hp}(B(\mathbf{0}, \gamma) - x) = \gamma^{r-2}$$

(478Qc). So $\limsup_{\|x\| \to \infty} \phi_x H$ is finite for every $H \in \mathrm{T}$. Take an ultrafilter $\mathcal{F}$ on $\mathbb{R}^r$ containing $\mathbb{R}^r \setminus B(\mathbf{0}, \gamma)$ for every $\gamma > 0$; then $\phi H = \lim_{x \to \mathcal{F}} \phi_x H$ is defined in $[0, \infty[$ for every $H \in \mathrm{T}$, and ϕ is additive. If $E \subseteq \mathbb{R}^r$ is bounded and either compact or open, then

$$\phi(\Psi E) = \lim_{x \to \mathcal{F}} \|x\|^{r-2} \operatorname{hp}(E - x) = c(E)$$

by 479B(ii). **Q**

(d) ϕ is inner regular with respect to the compact sets, in the sense that $\phi H = \sup\{\phi L : L \in \mathrm{T}$ is compact, $L \subseteq H\}$ for every $H \in \mathrm{T}$. **P** Note first that the set

$$\mathcal{H} = \{H : H \in \mathrm{T},\, \phi H = \sup\{\phi L : L \in \mathrm{T} \text{ is compact, } L \subseteq H\}\}$$

is a sublattice of T. Suppose that E, $H \subseteq \mathbb{R}^r$ are bounded sets which are either compact or open, and $\epsilon > 0$. Then there are a compact $K \subseteq E$ and a bounded open $G \supseteq H$ such that $c(E) \le \epsilon + \operatorname{cap} K$ and $c(G) \le \epsilon + c(H)$ (479E). Now ΨK is a closed subset of $\mathcal{C}$ included in $\mathcal{C}^+$, so is a compact subset of $\mathcal{C}^+$, while ΨG is open; thus $L = \Psi K \setminus \Psi G$ is a compact subset of $\Psi E \setminus \Psi H$, and of course $L \in \mathrm{T}$. Now

$$\begin{aligned}\phi(\Psi E \setminus \Psi H) &\le \phi L + \phi(\Psi E \setminus \Psi K) + \phi(\Psi G \setminus \Psi H)\\ &= \phi L + \phi(\Psi E) - \phi(\Psi K) + \phi(\Psi G) - \phi(\Psi H)\\ &= \phi L + c(E) - \operatorname{cap} K + c(G) - c(H) \le \phi L + 2\epsilon.\end{aligned}$$

As ϵ is arbitrary, $\Psi E \setminus \Psi H$ belongs to $\mathcal{H}$.

Since any member of T is expressible as a finite union of finite intersections of sets of this kind, $\mathrm{T} \subseteq \mathcal{H}$, as required. **Q**

(e) Let $\mathcal{L}$ be the family of compact subsets of $\mathcal{C}^+$. If $L \in \mathcal{L}$, it is a closed subset of $\mathcal{C}$ not containing $\emptyset$, so there must be a compact set $K \subseteq \mathbb{R}^r$ such that $L \subseteq \Psi K$. Thus every member of $\mathcal{L}$ is covered by a member of T, and we have a functional $\phi_1 : \mathcal{L} \to [0, \infty[$ defined by setting $\phi_1 L = \inf\{\phi E : E \in \mathrm{T},\, L \subseteq E\}$ for $L \in \mathcal{L}$.

I seek to apply 413J. Of course $\emptyset \in \mathcal{L}$ and $\mathcal{L}$ is closed under finite disjoint unions and countable intersections; moreover, if $\langle L_n \rangle_{n \in \mathbb{N}}$ is a non-increasing sequence in $\mathcal{L}$ with empty intersection, one of the L_n must be empty, so $\inf_{n \in \mathbb{N}} \phi_1 L_n = 0$. Now turn to condition (α) of the theorem:

$$\phi_1 L_1 = \phi_1 L_0 + \sup\{\phi_1 L : L \in \mathcal{L},\, L \subseteq L_1 \setminus L_0\} \text{ whenever } L_0,\, L_1 \in \mathcal{L} \text{ and } L_0 \subseteq L_1.$$

(i) If L_0, $L \in \mathcal{L}$ are disjoint, then $\phi_1(L_0 \cup L) \ge \phi_1 L_0 + \phi_1 L$. **P** The topology $\mathfrak{S}$ of $\mathcal{C}^+$ is generated by sets of the form ΨG, where $G \subseteq \mathbb{R}^r$ is open, and by sets of the form $\mathcal{C}^+ \setminus \Psi K$, where $K \subseteq \mathbb{R}^r$ is compact. It is therefore generated by

$$\{\Psi G \setminus \Psi K : G \subseteq \mathbb{R}^r \text{ is bounded and open, } K \subseteq \mathbb{R}^r \text{ is compact}\} \subseteq \mathrm{T}.$$

So $\mathfrak{S} \cap \mathrm{T}$ is a base for $\mathfrak{S}$ and disjoint compact sets in $\mathcal{C}^+$ can be separated by members of T (4A2F(h-i)); let $E_0 \in \mathrm{T}$ be such that $L_0 \subseteq E_0 \subseteq \mathcal{C}^+ \setminus L$. Now if $E \in \mathrm{T}$ and $E \supseteq L_0 \cup L$,

$$\phi E = \phi(E \cap E_0) + \phi(E \setminus E_0) \ge \phi_1 L_0 + \phi_1 L;$$

as E is arbitrary, $\phi_1(L_0 \cup L) \ge \phi_1 L_0 + \phi_1 L$. **Q**

(ii) If L_0, $L_1 \in \mathcal{L}$, $L_0 \subseteq L_1$ and $\epsilon > 0$, there is an $L \in \mathcal{L}$ such that $L \subseteq L_1 \cup L_0$ and $\phi_1 L_1 \le \phi_1 L_0 + \phi_1 L + 3\epsilon$. **P** Let E_0, $E_1 \in \mathrm{T}$ be such that $L_0 \subseteq E_0$, $L_1 \subseteq E_1$ and $\phi E_0 \le \phi_1 L_0 + \epsilon$. By (d), there is an $L' \in \mathcal{L} \cap \mathrm{T}$ such that $L' \subseteq E_1 \setminus E_0$ and $\phi L' \ge \phi(E_1 \setminus E_0) - \epsilon$. Set $L = L' \cap L_1$. Then $L \in \mathcal{L}$ and $L \subseteq L_1 \setminus L_0$. Let $E \in \mathrm{T}$ be such that $L \subseteq E$ and $\phi E \le \phi_1 L + \epsilon$. Then $L_1 \subseteq E_0 \cup E \cup ((E_1 \setminus E_0) \setminus L')$, so

$$\phi_1 L_1 \le \phi E_0 + \phi E + \phi(E_1 \setminus E_0) - \phi L' \le \phi_1 L_1 + \phi L + 3\epsilon,$$

as required. **Q**

Putting this together with (i), the final condition of 413J is satisfied.

(f) We therefore have a complete locally determined measure θ on $\mathcal{C}^+$ extending ϕ_1 and inner regular with respect to $\mathcal{L}$. For $E \subseteq \mathcal{C}^+$, θ measures E iff θ measures $E \cap L$ for every $L \in \mathcal{L}$ (412Ja); so θ measures all closed subsets of $\mathcal{C}^+$, and is a topological measure. Of course θ is inner regular with respect to the compact sets. If $C \in \mathcal{C}^+$, there is a bounded open set $G \subseteq \mathbb{R}^r$ meeting C, and now ΨG is an open set containing C and included in the compact set $\Psi\overline{G}$; accordingly

$$\theta(\Psi G) \le \theta(\Psi\overline{G}) = \phi_1(\Psi\overline{G}) = \phi(\Psi\overline{G}) = \operatorname{cap}\overline{G}$$

is finite. Thus θ is locally finite and is a Radon measure.

(g) As in (f), we have

$$\theta(\Psi K) = \phi_1(\Psi K) = \phi(\Psi K) = \operatorname{cap} K$$

for every compact $K \subseteq \mathbb{R}^r$. Next, $\theta(\Psi G) = c(G)$ for every open $G \subseteq \mathbb{R}^r$. **P** If G is bounded,

$$\begin{aligned}
\theta(\Psi G) &= \sup\{\theta L : L \subseteq \Psi G \text{ is compact}\} \\
&= \sup\{\phi_1 L : L \subseteq \Psi G \text{ is compact}\} \le \phi(\Psi G) = c(G) \\
&= \sup\{\operatorname{cap} K : K \subseteq G \text{ is compact}\} \\
&= \sup\{\theta(\Psi K) : K \subseteq G \text{ is compact}\} \le \theta(\Psi G).
\end{aligned}$$

If G is unbounded, then there is a non-decreasing sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ of bounded open sets with union G, so

$$\theta(\Psi G) = \theta(\bigcup_{n\in\mathbb{N}} \Psi G_n) = \sup_{n\in\mathbb{N}} \theta(\Psi G_n) = \sup_{n\in\mathbb{N}} c(G_n) = c(G). \ \mathbf{Q}$$

(h) Now suppose that $D \subseteq \mathbb{R}^r$ is any bounded set. We have

$$\theta^*(\Psi D) \le \inf\{\theta(\Psi G) : G \supseteq D \text{ is open}\} = \inf\{c(G) : G \supseteq D \text{ is open}\} = c(D).$$

? Suppose, if possible, that $\theta^*(\Psi D) < c(D)$. Let $G \supseteq D$ be a bounded open set. Then there is a compact $L \subseteq \Psi G \setminus \Psi D$ such that $\theta L > \theta(\Psi G) - c(D)$. Set $F = \bigcup_{C\in L} C$; then F is closed (4A2T(e-iii)) and disjoint from D, so $G \setminus F$ is open, $D \subseteq G \setminus F$ and $\Psi(G \setminus F)$ is disjoint from L. But this means that

$$c(D) \le c(G \setminus F) = \theta(\Psi(G \setminus F)) \le \theta(\Psi G) - \theta L < c(D),$$

which is absurd. **X** So $\theta^*(\Psi D) = c(D)$.

