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Chapter 46

Pointwise compact sets of measurable functions

This chapter collects results inspired by problems in functional analysis. §§461 and 466 look directly at measures on linear topological spaces. The primary applications are of course to Banach spaces, but as usual we quickly find ourselves considering weak topologies. In §461 I look at 'barycenters', or centres of mass, of probability measures, with the basic theorems on existence and location of barycenters of given measures and the construction of measures with given barycenters. In §466 I examine topological measures on linear spaces in terms of the classification developed in Chapter 41. A special class of normed spaces, those with 'Kadec norms', is particularly important, and in §467 I sketch the theory of the most interesting Kadec norms, the 'locally uniformly rotund' norms.

In the middle sections of the chapter, I give an account of the theory of pointwise compact sets of measurable functions, as developed by A.Bellow, M.Talagrand and myself. The first step is to examine pointwise compact sets of continuous functions (§462); these have been extensively studied because they represent an effective tool for investigating weakly compact sets in Banach spaces, but here I give only results which are important in measure theory, with a little background material. In §463 I present results on the relationship between the two most important topologies on spaces of measurable functions, *not* identifying functions which are equal almost everywhere: the pointwise topology and the topology of convergence in measure. These topologies have very different natures but nevertheless interact in striking ways. In particular, we have important theorems giving conditions under which a pointwise compact set of measurable functions will be compact for the topology of convergence in measure (463G, 463L).

The remaining two sections are devoted to some remarkable ideas due to Talagrand. The first, 'Talagrand's measure' (§464), is a special measure on $\mathcal{P}I$ (or $\ell^\infty(I)$), extending the usual measure of $\mathcal{P}I$ in a canonical way. In §465 I turn to the theory of 'stable' sets of measurable functions, showing how a concept arising naturally in the theory of pointwise compact sets led to a characterization of Glivenko-Cantelli classes in the theory of empirical measures.

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461 Barycenters and Choquet's theorem

One of the themes of this chapter will be the theory of measures on linear spaces, and the first fundamental concept is that of 'barycenter' of a measure, its centre of mass (461Aa). The elementary theory (461B-461E) uses non-trivial results from the theory of locally convex spaces, but is otherwise natural and straightforward. It is not always easy to be sure whether a measure has a barycenter in a given space, and I give a representative pair of results in this direction (461F, 461H). Deeper questions concern the existence and nature of measures on a given compact set with a given barycenter. The Riesz representation theorem is enough to tell us just which points can be barycenters of measures on compact sets (461I). A new idea (461K-461L) shows that the measures can be moved out towards the boundary of the compact set. We need a precise definition of 'boundary'; the set of extreme points seems to be the appropriate concept (461M). In some important cases, such representing measures on boundaries are unique (461P). I append a result identifying the extreme points of a particular class of compact convex sets of measures (461Q-461R).

461A Definitions (a) Let X be a Hausdorff locally convex linear topological space, and μ a probability measure on a subset A of X. Then $x^* \in X$ is a **barycenter** of μ if $\int_A g\,d\mu$ is defined and equal to $g(x^*)$ for every $g \in X^*$. μ can have at most one barycenter.

(b) Let X be any linear space over $\mathbb{R}$, and $C \subseteq X$ a convex set. Then a function $f : C \to \mathbb{R}$ is **convex** if $f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$ for all x, $y \in C$ and $t \in [0,1]$.

(c) Let X be a linear space over $\mathbb{R}$, $C \subseteq X$ a convex set, and $f : C \to \mathbb{R}$ a function. Then f is convex iff the set $\{(x,\alpha) : x \in C,\, \alpha \ge f(x)\}$ is convex in $X \times \mathbb{R}$.

461B Proposition Let X and Y be Hausdorff locally convex linear topological spaces, and $T : X \to Y$ a continuous linear operator. Suppose that $A \subseteq X$, $B \subseteq Y$ are such that $T[A] \subseteq B$, and let μ be a probability measure on A which has a barycenter x^* in X. Then Tx^* is the barycenter of the image measure μT^{-1} on B.

461C Lemma Let X be a Hausdorff locally convex linear topological space, C a convex subset of X, and $f : C \to \mathbb{R}$ a lower semi-continuous convex function. If $x \in C$ and $\gamma < f(x)$, there is a $g \in X^*$ such that $g(y) + \gamma - g(x) \le f(y)$ for every $y \in C$.

461D Theorem Let X be a Hausdorff locally convex linear topological space, $C \subseteq X$ a convex set and μ a probability measure on a subset A of C. Suppose that μ has a barycenter x^* in X which belongs to C. Then $f(x^*) \le \underline{\int}_A f\,d\mu$ for every lower semi-continuous convex function $f : C \to \mathbb{R}$.

461E Theorem Let X be a Hausdorff locally convex linear topological space, and μ a probability measure on X such that (i) the domain of μ includes the cylindrical σ-algebra of X (ii) there is a compact convex set $K \subseteq X$ such that $\mu^*K = 1$. Then μ has a barycenter in X, which belongs to K.

461F Theorem Let X be a complete locally convex linear topological space, and $A \subseteq X$ a bounded set. Let μ be a τ-additive topological probability measure on A. Then μ has a barycenter in X.

461G Lemma Let X be a normed space, and μ a probability measure on X such that every member of the dual X^* of X is integrable. Then $g \mapsto \int g\,d\mu : X^* \to \mathbb{R}$ is a bounded linear functional on X^*.

461H Proposition Let X be a reflexive Banach space, and μ a probability measure on X such that every member of X^* is μ-integrable. Then μ has a barycenter in X.

461I Theorem Let X be a Hausdorff locally convex linear topological space, and $K \subseteq X$ a compact set. Then the closed convex hull of K in X is just the set of barycenters of Radon probability measures on K.

461J Corollary: Kreĭn's theorem Let X be a complete Hausdorff locally convex linear topological space, and $K \subseteq X$ a weakly compact set. Then the closed convex hull $\overline{\Gamma(K)}$ of K is weakly compact.

461K Lemma Let X be a Hausdorff locally convex linear topological space, K a compact convex subset of X, and P the set of Radon probability measures on K. Define a relation $\preccurlyeq$ on P by saying that $\mu \preccurlyeq \nu$ if $\int f d\mu \le \int f d\nu$ for every continuous convex function $f : K \to \mathbb{R}$.

(a) $\preccurlyeq$ is a partial order on P.

(b) If $\mu \preccurlyeq \nu$ then $\int f d\mu \le \int f d\nu$ for every lower semi-continuous convex function $f : K \to \mathbb{R}$.

(c) If $\mu \preccurlyeq \nu$ then μ and ν have the same barycenter.

(d) If we give P its narrow topology, then $\preccurlyeq$ is closed in $P \times P$.

(e) For every $\mu \in P$ there is a $\preccurlyeq$-maximal $\nu \in P$ such that $\mu \preccurlyeq \nu$.

461L Lemma Let X be a Hausdorff locally convex linear topological space, K a compact convex subset of X, and P the set of Radon probability measures on K. Suppose that $\mu \in P$ is maximal for the partial order $\preccurlyeq$ of 461K.

(a) $\mu(\frac{1}{2}(M_1 + M_2)) = 0$ whenever M_1, M_2 are disjoint closed convex subsets of K.

(b) $\mu F = 0$ whenever $F \subseteq K$ is a Baire set (for the subspace topology of K) not containing any extreme point of K.

461M Theorem Let X be a Hausdorff locally convex linear topological space, K a compact convex subset of X and E the set of extreme points of K. Let $x \in X$. Then there is a probability measure μ on E with barycenter x. If K is metrizable we can take μ to be a Radon measure.

461N Lemma Let X be a Hausdorff locally convex linear topological space, K a compact convex subset of X, and P the set of Radon probability measures on K. Let E be the set of extreme points of K and suppose that $\mu \in P$ and $\mu^*E = 1$. Then μ is maximal in P for the partial order $\preccurlyeq$ of 461K.

461O Lemma Suppose that X is a Riesz space with a Hausdorff locally convex linear space topology, and $K \subseteq X$ a compact convex set such that every non-zero member of the positive cone X^+ is uniquely expressible as αx for some $x \in K$ and $\alpha \ge 0$. Let P be the set of Radon probability measures on K and $\preccurlyeq$ the partial order described in 461K. If μ, $\nu \in P$ have the same barycenter then they have a common upper bound in P.

461P Theorem Suppose that X is a Riesz space with a Hausdorff locally convex linear space topology, and $K \subseteq X$ a metrizable compact convex set such that every non-zero member of the positive cone X^+ is uniquely expressible as αx for some $x \in K$ and $\alpha \ge 0$. Let E be the set of extreme points of K, and x any point of K. Then there is a unique Radon probability measure μ on E such that x is the barycenter of μ.

461Q Proposition (a) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a sequentially order-continuous Boolean homomorphism. Let M_σ be the L-space of countably additive real-valued functionals on $\mathfrak{A}$, and Q the set

$$\{\nu : \nu \in M_\sigma,\ \nu \ge 0,\ \nu 1 = 1,\ \nu\pi = \nu\}.$$

If $\nu \in Q$, then the following are equiveridical: (i) ν is an extreme point of Q; (ii) $\nu a \in \{0, 1\}$ whenever $\pi a = a$; (iii) $\nu a \in \{0, 1\}$ whenever $a \in \mathfrak{A}$ is such that $\nu(a \triangle \pi a) = 0$.

(b) Let X be a set, Σ a σ-algebra of subsets of X, and $\phi : X \to X$ a (Σ, Σ)-measurable function. Let M_σ be the L-space of countably additive real-valued functionals on Σ, and $Q \subseteq M_\sigma$ the set of probability measures with domain Σ for which ϕ is inverse-measure-preserving. If $\mu \in Q$, then μ is an extreme point of Q iff ϕ is ergodic with respect to μ.

461R Corollary Let X be a compact Hausdorff space and $\phi : X \to X$ a continuous function. Let Q be the non-empty compact convex set of Radon probability measures μ on X such that ϕ is inverse-measure-preserving for μ, with its narrow topology and the convex structure defined by 234G and 234Xf. Then the extreme points of Q are those for which ϕ is ergodic.

