

Chapter 46

Pointwise compact sets of measurable functions

This chapter collects results inspired by problems in functional analysis. §§461 and 466 look directly at measures on linear topological spaces. The primary applications are of course to Banach spaces, but as usual we quickly find ourselves considering weak topologies. In §461 I look at ‘barycenters’, or centres of mass, of probability measures, with the basic theorems on existence and location of barycenters of given measures and the construction of measures with given barycenters. In §466 I examine topological measures on linear spaces in terms of the classification developed in Chapter 41. A special class of normed spaces, those with ‘Kadec norms’, is particularly important, and in §467 I sketch the theory of the most interesting Kadec norms, the ‘locally uniformly rotund’ norms.

In the middle sections of the chapter, I give an account of the theory of pointwise compact sets of measurable functions, as developed by A.Bellow, M.Talagrand and myself. The first step is to examine pointwise compact sets of continuous functions (§462); these have been extensively studied because they represent an effective tool for investigating weakly compact sets in Banach spaces, but here I give only results which are important in measure theory, with a little background material. In §463 I present results on the relationship between the two most important topologies on spaces of measurable functions, *not* identifying functions which are equal almost everywhere: the pointwise topology and the topology of convergence in measure. These topologies have very different natures but nevertheless interact in striking ways. In particular, we have important theorems giving conditions under which a pointwise compact set of measurable functions will be compact for the topology of convergence in measure (463G, 463L).

The remaining two sections are devoted to some remarkable ideas due to Talagrand. The first, ‘Talagrand’s measure’ (§464), is a special measure on $\mathcal{P}I$ (or $\ell^\infty(I)$), extending the usual measure of $\mathcal{P}I$ in a canonical way. In §465 I turn to the theory of ‘stable’ sets of measurable functions, showing how a concept arising naturally in the theory of pointwise compact sets led to a characterization of Glivenko-Cantelli classes in the theory of empirical measures.

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461 Barycenters and Choquet’s theorem

One of the themes of this chapter will be the theory of measures on linear spaces, and the first fundamental concept is that of ‘barycenter’ of a measure, its centre of mass (461Aa). The elementary theory (461B-461E) uses non-trivial results from the theory of locally convex spaces (§4A4), but is otherwise natural and straightforward. It is not always easy to be sure whether a measure has a barycenter in a given space, and I give a representative pair of results in this direction (461F, 461H). Deeper questions concern the existence and nature of measures on a given compact set with a given barycenter. The Riesz representation theorem is enough to tell us just which points can be barycenters of measures on compact sets (461I). A new idea (461K-461L) shows that the measures can be moved out towards the boundary of the compact set. We need a precise definition of ‘boundary’; the set of extreme points seems to be the appropriate concept (461M). In some important cases, such representing measures on boundaries are unique (461P). I append a result identifying the extreme points of a particular class of compact convex sets of measures (461Q-461R).

461A Definitions (a) Let X be a Hausdorff locally convex linear topological space, and μ a probability measure on a subset A of X . Then $x^* \in X$ is a **barycenter** or **resultant** of μ if $\int_A g d\mu$ is defined and equal to $g(x^*)$ for every $g \in X^*$. Because X^* separates the points of X (4A4Ec), μ can have at most one barycenter, so we may speak of ‘the’ barycenter of μ .

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(b) Let X be any linear space over \mathbb{R} , and $C \subseteq X$ a convex set (definition: 2A5E). Then a function $f : C \rightarrow \mathbb{R}$ is **convex** if $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$ for all $x, y \in C$ and $t \in [0, 1]$. (Compare 233G, 233Xd.)

(c) The following elementary remark is useful. Let X be a linear space over \mathbb{R} , $C \subseteq X$ a convex set, and $f : C \rightarrow \mathbb{R}$ a function. Then f is convex iff the set $\{(x, \alpha) : x \in C, \alpha \geq f(x)\}$ is convex in $X \times \mathbb{R}$ (cf. 233Xd).

461B Proposition Let X and Y be Hausdorff locally convex linear topological spaces, and $T : X \rightarrow Y$ a continuous linear operator. Suppose that $A \subseteq X$, $B \subseteq Y$ are such that $T[A] \subseteq B$, and let μ be a probability measure on A which has a barycenter x^* in X . Then Tx^* is the barycenter of the image measure μT^{-1} on B .

proof All we have to observe is that if $g \in Y^*$ then $gT \in X^*$ (4A4Bd), so that

$$g(Tx^*) = \int_A g(Tx)\mu(dx) = \int_B g(y)\nu(dy)$$

by 235G¹.

461C Lemma Let X be a Hausdorff locally convex linear topological space, C a convex subset of X , and $f : C \rightarrow \mathbb{R}$ a lower semi-continuous convex function. If $x \in C$ and $\gamma < f(x)$, there is a $g \in X^*$ such that $g(y) + \gamma - g(x) \leq f(y)$ for every $y \in C$.

proof Let D be the convex set $\{(x, \alpha) : x \in C, \alpha \geq f(x)\}$ in $X \times \mathbb{R}$ (461Ac). Then the closure \bar{D} of D in $X \times \mathbb{R}$ is also convex (2A5Eb). Now D is closed in $C \times \mathbb{R}$ (4A2B(d-i)), and $(x, \gamma) \notin D$, so $(x, \gamma) \notin \bar{D}$.

Consequently there is a continuous linear functional $h : X \times \mathbb{R} \rightarrow \mathbb{R}$ such that $h(x, \gamma) < \inf_{w \in \bar{D}} h(w)$ (4A4Eb). Now there are $g_0 \in X^*$, $\beta \in \mathbb{R}$ such that $h(y, \alpha) = g_0(y) + \beta\alpha$ for every $y \in X$ and $\alpha \in \mathbb{R}$ (4A4Be). So we have

$$g_0(x) + \beta\gamma = h(x, \gamma) < h(y, f(y)) = g_0(y) + \beta f(y)$$

for every $y \in C$. In particular, $g_0(x) + \beta\gamma < g_0(x) + \beta f(x)$, so $\beta > 0$. Setting $g = -\frac{1}{\beta}g_0$,

$$f(y) \geq \frac{1}{\beta}g_0(x) + \gamma - \frac{1}{\beta}g_0(y) = g(y) + \gamma - g(x)$$

for every $y \in C$, as required.

461D Theorem Let X be a Hausdorff locally convex linear topological space, $C \subseteq X$ a convex set and μ a probability measure on a subset A of C . Suppose that μ has a barycenter x^* in X which belongs to C . Then $f(x^*) \leq \int_A f d\mu$ for every lower semi-continuous convex function $f : C \rightarrow \mathbb{R}$.

proof Take any $\gamma < f(x^*)$. By 461C there is a $g \in X^*$ such that $g(y) + \gamma - g(x^*) \leq f(y)$ for every $y \in C$. Integrating with respect to μ ,

$$\int_A f d\mu \geq \gamma - g(x^*) + \int_A g d\mu = \gamma.$$

As γ is arbitrary, $f(x^*) \leq \int_A f d\mu$.

Remark Of course $\int_A f d\mu$ might be infinite.

461E Theorem Let X be a Hausdorff locally convex linear topological space, and μ a probability measure on X such that (i) the domain of μ includes the cylindrical σ -algebra of X (ii) there is a compact convex set $K \subseteq X$ such that $\mu^*K = 1$. Then μ has a barycenter in X , which belongs to K .

proof If $g \in X^*$, then $\gamma_g = \sup_{x \in K} |g(x)|$ is finite, and $\{x : |g(x)| \leq \gamma_g\}$ is a measurable set including K , so must be conegligible, and $\phi(g) = \int g d\mu$ is defined and finite. Now $\phi : X^* \rightarrow \mathbb{R}$ is a linear functional and $\phi(g) \leq \sup_{x \in K} g(x)$ for every $g \in X^*$; because K is compact and convex, there is an $x_0 \in K$ such that $\phi(g) = g(x_0)$ for every $g \in X^*$ (4A4Ef), so that x_0 is the barycenter of μ in X .

¹Formerly 235I.

461F Theorem Let X be a complete locally convex linear topological space, and $A \subseteq X$ a bounded set. Let μ be a τ -additive topological probability measure on A . Then μ has a barycenter in X .

proof (a) If $g \in X^*$, then $g|_A$ is continuous and bounded (3A5N(b-v)), therefore μ -integrable. For each neighbourhood G of 0 in X , set

$$F_G = \{y : y \in X, |g(y) - \int_A g d\mu| \leq 2\tau_G(g) \text{ for every } g \in X^*\}$$

where $\tau_G(g) = \sup_{x \in G} g(x)$ for $g \in X^*$. Then F_G is non-empty. **P** Set $H = \text{int}(G \cap (-G))$, so that H is an open neighbourhood of 0. Because A is bounded, there is an $m \geq 1$ such that $A \subseteq mH$. The set $\{x+H : x \in A\}$ is an open cover of A , and μ is a τ -additive topological measure, so there are $x_0, \dots, x_n \in A$ such that $\mu(A \setminus \bigcup_{i \leq n} (x_i + H)) \leq \frac{1}{m}$. Set

$$E_i = A \cap (x_i + H) \setminus \bigcup_{j < i} (x_j + H)$$

for $i \leq n$, and $y = \sum_{i=0}^n (\mu E_i) x_i$; set $E = A \setminus \bigcup_{i \leq n} (x_i + H)$. Then, for any $g \in X^*$,

$$\begin{aligned} |g(y) - \int_A g d\mu| &\leq \sum_{i=0}^n |\mu E_i g(x_i) - \int_{E_i} g d\mu| + \int_E |g| d\mu \\ &\leq \sum_{i=0}^n \int_{E_i} |g(x) - g(x_i)| \mu(dx) + m\tau_G(g) \mu E \end{aligned}$$

(because $E \subseteq mH$, so $g(x) \leq m\tau_G(g)$ and $g(-x) \leq m\tau_G(g)$ for every $x \in E$)

$$\leq \sum_{i=0}^n \tau_G(g) \mu E_i + m\tau_G(g) \mu E$$

(because if $i \leq n$ and $x \in E_i$, then $x - x_i$ and $x_i - x$ belong to G , so $|g(x) - g(x_i)| = |g(x - x_i)| \leq \tau_G(g)$)

$$\leq 2\tau_G(g).$$

As g is arbitrary, $y \in F_G$ and $F_G \neq \emptyset$. **Q**

(c) Since $F_{G \cap H} \subseteq F_G \cap F_H$ for all neighbourhoods G and H of 0, $\{F_G : G \text{ is a neighbourhood of } 0\}$ is a filter base and generates a filter \mathcal{F} on X . Now \mathcal{F} is Cauchy. **P** If G is any neighbourhood of 0, let $G_1 \subseteq G$ be a closed convex neighbourhood of 0, and set $H = \frac{1}{4}G_1$. **?** If $y, y' \in F_H$ and $y - y' \notin G$, then there is a $g \in X^*$ such that $g(y - y') > \tau_{G_1}(g)$ (4A4Eb again). But now

$$\begin{aligned} \tau_H(g) &= \frac{1}{4}\tau_{G_1}(g) < \frac{1}{4}(|g(y) - \int_A g d\mu| + |g(y') - \int_A g d\mu|) \\ &\leq \frac{1}{4}(2\tau_H(g) + 2\tau_H(g)). \quad \mathbf{X} \end{aligned}$$

This means that $F_H - F_H \subseteq G$; as G is arbitrary, \mathcal{F} is Cauchy. **Q**

(d) Because X is complete, \mathcal{F} has a limit x^* say. Take any $g \in X^*$. **?** If $g(x^*) \neq \int_A g d\mu$, set $G = \{x : |g(x)| \leq \frac{1}{3}|g(x^*) - \int_A g d\mu|\}$. Then G is a neighbourhood of 0 in X , and

$$\begin{aligned} 0 < |g(x^*) - \int_A g d\mu| &= \lim_{x \rightarrow \mathcal{F}} |g(x) - \int_A g d\mu| \\ &\leq \sup_{x \in F_G} |g(x) - \int_A g d\mu| \leq 2\tau_G(g) \leq \frac{2}{3}|g(x^*) - \int_A g d\mu|. \quad \mathbf{X} \end{aligned}$$

So $g(x^*) = \int_A g d\mu$; as g is arbitrary, x^* is the barycenter of μ .

461G Lemma Let X be a normed space, and μ a probability measure on X such that every member of the dual X^* of X is integrable. Then $g \mapsto \int g d\mu : X^* \rightarrow \mathbb{R}$ is a bounded linear functional on X^* .

proof Replacing μ by its completion if necessary, we may suppose that μ is complete, so that every member of X^* is Σ -measurable, where Σ is the domain of μ . (The point is that μ and its completion give rise to

the same integrals, by 212Fb.) Set $\phi(g) = \int g d\mu$ for $g \in X^*$. For each $n \in \mathbb{N}$ let E_n be a measurable envelope of $B_n = \{x : \|x\| \leq n\}$; replacing E_n by $\bigcap_{i \geq n} E_i$ if necessary, we may suppose that $\langle E_n \rangle_{n \in \mathbb{N}}$ is non-decreasing. If $n \in \mathbb{N}$ and $g \in X^*$ then $\{x : x \in E_n, |g(x)| > n\|g\|\}$ is a measurable subset of E_n disjoint from B_n , so must be negligible, and $|\int_{E_n} g| \leq n\|g\|$. We therefore have an element ϕ_n of X^{**} defined by setting $\phi_n(g) = \int_{E_n} g$ for every $g \in X^*$. But also $\phi(g) = \lim_{n \rightarrow \infty} \phi_n(g)$ for every g , because $\langle E_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence of measurable sets with union X . By the Uniform Boundedness Theorem (3A5Ha), $\{\phi_n : n \in \mathbb{N}\}$ is bounded in X^{**} , and $\phi \in X^{**}$.

461H Proposition Let X be a reflexive Banach space, and μ a probability measure on X such that every member of X^* is μ -integrable. Then μ has a barycenter in X .

proof By 461G, $g \mapsto \int g d\mu$ is a bounded linear functional on X^* ; but this means that it is represented by a member of X , which is the barycenter of μ .

461I Theorem Let X be a Hausdorff locally convex linear topological space, and $K \subseteq X$ a compact set. Then the closed convex hull of K in X is just the set of barycenters of Radon probability measures on K .

proof (a) If μ is a Radon probability measure on K with barycenter x^* , then

$$g(x^*) = \int_K g(x) \mu(dx) \leq \sup_{x \in K} g(x) \leq \sup_{x \in \overline{\Gamma(K)}} g(x)$$

for every $g \in X^*$; because $\overline{\Gamma(K)}$ is closed and convex, it must contain x^* (4A4Eb once more).

(b) Now suppose that $x^* \in \overline{\Gamma(K)}$. Let $W \subseteq C(X)$ be the set of functionals of the form $g + \alpha \chi X$, where $g \in X^*$ and $\alpha \in \mathbb{R}$. Set $U = \{g \upharpoonright K : g \in W\}$, so that U is a linear subspace of $C(K)$ containing χK .

If $g_1, g_2 \in W$ and $g_1 \upharpoonright K = g_2 \upharpoonright K$, then $\{x : g_1(x) = g_2(x)\}$ is a closed convex set including K , so contains x^* , and $g_1(x^*) = g_2(x^*)$; accordingly we have a functional $\phi : U \rightarrow \mathbb{R}$ defined by setting $\phi(g \upharpoonright K) = g(x^*)$ for every $g \in W$. Of course ϕ is linear; moreover, $\phi(f) \leq \sup_{x \in K} f(x)$ for every $f \in U$, by 4A4Eb yet again. Applying this to $\pm f$, we see that $|\phi(f)| \leq \|f\|_\infty$ for every $f \in U$. We therefore have an extension of ϕ to a continuous linear functional ψ on $C(K)$ such that $\|\psi\| \leq 1$, by the Hahn-Banach theorem (3A5Ab). Now

$$\psi(\chi K) = \phi(\chi K) = \chi X(x^*) = 1;$$

so if $0 \leq f \leq \chi K$ then

$$|1 - \psi(f)| = |\psi(\chi K - f)| \leq \|\chi K - f\|_\infty \leq 1,$$

and $\psi(f) \geq 0$. It follows that $\psi(f) \geq 0$ for every $f \in C(K)^+$. But this means that there is a Radon probability measure μ on K such that $\psi(f) = \int f d\mu$ for every $f \in C(K)$ (436J/436K). As $\mu K = \psi(\chi K) = 1$, μ is a probability measure; and for any $g \in X^*$

$$\int_K g d\mu = \psi(g \upharpoonright K) = \phi(g \upharpoonright K) = g(x^*),$$

so x^* is the barycenter of μ , as required.

461J Corollary: Krein's theorem Let X be a complete Hausdorff locally convex linear topological space, and $K \subseteq X$ a weakly compact set. Then the closed convex hull $\overline{\Gamma(K)}$ of K is weakly compact.

proof Give K the weak topology induced by $\mathfrak{T}_s(X, X^*)$. Let P be the set of Radon probability measures on K , so that P is compact in its narrow topology (437R(f-ii)). By 461F, every $\mu \in P$ has a barycenter $b(\mu)$ in K . If $g \in X^*$, $g(b(\mu)) = \int_K g d\mu$, while $g \upharpoonright K$ is continuous, so $\mu \mapsto \int_K g d\mu$ is continuous, by 437Kc. Accordingly $b : P \rightarrow X$ is continuous for the narrow topology on P and the weak topology on X , and $b[P]$ is weakly compact. But $b[P]$ is the weakly closed convex hull of K , by 461I applied to the weak topology on X . By 4A4Ed, $\Gamma(K)$ has the same closure for the original topology of X as it has for the weak topology, and $\overline{\Gamma(K)} = b[P]$ is weakly compact.

461K Lemma Let X be a Hausdorff locally convex linear topological space, K a compact convex subset of X , and P the set of Radon probability measures on K . Define a relation \preceq on P by saying that $\mu \preceq \nu$ if $\int f d\mu \leq \int f d\nu$ for every continuous convex function $f : K \rightarrow \mathbb{R}$.

- (a) \preceq is a partial order on P .
- (b) If $\mu \preceq \nu$ then $\int f d\mu \leq \int f d\nu$ for every lower semi-continuous convex function $f : K \rightarrow \mathbb{R}$.
- (c) If $\mu \preceq \nu$ then μ and ν have the same barycenter.
- (d) If we give P its narrow topology, then \preceq is closed in $P \times P$.
- (e) For every $\mu \in P$ there is a \preceq -maximal $\nu \in P$ such that $\mu \preceq \nu$.

proof (a) Write Ψ for the set of continuous convex functions from K to \mathbb{R} . Note that if $f, g \in \Psi$ and $\alpha \geq 0$ then $\alpha f, f + g$ and $f \vee g$ all belong to Ψ . Consequently $\Psi - \Psi$ is a Riesz subspace of $C(K)$. **P** $\Psi - \Psi$ is a linear subspace because Ψ is closed under addition and multiplication by positive scalars. If $f, g \in \Psi$ then

$$|f - g| = (f - g) \vee (g - f) = 2(f \vee g) - (f + g) \in \Psi;$$

by 352Ic, $\Psi - \Psi$ is a Riesz subspace. **Q**

It follows that $\Psi - \Psi$ is $\|\cdot\|_\infty$ -dense in $C(K)$. **P** Constant functions belong to Ψ , and if $x, y \in K$ are distinct there is an $f \in X^*$ such that $f(x) \neq f(y)$, in which case $f \upharpoonright K$ belongs to Ψ and separates y from x . By the Stone-Weierstrass theorem (281A), $\Psi - \Psi$ is dense. **Q**

The definition of \preceq makes it plain that it is reflexive and transitive. But it is also antisymmetric. **P** If $\mu \preceq \nu$ and $\nu \preceq \mu$, then $\int f d\mu = \int f d\nu$ for every $f \in \Psi$, therefore for every $f \in \Psi - \Psi$, therefore for every $f \in C(K)$, and $\mu = \nu$ by 416E(b-v). **Q**

So \preceq is a partial order.

(b)(i) Now suppose that $f : K \rightarrow \mathbb{R}$ is a lower semi-continuous convex function, and $x \in K$. Then $f(x) = \sup\{g(x) : g \in \Psi, g \leq f\}$. **P** If $\gamma < f(x)$ there is a $g \in X^*$ such that $g(y) + \gamma - g(x) \leq f(y)$ for every $y \in K$, by 461C. Now $g \upharpoonright K$ belongs to Ψ , $g \upharpoonright K \leq f$ and $(g \upharpoonright K)(x) = \gamma$. **Q**

(ii) It follows that if $f : K \rightarrow \mathbb{R}$ is lower semi-continuous and convex, $\int f d\mu = \sup\{\int g d\mu : g \in \Psi, g \leq f\}$ for every $\mu \in P$. **P** Because Ψ is closed under \vee , $A = \{g : g \in \Psi, g \leq f\}$ is upwards-directed. Because μ is τ -additive and $f = \sup A$, $\int f d\mu = \sup_{g \in A} \int g d\mu$ by 414Ab. **Q**

So if $\mu, \nu \in P$ and $\mu \preceq \nu$, then

$$\int f d\mu = \sup_{g \in \Psi, g \leq f} \int g d\mu \leq \sup_{g \in \Psi, g \leq f} \int g d\nu = \int f d\nu;$$

as f is arbitrary, (b) is true.

(c) By 461E, applied to the Radon probability measure on X extending μ , μ has a barycenter $x \in K$. If $g \in X^*$ then $g \upharpoonright K$ belongs to Ψ , so $\int_K g d\mu \leq \int_K g d\nu$; but the same applies to $-g$, so

$$g(x) = \int_K g d\mu = \int_K g d\nu.$$

As g is arbitrary, x is also the barycenter of ν .

(d) As noted in 437Kc, the narrow topology on P corresponds to the weak* topology on $C(K)^*$. So all the functionals $\mu \mapsto \int f d\mu$, for $f \in \Psi$, are continuous; it follows at once that \preceq is a closed subset of $P \times P$.

(e) Any non-empty upwards-directed $Q \subseteq P$ has an upper bound in P . **P** For $\nu \in Q$ set $V_\nu = \{\lambda : \nu \preceq \lambda\}$. Then every V_ν is closed, and because Q is upwards-directed the family $\{V_\nu : \nu \in Q\}$ has the finite intersection property. Because P is compact (437R(f-ii) again), $\bigcap_{\nu \in Q} V_\nu$ is non-empty; now any member of the intersection is an upper bound of Q . **Q**

By Zorn's lemma, every member of P is dominated by a maximal element of P .

461L Lemma Let X be a Hausdorff locally convex linear topological space, K a compact convex subset of X , and P the set of Radon probability measures on K . Suppose that $\mu \in P$ is maximal for the partial order \preceq of 461K.

- (a) $\mu(\frac{1}{2}(M_1 + M_2)) = 0$ whenever M_1, M_2 are disjoint closed convex subsets of K .
- (b) $\mu F = 0$ whenever $F \subseteq K$ is a Baire set (for the subspace topology of K) not containing any extreme point of K .

proof (a) Set $M = \{\frac{1}{2}(x + y) : x \in M_1, y \in M_2\}$. Set $q(x, y) = \frac{1}{2}(x + y)$ for $x \in M_1, y \in M_2$, so that $q : M_1 \times M_2 \rightarrow M$ is continuous. Let μ_M be the subspace measure on M induced by μ , so that μ_M is a Radon measure on M (416Rb). Let λ be a Radon measure on $M_1 \times M_2$ such that $\mu_M = \lambda q^{-1}$ (418L), and define $\psi : C(K) \rightarrow \mathbb{R}$ by writing

$$\psi(f) = \int_{K \setminus M} f \, d\mu + \int_{M_1 \times M_2} \frac{1}{2}(f(x) + f(y))\lambda(d(x, y)).$$

Then ψ is linear, $\psi(f) \geq 0$ whenever $f \in C(K)^+$, and

$$\psi(\chi K) = \mu(K \setminus M) + \lambda(M_1 \times M_2) = \mu(K \setminus M) + \mu_M(M) = 1.$$

Let $\nu \in P$ be such that $\int f \, d\nu = \psi(f)$ for every $f \in C(K)$ (436J/436K again). If $f : K \rightarrow \mathbb{R}$ is continuous and convex, then

$$\begin{aligned} \int f \, d\nu = \psi(f) &= \int_{K \setminus M} f \, d\mu + \int_{M_1 \times M_2} \frac{1}{2}(f(x) + f(y))\lambda(d(x, y)) \\ &\geq \int_{K \setminus M} f \, d\mu + \int_{M_1 \times M_2} f\left(\frac{1}{2}(x + y)\right)\lambda(d(x, y)) \\ &= \int_{K \setminus M} f \, d\mu + \int_{M_1 \times M_2} f \, d\lambda = \int_{K \setminus M} f \, d\mu + \int_M f \, d\mu_M \end{aligned} \tag{235G}$$

$$= \int_K f \times \chi(K \setminus M) \, d\mu + \int_K f \times \chi M \, d\mu \tag{131Fa}$$

$$= \int_K f \, d\mu.$$

So $\mu \preceq \nu$; as we are assuming that μ is maximal, $\mu = \nu$.

Because M_1 and M_2 are disjoint compact convex sets in the Hausdorff locally convex space X , there are a $g \in X^*$ and an $\alpha \in \mathbb{R}$ such that $g(x) < \alpha < g(y)$ whenever $x \in M_1$ and $y \in M_2$ (4A4Ee). Set $f(x) = |g(x) - \alpha|$ for $x \in K$; then f is a continuous convex function. If $x \in M_1$ and $y \in M_2$, then

$$\begin{aligned} f\left(\frac{1}{2}(x + y)\right) &= \left|g\left(\frac{1}{2}(x + y)\right) - \alpha\right| = \frac{1}{2}|(g(x) - \alpha) + (g(y) - \alpha)| \\ &< \frac{1}{2}(|g(x) - \alpha| + |g(y) - \alpha|) = \frac{1}{2}(f(x) + f(y)). \end{aligned}$$

Looking at the formulae above for $\psi(f)$, we see that we have

$$\begin{aligned} \int f \, d\nu &= \int_{K \setminus M} f \, d\mu + \int_{M_1 \times M_2} \frac{1}{2}(f(x) + f(y))\lambda(d(x, y)), \\ \int f \, d\mu &= \int_{K \setminus M} f \, d\mu + \int_{M_1 \times M_2} f\left(\frac{1}{2}(x + y)\right)\lambda(d(x, y)). \end{aligned}$$

Since these are equal, and $f\left(\frac{1}{2}(x + y)\right) < \frac{1}{2}(f(x) + f(y))$ for all $x \in M_1$ and $y \in M_2$, we must have $\mu M = \lambda(M_1 \times M_2) = 0$, as required.

(b)(i) Consider first the case in which F is a zero set for the subspace topology. Since $F \subseteq K$ is a closed G_δ set in K , $K \setminus F$ is expressible as a union $\bigcup_{n \in \mathbb{N}} F_n$ of compact sets. For any $n \in \mathbb{N}$, $z \in F$ and $y \in F_n$, there is a $g \in X^*$ such that $g(z) \neq g(y)$; since F and F_n are compact, there is a finite set $\Phi_n \subseteq X^*$ such that whenever $z \in F$ and $y \in F_n$ there is a $g \in \Phi_n$ such that $g(z) \neq g(y)$. Set $\Phi = \bigcup_{n \in \mathbb{N}} \Phi_n \cup \{0\}$; then Φ is countable; let $\langle g_n \rangle_{n \in \mathbb{N}}$ be a sequence running over Φ , and define $T : X \rightarrow \mathbb{R}^{\mathbb{N}}$ by setting $(Tx)(n) = g_n(x)$ for $n \in \mathbb{N}$, $x \in X$. Then $T[F] \cap T[F_n] = \emptyset$ for every n , so $F = K \cap T^{-1}[T[F]]$.

Now $T[F]$ is a compact subset of the metrizable compact convex set $T[K]$, and does not contain any extreme point of $T[K]$, by 4A4Gc.

Let \mathcal{U} be the set of convex open subsets of $\mathbb{R}^{\mathbb{N}}$. Because the topology of $\mathbb{R}^{\mathbb{N}}$ is locally convex, \mathcal{U} is a base for the topology of $\mathbb{R}^{\mathbb{N}}$; because it is separable and metrizable, \mathcal{U} includes a countable base \mathcal{U}_0 (4A2P(a-iii)), and $\mathcal{V} = \{T[K] \cap U : U \in \mathcal{U}_0\}$ is a countable base for the topology of $T[K]$ (4A2B(a-vi)). Set $\mathcal{M} = \{K \cap T^{-1}[\overline{V}] : V \in \mathcal{V}\}$, so that \mathcal{M} is a countable family of closed convex subsets of K .

If $z \in F$, then there are distinct $u, v \in T[K]$ such that $Tz = \frac{1}{2}(u + v)$. Now there must be $V, V' \in \mathcal{V}$, with disjoint closures, such that $u \in V$ and $v \in V'$, so that $z \in \frac{1}{2}(M + M')$, where $M = K \cap T^{-1}[\overline{V}]$ and $M' = K \cap T^{-1}[\overline{V'}]$ are disjoint members of \mathcal{M} . Thus

$$F \subseteq \bigcup \left\{ \frac{1}{2}(M + M') : M, M' \in \mathcal{M}, M \cap M' = \emptyset \right\}.$$

But (a) tells us that $\frac{1}{2}(M + M')$ is μ -negligible whenever $M, M' \in \mathcal{M}$ are disjoint. As \mathcal{M} is countable, $\mu F = 0$, as required.

(ii) Now consider the Baire measure $\mu \upharpoonright \mathcal{B}\mathfrak{a}(K)$, where $\mathcal{B}\mathfrak{a}(K)$ is the Baire σ -algebra of K . This is inner regular with respect to the zero sets (412D). If $F \in \mathcal{B}\mathfrak{a}(K)$ contains no extremal point of K , then (i) tells us that $\mu Z = 0$ for every zero set $Z \subseteq F$, so μF must also be 0.

461M Theorem Let X be a Hausdorff locally convex linear topological space, K a compact convex subset of X and E the set of extreme points of K . Let $x \in X$. Then there is a probability measure μ on E with barycenter x . If K is metrizable we can take μ to be a Radon measure.

proof Let P be the set of Radon probability measures on K and \preceq the partial order on P described in 461K. By 461Ke, there is a maximal element ν of P such that $\delta_x \preceq \nu$, where $\delta_x \in P$ is the Dirac measure on X concentrated at x . By 461Kc, x is the barycenter of ν .

Let $\lambda = \nu \upharpoonright \mathcal{B}\mathfrak{a}(K)$ be the Baire measure associated with ν . By 461Lb, $\lambda^* E = 1$. So the subspace measure λ_E on E is a probability measure on E . Let μ be the completion of λ_E .

If $g \in X^*$ then $g \upharpoonright K$ is continuous, therefore $\mathcal{B}\mathfrak{a}(K)$ -measurable, so

$$\begin{aligned} g(x) &= \int_K g \, d\nu = \int_K g \, d\lambda = \int_E g \, d\lambda_E \\ (214F) \quad &= \int_E g \, d\mu \end{aligned}$$

(212Fb). So x is the barycenter of μ .

Now suppose that K is metrizable. In this case E is a G_δ set in K . **P** Let ρ be a metric on K inducing its topology. Then

$$K \setminus E = \bigcup_{n \geq 1} \{tx + (1-t)y : x, y \in K, t \in [2^{-n}, 1 - 2^{-n}], \rho(x, y) \geq 2^{-n}\}$$

is K_σ , so its complement in K is a G_δ set in K . **Q** So E is analytic (423Eb) and μ is a Radon measure (433Cb).

461N Lemma Let X be a Hausdorff locally convex linear topological space, K a compact convex subset of X , and P the set of Radon probability measures on K . Let E be the set of extreme points of K and suppose that $\mu \in P$ and $\mu^* E = 1$. Then μ is maximal in P for the partial order \preceq of 461K.

proof (a) For $\mu \in P$ write $b(\mu)$ for the barycenter of μ . Then $b : P \rightarrow K$ is continuous for the narrow topology of P and the weak topology of X , as in 461J. For $f \in C(K)$ define $\bar{f} : K \rightarrow \mathbb{R}$ by setting $\bar{f}(x) = \sup \{ \int f \, d\mu : \mu \in P, b(\mu) = x \}$ for $x \in K$.

(i) Taking δ_x to be the Dirac measure on X concentrated at x , we see that

$$f(x) = \int f \, d\delta_x \leq \bar{f}(x)$$

for any $x \in K$.

(ii) For any $x \in K$ there is a $\mu \in P$ such that $b(\mu) = x$ and $\int f \, d\mu = \bar{f}(x)$. **P** The set $\{ \mu : \mu \in P, b(\mu) = x \}$ is compact, so its continuous image $\{ \int f \, d\mu : b(\mu) = x \}$ is compact and contains its supremum. **Q** \bar{f} is upper semi-continuous. **P** For any $\alpha \in \mathbb{R}$, the set

$$\{ (\mu, x) : \mu \in P, x \in K, b(\mu) = x, \int f \, d\mu \geq \alpha \}$$

is compact, so its projection onto the second coordinate is closed; but this projection is just $\{x : \bar{f}(x) \geq \alpha\}$.

Q

(iii) $\bar{f} : K \rightarrow \mathbb{R}$ is concave. **P** Suppose that $x, y \in K$ and $t \in [0, 1]$. Take $\mu, \nu \in P$ such that $b(\mu) = x$, $\bar{f}(x) = \int f d\mu$, $b(\nu) = y$ and $\bar{f}(y) = \int f d\nu$. Set $\lambda = t\mu + (1-t)\nu$. Then

$$\int_K g d\lambda = t \int_K g d\mu + (1-t) \int_K g d\nu = tg(x) + (1-t)g(y) = g(tx + (1-t)y)$$

for any $g \in X^*$, so $b(\lambda) = tx + (1-t)y$ and

$$\bar{f}(tx + (1-t)y) \geq \int f d\lambda = t \int f d\mu + (1-t) \int f d\nu = t\bar{f}(x) + (1-t)\bar{f}(y).$$

As x, y and t are arbitrary, \bar{f} is concave. **Q**

(iv) If $x \in K$ and $\bar{f}(x) > f(x)$ then $x \notin E$. **P** There is a $\mu \in P$ such that $b(\mu) = x$ and $\int f d\mu > f(x)$. We cannot have $\mu\{x\} = 1$ because $\int f d\mu \neq f(x)$, so there is a point y of the support of μ such that $y \neq x$. Let $g \in X^*$ be such that $g(y) > g(x)$ and set $G = \{z : z \in K, g(z) > g(x)\}$, $t = \mu G$. Then $t > 0$; also $\int_G g d\mu > tg(x) = t \int g d\mu$, so $t \neq 1$. Define $\nu_1, \nu_2 \in P$ by setting $\nu_1 H = \frac{1}{t}\mu(G \cap H)$ whenever $H \subseteq K$ and μ measures $G \cap H$, $\nu_2 H = \frac{1}{1-t}\mu(H \setminus G)$ whenever $H \subseteq K$ and μ measures $H \setminus G$. Let x_1, x_2 be the barycenters of ν_1, ν_2 respectively. For any $h \in X^*$,

$$\begin{aligned} h(tx_1 + (1-t)x_2) &= th(x_1) + (1-t)h(x_2) = t \int_K h d\nu_1 + (1-t) \int_K h d\nu_2 \\ &= \int_G h d\mu + \int_{K \setminus G} h d\mu = h(x). \end{aligned}$$

So $x = tx_1 + (1-t)x_2$. But both x_1 and x_2 belong to K and $g(x_1) > g(x)$, so $x_1 \neq x$, while $0 < t < 1$, so x is not an extreme point of K . **Q**

(b) Now take $\mu \in P$ such that $\mu^*E = 1$, and $\nu \in P$ such that $\mu \preceq \nu$. For any convex $f \in C(K)$, $f - \bar{f}$ is the sum of lower semi-continuous convex functions so is lower semi-continuous, and $\{x : f(x) = \bar{f}(x)\} = \{x : f(x) - \bar{f}(x) \geq 0\}$ is a G_δ set. By (a-iv), it includes E , so $\int f d\mu = \int f d\nu$. In addition, $\int f d\mu \leq \int f d\nu$ and $\int f d\nu \leq \int \bar{f} d\nu$, by 461Kb applied to $-\bar{f}$. But since $f \leq \bar{f}$, we have

$$\int f d\nu \leq \int \bar{f} d\nu \leq \int \bar{f} d\mu = \int f d\mu;$$

as f is arbitrary, $\nu \preceq \mu$; as ν is arbitrary, μ is maximal.

461O Lemma Suppose that X is a Riesz space with a Hausdorff locally convex linear space topology, and $K \subseteq X$ a compact convex set such that every non-zero member of the positive cone X^+ is uniquely expressible as αx for some $x \in K$ and $\alpha \geq 0$. Let P be the set of Radon probability measures on K and \preceq the partial order described in 461K. If $\mu, \nu \in P$ have the same barycenter then they have a common upper bound in P .

proof (a) If $X = \{0\}$ then K is either $\{0\}$ or empty and the result is immediate, so henceforth suppose that X is non-trivial. Because each non-zero member of X^+ is uniquely expressible as a multiple of a member of K , no distinct members of $K \setminus \{0\}$ can be multiples of each other; as K is convex, $0 \notin K$. If $z_0, \dots, z_r \in K$, $\gamma_0, \dots, \gamma_r \geq 0$ and $z = \sum_{k=0}^r \gamma_k z_k$ belongs to K , then $\sum_{k=0}^r \gamma_k = 1$. **P** Setting $\gamma = \sum_{k=0}^r \gamma_k$, $\gamma \neq 0$ and $\frac{1}{\gamma}z = \sum_{k=0}^r \frac{\gamma_k}{\gamma} z_k$ belongs to K ; accordingly $\frac{1}{\gamma}z = z$ and $\gamma = 1$. **Q**

(b) Take any $x \in K$, and let P_x be the set of elements of P with barycenter x . Write Q_x for the set of members of P_x with finite support. Then Q_x is dense in P_x . **P** Suppose that $\mu \in P_x$, $f_0, \dots, f_n \in C(K)$ and $\epsilon > 0$. Then there is a finite cover of K by relatively open convex sets on each of which every f_i has oscillation at most ϵ ; so we have a partition \mathcal{H} of K into finitely many non-empty Borel sets H such that every f_i has oscillation at most ϵ on the convex hull $\Gamma(H)$. If $H \in \mathcal{H}$ and $\mu H > 0$, let x_H be the barycenter of the measure $\mu_H \in P$ where $\mu_H F = \frac{1}{\mu H} \mu(F \cap H)$ whenever $F \subseteq K$ and μ measures $F \cap H$. For other $H \in \mathcal{H}$, take any point x_H of H . In all cases, $x_H \in \overline{\Gamma(H)}$ so $|f_i(y) - f_i(x_H)| \leq \epsilon$ whenever $i \leq n$ and $y \in H$.

Consider $\nu = \sum_{H \in \mathcal{H}} \mu H \cdot \delta_{x_H}$, where $\delta_{x_H} \in P$ is the Dirac measure on K concentrated at x_H . If $g \in X^*$ then

$$\int_K g d\nu = \sum_{H \in \mathcal{H}} \mu H \cdot g(x_H) = \sum_{H \in \mathcal{H}} \int_H g d\mu = \int_K g d\mu = g(x);$$

as g is arbitrary, $\nu \in P_x$ and $\nu \in Q_x$. Next, for $i \leq n$,

$$\begin{aligned} \left| \int f_i d\nu - \int f_i d\mu \right| &\leq \sum_{H \in \mathcal{H}} |f_i(x_H) \mu H - \int_H f_i d\mu| \\ &\leq \sum_{H \in \mathcal{H}} \int_H |f_i(x_H) - f_i(y)| \mu(dy) \leq \sum_{H \in \mathcal{H}} \epsilon \mu H = \epsilon. \end{aligned}$$

As f_0, \dots, f_n and ϵ are arbitrary, Q_x is dense in P_x . **Q**

(c) Suppose that $x \in K$ and $\mu, \nu \in Q_x$. Then they have a common upper bound in P . **P** Express μ, ν as $\sum_{i=0}^m \alpha_i \delta_{x_i}, \sum_{j=0}^n \beta_j \delta_{y_j}$ respectively, where all the α_i and β_j are strictly positive, all the x_i and y_j belong to K , and $\sum_{i=0}^m \alpha_i = \sum_{j=0}^n \beta_j = 1$. If $g \in X^*$ then

$$g(x) = \int_K g d\mu = \sum_{i=0}^m \alpha_i g(x_i) = \sum_{j=0}^n \beta_j g(y_j),$$

so $x = \sum_{i=0}^m \alpha_i x_i = \sum_{j=0}^n \beta_j y_j$. By the decomposition theorem 352Fd there is a family $\langle w_{ij} \rangle_{i \leq m, j \leq n}$ in X^+ such that $\alpha_i x_i = \sum_{j=0}^n w_{ij}$ for every $i \leq m$ and $\beta_j y_j = \sum_{i=0}^m w_{ij}$ for every $j \leq n$. Each w_{ij} is expressible as $\gamma_{ij} z_{ij}$ where $z_{ij} \in K$ and $\gamma_{ij} \geq 0$. Now

$$\sum_{j=0}^n \frac{\gamma_{ij}}{\alpha_i} z_{ij} = x_i \in K;$$

by (a), $\sum_{j=0}^n \gamma_{ij} = \alpha_i$, for every $i \leq m$. Similarly, $\sum_{i=0}^m \gamma_{ij} = \beta_j$ for $j \leq n$. Of course this means that $\sum_{i=0}^m \sum_{j=0}^n \gamma_{ij} = 1$.

Set $\lambda = \sum_{i=0}^m \sum_{j=0}^n \gamma_{ij} \delta_{z_{ij}} \in P$. If $f : K \rightarrow \mathbb{R}$ is continuous and convex,

$$\begin{aligned} \int f d\mu &= \sum_{i=0}^m \alpha_i f(x_i) = \sum_{i=0}^m \alpha_i f\left(\sum_{j=0}^n \frac{\gamma_{ij}}{\alpha_i} z_{ij}\right) \\ &\leq \sum_{i=0}^m \alpha_i \sum_{j=0}^n \frac{\gamma_{ij}}{\alpha_i} f(z_{ij}) = \sum_{i=0}^m \sum_{j=0}^n \gamma_{ij} f(z_{ij}) = \int f d\lambda. \end{aligned}$$

So $\mu \preceq \lambda$. Similarly, $\nu \preceq \lambda$ and we have the required upper bound for $\{\mu, \nu\}$. **Q**

(d) Now consider

$$\{(\mu, \nu, \lambda) : \mu, \nu, \lambda \in P, \mu \preceq \lambda, \nu \preceq \lambda\}.$$

This is a closed set in the compact set $P \times P \times P$ (461Kd), so its projection

$$R = \{(\mu, \nu) : \mu, \nu \text{ have a common upper bound in } P\}$$

is a closed set in $P \times P$.

If $x \in K$ then (c) tells us that R includes $Q_x \times Q_x$, so (b) tells us that R includes $P_x \times P_x$. Thus any two members of P_x have a common upper bound in P , as required.

461P Theorem Suppose that X is a Riesz space with a Hausdorff locally convex linear space topology, and $K \subseteq X$ a metrizable compact convex set such that every non-zero member of the positive cone X^+ is uniquely expressible as αx for some $x \in K$ and $\alpha \geq 0$. Let E be the set of extreme points of K , and x any point of K . Then there is a unique Radon probability measure μ on E such that x is the barycenter of μ .

proof By 461M, there is a Radon probability measure μ on E such that x is the barycenter of μ . Suppose that μ_1 is another measure with the same properties. Let ν, ν_1 be the Radon probability measures on K extending μ, μ_1 respectively. Then ν and ν_1 both have barycenter x and make E conegligible. By 461N, they are both maximal in P . By 461O, they must have a common upper bound in P , so they are equal. But this means that $\mu = \mu_1$.

461Q It is a nearly universal rule that when we encounter a compact convex set we should try to identify its extreme points. I look now at some sets which arose naturally in §437.

Proposition (a) Let \mathfrak{A} be a Dedekind σ -complete Boolean algebra and $\pi : \mathfrak{A} \rightarrow \mathfrak{A}$ a sequentially order-continuous Boolean homomorphism. Let M_σ be the L -space of countably additive real-valued functionals on \mathfrak{A} , and Q the set

$$\{\nu : \nu \in M_\sigma, \nu \geq 0, \nu 1 = 1, \nu \pi = \nu\}.$$

If $\nu \in Q$, then the following are equiveridical: (i) ν is an extreme point of Q ; (ii) $\nu a \in \{0, 1\}$ whenever $\pi a = a$; (iii) $\nu a \in \{0, 1\}$ whenever $a \in \mathfrak{A}$ is such that $\nu(a \triangle \pi a) = 0$.

(b) Let X be a set, Σ a σ -algebra of subsets of X , and $\phi : X \rightarrow X$ a (Σ, Σ) -measurable function. Let M_σ be the L -space of countably additive real-valued functionals on Σ , and $Q \subseteq M_\sigma$ the set of probability measures with domain Σ for which ϕ is inverse-measure-preserving. If $\mu \in Q$, then μ is an extreme point of Q iff ϕ is ergodic with respect to μ (definition: 372Ob²).

proof (a) I ought to remark at once that because \mathfrak{A} is Dedekind σ -complete, every countably additive functional on \mathfrak{A} is bounded (326M³), so that M_σ is the L -space of bounded countably additive functionals on \mathfrak{A} , as studied in 362A-362B.

(i) \Rightarrow (ii) Suppose that ν is an extreme point of Q and that $a \in \mathfrak{A}$ is such that $\pi a = a$. Set $\alpha = \nu a$. ? If $0 < \alpha < 1$, define $\nu_1 : \mathfrak{A} \rightarrow \mathbb{R}$ by setting

$$\nu_1 b = \frac{1}{\alpha} \nu(b \cap a)$$

for $b \in \mathfrak{A}$. Then ν_1 is a non-negative countably additive functional, and $\nu_1 1 = 1$. Moreover, for any $b \in \mathfrak{A}$,

$$\nu_1 \pi b = \frac{1}{\alpha} \nu(\pi b \cap a) = \frac{1}{\alpha} \nu(\pi b \cap \pi a) = \frac{1}{\alpha} \nu \pi(b \cap a) = \frac{1}{\alpha} \nu(b \cap a) = \nu_1 b,$$

so $\nu_1 \pi = \nu_1$ and $\nu_1 \in Q$. Since $\nu_1 a = 1$, $\nu_1 \neq \nu$. Similarly, $\nu_2 \in Q$, where $\nu_2 b = \frac{1}{1-\alpha} \nu(b \setminus a)$ for $b \in \mathfrak{A}$. Now $\nu = \alpha \nu_1 + (1-\alpha) \nu_2$ is a proper convex combination of members of Q and is not extreme. **X** So $\nu a \in \{0, 1\}$; as a is arbitrary, (ii) is true.

(ii) \Rightarrow (iii) Suppose that (ii) is true, and that $a \in \mathfrak{A}$ is such that $\nu(a \triangle \pi a) = 0$. Then

$$\nu(\pi^n a \triangle \pi^{n+1} a) = \nu \pi^n(a \triangle \pi a) = \nu(a \triangle \pi a) = 0$$

for every $n \in \mathbb{N}$, so $\nu(a \triangle \pi^n a) = 0$ for every $n \in \mathbb{N}$. Set

$$b_n = \sup_{m \geq n} \pi^m a \text{ for } n \in \mathbb{N}, \quad b = \inf_{n \in \mathbb{N}} b_n.$$

Because π is sequentially order-continuous and ν is countably additive,

$$\nu(a \triangle b_n) = 0 \text{ for every } n \in \mathbb{N}, \quad \nu(a \triangle b) = 0.$$

Now

$$\pi b_n = \sup_{m \geq n} \pi^{m+1} a = b_{n+1} \subseteq b_n$$

for every $n \in \mathbb{N}$, so

$$\pi b = \inf_{n \in \mathbb{N}} \pi b_n = \inf_{n \in \mathbb{N}} b_{n+1} = b.$$

Consequently $\nu a = \nu b \in \{0, 1\}$. As a is arbitrary, (iii) is true.

(iii) \Rightarrow (i) Suppose that (iii) is true, and that $\nu = \frac{1}{2}(\nu_1 + \nu_2)$ where $\nu_1, \nu_2 \in Q$. For $\alpha \geq 0$, set $\theta_\alpha = \nu_1 - \alpha \nu \in M_\sigma$. Then we have a corresponding element $a_\alpha = \llbracket \theta_\alpha > 0 \rrbracket$ in \mathfrak{A} such that

$$\theta_\alpha c > 0 \text{ whenever } 0 \neq c \subseteq a_\alpha, \quad \theta_\alpha c \leq 0 \text{ whenever } c \cap a_\alpha = 0$$

(326S⁴). Observe that if $b \in \mathfrak{A}$, then

²Formerly 372Pb.

³Formerly 326I.

⁴Formerly 326O.

$$\theta_\alpha b = \theta_\alpha a_\alpha - \theta_\alpha(a_\alpha \setminus b) + \theta_\alpha(b \setminus a_\alpha) \leq \theta_\alpha a_\alpha,$$

$$\theta_\alpha \pi b = \nu_1 \pi b - \alpha \nu \pi b = \nu_1 b - \alpha \nu b = \theta_\alpha b.$$

Now $\nu(a_\alpha \setminus \pi a_\alpha) = 0$. **P?** Otherwise, setting $c = a_\alpha \setminus \pi a_\alpha$, we must have $\theta_\alpha c > 0$, so

$$\theta_\alpha(c \cup \pi a_\alpha) > \theta_\alpha \pi a_\alpha = \theta_\alpha a_\alpha. \quad \mathbf{XQ}$$

Consequently

$$\nu(a_\alpha \Delta \pi a_\alpha) = \nu \pi a_\alpha - \nu a_\alpha + 2\nu(a_\alpha \setminus \pi a_\alpha) = 0$$

and $\nu a_\alpha \in \{0, 1\}$.

If $\alpha \leq \beta$ then $\theta_\beta \leq \theta_\alpha$ and $a_\beta \subseteq a_\alpha$. As $\theta_0 = \nu_1$, $\nu_1(1 \setminus a_0) = 0$, $\nu_1 a_0 = 1$ and $\nu a_0 \geq \frac{1}{2}$; accordingly $\nu a_0 = 1$. As $\theta_2 \leq 0$, $a_2 = 0$ and $\nu a_2 = 0$. So

$$\beta = \sup\{\alpha : \nu a_\alpha = 1\} = \sup\{\alpha : \nu a_\alpha > 0\}$$

is defined in $[0, 2]$.

Now $\nu_1 = \beta\nu$. **P** Let $c \in \mathfrak{A}$. **?** If $\nu_1 c > \beta\nu c$, take $\alpha > \beta$ such that $\nu_1 c > \alpha\nu c$. Then $\nu a_\alpha = 0$, but

$$0 < \theta_\alpha c \leq \theta_\alpha a_\alpha \leq \nu_1 a_\alpha \leq 2\nu a_\alpha. \quad \mathbf{X}$$

? If $\nu_1 c < \beta\nu c$, take $\alpha \in [0, \beta[$ such that $\nu_1 c < \alpha\nu c$. Then $\nu a_\alpha = 1$ so $\nu(1 \setminus a_\alpha) = 0$, but

$$0 > \theta_\alpha c \geq \theta_\alpha(c \setminus a_\alpha) \geq -\alpha\nu(c \setminus a_\alpha) \geq -\alpha\nu(1 \setminus a_\alpha). \quad \mathbf{X}$$

Thus $\nu_1 c = \beta\nu c$; as c is arbitrary, $\nu_1 = \beta\nu$. **Q**

Accordingly ν_1 is a multiple of ν and must be equal to ν . Similarly, $\nu_2 = \nu$. As ν_1 and ν_2 were arbitrary, ν is an extreme point of Q .

(b) In (a), set $\mathfrak{A} = \Sigma$ and $\pi E = \phi^{-1}[E]$ for $E \in \Sigma$; then ' ϕ is ergodic with respect to μ ' corresponds to condition (ii) of (a), so we have the result.

461R Corollary Let X be a compact Hausdorff space and $\phi : X \rightarrow X$ a continuous function. Let Q be the non-empty compact convex set of Radon probability measures μ on X such that ϕ is inverse-measure-preserving for μ , with its narrow topology and the convex structure defined by 234G and 234Xf. (See 437T.) Then the extreme points of Q are those for which ϕ is ergodic.

proof (a) If $\mu_0 \in Q$ is not extreme, let $\mathcal{B} = \mathcal{B}(X)$ be the Borel σ -algebra of X , so that ϕ is $(\mathcal{B}, \mathcal{B})$ -measurable, and write Q' for the set of ϕ -invariant Borel probability measures on X . Then 461Q tells us that the extreme points of Q' are just the measures for which ϕ is ergodic. If $\mu \in Q$, then $\mu \upharpoonright \mathcal{B} \in Q'$, and of course the function $\mu \mapsto \mu \upharpoonright \mathcal{B}$ is injective (416Eb) and preserves convex combinations. So $\mu_0 \upharpoonright \mathcal{B}$ is not extreme in Q' . By 461Q, ϕ is not $\mu_0 \upharpoonright \mathcal{B}$ -ergodic, and therefore not μ_0 -ergodic.

(b) If $\mu_0 \in Q$ and ϕ is not μ_0 -ergodic, let $E \in \text{dom } \mu_0$ be such that $0 < \mu_0 E < 1$ and $\phi^{-1}[E] = E$. Set $\alpha = \mu_0 E$ and $\beta = 1 - \alpha$, and let μ_1, μ_2 be the indefinite-integral measures over μ_0 defined by $\frac{1}{\alpha}\chi E$ and $\frac{1}{\beta}\chi(X \setminus E)$. Then μ_1 is a Radon probability measure on X (416Sa), so the image measure $\mu_1 \phi^{-1}$ also is a Radon measure (418I). The argument of part (b) of the proof of 461Q tells us that $\mu_1 \phi^{-1}$ agrees with μ_1 on Borel sets, so $\mu_1 = \mu_1 \phi^{-1}$ (416Eb) and $\mu_1 \in Q$. Similarly, $\mu_2 \in Q$, and $\mu_0 = \alpha\mu_1 + \beta\mu_2$, so μ_0 is not extreme in Q .

461X Basic exercises **>(a)** Let X be a Hausdorff locally convex linear topological space, $C \subseteq X$ a convex set, and $g : C \rightarrow \mathbb{R}$ a function. Show that the following are equiveridical: (i) g is convex and lower semi-continuous; (ii) there are a non-empty set $D \subseteq X^*$ and a family $\langle \beta_f \rangle_{f \in D}$ in \mathbb{R} such that $g(x) = \sup_{f \in D} f(x) + \beta_f$ for every $x \in C$. (Compare 233Hb.)

>(b) Let X be a Hausdorff locally convex linear topological space, $K \subseteq X$ a compact convex set, and x an extreme point of K . Let μ be a probability measure on X such that $\mu^* K = 1$ and x is the barycenter of μ . (i) Show that $\{y : y \in K, f(y) \neq f(x)\}$ is μ -negligible for every $f \in X^*$. (*Hint:* 461E.) (ii) Show that if μ is a Radon measure then $\mu\{x\} = 1$.

(c) For each $n \in \mathbb{N}$, define $e_n \in \mathbf{c}_0$ by saying that $e_n(n) = 1$, $e_n(i) = 0$ if $i \neq n$. Let μ be the point-supported Radon probability measure on \mathbf{c}_0 defined by saying that $\mu E = \sum_{n=0}^{\infty} 2^{-n-1} \chi E(2^n e_n)$ for every $E \subseteq \mathbf{c}_0$. (i) Show that every member of \mathbf{c}_0^* is μ -integrable. (*Hint*: \mathbf{c}_0^* can be identified with ℓ^1 .) (ii) Show that μ has no barycenter in \mathbf{c}_0 .

(d) Let I be an uncountable set, and $X = \{x : x \in \ell^\infty(I), \{i : x(i) \neq 0\} \text{ is countable}\}$. (i) Show that X is a closed linear subspace of $\ell^\infty(I)$. (ii) Show that there is a probability measure μ on X such that (a) $\mu\{x : \|x\| \leq 1\}$ is defined and equal to 1 (b) $\mu\{x : x(i) = 1\} = 1$ for every $i \in I$. (iii) Show that $\int f d\mu$ is defined for every $f \in X^*$. (*Hint*: for any $f \in X^*$, there is a countable set $J \subseteq X$ such that $f(x) = 0$ whenever $x|_J = 0$, so that $f =_{\text{a.e.}} f(\chi J)$.) (iv) Show that μ has no barycenter in X .

>(e) Let X be a complete Hausdorff locally convex linear topological space, and $K \subseteq X$ a compact set. Show that every extreme point of $\overline{\Gamma(K)}$ belongs to K . (*Hint*: show that it cannot be the barycenter of any measure on K which is not supported by a single point.)

>(f) Let X be a Hausdorff locally convex linear topological space, and $K \subseteq X$ a metrizable compact set. Show that $\overline{\Gamma(K)}$ is metrizable. (*Hint*: we may suppose that X is complete, so that $\overline{\Gamma(K)}$ is compact. Show that $\overline{\Gamma(K)}$ is a continuous image of the space of Radon probability measures on K , and use 437Rf.)

(g) Let X be a Hausdorff locally convex linear topological space and $K \subseteq X$ a compact set. Show that the Baire σ -algebra of K is just the subspace σ -algebra induced by the cylindrical σ -algebra of X .

>(h) Let X be a Hausdorff locally convex linear topological space, and $K \subseteq X$ a compact convex set; let E be the set of extreme points of K . Let $\langle f_n \rangle_{n \in \mathbb{N}}$ be a sequence in X^* such that $\sup_{x \in E, n \in \mathbb{N}} |f_n(x)|$ is finite and $\lim_{n \rightarrow \infty} f_n(x) = 0$ for every $x \in E$. Show that $\lim_{n \rightarrow \infty} f_n(x) = 0$ for every $x \in K$.

>(i) Let X be a Hausdorff locally convex linear topological space, and $K \subseteq X$ a metrizable compact convex set. Show that the algebra of Borel subsets of K is just the subspace algebra of the cylindrical σ -algebra of X .

(j) Let X be a Hausdorff locally convex linear topological space, and $K \subseteq X$ a compact set. Let us say that a point x of K is **extreme** if the only Radon probability measure on K with barycenter x is the Dirac measure on K concentrated at x . (Cf. 461Xb.) (i) Show that if X is complete, then $x \in X$ is an extreme point of K iff it is an extreme point of $\overline{\Gamma(K)}$. (ii) Writing E for the set of extreme points of K , show that any point of K is the barycenter of some probability measure on E . (iii) Show that if K is metrizable then E is a G_δ subset of K and any point of K is the barycenter of some Radon probability measure on E .

(k) Let G be an abelian group with identity e , and K the set of positive definite functions $h : G \rightarrow \mathbb{C}$ such that $h(e) = 1$. (i) Show that K is a compact convex subset of \mathbb{C}^G . (ii) Show that the extreme points of K are just the group homomorphisms from G to S^1 . (iii) Show that K generates the positive cone of a Riesz space. (*Hint*: 445N.)

(l) Let X be a compact metrizable space and G a subgroup of the group of autohomeomorphisms of X . Let M_σ be the space of signed Borel measures on X with its vague topology, and $Q \subseteq M_\sigma$ the set of G -invariant Borel probability measures on X . Show that every member of Q is uniquely expressible as the barycenter of a Radon probability measure on the set of extreme points of Q .

(m) Let X be a set, Σ a σ -algebra of subsets of X , and P the set of probability measures with domain Σ , regarded as a convex subset of the linear space of countably additive functionals on Σ . Show that $\mu \in P$ is an extreme point in P iff it takes only the values 0 and 1.

(n) Let \mathfrak{A} be a Boolean algebra and M the L -space of bounded finitely additive functionals on \mathfrak{A} , and $\pi : \mathfrak{A} \rightarrow \mathfrak{A}$ a Boolean homomorphism. (i) Show that $U = \{\nu : \nu \in M, \nu\pi = \nu\}$ is a closed Riesz subspace of M . (ii) Set $Q = \{\nu : \nu \in U, \nu \geq 0, \nu 1 = 1\}$. Show that if μ, ν are distinct extreme points of Q then $\mu \wedge \nu = 0$. (iii) Set $Q_\sigma = \{\nu : \nu \in Q, \nu \text{ is countably additive}\}$. Show that any extreme point of Q_σ is an extreme point of Q . (iv) Set $Q_\tau = \{\nu : \nu \in Q, \nu \text{ is countably additive}\}$. Show that any extreme point of Q_τ is an extreme point of Q .

>(o) Set $S^1 = \{z : z \in \mathbb{C}, |z| = 1\}$, and let $w \in S^1$ be such that $w^n \neq 1$ for any integer n . Define $\phi : S^1 \rightarrow S^1$ by setting $\phi(z) = wz$ for every $z \in S^1$. Show that the only Radon probability measure on S^1 for which ϕ is inverse-measure-preserving is the Haar probability measure μ of S^1 . (*Hint*: use 281N to show that $\lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=0}^n f(w^k z) = \int f d\mu$ for every $f \in C(S^1)$; now put 461R and 372H⁵ together.)

(p) Set $\phi(x) = 2 \min(x, 1-x)$ for $x \in [0, 1]$ (cf. 372Xp⁶). Show that there are many point-supported Radon measures on $[0, 1]$ for which ϕ is inverse-measure-preserving.

(q) Let X and Y be Hausdorff locally convex linear topological spaces, $A \subseteq X$ a convex set and $\phi : A \rightarrow Y$ a continuous function such that $\phi[A]$ is bounded and $\phi(tx + (1-t)y) = t\phi(x) + (1-t)\phi(y)$ for all $x, y \in A$ and $t \in [0, 1]$. Let μ be a topological probability measure on A with a barycenter $x^* \in A$. Show that $\phi(x^*)$ is the barycenter of the image measure $\mu\phi^{-1}$ on Y . (*Hint*: show first that if $\langle E_i \rangle_{i \in I}$ is a finite partition of A into non-empty convex sets measured by μ , $\alpha_i = \mu E_i$ for each $i \in I$ and $C = \{\sum_{i \in I} \alpha_i x_i : x_i \in E_i \text{ for every } i \in I\}$, then $x^* \in \overline{C}$.)

461Y Further exercises (a) Let X be a Hausdorff locally convex linear topological space. (i) Show that if M_0, \dots, M_n are non-empty compact convex subsets of X with empty intersection, then there is a continuous convex function $g : X \rightarrow \mathbb{R}$ such that $g(\sum_{i=0}^n \alpha_i x_i) < \sum_{i=0}^n \alpha_i g(x_i)$ whenever $x_i \in M_i$ and $\alpha_i > 0$ for every $i \leq n$. (ii) Show that if $K \subseteq X$ is compact and $x^* \in \overline{\Gamma(K)}$ then there is a Radon probability measure μ on K , with barycenter x^* , such that $\mu(\alpha_0 M_0 + \dots + \alpha_n M_n) = 0$ whenever M_0, \dots, M_n are compact convex subsets of K with empty intersection, $\alpha_i \geq 0$ for every $i \leq n$, and $\sum_{i=0}^n \alpha_i = 1$.

(b) In $\mathbb{R}^{[0,2]}$ let K be the set of those functions u such that (α) $0 \leq u(s) \leq u(t) \leq 1$ whenever $0 \leq s \leq t \leq 1$ (β) $|u(t+1)| \leq u(s') - u(s)$ whenever $0 \leq s < t < s' \leq 1$. (i) Show that K is a compact convex set. (ii) Show that the set E of extreme points of K is just the set of functions of the types $\mathbf{0}$, $\chi[0, 1]$, $\chi[s, 1] \pm \chi\{1+s\}$ and $\chi[s, 1] \pm \chi\{1+s\}$ for $0 < s < 1$. (iii) Set $w(s) = s$ for $s \in [0, 1]$, 0 for $s \in]1, 2[$. Show that if μ is any Radon probability measure on K with barycenter w then $\mu E = 0$.

(c) Write ν_{ω_1} for the usual measure on $Z = \{0, 1\}^{\omega_1}$. Fix any $z_0 \in Z$, and let U be the linear space $\{u : u \in C(Z), u(z_0) = \int u d\nu_{\omega_1}\}$. Let X be the Riesz space of signed tight Borel measures μ on Z such that $\mu\{z_0\} = 0$, with the topology generated by the functionals $\mu \mapsto \int u d\mu$ as u runs over U . Let $K \subseteq X$ be the set of tight Borel probability measures μ on Z such that $\mu\{z_0\} = 0$. (i) Show that K is compact and convex and that every member of $X^+ \setminus \{0\}$ is uniquely expressible as a positive multiple of a member of K . (ii) Show that the set E of extreme points of K can be identified, as topological space, with $Z \setminus \{z_0\}$, so is a Borel subset of K but not a Baire subset. (iii) Show that the restriction of ν_{ω_1} to the Borel σ -algebra of Z is the barycenter of more than one Baire measure on E .

(d) Let X be a set, Σ a σ -algebra of subsets of X , and Φ a set of (Σ, Σ) -measurable functions from X to itself. Let M_σ be the L -space of countably additive real-valued functionals on Σ , and $Q \subseteq M_\sigma$ the set of probability measures with domain Σ for which every member of Φ is inverse-measure-preserving. (i) Show that if $\mu \in Q$, then μ is an extreme point of Q iff $\mu E \in \{0, 1\}$ whenever $E \in \Sigma$ and $\mu(E \Delta \phi^{-1}[E]) = 0$ for every $\phi \in \Phi$. (ii) Show that if $\mu \in Q$ and Φ is countable and commutative, then μ is an extreme point of Q iff $\mu E \in \{0, 1\}$ whenever $E \in \Sigma$ and $E = \phi^{-1}[E]$ for every $\phi \in \Phi$.

(e) Let X be a non-empty Hausdorff space, and define $\phi : X^{\mathbb{N}} \rightarrow X^{\mathbb{N}}$ by setting $\phi(x)(n) = x(n+1)$ for $x \in X^{\mathbb{N}}$ and $n \in \mathbb{N}$. Let Q be the set of Radon probability measures on $X^{\mathbb{N}}$ for which ϕ is inverse-measure-preserving. Show that a Radon probability measure λ on $X^{\mathbb{N}}$ is an extreme point of Q iff it is a Radon product measure $\mu^{\mathbb{N}}$ for some Radon probability measure μ on X .

(f) Let G be a topological group. Show that the following are equiveridical: (i) G is amenable; (ii) whenever X is a Hausdorff locally convex linear topological space, and \bullet is a continuous action of G on X such that $x \mapsto a \bullet x$ is a linear operator for every $a \in G$, and $K \subseteq X$ is a non-empty compact convex set such

⁵Formerly 372I.

⁶Formerly 372Xm.

that $a \bullet x \in K$ whenever $a \in G$ and $x \in K$, then there is an $x \in K$ such that $a \bullet x = x$ for every $a \in G$; (iii) whenever X is a Hausdorff locally convex linear topological space, $K \subseteq X$ is a non-empty compact convex set, and \bullet is a continuous action of G on K such that $a \bullet (tx + (1-t)y) = t a \bullet x + (1-t) a \bullet y$ whenever $a \in G$, $x, y \in K$ and $t \in [0, 1]$, then there is an $x \in K$ such that $a \bullet x = x$ for every $a \in G$. Use this to simplify parts of the proof of 449C. (*Hint*: 493B.)

(g) Let G be an amenable topological group, and \bullet an action of G on a reflexive Banach space U , continuous for the given topology on G and the weak topology of U , such that $u \mapsto a \bullet u$ is a linear operator of norm at most 1 for every $a \in G$. Set $V = \{v : v \in U, a \bullet v = v \text{ for every } a \in G\}$. Show that $\{u + v - a \bullet u : u \in U, v \in V, a \in G\}$ is dense in U .

461 Notes and comments The results above are a little unusual in that we have studied locally convex spaces for several pages without encountering two topologies on the same space more than once (461J). In fact some of the most interesting properties of measures on locally convex spaces concern their relationships with strong and weak topologies, but I defer these ideas to later parts of the chapter. For the moment, we just have the basic results affirming (i) that barycenters exist (461E, 461F, 461H) (ii) that points can be represented as barycenters (461I, 461M). The last two can be thought of as refinements of the Kreĭn-Milman theorem. Any compact convex set K (in a locally compact Hausdorff space) is the closed convex hull of the set E of its extreme points. By 461M, given $x \in K$, we can actually find a measure on E with barycenter x ; and if K is metrizable we can do this with a Radon measure. Of course the second part of 461M is a straightforward consequence of the first. But I do not know of any proof of 461M which does not pass through 461K-461L.

Kreĭn's theorem (461J) is a fundamental result in the theory of linear topological spaces. The proof here, using the Riesz representation theorem and vague topologies, is a version of the standard one (e.g., BOURBAKI 87, II.4.1), written out to be a little heavier in the measure theory and a little lighter in the topological linear space theory than is usual. There are of course proofs which do not use measure theory.

In §437 I have already looked at an archetypal special case of 461I and 461M. If X is a compact Hausdorff space and P the compact convex set of Radon probability measures on X with the narrow (or vague) topology, then the set of extreme points of P can be identified with the set Δ of Dirac measures on X (437S, 437Xt). If we think of P as a subset of $C(X)^*$ with the weak* topology, so that the dual of the linear topological space $C(X)^*$ can be identified with $C(X)$, then any $\mu \in P$ is the barycenter of a Baire probability measure ν on Δ . In fact (because Δ here is compact) μ is the barycenter of a Radon measure on Δ , and this is just the image measure $\mu\delta^{-1}$.

I have put the phrase 'Choquet's theorem' into the title of this section. Actually it should perhaps be 'first steps in Choquet theory', because while the theory as a whole was dominated for many years by the work of G.Choquet the exact attribution of the results presented here is more complicated. See PHELPS 66 for a much fuller account. But certainly both the existence and uniqueness theorems 461M and 461P draw heavily on Choquet's ideas.

Theorem 461P, demanding an excursion through 461N-461O, seems fairly hard work for a relatively specialized result. But it provides a unified explanation for a good many apparently disparate phenomena. Of course the simplest example is when $X = C(Z)^*$ for some compact metrizable space Z and K is the set of positive linear functionals of norm 1, so that E can be identified with Z and we find ourselves back with the Riesz representation theorem. A less familiar case already examined is in 461Xk. At the next level we have such examples as 461Xl.