If $D \subseteq \mathbb{R}^r$ is any set, then it is expressible as the union of a non-decreasing sequence $\langle D_n\rangle_{n\in\mathbb{N}}$ of bounded sets, so

$$c(D) = \lim_{n\to\infty} c(D_n) = \lim_{n\to\infty} \theta^*(\Psi D_n) = \theta^*(\bigcup_{n\in\mathbb{N}} \Psi D_n) = \theta^*(\Psi D).$$

Thus θ has all the properties declared.

(i) To see that θ is unique, consider the base $\mathcal{V}$ for the topology of $\mathcal{C}^+$ consisting of sets of the form $\bigcap_{i\in I} \Psi G_i \setminus \Psi K$ where $\langle G_i\rangle_{i\in I}$ is a non-empty finite family of bounded open sets in $\mathbb{R}^r$ and $K \subseteq \mathbb{R}^r$ is compact. The conditions that θ must satisfy determine its value on any set of the form $\Psi(G \cup K) = \Psi G \cup \Psi K$ where $G \subseteq \mathbb{R}^r$ is open and $K \subseteq \mathbb{R}^r$ is compact, and therefore determine its values on $\mathcal{V}$. By 415H(iv), θ is fixed by these.

479X Basic exercises (a) Let ζ be a Radon measure on $\mathbb{R}^r$. Show that

$$\zeta\mathbb{R}^r = \lim_{\gamma\to\infty} \frac{2}{r\beta_r\gamma^2}\int_{B(\mathbf{0},\gamma)} W_\zeta d\mu.$$

>(b)(i) Show directly from 479B-479C that Choquet-Newton capacity c is invariant under isometries of $\mathbb{R}^r$. (ii) Show that $c(\alpha D) = \alpha^{r-2}c(D)$ whenever $\alpha \ge 0$ and $D \subseteq \mathbb{R}^r$.

(c) Suppose that ζ_1 and ζ_2 are totally finite Radon measures on $\mathbb{R}^r$. Show that $W_{\zeta_1 * \zeta_2} = \zeta_1 * W_{\zeta_2} = \zeta_2 * W_{\zeta_1}$.

(d) Show that there is a closed set $F \subseteq \mathbb{R}^r$ such that $\operatorname{hp}(F) < 1$ but $c(F) = \infty$. (*Hint*: look at the proof of 479Ma.)

(e) Let $K \subseteq \mathbb{R}^r$ be compact. Show that $\operatorname{int}\{x : \tilde W_K(x) < 1\}$ is the unbounded component of $\mathbb{R}^r \setminus \operatorname{supp}\lambda_K$. (*Hint*: setting $L = \operatorname{supp}\lambda_K$, show that $\operatorname{cap}(\operatorname{supp}\lambda_K) = \operatorname{cap}K$ so that $\tilde W_K = \tilde W_L$.)

(f) Let $A \subseteq \mathbb{R}^r$ be an analytic set such that $c(A) < \infty$. Show that $\tilde W_A = \sup\{W_\zeta : \zeta$ is a Radon measure on $\mathbb{R}^r$, A is ζ-conegligible, $W_\zeta \le 1\}$.

>(g) Show that there is a universally negligible set $D \subseteq B(\mathbf{0},1)$ such that $c(D) = 1$. (*Hint*: use the ideas of 439F to find D such that $\{\|x\| : x \in D\}$ is universally negligible and $x \mapsto \|x\| : D \to [0,1]$ is injective, but $\nu^*\{\frac{x}{\|x\|} : x \in D,\ \|x\| \ge 1-\delta\} = r\beta_r$ for every $\delta \in \,]0,1[$; compute $c(D)$ with the aid of 479D, 479P(c-iii-α) and 479P(c-vii).)

(h) Suppose that $D \subseteq D' \subseteq \mathbb{R}^r$ and $c(D') < \infty$. Show that $\int f d\lambda_D \le \int f d\lambda_{D'}$ for every lower semi-continuous superharmonic $f : \mathbb{R}^r \to [0,\infty]$.

(i) Let $D \subseteq \mathbb{R}^r$ be a bounded set. Show that $c(D) = \lim_{\|x\|\to\infty} \|x\|^{r-2}\operatorname{hp}^*(D-x)$. (*Hint*: 477Id.)

(j) Let $T : \mathbb{R}^r \to \mathbb{R}^r$ be an isometry, and $D \subseteq \mathbb{R}^r$ a set such that $c(D) < \infty$. Show that $\tilde W_{T[D]}Tx) = \tilde W_D(x)$ for every $x \in \mathbb{R}^r$ and that $\lambda_{T[D]}$ is the image measure $\lambda_D T^{-1}$.

(k) Let $D \subseteq \mathbb{R}^r$ be a set such that $c(D) < \infty$, and $\alpha > 0$. Show that $\tilde W_{\alpha D}(x) = \tilde W_D(\frac{1}{\alpha}x)$ for every $x \in \mathbb{R}^r$, and that $\lambda_{\alpha D} = \alpha^{r-2}\lambda_D T^{-1}$, where $T(x) = \alpha x$ for $x \in \mathbb{R}^r$.

(l) Show that $c(D) = \inf\{\zeta\mathbb{R}^r : \zeta$ is a Radon measure on $\mathbb{R}^r$, $W_\zeta \ge \chi D\} = \inf\{\operatorname{energy}(\zeta) : \zeta$ is a Radon measure on $\mathbb{R}^r$, $W_\zeta \ge \chi D\}$ for any $D \subseteq \mathbb{R}^r$.

(m) Let $K \subseteq \mathbb{R}^r$ be a compact set, with complement G, and Φ the set of continuous harmonic functions $f : G \to [0,1]$ such that $\lim_{\|x\|\to\infty} f(x) = 0$. Show that $\tilde W_K{\restriction}G$ is the greatest element of Φ. (*Hint*: 479Pb, 478Pc.)

(n)(i) Show that if G is a convex open set then $\operatorname{hp}(G-x) = 1$ for every $x \in \overline{G}$. (ii) Show that if $D \subseteq \mathbb{R}^r$ is a convex bounded set with non-empty interior, then $\tilde W_D$ is continuous.

(o) Show that if D, $D' \subseteq \mathbb{R}^r$ and $c(D \cup D') < \infty$, then $\tilde W_{D\cap D'} + \tilde W_{D\cup D'} \le \tilde W_D + \tilde W_{D'}$.

(p) Let $D \subseteq \mathbb{R}^r$ be a set such that $c(D) < \infty$, and set $\tilde D = \{x : \tilde W_D(x) = 1\}$. Show that (i) $D \setminus \tilde D$ is polar (ii) $\lambda_{\tilde D} = \lambda_D$. (*Hint*: reduce to the case in which $D = A$ is analytic; use 479Fg to show that $\tilde W_{\tilde A} \le \tilde W_A$; use 479J(b-v).)

(q) Let $\mathcal{A}$ be the set of subsets of $\mathbb{R}^r$ with finite Choquet-Newton capacity, and ρ the pseudometric $(D,D') \mapsto 2c(D\cup D') - c(D) - c(D')$ (432Xj). (i) Show that $\|U_{\lambda_D} - U_{\lambda_{D'}}\|_2^2 \le 2c_r\rho(D,D')$ for D, $D' \in \mathcal{A}$. (ii) Show that $\rho(D,D') = 0$ iff $\lambda_D = \lambda_{D'}$.

(r) Suppose that $D \subseteq \mathbb{R}^r$. Show that the following are equiveridical: (i) D is polar; (ii) there is some $x \in \mathbb{R}^r$ such that $\operatorname{hp}(D-x) = 0$; (iii) $\operatorname{hp}((D \setminus \{x\}) - x) = 0$ for every $x \in \mathbb{R}^r$.