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462 Pointwise compact sets of continuous functions

In preparation for the main work of this chapter, beginning in the next section, I offer a few pages on spaces of continuous functions under their 'pointwise' topologies (462Ab). There is an extensive general theory of such spaces, described in Arkhangel'skii 92; here I present only those fragments which seem directly relevant to the theory of measures on normed spaces and spaces of functions. In particular, I star the paragraphs 462C-462D, which are topology and functional analysis rather than measure theory. They are here because although this material is well known, and may be found in many places, I think that the ideas, as well as the results, are essential for any understanding of measures on linear topological spaces.

Measure theory enters the section in the proof of 462E, in the form of an application of the Riesz representation theorem, though 462E itself remains visibly part of functional analysis. In the rest of the section, however, we come to results which are pure measure theory. For (countably) compact spaces X, the Radon measures on $C(X)$ are the same for the pointwise and norm topologies (462I). This fact has extensive implications for the theory of separately continuous functions (462K) and for the theory of convex hulls in linear topological spaces (462L).

462A Definitions (a) A regular Hausdorff space X is **angelic** if whenever A is a subset of X which is relatively countably compact in X, then (i) its closure $\overline{A}$ is compact (ii) every point of $\overline{A}$ is the limit of a sequence in A.

(b) If X is any set and A a subset of $\mathbb{R}^X$, then the topology of **pointwise convergence** on A is that inherited from the usual product topology of $\mathbb{R}^X$. I shall commonly use the symbol $\mathfrak{T}_p$ for such a topology. In this context, I will say that a sequence or filter is **pointwise convergent** if it is convergent for the topology of pointwise convergence. Note that if A is a linear subspace of $\mathbb{R}^X$ then $\mathfrak{T}_p$ is a linear space topology on A.

***462B Proposition** Let $(X, \mathfrak{T})$ be an angelic regular Hausdorff space.
(a) Any subspace of X is angelic.
(b) If $\mathfrak{S}$ is a regular topology on X finer than $\mathfrak{T}$, then $\mathfrak{S}$ is angelic.
(c) Any countably compact subset of X is compact and sequentially compact.

***462C Theorem** Let X be a topological space such that there is a sequence $\langle X_n \rangle_{n\in\mathbb{N}}$ of relatively countably compact subsets of X, covering X, with the property that a function $f : X \to \mathbb{R}$ is continuous whenever $f{\restriction}X_n$ is continuous for every $n \in \mathbb{N}$. Then the space $C(X)$ of continuous real-valued functions on X is angelic in its topology of pointwise convergence.

***462D Theorem** Let U be any normed space. Then it is angelic in its weak topology.

462E Theorem Let X be a locally compact Hausdorff space, and $C_0(X)$ the Banach lattice of continuous real-valued functions on X which vanish at infinity. Write $\mathfrak{T}_p$ for the topology of pointwise convergence on $C_0(X)$.
(i) $C_0(X)$ is $\mathfrak{T}_p$-angelic.
(ii) A sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ in $C_0(X)$ is weakly convergent to $u \in C_0(X)$ iff it is $\mathfrak{T}_p$-convergent to u and norm-bounded.
(iii) A subset K of $C_0(X)$ is weakly compact iff it is norm-bounded and $\mathfrak{T}_p$-countably compact.

462F Lemma Let X be a topological space, and Q a relatively countably compact subset of X. Suppose that $K \subseteq C_b(X)$ is $\| \|_\infty$-bounded and $\mathfrak{T}_p$-countably compact, where $\mathfrak{T}_p$ is the topology of pointwise convergence on $C_b(X)$. Then the map $u \mapsto u{\restriction}Q : K \to C_b(Q)$ is continuous for $\mathfrak{T}_p$ on K and the weak topology of the Banach space $C_b(Q)$.

462G Proposition Let X be a countably compact topological space. Then a subset of $C_b(X)$ is weakly compact iff it is norm-bounded and compact for the topology of pointwise convergence.

462H Lemma Let X be a topological space, Q a relatively countably compact subset of X, and μ a totally finite measure on $C_b(X)$ which is Radon for the topology $\mathfrak{T}_p$ of pointwise convergence on $C_b(X)$. Let $T : C_b(X) \to C_b(Q)$ be the restriction map. Then the image measure $\nu = \mu T^{-1}$ on $C_b(Q)$ is Radon for the norm topology of $C_b(Q)$.

462I Theorem Let X be a countably compact topological space. Then the totally finite Radon measures on $C(X)$ are the same for the topology of pointwise convergence and the norm topology.

462J Corollary Let X be a countably compact Hausdorff space, and give $C(X)$ its topology of pointwise convergence. If μ is any Radon measure on $C(X)$, it is inner regular with respect to the family of compact metrizable subsets of $C(X)$.

462K Proposition Let X be a topological space, Y a Hausdorff space, $f : X \times Y \to \mathbb{R}$ a bounded separately continuous function, and ν a totally finite Radon measure on Y. Set $\phi(x) = \int f(x, y)\nu(dy)$ for every $x \in X$. Then $\phi{\restriction}Q$ is continuous for every relatively countably compact set $Q \subseteq X$.

462L Corollary Let X be a topological space such that

whenever $h \in \mathbb{R}^X$ is such that $h \restriction Q$ is continuous for every relatively countably compact $Q \subseteq X$, then h is continuous.

Write $\mathfrak{T}_p$ for the topology of pointwise convergence on $C(X)$. Let $K \subseteq C(X)$ be a $\mathfrak{T}_p$-compact set such that $\{h(x) : h \in K, x \in Q\}$ is bounded for any relatively countably compact set $Q \subseteq X$. Then the $\mathfrak{T}_p$-closed convex hull of K, taken in $C(X)$, is $\mathfrak{T}_p$-compact.

462Z Problem Let K be a compact Hausdorff space. Is $C(K)$, with the topology of pointwise convergence, necessarily a pre-Radon space?

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463 $\mathfrak{T}_p$ and $\mathfrak{T}_m$

We are now ready to start on the central ideas of this chapter with an investigation of sets of measurable functions which are compact for the topology of pointwise convergence. Because 'measurability' is, from the point of view of this topology on $\mathbb{R}^X$, a rather arbitrary condition, we are looking at compact subsets of a topologically irregular subspace of $\mathbb{R}^X$; there are consequently relatively few of them, and (under a variety of special circumstances, to be examined later in the chapter and also in Volume 5) they have some striking special properties.

The presentation here is focused on the relationship between the two natural topologies on any space of measurable functions, the 'pointwise' topology $\mathfrak{T}_p$ and the topology $\mathfrak{T}_m$ of convergence in measure (463A). In this section I begin with results which apply to any σ-finite measure space (463B-463H) before turning to some which apply to perfect measure spaces (463I-463L) – in particular, to Lebesgue measure. These lead to some interesting properties of separately continuous functions (463M-463N).

463A Preliminaries Let (X, Σ, μ) be a measure space, and $\mathcal{L}^0$ the space of all Σ-measurable functions from X to $\mathbb{R}$. On $\mathcal{L}^0$ we shall be concerned with two topologies. The first is the topology $\mathfrak{T}_p$ of pointwise convergence; the second is the topology $\mathfrak{T}_m$ of convergence in measure. Both are linear space topologies. $\mathfrak{T}_p$ is Hausdorff and locally convex.

Associated with the topology of pointwise convergence on $\mathbb{R}^X$ is the usual topology of $\mathcal{P}X$; the map $\chi : \mathcal{P}X \to \mathbb{R}^X$ is a homeomorphism between $\mathcal{P}X$ and its image $\{0,1\}^X \subseteq \mathbb{R}^X$.

A subset of $\mathcal{L}^0$ is open for $\mathfrak{T}_m$ iff it is of the form $\{f : f^\bullet \in G\}$ for some open set $G \subseteq L^0$; a subset K of $\mathcal{L}^0$ is compact, or separable, for $\mathfrak{T}_m$ iff $\{f^\bullet : f \in K\}$ is compact or separable for the topology of convergence in measure on L^0.

463B Lemma Let (X, Σ, μ) be a measure space, and $\mathcal{L}^0$ the space of Σ-measurable real-valued functions on X. Then every pointwise convergent sequence in $\mathcal{L}^0$ is convergent in measure to the same limit.

463C Proposition Let (X, Σ, μ) be a measure space, and $\mathcal{L}^0$ the space of Σ-measurable real-valued functions on X. Write $\mathfrak{T}_p$, $\mathfrak{T}_m$ for the topologies of pointwise convergence and convergence in measure on $\mathcal{L}^0$; for $A \subseteq \mathcal{L}^0$, write $\mathfrak{T}_p^{(A)}$, $\mathfrak{T}_m^{(A)}$ for the corresponding subspace topologies.

(a) If $A \subseteq \mathcal{L}^0$ and $\mathfrak{T}_p^{(A)}$ is metrizable, then the identity map from A to itself is $(\mathfrak{T}_p^{(A)}, \mathfrak{T}_m^{(A)})$-continuous.

(b) Suppose that μ is semi-finite. Then, for any $A \subseteq \mathcal{L}^0$, $\mathfrak{T}_m^{(A)}$ is Hausdorff iff whenever f, g are distinct members of A the set $\{x : f(x) \neq g(x)\}$ is non-negligible.

(c) Suppose that $K \subseteq \mathcal{L}^0$ is such that $\mathfrak{T}_p^{(K)}$ is compact and metrizable. Then $\mathfrak{T}_p^{(K)} = \mathfrak{T}_m^{(K)}$ iff $\mathfrak{T}_m^{(K)}$ is Hausdorff.

(d) Suppose that μ is σ-finite, and that $K \subseteq \mathcal{L}^0$ is $\mathfrak{T}_p$-sequentially compact. Then $\mathfrak{T}_p^{(K)} = \mathfrak{T}_m^{(K)}$ iff $\mathfrak{T}_m^{(K)}$ is Hausdorff, and in this case $\mathfrak{T}_p^{(K)}$ is compact and metrizable.