Another class of examples arising in §437 is explored in 461Q-461R, 461Xm-461Xn and 461Yd-461Ye. It is when we have an explicit listing of the extreme points, as in 461Yb and 461Ye, that we can begin to feel that we understand a compact convex set.

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462 Pointwise compact sets of continuous functions

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In preparation for the main work of this chapter, beginning in the next section, I offer a few pages on spaces of continuous functions under their ‘pointwise’ topologies (462Ab). There is an extensive general theory of such spaces, described in ARKHANGEL’SII 92; here I present only those fragments which seem directly relevant to the theory of measures on normed spaces and spaces of functions. In particular, I star the paragraphs 462C-462D, which are topology and functional analysis rather than measure theory. They are here because although this material is well known, and may be found in many places, I think that the ideas, as well as the results, are essential for any understanding of measures on linear topological spaces.

Measure theory enters the section in the proof of 462E, in the form of an application of the Riesz representation theorem, though 462E itself remains visibly part of functional analysis. In the rest of the section, however, we come to results which are pure measure theory. For (countably) compact spaces X , the Radon measures on $C(X)$ are the same for the pointwise and norm topologies (462I). This fact has extensive implications for the theory of separately continuous functions (462K) and for the theory of convex hulls in linear topological spaces (462L).

462A Definitions (a) A regular Hausdorff space X is **angelic** if whenever A is a subset of X which is relatively countably compact in X , then (i) its closure \bar{A} is compact (ii) every point of \bar{A} is the limit of a sequence in A .

(A **Fréchet-Urysohn** space is a topological space in which, for any set A , every point of the closure of A is a limit of a sequence in A . So (ii) here may be written as ‘every compact subspace of X is a Fréchet-Urysohn space’.)

(b) If X is any set and A a subset of \mathbb{R}^X , then the topology of **pointwise convergence** on A is that inherited from the usual product topology of \mathbb{R}^X ; that is, the coarsest topology on A for which the map $f \mapsto f(x) : A \rightarrow \mathbb{R}$ is continuous for every $x \in X$. I shall commonly use the symbol \mathfrak{T}_p for such a topology. In this context, I will say that a sequence or filter is **pointwise convergent** if it is convergent for the topology of pointwise convergence. Note that if A is a linear subspace of \mathbb{R}^X then \mathfrak{T}_p is a linear space topology on A (4A4Ba).

***462B Proposition** (PRYCE 71) Let (X, \mathfrak{T}) be an angelic regular Hausdorff space.

- (a) Any subspace of X is angelic.
- (b) If \mathfrak{S} is a regular topology on X finer than \mathfrak{T} , then \mathfrak{S} is angelic.
- (c) Any countably compact subset of X is compact and sequentially compact.

proof (a) Let Y be any subset of X . Then of course the subspace topology on Y is regular and Hausdorff. If $A \subseteq Y$ is relatively countably compact in Y , then A is relatively countably compact in X , so \bar{A} , the closure of A in X , is compact. Now if $x \in \bar{A}$, there is a sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ in A converging to x ; but $\langle x_n \rangle_{n \in \mathbb{N}}$ must have a cluster point in Y , and (because \mathfrak{T} is Hausdorff) this cluster point can only be x . Accordingly $\bar{A} \subseteq Y$ and is the closure of A in Y . Thus A is relatively compact in Y . Moreover, any point of \bar{A} is the limit of a sequence in A . As A is arbitrary, Y is angelic.

(b) By hypothesis, \mathfrak{S} is regular, and it is Hausdorff because it is finer than \mathfrak{T} . Now suppose that $A \subseteq X$ is \mathfrak{S} -relatively countably compact. Because the identity map from (X, \mathfrak{S}) to (X, \mathfrak{T}) is continuous, A is \mathfrak{T} -relatively countably compact (4A2G(f-iv)), and the \mathfrak{T} -closure \bar{A} of A is \mathfrak{T} -compact.

Let \mathcal{F} be any ultrafilter on X containing A . Then \mathcal{F} has a \mathfrak{T} -limit $x \in X$. **?** If \mathcal{F} is not \mathfrak{S} -convergent to x , there is an $H \in \mathfrak{S}$ such that $x \in H$ and $X \setminus H \in \mathcal{F}$, so that $A \setminus H \in \mathcal{F}$. Now x belongs to the \mathfrak{T} -closure of $A \setminus H$, because $A \setminus H \in \mathcal{F}$ and \mathcal{F} is \mathfrak{T} -convergent to x ; because \mathfrak{T} is angelic, there is a sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ in $A \setminus H$ which \mathfrak{T} -converges to x . But now $\langle x_n \rangle_{n \in \mathbb{N}}$ has a \mathfrak{S} -cluster point x' . x' must also be a \mathfrak{T} -cluster point of $\langle x_n \rangle_{n \in \mathbb{N}}$, so $x' = x$; but every x_n belongs to the \mathfrak{S} -closed set $X \setminus H$, so $x' \notin H$, which is impossible. **■**

Thus every ultrafilter on X containing A is \mathfrak{S} -convergent. Because \mathfrak{S} is regular, the \mathfrak{S} -closure \tilde{A} of A is \mathfrak{S} -compact (3A3De).

Again because \mathfrak{S} is finer than \mathfrak{T} , and \mathfrak{T} is Hausdorff, the two topologies must agree on $\tilde{A} = \bar{A}$. But now every point of \bar{A} is the \mathfrak{T} -limit of a sequence in A , so every point of \tilde{A} is the \mathfrak{S} -limit of a sequence in A . As A is arbitrary, \mathfrak{S} is angelic.

(c) If $K \subseteq X$ is countably compact, then of course it is relatively countably compact in its subspace topology, so (being angelic) must be compact in its subspace topology. If $\langle x_n \rangle_{n \in \mathbb{N}}$ is a sequence in K , let

x be any cluster point of $\langle x_n \rangle_{n \in \mathbb{N}}$. If $\{n : x_n = x\}$ is infinite, then this immediately provides us with a subsequence converging to x . Otherwise, take n such that $x \neq x_i$ for $i \geq n$. Since $x \in \overline{\{x_i : i \geq n\}}$, and $\{x_i : i \geq n\}$ is relatively countably compact, there is a sequence $\langle y_i \rangle_{i \in \mathbb{N}}$ in $\{x_i : i \geq n\}$ converging to x . The topology of X being Hausdorff, $\{y_i : i \in \mathbb{N}\}$ must be infinite, and $\langle y_i \rangle_{i \in \mathbb{N}}, \langle x_i \rangle_{i \in \mathbb{N}}$ have a common subsequence which converges to x . As $\langle x_i \rangle_{i \in \mathbb{N}}$ is arbitrary, K is sequentially compact.

***462C Theorem** (PRYCE 71) Let X be a topological space such that there is a sequence $\langle X_n \rangle_{n \in \mathbb{N}}$ of relatively countably compact subsets of X , covering X , with the property that a function $f : X \rightarrow \mathbb{R}$ is continuous whenever $f|X_n$ is continuous for every $n \in \mathbb{N}$. Then the space $C(X)$ of continuous real-valued functions on X is angelic in its topology of pointwise convergence.

proof Of course $C(X)$ is regular and Hausdorff under \mathfrak{T}_p , because \mathbb{R}^X is, so we need attend only to the rest of the definition in 462Aa. Let $A \subseteq C(X)$ be relatively countably compact for \mathfrak{T}_p .

(a) Since $\{f(x) : f \in A\}$, being a continuous image of A , must be relatively countably compact in \mathbb{R} (4A2G(f-iv)), therefore relatively compact (4A2Le), for every $x \in X$, the closure \overline{A} of A in \mathbb{R}^X is compact, by Tychonoff's theorem.

? Suppose, if possible, that $\overline{A} \not\subseteq C(X)$; let $g \in \overline{A}$ be a discontinuous function. By the hypothesis of the theorem, there is an $n \in \mathbb{N}$ such that $g|X_n$ is not continuous; take $x^* \in X_n$ such that $g|X_n$ is discontinuous at x^* . Let $\epsilon > 0$ be such that for every neighbourhood U of x^* in X_n there is a point $x \in U$ such that $|g(x) - g(x^*)| \geq \epsilon$.

Choose sequences $\langle f_i \rangle_{i \in \mathbb{N}}$ in A and $\langle x_i \rangle_{i \in \mathbb{N}}$ in X_n as follows. Given $\langle f_i \rangle_{i < m}$ and $\langle x_i \rangle_{i < m}$, choose $x_m \in X_n$ such that $|f_i(x_m) - f_i(x^*)| \leq 2^{-m}$ for every $i < m$ and $|g(x^*) - g(x_m)| \geq \epsilon$. Now choose $f_m \in A$ such that $|f_m(x^*) - g(x^*)| \leq 2^{-m}$ and $|f_m(x_i) - g(x_i)| \leq 2^{-m}$ for every $i \leq m$. Continue.

At the end of the induction, take a cluster point x of $\langle x_i \rangle_{i \in \mathbb{N}}$ in X and a cluster point f of $\langle f_i \rangle_{i \in \mathbb{N}}$ in $C(X)$. Because $|f_i(x_m) - f_i(x^*)| \leq 2^{-m}$ whenever $i < m$, $f_i(x) = f_i(x^*)$ for every i , and $f(x) = f(x^*)$. Because $|f_m(x^*) - g(x^*)| \leq 2^{-m}$ for every m , $f(x^*) = g(x^*)$. Because $|f_m(x_i) - g(x_i)| \leq 2^{-m}$ whenever $i \leq m$, $f(x_i) = g(x_i)$ for every i , $|g(x^*) - f(x_i)| \geq \epsilon$ for every i , and $|g(x^*) - f(x)| \geq \epsilon$; but this is impossible, because $f(x) = f(x^*) = g(x^*)$. **X**

Thus the compact set $\overline{A} \subseteq C(X)$ is the closure of A in $C(X)$, and A is relatively compact in $C(X)$.

(b) Now take any $g \in \overline{A}$. There are countable sets $D \subseteq X$, $B \subseteq A$ such that

whenever $I \subseteq B \cup \{g\}$ is finite, $n \in \mathbb{N}$, $\epsilon > 0$ and $x \in X_n$, there is a $y \in D \cap X_n$ such that $|f(y) - f(x)| \leq \epsilon$ for every $f \in I$;

whenever $J \subseteq D$ is finite and $\epsilon > 0$ there is an $f \in B$ such that $|f(x) - g(x)| \leq \epsilon$ for every $x \in J$.

P For any finite set $I \subseteq \mathbb{R}^X$ and $n \in \mathbb{N}$, the set $Q_{In} = \{\langle f(x) \rangle_{f \in I} : x \in X_n\}$ is a subset of the separable metrizable space \mathbb{R}^I , so is itself separable, and there is a countable dense set $D_{In} \subseteq X_n$ such that $Q'_{In} = \{\langle f(x) \rangle_{f \in I} : x \in D_{In}\}$ is dense in Q_{In} . Similarly, because $g \in \overline{A}$, we can choose for any finite set $J \subseteq X$ a sequence $\langle f_{Ji} \rangle_{i \in \mathbb{N}}$ in A such that $\lim_{i \rightarrow \infty} f_{Ji}(x) = g(x)$ for every $x \in J$.

Now construct $\langle D_m \rangle_{m \in \mathbb{N}}, \langle B_m \rangle_{m \in \mathbb{N}}$ inductively by setting

$$D_m = \bigcup \{D_{In} : n \in \mathbb{N}, I \subseteq \{g\} \cup \bigcup_{i < m} B_i \text{ is finite}\},$$

$$B_m = \{f_{Jk} : k \in \mathbb{N}, J \subseteq \bigcup_{i < m} D_i \text{ is finite}\}.$$

At the end of the induction, set $D = \bigcup_{m \in \mathbb{N}} D_m$ and $B = \bigcup_{m \in \mathbb{N}} B_m$. Since the construction clearly ensures that $\langle D_m \rangle_{m \in \mathbb{N}}$ and $\langle B_m \rangle_{m \in \mathbb{N}}$ are non-decreasing sequences of countable sets, D and B are countable, and we shall have $D_{In} \subseteq D$ whenever $n \in \mathbb{N}$ and $I \subseteq B \cup \{g\}$ is finite, while $f_{Ji} \in B$ whenever $i \in \mathbb{N}$ and $J \subseteq D$ is finite. Thus we have suitable sets D and B . **Q**

By the second condition on D and B , there must be a sequence $\langle f_i \rangle_{i \in \mathbb{N}}$ in B such that $g(x) = \lim_{i \rightarrow \infty} f_i(x)$ for every $x \in D$. In fact $g(y) = \lim_{i \rightarrow \infty} f_i(y)$ for every $y \in X$. **P?** Otherwise, there is an $\epsilon > 0$ such that $J = \{i : |g(y) - f_i(y)| \geq \epsilon\}$ is infinite. Let n be such that $y \in X_n$. For each $m \in \mathbb{N}$, $I_m = \{f_i : i \leq m\}$ is a finite subset of B , so there is an $x_m \in D_{I_m n}$ such that $|f(x_m) - f(y)| \leq 2^{-m}$ for every $f \in I_m \cup \{g\}$. Let $x^* \in X$ be a cluster point of $\langle x_m \rangle_{m \in \mathbb{N}}$, and $h \in C(X)$ a cluster point of $\langle f_i \rangle_{i \in \mathbb{N}}$. Then

because $g(x) = \lim_{i \rightarrow \infty} f_i(x)$ for every $x \in D$, $g(x_m) = h(x_m)$ for every $m \in \mathbb{N}$, and $g(x^*) = h(x^*)$;

because $|g(y) - f_i(y)| \geq \epsilon$ for every $i \in J$, $|g(y) - h(y)| \geq \epsilon$;

because $|f_i(x_m) - f_i(y)| \leq 2^{-m}$ whenever $i \leq m$, $f_i(x^*) = f_i(y)$ for every $i \in \mathbb{N}$, and $h(x^*) = h(y)$;

because $|g(x_m) - g(y)| \leq 2^{-m}$ for every m , $g(x^*) = g(y)$;

but this means that $g(y) = g(x^*) = h(x^*) = h(y) \neq g(y)$, which is absurd. **XQ**

So $g = \lim_{i \rightarrow \infty} f_i$ for \mathfrak{T}_p . As g is arbitrary, A has both properties required in 462Aa; as A is arbitrary, $C(X)$ is angelic.

Remark For a slight strengthening of this result, see 462Ya.

***462D Theorem** Let U be any normed space. Then it is angelic in its weak topology.

proof Write X for the unit ball of the dual space U^* , with its weak* topology. Then X is compact (3A5F). We have a natural map $u \mapsto \hat{u} : U \rightarrow \mathbb{R}^X$ defined by setting $\hat{u}(x) = x(u)$ for $x \in X$ and $u \in U$. By the definition of the weak* topology, $\hat{u} \in C(X)$ for every $u \in U$. The weak topology of U is normally defined in terms of all functionals $u \mapsto f(u)$, for $f \in U^*$; but as every member of U^* is a scalar multiple of some $x \in X$, we can equally regard the weak topology of U as defined just by the functionals $u \mapsto x(u) = \hat{u}(x)$, for $x \in X$. But this means that the map $u \mapsto \hat{u}$ is a homeomorphism between U , with its weak topology, and its image \hat{U} in $C(X)$, with the topology of pointwise convergence.

Since $C(X)$ is \mathfrak{T}_p -angelic (462C), so is \hat{U} (462Ba), and U is angelic in its weak topology.

462E Theorem Let X be a locally compact Hausdorff space, and $C_0(X)$ the Banach lattice of continuous real-valued functions on X which vanish at infinity (436I). Write \mathfrak{T}_p for the topology of pointwise convergence on $C_0(X)$.

(i) $C_0(X)$ is \mathfrak{T}_p -angelic.

(ii) A sequence $\langle u_n \rangle_{n \in \mathbb{N}}$ in $C_0(X)$ is weakly convergent to $u \in C_0(X)$ iff it is \mathfrak{T}_p -convergent to u and norm-bounded.

(iii) A subset K of $C_0(X)$ is weakly compact iff it is norm-bounded and \mathfrak{T}_p -countably compact.

proof (a) Let $X^* = X \cup \{x_\infty\}$ be the one-point compactification of X (3A3O). Then $C(X^*)$ is angelic in its topology \mathfrak{T}_p^* of pointwise convergence, by 462C. Set $V = \{g : g \in C(X^*), g(x_\infty) = 0\}$. By 462Ba, V is angelic in the subspace topology induced by \mathfrak{T}_p^* . Now observe that we have a natural bijection $g \mapsto g|_X : V \rightarrow C_0(X)$, and that this is a homeomorphism for the topologies of pointwise convergence on V and $C_0(X)$. So $C_0(X)$ is angelic under \mathfrak{T}_p .

(b) Since all the maps $u \mapsto u(x)$, where $x \in X$, are bounded linear functionals on $C_0(X)$, \mathfrak{T}_p is coarser than the weak topology \mathfrak{T}_s ; so a \mathfrak{T}_s -convergent sequence is \mathfrak{T}_p -convergent to the same limit, and a \mathfrak{T}_s -compact set is \mathfrak{T}_p -compact, therefore \mathfrak{T}_p -countably compact.

(c) If $K \subseteq C_0(X)$ is \mathfrak{T}_s -compact, then $f[K] \subseteq \mathbb{R}$ must be compact, therefore bounded, for every $f \in C_0(X)^*$; by the Uniform Boundedness Theorem (3A5Hb), K is norm-bounded. Applying this to $\{u\} \cup \{u_n : n \in \mathbb{N}\}$, we see that any \mathfrak{T}_s -convergent sequence $\langle u_n \rangle_{n \in \mathbb{N}}$ with limit u is norm-bounded.

(d) Suppose that $\langle u_n \rangle_{n \in \mathbb{N}}$ is norm-bounded and \mathfrak{T}_p -convergent to u . By Lebesgue's Dominated Convergence Theorem, $\lim_{n \rightarrow \infty} \int u_n d\mu = \int u d\mu$ for every totally finite Radon measure μ on X . But by the Riesz Representation Theorem (in the form 436K), this says just that $\lim_{n \rightarrow \infty} f(u_n) = f(u)$ for every positive linear functional f on $C_0(X)$. Since every member of $C_0(X)^*$ is expressible as the difference of two positive linear functionals (356Dc), $\lim_{n \rightarrow \infty} f(u_n) = f(u)$ for every $f \in U^*$, that is, $\langle u_n \rangle_{n \in \mathbb{N}}$ is \mathfrak{T}_s -convergent to u .

Putting this together with (b) and (c), we see that (ii) is true.

(e) Now suppose that $K \subseteq C_0(X)$ is norm-bounded and \mathfrak{T}_p -countably compact. Any sequence $\langle u_n \rangle_{n \in \mathbb{N}}$ in K has a subsequence which is \mathfrak{T}_p -convergent to a point of K (462Bc), and this subsequence is also \mathfrak{T}_s -convergent, by (c). This means that K is sequentially compact, therefore countably compact, in $C_0(X)$ for the topology \mathfrak{T}_s . Since \mathfrak{T}_s is angelic (462D), K is \mathfrak{T}_s -compact, by 462Bc again.

462F Lemma Let X be a topological space, and Q a relatively countably compact subset of X . Suppose that $K \subseteq C_b(X)$ is $\|\cdot\|_\infty$ -bounded and \mathfrak{T}_p -countably compact, where \mathfrak{T}_p is the topology of pointwise convergence on $C_b(X)$. Then the map $u \mapsto u|_Q : K \rightarrow C_b(Q)$ is continuous for \mathfrak{T}_p on K and the weak topology of the Banach space $C_b(Q)$.

proof We have a natural map $x \mapsto \hat{x} : X \rightarrow \mathbb{R}^K$ defined by writing $\hat{x}(u) = u(x)$ for every $u \in K$ and $x \in X$. By the definition of \mathfrak{T}_p , $\hat{x} \in C(K)$ for every $x \in X$, if we take $C(K)$ to be the space of \mathfrak{T}_p -continuous real-valued functions on K ; and $x \mapsto \hat{x} : X \rightarrow C(K)$ is continuous for the given topology on X and the topology of pointwise convergence on $C(K)$ because $K \subseteq C(X)$. It follows that $\{\hat{x} : x \in Q\}$ is relatively countably compact for the topology of pointwise convergence on $C(K)$ (4A2G(f-iv)). But now $Z = \overline{\{\hat{x} : x \in Q\}}$ must be actually compact for the topology of pointwise convergence on $C(K)$, by 462C.

Next, consider the natural map $u \mapsto \hat{u} : K \rightarrow \mathbb{R}^Z$ defined by setting $\hat{u}(f) = f(u)$ for $f \in Z$ and $u \in K$. Just as in the last paragraph, this is a continuous function from K to $C(Z)$, if we give K , Z and $C(Z)$ their topologies of pointwise convergence. So $L = \{\hat{u} : u \in K\}$ is countably compact for the topology of pointwise convergence on $C(Z)$ (4A2G(f-vi)). Moreover, it is norm-bounded, because

$$\begin{aligned} \sup_{\phi \in L} \|\phi\|_\infty &= \sup_{u \in K, f \in Z} |\hat{u}(f)| = \sup_{u \in K, f \in Z} |f(u)| = \sup_{u \in K, x \in Q} |\hat{x}(u)| \\ &= \sup_{u \in K, x \in Q} |u(x)| \leq \sup_{u \in K, x \in X} |u(x)| = \sup_{u \in K} \|u\|_\infty \end{aligned}$$

is finite. So 462E(iii) tells us that L is weakly compact in $C(Z)$. (Note that $C(Z) = C_0(Z)$ because Z is compact.) Since the weak topology on $C(Z)$ is finer than the pointwise topology, while the pointwise topology is Hausdorff, the two topologies on L coincide; it follows that $u \mapsto \hat{u} : K \rightarrow C(Z)$ is continuous for \mathfrak{T}_p and the weak topology on $C(Z)$.

Now we have an operator $T : C(Z) \rightarrow \mathbb{R}^Q$ defined by setting

$$(T\phi)(x) = \phi(\hat{x})$$

for $\phi \in C(Z)$ and $x \in Q$. Because $x \mapsto \hat{x} : Q \rightarrow Z$ is continuous, $T\phi \in C(Q)$ for every $\phi \in C(Z)$, and of course T , regarded as a linear operator from $C(Z)$ to $C_b(Q)$, has norm at most 1. So T is continuous for the weak topologies of $C(Z)$ and $C_b(Q)$ (2A5If), and $u \mapsto T\hat{u} : K \rightarrow C_b(Q)$ is continuous for \mathfrak{T}_p and the weak topology of $C_b(Q)$.

But if $u \in K$ and $x \in Q$,

$$(T\hat{u})(x) = \hat{u}(\hat{x}) = \hat{x}(u) = u(x),$$

so $T\hat{u} = u|_Q$. Accordingly $u \mapsto u|_Q : K \rightarrow C_b(Q)$ is continuous for \mathfrak{T}_p and the weak topology on $C_b(Q)$.

462G Proposition Let X be a countably compact topological space. Then a subset of $C_b(X)$ is weakly compact iff it is norm-bounded and compact for the topology \mathfrak{T}_p of pointwise convergence.

proof A weakly compact subset of $C_b(X)$ is norm-bounded and \mathfrak{T}_p -compact by the same arguments as in (b)-(c) of the proof of 462E. In the other direction, taking $Q = X$ in 462F, we see that a norm-bounded \mathfrak{T}_p -compact set is weakly compact.

462H Lemma Let X be a topological space, Q a relatively countably compact subset of X , and μ a totally finite measure on $C_b(X)$ which is Radon for the topology \mathfrak{T}_p of pointwise convergence on $C_b(X)$. Let $T : C_b(X) \rightarrow C_b(Q)$ be the restriction map. Then the image measure $\nu = \mu T^{-1}$ on $C_b(Q)$ is Radon for the norm topology of $C_b(Q)$.

proof (a) T is almost continuous for \mathfrak{T}_p and the weak topology of $C_b(Q)$. **P** If $E \in \text{dom } \mu$ and $\mu E > \gamma \geq 0$, then there is a \mathfrak{T}_p -compact set $K \subseteq E$ such that $\mu K > \gamma$. Since all the balls $\{f : f \in C_b(X), \|f\|_\infty \leq k\}$ are \mathfrak{T}_p -closed, we may suppose that K is norm-bounded. Now $T|_K$ is continuous for \mathfrak{T}_p and the weak topology, by 462F. **Q** By 418I, ν is a Radon measure for the weak topology of $C_b(Q)$.

(b) I show next that if $F \in \text{dom } \nu$, $\nu F > 0$ and $\epsilon > 0$, there is some $g \in C_b(Q)$ such that $\nu(F \cap B(g, \epsilon)) > 0$, where $B(g, \epsilon) = \{h : \|h - g\|_\infty \leq \epsilon\}$. **P** Since all the balls $B(g, \epsilon)$ are convex and norm-closed, they are weakly closed (3A5Ee) and measured by ν . **?** Suppose, if possible, that $F \cap B(g, \epsilon)$ is ν -negligible for every

$g \in C_b(Q)$. Set $E = T^{-1}[F]$. As in (a), there is a $\|\cdot\|_\infty$ -bounded \mathfrak{T}_p -compact set $K \subseteq E$ such that $\mu K > 0$. Choose $\langle K_n \rangle_{n \in \mathbb{N}}$ and $\langle f_n \rangle_{n \in \mathbb{N}}$ as follows. $K_0 = K$. Given that $K_n \subseteq E$ is non-negligible and \mathfrak{T}_p -compact, and that $\langle f_i \rangle_{i < n}$ is a finite sequence in $C_b(X)$, then the convex hull

$$\Gamma_n = \left\{ \sum_{i=0}^{n-1} \alpha_i T f_i : \alpha_i \geq 0 \text{ for every } i < n, \sum_{i=0}^{n-1} \alpha_i = 1 \right\}$$

of the finite set $\{T f_i : i < n\}$ is norm-compact in $C_b(Q)$, so there is a finite set $D_n \subseteq \Gamma_n$ such that for every $g \in \Gamma_n$ there is a $g' \in D_n$ such that $\|g - g'\|_\infty \leq \frac{1}{2}\epsilon$. Now

$$H_n = \{f : \|Tf - g\|_\infty > \epsilon \text{ for every } g \in D_n\}$$

is a \mathfrak{T}_p -open set and

$$E \setminus H_n = \bigcup_{g \in D_n} T^{-1}[F \cap B(g, \epsilon)]$$

is μ -negligible, so $\mu(K_n \cap H_n) > 0$ and we can find a non-negligible \mathfrak{T}_p -closed set $K_{n+1} \subseteq K_n \cap H_n$; choose $f_n \in K_{n+1}$. Continue.

At the end of the induction, let $f^* \in K$ be a cluster point of $\langle f_n \rangle_{n \in \mathbb{N}}$ for \mathfrak{T}_p . Since $T|_K : K \rightarrow C_b(Q)$ is continuous for \mathfrak{T}_p and the weak topology of $C_b(Q)$, Tf^* is a cluster point of $\langle Tf_n \rangle_{n \in \mathbb{N}}$ for the weak topology on $C_b(Q)$. The set $\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n$ is convex, so its norm-closure $\bar{\Gamma}$ is also convex (2A5Eb), therefore closed for the weak topology (3A5Ee), and contains Tf^* . So there is a $g \in \Gamma$ such that $\|Tf^* - g\|_\infty \leq \frac{1}{2}\epsilon$. Now there is some n such that $g \in \Gamma_n$. Let $g' \in D_n$ be such that $\|g - g'\|_\infty \leq \frac{1}{2}\epsilon$, so that $\|Tf^* - g'\|_\infty \leq \epsilon$. But $f_i \in K_{n+1}$ for every $i \geq n$, so $f^* \in K_{n+1} \subseteq H_n$ and $\|Tf^* - g'\|_\infty > \epsilon$, which is impossible. **XQ**

(c) What this means is that if we take \mathcal{K}_n to be the family of subsets of $C_b(Q)$ which can be covered by finitely many balls of radius at most 2^{-n} , then ν is inner regular with respect to \mathcal{K}_n (see 412Aa), and therefore with respect to $\mathcal{K} = \bigcap_{n \in \mathbb{N}} \mathcal{K}_n$ (412Ac). But \mathcal{K} is just the set of subsets of $C_b(Q)$ which are totally bounded for the norm-metric ρ on $C_b(Q)$.

At the same time, ν is inner regular with respect to the ρ -closed sets in $C_b(Q)$. **P** If $\nu F > \gamma$, there is a $\|\cdot\|_\infty$ -bounded \mathfrak{T}_p -compact set $K \subseteq T^{-1}[F]$ such that $\mu K \geq \gamma$; now $T[K]$ is weakly compact, therefore weakly closed and ρ -closed in $C_b(Q)$, while $T[K] \subseteq F$ is measured by ν and

$$\nu T[K] = \mu T^{-1}[T[K]] \geq \mu K \geq \gamma. \quad \mathbf{Q}$$

(d) By 412Ac again, ν must be inner regular with respect to the family of ρ -closed ρ -totally bounded sets; because $C_b(Q)$ is ρ -complete, these are the ρ -compact sets. Next, every ρ -compact set is weakly compact, therefore weakly closed, and is measured by ν , by (a); and ν , being the image of a complete totally finite measure, is complete and totally finite. Consequently every ρ -closed set is measured by ν (use 412Ja) and ν is a ρ -Radon measure, as claimed.

462I Theorem Let X be a countably compact topological space. Then the totally finite Radon measures on $C(X)$ are the same for the topology of pointwise convergence and the norm topology.

proof Write \mathfrak{T}_p for the topology of pointwise convergence on $C(X)$ and \mathfrak{T}_∞ for the norm topology. Because $\mathfrak{T}_p \subseteq \mathfrak{T}_\infty$ and \mathfrak{T}_p is Hausdorff, every totally finite \mathfrak{T}_∞ -Radon measure is \mathfrak{T}_p -Radon (418I). On the other hand, 462H, with $Q = X$, tells us that every \mathfrak{T}_p -Radon measure is \mathfrak{T}_∞ -Radon.

462J Corollary Let X be a countably compact Hausdorff space, and give $C(X)$ its topology of pointwise convergence. If μ is any Radon measure on $C(X)$, it is inner regular with respect to the family of compact metrizable subsets of $C(X)$.

proof In the language of 462I, μ is inner regular with respect to the family of \mathfrak{T}_∞ -compact sets; but as $\mathfrak{T}_p \subseteq \mathfrak{T}_\infty$, the two topologies agree on all such sets, and they are compact and metrizable for \mathfrak{T}_p .

462K Proposition Let X be a topological space, Y a Hausdorff space, $f : X \times Y \rightarrow \mathbb{R}$ a bounded separately continuous function, and ν a totally finite Radon measure on Y . Set $\phi(x) = \int f(x, y)\nu(dy)$ for every $x \in X$. Then $\phi|_Q$ is continuous for every relatively countably compact set $Q \subseteq X$.

proof For $y \in Y$, set $u_y(x) = f(x, y)$ for every $x \in X$. Then every u_y is continuous and bounded, because f is bounded and continuous in the first variable, and $y \mapsto u_y : Y \rightarrow C_b(X)$ is continuous, if we give $C_b(X)$

the topology \mathfrak{T}_p of pointwise convergence, because f is continuous in the second variable. We therefore have a \mathfrak{T}_p -Radon image measure μ on $C_b(X)$, by 418I.

Let $T : C_b(X) \rightarrow C_b(Q)$ be the restriction map. By 462H, the image measure $\lambda = \mu T^{-1}$ is a Radon measure for the norm topology of $C_b(Q)$. Now recall that f is bounded. If $|f(x, y)| \leq M$ for all $x \in X$, $y \in Y$, then $\|u_y\|_\infty \leq M$ for every $y \in Y$, and the ball $B(0, M)$ in $C_b(Q)$ is λ -conegligible. By 461F, applied to the subspace measure on $B(0, M)$, ν has a barycenter h in $C_b(Q)$. Now we can compute h by the formulae

$$\begin{aligned} h(x) &= \int_{C_b(Q)} g(x) \lambda(dg) \\ &\text{(because } g \mapsto g(x) \text{ belongs to } C_b(X)^*) \\ &= \int_{C_b(X)} u(x) \mu(du) = \int_Y u_y(x) \nu(dy) \\ &\text{(by 235G)} \\ &= \int_Y f(x, y) \nu(dy) = \phi(x), \end{aligned}$$

for every $x \in Q$. So $\phi = h$ is continuous.

462L Corollary Let X be a topological space such that

whenever $h \in \mathbb{R}^X$ is such that $h \upharpoonright Q$ is continuous for every relatively countably compact $Q \subseteq X$, then h is continuous.

Write \mathfrak{T}_p for the topology of pointwise convergence on $C(X)$. Let $K \subseteq C(X)$ be a \mathfrak{T}_p -compact set such that $\{h(x) : h \in K, x \in Q\}$ is bounded for any relatively countably compact set $Q \subseteq X$. Then the \mathfrak{T}_p -closed convex hull of K , taken in $C(X)$, is \mathfrak{T}_p -compact.

proof If K is empty, this is trivial; suppose that $K \neq \emptyset$. Since $\sup_{h \in K} |h(x)|$ is finite for every $x \in X$, the closed convex hull $\overline{\Gamma(K)}$ of K , taken in \mathbb{R}^X , is closed and included in a product of closed bounded intervals, therefore compact. If $h \in \overline{\Gamma(K)}$, then there is a Radon probability measure μ on K such that h is the barycenter of μ (461I), so that $h(x) = \int f(x) \mu(df)$ for every $x \in X$.

If $Q \subseteq X$ is relatively countably compact, then $h \upharpoonright Q$ is continuous. **P** Of course we may suppose that Q is non-empty. Consider its closure $Z = \overline{Q}$. We have a continuous linear operator $T : \mathbb{R}^X \rightarrow \mathbb{R}^Z$ defined by setting $Tf = f \upharpoonright Z$ for every $f \in \mathbb{R}^X$. $L = T[K]$ is compact in \mathbb{R}^Z , and $L \subseteq C(Z)$; moreover,

$$\sup_{g \in L} \|g\|_\infty = \sup_{f \in K, x \in Z} |f(x)| = \sup_{f \in K, x \in Q} |f(x)|$$

is finite. Since $T \upharpoonright K : K \rightarrow L$ is continuous, the image measure $\nu = \mu(T \upharpoonright K)^{-1}$ on L is a Radon measure. If $x \in Z$, then

$$h(x) = \int_K f(x) \mu(df) = \int_K (Tf)(x) \mu(df) = \int_L g(x) \nu(dg).$$

The map $(x, g) \mapsto g(x) : Z \times L \rightarrow \mathbb{R}$ is separately continuous, because $L \subseteq C(Z)$ is being given its topology of pointwise convergence, and bounded. Also every sequence in Q has a cluster point in X which must also belong to Z , and Q is relatively countably compact in Z . By 462K, $h \upharpoonright Q$ is continuous, as required.

Q

Thus the \mathfrak{T}_p -compact set $\overline{\Gamma(K)}$ is included in $C(X)$, and must be the closed convex hull of K in $C(X)$.

Remark The hypothesis

whenever $h \in \mathbb{R}^X$ is such that $h \upharpoonright Q$ is continuous for every relatively countably compact $Q \subseteq X$, then h is continuous

is clumsy, but seems the best way to cover the large number of potential applications of the ideas here. Besides the obvious case of countably compact spaces X , we have all first-countable spaces (for which, of course, the other hypotheses can be relaxed, as in 462Xc), and all k -spaces. (A k -space is a topological space X such that a set $G \subseteq X$ is open iff $G \cap K$ is relatively open in K for every compact set $K \subseteq X$; see ENGELKING 89, 3.3.18 *et seq.* In particular, all locally compact spaces are k -spaces.)

462X Basic exercises (a) (i) Show that the Sorgenfrey line (415Xc) is angelic. (ii) Show that any metrizable space is angelic. (iii) Show that the one-point compactification of an angelic locally compact Hausdorff space is angelic. (iv) Find a first-countable regular Hausdorff space which is not angelic.

>(b) Let X be any countably compact topological space. Show that a norm-bounded sequence in $C_b(X)$ which is pointwise convergent is weakly convergent.

(c) Let X be a first-countable topological space, (Y, \mathbb{T}, ν) a totally finite measure space, and $f : X \times Y \rightarrow \mathbb{R}$ a bounded function such that $y \mapsto f(x, y)$ is measurable for every $x \in X$, and $x \mapsto f(x, y)$ is continuous for almost every $y \in Y$. Show that $x \mapsto \int f(x, y)\nu(dy)$ is continuous.

(d) Give an example of a \mathfrak{T}_p -compact subset K of $C([0, 1])$ such that the convex hull of K is not relatively compact in $C([0, 1])$.

462Y Further exercises (a) Let X be a topological space such that there is a sequence $\langle X_n \rangle_{n \in \mathbb{N}}$ of relatively countably compact subsets of X , covering X , with the property that a function $f : X \rightarrow \mathbb{R}$ is continuous whenever $f \upharpoonright X_n$ is continuous for every $n \in \mathbb{N}$. Let \mathfrak{T}_p be the topology of pointwise convergence on $C(X)$. Show that, for a set $K \subseteq C(X)$, the following are equiveridical: (i) $\phi[K]$ is bounded for every \mathfrak{T}_p -continuous function $\phi : C(X) \rightarrow \mathbb{R}$; (ii) whenever $\langle f_n \rangle_{n \in \mathbb{N}}$ is a sequence in K and $A \subseteq X$ is countable, there is a cluster point of $\langle f_n \upharpoonright A \rangle_{n \in \mathbb{N}}$ in $C(A)$ for the topology of pointwise convergence on $C(A)$; (iii) K is relatively compact in $C(X)$ for \mathfrak{T}_p . (See ASANOV & VELICHKO 81.)

(b) Let U be a metrizable locally convex linear topological space. Show that it is angelic in its weak topology. (*Hint*: start with the case in which U is complete, using Grothendieck's theorem and the full strength of 462C, with $X = U^*$.)

(c) In 462K, show that the conclusion remains valid for any totally finite τ -additive topological measure ν on Y which is inner regular with respect to the relatively countably compact subsets of Y .

(d) Show that if X is any compact topological space (more generally, any topological space such that X^n is Lindelöf for every $n \in \mathbb{N}$), then $C(X)$, with its topology of pointwise convergence, is countably tight.

(e)(i) Let X and Y be Polish spaces, and write $B_1(X; Y)$ for the set of functions $f : X \rightarrow Y$ such that $f^{-1}[H]$ is G_δ in X for every closed set $H \subseteq Y$ (KURATOWSKI 66, §31). Show that $B_1(X; Y)$, with the topology of pointwise convergence inherited from Y^X , is angelic. (*Hint*: BOURGAIN FREMLIN & TALAGRAND 78.) **(ii)** Let X be a Polish space. Show that the space $\tilde{C}^1(X)$ of 438P-438Q is angelic.

462Z Problem Let K be a compact Hausdorff space. Is $C(K)$, with the topology of pointwise convergence, necessarily a pre-Radon space? (Compare 454S.)

462 Notes and comments The theory of pointwise convergence in spaces of continuous functions is intimately connected with the theory of separately continuous functions of two variables. For if X and Y are topological spaces, and $f : X \times Y \rightarrow \mathbb{R}$ is any separately continuous function, then we have natural maps $x \mapsto f_x : X \rightarrow C(Y)$ and $y \mapsto f^y : Y \rightarrow C(X)$, writing $f_x(y) = f^y(x) = f(x, y)$, which are continuous if $C(X)$ and $C(Y)$ are given their topologies of pointwise convergence; and if X is a topological space and Y is any subset of $C(X)$ with its topology of pointwise convergence, the map $(x, y) \mapsto y(x) : X \times Y \rightarrow \mathbb{R}$ is separately continuous. I include a back-and-forth shuffle between $C(X)$ and separately continuous functions in 462H-462K-462L as a demonstration of the principle that all the theorems here can be expressed in both languages.

462Yb is a compendium of Šmulian's theorem with part of Eberlein's theorem; 462E and 462L can be thought of as the centre of Krein's theorem. There are many alternative routes to these results, which may be found in KÖTHE 69 or GROTHENDIECK 92. In particular, 462E can be proved without using measure theory; see, for instance, FREMLIN 74, A2F.

Topological spaces homeomorphic to compact uniformly bounded subsets of $C(X)$, where X is some compact space and $C(X)$ is given its topology of pointwise convergence, are called **Eberlein compacta**; see 467O-467P.

A positive answer to A.Bellow's problem (463Za below) would imply a positive answer to 462Z; so if the continuum hypothesis, for instance, is true, then $C(K)$ is pre-Radon in its topology of pointwise convergence for any compact space K .

Version of 1.2.13

463 \mathfrak{T}_p and \mathfrak{T}_m

We are now ready to start on the central ideas of this chapter with an investigation of sets of measurable functions which are compact for the topology of pointwise convergence. Because 'measurability' is, from the point of view of this topology on \mathbb{R}^X , a rather arbitrary condition, we are looking at compact subsets of a topologically irregular subspace of \mathbb{R}^X ; there are consequently relatively few of them, and (under a variety of special circumstances, to be examined later in the chapter and also in Volume 5) they have some striking special properties.

The presentation here is focused on the relationship between the two natural topologies on any space of measurable functions, the 'pointwise' topology \mathfrak{T}_p and the topology \mathfrak{T}_m of convergence in measure (463A). In this section I begin with results which apply to any σ -finite measure space (463B-463H) before turning to some which apply to perfect measure spaces (463I-463L) – in particular, to Lebesgue measure. These lead to some interesting properties of separately continuous functions (463M-463N).

463A Preliminaries Let (X, Σ, μ) be a measure space, and $\mathcal{L}^0 = \mathcal{L}^0(\Sigma)$ the space of all Σ -measurable functions from X to \mathbb{R} , so that \mathcal{L}^0 is a linear subspace of \mathbb{R}^X . On \mathcal{L}^0 we shall be concerned with two very different topologies. The first is the topology \mathfrak{T}_p of pointwise convergence (462Ab); the second is the topology \mathfrak{T}_m of (local) convergence in measure (245A). Both are linear space topologies. **P** For \mathfrak{T}_p I have already noted this in 462Ab. For \mathfrak{T}_m , repeat the argument of 245Da; \mathfrak{T}_m is defined by the functionals $f \mapsto \int_F \min(1, |f|) d\mu$, where $\mu F < \infty$, and these are F-seminorms (definition: 2A5B). **Q** \mathfrak{T}_p is Hausdorff (3A3Id) and locally convex (4A4Ce); only in exceptional circumstances is either true of \mathfrak{T}_m . However, \mathfrak{T}_m can easily be pseudometrizable (if, for instance, μ is σ -finite, as in 245Eb), while \mathfrak{T}_p is not, except in nearly trivial cases.

Associated with the topology of pointwise convergence on \mathbb{R}^X is the usual topology of $\mathcal{P}X$ (4A2A); the map $\chi : \mathcal{P}X \rightarrow \mathbb{R}^X$ is a homeomorphism between $\mathcal{P}X$ and its image $\{0, 1\}^X \subseteq \mathbb{R}^X$.

\mathfrak{T}_m is intimately associated with the topology of convergence in measure on $L^0 = L^0(\mu)$ (§245). A subset of \mathcal{L}^0 is open for \mathfrak{T}_m iff it is of the form $\{f : f^\bullet \in G\}$ for some open set $G \subseteq L^0$; consequently, a subset K of \mathcal{L}^0 is compact, or separable, for \mathfrak{T}_m iff $\{f^\bullet : f \in K\}$ is compact or separable for the topology of convergence in measure on L^0 .

It turns out that the identity map from $(\mathcal{L}^0, \mathfrak{T}_p)$ to $(\mathcal{L}^0, \mathfrak{T}_m)$ is sequentially continuous (463B). Only in nearly trivial cases is it actually continuous (463Xa(i)), and it is similarly rare for the reverse map from $(\mathcal{L}^0, \mathfrak{T}_m)$ to $(\mathcal{L}^0, \mathfrak{T}_p)$ to be continuous (463Xa(ii)). If, however, we relativise both topologies to a \mathfrak{T}_p -compact subset of \mathcal{L}^0 , the situation becomes very different, and there are many important cases in which the topologies are comparable.

463B Lemma Let (X, Σ, μ) be a measure space, and \mathcal{L}^0 the space of Σ -measurable real-valued functions on X . Then every pointwise convergent sequence in \mathcal{L}^0 is convergent in measure to the same limit.

proof 245Ca.

463C Proposition (IONESCU TULCEA 73) Let (X, Σ, μ) be a measure space, and \mathcal{L}^0 the space of Σ -measurable real-valued functions on X . Write $\mathfrak{T}_p, \mathfrak{T}_m$ for the topologies of pointwise convergence and convergence in measure on \mathcal{L}^0 ; for $A \subseteq \mathcal{L}^0$, write $\mathfrak{T}_p^{(A)}, \mathfrak{T}_m^{(A)}$ for the corresponding subspace topologies.

(a) If $A \subseteq \mathcal{L}^0$ and $\mathfrak{T}_p^{(A)}$ is metrizable, then the identity map from A to itself is $(\mathfrak{T}_p^{(A)}, \mathfrak{T}_m^{(A)})$ -continuous.

(b) Suppose that μ is semi-finite. Then, for any $A \subseteq \mathcal{L}^0$, $\mathfrak{T}_m^{(A)}$ is Hausdorff iff whenever f, g are distinct members of A the set $\{x : f(x) \neq g(x)\}$ is non-negligible.

(c) Suppose that $K \subseteq \mathcal{L}^0$ is such that $\mathfrak{T}_p^{(K)}$ is compact and metrizable. Then $\mathfrak{T}_p^{(K)} = \mathfrak{T}_m^{(K)}$ iff $\mathfrak{T}_m^{(K)}$ is Hausdorff.

(d) Suppose that μ is σ -finite, and that $K \subseteq \mathcal{L}^0$ is \mathfrak{T}_p -sequentially compact. Then $\mathfrak{T}_p^{(K)} = \mathfrak{T}_m^{(K)}$ iff $\mathfrak{T}_m^{(K)}$ is Hausdorff, and in this case $\mathfrak{T}_p^{(K)}$ is compact and metrizable.

(e) Suppose that $K \subseteq \mathcal{L}^0$ is such that $\mathfrak{T}_p^{(K)}$ is compact and metrizable. Then whenever $\epsilon > 0$ and $E \in \Sigma$ is a non-negligible measurable set, there is a non-negligible measurable set $F \subseteq E$ such that $|f(x) - f(y)| \leq \epsilon$ whenever $f \in K$ and $x, y \in F$.

proof (a) All we need is to remember that sequentially continuous functions from metrizable spaces are continuous (4A2Ld), and apply 463B.

(b) $\mathfrak{T}_m^{(A)}$ is Hausdorff iff for any distinct $f, g \in A$ there is a measurable set F of finite measure such that $\int_F \min(1, |f - g|) d\mu > 0$, that is, $\mu\{x : x \in F, f(x) \neq g(x)\} > 0$; because μ is semi-finite, this happens iff $\mu\{x : f(x) \neq g(x)\} > 0$.

(c) If $\mathfrak{T}_p^{(K)} = \mathfrak{T}_m^{(K)}$ then of course $\mathfrak{T}_m^{(K)}$ is Hausdorff, because $\mathfrak{T}_p^{(K)}$ is. If $\mathfrak{T}_m^{(K)}$ is Hausdorff then the identity map $(K, \mathfrak{T}_p^{(K)}) \rightarrow (K, \mathfrak{T}_m^{(K)})$ is an injective function from a compact space to a Hausdorff space and (by (a)) is continuous, therefore a homeomorphism, so the two topologies are equal.

(d) If $\mathfrak{T}_p^{(K)} = \mathfrak{T}_m^{(K)}$ then $\mathfrak{T}_m^{(K)}$ must be Hausdorff, just as in (c). So let us suppose that $\mathfrak{T}_m^{(K)}$ is Hausdorff. Note that, by 245Eb, the topology of convergence in measure on L^0 is metrizable; in terms of \mathcal{L}^0 , this says just that the topology of convergence in measure on \mathcal{L}^0 is pseudometrizable. So $\mathfrak{T}_m^{(K)}$ is Hausdorff and pseudometrizable, therefore metrizable (4A2La).

We are told that any sequence in K has a $\mathfrak{T}_p^{(K)}$ -convergent subsequence. But this subsequence is now $\mathfrak{T}_m^{(K)}$ -convergent (463B), so $\mathfrak{T}_m^{(K)}$ is sequentially compact; being metrizable, it is compact (4A2Lf). Moreover, the same is true of any $\mathfrak{T}_p^{(K)}$ -closed subset of K , so every $\mathfrak{T}_p^{(K)}$ -closed set is $\mathfrak{T}_m^{(K)}$ -compact, therefore $\mathfrak{T}_m^{(K)}$ -closed. Thus the identity map from $(K, \mathfrak{T}_m^{(K)})$ to $(K, \mathfrak{T}_p^{(K)})$ is continuous. Since $\mathfrak{T}_m^{(K)}$ is compact and $\mathfrak{T}_p^{(K)}$ is Hausdorff, the two topologies are equal; and, in particular, $\mathfrak{T}_p^{(K)}$ is compact and metrizable.

(e) Let ρ be a metric on K inducing the topology $\mathfrak{T}_p^{(K)}$. Let $D \subseteq K$ be a countable dense set. For each $n \in \mathbb{N}$, set

$$G_n = \{x : |f(x) - g(x)| \leq \frac{1}{3}\epsilon \text{ whenever } f, g \in D \text{ and } \rho(f, g) \leq 2^{-n}\}.$$

Because D is countable, G_n is measurable. Now $\bigcup_{n \in \mathbb{N}} G_n = X$. **P?** If $x \in X \setminus \bigcup_{n \in \mathbb{N}} G_n$, then for each $n \in \mathbb{N}$ we can find $f_n, g_n \in D$ such that $\rho(f_n, g_n) \leq 2^{-n}$ and $|f_n(x) - g_n(x)| \geq \frac{1}{3}\epsilon$. Because K is compact, there is a strictly increasing sequence $\langle n_k \rangle_{k \in \mathbb{N}}$ such that $\langle f_{n_k} \rangle_{k \in \mathbb{N}}$ and $\langle g_{n_k} \rangle_{k \in \mathbb{N}}$ are both convergent to f, g say. Now

$$\rho(f, g) = \lim_{k \rightarrow \infty} \rho(f_{n_k}, g_{n_k}) = 0, \quad |f(x) - g(x)| = \lim_{k \rightarrow \infty} |f_{n_k}(x) - g_{n_k}(x)| \geq \frac{1}{3}\epsilon,$$

so $f = g$ while $f(x) \neq g(x)$, which is impossible. **XQ**

There is therefore some $n \in \mathbb{N}$ such that $\mu(E \cap G_n) > 0$. Since K , being compact, is totally bounded for ρ , there is a finite set $D' \subseteq D$ such that every member of D is within a distance of 2^{-n} of some member of D' . Now there is a measurable set $F \subseteq E \cap G_n$ such that $\mu F > 0$ and $|g(x) - g(y)| \leq \frac{1}{3}\epsilon$ whenever $g \in D'$ and $x, y \in F$. So $|f(x) - f(y)| \leq \epsilon$ whenever $f \in D$ and $x, y \in F$. But as D is dense in K , $|f(x) - f(y)| \leq \epsilon$ whenever $f \in K$ and $x, y \in F$, as required.

463D Lemma Let (X, Σ, μ) be a measure space, and \mathcal{L}^0 the space of Σ -measurable real-valued functions on X . Write \mathfrak{T}_p for the topology of pointwise convergence on \mathcal{L}^0 . Suppose that $K \subseteq \mathcal{L}^0$ is \mathfrak{T}_p -compact and that there is no \mathfrak{T}_p -continuous surjection from any closed subset of K onto $\{0, 1\}^{\omega_1}$. If $E \in \Sigma$ has finite measure, then every sequence in K has a subsequence which is convergent almost everywhere in E .

proof (a) Let $\langle f_n \rangle_{n \in \mathbb{N}}$ be a sequence in K . Set $q(x) = \max(1, 2 \sup_{n \in \mathbb{N}} |f_n(x)|)$ for each $x \in X$. For any infinite $I \subseteq \mathbb{N}$, set

$$g_I = \liminf_{i \rightarrow I} f_i = \sup_{n \in \mathbb{N}} \inf_{i \in I, i \geq n} f_i,$$

$$h_I = \limsup_{i \rightarrow I} f_i = \inf_{n \in \mathbb{N}} \sup_{i \in I, i \geq n} f_i;$$

because $\sup_{f \in K} |f(x)|$ is surely finite for each $x \in X$, g_I and h_I are defined in \mathcal{L}^0 , and $g_I \leq h_I$. For $f \in \mathcal{L}^0$ set $\tau'(f) = \int_E \min(1, |f|/q)$, and for $I \in [\mathbb{N}]^\omega$ (the set of infinite subsets of \mathbb{N}) set $\Delta(I) = \tau'(h_I - g_I)$. Since $h_I - g_I \leq q$, $\Delta(I) = \int_E (h_I - g_I)/q$. If $I, J \in [\mathbb{N}]^\omega$ and $J \setminus I$ is finite, then $g_I \leq g_J \leq h_J \leq h_I$, so $\Delta(J) \leq \Delta(I)$, with equality only when $g_I = g_J$ a.e. on E and $h_I = h_J$ a.e. on E .

(b) There is a $J \in [\mathbb{N}]^\omega$ such that $\Delta(I) = \Delta(J)$ for every $I \in [J]^\omega$. **P** For $J \in [\mathbb{N}]^\omega$, set $\underline{\Delta}(J) = \inf\{\Delta(I) : I \in [J]^\omega\}$. Choose $\langle I_n \rangle_{n \in \mathbb{N}}$ in $[\mathbb{N}]^\omega$ inductively in such a way that $I_{n+1} \subseteq I_n$ and $\Delta(I_{n+1}) \leq \underline{\Delta}(I_n) + 2^{-n}$ for every n . If we now set

$$J = \{\min\{i : i \in I_n, i \geq n\} : n \in \mathbb{N}\},$$

$J \subseteq \mathbb{N}$ will be an infinite set and $J \setminus I_n$ is finite for every n . If $I \in [J]^\omega$ then, for every n ,

$$\Delta(J) \leq \Delta(I_{n+1}) \leq \underline{\Delta}(I_n) + 2^{-n} \leq \Delta(I_n \cap I) + 2^{-n} = \Delta(I) + 2^{-n};$$

as n and I are arbitrary, $\underline{\Delta}(J) = \Delta(J)$, as required. **Q**

Now for any $I \in [J]^\omega$ we have $g_I = g_J$ a.e. on E and $h_I = h_J$ a.e. on E .

(c) $\Delta(J) = 0$. P? Otherwise, $F = \{x : x \in E, g_J(x) < h_J(x)\}$ has positive measure. Write K_0 for $\bigcap_{n \in \mathbb{N}} \overline{\{f_i : i \in J, i \geq n\}}$, the closure being taken for \mathfrak{T}_p , so that K_0 is \mathfrak{T}_p -compact. Let \mathcal{A} be the family of sets $A \subseteq F$ such that whenever $L, M \subseteq A$ are finite and disjoint there is an $f \in K_0$ such that $f(x) = g_J(x)$ for $x \in L$ and $f(x) = h_J(x)$ for $x \in M$. Then \mathcal{A} has a maximal member A_0 say. If we define $\phi : \mathcal{L}^0 \rightarrow [0, 1]^{A_0}$ by setting $\phi(f)(x) = \frac{\text{med}(0, f(x) - g_J(x), h_J(x) - g_J(x))}{h_J(x) - g_J(x)}$ for $x \in A_0$ and $f \in \mathcal{L}^0$, $\phi[K_0]$ is a compact subset of $[0, 1]^{A_0}$, and whenever $L, M \subseteq A_0$ are finite there is a $g \in \phi[K_0]$ such that $g(x) = 0$ for $x \in L$ and $g(x) = 1$ for $x \in M$. This means that $\phi[K_0] \cap \{0, 1\}^{A_0}$ is dense in $\{0, 1\}^{A_0}$ and must therefore be the whole of $\{0, 1\}^{A_0}$. So $\{0, 1\}^{A_0}$ is a continuous image of a closed subset of K .

Since $\{0, 1\}^{\omega_1}$ is not a continuous image of a closed subset of K , it is not a continuous image of $\{0, 1\}^{A_0}$, and cannot be homeomorphic to $\{0, 1\}^A$ for any $A \subseteq A_0$. Thus no subset of A_0 can have cardinal ω_1 and A_0 is countable.

For each pair L, M of disjoint finite subsets of A_0 , we have a cluster point f_{LM} of $\langle f_j \rangle_{j \in J}$ such that $f_{LM}(x) = g_J(x)$ for $x \in L$ and $f_{LM}(x) = h_J(x)$ for $x \in M$. Let $I(L, M)$ be an infinite subset of J such that $\lim_{i \rightarrow \infty, i \in I(L, M)} f_i(x) = f_{LM}(x)$ for every $x \in A_0$. Then $g_{I(L, M)} = g_J$ and $h_{I(L, M)} = h_J$ almost everywhere in E . Because $\mu F > 0$ and A_0 has only countably many finite subsets, there is a $y \in F$ such that $g_{I(L, M)}(y) = g_J(y)$ and $h_{I(L, M)}(y) = h_J(y)$ whenever L and M are disjoint finite subsets of A_0 .

What this means is that if L and M are disjoint finite subsets of A_0 , then there are infinite sets $I', I'' \subseteq I(L, M)$ such that $\lim_{i \rightarrow \infty, i \in I'} f_i(y) = g_J(y)$ and $\lim_{i \rightarrow \infty, i \in I''} f_i(y) = h_J(y)$; so that there are $f', f'' \in K_0$ such that

$$f'(x) = g_J(x) \text{ for } x \in L \cup \{y\}, \quad f'(x) = h_J(x) \text{ for } x \in M,$$

$$f''(x) = g_J(x) \text{ for } x \in L, \quad f''(x) = h_J(x) \text{ for } x \in M \cup \{y\}.$$

But this means that $A_0 \cup \{y\} \in \mathcal{A}$, and also that $y \notin A_0$; and A_0 was chosen to be maximal. **XQ**

(d) So $\int_E (h_J - g_J)/q = 0$ and $g_J = h_J$ almost everywhere in E . But if we enumerate J in ascending order as $\langle n_i \rangle_{i \in \mathbb{N}}$, $g_J = \liminf_{i \rightarrow \infty} f_{n_i}$ and $h_J = \limsup_{i \rightarrow \infty} f_{n_i}$, so $\langle f_{n_i} \rangle_{i \in \mathbb{N}}$ converges almost everywhere in E .

463E Proposition Let (X, Σ, μ) be a measure space, and \mathcal{L}^0 the space of Σ -measurable real-valued functions on X . Write $\mathfrak{T}_p, \mathfrak{T}_m$ for the topologies of pointwise convergence and convergence in measure on \mathcal{L}^0 . Suppose that $K \subseteq \mathcal{L}^0$ is \mathfrak{T}_p -compact and that there is no \mathfrak{T}_p -continuous surjection from any closed subset of K onto $\omega_1 + 1$ with its order topology. Then the identity map from (K, \mathfrak{T}_p) to (K, \mathfrak{T}_m) is continuous.

proof (a) It is worth noting straight away that $\xi \mapsto \chi_\xi : \omega_1 + 1 \rightarrow \{0, 1\}^{\omega_1}$ is a homeomorphism between $\omega_1 + 1$ and a subspace of $\{0, 1\}^{\omega_1}$. So our hypothesis tells us that there is no continuous surjection from any closed subset of K onto $\{0, 1\}^{\omega_1}$, and therefore none onto $\{0, 1\}^A$ for any uncountable A .

(b) ? Suppose, if possible, that the identity map from (K, \mathfrak{T}_p) to (K, \mathfrak{T}_m) is not continuous at $f_0 \in K$. Then there are an $E \in \Sigma$, of finite measure, and an $\epsilon > 0$ such that $C = \{f : f \in K, \tau_E(f - f_0) \geq \epsilon\}$ meets every \mathfrak{T}_p -neighbourhood of f , where $\tau_E(f) = \int_E \min(1, |f|)$ for every $f \in \mathcal{L}^0$, and there is an ultrafilter \mathcal{F} on \mathcal{L}^0 which contains C and converges to f_0 for \mathfrak{T}_p . Consider the map $\psi : \mathcal{L}^0 \rightarrow L^0(\mu_E)$, where μ_E is the subspace measure on E , defined by setting $\psi(f) = (f \upharpoonright E)^\bullet$ for $f \in \mathcal{L}^0$. We know from 463D that every sequence in K has a subsequence convergent almost everywhere in E , so every sequence in $\psi[K]$ has a subsequence which is convergent for the topology of convergence in measure on $L^0(\mu_E)$. Since this is metrizable, $\psi[K]$ is relatively compact in $L^0(\mu_E)$ (4A2Le), and the image filter $\psi[[\mathcal{F}]]$ has a limit $v \in L^0(\mu_E)$. Let $f_1 \in \mathcal{L}^0$ be such that $\psi(f_1) = v$.

For any countable set $A \subseteq X$ there is a $g \in C$ such that $g \upharpoonright A = f_0 \upharpoonright A$ and $g = f_1$ almost everywhere in E . **P** If $X = \emptyset$ this is trivial, so we may, if necessary, enlarge A by one point so that it is not empty. Let $\langle x_n \rangle_{n \in \mathbb{N}}$ be a sequence running over A . Then for each $n \in \mathbb{N}$ the set

$$\{g : g \in C, |g(x_i) - f_0(x_i)| \leq 2^{-n} \text{ for every } i \leq n, \tau_E(g, f_1) \leq 2^{-n}\}$$

belongs to \mathcal{F} , so is not empty; take g_n in this set. Let $g \in K$ be any cluster point of $\langle g_n \rangle_{n \in \mathbb{N}}$. Since $\langle g_n \rangle_{n \in \mathbb{N}}$ converges to f_1 almost everywhere in E , $g = f_1$ a.e. on E and $\langle g_n \rangle_{n \in \mathbb{N}}$ converges to g almost everywhere in E . Consequently $\tau_E(g - f_0) = \lim_{n \rightarrow \infty} \tau_E(g_n - f_0)$, by the dominated convergence theorem, and $g \in C$. Since $\langle g_n(x_i) \rangle_{n \in \mathbb{N}}$ converges to $f_0(x_i)$ for every i , $g \upharpoonright A = f_0 \upharpoonright A$. So we have the result. **Q**

In particular, there is a $g \in C$ such that $g = f_1$ a.e. on E , so $\tau_E(f_1 - f_0) = \tau(g - f_0) \geq \epsilon$ and $F = \{x : x \in E, f_0(x) \neq f_1(x)\}$ has non-zero measure. Now choose $\langle g_\xi \rangle_{\xi < \omega_1}$ in K and $\langle x_\xi \rangle_{\xi < \omega_1}$ in F inductively so that

$$g_\xi \in C, \quad g_\xi = f_1 \text{ almost everywhere in } E, \quad g_\xi(x_\eta) = f_0(x_\eta) \text{ for } \eta < \xi$$

(choosing g_ξ),

$$x_\xi \in F, \quad g_\eta(x_\xi) = f_1(x_\xi) \text{ for } \eta \leq \xi$$

(choosing x_ξ). If we now set $A = \{x_\xi : \xi < \omega_1\}$,

$$K_1 = \bigcap_{\xi \leq \eta < \omega_1} \{f : f \in K, \text{ either } f(x_\xi) = f_0(x_\xi) \text{ or } f(x_\eta) = f_1(x_\eta)\},$$

then K_1 is a closed subset of K containing every g_ξ and also f_0 . But if we look at $\{f \upharpoonright A : f \in K_1\}$, this is homeomorphic to $\omega_1 + 1$; which is supposed to be impossible. **X**

So we conclude that the identity map from (K, \mathfrak{T}_p) to (K, \mathfrak{T}_m) is continuous.

463F Corollary Let (X, Σ, μ) be a measure space, and \mathcal{L}^0 the space of Σ -measurable real-valued functions on X . Write $\mathfrak{T}_p, \mathfrak{T}_m$ for the topologies of pointwise convergence and convergence in measure on \mathcal{L}^0 . Suppose that $K \subseteq \mathcal{L}^0$ is compact and countably tight for \mathfrak{T}_p . Then the identity map from (K, \mathfrak{T}_p) to (K, \mathfrak{T}_m) is continuous. If \mathfrak{T}_m is Hausdorff on K , the two topologies coincide on K .

proof Since $\omega_1 + 1$ is not countably tight (the top point ω_1 is not in the closure of any countable subset of ω_1), $\omega_1 + 1$ is not a continuous image of any closed subset of K (4A2Kb), and we can apply 463E to see that the identity map is continuous. It follows at once that if \mathfrak{T}_m is Hausdorff on K , then the topologies coincide.

463G Theorem (IONESCU TULCEA 74) Let (X, Σ, μ) be a σ -finite measure space, and K a convex set of measurable functions from X to \mathbb{R} such that (i) K is compact for the topology \mathfrak{T}_p of pointwise convergence (ii) $\{x : f(x) \neq g(x)\}$ is not negligible for any distinct $f, g \in K$. Then K is metrizable for \mathfrak{T}_p , which agrees with the topology of convergence in measure on K .

proof (a) Let $\langle f_n \rangle_{n \in \mathbb{N}}$ be a sequence in K . Then it has a pointwise convergent subsequence. **P** Because K is compact, we surely have $\sup_{f \in K} |f(x)| < \infty$ for every $x \in X$. Let $\langle X_k \rangle_{k \in \mathbb{N}}$ be a sequence of measurable sets of finite measure covering X , and set

$$Y_k = \{x : x \in X_k, |f_n(x)| \leq k \text{ for every } n \in \mathbb{N}\}$$

for $k \in \mathbb{N}$,

$$q = \sum_{k=0}^{\infty} \frac{1}{2^k(1+\mu Y_k)} \chi_{Y_k},$$

so that q is a strictly positive measurable function and

$$\|f_n \times q\|_2 \leq \sum_{k=0}^{\infty} \frac{k}{2^k} = 2$$

for every n .

Setting $g_n = f_n \times q$, the sequence $\langle g_n^\bullet \rangle_{n \in \mathbb{N}}$ of equivalence classes is a norm-bounded sequence in the Hilbert space $L^2 = L^2(\mu)$. It therefore has a weakly convergent subsequence $\langle g_{n(i)}^\bullet \rangle_{i \in \mathbb{N}}$ say (4A4Kb), with limit v .

? Suppose, if possible, that $\langle f_{n(i)} \rangle_{i \in \mathbb{N}}$ is not pointwise convergent. Then there must be some $x_0 \in X$ such that $\liminf_{i \rightarrow \infty} f_{n(i)}(x_0) < \limsup_{i \rightarrow \infty} f_{n(i)}(x_0)$; let $\alpha < \beta$ in \mathbb{R} be such that $I = \{i : f_{n(i)}(x_0) \leq \alpha\}$, $I' = \{i : f_{n(i)}(x_0) \geq \beta\}$ are both infinite. In this case v belongs to the weak closures of both $D = \{g_{n(i)}^\bullet : i \in I\}$ and $D' = \{g_{n(i)}^\bullet : i \in I'\}$. It must therefore belong to the *norm* closures of their convex hulls $\Gamma(D)$, $\Gamma(D')$ (4A4Ed). Accordingly we can find $v_n \in \Gamma(D)$, $v'_n \in \Gamma(D')$ such that $\|v - v_n\|_2 \leq 3^{-n}$, $\|v - v'_n\|_2 \leq 3^{-n}$ for every $n \in \mathbb{N}$.

Setting $A = \{f_{n(i)} : i \in I\}$, $A' = \{f_{n(i)} : i \in I'\}$, we see that there must be $h_n \in \Gamma(A)$, $h'_n \in \Gamma(A')$ such that $v_n = (h_n \times q)^\bullet$, $v'_n = (h'_n \times q)^\bullet$ for every $n \in \mathbb{N}$. Now if $g : X \rightarrow \mathbb{R}$ is a measurable function such that $g^\bullet = v$, and $\tilde{h} = g/q$, we have

$$\begin{aligned} \mu\{x : |\tilde{h}(x) - h_n(x)| \geq \frac{1}{2^n q(x)}\} &= \mu\{x : |g(x) - (h_n \times q)(x)| \geq 2^{-n}\} \\ &\leq 4^n \|v - v_n\|_2^2 \leq 2^{-n} \end{aligned}$$

for every $n \in \mathbb{N}$, and $h_n \rightarrow \tilde{h}$ a.e. Similarly, $h'_n \rightarrow \tilde{h}$ a.e.

At this point, recall that K is supposed to be convex, so all the h_n, h'_n belong to K . Let \mathcal{F} be any non-principal ultrafilter on \mathbb{N} . Because K is pointwise compact, $h = \lim_{n \rightarrow \mathcal{F}} h_n$ and $h' = \lim_{n \rightarrow \mathcal{F}} h'_n$ are both defined in K for the topology of pointwise convergence. For any x such that $\lim_{n \rightarrow \infty} h_n(x) = \tilde{h}(x)$, we surely have $h(x) = \tilde{h}(x)$; so $h =_{\text{a.e.}} \tilde{h}$. Similarly, $h' =_{\text{a.e.}} \tilde{h}$, and $h =_{\text{a.e.}} h'$.

Now at last we apply the hypothesis that distinct members of K are not equal almost everywhere, to see that $h = h'$. But if we look at what happens at the distinguished point x_0 above, we see that $f(x_0) \leq \alpha$ for every $f \in A$, so that $f(x_0) \leq \alpha$ for every $f \in \Gamma(A)$, $h_n(x_0) \leq \alpha$ for every $n \in \mathbb{N}$, and $h(x_0) \leq \alpha$; and similarly $h'(x_0) \geq \beta$. So $h \neq h'$, which is absurd. **X**

This contradiction shows that $\langle f_{n(i)} \rangle_{i \in \mathbb{N}}$ is pointwise convergent, and is an appropriate subsequence. **Q**

(b) Now 463Cd tells us that K is metrizable for \mathfrak{T}_p , and that \mathfrak{T}_p agrees on K with the topology of convergence in measure.