(s) Suppose that $\langle D_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of subsets of $\mathbb{R}^r$ such that $\inf_{n\in\mathbb{N}} c(D_n)$ is finite and $\bigcap_{n\in\mathbb{N}} D_n = \bigcap_{n\in\mathbb{N}} \overline{D_n} = F$ say. Show that $\langle\lambda_{D_n}\rangle_{n\in\mathbb{N}} \to \lambda_D$ for the narrow topology, and that $\langle c(D_n)\rangle_{n\in\mathbb{N}} \to c(F)$. (Compare 479Ye.)

(t) For $\omega\in\Omega$ set $\tau(\omega) = \sup\{t : \|\omega(t)\| \le 1\}$. (i) Show that $\tau : \Omega \to [0,\infty]$ is measurable. (ii) Show that if $r \le 2$ then $\tau = \infty$ a.e. (iii) Show that if $r \ge 3$ then τ is not a stopping time. (iv) Show that if $3 \le r \le 4$ then τ is finite a.e., but has infinite expectation. (v) Show that if $r \ge 5$ then τ has finite expectation. (*Hint*: show that if $r \ge 2$ then

$$\Pr(\tau \ge t) = \frac{1}{(\sqrt{2\pi})^r}\int e^{-\|x\|^2/2}\frac{1}{\max(1,(\sqrt{t})^{r-2}\|x\|^{r-2})}dx.)$$

479Y Further exercises (a)(i) Show that there is an open set $G \subseteq B(\mathbf{0},1)$, dense in $B(\mathbf{0},1)$, such that $c(G) < 1$. (ii) Show that $\operatorname{cap}(\operatorname{supp}\lambda_G) = 1$.

(b) In 479G, suppose that $0<\alpha<r$, $0<\beta<r$ and $\alpha+\beta>r$. Show that

$$k_{\alpha+\beta-r} = \frac{\Gamma(r-\frac{\alpha+\beta}{2})\Gamma(\frac{\alpha}{2})\Gamma(\frac{\beta}{2})}{(\sqrt{\pi})^r\Gamma(\frac{\alpha+\beta-r}{2})\Gamma(\frac{r-\alpha}{2})\Gamma(\frac{r-\beta}{2})}k_\alpha * k_\beta.$$

(c) Let $A \subseteq \mathbb{R}^r$ be an analytic set with $c(A) < \infty$. (i) Show that for every $\gamma > 0$ there is a Radon measure ζ_γ on $\mathbb{R}^r$ such that $\langle\frac{1}{r\beta_r\gamma}\mu_x^{(A)}\rangle_{x\in\partial B(\mathbf{0},\gamma)}$ is a disintegration of ζ_γ over the subspace measure $\nu_{\partial B(\mathbf{0},\gamma)}$. (ii) Show that $\lim_{\gamma\to\infty}\zeta_\gamma = \lambda_A$ for the total variation metric on $M_{\mathrm{R}}^+(\mathbb{R}^r)$.

(d) Set $c'(D) = \sup\{\zeta^*D : \zeta$ is a Radon measure on $\mathbb{R}^r$ such that $W_\zeta \le 1$ everywhere$\}$ for $D \subseteq \mathbb{R}^r$. Show that c' is a Choquet capacity on $\mathbb{R}^r$, extending Newtonian capacity for compact sets, which is different from Choquet-Newton capacity.

(e) Let $\langle D_n\rangle_{n\in\mathbb{N}}$ be a non-increasing sequence of subsets of $\mathbb{R}^r$ with finite Choquet-Newton capacity. For each $n\in\mathbb{N}$, set $\tilde{D}_n = \{x : \tilde{W}_{D_n}(x) = 1\}$, and set $A = \bigcap_{n\in\mathbb{N}}\tilde{D}_n$. Show that $c(A) = \inf_{n\in\mathbb{N}} c(D_n)$ and that λ_A is the limit of $\langle\lambda_{D_n}\rangle_{n\in\mathbb{N}}$ for the narrow topology on $M_{\mathrm{R}}^+(\mathbb{R}^r)$.

(f) Let $\mathcal{A}$, ρ be as in 479Xq, and let $(\bar{\mathcal{A}},\bar{\rho})$ be the corresponding metric space, identifying members of $\mathcal{A}$ which are zero distance apart. Show that $\bar{\mathcal{A}}$ is complete.

(g) Let $E \subseteq \mathbb{R}^r$ be a set of finite Lebesgue measure, and B_E the ball with centre 0 and the same measure as E. Show that $\operatorname{energy}(\mu\llcorner E) \le \operatorname{energy}(\mu\llcorner B_E)$.

(h) Prove 479V from 479U and 476Yb.

(i)(i) Show that c is alternating of all orders, that is,

$$\sum_{J\subseteq I,\#(J)\text{ is even}} c(D\cup\bigcup_{i\in J}D_i) \le \sum_{J\subseteq I,\#(J)\text{ is odd}} c(D\cup\bigcup_{i\in J}D_i)$$

whenever I is a non-empty finite set, $\langle D_i\rangle_{i\in I}$ is a family of subsets of $\mathbb{R}^r$ and D is another subset of $\mathbb{R}^r$. (Cf. 132Yf.) (ii) Show that if $c(D\cup\bigcup_{i\in I}D_i) < \infty$, then

$$\sum_{J\subseteq I,\#(J)\text{ is even}} \tilde{W}_{D\cup\bigcup_{i\in J}D_i} \le \sum_{J\subseteq I,\#(J)\text{ is odd}} \tilde{W}_{D\cup\bigcup_{i\in J}D_i}.$$

(j) Let us say that if X is a Polish space, a set $A \subseteq X$ is **projectively universally measurable** if $W[A]$ is universally measurable whenever Y is a Polish space and $W \subseteq X\times Y$ is analytic. Show that we can replace the word 'analytic' by the phrase 'projectively universally measurable' in all the theorems of this section.

(k) Suppose that $A \subseteq \mathbb{R}^r$ is analytic and non-empty, and $x\in\mathbb{R}^r$ is such that $\rho(x,A) = \delta > 0$. Show that $\operatorname{energy}(\mu_x^{(A)}) \le \frac{1}{\delta^{r-2}}$.

(l) Show that if $D \subseteq \mathbb{R}^r$ and $c(D) < \infty$, then $c(\{x : \tilde{W}_D(x) \ge \gamma\}) \le \frac{1}{\gamma}c(D)$ for every $\gamma > 0$.

(m) For a set $D \subseteq \mathbb{R}^r$ with $c(D) < \infty$, set $\mathrm{cl}_{\mathrm{cap}}D = \{x : \tilde{W}_D(x) = 1\}$. Show that $c(D) = c(\mathrm{cl}_{\mathrm{cap}}D) = c(D \cup \mathrm{cl}_{\mathrm{cap}}D)$.

479 Notes and comments Newtonian potential is another of the great concepts of mathematics, and is one of the points at which physical problems and intuitions have stimulated and illuminated the development of analysis. As with all the best ideas of mathematics, there is more than one route to it, and any proper understanding of it must include a matching of the different approaches. In the exposition here I start with a description of equilibrium measures in terms of harmonic measures (479B), themselves defined in 478P in terms of Brownian motion. We are led quickly to definitions of capacity and equilibrium potential (479C), with some elementary properties (479D). Moreover, some very striking further results (479E, 479W) are already accessible.

However we are still rather far from the original physical concept of 'capacity' of a conductor. If you have ever studied electrostatics, the ideas here may recall some basic physical principles. The kernel $x \mapsto \frac{1}{\|x\|^{r-2}}$ represents the potential energy field of a point mass or charge; the potential W_ζ represents the field due to a mass or charge with distribution ζ. The capacity of a set K is the largest charge that can be put on K without raising the potential of any point above 1 (479Na), and the infimum of the charges which raise the potential of every point of K to 1 (479P(c-v)). The result that λ_K is supported by ∂K (479B(i)) corresponds to the principle that the charge on a conductor always collects on the surface of the conductor; 479D(b-iii) corresponds to the principle that there is no electric field inside a conductor.

At the same time, the equilibrium measure is supposed to be the (unique) distribution of the charge, which on physical grounds ought to be the distribution with least energy, as in 479K. To reach these ideas, it seems that we need to know various non-trivial facts from classical analysis, which I set out in 479G-479I. The deepest of these is in 479Ib: for the Riesz kernels k_α, the convolution $\zeta * k_\alpha$ determines the totally finite Radon measure ζ. I do not know of any way of establishing this except through the Fourier analysis of 479H and the detailed calculations of 479G and 479Ia.

The ideas here are connected in so many ways that there is no clear flow to the logic, and we are more than usually in danger of using circular arguments. In my style of exposition, this complexity manifests itself in an exceptional density of detailed back-references; I hope that these will enable you to check the proofs effectively. On a larger scale, the laborious zigzag progression from the original notion of capacity of compact sets, as in 479K and 479U, through bounded analytic sets (479B, 479E) and general analytic sets (479M, 479N) to arbitrary sets (479P), displays a choice of path to which there are surely many alternatives.