(e) Suppose that $K \subseteq \mathcal{L}^0$ is such that $\mathfrak{T}_p^{(K)}$ is compact and metrizable. Then whenever $\epsilon > 0$ and $E \in \Sigma$ is a non-negligible measurable set, there is a non-negligible measurable set $F \subseteq E$ such that $|f(x) - f(y)| \leq \epsilon$ whenever $f \in K$ and x, $y \in F$.

463D Lemma Let (X, Σ, μ) be a measure space, and $\mathcal{L}^0$ the space of Σ-measurable real-valued functions on X. Write $\mathfrak{T}_p$ for the topology of pointwise convergence on $\mathcal{L}^0$. Suppose that $K \subseteq \mathcal{L}^0$ is $\mathfrak{T}_p$-compact and that there is no $\mathfrak{T}_p$-continuous surjection from any closed subset of K onto $\{0,1\}^{\omega_1}$. If $E \in \Sigma$ has finite measure, then every sequence in K has a subsequence which is convergent almost everywhere in E.

463E Proposition Let (X, Σ, μ) be a measure space, and $\mathcal{L}^0$ the space of Σ-measurable real-valued functions on X. Write $\mathfrak{T}_p$, $\mathfrak{T}_m$ for the topologies of pointwise convergence and convergence in measure on $\mathcal{L}^0$. Suppose that $K \subseteq \mathcal{L}^0$ is $\mathfrak{T}_p$-compact and that there is no $\mathfrak{T}_p$-continuous surjection from any closed subset of K onto ω_1+1 with its order topology. Then the identity map from $(K, \mathfrak{T}_p)$ to $(K, \mathfrak{T}_m)$ is continuous.

463F Corollary Let (X, Σ, μ) be a measure space, and $\mathcal{L}^0$ the space of Σ-measurable real-valued functions on X. Write $\mathfrak{T}_p$, $\mathfrak{T}_m$ for the topologies of pointwise convergence and convergence in measure on $\mathcal{L}^0$. Suppose that $K \subseteq \mathcal{L}^0$ is compact and countably tight for $\mathfrak{T}_p$. Then the identity map from $(K, \mathfrak{T}_p)$ to $(K, \mathfrak{T}_m)$ is continuous. If $\mathfrak{T}_m$ is Hausdorff on K, the two topologies coincide on K.

463G Theorem Let (X, Σ, μ) be a σ-finite measure space, and K a convex set of measurable functions from X to $\mathbb{R}$ such that (i) K is compact for the topology $\mathfrak{T}_p$ of pointwise convergence (ii) $\{x : f(x) \neq g(x)\}$ is not negligible for any distinct f, $g \in K$. Then K is metrizable for $\mathfrak{T}_p$, which agrees with the topology of convergence in measure on K.

463H Corollary Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a σ-finite topological measure space in which μ is strictly positive. Suppose that

whenever $h \in \mathbb{R}^X$ is such that $h \restriction Q$ is continuous for every relatively countably compact $Q \subseteq X$, then h is continuous.

If $K \subseteq C_b(X)$ is a norm-bounded $\mathfrak{T}_p$-compact set, then it is $\mathfrak{T}_p$-metrizable.

463I Lemma Let (X, Σ, μ) be a perfect probability space, and $\langle E_n \rangle_{n \in \mathbb{N}}$ a sequence in Σ. Suppose that there is an $\epsilon > 0$ such that

$$\epsilon \mu F \leq \liminf_{n \to \infty} \mu(F \cap E_n) \leq \limsup_{n \to \infty} \mu(F \cap E_n) \leq (1 - \epsilon)\mu F$$

for every $F \in \Sigma$. Then $\langle E_n \rangle_{n \in \mathbb{N}}$ has a subsequence $\langle E_{n_k} \rangle_{k \in \mathbb{N}}$ such that $\mu_* A = 0$ and $\mu^* A = 1$ for any cluster point A of $\langle E_{n_k} \rangle_{k \in \mathbb{N}}$ in $\mathcal{P}X$; in particular, $\langle E_{n_k} \rangle_{k \in \mathbb{N}}$ has no measurable cluster point.

463J Lemma Let (X, Σ, μ) be a perfect probability space, and $\langle E_n \rangle_{n \in \mathbb{N}}$ a sequence in Σ. Then

either $\langle \chi E_n \rangle_{n \in \mathbb{N}}$ has a subsequence which is convergent almost everywhere

or $\langle E_n \rangle_{n \in \mathbb{N}}$ has a subsequence with no measurable cluster point in $\mathcal{P}X$.

463K Fremlin's Alternative Let (X, Σ, μ) be a perfect σ-finite measure space, and $\langle f_n \rangle_{n \in \mathbb{N}}$ a sequence of real-valued measurable functions on X. Then

either $\langle f_n \rangle_{n \in \mathbb{N}}$ has a subsequence which is convergent almost everywhere

or $\langle f_n \rangle_{n \in \mathbb{N}}$ has a subsequence with no measurable cluster point in $\mathbb{R}^X$.

463L Corollary Let (X, Σ, μ) be a perfect σ-finite measure space. Write $\mathcal{L}^0 \subseteq \mathbb{R}^X$ for the space of real-valued Σ-measurable functions on X.

(a) If $K \subseteq \mathcal{L}^0$ is relatively countably compact for the topology $\mathfrak{T}_p$ of pointwise convergence on $\mathcal{L}^0$, then every sequence in K has a subsequence which is convergent almost everywhere. Consequently K is relatively compact in $\mathcal{L}^0$ for the topology $\mathfrak{T}_m$ of convergence in measure.

(b) If $K \subseteq \mathcal{L}^0$ is countably compact for $\mathfrak{T}_p$, then it is compact for $\mathfrak{T}_m$.

(c) Suppose that $K \subseteq \mathcal{L}^0$ is countably compact for $\mathfrak{T}_p$ and that $\mu\{x : f(x) \neq g(x)\} > 0$ for any distinct f, $g \in K$. Then the topologies $\mathfrak{T}_m$ and $\mathfrak{T}_p$ agree on K, so both are compact and metrizable.

463M Proposition Let $X_0, \ldots, X_n$ be countably compact topological spaces, each carrying a σ-finite perfect strictly positive measure which measures every Baire set. Let X be their product and $\mathcal{B}\mathfrak{a}(X_i)$ the Baire σ-algebra of X_i for each i. Then any separately continuous function $f : X \to \mathbb{R}$ is measurable with respect to the σ-algebra $\widehat{\bigotimes}_{i \leq n} \mathcal{B}\mathfrak{a}(X_i)$ generated by $\{\prod_{i \leq n} E_i : E_i \in \mathcal{B}\mathfrak{a}(X_i) \text{ for } i \leq n\}$.

463N Corollary Let $X_0, \ldots, X_n$ be Hausdorff spaces with product X. Then every separately continuous function $f : X \to \mathbb{R}$ is universally Radon-measurable.

463Z Problems (a) A.Bellow's problem Let (X, Σ, μ) be a probability space, and $K \subseteq \mathcal{L}^0$ a $\mathfrak{T}_p$-compact set such that $\{x : f(x) \neq g(x)\}$ is non-negligible for any distinct functions f, $g \in K$, as in 463G and 463Lc. Does it follow that K is metrizable for $\mathfrak{T}_p$?

(b) Let $X \subseteq [0, 1]$ be a set of outer Lebesgue measure 1, and μ the subspace measure on X, with Σ its domain. Let K be a $\mathfrak{T}_p$-compact subset of $\mathcal{L}^0$. Must K be $\mathfrak{T}_m$-compact?

(c) Let $X_0, \ldots, X_n$ be compact Hausdorff spaces and $f : X_0 \times \ldots \times X_n \to \mathbb{R}$ a separately continuous function. Must f be universally measurable?

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464 Talagrand's measure

An obvious question arising from 463I and its corollaries is, do we really need the hypothesis that the measure involved is perfect? A very remarkable construction by M.Talagrand (464D) shows that these results are certainly not true of all probability spaces (464E). Investigating the properties of this measure we are led to some surprising facts about additive functionals on algebras $\mathcal{P}I$ and the duals of ℓ^∞ spaces (464M, 464R).

464A The usual measure on $\mathcal{P}I$ Recall that for any set I we have a standard measure ν, a Radon measure for the usual topology on $\mathcal{P}I$.

(a) If $\langle I_j\rangle_{j\in J}$ is any partition of I, then ν can be identified with the product of the family $\langle \nu_j\rangle_{j\in J}$, where ν_j is the usual measure on $\mathcal{P}I_j$. It follows that if we have any family $\langle A_j\rangle_{j\in J}$ of subsets of $\mathcal{P}I$, and if for each j the set A_j is 'determined by coordinates in I_j' in the sense that, for $a \subseteq I$, $a \in A_j$ iff $a \cap I_j \in A_j$, then $\nu^*(\bigcap_{j\in J} A_j) = \prod_{j\in J} \nu^* A_j$.

(b) Similarly, if f_1, f_2 are non-negative real-valued functions on $\mathcal{P}I$, and if there are disjoint sets I_1, $I_2 \subseteq I$ such that $f_j(a) = f_j(a \cap I_j)$ for every $a \subseteq I$ and both j, then the upper integral $\overline{\int} f_1 + f_2\, d\nu$ is $\overline{\int} f_1 d\nu + \overline{\int} f_2 d\nu$.

(c) If $A \subseteq \mathcal{P}I$ is such that $b \in A$ whenever $a \in A$, $b \subseteq I$ and $a \triangle b$ is finite, then $\nu^* A$ must be either 0 or 1.

464B Lemma Let I be any set, and ν the usual measure on $\mathcal{P}I$.

(a)(i) There is a sequence $\langle m(n)\rangle_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that $\prod_{n=0}^\infty 1 - 2^{-m(n)} = \frac{1}{2}$.