463H Corollary Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a σ -finite topological measure space in which μ is strictly positive. Suppose that

whenever $h \in \mathbb{R}^X$ is such that $h \upharpoonright Q$ is continuous for every relatively countably compact $Q \subseteq X$,
then h is continuous.

If $K \subseteq C_b(X)$ is a norm-bounded \mathfrak{T}_p -compact set, then it is \mathfrak{T}_p -metrizable.

proof By 462L, the \mathfrak{T}_p -closed convex hull $\overline{\Gamma(K)}$ of K in $C(X)$ is \mathfrak{T}_p -compact. Because μ is strictly positive, $\mu\{x : f(x) \neq g(x)\} > 0$ whenever f and g are distinct continuous real-valued functions on X . So the result is immediate from 463G.

463I Lemma Let (X, Σ, μ) be a perfect probability space, and $\langle E_n \rangle_{n \in \mathbb{N}}$ a sequence in Σ . Suppose that there is an $\epsilon > 0$ such that

$$\epsilon \mu F \leq \liminf_{n \rightarrow \infty} \mu(F \cap E_n) \leq \limsup_{n \rightarrow \infty} \mu(F \cap E_n) \leq (1 - \epsilon) \mu F$$

for every $F \in \Sigma$. Then $\langle E_n \rangle_{n \in \mathbb{N}}$ has a subsequence $\langle E_{n_k} \rangle_{k \in \mathbb{N}}$ such that $\mu_* A = 0$ and $\mu^* A = 1$ for any cluster point A of $\langle E_{n_k} \rangle_{k \in \mathbb{N}}$ in $\mathcal{P}X$; in particular, $\langle E_{n_k} \rangle_{k \in \mathbb{N}}$ has no measurable cluster point.

proof (a) If we replace μ by its completion, we do not change μ^* and μ_* (212Ea, 413Eg), so we may suppose that μ is already complete.

(b) Suppose that $\langle E_n \rangle_{n \in \mathbb{N}}$ is actually stochastically independent, with $\mu E_n = \frac{1}{k}$ for every n , where $k \geq 2$ is an integer. In this case $\mu^* A = 1$ for any cluster point A of $\langle E_n \rangle_{n \in \mathbb{N}}$.

P (i) There is a non-principal ultrafilter \mathcal{F} on \mathbb{N} such that $A = \lim_{n \rightarrow \mathcal{F}} E_n$ in $\mathcal{P}X$ (4A2F(a-ii)); that is, $\chi A(x) = \lim_{n \rightarrow \mathcal{F}} \chi E_n(x)$ for every $x \in X$; that is, $A = \{x : x \in E_n, \{n : x \in E_n\} \in \mathcal{F}\}$.

(ii) We have a measurable function $\phi : X \rightarrow Y = \{0, 1\}^{\mathbb{N}}$ defined by setting $\phi(x)(n) = (\chi E_n)(x)$ for every $x \in X, n \in \mathbb{N}$. Because μ is complete and perfect and totally finite, and $\{0, 1\}^{\mathbb{N}}$ is compact and metrizable, the image measure $\nu = \mu\phi^{-1}$ is a Radon measure (451O). For any basic open set of the form $H = \{y : y(i) = \epsilon_i \text{ for every } i \leq n\}$, $\mu\phi^{-1}[H] = \tilde{\nu}H$, where $\tilde{\nu}$ is the product measure corresponding to the measure ν_0 on $\{0, 1\}$ for which $\nu_0\{1\} = \frac{1}{k}, \nu_0\{0\} = \frac{k-1}{k}$. Since $\tilde{\nu}$ also is a Radon measure (416U), $\tilde{\nu} = \nu$ (415H(v)).

Set $B = \{y : y \in Y, \{n : y(n) = 1\} \in \mathcal{F}\}$, so that $\phi^{-1}[B] = A$ and B is determined by coordinates in $\{n, n + 1, \dots\}$ for every $n \in \mathbb{N}$. By 451Pc, $\mu^* A = \nu^* B$; by the zero-one law (254Sa), $\nu^* B$ must be either 0 or 1. So $\mu^* A$ is either 0 or 1.

(iii) To see that $\mu^* A$ cannot be 0, we repeat the arguments of (ii) from the other side, as follows. Let λ_0 be the uniform probability measure on $\{0, 1, \dots, k - 1\}$, giving measure $\frac{1}{k}$ to each point; let λ be the corresponding product measure on $Z = \{0, \dots, k - 1\}^{\mathbb{N}}$. Let $\psi : Z \rightarrow Y$ be defined by setting $\psi(z)(n) = 1$ if $z(n) = 0, 0$ otherwise; then ψ is inverse-measure-preserving (254G). Since λ is a Radon measure and ψ is continuous, $\lambda\psi^{-1}$ is a Radon measure on Y and must be equal to ν . Accordingly $\nu^* B = \lambda^*\psi^{-1}[B]$, by 451Pc again or otherwise.

(iv) We have a measure space automorphism $\theta : Z \rightarrow Z$ defined by setting $\theta(z)(n) = z(n) +_k 1$ for every $z \in Z, n \in \mathbb{N}$, where $+_k$ is addition mod k . So, writing $C = \psi^{-1}[B]$, $\lambda^* C = \lambda^*\theta^i[C]$ for every $i \in \mathbb{N}$. Now, for $z \in Z$,

$$\{n : z(n) = 0\} \in \mathcal{F} \iff \{n : \psi(z)(n) = 1\} \in \mathcal{F} \iff \psi(z) \in B \iff z \in C.$$

But for any $z \in Z$, there is some $i < k$ such that $\{n : z(n) = i\} \in \mathcal{F}$, so that $\theta^{k-i}(z) \in C$. Thus $\bigcup_{i < k} \theta^i[C] = Z$ and $\sum_{i=0}^{k-1} \lambda^*\theta^i[C] \geq 1$ and $\lambda^* C > 0$. But this means that $\mu^* A = \nu^* B = \lambda^* C$ is non-zero, and $\mu^* A$ must be 1. **Q**

(c) Now return to the general case considered in (a). Note first that μ is atomless, because if $\mu F > 0$ there is some $n \in \mathbb{N}$ such that $0 < \mu(F \cap E_n) < \mu F$.

Let $k \geq 2$ be such that $\frac{1}{k} < \epsilon$. Then there are a strictly increasing sequence $\langle m(i) \rangle_{i \in \mathbb{N}}$ in \mathbb{N} and a stochastically independent sequence $\langle F_i \rangle_{i \in \mathbb{N}}$ in Σ such that $F_i \subseteq E_{m(i)}$ and $\mu F_i = \frac{1}{k}$ for every $i \in \mathbb{N}$. **P** Choose $\langle m(i) \rangle_{i \in \mathbb{N}}, F_i$ inductively, as follows. Let Σ_i be the (finite) algebra generated by $\{F_j : j < i\}$. Choose $m(i)$ such that $m(i) > m(j)$ for any $j < i$ and $\mu(F \cap E_{m(i)}) \geq \frac{1}{k}\mu F$ for every $F \in \Sigma_i$. List the atoms of Σ_i as G_{i0}, \dots, G_{ip_i} , and choose $F_{ir} \subseteq E_{m(i)} \cap G_{ir}$ such that $\mu F_{ir} = \frac{1}{k}\mu G_{ir}$, for each $r \leq p_i$; 215D tells us that this is possible. Set $F_i = \bigcup_{r \leq p_i} F_{ir}$; then $\mu(F_i \cap F) = \frac{1}{k}\mu F$ for every $F \in \Sigma_i$, and $F_i \subseteq E_{m(i)}$. Continue. It is easy to check that $\mu(F_{i_1} \cap \dots \cap F_{i_r}) = 1/k^r$ whenever $i_1 < \dots < i_r$, so that $\langle F_i \rangle_{i \in \mathbb{N}}$ is stochastically independent. **Q**

If A is a cluster point of $\langle E_{m(i)} \rangle_{i \in \mathbb{N}}$, then there is a non-principal ultrafilter \mathcal{F} on \mathbb{N} such that $A = \lim_{i \rightarrow \mathcal{F}} E_{m(i)}$ in $\mathcal{P}X$. In this case, $A \supseteq A'$, where $A' = \lim_{i \rightarrow \mathcal{F}} F_i$. But (b) tells us that $\mu^* A'$ must be 1, so $\mu^* A = 1$.

(d) Thus we have a subsequence $\langle E_{m(i)} \rangle_{i \in \mathbb{N}}$ of $\langle E_n \rangle_{n \in \mathbb{N}}$ such that any cluster point of $\langle E_{m(i)} \rangle_{i \in \mathbb{N}}$ has outer measure 1. But the same argument applies to $\langle X \setminus E_{m(i)} \rangle_{i \in \mathbb{N}}$ to show that there is a strictly increasing sequence $\langle i_k \rangle_{k \in \mathbb{N}}$ such that every cluster point of $\langle X \setminus E_{m(i_k)} \rangle_{k \in \mathbb{N}}$ has outer measure 1. Since complementation is a homeomorphism of $\mathcal{P}X$, $\mu^*(X \setminus A) = 1$, that is, $\mu_* A = 0$, for every cluster point A of $\langle E_{m(i_k)} \rangle_{k \in \mathbb{N}}$. So if we set $n_k = m(i_k)$, any cluster point of $\langle E_{n_k} \rangle_{k \in \mathbb{N}}$ will have inner measure 0 and outer measure 1, as claimed.

463J Lemma Let (X, Σ, μ) be a perfect probability space, and $\langle E_n \rangle_{n \in \mathbb{N}}$ a sequence in Σ . Then either $\langle \chi E_n \rangle_{n \in \mathbb{N}}$ has a subsequence which is convergent almost everywhere

or $\langle E_n \rangle_{n \in \mathbb{N}}$ has a subsequence with no measurable cluster point in $\mathcal{P}X$.

proof Consider the sequence $\langle \chi E_n^\bullet \rangle_{n \in \mathbb{N}}$ in the Hilbert space $L^2 = L^2(\mu)$. This is a norm-bounded sequence, so has a weakly convergent subsequence $\langle \chi E_{n_i}^\bullet \rangle_{i \in \mathbb{N}}$ with limit v say (4A4Kb again). Express v as g^\bullet where $g : X \rightarrow \mathbb{R}$ is Σ -measurable.

case 1 Suppose that $g(x) \in \{0, 1\}$ for almost every $x \in X$; set $F = \{x : g(x) = 1\}$. Then

$$\lim_{i \rightarrow \infty} \int_F \chi E_{n_i} = \int_F g = \mu F, \quad \lim_{i \rightarrow \infty} \int_{X \setminus F} \chi E_{n_i} = \int_{X \setminus F} g = 0.$$

So, replacing $\langle \chi E_{n_i} \rangle_{i \in \mathbb{N}}$ with a sub-subsequence if necessary, we may suppose that

$$|\mu F - \int_F \chi E_{n_i}| \leq 2^{-i}, \quad |\int_{X \setminus F} \chi E_{n_i}| \leq 2^{-i}$$

for every i . But as $0 \leq \chi E_{n_i} \leq 1$ everywhere, we have $\int |\chi F - \chi E_{n_i}| \leq 2^{-i+1}$ for every i , so that $\chi E_{n_i} \rightarrow \chi F$ a.e., and we have a subsequence of $\langle \chi E_n \rangle_{n \in \mathbb{N}}$ which is convergent almost everywhere.

case 2 Suppose that $\{x : g(x) \notin \{0, 1\}\}$ has positive measure. Note that because $\int_F g = \lim_{i \rightarrow \infty} \mu(F \cap E_{n_i})$ lies between 0 and μF for every $F \in \Sigma$, $0 \leq g \leq 1$ a.e., and $\mu\{x : 0 < g(x) < 1\} > 0$. There is therefore an $\epsilon > 0$ such that $\mu G > 0$, where $G = \{x : \epsilon \leq g(x) \leq 1 - \epsilon\}$.

Write μ_G for the subspace measure on G , and Σ_G for its domain; set $\nu = (\mu_G)^{-1} \mu_G$, so that ν is a probability measure. We know that μ_G is perfect (451Dc), so ν also is (see the definition in 451Ad). Now if $F \in \Sigma_G$,

$$\lim_{i \rightarrow \infty} \nu(F \cap E_{n_i}) = (\mu_G)^{-1} \int_F g$$

lies between $\epsilon \mu F / \mu G = \epsilon \nu F$ and $(1 - \epsilon) \nu F$.

By 463I, there is a strictly increasing sequence $\langle i(k) \rangle_{k \in \mathbb{N}}$ such that $B \notin \Sigma_G$ whenever B is a cluster point of $\langle G \cap E_{n_{i(k)}} \rangle_{k \in \mathbb{N}}$ in $\mathcal{P}G$. If A is any cluster point of $\langle E_{n_{i(k)}} \rangle_{k \in \mathbb{N}}$ in \mathbb{R}^X , then $A \cap G$ is a cluster point of $\langle G \cap E_{n_{i(k)}} \rangle_{k \in \mathbb{N}}$ in $\mathcal{P}G$, so cannot belong to Σ_G . Thus $A \notin \Sigma$.

So in this case we have a subsequence $\langle E_{n_{i(k)}} \rangle_{k \in \mathbb{N}}$ of $\langle E_n \rangle_{n \in \mathbb{N}}$ which has no measurable cluster point.

463K Fremlin's Alternative (FREMLIN 75A) Let (X, Σ, μ) be a perfect σ -finite measure space, and $\langle f_n \rangle_{n \in \mathbb{N}}$ a sequence of real-valued measurable functions on X . Then

either $\langle f_n \rangle_{n \in \mathbb{N}}$ has a subsequence which is convergent almost everywhere

or $\langle f_n \rangle_{n \in \mathbb{N}}$ has a subsequence with no measurable cluster point in \mathbb{R}^X .

proof (a) If $\mu X = 0$ then of course $\langle f_n \rangle_{n \in \mathbb{N}}$ itself is convergent a.e., so we may suppose that $\mu X > 0$. If there is any $x \in X$ such that $\sup_{n \in \mathbb{N}} |f_n(x)| = \infty$, then $\langle f_n \rangle_{n \in \mathbb{N}}$ has a subsequence with no cluster point in \mathbb{R}^X , measurable or otherwise; so we may suppose that $\langle f_n \rangle_{n \in \mathbb{N}}$ is bounded at each point of X .

(b) Let λ be the c.l.d. product of μ with Lebesgue measure on \mathbb{R} , and Λ its domain. Then λ is perfect (451Ic) and also σ -finite (251K). There is therefore a probability measure ν on $X \times \mathbb{R}$ with the same domain and the same negligible sets as λ (215B(vii)), so that ν also is perfect. For any function $h \in \mathbb{R}^X$, write

$$\Omega(h) = \{(x, \alpha) : x \in X, \alpha \leq h(x)\} \subseteq X \times \mathbb{R}$$

(compare 252N).

(c) By 463J, applied to the measure space $(X \times \mathbb{R}, \Lambda, \nu)$ and the sequence $\langle \chi \Omega(f_n) \rangle_{n \in \mathbb{N}}$, we have a strictly increasing sequence $\langle n(i) \rangle_{i \in \mathbb{N}}$ such that either $\langle \chi \Omega(f_{n(i)}) \rangle_{i \in \mathbb{N}}$ is convergent ν -a.e. or $\langle \Omega(f_{n(i)}) \rangle_{i \in \mathbb{N}}$ has no cluster point in Λ .

case 1 Suppose that $\langle \chi \Omega(f_{n(i)}) \rangle_{i \in \mathbb{N}}$ is convergent ν -a.e. Set

$$W = \{(x, \alpha) : \lim_{i \rightarrow \infty} \chi \Omega(f_{n(i)})(x, \alpha) \text{ is defined}\}.$$

Then W is λ -conegligible, so $W^{-1}[\{\alpha\}] = \{x : (x, \alpha) \in W\}$ is μ -conegligible for almost every α (apply 252D to the complement of W). Set $D = \{\alpha : W^{-1}[\{\alpha\}] \text{ is } \mu\text{-conegligible}\}$, and let $Q \subseteq D$ be a countable dense set; then $G = \bigcap_{\alpha \in Q} W^{-1}[\{\alpha\}]$ is μ -conegligible. But if $x \in G$, then for any $\alpha \in Q$ the set $\{i : f_{n(i)}(x) \geq \alpha\} = \{i : \chi \Omega(f_{n(i)})(x, \alpha) = 1\}$ is either finite or has finite complement in \mathbb{N} , so $\langle f_{n(i)}(x) \rangle_{i \in \mathbb{N}}$ must

be convergent in $[-\infty, \infty]$. Since $\langle f_n(x) \rangle_{n \in \mathbb{N}}$ is supposed to be bounded, $\langle f_{n(i)}(x) \rangle_{i \in \mathbb{N}}$ is convergent in \mathbb{R} . Thus in this case we have an almost-everywhere-convergent subsequence of $\langle f_n \rangle_{n \in \mathbb{N}}$.

case 2 Suppose that $\langle \Omega(f_{n(i)}) \rangle_{i \in \mathbb{N}}$ has no cluster point in Λ . Let h be any cluster point of $\langle f_{n(i)} \rangle_{i \in \mathbb{N}}$ in \mathbb{R}^X . Then there is a non-principal ultrafilter \mathcal{F} on \mathbb{N} such that $h = \lim_{i \rightarrow \mathcal{F}} f_{n(i)}$ in \mathbb{R}^X . Set $A = \lim_{i \rightarrow \mathcal{F}} \Omega(f_{n(i)})$, so that $A \notin \Lambda$. If $x \in X$ and $\alpha \in \mathbb{R}$, then

$$\alpha < h(x) \implies \{i : \alpha < f_{n(i)}(x)\} \in \mathcal{F} \implies (x, \alpha) \in A,$$

$$h(x) < \alpha \implies \{i : \alpha < f_{n(i)}(x)\} \notin \mathcal{F} \implies (x, \alpha) \notin A.$$

Thus $\Omega'(h) \subseteq A \subseteq \Omega(h)$, where $\Omega'(h) = \{(x, \alpha) : \alpha < h(x)\}$.

? If h is Σ -measurable, then

$$\Omega'(h) = \bigcup_{q \in \mathbb{Q}} \{x : h(x) > q\} \times]-\infty, q],$$

$$\Omega(h) = (X \times \mathbb{R}) \setminus \bigcup_{q \in \mathbb{Q}} \{x : h(x) < q\} \times [q, \infty[$$

belong to Λ , and $\lambda(\Omega(h) \setminus \Omega'(h)) = 0$ (because every vertical section of $\Omega(h) \setminus \Omega'(h)$ is negligible). But as $\Omega'(h) \subseteq A \subseteq \Omega(h)$, $A \in \Lambda$ (remember that product measures in this book are complete), which is impossible.

X

Thus h is not Σ -measurable. As h is arbitrary, $\langle f_{n(i)} \rangle_{i \in \mathbb{N}}$ has no measurable cluster point in \mathbb{R}^X .

So at least one of the envisaged alternatives must be true.

463L Corollary Let (X, Σ, μ) be a perfect σ -finite measure space. Write $\mathcal{L}^0 \subseteq \mathbb{R}^X$ for the space of real-valued Σ -measurable functions on X .

(a) If $K \subseteq \mathcal{L}^0$ is relatively countably compact for the topology \mathfrak{T}_p of pointwise convergence on \mathcal{L}^0 , then every sequence in K has a subsequence which is convergent almost everywhere. Consequently K is relatively compact in \mathcal{L}^0 for the topology \mathfrak{T}_m of convergence in measure.

(b) If $K \subseteq \mathcal{L}^0$ is countably compact for \mathfrak{T}_p , then it is compact for \mathfrak{T}_m .

(c) Suppose that $K \subseteq \mathcal{L}^0$ is countably compact for \mathfrak{T}_p and that $\mu\{x : f(x) \neq g(x)\} > 0$ for any distinct $f, g \in K$. Then the topologies \mathfrak{T}_m and \mathfrak{T}_p agree on K , so both are compact and metrizable.

proof (a) Since every sequence in K must have a \mathfrak{T}_p -cluster point in \mathcal{L}^0 , 463K tells us that every sequence in K has a subsequence which is convergent almost everywhere, therefore \mathfrak{T}_m -convergent. Now K is relatively sequentially compact in the pseudometrizable space $(\mathcal{L}^0, \mathfrak{T}_m)$, therefore relatively compact (4A2Le again).

(b) As in (a), every sequence $\langle f_n \rangle_{n \in \mathbb{N}}$ in K has a subsequence $\langle g_n \rangle_{n \in \mathbb{N}}$ which is convergent almost everywhere. But $\langle g_n \rangle_{n \in \mathbb{N}}$ has a \mathfrak{T}_p -cluster point g in K , and now $g(x) = \lim_{n \rightarrow \infty} g_n(x)$ for every x for which the limit is defined; accordingly $g_n \rightarrow g$ a.e., and g is a \mathfrak{T}_m -limit of $\langle g_n \rangle_{n \in \mathbb{N}}$ in K . Thus every sequence in K has a \mathfrak{T}_m -cluster point in K , and (because \mathfrak{T}_m is pseudometrizable) K is \mathfrak{T}_m -compact.

(c) The point is that K is sequentially compact under \mathfrak{T}_p . **P** Note that as K is countably compact, $\sup_{f \in K} |f(x)|$ is finite for every $x \in X$. (I am passing over the trivial case $K = \emptyset$.) If $\langle f_n \rangle_{n \in \mathbb{N}}$ is a sequence in K , then, by (a), it has a subsequence $\langle g_n \rangle_{n \in \mathbb{N}}$ which is convergent a.e. **?** If $\langle g_n \rangle_{n \in \mathbb{N}}$ is not \mathfrak{T}_p -convergent, then there are a point $x_0 \in X$ and two further subsequences $\langle g'_n \rangle_{n \in \mathbb{N}}$, $\langle g''_n \rangle_{n \in \mathbb{N}}$ of $\langle g_n \rangle_{n \in \mathbb{N}}$ such that $\lim_{n \rightarrow \infty} g'_n(x_0)$, $\lim_{n \rightarrow \infty} g''_n(x_0)$ exist and are different. Now $\langle g'_n \rangle_{n \in \mathbb{N}}$, $\langle g''_n \rangle_{n \in \mathbb{N}}$ must have cluster points $g', g'' \in K$ with $g'(x_0) \neq g''(x_0)$.

However,

$$g'(x) = \lim_{n \rightarrow \infty} g_n(x) = g''(x)$$

whenever the limit is defined, which is almost everywhere; so $g' =_{\text{a.e.}} g''$. And this contradicts the hypothesis that if two elements of K are equal a.e., they are identical. **X** Thus $\langle g_n \rangle_{n \in \mathbb{N}}$ is a \mathfrak{T}_p -convergent subsequence of $\langle f_n \rangle_{n \in \mathbb{N}}$. As $\langle f_n \rangle_{n \in \mathbb{N}}$ is arbitrary, K is \mathfrak{T}_p -sequentially compact. **Q**

Now 463Cd gives the result.

463M Proposition Let X_0, \dots, X_n be countably compact topological spaces, each carrying a σ -finite perfect strictly positive measure which measures every Baire set. Let X be their product and $\mathcal{B}\alpha(X_i)$ the

Baire σ -algebra of X_i for each i . Then any separately continuous function $f : X \rightarrow \mathbb{R}$ is measurable with respect to the σ -algebra $\widehat{\bigotimes}_{i \leq n} \mathcal{B}\mathfrak{a}(X_i)$ generated by $\{\prod_{i \leq n} E_i : E_i \in \mathcal{B}\mathfrak{a}(X_i) \text{ for } i \leq n\}$.

proof For $i \leq n$ let μ_i be a σ -finite perfect strictly positive measure on X_i such that $\mathcal{B}\mathfrak{a}(X_i) \subseteq \text{dom } \mu_i$; let μ be the product measure on X .

(a) The proof relies on the fact that

(*) if $g, g' : X \rightarrow \mathbb{R}$ are distinct separately continuous functions, then $\mu\{x : g(x) \neq g'(x)\} > 0$;

I seek to prove this, together with the stated result, by induction on n . The induction starts easily with $n = 0$, so that X can be identified with X_0 , a separately continuous function on X is just a continuous function on X_0 , and (*) is true because $\mu = \mu_0$ is strictly positive.

(b) For the inductive step to $n + 1$, given a separately continuous function $f : X_0 \times \dots \times X_{n+1} \rightarrow \mathbb{R}$, set $f_t(y) = f(y, t)$ for every $y \in Y = X_0 \times \dots \times X_n$ and $t \in X_{n+1}$, and $K = \{f_t : t \in X_{n+1}\}$. Then every f_t is separately continuous, therefore $\widehat{\bigotimes}_{i \leq n} \mathcal{B}\mathfrak{a}(X_i)$ -measurable, by the inductive hypothesis. So K consists of $\widehat{\bigotimes}_{i \leq n} \mathcal{B}\mathfrak{a}(X_i)$ -measurable functions. Moreover, again because f is separately continuous, the function $t \mapsto f_t(y)$ is continuous for every y , that is, $t \mapsto f_t : X_{n+1} \rightarrow \mathbb{R}^Y$ is continuous; it follows that K is countably compact (4A2G(f-vi)). Finally, by the inductive hypothesis (*), $\nu\{y : f_t(y) \neq f_{t'}(y)\} > 0$ whenever $t, t' \in X_{n+1}$ and $f_t \neq f_{t'}$, where ν is the product measure on Y .

Since ν is perfect (451Ic) and $\widehat{\bigotimes}_{i \leq n} \mathcal{B}\mathfrak{a}(X_i) \subseteq \text{dom } \nu$, we can apply 463Lc to see that K is metrizable for the topology of pointwise convergence. Let ρ be a metric on K inducing its topology, and $\langle g_i \rangle_{i \in \mathbb{N}}$ a sequence running over a dense subset of K . (I am passing over the trivial case $K = \emptyset = X_{n+1}$.) For $m, i \in \mathbb{N}$ set $E_{mi} = \{t : \rho(f_t, g_i) \leq 2^{-m}\}$. Because $t \mapsto f_t$ and $t \mapsto \rho(f_t, g_i)$ are continuous, $E_{mi} \in \mathcal{B}\mathfrak{a}(X_{n+1})$. Set $f^{(m)}(y, t) = g_i(y)$ for $t \in E_{mi} \setminus \bigcup_{j < i} E_{mj}$ for $m, i \in \mathbb{N}$, $y \in Y$ and $t \in X_{n+1}$. Then $f^{(m)} : X \rightarrow \mathbb{R}$ is $\widehat{\bigotimes}_{i \leq n+1} \mathcal{B}\mathfrak{a}(X_i)$ -measurable because every g_i is $\widehat{\bigotimes}_{i \leq n} \mathcal{B}\mathfrak{a}(X_i)$ -measurable and every E_{mi} belongs to $\mathcal{B}\mathfrak{a}(X_{n+1})$. Also $\langle f^{(m)} \rangle_{m \in \mathbb{N}} \rightarrow f$ at every point, because $\rho(f_t^{(m)}, f_t) \leq 2^{-m}$ for every $m \in \mathbb{N}$ and $t \in X_{n+1}$. So f is $\widehat{\bigotimes}_{i \leq n+1} \mathcal{B}\mathfrak{a}(X_i)$ -measurable.

(c) We still have to check that (*) is true at the new level. But if $h, h' : X \rightarrow \mathbb{R}$ are distinct separately continuous functions, then there are $t_0 \in X_{n+1}$, $y_0 \in Y$ such that $h(y_0, t_0) \neq h'(y_0, t_0)$. Let G be an open set containing t_0 such that $h(y_0, t) \neq h'(y_0, t)$ whenever $t \in G$. Then $\nu\{y : h(y, t) \neq h'(y, t)\} > 0$ for every $t \in G$, by the inductive hypothesis, so

$$\mu\{y, t : h(y, t) \neq h'(y, t)\} = \int \nu\{y : h(y, t) \neq h'(y, t)\} \mu_{n+1}(dt) > 0$$

because μ_{n+1} is strictly positive. Thus the induction continues.

463N Corollary Let X_0, \dots, X_n be Hausdorff spaces with product X . Then every separately continuous function $f : X \rightarrow \mathbb{R}$ is universally Radon-measurable in the sense of 434Ec.

proof Let μ be a Radon measure on X and Σ its domain.

(a) Suppose first that the support C of μ is compact. For each $i \leq n$, let $\pi_i : X \rightarrow X_i$ be the coordinate projection, and $\mu_i = \mu \pi_i^{-1}$ the image Radon measure; let Z_i be the support of μ_i and $Z = \prod_{i \leq n} Z_i$. Note that $\pi_i[C]$ is compact and μ_i -conegligible, so that $Z_i \subseteq \pi_i[C]$ is compact, for each i . At the same time, $\pi^{-1}[Z_i]$ is μ -conegligible for each i , so that Z is μ -conegligible.

By 463M, $f \upharpoonright Z$ is $\widehat{\bigotimes}_{i \leq n} \mathcal{B}\mathfrak{a}(Z_i)$ -measurable; because Z is conegligible, f is Σ -measurable.

(b) In general, if $C \subseteq X$ is compact, then we can apply (a) to the measure $\mu \llcorner C$ (234M) to see that $f \upharpoonright C$ is Σ -measurable. As μ is complete and locally determined and inner regular with respect to the compact sets, f is Σ -measurable (see 412Ja).

As μ is arbitrary, f is universally Radon-measurable.

463X Basic exercises >(a) Let (X, Σ, μ) be a measure space, \mathcal{L}^0 the space of Σ -measurable real-valued functions on X , \mathfrak{T}_p the topology of pointwise convergence on \mathcal{L}^0 and \mathfrak{T}_m the topology of convergence in measure on \mathcal{L}^0 . (i) Show that $\mathfrak{T}_m \subseteq \mathfrak{T}_p$ iff for every measurable set E of finite measure there is a countable set $D \subseteq E$ such that $\mu^* D = \mu E$. (ii) Show that $\mathfrak{T}_p \subseteq \mathfrak{T}_m$ iff $0 < \mu^* \{x\} < \infty$ for every $x \in X$.

(b) Let (X, Σ, μ) be a σ -finite measure space, and $K \subseteq \mathcal{L}^0$ a \mathfrak{T}_p -countably compact set. Show that the following are equiveridical: (i) every sequence in K has a subsequence which converges almost everywhere; (ii) K is \mathfrak{T}_m -compact; (iii) K is totally bounded for the uniformity associated with the linear space topology \mathfrak{T}_m . Show that if moreover the topology on K induced by \mathfrak{T}_m is Hausdorff, then K is \mathfrak{T}_p -metrizable.

(c)(i) Show that there is a set of Borel measurable functions on $[0, 1]$ which is countably tight, compact and non-metrizable for the topology of pointwise convergence. (ii) Show that there is a strictly localizable measure space (X, Σ, μ) with a set K of measurable functions which is countably tight, compact, Hausdorff and non-metrizable for both the topology of pointwise convergence and the topology of convergence in measure. (*Hint*: the one-point compactification of any discrete space is countably tight.)

(d) Let X be a topological space and $K \subseteq C(X)$ a convex \mathfrak{T}_p -compact set. Show that if there is a strictly positive σ -finite topological measure on X , then K is \mathfrak{T}_p -metrizable.

(e) Use Komlós's theorem (276H) to shorten the proof of 463G.

(f) Let (X, Σ, μ) be any complete σ -finite measure space. Show that if $A \subseteq \mathcal{L}^0$ is \mathfrak{T}_m -relatively compact, and $\sup_{f \in A} |f(x)|$ is finite for every $x \in X$, then A is \mathfrak{T}_p -relatively countably compact in \mathcal{L}^0 .

(g) Let K be the set of non-decreasing functions from $[0, 1]$ to $\{0, 1\}$. Show that K , with its topology of pointwise convergence, is homeomorphic to the split interval (419L). Show that (for any Radon measure μ on $[0, 1]$) the identity map from (K, \mathfrak{T}_p) to (K, \mathfrak{T}_m) is continuous.

>(h) Let K be the set of non-decreasing functions from ω_1 to $\{0, 1\}$. Show that if μ is the countable-cocountable measure on ω_1 then K is a \mathfrak{T}_p -compact set of measurable functions and is also \mathfrak{T}_m -compact, but the identity map from (K, \mathfrak{T}_p) to (K, \mathfrak{T}_m) is not continuous.

>(i) Let K be the set of functions $f : [0, 1] \rightarrow \mathbb{R}$ such that $\max(\|f\|_\infty, \text{Var}_{[0,1]} f) \leq 1$, where $\text{Var}_{[0,1]} f$ is the variation of f (224A). Show that K is \mathfrak{T}_p -compact and that (for any Radon measure on $[0, 1]$) the identity map from (K, \mathfrak{T}_p) to (K, \mathfrak{T}_m) is continuous.

>(j) Let A be the set of functions $f : [0, 1] \rightarrow [0, 1]$ such that $\int |f| d\mu \cdot \text{Var}_{[0,1]} f \leq 1$, where μ is Lebesgue measure. Show that every member of A is measurable and that every sequence in A has a subsequence which converges almost everywhere to a member of A , but that $f : A \rightarrow [0, 1]$ is not \mathfrak{T}_p -continuous, while A is \mathfrak{T}_p -dense in $[0, 1]^{[0,1]}$.

(k) Let X be a Hausdorff space and $K \subseteq C(X)$ a \mathfrak{T}_p -compact set. Show that if there is a strictly positive σ -finite Radon measure on X then K is \mathfrak{T}_p -metrizable.

(l) Let (X, Σ, μ) be a localizable measure space and $K \subseteq \mathcal{L}^0$ a non-empty \mathfrak{T}_p -compact set. Show that $\sup\{f^\bullet : f \in K\}$ is defined in $L^0(\mu)$.

463Y Further exercises (a) Let (X, Σ, μ) be a probability space and V a Banach space. A function $\phi : X \rightarrow V$ is **scalarly measurable** (often called **weakly measurable**) if $h\phi : X \rightarrow \mathbb{R}$ is Σ -measurable for every $h \in V^*$. ϕ is **Pettis integrable**, with **indefinite Pettis integral** $\theta : \Sigma \rightarrow V$, if $\int_E h\phi d\mu$ is defined and equal to $h(\theta E)$ for every $E \in \Sigma$ and every $h \in V^*$. (i) Show that if ϕ is scalarly measurable, then $K = \{h\phi : h \in V^*, \|h\| \leq 1\}$ is a \mathfrak{T}_p -compact subset of \mathcal{L}^0 . (ii) Show that if ϕ is scalarly measurable, then it is Pettis integrable iff every function in K is integrable and $f \mapsto \int_E f : K \rightarrow \mathbb{R}$ is \mathfrak{T}_p -continuous for every $E \in \Sigma$. (*Hint*: 4A4Cg.) (iii) In particular, if ϕ is bounded and scalarly measurable and the identity map from (K, \mathfrak{T}_p) to (K, \mathfrak{T}_m) is continuous, then ϕ is Pettis integrable. (See TALAGRAND 84, chap. 4.)

(b) Show that any Bochner integrable function (253Yf) is Pettis integrable.

(c) Let μ be Lebesgue measure on $[0, 1]$, and define $\phi : [0, 1] \rightarrow L^\infty(\mu)$ by setting $\phi(t) = \chi_{[0,t]}^\bullet$ for every $t \in [0, 1]$. (i) Show that if $h \in L^\infty(\mu)^*$ and $\|h\| \leq 1$, then $h\phi$ has variation at most 1. (ii) Show that $K = \{h\phi : h \in L^\infty(\mu)^*, \|h\| \leq 1\}$ is a \mathfrak{T}_p -compact set of Lebesgue measurable functions, and that the identity map from (K, \mathfrak{T}_p) to (K, \mathfrak{T}_m) is continuous, so that ϕ is Pettis integrable. (iii) Show that ϕ is not Bochner integrable.

(d) Let (X, Σ, μ) be a σ -finite measure space and suppose that μ is inner regular with respect to some family $\mathcal{E} \subseteq \Sigma$ with cardinal at most ω_1 . (Subject to the continuum hypothesis, this is true for any subset of \mathbb{R} , for instance.) Show that if $K \subseteq \mathcal{L}^0$ is \mathfrak{T}_p -compact then it is \mathfrak{T}_m -compact. (See 536C, or TALAGRAND 84, 9-3-3.)

(e) Assume that the continuum hypothesis is true; let \preccurlyeq be a well-ordering of $[0, 1]$ with order type ω_1 (4A1Ad). Let (Z, ν) be the Stone space of the measure algebra of Lebesgue measure on $[0, 1]$, and $q : Z \rightarrow [0, 1]$ the canonical inverse-measure-preserving map (416V). Let $g : [0, 1] \rightarrow [0, \infty[$ be any function. Show that there is a function $f : [0, 1] \times Z \rightarrow [0, \infty[$ such that (α) f is continuous in the second variable (β) $f(t, z) = 0$ whenever $q(z) \preccurlyeq t$ (γ) $\int f(t, z)\nu(dz) = g(t)$ for every $t \in [0, 1]$. Show that f is universally measurable in the first variable, but need not be $\tilde{\Lambda}$ -measurable, where $\tilde{\Lambda}$ is the domain of the product Radon measure on $[0, 1] \times Z$. Setting $f_z(t) = f(t, z)$, show that $K = \{f_z : z \in Z\}$ is a \mathfrak{T}_p -compact set of Lebesgue measurable functions and that g belongs to the \mathfrak{T}_p -closed convex hull of K in $\mathbb{R}^{[0,1]}$.

463Z Problems (a) A.Bellow's problem Let (X, Σ, μ) be a probability space, and $K \subseteq \mathcal{L}^0$ a \mathfrak{T}_p -compact set such that $\{x : f(x) \neq g(x)\}$ is non-negligible for any distinct functions $f, g \in K$, as in 463G and 463Lc. Does it follow that K is metrizable for \mathfrak{T}_p ?

A positive answer would displace several of the arguments of this section, and have other consequences (see 462Z, for instance). It is known that under any of a variety of special axioms (starting with the continuum hypothesis) there is indeed a positive answer; see §536 in Volume 5, or TALAGRAND 84, chap. 12.

(b) Let $X \subseteq [0, 1]$ be a set of outer Lebesgue measure 1, and μ the subspace measure on X , with Σ its domain. Let K be a \mathfrak{T}_p -compact subset of \mathcal{L}^0 . Must K be \mathfrak{T}_m -compact?

(c) Let X_0, \dots, X_n be compact Hausdorff spaces and $f : X_0 \times \dots \times X_n \rightarrow \mathbb{R}$ a separately continuous function. Must f be universally measurable?

463 Notes and comments The relationship between the topologies \mathfrak{T}_p and \mathfrak{T}_m is complex, and I do not think that the results here are complete; in particular, we have a remarkable outstanding problem in 463Za. Much of the work presented here has been stimulated by problems concerning the integration of vector-valued functions. I am keeping this theory firmly in the 'further exercises' (463Ya-463Yc), but it is certainly the most important source of examples of pointwise compact sets of measurable functions. In particular, since the set $\{h\phi : h \in V^*, \|h\| \leq 1\}$ is necessarily convex whenever V is a Banach space and $\phi : X \rightarrow V$ is a function, we are led to look at the special properties of convex sets, as in 463G. There are obvious connexions with the theory of measures on linear topological spaces, which I will come to in §466.

The dichotomy in 463K shows that sets of measurable functions on perfect measure spaces are either 'good' (relatively countably compact for \mathfrak{T}_p , relatively compact for \mathfrak{T}_m) or 'bad' (with neither property). It is known that the result is not true for arbitrary σ -finite measure spaces (see §464 below), but it is not clear whether there are important non-perfect spaces in which it still applies in some form; see 463Zb.

Just as in §462, many questions concerning the topology \mathfrak{T}_p on \mathbb{R}^X can be re-phrased as questions about real-valued functions on products $X \times K$ which are continuous in the second variable. For the topology of pointwise convergence on sets of measurable functions, we find ourselves looking at functions which are measurable in the first variable. In this way we are led to such results as 463M-463N and 463Ye. Concerning 463M and 463Zc, it is the case that if X and Y are *any* compact Hausdorff spaces, and $f : X \times Y \rightarrow \mathbb{R}$ is separately continuous, then f is Borel measurable (BURKE & POL 05, 5.2).

A substantial proportion of the questions which arise naturally in this topic are known to be undecidable without using special axioms. I am avoiding such questions in this volume, but it is worth noting that the continuum hypothesis, in particular, has many striking consequences here, of which 463Ye is a sample. It also decides 463Za and 463Zb (see 463Yd).

464 Talagrand's measure

An obvious question arising from 463I and its corollaries is, do we really need the hypothesis that the measure involved is perfect? A very remarkable construction by M.Talagrand (464D) shows that these results are certainly not true of all probability spaces (464E). Investigating the properties of this measure we are led to some surprising facts about additive functionals on algebras $\mathcal{P}I$ and the duals of ℓ^∞ spaces (464M, 464R).

464A The usual measure on $\mathcal{P}I$ Recall from 254J and 416U that for any set I we have a standard measure ν , a Radon measure for the usual topology on $\mathcal{P}I$, defined by saying that $\nu\{a : a \subseteq I, a \cap J = c\} = 2^{-\#(J)}$ whenever $J \subseteq I$ is a finite set and $c \subseteq J$, or by copying from the usual product measure on $\{0, 1\}^I$ by means of the bijection $a \mapsto \chi a : \mathcal{P}I \rightarrow \{0, 1\}^I$. We shall need a couple of simple facts about these measures.

(a) If $\langle I_j \rangle_{j \in J}$ is any partition of I , then ν can be identified with the product of the family $\langle \nu_j \rangle_{j \in J}$, where ν_j is the usual measure on $\mathcal{P}I_j$, and we identify $\mathcal{P}I$ with $\prod_{j \in J} \mathcal{P}I_j$ by matching $a \subseteq I$ with $\langle a \cap I_j \rangle_{j \in J}$; this is the 'associative law' 254N. It follows that if we have any family $\langle A_j \rangle_{j \in J}$ of subsets of $\mathcal{P}I$, and if for each j the set A_j is 'determined by coordinates in I_j ' in the sense that, for $a \subseteq I$, $a \in A_j$ iff $a \cap I_j \in A_j$, then $\nu^*(\bigcap_{j \in J} A_j) = \prod_{j \in J} \nu^* A_j$ (use 254Lb).

(b) Similarly, if f_1, f_2 are non-negative real-valued functions on $\mathcal{P}I$, and if there are disjoint sets $I_1, I_2 \subseteq I$ such that $f_j(a) = f_j(a \cap I_j)$ for every $a \subseteq I$ and both j , then the upper integral $\overline{\int} f_1 + f_2 d\nu$ is $\overline{\int} f_1 d\nu + \overline{\int} f_2 d\nu$. **P** We may suppose that $I_2 = I \setminus I_1$. For each j , define $g_j : \mathcal{P}I_j \rightarrow [0, \infty[$ by setting $g_j = f_j \upharpoonright \mathcal{P}I_j$, so that $f_j(a) = g_j(a \cap I_j)$ for every $a \subseteq I$. Let ν_j be the usual measure on $\mathcal{P}I_j$, so that we can identify ν with the product measure $\nu_1 \times \nu_2$, if we identify $\mathcal{P}I$ with $\mathcal{P}I_1 \times \mathcal{P}I_2$; that is, we think of a subset of I as a pair (a_1, a_2) where $a_j \subseteq I_j$ for both j .

Now we have

$$\begin{aligned} \overline{\int} f_1 + f_2 d\nu &= \overline{\int} g_1 d\nu_1 + \overline{\int} g_2 d\nu_2 \\ (253K) \qquad &= \overline{\int} g_1 d\nu_1 \cdot \overline{\int} \chi(\mathcal{P}I_2) d\nu_2 + \overline{\int} \chi(\mathcal{P}I_1) d\nu_1 \cdot \overline{\int} g_2 d\nu_2 \\ &= \overline{\int} f_1 d\nu + \overline{\int} f_2 d\nu \end{aligned}$$

by 253J, because we can think of $f_1(a_1, a_2)$ as $g_1(a_1) \cdot (\chi \mathcal{P}I_2)(a_2)$ for all a_1, a_2 . **Q**

(c) If $A \subseteq \mathcal{P}I$ is such that $b \in A$ whenever $a \in A$, $b \subseteq I$ and $a \Delta b$ is finite, then $\nu^* A$ must be either 0 or 1; this is the zero-one law 254Sa, applied to the set $\{\chi a : a \in A\} \subseteq \{0, 1\}^I$ and the usual measure on $\{0, 1\}^I$.

464B Lemma Let I be any set, and ν the usual measure on $\mathcal{P}I$.

(a)(i) There is a sequence $\langle m(n) \rangle_{n \in \mathbb{N}}$ in \mathbb{N} such that $\prod_{n=0}^\infty 1 - 2^{-m(n)} = \frac{1}{2}$.

(ii) Given such a sequence, write X for $\prod_{n \in \mathbb{N}} (\mathcal{P}I)^{m(n)}$, and let λ be the product measure on X . We have a function $\phi : X \rightarrow \mathcal{P}I$ defined by setting

$$\phi(\langle \langle a_{ni} \rangle_{i < m(n)} \rangle_{n \in \mathbb{N}}) = \bigcup_{n \in \mathbb{N}} \bigcap_{i < m(n)} a_{ni}$$

whenever $\langle \langle a_{ni} \rangle_{i < m(n)} \rangle_{n \in \mathbb{N}} \in X$. Now ϕ is inverse-measure-preserving for λ and ν .

(b) The map

$$(a, b, c) \mapsto (a \cap b) \cup (a \cap c) \cup (b \cap c) : (\mathcal{P}I)^3 \rightarrow \mathcal{P}I$$

is inverse-measure-preserving for the product measure on $(\mathcal{P}I)^3$.

proof (a)(i) Choose $m(n)$ inductively so that, for each n in turn, $m(n)$ is minimal subject to the requirement $\prod_{k=0}^n 1 - 2^{-m(k)} > \frac{1}{2}$.

(ii) If $t \in I$, then

$$\{\mathfrak{r} : \mathfrak{r} \in X, t \notin \phi(\mathfrak{r})\} = \{ \langle \langle a_{ni} \rangle_{i < m(n)} \rangle_{n \in \mathbb{N}} : t \notin \bigcup_{n \in \mathbb{N}} \bigcap_{i < m(n)} a_{ni} \}$$

has measure

$$\prod_{n=0}^{\infty} \lambda_n \{ \langle a_i \rangle_{i < m(n)} : t \notin \bigcap_{i < m(n)} a_i \},$$

where λ_n is the product measure on $(\mathcal{P}I)^{m(n)}$ for each n . But this is just

$$\prod_{n=0}^{\infty} 1 - 2^{-m(n)} = \frac{1}{2},$$

by the choice of $\langle m(n) \rangle_{n \in \mathbb{N}}$. Accordingly $\lambda\{\mathfrak{r} : t \in \phi(\mathfrak{r})\} = \frac{1}{2}$ for every $t \in I$. Next, if we identify X with $\mathcal{P}(\{(t, n, i) : t \in I, n \in \mathbb{N}, i < m(n)\})$, each set $E_t = \{\mathfrak{r} : t \in \phi(\mathfrak{r})\}$ is determined by coordinates in $J_t = \{(t, n, i) : n \in \mathbb{N}, i < m(n)\}$. Since the sets J_t are disjoint, the sets E_t , for different t , are stochastically independent (464Aa), so if $J \subseteq I$ is finite,

$$\lambda\{\mathfrak{r} : J \subseteq \phi(\mathfrak{r})\} = \prod_{t \in J} \lambda E_t = 2^{-\#(J)} = \nu\{a : J \subseteq a\}.$$

This shows that $\lambda\phi^{-1}[F] = \nu F$ whenever F is of the form $\{a : J \subseteq a\}$ for some finite $J \subseteq I$. By the Monotone Class Theorem (136C), $\lambda\phi^{-1}[F] = \nu F$ for every F belonging to the σ -algebra generated by sets of this form. But this σ -algebra certainly contains all sets of the form $\{a : a \cap J = c\}$ where $J \subseteq I$ is finite and $c \subseteq J$, which are the sets corresponding to the basic cylinder sets in the product $\{0, 1\}^I$. By 254G, ϕ is inverse-measure-preserving.

(b) This uses the same idea as (a-ii). Writing ν^3 for the product measure on $(\mathcal{P}I)^3$, then, for any $t \in I$,

$$\begin{aligned} & \nu^3\{(a, b, c) : t \in (a \cap b) \cup (a \cap c) \cup (b \cap c)\} \\ &= \nu^3\{(a, b, c) : t \in a \cap b\} + \nu^3\{(a, b, c) : t \in a \cap c\} \\ & \quad + \nu^3\{(a, b, c) : t \in b \cap c\} - 2\nu^3\{(a, b, c) : t \in a \cap b \cap c\} \\ &= 3 \cdot \frac{1}{4} - 2 \cdot \frac{1}{8} = \frac{1}{2}. \end{aligned}$$

Once again, these sets are independent for different t , and this is all we need to know in order to be sure that the map is inverse-measure-preserving.

464C Lemma Let I be any set, and let ν be the usual measure on $\mathcal{P}I$.

(a) (see SIERPIŃSKI 1945) If $\mathcal{F} \subseteq \mathcal{P}I$ is any filter containing every cofinite set, then $\nu_*\mathcal{F} = 0$ and $\nu^*\mathcal{F}$ is either 0 or 1. If \mathcal{F} is a non-principal ultrafilter then $\nu^*\mathcal{F} = 1$.

(b) (TALAGRAND 80⁷) If $\langle \mathcal{F}_n \rangle_{n \in \mathbb{N}}$ is a sequence of filters on I , all of outer measure 1, then $\bigcap_{n \in \mathbb{N}} \mathcal{F}_n$ also has outer measure 1.

proof (a) That $\nu^*\mathcal{F} \in \{0, 1\}$ is immediate from 464Ac. If \mathcal{F} is an ultrafilter, then $\mathcal{P}I = \mathcal{F} \cup \{I \setminus a : a \in \mathcal{F}\}$; but as $a \mapsto I \setminus a$ is a measure space automorphism of $(\mathcal{P}I, \nu)$, $\nu^*\{I \setminus a : a \in \mathcal{F}\} = \nu^*\mathcal{F}$, and both must be at least $\frac{1}{2}$, so $\nu^*\mathcal{F} = 1$. Equally, $\nu^*(\mathcal{P}I \setminus \mathcal{F}) = \nu^*\{I \setminus a : a \in \mathcal{F}\} = 1$, so $\nu_*\mathcal{F} = 0$. Returning to a general filter containing every cofinite set, this is included in a non-principal ultrafilter, so also has inner measure 0.

(b) Let $\langle m(n) \rangle_{n \in \mathbb{N}}$ be a sequence in \mathbb{N} such that $\prod_{n=0}^{\infty} 1 - 2^{-m(n)} = \frac{1}{2}$, and let X, λ and $\phi : X \rightarrow \mathcal{P}I$ be as in 464Ba. Consider the set $D = \prod_{n \in \mathbb{N}} \mathcal{F}_n^{m(n)}$ as a subset of X . By 254Lb, $\lambda^*D = 1$. If we set $\mathcal{F} = \bigcap_{n \in \mathbb{N}} \mathcal{F}_n$, then we see that whenever $\mathfrak{r} = \langle \langle a_{ni} \rangle_{i < m(n)} \rangle_{n \in \mathbb{N}}$ belongs to D ,

$$\phi(\mathfrak{r}) \supseteq \bigcap_{i < m(n)} a_{ni} \in \mathcal{F}_n$$

⁷The date of this paper is misleading, as there was an unusual backlog in the journal; in reality it preceded FREMLIN & TALAGRAND 79.

for every n , so that $\phi(\mathfrak{r}) \in \mathcal{F}$. Thus $D \subseteq \phi^{-1}[\mathcal{F}]$ and

$$\nu^*\mathcal{F} \geq \lambda^*\phi^{-1}[\mathcal{F}] \geq \lambda^*D = 1$$

because ϕ is inverse-measure-preserving (413Eh).

As $\langle \mathcal{F}_n \rangle_{n \in \mathbb{N}}$ is arbitrary, we have the result.

464D Construction (TALAGRAN 80) Let I be any set, and ν the usual Radon measure on $\mathcal{P}I$, with \mathbb{T} its domain. Let Σ be the set

$$\{E : E \subseteq \mathcal{P}I, \text{ there are a set } F \in \mathbb{T} \text{ and a filter } \mathcal{F} \text{ on } I \\ \text{ such that } \nu^*\mathcal{F} = 1 \text{ and } E \cap \mathcal{F} = F \cap \mathcal{F}\}.$$

Then there is a unique extension of ν to a complete probability measure μ , with domain Σ , defined by saying that $\mu E = \nu F$ whenever $E \in \Sigma$, $F \in \mathbb{T}$ and there is a filter \mathcal{F} on I such that $\nu^*\mathcal{F} = 1$ and $E \cap \mathcal{F} = F \cap \mathcal{F}$. (By 464C, we can apply 417A with

$$A = \{\mathcal{P}I \setminus \mathcal{F} : \mathcal{F} \text{ is a filter on } I \text{ such that } \nu^*\mathcal{F} = 1\}.)$$

Definition This measure μ is **Talagrand's measure** on $\mathcal{P}I$.

464E Example If μ is Talagrand's measure on $X = \mathcal{P}\mathbb{N}$, and Σ its domain, then there is a set $K \subseteq \mathbb{R}^X$, consisting of Σ -measurable functions and separable and compact for the topology \mathfrak{T}_p of pointwise convergence, such that K is not compact for the topology of convergence in measure, that is, there is a sequence in K with no subsequence which is convergent almost everywhere, even though every cluster point is Σ -measurable.

proof Take K to be $\{\chi_{\mathcal{F}} : \mathcal{F} \text{ is an ultrafilter on } \mathbb{N}\}$. Then K is \mathfrak{T}_p -compact (in fact, is precisely the set of Boolean homomorphisms from $\mathcal{P}\mathbb{N}$ to $\{0, 1\}$, identified with the Stone space of $\mathcal{P}\mathbb{N}$ in 311E). As such, it is separable, the countable set of principal ultrafilters being dense. By 464Ca, we see that any non-principal ultrafilter \mathcal{F} on \mathbb{N} belongs to Σ , and $\mu\mathcal{F} = 1$. On the other hand, all the principal ultrafilters $\mathcal{F}_n = \{a : n \in a \subseteq \mathbb{N}\}$ are measured by ν and therefore by μ , and form a stochastically independent sequence of sets of measure $\frac{1}{2}$. So K consists of Σ -measurable functions; but the sequence $\langle \chi_{\mathcal{F}_n} \rangle_{n \in \mathbb{N}}$ has no subsequence which is convergent almost everywhere, and K is not relatively compact for the topology of convergence in measure.

Remark In this example, a very large number of members of K are equal almost everywhere; indeed, all non-principal ultrafilters are equal a.e., and if we look at $\{f^\bullet : f \in K\}$ in $L^0(\mu)$, it is a countable discrete set. Given that K is a set of measurable functions homeomorphic to $\beta\mathbb{N}$, something like this has to happen (see 536D⁸ in Volume 5). Looking at this from a different angle, if we wish to extend the usual measure ν on $\mathcal{P}\mathbb{N}$ to measure every ultrafilter, there will have to be two distinct ultrafilters $\mathcal{F}_1, \mathcal{F}_2$ such that $\mathcal{F}_1 \Delta \mathcal{F}_2$ is negligible for the extended measure.

464F The L -space $\ell^\infty(I)^*$ For the next step, we shall need to recall some facts from Volume 3. Let I be any set.

(a) The space $\ell^\infty(I)$ of bounded real-valued functions on I is an M -space (354Ha), so its dual $\ell^\infty(I)^\sim = \ell^\infty(I)^*$ is an L -space (356N), that is, a Banach lattice such that $\|f + g\| = \|f\| + \|g\|$ for all non-negative $f, g \in \ell^\infty(I)^*$. Since $\ell^\infty(I)$ can be identified with the space $L^\infty(\mathcal{P}I)$ as described in §363 (see 363Ha), we can identify $\ell^\infty(I)^*$ with the L -space M of bounded finitely additive functionals on $\mathcal{P}I$ (363K), matching any $f \in \ell^\infty(I)^*$ with the functional $a \mapsto f(\chi_a) : \mathcal{P}I \rightarrow \mathbb{R}$ in M .

(b) In §§361-363 I examined some of the bands in M . The most significant ones for our present purposes are the band M_τ of completely additive functionals (362Bb) and its complement M_τ^\perp (352P); because M is Dedekind complete, we have $M = M_\tau \oplus M_\tau^\perp$ (353J⁹). In fact M_τ is just the set of those $\theta \in M$ such that

⁸Formerly 536C.

⁹Formerly 353I.

$\theta a = \sum_{t \in a} \theta\{t\}$ for every $a \subseteq I$, while M_τ^\perp is the set of those $\theta \in M$ such that $\theta\{t\} = 0$ for every $t \in I$. **P** For any $\theta \in M$, we can set $\alpha_t = \theta\{t\}$ for each $t \in I$; in this case,

$$|\sum_{t \in J} \alpha_t| = |\theta J| \leq \|\theta\|$$

for every finite set $J \subseteq I$, so $\sum_{t \in I} |\alpha_t|$ is finite, and we have a functional θ_1 defined by setting $\theta_1 a = \sum_{t \in a} \alpha_t$ for every $a \subseteq I$. It is easy to check that $\theta_1 \in M_\tau$. If we write $\theta_2 = \theta - \theta_1$ then $\theta_2\{t\} = 0$ for every $t \in I$. So if $\phi \in M_\tau$ and $0 \leq \phi \leq |\theta_2|$, we still have $\phi\{t\} = 0$ for every $t \in I$, and $\phi a = 0$ for every finite $a \subseteq I$ and $\phi = 0$; thus $\theta_2 \in M_\tau^\perp$. Now $\theta \in M_\tau$ iff $\theta_2 = 0$, that is, iff $\theta = \theta_1$ and $\theta a = \sum_{t \in a} \theta\{t\}$ for every $a \subseteq I$; while $\theta \in M_\tau^\perp$ iff $\theta_1 = 0$, that is, $\theta\{t\} = 0$ for every $t \in I$. **Q**

Observe that if $\theta \in M_\tau^\perp$ and $a, b \subseteq I$ are such that $a \Delta b$ is finite, then $\theta a = \theta b$, because $\theta(a \setminus b) = \theta(b \setminus a) = 0$.

(c) It will be useful to have an elementary fact out in the open. If $\theta \in M^+ \setminus \{0\}$, then $\{a : a \subseteq I, \theta a = \theta I\}$ is a filter; this is because $\{a : \theta a = 0\}$ is a proper ideal in $\mathcal{P}I$.

464G We also need a new result not exactly covered by those in Chapters 35 and 36.

Lemma Let \mathfrak{A} be any Boolean algebra. Write M for the L -space of bounded additive functionals on \mathfrak{A} , and M^+ for its positive cone, the set of non-negative additive functionals. Suppose that $\Delta : M^+ \rightarrow [0, \infty[$ is a functional such that

- (α) Δ is non-decreasing,
- (β) $\Delta(\alpha\theta) = \alpha\Delta(\theta)$ whenever $\theta \in M^+, \alpha \geq 0$,
- (γ) $\Delta(\theta_1 + \theta_2) = \Delta(\theta_1) + \Delta(\theta_2)$ whenever $\theta_1, \theta_2 \in M^+$ are such that, for some $e \subseteq I$, $\theta_1(1 \setminus e) = \theta_2 e = 0$,
- (δ) $|\Delta(\theta_1) - \Delta(\theta_2)| \leq \|\theta_1 - \theta_2\|$ for all $\theta_1, \theta_2 \in M^+$.

Then there is a non-negative $h \in M^*$ extending Δ .

proof (a) If $\theta_1, \theta_2 \in M^+$ are such that $\theta_1 \wedge \theta_2 = 0$ in M^+ , then $\Delta(\theta_1 + \theta_2) = \Delta(\theta_1) + \Delta(\theta_2)$. **P** Let $\epsilon > 0$. Then there is an $e \in \mathfrak{A}$ such that $\theta_1(1 \setminus e) + \theta_2 e \leq \epsilon$ (362Ba). Set $\theta'_1 a = \theta_1(a \cap e)$, $\theta'_2(a) = \theta_2(a \setminus e)$ for $a \in \mathfrak{A}$; then θ'_1 and θ'_2 belong to M^+ and $\theta'_1(1 \setminus e) = \theta'_2 e = 0$, so $\Delta(\theta'_1 + \theta'_2) = \Delta(\theta'_1) + \Delta(\theta'_2)$, by hypothesis (γ). On the other hand,

$$0 \leq \theta_1 a - \theta'_1 a = \theta(a \setminus e) \leq \theta(1 \setminus e) \leq \epsilon$$

for every $a \in \mathfrak{A}$, so $\|\theta_1 - \theta'_1\| \leq \epsilon$. Similarly, $\|\theta_2 - \theta'_2\| \leq \epsilon$ and $\|(\theta_1 + \theta_2) - (\theta'_1 + \theta'_2)\| \leq 2\epsilon$. But this means that we can use the hypothesis (δ) to see that

$$|\Delta(\theta_1 + \theta_2) - \Delta(\theta_1) - \Delta(\theta_2)| \leq |\Delta(\theta'_1 + \theta'_2) - \Delta(\theta'_1) - \Delta(\theta'_2)| + 4\epsilon = 4\epsilon.$$

As ϵ is arbitrary, $\Delta(\theta_1 + \theta_2) = \Delta(\theta_1) + \Delta(\theta_2)$. **Q**

(b) Now recall that M , being a Dedekind complete Riesz space, can be identified with an order-dense solid linear subspace of $L^0(\mathfrak{C})$ for some Boolean algebra \mathfrak{C} (368H). Inside $L^0(\mathfrak{C})$ we have the order-dense Riesz subspace $S(\mathfrak{C})$ (364Ja). Write S_1 for $M \cap S(\mathfrak{C})$, so that S_1 is an order-dense Riesz subspace of M (352Nc, 353A).

If $\theta_1, \theta_2 \in S_1^+ = S_1 \cap M^+$, then $\Delta(\theta_1 + \theta_2) = \Delta(\theta_1) + \Delta(\theta_2)$. **P** We can express θ_1 as $\sum_{i=0}^m \alpha_i \chi c_i$, where c_0, \dots, c_m are disjoint members of \mathfrak{C} , and $\alpha_i \geq 0$ for each i (361Ec); adding a term $0 \cdot \chi(1 \setminus \sup_{i \leq m} c_i)$ if necessary, we may suppose that $\sup_{i \leq m} c_i = 1$. Similarly, we can express θ_2 as $\sum_{j=0}^n \beta_j \chi d_j$ where $d_0, \dots, d_n \in \mathfrak{C}$ are disjoint, $\beta_j \geq 0$ for every j and $\sup_{j \leq n} d_j = 1$. In this case, $\theta_1 = \sum_{i \leq m, j \leq n} \alpha_i \chi(c_i \cap d_j)$ and $\theta_2 = \sum_{i \leq m, j \leq n} \beta_j \chi(c_i \cap d_j)$. Re-enumerating $\{c_i \cap d_j : i \leq m, j \leq n\}$ as $\{e_i : i \leq k\}$ we have expressions of θ_1, θ_2 in the form $\sum_{i \leq k} \gamma_i \chi e_i, \sum_{i \leq k} \delta_i \chi e_i$, while e_0, \dots, e_k are disjoint.

Setting $\theta_1^{(r)} = \sum_{i=0}^r \gamma_i \chi e_i$ for $r \leq k$, we see that $\theta_1^{(r)} \wedge \gamma_{r+1} \chi e_{r+1} = 0$ for $r < k$, so (a) above, together with the hypothesis (β), tell us that

$$\Delta(\theta_1^{(r+1)}) = \Delta(\theta_1^{(r)}) + \Delta(\gamma_{r+1} \chi e_{r+1}) = \Delta(\theta_1^{(r)}) + \gamma_{r+1} \Delta(\chi e_{r+1})$$

for $r < k$. Accordingly $\Delta(\theta_1) = \sum_{i=0}^k \gamma_i \Delta(\chi e_i)$. Similarly, $\Delta(\theta_2) = \sum_{i=0}^k \delta_i \Delta(\chi e_i)$ and

$$\Delta(\theta_1 + \theta_2) = \sum_{i=0}^k (\gamma_i + \delta_i) \Delta(\chi_{e_i}) = \Delta(\theta_1) + \Delta(\theta_2),$$

as claimed. **Q**

(c) Consequently, $\Delta(\theta_1 + \theta_2) = \Delta(\theta_1) + \Delta(\theta_2)$ for all $\theta_1, \theta_2 \in M^+$. **P** Let $\epsilon > 0$. Because S_1 is order-dense in M , and the norm of M is order-continuous (354N), S_1 is norm-dense (354Ef), and there are $\theta'_1, \theta'_2 \in S_1^+$ such that $\|\theta_j - \theta'_j\| \leq \epsilon$ for both j (354Be). But now, just as in (a),

$$|\Delta(\theta_1 + \theta_2) - \Delta(\theta_1) - \Delta(\theta_2)| \leq |\Delta(\theta'_1 + \theta'_2) - \Delta(\theta'_1) - \Delta(\theta'_2)| + 4\epsilon = 4\epsilon.$$

As ϵ is arbitrary, we have the result. **Q**

(d) Now (c) and the hypothesis (β) are sufficient to ensure that Δ has an extension to a positive linear functional (355D).

464H The next lemma contains the key ideas needed for the rest of the section.

Lemma Let I be any set, and M the L -space of bounded additive functionals on $\mathcal{P}I$; let ν be the usual measure on $\mathcal{P}I$. For $\theta \in M^+$, set

$$\Delta(\theta) = \bar{\int} \theta d\nu.$$

(a) For every $\theta \in M^+$, $\frac{1}{2}\theta I \leq \Delta(\theta) \leq \theta I$.

(b) There is a non-negative $h \in M^*$ such that $h(\theta) = \Delta(\theta)$ for every $\theta \in M^+$.

(c) If $\theta \in (M_\tau^\perp)^+$, where $M_\tau \subseteq M$ is the band of completely additive functionals, then $\theta \leq \Delta(\theta)$ ν -a.e., and $\nu^*\{a : \alpha \leq \theta a \leq \Delta(\theta)\} = 1$ for every $\alpha < \Delta(\theta)$.

(d) Suppose that $\theta \in (M_\tau^\perp)^+$ and $\beta, \gamma \in [0, 1]$ are such that $\theta I = 1$ and $\beta\theta' I \leq \Delta(\theta') \leq \gamma\theta' I$ whenever $\theta' \leq \theta$ in M^+ . Then, for any $\alpha < \beta$,

(i) for any finite set $K \subseteq \mathcal{P}I$, the set

$$\{a : a \subseteq I, \alpha\theta b \leq \theta(a \cap b) \leq \gamma\theta b \text{ for every } b \in K\}$$

has outer measure 1 in $\mathcal{P}I$;

(ii) if $\alpha \geq \frac{1}{2}$, the set

$$R = \{(a, b, c) : a, b, c \subseteq I, \theta((a \cap b) \cup (a \cap c) \cup (b \cap c)) \geq 2\alpha^2 + (1 - 2\alpha)\gamma^2\}$$

has outer measure 1 in $(\mathcal{P}I)^3$;

(iii) if $\alpha \geq \frac{1}{2}$, then $2\alpha^2 + (1 - 2\alpha)\gamma^2 \leq \gamma$.

(e) Any $\theta \in M^+$ can be expressed as $\theta_1 + \theta_2$ where $\Delta(\theta_1) = \frac{1}{2}\theta_1 I$ and $\Delta(\theta_2) = \theta_2 I$.

(f) Suppose that $0 \leq \theta' \leq \theta$ in M .

(i) If $\Delta(\theta) = \frac{1}{2}\theta I$, then $\Delta(\theta') = \frac{1}{2}\theta' I$.

(ii) If $\Delta(\theta) = \theta I$, then $\Delta(\theta') = \theta' I$.

proof (a) Of course

$$\Delta(\theta) \leq \int \theta I d\nu = \theta I.$$

On the other hand, because $a \mapsto I \setminus a : \mathcal{P}I \rightarrow \mathcal{P}I$ is an automorphism of the measure space $(\mathcal{P}I, \nu)$, $\bar{\int} f(I \setminus a) \nu(da) = \bar{\int} f(a) \nu(da)$ for any real-valued function f (cf. 235Xn). In particular,

$$\Delta(\theta) = \bar{\int} \theta(a) \nu(da) = \bar{\int} \theta(I \setminus a) \nu(da).$$

So

$$2\Delta(\theta) = \bar{\int} \theta(a) \nu(da) + \bar{\int} \theta(I \setminus a) \nu(da) \geq \bar{\int} \theta(a) + \theta(I \setminus a) \nu(da)$$

(133J(b-ii))

$$= \bar{\int} \theta I \nu(da) = \theta I,$$

and $\Delta(\theta) \geq \frac{1}{2}\theta I$.

(b) I use 464G with $\mathfrak{A} = \mathcal{P}I$. Examine the conditions (α) - (δ) there.

(α) Of course Δ is non-decreasing (133Jc).

(β) $\Delta(\alpha\theta) = \alpha\Delta(\theta)$ for every $\theta \in M^+$ and every $\alpha \geq 0$, by 133J(b-iii).

(γ) If $\theta_1, \theta_2 \in M^+$ then $\Delta(\theta_1 + \theta_2) \leq \Delta(\theta_1) + \Delta(\theta_2)$ by 133J(b-ii), as in (a) above. If $\theta_1, \theta_2 \in M^+$ and $e \subseteq I$ are such that $\theta_1(I \setminus e) = \theta_2 e = 0$, then $\theta_1 a = \theta_1(a \cap e)$, $\theta_2(a) = \theta_2(a \setminus e)$ for every $a \subseteq I$. So $\Delta(\theta_1 + \theta_2) = \Delta(\theta_1) + \Delta(\theta_2)$ by 464Ab.

(δ) For every $a \subseteq I$, $\theta_2 a \leq \theta_1 a + \|\theta_2 - \theta_1\|$, so $\Delta(\theta_2) \leq \Delta(\theta_1) + \|\theta_2 - \theta_1\|$. Similarly, $\Delta(\theta_1) \leq \Delta(\theta_2) + \|\theta_1 - \theta_2\|$. So $|\Delta(\theta_1) - \Delta(\theta_2)| \leq \|\theta_1 - \theta_2\|$, and Δ satisfies condition (δ) of 464G.

