Chapter 45

Perfect measures and disintegrations

One of the most remarkable features of countably additive measures is that they provide us with a framework for probability theory, as described in Chapter 27. The extraordinary achievements of probability theory since Kolmogorov are to a large extent possible because of the rich variety of probability measures which can be constructed. We have already seen image measures (234C[1]) and product measures (§254). The former are elementary, but a glance at the index will confirm that they have many surprises to offer; the latter are obviously fundamental to any idea of what probability theory means. In this chapter I will look at some further constructions. The most important are those associated with 'disintegrations' or 'regular conditional probabilities' (§§452-453) and methods for confirming the existence of measures on product spaces with given images on subproducts (§454, 455A). We find that these constructions have to be based on measure spaces of special types; the measures involved in the principal results are the Radon measures of Chapter 41 (of course), the compact and perfect measures of Chapter 34, and an intermediate class, the 'countably compact' measures of Marczewski 53 (451B). So the first section of this chapter is a systematic discussion of compact, countably compact and perfect measures.

A 'disintegration', when present, is likely to provide us with a particularly effective instrument for studying a measure, analogous to Fubini's theorem for product measures (see 452F). §§452-453 therefore concentrate on theorems guaranteeing the existence of disintegrations compatible with some pre-existing structure, typically an inverse-measure-preserving function (452I, 452O, 453K) or a product structure (452M). Both depend on the existence of suitable liftings, and for the topological version in §453 we need a 'strong' lifting, so much of that section is devoted to the study of such liftings.

One of the central concerns of probability theory is to understand 'stochastic processes', that is, models of systems evolving randomly over time. If we think of our state space as consisting of functions, so that a whole possible history is described by a random function of time, it is natural to think of our functions as members of some set $\prod_{n\in\mathbb{N}} Z_n$ (if we think of observations as being taken at discrete time intervals) or $\prod_{t\in[0,\infty[} Z_t$ (if we regard our system as evolving continuously), where Z_t represents the set of possible states of the system at time t. We are therefore led to consider measures on such product spaces, and the new idea is that we may have some definite intuition concerning the joint distribution of *finite* strings $(f(t_0),\dots,f(t_n))$ of values of our random function, that is to say, we may think we know something about the image measures on finite products $\prod_{i\le n} Z_{t_i}$. So we come immediately to a fundamental question: given a (probability) measure μ_J on $\prod_{i\in J} Z_i$ for each finite $J\subseteq T$, when will there be a measure on $\prod_{i\in T} Z_i$ compatible with every μ_J? In §454 I give the most important generally applicable existence theorems for such measures, and in 455A-455E I show how they can be applied to a general construction for models of Markov processes. These models enable us to discuss the Markov property either in terms of disintegrations or in terms of conditional expectations (455C, 455O), and for Lévy processes, in terms of inverse-measure-preserving functions (455U).

The abstract theory of §454 yields measures on product spaces which, from the point of view of a probabilist, are unnaturally large, often much larger than intuition suggests. Some of the most powerful results in the theory of Markov processes, such as the strong Markov property (455O), depend on moving to much smaller spaces; most notably the space of càdlàg functions (455G), but the larger space of càllàl functions is also of interest. The most important example, Brownian motion, will have to wait for Chapter 47, but I give the basic general theory of Lévy processes in complete metric groups.

One of the defining characteristics of Brownian motion is the fact that all its finite-dimensional marginals are Gaussian distributions. Stochastic processes with this property form a particularly interesting class,

[1]Formerly 112E.

which I examine in §456. From the point of view of this volume, one of their most striking properties is Talagrand's theorem that, regarded as measures on powers $\mathbb{R}^I$, they are τ-additive (456O).

The next two sections look again at some of the ideas of the previous sections when interpreted as answers to questions of the form 'can all the measures in such-and-such a family be simultaneously extended to a single measure?' If we seek only a *finitely* additive common extension, there is a reasonably convincing general result (457A); but countably additive measures remain puzzling even in apparently simple circumstances (457Z). In §458 I introduce 'relatively independent' families of σ-algebras, with the associated concept of 'relative product' of measures, and the corresponding concepts for probability algebras. Finally, in §459, I give some basic results on symmetric measures and exchangeable random variables, with De Finetti's theorem (459C) and corresponding theorems on representing permutation-invariant measures on products as mixtures of product measures (459E, 459H).

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451 Perfect, compact and countably compact measures

In §§342-343 I introduced 'compact' and 'perfect' measures as part of a study of the representation of homomorphisms of measure algebras by functions between measure spaces. An intermediate class of 'countably compact' measures has appeared in the exercises. It is now time to collect these ideas together in a more systematic way. In this section I run through the standard properties of compact, countably compact and perfect measures (451A-451J), with a couple of simple examples of their interaction with topologies (451M-451P). An example of a perfect measure space which is not countably compact is in 451U. Some new ideas, involving non-trivial set theory, show that measurable functions from compact totally finite measure spaces to metrizable spaces have 'essentially separable ranges' (451R); consequently, any measurable function from a Radon measure space to a metrizable space is almost continuous (451T).

451B Definition Let (X,Σ,μ) be a measure space. Then (X,Σ,μ), or μ, is **countably compact** if μ is inner regular with respect to some countably compact class of sets.

451C Proposition Any semi-finite countably compact measure is perfect.

451D Proposition Let (X,Σ,μ) be a measure space, and $E\in\Sigma$; let μ_E be the subspace measure on E.

(a) If μ is compact, so is μ_E.

(b) If μ is countably compact, so is μ_E.

(c) If μ is perfect, so is μ_E.

451E Proposition Let (X,Σ,μ) be a perfect measure space.

(a) If (Y,T,ν) is another measure space and $f:X\to Y$ is an inverse-measure-preserving function, then ν is perfect.

(b) In particular, $\mu\restriction\mathrm{T}$ is perfect for any σ-subalgebra T of Σ.

451F Lemma Let (X,Σ,μ) be a semi-finite measure space. Then the following are equiveridical:

(i) μ is perfect;

(ii) $\mu\restriction\mathrm{T}$ is compact for every countably generated σ-subalgebra T of Σ;

(iii) $\mu\restriction\mathrm{T}$ is perfect for every countably generated σ-subalgebra T of Σ;

(iv) for every countable set $\mathcal{E}\subseteq\Sigma$ there is a σ-algebra $\mathrm{T}\supseteq\mathcal{E}$ such that $\mu\restriction\mathrm{T}$ is perfect.

451G Proposition Let (X,Σ,μ) be a measure space. Let $(X,\hat\Sigma,\hat\mu)$ be its completion and $(X,\tilde\Sigma,\tilde\mu)$ its c.l.d. version. Then

(a)(i) if μ is compact, so are $\hat\mu$ and $\tilde\mu$;

(ii) if μ is semi-finite and either $\hat\mu$ or $\tilde\mu$ is compact, then μ is compact.

(b)(i) If μ is countably compact, so are $\hat\mu$ and $\tilde\mu$;

(ii) if μ is semi-finite and either $\hat{\mu}$ or $\tilde{\mu}$ is countably compact, then μ is countably compact.

(c)(i) If μ is perfect, so are $\hat{\mu}$ and $\tilde{\mu}$;

(ii) if $\hat{\mu}$ is perfect, then μ is perfect;

(iii) if μ is semi-finite and $\tilde{\mu}$ is perfect, then μ is perfect.

451H Lemma Let $\langle X_i\rangle_{i\in I}$ be a family of sets with product X. Suppose that $\mathcal{K}_i \subseteq \mathcal{P}X_i$ for each $i \in I$, and set $\mathcal{K} = \{\pi_i^{-1}[K] : i \in I,\, K \in \mathcal{K}_i\}$, where $\pi_i : X \to X_i$ is the coordinate map for each $i \in I$. Then

(a) if every $\mathcal{K}_i$ is a compact class, so is $\mathcal{K}$;

(b) if every $\mathcal{K}_i$ is a countably compact class, so is $\mathcal{K}$.