(ii) Given such a sequence, write X for $\prod_{n\in\mathbb{N}}(\mathcal{P}I)^{m(n)}$, and let λ be the product measure on X. We have a function $\phi : X \to \mathcal{P}I$ defined by setting

$$\phi(\langle\langle a_{ni}\rangle_{i<m(n)}\rangle_{n\in\mathbb{N}}) = \bigcup_{n\in\mathbb{N}} \bigcap_{i<m(n)} a_{ni}$$

whenever $\langle\langle a_{ni}\rangle_{i<m(n)}\rangle_{n\in\mathbb{N}} \in X$. Now ϕ is inverse-measure-preserving for λ and ν.

(b) The map

$$(a, b, c) \mapsto (a \cap b) \cup (a \cap c) \cup (b \cap c) : (\mathcal{P}I)^3 \to \mathcal{P}I$$

is inverse-measure-preserving for the product measure on $(\mathcal{P}I)^3$.

464C Lemma Let I be any set, and let ν be the usual measure on $\mathcal{P}I$.

(a) If $\mathcal{F} \subseteq \mathcal{P}I$ is any filter containing every cofinite set, then $\nu_*\mathcal{F} = 0$ and $\nu^*\mathcal{F}$ is either 0 or 1. If $\mathcal{F}$ is a non-principal ultrafilter then $\nu^*\mathcal{F} = 1$.

(b) If $\langle \mathcal{F}_n\rangle_{n\in\mathbb{N}}$ is a sequence of filters on I, all of outer measure 1, then $\bigcap_{n\in\mathbb{N}} \mathcal{F}_n$ also has outer measure 1.

464D Construction Let I be any set, and ν the usual Radon measure on $\mathcal{P}I$, with T its domain. Let Σ be the set

$$\{E : E \subseteq \mathcal{P}I, \text{ there are a set } F \in \mathrm{T} \text{ and a filter } \mathcal{F} \text{ on } I \text{ such that } \nu^*\mathcal{F} = 1 \text{ and } E \cap \mathcal{F} = F \cap \mathcal{F}\}.$$

Then there is a unique extension of ν to a complete probability measure μ, with domain Σ, defined by saying that $\mu E = \nu F$ whenever $E \in \Sigma$, $F \in \mathrm{T}$ and there is a filter $\mathcal{F}$ on I such that $\nu^*\mathcal{F} = 1$ and $E \cap \mathcal{F} = F \cap \mathcal{F}$.

Definition This measure μ is **Talagrand's measure** on $\mathcal{P}I$.

464E Example If μ is Talagrand's measure on $X = \mathcal{P}\mathbb{N}$, and Σ its domain, then there is a set $K \subseteq \mathbb{R}^X$, consisting of Σ-measurable functions and separable and compact for the topology $\mathfrak{T}_p$ of pointwise convergence, such that K is not compact for the topology of convergence in measure.

464F The L-space $\ell^\infty(I)^*$ Let I be any set.

(a) $\ell^\infty(I)$ is an M-space, so $\ell^\infty(I)^\sim = \ell^\infty(I)^*$ is an L-space. We can identify $\ell^\infty(I)^*$ with the L-space M of bounded finitely additive functionals on $\mathcal{P}I$, matching any $f \in \ell^\infty(I)^*$ with the functional $a \mapsto f(\chi a) : \mathcal{P}I \to \mathbb{R}$ in M.

(b) Write M_τ for the band of completely additive functionals on $\mathcal{P}I$. M_τ is just the set of those $\theta \in M$ such that $\theta a = \sum_{t\in a}\theta\{t\}$ for every $a \subseteq I$, while $M_\tau^\perp$ is the set of those $\theta \in M$ such that $\theta\{t\} = 0$ for every $t \in I$.

Observe that if $\theta \in M_\tau^\perp$ and a, $b \subseteq I$ are such that $a \triangle b$ is finite, then $\theta a = \theta b$.

(c) If $\theta \in M^+ \setminus \{0\}$, then $\{a : a \subseteq I, \theta a = \theta I\}$ is a filter.

464G Lemma Let $\mathfrak{A}$ be any Boolean algebra. Write M for the L-space of bounded additive functionals on $\mathfrak{A}$, and M^+ for its positive cone. Suppose that $\Delta : M^+ \to [0, \infty[$ is a functional such that

(α) Δ is non-decreasing,

(β) $\Delta(\alpha\theta) = \alpha\Delta(\theta)$ whenever $\theta \in M^+$, $\alpha \geq 0$,

(γ) $\Delta(\theta_1 + \theta_2) = \Delta(\theta_1) + \Delta(\theta_2)$ whenever θ_1, $\theta_2 \in M^+$ are such that, for some $e \subseteq I$, $\theta_1(1 \setminus e) = \theta_2 e = 0$,

(δ) $|\Delta(\theta_1) - \Delta(\theta_2)| \leq \|\theta_1 - \theta_2\|$ for all θ_1, $\theta_2 \in M^+$.

Then there is a non-negative $h \in M^*$ extending Δ.

464H Lemma Let I be any set, and M the L-space of bounded additive functionals on $\mathcal{P}I$; let ν be the usual measure on $\mathcal{P}I$. For $\theta \in M^+$, set

$$\Delta(\theta) = \overline{\int}\theta\,d\nu.$$

(a) For every $\theta \in M^+$, $\frac{1}{2}\theta I \leq \Delta(\theta) \leq \theta I$.

(b) There is a non-negative $h \in M^*$ such that $h(\theta) = \Delta(\theta)$ for every $\theta \in M^+$.

(c) If $\theta \in (M_\tau^\perp)^+$, where $M_\tau \subseteq M$ is the band of completely additive functionals, then $\theta \leq \Delta(\theta)$ ν-a.e., and $\nu^*\{a : \alpha \leq \theta a \leq \Delta(\theta)\} = 1$ for every $\alpha < \Delta(\theta)$.

(d) Suppose that $\theta \in (M_\tau^\perp)^+$ and β, $\gamma \in [0, 1]$ are such that $\theta I = 1$ and $\beta\theta' I \leq \Delta(\theta') \leq \gamma\theta' I$ whenever $\theta' \leq \theta$ in M^+. Then, for any $\alpha < \beta$,

(i) for any finite set $K \subseteq \mathcal{P}I$, the set

$$\{a : a \subseteq I, \alpha\theta b \leq \theta(a \cap b) \leq \gamma\theta b \text{ for every } b \in K\}$$

has outer measure 1 in $\mathcal{P}I$;

(ii) if $\alpha \geq \frac{1}{2}$, the set

$$R = \{(a, b, c) : a, b, c \subseteq I, \theta((a \cap b) \cup (a \cap c) \cup (b \cap c)) \geq 2\alpha^2 + (1 - 2\alpha)\gamma^2\}$$

has outer measure 1 in $(\mathcal{P}I)^3$;

(iii) if $\alpha \geq \frac{1}{2}$, then $2\alpha^2 + (1-2\alpha)\gamma^2 \leq \gamma$.
(e) Any $\theta \in M^+$ can be expressed as $\theta_1 + \theta_2$ where $\Delta(\theta_1) = \frac{1}{2}\theta_1 I$ and $\Delta(\theta_2) = \theta_2 I$.
(f) Suppose that $0 \leq \theta' \leq \theta$ in M.
(i) If $\Delta(\theta) = \frac{1}{2}\theta I$, then $\Delta(\theta') = \frac{1}{2}\theta' I$.
(ii) If $\Delta(\theta) = \theta I$, then $\Delta(\theta') = \theta' I$.

464I Measurable and purely non-measurable functionals As before, let I be any set, ν the usual measure on $\mathcal{P}I$, T its domain, and M the L-space of bounded additive functionals on $\mathcal{P}I$. I say that $\theta \in M$ is **measurable** if it is T-measurable when regarded as a real-valued function on $\mathcal{P}I$, and **purely non-measurable** if $\{a : a \subseteq I, |\theta|(a) = |\theta|(I)\}$ has outer measure 1.

464J Examples Let I, ν and M be as in 464I.

(a) Any $\theta \in M_\tau$ is measurable.

(b) Let $\langle t_n \rangle_{n\in\mathbb{N}}$ be any sequence of distinct points in I. Then if we take any non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$, and set $\theta a = \lim_{n\to\mathcal{F}} \dfrac{1}{n}\#(\{i : i < n,\, t_i \in a\})$ for $a \subseteq I$, θ will be constant ν-almost everywhere, and measurable. $\theta \in M_\tau^\perp$.

(c) If $\mathcal{F}$ is any non-principal ultrafilter on I, and we set $\theta a = 1$ for $a \in \mathcal{F}$, 0 otherwise, then θ is an additive functional which is purely non-measurable.

464K The space M_{m}: Lemma Let I be any set, ν the usual measure on $\mathcal{P}I$, and M the L-space of bounded additive functionals on $\mathcal{P}I$. Write M_{m} for the set of measurable $\theta \in M$, M_τ for the space of completely additive functionals on $\mathcal{P}I$ and $\Delta(\theta) = \overline{\int}\theta\, d\nu$ for $\theta \in M^+$.
(a) If $\theta \in M_{\mathrm{m}} \cap M_\tau^\perp$ and $b \subseteq I$, then $\theta(a \cap b) = \frac{1}{2}\theta b$ for ν-almost every $a \subseteq I$.
(b) $|\theta| \in M_{\mathrm{m}}$ for every $\theta \in M_{\mathrm{m}}$.
(c) A functional $\theta \in M^+$ is measurable iff $\Delta(\theta) = \frac{1}{2}\theta I$.
(d) M_{m} is a solid linear subspace of M.

464L The space M_{pnm}: Lemma Let I be any set, ν the usual measure on $\mathcal{P}I$, and M the L-space of bounded additive functionals on $\mathcal{P}I$. This time, write M_{pnm} for the set of those members of M which are purely non-measurable.
(a) If $\theta \in M^+$, then θ is purely non-measurable iff $\Delta(\theta) = \theta I$.
(b) M_{pnm} is a solid linear subspace of M.