(ϵ) Accordingly 464G tells us that Δ has an extension to a member of M^* .

(c) If $\theta \in M_\tau^\perp$, then $\theta a = \theta b$ whenever $a \Delta b$ is finite (464Fb), so all the sets $\{a : \theta a > \alpha\}$ have outer measure either 0 or 1, by 464Ac. But this means that if f is a measurable function and $\theta \leq_{\text{a.e.}} f$ and $\int f = \Delta(\theta)$, as in 133J(a-i), $\{a : f(a) \leq \Delta(\theta)\}$ has positive measure and meets $A = \{a : \theta a > \Delta(\theta)\}$ in a negligible set; so A cannot have full outer measure and is negligible.

On the other hand, if $\alpha < \Delta(\theta)$, then θ cannot be dominated a.e. by $\alpha\chi(\mathcal{P}I)$, so $\{a : \theta a > \alpha\}$ is not negligible and has outer measure 1. Consequently $\nu^*\{a : \alpha < \theta a \leq \Delta(\theta)\} = 1$.

(d)(i) The point is just that for any $b \subseteq I$, the functional θ_b , defined by saying that $\theta_b(a) = \theta(a \cap b)$ for every $a \subseteq I$, belongs to M^+ and is dominated by θ , so that $\beta\theta_b I \leq \Delta(\theta_b) \leq \gamma\theta_b I$, and $\{a : \alpha\theta_b I \leq \theta_b a \leq \gamma\theta_b I\}$ has outer measure 1, by (c). (If $\alpha\theta_b I = \Delta(\theta_b)$, this is because $\theta_b I = 0$, and the result is trivial; otherwise, $\alpha\theta_b I < \Delta(\theta_b) \leq \gamma\theta_b I$.) But this just says that, for any $b \subseteq I$,

$$\{a : \alpha\theta b \leq \theta(a \cap b) \leq \gamma\theta b\}$$

has outer measure 1.

Now, given any finite set $K \subseteq \mathcal{P}I$, let \mathcal{B} be the subalgebra of $\mathcal{P}I$ generated by K , and b_0, \dots, b_n the atoms of \mathcal{B} . Then all the sets

$$A_i = \{a : \alpha\theta b_i \leq \theta(a \cap b_i) \leq \gamma\theta b_i\}$$

have outer measure 1; because each A_i is determined by coordinates in b_i , and b_0, \dots, b_n are disjoint, $A = \bigcap_{i \leq n} A_i$ still has outer measure 1, by 464Aa. But if $a \in A$ and $b \in K$, then $b = \bigcup_{i \in J} b_i$ for some $J \subseteq \{0, \dots, n\}$ and

$$\begin{aligned} \alpha\theta b &= \sum_{i \in J} \alpha\theta b_i \leq \sum_{i \in J} \theta(a \cap b_i) \\ &= \theta(a \cap b) \leq \sum_{i \in J} \gamma\theta b_i = \gamma\theta b. \end{aligned}$$

So $\{a : \alpha\theta b \leq \theta(a \cap b) \leq \gamma\theta b$ for every $b \in K\}$ includes A and has outer measure 1.

(ii) ? Suppose, if possible, otherwise; then $\nu_*^3(\mathcal{P}I \setminus R) > 0$, where ν^3 is the product measure on $(\mathcal{P}I)^3$, and there is a measurable set $W \subseteq (\mathcal{P}I)^3 \setminus R$ such that $\nu^3 W > 0$. For $a \subseteq I$, set $W_a = \{(b, c) : (a, b, c) \in W\}$; then $\nu^3 W = \int \nu^2 W_a \nu(da)$, where ν^2 is the product measure on $(\mathcal{P}I)^2$, by Fubini's theorem (252D). Set $E = \{a : \nu^2 W_a \text{ is defined and not zero}\}$; then $\nu E > 0$. Since $\{a : \alpha \leq \theta a \leq \gamma\}$ has outer measure 1, there is an $a \in E$ such that $\alpha \leq \theta a \leq \gamma$.

For $b \subseteq I$, set $W_{ab} = \{c : (b, c) \in W_a\} = \{c : (a, b, c) \in W\}$. Then $0 < \nu^2 W_a = \int \nu W_{ab} \nu(db)$, so $F = \{b : \nu W_{ab} \text{ is defined and not } 0\}$ has non-zero measure. But also, by (i),

$$\{b : \theta b \geq \alpha, \theta(a \cap b) \leq \gamma\theta a\}$$

has outer measure 1, so we can find a $b \in F$ such that $\theta b \geq \alpha$ and $\theta(a \cap b) \leq \gamma\theta a$.

By (i) again,

$$\{c : \theta(c \cap (a \Delta b)) \geq \alpha\theta(a \Delta b)\}$$

has outer measure 1, so meets W_{ab} ; accordingly we have a $c \subseteq I$ such that $(a, b, c) \in W$ while $\theta(c \cap (a \Delta b)) \geq \alpha\theta(a \Delta b)$.

Now calculate, for this triple (a, b, c) ,

$$\begin{aligned} \theta((a \cap b) \cup (a \cap c) \cup (b \cap c)) &= \theta(a \cap b) + \theta(c \cap (a \Delta b)) \geq \theta(a \cap b) + \alpha\theta(a \Delta b) \\ \text{(by the choice of } c) & \\ &= \alpha(\theta a + \theta b) + (1 - 2\alpha)\theta(a \cap b) \\ &\geq \alpha(\theta a + \alpha) + (1 - 2\alpha)\gamma\theta a \\ \text{(by the choice of } b, \text{ recalling that } 1 - 2\alpha \leq 0) & \\ &\geq 2\alpha^2 + (1 - 2\alpha)\gamma^2 \end{aligned}$$

by the choice of a . But this means that $(a, b, c) \in W \cap R$, which is supposed to be impossible. **X**

(iii) Now recall that the map $(a, b, c) \mapsto (a \cap b) \cup (a \cap c) \cup (b \cap c)$ is inverse-measure-preserving (464Bb). Since $\theta a \leq \Delta(\theta) \leq \gamma$ for ν -almost every a , we must have $\theta((a \cap b) \cup (a \cap c) \cup (b \cap c)) \leq \gamma$ for ν^3 -almost every (a, b, c) . But as R is not negligible, there must be some $(a, b, c) \in R$ such that $\theta((a \cap b) \cup (a \cap c) \cup (b \cap c)) \leq \gamma$, and $2\alpha^2 + (1 - 2\alpha)\gamma^2 \leq \gamma$.

(e)(i) M^* , being the dual of an L -space, is an M -space (356Pb), so can be represented as $C(Z)$ for some compact Hausdorff space Z (354L). The functional h of (b) above therefore corresponds to a function $w \in C(Z)$. Any $\theta \in M^+$ acts on M^* as a positive linear functional, so corresponds to a Radon measure μ_θ on Z (436J/436K); we have $\Delta(\theta) = h(\theta) = \int w d\mu_\theta$. The inequalities $\frac{1}{2}\theta I \leq \Delta(\theta) \leq \theta I$ become $\frac{1}{2}\mu_\theta Z \leq \int w d\mu_\theta \leq \mu_\theta Z$, because the constant function χZ corresponds to the standard order unit of M^* (356Pb again), so that

$$\mu_\theta Z = \int \chi Z d\mu_\theta = \|\theta\| = \theta I$$

for every $\theta \geq 0$. Since $0 \leq h(\theta) \leq \|\theta\|$ for every $\theta \geq 0$, $\|w\|_\infty = \|h\| \leq 1$ and $0 \leq w \leq \chi Z$.

(ii) Now suppose that $\beta < \gamma$ and that $G = \{z : z \in Z, \beta < w(z) < \gamma\}$ is non-empty. In this case there is a non-zero $\theta_0 \in M^+$ such that $\beta\theta' I \leq \Delta(\theta') \leq \gamma\theta' I$ whenever $0 \leq \theta' \leq \theta_0$.

P(α) We have a solid linear subspace $V = \{v : v \in C(Z), v(z) = 0 \text{ for every } z \in G\}$ of $C(Z)$. Consider $U = \{\theta : \theta \in M, (\theta|v) = 0 \text{ for every } v \in V\}$, where I write $(|)$ for the duality between M and $C(Z)$ corresponding to the identification of $C(Z)$ with M^* .

(β) If $\theta \in U \cap M^+$, then $\beta\theta I \leq \Delta(\theta) \leq \gamma\theta I$. To see this, observe that $\int v d\mu_\theta = (\theta|v) = 0$ for every $v \in V$, so

$$\begin{aligned} \mu_\theta(Z \setminus \overline{G}) &= \sup\{\mu_\theta K : K \subseteq Z \setminus \overline{G} \text{ is compact}\} \\ &= \sup\left\{\int v d\mu_\theta : v \in C(Z), 0 \leq v \leq \chi(Z \setminus \overline{G})\right\} \end{aligned}$$

(because whenever $K \subseteq Z \setminus \overline{G}$ is compact there is a $v \in C(Z)$ such that $\chi K \leq v \leq \chi(Z \setminus \overline{G})$, by 4A2F(h-ii))
= 0.

Because w is continuous, $\beta \leq w(z) \leq \gamma$ for every $z \in \overline{G}$; thus $\beta \leq w \leq \gamma$ μ_θ -a.e. and $\int w d\mu_\theta$ must belong to $[\beta\mu_\theta Z, \gamma\mu_\theta Z] = [\beta\theta I, \gamma\theta I]$.

(γ) The dual $M^* = M^\times$ of M is perfect (356Lb), and $C(Z)$ is perfect; moreover, M is perfect (356Pa), so the duality $(|)$ identifies M with $C(Z)^\times$. Now V^\perp , taken in $C(Z)$, contains any continuous function zero on $Z \setminus G$, so is not $\{0\}$; since V^\perp , like $C(Z)$, must be perfect (356La), $(V^\perp)^\times$ is non-trivial. Take any $\psi > 0$ in $(V^\perp)^\times$. Being perfect, $C(Z)$ is Dedekind complete (356K), so there is a band projection $P : C(Z) \rightarrow V^\perp$ (353J again). Now ψP is a positive element of $C(Z)^\times$ which is zero on V , and must correspond to a non-zero element θ_0 of $U \cap M^+$.

(δ) If $0 \leq \theta' \leq \theta_0$ in M , then, for any $v \in V$,

$$|(\theta'|v)| \leq (\theta'|v) \leq (\theta_0|v) = 0,$$

because $|v| \in V$. So $\theta' \in U$ and $\beta\theta'I \leq \Delta(\theta') \leq \gamma\theta'I$, by (β). Thus θ_0 has the required property. **Q**

(iii) It follows at once that $w(z) \geq \frac{1}{2}$ for every $z \in Z$. **P?** If $w(z_0) < \frac{1}{2}$, then we can apply (ii) with $\beta = -1$, $\gamma \in]w(z_0), \frac{1}{2}[$ to see that there is a non-zero $\theta \in M^+$ such that $\Delta(\theta) \leq \gamma\theta I < \frac{1}{2}\theta I$, which is impossible, by (a). **XQ**

(iv) But we find also that $w(z) \notin]\frac{1}{2}, 1[$ for any $z \in Z$. **P?** If $w(z_0) = \delta \in]\frac{1}{2}, 1[$, then $2\delta^2 + (1-2\delta)\delta^2 > \delta$ (because $\delta(2\delta - 1)(1 - \delta) > 0$). We can therefore find α, β and γ such that $\frac{1}{2} \leq \alpha < \beta < \delta < \gamma$ and $2\alpha^2 + (1 - 2\alpha)\gamma^2 > \gamma$. But now $\{z : \beta < w(z) < \gamma\}$ is non-empty, so by (ii) there is a non-zero $\theta \in M^+$ such that $\beta\theta'I \leq \Delta(\theta') \leq \gamma\theta'I$ whenever $0 \leq \theta' \leq \theta$. Multiplying θ by a suitable scalar if necessary, we can arrange that θI should be 1. But this is impossible, by (d-iii). **XQ**

(v) Thus w takes only the values $\frac{1}{2}$ and 1; let H_1 and H_2 be the corresponding open-and-closed subsets of Z .

Take $\theta \in M^+$. For $u \in C(Z)$, set $\phi(u) = \int u d\mu_\theta$ and $\phi_j(u) = \int_{H_j} u d\mu_\theta$ for each j . Then each ϕ_j is a positive linear functional on $C(Z)$ and $\phi_j \leq \phi$. But ϕ is the image of θ under the canonical isomorphism from M to $C(Z)^\times \cong M^{\times\times}$, and $C(Z)^\times$ is solid in $C(Z)^\sim$ (356B), so both ϕ_1 and ϕ_2 belong to the image of M , and correspond to $\theta_1, \theta_2 \in M$. For any $u \in C(Z) \cong M^*$,

$$(\theta_1 + \theta_2|u) = \phi_1(u) + \phi_2(u) = (\theta|u),$$

so $\theta = \theta_1 + \theta_2$. We have

$$\begin{aligned} \Delta(\theta_j) &= \phi_j(w) = \int_{H_j} w d\mu_\theta \\ &= \frac{1}{2}\mu_\theta(H_1) = \frac{1}{2}\theta_1 I \text{ if } j = 1, \\ &= \mu_\theta(H_2) = \theta_2 I \text{ if } j = 2. \end{aligned}$$

So we have a suitable decomposition $\theta = \theta_1 + \theta_2$.

(f) This is easy. Set $\theta'' = \theta - \theta'$; then

$$\begin{aligned} \frac{1}{2}\theta' I &\leq \Delta(\theta') \leq \theta' I, \\ \frac{1}{2}\theta'' I &\leq \Delta(\theta'') \leq \theta'' I \end{aligned}$$

by (a), while $\Delta(\theta') + \Delta(\theta'') = \Delta(\theta)$ by (b), and of course $\theta' I + \theta'' I = \theta I$. But this means that

$$\Delta(\theta') - \frac{1}{2}\theta' I \leq \Delta(\theta) - \frac{1}{2}\theta I, \quad \theta' I - \Delta(\theta') \leq \theta I - \Delta(\theta),$$

and the results follow.

464I Measurable and purely non-measurable functionals As before, let I be any set, ν the usual measure on $\mathcal{P}I$, T its domain, and M the L -space of bounded additive functionals on $\mathcal{P}I$. Following FREMLIN & TALAGRAND 79, I say that $\theta \in M$ is **measurable** if it is T -measurable when regarded as a real-valued function on $\mathcal{P}I$, and **purely non-measurable** if $\{a : a \subseteq I, |\theta|(a) = |\theta|(I)\}$ has outer measure 1. (Of course the zero functional is both measurable and purely non-measurable.)

464J Examples Before going farther, I had better offer some examples of measurable and purely non-measurable functionals. Let I, ν and M be as in 464I.

(a) Any $\theta \in M_\tau$ is measurable, where M_τ is the space of completely additive functionals on $\mathcal{P}I$. **P** By 464Fb, θ can be expressed as a sum of point masses; say $\theta a = \sum_{t \in a} \alpha_t$ for some family $\langle \alpha_t \rangle_{t \in I}$ in \mathbb{R} . Since $\sum_{t \in I} |\alpha_t|$ must be finite, $\{t : \alpha_t \neq 0\}$ is countable, and we can express θ as the limit of a sequence of finite sums $\sum_{t \in K} \alpha_t \hat{t}$, where $\hat{t}(a) = 1$ if $t \in a$, 0 otherwise. But of course every \hat{t} is a measurable function, so $\sum_{t \in K} \alpha_t \hat{t}$ is measurable for every finite set K , and θ is measurable. **Q**

(b) For a less elementary measurable functional, consider the following construction. Let $\langle t_n \rangle_{n \in \mathbb{N}}$ be any sequence of distinct points in I . Then $\lim_{n \rightarrow \infty} \frac{1}{n} \#(\{i : i < n, t_i \in a\}) = \frac{1}{2}$ for ν -almost every $a \subseteq I$. **P** Set

$f_n(a) = 1$ if $t_n \in a$, 0 otherwise. Then $\langle f_n \rangle_{n \in \mathbb{N}}$ is an independent sequence of random variables. By any of the versions of the Strong Law of Large Numbers in §273 (273D, 273H, 273I), $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} f_i = \frac{1}{2}$ a.e., which is what was claimed. **Q** So if we take any non-principal ultrafilter \mathcal{F} on \mathbb{N} , and set $\theta a = \lim_{n \rightarrow \mathcal{F}} \frac{1}{n} \#(\{i : i < n, t_i \in a\})$ for $a \subseteq I$, θ will be constant ν -almost everywhere, and measurable; and it is easy to check that θ is additive. Note that $\theta\{t\} = 0$ for every t , so $\theta \in M_\tau^\perp$, by 464Fb.

(c) If \mathcal{F} is any non-principal ultrafilter on I , and we set $\theta a = 1$ for $a \in \mathcal{F}$, 0 otherwise, then θ is an additive functional which is purely non-measurable, by 464Ca.

For further remarks on where to look for measurable and purely non-measurable functionals, see 464P-464Q below.

464K The space M_m : Lemma Let I be any set, ν the usual measure on $\mathcal{P}I$, and M the L -space of bounded additive functionals on $\mathcal{P}I$. Write M_m for the set of measurable $\theta \in M$, M_τ for the space of completely additive functionals on $\mathcal{P}I$ and $\Delta(\theta) = \int \theta d\nu$ for $\theta \in M^+$, as in 464H.

(a) If $\theta \in M_m \cap M_\tau^\perp$ and $b \subseteq I$, then $\theta(a \cap b) = \frac{1}{2}\theta b$ for ν -almost every $a \subseteq I$.

(b) $|\theta| \in M_m$ for every $\theta \in M_m$.

(c) A functional $\theta \in M^+$ is measurable iff $\Delta(\theta) = \frac{1}{2}\theta I$.

(d) M_m is a solid linear subspace of M .

proof (a)(i) $\theta = \frac{1}{2}\theta I$ ν -a.e. **P** For any $\alpha \in \mathbb{R}$, $A_\alpha = \{a : \theta a < \alpha\}$ is measurable; but also $a' \in A$ whenever $a \in A$ and $a\Delta a'$ is finite, by 464Fb, so νA must be either 1 or 0, by 464Ac. Setting $\delta = \sup\{\alpha : \nu A_\alpha = 0\}$, we see that $\nu A_\delta = 0$, $\nu A_{\delta+2^{-n}} = 1$ for every $n \in \mathbb{N}$, so that $\theta = \delta$ a.e. Also, because $a \mapsto I \setminus a$ is a measure space automorphism, $\theta(I \setminus a) = \delta$ for almost every a , so there is some a such that $\theta a = \theta(I \setminus a) = \delta$, and $\delta = \frac{1}{2}\theta I$. **Q**

(ii) $\theta(a \cap b) = \frac{1}{2}\theta b$ for almost every a . **P** We know that $\theta a = \frac{1}{2}\theta I$ for almost every a . But $a \mapsto a\Delta b : \mathcal{P}I \rightarrow \mathcal{P}I$ is inverse-measure-preserving, so $\theta(a\Delta b) = \frac{1}{2}\theta I$ for almost every a . This means that $\theta a = \theta(a\Delta b)$ for almost every a , and

$$\theta(a \cap b) = \frac{1}{2}(\theta b + \theta a - \theta(a\Delta b)) = \frac{1}{2}\theta b$$

for almost every a . **Q**

(b)(i) If $\theta \in M_m \cap M_\tau^\perp$, then θ^+ , taken in M , is measurable. **P** For any $n \in \mathbb{N}$ we can find $b_n \subseteq I$ such that $\theta^- b_n + \theta^+(I \setminus b_n) \leq 2^{-n}$, so that

$$|\theta^+ a - \theta^+(a \cap b_n)| = |\theta^+ a - \theta^+(a \cap b_n) + \theta^-(a \cap b_n)| \leq \theta^+(I \setminus b_n) + \theta^- b_n \leq 2^{-n}$$

for every $a \subseteq I$. But as $a \mapsto \theta^+(a \cap b_n)$ is constant a.e. for every n , by (a), so is θ^+ , and θ^+ is measurable. **Q**

Consequently $|\theta| = 2\theta^+ - \theta$ is measurable.

(ii) Now take an arbitrary $\theta \in M_m$. Because M is Dedekind complete (354N, 354Ee), $M = M_\tau + M_\tau^\perp$ (J once more), and we can express θ as $\theta_1 + \theta_2$ where $\theta_1 \in M_\tau$ and $\theta_2 \in M_\tau^\perp$; moreover, $|\theta| = |\theta_1| + |\theta_2|$ (352Fb). Now θ_1 is measurable, by 464Ja, so $\theta_2 = \theta - \theta_1$ is measurable; as $\theta_2 \in M_\tau^\perp$, (i) tells us that $|\theta_2|$ is measurable. On the other hand, $|\theta_1|$ belongs to M_τ and is measurable, so $|\theta| = |\theta_1| + |\theta_2|$ is measurable.

Thus (b) is true.

(c) Let f be a ν -integrable function such that $\theta \leq_{\text{a.e.}} f$ and $\int f d\nu = \Delta(\theta)$. Then

$$\theta I - f(a) \leq \theta I - \theta a = \theta(I \setminus a)$$

for almost every a , so

$$\theta I - \Delta(\theta) = \int \theta I - f(a) \nu(da) \leq \int \theta(I \setminus a) \nu(da) = \int \theta(a) \nu(da)$$

because $a \mapsto I \setminus a$ is a measure space automorphism, as in the proof of 464Ha. So if $\Delta(\theta) = \frac{1}{2}\theta I$ then $\int \theta d\nu = \int \theta d\nu$ and θ is ν -integrable (133Jd), therefore (because ν is complete) $(\text{dom } \nu)$ -measurable. On the other hand, if θ is measurable, then

$$\Delta(\theta) = \int \theta \, d\nu = \int \theta(I \setminus a)\nu(da) = \theta I - \int \theta \, d\nu = \theta I - \Delta(\theta),$$

so surely $\Delta(\theta) = \frac{1}{2}\theta I$.

(d) Of course M_m is a linear subspace. If $\theta_0 \in M_m$ and $|\theta| \leq |\theta_0|$, then $|\theta_0| \in M_m$, by (b), so $\Delta(|\theta_0|) = \frac{1}{2}|\theta_0|(I)$, by (c). Because $\theta^+ \leq |\theta| \leq |\theta_0|$, $\Delta(\theta^+) = \frac{1}{2}\theta^+ I$ (464H(f-i)), and θ^+ is measurable, by (c) in the reverse direction. Similarly, θ^- is measurable, and $\theta = \theta^+ - \theta^-$ is measurable. As θ and θ_0 are arbitrary, M_m is solid.

464L The space M_{pnm} : Lemma Let I be any set, ν the usual measure on $\mathcal{P}I$, and M the L -space of bounded additive functionals on $\mathcal{P}I$. This time, write M_{pnm} for the set of those members of M which are purely non-measurable in the sense of 464I.

(a) If $\theta \in M^+$, then θ is purely non-measurable iff $\Delta(\theta) = \theta I$.

(b) M_{pnm} is a solid linear subspace of M .

proof (a)(i) If θ is purely non-measurable, and $f \geq \theta$ is integrable, then $\{a : f(a) \geq \theta I\}$ is a measurable set including $\{a : \theta a = \theta I\}$, so has measure 1, and $\int f \geq \theta I$; as f is arbitrary, $\Delta(\theta) = \theta I$.

(ii) If $\Delta(\theta) = \theta I$, then $\Delta(\theta') = \theta' I$ whenever $0 \leq \theta' \leq \theta$, by 464Hf. But this means that θ' cannot be measurable whenever $0 < \theta' \leq \theta$, by 464Kc above, so that $\theta' \notin M_\tau$ whenever $0 < \theta' \leq \theta$, by 464Ja. Thus $\theta \in M_\tau^\perp$.

By 464Hc, $\nu^*\{a : \alpha \leq \theta a \leq \Delta(\theta)\} = 1$ for every $\alpha < \Delta(\theta)$. Let $\langle m(n) \rangle_{n \in \mathbb{N}}$ be a sequence in \mathbb{N} such that $\prod_{n=0}^\infty 1 - 2^{-m(n)} = \frac{1}{2}$, and define X , λ and ϕ as in 464Ba. Set $\eta_n = 2^{-n}/m(n) > 0$ for each n . Consider the sets $A_n = \{a : \theta a \geq (1 - \eta_n)\theta I\}$ for each $n \in \mathbb{N}$. Then $\nu^* A_n = 1$ for each n , and $\lambda^*(\prod_{n \in \mathbb{N}} A_n^{m(n)}) = 1$. Because ϕ is inverse-measure-preserving, $\nu^*(\phi[\prod_{n \in \mathbb{N}} A_n^{m(n)}]) = 1$ (413Eh again). But if $\mathfrak{r} = \langle \langle a_{ni} \rangle_{i < m(n)} \rangle_{n \in \mathbb{N}}$ belongs to $\prod_{n \in \mathbb{N}} A_n^{m(n)}$, then

$$\theta(\bigcap_{i < m(n)} a_{ni}) \geq \theta I - m(n)\eta_n \theta I = (1 - 2^{-n})\theta I$$

for each n , and

$$\theta(\phi(\mathfrak{r})) \geq \sup_{n \in \mathbb{N}} \theta(\bigcap_{i < m(n)} a_{ni}) = \theta I.$$

Thus the filter $\{a : \theta a = \theta I\}$ includes $\phi[\prod_{n \in \mathbb{N}} A_n^{m(n)}]$ and has outer measure 1, so that θ is purely non-measurable.

(b)(i) If $\theta \in M_{\text{pnm}}$ and $|\theta'| \leq |\theta|$, then $\Delta(|\theta|) = |\theta|(I)$, by the definition in 464I, so $\Delta(|\theta'|) = |\theta'|(I)$, by 464H(f-ii), and $\theta' \in M_{\text{pnm}}$, by (a) above. Thus M_{pnm} is solid.

(ii) If $\theta_1, \theta_2 \in M_{\text{pnm}}$, then

$$\Delta(|\theta_1| + |\theta_2|) = \Delta(|\theta_1|) + \Delta(|\theta_2|) = |\theta_1|(I) + |\theta_2|(I) = (|\theta_1| + |\theta_2|)(I)$$

(using 464Hb), and $|\theta_1| + |\theta_2| \in M_{\text{pnm}}$; as $|\theta_1 + \theta_2| \leq |\theta_1| + |\theta_2|$, $\theta_1 + \theta_2 \in M_{\text{pnm}}$. Thus M_{pnm} is closed under addition.

(iii) It follows from (ii) that if $\theta \in M_{\text{pnm}}$ then $n\theta \in M_{\text{pnm}}$ for every integer $n \geq 1$, and then from (i) that $\alpha\theta \in M_{\text{pnm}}$ for every $\alpha \in \mathbb{R}$; so that M_{pnm} is closed under scalar multiplication, and is a linear subspace.

464M Theorem (FREMLIN & TALAGRAND 79) Let I be any set. Write M for the L -space of bounded finitely additive functionals on $\mathcal{P}I$, and M_m, M_{pnm} for the spaces of measurable and purely non-measurable functionals, as in 464K-464L. Then M_m and M_{pnm} are complementary bands in M .

proof (a) We know from 464K and 464L that these are both solid linear subspaces of M . Next, $M_m \cap M_{\text{pnm}} = \{0\}$. **P** If θ belongs to the intersection, then $\Delta(|\theta|) = \frac{1}{2}|\theta|(I) = |\theta|(I)$, by 464Kc and 464La; so $\theta = 0$. **Q**

(b) Now recall that every element of M^+ is expressible in the form $\theta_1 + \theta_2$ where $\theta_1 \in M_m^+$ and $\theta_2 \in M_{\text{pnm}}^+$; this is 464He, using 464Kc and 464La again. Because M_m and M_{pnm} are linear subspaces, with intersection $\{0\}$, $M = M_m \oplus M_{\text{pnm}}$. Now $M_{\text{pnm}} \subseteq M_m^\perp$, so $M_m + M_m^\perp = M$ and $M_m = M_m^{\perp\perp}$ is a complemented band (352Ra); similarly, M_{pnm} is a complemented band. Since $(M_m + M_{\text{pnm}})^\perp = \{0\}$, M_m and M_{pnm} are complementary bands (see 352S).

464N Corollary (FREMLIN & TALAGRAND 79) Let I be any set, and let μ be Talagrand's measure on $\mathcal{P}I$; write Σ for its domain. Then every bounded additive functional on $\mathcal{P}I$ is Σ -measurable.

proof Defining M , M_m and M_{pnm} as in 464K-464M, we see that every functional in M_m is Σ -measurable because it is (by definition) $(\text{dom } \nu)$ -measurable, where ν is the usual measure on $\mathcal{P}I$. If $\theta \in M_{\text{pnm}}^+$, then $\mathcal{F} = \{a : \theta a = \theta I\}$ is a non-measurable filter; but this means that $\mu\mathcal{F} = 1$, by the construction of μ , so that $\theta = \theta I$ μ -a.e. So if θ is any member of M_{pnm} , both θ^+ and θ^- are Σ -measurable, and $\theta = \theta^+ - \theta^-$ also is.

464O Remark Note that we have a very simple description of the behaviour of additive functionals as seen by the measure μ . Since $M_\tau \subseteq M_m$, we have a three-part band decomposition $M = M_\tau \oplus (M_m \cap M_\tau^\perp) \oplus M_{\text{pnm}}$.

(i) Functionals in M_τ are T -measurable, where T is the domain of ν , therefore Σ -measurable, just because they can be built up from the functionals $a \mapsto \chi a(t)$, as in 464Fb.

(ii) A functional in M_m is T -measurable, by definition; but a functional θ in $M_m \cap M_\tau^\perp$ is actually constant, with value $\frac{1}{2}\theta I$, ν -almost everywhere, by 464Ka. Thus the almost-constant nature of the functionals described in 464Jb is typical of measurable functionals in M_τ^\perp .

(iii) Finally, a functional $\theta \in M_{\text{pnm}}$ is equal to θI μ -almost everywhere; once again, this follows from the definition of 'purely non-measurable' and the construction of μ for $\theta \geq 0$, and from the fact that M_{pnm} is solid for other θ .

(iv) Thus we see that any $\theta \in M_\tau^\perp = (M_m \cap M_\tau^\perp) \oplus M_{\text{pnm}}$ is constant μ -a.e. We also have $\int \theta d\mu = \Delta(\theta)$ for every $\theta \geq 0$ (look at $\theta \in M_m$ and $\theta \in M_{\text{pnm}}$ separately, using 464La for the latter), so that if $h \in M^*$ is the linear functional of 464Hb, then $\int \theta d\mu = h(\theta)$ for every $\theta \in M$.

464P More on purely non-measurable functionals (a) We can discuss non-negative additive functionals on $\mathcal{P}I$ in terms of the Stone-Ćech compactification βI of I , as follows. For any set $A \subseteq \beta I$ set $H_A = \{a : a \subseteq I, A \subseteq \widehat{a}\}$, where $\widehat{a} \subseteq \beta I$ is the open-and-closed set corresponding to $a \subseteq I$. If $A \neq \emptyset$, H_A is a filter on I . Write \mathcal{A} for the family of those sets $A \subseteq \beta I$ such that $\nu^*H_A = 1$, where ν is the usual measure on $\mathcal{P}I$. Then \mathcal{A} is a σ -ideal. **P** Of course $A \in \mathcal{A}$ whenever $A \subseteq B \in \mathcal{A}$, since then $H_A \supseteq H_B$. If $\langle A_n \rangle_{n \in \mathbb{N}}$ is a sequence in \mathcal{A} with union A , then $\nu^*(\bigcap_{n \in \mathbb{N}} H_{A_n}) = 1$, by 464C; but $H_A \supseteq \bigcap_{n \in \mathbb{N}} H_{A_n}$. **Q** Note that if $A \in \mathcal{A}$ then $\overline{A} \in \mathcal{A}$, because $H_A = H_{\overline{A}}$. We see also that $\{z\} \in \mathcal{A}$ for every $z \in \beta I \setminus I$ (since $H_{\{z\}}$ is a non-principal ultrafilter, as in 464Jc), while $\{t\} \notin \mathcal{A}$ for any $t \in I$ (since $H_{\{t\}} = \{a : t \in a\}$ has measure $\frac{1}{2}$).

Because βI can be identified with the Stone space of $\mathcal{P}I$ (4A2I(b-i)), we have a one-to-one correspondence between non-negative additive functionals θ on $\mathcal{P}I$ and Radon measures μ_θ on βI , defined by writing $\mu_\theta(\widehat{a}) = \theta a$ whenever $a \subseteq I$ and $\theta \in M^+$ (416Qb). (This is not the same as the measure μ_θ of part (e) of the proof of 464H, which is on a much larger space.) Now suppose that θ is a non-negative additive functional on $\mathcal{P}I$. Then $\mathcal{F}_\theta = \{a : \theta a = \theta I\}$ is either $\mathcal{P}I$ or a filter on I . If we set $F_\theta = \bigcap \{\widehat{a} : a \in \mathcal{F}_\theta\}$, then $\mathcal{F}_\theta = H_{F_\theta}$ (4A2I(b-iii)). Since $a \in \mathcal{F}_\theta$ iff $\theta a = \theta I$,

$$\begin{aligned} F_\theta &= \bigcap \{\widehat{a} : a \in \mathcal{F}_\theta\} = \bigcap \{\widehat{a} : \theta a = \theta I\} \\ &= \bigcap \{\widehat{a} : \mu_\theta \widehat{a} = \mu_\theta(\beta I)\} = \beta I \setminus \bigcup \{\widehat{a} : \mu_\theta \widehat{a} = 0\}. \end{aligned}$$

But this is just the support of μ_θ (411N), because $\{\widehat{a} : a \subseteq I\}$ is a base for the topology of βI .

Thus we see that $\theta \in M^+$ is purely non-measurable iff the support F_θ of the measure μ_θ belongs to \mathcal{A} . If you like, θ is purely non-measurable iff the support of μ_θ is 'small'.

(b) Yet another corollary of 464C is the following. Since M is a set of real-valued functions on $\mathcal{P}I$, it has the corresponding topology \mathfrak{T}_p of pointwise convergence as a subspace of $\mathbb{R}^{\mathcal{P}I}$. Now if $C \subseteq M_{\text{pnm}}$ is countable, its \mathfrak{T}_p -closure \overline{C} is included in M_{pnm} . **P** It is enough to consider the case in which C is non-empty and $0 \notin C$. For each $\theta \in C$, $\mathcal{F}_{|\theta|} = \{a : |\theta|(a) = |\theta|(I)\}$ is a filter with outer measure 1, so $\mathcal{F} = \{a : |\theta|(a) = |\theta|(I) \text{ for every } \theta \in C\}$ also has outer measure 1, by 464Cb. Now suppose that $\theta_0 \in \overline{C}$. If $a \in \mathcal{F}$, then $|\theta|(I \setminus a) = 0$ for every $\theta \in C$, that is, $\theta b = 0$ whenever $\theta \in C$ and $b \subseteq I \setminus a$ (362Ba). But this means that $\theta_0 b = 0$ whenever $b \subseteq I \setminus a$, so $|\theta_0|(I \setminus a) = 0$ and $|\theta_0|(a) = |\theta_0|(I)$. Thus $\mathcal{F}_{|\theta_0|}$ includes \mathcal{F} , so has outer measure 1, and θ_0 also is purely non-measurable. **Q**

(c) If $\theta \in M$ is such that $\theta a = 0$ for every countable set $a \subseteq I$, then $\theta \in M_{\text{pnm}}$. **P** ν is inner regular with respect to the family \mathcal{W} of sets which are determined by coordinates in countable subsets of I , by 254Ob. But if $W \in \mathcal{W}$ and $\nu W > 0$, let $J \subseteq I$ be a countable set such that W is determined by coordinates in J ; then $|\theta|(J) = 0$, so if a is any member of W we shall have $a \cup (I \setminus J) \in W \cap \mathcal{F}_{|\theta|}$. As W is arbitrary, $\nu^* \mathcal{F}_{|\theta|} = 1$ and θ is purely non-measurable. **Q**

In particular, if $\theta \in M_\sigma \cap M_\tau^\perp$, where M_σ is the space of countably additive functionals on $\mathcal{P}I$ (362B), then $\theta \in M_{\text{pnm}}$. (For ‘ordinary’ sets I , $M_\sigma = M_\tau$; see 438Xa. But this observation is peripheral to the concerns of the present section.)

In the language of (a) above, we have a closed set in βI , being $F = \beta I \setminus \bigcup \{\hat{a} : a \in [I]^{\leq \omega}\}$; and if θ is such that the support of μ_θ is included in F , then θ is purely non-measurable.

464Q More on measurable functionals (a) We know that M_{m} is a band in M , and that it includes the band M_τ . So it is natural to look at the band $M_{\text{m}} \cap M_\tau^\perp$.

(b) If θ is any non-zero non-negative functional in $M_{\text{m}} \cap M_\tau^\perp$, we can find a family $\langle a_\xi \rangle_{\xi < \omega_1}$ in $\mathcal{P}I$ which is independent in the sense that $\theta(\bigcap_{\xi \in K} a_\xi) = 2^{-\#(K)} \theta I$ for every non-empty finite $K \subseteq I$. **P** Choose the a_ξ inductively, observing that at the inductive step we have to satisfy only countably many conditions of the form $\theta(a_\xi \cap b) = \frac{1}{2} \theta b$, where b runs over the subalgebra generated by $\{a_\eta : \eta < \xi\}$, and that each such condition is satisfied ν -a.e., by 464Ka; so that ν -almost any a will serve for a_ξ . **Q**

In terms of the associated measure μ_θ on βI , this means that μ_θ has Maharam type at least ω_1 (use 331Ja). If $\theta I = 1$, so that μ_θ is a probability measure, then $\langle (\hat{a}_\xi)^* \rangle_{\xi < \omega_1}$ is an uncountable stochastically independent family in the measure algebra of μ_θ (325Xf),

Turning this round, we see that if λ is a Radon measure on βI , of countable Maharam type, and $\lambda I = 0$, then the corresponding functional on $\mathcal{P}I$ is purely non-measurable.

[For a stronger result in this direction, see 521T in Volume 5.]

(c) Another striking property of measurable additive functionals is the following. If $\theta \in M_{\text{m}} \cap M_\tau^\perp$, and $n \in \mathbb{N}$, then $\theta(a_0 \cap a_1 \cap \dots \cap a_n) = 2^{-n-1} \theta I$ for ν^{n+1} -almost every $a_0, \dots, a_n \subseteq I$, where ν^{n+1} is the product measure on $(\mathcal{P}I)^{n+1}$. **P** For $K \subseteq \{0, \dots, n\}$ write $\psi_K(a_0, \dots, a_n) = I \cap \bigcap_{i \in K} a_i \setminus \bigcup_{i \leq n, i \notin K} a_i$ for $a_0, \dots, a_n \subseteq I$; for $\mathcal{S} \subseteq \mathcal{P}\{0, \dots, n\}$ write $\phi_{\mathcal{S}}(\mathfrak{x}) = \bigcup_{K \in \mathcal{S}} \psi_K(\mathfrak{x})$ for $\mathfrak{x} \in (\mathcal{P}I)^{n+1}$. Then, for any $t \in I$,

$$\nu^{n+1}\{\mathfrak{x} : t \in \phi_{\mathcal{S}}(\mathfrak{x})\} = \sum_{K \in \mathcal{S}} \nu^{n+1}\{\mathfrak{x} : t \in \psi_K(\mathfrak{x})\} = 2^{-n-1} \#(\mathcal{S}).$$

Moreover, for different t , these sets are independent. So if $\#(\mathcal{S}) = 2^n$, that is, if \mathcal{S} is just half of $\mathcal{P}\{0, \dots, n\}$, then $\phi_{\mathcal{S}}$ will be inverse-measure-preserving, by the arguments in 464B. (In fact 464Bb is the special case $n = 2$, $\mathcal{S} = \{K : \#(K) \geq 2\}$.)

Accordingly we shall have $\theta(\phi_{\mathcal{S}}(\mathfrak{x})) = \frac{1}{2} \theta I$ for ν^{n+1} -almost every \mathfrak{x} whenever $\mathcal{S} \subseteq \mathcal{P}\{0, \dots, n\}$ has 2^n members, as in 464Ka. Since there are only finitely many sets \mathcal{S} , the set E is ν^{n+1} -conegligible, where

$$E = \{\mathfrak{x} : \mathfrak{x} \in (\mathcal{P}I)^{n+1}, \theta(\phi_{\mathcal{S}}(\mathfrak{x})) = \frac{1}{2} \theta I \text{ whenever } \mathcal{S} \subseteq \mathcal{P}\{0, \dots, n\} \text{ and } \#(\mathcal{S}) = 2^n\}.$$

But given $\mathfrak{x} = (a_0, \dots, a_n) \in E$, let $K_1, \dots, K_{2^{n+1}}$ be a listing of $\mathcal{P}\{0, \dots, n\}$ in such an order that $\theta(\psi_{K_i}(\mathfrak{x})) \leq \theta(\psi_{K_j}(\mathfrak{x}))$ whenever $i \leq j$, and consider $\mathcal{S} = \{K_1, \dots, K_{2^n}\}$; since

$$\sum_{i=1}^{2^n} \theta(\psi_{K_i}(\mathfrak{x})) = \theta(\phi_{\mathcal{S}}(\mathfrak{x})) = \frac{1}{2} \theta I = \frac{1}{2} \sum_{i=0}^{2^{n+1}} \theta(\psi_{K_i}(\mathfrak{x})),$$

we must have $\theta(\psi_{K_i}(\mathfrak{x})) = 2^{-n-1} \theta I$ for every i . In particular, $\theta(\psi_{\{0, \dots, n\}}(\mathfrak{x})) = 2^{-n-1} \theta I$, that is, $\theta(\bigcap_{i \leq n} a_i) = 2^{-n-1} \theta I$. As this is true whenever $a_0, \dots, a_n \in E$, we have the result. **Q**

464R A note on $\ell^\infty(I)$ As already noted in 464F, we have, for any set I , a natural additive map $\chi : \mathcal{P}I \rightarrow \ell^\infty(I) \cong L^\infty(\mathcal{P}I)$, giving rise to an isomorphism between the L -space M of bounded additive functionals on $\mathcal{P}I$ and $\ell^\infty(I)^*$. If we write $\tilde{\mu}$ for the image measure $\mu \chi^{-1}$ on $\ell^\infty(I)$, where μ is Talagrand’s measure on $\mathcal{P}I$, and $\tilde{\Sigma}$ for the domain of $\tilde{\mu}$, then every member of $\ell^\infty(I)^*$ is $\tilde{\Sigma}$ -measurable, by 464N. Thus $\tilde{\Sigma}$ includes the cylindrical σ -algebra of $\ell^\infty(I)$ (4A3U). We also have a band decomposition $\ell^\infty(I)^* = \ell^\infty(I)_{\text{m}}^* \oplus \ell^\infty(I)_{\text{pnm}}^*$ corresponding to the decomposition $M = M_{\text{m}} \oplus M_{\text{pnm}}$ (464M).

In this context, M_τ corresponds to $\ell^\infty(I)^\times$ (363K, as before). Since we can identify $\ell^\infty(I)$ with $\ell^1(I)^* = \ell^1(I)^\times$ (243X1), and $\ell^1(I)$, like any L -space, is perfect, $\ell^\infty(I)^\times$ is the canonical image of $\ell^1(I)$ in $\ell^\infty(I)^*$. Because any functional in M_τ^\perp is μ -almost constant (464O), any functional in $(\ell^\infty(I)^\times)^\perp$ will be $\tilde{\mu}$ -almost constant.

464X Basic exercises (a) Let I be any set and λ a Radon measure on βI . Show that if the support of λ is a separable subset of $\beta I \setminus I$, then the corresponding additive functional on $\mathcal{P}I$ is purely non-measurable.

(b) Let I be any set, and $\tilde{\mu}$ the image of Talagrand's measure on $\ell^\infty(I)$, as in 464R. Show that $\tilde{\mu}$ has a barycenter in $\ell^\infty(I)$ iff I is finite.

464Y Further exercises (a) Show that there is a sequence $\langle \mathcal{F}_n \rangle_{n \in \mathbb{N}}$ of distinct non-principal ultrafilters on \mathbb{N} with the following property: if we define $h(a) = \{n : a \in \mathcal{F}_n\}$ for $a \subseteq \mathbb{N}$, then $\{h(a) : a \subseteq \mathbb{N}\}$ is negligible for the usual measure on $\mathcal{P}\mathbb{N}$.

(b) Let μ be Talagrand's measure on $\mathcal{P}\mathbb{N}$, and λ the corresponding product measure on $X = \mathcal{P}\mathbb{N} \times \mathcal{P}\mathbb{N}$. Define $\Phi : X \rightarrow \ell^\infty$ by setting $\Phi(a, b) = \chi_a - \chi_b$ for all $a, b \subseteq \mathbb{N}$. Show that Φ is Pettis integrable (463Ya) with indefinite Pettis integral Θ defined by setting $(\Theta E)(n) = \int_E f_n d\lambda$, where $f_n(a, b) = \chi_a(n) - \chi_b(n)$. Show that $K = \{h\Phi : h \in \ell^{\infty*}, \|h\| \leq 1\}$ contains every f_n , and in particular is not \mathfrak{T}_m -compact, so the identity map from (K, \mathfrak{T}_p) to (K, \mathfrak{T}_m) is not continuous.

(c) Let I be any set. Write $\mathbf{c}_0(I)$ for the closed linear subspace of $\ell^\infty(I)$ consisting of those $x \in \mathbb{R}^I$ such that $\{t : t \in I, |x(t)| \geq \epsilon\}$ is finite for every $\epsilon > 0$; that is, $C_0(I)$ if I is given its discrete topology (436I). Show that, in 464R, M_τ^\perp can be identified as Banach lattice with $(\ell^\infty(I)/\mathbf{c}_0(I))^*$.

(d)(i) Let $\theta : \mathcal{P}\mathbb{N} \rightarrow \mathbb{R}$ be an additive functional which is T-measurable in the sense of 464I. Show that $\{\theta\{n\} : n \in \mathbb{N}\}$ is bounded. **(ii)** Let $\theta : \mathcal{P}\mathbb{N} \rightarrow \mathbb{R}$ be an additive functional which is universally measurable for the usual topology of $\mathcal{P}\mathbb{N}$. Show that θ is bounded. **(iii)** Let \mathfrak{A} be a Dedekind σ -complete Boolean algebra and $\theta : \mathfrak{A} \rightarrow \mathbb{R}$ an additive functional which is universally measurable for the order-sequential topology on \mathfrak{A} (definition: 393L). Show that θ is bounded and θ^+ is universally measurable.

(e) Show that there is a T-measurable finitely additive functional $\theta : \mathcal{P}\mathbb{N} \rightarrow \mathbb{R}$ which is not bounded.

464Z Problem Let I be an infinite set, and $\tilde{\mu}$ the image on $\ell^\infty(I)$ of Talagrand's measure (464R). Is $\tilde{\mu}$ a topological measure for the weak topology of $\ell^\infty(I)$?

464 Notes and comments The central idea of this section appears in 464B: the algebraic structure of $\mathcal{P}I$ leads to a variety of inverse-measure-preserving functions ϕ from powers $(\mathcal{P}I)^K$ to $\mathcal{P}I$. The simplest of these is the measure space automorphism $a \mapsto I \setminus a$, as used in the proofs of 464Ca, 464Ha, 464Ka and 464Kc. Then we have the map $(a, b, c) \mapsto (a \cap b) \cup (a \cap c) \cup (b \cap c)$, as in 464Bb, and the generalization of this in the argument of 464Qc; and, most important of all, the map $\langle \langle a_{ni} \rangle_{i < m(n)} \rangle_{n \in \mathbb{N}} \mapsto \bigcup_{n \in \mathbb{N}} \bigcap_{i < m(n)} a_{ni}$ of 464Ba. In each case we can use probabilistic intuitions to guide us to appropriate formulae, since the events $t \in \phi(\mathfrak{x})$ are always independent, so if they have probability $\frac{1}{2}$ for every $t \in I$, the function ϕ will be inverse-measure-preserving. Of course this depends on the analysis of product measures in §254. It means also that we must use the 'ordinary' product measure defined there; but happily this coincides with the 'Radon' product measure of §417 (416U).

Talagrand devised his measure when seeking an example of a pointwise compact set of measurable functions which is not compact for the topology of convergence in measure, as in 464E. The remarkable fact that it is already, in effect, a measure on the cylindrical σ -algebra of ℓ^∞ (464R) became apparent later, and requires a much more detailed analysis. An alternative argument not explicitly involving the Riesz space structure of the space M of bounded additive functionals may be found in FREMLIN & TALAGRAND 79. The proof I give here depends on the surprising fact that, for non-negative additive functionals, the upper integral $\bar{\int} d\nu$ is additive (464Hb), even though the functionals may be very far from being measurable. Once we know this, we can apply the theory of Banach lattices to investigate the corresponding linear functional

on the L -space M . There is a further key step in 464Hd. We there have a non-negative $\theta \in M_\tau^\perp$ such that $\theta I = 1$ and $\beta\theta'I \leq \int \theta' d\nu \leq \gamma\theta'I$ whenever $0 \leq \theta' \leq \theta$. It is easy to deduce that $\{a : \alpha\theta b \leq \theta(a \cap b) \leq \gamma\theta b \text{ for every } b \in K\}$ has outer measure 1 for every finite $K \subseteq \mathcal{P}I$ and $\alpha < \beta$. If α and γ are very close, this means that there will be many families (a, b, c) in $\mathcal{P}I$ which, as measured by θ , look like independent sets of measure γ , so that $\theta((a \cap b) \cup (a \cap c) \cup (b \cap c)) \simeq 3\gamma^2 - 2\gamma^3$. But since we must also have $\theta((a \cap b) \cup (a \cap c) \cup (b \cap c)) \leq \gamma$ almost everywhere, we can get information about the possible values of γ .

Once having noted this remarkable dichotomy between ‘measurable’ and ‘purely non-measurable’ functionals, it is natural to look for other ways in which they differ. The position seems to be that ‘simple’ Radon measures on $\beta I \setminus I$ (e.g., all measures with separable support (464Xa) or countable Maharam type (464Qb)) have to correspond to purely non-measurable functionals. Of course the simplest possible measures on βI are those concentrated on I , which give rise to functionals which belong to M_τ and are therefore measurable; it is functionals in $M_m \cap M_\tau^\perp$ which have to give rise to ‘complicated’ Radon measures.

The fact that every element of M_τ^\perp is almost constant for Talagrand’s measure leads to an interesting Pettis integrable function (464Yb). The suggestion that Talagrand’s measure on ℓ^∞ might be a topological measure for the weak topology (464Z) is a bold one, but no more outrageous than the suggestion that it might measure every continuous linear functional once seemed. Talagrand’s measure does of course measure every Baire set for the weak topology (4A3V). I note here that the usual measure on $\{0, 1\}^I$, when transferred to $\ell^\infty(I)$, is actually a Radon measure for the weak* topology $\mathfrak{T}_s(\ell^\infty, \ell^1)$, because on $\{0, 1\}^I$ this is just the ordinary topology.

Version of 22.3.16

465 Stable sets

The structure of general pointwise compact sets of measurable functions is complex and elusive. One particular class of such sets, however, is relatively easy to describe, and has a variety of remarkable properties, some of them relevant to important questions arising in the theory of empirical measures. In this section I outline the theory of ‘stable’ sets of measurable functions from TALAGRAND 84 and TALAGRAND 87.

The first steps are straightforward enough. The definition of stable set (465B) is not obvious, but given this the basic properties of stable sets listed in 465C are natural and easy to check, and we come quickly to the fact that (for complete locally determined spaces) pointwise bounded stable sets are relatively pointwise compact sets of measurable functions (465D). A less transparent, but still fairly elementary, argument leads to the next reason for looking at stable sets: the topology of pointwise convergence on a stable set is finer than the topology of convergence in measure (465G).

At this point we come to a remarkable fact: a uniformly bounded set A of functions on a complete probability space is stable if and only if certain laws of large numbers apply ‘nearly uniformly’ on A . These laws are expressed in conditions (ii), (iv) and (v) of 465M. For singleton sets A , they can be thought of as versions of the strong law of large numbers described in §273. To get the full strength of 465M a further idea in this direction needs to be added, described in 465H here.

The theory of stable sets applies in the first place to sets of true functions. There is however a corresponding notion applicable in function spaces, which I explore briefly in 465O-465R. Finally, I mention the idea of ‘R-stable’ set (465S-465U), obtained by using τ -additive product measures instead of c.l.d. product measures in the definition.

465A Notation Throughout this section, I will use the following notation.

(a) If X is a set and Σ a σ -algebra of subsets of X , I will write $\mathcal{L}^0(\Sigma)$ for the space of Σ -measurable functions from X to \mathbb{R} , as in §463; $\mathcal{L}^\infty(\Sigma)$ will be the space of bounded functions in $\mathcal{L}^0(\Sigma)$.

(b) I will identify \mathbb{N} with the set of finite ordinals, so that each $n \in \mathbb{N}$ is the set of its predecessors, and a power X^n becomes identified with the set of functions from $\{0, \dots, n-1\}$ to X .

(c) If (X, Σ, μ) is any measure space, then for finite sets I (in particular, if $I = k \in \mathbb{N}$) I write μ^I for the c.l.d. product measure on X^I , as defined in 251W. (For definiteness, let us take μ^0 to be the unique probability measure on $X^0 = \{\emptyset\}$.) If (X, Σ, μ) is a probability space, then for any set I μ^I is to be the product probability measure on X^I , as defined in §254.

(d) We shall have occasion to look at free powers of algebras of sets. If X is a set and Σ is an algebra of subsets of X , then for any set I write $\bigotimes_I \Sigma$ for the algebra of subsets of X^I generated by sets of the form $\{w : w(i) \in E\}$ where $i \in I$ and $E \in \Sigma$, and $\widehat{\bigotimes}_I \Sigma$ for the σ -algebra generated by $\bigotimes_I \Sigma$.

(e) Now for a new idea, which will be used in almost every paragraph of the section. If X is a set, $A \subseteq \mathbb{R}^X$ a set of real-valued functions defined on X , $E \subseteq X$, $\alpha < \beta$ in \mathbb{R} and $k \geq 1$, write

$$D_k(A, E, \alpha, \beta) = \bigcup_{f \in A} \{w : w \in E^{2k}, f(w(2i)) \leq \alpha, \\ f(w(2i + 1)) \geq \beta \text{ for } i < k\}.$$

(f) In this context it will be useful to have a special notation. If X is a set, $k \geq 1$, $u \in X^k$ and $v \in X^k$, then I will write $u \# v = (u(0), v(0), u(1), v(1), \dots, u(k - 1), v(k - 1)) \in X^{2k}$. Note that if (X, Σ, μ) is a measure space then $(u, v) \mapsto u \# v$ is an isomorphism between the c.l.d. product $(X^k, \mu^k) \times (X^k, \mu^k)$ and (X^{2k}, μ^{2k}) (see 251Wh).

We are now ready for the main definition.

465B Definition Let (X, Σ, μ) be a semi-finite measure space. Following TALAGRAND 84, I say that a set $A \subseteq \mathbb{R}^X$ is **stable** if whenever $E \in \Sigma$, $0 < \mu E < \infty$ and $\alpha < \beta$ in \mathbb{R} , there is some $k \geq 1$ such that $(\mu^{2k})^* D_k(A, E, \alpha, \beta) < (\mu E)^{2k}$.

Remark I hope that the next few results will show why this concept is important. It is worth noting at once that these sets D_k need not be measurable, and that some of the power of the definition derives precisely from the fact that quite naturally arising sets A can give rise to non-measurable sets $D_k(A, E, \alpha, \beta)$. If, however, the set A is countable, then all the corresponding D_k will be measurable; this will be important in the results following 465R.

465C I start with a list of the ‘easy’ properties of stable sets, derivable more or less directly from the definition.

Proposition Let (X, Σ, μ) be a semi-finite measure space.

- (a) Let $A \subseteq \mathbb{R}^X$ be a stable set.
 - (i) Any subset of A is stable.
 - (ii) \overline{A} , the closure of A in \mathbb{R}^X for the topology of pointwise convergence, is stable.
 - (iii) $\gamma A = \{\gamma f : f \in A\}$ is stable, for any $\gamma \in \mathbb{R}$.
 - (iv) If $g \in \mathcal{L}^0 = \mathcal{L}^0(\Sigma)$, then $A + g = \{f + g : f \in A\}$ is stable.
 - (v) If $g \in \mathcal{L}^0$, then $A \times g = \{f \times g : f \in A\}$ is stable.
 - (vi) Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous non-decreasing function. Then $\{hf : f \in A\}$ is stable.
- (b)(i) Suppose that $A \subseteq \mathbb{R}^X$, $E \in \Sigma$, $n \geq 1$ and $\alpha < \beta$ are such that $0 < \mu E < \infty$ and $(\mu^{2n})^* D_n(A, E, \alpha, \beta) < (\mu E)^{2n}$. Then

$$\lim_{k \rightarrow \infty} \frac{1}{(\mu E)^{2k}} (\mu^{2k})^* D_k(A, E, \alpha, \beta) = 0.$$

- (ii) If $A, B \subseteq \mathbb{R}^X$ are stable, then $A \cup B$ is stable.
- (iii) If $A \subseteq \mathcal{L}^0$ is finite it is stable.
- (iv) If $A \subseteq \mathbb{R}^X$ is stable, so is $\{f^+ : f \in A\} \cup \{f^- : f \in A\}$.
- (c) Let A be a subset of \mathbb{R}^X .
 - (i) If $\hat{\mu}, \tilde{\mu}$ are the completion and c.l.d. version of μ , then A is stable with respect to one of the measures $\mu, \hat{\mu}, \tilde{\mu}$ iff it is stable with respect to the others.
 - (ii) Let ν be an indefinite-integral measure over μ (234J). If A is stable with respect to μ , it is stable with respect to ν and with respect to $\nu \upharpoonright \Sigma$.
 - (iii) If A is stable, and $Y \subseteq X$ is such that the subspace measure μ_Y is semi-finite, then $A_Y = \{f \upharpoonright Y : f \in A\}$ is stable in \mathbb{R}^Y with respect to the measure μ_Y .
 - (iv) A is stable iff $A_E = \{f \upharpoonright E : f \in A\}$ is stable in \mathbb{R}^E with respect to the subspace measure μ_E whenever $E \in \Sigma$ has finite measure.

(v) A is stable iff $A_n = \{\text{med}(-n\chi_X, f, n\chi_X) : f \in A\}$ is stable for every $n \in \mathbb{N}$.

(d) Suppose that μ is σ -finite, (Y, \mathbb{T}, ν) is another measure space and $\phi : Y \rightarrow X$ is inverse-measure-preserving. If $A \subseteq \mathbb{R}^X$ is stable with respect to μ , then $B = \{f\phi : f \in A\}$ is stable with respect to ν .

proof (a)(i) This is immediate from the definition in 465B, since $D_k(B, E, \alpha, \beta) \subseteq D_k(A, E, \alpha, \beta)$ for all k, E, α, β and $B \subseteq A$.

(ii) Given E such that $0 < \mu E < \infty$, and $\alpha < \beta$, take α', β' such that $\alpha < \alpha' < \beta' < \beta$. Then it is easy to see that $D_k(\bar{A}, E, \alpha, \beta) \subseteq D_k(A, E, \alpha', \beta')$, so

$$(\mu^{2k})^* D_k(\bar{A}, E, \alpha, \beta) \leq (\mu^{2k})^* D_k(A, E, \alpha', \beta') < (\mu E)^{2k}$$

for some $k \geq 1$. As E, α and β are arbitrary, \bar{A} is stable.

(iii)(\alpha) If $\gamma > 0$, $D_k(\gamma A, E, \alpha, \beta) = D_k(A, E, \alpha/\gamma, \beta/\gamma)$ for all k, E, α and β , so the result is elementary. Similarly, if $\gamma = 0$, then $D_k(\gamma A, E, \alpha, \beta) = \emptyset$ whenever $k \geq 1, E \in \Sigma$ and $\alpha < \beta$, so again we see that γA is stable.

(\beta) If $\gamma = -1$ then, for any k, E, α and β ,

$$\begin{aligned} D_k(-A, E, \alpha, \beta) &= \bigcup_{f \in A} \{w : w \in E^{2k}, f(w(2i)) \geq -\alpha, \\ &\quad f(w(2i+1)) \leq -\beta \forall i < k\} \\ &= \phi[D_k(A, E, -\beta, -\alpha)], \end{aligned}$$

where $\phi : X^{2k} \rightarrow X^{2k}$ is the measure space automorphism defined by setting

$$\phi(w) = (w(1), w(0), w(3), w(2), \dots, w(2k-1), w(2k-2))$$

for $w \in X^{2k}$. So, given $E \in \Sigma$ and $\alpha < \beta$, there is a $k \geq 1$ such that

$$(\mu^{2k})^* D_k(-A, E, \alpha, \beta) = (\mu^{2k})^* D_k(A, E, -\beta, -\alpha) < (\mu E)^{2k}.$$

So $-A$ is stable.

(\gamma) Together with **(\alpha)** this shows that γA is stable for every $\gamma \in \mathbb{R}$.

(iv) Take E such that $0 < \mu E < \infty$, and $\alpha < \beta$. Set $\eta = \frac{1}{2}(\beta - \alpha) > 0$. Then there is a $\gamma \in \mathbb{R}$ such that $F = \{x : x \in E, \gamma \leq g(x) \leq \gamma + \eta\}$ has non-zero measure. Set $\alpha' = \alpha - \gamma, \beta' = \beta - \gamma - \eta$. Then $D_k(A + g, F, \alpha, \beta) \subseteq D_k(A, F, \alpha', \beta')$, while $\alpha' < \beta'$. So if we take $k \geq 1$ such that $(\mu^{2k})^* D_k(A, F, \alpha', \beta') < (\mu F)^{2k}$, then

$$(\mu^{2k})^* D_k(A + g, E, \alpha, \beta) \leq (\mu^{2k})^* D_k(A + g, F, \alpha, \beta) + \mu^{2k}(E^{2k} \setminus F^{2k}) < (\mu E)^{2k}.$$

(v) Let $E \in \Sigma, \alpha < \beta$ be such that $0 < \mu E < \infty$. Set

$$E_0 = \{x : x \in E, g(x) = 0\}, \quad E_1 = \{x : x \in E, g(x) > 0\},$$

$$E_2 = \{x : x \in E, g(x) < 0\}.$$

(\alpha) Suppose that $\mu E_0 > 0$. Then $D_1(A \times g, E, \alpha, \beta)$ does not meet E_0^2 , so $\mu^* D_1(A \times g, E, \alpha, \beta) < \mu E$.

(\beta) Suppose that $\mu E_1 > 0$. Let $\eta > 0$ be such that

$$\max(\alpha, \frac{\alpha}{1+\eta}) = \alpha' < \beta' = \min(\beta, \frac{\beta}{1+\eta}).$$

Let $\gamma > 0$ be such that $\mu F > 0$, where $F = \{x : x \in E, \gamma \leq g(x) \leq \gamma(1+\eta)\}$. If $x \in F$ then

$$f(x)g(x) \leq \alpha \implies f(x) \leq \frac{\alpha}{g(x)} \leq \frac{\alpha'}{\gamma},$$

$$f(x)g(x) \geq \beta \implies f(x) \geq \frac{\beta}{g(x)} \geq \frac{\beta'}{\gamma}.$$

So

$$D_k(A \times g, F, \alpha, \beta) \subseteq D_k(A, F, \frac{\alpha'}{\gamma}, \frac{\beta'}{\gamma})$$

for every k . Now, because A is stable, there is some $k \geq 1$ such that

$$(\mu^{2k})^* D_k(A, F, \frac{\alpha'}{\gamma}, \frac{\beta'}{\gamma}) < (\mu F)^{2k},$$

and in this case $(\mu^{2k})^* D_k(A \times g, E, \alpha, \beta) < (\mu E)^{2k}$, just as in the argument for (a-iv) above.

(γ) If $\mu E_2 > 0$, then we know from (iii) above that $-A$ is stable, so (β) here tells us that there is a $k \geq 1$ such that $(\mu^{2k})^* D_k((-A) \times (-g), E, \alpha, \beta) < (\mu E)^{2k}$.

Since one of these three cases must occur, and since E, α and β are arbitrary, $A \times g$ is stable.

(vi) Write $B = \{hf : f \in A\}$. Suppose that $0 < \mu E < \infty$ and $\alpha < \beta$. If either $\alpha < h(\gamma)$ for every $\gamma \in \mathbb{R}$ or $h(\gamma) < \beta$ for every $\gamma \in \mathbb{R}$,

$$\mu^* D_1(B, E, \alpha, \beta) = \mu \emptyset = 0 < \mu E.$$

Otherwise, because h is continuous, the Intermediate Value Theorem tells us that there are $\alpha' < \beta'$ such that $\alpha < h(\alpha') < h(\beta') < \beta$. In this case $D_k(B, E, \alpha, \beta) \subseteq D_k(A, E, \alpha', \beta')$ for every k . Because A is stable, there is some $k \geq 1$ such that $(\mu^{2k})^* D_k(A, E, \alpha', \beta') < (\mu E)^{2k}$, so that $(\mu^{2k})^* D_k(B, E, \alpha, \beta) < (\mu E)^{2k}$. As E, α and β are arbitrary, B is stable.

(b)(i) For any $m \geq 1$ and $l < n$, if we identify $X^{2(mn+l)}$ with $(X^{2n})^m \times X^{2l}$, we see that $D_{mn+l}(A, E, \alpha, \beta)$ becomes identified with a subset of $D_n(A, E, \alpha, \beta)^m \times E^{2l}$. (If $w \in D_{mn+l}(A, E, \alpha, \beta)$, there is an $f \in A$ such that $f(w(2i)) \leq \alpha, f(w(2i+1)) \geq \beta$ for $i < mn+l$. Now $(w(2rn), w(2rn+1), \dots, w(2rn+2n-1)) \in D_n(A, E, \alpha, \beta)$ for $r < m$.) So

$$\begin{aligned} & \frac{1}{(\mu E)^{2(mn+l)}} (\mu^{2(mn+l)})^* D_{mn+l}(A, E, \alpha, \beta) \\ & \leq \frac{1}{(\mu E)^{2(mn+l)}} ((\mu^{2n})^* D_n(A, E, \alpha, \beta))^m (\mu E)^{2l} \\ (251Wm) \quad & = \left(\frac{1}{(\mu E)^{2n}} (\mu^{2n})^* D_n(A, E, \alpha, \beta)\right)^m \rightarrow 0 \end{aligned}$$

as $m \rightarrow \infty$.

(ii) Note that $D_k(A \cup B, E, \alpha, \beta) = D_k(A, E, \alpha, \beta) \cup D_k(B, E, \alpha, \beta)$ for all E, k, α and β . Now, given that $0 < \mu E < \infty$ and $\alpha < \beta$, there are $m, n \geq 1$ such that $(\mu^{2m})^* D_m(A, E, \alpha, \beta) < (\mu E)^{2m}$ and $(\mu^{2n})^* D_n(A, E, \alpha, \beta) < (\mu E)^{2n}$. So, by (i) just above,

$$\begin{aligned} & \frac{1}{(\mu E)^{2k}} (\mu^{2k})^* D_k(A \cup B, E, \alpha, \beta) \\ & \leq \frac{1}{(\mu E)^{2k}} ((\mu^{2k})^* D_k(A, E, \alpha, \beta) + (\mu^{2k})^* D_k(B, E, \alpha, \beta)) \rightarrow 0 \end{aligned}$$

as $k \rightarrow \infty$, and there is some k such that $(\mu^{2k})^* D_k(A \cup B, E, \alpha, \beta) < (\mu E)^{2k}$. As E, α and β are arbitrary, $A \cup B$ is stable.

(iii) If A is empty, or contains only the constant function $\mathbf{0}$ with value 0, this is trivial. Now (a-iv) tells us that $\{g\} = \{\mathbf{0}\} + g$ is stable for every $g \in \mathcal{L}^0$, and from (ii) here it follows that any finite subset of \mathcal{L}^0 is stable.

(iv) From (a-vi) we see that $\{f^+ : f \in A\}$ is stable; now from (a-iii) and (ii) here we see that $\{f^- : f \in A\} = \{f^+ : f \in -A\}$ and $\{f^+ : f \in A\} \cup \{f^- : f \in A\}$ are stable.

(c)(i) The product measures $\mu^{2k}, \hat{\mu}^{2k}$ and $\tilde{\mu}^{2k}$ are all the same (251Wn), so this follows immediately from the definition in 465B.

(ii) Let h be a Radon-Nikodým derivative of ν with respect to μ (234J). Suppose that $0 < \nu E < \infty$ and $\alpha < \beta$. Then there is an $F \in \Sigma$ such that $F \subseteq E \cap \text{dom } h$, $h(x) > 0$ for every $x \in F$, and $0 < \mu F < \infty$. There is a $k \geq 1$ such that $(\mu^{2k})^* D_k(A, F, \alpha, \beta) < (\mu F)^{2k}$, that is, there is a $W \subseteq F^{2k} \setminus D_k(A, F, \alpha, \beta)$ such that $\mu^{2k} W > 0$. In this case, $\nu^{2k} W > 0$. **P** Set $\tilde{h}(w) = \prod_{i=0}^{2k-1} h(w(i))$ for $w \in (\text{dom } h)^{2k}$. Then ν^{2k} is the indefinite integral of \tilde{h} with respect to μ^{2k} (253I, extended by induction to the product of more than two factors), and $\tilde{h}(w) > 0$ for every $w \in F^{2k}$. **Q** Since $W \subseteq E^{2k} \setminus D_k(A, E, \alpha, \beta)$, $(\nu^{2k})^* D_k(A, E, \alpha, \beta) < (\nu E)^{2k}$; as E, α and β are arbitrary, A is stable with respect to ν .

Since ν is the completion of its restriction $\nu|_\Sigma$ (234Lb), A is also stable with respect to $\nu|_\Sigma$, by (i) just above.