451I Theorem Let (X, Σ, μ) and (Y, T, ν) be measure spaces, with c.l.d. product $(X \times Y, \Lambda, \lambda)$.

(a) If μ and ν are compact, so is λ.

(b) If μ and ν are countably compact, so is λ.

(c) If μ and ν are perfect, so is λ.

451J Theorem Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of probability spaces, with product (X, Σ, μ).

(a) If every μ_i is compact, so is μ.

(b) If every μ_i is countably compact, so is μ.

(c) If every μ_i is perfect, so is μ.

451K Proposition Let $\langle X_i\rangle_{i\in I}$ be a family of sets with product X, and Σ_i a σ-algebra of subsets of X_i for each i. Let λ be a perfect totally finite measure with domain $\widehat{\bigotimes}_{i\in I}\Sigma_i$. Set $\pi_J(x) = x{\restriction}J$ for $x \in X$ and $J \subseteq I$.

(a) Let $\mathcal{K}$ be the set $\{V : V \subseteq X,\, \pi_J[V] \in \widehat{\bigotimes}_{i\in J}\Sigma_i$ for every $J \subseteq I\}$. Then λ is inner regular with respect to $\mathcal{K}$.

(b) Let $\hat{\lambda}$ be the completion of λ.

(i) For any $J \subseteq I$, the completion of the image measure $\lambda\pi_J^{-1}$ on $\prod_{i\in J} X_i$ is the image measure $\hat{\lambda}\pi_J^{-1}$.

(ii) If W is measured by $\hat{\lambda}$ and W is determined by coordinates in $J \subseteq I$, then there is a $V \in \widehat{\bigotimes}_{i\in I}\Sigma_i$ such that $V \subseteq W$, V is determined by coordinates in J and $W \setminus V$ is λ-negligible.

***451L Proposition** Let (X, Σ, μ) be a strictly localizable measure space. Let us say that a family $\mathcal{E} \subseteq \Sigma$ is **μ-centered** if $\mu(\bigcap \mathcal{E}_0) > 0$ for every non-empty finite $\mathcal{E}_0 \subseteq \mathcal{E}$.

(i) Suppose that μ is inner regular with respect to some $\mathcal{K} \subseteq \Sigma$ such that every μ-centered subset of $\mathcal{K}$ has non-empty intersection. Then μ is compact.

(ii) Suppose that μ is inner regular with respect to some $\mathcal{K} \subseteq \Sigma$ such that every countable μ-centered subset of $\mathcal{K}$ has non-empty intersection. Then μ is countably compact.

451M Proposition Let (X, Σ) be a standard Borel space. Then any semi-finite measure μ with domain Σ is compact, therefore perfect.

451N Proposition Let (X, Σ, μ) be a perfect measure space and $\mathfrak{T}$ a T_0 topology on X with a countable network consisting of measurable sets. Then μ is inner regular with respect to the compact sets.

451O Corollary Let (X, Σ, μ) be a complete perfect measure space, Y a Hausdorff space with a countable network consisting of Borel sets and $f : X \to Y$ a measurable function. If the image measure μf^{-1} is locally finite, it is a Radon measure.

451P Corollary Let (X, Σ, μ) be a perfect measure space, Y a separable metrizable space, and $f : X \to Y$ a measurable function.

(a) If $E \in \Sigma$ and $\gamma < \mu E$, there is a compact set $K \subseteq f[E]$ such that $\mu(E \cap f^{-1}[K]) \geq \gamma$.

(b) If $\nu = \mu f^{-1}$ is the image measure, then $\mu_* f^{-1}[B] = \nu_* B$ for every $B \subseteq Y$.

(c) If moreover μ is σ-finite, then $\mu^* f^{-1}[B] = \nu^* B$ for every $B \subseteq Y$.

451Q Lemma Let (X, Σ, μ) be a semi-finite compact measure space, and $\langle E_i \rangle_{i \in I}$ a disjoint family of subsets of X such that $\bigcup_{i \in J} E_i \in \Sigma$ for every $J \subseteq I$. Then $\mu(\bigcup_{i \in I} E_i) = \sum_{i \in I} \mu E_i$.

451R Lemma Let (X, Σ, μ) be a totally finite compact measure space, Y a metrizable space, and $f : X \to Y$ a measurable function. Then there is a closed separable subspace Y_0 of Y such that $f^{-1}[Y \setminus Y_0]$ is negligible.

451S Proposition Let (X, Σ, μ) be a semi-finite compact measure space, Y a metrizable space and $f : X \to Y$ a measurable function.

(a) The image measure $\nu = \mu f^{-1}$ is tight.

(b) If ν is locally finite and μ is complete and locally determined, ν is a Radon measure.

451T Theorem Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space and Y a metrizable space. Then a function $f : X \to Y$ is measurable iff it is almost continuous.

451U Example There is a perfect completion regular quasi-Radon probability space which is not countably compact.

***451V Weakly α-favourable spaces** For any measure space (X, Σ, μ) we can imagine an infinite game for two players, whom I will call 'Empty' and 'Nonemepty'. Empty chooses a non-negligible measurable set E_0; Nonempty chooses a non-negligible measurable set $F_0 \subseteq E_0$; Empty chooses a non-negligible measurable set $E_1 \subseteq F_0$; Nonempty chooses a non-negligible measurable set $F_1 \subseteq E_1$, and so on. At the end of the game, Empty wins if $\bigcap_{n \in \mathbb{N}} E_n = \bigcap_{n \in \mathbb{N}} F_n$ is empty; otherwise Nonempty wins. (If $\mu X = 0$, so that Empty has no legal initial move, I declare Nonempty the winner by default.)

A **strategy** for Nonempty is a function $\sigma : \bigcup_{n \in \mathbb{N}} (\Sigma \setminus \mathcal{N})^{n+1} \to \Sigma \setminus \mathcal{N}$, where $\mathcal{N}$ is the ideal of negligible sets, such that $\sigma(E_0, \ldots, E_n) \subseteq E_n$ for all $E_0, \ldots, E_n \in \Sigma \setminus \mathcal{N}$. σ is a **winning strategy** if $\bigcap_{n \in \mathbb{N}} E_n \neq \emptyset$ whenever $\langle E_n \rangle_{n \in \mathbb{N}}$ is a sequence in $\Sigma \setminus \mathcal{N}$ such that $E_{n+1} \subseteq \sigma(E_0, \ldots, E_n)$ for every $n \in \mathbb{N}$.

Now we say that the measure space (X, Σ, μ) is **weakly α-favourable** if there is a winning strategy for Nonempty.

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452 Integration and disintegration of measures

A standard method of defining measures is through a formula

$$\mu E = \int \mu_y E \, \nu(dy)$$

where (Y, T, ν) is a measure space and $\langle \mu_y \rangle_{y \in Y}$ is a family of measures on another set X. In practice these constructions commonly involve technical problems concerning the domain of μ, which is why I have hardly used them so far in this treatise. There are not-quite-trivial examples in 417Yb, 434R and 436F, and the indefinite-integral measures of §234 can also be expressed in this way; for a case in which this approach is worked out fully, see 453N. But when a formula of this kind is valid, as in Fubini's theorem, it is likely to be so useful that it dominates further investigation of the topic. In this section I give one of the two most important theorems guaranteeing the existence of appropriate families $\langle \mu_y \rangle_{y \in Y}$ when μ and ν are given (452I); the other will follow in the next section. They both suppose that we are provided with a suitable function $f : X \to Y$, and rely heavily on the Lifting Theorem and on considerations of inner regularity from Chapter 41.

The formal definition of a 'disintegration' (which is nearly the same thing as a 'regular conditional probability') is in 452E. The main theorem depends, for its full generality, on the concept of 'countably compact measure'. It can be strengthened when μ is actually a Radon measure (452O).

The greater part of the section is concerned with general disintegrations, in which the measures μ_y are supposed to be measures on X and are not necessarily related to any particular structure on X. However a natural, and obviously important, class of applications has $X = Y \times Z$ and each μ_y based on the section $\{y\} \times Z$, so that it can be regarded as a measure on Z. Mostly there is very little more to be said in this case (see 452B-452D); but in 452M we find that there is an interesting variation in the way that countable compactness can be used.