464M Theorem Let I be any set. Write M for the L-space of bounded finitely additive functionals on $\mathcal{P}I$, and M_{m}, M_{pnm} for the spaces of measurable and purely non-measurable functionals. Then M_{m} and M_{pnm} are complementary bands in M.

464N Corollary Let I be any set, and let μ be Talagrand's measure on $\mathcal{P}I$; write Σ for its domain. Then every bounded additive functional on $\mathcal{P}I$ is Σ-measurable.

464O Remark Since $M_\tau \subseteq M_{\mathrm{m}}$, we have a three-part band decomposition $M = M_\tau \oplus (M_{\mathrm{m}} \cap M_\tau^\perp) \oplus M_{\mathrm{pnm}}$.
(i) Functionals in M_τ are T-measurable, where T is the domain of ν, therefore Σ-measurable.
(ii) A functional θ in $M_{\mathrm{m}} \cap M_\tau^\perp$ is constant, with value $\frac{1}{2}\theta I$, ν-almost everywhere.
(iii) Finally, a functional $\theta \in M_{\mathrm{pnm}}$ is equal to θI μ-almost everywhere.
(iv) Thus any $\theta \in M_\tau^\perp = (M_{\mathrm{m}} \cap M_\tau^\perp) \oplus M_{\mathrm{pnm}}$ is constant μ-a.e. $\int \theta\, d\mu = \Delta(\theta)$ for every $\theta \geq 0$, so that if $h \in M^*$ is the linear functional of 464Hb, then $\int \theta\, d\mu = h(\theta)$ for every $\theta \in M$.

464P More on purely non-measurable functionals (a) We can discuss non-negative additive functionals on $\mathcal{P}I$ in terms of the Stone-Čech compactification βI of I. For any set $A \subseteq \beta I$ set $H_A = \{a : a \subseteq I,\, A \subseteq \widehat{a}\}$, where $\widehat{a} \subseteq \beta I$ is the open-and-closed set corresponding to $a \subseteq I$. If $A \neq \emptyset$, H_A is a filter on I.

Write $\mathcal{A}$ for the family of those sets $A \subseteq \beta I$ such that $\nu^* H_A = 1$, where ν is the usual measure on $\mathcal{P}I$. Then $\mathcal{A}$ is a σ-ideal. Note that if $A \in \mathcal{A}$ then $\overline{A} \in \mathcal{A}$. $\{z\} \in \mathcal{A}$ for every $z \in \beta I \setminus I$, while $\{t\} \notin \mathcal{A}$ for any $t \in I$.

We have a one-to-one correspondence between non-negative additive functionals θ on $\mathcal{P}I$ and Radon measures μ_θ on βI, defined by writing $\mu_\theta(\widehat{a}) = \theta a$ whenever $a \subseteq I$ and $\theta \in M^+$. Now suppose that θ is a non-negative additive functional on $\mathcal{P}I$. Then $\mathcal{F}_\theta = \{a : \theta a = \theta I\}$ is either $\mathcal{P}I$ or a filter on I. If we set $F_\theta = \bigcap\{\widehat{a} : a \in \mathcal{F}_\theta\}$, then $\mathcal{F}_\theta = H_{F_\theta}$.

$\theta \in M^+$ is purely non-measurable iff the support of the measure μ_θ belongs to $\mathcal{A}$.

(b) Since M is a set of real-valued functions on $\mathcal{P}I$, it has the corresponding topology $\mathfrak{T}_p$ of pointwise convergence as a subspace of $\mathbb{R}^{\mathcal{P}I}$. Now if $C \subseteq M_{\mathrm{pnm}}$ is countable, its $\mathfrak{T}_p$-closure $\overline{C}$ is included in M_{pnm}.

(c) If $\theta \in M$ is such that $\theta a = 0$ for every countable set $a \subseteq I$, then $\theta \in M_{\mathrm{pnm}}$.

In particular, if $\theta \in M_\sigma \cap M_\tau^\perp$, where M_σ is the space of countably additive functionals on $\mathcal{P}I$, then $\theta \in M_{\mathrm{pnm}}$.

In the language of (a) above, we have a closed set in βI, being $F = \beta I \setminus \bigcup\{\widehat{a} : a \in [I]^{\le\omega}\}$; and if θ is such that the support of μ_θ is included in F, then θ is purely non-measurable.

464Q More on measurable functionals (a) We know that M_{m} is a band in M, and that it includes the band M_τ. So it is natural to look at the band $M_{\mathrm{m}} \cap M_\tau^\perp$.

(b) If θ is any non-zero non-negative functional in $M_{\mathrm{m}} \cap M_\tau^\perp$, we can find a family $\langle a_\xi \rangle_{\xi<\omega_1}$ in $\mathcal{P}I$ which is independent in the sense that $\theta(\bigcap_{\xi\in K} a_\xi) = 2^{-\#(K)}\theta I$ for every non-empty finite $K \subseteq I$.

In terms of the associated measure μ_θ on βI, this means that μ_θ has Maharam type at least ω_1. If $\theta I = 1$, then $\langle (\widehat{a}_\xi)^\bullet \rangle_{\xi<\omega_1}$ is an uncountable stochastically independent family in the measure algebra of μ_θ,

Turning this round, we see that if λ is a Radon measure on βI, of countable Maharam type, and $\lambda I = 0$, then the corresponding functional on $\mathcal{P}I$ is purely non-measurable.

(c) If $\theta \in M_{\mathrm{m}} \cap M_\tau^\perp$, and $n \in \mathbb{N}$, then $\theta(a_0 \cap a_1 \cap \ldots \cap a_n) = 2^{-n-1}\theta I$ for ν^{n+1}-almost every $a_0, \ldots, a_n \subseteq I$, where ν^{n+1} is the product measure on $(\mathcal{P}I)^{n+1}$.

464R A note on $\ell^\infty(I)$ If we write $\tilde{\mu}$ for the image measure $\mu\chi^{-1}$ on $\ell^\infty(I)$, where μ is Talagrand's measure on $\mathcal{P}I$, and $\tilde{\Sigma}$ for the domain of $\tilde{\mu}$, then $\tilde{\Sigma}$ includes the cylindrical σ-algebra of $\ell^\infty(I)$.

M_τ corresponds to $\ell^\infty(I)^\times$. Any functional in $(\ell^\infty(I)^\times)^\perp$ will be $\tilde{\mu}$-almost constant.

464Z Problem Let I be an infinite set, and $\tilde{\mu}$ the image on $\ell^\infty(I)$ of Talagrand's measure. Is $\tilde{\mu}$ a topological measure for the weak topology of $\ell^\infty(I)$?

Version of 22.3.16

465 Stable sets

The structure of general pointwise compact sets of measurable functions is complex and elusive. One particular class of such sets, however, is relatively easy to describe, and has a variety of remarkable properties, some of them relevant to important questions arising in the theory of empirical measures. In this section I outline the theory of 'stable' sets of measurable functions from Talagrand 84 and Talagrand 87.

The first steps are straightforward enough. The definition of stable set (465B) is not obvious, but given this the basic properties of stable sets listed in 465C are natural and easy to check, and we come quickly to the fact that (for complete locally determined spaces) pointwise bounded stable sets are relatively pointwise compact sets of measurable functions (465D). A less transparent, but still fairly elementary, argument leads to the next reason for looking at stable sets: the topology of pointwise convergence on a stable set is finer than the topology of convergence in measure (465G).

At this point we come to a remarkable fact: a uniformly bounded set A of functions on a complete probability space is stable if and only if certain laws of large numbers apply 'nearly uniformly' on A. These laws are expressed in conditions (ii), (iv) and (v) of 465M. For singleton sets A, they can be thought of as versions of the strong law of large numbers described in §273. To get the full strength of 465M a further idea in this direction needs to be added, described in 465H here.

The theory of stable sets applies in the first place to sets of true functions. There is however a corresponding notion applicable in function spaces, which I explore briefly in 465O-465R. Finally, I mention the idea of 'R-stable' set (465S-465U), obtained by using τ-additive product measures instead of c.l.d. product measures in the definition.

465A Notation (a) If X is a set and Σ a σ-algebra of subsets of X, I will write $\mathcal{L}^0(\Sigma)$ for the space of Σ-measurable functions from X to $\mathbb{R}$; $\mathcal{L}^\infty(\Sigma)$ will be the space of bounded functions in $\mathcal{L}^0(\Sigma)$.

(b) I will identify $\mathbb{N}$ with the set of finite ordinals, so that a power X^n becomes identified with the set of functions from $\{0,\dots,n-1\}$ to X.

(c) If (X,Σ,μ) is any measure space, then for finite sets I I write μ^I for the product measure on X^I. If (X,Σ,μ) is a probability space, then for any set I μ^I is to be the product probability measure on X^I.

(d) If X is a set and Σ is an algebra of subsets of X, then for any set I write $\bigotimes_I \Sigma$ for the algebra of subsets of X^I generated by sets of the form $\{w : w(i) \in E\}$ where $i \in I$ and $E \in \Sigma$, and $\widehat{\bigotimes}_I \Sigma$ for the σ-algebra generated by $\bigotimes_I \Sigma$.

(e) If X is a set, $A \subseteq \mathbb{R}^X$, $E \subseteq X$, $\alpha < \beta$ in $\mathbb{R}$ and $k \ge 1$, write

$$D_k(A,E,\alpha,\beta) = \bigcup_{f\in A} \{w : w \in E^{2k}, f(w(2i)) \le \alpha,$$

$$f(w(2i+1)) \ge \beta \text{ for } i < k\}.$$

(f) If X is a set, $k \ge 1$, $u \in X^k$ and $v \in X^k$, then I will write $u\#v = (u(0),v(0),u(1),v(1),\dots,u(k-1),v(k-1)) \in X^{2k}$. Note that if (X,Σ,μ) is a measure space then $(u,v) \mapsto u\#v$ is an isomorphism between the c.l.d. product $(X^k,\mu^k)\times(X^k,\mu^k)$ and (X^{2k},μ^{2k}).