Of course we cannot expect all the properties of Newtonian capacity to have recognizable forms in such a general context as that of 479P (see 479Xg, which shows that we cannot hope to replicate the ideas of 479Na-479Nc), but the elementary results if 479D mostly extend (479Pc). More importantly, we have a quite new characterization of equilibrium potentials (479Pb). With these techniques available, we can learn a good deal more about Brownian motion. 479R is a curious and striking fact to go with 477K, 477L and 478M. It is not a surprise that capacity and Hausdorff dimension should be linked, but it is notable that the phase change is at dimension $r-2$ (479Q); this goes naturally with 479P(c-vii). I know of no such dramatic difference between four and five dimensions, but for some purposes 479Xt marks a significant change.

My treatment is an unconventional one, so perhaps I should indicate points where you should expect other authors to diverge from it. While the notions of Newtonian capacity, equilibrium measure and equilibrium potential are solidly established for compact sets in $\mathbb{R}^r$ (at least up to scalar factors, and for $r \ge 3$), the extension to general bounded analytic sets is not I think standard. (I try to signal this by writing $c(A)$ in place of $\operatorname{cap} A$, after 479E, for sets which are not guaranteed to be compact, even when the definition in 479C(a-i) is applicable. The fact that this step gives very little extra trouble is a demonstration of the power of the Brownian-motion approach.) The further extension of Newtonian capacity, defined on compact sets, to a Choquet capacity, defined on every subset of $\mathbb{R}^r$ (479Ed), is surely not standard, which is why I give the extension a different name. (While Choquet certainly considered the capacity which I here call 'Choquet-Newton capacity', I fear that the phrase has no real historical justification; but I hope it will convey some of the right ideas.) You may have noticed that I give essentially nothing concerning differential equations, which have traditionally been one of the central concerns of potential theory; there are hints in 479Xm and 479T.

A weakness in the formulae of 479B is that they are not self-evidently translation-invariant. Of course it is easy to show that in fact we have an isometry-invariant construction (479Xb), and this can also be seen from the descriptions of capacity and equilibrium potentials in 479N and 479Pb. Because the capacity c is countably subadditive, it is easy to build a dense open subset G of $\mathbb{R}^r$ such that $c(G)$ is finite (see 479Ya), and for such a set we cannot ask that λ_G should be describable as a limit of $\|x\|^{r-2}\mu_x^{(G)}$ as $\|x\| \to \infty$. But if we start from 479B(i) rather than 479B(ii), we do have an averaged form, with

$$\lambda_A = \lim_{\gamma\to\infty} \frac{1}{r\beta_r\gamma}\int_{\partial B(\mathbf{0},\gamma)} \mu_x^{(A)}\nu(dx)$$

whenever A is an analytic set and $c(A) < \infty$ (479Yc; see also 479J(b-vi) and 479Xa).

The factor $(\sqrt{2\pi})^r$ in 479H repeats that of 283Wg[13]. The appearance of $\sqrt{2\pi}$ in 283M, but not 445G, is proof that the conventions of Chapter 28 are not reconcilable with those of §445.

In 479O I describe one of the important notions of 'small' set in Euclidean space, to go with 'negligible' and 'meager'. I have no space to deal with it properly here, but the applications in the proofs of 479P, 479R and 479S will give an idea of its uses; another is in 479Xm. As another example of the logical complexity of the patterns here, consider the problem of either proving 479Pb without 479O, or extending 479O to cover 479Xr without passing through a version of 479P.

Quite a lot of the work here is caused by the need to accommodate discontinuous equilibrium potentials (479S). This has been an important theme in general potential theory. 479Pb shows that the problem is essentially geometric: if a compact set K has a sufficiently narrow spike at e, then a Brownian path starting at e can easily fail to enter K again.

As I have written the theory out, 479T-479U are rather separate from the rest, being closer in spirit to the work of §§473-475. They explore some more of the basic principles of potential theory. Note, in particular, the formula of 479Ta, which amounts to saying that (under the right conditions) a function g is a multiple of $k_{r-2} * \nabla^2 g$; of course this can be thought of as a method of finding a particular solution of the equation $\nabla^2 g = f$; equally, it gives an approach to the problem of expressing a given g as a potential W_ζ. From 479Tb and 479Tc we see that in the sense of distributional derivatives we can think of $r(r-2)\beta_r\zeta$ as representing the Laplacian $-\nabla^2 W_\zeta$; recall that as W_ζ is superharmonic (479Fb), we expect $\nabla^2 W_\zeta$ to be negative (478E).

I give a bit of space to 479V because it links the material here to that of §476, and this book is about such linkages, and because it supports my thesis that capacity is a geometrical concept. 479W is characteristic of Choquet capacities which are alternating of all orders (479Yi). I spell it out here because it calls on the Fell topology, which is important elsewhere in this volume.

It is natural to ask which of the ideas here applying to analytic sets can be extended to wider classes. If you look back to where analytic sets first entered the discussion, in the theory of hitting times (455M), you will see that we needed a class of universally measurable sets which would be invariant under various operations, notably projections (479Yj). In Volume 5 we shall meet axiom systems in which there are various interesting possibilities.

This section is firmly directed at Euclidean space of three or more dimensions. The harmonic and Fourier analysis of 479G-479I applies unchanged to dimensions 1 and 2; so does 479Tb. On the line, Brownian hitting probabilities are trivial; in the plane, they are very different from hitting probabilities in higher dimensions, but still of considerable interest. Theorems 479B, 479E and 479W still work, but 'capacity', if defined by the formulation of 479Ca, is bounded by 1. The geometric nature of the results changes dramatically, and 479I cannot be applied in the same way, since we no longer have $0 < r-2$.

Version of 16.2.10

Concordance

[13] Formerly 283Wj.

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this chapter, and which have since been changed.

479Xe Exercise 479Xe on Choquet-Newton capacity, referred to in the 2008 edition of Volume 5, is now 479Xi.

Version of 9.2.13

References for Volume 4

Aldaz J.M. [95] 'On compactness and Loeb measures', Proc. Amer. Math. Soc. 123 (1995) 173-175. [439Xa.]

Aldaz J.M. [97] 'Borel extensions of Baire measures', Fund. Math. 154 (1997) 275-293. [434Yn.]

Aldaz J.M. & Render H. [00] 'Borel measure extensions of measures defined on sub-σ-algebras', Advances in Math. 150 (2000) 233-263. [432D.]

Alexandroff P.S. & Urysohn P.S. [1929] *Mémoire sur les espaces topologiques compacts*, Verhandelingen Akad. Wetensch. Amsterdam 14 (1929) 1-96. [419L.]

Anderson I. [87] *Combinatorics of Finite Sets.* Oxford U.P., 1987. [4A4N.]

Andretta A. & Camerlo R. [13] 'The descriptive set theory of the Lebesgue density theorem', Advances in Math. 234 (2013). [475Yg.]

Arkhangel'skii A.V. [92] *Topological Function Spaces.* Kluwer, 1992. [§462 *intro.*, §467 *notes.*]

Aronov B., Basu S., Pach J. & Sharir M. [03] (eds.) *Discrete and Computational Geometry; the Goodman-Pollack Festschrift,* Springer, 2003.

Asanov M.O. & Velichko N.V. [81] 'Compact sets in $C_p(X)$', Comm. Math. Helv. 22 (1981) 255-266. [462Ya.]

Austin T. [10a] 'On the norm convergence of nonconventional ergodic averages', Ergodic Theory and Dynamical Systems 30 (2010) 321-338. [497M.]

Austin T. [10b] 'Deducing the multidimensional Szemerédi theorem from the infinitary hypergraph removal lemma', J. d'Analyse Math. 111 (2010) 131-150. [497M.]

Balogh Z. [96] 'A small Dowker space in ZFC', Proc. Amer. Math. Soc. 124 (1996) 2555-2560. [439O.]

Banach S. [1923] 'Sur le problème de la mesure', Fundamenta Math. 4 (1923) 7-33. [449O.]

Becker H. & Kechris A.S. [96] *The descriptive set theory of Polish group actions.* Cambridge U.P., 1996 (London Math. Soc. Lecture Note Series 232). [424H, 448P, §448 *notes.*]

Bellow A. *see* Ionescu Tulcea A.

Bellow A. & Kölzow D. [76] *Measure Theory, Oberwolfach 1975.* Springer, 1976 (Lecture Notes in Mathematics 541).

Bergelson V. [85] 'Sets of recurrence of $\mathbb{Z}^m$-actions and properties of sets of differences in $\mathbb{Z}^m$', J. London Math. Soc. (2) 31 (1985) 295-304. [491Yc.]

Bergman G.M. [06] 'Generating infinite symmetric groups', Bull. London Math. Soc. 38 (2006) 429-440. [§494 *notes.*]

Billingsley P. [99] *Convergence of Probability Measures.* Wiley, 1999. [§4A3 *notes.*]

Blackwell D. [56] 'On a class of probability spaces', pp. 1-6 in Neyman 56. [419K, 452P.]

Bochner S. [1933] 'Monotone Funktionen, Stieltjessche Integrale, und harmonische Analyse', Math. Ann. 108 (1933) 378-410. [445N.]