452A Lemma Let (Y,T,ν) be a measure space, X a set, and $\langle\mu_y\rangle_{y\in Y}$ a family of measures on X. Let $\mathcal{A}$ be the family of subsets A of X such that $\theta E=\int\mu_y E\,\nu(dy)$ is defined in $\mathbb{R}$. Suppose that $X\in\mathcal{A}$.

(a) $\mathcal{A}$ is a Dynkin class.

(b) If Σ is any σ-subalgebra of $\mathcal{A}$ then $\mu=\theta\restriction\Sigma$ is a measure on X.

(c) Suppose now that every μ_y is complete. If, in (b), $\hat\mu$ is the completion of μ and $\hat\Sigma$ its domain, then $\hat\Sigma\subseteq\mathcal{A}$ and $\hat\mu=\theta\restriction\hat\Sigma$.

452B Theorem (a) Let X be a set, (Y,T,ν) a measure space, and $\langle\mu_y\rangle_{y\in Y}$ a family of measures on X such that $\int\mu_y X\,\nu(dy)$ is defined and finite. Let $\mathcal{E}$ be a family of subsets of X, closed under finite intersections, such that $\int\mu_y E\,\nu(dy)$ is defined in $\mathbb{R}$ for every $E\in\mathcal{E}$.

(i) If Σ is the σ-algebra of subsets of X generated by $\mathcal{E}$, we have a totally finite measure μ on X, with domain Σ, given by the formula $\mu E=\int\mu_y E\,\nu(dy)$ for every $E\in\Sigma$.

(ii) If $\hat\mu$ is the completion of μ and $\hat\Sigma$ its domain, then $\int\hat\mu_y E\,\nu(dy)$ is defined and equal to $\hat\mu E$ for every $E\in\hat\Sigma$, where $\hat\mu_y$ is the completion of μ_y for each $y\in Y$.

(b) Let Z be a set, (Y,T,ν) a measure space, and $\langle\mu_y\rangle_{y\in Y}$ a family of measures on Z such that $\int\mu_y Z\,\nu(dy)$ is defined and finite. Let $\mathcal{H}$ be a family of subsets of Z, closed under finite intersections, such that $\int\mu_y H\,\nu(dy)$ is defined in $\mathbb{R}$ for every $H\in\mathcal{H}$.

(i) If Υ is the σ-algebra of subsets of Z generated by $\mathcal{H}$, we have a totally finite measure μ on $Y\times Z$, with domain $\mathrm{T}\widehat{\otimes}\Upsilon$, defined by setting $\mu E=\int\mu_y E[\{y\}]\nu(dy)$ for every $E\in\mathrm{T}\widehat{\otimes}\Upsilon$.

(ii) If $\hat\mu$ is the completion of μ and $\hat\Sigma$ its domain, then $\int\hat\mu_y E[\{y\}]\nu(dy)$ is defined and equal to $\hat\mu E$ for every $E\in\hat\Sigma$, where $\hat\mu_y$ is the completion of μ_y for each $y\in Y$.

452C Theorem (a) Let Y be a topological space, ν a τ-additive topological measure on Y, $(X,\mathfrak{T})$ a topological space, and $\langle\mu_y\rangle_{y\in Y}$ a family of τ-additive topological measures on X such that $\int\mu_y X\,\nu(dy)$ is defined and finite. Suppose that there is a base $\mathcal{U}$ for $\mathfrak{T}$, closed under finite unions, such that $y\mapsto\mu_y U$ is lower semi-continuous for every $U\in\mathcal{U}$.

(i) We can define a τ-additive Borel measure μ on X by writing $\mu E=\int\mu_y E\,\nu(dy)$ for every Borel set $E\subseteq X$.

(ii) If $\hat\mu$ is the completion of μ and $\hat\Sigma$ its domain, then $\int\hat\mu_y E\,\nu(dy)$ is defined and equal to $\hat\mu E$ for every $E\in\hat\Sigma$, where $\hat\mu_y$ is the completion of μ_y for each $y\in Y$.

(b) Let Y be a topological space, ν a τ-additive topological measure on Y, $(Z,\mathfrak{U})$ a topological space, and $\langle\mu_y\rangle_{y\in Y}$ a family of τ-additive topological measures on Z such that $\int\mu_y Z\,\nu(dy)$ is defined and finite. Suppose that there is a base $\mathcal{V}$ for $\mathfrak{U}$, closed under finite unions, such that $y\mapsto\mu_y V$ is lower semi-continuous for every $V\in\mathcal{V}$.

(i) We can define a τ-additive Borel measure μ on $Y\times Z$ by writing $\mu E=\int\mu_y E[\{y\}]\nu(dy)$ for every Borel set $E\subseteq Y\times Z$.

(ii) If $\hat\mu$ is the completion of μ and $\hat\Sigma$ its domain, then $\int\hat\mu_y E[\{y\}]\nu(dy)$ is defined and equal to $\hat\mu E$ for every $E\in\hat\Sigma$, where $\hat\mu_y$ is the completion of μ_y for each $y\in Y$.

452D Theorem (a) Let $(Y,\mathfrak{S},\mathrm{T},\nu)$ be a Radon measure space, $(X,\mathfrak{T})$ a topological space, and $\langle\mu_y\rangle_{y\in Y}$ a uniformly tight family of Radon measures on X such that $\int\mu_y X\,\nu(dy)$ is defined and finite. Suppose that there is a base $\mathcal{U}$ for $\mathfrak{T}$, closed under finite unions, such that $y\mapsto\mu_y U$ is lower semi-continuous for every $U\in\mathcal{U}$. Then we have a totally finite Radon measure $\tilde\mu$ on X defined by saying that that $\tilde\mu E=\int\mu_y E\,\nu(dy)$ whenever $\tilde\mu$ measures E.

(b) Let $(Y,\mathfrak{S},\mathrm{T},\nu)$ be a Radon measure space, $(Z,\mathfrak{U})$ a topological space, and $\langle\mu_y\rangle_{y\in Y}$ a uniformly tight family of Radon measures on Z such that $\int\mu_y Z\,\nu(dy)$ is defined and finite. Suppose that there is a base $\mathcal{V}$ for $\mathfrak{U}$, closed under finite unions, such that $y\mapsto\mu_y V$ is lower semi-continuous for every $V\in\mathcal{V}$. Then we have a totally finite Radon measure $\tilde\mu$ on $Y\times Z$ such that $\tilde\mu E=\int\mu_y E[\{y\}]\nu(dy)$ whenever $\tilde\mu$ measures E.

452E Definition Let (X,Σ,μ) and (Y,T,ν) be measure spaces. A **disintegration** of μ over ν is a family $\langle\mu_y\rangle_{y\in Y}$ of measures on X such that $\int\mu_y E\,\nu(dy)$ is defined in $[0,\infty]$ and equal to μE for every $E\in\Sigma$. If $f:X\to Y$ is an inverse-measure-preserving function, a disintegration $\langle\mu_y\rangle_{y\in Y}$ of μ over ν is **consistent** with f if, for each $F\in\mathrm{T}$, $\mu_y f^{-1}[F]=1$ for ν-almost every $y\in F$. $\langle\mu_y\rangle_{y\in Y}$ is **strongly consistent** with f if, for almost every $y\in Y$, μ_y is a probability measure for which $f^{-1}[\{y\}]$ is conegligible.

452F Proposition Let (X, Σ, μ) and (Y, T, ν) be measure spaces and $\langle \mu_y \rangle_{y \in Y}$ a disintegration of μ over ν. Then $\iint f(x)\mu_y(dx)\nu(dy)$ is defined and equal to $\int f d\mu$ for every $[-\infty, \infty]$-valued function f such that $\int f d\mu$ is defined in $[-\infty, \infty]$.