(iii) Writing Σ_Y for the subspace σ -algebra, take $F \in \Sigma_Y$ such that $\mu_Y F = \mu^* F$ is finite and non-zero, and $\alpha < \beta$ in \mathbb{R} . Let $E \in \Sigma$ be a measurable envelope of F . Then there is a $k \geq 1$ such that $(\mu^{2k})^* D_k(A, E, \alpha, \beta) < (\mu E)^{2k}$. Consider $D_k(A_Y, F, \alpha, \beta) = F^{2k} \cap D_k(A, E, \alpha, \beta)$. The identity map from F to E is inverse-measure-preserving for the subspace measures μ_F and μ_E (214Ce), so the identity map from F^{2k} to E^{2k} is inverse-measure-preserving for the product measures μ_F^{2k} and μ_E^{2k} (apply 254H to appropriate normalizations of μ_E, μ_F). Also μ_E^{2k} is the subspace measure on E^{2k} induced by μ^{2k} (251W1), and similarly μ_F^{2k} is the subspace measure on F^{2k} induced by μ_Y^{2k} , so

$$(413Eh) \quad \begin{aligned} (\mu_Y^{2k})^* D_k(A_Y, F, \alpha, \beta) &= (\mu_F^{2k})^* D_k(A_Y, F, \alpha, \beta) \leq (\mu_E^{2k})^* D_k(A, E, \alpha, \beta) \\ &= (\mu^{2k})^* D_k(A, E, \alpha, \beta) < (\mu E)^{2k} = (\mu_Y F)^{2k}. \end{aligned}$$

As F, α and β are arbitrary, A_Y is stable, as claimed.

(iv) If A is stable, then (iii) just above tells us that A_E will be stable for every $E \in \Sigma$. Conversely, if A_E is stable for every E of finite measure, take $E \in \Sigma$ such that $0 < \mu E < \infty$ and $\alpha < \beta$ in \mathbb{R} . Then there is a $k \geq 1$ such that

$$(\mu E)^{2k} = (\mu_E E)^{2k} > (\mu_E^{2k})^* D_k(A_E, E, \alpha, \beta) = (\mu^{2k})^* D_k(A, E, \alpha, \beta).$$

As E, α and β are arbitrary, A is stable.

(v) If A is stable, then every A_n is stable, because $A_n = \{h_n f : f \in A\}$ where $h_n(\alpha) = \text{med}(-n, \alpha, n)$ and we can apply (a-v). Conversely, if every A_n is stable, and we are given a set E of non-zero finite measure and $\alpha < \beta$, take $n \in \mathbb{N}$ such that $-n \leq \alpha < \beta \leq n$. Then $D_k(A, E, \alpha, \beta) = D_k(A_n, E, \alpha, \beta)$ for every k , and we have a $k \geq 1$ such that

$$(\mu^{2k})^* D_k(A, E, \alpha, \beta) = (\mu^{2k})^* D_k(A_n, E, \alpha, \beta) < (\mu E)^{2k}.$$

So A is stable.

(d) Let $\langle X_n \rangle_{n \in \mathbb{N}}$ be a sequence of measurable sets of finite measure covering X . Take $F \in \mathbb{T}$ such that $\nu F > 0$, and $\alpha < \beta$ in \mathbb{R} . Let $n \in \mathbb{N}$ be such that $F_0 = F \cap \phi^{-1}[X_n]$ has non-zero measure. Set $\lambda E = \nu(F_0 \cap \phi^{-1}[E])$ for $E \in \Sigma$; then λ is countably additive and $\lambda \leq \mu$. By the Radon-Nikodým theorem, there is a Σ -measurable $h : X \rightarrow [0, 1]$ such that $\int_E h = \lambda E$ for every $E \in \Sigma$. Set $G = \{x : x \in X_n, h(x) > 0\}$; then $\nu(F_0 \setminus \phi^{-1}[G]) = \int_{X_n \setminus G} h = 0$.

Because A is stable, there is a $k \geq 1$ such that $(\mu^{2k})^* D_k(A, G, \alpha, \beta) < (\mu G)^{2k}$. Let $V \in \widehat{\otimes}_{2k} \Sigma$ be such that $V \subseteq G^{2k} \setminus D_k(A, G, \alpha, \beta)$ and $\mu^{2k} V > 0$. Then $\lambda^{2k} V = \int_V h^\# d\mu^{2k}$ where $h^\#(w) = \prod_{i < 2k} h(w(i))$ for $w \in X^{2k}$ (253I). Because $h^\#(w) > 0$ for every $w \in G^{2k}$, $\lambda^{2k} V > 0$.

Note next that if we define $\psi : Y^{2k} \rightarrow X^{2k}$ by setting $\psi(\langle y_i \rangle_{i < 2k}) = \langle \phi(y_i) \rangle_{i < 2k}$ when $y_i \in Y$ for $i < 2k$, then

$$\{W : W \in \widehat{\otimes}_{2k} \Sigma, \psi^{-1}[W] \in \widehat{\otimes}_{2k} \mathbb{T}, \lambda^{2k} W = \nu^{2k}(F_0^{2k} \cap \psi^{-1}[W])\}$$

is a Dynkin class containing $\prod_{i < 2k} E_i$ whenever $E_i \in \Sigma$ for every $i < 2k$, and therefore contains V , by the Monotone Class Theorem (136B). So

$$\nu^{2k}(F^{2k} \cap \psi^{-1}[V]) \geq \nu^{2k}(F_0^{2k} \cap \psi^{-1}[V]) > 0.$$

On the other hand, if $v \in D_k(B, F, \alpha, \beta)$ then there is an $f \in A$ such that $f\phi(v(2i)) \leq \alpha$ and $f\phi(v(2i+1)) \geq \beta$ for every $i < k$, that is, $\psi(v) \in D_k(A, X, \alpha, \beta)$, and $\psi(v) \notin V$. But this means that

$$(\nu^{2k})^* D_k(B, F, \alpha, \beta) \leq \nu^{2k}(F^{2k} \setminus \psi^{-1}[V]) < \nu^{2k} F^{2k} = (\nu F)^{2k}.$$

As F , α and β are arbitrary, B is stable.

465D Now for the first result connecting the notion of ‘stable’ set with the concerns of this chapter.

Proposition Let (X, Σ, μ) be a complete locally determined measure space, and $A \subseteq \mathbb{R}^X$ a stable set.

(a) $A \subseteq \mathcal{L}^0 = \mathcal{L}^0(\Sigma)$ (that is, every member of A is Σ -measurable).

(b) If $\{f(x) : f \in A\}$ is bounded for each $x \in X$, then A is relatively compact in \mathcal{L}^0 for the topology of pointwise convergence.

proof (a) ? Suppose, if possible, that there is a non-measurable $f \in A$. By 413G, there are $\alpha < \beta$ and $E \in \Sigma$ such that $0 < \mu E < \infty$ and $\mu^* P = \mu^* Q = \mu E$ where $P = \{x : x \in E, f(x) \leq \alpha\}$ and $Q = \{x : x \in E, f(x) \geq \beta\}$.

Now suppose that $k \geq 1$. Then $D_k(\{f\}, E, \alpha, \beta) \supseteq (P \times Q)^k$, so

$$(\mu^{2k})^* D_k(\{f\}, E, \alpha, \beta) = (\mu^* P \cdot \mu^* Q)^k = (\mu E)^{2k}$$

(251Wm again). Since this is true for every k , $\{f\}$ is not stable, and (by 465C(a-i)) A cannot be stable; which contradicts our hypothesis. **X**

(b) Because $\{f(x) : x \in A\}$ is bounded for each x , \overline{A} , the closure of A in \mathbb{R}^X , is compact for the topology of pointwise convergence. But \overline{A} is stable, by 465C(a-ii), so is included in \mathcal{L}^0 , by (a).

465E The topology $\mathfrak{T}_s(\mathcal{L}^2, \mathcal{L}^2)$ Some of the arguments below will rely on ideas of compactness in function spaces. There are of course many ways of expressing the method, but a reasonably accessible one uses the Hilbert space L^2 , as follows. Let (X, Σ, μ) be any measure space. Then $L^2 = L^2(\mu)$ is a Hilbert space with a corresponding weak topology $\mathfrak{T}_s(L^2, L^2)$ defined by the functionals $u \mapsto (u|v)$ for $v \in L^2$. In the present section it will be more convenient to regard this as a topology $\mathfrak{T}_s(\mathcal{L}^2, \mathcal{L}^2)$ on the space $\mathcal{L}^2 = \mathcal{L}^2(\mu)$ of square-integrable real-valued functions, defined by the functionals $f \mapsto \int f \times g$ for $g \in \mathcal{L}^2$. The essential fact we need is that norm-bounded sets are relatively weakly compact. In L^2 , this is because Hilbert spaces are reflexive (4A4Ka). In \mathcal{L}^2 , given an ultrafilter \mathcal{F} containing a $\|\cdot\|_2$ -bounded set $B \subseteq \mathcal{L}^2$, $v = \lim_{f \rightarrow \mathcal{F}} f^\bullet$ must be defined in L^2 for $\mathfrak{T}_s(L^2, L^2)$, and now there is a $g \in \mathcal{L}^2$ such that $g^\bullet = v$; in which case

$$\lim_{f \rightarrow \mathcal{F}} \int f \times h = \lim_{f \rightarrow \mathcal{F}} (f^\bullet | h^\bullet) = (g^\bullet | h^\bullet) = \int g \times h$$

for every $h \in \mathcal{L}^2$. Note that we are free to take g to be a Σ -measurable function with domain X (241Bk).

465F Lemma Let (X, Σ, μ) be a measure space, and $B \subseteq \mathcal{L}^2 = \mathcal{L}^2(\mu)$ a $\|\cdot\|_2$ -bounded set. Suppose that $h \in \mathcal{L}^2$ belongs to the closure of B for $\mathfrak{T}_s(\mathcal{L}^2, \mathcal{L}^2)$. Then for any $\delta > 0$ and $k \geq 1$ the set

$$W = \bigcup_{f \in B} \{w : w \in X^k, w(i) \in \text{dom } f \cap \text{dom } h \text{ and } f(w(i)) \geq h(w(i)) - \delta \text{ for every } i < k\}$$

is μ^k -conegligible in X^k .

proof (a) Since completing the measure μ does not change the space \mathcal{L}^2 (244Xa) nor the product measure μ^k (251Wn), we may suppose that μ is complete.

(b) The first substantive fact to note is that there is a sequence $\langle f_n \rangle_{n \in \mathbb{N}}$ in B converging to h for $\mathfrak{T}_s = \mathfrak{T}_s(\mathcal{L}^2, \mathcal{L}^2)$. **P** Setting $C = \{f^\bullet : f \in B\}$, C is a bounded set in $L^2 = L^2(\mu)$ and h^\bullet belongs to the $\mathfrak{T}_s(L^2, L^2)$ -closure of C . But L^2 , being a normed space, is angelic in its weak topology (462D), and C is relatively compact in L^2 , so there is a sequence in C converging to h^\bullet . We can represent this sequence as $\langle f_n^\bullet \rangle_{n \in \mathbb{N}}$ where $f_n \in B$ for every n , and now $\langle f_n \rangle_{n \in \mathbb{N}} \rightarrow h$ for \mathfrak{T}_s . **Q**

(c) The second component of the proof is the following simple idea. Suppose that $\langle E_n \rangle_{n \in \mathbb{N}}$ is a sequence in Σ such that $\bigcap_{n \in I} E_n$ is negligible for every infinite set $I \subseteq \mathbb{N}$. For $m \geq 1$ and $I \subseteq \mathbb{N}$ set

$$V_m(I) = \bigcap_{n \in I} \{w : w \in X^m, \exists i < m, w(i) \in E_n\}.$$

Then $\mu^m V_m(I) = 0$ for every infinite $I \subseteq \mathbb{N}$. **P** Induce on m . For $m = 1$ this is just the original hypothesis on $\langle E_n \rangle_{n \in \mathbb{N}}$. For the inductive step to $m + 1$, identify μ^{m+1} with the product of μ^m and μ , and observe that

$$V_{m+1}(I)^{-1}[\{x\}] = \{w : (w, x) \in V_{m+1}(I)\} = V_m(\{n : n \in I, x \notin E_n\})$$

for every $x \in X$. Now, setting $I_x = \{n : n \in I, x \notin E_n\}$, $F = \{x : I_x \text{ is finite}\}$, we have

$$F = \bigcup_{r \in \mathbb{N}} \bigcap_{n \in I \setminus r} E_n,$$

so F is negligible, while if $x \notin F$ then $V_m(I_x)$ is negligible, by the inductive hypothesis. But this means that almost every horizontal section of $V_{m+1}(I)$ is negligible, and $V_{m+1}(I)$, being measurable, must be negligible, by Fubini's theorem (252F). Thus the induction proceeds. **Q**

(d) Now let us return to the main line of the argument from (b). For each $n \in \mathbb{N}$, set $E_n = \{x : x \in \text{dom } f_n \cap \text{dom } h, f_n(x) < h(x) - \delta\}$. If $I \subseteq \mathbb{N}$ is infinite, then $\bigcap_{n \in I} E_n$ is negligible. **P** Setting $G = \bigcap_{n \in I} E_n$, μG is finite (because $\delta \chi E_n \leq_{\text{a.e.}} |h - f_n|$, so $\mu E_n < \infty$ for every n) and $\int_G f_n \leq \int_G h - \delta \mu G$ for every $n \in I$. But $\lim_{n \rightarrow \infty} \int_G f_n = \int_G h$ and I is infinite, so

$$\delta \mu G \leq \inf_{n \in I} \left| \int_G h - \int_G f_n \right| = 0. \quad \mathbf{Q}$$

By (c), it follows that

$$V = \bigcap_{n \in \mathbb{N}} \{w : w \in X^k, \exists i < k, w(i) \in E_n\}$$

is negligible. But if we set $Y = \bigcap_{n \in \mathbb{N}} \text{dom } f_n \cap \text{dom } h$, Y is a conegligible subset of X , Y^k is a conegligible subset of X^k , and $Y^k \setminus V \subseteq W$, so W is conegligible, as required.

465G Theorem Let (X, Σ, μ) be a semi-finite measure space, and $A \subseteq \mathcal{L}^0 = \mathcal{L}^0(\Sigma)$ a stable set of measurable functions. Let \mathfrak{T}_p and \mathfrak{T}_m be the topologies of pointwise convergence and convergence in measure, as in §463. Then the identity map from A to itself is $(\mathfrak{T}_p, \mathfrak{T}_m)$ -continuous.

proof ? Suppose, if possible, otherwise.

(a) We must have an $f_0 \in A$, a set $F \in \Sigma$ of finite measure, and an $\epsilon > 0$ such that for every \mathfrak{T}_p -neighbourhood U of f_0 there is an $f \in A \cap U$ such that $\int \chi_F \wedge |f - f_0| d\mu \geq 2\epsilon$. Set

$$B = \{\chi_F \wedge (f - f_0)^+ : f \in A\} \cup \{\chi_F \wedge (f - f_0)^- : f \in A\}.$$

Then

$$B = \{\chi_F - (\chi_F - (f - f_0)^+)^+ : f \in A\} \cup \{\chi_F - (\chi_F - (f - f_0)^-)^+ : f \in A\}$$

is stable, by 465C(a-iv), 465C(b-iv) and 465C(a-iii), used repeatedly. Setting $B' = \{f : f \in B, \int f \geq \epsilon\}$, B' is again stable (465C(a-i)). Our hypothesis is that f_0 is in the \mathfrak{T}_p -closure of

$$\begin{aligned} \{f : f \in A, \int \chi_F \wedge |f - f_0| \geq 2\epsilon\} &\subseteq \{f : f \in A, \int \chi_F \wedge (f - f_0)^+ \geq \epsilon\} \\ &\cup \{f : f \in A, \int \chi_F \wedge (f - f_0)^- \geq \epsilon\}; \end{aligned}$$

since $f \mapsto \chi_F \wedge (f - f_0)^+$ and $f \mapsto \chi_F \wedge (f - f_0)^-$ are \mathfrak{T}_p -continuous, 0 belongs to the \mathfrak{T}_p -closure of B' .

(b) Let \mathcal{F} be an ultrafilter on B' which \mathfrak{T}_p -converges to 0. Because B' is $\|\cdot\|_2$ -bounded (since $0 \leq f \leq \chi_F$ for every $f \in B'$), \mathcal{F} also has a $\mathfrak{T}_s(\mathcal{L}^2, \mathcal{L}^2)$ -limit h say, as noted in 465E; and we can suppose that h is measurable and defined everywhere. We must have

$$\int_F h = \lim_{f \rightarrow \mathcal{F}} \int_F f \geq \epsilon > 0.$$

So there is a $\delta > 0$ such that $E = \{x : x \in F, h(x) \geq 3\delta\}$ has measure greater than 0.

(c) Because B' is stable, there must be some $k \geq 1$ such that $(\mu^{2k})^* D_k(B', E, \delta, 2\delta) < (\mu E)^{2k}$. Let $W \subseteq E^{2k} \setminus D_k(B', E, \delta, 2\delta)$ be a measurable set of positive measure. By Fubini's theorem, there must be a $u \in X^k$ such that $\mu^k V$ is defined and greater than 0, where

$$V = \{v : v \in X^k, u\#v \in W\}.$$

Set $C = \{f : f \in B', f(u(i)) \leq \delta \text{ for every } i < k\}$; then $C \in \mathcal{F}$, because $\mathcal{F} \rightarrow 0$ for \mathfrak{T}_p . Accordingly h belongs to the $\mathfrak{T}_s(\mathcal{L}^2, \mathcal{L}^2)$ -closure of C . But now 465F tells us that there must be a $v \in V$ and an $f \in C$ such that $f(v(i)) \geq h(v(i)) - \delta$ for every $i < k$.

Consider $w = u\#v$. We know that $w \in W$ (because $v \in V$), so, in particular, $w \in E^{2k}$ and $h(w(i)) \geq 3\delta$ for every $i < 2k$; accordingly

$$f(w(2i + 1)) = f(v(i)) \geq h(v(i)) - \delta \geq 2\delta$$

for every $i < k$. On the other hand, $f(w(2i)) = f(u(i)) \leq \delta$ for every $i < k$, because $f \in C$. But this means that f witnesses that $w \in D_k(B', E, \delta, 2\delta)$, which is supposed to be disjoint from W . **X**

This contradiction shows that the theorem is true.

465H We shall need some interesting and important general facts concerning powers of measures. I start with a useful elaboration of the strong law of large numbers.

Theorem Let (X, Σ, μ) be any probability space. For $n \in \mathbb{N}$, write Λ_n for the domain of the product measure μ^n . For $w \in X^{\mathbb{N}}$, $k \geq 1$, $n \geq 1$ write ν_{wk} for the probability measure with domain $\mathcal{P}X$ defined by writing

$$\nu_{wk}(E) = \frac{1}{k} \#(\{i : i < k, w(i) \in E\})$$

for $E \subseteq X$, and ν_{wk}^n for the corresponding product measure on X^n .

Then whenever $n \geq 1$ and $f : X^n \rightarrow \mathbb{R}$ is bounded and Λ_n -measurable, $\lim_{k \rightarrow \infty} \int f d\nu_{wk}^n$ exists, and is equal to $\int f d\mu^n$, for $\mu^{\mathbb{N}}$ -almost every $w \in X^{\mathbb{N}}$.

proof (a) For $k \geq n$ and $w \in X^{\mathbb{N}}$, set

$$h_k(w) = \frac{(k-n)!}{k!} \sum_{\pi: n \rightarrow k \text{ is injective}} f(w\pi).$$

Setting $M = \sup_{v \in X^n} |f(v)|$, we have

$$\begin{aligned} |h_k(w) - \int f d\nu_{wk}^n| &= \left| \frac{(k-n)!}{k!} \sum_{\pi: n \rightarrow k \text{ is injective}} f(w\pi) - \frac{1}{k^n} \sum_{\pi: n \rightarrow k} f(w\pi) \right| \\ &\leq \frac{Mk!}{(k-n)!} \left(\frac{(k-n)!}{k!} - \frac{1}{k^n} \right) + \frac{M}{k^n} \left(k^n - \frac{k!}{(k-n)!} \right) \\ &= 2M \left(1 - \frac{k!}{k^n(k-n)!} \right) \rightarrow 0 \end{aligned}$$

as $k \rightarrow \infty$, for every $w \in X^{\mathbb{N}}$.

(b) Write Λ for the domain of $\mu^{\mathbb{N}}$, and for $k \geq n$ set

$$\mathbb{T}_k = \{W : W \in \Lambda, w\pi \in W \text{ whenever } w \in W \text{ and } \pi \in S_k\},$$

where S_k is the set of permutations $\psi : \mathbb{N} \rightarrow \mathbb{N}$ such that $\psi(i) = i$ for $i \geq k$. Then $\langle \mathbb{T}_k \rangle_{k \geq n}$ is a non-increasing sequence of σ -subalgebras of Λ . For any injective function $\pi : n \rightarrow k$ there are just $(k-n)!$ extensions of π to a member of S_k . So

$$h_k(w) = \frac{1}{k!} \sum_{\pi \in S_k} f(w\pi \upharpoonright n)$$

for every w . Observe that $h_k(w\psi) = h_k(w)$ whenever $\psi \in S_k$ and $w \in X^{\mathbb{N}}$, so h_k is \mathbb{T}_k -measurable. (Of course we need to look back at the definition of h_k to confirm that it is Λ -measurable.)

(c) For any $k \geq n$, h_k is a conditional expectation of h_n on \mathbb{T}_k . **P** If $W \in \mathbb{T}_k$, then

$$\int_W h_k(w) dw = \frac{1}{k!} \sum_{\pi \in S_k} \int_W f(w\pi \upharpoonright n) dw = \frac{1}{k!} \sum_{\pi \in S_k} \int_W g(w\pi) dw$$

where $g(w) = f(w \upharpoonright n)$ for $w \in X^{\mathbb{N}}$. Now observe that for every $\pi \in S_k$ the map $w \mapsto w\pi$ is a measure space automorphism of $X^{\mathbb{N}}$ which leaves W unchanged, because $W \in \mathbb{T}_k$; so that $\int_W g(w\pi) dw = \int_W g(w) dw$, by

235Gc. So $\int_W h_k = \int_W g$. But (since $W \in T_k \subseteq T_n$) $\int_W h_n$ is also equal to $\int_W g$, and $\int_W h_k = \int_W h_n$. As W is arbitrary, h_k is a conditional expectation of h_n on T_k . **Q**

(d) By the reverse martingale theorem (275K), $h_\infty(w) = \lim_{k \rightarrow \infty} h_k(w)$ is defined for almost every $w \in X^\mathbb{N}$. Accordingly $\lim_{k \rightarrow \infty} \int f d\nu_{wk}^n$ is defined for almost every w .

(e) To see that the limit is $\int f d\mu^n$, observe that if $W \in T_\infty = \bigcap_{k \in \mathbb{N}} T_k$ then $\mu^\mathbb{N}W$ must be either 0 or 1. **P** Set $\gamma = \mu^\mathbb{N}W$. Let $\epsilon > 0$. Then there is a $V \in \bigotimes_{\mathbb{N}} \Sigma$ (notation: 465Ad) such that $\mu^\mathbb{N}(W \Delta V) \leq \epsilon$ (254Fe). There is some k such that V is determined by coordinates in k . If we set $\pi(i) = 2k - i$ for $i < 2k$, i for $i \geq 2k$, then $V' = \{w\pi : w \in V\}$ is determined by coordinates in $2k \setminus k$, so $\mu^\mathbb{N}(V \cap V') = (\mu^\mathbb{N}V)^2$. On the other hand, because $W \in T_{2k}$, the measure space automorphism $w \mapsto w\pi$ does not move W , and $\mu^\mathbb{N}(W \setminus V') = \mu^\mathbb{N}(W \setminus V)$. Accordingly

$$\gamma = \mu^\mathbb{N}(W \cap W) \leq \mu^\mathbb{N}(V \cap V') + 2\mu^\mathbb{N}(W \setminus V) \leq (\mu^\mathbb{N}V)^2 + 2\epsilon \leq (\gamma + \epsilon)^2 + 2\epsilon.$$

As ϵ is arbitrary, $\gamma \leq \gamma^2$ and $\gamma \in \{0, 1\}$. **Q**

(e) Now 275K tells us that h_∞ is T_∞ -measurable, therefore essentially constant, and must be equal to its expectation almost everywhere. But, setting $W = X^\mathbb{N}$ in the proof of (c), we see that

$$\int h_k(w)dw = \int f(w \upharpoonright n)dw = \int f d\mu^n$$

for every $k \geq n$, so

$$\begin{aligned} \lim_{k \rightarrow \infty} \int f d\nu_{wk}^n &= \lim_{k \rightarrow \infty} h_k(w) = h_\infty(w) \\ &= \int h_\infty(w)dw = \lim_{k \rightarrow \infty} \int h_k(w)dw = \int f d\mu^n \end{aligned}$$

for almost every w , as claimed.

465I Now for a string of lemmas, working towards the portmanteau Theorem 465M. The first is elementary.

Lemma Let X be a set, and Σ a σ -algebra of subsets of X . For $w \in X^\mathbb{N}$ and $k \geq 1$, write ν_{wk} for the probability measure with domain $\mathcal{P}X$ defined by writing

$$\nu_{wk}(E) = \frac{1}{k} \#(\{i : i < k, w(i) \in E\})$$

for $E \subseteq X$. Then for any $k \in \mathbb{N}$ and any set I , $w \mapsto \nu_{wk}^I(W)$ is $\widehat{\bigotimes}_{\mathbb{N}} \Sigma$ -measurable (notation: 465Ad) for every $W \in \widehat{\bigotimes}_I \Sigma$.

proof Write \mathcal{W} for the set of subsets W of X^I such that $w \mapsto \nu_{wk}^I(W)$ is $\widehat{\bigotimes}_{\mathbb{N}} \Sigma$ -measurable. Then $X^I \in \mathcal{W}$, $W' \setminus W \in \mathcal{W}$ whenever $W, W' \in \mathcal{W}$ and $W \subseteq W'$, and $\bigcup_{n \in \mathbb{N}} W_n \in \mathcal{W}$ whenever $\langle W_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence in \mathcal{W} . Write \mathcal{V} for the set of Σ -cylinders in X^I , that is, sets expressible in the form $\{v : v(i) \in E_i \text{ for every } i \in J\}$, where $J \subseteq I$ is finite and $E_i \in \Sigma$ for $i \in J$. Then $\mathcal{V} \subseteq \mathcal{W}$. **P** If $J \subseteq I$ is finite and $E_i \in \Sigma$ for $i \in J$, then

$$w \mapsto \nu_{wk} E_i = \frac{1}{k} \sum_{j=0}^{k-1} \chi_{E_i}(w(j))$$

is $\widehat{\bigotimes}_{\mathbb{N}}$ -measurable for every $i \in J$. So

$$w \mapsto \nu_{wk}^I \{v : v(i) \in E_i \forall i \in J\} = \prod_{i \in J} \nu_{wk} E_i$$

is also measurable. **Q**

Because $V \cap V' \in \mathcal{V}$ for all $V, V' \in \mathcal{V}$, the Monotone Class Theorem (136B) tells us that \mathcal{W} must include the σ -algebra generated by \mathcal{V} , which is $\widehat{\bigotimes}_I \Sigma$.

465J The next three lemmas are specifically adapted to the study of stable sets of functions.

Lemma Let (X, Σ, μ) be a probability space. For any $n \in \mathbb{N}$ and $W \subseteq X^n$ I say that W is **symmetric** if $w\pi \in W$ whenever $w \in W$ and $\pi : n \rightarrow n$ is a permutation. For each n , write Λ_n for the domain of the product measure μ^n .

(a) Suppose that for each $n \geq 1$ we are given $W_n \in \Lambda_n$, and that $W_{m+n} \subseteq W_m \times W_n$ for all $m, n \geq 1$, identifying X^{m+n} with $X^m \times X^n$. Then $\lim_{n \rightarrow \infty} (\mu^n W_n)^{1/n}$ is defined and equal to $\delta = \inf_{n \geq 1} (\mu^n W_n)^{1/n}$.

(b) Now suppose that each W_n is symmetric. Then there is an $E \in \Sigma$ such that $\mu E = \delta$ and $E^n \setminus W_n$ is negligible for every $n \in \mathbb{N}$.

(c) Next, let $\langle D_n \rangle_{n \geq 1}$ be a sequence of sets such that

$$D_n \subseteq X^n \text{ is symmetric for every } n \geq 1,$$

$$\text{whenever } 1 \leq m \leq n \text{ and } v \in D_n \text{ then } v \upharpoonright m \in D_m.$$

Then $\delta = \lim_{n \rightarrow \infty} ((\mu^n)^* D_n)^{1/n}$ is defined and there is an $E \in \Sigma$ such that $\mu E = \delta$ and $(\mu^n)^*(D_n \cap E^n) = (\mu E)^n$ for every $n \in \mathbb{N}$.

proof (a) For any $\eta > 0$, there is an $m \geq 1$ such that $\mu^m W_m \leq (\delta + \eta)^m$. If $n = mk + i$, where $k \geq 1$ and $i < m$, then (identifying X^n with $(X^m)^k \times X^i$) $W_n \subseteq (W_m)^k \times X^i$, so

$$\mu^n W_n \leq (\delta + \eta)^{mk} \leq \gamma(\delta + \eta)^{mk+i} = \gamma(\delta + \eta)^n,$$

where

$$\gamma = \max_{i < m} \left(\frac{1}{\delta + \eta}\right)^i.$$

So

$$\limsup_{n \rightarrow \infty} (\mu^n W_n)^{1/n} \leq (\delta + \eta) \limsup_{n \rightarrow \infty} \gamma^{1/n} = \delta + \eta.$$

As η is arbitrary,

$$\delta \leq \liminf_{n \rightarrow \infty} (\mu^n W_n)^{1/n} \leq \limsup_{n \rightarrow \infty} (\mu^n W_n)^{1/n} \leq \delta,$$

and $\lim_{n \rightarrow \infty} (\mu^n W_n)^{1/n} = \delta$.

(b) It is enough to consider the case $\delta > 0$.

(i) Consider the family \mathbb{V} of sequences $\langle V_n \rangle_{n \geq 1}$ such that

$$\text{for each } n \geq 1, V_n \in \Lambda_n \text{ is symmetric and } \mu^n V_n \geq \delta^n,$$

$$\text{if } 1 \leq m \leq n \text{ then } v \upharpoonright m \in V_m \text{ for every } v \in V_n.$$

Observe that $\mathbf{W} = \langle V_n \rangle_{n \geq 1} \in \mathbb{V}$. Order \mathbb{V} by saying that $\langle V_n \rangle_{n \geq 1} \leq \langle V'_n \rangle_{n \geq 1}$ if $V_n \subseteq V'_n$ for every n . For $\mathbf{V} = \langle V_n \rangle_{n \geq 1}$ in \mathbb{V} , set $\theta(\mathbf{V}) = \sum_{n=1}^{\infty} 2^{-n} \mu^n V_n$. Any non-increasing sequence $\langle \langle V_{kn} \rangle_{n \geq 1} \rangle_{k \in \mathbb{N}}$ in \mathbb{V} has a lower bound $\langle \bigcap_{k \in \mathbb{N}} V_{kn} \rangle_{n \geq 1}$ in \mathbb{V} , so there must be a $\mathbf{W}' = \langle W'_n \rangle_{n \geq 1} \in \mathbb{V}$ such that $\mathbf{W}' \leq \mathbf{W}$ and $\theta(\mathbf{V}) = \theta(\mathbf{W}')$ whenever $\mathbf{V} \in \mathbb{V}$ and $\mathbf{V} \leq \mathbf{W}'$; that is, whenever $\langle V_n \rangle_{n \geq 1} \in \mathbb{V}$ and $V_n \subseteq W'_n$ for every $n \geq 1$, then $\mu^n V_n = \mu^n W'_n$ for every n .

(ii) For $x \in X$, $n \geq 1$ set $V_n^{(x)} = \{w : (x, w) \in W'_{n+1}\}$. Then $V_n^{(x)} \in \Lambda_n$ for almost every x ; let $X_1 \subseteq X$ be a conegligible set such that $V_n^{(x)} \in \Lambda_n$ for every $x \in X_1$ and every $n \geq 1$. Every $V_n^{(x)}$ is symmetric, and if $1 \leq m \leq n$ and $v \in V_n^{(x)}$ then $v \upharpoonright m \in V_m^{(x)}$. It follows that if $m, n \geq 1$ then $V_{m+n}^{(x)}$ becomes identified with a subset of $V_m^{(x)} \times V_n^{(x)}$.

From (a) we see that $\delta_x = \lim_{n \rightarrow \infty} (\mu^n V_n^{(x)})^{1/n}$ is defined for every $x \in X_1$. The map $x \mapsto \mu^n V_n^{(x)} : X_1 \rightarrow [0, 1]$ is measurable for each n , by Fubini's theorem (252D), so $x \mapsto \delta_x$ is also measurable. Since $V_n^{(x)} \subseteq W'_n \subseteq W_n$ for every x and n , $\delta_x \leq \delta$ for every $x \in X_1$.

(iii) Set $E = \{x : x \in X_1, \delta_x = \delta\}$. Then $\mu E \geq \delta$. **P?** Otherwise, there is some $\beta < \delta$ such that $\mu F < \beta$, where $F = \{x : x \in X_1, \delta_x \geq \beta\}$. Now

$$\begin{aligned} X_1 \setminus F &= \{x : x \in X_1, \lim_{n \rightarrow \infty} (\mu^n V_n^{(x)})^{1/n} < \beta\} \\ &\subseteq \bigcup_{m \in \mathbb{N}} \bigcap_{n \geq m} \{x : x \in X_1, \mu^n V_n^{(x)} \leq \beta^n\}, \end{aligned}$$

so there is some $m \in \mathbb{N}$ such that $\mu H \geq 1 - \beta$, where

$$H = \bigcap_{n \geq m} \{x : x \in X_1, \mu^n V_n^{(x)} \leq \beta^n\}.$$

Set $\gamma_n = \mu^n W'_n \geq \delta^n$ for each n . Then, for any $n \geq m$,

$$\begin{aligned} \gamma_{n+1} &= \mu^{n+1} W'_{n+1} = \int_{X_1} \mu^n V_n^{(x)} \mu(dx) \\ &= \int_H \mu^n V_n^{(x)} \mu(dx) + \int_{X_1 \setminus H} \mu^n V_n^{(x)} \mu(dx) \leq \beta^n + \beta \gamma_n \end{aligned}$$

because $V_n^{(x)} \subseteq W'_n$ for every x , and $\mu(X_1 \setminus H) \leq \beta$. An easy induction shows that $\gamma_{m+k} \leq k\beta^{m+k-1} + \beta^k \gamma_m$ for every $k \in \mathbb{N}$. But this means that

$$\delta^k = \delta^{-m} \delta^{m+k} \leq \delta^{-m} \gamma_{m+k} \leq \beta^k \delta^{-m} (k\beta^{m-1} + \gamma_m)$$

for every k ; setting $\eta = (\delta - \beta)/\beta > 0$,

$$\frac{k(k-1)}{2} \eta \leq (1 + \eta)^k = \left(\frac{\delta}{\beta}\right)^k \leq \delta^{-m} (k\beta^{m-1} + \gamma_m)$$

for every k , which is impossible. **XQ**

(iv) Next, for any $x \in E$, $\mathbf{V}^{(x)} = \langle V_n^{(x)} \rangle_{n \geq 1} \in \mathbb{V}$ and $\mathbf{V}^{(x)} \leq \mathbf{W}'$, so $\mu^n (W'_n \setminus V_n^{(x)}) = 0$ for every $n \geq 1$. This means that, given $n \geq 1$, every vertical section of $(E \times W'_n) \setminus W'_{n+1}$ (regarded as a subset of $X \times X^n$) is negligible; so $(E \times W'_n) \setminus W'_{n+1}$ is negligible. We are assuming that $\delta > 0$, so

$$E \subseteq \{x : x \in X_1, V_1^{(x)} \neq \emptyset\} \subseteq \{x : \{w : (x, w) \in W'_2\} \neq \emptyset\} \subseteq W'_1.$$

Now a simple induction shows that $E^n \setminus W'_n$ is negligible for every $n \geq 1$, so that $E^n \setminus W_n$ is negligible for every n , and we have an appropriate E . (Of course $\mu E = \delta$ exactly, because $(\mu E)^n \leq \mu W_n$ for every n .)

(c) For each $n \in \mathbb{N}$ let $V_n \in \Lambda_n$ be a measurable envelope of D_n in X^n . Define $\langle W_n \rangle_{n \geq 1}$ inductively by saying

$$W_1 = V_1,$$

$$\begin{aligned} W_{n+1} &= \{w : w \in X^{n+1}, w\pi \upharpoonright n \in W_n, \\ &\quad w\pi \in V_{n+1} \text{ for every permutation } \pi : n+1 \rightarrow n+1\} \end{aligned}$$

for each $n \geq 1$. Then an easy induction on n shows that W_n is measurable and symmetric and that $D_n \subseteq W_n \subseteq V_n$, so that W_n is a measurable envelope of D_n and $\mu^n W_n = (\mu^n)^* D_n$.

Now $\langle W_n \rangle_{n \geq 1}$ satisfies the hypotheses of (a)-(b), so

$$\delta = \lim_{n \rightarrow \infty} (\mu^n W_n)^{1/n} = \lim_{n \rightarrow \infty} ((\mu^n)^* D_n)^{1/n}$$

is defined and there is a set $E \in \Sigma$, of measure δ , such that $E^n \setminus W_n$ is negligible for every n ; but this means that

$$(\mu^n)^*(E^n \cap D_n) = \mu^n(E^n \cap W_n) = \delta^n$$

for every n , as required.

465K Lemma Let (X, Σ, μ) be a complete probability space, and $A \subseteq [0, 1]^X$ a stable set. Suppose that $\epsilon > 0$ is such that $\int f d\mu \leq \epsilon^2$ for every $f \in A$. Then there are an $n \geq 1$ and a $W \in \widehat{\bigotimes}_n \Sigma$ (notation: 465Ad) and a $\gamma > \mu^n W$ such that $\int f d\nu \leq 3\epsilon$ whenever $f \in A$ and ν is a probability measure on X with domain including Σ such that $\nu^n W \leq \gamma$.

proof (a) For $n \in \mathbb{N}$ write $\tilde{C}_n = \bigcup_{f \in A} \{x : f(x) \geq \epsilon\}^n$. Then $\langle \tilde{C}_n \rangle_{n \in \mathbb{N}}$ satisfies the conditions of 465Jc, so $\delta = \lim_{n \rightarrow \infty} ((\mu^n)^* \tilde{C}_n)^{1/n}$ is defined, and there is an $E \in \Sigma$ such that $\mu E = \delta$ and $(\mu^n)^*(E^n \cap \tilde{C}_n) = \delta^n$ for every $n \in \mathbb{N}$. Now, for $B \subseteq [0, 1]^X$ and $n \in \mathbb{N}$, write $C_n(B) = \bigcup_{f \in B} \{x : x \in E, f(x) \geq \epsilon\}^n$, so that $C_n(A) = E^n \cap \tilde{C}_n$, and $(\mu^n)^* C_n(A) = \delta^n$ for every n . For any $B \subseteq [0, 1]^X$, $\langle C_n(B) \rangle_{n \in \mathbb{N}}$ also satisfies the conditions of 465Jc, and $\delta_B = \lim_{n \rightarrow \infty} ((\mu^n)^* C_n(B))^{1/n}$ is defined; we have $\delta_A = \delta$.

(b) If $B, B' \subseteq [0, 1]^X$, then

$$(\mu^n)^*C_n(B) \leq (\mu^n)^*C_n(B \cup B') \leq (\mu^n)^*C_n(B) + (\mu^n)^*C_n(B')$$

for every n , so $\delta_B \leq \delta_{B \cup B'} \leq \max(\delta_B, \delta_{B'})$. It follows that if

$$\mathcal{G} = \{G : G \subseteq [0, 1]^X \text{ is } \mathfrak{T}_p\text{-open, } \delta_{G \cap A} < \delta\},$$

where \mathfrak{T}_p is the usual topology of $[0, 1]^X$, no finite subfamily of \mathcal{G} can cover A . Accordingly, since the \mathfrak{T}_p -closure \bar{A} of A is \mathfrak{T}_p -compact, there is an $h \in \bar{A}$ such that $\delta_{G \cap A} = \delta$ for every \mathfrak{T}_p -open set G containing h .

(c) At this point recall that every function in \bar{A} is measurable (465C(a-ii), 465Da) and that $f : \bar{A} \rightarrow [0, 1]$ is \mathfrak{T}_p -continuous (465G). So $\int h \leq \epsilon^2$ and $\mu\{x : h(x) \geq \epsilon\} \leq \epsilon$.

¶ Suppose, if possible, that $\delta > \epsilon$. Then there is some $\eta > 0$ such that $\mu F > 0$, where $F = \{x : x \in E, h(x) < \epsilon - \eta\}$. For $k \in \mathbb{N}$ and $u \in F^k$ set $G_u = \{f : f \in [0, 1]^X, f(u(i)) < \epsilon - \eta \text{ for every } i < k\}$. Then G_u is an open neighbourhood of h , so $\delta_{G_u \cap A} = \delta$ and $(\mu^k)^*(C_k(G_u \cap A)) \geq \delta^k$. But because $F \subseteq E$ and $C_k(G_u \cap A) \subseteq E^k$ and $\mu E = \delta$, this means that $(\mu^k)^*(F^k \cap C_k(G_u \cap A)) = (\mu F)^k$.

In the notation of 465Af,

$$C_k(G_u \cap A) \cap F^k \subseteq \{v : u \# v \in D_k(A, F, \epsilon - \eta, \epsilon)\}$$

for any $u \in F^k$. So $(\mu^k)^*\{v : u \# v \in D_k(A, F, \epsilon - \eta, \epsilon)\} = (\mu F)^k$ for every $u \in F^k$. But this means that $(\mu^{2k})^*D_k(A, F, \epsilon - \eta, \epsilon) = (\mu F)^{2k}$. Since this is so for every $k \geq 1$, A is not stable. **X**

(d) Thus $\delta \leq \epsilon$. There is therefore some $n \geq 1$ such that $(\mu^n)^*\tilde{C}_n < (2\epsilon)^n$. Let $W \in \widehat{\otimes}_n \Sigma$ be a measurable envelope of \tilde{C}_n , and try $\gamma = (2\epsilon)^n$. If ν is any probability measure on X with domain including Σ such that $\nu^n W \leq \gamma$, then for any $f \in A$ we have

$$\{x : f(x) \geq \epsilon\}^n \subseteq \tilde{C}_n \subseteq W, \quad (\nu\{x : f(x) \geq \epsilon\})^n \leq \nu^n W \leq (2\epsilon)^n,$$

so that $\nu\{x : f(x) \geq \epsilon\} \leq 2\epsilon$. As $0 \leq f(x) \leq 1$ for every $x \in X$, $\int f d\nu \leq 3\epsilon$, as required.

465L Lemma (TALAGRAN 87) Let (X, Σ, μ) be a complete probability space, and $A \subseteq [0, 1]^X$ a set which is not stable. Then there are measurable functions $h_0, h_1 : X \rightarrow [0, 1]$ such that $\int h_0 d\mu < \int h_1 d\mu$ and $(\mu^{2k})^*\tilde{D}_k = 1$ for every $k \geq 1$, where

$$\begin{aligned} \tilde{D}_k = \bigcup_{f \in A} \{w : w \in X^{2k}, f(w(2i)) \leq h_0(w(2i)), \\ f(w(2i+1)) \geq h_1(w(2i+1)) \text{ for every } i < k\}. \end{aligned} \tag{*}$$

proof The proof divides into two cases.

case 1 Suppose that there is an ultrafilter \mathcal{F} on A such that the \mathfrak{T}_p -limit g_0 of \mathcal{F} is not measurable. Let $h'_0, h'_1 : X \rightarrow [0, 1]$ be measurable functions such that $h'_0 \leq g_0 \leq h'_1$ and $\int h'_0 = \underline{\int} g_0, \int h'_1 = \bar{\int} g_0$ (133Ja). Then $\delta = \frac{1}{5} \int h'_1 - h'_0 > 0$ (133Jd). Set $h_0 = h'_0 + 2\delta\chi_X, h_1 = h'_1 - 2\delta\chi_X$, so that $\int h_0 < \int h_1$.

Set $Q_0 = \{x : x \in X, g_0(x) \leq h_0(x) - \delta\}$. Then $\mu^*Q_0 = 1$, by 133J(a-i). Similarly, $\mu^*Q_1 = 1$, where $Q_1 = \{x : g_0(x) \geq h_1(x) + \delta\}$.

If $k \geq 1, u \in Q_0^k$ and $v \in Q_1^k$, then there is an $f \in A$ such that $|f(u(i)) - g_0(u(i))| \leq \delta, |f(v(i)) - g_0(v(i))| \leq \delta$ for every $i < k$. But this means that $f(u(i)) \leq h_0(u(i))$ and $f(v(i)) \geq h_1(v(i))$ for every $i < k$, and $u \# v \in \tilde{D}_k$. Thus

$$\tilde{D}_k \supseteq Q_0 \times Q_1 \times Q_0 \times Q_1 \times \dots \times Q_0 \times Q_1,$$

which has full outer measure, by 254L. As k is arbitrary, we have found appropriate h_0, h_1 in this case.

case 2 Now suppose that for every ultrafilter \mathcal{F} on A , the \mathfrak{T}_p -limit of \mathcal{F} is measurable.

(i) We are supposing that A is not stable, so there are $E \in \Sigma$ and $\alpha < \beta$ such that $\mu E > 0$ and $(\mu^{2k})^*D_k(A, E, \alpha, \beta) = (\mu E)^{2k}$ for every $k \geq 1$. The first thing to note is that if \mathcal{I} is the set of those $B \subseteq A$

for which there is some $k \in \mathbb{N}$ such that $(\mu^{2k})^* D_k(B, E, \alpha, \beta) < (\mu E)^{2k}$, then \mathcal{I} is an ideal of subsets of A . **P** Of course $\emptyset \in \mathcal{I}$ and $B \in \mathcal{I}$ whenever $B \subseteq B' \in \mathcal{I}$. Also 465C(b-i) tells us that, if $B \in \mathcal{I}$, then $\lim_{k \rightarrow \infty} \frac{1}{(\mu E)^{2k}} (\mu^{2k})^* D_k(B, E, \alpha, \beta) = 0$. It follows easily (as in the proof of 465C(b-ii)) that $B \cup B' \in \mathcal{I}$ for all $B, B' \in \mathcal{I}$. **Q**

(ii) \mathcal{I} is a proper ideal of subsets of A (by the choice of E, k, α and β), so there is an ultrafilter \mathcal{F} on A such that $\mathcal{F} \cap \mathcal{I} = \emptyset$. Let g_0 be the \mathfrak{T}_p -limit of \mathcal{F} . Then g_0 is measurable.

Set $\delta = \frac{1}{3}(\beta - \alpha)\mu E > 0$, and define h_0, h_1 by setting

$$\begin{aligned} h_0(x) &= \alpha \text{ if } x \in E, \\ &= g_0(x) + \delta \text{ if } x \in X \setminus E, \\ h_1(x) &= \beta \text{ if } x \in E, \\ &= g_0(x) - \delta \text{ if } x \in X \setminus E. \end{aligned}$$

Then

$$\int h_1 - \int h_0 \geq (\beta - \alpha)\mu E - 2\delta\mu(X \setminus E) > 0.$$

(iii) We shall need to know a little more about sets of the form $D_k(B, E, \alpha, \beta)$ for $B \in \mathcal{F}$. In fact, if $B \subseteq A$ and $B \notin \mathcal{I}$, then for any finite sets I and J

$$(\mu^I \times \mu^J)^* D_{I,J}(B, E, \alpha, \beta) = (\mu E)^{\#(I) + \#(J)},$$

where $D_{I,J}(B, E, \alpha, \beta)$ is

$$\bigcup_{f \in B} \{(u, v) : u \in E^I, v \in E^J, f(u(i)) \leq \alpha \text{ for } i \in I, f(v(j)) \geq \beta \text{ for } j \in J\}.$$

P We may suppose that $I = k$ and $J = l$ where $k, l \in \mathbb{N}$. Take $m = \max(1, k, l)$. Then we have an inverse-measure-preserving map $\phi : X^{2m} \rightarrow X^k \times X^l$ defined by saying that $\phi(w) = (u, v)$ where $u(i) = w(2i)$ for $i < k$ and $v(i) = w(2i + 1)$ for $i < l$. **?** If $(\mu^k \times \mu^l)^* D_{k,l}(B, E, \alpha, \beta) < (\mu E)^{k+l}$, there is a non-negligible measurable set $V \subseteq (E^k \times E^l) \setminus D_{k,l}(B, E, \alpha, \beta)$. Now $\phi^{-1}[V]$ is non-negligible and depends only on coordinates in $\{2i : i < k\} \cup \{2i + 1 : i < l\}$, so

$$\mu^{2m}(E^{2m} \cap \phi^{-1}[V]) = \mu^{2m}(\phi^{-1}[V]) \cdot (\mu E)^{2m-k-l} > 0.$$

But $\phi^{-1}[V] \cap D_m(B, E, \alpha, \beta) = \emptyset$, so $(\mu^{2m})^* D_m(B, E, \alpha, \beta) < (\mu E)^{2m}$, and $B \in \mathcal{I}$. **X** So $(\mu^k \times \mu^l)^* D_{k,l}(B, E, \alpha, \beta) = (\mu E)^{k+l}$, as required. **Q**

(iv) **?** Suppose, if possible, that $k \geq 1$ is such that $(\mu^{2k})^* \tilde{D}_k < 1$, where \tilde{D}_k is defined from h_0 and h_1 by the formula (*) in the statement of the lemma. Let $W \subseteq X^{2k}$ be a measurable set of positive measure disjoint from \tilde{D}_k . For $I, J \subseteq k$ write W_{IJ} for

$$\begin{aligned} \{w : w \in W, w(2i) \in E \text{ for } i \in I, w(2i) \notin E \text{ for } i \in k \setminus I, \\ w(2i + 1) \in E \text{ for } i \in J, w(2i + 1) \notin E \text{ for } i \in k \setminus J\}. \end{aligned}$$

Then there are $I, J \subseteq k$ such that $\mu^{2k} W_{IJ} > 0$.

We can identify X^{2k} with $X^I \times X^J \times X^{k \setminus I} \times X^{k \setminus J}$, matching any $w \in X^{2k}$ with (w_0, w_1, w_2, w_3) where

$$\begin{aligned} w_0(i) &= w(2i) \text{ for } i \in I, \\ w_1(i) &= w(2i + 1) \text{ for } i \in J, \\ w_2(i) &= w(2i) \text{ for } i \in k \setminus I, \\ w_3(i) &= w(2i + 1) \text{ for } i \in k \setminus J. \end{aligned}$$

Write \tilde{W} for the image of W_{IJ} under this matching. The condition $W_{IJ} \cap \tilde{D}_k = \emptyset$ translates into

- (†) whenever $(w_0, w_1, w_2, w_3) \in \tilde{W}$, $f \in A$,
 either there is an $i \in I$ such that $f(w_0(i)) > \alpha$
 or there is an $i \in J$ such that $f(w_1(i)) < \beta$
 or there is an $i \in k \setminus I$ such that $f(w_2(i)) > g_0(w_2(i)) + \delta$

or there is an $i \in k \setminus J$ such that $f(w_3(i)) < g_0(w_3(i)) - \delta$.

(v) By Fubini's theorem, applied to $(X^I \times X^J) \times (X^{k \setminus I} \times X^{k \setminus J})$, we can find $w_2 \in X^{k \setminus I}$, $w_3 \in X^{k \setminus J}$ such that $(\mu^I \times \mu^J)(V)$ is defined and greater than 0, where $V = \{(w_0, w_1) : (w_0, w_1, w_2, w_3) \in \tilde{W}\}$. Set

$$B = \{f : f \in A, |f(w_2(i)) - g_0(w_2(i))| \leq \delta \text{ for } i \in k \setminus I, \\ |f(w_3(i)) - g_0(w_3(i))| \leq \delta \text{ for } i \in k \setminus J\}.$$

Then $B \in \mathcal{F}$, because $\mathcal{F} \rightarrow g_0$ for \mathfrak{T}_p . So $(\mu^I \times \mu^J)^* D_{I,J}(B, E, \alpha, \beta) = (\mu E)^{\#(I)+\#(J)}$, by (iii) above. Since $W_{I,J} \subseteq W$, V is included in $E^I \times E^J$ and meets $D_{I,J}(B, E, \alpha, \beta)$; that is, there are $f \in B$ and $(w_0, w_1) \in V$ such that $f(w_0(i)) \leq \alpha$ for $i \in I$ and $f(w_1(i)) \geq \beta$ for $i \in J$. But because $f \in B$ we also have $f(w_2(i)) \leq g_0(w_2(i)) + \delta$ for $i \in k \setminus I$ and $f(w_3(i)) \geq g_0(w_3(i)) - \delta$ for $i \in k \setminus J$; which contradicts the list in (†) above. **X**

(vi) Thus $(\mu^{2k})^* \tilde{D}_k = 1$ for every k , and in this case also we have an appropriate pair h_0, h_1 .

465M Theorem (TALAGRAN 82, TALAGRAN 87) Let (X, Σ, μ) be a complete probability space, and A a non-empty uniformly bounded set of real-valued functions defined on X . Then the following are equiveridical.

(i) A is stable.

(ii) Every function in A is measurable, and $\lim_{k \rightarrow \infty} \sup_{f \in A} |\frac{1}{k} \sum_{i=0}^{k-1} f(w(i)) - \int f| = 0$ for almost every $w \in X^{\mathbb{N}}$.

(iii) Every function in A is measurable, and for every $\epsilon > 0$ there are a finite subalgebra T of Σ in which every atom is non-negligible and a sequence $\langle h_k \rangle_{k \geq 1}$ of measurable functions on $X^{\mathbb{N}}$ such that

$$h_k(w) \geq \sup_{f \in A} \frac{1}{k} \sum_{i=0}^{k-1} |f(w(i)) - \mathbb{E}(f|T)(w(i))|$$

for every $w \in X^{\mathbb{N}}$ and $k \geq 1$, and

$$\limsup_{k \rightarrow \infty} h_k(w) \leq \epsilon$$

for almost every $w \in X^{\mathbb{N}}$. (Here $\mathbb{E}(f|T)$ is the (unique) conditional expectation of f on T .)

(iv) $\lim_{k,l \rightarrow \infty} \sup_{f \in A} |\frac{1}{k} \sum_{i=0}^{k-1} f(w(i)) - \frac{1}{l} \sum_{i=0}^{l-1} f(w(i))| = 0$ for almost every $w \in X^{\mathbb{N}}$.

(v) $\lim_{k,l \rightarrow \infty} \int \sup_{f \in A} |\frac{1}{k} \sum_{i=0}^{k-1} f(w(i)) - \frac{1}{l} \sum_{i=0}^{l-1} f(w(i))| \mu^{\mathbb{N}}(dw) = 0$.

proof All the statements (i)-(v) are unaffected by translations (by constant functions) and non-zero scalar multiplications of the set A , so it will be enough to consider the case in which $A \subseteq [0, 1]^X$.

As in 465H-465I, I write $\nu_{wk}(E) = \frac{1}{k} \#(\{i : i < k, w(i) \in E\})$ for $w \in X^{\mathbb{N}}$, $k \geq 1$ and $E \subseteq X$; so for any function $f : X \rightarrow \mathbb{R}$, $\int f d\nu_{wk} = \frac{1}{k} \sum_{i=0}^{k-1} f(w(i))$.

(a)(i) \Rightarrow (iii)(α) If A is stable, then every function in A is measurable, by 465Da. Let $\epsilon > 0$. Set $\eta = \frac{1}{108} \epsilon^2 > 0$. By 465C(a-ii) and 465Db, the \mathfrak{T}_p -closure \bar{A} of A in \mathbb{R}^X is a stable \mathfrak{T}_p -compact set of measurable functions, and by 465G it is \mathfrak{T}_m -compact; because A is uniformly bounded, it must be totally bounded for the pseudometric induced by $\|\cdot\|_1$. So there are $f_0, \dots, f_m \in A$ such that for every $f \in A$ there is an $i \leq m$ such that $\|f - f_i\|_1 \leq \eta$. Let T_0 be the finite subalgebra of Σ generated by the sets $\{x : j\eta \leq f_i(x) < (j+1)\eta\}$ for $i \leq m$ and $j \leq \frac{1}{\eta}$. Then T_0 may have negligible atoms, but if we absorb these into non-negligible atoms we get a finite subalgebra T of Σ such that $|f_i(x) - \mathbb{E}(f_i|T)(x)| \leq \eta$ for almost every $x \in X$, every $i \leq m$. (Because T is a finite algebra with non-negligible atoms, two T -measurable functions which are equal almost everywhere must be identical, and we have unique conditional expectations with respect to T .) Since $\|\mathbb{E}(f|T) - \mathbb{E}(g|T)\|_1 \leq \|f - g\|_1$ for all integrable functions f and g (242Je), $\|f - \mathbb{E}(f|T)\|_1 \leq 3\eta$ for every $f \in A$.

(β) Set $A' = \{f - \mathbb{E}(f|T) : f \in A\}$. Then A' is stable. **P** Suppose that $\mu E > 0$ and $\alpha < \beta$. The set $B = \{\mathbb{E}(f|T) : f \in A\}$ is a uniformly bounded subset of a finite-dimensional space of functions, so is

$\|\cdot\|_\infty$ -compact. So there are $g_0, \dots, g_r \in B$ such that $B \subseteq \bigcup_{i \leq r} \{g : \|g - g_i\|_\infty \leq \frac{1}{3}(\beta - \alpha)\}$. By 465C(a-iv) and 465C(b-ii), $C = \bigcup_{i \leq r} A - g_i$ is stable. So there is a $k \geq 1$ such that

$$(\mu^{2k})^* D_k(C, E, \frac{2}{3}\alpha + \frac{1}{3}\beta, \frac{1}{3}\alpha + \frac{2}{3}\beta) < (\mu E)^{2k}.$$

But for every $g \in A'$ there is an $h \in C$ such that $\|g - h\|_\infty \leq \frac{1}{3}(\beta - \alpha)$, so

$$D_k(A', E, \alpha, \beta) \subseteq D_k(C, E, \frac{2}{3}\alpha + \frac{1}{3}\beta, \frac{1}{3}\alpha + \frac{2}{3}\beta), \quad (\mu^{2k})^* D_k(A', E, \alpha, \beta) < (\mu E)^{2k}.$$

As E, α and β are arbitrary, A' is stable. **Q**

(γ) By 465C(b-iv),

$$A'' = \{g^+ : g \in A'\} \cup \{g^- : g \in A'\}$$

is stable. By 465K there are an $n \geq 1$, a $W \in \widehat{\bigotimes}_n \Sigma$ and a $\gamma > \mu^n W$ such that $\int h d\nu \leq 3\sqrt{3}\eta = \frac{1}{2}\epsilon$ whenever $h \in A''$ and ν is a probability measure on X , with domain including Σ , such that $\nu^n W \leq \gamma$. So $\int |g| d\nu \leq \epsilon$ whenever $g \in A'$ and ν is such a measure. If $w \in X^{\mathbb{N}}$ and $k \in \mathbb{N}$, ν_{wk} is a probability measure on X ; set $q_k(w) = \nu_{wk}^n(W)$. Applying 465H to χW , we see that $\lim_{k \rightarrow \infty} q_k(w) = \mu^n W$ for almost every $w \in X^{\mathbb{N}}$. Also, because $W \in \widehat{\bigotimes}_n \Sigma$, every q_k is measurable, by 465I.

Set $h_k(w) = 1$ if $q_k(w) > \gamma$, ϵ if $q_k(w) \leq \gamma$. Then every h_k is measurable and $\lim_{k \rightarrow \infty} h_k(w) = \epsilon$ for almost every w .

For any $w \in X^{\mathbb{N}}$ and any $f \in A, g = f - \mathbb{E}(f|\mathbb{T}) \in A'$ and $\|g\|_\infty \leq 1$. So either $h_k(w) = 1$ and certainly $\int |g| d\nu_{wk} \leq h_k(w)$, or $h_k(w) = \epsilon$, $\nu_{wk}^n(W) \leq \gamma$ and $\int |g| d\nu_{wk} \leq \epsilon$. Thus we have

$$\frac{1}{k} \sum_{i=0}^{k-1} |f(w(i)) - \mathbb{E}(f|\mathbb{T})(w(i))| = \int |f - \mathbb{E}(f|\mathbb{T})| d\nu_{wk} \leq h_k(w)$$

for every $w \in X^{\mathbb{N}}$ and every $f \in A$, as required by (iii).

(b)(iii) \Rightarrow (ii)&(v) Set

$$g_k(w) = \sup_{f \in A} \left| \frac{1}{k} \sum_{i=0}^{k-1} f(w(i)) - \int f d\mu \right|,$$

$$g'_{kl}(w) = \sup_{f \in A} \left| \frac{1}{k} \sum_{i=0}^{k-1} f(w(i)) - \frac{1}{l} \sum_{i=0}^{l-1} f(w(i)) \right|,$$

for $w \in X^{\mathbb{N}}$ and $k, l \geq 1$. Let $\epsilon > 0$. Let \mathbb{T} and $\langle h_k \rangle_{k \geq 1}$ be as in (iii); as $A \subseteq [0, 1]^X$, we can suppose that $h_k(w) \leq 2$ for every k and w . Let E_0, \dots, E_r be the atoms of \mathbb{T} . For $w \in X^{\mathbb{N}}, k \geq 1, j \leq r$ set $q_{kj}(w) = |\mu E_j - \nu_{wk} E_j|$. Then for any $f \in A, \mathbb{E}(f|\mathbb{T})$ is expressible as $\sum_{j=0}^r \alpha_j \chi E_j$ where $\alpha_j \in [0, 1]$ for every j (remember that $A \subseteq [0, 1]^X$), so

$$\begin{aligned} \left| \frac{1}{k} \sum_{i=0}^{k-1} f(w(i)) - \int f d\mu \right| &= \left| \int f d\nu_{wk} - \int f d\mu \right| \\ &\leq \left| \int f d\nu_{wk} - \int \mathbb{E}(f|\mathbb{T}) d\nu_{wk} \right| + \left| \int \mathbb{E}(f|\mathbb{T}) d\nu_{wk} - \int \mathbb{E}(f|\mathbb{T}) d\mu \right| \\ &\leq \frac{1}{k} \sum_{i=0}^{k-1} |f(w(i)) - \mathbb{E}(f|\mathbb{T})(w(i))| + \sum_{j=0}^r \alpha_j |\nu_{wk} E_j - \mu E_j| \\ &\leq h_k(w) + \sum_{j=0}^r q_{kj}(w), \\ \left| \frac{1}{k} \sum_{i=0}^{k-1} f(w(i)) - \frac{1}{l} \sum_{i=0}^{l-1} f(w(i)) \right| \\ &\leq h_k(w) + \sum_{j=0}^r q_{kj}(w) + h_l(w) + \sum_{j=0}^r q_{lj}(w). \end{aligned}$$

Taking the supremum over f , we have

$$g_k(w) \leq h_k(w) + \sum_{j=0}^r q_{kj}(w),$$

$$g'_{kl}(w) \leq h_k(w) + \sum_{j=0}^r q_{kj}(w) + \sum_{j=0}^r q_{lj}(w) + h_l(w).$$

But, for each $j \leq r$, $\lim_{k \rightarrow \infty} q_{kj}(w) = 0$ for almost every w , by 273J applied to χE_j . So

$$\limsup_{k \rightarrow \infty} g_k(w) \leq \limsup_{k \rightarrow \infty} h_k(w) \leq \epsilon$$

for almost every w . At the same time,

$$\limsup_{k, l \rightarrow \infty} \int g'_{kl} \leq \limsup_{k, l \rightarrow \infty} \int h_k + \sum_{j=0}^r \int q_{kj} + \sum_{j=0}^r \int q_{lj} + \int h_l \leq 2\epsilon.$$

As ϵ is arbitrary, $\{w : \limsup_{k \rightarrow \infty} g_k(w) \geq 2^{-i}\}$ is negligible for every $i \in \mathbb{N}$, and $\lim_{k \rightarrow \infty} g_k = 0$ almost everywhere, as required, while equally $\lim_{k, l \rightarrow \infty} \int g'_{kl} = 0$. Thus (ii) and (v) are true.

(c)(ii) \Rightarrow (iv) is trivial, since

$$\left| \frac{1}{k} \sum_{i=0}^{k-1} f(w(i)) - \frac{1}{l} \sum_{i=0}^{l-1} f(w(i)) \right| \leq \left| \frac{1}{k} \sum_{i=0}^{k-1} f(w(i)) - \int f \right| + \left| \frac{1}{l} \sum_{i=0}^{l-1} f(w(i)) - \int f \right|.$$

(d) not-(i) \Rightarrow not-(iv) & not-(v) Now suppose that A is not stable.

(α) In this case, by 465L, there are measurable functions $h_0, h_1 : X \rightarrow [0, 1]$ such that $\int h_0 d\mu < \int h_1 d\mu$ and $(\mu^{2k})^* \tilde{D}_k = 1$ for every $k \in \mathbb{N}$, where

$$\tilde{D}_k = \bigcup_{f \in A} \{w : w \in X^{2k}, f(w(2i)) \leq h_0(w(2i)), \\ f(w(2i+1)) \geq h_1(w(2i+1)) \text{ for every } i < k\}.$$

(β) Set $\delta = \frac{1}{4} \int h_1 - h_0 > 0$. Let $k_0 \geq 1$ be so large that

$$\mu^k \{w : w \in X^k, |\int h_j - \frac{1}{k} \sum_{i=0}^{k-1} h_j(w(i))| \leq \delta\} \geq \frac{1}{2}$$

for both j and for every $k \geq k_0$ (273J or 465H). The point is that

$$\mu^k \{w : w \in X^k, |\int h_j - \frac{1}{k} \sum_{i=0}^{k-1} h_j(w(i))| \leq \delta\} \\ = \mu^{\mathbb{N}} \{w : w \in X^{\mathbb{N}}, |\int h_j - \frac{1}{k} \sum_{i=0}^{k-1} h_j(w(i))| \leq \delta\} \rightarrow 1 \text{ as } k \rightarrow \infty.$$

Let $\langle k_n \rangle_{n \geq 1}$ be such that $k_n \geq \frac{2}{\delta} \sum_{i=0}^{n-1} k_i$ for every $n \geq 1$. Since $\delta \leq \frac{1}{4}$, every k_n is at least as large as k_0 .

(γ) Set $m_n = 2 \sum_{i < n} k_i$ for each $n \in \mathbb{N}$. Then we have a measure space isomorphism $\phi : \prod_{n \in \mathbb{N}} X^{2k_n} \rightarrow X^{\mathbb{N}}$ defined by setting

$$\phi(w)(m_n + i) = w(n)(2i), \quad \phi(w)(m_n + k_n + i) = w(n)(2i + 1)$$

for $n \in \mathbb{N}$ and $i < k_n$. For each $n \in \mathbb{N}$, \tilde{D}_{k_n} has outer measure 1 in X^{2k_n} , so $\tilde{D} = \prod_{n \in \mathbb{N}} \tilde{D}_{k_n}$ has outer measure 1 in $\prod_{n \in \mathbb{N}} X^{2k_n}$, and $\phi[\tilde{D}]$ has outer measure 1 in $X^{\mathbb{N}}$. Note that $\phi[\tilde{D}]$ is just the set of $w \in X^{\mathbb{N}}$ such that, for every $n \in \mathbb{N}$, there is an $f \in A$ such that

$$f(w(i)) \leq h_0(w(i)) \text{ for } m_n \leq i < m_n + k_n, \\ f(w(i)) \geq h_1(w(i)) \text{ for } m_n + k_n \leq i < m_n + 2k_n = m_{n+1}.$$

If we set

$$V_{n_0} = \{w : w \in X^{\mathbb{N}}, |\frac{1}{k_n} \sum_{i=m_n}^{m_n+k_n-1} h_0(w(i)) - \int h_0| \leq \delta\},$$

$$V_{n1} = \{w : w \in X^{\mathbb{N}}, |\frac{1}{k_n} \sum_{i=m_n+k_n}^{m_n+2k_n-1} h_1(w(i)) - \int h_1| \leq \delta\},$$

every V_{n0} and V_{n1} has measure at least $\frac{1}{2}$, because $k_n \geq k_0$.

(δ) Now suppose that $n \in \mathbb{N}$ and $w \in V_{n0} \cap V_{n1} \cap \phi[\tilde{D}]$. Then

$$\sup_{f \in A} \left| \frac{1}{m_n+2k_n} \sum_{i=0}^{m_n+2k_n-1} f(w(i)) - \frac{1}{m_n+k_n} \sum_{i=0}^{m_n+k_n-1} f(w(i)) \right| \geq \frac{\delta}{(2+\delta)(1+\delta)}.$$

P Since $w \in \phi[\tilde{D}]$, there must be an $f \in A$ such that

$$\begin{aligned} f(w(i)) &\leq h_0(w(i)) \text{ for } m_n \leq i < m_n + k_n, \\ f(w(i)) &\geq h_1(w(i)) \text{ for } m_n + k_n \leq i < m_n + 2k_n = m_{n+1}. \end{aligned}$$

Set $s = \sum_{i=0}^{m_n-1} f(w(i))$, $t = \sum_{i=m_n}^{m_n+k_n-1} f(w(i))$ and $t' = \sum_{i=m_n+k_n}^{m_n+2k_n-1} f(w(i))$. Then

$$\begin{aligned} \frac{1}{m_n+2k_n} \sum_{i=0}^{m_n+2k_n-1} f(w(i)) - \frac{1}{m_n+k_n} \sum_{i=0}^{m_n+k_n-1} f(w(i)) &= \frac{s+t+t'}{m_n+2k_n} - \frac{s+t}{m_n+k_n} = \frac{(t'-t-s)k_n + t'm_n}{(m_n+2k_n)(m_n+k_n)} \\ &\geq \frac{(t'-t-m_n)k_n}{(m_n+2k_n)(m_n+k_n)} \geq \frac{t'-t-\delta k_n}{(2+\delta)(1+\delta)k_n} \end{aligned}$$

because $m_n \leq \delta k_n$, by the choice of k_n .