Bogachev V.I. [07] *Measure theory.* Springer, 2007. [437Q, 437Xs, 437Yo, 437Yu, 457K.]

Bollobás B. [79] *Graph Theory.* Springer, 1979. [4A1G, 4A4N.]

Bongiorno B., Piazza L.di & Preiss D. [00] 'A constructive minimal integral which includes Lebesgue integrable functions and derivatives', J. London Math. Soc. (2) 62 (2000) 117-126. [483Yi.]

Borodulin-Nadzieja P. & Plebanek G. [05] 'On compactness of measures on Polish spaces', Illinois J. Math. 49 (2005) 531-545. [451L.]

Bourbaki N. [65] *Intégration,* chaps. 1-4. Hermann, 1965 (Actualités Scientifiques et Industrielles 1175). [§416 *notes*, §436 *notes.*]

Bourbaki N. [66] *Espaces Vectoriels Topologiques*, chaps. 1-2, 2^e^ éd. Hermann, 1966 (Actualités Scientifiques et Industrielles 1189). [4A2J.]

Bourbaki N. [69] *Intégration,* chap. 9. Hermann, 1969 (Actualités Scientifiques et Industrielles 1343). [§436 *notes.*]

Bourbaki N. [87] *Topological Vector Spaces.* Springer, 1987. [§461 *notes*, §4A4.]

Bourgain J., Fremlin D.H. & Talagrand M. [78] 'Pointwise compact sets of Baire-measurable functions', Amer. J. Math. 100 (1978) 845-886. [462Ye.]

Brodskiĭ M.L. [1949] 'On some properties of sets of positive measure', Uspehi Matem. Nauk (N.S.) 4 (1949) 136-138. [498B.]

Brook R.B. [70] 'A construction of the greatest ambit', Math. Systems Theory 4 (1970) 243-248. [449D.]

Burke D.K. & Pol R. [05] 'Note on separate continuity and the Namioka property', Top. Appl. 152 (2005) 258-268. [§463 *notes*.]

Burke M.R., Macheras N.D. & Strauss W. [p21] 'The strong marginal lifting problem for hyperstonian spaces', preprint (2021).

Carrington D.C. [72] 'The generalised Riemann-complete integral', PhD thesis, Cambridge, 1972. [481L.]

Čech E. [66] *Topological Spaces.* Wiley, 1966. [§4A2, 4A3S.]

Chacon R.V. [69] 'Weakly mixing transformations which are not strongly mixing', Proc. Amer. Math. Soc. 22 (1969) 559-562. [494F.]

Choquet G. [55] 'Theory of capacities', Ann. Inst. Fourier (Grenoble) 5 (1955) 131-295. [432K.]

Chung K.L. [95] *Green, Brown and Probability.* World Scientific, 1995.

Ciesielski K. & Pawlikowski J. [03] 'Covering Property Axiom $\mathrm{CPA}_{\mathrm{cube}}$ and its consequences', Fundamenta Math. 176 (2003) 63-75. [498C.]

Császár Á. [78] *General Topology.* Adam Hilger, 1978. [§4A2, §4A5.]

Davies R.O. [70] 'Increasing sequences of sets and Hausdorff measure', Proc. London Math. Soc. (3) 20 (1970) 222-236. [471G.]

Davies R.O. [71] 'Measures not approximable or not specifiable by means of balls', Mathematika 18 (1971) 157-160. [§466 *notes*.]

Davis W.J., Ghoussoub N. & Lindenstrauss J. [81] 'A lattice-renorming theorem and applications to vector-valued processes', Trans. Amer. Math. Soc. 263 (1981) 531-540. [§467 *notes*.]

de Caux P. [76] 'A collectionwise normal, weakly θ-refinable Dowker space which is neither irreducible nor realcompact', Topology Proc. 1 (1976) 67-77. [439O.]

Dellacherie C. [80] 'Un cours sur les ensembles analytiques', pp. 183-316 in Rogers 80. [§432 *notes*.]

de Maria J.L. & Rodriguez-Salinas B. [91] 'The space (ℓ_∞/c_0),weak) is not a Radon space', Proc. Amer. Math. Soc. 112 (1991) 1095-1100. [466I.]

Deville R., Godefroy G. & Zizler V. [93] *Smoothness and Renormings in Banach Spaces.* Pitman, 1993. [§467 *intro.*, 467Ye, §467 *notes*.]

Diaconis, P. & Freedman, D. [80] 'Finite exchangeable sequences.' Ann. of Probability 8 (1980) 745-764. [459Xd.]

Droste M., Holland C. & Ulbrich G. [08] 'On full groups of measure preserving and ergodic transformations with uncountable cofinalities', Bull. London Math. Soc. 40 (2008) 463-472. [494Q.]

Dubins L.E. & Freedman D.A. [79] 'Exchangeable processes need not be mixtures of independent, identically distributed random variables', Z. Wahrscheinlichkeitstheorie und verw. Gebiete 48 (1979) 115-132. [459K.]

Dunford N. & Schwartz J.T. [57] *Linear Operators I.* Wiley, 1957 (reprinted 1988). [§4A4.]

Du Plessis, N. [70] *Introduction to Potential Theory.* Oliver & Boyd, 1970.

Eggleston H.G. [54] 'Two measure properties of Cartesian product sets', Quarterly J. Math. (2) 5 (1954) 108-115. [498B.]

Emerson W. & Greenleaf F. [67] 'Covering properties and Følner conditions for locally compact groups', Math. Zeitschrift 102 (1967) 370-384. [449J.]

Enderton H.B. [77] *Elements of Set Theory.* Academic, 1977. [4A1A.]

Engelking R. [89] *General Topology.* Heldermann, 1989 (Sigma Series in Pure Mathematics 6). [Vol. 4 *intro.*, 415Yb, §434 *notes*, §435 *notes*, 462L, §4A2, 4A4B.]

Evans L.C. & Gariepy R.F. [92] *Measure Theory and Fine Properties of Functions.* CRC Press, 1992. [Chap. 47 *intro.*, §473 *intro.*, §473 *notes*, §474 *notes*, §475 *notes*.]

Falconer K.J. [90] *Fractal Geometry.* Wiley, 1990. [§471 *intro.*]

Federer H. [69] *Geometric Measure Theory.* Springer, 1969 (reprinted 1996). [§441 *notes*, 442Ya, §443 *notes*, Chap. 47 *intro.*, 471Yk, §471 *notes*, §472 *notes*, §475 *notes*.]

Fernique X. [97] *Fonctions aléatoires gaussiennes, vecteurs aléatoires gaussiens*, CRM Publications, 1997. [§456 *notes*.]

Folland G.B. [95] *A Course in Abstract Harmonic Analysis.* CRC Press, 1995 [§4A5, §4A6.]

Følner E. [55] 'On groups with full Banach mean value', Math. Scand. 3 (1955) 243-254. [§449 *notes*.]

Ford L.R. & Fulkerson D.R. [56] 'Maximal flow through a network', Canadian J. Math. 8 (1956) 399-404. [4A4N.]

Frankl P. & Rödl V. [02] 'Extremal problems on set systems', Random Structures Algorithms 20 (2002) 131-164. [497L.]

Fremlin D.H. [74] *Topological Riesz Spaces and Measure Theory.* Cambridge U.P., 1974. [§462 *notes*.]

Fremlin D.H. [75a] 'Pointwise compact sets of measurable functions', Manuscripta Math. 15 (1975) 219-242. [463K.]

Fremlin D.H. [75b] 'Topological measure spaces: two counter-examples', Math. Proc. Cambridge Phil. Soc. 78 (1975) 95-106. [419C, 419D.]

Fremlin D.H. [76] 'Products of Radon measures: a counterexample', Canadian Math. Bull. 19 (1976) 285-289. [419E.]

Fremlin D.H. [81] 'Measurable functions and almost continuous functions', Manuscripta Math. 33 (1981) 387-405. [451T.]

Fremlin D.H. [84] *Consequences of Martin's Axiom.* Cambridge U.P., 1984. [§434 *notes*, 439Yd, 4A2E.]

Fremlin D.H. [93] 'Real-valued-measurable cardinals', pp. 151-304 in Judah 93. [§438 *notes*.]

Fremlin D.H. [95] 'The generalized McShane integral', Illinois J. Math. 39 (1995) 39-67. [481N.]

Fremlin D.H. [00] 'Weakly α-favourable measure spaces', Fundamenta Math. 165 (2000) 67-94. [451V.]

Fremlin D.H. [n05] 'Strictly localizable Borel measures', note of 7.5.05 (`http://www.essex.ac.uk/maths/people/fremlin/preprints.htm`). [434Yr.]

Fremlin D.H. [n15] 'The Sorgenfrey line is not a Prokhorov space', note of 5.8.15 (`http://www.essex.ac.uk/maths/people/fremlin/preprints.htm`). [439Yk.]

Fremlin D.H. & Grekas S. [95] 'Products of completion regular measures', Fund. Math. 147 (1995) 27-37. [434Q.]