Remark When $X = Y \times Z$ and our disintegration is a family $\langle \mu'_y \rangle_{y \in Y}$ of measures on X defined from a family $\langle \mu_y \rangle_{y \in Y}$ of probability measures on Z we can more naturally write $\int f(y, z)\mu_y(dz)$ in place of $\int f(x)\mu'_y(dx)$, and we get

$$\iint f(y, z)\mu_y(dz)\nu(dy) = \int f d\mu \text{ whenever the latter is defined in } [-\infty, \infty].$$

452G Proposition Let (X, Σ, μ) and (Y, T, ν) be measure spaces, $f : X \to Y$ an inverse-measure-preserving function, and $\langle \mu_y \rangle_{y \in Y}$ a disintegration of μ over ν.

(a) If $\langle \mu_y \rangle_{y \in Y}$ is consistent with f, and $F \in \mathrm{T}$, then $\mu_y f^{-1}[F] = \chi F(y)$ for ν-almost every $y \in Y$; in particular, almost every μ_y is a probability measure.

(b) If $\langle \mu_y \rangle_{y \in Y}$ is strongly consistent with f it is consistent with f.

(c) If ν is countably separated and $\langle \mu_y \rangle_{y \in Y}$ is consistent with f, then it is strongly consistent with f.

452H Lemma Let (X, Σ, μ) and (Y, T, ν) be probability spaces, and $T : L^\infty(\mu) \to L^\infty(\nu)$ a positive linear operator such that $T(\chi X^\bullet) = \chi Y^\bullet$ and $\int Tu = \int u$ whenever $u \in L^\infty(\mu)^+$. Let $\mathcal{K}$ be a countably compact class of subsets of X, closed under finite unions and countable intersections, such that μ is inner regular with respect to $\mathcal{K}$. Then there is a disintegration $\langle \mu_y \rangle_{y \in Y}$ of μ over ν such that

(i) μ_y is a complete probability measure on X, inner regular with respect to $\mathcal{K}$ and measuring every member of $\mathcal{K}$, for every $y \in Y$;

(ii) setting $h_g(y) = \int g \, d\mu_y$ whenever $g \in \mathcal{L}^\infty(\mu)$ and $y \in Y$ are such that the integral is defined, $h_g \in \mathcal{L}^\infty(\nu)$ and $T(g^\bullet) = h_g^\bullet$ for every $g \in \mathcal{L}^\infty(\mu)$.

452I Theorem Let (X, Σ, μ) be a non-empty countably compact measure space, (Y, T, ν) a σ-finite measure space, and $f : X \to Y$ an inverse-measure-preserving function. Then there is a disintegration $\langle \mu_y \rangle_{y \in Y}$ of μ over ν, consistent with f, such that μ_y is a complete probability measure on X for every $y \in Y$. Moreover,

(i) if $\mathcal{K}$ is a countably compact class of subsets of X such that μ is inner regular with respect to $\mathcal{K}$, then we can arrange that $\mathcal{K} \subseteq \operatorname{dom} \mu_y$ for every $y \in Y$;

(ii) if, in (i), $\mathcal{K}$ is closed under finite unions and countable intersections, then we can arrange that $\mathcal{K} \subseteq \operatorname{dom} \mu_y$ and μ_y is inner regular with respect to $\mathcal{K}$ for every $y \in Y$.

452K Example Set $Y = [0, 1]$, and let ν be Lebesgue measure on Y, with domain T. Let $X \subseteq [0, 1]$ have outer measure 1 and inner measure 0; let μ be the subspace measure on X. Set $f(x) = x$ for $x \in X$. Then there is no disintegration $\langle \mu_y \rangle_{y \in Y}$ of μ over ν which is consistent with f.

452L Definition Let $\langle X_i \rangle_{i \in I}$ be a family of sets, and λ a measure on $X = \prod_{i \in I} X_i$. For each $i \in I$ set $\pi_i(x) = x(i)$ for $x \in X$. Then the image measure $\lambda \pi_i^{-1}$ is the **marginal measure** of λ on X_i.

452M Theorem Let Y and Z be sets and $\mathrm{T} \subseteq \mathcal{P}Y$, $\Upsilon \subseteq \mathcal{P}Z$ σ-algebras. Let μ be a non-zero totally finite measure with domain $\mathrm{T} \widehat{\otimes} \Upsilon$, and ν the marginal measure of μ on Y. Suppose that the marginal measure λ of μ on Z is inner regular with respect to a countably compact class $\mathcal{K} \subseteq \mathcal{P}Z$ which is closed under finite unions and countable intersections. Then there is a family $\langle \mu_y \rangle_{y \in Y}$ of complete probability measures on Z, all measuring every member of $\mathcal{K}$ and inner regular with respect to $\mathcal{K}$, such that

$$\mu E = \int \mu_y E[\{y\}]\nu(dy)$$

for every $E \in \mathrm{T} \widehat{\otimes} \Upsilon$, and

$$\int f d\mu = \iint f(y, z)\mu_y(dz)\nu(dy)$$

whenever f is a $[-\infty, \infty]$-valued function such that $\int f d\mu$ is defined in $[-\infty, \infty]$.

452N Corollary Let Y and Z be sets and $\mathrm{T} \subseteq \mathcal{P}Y$, $\Upsilon \subseteq \mathcal{P}Z$ σ-algebras. Let μ be a probability measure with domain $\mathrm{T}\widehat{\otimes}\Upsilon$, and ν the marginal measure of μ on Y. Suppose that

either Υ is the Baire σ-algebra with respect to a compact Hausdorff topology on Z

or Υ is the Borel σ-algebra with respect to an analytic Hausdorff topology on Z

or (Z, Υ) is a standard Borel space.

Then there is a family $\langle \mu_y \rangle_{y\in Y}$ of probability measures on Z, all with domain Υ, such that

$$\mu E = \int \mu_y E[\{y\}]\nu(dy)$$

for every $E \in \mathrm{T}\widehat{\otimes}\Upsilon$, and

$$\int f d\mu = \iint f(y,z)\mu_y(dz)\nu(dy)$$

whenever f is a $[-\infty, \infty]$-valued function such that $\int f d\mu$ is defined in $[-\infty, \infty]$.

452O Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space, (Y, T, ν) a strictly localizable measure space, and $f : X \to Y$ an inverse-measure-preserving function. Then there is a disintegration $\langle \mu_y \rangle_{y\in Y}$ of μ over ν, consistent with f, such that every μ_y is a Radon measure on X.

452P Corollary Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space, $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ an analytic Radon measure space and $f : X \to Y$ an inverse-measure-preserving function. Then there is a disintegration $\langle \mu_y \rangle_{y\in Y}$ of μ over ν, strongly consistent with f, such that every μ_y is a Radon measure on X.

452Q Disintegrations and conditional expectations: Proposition Let (X, Σ, μ) and (Y, T, ν) be probability spaces and $f : X \to Y$ an inverse-measure-preserving function. Suppose that $\langle \mu_y \rangle_{y\in Y}$ is a disintegration of μ over ν which is consistent with f, and that g is a μ-integrable real-valued function.

(a) Setting $h_0(y) = \int g\,d\mu_y$ whenever $y \in Y$ and the integral is defined in $\mathbb{R}$, h_0 is a Radon-Nikodým derivative of the functional $F \mapsto \int_{f^{-1}[F]} g\,d\mu : \mathrm{T} \to \mathbb{R}$.

(b) Now suppose that ν is complete. Setting $h_1(x) = \int g\,d\mu_{f(x)}$ whenever $x \in X$ and the integral is defined in $\mathbb{R}$, then h_1 is a conditional expectation of g on the σ-algebra $\Sigma_0 = \{f^{-1}[F] : F \in \mathrm{T}\}$.

***452R Theorem** Let (X, Σ, μ) be a countably compact measure space, (Y, T, ν) a strictly localizable measure space, and $f : X \to Y$ an inverse-measure-preserving function. Then ν is countably compact.

***452S Corollary** If (X, Σ, μ) is a countably compact totally finite measure space, and T is any σ-subalgebra of Σ, then $\mu\restriction\mathrm{T}$ is countably compact.