To estimate t and t' , we have

$$t = \sum_{i=m_n}^{m_n+k_n-1} f(w(i)) \leq \sum_{i=m_n}^{m_n+k_n-1} h_0(w(i)) \leq k_n(\int h_0 + \delta)$$

because $w \in V_{n0}$. On the other hand,

$$t' = \sum_{i=m_n+k_n}^{m_n+2k_n-1} f(w(i)) \geq \sum_{i=m_n+k_n}^{m_n+2k_n-1} h_1(w(i)) \geq k_n(\int h_1 - \delta)$$

because $w \in V_{n1}$. So

$$t' - t \geq k_n(\int h_1 - h_0 - 2\delta) = 2k_n\delta,$$

and

$$\begin{aligned} \frac{1}{m_n+2k_n} \sum_{i=0}^{m_n+2k_n-1} f(w(i)) - \frac{1}{m_n+k_n} \sum_{i=0}^{m_n+k_n-1} f(w(i)) &\geq \frac{2\delta k_n - \delta k_n}{(2+\delta)(1+\delta)k_n} = \frac{\delta}{(2+\delta)(1+\delta)}. \quad \mathbf{Q} \end{aligned}$$

(ϵ) The V_{n0} and V_{n1} are all independent. So $\mu^{\mathbb{N}}(V_{n0} \cap V_{n1}) \geq \frac{1}{4}$ for every n , and

$$W = \{w : w \in X^{\mathbb{N}}, w \in V_{n0} \cap V_{n1} \text{ for infinitely many } n\}$$

has measure 1 (by the Borel-Cantelli lemma, 273K, or otherwise). Accordingly $W \cap \phi[\tilde{D}]$ has outer measure 1 in $X^{\mathbb{N}}$. But if $w \in W \cap \phi[\tilde{D}]$, then (δ) tells us that

$$\limsup_{k,l \rightarrow \infty} \sup_{f \in A} \left| \frac{1}{k} \sum_{i=0}^{k-1} f(w(i)) - \frac{1}{l} \sum_{i=0}^{l-1} f(w(i)) \right| \geq \frac{\delta}{(2+\delta)(1+\delta)},$$

because there are infinitely many n such that $w \in V_{n0} \cap V_{n1} \cap \phi[\tilde{D}]$. So (iv) must be false.

(ζ) We see also that, for any $n \in \mathbb{N}$,

$$\overline{\int} g'_{m_n+k_n, m_n+2k_n} \geq \frac{\delta}{(2+\delta)(1+\delta)} \mu^{\mathbb{N}}(V_{n0} \cap V_{n1}) \geq \frac{\delta}{4(2+\delta)(1+\delta)},$$

writing

$$g'_{kl}(w) = \sup_{f \in A} \left| \frac{1}{k} \sum_{i=0}^{k-1} f(w(i)) - \frac{1}{l} \sum_{i=0}^{l-1} f(w(i)) \right|$$

for $k, l \in \mathbb{N}$. So $\limsup_{k,l \rightarrow \infty} \overline{\int} g'_{kl} > 0$, and (v) is false.

(e) Assembling these, we have

$$(i) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (iv) \Rightarrow (i), \quad (iii) \Rightarrow (v) \Rightarrow (i),$$

so the five assertions are equiveridical.

Remark If (X, Σ, μ) is a probability space, a set $A \subseteq \mathcal{L}^1(\mu)$ is a **Glivenko-Cantelli class** if $\sup_{f \in A} \left| \frac{1}{k} \sum_{i=0}^{k-1} f(w(i)) - \int f \right| \rightarrow 0$ as $k \rightarrow \infty$ for $\mu^{\mathbb{N}}$ -almost every $w \in X^{\mathbb{N}}$. Compare 273Xi.

465N Theorem Let (X, Σ, μ) be a semi-finite measure space.

(a) (TALAGRAN 84) Let $A \subseteq \mathbb{R}^X$ be a stable set. Suppose that there is a measurable function $g : X \rightarrow [0, \infty[$ such that $|f(x)| \leq g(x)$ whenever $x \in X$ and $f \in A$. Then the convex hull $\Gamma(A)$ of A in \mathbb{R}^X is stable.

(b) If $A \subseteq \mathbb{R}^X$ is stable, then $|A| = \{|f| : f \in A\}$ is stable.

(c) Let $A, B \subseteq \mathbb{R}^X$ be two stable sets such that $\{f(x) : f \in A \cup B\}$ is bounded for every $x \in X$. Then $A + B = \{f_1 + f_2 : f_1 \in A, f_2 \in B\}$ is stable.

(d) Suppose that μ is complete and locally determined. Let $A \subseteq \mathbb{R}^X$ be a stable set such that $\{f(x) : f \in A\}$ is bounded for every $x \in X$. Then $\Gamma(A)$ is relatively compact in $\mathcal{L}^0(\Sigma)$ for the topology of pointwise convergence.

proof (a)(i) Consider first the case in which $\mu X = 1$ and $A \subseteq [-1, 1]^X$. In this case,

$$\sup_{f \in \Gamma(A)} \left| \frac{1}{k} \sum_{i=0}^{k-1} f(w(i)) - \int f \right| = \sup_{f \in A} \left| \frac{1}{k} \sum_{i=0}^{k-1} f(w(i)) - \int f \right|$$

for every $k \geq 1$ and $w \in X^{\mathbb{N}}$. So $\Gamma(A)$ satisfies condition (ii) of 465M whenever A does, and we have the result.

(ii) Now suppose just that $\mu X = 1$. Set $A' = \left\{ \frac{f}{g + \chi_X} : f \in A \right\}$. Then A' is stable (465C(a-v)), so $\Gamma(A')$ is stable, by (i), and $\Gamma(A) = \{f \times (g + \chi_X) : f \in A'\}$ is stable.

(iii) If $\mu X = 0$, the result is trivial. If $\mu X < \infty$, apply (ii) to a multiple of the measure μ . For the general case, write $A_E = \{f \upharpoonright E : f \in A\}$ for $E \subseteq X$. Then A_E is stable for the subspace measure on E , by 465Cm. It follows that $\Gamma(A_E)$ is stable whenever $\mu E < \infty$. But $\Gamma(A_E) = (\Gamma(A))_E$, so 465Cn tells us that $\Gamma(A)$ is stable.

(b)(i) I begin with a basic special case of (c). If $A, B \subseteq \mathbb{R}^X$ are stable and uniformly bounded, then $A + B$ is stable. **P** Putting 465C(b-ii), (a) of this theorem, 465C(a-iii) and 465C(a-i) together, we see that $A \cup B, \Gamma(A \cup B)$ and $A + B \subseteq 2\Gamma(A \cup B)$ are stable. **Q**

(ii) Adding this to 465C(b-iv), $|A| \subseteq \{f^+ : f \in A\} + \{f^- : f \in A\}$ is stable whenever $A \subseteq \mathbb{R}^X$ is stable and uniformly bounded.

(iii) For the general case, use 465C(c-v). For $n \in \mathbb{N}$, set $A_n = \{\text{med}(-n\chi_X, f, n\chi_X) : f \in A\}$,

$$|A|_n = \{\text{med}(-n\chi_X, f, n\chi_X) : f \in |A|\} = \{\text{med}(-n\chi_X, |f|, n\chi_X) : f \in A\} = |A_n|.$$

Now 465C(c-v) and (i)-(ii) above tell us that if A is stable, so are every A_n and every $|A_n| = |A|_n$, so that A is stable.

(c)(i) By 465Cn, it is enough to consider the case of totally finite μ , so let us suppose from now on that $\mu X < \infty$. We may also suppose that neither A nor B is empty; finally, by 465C(c-i), we can suppose that μ is complete, so that $A \cup B \subseteq \mathcal{L}^0(\Sigma)$ (465Da).

I introduce some temporary notation: if $E \subseteq X, k \geq 1, \epsilon > 0$ and $A \subseteq \mathbb{R}^X$, set

$$\tilde{D}_k(A, E, \epsilon) = \bigcup_{f \in A} \{u : u \in E^k, |f(u(i))| \geq \epsilon \text{ for } i < k\}.$$

(ii) We need to know that if $A \subseteq \mathbb{R}^X$ and every $f \in A$ is zero a.e., then A is stable iff whenever $E \in \Sigma$, $0 < \mu E < \infty$ and $\epsilon > 0$ there is a $k \geq 1$ such that $(\mu^k)^* \tilde{D}_k(A, E, \epsilon) < (\mu E)^k$.

P(α) If A is stable, then $|A|$ is stable, by (b), so if $0 < \mu E < \infty$ and $\epsilon > 0$ there is a $k \geq 1$ such that $(\mu^{2k})^* D_{2k}(|A|, E, 0, \epsilon) < (\mu E)^{2k}$. Let $W \in \widehat{\otimes}_{2k} \Sigma$ be such that $D_{2k}(|A|, E, 0, \epsilon) \subseteq W \subseteq E^{2k}$ and $\mu^{2k} W < (\mu E)^{2k}$. Because $(u, v) \mapsto u \# v$ is a measure space isomorphism,

$$\mu^{2k} W = \int \mu^k \{u : u \# v \in W\} \mu^k(dv),$$

so if we set $V = \{v : v \in E^k, \mu^k \{u : u \# v \in W\} = (\mu E)^k\}$ we must have $\mu^k V < (\mu E)^k$. If $v \in \tilde{D}_k(A, E, \epsilon)$, there is an $f \in A$ such that $|f(v(i))| \geq \epsilon$ for every $i < k$; now

$$\{u : u \# v \in W\} \supseteq \{u : u \in E^k, f(u(i)) = 0 \text{ for every } i < k\}$$

has measure $(\mu E)^k$, because $f = 0$ a.e. So $\tilde{D}_k(A, E, \epsilon) \subseteq V$ and $(\mu^k)^* \tilde{D}_k(A, E, \epsilon) < (\mu E)^k$. As E and ϵ are arbitrary, A satisfies the condition.

(β) Now suppose that A satisfies the condition. Take $E \in \Sigma$ such that $\mu E > 0$, and $\alpha < \beta$ in \mathbb{R} . If $\beta > 0$, set $\epsilon = \beta$; otherwise, set $\epsilon = -\alpha$. Then there is a $k \in \mathbb{N}$ such that $(\mu^k)^* \tilde{D}_k(A, E, \epsilon) < (\mu E)^k$. As $D_k(A, E, \alpha, \beta)$ is included in $\{(u, v) : u \in E^k, v \in \tilde{D}_k(A, E, \epsilon)\}$ (if $\beta > 0$) or $\{(u, v) : u \in \tilde{D}_k(A, E, \epsilon), v \in E^k\}$ (if $\beta \leq 0$), $(\mu^{2k})^* D_k(A, E, \alpha, \beta) < (\mu E)^{2k}$. As E, α and β are arbitrary, A is stable. **Q**

(iii) Suppose that A and B are stable sets such that $f = 0$ a.e. for every $f \in A \cup B$. Then $A + B$ is stable. **P** Set $A' = \{|f| \wedge \chi X : f \in A\}$, $B' = \{|f| \wedge \chi X : f \in B\}$. Then A' and B' are stable, so $A' + B'$ is stable, by (b)(i) above. But now observe that if $u \in \tilde{D}_k(A + B, E, \epsilon)$, where $E \subseteq X$, $\epsilon > 0$ and $k \geq 1$, then there are $f_1 \in A$, $f_2 \in B$ such that $|f_1(u(i)) + f_2(u(i))| \geq \epsilon$ for every $i < k$. In this case, setting $f'_j = |f_j| \wedge \chi X$ for both $j, g = f'_1 + f'_2$ belongs to $A' + B'$ and $g(u(i)) \geq \min(1, \epsilon)$ for every $i < k$. This shows that $\tilde{D}_k(A + B, E, \epsilon) \subseteq \tilde{D}_k(A' + B', E, \min(1, \epsilon))$. Also every function in either $A + B$ or $A' + B'$ is zero a.e. So (ii) tells us that $A + B$ also is stable. **Q**

(iv) Suppose that $A, B \subseteq \mathbb{R}^X$ are stable, that $|f| \leq \chi X$ for every $f \in A$, and that $g = 0$ a.e. for every $g \in B$. Then $A + B$ is stable. **P** For $g \in B$ set $g'(x) = \text{med}(-2, g(x), 2)$ for $x \in X$; set $B' = \{g' : g \in B\}$. Then B' is stable, by 465C(a-vi), and both A and B' are uniformly bounded, so $A + B'$ is stable. Take $E \in \Sigma$ such that $\mu E > 0$, and $\alpha < \beta$ in \mathbb{R} .

If $\beta > 1$, then, by (ii), there is a $k \geq 1$ such that $(\mu^k)^* \tilde{D}_k(B, E, \beta - 1) < (\mu E)^k$. Now if $w \in D_k(A + B, E, \alpha, \beta)$ there are $f \in A, g \in B$ such that $f(w(2i + 1)) + g(w(2i + 1)) \geq \beta$ for every $i < k$; accordingly $g(w(2i + 1)) \geq \beta - 1$ for $i < k$ and $w = u \# v$ for some $u \in E^k, v \in \tilde{D}_k(B, E, \beta - 1)$. So

$$(\mu^{2k})^* D_k(A + B, E, \alpha, \beta) \leq (\mu E)^k \cdot (\mu^k)^* \tilde{D}(B, E, \beta - 1) < (\mu E)^{2k}.$$

Similarly, if $\alpha < -1$, then

$$(\mu^{2k})^* D_k(A + B, E, \alpha, \beta) \leq (\mu E)^k \cdot (\mu^k)^* \tilde{D}(B, E, -1 - \alpha) < (\mu E)^{2k}$$

for some k .

On the other hand, if $-1 \leq \alpha < \beta \leq 1$, there is a $k \geq 1$ such that $(\mu^{2k})^* D_k(A + B', E, \alpha, \beta) < (\mu E)^{2k}$. If now $w \in D_k(A + B, E, \alpha, \beta)$, take $f \in A$ and $g \in B$ such that $f(w(2i)) + g(w(2i)) \leq \alpha$ and $f(w(2i + 1)) + g(w(2i + 1)) \geq \beta$ for $i < k$. In this case, for each $i < k$,

— either $g'(w(2i)) \leq g(w(2i))$ and $f(w(2i)) + g'(w(2i)) \leq \alpha$, or $g'(w(2i)) = -2$ and $f(w(2i)) + g'(w(2i)) \leq -1 \leq \alpha$,

— either $g'(w(2i + 1)) \geq g(w(2i + 1))$ and $f(w(2i + 1)) + g'(w(2i + 1)) \geq \beta$, or $g'(w(2i + 1)) = 2$ and $f(w(2i + 1)) + g'(w(2i + 1)) \geq 1 \geq \beta$.

So $w \in D_k(A + B', E, \alpha, \beta)$. Accordingly $(\mu^{2k})^* D_k(A + B, E, \alpha, \beta) < (\mu E)^{2k}$.

As E, α and β are arbitrary, $A + B$ is stable. **Q**

(v) Now suppose that $|f| \leq_{\text{a.e.}} \chi X$ for every $f \in A \cup B$. For $f \in A \cup B$ and $x \in X$, set $f_0(x) = \text{med}(-1, f(x), 1)$, $f_1(x) = \max(0, f(x) - 1)$ and $f_2(x) = \max(0, -1 - f(x))$; then $f = f_0 + f_1 - f_2$, $|f_0| \leq \chi X$ and f_1, f_2 are zero a.e. Also $A_0 = \{f_0 : f \in A\}$, $A_1 = \{f_1 : f \in A\}$, $A_2 = \{f_2 : f \in A\}$, $B_0 = \{f_0 : f \in B\}$, $B_1 = \{f_1 : f \in B\}$ and $B_2 = \{f_2 : f \in B\}$ are all stable, by 465C(a-vi). Accordingly $A_0 + B_0$ is stable, by (ii); by (iii), $A_1 - A_2 + B_1 - B_2$ is stable; by (iv),

$$A + B \subseteq A_0 + B_0 + A_1 - A_2 + B_1 - B_2$$

is stable.

(vi) Finally, turn to the hypothesis stated in the proposition: that A and B are stable and pointwise bounded. Let $h : X \rightarrow [0, \infty[$ be such that $|f(x)| \leq h(x)$ for every $f \in A \cup B$ and $x \in X$; note that I do *not* assume here that h is measurable. However, we are supposing that μ is totally finite, so there must be a sequence $\langle f_n \rangle_{n \in \mathbb{N}}$ in $A \cup B$ such that $|f| \leq_{\text{a.e.}} \sup_{n \in \mathbb{N}} |f_n|$ for every $f \in A \cup B$. **P** For each $q \in \mathbb{Q}$, choose a countable set $C_q \subseteq A \cup B$ such that $\{x : |f(x)| \geq q\} \setminus \bigcup_{g \in C_q} \{x : |g(x)| \geq q\}$ is negligible for any $f \in A \cup B$ (215B(iv)); let $\langle f_n \rangle_{n \in \mathbb{N}}$ run over $\bigcup_{q \in \mathbb{Q}} C_q$. **Q** Set $h_1 = \chi_X + \sup_{n \in \mathbb{N}} |f_n|$; then h_1 is finite-valued, strictly positive and measurable, and $|f| \leq_{\text{a.e.}} h_1$ for every $f \in A \cup B$. By 465C(a-v), $A_1 = \{f/h_1 : f \in A\}$ and $B_1 = \{f/h_1 : f \in B\}$ are stable; by (v) here, $A_1 + A_2$ is stable; by 465C(a-v) again, $A + B = \{g \times h_1 : g \in A_1 + A_2\}$ is stable. So we're done.

(d) Since A is pointwise bounded, the closure $\overline{\Gamma(A)}$ of $\Gamma(A)$ in \mathbb{R}^X for the topology of pointwise convergence is compact. **Z** Suppose, if possible, that there is a $g \in \overline{\Gamma(A)} \setminus \mathcal{L}^0(\Sigma)$. Then there must be a measurable set E of finite measure and $\alpha < \beta$ in \mathbb{R} such that $\mu^*P = \mu^*Q = \mu E > 0$, where

$$P = \{x : x \in E, g(x) \leq \alpha\}, \quad Q = \{x : x \in E, g(x) \geq \beta\}$$

by 413G again. Set $Y_n = \{x : x \in E, |f(x)| \leq n \text{ for every } f \in A\}$; then $\langle Y_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence with union E , so there is an $n \in \mathbb{N}$ such that $\mu^*(P \cap Y_n)$ and $\mu^*(Q \cap Y_n)$ are both at least $\frac{2}{3}\mu E$. Let F', F'' be measurable envelopes of $P \cap Y_n$ and $Q \cap Y_n$ respectively, and $Y = F' \cap F'' \cap Y_n$; then

$$\begin{aligned} 0 < \mu(F' \cap F'') &= \mu^*(F' \cap F'' \cap P \cap Y_n) \\ &= \mu^*(F' \cap F'' \cap Q \cap Y_n) = \mu^*Y = \mu^*(P \cap Y) = \mu^*(Q \cap Y). \end{aligned}$$

Let μ_Y be the subspace measure on Y and Σ_Y its domain, and consider the set $A_Y = \{f|_Y : f \in A\}$. With respect to the measure μ_Y , this is stable (465Cm). Also it is uniformly bounded. So $\Gamma(A_Y)$ is μ_Y -stable, by (a) of this theorem. As μ_Y is complete and totally finite, the closure $\overline{\Gamma(A_Y)}$ for the topology of pointwise convergence in \mathbb{R}^Y is included in $\mathcal{L}^0(\Sigma_Y)$ (465Da). Since $f \mapsto f|_Y : \mathbb{R}^X \rightarrow \mathbb{R}^Y$ is linear and continuous for the topologies of pointwise convergence, $g|_Y \in \overline{\Gamma(A_Y)}$ and $g|_Y$ is Σ_Y -measurable. But

$$\mu_Y^*(P \cap Y) = \mu^*(P \cap Y) = \mu^*(Q \cap Y) = \mu_Y^*(Q \cap Y) = \mu_Y Y \in]0, \infty[$$

(214Cd), so this is impossible. **X**

Thus $\overline{\Gamma(A)} \subseteq \mathcal{L}^0(\Sigma)$ and $\Gamma(A)$ is relatively compact.

465O Stable sets in L^0 The notion of ‘stability’ as defined in 465B is applicable only to true functions; in such examples as 465Xj, the irregularity of the set A is erased entirely if we look at its image in the space L^0 of equivalence classes of measurable functions. We do, however, have a corresponding concept for subsets of function spaces, which can be expressed in the language of §325. If $(\mathfrak{A}, \bar{\mu})$ is a semi-finite measure algebra, and $k \geq 1$, I write $(\widehat{\otimes}_k \mathfrak{A}, \bar{\mu}^k)$ for the localizable measure algebra free product of k copies of $(\mathfrak{A}, \bar{\mu})$, as described in 325H. If $Q \subseteq L^0(\mathfrak{A})$, $k \geq 1$, $a \in \mathfrak{A}$ has finite measure and $\alpha < \beta$ in \mathbb{R} , set

$$\begin{aligned} d_k(Q, a, \alpha, \beta) &= \sup_{v \in Q} ((a \cap [v \leq \alpha]) \otimes (a \cap [v \geq \beta]) \otimes \dots \\ &\quad \otimes (a \cap [v \leq \alpha]) \otimes (a \cap [v \geq \beta])) \end{aligned}$$

in $\widehat{\otimes}_{2k} \mathfrak{A}$, taking k repetitions of the formula $(a \cap [v \leq \alpha]) \otimes (a \cap [v \geq \beta])$ to match the corresponding formula

$$D_k(A, E, \alpha, \beta) = \bigcup_{f \in A} ((E \cap \{x : f(x) \leq \alpha\}) \times (E \cap \{x : f(x) \geq \beta\}))^k.$$

(Note that the supremum $\sup_{v \in Q} \dots$ is defined because $a^{\otimes 2k} = a \otimes \dots \otimes a$ has finite measure in the measure algebra $(\widehat{\otimes}_{2k} \mathfrak{A}, \bar{\mu}^{2k})$. Of course I mean to take $d_k(Q, a, \alpha, \beta) = 0$ if $Q = \emptyset$.) Now we can say that Q is **stable** if whenever $0 < \bar{\mu}a < \infty$ and $\alpha < \beta$ there is a $k \geq 1$ such that $\bar{\mu}^{2k}d_k(Q, a, \alpha, \beta) < (\bar{\mu}a)^{2k}$; that is, $d_k(Q, a, \alpha, \beta) \neq a \otimes \dots \otimes a$.

We have the following relationships between the two concepts of stability.

465P Theorem Let (X, Σ, μ) be a semi-finite measure space, with measure algebra $(\mathfrak{A}, \bar{\mu})$.

(a) Suppose that $A \subseteq \mathcal{L}^0(\Sigma)$ and that $Q = \{f^\bullet : f \in A\} \subseteq L^0(\mu)$, identified with $L^0 = L^0(\mathfrak{A})$ (364Ic). Then Q is stable in the sense of 465O iff every countable subset of A is stable in the sense of 465B.

(b) Suppose that μ is complete and strictly localizable and Q is a stable subset of $L^\infty(\mu)$, identified with $L^\infty(\mathfrak{A})$ (363I). Then there is a stable set $B \subseteq \mathcal{L}^\infty(\Sigma)$ such that $Q = \{f^\bullet : f \in B\}$.

proof (a)(i) Suppose that all countable subsets of A are stable, and take $a \in \mathfrak{A}$ such that $0 < \bar{\mu}a < \infty$ and $\alpha < \beta$ in \mathbb{R} . For each $k \in \mathbb{N}$ there is a countable set $Q_k \subseteq Q$ such that $d_k(Q_k, a, \alpha, \beta) = d_k(Q, a, \alpha, \beta)$, because $a^{\otimes 2k}$ has finite measure in $\widehat{\otimes}_{2k} \mathfrak{A}$. Now there is a countable set $A' \subseteq A$ such that $\{f^\bullet : f \in A'\} = \bigcup_{k \in \mathbb{N}} Q_k$. Let $E \in \Sigma$ be such that $E^\bullet = a$. As $\mu E = \bar{\mu}a \in]0, \infty[$ and A' is stable, there is a $k \geq 1$ such that $(\mu^{2k})^* D_k(A', E, \alpha, \beta) < (\mu E)^{2k}$. Because A' is countable, $D_k(A', E, \alpha, \beta)$ is measurable. But 325He tells us that we have an order-continuous measure-preserving Boolean homomorphism π from the measure algebra of μ^{2k} to $\widehat{\otimes}_{2k} \mathfrak{A}$, such that $\pi(\prod_{i < 2k} F_i)^\bullet = F_0^\bullet \otimes \dots \otimes F_{2k-1}^\bullet$ for all $F_0, \dots, F_{2k-1} \in \Sigma$; accordingly

$$\begin{aligned} \bar{\mu}^{2k} d_k(Q, a, \alpha, \beta) &= \bar{\mu}^{2k} d_k(Q_k, a, \alpha, \beta) \leq \bar{\mu}^{2k} d_k\left(\bigcup_{i \in \mathbb{N}} Q_i, a, \alpha, \beta\right) \\ &= \bar{\mu}^{2k} (\pi D_k(A', E, \alpha, \beta))^\bullet = \mu^{2k} D_k(A', E, \alpha, \beta) \\ &< (\mu E)^{2k} = \bar{\mu}^{2k} a^{\otimes 2k}. \end{aligned}$$

As a, α and β are arbitrary, Q is stable.

(ii) Now suppose that Q is stable and that A' is any countable subset of A . Take $E \in \Sigma$ such that $0 < \mu E < \infty$, and $\alpha < \beta$ in \mathbb{R} . Set $a = E^\bullet \in \mathfrak{A}$. This time, writing Q' for $\{f^\bullet : f \in A'\}$, we have

$$\pi D_k(A', E, \alpha, \beta)^\bullet = d_k(Q', a, \alpha, \beta) \subseteq d_k(Q, a, \alpha, \beta)$$

for every $k \geq 1$. There is some k such that $d_k(Q, a, \alpha, \beta) \neq a^{\otimes 2k}$, and in this case $\mu^{2k} D_k(A', E, \alpha, \beta) < (\mu E)^{2k}$; as E, α and β are arbitrary, A' is stable.

(b)(i) If $\mu X = 0$ this is trivial; suppose that $\mu X > 0$. Let $\langle E_i \rangle_{i \in I}$ be a decomposition of X into sets of finite measure. Amalgamating any negligible E_i into other non-negligible ones, we may suppose that $\mu E_i > 0$ for each i . Writing μ_i for the subspace measure on E_i , we have a consistent lifting ϕ_i for μ_i (346J). Set $\phi E = \bigcup_{i \in I} \phi_i(E \cap X_i)$ for $E \in \Sigma$; then ϕ is a lifting for μ . Let θ be the corresponding lifting from \mathfrak{A} to Σ (341B) and $T : L^\infty(\mathfrak{A}) \rightarrow \mathcal{L}^\infty(\Sigma)$ the associated linear operator, defined by saying that $T(\chi a) = \chi(\theta a)$ for every $a \in \mathfrak{A}$ (363F). Since $\theta(a)^\bullet = a$ for every $a \in \mathfrak{A}$, $(Tv)^\bullet = v$ for every $v \in L^\infty = L^\infty(\mathfrak{A})$, identified with $L^\infty(\mu)$ (cf. 363Xe).

(ii) We need to know that if $v \in L^\infty$ and $\alpha < \alpha'$, then $\{x : (Tv)(x) \leq \alpha\} \subseteq \phi(\{x : (Tv)(x) \leq \alpha'\})$. **P** Let $v' \in S(\mathfrak{A})$ be such that $\|v - v'\|_\infty \leq \frac{1}{2}(\alpha' - \alpha)$ (363C), and set $\gamma = \frac{1}{2}(\alpha + \alpha')$. Express v' as $\sum_{i=0}^n \alpha_i \chi a_i$ where $a_0, \dots, a_n \in \mathfrak{A}$ are disjoint. Then $Tv' = \sum_{i=0}^n \alpha_i \chi(\theta a_i)$. Now

$$\|Tv - Tv'\|_\infty \leq \|v - v'\|_\infty \leq \gamma - \alpha = \alpha' - \gamma,$$

so

$$\begin{aligned} \{x : (Tv)(x) \leq \alpha\} &\subseteq \{x : (Tv')(x) \leq \gamma\} \\ &= \bigcup \{\theta a_i : i \leq n, \alpha_i \leq \gamma\} = \phi\left(\bigcup \{\theta a_i : i \leq n, \alpha_i \leq \gamma\}\right) \end{aligned}$$

(because $\phi(\theta a) = \theta a$ for every $a \in \mathfrak{A}$)

$$= \phi(\{x : (Tv')(x) \leq \gamma\}) \subseteq \phi(\{x : (Tv)(x) \leq \alpha'\}),$$

as claimed. **Q**

Similarly, if $\beta' < \beta$ then $\{x : (Tv)(x) \geq \beta\} \subseteq \phi(\{x : (Tv)(x) \geq \beta'\})$.

(iii) Set $B = T[Q]$, so that $B \subseteq \mathcal{L}^\infty(\Sigma)$ and $Q = \{f^\bullet : f \in B\}$. Then B is stable. **P** Let $E \in \Sigma$ be such that $0 < \mu E < \infty$, and $\alpha < \beta$. Let $i \in I$ be such that $\mu(E \cap E_i) > 0$, and $\alpha', \beta' \in \mathbb{R}$ such that $\alpha < \alpha' < \beta' < \beta$. Setting $a = (E \cap E_i)^\bullet$, we have $0 < \bar{\mu}a < \infty$, so there is some $k \in \mathbb{N}$ such that $d_k(Q, a, \alpha', \beta') \neq a^{\otimes 2k}$. Let π be the measure-preserving Boolean homomorphism from the measure algebra

of μ^{2k} to $\widehat{\bigotimes}_{2k} \mathfrak{A}$ described in part (a) of this proof; as noted in 325He, the present context is enough to ensure that π is an isomorphism. So there is a $W \in \text{dom } \mu^{2k}$ such that $\pi W^\bullet = d_k(Q, a, \alpha', \beta')$; since

$$d_k(Q, a, \alpha', \beta') \subseteq a^{\otimes 2k} \subseteq (E_i^{2k})^\bullet,$$

we may suppose that $W \subseteq E_i^{2k}$. If $f \in B$, then

$$\pi D_k(\{f\}, E, \alpha', \beta')^\bullet = d_k(\{f^\bullet\}, a, \alpha', \beta') \subseteq d_k(Q, a, \alpha', \beta'),$$

so $D_k(\{f\}, E, \alpha', \beta') \setminus W$ is negligible.

At this point, recall that ϕ_i was supposed to be a consistent lifting for μ_i . So we have a lifting ϕ' of μ_i^{2k} such that $\phi'(\prod_{j < 2k} F_j) = \prod_{j < 2k} \phi_i F_j$ for all $F_0, \dots, F_{2k-1} \in \Sigma_i$. In particular, if $f \in B$ and we set

$$F_{2j} = \{x : x \in E \cap E_i, f(x) \leq \alpha\}, \quad F_{2j+1} = \{x : x \in E \cap E_i, f(x) \geq \beta\},$$

$$F'_{2j} = \{x : x \in E \cap E_i, f(x) \leq \alpha'\}, \quad F'_{2j+1} = \{x : x \in E \cap E_i, f(x) \geq \beta'\},$$

for $j < k$, we shall have

$$\prod_{j < 2k} F_j \subseteq \prod_{j < 2k} \phi F'_j$$

(by (ii) above, because $f = Tv$ for some v)

$$= \prod_{j < 2k} \phi_i F'_j = \phi'(\prod_{j < 2k} F'_j) \subseteq \phi'W$$

because $\prod_{j < 2k} F'_j = D_k(\{f\}, E, \alpha', \beta')$. As f is arbitrary, $D_k(B, E \cap E_i, \alpha, \beta) \subseteq \phi'W$. But now

$$(\mu^{2k})^* D_k(B, E \cap E_i, \alpha, \beta) \leq \mu^{2k}(\phi'W) = \mu_i^{2k}(\phi'W)$$

(251W1)

$$\begin{aligned} &= \mu_i^{2k}W = \mu^{2k}W = \bar{\mu}^{2k}d_k(Q, a, \alpha', \beta') \\ &< \bar{\mu}^{2k}(a^{\otimes 2k}) = \mu(E \cap E_i)^{2k}. \end{aligned}$$

As usual, it follows that $(\mu^{2k})^* D_k(B, E, \alpha, \beta) < (\mu E)^{2k}$; as E, α and β are arbitrary, B is stable, as claimed.

Q

465Q Remarks Using 465Pa, we can work through the first part of this section to get a list of properties of stable subsets of L^0 . For instance, the convex hull of an order-bounded stable set in L^0 is stable, as in 465Na. It is harder to relate such results as 465M to the idea of stability in L^0 , but the argument of 465Pb gives a line to follow: if (X, Σ, μ) is complete and strictly localizable, there is a linear operator $T : L^\infty(\mu) \rightarrow \mathcal{L}^\infty(\Sigma)$, defined from a lifting, such that, for $Q \subseteq L^\infty$, $T[Q]$ is stable iff Q is stable. So when μ is a complete probability measure, we can look at the averages $\psi_{wk}(v) = \frac{1}{k} \sum_{i=0}^{k-1} (Tv)(w(i))$ for $v \in Q$, $w \in X^{\mathbb{N}}$ to devise criteria for stability of Q in terms of the linear functionals ψ_{wk} .

Working in L^1 , however, we can seek results of a different type, as follows.

465R Theorem (TALAGRAN 84) Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be measure algebras, and $T : L^1(\mathfrak{A}, \bar{\mu}) \rightarrow L^1(\mathfrak{B}, \bar{\nu})$ a bounded linear operator. If Q is stable and order-bounded in $L^1(\mathfrak{A}, \bar{\mu})$, then $T[Q] \subseteq L^1(\mathfrak{B}, \bar{\nu})$ is stable.

proof (a) To begin with (down to (d) below) let us suppose that

$(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ are the measure algebras of measure spaces (X, Σ, μ) and (Y, \mathfrak{T}, ν) , so that we can identify $L^1_{\bar{\mu}}, L^1_{\bar{\nu}}$ with $L^1(\mu), L^1(\nu)$ (365B),

Q is countable,

so that Q can be expressed as $\{f^\bullet : f \in A\}$, where $A \subseteq \mathcal{L}^0(\Sigma)$ is countable and stable (465P),

T is positive,

μ and ν are totally finite,
 $T(\chi X^\bullet) = \chi Y^\bullet$,
 $Q \subseteq L^\infty(\mathfrak{A})$ is $\|\cdot\|_\infty$ -bounded,

so that we may take $A \subseteq \mathcal{L}^\infty$ to be $\|\cdot\|_\infty$ -bounded,

$\nu Y = 1$,
 $\mu X = 1$,
and that $\|T\| \leq 1$.

(b) The idea of the argument is that for any $n \geq 1$ we have a positive linear operator $U_n : L^1(\mu^n) \rightarrow L^1(\nu^n)$ defined as follows.

If $f_0, \dots, f_{n-1} \in L^1(\mu)$, set $(f_0 \otimes \dots \otimes f_{n-1})(w) = \prod_{i=0}^{n-1} f_i(w(i))$ whenever $w \in \prod_{i < n} \text{dom } f_i$. Now we can define $u_0 \otimes u_1 \otimes \dots \otimes u_{n-1} \in L^1(\mu^n)$, for $u_0, \dots, u_{n-1} \in L^1(\mu)$, by saying that $f_0^\bullet \otimes \dots \otimes f_{n-1}^\bullet = (f_0 \otimes \dots \otimes f_{n-1})^\bullet$ for all $f_0, \dots, f_{n-1} \in L^1(\mu)$, as in 253E.

Define the operators U_n inductively. $U_1 = T$. Given that $U_n : L^1(\mu^n) \rightarrow L^1(\nu^n)$ is a positive linear operator, then we have a bilinear operator $\psi : L^1(\mu^n) \times L^1(\mu) \rightarrow L^1(\nu^{n+1})$ defined by saying that $\psi(q, u) = U_n q \otimes T u$ for $q \in L^1(\mu^n)$, $u \in L^1(\mu)$, where $\otimes : L^1(\nu^n) \times L^1(\nu) \rightarrow L^1(\nu^n \times \nu) \cong L^1(\nu^{n+1})$ is the operator of 253E. By 253F, there is a (unique) bounded linear operator $U_{n+1} : L^1(\mu^{n+1}) \rightarrow L^1(\nu^{n+1})$ such that $U_{n+1}(q \otimes u) = \psi(q, u)$ for all $q \in L^1(\mu^n)$, $u \in L^1(\mu)$. To see that U_{n+1} is positive, use 253Gc. (Remember that we are supposing that T is positive.) Continue.

Now it is easy to check that

$$U_n(u_0 \otimes \dots \otimes u_{n-1}) = T u_0 \otimes \dots \otimes T u_{n-1}$$

for all $u_0, \dots, u_{n-1} \in L^1(\mu)$. Moreover, $\|U_{n+1}\| \leq \|U_n\| \|T\|$ for every n (see 253F), so $\|U_n\| \leq 1$ for every n .

For $i < n \in \mathbb{N}$, we have a natural operator $R_{ni} : L^1(\mu) \rightarrow L^1(\mu^n)$, defined by saying that $R_{ni} f^\bullet = (f \pi_{ni})^\bullet$ for every $f \in L^1(\mu)$, where $\pi_{ni}(w) = w(i)$ for $w \in X^n$. Similarly, we have an operator $S_{ni} : L^1(\nu) \rightarrow L^1(\nu^n)$. Observe that

$$R_{ni} u = e \otimes \dots \otimes e \otimes u \otimes e \otimes \dots \otimes e$$

where $e = \chi X^\bullet$ and the u is put in the position corresponding to the coordinate i . Since $T e = (\chi Y)^\bullet = e'$ say,

$$U_n R_{ni} u = e' \otimes \dots \otimes T u \otimes \dots \otimes e' = S_{ni} T u$$

for every $u \in L^1(\mu)$.

(c) Let $B \subseteq \mathcal{L}^\infty(\mathbb{T})$ be a countable $\|\cdot\|_\infty$ -bounded set such that $T[Q] = \{g^\bullet : g \in B\}$. ($T[Q]$ is $\|\cdot\|_\infty$ -bounded because T is positive and $T(\chi X)^\bullet = (\chi Y)^\bullet$.) I seek to show that B is stable by using the criterion 465M(v). Let $\epsilon > 0$. Then there is an $m \geq 1$ such that $\int f_{kl}(w) \mu^{\mathbb{N}}(dw) \leq \epsilon$ for any $k, l \geq m$, writing

$$f_{kl}(w) = \sup_{f \in A} \left| \frac{1}{k} \sum_{i=0}^{k-1} f(w(i)) - \frac{1}{l} \sum_{i=0}^{l-1} f(w(i)) \right|$$

for $w \in X^{\mathbb{N}}$; note that f_{kl} is measurable because A is countable.

Take any $k, l \geq m$ and consider

$$g_{kl}(z) = \sup_{g \in B} \left| \frac{1}{k} \sum_{i=0}^{k-1} g(z(i)) - \frac{1}{l} \sum_{i=0}^{l-1} g(z(i)) \right|$$

for $z \in Y^{\mathbb{N}}$. I claim that $\int g_{kl} d\nu^{\mathbb{N}} \leq \epsilon$. **P** Set $n = \max(k, l)$. Then $\int g_{kl} d\nu^{\mathbb{N}} = \int \tilde{g} d\nu^n$, where

$$\tilde{g}(z) = \sup_{g \in B} \left| \frac{1}{k} \sum_{i=0}^{k-1} g(z(i)) - \frac{1}{l} \sum_{i=0}^{l-1} g(z(i)) \right|$$

for $z \in Y^n$. If we look at \tilde{g}^\bullet in $L^1(\nu^n)$, we see that it is

$$\sup_{g \in B} \left| \frac{1}{k} \sum_{i=0}^{k-1} S_{ni} g^\bullet - \frac{1}{l} \sum_{i=0}^{l-1} S_{ni} g^\bullet \right|$$

where $S_{ni} : L^1(\nu) \rightarrow L^1(\nu^n)$ is defined in (b) above. Thus

$$\tilde{g}^\bullet = \sup_{v \in T[Q]} \left| \frac{1}{k} \sum_{i=0}^{k-1} S_{ni}v - \frac{1}{l} \sum_{i=0}^{l-1} S_{ni}v \right|.$$

Similarly, setting

$$\tilde{f}(w) = \sup_{f \in A} \left| \frac{1}{k} \sum_{i=0}^{k-1} f(w(i)) - \frac{1}{l} \sum_{i=0}^{l-1} f(w(i)) \right|$$

for $w \in X^n$,

$$\tilde{f}^\bullet = \sup_{u \in Q} \left| \frac{1}{k} \sum_{i=0}^{k-1} R_{ni}u - \frac{1}{l} \sum_{i=0}^{l-1} R_{ni}u \right|.$$

Now consider $U_n \tilde{f}^\bullet$. For any $v \in T[Q]$, we can express v as Tu where $u \in Q$, so

$$\begin{aligned} \left| \frac{1}{k} \sum_{i=0}^{k-1} S_{ni}v - \frac{1}{l} \sum_{i=0}^{l-1} S_{ni}v \right| &= \left| \frac{1}{k} \sum_{i=0}^{k-1} S_{ni}Tu - \frac{1}{l} \sum_{i=0}^{l-1} S_{ni}Tu \right| \\ &= \left| \frac{1}{k} \sum_{i=0}^{k-1} U_n R_{ni}u - \frac{1}{l} \sum_{i=0}^{l-1} U_n R_{ni}u \right| \end{aligned}$$

(because $U_n R_{ni} = S_{ni}T$, as noted in (b) above)

$$\begin{aligned} &= \left| U_n \left(\frac{1}{k} \sum_{i=0}^{k-1} R_{ni}u - \frac{1}{l} \sum_{i=0}^{l-1} R_{ni}u \right) \right| \\ &\leq U_n \left(\left| \frac{1}{k} \sum_{i=0}^{k-1} R_{ni}u - \frac{1}{l} \sum_{i=0}^{l-1} R_{ni}u \right| \right) \end{aligned}$$

(because U_n is positive)

$$\leq U_n \tilde{f}^\bullet.$$

As v is arbitrary, $\tilde{g}^\bullet \leq U_n \tilde{f}^\bullet$, and

$$\int g_{kl} d\nu^{\mathbb{N}} = \int \tilde{g} d\nu^n = \|\tilde{g}^\bullet\|_1 \leq \|U_n \tilde{f}^\bullet\|_1 \leq \|\tilde{f}^\bullet\|_1$$

(because $\|U_n\| \leq 1$)

$$\leq \epsilon$$

because $k, l \geq m$. **Q**

(d) As ϵ is arbitrary, B satisfies the criterion 465M(v), and is stable. So $T[Q]$ is stable, by 465Pa in the other direction.

(e) Now let us seek to unwind the list of special assumptions used in the argument above. Suppose we drop the last two, and assume only that

- ($\mathfrak{A}, \bar{\mu}$) and ($\mathfrak{B}, \bar{\nu}$) are the measure algebras of measure spaces (X, Σ, μ) and (Y, T, ν) ,
- Q is countable,
- T is positive,
- μ and ν are totally finite,
- $T(\chi X^\bullet) = \chi Y^\bullet$,
- $Q \subseteq L^\infty(\mathfrak{A})$ is $\|\cdot\|_\infty$ -bounded,
- $\nu Y = 1$.

Then $T[Q]$ is stable. **P** Define $\mu_1 : \Sigma \rightarrow [0, \infty[$ by setting $\mu_1 E = \int T(\chi E^\bullet)$ for every $E \in \Sigma$. Then μ_1 is countably additive because T is (sequentially) order-continuous (355Ka). If $\mu E = 0$ then $\chi E^\bullet = 0$ in $L^1(\mu)$ and $\mu_1 E = 0$, so μ_1 is truly continuous with respect to μ (232Bb) and has a Radon-Nikodým derivative (232E). By 465C(c-ii), A is stable with respect to μ_1 , while $\mu_1 X = \nu Y = 1$, because $T(\chi X)^\bullet = \chi Y^\bullet$.

Let $(\mathfrak{A}_1, \bar{\mu}_1)$ be the measure algebra of μ_1 . If $E \in \Sigma$ and $\mu_1 E = 0$, then $T(\chi E^\bullet) = 0$. Accordingly we can define an additive function $\theta : \mathfrak{A}_1 \rightarrow L^1(\nu)$ by setting $\theta E^\bullet = T(\chi E^\bullet)$ for every $E \in \Sigma$. (Note that the two \bullet 's here must be interpreted differently. In the formula θE^\bullet , the equivalence class E^\bullet is to be taken in \mathfrak{A}_1 . In the formula $\chi E^\bullet = (\chi E)^\bullet$, the equivalence class is to be taken in $L^0(\mu)$. In the rest of this proof I will pass over such points without comment; I hope the context will always make it clear how each \bullet is to be read.) Because T is positive, θ is non-negative, and by the definition of μ_1 we have $\|\theta a\|_1 = \bar{\mu}_1 a$ for every $a \in \mathfrak{A}_1$. So we have a positive linear operator $T_1 : L^1(\mathfrak{A}_1, \bar{\mu}_1) \rightarrow L^1(\nu)$ defined by setting

$$T_1(\chi E^\bullet) = \theta E^\bullet = T(\chi E^\bullet)$$

for every $E \in \Sigma$ (365J). As $\|T_1(\chi E^\bullet)\|_1 = \mu_1 E$ for every $E \in \Sigma$, $\|T_1\| = 1$ (365I(ii- β)).

If $f : X \rightarrow \mathbb{R}$ is simple (that is, a linear combination of indicator functions of sets in Σ), then $T_1 f^\bullet = T f^\bullet$. So this is also true for every $f \in \mathcal{L}^\infty(\Sigma)$; in particular, it is true for every $f \in A$, so that $T[Q] = T_1[Q_1]$, where $Q_1 = \{f^\bullet : f \in A\} \subseteq L^1(\mu_1)$. But μ_1 , Q_1 and T_1 satisfy all the conditions of (a), so (b)-(d) tell us that $T_1[Q_1]$ is stable, and $T[Q]$ is stable, as required. **Q**

(f) The next step is to drop the condition ' $\nu Y = 1$ '. But this is elementary, since we are still assuming that ν is totally finite, and multiplying ν by a non-zero scalar doesn't change $L^0(\nu)$ or the stability of any of its subsets, while the case $\nu Y = 0$ is trivial. So we conclude that if

- ($\mathfrak{A}, \bar{\mu}$) and ($\mathfrak{B}, \bar{\nu}$) are the measure algebras of measure spaces (X, Σ, μ) and (Y, T, ν) ,
- Q is countable,
- T is positive,
- μ and ν are totally finite,
- $T(\chi X^\bullet) = \chi Y^\bullet$,
- $Q \subseteq L^\infty(\mathfrak{A})$ is $\|\cdot\|_\infty$ -bounded,

then $T[Q]$ is stable.

(g) We can now attack what remains. We find that if

- ($\mathfrak{A}, \bar{\mu}$) and ($\mathfrak{B}, \bar{\nu}$) are the measure algebras of measure spaces (X, Σ, μ) and (Y, T, ν) ,
- Q is countable,
- T is positive,

then $T[Q]$ is stable. **P** At this point recall that we are supposing that Q is order-bounded. Let $u_0 \in L^1(\mu)$ be such that $|u| \leq u_0$ for every $u \in Q$; let $f_0 \in \mathcal{L}^0(\Sigma)^+$ be such that $f_0^\bullet = u_0$. Setting $A' = \{\text{med}(-f_0, f, f_0) : f \in A\}$, A' is still stable, because its image in $L^1(\mu)$ is still Q , or otherwise. Set $v_0 = T u_0$, and let $g_0 \in \mathcal{L}^0(\mathsf{T})^+$ be such that $g_0^\bullet = v_0$. Because T is positive, $|T u| \leq T|u| \leq v_0$ for every $u \in Q$. So we can represent $T[Q]$ as $\{g^\bullet : g \in B\}$, where $B \subseteq \mathcal{L}^0(\mathsf{T})$ is a countable set and $|g| \leq g_0$ for every $g \in B$. Set $F_0 = \{y : g_0(y) \neq 0\}$.

Define measures μ_1, ν_1 by setting $\mu_1 E = \int_E f_0 d\mu$ for $E \in \Sigma$, $\nu_1 F = \int_F g_0 d\nu$ for $F \in \mathsf{T}$. Then both μ_1 and ν_1 are totally finite. By 465C(c-ii), A' is stable with respect to μ_1 . Set $A_1 = \{\frac{f}{f_0} : f \in A'\}$, interpreting $\frac{f}{f_0}(x)$ as 0 if $f_0(x) = 0$; then A_1 is stable with respect to μ_1 , by 465C(a-v), and $\|f\|_\infty \leq 1$ for every $f \in A_1$. Take $Q_1 = \{f^\bullet : f \in A_1\} \subseteq L^1(\mu_1)$, so that Q_1 is stable.

We have a norm-preserving positive linear operator $R : L^1(\mu_1) \rightarrow L^1(\mu)$ defined by setting $R f^\bullet = (f \times f_0)^\bullet$ for every $f \in \mathcal{L}^1(\mu_1)$ (use 235A). Observe that $R[Q_1] = Q$ and $R(\chi X)^\bullet = u_0$. Similarly, we have a norm-preserving positive linear operator $S : L^1(\nu_1) \rightarrow L^1(\nu)$ defined by setting $S g^\bullet = (g \times g_0)^\bullet$ for $g \in \mathcal{L}^1(\nu_1)$. The set of values of S is just

$$\{g^\bullet : g \in \mathcal{L}^1(\nu), g(y) = 0 \text{ whenever } g_0(y) = 0\},$$

which is the band in $L^1(\nu)$ generated by v_0 . So

$$\{u : u \in L^1(\mu_1), TR|u| \in S[L^1(\nu_1)]\}$$

is a band in $L^1(\nu_1)$ containing χX^\bullet , and must be the whole of $L^1(\mu_1)$. Thus we have a positive linear operator $T_1 = S^{-1}TR : L^1(\mu_1) \rightarrow L^1(\nu_1)$, and $T_1(\chi X)^\bullet = \chi Y^\bullet$ in $L^1(\nu_1)$.

By (f), $T_1[Q_1]$ is stable in $L^1(\nu_1)$. Observe that $T_1[Q_1] = \{S^{-1}g^\bullet : g \in B\} = \{g^\bullet : g \in B_1\}$, where $B_1 = \{\frac{g}{g_0} : g \in B\}$, interpreting $\frac{g}{g_0}(y)$ as 0 if $y \in Y \setminus F_0$. Consequently B_1 and $B = \{g \times g_0 : g \in B_1\}$ are

stable with respect to ν_1 . By 465C(c-ii), once more, B is stable with respect to ν_0 , where

$$\nu_0 F = \int_F \frac{1}{g_0} d\nu_1 = \nu(F \cap F_0)$$

for any $F \in \mathbb{T}$. But because $g(y) = 0$ whenever $g \in B$ and $y \in Y \setminus F_0$,

$$(\nu^{2k})^* D_k(B, F, \alpha, \beta) = (\nu^{2k})^* D_k(B, F \cap F_0, \alpha, \beta) = (\nu_0^{2k})^* D_k(B, F, \alpha, \beta)$$

whenever $F \in \mathbb{T}$, $\alpha < \beta$ and $k \geq 1$; so B is also stable with respect to ν , and $Q = \{g^\bullet : g \in B\}$ is stable in $L^1(\nu)$. **Q**

(h) The worst is over. If we are not told that T is positive, we know that it is expressible as the difference of positive linear operators T_1 and T_2 (371D); now $T_1[Q]$ and $T_2[Q]$ will be stable, by the work above, so $T[Q] \subseteq T_1[Q] - T_2[Q]$ is stable, by 465Nc. If we are not told that Q is countable, we refer to 465P to see that we need only check that countable subsets of $T[Q]$ are stable, and these are images of countable subsets of Q . Finally, the identification of the abstract measure algebras $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ with the measure algebras of measure spaces is Theorem 321J.

***465S R-stable sets** The theory above has been developed in the context of general measure (or probability) spaces and the ‘ordinary’ product measure of measure spaces. For τ -additive measures – in particular, for Radon measures – we have an alternative product measure, as described in §417. If $(X, \mathfrak{T}, \Sigma, \mu)$ is a semi-finite τ -additive topological measure space such that μ is inner regular with respect to the Borel sets, write $\tilde{\mu}^I$ for the τ -additive product measure on X^I , as described in 417C (for the product of two spaces) and 417E (for the product of any family of probability spaces); we can extend the construction of 417C to arbitrary finite products (417D). Now say that $A \subseteq \mathbb{R}^X$ is **R-stable** if whenever $0 < \mu E < \infty$ and $\alpha < \beta$ there is a $k \geq 1$ such that $(\tilde{\mu}^{2k})^* D_k(A, E, \alpha, \beta) < (\mu E)^{2k}$. Because we have a version of Fubini’s theorem for the products of τ -additive topological measures (417G), all the arguments of this section can be applied to R-stable sets, yielding criteria for R-stability exactly like those in 465M.

Because the τ -additive product measure extends the c.l.d. product measure, stable sets are always R-stable. (We must have

$$(\tilde{\mu}^{2k})^* D_k(A, E, \alpha, \beta) \leq (\mu^{2k})^* D_k(A, E, \alpha, \beta)$$

for all k, A, E, α and β .) For an example of an R-stable set which is not stable, see 465U.

The concept of ‘R-stability’ is used in TALAGRAND 84 in applications to the integration of vector-valued functions. I give one result, however, to show how it is relevant to a question with a natural expression in the language of this chapter.

***465T Proposition** (TALAGRAND 84) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a semi-finite τ -additive topological measure space such that μ is inner regular with respect to the Borel sets. If $A \subseteq C(X)$ is such that every countable subset of A is R-stable, then A is R-stable.

proof For any $\alpha < \beta$ and $k \geq 1$,

$$D'_k(A, X, \alpha, \beta) = \bigcup_{f \in A} \{w : w \in X^{2k}, f(w(2i)) < \alpha, \\ f(w(2i+1)) > \beta \text{ for } i < k\}$$

is open. Suppose that $0 < \mu E < \infty$. Because all the product measures $\tilde{\mu}^{2k}$ are τ -additive, we can find a countable set $A' \subseteq A$ such that

$$\tilde{\mu}^{2k} D'_k(A, E, \alpha, \beta) = \tilde{\mu}^{2k} D'_k(A', E, \alpha, \beta)$$

for every $k \geq 1$ and all rational α, β . Now, if $\alpha < \beta$, there are rational α', β' such that $\alpha < \alpha' < \beta' < \beta$, and a $k \geq 1$ such that $\tilde{\mu}^{2k} D_k(A', E, \alpha', \beta') < (\mu E)^{2k}$; in which case

$$(\tilde{\mu}^{2k})^* D_k(A, E, \alpha, \beta) \leq \tilde{\mu}^{2k} D'_k(A, E, \alpha', \beta') = \tilde{\mu}^{2k} D'_k(A', E, \alpha', \beta') \\ \leq \tilde{\mu}^{2k} D_k(A', E, \alpha', \beta') < (\mu E)^{2k}.$$

As E, α and β are arbitrary, A is R-stable.

***465U** I come now to the promised example of an R-stable set which is not stable. I follow the construction in TALAGRAND 88, which displays an interesting characteristic related to 465O-465P above.

Example There is a Radon probability space with an R-stable set of continuous functions which is not stable.

proof (a) Let (X, Σ, μ) be an atomless probability space (e.g., the unit interval with Lebesgue measure). Define $\langle r_n \rangle_{n \in \mathbb{N}}$ by setting $r_0 = 1$, $r_1 = 2$, $r_{n+1} = 2^{nr_n}$ for $n \geq 1$; then $2^n \leq r_n < r_{n+1}$ for every n . Let $\langle \Sigma_n \rangle_{n \in \mathbb{N}}$ be an increasing sequence of finite subalgebras of Σ , each Σ_n having r_n atoms of the same size; this is possible because r_{n+1} is always a multiple of r_n . Write \mathcal{H}_n for the set of atoms of Σ_n . Next, for each $n \in \mathbb{N}$, let $\langle G_H \rangle_{H \in \mathcal{H}_n}$ be an independent family in Σ_{n+1} of sets of measure 2^{-n} ; such a family exists because r_{n+1} is a multiple of 2^{nr_n} .

Let \mathcal{E} be the family of all sets expressible in the form $E = \bigcup_{i \in \mathbb{N}} H_i$ where, for some strictly increasing sequence $\langle n_i \rangle_{i \in \mathbb{N}}$ in \mathbb{N} , $H_i \in \mathcal{H}_{n_i}$ and $H_j \subseteq G_{H_i}$ whenever $i < j$ in \mathbb{N} . Set $A = \{\chi E : E \in \mathcal{E}\} \subseteq \mathcal{L}^0(\Sigma)$.

(b) A is stable. **P** Suppose that $F \in \Sigma$ and that $\mu F > 0$. Take $n \in \mathbb{N}$ so large that $3 \cdot 2^{-n} < (\mu F)^2$. Set $\mathcal{H} = \{H : H \in \mathcal{H}_n, \mu(H \cap F) > 0\}$; enumerate \mathcal{H} as $\langle H_i \rangle_{i < m}$; set $F_1 = F \cap \bigcup_{i < m} H_i$, $V = \prod_{j < m} (H_j \cap F_1) \subseteq F_1^m$. Because $\mu F_1 = \mu F$, $m \geq r_n \mu F \geq 3$.

Consider

$$U_k = \bigcup_{H \in \mathcal{H}_k} (H \times H) \cup (H \times G_H) \cup (G_H \times H) \subseteq X^2$$

for $k \in \mathbb{N}$. Then

$$\mu^2 U_k \leq r_k \left(\frac{1}{r_k^2} + \frac{1}{2^k r_k} + \frac{1}{2^k r_k} \right) \leq 3 \cdot 2^{-k},$$

so

$$\mu^2(F_1^2 \setminus \bigcup_{k > n} U_k) \geq (\mu F)^2 - 3 \sum_{k=n+1}^{\infty} 2^{-k} > 0.$$

Set

$$V = \{w : w \in F_1^{2m}, w(2i) \in H_i \text{ for every } i < m, (w(1), w(3)) \notin \bigcup_{k > n} U_k\}.$$

Then

$$\mu^{2m}(V) \geq \prod_{i=0}^{m-1} \mu(F_1 \cap H_i) \cdot \mu^2(F_1^2 \setminus \bigcup_{k > n} U_k) \cdot (\mu F_1)^{m-2} > 0.$$

? Suppose, if possible, that there is a point $w \in V \cap D_m(A, F, 0, 1)$. Then there is an $E \in \mathcal{E}$ such that $w(2i) \notin E$, $w(2i+1) \in E$ for $i < m$. Express E as $\bigcup_{j \in \mathbb{N}} E_j$ where $E_j \in \mathcal{H}_{n_j}$ for every $j \in \mathbb{N}$, where $\langle n_j \rangle_{j \in \mathbb{N}}$ is strictly increasing, and $E_k \subseteq G_{E_j}$ whenever $j < k$. Because $w(2i) \in F_1 \cap H_i \setminus E$, E_j cannot include H_i for any $i < m$, $j \in \mathbb{N}$; so $E_j \cap H_i = \emptyset$ whenever $i < m$ and $n_j \leq n$. Because $w(1), w(3) \in E$, there are $j_0, j_1 \in \mathbb{N}$ such that $w(1) \in E_{j_0}$ and $w(3) \in E_{j_1}$. Since both $w(1)$ and $w(3)$ belong to $F_1 \subseteq \bigcup_{i < m} H_i$, n_{j_0} and n_{j_1} are both greater than n . But now

- if $j_0 = j_1$, then $(w(1), w(3)) \in E_{j_0}^2$,
- if $j_0 < j_1$, then $(w(1), w(3)) \in E_{j_0} \times E_{j_1} \subseteq E_{j_0} \times G_{E_{j_0}}$,
- if $j_1 < j_0$, then $(w(1), w(3)) \in E_{j_0} \times E_{j_1} \subseteq G_{E_{j_1}} \times E_{j_1}$,

so in any case $(w(1), w(3)) \in U_k$ where $k = \min(n_{j_0}, n_{j_1}) > n$, which is impossible. **X**

Thus $D_m(A, F, 0, 1)$ does not meet V , and $(\mu^{2m})^* D_m(A, F, 0, 1) < (\mu F)^{2m}$.

Now if $\alpha < \beta$, then $D_m(A, F, \alpha, \beta) = D_m(A, F, 0, 1)$ if $0 \leq \alpha < \beta \leq 1$, \emptyset otherwise; so in all cases $(\mu^{2m})^* D_m(A, F, \alpha, \beta) < (\mu F)^{2m}$. As F also is arbitrary, A is stable, as claimed. **Q**

(c) Now let $(Z, \mathfrak{S}, T, \nu)$ be the Stone space of the measure algebra of (X, Σ, μ) (321K), so that ν is a Radon measure (411Pe). For $E \in \Sigma$, write E^* for the corresponding open-and-closed set in Z , so that $E \mapsto E^* : \Sigma \rightarrow T$ is a measure-preserving Boolean homomorphism. Set $A^* = \{\chi E^* : E \in \mathcal{E}\} \subseteq C(Z)$. Write ν^m for the c.l.d. product measure on Z^m for $m \geq 1$. We already know that ν^2 is not a topological measure (419E, 419Xc).

(d) The point is that $(\nu^{2m})^* D_m(A^*, Z, 0, 1) = 1$ for every $m \geq 1$. **P?** Suppose, if possible, otherwise. Then there is a set $\tilde{W} \subseteq Z^{2m}$ such that $\nu^{2m}(\tilde{W}) > 0$ and $\tilde{W} \cap D_m(A^*, Z, 0, 1) \neq \emptyset$. There is an $\epsilon > 0$ such that

$$\tilde{V} = \{v : v \in Z^m, \nu^m\{u : u \in Z^m, u \# v \in \tilde{W}\} \text{ is defined and greater than } m\epsilon\}$$

has non-zero inner measure for ν^m . Now there are sets $\tilde{F}_{ij} \in \mathbb{T}$, for $i \in \mathbb{N}$ and $j < m$, such that

$$Z^m \setminus \tilde{V} \subseteq \bigcup_{i \in \mathbb{N}} \prod_{j < m} \tilde{F}_{ij}, \quad \sum_{i=0}^{\infty} \prod_{j=0}^m \nu \tilde{F}_{ij} < 1.$$

Enlarging the \tilde{F}_{ij} slightly if need be, we may suppose that they are all open-and-closed (322Rc), therefore expressible as F_{ij}^* where $F_{ij} \in \Sigma$ for $i \in \mathbb{N}$, $j < m$. Set $V = X^m \setminus \bigcup_{i \in \mathbb{N}} \prod_{j < m} F_{ij}$, so that

$$\mu^m V = \nu^m(Z^m \setminus \bigcup_{i \in \mathbb{N}} \prod_{j < m} F_{ij}^*) > 0.$$

I seek to choose $\langle n_k \rangle_{k \in \mathbb{N}}$ in \mathbb{N} , $\langle H_k \rangle_{k \in \mathbb{N}}$ and $\langle E_{kj} \rangle_{k \in \mathbb{N}, j < m}$ in Σ inductively, in such a way that

$$2^{-n_0} \leq \frac{1}{2}\epsilon,$$

$$E_{kj} \subseteq E_{ij} \cap G_{H_i} \text{ for } i < k \text{ and } j < m,$$

$$\prod_{j < m} E_{kj} \cap \prod_{j < m} F_{kj} = \emptyset,$$

$$\mu^m(V \cap G_{H_k}^m \cap \prod_{j < m} E_{kj}) > 0,$$

$$n_i < n_k \text{ for every } i < k,$$

$$H_k \in \mathcal{H}_{n_k},$$

if $k = i_k m + j_k$, where $i_k \in \mathbb{N}$ and $j_k < m$, then $\mu(H_k \cap E_{kj_k}) > 0$,

for every $k \in \mathbb{N}$. The induction proceeds as follows. Set $E'_{kj} = X$ if $k = 0$, $G_{H_{k-1}} \cap E_{k-1,j}$ otherwise, so that $\mu^m(V \cap \prod_{j < m} E'_{kj}) > 0$. Because $V \cap \prod_{j < m} F_{kj}$ is empty, we can find $E_{kj} \subseteq E'_{kj}$, for $j < m$, such that $\mu^m(V \cap \prod_{j < m} E_{kj}) > 0$ and $\prod_{j < m} E_{kj} \cap \prod_{j < m} F_{kj} = \emptyset$. Set $\eta = \mu E_{kj_k}$, $\delta = \mu^m(V \cap \prod_{j < m} E_{kj})$, so that η and δ are both strictly positive.

Now take n_k so large that

$$2^{-n_k} \leq \frac{1}{2}\epsilon, \quad n_k > n_i \text{ for } i < k, \quad (1 - 2^{-mn_k})^{\eta r_{n_k}} < \delta.$$

(This is possible because $\lim_{n \rightarrow \infty} 2^{-mn} r_n = \infty$.) Set $\mathcal{H} = \{H : H \in \mathcal{H}_{n_k}, \mu(H \cap E_{kj_k}) > 0\}$; then $\#\mathcal{H} \geq \eta r_{n_k}$. Consider the family $\langle G_H^m \rangle_{H \in \mathcal{H}}$. These are stochastically independent sets of measure 2^{-mn_k} , so their union has measure $1 - (1 - 2^{-mn_k})^{\#\mathcal{H}} > 1 - \delta$, and there is an $H_k \in \mathcal{H}$ such that $\mu^m(V \cap G_{H_k}^m \cap \prod_{j < m} E_{kj}) > 0$. Thus the induction continues.

Look at the sequence $\langle H_k \rangle_{k \in \mathbb{N}}$ and its union E . We have $H_k \in \mathcal{H}_{n_k}$ for every k ; moreover, if $i < k$, then $\mu(H_k \cap E_{kj_k}) > 0$, while $E_{kj_k} \subseteq G_{H_i}$; since H_k is an atom of Σ_{n_k} , while $G_{H_i} \in \Sigma_{n_k}$, $H_k \subseteq G_{H_i}$. Thus $E \in \mathcal{E}$. Next, whenever $i \leq k \in \mathbb{N}$,

$$\prod_{j < m} E_{kj} \cap \prod_{j < m} F_{ij} \subseteq \prod_{j < m} E_{ij} \cap \prod_{j < m} F_{ij} = \emptyset,$$

so $\prod_{j < m} E_{kj} \cap \bigcup_{i \leq k} \prod_{j < m} F_{ij} = \emptyset$. At the same time, we know that

$$E^m \cap \prod_{j < m} E_{kj} \supseteq \prod_{j < m} H_{km+j} \cap E_{kj} \supseteq \prod_{j < m} H_{km+j} \cap E_{km+j,j}$$

has non-zero measure. So $\mu^m(E^m \setminus \bigcup_{i \leq k} \prod_{j < m} F_{ij}) > 0$.

Moving back to Z , this translates into

$$\nu^m((E^*)^m \setminus \bigcup_{i \leq k} \prod_{j < m} F_{ij}^*) > 0.$$

But this means that $(E^*)^m$ is not included in $\bigcup_{i \leq k} \prod_{j < m} F_{ij}^*$, for any $k \in \mathbb{N}$. Because E^* is compact and every F_{ij}^* is open, $(E^*)^m$ is not included in $\bigcup_{i \in \mathbb{N}} \prod_{j < m} F_{ij}^*$, and there is some $v \in (E^*)^m \cap \tilde{V}$.

By the definition of \tilde{V} ,

$$\begin{aligned} \nu^m\{u : u \# v \in \tilde{W}\} &> m\epsilon \geq m \sum_{k=0}^{\infty} 2^{-n_0-k} \geq m \sum_{k=0}^{\infty} 2^{-n_k} \\ &= m \sum_{k=0}^{\infty} \mu H_k \geq m\mu E = m\nu E^*. \end{aligned}$$

So there must be some u such that $u\#v \in \tilde{W}$ and $u(j) \notin E^*$ for every $j < m$. But now, setting $w = u\#v$, we have $w(2j) \notin E^*$, $w(2j+1) \in E^*$ for $j < m$, and $w \in D_m(A^*, Z, 0, 1) \cap \tilde{W}$; which is supposed to be impossible. **XQ**

(e) This shows that A^* is not stable. It is, however, R-stable. **P** We have a measure algebra isomorphism between the measure algebras of μ and ν defined by the map $E \mapsto E^* : \Sigma \rightarrow \mathsf{T}$. The corresponding isomorphism between $L^0(\mu)$ and $L^0(\nu)$ takes $\{f^\bullet : f \in A\}$ to $\{h^\bullet : h \in A^*\}$. By 465Pa and (b) above, $\{f^\bullet : f \in A\}$ is stable in $L^0(\mu)$, so $\{h^\bullet : h \in A^*\}$ is stable in $L^0(\nu)$, and every countable subset of A^* is stable. Since $A^* \subseteq C(X)$, it follows that A^* is stable (465T). **Q**

***465V Remark** This example is clearly related to 419E. The argument here is significantly deeper, but it does have an idea in common with that in 419E, besides the obvious point that both involve the Stone spaces of atomless probability spaces. Suppose that, in the context of 465U, we take \mathcal{E}_0 to be the family of sets E expressible as the union of a finite chain H_0, \dots, H_k where $H_i \in \mathcal{H}_{n_i}$ for $i \leq k$ and $H_j \subseteq G_{H_i}$ for $i < j \leq k$. Then we find, on repeating the argument of (b) in the proof of 465U, that the countable set $A_0^* = \{\chi E^* : E \in \mathcal{E}_0\}$ is stable, so that, setting $W_m = D_m(A_0^*, Z, 0, 1)$, $\nu^{2m}W_m$ is small for large m . On the other hand, setting

$$\tilde{W}_m = \bigcup \{V : V \subseteq Z^m \text{ is open, } V \setminus W_m \text{ is negligible}\},$$

we see that $D_m(A^*, Z, 0, 1) \subseteq \tilde{W}_m$, so that $(\nu^{2m})^*\tilde{W}_m = 1$ for all $m \geq 1$. Of course, writing $\tilde{\nu}^{2m}$ for the Radon product measure on Z^{2m} , we have $\tilde{\nu}^{2m}(\tilde{W}_m) = \nu^{2m}W_m < 1$ for large m , just as in 419E.