Fremlin D.H. & Talagrand M. [78] 'On the representation of increasing linear functionals on Riesz spaces by measures', Mathematika 25 (1978) 213-215. [439I.]

Fremlin D.H. & Talagrand M. [79] 'A decomposition theorem for additive set functions and applications to Pettis integral and ergodic means', Math. Zeitschrift 168 (1979) 117-142. [§464.]

Fristedt B. & Gray L. [97] *A Modern Approach to Probability Theory.* Birkhäuser, 1997. [§455 *notes*, §495 *notes*.]

Frolík Z. [61] 'On analytic spaces', Bull. Acad. Polon. Sci. 9 (1961) 721-726. [422F.]

Frolík Z. [88] (ed) *General Topology and its Relations to Modern Analysis and Algebra VI* (Proc. Sixth Prague Topological Symposium), Heldermann, 1988.

Furstenberg H. [77] 'Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions', J. d'Analyse Math. 31 (1977), 204-256. [§497 *intro.*, §497 *notes*.]

Furstenberg H. [81] *Recurrence in Ergodic Theory and Combinatorial Number Theory.* Princeton U.P., 1981. [§497 *intro.*, 497N.]

Furstenberg H. & Katznelson Y. [85] 'An ergodic Szemerédi theorem for IP-systems and combinatorial theory', J. d'Analyse Math. 45 (1985) 117-168. [§497 *intro.*]

Gaal S.A. [64] *Point Set Topology.* Academic, 1964. [§4A2.]

Gardner R.J. [75] 'The regularity of Borel measures and Borel measure-compactness', Proc. London Math. Soc. (3) 30 (1975) 95-113. [438M.]

Gardner R.J. & Pfeffer W.F. [84] 'Borel measures', pp. 961-1043 in Kunen & Vaughan 84. [434G.]

Giordano T. & Pestov V. [02] 'Some extremely amenable groups', C.R.A.S. Paris (Sér. I) 334 (2002) 273-278. [494I.]

Glasner S. [98] 'On minimal actions of Polish groups', Topology and its Applications 85 (1998) 119-125. [§493 *notes*.]

Gordon R.A. [94] *The Integrals of Lebesgue, Denjoy, Perron and Henstock.* Amer. Math. Soc., 1994 (Graduate Studies in Mathematics 4). [481Q, §483 *notes*.]

Greenleaf F.P. [69] *Invariant Means on Topological Groups and Their Applications.* van Nostrand, 1969. [§449 *notes*.]

Grothendieck A. [92] *Topological Vector Spaces.* Gordon & Breach, 1992. [§462 *notes*.]

Grzegorek E. [81] 'On some results of Darst and Sierpiński concerning universal null and universally negligible sets', Bull. Acad. Polon. Sci. (Math.) 29 (1981) 1-5. [439F, §439 *notes*.]

Halmos P.R. [1944] 'In general a measure-preserving transformation is mixing', Annals of Math. 45 (1944) 786-792. [494E.]

Halmos P.R. [50] *Measure Theory.* Springer, 1974. [§441 *notes*, §442 *notes*, 443M, §443 *notes*.]

Halmos P.R. [56] *Lectures on ergodic theory.* Chelsea, 1956. [494A.]

Hansell R.W. [01] 'Descriptive sets and the topology of non-separable Banach spaces', Serdica Math. J. 27 (2001) 1-66. [466D.]

Hart J.E. & Kunen K. [99] 'Orthogonal continuous functions', Real Analysis Exchange 25 (1999) 653-659. [416Yh.]

Hartman S. & Mycielski J. [58] 'On the imbedding of topological groups into connected topological groups', Colloq. Math. 5 (1958) 167-169. [493Ya.]

Haydon R. [74] 'On compactness in spaces of measures and measure-compact spaces', Proc. London Math. Soc. (3) 29 (1974) 1-6. [438J.]

Henry J.P. [69] 'Prolongement des mesures de Radon', Ann. Inst. Fourier 19 (1969) 237-247. [416N.]

Henstock R. [63] *Theory of Integration.* Butterworths, 1963. [Chap. 48 *intro.*, 481J.]

Henstock R. [91] *The General Theory of Integration.* Oxford U.P., 1991. [§481 *notes*.]

Herer W. & Christensen J.P.R. [75] 'On the existence of pathological submeasures and the construction of exotic topological groups', Math. Ann. 213 (1975) 203-210. [§493 *notes*.]

Herglotz G. [1911] 'Über Potenzreihen mit positivem reellen Teil im Einheitskreis', Leipziger Berichte 63 (1911) 501-511. [445N.]

Hewitt E. & Ross K.A. [63] *Abstract Harmonic Analysis I.* Springer, 1963 and 1979. [§442 *notes*, §444 *intro.*, §444 *notes*, §4A5, §4A6.]

Hewitt E. & Savage L.J. [55] 'Symmetric measures on Cartesian products', Trans. Amer. Math. Soc. 80 (1955) 470-501. [459Xe.]

Howroyd J.D. [95] 'On dimension and on the existence of sets of finite positive Hausdorff measure', Proc. London Math. Soc. (3) 70 (1995) 581-604. [471R, 471S.]

Ionescu Tulcea A. [73] 'On pointwise convergence, compactness and equicontinuity I', Z. Wahrscheinlichkeitstheorie und verw. Gebiete 26 (1973) 197-205. [463C.]

Ionescu Tulcea A. [74] 'On pointwise convergence, compactness, and equicontinuity II', Advances in Math. 12 (1974) 171-177. [463G.]

Ionescu Tulcea A. & Ionescu Tulcea C. [67] 'On the existence of a lifting commuting with the left translations of an arbitrary locally compact group', pp. 63-97 in LeCam & Neyman 67. [§447 *intro.*, 447I, §447 *notes*.]

Ionescu Tulcea A. & Ionescu Tulcea C. [69] *Topics in the Theory of Lifting.* Springer, 1969. [§453 *intro.*]

Ismail M. & Nyikos P.J. [80] 'On spaces in which countably compact sets are closed, and hereditary properties', Topology and its Appl. 11 (1980) 281-292. [§434 *notes*.]

Jameson G.J.O. [74] *Topology and Normed Spaces.* Chapman & Hall, 1974. [§4A4.]

Jayne J.E. [76] 'Structure of analytic Hausdorff spaces', Mathematika 23 (1976) 208-211. [422Yc.]

Jayne J.E. & Rogers C.A. [95] 'Radon measures on Banach spaces with their weak topologies', Serdica Math. J. 21 (1995) 283-334. [466H, §466 *notes*.]

Jech T. [78] *Set Theory.* Academic, 1978. [423S, §4A1.]

Jech T. [03] *Set Theory, Millennium Edition.* Springer, 2003. [§4A1.]

Johnson R.A. [82] 'Products of two Borel measures', Trans. Amer. Math. Soc. 269 (1982) 611-625. [434R.]

Judah H. [93] (ed.) *Proceedings of the Bar-Ilan Conference on Set Theory and the Reals, 1991.* Amer. Math. Soc. (Israel Mathematical Conference Proceedings 6), 1993.

Juhász I., Kunen K. & Rudin M.E. [76] 'Two more hereditarily separable non-Lindelöf spaces', Canadian J. Math. 29 (1976) 998-1005. [§437 *notes*, 439K, §439 *notes*.]

Just W. & Weese M. [96] *Discovering Modern Set Theory I.* Amer. Math. Soc., 1996 (Graduate Studies in Mathematics 8). [4A1A.]

Just W. & Weese M. [97] *Discovering Modern Set Theory II.* Amer. Math. Soc., 1997 (Graduate Studies in Mathematics 18). [§4A1.]

Kampen, E.R.van [1935] 'Locally bicompact abelian groups and their character groups', Ann. of Math. (2) 36 (1935) 448-463. [445U.]

Kaplansky I. [71] *Lie Algebras and Locally Compact Groups.* Univ. of Chicago Press, 1971. [§446 *notes.*]

Kechris A.S. [95] *Classical Descriptive Set Theory.* Springer, 1995. [Chap. 42 *intro.*, 423T, §423 *notes*, §448 *notes*, §4A2, 4A3S, 4A3T.]

Kellerer H.G. [84] 'Duality theorems for marginal problems', Zeitschrift für Wahrscheinlichkeitstheorie verw. Gebiete 67 (1984) 399-432. [457M.]

Kelley J.L. [55] *General Topology.* Van Nostrand, 1955. [438R.]

Kelley J.L. & Srinivasan T.P. [71] 'Pre-measures on lattices of sets', Math. Annalen 190 (1971) 233-241. [413Xr.]

Kittrell J. & Tsankov T. [09] 'Topological properties of full groups', Ergodic Theory and Dynamical Systems 2009 (doi:10.1017/S0143385709000078). [494O, §494 *notes.*]

Kolmogorov A.N. [1933] *Grundbegriffe der Wahrscheinlichkeitsrechnung.* Springer, 1933; translated as *Foundations of Probability Theory,* Chelsea, 1950. [454D.]