452T Theorem Let X be a locally compact Hausdorff space, G a compact Hausdorff topological group and $\bullet$ a continuous action of G on X. Suppose that μ is a G-invariant Radon probability measure on X. For $x \in X$, write $f(x)$ for the corresponding orbit $\{a\bullet x : a \in G\}$ of the action. Let $Y = f[X]$ be the set of orbits, with the topology $\{W : W \subseteq Y,\ f^{-1}[W] \text{ is open in } X\}$. Write ν for the image measure μf^{-1} on Y.

(a) Y is locally compact and Hausdorff, and ν is a Radon probability measure.

(b) For each $\boldsymbol{y} \in Y$, there is a unique G-invariant Radon probability $\mu_{\boldsymbol{y}}$ on X such that $\mu_{\boldsymbol{y}}(\boldsymbol{y}) = 1$.

(c) $\langle \mu_{\boldsymbol{y}} \rangle_{\boldsymbol{y}\in Y}$ is a disintegration of μ over ν, strongly consistent with f.

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453 Strong liftings

The next step involves the concept of 'strong' lifting on a topological measure space (453A); I devote a few pages to describing the principal cases in which strong liftings are known to exist (453B-453J). When we have *Radon* measures μ and ν, with an *almost continuous* inverse-measure-preserving function between them, and a *strong* lifting for ν, we can hope for a disintegration $\langle \mu_y \rangle_{y\in Y}$ such that (almost) every μ_y lives on the appropriate fiber. This is the content of 453K. I end the section with a note on the relation between strong liftings and Stone spaces (453M) and with V.Losert's example of a space with no strong lifting (453N).

453A Definition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a topological measure space. A lifting $\phi : \Sigma \to \Sigma$ is **strong** or **of local type** if $\phi G \supseteq G$ for every open set $G \subseteq X$, that is, if $\phi F \subseteq F$ for every closed set $F \subseteq X$. I will say that ϕ is **almost strong** if $\bigcup_{G \in \mathfrak{T}} G \setminus \phi G$ is negligible.

Similarly, if $\mathfrak{A}$ is the measure algebra of μ, a lifting $\theta : \mathfrak{A} \to \Sigma$ is **strong** if $\theta G^\bullet \supseteq G$ for every open set $G \subseteq X$, and **almost strong** if $\bigcup_{G \in \mathfrak{T}} G \setminus \theta G^\bullet$ is negligible.

453B Theorem Let X be a topological group with a Haar measure μ, and Σ its algebra of Haar measurable sets.

(a) If $\phi : \Sigma \to \Sigma$ is a left-translation-invariant lifting, then ϕ is strong.

(b) μ has a strong lifting.

453C Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a topological measure space and $\phi : \Sigma \to \Sigma$ a lifting. Write $\mathcal{L}^\infty$ for the space of bounded Σ-measurable real-valued functions on X, so that $\mathcal{L}^\infty$ can be identified with $L^\infty(\Sigma)$ and the Boolean homomorphism $\phi : \Sigma \to \Sigma$ gives rise to a Riesz homomorphism $T : \mathcal{L}^\infty \to \mathcal{L}^\infty$.

(a) If ϕ is a strong lifting, then $Tf = f$ for every bounded continuous function $f : X \to \mathbb{R}$.

(b) If $(X, \mathfrak{T})$ is completely regular and $Tf = f$ for every $f \in C_b(X)$, then ϕ is strong.

453D Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a topological measure space.

(a) If μ has a strong lifting it is strictly positive.

(b) If μ is strictly positive and complete, and has an almost strong lifting, it has a strong lifting.

(c) If μ has an almost strong lifting it is τ-additive, so has a support.

(d) If μ is complete and $\mu X > 0$ and the subspace measure μ_E has an almost strong lifting for some conegligible set $E \subseteq X$, then μ has an almost strong lifting.

453E Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a complete strictly localizable topological measure space with an almost strong lifting, and $A \subseteq X$ a non-negligible set. Then the subspace measure μ_A has an almost strong lifting.

453F Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a complete strictly localizable topological measure space.

(a) If $\mathfrak{T}$ has a countable network, any lifting for μ is almost strong.

(b) Suppose that $\mu X > 0$ and μ is inner regular with respect to

$$\mathcal{K} = \{K : K \in \Sigma,\ \mu_K \text{ has an almost strong lifting}\},$$

where μ_K is the subspace measure on K. Then μ has an almost strong lifting.

453G Corollary (a) A non-zero quasi-Radon measure on a separable metrizable space has an almost strong lifting.

(b) A non-zero Radon measure μ on an analytic Hausdorff space X has an almost strong lifting.

453H Lemma Let (X, Σ, μ) be a complete locally determined measure space and $\mathfrak{T}$ a topology on X generated by a family $\mathcal{U} \subseteq \Sigma$. Suppose that $\phi : \Sigma \to \Sigma$ is a lifting such that $\phi U \supseteq U$ for every $U \in \mathcal{U}$. Then μ is a τ-additive topological measure, and ϕ is a strong lifting.

453I Proposition Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ be a family of topological probability spaces such that every $\mathfrak{T}_i$ has a countable network and every μ_i is strictly positive. Let λ be the (ordinary) complete product measure on $X = \prod_{i \in I} X_i$. Then λ is a τ-additive topological measure and has a strong lifting.

453J Corollary Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ be a family of quasi-Radon probability spaces such that every $\mathfrak{T}_i$ has a countable network consisting of measurable sets and every μ_i is strictly positive. Then the ordinary product measure λ on $X = \prod_{i \in I} X_i$ is quasi-Radon and has a strong lifting. If every X_i is compact and Hausdorff, then λ is a Radon measure.

453K Theorem Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be Radon measure spaces and $f : X \to Y$ an almost continuous inverse-measure-preserving function. Suppose that ν has an almost strong lifting. Then there is a disintegration $\langle \mu_y \rangle_{y \in Y}$ of μ over ν such that every μ_y is a Radon measure and $\mu_y X = \mu_y f^{-1}[\{y\}] = 1$ for almost every $y \in Y$.

453M Strong liftings and Stone spaces Let $(X,\mathfrak{T},\Sigma,\mu)$ be a quasi-Radon measure space, and $(Z,\mathfrak{S},\mathrm{T},\nu)$ the Stone space of the measure algebra $(\mathfrak{A},\bar{\mu})$ of μ. For $E\in\Sigma$ let $E^*\subseteq Z$ be the open-and-closed set corresponding to the equivalence class $E^\bullet\in\mathfrak{A}$. Let R be the relation

$$\bigcap_{F\subseteq X\text{ is closed}}\{(z,x):z\in Z\setminus F^*\text{ or }x\in F\}\subseteq Z\times X.$$

For every lifting $\phi:\Sigma\to\Sigma$ we have a unique function $g_\phi:X\to Z$ such that $\phi E=g_\phi^{-1}[E^*]$ for every $E\in\Sigma$.

(a) ϕ is strong iff $(g_\phi(x),x)\in R$ for every $x\in X$.

(b) If $\mathfrak{T}$ is Hausdorff, so that R is the graph of a function f, then ϕ is strong iff $fg_\phi(x)=x$ for every $x\in X$.

453N Losert's example There is a compact Hausdorff space with a strictly positive completion regular Radon probability measure which has no strong lifting.

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454 Measures on product spaces

A central concern of probability theory is the study of 'processes', that is, families $\langle X_t\rangle_{t\in T}$ of random variables thought of as representing the evolution of a system in time. Kolmogorov's successful representation of such processes as measurable functions on an abstract probability space was one of the foundations on which the modern concept of 'random variable' was built. In this section I give a version of Kolmogorov's theorem on the extension of consistent families of measures on subproducts to a measure on the whole product (454D). It turns out that some restriction on the marginal measures is necessary, and 'perfectness' seems to be an appropriate hypothesis, necessarily satisfied if the factor spaces are standard Borel spaces or the marginal measures are Radon measures. If we have marginal measures with stronger properties then we shall be able to infer corresponding properties of the measure on the product space (454A, generalizing 451J).

The apparatus here makes it easy to describe joint distributions of arbitrary families of real-valued random variables (454J-454P), extending the ideas of §271. For the sake of the theorem that almost all Brownian paths are continuous (477B) I briefly investigate measures on $C(T)$, where T is a Polish space (454Q-454S).