Both $A \subseteq \mathcal{L}^0(\Sigma)$ and $A^* \subseteq \mathcal{L}^0(\mathsf{T})$ are relatively pointwise compact. Note that while I took (X, Σ, μ) and (Z, T, ν) to be quite separate, it is entirely possible for them to be actually the same space. In this case it is natural to take every Σ_n to consist of open-and-closed sets, so that every member of \mathcal{E} is open, and E^* becomes identified with the closure of E for $E \in \mathcal{E}$.

465X Basic exercises (a) Let (X, Σ, μ) be a semi-finite measure space, and $\langle f_n \rangle_{n \in \mathbb{N}}$ a sequence of measurable real-valued functions on X which converges a.e. Show that $\{f_n : n \in \mathbb{N}\}$ is stable.

(b) Let \mathcal{C} be the family of convex sets in \mathbb{R}^r , where $r \geq 1$ is an integer. Show that $\{\chi C : C \in \mathcal{C}\}$ is stable with respect to Lebesgue measure on \mathbb{R}^r , but that if $r \geq 2$ there is a Radon probability measure ν on \mathbb{R}^r such that $\{\chi C : C \in \mathcal{C}, C \text{ is closed}\}$ is not stable with respect to ν .

(c) Show that, for any $M \geq 0$, the set of functions $f : \mathbb{R} \rightarrow \mathbb{R}$ of variation at most M is stable with respect to any Radon measure on \mathbb{R} . (*Hint*: show that if μ is a Radon measure on \mathbb{R} and $E \in \text{dom } \mu$ has non-zero finite measure, and $(2k-1)(\beta - \alpha) > M$, then $(\mu^{2k})^*D_k(A, E, \alpha, \beta) < (\mu E)^{2k}$.)

(d) Let (X, Σ, μ) be a semi-finite measure space, and $A \subseteq \mathbb{R}^X$. Show that A is stable iff $\{f^+ : f \in A\}$ and $\{f^- : f \in A\}$ are both stable.

(f) Let (X, Σ, μ) be a semi-finite measure space and $A \subseteq \mathbb{R}^X$. Suppose that μ is inner regular with respect to the family $\{F : F \in \Sigma, \{f \times \chi F : f \in A\} \text{ is stable}\}$. Show that A is stable.

(g) Let (X, Σ, μ) be a totally finite measure space and T a σ -subalgebra of Σ . Let $A \subseteq \mathcal{L}^0(\mathsf{T})$ be any set. (i) Show that if A is $\mu \upharpoonright \mathsf{T}$ -stable then it is μ -stable. (ii) Give an example to show that A can be μ -stable and pointwise compact without being $\mu \upharpoonright \mathsf{T}$ -stable. (*Hint*: take μ to be Lebesgue measure on $[0, 1]$ and T the countable-cocountable algebra.)

(h) Let (X, Σ, μ) be a semi-finite measure space and A a subset of \mathbb{R}^X . Suppose that for every $\epsilon > 0$ there is a stable set $B \subseteq \mathbb{R}^X$ such that for every $f \in A$ there is a $g \in B$ such that $\|f - g\|_\infty \leq \epsilon$. Show that A is stable.

>(i) Let (X, Σ, μ) be a semi-finite measure space and $\langle E_n \rangle_{n \in \mathbb{N}}$ a sequence in Σ . (i) Suppose that whenever $F \in \Sigma$ and $\mu F < \infty$ there is a $k \geq 1$ such that $\sum_{n=0}^{\infty} (\mu(F \cap E_n) \mu(F \setminus E_n))^k$ is finite. Show that $\langle \chi E_n \rangle_{n \in \mathbb{N}}$ is stable. (*Hint*: 465Cc.) (ii) Suppose that $\mu X = 1$, that $\langle E_n \rangle_{n \in \mathbb{N}}$ is independent, and that $\sum_{n=0}^{\infty} ((1 - \mu E_n) \mu E_n)^k = \infty$ for every $k \geq 1$. Show that $\langle \chi E_n \rangle_{n \in \mathbb{N}}$ is not stable. (*Hint*: 273K.)

>(j) Show that there is a disjoint family \mathcal{I} of finite subsets of $[0, 1]$ such that $A = \{\chi_I : I \in \mathcal{I}\}$ is not stable with respect to Lebesgue measure, though A is pointwise compact and the identity map on A is continuous for the topology of pointwise convergence and the topology of convergence in measure. (*Hint*: show by induction on m that if $m \geq 1$ and $K \subseteq [0, 1]^m$ is a compact set with non-zero measure, then $K \cap ([0, 1] \setminus D)^m \neq \emptyset$ whenever $\#(D) < c$; now put this together with 4A3Fa.)

(k) Let (X, Σ, μ) be a semi-finite measure space and $A \subseteq \mathbb{R}^X$ a stable set such that $\{f(x) : f \in A\}$ is bounded for every $x \in X$. Let \bar{A} be the closure of A for the topology of pointwise convergence. Show that $\{f^\bullet : f \in \bar{A}\}$ is just the closure of $\{f^\bullet : f \in A\} \subseteq L^0(\mu)$ for the topology of convergence in measure.

(l) Let (X, Σ, μ) be a probability space. Show that $A \subseteq \mathbb{R}^X$ is stable iff

$$\inf_{m \in \mathbb{N}} ((\mu^{2m})^* D_m(A, X, \alpha, \beta))^{1/m} = 0$$

whenever $\alpha < \beta$ in \mathbb{R} .

(m) Let (X, Σ, μ) be a semi-finite measure space. Show that a countable set $A \subseteq \mathcal{L}^0(\Sigma)$ is not stable iff there are $E \in \Sigma$ and $\alpha < \beta$ such that $0 < \mu E < \infty$ and $\mu^m \tilde{D}_m(A, E, \alpha, \beta) = (\mu E)^m$ for every $m \geq 1$, where $\tilde{D}_m(A, E, \alpha, \beta)$ is the set of those $w \in E^m$ such that for every $I \subseteq m$ there is an $f \in A$ such that $f(w(i)) \leq \alpha$ for $i \in I$, $f(w(i)) \geq \beta$ for $i \in m \setminus I$. (*Hint*: see part (iii) of case 2 of the proof of 465L.)

(n) Let (X, Σ, μ) be a semi-finite measure space, and $A \subseteq \mathcal{L}^0(\Sigma)$ a set which is compact and metrizable for the topology of pointwise convergence. Show that A is stable. (*Hint*: otherwise, apply the ideas of case 2 in the proof of 465L to a countable dense subset of A to obtain a sequence which contradicts the conclusion of 465Xa.)

(o) Let (X, Σ, μ) be a probability space and $A \subseteq \mathcal{L}^0(\Sigma)$ a uniformly bounded set of functions. Show that A is stable iff

$$\lim_{n \rightarrow \infty} \int \sup_{f \in A, k, l \geq n} \left| \frac{1}{k} \sum_{i=0}^{k-1} f(w(i)) - \frac{1}{l} \sum_{i=0}^{l-1} f(w(i)) \right| \mu^{\mathbb{N}}(dw) = 0.$$

(p) Show that there is a set $A \subseteq \mathbb{R}^{[0,1]}$ such that $\sup_{f \in A} |f(x)|$ is finite for every x , and A is stable with respect to Lebesgue measure, but its convex hull $\Gamma(A)$ is not. (*Hint*: take \mathcal{I} as in 465Xj and $A = \{\frac{1}{\#(I)} \chi\{x\} : x \in I \in \mathcal{I} \setminus \{\emptyset\}\}$.)

(q) Show that there is a sequence $\langle f_n \rangle_{n \in \mathbb{N}}$ of functions from $[0, 1]$ to \mathbb{N} such that $\{f_n : n \in \mathbb{N}\}$ is stable for Lebesgue measure on $[0, 1]$, but $\{f_m - f_n : m, n \in \mathbb{N}\}$ is not.

(r) Let (X, Σ, μ) be a semi-finite measure space and $Q \subseteq L^0(\mu)$ a set which is stable in the sense of 465O. Show that the closure of Q (for the topology of convergence in measure) is stable. (*Hint*: 465Xk.)

(s) Let (X, Σ, μ) be a probability space, \mathbb{T} a σ -subalgebra of Σ , and $A \subseteq \mathbb{R}^X$ a countable stable set of Σ -measurable functions such that $\int \sup_{f \in A} |f(x)| \mu(dx) < \infty$. Show that if for each $f \in A$ we choose a conditional expectation g_f of f on \mathbb{T} (requiring each g_f to be \mathbb{T} -measurable and defined everywhere on X), then $\{g_f : f \in A\}$ is stable.

(t) Give an example of a probability algebra $(\mathfrak{A}, \bar{\mu})$, a conditional expectation operator $P : L_{\bar{\mu}}^1 \rightarrow L_{\bar{\mu}}^1$ (365Q), and a uniformly integrable stable set $A \subseteq L_{\bar{\mu}}^1$ such that $P[A]$ is not stable. (*Hint*: start by picking a sequence $\langle v_n \rangle_{n \in \mathbb{N}}$ in $P[L_{\bar{\mu}}^1]$ which is norm-convergent to 0 but not stable, and express this as $\langle P u_n \rangle_{n \in \mathbb{N}}$ where $\bar{\mu}[u_n > 0] \leq 2^{-n}$ for every n .)

465Y Further exercises (a) Let (X, Σ, μ) be a semi-finite measure space. A set $A \subseteq \mathbb{R}^X$ is said to have the **Bourgain property** if whenever $E \in \Sigma$, $\mu E > 0$ and $\epsilon > 0$, there are non-negligible measurable sets $F_0, \dots, F_n \subseteq E$ such that for every $f \in A$ there is an $i \leq n$ such that the oscillation $\sup_{x, y \in F_i} |f(x) - f(y)|$ of f on F_i is at most ϵ . Show that in this case A is stable.

(b) Let X be a topological space, and μ a τ -additive effectively locally finite topological measure on X . Show that any equicontinuous subset of $C(X)$ has the Bourgain property, so is stable.

(c) Find an integrable continuous function $f : [0, 1]^2 \rightarrow [0, \infty[$ such that, in the notation of 465H, $\limsup_{k \rightarrow \infty} \int f d\nu_{wk}^2 = \infty$ for almost every $w \in [0, 1]^{\mathbb{N}}$, if $[0, 1]^{\mathbb{N}}$ is given its usual measure.

(d) Show that in 465M we may replace the condition ‘ A is uniformly bounded’ by the condition ‘ $|f| \leq f_0$ for every $f \in A$, where f_0 is integrable’. (*Hint*: TALAGRAND 87.)

(e) Let (X, Σ, μ) be a probability space, and $A \subseteq \mathbb{R}^X$ a uniformly bounded set. Show that A is stable iff for every $\epsilon > 0$ there are a stable set $B \subseteq \mathbb{R}^X$, a sequence $\langle h_k \rangle_{k \in \mathbb{N}}$ of measurable functions on $X^{\mathbb{N}}$, and a family $\langle g_f \rangle_{f \in A}$ in B such that

$$h_k(w) \geq \frac{1}{k} \sum_{i=0}^{k-1} |f(w(i)) - g_f(w(i))| \text{ for every } w \in X^{\mathbb{N}}, k \geq 1 \text{ and } f \in A,$$

$$\limsup_{k \rightarrow \infty} h_k(w) \leq \epsilon \text{ for almost every } w \in X^{\mathbb{N}}.$$

(f) Let (X, Σ, μ) be a semi-finite measure space, and $A, B \subseteq \mathbb{R}^X$ pointwise bounded stable sets. Show that $\{f \times g : f \in A, g \in B\}$ is stable.

(g) Let (X, Σ, μ) and (Y, T, ν) be semi-finite measure spaces, with c.l.d. product $(X \times Y, \Lambda, \lambda)$. Suppose that $A \subseteq \mathbb{R}^X$ and $B \subseteq \mathbb{R}^Y$ are pointwise bounded stable sets. Show that $\{f \otimes g : f \in A, g \in B\}$ is stable with respect to λ , where $(f \otimes g)(x, y) = f(x)g(y)$.

(h) Let (X, Σ, μ) be a semi-finite measure space, $A \subseteq \mathbb{R}^X$ a uniformly bounded stable set and $h : \mathbb{R} \rightarrow \mathbb{R}$ a continuous function. Show that $\{hf : f \in A\}$ is stable.

(i) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be an effectively locally finite τ -additive topological measure space. Show that a countable \mathbb{R} -stable subset of \mathbb{R}^X is stable.

(j) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a complete τ -additive topological probability space such that μ is inner regular with respect to the Borel sets, and $A \subseteq [0, 1]^X$ an \mathbb{R} -stable set. Suppose that $\epsilon > 0$ is such that $\int f d\mu \leq \epsilon^2$ for every $f \in A$. Show that there are an $n \geq 1$ and a Borel set $W \subseteq X^n$ and a $\gamma > \tilde{\mu}^n W$ (writing $\tilde{\mu}^n$ for the τ -additive product measure on X^n) such that $\int f d\nu \leq 3\epsilon$ whenever $f \in A$, $\nu : \mathcal{P}X \rightarrow [0, 1]$ is a point-supported probability measure and $\nu^n W \leq \gamma$.

(k) Set $X = \prod_{n=2}^{\infty} \mathbb{Z}_n$, where each \mathbb{Z}_n is the cyclic group of order n with its discrete topology; let μ be the Haar probability measure on X . Let \bullet_l be the left shift action of X on \mathbb{R}^X , so that $(a \bullet_l f)(x) = f(x - a)$ for $a, x \in X$ and $f \in \mathbb{R}^X$. (i) Show that for any $n \in \mathbb{N}$ there is a compact negligible set $K_n \subseteq X$ such that for every $I \in [X]^n$ there are uncountably many $a \in X$ such that $I \subseteq K_n + a$. (ii) Show that there is a negligible set $E \subseteq X$ such that $\{a \bullet_l \chi_E : a \in X\}$ is dense in $\{0, 1\}^X$ for the topology of pointwise convergence. (iii) Show that if $f \in C(X)$ then $\{a \bullet_l f : a \in X\}$ is stable. (iv) Show that there is a sequence $\langle f_n \rangle_{n \in \mathbb{N}}$ in $C(X)$ such that $\{f_n : n \in \mathbb{N}\}$ is stable but $\{a \bullet_l f_n : a \in X, n \in \mathbb{N}\}$ is not. (v) Find expressions of these results when X is replaced by the circle group or by \mathbb{R} .

465 Notes and comments The definition in 465B arose naturally when M. Talagrand and I were studying pointwise compact sets of measurable functions; we found that in many cases a set of functions was relatively pointwise compact because it was stable (465Db). Only later did it appear that the concept was connected with Glivenko-Cantelli classes in the theory of empirical measures, as explained in TALAGRAND 87.

It is not the case that all pointwise compact sets of measurable functions are stable. In fact I have already offered examples in 463Xh and 464E above. In both cases it is easy to check from the definition in 465B that they are not stable, as can also be deduced from 465G. Another example is in 465Xj. You will observe however that all these examples are ‘pathological’ in the sense that either the measure space is irregular (from some points of view, indeed, any measure space not isomorphic to Lebesgue measure on the unit interval can be dismissed as peripheral), or the set of functions is uninteresting. It is clear from 465R and

465T, for instance, that we should start with countable sets. So it is natural to ask: if we have a *separable* pointwise compact set of real-valued measurable functions on the unit interval, must it be stable? If the continuum hypothesis is true, yes (536F¹⁰), but the question is undecided by the ordinary axioms of ZFC (SHELAH & FREMLIN 93). (If we ask for ‘metrizable’, instead of ‘separable’, we get a positive answer; see 465Xn.)

At the very beginning of our work on this topic, one of the leading questions was: under what circumstances will an independent family $\langle E_n \rangle_{n \in \mathbb{N}}$ of sets give rise to a relatively pointwise compact set $\{\chi E_n : n \in \mathbb{N}\}$ in \mathcal{L}^0 ? There remain interesting open questions here. But for stability, rather than relative compactness, we have a straightforward answer (465Xi).

The curious phrasing of the statement of 465M(iii), with the auxiliary functions h_k , turns on the fact that all the expressions ‘ $\sup_{f \in A} \dots$ ’ here give rise to functions which need not be measurable. Thus the simple pointwise convergence described in (ii) and (iv) is not at all the same thing as the convergence in (v), which may be thought of as a kind of $\|\cdot\|_1$ -convergence if we write $\|g\|_1 = \int |g|$ for arbitrary real-valued functions g . (Since the sets A here are uniformly bounded, it may equally be thought of as convergence in measure.) Similarly, 465M(iii) is saying much more than just

$$\lim_{k \rightarrow \infty} \sup_{f \in A} \frac{1}{k} \sum_{i=0}^{k-1} |f(w(i)) - \mathbb{E}(f|T)(w(i))| = 0 \text{ a.e.,}$$

though of course for countable sets A these distinctions disappear. On the other hand, since the convergence is certainly not monotonic in k , $\|\cdot\|_1$ -convergence does not imply pointwise convergence. So we can look for something stronger than either (iv) or (v) of 465M, as in 465Xo. But this is still nowhere near the strength of 465M(iii), in which $|\sum \dots|$ is replaced by $\sum |\dots|$. For further variations, see TALAGRAND 87 and TALAGRAND 96.

If we wish to adapt the ideas here to spaces of equivalence classes of functions rather than spaces of true functions, we find that problems of measurability evaporate, and that (because the definition of stability looks only at sets of finite measure) all the relevant suprema can be interpreted as suprema of countable sets. Consequently a subset of $L^0(\mu)$ or $L^0(\mathfrak{A})$ is stable if all its countable subsets are stable (465Pa). It is remarkable that, for strictly localizable measures μ , we can lift any stable set in L^0 to a stable set in \mathcal{L}^0 (465Pb, 465Q). By moving to function spaces we get a language in which to express a new kind of permanence property of stable sets (465R, 465Xs). See also TALAGRAND 89.

The definition of ‘stable set’ of functions seems to be utterly dependent on the underlying measure space. But 465R tells us that in fact the property of being an order-bounded stable subset of $L^1(\bar{\mu})$ is invariant under normed space automorphisms. (‘Order-boundedness’ is a normed space invariant in L^1 spaces by the Chacon-Krengel theorem, 371D.) Since stability can be defined in terms of order-bounded sets (465Xh), we could, for instance, develop a theory of stable sets in abstract L -spaces.

The theory of stable sets is of course bound intimately to the theory of product measures, and such results as 465J have independent interest as theorems about sets in product spaces. So any new theory of product measures will give rise to a new theory of stable sets. In particular, the τ -additive product measures in §417 lead to R-stable sets (465S). It is instructive to work through the details, observing how the properties of the product are employed. Primarily, of course, we need ‘associative’ and ‘commutative’ laws, and Fubini’s theorem; but some questions of measurability need to be re-examined, as in 465Yj.

I have starred 465U because it involves the notion of R-stability. In fact this appears only in the final stage, and the construction, as set out in part (a) of the proof, is an instructive challenge to any intuitive concept of what stable sets are like.

Version of 2.8.13

466 Measures on linear topological spaces

In this section I collect a number of results on the special properties of topological measures on linear topological spaces. The most important is surely Phillips’ theorem (466A-466B): on any Banach space, the weak and norm topologies give rise to the same totally finite Radon measures. This is not because

¹⁰Later editions only.

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the weak and norm topologies have the same Borel σ -algebras, though this does happen in interesting cases (466C-466E, §467). When the Borel σ -algebras are different, we can still ask whether the Borel measures are ‘essentially’ the same, that is, whether every (totally finite) Borel measure for the weak topology extends to a Borel measure for the norm topology. A construction due to M. Talagrand (466H, 466Ia) gives a negative answer to the general question.

Just as in \mathbb{R}^r , a totally finite quasi-Radon measure on a locally convex linear topological space is determined by its characteristic function (466K). I end the section with a note on measurability conditions sufficient to ensure that a linear operator between Banach spaces is continuous (466L-466M), and with brief remarks on Gaussian measures (466N-466O).

466A Theorem Let (X, \mathfrak{T}) be a metrizable locally convex linear topological space and μ a σ -finite measure on X which is quasi-Radon for the weak topology $\mathfrak{T}_s(X, X^*)$. Then the support of μ is separable, so μ is quasi-Radon for the original topology \mathfrak{T} . If X is complete and μ is locally finite with respect to \mathfrak{T} , then μ is Radon for \mathfrak{T} .

proof (a) Let $\langle V_n \rangle_{n \in \mathbb{N}}$ be a sequence running over a base of neighbourhoods of 0 in X , and for $n \in \mathbb{N}$ set $V_n^\circ = \{f : f \in X^*, f(x) \leq 1 \text{ for every } x \in V_n\}$. Then each V_n° is a convex \mathfrak{T}_s -compact subset of X^* (4A4Bf), and $X^* = \bigcup_{n \in \mathbb{N}} V_n^\circ$. Let $Z \subseteq X$ be the support of μ , and for $n \in \mathbb{N}$ set $K_n = \{f \upharpoonright Z : f \in V_n^\circ\}$. Since the map $f \mapsto f \upharpoonright Z : X^* \rightarrow C(Z)$ is linear and continuous for \mathfrak{T}_s and the topology \mathfrak{T}_p of pointwise convergence on Z , each K_n is convex and \mathfrak{T}_p -compact. Next, the subspace measure μ_Z on Z is σ -finite and strictly positive, so by 463G each K_n is \mathfrak{T}_p -metrizable.

Let \mathcal{U} be the base for \mathfrak{T}_p consisting of sets of the form

$$U(I, h, \epsilon) = \{f : f \in C(Z), |f(x) - h(x)| < \epsilon \text{ for every } x \in I\},$$

where $I \subseteq Z$ is finite, $h \in \mathbb{R}^I$ and $\epsilon > 0$. For each $n \in \mathbb{N}$, write $\mathcal{V}_n = \{K_n \cap U : U \in \mathcal{U}\}$, so that \mathcal{V}_n is a base for the subspace topology of K_n (4A2B(a-vi)). Since this is compact and metrizable, therefore second-countable, there is a countable base $\mathcal{V}'_n \subseteq \mathcal{V}_n$ (4A2Ob). Now there is a countable set $\mathcal{U}' \subseteq \mathcal{U}$ such that $\mathcal{V}'_n \subseteq \{K_n \cap U : U \in \mathcal{U}'\}$ for every $n \in \mathbb{N}$, and a countable set $D \subseteq Z$ such that

$$\mathcal{U}' \subseteq \{U(I, h, \epsilon) : I \subseteq D \text{ is finite, } h \in \mathbb{R}^I, \epsilon > 0\}.$$

Let $D' \subseteq X$ be the linear span of D over the rationals. Then D' is again countable, and its \mathfrak{T} -closure Y is a linear subspace of X (4A4Bg). **?** If $Z \not\subseteq Y$, then $\mu(X \setminus Y) > 0$. But, writing

$$\begin{aligned} Y^\circ &= \{f : f \in X^*, f(x) \leq 1 \text{ for every } x \in Y\} \\ &= \{f : f \in X^*, f(x) = 0 \text{ for every } x \in Y\}, \end{aligned}$$

$X \setminus Y$ must be $\bigcup_{f \in Y^\circ} \{x : |f(x)| > 0\}$, by 4A4Eb. Because μ is τ -additive, there must be an $f \in Y^\circ$ such that $\mu\{x : |f(x)| > 0\} > 0$. Let $n \in \mathbb{N}$ be such that $f \in V_n^\circ$. Since $f(x) = 0$ for every $x \in D$, $f \upharpoonright Z$ belongs to the same members of \mathcal{V}'_n as the zero function; since \mathcal{V}'_n is a base for the Hausdorff subspace topology of K_n , $f \upharpoonright Z$ actually is the zero function, and $\{x : |f(x)| > 0\} \subseteq X \setminus Z$, so $\mu Z < \mu X$, contrary to the choice of Z . **X**

Thus $Z \subseteq Y$. Because $Y = \overline{D'}$ is \mathfrak{T} -separable, and \mathfrak{T} is metrizable, Z also is \mathfrak{T} -separable (4A2P(a-iv)).

(b) The subspace topology \mathfrak{T}_Y induced on Y by \mathfrak{T} is a separable metrizable locally convex linear space topology, so the Borel σ -algebras of \mathfrak{T}_Y and the associated weak topology $\mathfrak{T}_s(Y, Y^*)$ are equal (4A3W). But $\mathfrak{T}_s(Y, Y^*)$ is just the subspace topology induced on Y by \mathfrak{T}_s (4A4Ea). Accordingly the subspace measure μ_Y on Y is $\mathfrak{T}_s(Y, Y^*)$ -quasi-Radon (415B). Since every \mathfrak{T}_Y -open set is $\mathfrak{T}_s(Y, Y^*)$ -Borel, μ_Y is a topological measure for \mathfrak{T}_Y . Since \mathfrak{T}_Y is finer than $\mathfrak{T}_s(Y, Y^*)$, μ_Y is effectively locally finite for \mathfrak{T}_Y and inner regular with respect to the \mathfrak{T}_Y -closed sets, and therefore is a quasi-Radon measure for \mathfrak{T}_Y (415D(i)).

By 415J, there is a measure $\tilde{\mu}$ on X , quasi-Radon for \mathfrak{T} , such that $\tilde{\mu}E = \mu_Y(E \cap Y)$ whenever $\tilde{\mu}$ measures E . But as Y is \mathfrak{T} -closed and μ -conegligible, and μ is complete, we have $\mu = \tilde{\mu}$, and μ is quasi-Radon for \mathfrak{T} .

(c) If X is complete and μ is locally finite with respect to \mathfrak{T} , then (X, \mathfrak{T}) is a pre-Radon space (434Jg), so μ is a Radon measure for \mathfrak{T} (434Jb).

466B Corollary (Compare 462I.) If X is a Banach space and μ is a totally finite measure on X which is quasi-Radon for the weak topology of X , it is a Radon measure for both the norm topology and the weak topology.

proof By 466A, μ is a Radon measure for the norm topology; by 418I, or otherwise, it is a Radon measure for the weak topology.

Remark Thus Banach spaces, with their weak topologies, are pre-Radon.

466C Definition A normed space X has a **Kadec norm** (also called **Kadec-Klee norm**) if the norm and weak topologies coincide on the sphere $\{x : \|x\| = 1\}$. Of course they will then also coincide on any sphere $\{x : \|x - y\| = \alpha\}$.

Example For any set I and any $p \in [1, \infty[$, the Banach space $\ell^p(I)$ has a Kadec norm. **P** Set $S = \{x : \|x\|_p = 1\}$. If $x \in S$ and $\epsilon > 0$, take $\eta \in]0, 1]$ such that $2\eta + (2p\eta)^{1/p} \leq \epsilon$. Let $J \subseteq I$ be a finite set such that $\sum_{i \in I \setminus J} |x(i)|^p \leq \eta^p$. Set $H = \{y : y \in \ell^p(I), \sum_{i \in J} |y(i) - x(i)|^p < \eta^p\}$; then H is open for the weak topology of $\ell^p(I)$. If $y \in H \cap S$, then, writing x_J for $x \times \chi_J$, etc.,

$$\begin{aligned} \|x_J\|_p &\geq 1 - \|x_{I \setminus J}\|_p \geq 1 - \eta, \\ \|y_J\|_p &\geq \|x_J\|_p - \eta \geq 1 - 2\eta, \quad \|y_J\|_p^p \geq 1 - 2p\eta, \\ \|y - x\|_p &\leq \|y_J - x_J\|_p + \|x_{I \setminus J}\|_p + \|y_{I \setminus J}\|_p \\ &\leq \eta + \eta + (1 - \|y_J\|_p^p)^{1/p} \leq 2\eta + (2p\eta)^{1/p} \leq \epsilon. \end{aligned}$$

Thus $\{y : y \in S, \|y - x\|_p \leq \epsilon\}$ is a neighbourhood of x for the subspace weak topology on S ; as x and ϵ are arbitrary, the weak and norm topologies agree on S . **Q**

For further examples, see 467B *et seq.*

466D Proposition (HANSELL 01) Let X be a normed space with a Kadec norm. Then there is a network for the norm topology on X expressible in the form $\bigcup_{n \in \mathbb{N}} \mathcal{V}_n$, where for each $n \in \mathbb{N}$ \mathcal{V}_n is an isolated family for the weak topology and $\bigcup \mathcal{V}_n$ is the difference of two closed sets for the weak topology.

proof Let \mathcal{U} be a σ -disjoint base for the norm topology of X (4A2L(g-ii)); express it as $\bigcup_{n \in \mathbb{N}} \mathcal{U}_n$ where every \mathcal{U}_n is disjoint. For rational numbers q, q' with $0 < q < q'$, set $S_{qq'} = \{x : q < \|x\| \leq q'\}$, and for $A \subseteq X$ write $W(A, q, q')$ for the interior of $A \cap S_{qq'}$ taken in the subspace weak topology of $S_{qq'}$. Set $\mathcal{V}_{nqq'} = \{W(U, q, q') : U \in \mathcal{U}_n\}$, so that $\mathcal{V}_{nqq'}$ is a disjoint family of relatively-weakly-open subsets of $S_{qq'}$, and is an isolated family for the weak topology. Now $\bigcup_{n \in \mathbb{N}, q, q' \in \mathbb{Q}, 0 < q < q'} \mathcal{V}_{nqq'}$ is a network for the norm topology on $X \setminus \{0\}$. **P** If $x \in X \setminus \{0\}$ and $\epsilon > 0$, then take $n \in \mathbb{N}, U \in \mathcal{U}_n$ such that $x \in U \subseteq \{y : \|y - x\| \leq \epsilon\}$. Let $\delta > 0$ be such that $\{y : \|y - x\| \leq \delta\} \subseteq U$. Next, because $\|\cdot\|$ is a Kadec norm, there is a weak neighbourhood V of 0 such that $\|y - x\| \leq \frac{1}{2}\delta$ whenever $y \in x - V$ and $\|y\| = \|x\|$. Let V' be an open weak neighbourhood of 0 such that $V' + V' \subseteq V$. Let $\eta \in]0, 1[$ be such that $\eta\|x\| \leq \frac{1}{2}\delta$ and $y \in V'$ whenever $\|y\| \leq \eta\|x\|$. If $y \in x - V'$ and $(1 - \eta)\|x\| \leq \|y\| \leq (1 + \eta)\|x\|$, then

$$\begin{aligned} \|y - \frac{\|x\|}{\|y\|}y\| &= |1 - \frac{\|x\|}{\|y\|}| \|y\| = |\frac{\|y\|}{\|x\|} - 1| \|x\| \leq \eta\|x\| \leq \frac{1}{2}\delta, \\ x - \frac{\|x\|}{\|y\|}y &= (x - y) + (y - \frac{\|x\|}{\|y\|}y) \in V' + V' \subseteq V, \end{aligned}$$

so $\|x - \frac{\|x\|}{\|y\|}y\| \leq \frac{1}{2}\delta$ and $\|x - y\| \leq \delta$ and $y \in U$. This means that if we take $q, q' \in \mathbb{Q}$ such that $(1 - \eta)\|x\| \leq q \leq \|x\| \leq q' \leq (1 + \eta)\|x\|$, then $(x - V') \cap S_{qq'} \subseteq U$ and $x \in W(U, q, q') \in \mathcal{V}_{nqq'}$. Since of course $W(U, q, q') \subseteq U \subseteq \{y : \|y - x\| \leq \epsilon\}$, and x and ϵ are arbitrary, we have the result. **Q**

To get a σ -isolated family for the weak topology which is a network for the norm topology on the whole of X , we just have to add the singleton set $\{0\}$. To see that the union of each of our isolated families is the difference of two weakly open sets, observe that $\bigcup \mathcal{V}_{nqq'}$ is a relatively weakly open subset of $S_{qq'}$, which is the difference of the weakly open sets $\{x : \|x\| > q\}$ and $\{x : \|x\| > q'\}$.

466E Corollary Let X be a normed space with a Kadec norm.

- (a) The norm and weak topologies give rise to the same Borel σ -algebras.
- (b) The weak topology has a σ -isolated network, so is hereditarily weakly θ -refinable.

proof (a) Write $\mathcal{B}_{\|\cdot\|}$, $\mathcal{B}_{\mathfrak{T}_s}$ for the Borel σ -algebras for the weak and norm topologies. Of course $\mathcal{B}_{\mathfrak{T}_s} \subseteq \mathcal{B}_{\|\cdot\|}$. Let $\langle \mathcal{V}_n \rangle_{n \in \mathbb{N}}$ be a sequence covering a network for the norm topology as in 466D. Because \mathcal{V}_n is (for the weak topology) isolated and its union belongs to $\mathcal{B}_{\mathfrak{T}_s}$, $\bigcup \mathcal{W} \in \mathcal{B}_{\mathfrak{T}_s}$ for every $n \in \mathbb{N}$ and $\mathcal{W} \subseteq \mathcal{V}_n$. But this means that $\bigcup \mathcal{W} \in \mathcal{B}_{\mathfrak{T}_s}$ for every $\mathcal{W} \subseteq \mathcal{V} = \bigcup_{n \in \mathbb{N}} \mathcal{V}_n$; and as \mathcal{V} is a network for the norm topology, every norm-open set belongs to $\mathcal{B}_{\mathfrak{T}_s}$, and $\mathcal{B}_{\|\cdot\|} \subseteq \mathcal{B}_{\mathfrak{T}_s}$. Thus the two Borel σ -algebras are equal.

(b) Of course \mathcal{V} is also a network for the weak topology, so the weak topology has a σ -isolated network; by 438Ld, it is hereditarily weakly θ -refinable.

466F Proposition Let X be a Banach space with a Kadec norm. Then the following are equiveridical:

- (i) X is a Radon space in its norm topology;
- (ii) X is a Radon space in its weak topology;
- (iii) the weight of X (for the norm topology) is measure-free in the sense of §438.

proof (a)(i) \Leftrightarrow (ii) By 466Ea, the norm and weak topologies give rise to the same algebra \mathcal{B} of Borel sets. If X is a Radon space in its norm topology, then any totally finite measure with domain \mathcal{B} is inner regular with respect to the norm-compact sets, therefore inner regular with respect to the weakly compact sets, and X is Radon in its weak topology. If X is a Radon space in its weak topology, then any totally finite measure μ with domain \mathcal{B} has a completion $\hat{\mu}$ which is a Radon measure for the weak topology, therefore also for the norm topology, by 466B; as μ is arbitrary, X is a Radon space for the norm topology.

(b)(i) \Leftrightarrow (iii) is a special case of 438H.

466G Definition A partially ordered set X has the σ -interpolation property or countable separation property if whenever A, B are non-empty countable subsets of X and $x \leq y$ for every $x \in A, y \in B$, then there is a $z \in X$ such that $x \leq z \leq y$ for every $x \in A$ and $y \in B$. A Dedekind σ -complete partially ordered set (314Ab) always has the σ -interpolation property.

466H Proposition (JAYNE & ROGERS 95) Let X be a Riesz space with a Riesz norm, given its weak topology $\mathfrak{T}_s = \mathfrak{T}_s(X, X^*)$. Suppose that (a) X has the σ -interpolation property (b) there is a strictly increasing family $\langle p_\xi \rangle_{\xi < \omega_1}$ in X . Then there is a \mathfrak{T}_s -Borel probability measure μ on X such that

- (i) μ is not inner regular with respect to the \mathfrak{T}_s -closed sets;
- (ii) μ is not τ -additive for the topology \mathfrak{T}_s ;
- (iii) μ has no extension to a norm-Borel measure on X .

Accordingly (X, \mathfrak{T}_s) is not a Radon space (indeed, is not Borel-measure-complete).

proof (a) Let K be the set

$$\{f : f \in X^*, f \geq 0, \|f\| \leq 1\} = \{f : \|f\| \leq 1\} \cap \bigcap_{x \in X^+} \{f : f(x) \geq 0\},$$

so that K is a weak*-closed subset of the unit ball of X^* and is weak*-compact. Because X^* is a solid linear subspace of the order-bounded dual X^\sim of X (356Da), every member of X^* is the difference of two non-negative members of X^* , and K spans X^* . We shall need to know that if $x < y$ in X , there is an $f \in K$ such that $f(x) < f(y)$; set $f = \frac{1}{\|g\|}|g|$ where $g \in X^*$ is such that $g(x) \neq g(y)$. (Recall that the norm of X^* is a Riesz norm, as also noted in 356Da.)

Set

$$A = \bigcup_{\xi < \omega_1} \{x : x \in X, x \leq p_\xi\};$$

then every sequence in A has an upper bound in A , but A has no greatest member. It follows that if $B \subseteq K$ is countable there is an $x \in A$ such that $f(x) = \sup_{y \in A} f(y)$ for every $f \in B$, so that $f(x) = f(y)$ whenever $x \leq y \in A$ and $f \in B$.

Let \mathcal{I} be the family of those sets $E \subseteq X$ such that $A \cap E$ is bounded above in A . Then \mathcal{I} is a σ -ideal of subsets of X . **P** Of course $\emptyset \in \mathcal{I}$. If $\langle E_n \rangle_{n \in \mathbb{N}}$ is a sequence in \mathcal{I} and $E \subseteq \bigcup_{n \in \mathbb{N}} E_n$, there is for each $n \in \mathbb{N}$

an $x_n \in A$ which is an upper bound for $E_n \cap A$. Let $x \in A$ be an upper bound for $\{x_n : n \in \mathbb{N}\}$; then x is an upper bound for $E \cap A$ in A . So $E \in \mathcal{I}$. **Q**

(b) For $G \in \mathfrak{T}_s$ and $k \in \mathbb{N}$, let $W(G, k)$ be the set of those $x \in X$ for which there are $f_0, \dots, f_k \in K$ such that $\{y : y \in X, |f_i(y) - f_i(x)| \leq 2^{-k}$ for every $i \leq k\}$ is included in G . Because K spans X^* , $G = \bigcup_{k \in \mathbb{N}} W(G, k)$. So if $G \in \mathfrak{T}_s \setminus \mathcal{I}$ there is a $k \in \mathbb{N}$ such that $W(G, k) \notin \mathcal{I}$.

(c) (The key.) If $\langle G_n \rangle_{n \in \mathbb{N}}$ is any sequence in $\mathfrak{T}_s \setminus \mathcal{I}$, then $\bigcap_{n \in \mathbb{N}} G_n \notin \mathcal{I}$. **P** Start from any $z^* \in A$. For each $n \in \mathbb{N}$, take $k_n \in \mathbb{N}$ such that $W(G_n, k_n) \notin \mathcal{I}$. For each $z \in A$ and $n \in \mathbb{N}$, choose $w_{zn} \in A \cap W(G_n, k_n)$ such that $w_{zn} \geq z$; now choose a family $\langle f_{nzi} \rangle_{z \in A, i \leq k_n}$ in K such that

$$\{y : |f_{nzi}(y - w_{zn})| \leq 2^{-k_n} \text{ for every } i \leq k_n\} \subseteq G_n.$$

Let \mathcal{F} be any ultrafilter on A containing $\{x : x \in A, x \geq z\}$ for every $z \in A$, and write $f_{ni} = \lim_{z \rightarrow \mathcal{F}} f_{nzi}$ for $n \in \mathbb{N}$ and $i \leq k_n$, the limit being taken for the weak* topology on K . Let $z_1^* > z^*$ be such that $z_1^* \in A$ and $f_{ni}(x - z_1^*) = 0$ whenever $x \in A, x \geq z_1^*, n \in \mathbb{N}$ and $i \leq k_n$.

Choose sequences $\langle x_n \rangle_{n \in \mathbb{N}}, \langle y_n \rangle_{n \in \mathbb{N}}$ and $\langle z_n \rangle_{n \in \mathbb{N}}$ in A inductively, as follows. Set $y_0 = z_1^*$. Given that $y_n \geq z_1^*$, the set

$$C_n = \{z : z \in A, z \geq y_n, |(f_{nzi} - f_{ni})(y_n - z_1^*)| \leq 2^{-k_n} \text{ for every } i \leq k_n\}$$

belongs to \mathcal{F} , so is not empty; take $z_n \in C_n$. Because $y_n \geq z_1^*$, we have $f_{ni}(y_n - z_1^*) = 0$ and $|f_{nzn_i}(y_n - z_1^*)| \leq 2^{-k_n}$, for every $i \leq k_n$. Set $x_n = w_{z_n n}$, so that $x_n \in A, x_n \geq z_n$ and $\{y : |f_{nzn_i}(y - x_n)| \leq 2^{-k_n}$ for every $i \leq k_n\}$ is included in G_n . Now let $y_{n+1} \in A$ be such that $y_{n+1} \geq x_n$ and $f_{nzn_i}(y - y_{n+1}) = 0$ whenever $y \in A$ and $y \geq y_{n+1}$. Of course

$$y_{n+1} \geq x_n \geq z_n \geq y_n \geq z_1^*.$$

Continue.

At the end of the induction, let z_2^* be an upper bound for $\{y_n : n \in \mathbb{N}\}$ in A . For $n \in \mathbb{N}$, set

$$u_n = z_1^* + \sum_{j=0}^n x_j - y_j, \quad v_n = u_n + z_2^* - y_{n+1}.$$

Then $\langle u_n \rangle_{n \in \mathbb{N}}$ is non-decreasing and $u_n \leq v_n \leq z_2^*$ for every $n \in \mathbb{N}$; moreover,

$$v_n - v_{n+1} = y_{n+1} - x_{n+1} - y_{n+1} + y_{n+2} \geq 0$$

for every n , so $\langle v_n \rangle_{n \in \mathbb{N}}$ is non-increasing, and $u_m \leq v_n$ for all $m, n \in \mathbb{N}$. Because X has the σ -interpolation property, there is an $x \in X$ such that $u_n \leq x \leq v_n$ for every $n \in \mathbb{N}$. Since $z_1^* \leq x \leq z_2^*, x \in A$ and $x > z^*$.

Take any $n \in \mathbb{N}$. Then

$$0 \leq f_{nzn_i}(x - u_n) \leq f_{nzn_i}(v_n - u_n) = f_{nzn_i}(z_2^* - y_{n+1}) = 0$$

for every $i \leq k_n$. On the other hand,

$$x_n - u_n = (y_n - z_1^*) - \sum_{j=0}^{n-1} (x_j - y_j)$$

lies between 0 and $y_n - z_1^*$, so

$$0 \leq f_{nzn_i}(x_n - u_n) \leq 2^{-k_n}$$

for every $i \leq k_n$, and $|f_{nzn_i}(x - x_n)| \leq 2^{-k_n}$ for every i . Thus $x \in G_n$. As n is arbitrary, $x \in \bigcap_{n \in \mathbb{N}} G_n$.

As $x > z^*$, this shows that z^* is not an upper bound of $\bigcap_{n \in \mathbb{N}} G_n \cap A$. As z^* is arbitrary, $\bigcap_{n \in \mathbb{N}} G_n \notin \mathcal{I}$.

Q

(d) Set

$$\mathcal{K}_0 = \{G \setminus H : G, H \in \mathfrak{T}_s, G \notin \mathcal{I}, H \in \mathcal{I}\}.$$

Then $\bigcap_{n \in \mathbb{N}} E_n \neq \emptyset$ for any sequence $\langle E_n \rangle_{n \in \mathbb{N}}$ in \mathcal{K}_0 . **P** Express each E_n as $G_n \setminus H_n$ where G_n, H_n are \mathfrak{T}_s -open, $G_n \notin \mathcal{I}$ and $H_n \in \mathcal{I}$. Then $\bigcup_{n \in \mathbb{N}} H_n \in \mathcal{I}$, as noted in (a), while $\bigcap_{n \in \mathbb{N}} G_n \notin \mathcal{I}$, by (c); so $\bigcap_{n \in \mathbb{N}} E_n = \bigcap_{n \in \mathbb{N}} G_n \setminus \bigcup_{n \in \mathbb{N}} H_n$ is non-empty. **Q**

It follows that $\mathcal{K} = \mathcal{K}_0 \cup \{\emptyset\}$ is a countably compact class in the sense of 413M. Moreover, $E \cap E' \in \mathcal{K}$ for all $E, E' \in \mathcal{K}$ (using (a) and (c) again), so if we define $\phi_0 : \mathcal{K} \rightarrow \{0, 1\}$ by writing $\phi_0(E) = 1$ for $E \in \mathcal{K}_0$ and $\phi_0(\emptyset) = 0$, then \mathcal{K} and ϕ_0 will satisfy all the conditions of 413N. There is therefore a measure $\hat{\mu}$ on X extending ϕ_0 and inner regular with respect to \mathcal{K}_δ , the family of sets expressible as intersections of sequences

in \mathcal{K} . The domain of $\hat{\mu}$ must include every member of \mathcal{K} ; but if $G \in \mathfrak{T}_s$ then either G or $X \setminus G$ belongs to \mathcal{K}_0 , so is measured by $\hat{\mu}$, and $\hat{\mu}$ is a topological measure.

We need to observe that, because ϕ_0 takes only the values 0 and 1, $\hat{\mu}E \leq 1$ for every $E \in \mathcal{K}_\delta$, and $\hat{\mu}X \leq 1$; since $\phi_0X = 1$, $\hat{\mu}X = 1$ and $\hat{\mu}$ is a probability measure.

(e) We may therefore take μ to be the restriction of $\hat{\mu}$ to the algebra \mathcal{B} of \mathfrak{T}_s -Borel sets, and μ is a \mathfrak{T}_s -Borel probability measure. Now μ is not inner regular with respect to the \mathfrak{T}_s -closed sets. **P** For each $\xi < \omega_1$, $p_\xi < p_{\xi+1} < p_{\xi+2}$, so there are $g_\xi, h_\xi \in K$ such that $g_\xi(p_\xi) < g_\xi(p_{\xi+1})$ and $h_\xi(p_{\xi+1}) < h_\xi(p_{\xi+2})$. Let $D \subseteq \omega_1$ be any set such that D and $\omega_1 \setminus D$ are both uncountable, and set

$$G = \bigcup_{\xi \in D} \{x : g_\xi(p_\xi) < g_\xi(x), h_\xi(x) < h_\xi(p_{\xi+2})\}.$$

Then $G \in \mathfrak{T}_s$, and $p_{\xi+1} \in G$ for every $\xi \in D$, so $G \notin \mathcal{I}$ and $\mu G = \phi_0 G = 1$. On the other hand, if $\eta \in \omega_1 \setminus D$, then for every $\xi \in D$ either $\xi < \eta$ and $h_\xi(p_{\xi+2}) \leq h_\xi(p_{\eta+1})$, or $\xi > \eta$ and $g_\xi(p_{\eta+1}) \leq g_\xi(p_\xi)$; thus $p_{\eta+1} \notin G$ for any $\eta \in \omega_1 \setminus D$, and $X \setminus G \notin \mathcal{I}$. But this means that if $F \subseteq G$ is closed then $X \setminus F \in \mathcal{K}_0$ and $\mu F = 0$. Thus $\mu G > \sup_{F \subseteq G \text{ is closed}} \mu F$. **Q**

(f) Because \mathfrak{T}_s is regular, μ cannot be τ -additive, by 414Mb. It follows at once that (X, \mathfrak{T}_s) is not Borel-measure-complete, and in particular is not a Radon space. To see that μ has no extension to a norm-Borel measure, we need to look again at the set A . For each $\xi < \omega_1$, set $F_\xi = \{x : x \leq p_\xi\}$. Then every F_ξ is norm-closed (354Bc) and $\langle F_\xi \rangle_{\xi < \omega_1}$ is an increasing family with union A . Consequently, A is norm-closed (4A2Ld, 4A2Ka). At the same time, every F_ξ is convex (cf. 351Ce), so A is also convex. It follows that A , like every F_ξ , is \mathfrak{T}_s -closed (3A5Ee). So μ measures A and every F_ξ . Because $X \setminus F_\xi$ is a \mathfrak{T}_s -open set not belonging to \mathcal{I} , $\mu F_\xi = 0$, for every $\xi < \omega_1$; because $X \setminus A$ is a \mathfrak{T}_s -open set belonging to \mathcal{I} , $\mu A = 1$.

But ω_1 is a measure-free cardinal (438Cd), so 438I tells us that $\lambda A = \sup_{\xi < \omega_1} \lambda F_\xi$ for any semi-finite norm-Borel measure λ on X . Thus μ has no extension to a norm-Borel measure, and the proof is complete.

466I Examples The following spaces satisfy the hypotheses of 466H.

(a) (TALAGRAN 78A, or TALAGRAN 84, 16-1-2) $X = \ell^\infty(I)$ or $\{x : x \in \ell^\infty(I), \{i : x(i) \neq 0\} \text{ is countable}\}$, where I is uncountable. **P** X has the σ -interpolation property because it is Dedekind complete, and if $\langle i_\xi \rangle_{\xi < \omega_1}$ is any family of distinct elements of I , we can set $p_\xi(i_\eta) = 1$ for $\eta \leq \xi$, $p_\xi(i) = 0$ for all other $i \in I$ to obtain a strictly increasing family $\langle p_\xi \rangle_{\xi < \omega_1}$ in X . **Q**

(b) (DE MARIA & RODRIGUEZ-SALINAS 91) $X = \ell^\infty/\mathbf{c}_0$, where \mathbf{c}_0 is the space of real sequences converging to 0.

P (i) To see that X has the σ -interpolation property, let $A, B \subseteq X$ be non-empty countable sets such that $u \leq v$ for all $u \in A, v \in B$. Let $\langle x_n \rangle_{n \in \mathbb{N}}, \langle y_n \rangle_{n \in \mathbb{N}}$ be sequences in ℓ^∞ such that $A = \{x_n^\bullet : n \in \mathbb{N}\}$ and $B = \{y_n^\bullet : n \in \mathbb{N}\}$. Set $\tilde{x}_n = \sup_{i \leq n} x_i, \tilde{y}_n = \inf_{i \leq n} y_i$ for $n \in \mathbb{N}$; then $\tilde{x}_n^\bullet \leq \tilde{y}_n^\bullet$, so $(\tilde{x}_n - \tilde{y}_n)^+ \in \mathbf{c}_0$. Set

$$k_n = \max(\{n\} \cup \{i : \tilde{x}_n(i) \geq \tilde{y}_n(i) + 2^{-n}\})$$

for $n \in \mathbb{N}$, and define $x \in \ell^\infty$ by writing

$$\begin{aligned} x(i) &= 0 \text{ if } i \leq k_0, \\ &= \tilde{x}_n(i) \text{ if } k_n < i \leq k_{n+1}. \end{aligned}$$

Then it is easy to check that $u \leq x^\bullet \leq v$ for every $u \in A, v \in B$; as A and B are arbitrary, X has the σ -interpolation property.

(ii) To see that X has the other property, recall that there is a family $\langle I_\xi \rangle_{\xi < \omega_1}$ of infinite subsets of \mathbb{N} such that $I_\xi \setminus I_\eta$ is finite if $\eta \leq \xi$, infinite if $\xi < \eta$ (4A1Fa). Setting $p_\xi = \chi(\mathbb{N} \setminus I_\xi)^\bullet$, we have a strictly increasing family $\langle p_\xi \rangle_{\xi < \omega_1}$ in X . **Q**

466J Theorem Let X be a linear topological space and Σ its cylindrical σ -algebra. If μ and ν are probability measures with domain Σ such that $\int e^{if(x)} \mu(dx) = \int e^{if(x)} \nu(dx)$ for every $f \in X^*$, then $\mu = \nu$.

proof Define $T : X \rightarrow \mathbb{R}^{X^*}$ by setting $(Tx)(f) = f(x)$ for $f \in X^*, x \in X$. Then T is linear and continuous for the weak topology of X . So if $F \subseteq \mathbb{R}^{X^*}$ is a Baire set for the product topology of \mathbb{R}^{X^*} , $T^{-1}[F]$ is a

Baire set for the weak topology of X (4A3Kc) and belongs to Σ (4A3V). We therefore have Baire measures μ', ν' on \mathbb{R}^{X^*} defined by saying that $\mu'F = \mu T^{-1}[F]$ and $\nu'F = \nu T^{-1}[F]$ for every Baire set $F \subseteq \mathbb{R}^{X^*}$.

If $h : \mathbb{R}^{X^*} \rightarrow \mathbb{R}$ is a continuous linear functional, it can be expressed in the form $h(z) = \sum_{i=0}^n \alpha_i z(f_i)$ where $f_0, \dots, f_n \in X^*$ and $\alpha_0, \dots, \alpha_n \in \mathbb{R}$. So

$$h(Tx) = \sum_{i=0}^n \alpha_i (Tx)(f_i) = \sum_{i=0}^n \alpha_i f_i(x) = f(x)$$

for every $x \in X$, where $f = \sum_{i=0}^n \alpha_i f_i$. This means that

$$\int e^{ih(z)} \mu'(dz) = \int e^{ih(Tx)} \mu(dx) = \int e^{if(x)} \mu(dx) = \int e^{if(x)} \nu(dx) = \int e^{ih(z)} \nu'(dz).$$

As h is arbitrary, $\mu' = \nu'$ (454Pa).

Now let Σ' be the family of subsets of X of the form $T^{-1}[F]$ where $F \subseteq \mathbb{R}^{X^*}$ is a Baire set. This is a σ -algebra and contains all sets of the form $\{x : f(x) \geq \alpha\}$ where $f \in X^*$ and $\alpha \in \mathbb{R}$. So every member of X^* is Σ' -measurable and Σ' must include the cylindrical σ -algebra of X . Since μ and ν agree on Σ' they must be identical.

466K Proposition If X is a locally convex linear topological space and μ, ν are quasi-Radon probability measures on X such that $\int e^{if(x)} \mu(dx) = \int e^{if(x)} \nu(dx)$ for every $f \in X^*$, then $\mu = \nu$.

proof Write \mathfrak{T} for the given topology on X and $\mathfrak{T}_s = \mathfrak{T}_s(X, X^*)$ for the weak topology. By 466J, μ and ν must agree on the cylindrical σ -algebra Σ of X . Since Σ includes a base for \mathfrak{T}_s , every weakly open set G is the union of an upwards-directed family of open sets belonging to Σ ; as μ and ν are τ -additive, $\mu G = \nu G$. Consequently μ and ν agree on \mathfrak{T}_s -closed sets, and therefore on \mathfrak{T} -closed convex sets (4A4Ed). Write \mathcal{H} for the family of \mathfrak{T} -open sets which are expressible as the union of a non-decreasing sequence of \mathfrak{T} -closed convex sets; then μ and ν agree on \mathcal{H} . If τ is a \mathfrak{T} -continuous seminorm on X , $x_0 \in X$ and $\alpha > 0$, then $\{x : \tau(x - x_0) < \alpha\} \in \mathcal{H}$; and sets of this kind constitute a base for \mathfrak{T} (4A4Cb). Also the intersection of two members of \mathcal{H} belongs to \mathcal{H} . By 415H(v), $\mu = \nu$.

Remark This generalizes 285M and 454Xl, which are the special cases $X = \mathbb{R}^r$ (for finite r) and $X = \mathbb{R}^I$; see also 445Xq.

466L Proposition Suppose that X and Y are Banach spaces and that $T : X \rightarrow Y$ is a linear operator such that $gT : X \rightarrow \mathbb{R}$ is universally Radon-measurable, in the sense of 434Ec, for every $g \in Y^*$. Then T is continuous.

proof ? Suppose, if possible, otherwise. Then there is a $g \in Y^*$ such that gT is not continuous (4A4Ib). For each $n \in \mathbb{N}$, take $x_n \in X$ such that $\|x_n\| = 2^{-n}$ and $g(Tx_n) > 2n$. Define $h : \{0, 1\}^{\mathbb{N}} \rightarrow X$ by setting $h(t) = \sum_{n=0}^{\infty} t(n)x_n$ (4A4Ie). Then h is continuous, because $\|h(t) - h(t')\| \leq \sum_{n=0}^{\infty} 2^{-n} |t(n) - t'(n)|$ for all $t, t' \in \{0, 1\}^{\mathbb{N}}$. Let ν be the usual measure on $\{0, 1\}^{\mathbb{N}}$; then the image measure $\mu = \nu h^{-1}$ is a Radon measure on X (418I), so gT must be dom μ -measurable, and $\phi = gTh$ is dom ν -measurable. In this case there is an $m \in \mathbb{N}$ such that $E = \{t : |\phi(t)| \leq m\}$ has measure greater than $\frac{1}{2}$. But as $g(Tx_m) > 2m$, we see that if $t \in E$ then $t' \notin E$, where t' differs from t at the m th coordinate only, so that $|\phi(t) - \phi(t')| = g(Tx_m)$. Since the map $t \mapsto t'$ is an automorphism of the measure space $(\{0, 1\}^{\mathbb{N}}, \nu)$, $\nu E \leq \frac{1}{2}$, which is impossible. **X**

466M Corollary If X is a Banach space, Y is a separable Banach space, and $T : X \rightarrow Y$ is a linear operator such that the graph of T is a Souslin-F set in $X \times Y$, then T is continuous.

proof It will be enough to show that $T \upharpoonright Z$ is continuous for every separable closed linear subspace Z of X (because then it must be sequentially continuous, and we can use 4A2Ld). Write $\Gamma \subseteq X \times Y$ for the graph of T . If $H \subseteq Y$ is open, then $\Gamma \cap (Z \times H)$ is a Souslin-F set in the Polish space $Z \times Y$, so is analytic (423Eb), and its projection $(T \upharpoonright Z)^{-1}[H]$ also is analytic (423Bb), therefore universally measurable (434Dc). Thus $T \upharpoonright Z : Z \rightarrow Y$ is a universally measurable function, and $gT \upharpoonright Z$ must be universally measurable for any $g \in Y^*$ (434Df). By 466L, $T \upharpoonright Z$ is continuous; as Z is arbitrary, T is continuous.

466N Gaussian measures Some of the ideas of §456 can be adapted to the present context, as follows.

Definition If X is a linear topological space, I will say that a probability measure μ on X is a **centered Gaussian measure** if its domain includes the cylindrical σ -algebra of X and every continuous linear

functional on X is either zero almost everywhere or a normal random variable with zero expectation. (Thus a ‘centered Gaussian distribution’ on \mathbb{R}^I , as defined in 456A, is a distribution in the sense of 454K which is a centered Gaussian measure when \mathbb{R}^I is thought of as a linear topological space.)

Warning! many authors reserve the phrase ‘Gaussian measure’ for strictly positive measures.

466O Proposition Let X be a separable Banach space, and μ a probability measure on X . Suppose that there is a linear subspace W of X^* , separating the points of X , such that every element of W is $\text{dom } \mu$ -measurable and either zero a.e. or a normal random variable with zero expectation. Then μ is a centered Gaussian measure with respect to the norm topology of X .

proof (a) As W separates the points of X , $X \setminus \{0\} = \bigcup_{f \in W} \{x : f(x) \neq 0\}$. Because X is Polish, therefore hereditarily Lindelöf, there is a countable set $I \subseteq W$ still separating the points of X . Let W_0 be the linear subspace of X^* generated by I .

Define $T : X \rightarrow \mathbb{R}^I$ by setting $(Tx)(f) = f(x)$ for $f \in I$. Then T is an injective linear operator, and is continuous for $\mathfrak{T}_s(X, W_0)$ and the usual topology of \mathbb{R}^I . Let λ be the distribution of the family $\langle f \rangle_{f \in I}$; T is inverse-measure-preserving for $\hat{\mu}$ and λ , where $\hat{\mu}$ is the completion of μ (454J(iv)). If $g : \mathbb{R}^I \rightarrow \mathbb{R}$ is a continuous linear functional, then $gT \in W_0$ (use 4A4Be); now the distribution of g , with respect to the probability measure λ , is just the distribution of gT with respect to $\hat{\mu}$ and μ , and is therefore either normal or the Dirac measure concentrated at 0. So λ is a centered Gaussian distribution in the sense of 456Ab. Because I is countable, λ is a Radon measure (454J(iii)).

(b) If $\epsilon > 0$, there is a norm-compact $K \subseteq X$ such that $\hat{\mu}K$ is defined and is at least $1 - \epsilon$. **P** As X and \mathbb{R}^I are analytic (423B), there is a Radon measure μ' on X such that $\lambda = \mu'T^{-1}$ (432G). Of course $\mu'X = \lambda\mathbb{R}^I = 1$. There is a compact set $K \subseteq X$ such that $\mu'K \geq 1 - \epsilon$; now $T[K]$ is compact and $K = T^{-1}[T[K]]$, so

$$\hat{\mu}K = \mu T^{-1}[T[K]] = \lambda T[K] = \mu'K \geq 1 - \epsilon. \quad \mathbf{Q}$$

(c) Now suppose that $g \in X^*$. Then there is a sequence $\langle g_n \rangle_{n \in \mathbb{N}}$ in W_0 such that $\langle g_n \rangle_{n \in \mathbb{N}} \rightarrow g$ μ -a.e. **P** For each $n \in \mathbb{N}$, there is a compact set $K_n \subseteq X$ such that $\hat{\mu}K_n \geq 1 - 2^{-n-1}$; we can suppose that $K_{n+1} \supseteq K_n$ for each n . W_0 is dense in X^* for the weak*-topology $\mathfrak{T}_s(X^*, X)$ (4A4Eh); being convex, it is dense for the Mackey topology $\mathfrak{T}_k(X^*, X)$ (4A4F), and there is a $g_n \in W_0$ such that $\sup_{x \in K_n} |g_n(x) - g(x)| \leq 2^{-n}$. Now $g(x) = \lim_{n \rightarrow \infty} g_n(x)$ for every x in the μ -conegligible set $\bigcup_{n \in \mathbb{N}} K_n$. **Q**

Set $\sigma_n = \sqrt{\text{Var}(g_n)}$ for each n . Then $\{\sigma_n : n \in \mathbb{N}\}$ is bounded. **P** Set $M = \sup_{x \in K_0} |g(x)|$. If $n \in \mathbb{N}$ and $\sigma_n \neq 0$, $|g_n(x)| \leq M + 1$ for every $x \in K_0$, and

$$\frac{1}{2} \leq \mu K_0 \leq \Pr(|g_n| \leq M + 1) = \frac{1}{\sigma_n \sqrt{2\pi}} \int_{-M-1}^{M+1} e^{-t^2/2\sigma_n^2} dt \leq \frac{2(M+1)}{\sigma_n \sqrt{2\pi}},$$

so $\sigma_n \leq \frac{4(M+1)}{\sqrt{2\pi}}$. **Q**

We therefore have a strictly increasing sequence $\langle \sigma_{n_k} \rangle_{k \in \mathbb{N}}$ such that $\sigma = \lim_{k \rightarrow \infty} \sigma_{n_k}$ is defined in $[0, \infty[$. For each k , let ν_k be the distribution of g_{n_k} and φ_k its characteristic function; let ν , φ be the distribution and characteristic function of g . Since $\langle g_{n_k} \rangle_{k \in \mathbb{N}} \rightarrow g$ a.e.,

$$\int h d\nu = \int hg d\mu = \lim_{k \rightarrow \infty} \int hg_{n_k} d\mu = \lim_{k \rightarrow \infty} \int h d\nu_k$$

for every bounded continuous function $h : \mathbb{R} \rightarrow \mathbb{R}$, and $\varphi(t) = \lim_{k \rightarrow \infty} \varphi_k(t)$ for every $t \in \mathbb{R}$, by 285L. But, for each k ,

$$\varphi_k(t) = \exp\left(-\frac{1}{2}\sigma_{n_k}^2 t^2\right)$$

by 285E if $\sigma_{n_k} > 0$ and by direct calculation if $\sigma_{n_k} = 0$, as then $g_{n_k} = 0$ almost everywhere.

Accordingly $\varphi(t) = \exp(-\frac{1}{2}\sigma^2 t^2)$ for every t . But this means that ν is either the Dirac measure concentrated at 0 (if $\sigma = 0$) or a normal distribution with zero expectation (if $\sigma > 0$).

(d) As g is arbitrary, μ is a centered Gaussian measure.

466X Basic exercises (a) Let (X, \mathfrak{T}) be a metrizable locally convex linear topological space and μ a totally finite measure on X which is quasi-Radon for the topology \mathfrak{T} . Show that μ is quasi-Radon for the weak topology $\mathfrak{T}_s(X, X^*)$.

>(b) Let $\langle e_n \rangle_{n \in \mathbb{N}}$ be the usual orthonormal basis of ℓ^2 . Give ℓ^2 the Radon probability measure ν such that $\nu\{e_n\} = 2^{-n-1}$ for every n . Let I be an uncountable set, and set $X = (\ell^2)^I$ with the product linear structure and the product topology \mathfrak{T} , each copy of ℓ^2 being given its norm topology. (i) Let λ be the τ -additive product of copies of ν (417F). Show that λ is quasi-Radon for \mathfrak{T} but is not inner regular with respect to the \mathfrak{T}_s -closed sets. (ii) Write \mathfrak{T}_s for the weak topology of X . Let λ_s be the τ -additive product measure of copies of ν when each copy of ℓ^2 is given its weak topology instead of its norm topology. Show that λ_s is quasi-Radon for \mathfrak{T}_s but does not measure every \mathfrak{T} -Borel set. (*Hint*: setting $E = \{e_n : n \in \mathbb{N}\}$, $\lambda(E^I) = 1$ and E^I is relatively \mathfrak{T}_s -compact.)

>(c) Let X be a metrizable locally convex linear topological space and μ a τ -additive totally finite measure on X with domain the cylindrical σ -algebra of X . Show that μ has an extension to a quasi-Radon measure on X . (*Hint*: 4A3V, 415N.)

(d) Let X be a metrizable locally convex linear topological space which is Lindelöf in its weak topology, and Σ the cylindrical σ -algebra of X . Show that any totally finite measure with domain Σ has an extension to a quasi-Radon measure on X .

>(e) Let X be a separable Banach space. Show that it is a Radon space when given its weak topology.

(f) Let K be a compact metrizable space, and $C(K)$ the Banach space of continuous real-valued functions on K . Show that the σ -algebra of subsets of $C(K)$ generated by the functionals $x \mapsto x(t) : C(K) \rightarrow \mathbb{R}$, for $t \in K$, is just the cylindrical σ -algebra of $C(K)$. (*Hint*: 4A2Pe.) Examine the connexions between this and 454Sa, 462Z and 466Xd.

(g) Let K be a scattered compact Hausdorff space. Show that the weak topology and the topology of pointwise convergence on $C(K)$ have the same Borel σ -algebras.

(h) Re-write part (d) of the proof of 466H to avoid any appeal to results from §413.

(i) Let X be a locally convex linear topological space and μ, ν two totally finite quasi-Radon measures on X . Show that if μ and ν give the same measure to every half-space $\{x : f(x) \geq \alpha\}$, where $f \in X^*$ and $\alpha \in \mathbb{R}$, then $\mu = \nu$.

(j) Let X be a Hilbert space and μ, ν two totally finite Radon measures on X . Show that if μ and ν give the same measure to every ball $B(x, \delta)$, where $x \in X$ and $\delta \geq 0$, then $\mu = \nu$. (*Hint*: every open half-space is the union of a non-decreasing sequence of balls.)

(k) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function such that $f(1) = 1$ and $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Show that the following are equiveridical: (i) $f(x) = x$ for every $x \in \mathbb{R}$; (ii) f is continuous at some point; (iii) f is bounded on some non-empty open set; (iv) f is bounded on some Lebesgue measurable set of non-zero measure; (v) f is Lebesgue measurable; (vi) f is Borel measurable; *(vii) f is bounded on some non-meager G_δ set; *(viii) f is $\hat{\mathcal{B}}$ -measurable, where $\hat{\mathcal{B}}$ is the Baire-property algebra of \mathbb{R} . (*Hint*: 443Db.)

(l) Set $X = \{x : x \in \mathbb{R}^{\mathbb{N}}, \{n : x(n) \neq 0\} \text{ is finite}\}$, and give X any norm. Show that any linear operator from X to any normed space is universally measurable.

(m) Let X be a linear topological space and μ a centered Gaussian measure on X . (i) Let Y be another linear topological space and $T : X \rightarrow Y$ a continuous linear operator. Show that the image measure μT^{-1} is a centered Gaussian measure on Y . (ii) Show that $X^* \subseteq \mathcal{L}^2(\mu)$. (iii) Let us say that the **covariance matrix** of μ is the family $\langle \sigma_{fg} \rangle_{f, g \in X^*}$, where $\sigma_{fg} = \int f \times g d\mu$ for $f, g \in X^*$. Suppose that ν is another centered Gaussian measure on X with the same covariance matrix. Show that μ and ν agree on the cylindrical σ -algebra of X .

(n) Let $\langle X_i \rangle_{i \in I}$ be a family of linear topological spaces with product X . Suppose that for each i we have a centered Gaussian measure μ_i on X_i . Show that the product probability measure $\prod_{i \in I} \mu_i$ is a centered Gaussian measure on X .

(o) Let X be a linear topological space. Show that the convolution of two quasi-Radon centered Gaussian measures on X is a centered Gaussian measure.

(p) Let X be a separable Banach space, and μ a complete measure on X . Show that the following are equivalent: (i) μ is a centered Gaussian measure on X ; (ii) μ extends a centered Gaussian Radon measure on X ; (iii) there are a set I , an injective continuous linear operator $T : X \rightarrow \mathbb{R}^I$ and a centered Gaussian distribution λ on \mathbb{R}^I such that T is inverse-measure-preserving for μ and λ ; (iv) whenever I is a set and $T : X \rightarrow \mathbb{R}^I$ is a continuous linear operator there is a centered Gaussian distribution λ on \mathbb{R}^I such that T is inverse-measure-preserving for μ and λ .

(q) Let X be a Banach space, and μ a Radon measure on X . Show that, with respect to μ , the unit ball of X^* is a stable set of functions in the sense of §465. (*Hint*: 465Yb.)

(r) Let I be an infinite set. Show that Talagrand's measure, interpreted as a measure on $\ell^\infty(I)$ (464R), is not τ -additive for the weak topology.

466Y Further exercises (a) Give an example of a Hausdorff locally convex linear topological space (X, \mathfrak{T}) with a probability measure μ on X which is a Radon measure for the weak topology $\mathfrak{T}_s(X, X^*)$ but not for the topology \mathfrak{T} . (*Hint*: take $C = C([0, 1])$ and $X = C^*$ with the Mackey topology for the dual pair (X, C) , that is, the topology of uniform convergence on weakly compact subsets of C .)