Köthe G. [69] *Topological Vector Spaces I.* Springer, 1969. [§462 *notes*, §4A4, 4A6C.]

Koumoullis G. & Prikry K. [83] 'The Ramsey property and measurable selectors', J. London Math. Soc. (2) 28 (1983) 203-210. [451T.]

Kuipers L. & Niederreiter H. [74] *Uniform Distribution of Sequences.* Wiley, 1974. [491Z.]

Kunen K. [80] *Set Theory.* North-Holland, 1980. [§4A1, 4A2E.]

Kunen K. & Vaughan J.E. [84] (eds.) *Handbook of Set-Theoretic Topology.* North-Holland, 1984.

Kuratowski K. [66] *Topology,* vol. I. Academic, 1966. [423T, 462Ye, §4A2, 4A3S.]

Kurzweil J. [57] 'Generalized ordinary differential equations and continuous dependence on a parameter', Czechoslovak Math. J. 7 (1957) 418-446. [Chap. 48 *intro.*]

Kwiatowska A. & Solecki S. [11] 'Spatial models of Boolean actions and groups of isometries', Ergodic Theory Dynam. Systems 31 (2011) 405-421. [§448 *notes.*]

Laczkovich M. [02] 'Paradoxes in measure theory', pp. 83-123 in Pap 02. [§449 *notes.*]

LeCam L.M. & Neyman J. [67] (eds.) *Proc. Fifth Berkeley Symposium in Mathematical Statistics and Probability*, vol. II. Univ. of California Press, 1967.

Levy A. [79] *Basic Set Theory.* Springer, 1979. [§4A1.]

Loève M. [77] *Probability Theory I.* Springer, 1977. [§495 *notes.*]

Losert V. [79] 'A measure space without the strong lifting property', Math. Ann. 239 (1979) 119-128. [453N, §453 *notes.*]

Lukeš J., Malý J. & Zajíček L. [86] *Fine topology methods in real analysis and potential theory,* Springer, 1986 (Lecture Notes in Mathematics 1189). [414Ye.]

Mackey G.W. [62] 'Point realizations of transformation groups', Illinois J. Math. 6 (1962) 327-335. [448S.]

Marczewski E. [53] 'On compact measures', Fundamenta Math. 40 (1953) 113-124. [413T, Chap. 45 *intro.*, §451 *intro.*, 451J.]

Marczewski E. & Ryll-Nardzewski C. [53] 'Remarks on the compactness and non-direct products of measures', Fundamenta Math. 40 (1953) 165-170. [454C.]

Mařík J. [57] 'The Baire and Borel measure', Czechoslovak Math. J. 7 (1957) 248-253. [435C.]

Mattila P. [95] *Geometry of Sets and Measures in Euclidean Spaces.* Cambridge U.P., 1995. [Chap. 47 *intro.*, §471 *intro.*, §472 *notes*, §475 *notes.*]

Mauldin R.D. & Stone A.H. [81] 'Realization of maps', Michigan Math. J. 28 (1981) 369-374. [418T.]

Maurey B. [79] 'Construction des suites symétriques', C.R.A.S. (Paris) 288 (1979) 679-681. [492H.]

McShane E.J. [73] 'A unified theory of integration', Amer. Math. Monthly 80 (1973) 349-359. [481M.]

Meester R. & Roy R. [96] *Continuum percolation.* Cambridge U.P., 1996. [§495 *notes.*]

Milman V.D. & Schechtman G. [86] *Asymptotic Theory of Finite Dimensional Normed Spaces.* Springer, 1986 (Lecture Notes in Mathematics 1200). [492G, §492 *notes.*]

Moltó A., Orihuela J., Troyanski S. & Valdivia M. [09] *A Non-linear Transfer Technique for Renorming.* Springer, 2009 (Lecture Notes in Mathematics 1951). [§467 *notes.*]

Montgomery D. & Zippin L. [55] *Topological Transformation Groups.* Interscience, 1955. [§446 *notes.*]

Moran W. [68] 'The additivity of measures on completely regular spaces', J. London Math. Soc. 43 (1968) 633-639. [439P.]

Moran W. [69] 'Measures and mappings on topological spaces', Proc. London Math. Soc. (3) 19 (1969) 493-508. [435Xk.]

Moran W. [70] 'Measures on metacompact spaces', Proc. London Math. Soc. (3) 20 (1970) 507-524. [438J.]

Moschovakis Y.N. [80] *Descriptive Set Theory.* Springer, 1980. [Chap. 42 *intro.*, §423 *notes.*]

Muldowney P. [87] *A General Theory of Integration in Function Spaces.* Longmans, 1987 (Pitman Research Notes in Mathematics 153). [481P, §482 *notes.*]

Mushtari D.H. [96] *Probabilities and Topologies on Linear Spaces.* Kazan Mathematics Foundation, 1996.

Musiał K. [76] 'Inheritness of compactness and perfectness of measures by thick subsets', pp. 31-42 in Bellow & Kölzow 76. [451U.]

Nadkarni M.G. [90] 'On the existence of a finite invariant measure', Proc. Indian Acad. Sci., Math. Sci. 100 (1990) 203-220. [448P, §448 *notes.*]

Nagle B., Rödl V. & Schacht M. [06] 'The counting lemma for regular k-uniform hypergraphs', Random Structures and Algorithms 28 (2006) 113-179. [497J.]

Neyman J. [56] (ed.) *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954-55,* vol. II. Univ. of California, 1956.

Oxtoby J.C. [70] 'Homeomorphic measures on metric spaces', Proc. Amer. Math. Soc. 24 (1970) 419-423. [434Yq.]

Pachl J.K. [78] 'Disintegration and compact measures', Math. Scand. 43 (1978) 157-168. [452I, 452S, 452Ye, §452 *notes.*]

Pachl J.K. [79] 'Two classes of measures', Colloquium Math. 42 (1979) 331-340. [452R.]

Pap E. [02] (ed.) *Handbook of Measure Theory.* North-Holland, 2002.

Paterson A.L.T. [88] *Amenability.* Amer. Math. Soc., 1988. [449K, §449 *notes.*]

Perron O. [1914] 'Über den Integralbegriff', Sitzungsberichte der Heidelberger Akad. Wiss. A14 (1914) 1-16. [Chap. 48 *intro.*, 483J.]

Pestov V. [99] 'Topological groups: where to from here?', Topology Proc. 24 (1999) 421-502. [§493 *notes.*]

Pestov V. [02] 'Ramsey-Milman phenomenon, Urysohn metric spaces, and extremely amenable groups', Israel J. Math. 127 (2002) 317-357. [493E, 493Ya, §493 *notes.*]

Pfeffer W.F. [91a] 'The Gauss-Green theorem', Advances in Math. 87 (1991) 93-147. [484Xb.]

Pfeffer W.F. [91b] 'Lectures on geometric integration and the divergence theorem', Rend. Ist. Mat. Univ. Trieste 23 (1991) 263-314. [484B.]

Pfeffer W.F. [01] *Derivation and Integration.* Cambridge U.P., 2001. [§484 *notes.*]

Phelps R.R. [66] *Lectures on Choquet's Theorem.* van Nostrand, 1966. [§461 *notes.*]

Plewik S. & Voigt B. [91] 'Partitions of reals: a measurable approach', J. Combinatorial Theory (A) 58 (1991) 136-140. [443Yl.]

Pontryagin L.S. [1934] 'The theory of topological commutative groups', Ann. Math. 35 (1934) 361-388. [445U.]

Preiss D. [73] 'Metric spaces in which Prohorov's theorem is not valid', Z. Wahrscheinlichkeitstheorie und verw. Gebiete 27 (1973) 109-116. [439S.]

Preiss D. & Tišer J. [91] 'Measures in Banach spaces are determined by their values on balls', Mathematika 38 (1991) 391-397. [§466 *notes.*]

Pryce J.D. [71] 'A device of R.J.Whitley's applied to pointwise compactness in spaces of continuous functions', Proc. London Math. Soc. (3) 23 (1971) 532-546. [462B, 462C.]

Radon J. [1913] 'Theorie und Anwendungen der absolut additivien Mengenfunktionen,' Sitzungsbereichen der Kais. Akad. der Wiss. in Wien 122 (1913) 28-33. [§416 *notes.*]

Ramachandran D. [02] 'Perfect measures and related topics', pp. 765-786 in Pap 02. [§451 *notes.*]

Rao B.V. [69] 'On discrete Borel spaces and projective hierarchies', Bull. Amer. Math. Soc. 75 (1969) 614-617. [419F.]

Rao B.V. & Srivastava S.M. [94] 'Borel isomorphism theorem', privately circulated, 1994. [424C.]

Rao M. [77] *Brownian motion and classical potential theory.* Aarhus Universitet Matematisk Institut, 1977 (Lecture Notes Series 47); `http://www1.essex.ac.uk/maths/people/fremlin/rao.htm`.

Ressel P. [77] 'Some continuity and measurability results on spaces of measures', Math. Scand. 40 (1977) 69-78. [417C, 437M.]