454A Theorem Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a non-empty family of totally finite measure spaces. Set $X=\prod_{i\in I}X_i$ and let μ be a measure on X which is inner regular with respect to the σ-algebra $\widehat{\bigotimes}_{i\in I}\Sigma_i$ generated by $\{\pi_i^{-1}[E]:i\in I,\,E\in\Sigma_i\}$, where $\pi_i:X\to X_i$ is the coordinate map for each $i\in I$. Suppose that every π_i is inverse-measure-preserving.

(a) If $\mathcal{K}\subseteq\mathcal{P}X$ is a family of sets which is closed under finite unions and countable intersections, and μ_i is inner regular with respect to $\mathcal{K}_i=\{K:K\subseteq X_i,\,\pi_i^{-1}[K]\in\mathcal{K}\}$ for every $i\in I$, then μ is inner regular with respect to $\mathcal{K}$.

(b)(i) If every μ_i is a compact measure, so is μ;

(ii) if every μ_i is a countably compact measure, so is μ;

(iii) if every μ_i is a perfect measure, so is μ.

454B Corollary Let $\langle X_i\rangle_{i\in I}$ be a family of Polish spaces with product X. Then any totally finite Baire measure on X is a compact measure.

454C Theorem Let (X,Σ,μ) be a perfect totally finite measure space and (Y,T,ν) any totally finite measure space. Let $\Sigma\otimes\mathrm{T}$ be the algebra of subsets of $X\times Y$ generated by $\{E\times F:E\in\Sigma,\,F\in\mathrm{T}\}$. If $\lambda_0:\Sigma\otimes\mathrm{T}\to[0,\infty[$ is a non-negative finitely additive functional such that $\lambda_0(E\times Y)=\mu E$ and $\lambda_0(X\times F)=\nu F$ whenever $E\in\Sigma$ and $F\in\mathrm{T}$, then λ_0 has a unique extension to a measure defined on $\Sigma\widehat{\otimes}\mathrm{T}$.

454D Theorem Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of totally finite perfect measure spaces. Set $X=\prod_{i\in I}X_i$, and write $\bigotimes_{i\in I}\Sigma_i$ for the algebra of subsets of X generated by $\{\pi_i^{-1}[E]:i\in I,\,E\in\Sigma_i\}$, where $\pi_i:X\to X_i$ is the coordinate map for each $i\in I$. Suppose that $\lambda_0:\bigotimes_{i\in I}\Sigma_i\to[0,\infty[$ is a non-negative finitely additive functional such that $\lambda_0\pi_i^{-1}[E]=\mu_iE$ whenever $i\in I$ and $E\in\Sigma_i$. Then λ_0 has a unique extension to a measure λ with domain $\widehat{\bigotimes}_{i\in I}\Sigma_i$, and λ is perfect.

454E Corollary Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of perfect measure spaces. Let $\mathcal{C}$ be the family of subsets of $X=\prod_{i\in I}X_i$ expressible in the form $X\cap\bigcap_{i\in J}\pi_i^{-1}[E_i]$ where $J\subseteq I$ is finite and $E_i\in\Sigma_i$ for every $i\in I$, writing $\pi_i(x)=x(i)$ for $x\in X$, $i\in I$. Suppose that $\lambda_0:\mathcal{C}\to\mathbb{R}$ is a functional such that (i) $\lambda_0\pi_i^{-1}[E]=\mu_iE$ whenever $i\in I$ and $E\in\Sigma_i$ (ii) $\lambda_0C=\lambda_0(C\cap\pi_i^{-1}[E])+\lambda_0(C\setminus\pi_i^{-1}[E])$ whenever $C\in\mathcal{C}$, $i\in I$ and $E\in\Sigma_i$. Then λ_0 has a unique extension to a measure on $\widehat{\bigotimes}_{i\in I}\Sigma_i$, which is necessarily perfect.

454F Corollary Let $\langle(X_i,\Sigma_i)\rangle_{i\in I}$ be a family of standard Borel spaces. Set $X=\prod_{i\in I}X_i$, and let $\bigotimes_{i\in I}\Sigma_i$ be the algebra of subsets of X generated by $\{\pi_i^{-1}[E]:i\in I,\,E\in\Sigma_i\}$, where $\pi_i:X\to X_i$ is the coordinate map for each i. Let $\lambda_0:\bigotimes_{i\in I}\Sigma_i\to[0,\infty[$ be a non-negative finitely additive functional such that all the marginal functionals $E\mapsto\lambda_0\pi_i^{-1}[E]:\Sigma_i\to[0,\infty[$ are countably additive. Then λ_0 has a unique extension to a measure defined on $\widehat{\bigotimes}_{i\in I}\Sigma_i$, which is a compact measure.

454G Corollary Let $\langle X_i\rangle_{i\in I}$ be a family of sets, and Σ_i a σ-algebra of subsets of X_i for each $i\in I$. Suppose that for each finite set $J\subseteq I$ we are given a totally finite measure μ_J on $Z_J=\prod_{i\in J}X_i$ with domain $\widehat{\bigotimes}_{i\in J}\Sigma_i$ such that (i) whenever J, K are finite subsets of I and $J\subseteq K$, then the canonical projection from Z_K to Z_J is inverse-measure-preserving (ii) every marginal measure $\mu_{\{i\}}$ on $Z_{\{i\}}\cong X_i$ is perfect. Then there is a unique measure μ defined on $\widehat{\bigotimes}_{i\in I}\Sigma_i$ such that the canonical projection $\tilde\pi_J:\prod_{i\in I}X_i\to Z_J$ is inverse-measure-preserving for every finite $J\subseteq I$.

454H Corollary Let $\langle(X_n,\Sigma_n)\rangle_{n\in\mathbb{N}}$ be a sequence of standard Borel spaces. For each $n\in\mathbb{N}$ set $Z_n=\prod_{i<n}X_i$ and $\mathrm{T}_n=\widehat{\bigotimes}_{i<n}\Sigma_i$. For $n\in\mathbb{N}$, $W\in\mathrm{T}_{n+1}$ and $z\in Z_n$ write $W[\{z\}]=\{\xi:\xi\in X_n,\,(z,\xi)\in W\}$; set $X=\prod_{n\in\mathbb{N}}X_n$ and write $\tilde\pi_n$ for the canonical projection of X onto Z_n. Suppose that for each $n\in\mathbb{N}$ and $z\in Z_n$ we are given a probability measure ν_z on X_n with domain Σ_n such that $z\mapsto\nu_z(E)$ is T_n-measurable for every $E\in\Sigma_n$.

(a) We have a sequence $\langle\mu_n\rangle_{n\in\mathbb{N}}$ of probability measures such that, for each $n\in\mathbb{N}$, μ_n has domain T_n and

$$\mu_{n+1}(W)=\int\nu_zW[\{z\}]\mu_n(dz)$$

for every $W\in\mathrm{T}_{n+1}$, and

$$\int fd\mu_{n+1}=\iint\dots\iint f(\xi_0,\dots,\xi_n)\nu_{(\xi_0,\dots,\xi_{n-1})}(d\xi_n)\\ \nu_{(\xi_0,\dots,\xi_{n-2})}(d\xi_{n-1})\dots\nu_{\xi_0}(d\xi_1)\nu_\emptyset(d\xi_0)$$

for every $n\in\mathbb{N}$ and μ_{n+1}-integrable real-valued function f.

(b) There is a unique probability measure μ on $X=\prod_{n\in\mathbb{N}}X_n$, with domain $\widehat{\bigotimes}_{n\in\mathbb{N}}\Sigma_n$, such that μ_n is the image measure $\mu\tilde\pi_n^{-1}$ on Z_n for every $n\in\mathbb{N}$

454J Distributions of random processes: Proposition Let (Ω,Σ,μ) be a probability space and $\langle X_i\rangle_{i\in I}$ a family of real-valued random variables on Ω.