(b) Let X be a normed space and \mathfrak{T} a linear space topology on X such that the unit ball of X is \mathfrak{T} -closed and the topology on the unit sphere S induced by \mathfrak{T} is finer than the norm topology on S . (i) Show that every norm-Borel subset of X is \mathfrak{T} -Borel. (ii) Show that if \mathfrak{T} is coarser than the norm topology, then it has a σ -isolated network.

(c) (i) Let X be a Banach space. Set $S_2 = \bigcup_{n \in \mathbb{N}} \{0, 1\}^n$ and suppose that $\langle K_\sigma \rangle_{\sigma \in S_2}$ is a family of non-empty weakly compact convex subsets of X such that $K_\sigma \subseteq K_\tau$ whenever $\sigma, \tau \in S_2$ and σ extends τ . (a) Show that there is a weakly Radon probability measure on X giving measure at least 2^{-n} to K_σ whenever $n \in \mathbb{N}$ and $\sigma \in \{0, 1\}^n$. (b) Show that there are a $\sigma \in S_2$ and $x \in K_{\sigma \frown \langle 0 \rangle}$, $y \in K_{\sigma \frown \langle 1 \rangle}$ such that $\|x - y\| \leq 1$. (ii) Let X be a locally convex Hausdorff linear topological space, and $\langle A_\sigma \rangle_{\sigma \in S_2}$ a family of non-empty relatively weakly compact subsets of X such that $A_\sigma \subseteq A_\tau$ whenever $\sigma, \tau \in S_2$ and σ extends τ . For $\sigma \in S_2$, set $C_\sigma = A_{\sigma \frown \langle 1 \rangle} - A_{\sigma \frown \langle 0 \rangle}$. Show that $0 \in \bigcup_{\sigma \in S_2} \overline{C_\sigma}$.

(d) Find Banach spaces X and Y and a linear operator from X to Y which is not continuous but whose graph is an F_σ set in $X \times Y$.

(e) Let X be a complete Hausdorff locally convex linear topological space and μ a Radon probability measure on X . Suppose that there is a linear subspace W of X^* , separating the points of X , such that every member of W is either zero a.e. or a normal random variable with zero expectation. Show that μ is a centered Gaussian measure.

466Z Problems (a) Does every probability measure defined on the $\mathfrak{T}_s(\ell^\infty, (\ell^\infty)^*)$ -Borel sets of $\ell^\infty = \ell^\infty(\mathbb{N})$ extend to a measure defined on the $\|\cdot\|_\infty$ -Borel sets?

It is by no means obvious that the Borel sets of ℓ^∞ are different for the weak and norm topologies; for a proof see TALAGRAND 78B.

(b) Assume that \mathfrak{c} is measure-free. Does it follow that ℓ^∞ , with its weak topology, is a Radon space?

Note that a positive answer to 464Z (with $I = \mathbb{N}$) would settle this, since Talagrand's measure, when interpreted as a measure on ℓ^∞ , cannot agree on the weakly Borel sets with any Radon measure on ℓ^∞ (466Xq, 466Xr).

466 Notes and comments I have given a proof of 466A using the machinery of §463; when the measure μ is known to be a Radon measure for the weak topology, rather than just a quasi-Radon measure or a τ -additive measure on the cylindrical algebra (466Xc), the theorem is older than this, and for an instructive alternative approach see TALAGRAND 84, 12-1-4. Another proof is in JAYNE & ROGERS 95.

On any Banach space we have at least three important σ -algebras: the norm-Borel σ -algebra (generated by the norm-open sets), the weak-Borel algebra (generated by the weakly open sets) and the cylindrical algebra (generated by the continuous linear functionals). (Note that the Baire σ -algebras corresponding to the norm and weak topologies are the norm-Borel algebra (4A3Kb) and the cylindrical algebra (4A3V).) If our Banach space is naturally represented as a subspace of some \mathbb{R}^I (e.g., because it is a space of continuous functions), then we have in addition the σ -algebra generated by the functionals $x \mapsto x(i)$ for $i \in I$, and the Borel algebra for the topology of pointwise convergence. We correspondingly have natural questions concerning when these algebras coincide, as in 466E and 4A3W and 466Xf, and when a measure on one of the algebras leads to a measure on another, as in 466A-466B and 466Xc.

The question of which Banach spaces are Radon spaces in their norm topologies is, if not exactly ‘solved’, at least reducible to a classical problem in set theory by the results in §438. It seems much harder to decide which non-separable Banach spaces are Radon spaces in their weak topologies. We have a simple positive result for spaces with Kadec norms (466F), and after some labour a negative result for a couple of standard examples (466I), but no effective general criterion is known. Even the case of ℓ^∞ seems still to be open in ‘ordinary’ set theories (466Zb). ℓ^∞ is of particular importance in this context because the dual of any separable Banach space is isometrically isomorphic to a linear subspace of ℓ^∞ (4A4Id). So a positive answer to either question in 466Z would have very interesting consequences – and would be correspondingly surprising.

466L and 466M belong to a large family of results of the general form: if, between spaces with both topological and algebraic structures, we have a homomorphism (for the algebraic structures) which is not continuous, then it is wildly irregular. I hope to return to some of these ideas in Volume 5. For the moment I just give a version of the classical result that an additive function $f : \mathbb{R} \rightarrow \mathbb{R}$ which is Lebesgue measurable must be continuous (466Xk). The definition of ‘universally measurable’ function which I gave in §434 has a number of paradoxical aspects. I have already remarked that in some contexts we might prefer to use the notion of ‘universally Radon-measurable’ function; this is also appropriate for 466L. But when our space X , for any reason, has few Borel measures, as in 466Xl, there are correspondingly many universally measurable functions defined on X . Of course 466M can also be thought of as a generalization of the closed graph theorem; but note that, unlike the closed graph theorem, it needs a separable codomain (466Yd).

The point of 466O is that the most familiar separable Banach spaces are presented with continuous linear embeddings into $\mathbb{R}^{\mathbb{N}}$, and of course any separable Banach space X has such a presentation. We can now describe the centered Gaussian Radon measures on X in terms of centered Gaussian distributions on $\mathbb{R}^{\mathbb{N}}$, as in 466Xp. But perhaps the most important centered Gaussian measure is Wiener measure (477Yj), which is not in fact on a Banach space.

A curious geometric question concerning measures on metric spaces is the following. If two totally finite Radon measures on a metric space agree on balls, must they be identical? It is known that (even for compact spaces) the answer, in general, is ‘no’ (DAVIES 71); in Hilbert spaces the answer is ‘yes’ (466Xj); and in fact the same is true in any normed space (PREISS & TIŠER 91).

Version of 13.1.10

*467 **Locally uniformly rotund norms**

In the last section I mentioned Kadec norms. These are interesting in themselves, but the reason for including them in this book is that in a normed space with a Kadec norm the weak topology has the same Borel sets as the norm topology (466Ea). The same will evidently be true of any space which has an equivalent Kadec norm. Now Kadec norms themselves are not uncommon, but equivalent Kadec norms appear in a striking variety of cases. Here I describe the principal class of spaces (the ‘weakly K-countably determined’ Banach spaces, 467H) which have equivalent Kadec norms. In fact they have ‘locally uniformly rotund’ norms, which are much easier to do calculations with.

Almost everything here is pure functional analysis, mostly taken from DEVILLE GODEFROY & ZIZLER 93, which is why I have starred the section. The word ‘measure’ does not appear until 467P. At that point,

however, we find ourselves with a striking result (Schachermayer's theorem) which appears to need the structure theory of weakly compactly generated Banach spaces developed in 467C-467M.

467A Definition Let X be a linear space with a norm $\|\cdot\|$. $\|\cdot\|$ is **locally uniformly rotund** or **locally uniformly convex** if whenever $\|x\| = 1$ and $\epsilon > 0$, there is a $\delta > 0$ such that $\|x - y\| \leq \epsilon$ whenever $\|y\| = 1$ and $\|x + y\| \geq 2 - \delta$.

If X has a locally uniformly rotund norm, then every subspace of X has a locally uniformly rotund norm. Of course any uniformly convex norm (definition: 2A4K) is locally uniformly rotund.

467B Proposition A locally uniformly rotund norm is a Kadec norm.

proof Let X be a linear space with a locally uniformly rotund norm $\|\cdot\|$. Set $S_X = \{x : \|x\| = 1\}$. Suppose that G is open for the norm topology and that $x \in G \cap S_X$. Then there is an $\epsilon > 0$ such that $G \supseteq B(x, \epsilon) = \{y : \|y - x\| \leq \epsilon\}$. Let $\delta > 0$ be such that $\|x - y\| \leq \epsilon$ whenever $\|y\| = 1$ and $\|x + y\| \geq 2 - \delta$. Now there is an $f \in X^*$ such that $f(x) = \|f\| = 1$ (3A5Ac). So $V = \{y : f(y) > 1 - \delta\}$ is open for the weak topology. But if $y \in V \cap S_X$, then $\|x + y\| \geq f(x + y) \geq 2 - \delta$, so $\|x - y\| \leq \epsilon$ and $y \in G$. As x is arbitrary, $G \cap S_X$ is open for the weak topology on S_X ; as G is arbitrary, the norm and weak topologies agree on S_X .

467C A technical device (a) I will use the following notation for the rest of the section. Let X be a linear space and $p : X \rightarrow [0, \infty[$ a seminorm. Define $q_p : X \times X \rightarrow [0, \infty[$ by setting

$$q_p(x, y) = 2p(x)^2 + 2p(y)^2 - p(x + y)^2 = (p(x) - p(y))^2 + (p(x) + p(y))^2 - p(x + y)^2$$

for $x \in X$.

(b) A norm $\|\cdot\|$ on X is locally uniformly rotund iff whenever $x \in X$ and $\epsilon > 0$ there is a $\delta > 0$ such that $\|x - y\| \leq \epsilon$ whenever $q_{\|\cdot\|}(x, y) \leq \delta$.

P(i) Suppose that $\|\cdot\|$ is locally uniformly rotund, $x \in X$ and $\epsilon > 0$. (α) If $x = 0$ then $q_{\|\cdot\|}(x, y) = \|y\|^2 = \|x - y\|^2$ for every y so we can take $\delta = \epsilon^2$. (β) If $x \neq 0$ set $x' = \frac{1}{\|x\|}x$. Let $\eta > 0$ be such that $\|x' - y'\| \leq \frac{1}{2}\epsilon\|x\|$ whenever $\|y'\| = 1$ and $\|x' + y'\| \geq 2 - \eta$. Let $\delta > 0$ be such that

$$\delta + 2\sqrt{\delta}\|x\| \leq \eta\|x\|^2, \quad \sqrt{\delta} < \|x\|, \quad \sqrt{\delta} \leq \frac{1}{2}\epsilon\|x\|^2.$$

Now if $q_{\|\cdot\|}(x, y) \leq \delta$, we must have

$$(\|x\| - \|y\|)^2 \leq \delta < \|x\|^2, \quad (\|x\| + \|y\|)^2 - \|x + y\|^2 \leq \delta,$$

so that $y \neq 0$ and

$$\left| \frac{1}{\|x\|} - \frac{1}{\|y\|} \right| \|y\| = \frac{\|y\| - \|x\|}{\|x\|} \leq \frac{\sqrt{\delta}}{\|x\|},$$

$$\|x\| + \|y\| - \|x + y\| \leq \frac{\delta}{\|x\| + \|y\| + \|x + y\|} \leq \frac{\delta}{\|x\|}.$$

Set $y' = \frac{1}{\|y\|}y$, $y'' = \frac{1}{\|x\|}y$. Then $\|y'\| = 1$, and

$$\|y' - y''\| = \left| \frac{1}{\|y\|} - \frac{1}{\|x\|} \right| \|y\| \leq \frac{\sqrt{\delta}}{\|x\|} \leq \frac{1}{2}\epsilon\|x\|.$$

Accordingly

$$\begin{aligned} \|x' + y'\| &\geq \|x' + y''\| - \|y' - y''\| \geq \frac{1}{\|x\|}\|x + y\| - \frac{\sqrt{\delta}}{\|x\|} \\ &\geq \frac{1}{\|x\|}(\|x\| + \|y\| - \frac{\delta}{\|x\|}) - \frac{\sqrt{\delta}}{\|x\|} = 1 + \frac{\|y\|}{\|x\|} - \frac{\delta}{\|x\|^2} - \frac{\sqrt{\delta}}{\|x\|} \\ &\geq 1 + \frac{\|x\| - \sqrt{\delta}}{\|x\|} - \frac{\delta}{\|x\|^2} - \frac{\sqrt{\delta}}{\|x\|} = 2 - \frac{\delta}{\|x\|^2} - \frac{2\sqrt{\delta}}{\|x\|} \geq 2 - \eta. \end{aligned}$$

But this means that $\|x' - y'\| \leq \frac{1}{2}\epsilon\|x\|$, so that $\|x' - y''\| \leq \epsilon\|x\|$ and $\|x - y\| \leq \epsilon$. As x and ϵ are arbitrary, the condition is satisfied.

(ii) Suppose the condition is satisfied. If $\|x\| = 1$ and $\epsilon > 0$, take $\delta \in]0, 2]$ such that $\|x - y\| \leq \epsilon$ whenever $q(x, y) \leq 4\delta$; then if $\|y\| = 1$ and $\|x + y\| \geq 2 - \delta$, $q(x, y) = 4 - \|x + y\|^2 \leq 4\delta$ and $\|x - y\| \leq \epsilon$. As x and ϵ are arbitrary, $\|\cdot\|$ is locally uniformly rotund. **Q**

(c) We have the following elementary facts. Let X be a linear space.

(i) For any seminorm p on X , $q_p(x, y) \geq (p(x) - p(y))^2 \geq 0$ for all $x, y \in X$. **P** $(p(x) + p(y))^2 - p(x + y)^2 \geq 0$ because $p(x + y) \leq p(x) + p(y)$. **Q**

(ii) Suppose that $\langle p_i \rangle_{i \in I}$ is a family of seminorms on X such that $\sum_{i \in I} p_i(x)^2$ is finite for every $x \in X$. Set $p(x) = \sqrt{\sum_{i \in I} p_i(x)^2}$ for $x \in X$; then p is a seminorm on X and $q_p = \sum_{i \in I} q_{p_i}$. **P** Of course $p(\alpha x) = |\alpha|p(x)$ for $\alpha \in \mathbb{R}$ and $x \in X$. If $x \in X$, then $p(x) = \|\phi(x)\|_2$, where $\phi(x) = \langle p_i(x) \rangle_{i \in I} \in \ell^2(I)$. Now for $x, y \in X$,

$$0 \leq \phi(x + y) \leq \phi(x) + \phi(y)$$

in $\ell^2(I)$, so

$$p(x + y) = \|\phi(x + y)\|_2 \leq \|\phi(x) + \phi(y)\|_2 \leq \|\phi(x)\|_2 + \|\phi(y)\|_2 = p(x) + p(y).$$

Thus p is a seminorm. Now the calculation of $q_p = \sum_{i \in I} q_{p_i}$ is elementary. **Q** In particular, $q_p \geq q_{p_i}$ for every $i \in I$.

(iii) If $\|\cdot\|$ is an inner product norm on X , then $q_{\|\cdot\|}(x, y) = \|x - y\|^2$ for all $x, y \in X$. **P**

$$\begin{aligned} 2\|x\|^2 + 2\|y\|^2 - \|x + y\|^2 &= 2(x|x) + 2(y|y) - (x + y|x + y) \\ &= (x|x) + (y|y) - (x|y) - (y|x) = (x - y|x - y). \end{aligned} \quad \mathbf{Q}$$

467D Lemma Let $(X, \|\cdot\|)$ be a normed space. Suppose that there are a space Y with a locally uniformly rotund norm $\|\cdot\|_Y$ and a bounded linear operator $T : Y \rightarrow X$ such that $T[Y]$ is dense in X and, for every $x \in X$ and $\gamma > 0$, there is a $z \in Y$ such that $\|x - Tz\|^2 + \gamma\|z\|_Y^2 = \inf_{y \in Y} \|x - Ty\|^2 + \gamma\|y\|_Y^2$. Then X has an equivalent locally uniformly rotund norm.

proof (a) For each $n \in \mathbb{N}$, $x \in X$ set

$$p_n(x) = \sqrt{\inf_{y \in Y} \|x - Ty\|^2 + 2^{-n}\|y\|_Y^2}.$$

Then $p_n : X \rightarrow [0, \infty[$ is a norm on X , equivalent to $\|\cdot\|$. **P** (i) The functionals $(x, y) \mapsto \|x - Ty\|$, $(x, y) \mapsto 2^{-n/2}\|y\|_Y$ from $X \times Y$ to $[0, \infty[$ are both seminorms, so the functional $(x, y) \mapsto \phi(x, y) = \sqrt{\|x - Ty\|^2 + 2^{-n}\|y\|_Y^2}$ also is, by 467C(c-ii). (ii) If $x \in X$ and $\alpha \in \mathbb{R}$, take $z \in Y$ such that $p_n(x) = \phi(x, z)$; then

$$p_n(\alpha x) \leq \phi(\alpha x, \alpha z) = |\alpha|\phi(x, z) = |\alpha|p_n(x).$$

If $\alpha \neq 0$, apply the same argument to see that $p_n(x) \leq |\alpha|^{-1}p_n(\alpha x)$, so that $p_n(\alpha x) = |\alpha|p_n(x)$. (iii) Now take any $x_1, x_2 \in X$. Let $z_1, z_2 \in Y$ be such that $p_n(x_i) = \phi(x_i, z_i)$ for both i . Then

$$\begin{aligned} p_n(x_1 + x_2) &= 2p_n\left(\frac{1}{2}x_1 + \frac{1}{2}x_2\right) \leq 2\phi\left(\frac{1}{2}x_1 + \frac{1}{2}x_2, \frac{1}{2}z_1 + \frac{1}{2}z_2\right) \\ &\leq 2\left(\frac{1}{2}\phi(x_1, z_1) + \frac{1}{2}\phi(x_2, z_2)\right) = p_n(x_1) + p_n(x_2). \end{aligned}$$

Thus p_n is a seminorm. (iv) $p_n(x) \leq \phi(x, 0) = \|x\|$ for every $x \in X$. (vi) For any $x \in X$ and $y \in Y$, either $\|Ty\| \geq \frac{1}{2}\|x\|$ and

$$\phi(x, y) \geq 2^{-n/2}\|y\|_Y \geq \frac{1}{2 \cdot 2^{n/2}\|T\|}\|x\|,$$

or $\|Ty\| \leq \frac{1}{2}\|x\|$ and

$$\phi(x, y) \geq \|x - Ty\| \geq \frac{1}{2}\|x\|;$$

this shows that $p_n(x) \geq \min(\frac{1}{2}, \frac{1}{2}2^{-n/2}\|T\|^{-1})\|x\|$. (I am passing over the trivial case $X = \{0\}$, $\|T\| = 0$.) In particular, $p_n(x) = 0$ only when $x = 0$. Thus p_n is a norm equivalent to p . **Q**

(b) For any $x \in X$, $\lim_{n \rightarrow \infty} p_n(x) = 0$. **P** Let $\epsilon > 0$. Let $y \in Y$ be such that $\|x - Ty\| \leq \epsilon$; then

$$\limsup_{n \rightarrow \infty} p_n(x)^2 \leq \limsup_{n \rightarrow \infty} \|x - Ty\|^2 + 2^{-n}\|y\|_Y^2 \leq \epsilon^2.$$

As ϵ is arbitrary, $\lim_{n \rightarrow \infty} p_n(x) = 0$. **Q**

(c) Set $\|x\|' = \sqrt{\sum_{n=0}^{\infty} 2^{-n}p_n(x)^2}$ for $x \in X$. The sum is always finite because $p_n(x) \leq \|x\|$ for every n , so $\| \cdot \|' \leq \sqrt{2}\| \cdot \|$ is a seminorm; and it is a norm equivalent to $\| \cdot \|$ because p_0 is. Now $\| \cdot \|'$ is locally uniformly rotund. **P** Take $x \in X$ and $\epsilon > 0$. Let $n \in \mathbb{N}$ be such that $p_n(x) \leq \frac{1}{4}\epsilon$. Choose $y \in Y$ such that $p_n(x)^2 = \|x - Ty\|^2 + 2^{-n}\|y\|_Y^2$. (This is where we really use the hypothesis that the infimum in the definition of p_n is attained.) Let $\delta > 0$ be such that $2^n\delta \leq (\frac{1}{4}\epsilon)^2$ and $\|T\|\|y' - y\| \leq \frac{1}{4}\epsilon$ whenever $q_{\| \cdot \|_Y}(y', y) \leq 2^{2n}\delta$.

If $q_{\| \cdot \|'}(x, x') \leq \delta$, then $q_{2^{-n/2}p_n}(x, x') \leq \delta$, by 467C(c-ii), that is, $q_{p_n}(x, x') \leq 2^n\delta$. Let $y' \in Y$ be such that $p_n(x')^2 = \|x' - Ty'\|^2 + 2^{-n}\|y'\|_Y^2$. Then

$$\begin{aligned} p_n(x + x')^2 &\leq \|x + x' - Ty - Ty'\|^2 + 2^{-n}\|y + y'\|_Y^2 \\ &\leq (\|x - Ty\| + \|x' - Ty'\|)^2 + 2^{-n}\|y + y'\|_Y^2, \end{aligned}$$

so

$$\begin{aligned} q_{p_n}(x, x') &= 2p_n(x)^2 + 2p_n(x')^2 - p_n(x + x')^2 \\ &\geq 2(\|x - Ty\|^2 + 2^{-n}\|y\|_Y^2) + 2(\|x' - Ty'\|^2 + 2^{-n}\|y'\|_Y^2) \\ &\quad - (\|x - Ty\| + \|x' - Ty'\|)^2 - 2^{-n}\|y + y'\|_Y^2 \\ &= (\|x - Ty\| - \|x' - Ty'\|)^2 + 2^{-n}(2\|y\|_Y^2 + 2\|y'\|_Y^2 - \|y + y'\|_Y^2). \end{aligned}$$

This means that

$$q_{\| \cdot \|_Y}(y, y') \leq 2^n q_{p_n}(x, x') \leq 2^{2n}\delta,$$

so $\|T\|\|y - y'\|_Y \leq \frac{1}{4}\epsilon$, while also

$$\|x - Ty\| + \|x' - Ty'\| \leq 2\|x - Ty\| + \sqrt{q_{p_n}(x, x')} \leq 2p_n(x) + 2^{n/2}\sqrt{\delta} \leq \frac{3}{4}\epsilon.$$

Finally

$$\frac{1}{\sqrt{2}}\|x - x'\|' \leq \|x - x'\| \leq \|T\|\|y - y'\|_Y + \|x - Ty\| + \|x' - Ty'\| \leq \frac{1}{4}\epsilon + \frac{3}{4}\epsilon = \epsilon.$$

As x and ϵ are arbitrary, this shows that $\| \cdot \|'$ is locally uniformly rotund. **Q**

This completes the proof.

467E Theorem Let X be a separable normed space. Then it has an equivalent locally uniformly rotund norm.

proof (a) It is enough to show that the completion of X has an equivalent locally uniformly rotund norm; since the completion of X is separable, we may suppose that X itself is complete. Let $\langle x_i \rangle_{i \in \mathbb{N}}$ be a sequence in X running over a dense subset of X . Define $T : \ell^2 \rightarrow X$ by setting

$$Ty = \sum_{i=0}^{\infty} \frac{y(i)}{2^i(1+\|x_i\|)} x_i$$

for $y \in \ell^2 = \ell^2(\mathbb{N})$; then Ty is always defined (4A4Ie); T is a linear operator and

$$\|Ty\| \leq \sum_{i=0}^{\infty} 2^{-i}|y(i)| \leq \sqrt{\sum_{i=0}^{\infty} 2^{-2i}} \|y\|_2$$

for every $y \in \ell^2$, by Cauchy's inequality (244Eb). So T is a bounded linear operator.

(b) T satisfies the conditions of 467D. **P** $T[\ell^2]$ is dense because it contains every x_i . Given $x \in X$, $\gamma > 0$ and $\alpha \geq 0$, the function $y \mapsto \sqrt{\|x - Ty\|^2 + \gamma\|y\|_2^2}$ is convex and norm-continuous, so the set

$$C_\alpha(x) = \{y : y \in \ell^2, \|x - Ty\|^2 + \gamma\|y\|_2^2 \leq \alpha^2\}$$

is convex and norm-closed. Consequently, $C_\alpha(x)$ is weakly closed (4A4Ed); since $\|y\|_2 \leq \gamma^{-1/2}\alpha$ for every $y \in C_\alpha(x)$, $C_\alpha(x)$ is weakly compact (4A4Ka). Set $\beta = \inf_{y \in \ell^2} \|x - Ty\|^2 + \gamma\|y\|_2^2$. Then $\{C_\alpha(x) : \alpha > \beta\}$ is a downwards-directed set of non-empty weakly compact sets, so has non-empty intersection; taking any $z \in \bigcap_{\alpha > \beta} C_\alpha(x)$, $\beta = \|x - Tz\|^2 + \gamma\|z\|_2^2$. **Q**

(c) So 467D gives the result.

467F Lemma Let $(X, \|\cdot\|)$ be a Banach space, and $\langle T_i \rangle_{i \in I}$ a family of bounded linear operators from X to itself such that

- (i) for each $i \in I$, the subspace $T_i[X]$ has an equivalent locally uniformly rotund norm,
- (ii) for each $x \in X$, $\epsilon > 0$ there is a finite set $J \subseteq I$ such that $\|x - \sum_{i \in J} T_i x\| \leq \epsilon$,
- (iii) for each $x \in X$, $\epsilon > 0$ the set $\{i : i \in I, \|T_i x\| \geq \epsilon\}$ is finite.

Then X has an equivalent locally uniformly rotund norm.

proof (a) Let $\|\cdot\|_i$ be a locally uniformly rotund norm on $X_i = T_i[X]$ equivalent to $\|\cdot\|$ on X_i . Reducing $\|\cdot\|_i$ by a scalar multiple if necessary, we may suppose that $\|T_i x\|_i \leq \|x\|$ for every $x \in X$ and $i \in I$. By (iii), $\sup_{i \in I} \|T_i x\|$ is finite for every $x \in X$; by the Uniform Boundedness Theorem (3A5Ha), $M = \sup_{i \in I} \sup_{\|x\| \leq 1} \|T_i x\|$ is finite. (This is where we need to suppose that X is complete.) For finite sets $J \subseteq I$ and $k \geq 1$, set

$$p_{Jk}(x) = \sqrt{\sum_{i \in J} \|T_i x\|_i^2 + \frac{1}{k} \sum_{K \subseteq J} \|x - \sum_{i \in K} T_i x\|^2};$$

for $n \in \mathbb{N}$ and $k \geq 1$ set

$$p_k^{(n)}(x) = \sup\{p_{Jk}(x) : J \subseteq I, \#(J) \leq n\}.$$

By 467C(c-ii), as usual, all the p_{Jk} are seminorms, and it follows at once that the $p_k^{(n)}$ are seminorms. Observe that if $K \subseteq I$ is finite, then $\|x - \sum_{i \in K} T_i x\| \leq (1 + M\#(K))\|x\|$ for every x , so if $J \subseteq I$ is finite then

$$p_{Jk}(x) \leq \sqrt{\#(J) + 2\#(J)(1 + M\#(J))}\|x\|,$$

and $p_k^{(n)}(x) \leq \sqrt{n + 2^n(1 + Mn)}\|x\|$ whenever $n \in \mathbb{N}$ and $k \geq 1$. Setting $\beta_{nk} = 2^{2n+k}$ for $n, k \in \mathbb{N}$,

$$\|x\|' = \sqrt{\sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \beta_{nk}^{-1} p_k^{(n)}(x)^2}$$

is finite for every $x \in X$, so that $\|\cdot\|'$ is a seminorm on X ; moreover, $\|x\|' \leq \beta\|x\|$ for every $x \in X$, where

$$\beta = \sqrt{\sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \beta_{nk}^{-1} (n + 2^n(1 + Mn))}$$

is finite. Since we also have

$$\|x\|' \geq \frac{1}{\sqrt{2}} p_1^{(0)}(x) = \frac{1}{\sqrt{2}} p_{01}(x) = \frac{1}{\sqrt{2}} \|x\|$$

for every x , $\|\cdot\|'$ is a norm on X equivalent to $\|\cdot\|$.

(b) Now $\|\cdot\|'$ is locally uniformly rotund. **P** Take $x \in X$ and $\epsilon > 0$. Let $K \subseteq I$ be a finite set such that $\|x - \sum_{i \in K} T_i x\| \leq \frac{1}{4}\epsilon$; we may suppose that $T_i x \neq 0$ for every $i \in K$. Set $\alpha_1 = \min_{i \in K} \|T_i x\|_i$, $J = \{i : i \in I, \|T_i x\|_i \geq \alpha_1\}$ and $\alpha_0 = \sup_{i \in I \setminus J} \|T_i x\|_i$. (For completeness, if $K = \emptyset$, take $J = \emptyset$, $\alpha_0 = \sup_{i \in I} \|T_i x\|_i$ and $\alpha_1 = \alpha_0 + 1$.) Then J is finite and $\alpha_0 < \alpha_1$, by hypothesis (iii) of the lemma. Set $n = \#(J)$. Let k be so large that $\frac{2^n(Mn+1)}{k} \|x\|^2 < \frac{1}{2}(\alpha_1^2 - \alpha_0^2)$. Let $\eta > 0$ be such that

$$(\beta_{nk} + 1)\eta \leq \min\left(\frac{\epsilon^2}{16k}, \alpha_1^2 - \alpha_0^2\right),$$

$$\|T_i x - z\| \leq \frac{\epsilon}{1+4n} \text{ whenever } i \in K, z \in X_i \text{ and } q_{\|\cdot\|_i}(T_i x, z) \leq (\beta_{nk} + 1)\eta;$$

this is where we use the hypothesis that every $\|\cdot\|_i$ is locally uniformly rotund (and equivalent to $\|\cdot\|$ on X_i).

Now suppose that $y \in X$ and $q_{\|\cdot\|'}(x, y) \leq \eta$. Then $q_{p_k^{(n)}}(x, y) \leq \beta_{nk}\eta$, by 467C(c-ii). Let $L \in [I]^n$ be such that $p_k^{(n)}(x+y)^2 \leq p_{Lk}(x+y)^2 + \eta$. Then

$$\begin{aligned} q_{p_{Lk}}(x, y) &= 2p_{Lk}(x)^2 + 2p_{Lk}(y)^2 - p_{Lk}(x+y)^2 \\ &\leq 2p_k^{(n)}(x)^2 + 2p_k^{(n)}(y)^2 - p_k^{(n)}(x+y)^2 + \eta \leq (\beta_{nk} + 1)\eta. \end{aligned}$$

We also have

$$\begin{aligned} 2p_{Lk}(x)^2 &\geq 2p_{Lk}(x)^2 + 2p_{Lk}(y)^2 - 2p_k^{(n)}(y)^2 \geq p_{Lk}(x+y)^2 - 2p_k^{(n)}(y)^2 \\ &\geq p_k^{(n)}(x+y)^2 - \eta - 2p_k^{(n)}(y)^2 \geq 2p_k^{(n)}(x)^2 - (\beta_{nk} + 1)\eta, \end{aligned}$$

so

$$\begin{aligned} \sum_{i \in J} \|T_i x\|_i^2 &\leq p_{Jk}(x)^2 \leq p_k^{(n)}(x)^2 \leq p_{Lk}(x)^2 + \frac{1}{2}(\beta_{nk} + 1)\eta \\ &\leq \sum_{i \in L} \|T_i x\|_i^2 + \frac{2^n(Mn+1)}{k} \|x\|^2 + \frac{1}{2}(\beta_{nk} + 1)\eta \\ &< \sum_{i \in L} \|T_i x\|_i^2 + \alpha_1^2 - \alpha_0^2. \end{aligned}$$

Since $\#(L) = \#(J)$ and

$$\|T_i x\|_i^2 \leq \alpha_0^2 < \alpha_1^2 \leq \|T_j x\|_j^2$$

whenever $j \in J$ and $i \in I \setminus J$, we must actually have $L = J$. In particular, $K \subseteq L$. But this means that (by 467C(c-ii) again)

$$q_{\|\cdot\|'}(T_i x, T_i y) \leq q_{p_{Lk}}(x, y) \leq (\beta_{nk} + 1)\eta$$

and (by the choice of η) $\|T_i x - T_i y\| \leq \frac{\epsilon}{1+4n}$ for every $i \in K$, so that $\|\sum_{i \in K} T_i x - \sum_{i \in K} T_i y\| \leq \frac{1}{4}\epsilon$.

The last element we need is that, setting $\tilde{p}(z) = \frac{1}{\sqrt{k}}\|z - \sum_{i \in K} T_i z\|$, \tilde{p} is a seminorm on X and is one of the constituents of p_{Lk} ; so that

$$\begin{aligned} \frac{1}{k}(\|x - \sum_{i \in K} T_i x\| - \|y - \sum_{i \in K} T_i y\|)^2 &\leq q_{\tilde{p}}(x, y) \leq q_{p_{Lk}}(x, y) \\ &\leq (\beta_{nk} + 1)\eta \leq \frac{\epsilon^2}{16k}, \end{aligned}$$

and $\|x - \sum_{i \in K} T_i x\| - \|y - \sum_{i \in K} T_i y\| \leq \frac{1}{4}\epsilon$. It follows that

$$\|y - \sum_{i \in K} T_i y\| \leq \frac{1}{4}\epsilon + \|x - \sum_{i \in K} T_i x\| \leq \frac{1}{2}\epsilon.$$

Putting these together,

$$\|x - y\| \leq \|x - \sum_{i \in K} T_i x\| + \sum_{i \in K} \|T_i x - T_i y\| + \|y - \sum_{i \in K} T_i y\| \leq \epsilon.$$

And this is true whenever $q_{\|\cdot\|'}(x, y) \leq \eta$. As x and ϵ are arbitrary, $\|\cdot\|'$ is locally uniformly rotund. **Q**

467G Theorem Let X be a Banach space. Suppose that there are an ordinal ζ and a family $\langle P_\xi \rangle_{\xi \leq \zeta}$ of bounded linear operators from X to itself such that

- (i) if $\xi \leq \eta \leq \zeta$ then $P_\xi P_\eta = P_\eta P_\xi = P_\xi$;
- (ii) $P_0(x) = 0$ and $P_\zeta(x) = x$ for every $x \in X$;
- (iii) if $\xi \leq \zeta$ is a non-zero limit ordinal, then $\lim_{\eta \uparrow \xi} P_\eta(x) = P_\xi(x)$ for every $x \in X$;
- (iv) if $\xi < \zeta$ then $X_\xi = \{(P_{\xi+1} - P_\xi)(x) : x \in X\}$ has an equivalent locally uniformly rotund norm.

Then X has an equivalent locally uniformly rotund norm.

Remark A A family $\langle P_\xi \rangle_{\xi \leq \zeta}$ satisfying (i), (ii) and (iii) here is called a **projectional resolution of the identity**.

proof For $\xi < \zeta$ set $T_\xi = P_{\xi+1} - P_\xi$. From condition (i) we see easily that $T_\xi T_\eta = T_\xi$ if $\xi = \eta$, 0 otherwise; and that $T_\xi P_\eta = T_\xi$ if $\xi < \eta$, 0 otherwise.

I seek to show that the conditions of 467F are satisfied by $\langle T_\xi \rangle_{\xi < \zeta}$. Condition (i) of 467F is just condition (iv) here. Let Z be the set of those $x \in X$ for which conditions (ii) and (iii) of 467F are satisfied; that is,

for each $\epsilon > 0$ there is a finite set $J \subseteq \zeta$ such that $\|x - \sum_{\xi \in J} T_\xi x\| \leq \epsilon$, and $\{\xi : \|T_\xi x\| \geq \epsilon\}$ is finite.

Then Z is a linear subspace of X . For $\xi \leq \zeta$, set $Y_\xi = P_\xi[X]$. Then $Y_\xi \subseteq Z$. **P** Induce on ξ . Since $Y_0 = \{0\}$, the induction starts. For the inductive step to a successor ordinal $\xi + 1 \leq \zeta$, $Y_{\xi+1} = Y_\xi + X_\xi \subseteq Z$. For the inductive step to a non-zero limit ordinal $\xi \leq \zeta$, given $x \in Y_\xi$ and $\epsilon > 0$, we know that there is a $\xi' < \xi$ such that $\|P_\eta x - P_\xi x\| \leq \frac{1}{3}\epsilon$ whenever $\xi' \leq \eta \leq \xi$. So $\|T_\eta x\| \leq \frac{2}{3}\epsilon$ whenever $\xi' \leq \eta < \xi$, and

$$\{\eta : \|T_\eta x\| \geq \epsilon\} = \{\eta : \eta < \xi', \|T_\eta x\| \geq \epsilon\} = \{\eta : \|T_\eta P_{\xi'} x\| \geq \epsilon\}$$

is finite, by the inductive hypothesis. Moreover, there is a finite set $J \subseteq \xi'$ such that $\|P_{\xi'} x - \sum_{\eta \in J} T_\eta P_{\xi'} x\| \leq \frac{2}{3}\epsilon$, and now $\|x - \sum_{\eta \in J} T_\eta x\| \leq \epsilon$. As x and ϵ are arbitrary, $Y_\xi \subseteq Z$. **Q**

In particular, $X = Y_\zeta \subseteq Z$ and conditions (ii) and (iii) of 467F are satisfied. So 467F gives the result.

467H Definitions (a) A topological space X is **K-countably determined** or a **Lindelöf- Σ** space if there are a subset A of $\mathbb{N}^{\mathbb{N}}$ and an usco-compact relation $R \subseteq A \times X$ such that $R[A] = X$. Observe that all K-analytic Hausdorff spaces (§422) are K-countably determined.

(b) A normed space X is **weakly K-countably determined** if it is K-countably determined in its weak topology.

(c) Let X be a normed space and Y, W closed linear subspaces of X, X^* respectively. I will say that (Y, W) is a **projection pair** if $X = Y \oplus W^\circ$ and $\|y + z\| \geq \|y\|$ for every $y \in Y, z \in W^\circ$, where

$$\begin{aligned} W^\circ &= \{z : z \in X, f(z) \leq 1 \text{ for every } f \in W\} \\ &= \{z : z \in X, f(z) = 0 \text{ for every } f \in W\}. \end{aligned}$$

467I Lemma (a) If X is a weakly K-countably determined normed space, then any closed linear subspace of X is weakly K-countably determined.

(b) If X is a weakly K-countably determined normed space, Y is a normed space, and $T : X \rightarrow Y$ is a continuous linear surjection, then Y is weakly K-countably determined.

(c) If X is a Banach space and $Y \subseteq X$ is a dense linear subspace which is weakly K-countably determined, then X is weakly K-countably determined.

proof (a) Let $A \subseteq \mathbb{N}^{\mathbb{N}}, R \subseteq A \times X$ be such that R is usco-compact (for the weak topology on X) and $R[A] = X$. Let Y be a (norm-)closed linear subspace of X ; then Y is closed for the weak topology (3A5Ee). Also the weak topology on Y is just the subspace topology induced by the weak topology of X (4A4Ea). Set $R' = R \cap (A \times Y)$. Then R' is usco-compact whether regarded as a subset of $A \times X$ or as a subset of $A \times Y$ (422Db, 422Dg). Since $Y = R'[A]$, Y is weakly K-countably determined.

(b) Let $A \subseteq \mathbb{N}^{\mathbb{N}}, R \subseteq A \times X$ be such that R is usco-compact for the weak topology on X and $R[A] = X$. Because T is continuous for the weak topologies on X and Y (3A5Ec),

$$R_1 = \{(\phi, y) : \text{there is some } x \in X \text{ such that } (\phi, x) \in R \text{ and } Tx = y\}$$

is usco-compact in $A \times Y$ (422Dc, 422Df). Also $R_1[A] = T[R[A]] = Y$. So Y is weakly K-countably determined.

(c) Let $A \subseteq \mathbb{N}^{\mathbb{N}}, R \subseteq A \times Y$ be such that R is usco-compact (for the weak topology on Y) and $R[A] = Y$. Then, as in (a), R is usco-compact when regarded as a subset of $A \times X$. By 422Dd, the set

$$R_1 = \{(\langle \phi_n \rangle_{n \in \mathbb{N}}, \langle y_n \rangle_{n \in \mathbb{N}}) : (\phi_n, y_n) \in R \text{ for every } n \in \mathbb{N}\}$$

is usco-compact in $A^{\mathbb{N}} \times Y^{\mathbb{N}}$. Now examine

$$R_2 = \{(\langle y_n \rangle_{n \in \mathbb{N}}, x) : x \in X, y_n \in Y \text{ and } \|y_n - x\| \leq 2^{-n} \text{ for every } n \in \mathbb{N}\}.$$

Then R_2 is usco-compact in $Y^{\mathbb{N}} \times X$. **P** (Remember that we are using weak topologies on X and Y throughout.) If $\mathbf{y} = \langle y_n \rangle_{n \in \mathbb{N}}$ is a sequence in Y and $(\mathbf{y}, x) \in R_2$, then $x = \lim_{n \rightarrow \infty} y_n$; so $R_2\{\mathbf{y}\}$ has at most one member and is certainly compact. Let $F \subseteq X$ be a weakly closed set and $\mathbf{y} \in Y^{\mathbb{N}} \setminus R_2^{-1}[F]$.

case 1 If there are $m, n \in \mathbb{N}$ such that $\|y_m - y_n\| > 2^{-m} + 2^{-n}$, let $f \in Y^*$ be such that $\|f\| \leq 1$ and $f(y_m - y_n) > 2^{-m} + 2^{-n}$. Then $G = \{\mathbf{z} : f(z_m - z_n) > 2^{-m} + 2^{-n}\}$ is an open set in $Y^{\mathbb{N}}$ containing \mathbf{y} and disjoint from $R_2^{-1}[F]$, because $R_2[G]$ is empty.

case 2 Otherwise, \mathbf{y} is a Cauchy sequence and (because X is a Banach space) has a limit $x \in X$, which does not belong to F . Let $\delta > 0$ and $f_0, \dots, f_r \in X^*$ be such that $\{w : |f_i(w) - f_i(x)| \leq \delta \text{ for every } i \leq r\}$ does not meet F . Let $n \in \mathbb{N}$ be such that $2^{-n}\|f_i\| \leq \frac{1}{3}\delta$ for every $i \leq r$. Then $G = \{\mathbf{z} : |f_i(z_n) - f_i(y_n)| < \frac{1}{3}\delta \text{ for every } i \leq r\}$ is an open set in $Y^{\mathbb{N}}$ containing \mathbf{y} . If $\mathbf{z} \in G$ and $(\mathbf{z}, w) \in R_2$, then $\|z_n - w\| \leq 2^{-n}$ so

$$\begin{aligned} |f_i(w) - f_i(x)| &\leq |f_i(w) - f_i(z_n)| + |f_i(z_n) - f_i(y_n)| + |f_i(y_n) - f_i(x)| \\ &\leq 2^{-n}\|f_i\| + \frac{1}{3}\delta + 2^{-n}\|f_i\| \leq \delta \end{aligned}$$

for every $i \leq r$, and $w \notin F$. Thus again $G \cap R_2^{-1}[F]$ is empty.

This shows that there is always an open set containing \mathbf{y} and disjoint from $R_2^{-1}[F]$. As \mathbf{y} is arbitrary, $R_2^{-1}[F]$ is closed. As F is arbitrary, R_2 is usco-compact. **Q**

It follows that $R_2 R_1 \subseteq A^{\mathbb{N}} \times X$ is usco-compact (422Df), while

$$(R_2 R_1)[A^{\mathbb{N}}] = R_2[R_1[A^{\mathbb{N}}]] = R_2[Y^{\mathbb{N}}] = X$$

because Y is dense in X . Finally, $A^{\mathbb{N}}$ is homeomorphic to a subset of $\mathbb{N}^{\mathbb{N}}$ because it is a subspace of $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \cong \mathbb{N}^{\mathbb{N}}$. So X is weakly K-countably determined.

467J Lemma Let X be a weakly K-countably determined Banach space. Then there is a family \mathcal{M} of subsets of $X \cup X^*$ such that

- (i) whenever $B \subseteq X \cup X^*$ there is an $M \in \mathcal{M}$ such that $B \subseteq M$ and $\#(M) \leq \max(\omega, \#(B))$;
- (ii) whenever $\mathcal{M}' \subseteq \mathcal{M}$ is upwards-directed, then $\bigcup \mathcal{M}' \in \mathcal{M}$;
- (iii) whenever $M \in \mathcal{M}$ then $(\overline{M \cap X}, \overline{M \cap X^*})$ (where the closures are taken for the norm topologies) is a projection pair of subspaces of X and X^* .

proof (a) Let $A \subseteq \mathbb{N}^{\mathbb{N}}$, $R \subseteq A \times X$ be such that R is usco-compact in $A \times X$ and $R[A] = X$. Set $S = \bigcup_{n \in \mathbb{N}} \mathbb{N}^n$ and for $\sigma \in S$ set $F_\sigma = R[I_\sigma]$, where $I_\sigma = \{\phi : \sigma \subseteq \phi \in \mathbb{N}^{\mathbb{N}}\}$; set $S_0 = \{\sigma : \sigma \in S, F_\sigma \neq \emptyset\}$.

Let \mathcal{M} be the family of those sets $M \subseteq X \cup X^*$ such that (α) whenever $x, y \in M \cap X$ and $q \in \mathbb{Q}$ then $x + y$ and qx belong to M (β) whenever $f, g \in M \cap X^*$ and $q \in \mathbb{Q}$ then $f + g$ and qf belong to M (γ) $\|x\| = \max\{f(x) : f \in M \cap X^*, \|f\| \leq 1\}$ for every $x \in M \cap X$ (δ) $\sup_{x \in F_\sigma} f(x) = \sup_{x \in F_\sigma \cap M} f(x)$ for every $f \in M \cap X^*$, $\sigma \in S_0$.

(b) For each $x \in X$, choose $h_x \in X^*$ such that $\|h_x\| \leq 1$ and $h_x(x) = \|x\|$; for each $f \in X^*$ and $\sigma \in S_0$ choose a countable set $C_{f\sigma} \subseteq F_\sigma$ such that $\sup\{f(x) : x \in C_{f\sigma}\} = \sup\{f(x) : x \in F_\sigma\}$. Given $B \subseteq X$, define $\langle B_n \rangle_{n \in \mathbb{N}}$ by setting

$$\begin{aligned} B_{n+1} &= B_n \cup \{x + y : x, y \in B_n \cap X\} \cup \{qx : q \in \mathbb{Q}, x \in B_n \cap X\} \\ &\quad \cup \{f + g : f, g \in B_n \cap X^*\} \cup \{qf : q \in \mathbb{Q}, f \in B_n \cap X^*\} \\ &\quad \cup \{h_x : x \in B_n \cap X\} \cup \bigcup \{C_{f\sigma} : f \in B_n \cap X^*, \sigma \in S_0\}, \end{aligned}$$

for each $n \in \mathbb{N}$. Then $M = \bigcup_{n \in \mathbb{N}} B_n$ belongs to \mathcal{M} and has cardinal at most $\max(\omega, \#(B))$.

(c) The definition of \mathcal{M} makes it plain that if $\mathcal{M}' \subseteq \mathcal{M}$ is upwards-directed then $\bigcup \mathcal{M}'$ belongs to \mathcal{M} .

(d) Now take $M \in \mathcal{M}$ and set $Y = \overline{M \cap X}$, $W = \overline{M \cap X^*}$. These are linear subspaces (2A5Ec). If $y \in M \cap X$ and $z \in W^\circ$, then there is an $f \in M \cap X^*$ such that $\|f\| \leq 1$ and $f(y) = \|y\|$, so that

$$\|y + z\| \geq f(y + z) = f(y) = \|y\|.$$

Because the function $y \mapsto \|y+z\| - \|y\|$ is continuous, $\|y+z\| \geq \|y\|$ for every $y \in Y$ and $z \in W^\circ$. In particular, if $y \in Y \cap W^\circ$, $\|y\| \leq \|y-y\| = 0$ and $y = 0$, so $Y + W^\circ = Y \oplus W^\circ$. If $x \in \overline{Y + W^\circ}$, then there are sequences $\langle y_n \rangle_{n \in \mathbb{N}}$ in Y and $\langle z_n \rangle_{n \in \mathbb{N}}$ in W° such that $x = \lim_{n \rightarrow \infty} y_n + z_n$; now $\|y_m - y_n\| \leq \|(y_m + z_m) - (y_n + z_n)\| \rightarrow 0$ as $n \rightarrow \infty$, so (because X is a Banach space) $\langle y_n \rangle_{n \in \mathbb{N}}$ is convergent to y say; in this case, $y \in Y$ and $x - y = \lim_{n \rightarrow \infty} z_n$ belongs to W° , so $x \in Y + W^\circ$. This shows that $Y \oplus W^\circ$ is a closed linear subspace of X .

(e) ? Suppose, if possible, that $Y \oplus W^\circ \neq X$. Then there is an $x_0 \in X \setminus (Y \oplus W^\circ)$. By 4A4Eb, there is an $f \in X^*$ such that $f(x_0) > 0$ and $f(y) = f(z) = 0$ whenever $y \in Y$ and $z \in W^\circ$; multiplying f by a scalar if necessary, we can arrange that $f(x_0) = 1$. By 4A4Eg, f belongs to the weak* closure of W in X^* .

Let $\phi \in A$ be such that $(\phi, x_0) \in R$. Then $K = R[\{\phi\}]$ is weakly compact. Now the weak* closure of W is also its closure for the Mackey topology of uniform convergence on weakly compact subsets of X (4A4F). So there is a $g \in W$ such that $|g(x) - f(x)| \leq \frac{1}{7}$ for every $x \in K$. Next, because K is bounded, and g belongs to the norm closure of $M \cap X^*$, there is an $h \in M \cap X^*$ such that $|h(x) - g(x)| \leq \frac{1}{7}$ for every $x \in K$. This means that $|h(x) - f(x)| \leq \frac{2}{7}$ for every $x \in K$, and K is included in the weakly open set $G = \{x : |h(x) - f(x)| < \frac{3}{7}\}$, that is, ϕ does not belong to $R^{-1}[X \setminus G]$, which is relatively closed in A , because R is usco-compact regarded as a relation between A and X with the weak topology. There is therefore a $\sigma \in S$ such that $\phi \in I_\sigma$ and $I_\sigma \cap R^{-1}[X \setminus G] = \emptyset$, that is, $F_\sigma \subseteq G$. In this case, $x_0 \in F_\sigma$, so $\sigma \in S_0$, while $h(x) - f(x) < \frac{3}{7}$ for every $x \in F_\sigma$. But, because $M \in \mathcal{M}$, there is a $y \in M \cap X \cap F_\sigma$ such that $h(y) \geq h(x_0) - \frac{1}{7}$, and as $y \in Y$ we must now have

$$\begin{aligned} 0 = f(y) &= h(y) - (h(y) - f(y)) > h(x_0) - \frac{1}{7} - \frac{3}{7} \\ &= f(x_0) - (f(x_0) - h(x_0)) - \frac{4}{7} \geq 1 - \frac{3}{7} - \frac{4}{7} = 0, \end{aligned}$$

which is absurd. **X**

Thus $X = Y \oplus W^\circ$ and (Y, W) is a projection pair. This completes the proof.

467K Theorem Let X be a weakly K-countably determined Banach space. Then it has an equivalent locally uniformly rotund norm.

proof Since the completion of X is weakly K-countably determined (467Ic), we may suppose that X is complete. The proof proceeds by induction on the weight of X .

(a) The induction starts by observing that if $w(X) \leq \omega$ then X is separable (4A2Li/4A2P(a-i)) so has an equivalent locally uniformly rotund norm by 467E.

(b) So let us suppose that $w(X) = \kappa > \omega$ and that any weakly K-countably determined Banach space of weight less than κ has an equivalent locally uniformly rotund norm.

Let \mathcal{M} be a family of subsets of $X \cup X^*$ as in 467J. Then there is a non-decreasing family $\langle M_\xi \rangle_{\xi \leq \kappa}$ in \mathcal{M} such that $\#(M_\xi) \leq \max(\omega, \#(\xi))$ for every $\xi \leq \kappa$, M_κ is dense in X , and $M_\xi = \bigcup_{\eta < \xi} M_\eta$ for every limit ordinal $\xi \leq \kappa$. **P** By 4A2Li, there is a dense subset of X with cardinal κ ; enumerate it as $\langle x_\xi \rangle_{\xi < \kappa}$. Choose M_ξ inductively, as follows. $M_0 = \emptyset$. Given M_ξ with $\#(M_\xi) \leq \max(\omega, \#(\xi))$, then by 467J(i) there is an $M_{\xi+1} \in \mathcal{M}$ such that $M_{\xi+1} \supseteq M_\xi \cup \{x_\xi\}$ and

$$\#(M_{\xi+1}) \leq \max(\omega, \#(M_\xi \cup \{x_\xi\})) \leq \max(\omega, \#(\xi + 1)).$$

Given that $\langle M_\eta \rangle_{\eta < \xi}$ is a non-decreasing family in \mathcal{M} with $\#(M_\eta) \leq \max(\omega, \#(\eta))$ for every $\eta < \xi$, set $M_\xi = \bigcup_{\eta < \xi} M_\eta$; then 467J(ii) tells us that $M_\xi \in \mathcal{M}$, while $\#(M_\xi) \leq \max(\omega, \#(\xi))$, as required by the inductive hypothesis. **Q**

At the end of the induction, $M_\kappa \supseteq \{x_\xi : \xi < \kappa\}$ will be dense in X .

(c) For each $\xi < \kappa$, set $Y_\xi = \overline{M_\xi} \cap \overline{X}$, $W_\xi = \overline{M_\xi} \cap \overline{X^*}$. Then (Y_ξ, W_ξ) is a projection pair, by 467J(iii). Since $X = Y_\xi \oplus W_\xi^\circ$, we have a projection $P_\xi : X \rightarrow Y_\xi$ defined by saying that $P_\xi(y+z) = y$ whenever $y \in Y_\xi$ and $z \in W_\xi^\circ$. Now $P_\xi P_\eta = P_\eta P_\xi = P_\xi$ whenever $\xi \leq \eta$. **P** The point is that $Y_\xi \subseteq Y_\eta$ and $W_\xi \subseteq W_\eta$, so $W_\eta^\circ \subseteq W_\xi^\circ$. If $x \in X$, express it as $y + z_1$ where $y \in Y_\xi$ and $z_1 \in W_\xi^\circ$, and express z_1 as $y' + z$ where $y' \in Y_\eta$ and $z' \in W_\eta^\circ$. Then

$$P_\xi x = y \in Y_\xi \subseteq Y_\eta$$

so $P_\eta P_\xi x = P_\xi x$. On the other hand, $x = y + y' + z$, $y + y' \in Y_\eta$ and $z \in W_\eta^\circ$, so $P_\eta x = y + y'$; and as $y' = z_1 - z$ belongs to W_ξ° , $P_\xi(y + y') = y$, so $P_\xi P_\eta x = P_\xi x$. **Q**

Note that the condition

$$\|y + z\| \geq \|y\| \text{ whenever } y \in Y_\xi, z \in W_\xi^\circ$$

ensures that $\|P_\xi\| \leq 1$ for every ξ .

Next, if $\xi \leq \kappa$ is a non-zero limit ordinal, $P_\xi x = \lim_{\eta \uparrow \xi} P_\eta x$. **P** We know that

$$P_\xi x \in Y_\xi = \overline{M_\xi \cap X} = \overline{\bigcup_{\eta < \xi} M_\eta \cap X}.$$

So, given $\epsilon > 0$, there is an $x' \in \bigcup_{\eta < \xi} M_\eta$ such that $\|P_\xi x - x'\| \leq \frac{1}{2}\epsilon$. Let $\eta < \xi$ be such that $x' \in M_\eta$. If $\eta \leq \eta' \leq \xi$, then

$$\|P_\xi x - P_{\eta'} x\| = \|P_\xi(P_\xi x - x') - P_{\eta'}(P_\xi x - x')\|$$

(because $x' \in Y_\eta$, so $P_\xi x' = P_{\eta'} x' = x'$)

$$\leq 2\|P_\xi x - x'\| \leq \epsilon.$$

As ϵ is arbitrary, $P_\xi x = \lim_{\eta \uparrow \xi} P_\eta x$. **Q**

(d) Now observe that every Y_ξ is weakly K-countably determined (467Ia), while $w(Y_\xi) \leq \max(\omega, \#(\xi)) < \kappa$ for every $\xi < \kappa$ (using 4A2Li, as usual). So the inductive hypothesis tells us that $Y_\xi = P_\xi[X]$ has an equivalent locally uniformly rotund norm for every $\xi < \kappa$. By 467G, $X = P_\kappa[X]$ has an equivalent locally uniformly rotund norm. Thus the induction proceeds.

467L Weakly compactly generated Banach spaces The most important class of weakly K-countably determined spaces is the following. A normed space X is **weakly compactly generated** if there is a sequence $\langle K_n \rangle_{n \in \mathbb{N}}$ of weakly compact subsets of X such that $\bigcup_{n \in \mathbb{N}} K_n$ is dense in X .

467M Proposition (TALAGRAN 75) A weakly compactly generated Banach space is weakly K-countably determined.

proof Let X be a Banach space with a sequence $\langle K_n \rangle_{n \in \mathbb{N}}$ of weakly compact subsets of X such that $\bigcup_{n \in \mathbb{N}} K_n$ is dense in X . Set

$$L_n = \{ \sum_{i=0}^n \alpha_i x_i : |\alpha_i| \leq 1, x_i \in \bigcup_{j \leq n} K_j \text{ for every } i \leq n \}$$

for $n \in \mathbb{N}$. Then every L_n is weakly compact, and $Y = \bigcup_{n \in \mathbb{N}} L_n$ is a linear subspace of X including $\bigcup_{n \in \mathbb{N}} K_n$, therefore dense. Now Y is a countable union of weakly compact sets, therefore K-analytic for its weak topology (422Gc, 422Hc); in particular, it is weakly K-countably determined. By 467Ic, X is weakly K-countably determined.

467N Theorem Let X be a Banach lattice with an order-continuous norm (§354). Then it has an equivalent locally uniformly rotund norm.

proof (a) Consider first the case in which X has a weak order unit e . Then the interval $[0, e]$ is weakly compact. **P** We have $X^* = X^\times$ and $X^{**} = X^{\times \sim}$ (356D). The canonical identification of X with its image in X^{**} is an order-continuous Riesz homomorphism from X onto a solid order-dense Riesz subspace of $X^{\times \times}$ (356I) and therefore of $X^{\times \sim} = X^{**}$. In particular, $[0, e] \subseteq X$ is matched with an interval $[0, \hat{e}] \subseteq X^{**}$. But $[0, \hat{e}] = \{ \theta : \theta \in X^{**}, 0 \leq \theta(f) \leq f(e) \text{ for every } f \in (X^*)^+ \}$ is weak*-closed and norm-bounded in X^{**} , therefore weak*-compact; as the weak* topology on X^{**} corresponds to the weak topology of X , $[0, e]$ is weakly compact. **Q**

Now, for each $n \in \mathbb{N}$, $K_n = [-ne, ne] = n[0, e] - n[0, e]$ is weakly compact. If $x \in X^+$, then $\langle x \wedge ne \rangle_{n \in \mathbb{N}}$ converges to x , because the norm of X is order-continuous; so for any $x \in X$, $\langle x^+ \wedge ne - x^- \wedge ne \rangle_{n \in \mathbb{N}}$ converges to x . Thus $\bigcup_{n \in \mathbb{N}} K_n$ is dense in X and X is weakly compactly generated. By 467M and 467K, X has an equivalent locally uniformly rotund norm.

(b) For the general case, let $\langle x_i \rangle_{i \in I}$ be a maximal disjoint family in X^+ . For each $i \in I$ let X_i be the band in X generated by x_i , and $T_i : X \rightarrow X_i$ the band projection onto X_i (354Ee, 353Ib¹¹). Then $\langle T_i \rangle_{i \in I}$ satisfies the conditions of 467F. **P** (i) Each T_i is a continuous linear operator because the given norm $\|\cdot\|$ of X is a Riesz norm. Next, each X_i has a weak order unit x_i , and the norm on X_i is order-continuous, so (a) tells us that there is an equivalent locally uniformly rotund norm on $X_i = T_i[X]$. (ii) If $x \in X$, set $x' = \sup_{i \in I} T_i|x|$; then $(|x| - x') \wedge x_i = 0$ for every i , so, by the maximality of $\langle x_i \rangle_{i \in I}$, $x' = |x|$. If $\epsilon > 0$ then, because the norm of X is order-continuous, there is a finite $J \subseteq I$ such that

$$\|x - \sum_{j \in J} T_j x\| = \|x' - \sup_{j \in J} T_j|x|\| \leq \epsilon.$$

Moreover, if $i \in I \setminus J$, then

$$\|T_i x\| \leq \|x - \sum_{j \in J} T_j x\| \leq \epsilon.$$

Thus conditions (ii) and (iii) of 467F are satisfied. **Q**

Accordingly 467F tells us that X has an equivalent locally uniformly rotund norm.

467O Eberlein compacta: Definition A topological space K is an **Eberlein compactum** if it is homeomorphic to a weakly compact subset of a Banach space.

467P Proposition Let K be a compact Hausdorff space.

(a) The following are equiveridical:

(i) K is an Eberlein compactum;

(ii) there is a set $L \subseteq C(K)$, separating the points of K , which is compact for the topology of pointwise convergence.

(b) Suppose that K is an Eberlein compactum.

(i) K has a σ -isolated network, so is hereditarily weakly θ -refinable.

(ii) (SCHACHERMAYER 77) If $w(K)$ is measure-free, K is a Radon space.

proof (a)(i) \Rightarrow (ii) If K is an Eberlein compactum, we may suppose that it is a weakly compact subset of a Banach space X . Set $L = \{f|K : f \in X^*, \|f\| \leq 1\}$; since the map $f \mapsto f|K : X^* \rightarrow C(K)$ is continuous for the weak* topology of X^* and the topology \mathfrak{T}_p of pointwise convergence on $C(K)$, L is \mathfrak{T}_p -compact; and L separates the points of K because X^* separates the points of X .

(ii) \Rightarrow (i) If $L \subseteq C(K)$ is \mathfrak{T}_p -compact and separates the points of K , set $L_n = \{f : f \in L, \|f\|_\infty \leq n\}$ for each $n \in \mathbb{N}$. Then L_n is \mathfrak{T}_p -compact for each n . Set $L' = \{0\} \cup \bigcup_{n \in \mathbb{N}} 2^{-n} L_n$; then $L' \subseteq C(K)$ is norm-bounded and \mathfrak{T}_p -compact and separates the points of K . Now define $x \mapsto \hat{x} : K \rightarrow \mathbb{R}^{L'}$ by setting $\hat{x}(f) = f(x)$ for $x \in K$ and $f \in L'$. Then $\hat{x} \in C(L')$ for every x and $x \mapsto \hat{x}$ is continuous for the given topology of K and the topology of pointwise convergence on $C(L')$; so the image $\hat{K} = \{\hat{x} : x \in K\}$ is \mathfrak{T}_p -compact. Since it is also bounded, it is weakly compact (462E). But $x \mapsto \hat{x}$ is injective, because L' separates the points of K ; so K is homeomorphic to \hat{K} , and is an Eberlein compactum.

(b) Again suppose K is actually a weakly compact subset of a Banach space X . As in 467M, set $L_n = \{\sum_{i=0}^n \alpha_i x_i : |\alpha_i| \leq 1, x_i \in K \text{ for every } i \leq n\}$ for each $n \in \mathbb{N}$. Then $Y = \overline{\bigcup_{n \in \mathbb{N}} L_n}$ is a weakly compactly generated Banach space. (I am passing over the trivial case $K = \emptyset$.) So Y has an equivalent locally uniformly rotund norm (467M, 467K), which is a Kadec norm (467B), and Y , with the weak topology, has a σ -isolated network (466Eb). It follows at once that K has a σ -isolated network (4A2B(a-ix)), so is hereditarily weakly θ -refinable (438Ld); and if $w(K)$ is measure-free, K is Borel-measure-complete (438M), therefore Radon (434Jf, 434Ka).

467X Basic exercises (a)(i) Show that a continuous image of a K -countably determined space is K -countably determined. (ii) Show that the product of a sequence of K -countably determined spaces is K -countably determined. (iii) Show that any K -countably determined topological space is Lindelöf. (*Hint*: 422De.) (iv) Show that any Souslin-F subset of a K -countably determined topological space is K -countably determined. (*Hint*: 422Hc.)

¹¹Formerly 353Hb.

(b) Let X be a σ -compact Hausdorff space. Show that a subspace Y of X is K -countably determined iff there is a countable family \mathcal{K} of compact subsets of X such that $\bigcap\{K : y \in K \in \mathcal{K}\} \subseteq Y$ for every $y \in Y$.

(c) Show that if X and Y are weakly K -countably determined normed spaces, then $X \times Y$, with an appropriate norm, is weakly K -countably determined.

(d) Show that a normed space X is weakly compactly generated iff there is a weakly compact set $K \subseteq X$ such that the linear subspace of X generated by K is dense in X .

(e) Show that any separable normed space is weakly compactly generated.

(f) Show that any reflexive Banach space is weakly compactly generated.

>(g) Show that if X is a weakly compactly generated Banach space, then it is K -analytic in its weak topology. (*Hint*: in 467M, use the proof of 467Ic.)

(h) Show that if X is a Banach space and there is a set $A \subseteq X$ such that A is K -countably determined for the weak topology and the linear subspace generated by A is dense, then X is weakly K -countably determined.

(i) Show that the one-point compactification of any discrete space is an Eberlein compactum.

>(j) Let K be an Eberlein compactum, and μ a Radon measure on K . Show that μ is completion regular and inner regular with respect to the compact metrizable subsets of K . (*Hint*: 466B.)

467Y Further exercises (a) Let $\langle X_n \rangle_{n \in \mathbb{N}}$ be a sequence of weakly K -countably determined normed spaces. Investigate normed subspaces of $\prod_{n \in \mathbb{N}} X_n$ which will be weakly K -countably determined.

(b)(i) Show that if (Ω, Σ, μ) is a probability space, then $L^1(\mu)$ has a locally uniformly rotund Riesz norm. (*Hint*: apply the construction of 467D with $Y = L^2(\mu)$ and T the identity operator; show that all the norms p_n are Riesz norms.) (ii) Show that if X is any L -space then it has a locally uniformly rotund Riesz norm. (*Hint*: apply the construction of 467F/467N, noting that if the T_i in 467F are band projections and $\|\cdot\|$ and all the $\|\cdot\|_i$ are Riesz norms, then all the norms in the proof of 467F are Riesz norms.)

(c) Let $(X, \|\cdot\|)$ be a Banach space, and \mathfrak{T} a linear space topology on X such that the unit ball of X is \mathfrak{T} -closed. Suppose that $\langle T_i \rangle_{i \in I}$ is a family of bounded linear operators from X to itself such that

- (i) for each $i \in I$, T_i is \mathfrak{T} -continuous as well as norm-continuous,
- (ii) for each $i \in I$, the subspace $T_i[X]$ has an equivalent locally uniformly rotund norm for which the unit ball is closed for the topology on $T_i[X]$ induced by \mathfrak{T} ,
- (iii) for each $x \in X$, $\epsilon > 0$ there is a finite set $J \subseteq I$ such that $\|x - \sum_{i \in J} T_i x\| \leq \epsilon$,
- (iv) for each $x \in X$, $\epsilon > 0$ the set $\{i : i \in I, \|T_i x\| \geq \epsilon\}$ is finite.

Show that X has an equivalent locally uniformly rotund norm for which the unit ball is \mathfrak{T} -closed.

(d) Let X be a normed space with a locally uniformly rotund norm, and \mathfrak{T} a linear space topology on X such that the unit ball of X is \mathfrak{T} -closed. Show that every norm-Borel subset of X is \mathfrak{T} -Borel.

(e) Let κ be any cardinal, and K a dyadic space. (i) Show that $C(K)$ has a locally uniformly rotund norm, equivalent to the usual supremum norm $\|\cdot\|_\infty$, for which the unit ball is closed for the topology \mathfrak{T}_p of pointwise convergence. (See DEVILLE GODEFROY & ZIZLER 93, VII.1.10.) (ii) Show that the norm topology on $C(K)$, the weak topology on $C(K)$ and \mathfrak{T}_p give rise to the same Borel σ -algebras. (iii) Show that \mathfrak{T}_p has a σ -isolated network. (iv) Show that if $w(K)$ is measure-free, then $(C(K), \mathfrak{T}_p)$ is Radon, and every \mathfrak{T}_p -Radon measure on $C(K)$ is norm-Radon.

467 Notes and comments The purpose of this section has been to give an idea of the scope of Proposition 466F. ‘Local uniform rotundity’ has an important place in the geometrical theory of Banach spaces, but for the many associated ideas I refer you to DEVILLE GODEFROY & ZIZLER 93. From our point of view, Theorem 467E is therefore purely accessory, since we know by different arguments that on separable Banach spaces the weak and norm topologies have the same Borel σ -algebras (4A3W). We need it to provide the first step in the inductive proof of 467K.

Since the concept of ‘K-analytic’ space is one of the fundamental ideas of Chapter 43, it is natural here to look at ‘K-countably determined’ spaces, especially as many of the ideas of §422 are directly applicable (467Xa). But the goal of this part of the argument is Schachermayer’s theorem 467P(b-ii), which uses ‘weakly compactly generated’ spaces (467L). ‘Eberlein compacta’ are of great interest in other ways; they are studied at length in ARKHANGEL’SII 92.

I mention order-continuous norms here (467N) because they are prominent in the theory of Banach lattices in Volume 3. Note that the methods here do *not* suffice in general to arrange that the locally uniformly rotund norm found on X is a Riesz norm; though see 467Yb. It is in fact the case that every Banach lattice with an order-continuous norm has an equivalent locally uniformly rotund Riesz norm, but this requires further ideas (see DAVIS GHOUSSOUB & LINDENSTRAUSS 81).

The general question of identifying Banach spaces with equivalent Kadec norms remains challenging. For a recent survey see MOLTÓ ORIHUELA TROYANSKI & VALDIVIA 09.

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