Riesz F. & Sz.-Nagy B. [55] *Functional Analysis.* Frederick Ungar, 1955. [§494 *notes*.]

Rogers C.A. [70] *Hausdorff Measures.* Cambridge, 1970. [§471 *notes*.]

Rogers C.A. [80] (ed.) *Analytic Sets.* Academic, 1980. [Chap. 42 *intro*., §422 *notes*, §423 *notes*.]

Rogers L.C.J. & Williams D. [94] *Diffusions, Markov Processes and Martingales,* vol. I. Wiley, 1994. [§455 *notes*.]

Rokhlin V.A. [1948] 'A "general" measure-preserving transformation is not mixing', Doklady Akademii Nauk SSSR 60 (1948) 349-351. [494E.]

Rosendal C. [09] 'The generic isometry and measure preserving homeomorphism are conjugate to their powers', Fundamenta Math. 205 (2009) 1-27. [494Ye.]

Rosendal C. & Solecki S. [07] 'Automatic continuity of homomorphisms and fixed points on metric compacta', Israel J. Math. 162 (2007) 349-371. [494Z, §494 *notes*.]

Rudin M.E. [71] 'A normal space X for which $X \times I$ is not normal', Fund. Math. 73 (1971) 179-186. [439O.]

Rudin M.E. [84] 'Dowker spaces', pp. 761-780 in Kunen & Vaughan 84. [439O.]

Rudin W. [67] *Fourier Analysis on Groups.* Wiley, 1967 and 1990. [§445 *intro*.]

Rudin W. [91] *Functional Analysis.* McGraw-Hill, 1991. [§4A4, §4A6.]

Ryll-Nardzewski C. [53] 'On quasi-compact measures', Fundamenta Math. 40 (1953) 125-130. [451C.]

Sazonov V.V. [66] 'On perfect measures', A.M.S. Translations (2) 48 (1966) 229-254. [451F.]

Schachermayer W. [77] 'Eberlein-compacts et espaces de Radon', C.R.A.S. (Paris) (A) 284 (1977) 405-407. [467P.]

Schaefer H.H. [71] *Topological Vector Spaces.* Springer, 1971. [§4A4.]

Schwartz L. [73] *Radon Measures on Arbitrary Topological Spaces and Cylindrical Measures.* Oxford U.P., 1973. [§434 *notes*.]

Shelah S. & Fremlin D.H. [93] 'Pointwise compact and stable sets of measurable functions', J. Symbolic Logic 58 (1993) 435-455. [§465 *notes*.]

Sierpiński W. [1945] 'Sur une suite infinie de fonctions de classe 1 dont toute fonction d'accumulation est non mesurable', Fundamenta Math. 33 (1945) 104-105. [464C.]

Solecki S. [01] 'Haar null and non-dominating sets', Fundamenta Math. 170 (2001) 197-217. [444Ye.]

Solymosi J. [03] 'Note on a generalization of Roth's theorem', pp. 825-827 in Aronov Basu Pach & Sharir 03. [497M.]

Souslin M. [1917] 'Sur une définition des ensembles mesurables B sans nombres infinis', C.R.Acad.Sci. (Paris) 164 (1917) 88-91. [421D, §421 *notes*.]

Steen L.A. & Seebach J.A. [78] *Counterexamples in Topology.* Springer, 1978. [434Ya, §439 *notes*.]

Steinlage R.C. [75] 'On Haar measure in locally compact T_2 spaces', Amer. J. Math. 97 (1975) 291-307. [441C.]

Strassen V. [65] 'The existence of probability measures with given marginals', Ann. Math. Statist. 36 (1965) 423-439. [457D.]

Sullivan J.M. [94] 'Sphere packings give an explicit bound for the Besicovitch covering theorem', J. Geometric Analysis 4 (1994) 219-231. [§472 *notes*.]

Świerczkowski S. [58] 'On a free group of rotations of the Euclidean space', Indagationes Math. 20 (1958) 376-378. [449Yi.]

Szemerédi, E. [75] 'On sets of integers containing no k elements in arithmetic progression', Acta Arithmetica 27 (1975) 199-245. [497L.]

Talagrand M. [75] 'Sur une conjecture de H.H.Corson', Bull. Sci. Math. 99 (1975) 211-212. [467M.]

Talagrand M. [78a] 'Sur un théorème de L.Schwartz', C.R.Acad. Sci. Paris 286 (1978) 265-267. [466I.]

Talagrand M. [78b] 'Comparaison des boreliens pour les topologies fortes et faibles', Indiana Univ. Math. J. 21 (1978) 1001-1004. [466Za.]

Talagrand M. [80] 'Compacts de fonctions mesurables et filtres non-mesurables', Studia Math. 67 (1980) 13-43. [464C, 464D.]

Talagrand M. [81] 'La τ-régularité de mesures gaussiennes', Z. Wahrscheinlichkeitstheorie und verw. Gebiete 57 (1981) 213-221. [§456 *intro*., 456O.]

Talagrand M. [82] 'Closed convex hull of set of measurable functions, Riemann-measurable functions and measurability of translations', Ann. Insitut Fourier (Grenoble) 32 (1982) 39-69. [465M.]

Talagrand M. [84] *Pettis integral and measure theory.* Mem. Amer. Math. Soc. 307 (1984). [463Ya, 463Yd, 463Za, §465 *intro.*, 465B, 465N, 465R-465T, 466I, §466 *notes*.]

Talagrand M. [87] 'The Glivenko-Cantelli problem', Ann. of Probability 15 (1987) 837-870. [§465 *intro.*, 465L, 465M, *465Yd*, §465 *notes*.]

Talagrand M. [88] 'On liftings and the regularization of stochastic processes', Prob. Theory and Related Fields 78 (1988) 127-134. [465U.]

Talagrand M. [89] 'A "regular oscillation" property of stable sets of measurable functions', Contemporary Math. 94 (1989) 309-313. [§465 *notes*.]

Talagrand M. [95] 'Concentration of measure and isoperimetric inequalities in product spaces', Publ. Math. Inst. Hautes Études Scientifiques 81 (1995) 73-205. [492D.]

Talagrand M. [96] 'The Glivenko-Cantelli problem, ten years later', J. Theoretical Probability 9 (1996) 371-384. [§465 *notes*.]

Tamanini I. & Giacomelli C. [89] 'Approximation of Caccioppoli sets, with applications to problems in image segmentation', Ann. Univ. Ferrara, Sez. VII (N.S.) 35 (1989) 187-213. [484B.]

Tao T. [07] 'A correspondence principle between (hyper)graph theory and probability theory, and the (hyper)graph removal lemma', J. d'Analyse Math. 103 (2007) 1-45' [459I, §497.]

Tao T. [l08] *Topics in Ergodic Theory,* lecture notes. `http://en.wordpress.com/tag/254a-ergodic-theory`, 2008. [494F.]

Taylor A.E. [64] *Introduction to Functional Analysis.* Wiley, 1964. [§4A4.]

Taylor S.J. [53] 'The Hausdorff α-dimensional measure of Brownian paths in n-space', Proc. Cambridge Philosophical Soc. 49 (1953) 31-39. [477L.]

Topsøe F. [70a] 'Compactness in spaces of measures', Studia Math. 36 (1970) 195-222. [413J, 416K.]

Topsøe F. [70b] *Topology and Measure.* Springer, 1970 (Lecture Notes in Mathematics 133). [437J.]

Törnquist A. [11] 'On the pointwise implementation of near-actions', Trans. Amer. Math. Soc. 363 (2011) 4929-4944. [425D, 425Ya.]

Ulam S. [1930] 'Zur Masstheorie in der allgemeinen Mengenlehre', Fund. Math. 16 (1930) 140-150. [419G, 438C.]

Uspenskiĭ V.V. [88] 'Why compact groups are dyadic', pp. 601-610 in Frolík 88. [4A5T.]

Vasershtein L.N. [69] 'Markov processes over denumerable products of spaces describing large systems of automata', Problems of Information Transmission 5 (1969) 47-52. [457L.]

Veech W.A. [71] 'Some questions of uniform distribution', Annals of Math. (2) 94 (1971) 125-138. [491H.]

Veech W.A. [77] 'Topological dynamics', Bull. Amer. Math. Soc. 83 (1977) 775-830. [493H.]

Винокуров В.Г. & Махкамов Б.М. [73] 'О пространствах с совершенной, но не компактной мерой', Научые записки Ташкентского института народного хозяйства 71 (1973) 97-103. [451U.]

Wage M.L. [80] 'Products of Radon spaces', Russian Math. Surveys 35:3 (1980) 185-187. [§434 *notes*.]

Wagon S. [85] *The Banach-Tarski Paradox.* Cambridge U.P., 1985. [449Yj.]

Weil A. [1940] *L'intégration dans les groupes topologiques et ses applications,* Hermann, 1940 (Actualités Scientifiques et Industrielles 869). [445N.]

Wheeler R.F. [83] 'A survey of Baire measures and strict topologies', Expositiones Math. 1 (1983) 97-190. [§435 *notes*, §437 *notes*.]