465B Definition Let (X,Σ,μ) be a semi-finite measure space. I say that a set $A \subseteq \mathbb{R}^X$ is **stable** if whenever $E \in \Sigma$, $0 < \mu E < \infty$ and $\alpha < \beta$ in $\mathbb{R}$, there is some $k \ge 1$ such that $(\mu^{2k})^* D_k(A,E,\alpha,\beta) < (\mu E)^{2k}$.

465C Proposition Let (X,Σ,μ) be a semi-finite measure space.

(a) Let $A \subseteq \mathbb{R}^X$ be a stable set.

(i) Any subset of A is stable.

(ii) $\overline{A}$, the closure of A in $\mathbb{R}^X$ for the topology of pointwise convergence, is stable.

(iii) $\gamma A = \{\gamma f : f \in A\}$ is stable, for any $\gamma \in \mathbb{R}$.

(iv) If $g \in \mathcal{L}^0 = \mathcal{L}^0(\Sigma)$, then $A + g = \{f + g : f \in A\}$ is stable.

(v) If $g \in \mathcal{L}^0$, then $A \times g = \{f \times g : f \in A\}$ is stable.

(vi) Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous non-decreasing function. Then $\{hf : f \in A\}$ is stable.

(b)(i) Suppose that $A \subseteq \mathbb{R}^X$, $E \in \Sigma$, $n \ge 1$ and $\alpha < \beta$ are such that $0 < \mu E < \infty$ and $(\mu^{2n})^* D_n(A,E,\alpha,\beta) < (\mu E)^{2n}$. Then

$$\lim_{k\to\infty} \frac{1}{(\mu E)^{2k}} (\mu^{2k})^* D_k(A,E,\alpha,\beta) = 0.$$

(ii) If A, $B \subseteq \mathbb{R}^X$ are stable, then $A \cup B$ is stable.

(iii) If $A \subseteq \mathcal{L}^0$ is finite it is stable.

(iv) If $A \subseteq \mathbb{R}^X$ is stable, so is $\{f^+ : f \in A\} \cup \{f^- : f \in A\}$.

(c) Let A be a subset of $\mathbb{R}^X$.

(i) If $\hat\mu$, $\tilde\mu$ are the completion and c.l.d. version of μ, then A is stable with respect to one of the measures μ, $\hat\mu$, $\tilde\mu$ iff it is stable with respect to the others.

(ii) Let ν be an indefinite-integral measure over μ. If A is stable with respect to μ, it is stable with respect to ν and with respect to $\nu\restriction\Sigma$.

(iii) If A is stable, and $Y \subseteq X$ is such that the subspace measure μ_Y is semi-finite, then $A_Y = \{f{\restriction}Y : f \in A\}$ is stable in $\mathbb{R}^Y$ with respect to the measure μ_Y.

(iv) A is stable iff $A_E = \{f{\restriction}E : f \in A\}$ is stable in $\mathbb{R}^E$ with respect to the subspace measure μ_E whenever $E \in \Sigma$ has finite measure.

(v) A is stable iff $A_n = \{\operatorname{med}(-n\chi X, f, n\chi X) : f \in A\}$ is stable for every $n \in \mathbb{N}$.

(d) Suppose that μ is σ-finite, (Y, T, ν) is another measure space and $\phi : Y \to X$ is inverse-measure-preserving. If $A \subseteq \mathbb{R}^X$ is stable with respect to μ, then $B = \{f\phi : f \in A\}$ is stable with respect to ν.

465D Proposition Let (X, Σ, μ) be a complete locally determined measure space, and $A \subseteq \mathbb{R}^X$ a stable set.

(a) $A \subseteq \mathcal{L}^0$.

(b) If $\{f(x) : f \in A\}$ is bounded for each $x \in X$, then A is relatively compact in $\mathcal{L}^0$ for the topology of pointwise convergence.

465E The topology $\mathfrak{T}_s(\mathcal{L}^2, \mathcal{L}^2)$ Let (X, Σ, μ) be any measure space. Then $L^2 = L^2(\mu)$ is a Hilbert space with a corresponding weak topology $\mathfrak{T}_s(L^2, L^2)$. In the present section it will be more convenient to regard this as a topology $\mathfrak{T}_s(\mathcal{L}^2, \mathcal{L}^2)$ on the space $\mathcal{L}^2 = \mathcal{L}^2(\mu)$ of square-integrable real-valued functions. The essential fact we need is that norm-bounded sets are relatively weakly compact.

465F Lemma Let (X, Σ, μ) be a measure space, and $B \subseteq \mathcal{L}^2 = \mathcal{L}^2(\mu)$ a $\| \|_2$-bounded set. Suppose that $h \in \mathcal{L}^2$ belongs to the closure of B for $\mathfrak{T}_s(\mathcal{L}^2, \mathcal{L}^2)$. Then for any $\delta > 0$ and $k \ge 1$ the set

$$W = \bigcup_{f \in B} \{w : w \in X^k, w(i) \in \operatorname{dom} f \cap \operatorname{dom} h$$
$$\text{and } f(w(i)) \ge h(w(i)) - \delta \text{ for every } i < k\}$$

is μ^k-conegligible in X^k.

465G Theorem Let (X, Σ, μ) be a semi-finite measure space, and $A \subseteq \mathcal{L}^0$ a stable set of measurable functions. Let $\mathfrak{T}_p$ and $\mathfrak{T}_m$ be the topologies of pointwise convergence and convergence in measure. Then the identity map from A to itself is $(\mathfrak{T}_p, \mathfrak{T}_m)$-continuous.

465H Theorem Let (X, Σ, μ) be any probability space. For $n \in \mathbb{N}$, write Λ_n for the domain of the product measure μ^n. For $w \in X^{\mathbb{N}}$, $k \ge 1$, $n \ge 1$ write ν_{wk} for the probability measure with domain $\mathcal{P}X$ defined by writing

$$\nu_{wk}(E) = \frac{1}{k}\#(\{i : i < k,\, w(i) \in E\})$$

for $E \subseteq X$, and ν_{wk}^n for the corresponding product measure on X^n.

Then whenever $n \ge 1$ and $f : X^n \to \mathbb{R}$ is bounded and Λ_n-measurable, $\lim_{k\to\infty} \int f d\nu_{wk}^n$ exists, and is equal to $\int f d\mu^n$, for $\mu^{\mathbb{N}}$-almost every $w \in X^{\mathbb{N}}$.

465I Lemma Let X be a set, and Σ a σ-algebra of subsets of X. For $w \in X^{\mathbb{N}}$ and $k \ge 1$, write ν_{wk} for the probability measure with domain $\mathcal{P}X$ defined by writing

$$\nu_{wk}(E) = \frac{1}{k}\#(\{i : i < k,\, w(i) \in E\})$$

for $E \subseteq X$. Then for any $k \in \mathbb{N}$ and any set I, $w \mapsto \nu_{wk}^I(W)$ is $\widehat{\bigotimes}_{\mathbb{N}}\Sigma$-measurable for every $W \in \widehat{\bigotimes}_I\Sigma$.

465J Lemma Let (X, Σ, μ) be a probability space. For any $n \in \mathbb{N}$ and $W \subseteq X^n$ I say that W is **symmetric** if $w\pi \in W$ whenever $w \in W$ and $\pi : n \to n$ is a permutation. For each n, write Λ_n for the domain of the product measure μ^n.

(a) Suppose that for each $n \ge 1$ we are given $W_n \in \Lambda_n$, and that $W_{m+n} \subseteq W_m \times W_n$ for all m, $n \ge 1$, identifying X^{m+n} with $X^m \times X^n$. Then $\lim_{n\to\infty}(\mu^n W_n)^{1/n}$ is defined and equal to $\delta = \inf_{n\ge1}(\mu^n W_n)^{1/n}$.

(b) Now suppose that each W_n is symmetric. Then there is an $E \in \Sigma$ such that $\mu E = \delta$ and $E^n \setminus W_n$ is negligible for every $n \in \mathbb{N}$.

(c) Next, let $\langle D_n \rangle_{n \geq 1}$ be a sequence of sets such that

$D_n \subseteq X^n$ is symmetric for every $n \geq 1$,

whenever $1 \leq m \leq n$ and $v \in D_n$ then $v \restriction m \in D_m$.

Then $\delta = \lim_{n\to\infty}((\mu^n)^* D_n)^{1/n}$ is defined and there is an $E \in \Sigma$ such that $\mu E = \delta$ and $(\mu^n)^*(D_n \cap E^n) = (\mu E)^n$ for every $n \in \mathbb{N}$.

465K Lemma Let (X, Σ, μ) be a complete probability space, and $A \subseteq [0,1]^X$ a stable set. Suppose that $\epsilon > 0$ is such that $\int f d\mu \leq \epsilon^2$ for every $f \in A$. Then there are an $n \geq 1$ and a $W \in \widehat{\bigotimes}_n \Sigma$ and a $\gamma > \mu^n W$ such that $\int f d\nu \leq 3\epsilon$ whenever $f \in A$ and ν is a probability measure on X with domain including Σ such that $\nu^n W \leq \gamma$.

465L Lemma (Talagrand 87) Let (X, Σ, μ) be a complete probability space, and $A \subseteq [0,1]^X$ a set which is not stable. Then there are measurable functions h_0, $h_1 : X \to [0,1]$ such that $\int h_0\, d\mu < \int h_1\, d\mu$ and $(\mu^{2k})^* \tilde{D}_k = 1$ for every $k \geq 1$, where

$$\tilde{D}_k = \bigcup_{f \in A} \{w : w \in X^{2k}, f(w(2i)) \leq h_0(w(2i)),$$
$$f(w(2i+1)) \geq h_1(w(2i+1)) \text{ for every } i < k\}.$$

465M Theorem Let (X, Σ, μ) be a complete probability space, and A a non-empty uniformly bounded set of real-valued functions defined on X. Then the following are equiveridical.

(i) A is stable.