(i) There is a unique complete probability measure ν on $\mathbb{R}^I$, measuring every Baire set and inner regular with respect to the zero sets, such that

$$\nu\{x:x\in\mathbb{R}^I,\,x(i_r)\le\alpha_r\text{ for every }r\le n\}=\Pr(X_{i_r}\le\alpha_r\text{ for every }r\le n)$$

whenever $i_0,\dots,i_n\in I$ and $\alpha_0,\dots,\alpha_n\in\mathbb{R}$.

(ii) If $i_0,\dots,i_n\in I$ and $\tilde\pi(x)=(x(i_0),\dots,x(i_n))$ for $x\in\mathbb{R}^I$, then the image measure $\nu\tilde\pi^{-1}$ on $\mathbb{R}^{n+1}$ is the joint distribution of $X_{i_0},\dots,X_{i_n}$.

(iii) ν is a compact measure. If I is countable then ν is a Radon measure.

(iv) If every X_i is defined everywhere on Ω, then the function $\omega \mapsto \langle X_i(\omega)\rangle_{i\in I} : \Omega \to \mathbb{R}^I$ is inverse-measure-preserving for $\hat{\mu}$ and ν, where $\hat{\mu}$ is the completion of μ.

454K Definition In the context of 454J, I will call ν the (**joint**) **distribution** of the process $\langle X_i\rangle_{i\in I}$.

454L Independence: Theorem Let (Ω,Σ,μ) be a probability space and $\langle X_i\rangle_{i\in I}$ a family of real-valued random variables on Ω, with distribution ν on $\mathbb{R}^I$. Then $\langle X_i\rangle_{i\in I}$ is independent iff ν is the c.l.d. product of the marginal measures on $\mathbb{R}$.

454M Proposition Let I be a set, and suppose that for each finite $J\subseteq I$ we are given a Radon probability measure ν_J on $\mathbb{R}^J$ such that whenever K is a finite subset of I and $J\subseteq K$, then the canonical projection from $\mathbb{R}^K$ to $\mathbb{R}^J$ is inverse-measure-preserving. Then there is a unique complete probability measure ν on $\mathbb{R}^I$, measuring every Baire set and inner regular with respect to the zero sets, such that the canonical projection from $\mathbb{R}^I$ to $\mathbb{R}^J$ is inverse-measure-preserving for every finite $J\subseteq I$.

454N Proposition Let Ω be a Hausdorff space, μ and ν two Radon probability measures on Ω, and $\langle X_i\rangle_{i\in I}$ a family of continuous functions separating the points of Ω. If μ and ν give $\langle X_i\rangle_{i\in I}$ the same distribution, they are equal.

454O What distributions determine: Proposition Let (Ω,Σ,μ), (Ω',Σ',μ') probability spaces, $\langle X_i\rangle_{i\in I}$ a family of random variables on Ω and $\langle X_i'\rangle_{i\in I}$ a family of random variables on Ω', both with the same distribution ν on R^I. Suppose that $\langle I_j\rangle_{j\in J}$ is a family of countable subsets of I, and that for each $j\in I$ we have a Borel measurable function $f_j:\mathbb{R}^{I_j}$ to $\mathbb{R}$. For $j\in J$ define Y_j, Y_j' by saying that

$$Y_j(\omega)=f_j(\langle X_i(\omega)\rangle_{i\in I_j}) \text{ for } \omega\in\Omega\cap\bigcap_{i\in I_j}\operatorname{dom}X_i,$$

$$Y_j'(\omega')=f_j(\langle X_i'(\omega')\rangle_{i\in I_j}) \text{ for } \omega'\in\Omega'\cap\bigcap_{i\in I_j}\operatorname{dom}X_i'.$$

Then $\langle Y_j\rangle_{j\in J}$ and $\langle Y_j'\rangle_{j\in J}$ have the same distribution.

454P Theorem Let I be a set.

(a) Let ν and ν' be Baire probability measures on $\mathbb{R}^I$ such that $\int e^{if(x)}\nu(dx)=\int e^{if(x)}\nu'(dx)$ for every continuous linear functional $f:\mathbb{R}^I\to\mathbb{R}$. Then $\nu=\nu'$.

(b) Let $\langle X_j\rangle_{j\in I}$ and $\langle Y_j\rangle_{j\in I}$ be two families of random variables such that

$$\mathbb{E}(\exp(i\textstyle\sum_{r=0}^n\alpha_rX_{j_r}))=\mathbb{E}(\exp(i\textstyle\sum_{r=0}^n\alpha_rY_{j_r}))$$

whenever $j_0,\dots,j_n\in I$ and $\alpha_0,\dots,\alpha_n\in\mathbb{R}$. Then $\langle X_j\rangle_{j\in I}$ and $\langle Y_j\rangle_{j\in I}$ have the same distribution.

454Q Continuous processes: Lemma Let T be a separable metrizable space and (X,Σ,μ) a semi-finite measure space. Let $\mathfrak{T}$ be a topology on X such that μ is inner regular with respect to the closed sets.

(a) Let $\phi:X\times T\to\mathbb{R}$ be a function such that (i) for each $x\in X$, $t\mapsto\phi(x,t)$ is continuous (ii) for each $t\in T$, $x\mapsto\phi(x,t)$ is Σ-measurable. Then μ is inner regular with respect to $\mathcal{K}=\{K:K\subseteq X,\ \phi\restriction K\times T$ is continuous$\}$.

(b) Let $\theta:X\to C(T)$ be a function such that $x\mapsto\theta(x)(t)$ is Σ-measurable for every $t\in T$. Give $C(T)$ the topology $\mathfrak{T}_c$ of uniform convergence on compact subsets of T. Then θ is almost continuous.

454R Proposition Let T be an analytic metrizable space, and μ a probability measure on $C(T)$ with domain the σ-algebra Σ generated by the evaluation functionals $f\mapsto f(t):C(T)\to\mathbb{R}$ for $t\in T$. Give $C(T)$ the topology $\mathfrak{T}_c$ of uniform convergence on compact subsets of T. Then the completion of μ is a $\mathfrak{T}_c$-Radon measure.

454S Corollary Let T be an analytic metrizable space.

(a) $C(T)$, with either the topology $\mathfrak{T}_p$ of uniform convergence on finite subsets of T or the topology $\mathfrak{T}_c$ of uniform convergence on compact subsets of T, is a measure-compact Radon space.

(b) Let μ be a Baire probability measure on $\mathbb{R}^T$ such that $\mu^*C(T) = 1$. Then the subspace measure $\hat{\mu}_C$ on $C(T)$ induced by the completion of μ is a Radon measure on $C(T)$ if $C(T)$ is given either $\mathfrak{T}_p$ or $\mathfrak{T}_c$. μ itself is τ-additive and has a unique extension $\tilde{\mu}$ which is a Radon measure on $\mathbb{R}^T$; $\hat{\mu}_C$ is the subspace measure on $C(T)$ induced by $\tilde{\mu}$.

***454T Convergence of distributions (a)** Let I be a set. Write M for the set of distributions on $\mathbb{R}^I$, that is, the set of completions of probability measures with domain $\mathcal{B}\mathfrak{a}(\mathbb{R}^I)$. For any $\nu \in M$, the integral $\int f\,d\nu$ is defined for every bounded continuous function $f : \mathbb{R}^I \to \mathbb{R}$. I will say that the **vague topology** on M is the topology generated by the functionals $\nu \mapsto \int f d\nu$ as f runs over the space $C_b(\mathbb{R}^I)$ of bounded continuous real-valued functions on $\mathbb{R}^I$.

(b) The vague topology on M is Hausdorff.

***454U Theorem** Let (Ω, Σ, μ) be a probability space, and I a set. Let M be the set of distributions on $\mathbb{R}^I$; for a family $\boldsymbol{X} = \langle X_i \rangle_{i \in I}$ of real-valued random variables on Ω, let $\nu_{\boldsymbol{X}}$ be its distribution. Then the function $\boldsymbol{X} \mapsto \nu_{\boldsymbol{X}} : \mathcal{L}^0(\mu)^I \to M$ is continuous for the product topology on $\mathcal{L}^0(\mu)^I$ corresponding to the topology of convergence in measure on $\mathcal{L}^0(\mu)$ and the vague topology on M.