(ii) Every function in A is measurable, and $\lim_{k\to\infty} \sup_{f\in A} |\frac{1}{k}\sum_{i=0}^{k-1} f(w(i)) - \int f| = 0$ for almost every $w \in X^{\mathbb{N}}$.

(iii) Every function in A is measurable, and for every $\epsilon > 0$ there are a finite subalgebra T of Σ in which every atom is non-negligible and a sequence $\langle h_k \rangle_{k\geq 1}$ of measurable functions on $X^{\mathbb{N}}$ such that

$$h_k(w) \geq \sup_{f\in A} \frac{1}{k}\sum_{i=0}^{k-1} \bigl|f(w(i)) - \mathbb{E}(f|\mathrm{T})(w(i))\bigr|$$

for every $w \in X^{\mathbb{N}}$ and $k \geq 1$, and

$$\limsup_{k\to\infty} h_k(w) \leq \epsilon$$

for almost every $w \in X^{\mathbb{N}}$. (Here $\mathbb{E}(f|\mathrm{T})$ is the conditional expectation of f on T.)

(iv) $\lim_{k,l\to\infty} \sup_{f\in A} |\frac{1}{k}\sum_{i=0}^{k-1} f(w(i)) - \frac{1}{l}\sum_{i=0}^{l-1} f(w(i))| = 0$ for almost every $w \in X^{\mathbb{N}}$.

(v) $\lim_{k,l\to\infty} \overline{\int} \sup_{f\in A} |\frac{1}{k}\sum_{i=0}^{k-1} f(w(i)) - \frac{1}{l}\sum_{i=0}^{l-1} f(w(i))| \mu^{\mathbb{N}}(dw) = 0$.

465N Theorem Let (X, Σ, μ) be a semi-finite measure space.

(a) Let $A \subseteq \mathbb{R}^X$ be a stable set. Suppose that there is a measurable function $g : X \to [0, \infty[$ such that $|f(x)| \leq g(x)$ whenever $x \in X$ and $f \in A$. Then the convex hull $\Gamma(A)$ of A in $\mathbb{R}^X$ is stable.

(b) If $A \subseteq \mathbb{R}^X$ is stable, then $|A| = \{|f| : f \in A\}$ is stable.

(c) Let A, $B \subseteq \mathbb{R}^X$ be two stable sets such that $\{f(x) : f \in A \cup B\}$ is bounded for every $x \in X$. Then $A + B = \{f_1 + f_2 : f_1 \in A,\ f_2 \in B\}$ is stable.

(d) Suppose that μ is complete and locally determined. Let $A \subseteq \mathbb{R}^X$ be a stable set such that $\{f(x) : f \in A\}$ is bounded for every $x \in X$. Then $\Gamma(A)$ is relatively compact in $\mathcal{L}^0(\Sigma)$ for the topology of pointwise convergence.

465O Stable sets in L^0 If $(\mathfrak{A},\bar{\mu})$ is a semi-finite measure algebra, and $k\ge 1$, I write $(\widehat{\bigotimes}_k\mathfrak{A},\bar{\mu}^k)$ for the localizable measure algebra free product of k copies of $(\mathfrak{A},\bar{\mu})$. If $Q\subseteq L^0(\mathfrak{A})$, $k\ge 1$, $a\in\mathfrak{A}$ has finite measure and $\alpha<\beta$ in $\mathbb{R}$, set

$$d_k(Q,a,\alpha,\beta)=\sup_{v\in Q}\big((a\cap[\![v\le\alpha]\!])\otimes(a\cap[\![v\ge\beta]\!])\otimes\ldots$$
$$\otimes(a\cap[\![v\le\alpha]\!])\otimes(a\cap[\![v\ge\beta]\!])\big)$$

in $\widehat{\bigotimes}_{2k}\mathfrak{A}$, taking k repetitions of the formula $(a\cap[\![v\le\alpha]\!])\otimes(a\cap[\![v\ge\beta]\!])$. Q is **stable** if whenever $0<\bar{\mu}a<\infty$ and $\alpha<\beta$ there is a $k\ge 1$ such that $\bar{\mu}^{2k}d_k(Q,a,\alpha,\beta)<(\bar{\mu}a)^{2k}$.

465P Theorem Let (X,Σ,μ) be a semi-finite measure space, with measure algebra $(\mathfrak{A},\bar{\mu})$.

(a) Suppose that $A\subseteq\mathcal{L}^0(\Sigma)$ and that $Q=\{f^\bullet:f\in A\}\subseteq L^0(\mu)$, identified with $L^0=L^0(\mathfrak{A})$. Then Q is stable iff every countable subset of A is stable.

(b) Suppose that μ is complete and strictly localizable and Q is a stable subset of $L^\infty(\mu)$, identified with $L^\infty(\mathfrak{A})$ (363I). Then there is a stable set $B\subseteq\mathcal{L}^\infty(\Sigma)$ such that $Q=\{f^\bullet:f\in B\}$.

465R Theorem Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be measure algebras, and $T:L^1(\mathfrak{A},\bar{\mu})\to L^1(\mathfrak{B},\bar{\nu})$ a bounded linear operator. If Q is stable and order-bounded in $L^1(\mathfrak{A},\bar{\mu})$, then $T[Q]\subseteq L^1(\mathfrak{B},\bar{\nu})$ is stable.

***465S R-stable sets** If $(X,\mathfrak{T},\Sigma,\mu)$ is a semi-finite τ-additive topological measure space such that μ is inner regular with respect to the Borel sets, write $\tilde{\mu}^I$ for the τ-additive product measure on X^I. $A\subseteq\mathbb{R}^X$ is **R-stable** if whenever $0<\mu E<\infty$ and $\alpha<\beta$ there is a $k\ge 1$ such that $(\tilde{\mu}^{2k})^*D_k(A,E,\alpha,\beta)<(\mu E)^{2k}$.

Because the τ-additive product measure extends the c.l.d. product measure, stable sets are always R-stable.

***465T Proposition** Let $(X,\mathfrak{T},\Sigma,\mu)$ be a semi-finite τ-additive topological measure space such that μ is inner regular with respect to the Borel sets. If $A\subseteq C(X)$ is such that every countable subset of A is R-stable, then A is R-stable.

***465U Example** There is a Radon probability space with an R-stable set of continuous functions which is not stable.

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466 Measures on linear topological spaces

In this section I collect a number of results on the special properties of topological measures on linear topological spaces. The most important is surely Phillips' theorem (466A-466B): on any Banach space, the weak and norm topologies give rise to the same totally finite Radon measures. This is not because the weak and norm topologies have the same Borel σ-algebras, though this does happen in interesting cases (466C-466E, §467). When the Borel σ-algebras are different, we can still ask whether the Borel measures are 'essentially' the same, that is, whether every (totally finite) Borel measure for the weak topology extends to a Borel measure for the norm topology. A construction due to M.Talagrand (466H, 466Ia) gives a negative answer to the general question.

Just as in $\mathbb{R}^r$, a totally finite quasi-Radon measure on a locally convex linear topological space is determined by its characteristic function (466K). I end the section with a note on measurability conditions sufficient to ensure that a linear operator between Banach spaces is continuous (466L-466M), and with brief remarks on Gaussian measures (466N-466O).

466A Theorem Let $(X,\mathfrak{T})$ be a metrizable locally convex linear topological space and μ a σ-finite measure on X which is quasi-Radon for the weak topology $\mathfrak{T}_s(X,X^*)$. Then the support of μ is separable, so μ is quasi-Radon for the original topology $\mathfrak{T}$. If X is complete and μ is locally finite with respect to $\mathfrak{T}$, then μ is Radon for $\mathfrak{T}$.

466B Corollary If X is a Banach space and μ is a totally finite measure on X which is quasi-Radon for the weak topology of X, it is a Radon measure for both the norm topology and the weak topology.

466C Definition A normed space X has a **Kadec norm** (also called **Kadec-Klee norm**) if the norm and weak topologies coincide on the sphere $\{x : \|x\| = 1\}$. Of course they will then also coincide on any sphere $\{x : \|x - y\| = \alpha\}$.

466D Proposition Let X be a normed space with a Kadec norm. Then there is a network for the norm topology on X expressible in the form $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$, where for each $n \in \mathbb{N}$ $\mathcal{V}_n$ is an isolated family for the weak topology and $\bigcup \mathcal{V}_n$ is the difference of two closed sets for the weak topology.

466E Corollary Let X be a normed space with a Kadec norm.
(a) The norm and weak topologies give rise to the same Borel σ-algebras.
(b) The weak topology has a σ-isolated network, so is hereditarily weakly θ-refinable.

466F Proposition Let X be a Banach space with a Kadec norm. Then the following are equiveridical:
(i) X is a Radon space in its norm topology;
(ii) X is a Radon space in its weak topology;
(iii) the weight of X (for the norm topology) is measure-free.

466G Definition A partially ordered set X has the **σ-interpolation property** if whenever A, B are non-empty countable subsets of X and $x \le y$ for every $x \in A$, $y \in B$, then there is a $z \in X$ such that $x \le z \le y$ for every $x \in A$ and $y \in B$.

466H Proposition Let X be a Riesz space with a Riesz norm, given its weak topology $\mathfrak{T}_s = \mathfrak{T}_s(X, X^*)$. Suppose that (α) X has the σ-interpolation property (β) there is a strictly increasing family $\langle p_\xi \rangle_{\xi<\omega_1}$ in X. Then there is a $\mathfrak{T}_s$-Borel probability measure μ on X such that
(i) μ is not inner regular with respect to the $\mathfrak{T}_s$-closed sets;
(ii) μ is not τ-additive for the topology $\mathfrak{T}_s$;
(iii) μ has no extension to a norm-Borel measure on X.
Accordingly $(X, \mathfrak{T}_s)$ is not a Radon space (indeed, is not Borel-measure-complete).

466I Examples The following spaces satisfy the hypotheses of 466H.

(a) $X = \ell^\infty(I)$ or $\{x : x \in \ell^\infty(I), \{i : x(i) \ne 0\}$ is countable$\}$, where I is uncountable.

(b) $X = \ell^\infty/\boldsymbol{c}_0$.

466J Theorem Let X be a linear topological space and Σ its cylindrical σ-algebra. If μ and ν are probability measures with domain Σ such that $\int e^{if(x)}\mu(dx) = \int e^{if(x)}\nu(dx)$ for every $f \in X^*$, then $\mu = \nu$.

466K Proposition If X is a locally convex linear topological space and μ, ν are quasi-Radon probability measures on X such that $\int e^{if(x)}\mu(dx) = \int e^{if(x)}\nu(dx)$ for every $f \in X^*$, then $\mu = \nu$.

466L Proposition Suppose that X and Y are Banach spaces and that $T : X \to Y$ is a linear operator such that $gT : X \to \mathbb{R}$ is universally Radon-measurable for every $g \in Y^*$. Then T is continuous.

466M Corollary If X is a Banach space, Y is a separable Banach space, and $T : X \to Y$ is a linear operator such that the graph of T is a Souslin-F set in $X \times Y$, then T is continuous.

466N Gaussian measures: Definition If X is a linear topological space, a probability measure μ on X is a **centered Gaussian measure** if its domain includes the cylindrical σ-algebra of X and every continuous linear functional on X is either zero almost everywhere or a normal random variable with zero expectation.

466O Proposition Let X be a separable Banach space, and μ a probability measure on X. Suppose that there is a linear subspace W of X^*, separating the points of X, such that every element of W is $\operatorname{dom}\mu$-measurable and either zero a.e. or a normal random variable with zero expectation. Then μ is a centered Gaussian measure with respect to the norm topology of X.

466Z Problems (a) Does every probability measure defined on the $\mathfrak{T}_s(\ell^\infty,(\ell^\infty)^*)$-Borel sets of ℓ^∞ extend to a measure defined on the $\| \|_\infty$-Borel sets?

(b) Assume that $\mathfrak{c}$ is measure-free. Does it follow that ℓ^∞, with its weak topology, is a Radon space?

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*467 Locally uniformly rotund norms

In the last section I mentioned Kadec norms. These are interesting in themselves, but the reason for including them in this book is that in a normed space with a Kadec norm the weak topology has the same Borel sets as the norm topology. The same will evidently be true of any space which has an equivalent Kadec norm. Now Kadec norms themselves are not uncommon, but equivalent Kadec norms appear in a striking variety of cases. Here I describe the principal class of spaces (the 'weakly K-countably determined' Banach spaces, 467H) which have equivalent Kadec norms. In fact they have 'locally uniformly rotund' norms, which are much easier to do calculations with.

Almost everything here is pure functional analysis, mostly taken from Deville Godefroy & Zizler 93, which is why I have starred the section. The word 'measure' does not appear until 467P. At that point, however, we find ourselves with a striking result (Schachermayer's theorem) which appears to need the structure theory of weakly compactly generated Banach spaces developed in 467C-467M.

467A Definition Let X be a linear space with a norm $\| \|$. $\| \|$ is **locally uniformly rotund** or **locally uniformly convex** if whenever $\|x\| = 1$ and $\epsilon > 0$, there is a $\delta > 0$ such that $\|x - y\| \le \epsilon$ whenever $\|y\| = 1$ and $\|x + y\| \ge 2 - \delta$.

If X has a locally uniformly rotund norm, then every subspace of X has a locally uniformly rotund norm.

467B Proposition A locally uniformly rotund norm is a Kadec norm.

467C A technical device (a) I will use the following notation for the rest of the section. Let X be a linear space and $p : X \to [0,\infty[$ a seminorm. Define $q_p : X \times X \to [0,\infty[$ by setting

$$q_p(x,y) = 2p(x)^2 + 2p(y)^2 - p(x+y)^2 = (p(x) - p(y))^2 + (p(x) + p(y))^2 - p(x+y)^2$$

for $x \in X$.

(b) A norm $\| \|$ on X is locally uniformly rotund iff whenever $x \in X$ and $\epsilon > 0$ there is a $\delta > 0$ such that $\|x - y\| \le \epsilon$ whenever $q_{\| \|}(x,y) \le \delta$.

(c) Let X be a linear space.

(i) For any seminorm p on X, $q_p(x,y) \ge (p(x) - p(y))^2 \ge 0$ for all x, $y \in X$.

(ii) Suppose that $\langle p_i\rangle_{i\in I}$ is a family of seminorms on X such that $\sum_{i\in I} p_i(x)^2$ is finite for every $x \in X$. Set $p(x) = \sqrt{\sum_{i\in I} p_i(x)^2}$ for $x \in X$; then p is a seminorm on X and $q_p = \sum_{i\in I} q_{p_i}$. $q_p \ge q_{p_i}$ for every $i \in I$.

(iii) If $\| \|$ is an inner product norm on X, then $q_{\| \|}(x,y) = \|x - y\|^2$ for all x, $y \in X$.

467D Lemma Let $(X, \| \|)$ be a normed space. Suppose that there are a space Y with a locally uniformly rotund norm $\| \|_Y$ and a bounded linear operator $T : Y \to X$ such that $T[Y]$ is dense in X and, for every $x \in X$ and $\gamma > 0$, there is a $z \in Y$ such that $\|x - Tz\|^2 + \gamma\|z\|_Y^2 = \inf_{y\in Y} \|x - Ty\|^2 + \gamma\|y\|_Y^2$. Then X has an equivalent locally uniformly rotund norm.

467E Theorem Let X be a separable normed space. Then it has an equivalent locally uniformly rotund norm.

467F Lemma Let $(X, \| \|)$ be a Banach space, and $\langle T_i \rangle_{i \in I}$ a family of bounded linear operators from X to itself such that

(i) for each $i \in I$, the subspace $T_i[X]$ has an equivalent locally uniformly rotund norm,

(ii) for each $x \in X$, $\epsilon > 0$ there is a finite set $J \subseteq I$ such that $\|x - \sum_{i \in J} T_i x\| \le \epsilon$,

(iii) for each $x \in X$, $\epsilon > 0$ the set $\{i : i \in I, \|T_i x\| \ge \epsilon\}$ is finite.

Then X has an equivalent locally uniformly rotund norm.

467G Theorem Let X be a Banach space. Suppose that there are an ordinal ζ and a family $\langle P_\xi \rangle_{\xi \le \zeta}$ of bounded linear operators from X to itself such that

(i) if $\xi \le \eta \le \zeta$ then $P_\xi P_\eta = P_\eta P_\xi = P_\xi$;

(ii) $P_0(x) = 0$ and $P_\zeta(x) = x$ for every $x \in X$;

(iii) if $\xi \le \zeta$ is a non-zero limit ordinal, then $\lim_{\eta \uparrow \xi} P_\eta(x) = P_\xi(x)$ for every $x \in X$;

(iv) if $\xi < \zeta$ then $X_\xi = \{(P_{\xi+1} - P_\xi)(x) : x \in X\}$ has an equivalent locally uniformly rotund norm.

Then X has an equivalent locally uniformly rotund norm.

Remark A family $\langle P_\xi \rangle_{\xi \le \zeta}$ satisfying (i), (ii) and (iii) here is called a **projectional resolution of the identity**.

467H Definitions (a) A topological space X is **K-countably determined** or a **Lindelöf-Σ** space if there are a subset A of $\mathbb{N}^{\mathbb{N}}$ and an usco-compact relation $R \subseteq A \times X$ such that $R[A] = X$.

(b) A normed space X is **weakly K-countably determined** if it is K-countably determined in its weak topology.

(c) Let X be a normed space and Y, W closed linear subspaces of X, X^* respectively. I will say that (Y, W) is a **projection pair** if $X = Y \oplus W^\circ$ and $\|y + z\| \ge \|y\|$ for every $y \in Y$, $z \in W^\circ$.

467I Lemma (a) If X is a weakly K-countably determined normed space, then any closed linear subspace of X is weakly K-countably determined.

(b) If X is a weakly K-countably determined normed space, Y is a normed space, and $T : X \to Y$ is a continuous linear surjection, then Y is weakly K-countably determined.

(c) If X is a Banach space and $Y \subseteq X$ is a dense linear subspace which is weakly K-countably determined, then X is weakly K-countably determined.

467J Lemma Let X be a weakly K-countably determined Banach space. Then there is a family $\mathcal{M}$ of subsets of $X \cup X^*$ such that

(i) whenever $B \subseteq X \cup X^*$ there is an $M \in \mathcal{M}$ such that $B \subseteq M$ and $\#(M) \le \max(\omega, \#(B))$;

(ii) whenever $\mathcal{M}' \subseteq \mathcal{M}$ is upwards-directed, then $\bigcup \mathcal{M}' \in \mathcal{M}$;

(iii) whenever $M \in \mathcal{M}$ then $(\overline{M \cap X}, \overline{M \cap X^*})$ is a projection pair of subspaces of X and X^*.

467K Theorem Let X be a weakly K-countably determined Banach space. Then it has an equivalent locally uniformly rotund norm.

467L Weakly compactly generated Banach spaces A normed space X is **weakly compactly generated** if there is a sequence $\langle K_n \rangle_{n \in \mathbb{N}}$ of weakly compact subsets of X such that $\bigcup_{n \in \mathbb{N}} K_n$ is dense in X.

467M Proposition A weakly compactly generated Banach space is weakly K-countably determined.

467N Theorem Let X be a Banach lattice with an order-continuous norm. Then it has an equivalent locally uniformly rotund norm.

467O Eberlein compacta: Definition A topological space K is an **Eberlein compactum** if it is homeomorphic to a weakly compact subset of a Banach space.

467P Proposition Let K be a compact Hausdorff space.

(a) The following are equiveridical:

(i) K is an Eberlein compactum;

(ii) there is a set $L \subseteq C(K)$, separating the points of K, which is compact for the topology of pointwise convergence.

(b) Suppose that K is an Eberlein compactum.

(i) K has a σ-isolated network, so is hereditarily weakly θ-refinable.

(ii) If $w(K)$ is measure-free, K is a Radon space.