***454V (a)(i)** If $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra, I is a set, and $u \in L^0(\mathfrak{A})^I$, we have a sequentially order-continuous Boolean homomorphism $E \mapsto [\![u \in E]\!] : \mathcal{B}\mathfrak{a}(\mathbb{R}^I) \to \mathfrak{A}$ defined by saying that

$$[\![u \in \{x : x \in \mathbb{R}^I,\, x(i) \le \alpha\}]\!] = [\![u(i) \le \alpha]\!]$$

whenever $i \in I$ and $\alpha \in \mathbb{R}$.

(ii) If $h : \mathbb{R}^I \to \mathbb{R}$ is a Baire measurable function, there is a function $\bar{h} : L^0(\mathfrak{A})^I \to L^0(\mathfrak{A})$ defined by saying that $[\![\bar{h}(u) \in E]\!] = [\![u \in h^{-1}[E]]\!]$ for every Borel set $E \subseteq \mathbb{R}$.

(b) Suppose that $(\mathfrak{A}, \bar{\mu})$ is a probability algebra, I is a set and $u \in L^0(\mathfrak{A})^I$. Then there is a unique complete probability measure ν on $\mathbb{R}^I$, measuring every Baire set and inner regular with respect to the zero sets, such that

$$\nu\{x : x \in \mathbb{R}^I,\, x(i) \in E_i \text{ for every } i \in J\} = \bar{\mu}(\inf_{i \in J} [\![u(i) \in E_i]\!])$$

whenever $J \subseteq I$ is finite and $E_i \subseteq \mathbb{R}$ is a Borel set for every $i \in J$.

(c) In this context, I will call ν the **(joint) distribution** of u.

(d) Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{A}', \bar{\mu}')$ be probability algebras, and $u \in L^0(\mathfrak{A})^I$, $u' \in L^0(\mathfrak{A}')^I$ families with the same distribution. Suppose that $\langle h_j \rangle_{j \in J}$ is a family of Baire measurable functions from $\mathbb{R}^I$ to $\mathbb{R}$. Then $\langle \bar{h}_j(u) \rangle_{j \in J}$ and $\langle \bar{h}_j(u') \rangle_{j \in J}$ have the same distribution.

(e) Similarly, if $(\mathfrak{A}, \bar{\mu})$ is a probability algebra, I a set, and we write ν_u for the distribution of $u \in L^0(\mathfrak{A})^I$, $u \mapsto \nu_u$ is continuous for the product topology on $L^0(\mathfrak{A})^I$ corresponding to the topology of convergence in measure on $L^0(\mathfrak{A})$ and the vague topology on the space M of distributions on $\mathbb{R}^I$.

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455 Markov and Lévy processes

For a 'Markov' process, in which the evolution of the system after a time t depends only on the state at time t, the general theory of §454 leads to a straightforward existence theorem (at least for random variables taking values in standard Borel spaces) dependent only on a natural consistency condition on the transitional probabilities (455A, 455E). The formulation leads naturally to descriptions of the 'Markov property' (for stopping times taking only countably many values) in terms of disintegrations and conditional expectations (455C, 455Ec). With appropriate continuity conditions, we find that the process can be represented either by a Radon measure (455H) or by a measure on the set of càdlàg paths (455Gc) for which we have a formulation of the strong Markov property (for general stopping times) in terms of disintegrations (455O). These conditions are satisfied by Lévy processes (455P-455R). For these, we have an alternative expression of the strong Markov property in terms of inverse-measure-preserving functions (455U). By far the most important example of a continuous-time Markov process is Brownian motion, but I defer discussion of this to §477.

455A Theorem Let T be a totally ordered set with least element t^*, and for each $t \in T$ let Ω_t be a non-empty set and T_t a σ-algebra of subsets of Ω_t containing all singleton subsets of Ω_t. Set $\Omega = \prod_{t\in T}\Omega_t$ and for $t \in T$, $\omega \in \Omega$ set $X_t(\omega) = \omega(t)$. Fix $x^* \in \Omega_{t^*}$. Suppose that we are given, for each pair $s < t$ in T, a family $\langle\nu_x^{(s,t)}\rangle_{x\in\Omega_s}$ of perfect probability measures on Ω_t, all with domain T_t, and suppose that

(†) whenever $s < t < u$ in T and $x \in \Omega_s$, then $\langle\nu_y^{(t,u)}\rangle_{y\in\Omega_t}$ is a disintegration of $\nu_x^{(s,u)}$ over $\nu_x^{(s,t)}$.

For $J \subseteq T$ write π_J for the canonical map from Ω onto $Z_J = \prod_{t\in J}\Omega_t$. Then there is a unique probability measure μ on Ω, with domain $\widehat{\bigotimes}_{t\in T}\mathrm{T}_t$, such that, writing λ_J for the image measure $\mu\pi_J^{-1}$,

$$\begin{aligned}\int f d\lambda_J &= \int f(\omega(t^*),\omega(t_1),\dots,\omega(t_n))\mu(d\omega)\\ &= \int\dots\iint f(x^*,x_1,\dots,x_n)\nu_{x_{n-1}}^{(t_{n-1},t_n)}(dx_n)\\ &\qquad\qquad\nu_{x_{n-2}}^{(t_{n-2},t_{n-1})}(dx_{n-1})\dots\nu_{x^*}^{(t^*,t_1)}(dx_1)\end{aligned}$$

whenever $t^* < t_1 < \dots < t_n$, $J = \{t^*,t_1,\dots,t_n\}$ and f is λ_J-integrable. μ is perfect, and the marginal measure $\mu_t = \mu X_t^{-1}$ is equal to $\nu_{x^*}^{(t^*,t)}$, if $t > t^*$, while $\mu_{t^*}\{x^*\} = 1$.

455B Lemma Suppose that T, t^*, $\langle(\Omega_t,\mathrm{T}_t)\rangle_{t\in T}$, Ω, x^* and $\langle\nu_x^{(s,t)}\rangle_{s<t,x\in\Omega_s}$ are as in 455A.

(a) Suppose that μ is constructed from x^* and $\langle\nu_x^{(s,t)}\rangle_{s<t,x\in\Omega_s}$ as in 455A. If $F \in \widehat{\bigotimes}_{t\in T}\mathrm{T}_t$ is determined by coordinates in $[t^*,t_0]$ and $H^* = \{\omega : \omega(t_i) \in E_i \text{ for } 1 \le i \le n\}$ where $t_0 < t_1 \dots < t_n$ and $E_i \in \mathrm{T}_{t_i}$ for $1 \le i \le n$, then

$$\mu(H^*\cap F) = \int_F\int\dots\int\chi H(y_1,\dots,y_n)\nu_{y_{n-1}}^{(t_{n-1},t_n)}(dy_n)\dots\nu_{\omega(t_0)}^{(t_0,t_1)}(dy_1)\mu(d\omega) \qquad (*)$$

where $H = \prod_{1\le i\le n}E_i$.

(b) Suppose that $\omega \in \Omega$ and $a \in T\cup\{\infty\}$, where ∞ is taken to be greater than every element of T. For $s < t$ in T and $x \in \Omega_s$ set

$$\begin{aligned}\nu_{\omega ax}^{(s,t)} &= \nu_x^{(s,t)} \text{ if } a < s,\\ &= \nu_{\omega(a)}^{(a,t)} \text{ if } s \le a < t,\\ &= \delta_{\omega(t)}^{(t)} \text{ if } t \le a,\end{aligned}$$

here writing $\delta_x^{(t)}$ for the probability measure with domain T_t such that $\delta_x^{(t)}(\{x\}) = 1$.

(i) $\nu_{\omega ax}^{(s,t)}$ is always a perfect probability measure with domain T_t, and $\langle\nu_{\omega ay}^{(t,u)}\rangle_{y\in\Omega_t}$ is a disintegration of $\nu_{\omega ax}^{(s,u)}$ over $\nu_{\omega ax}^{(s,t)}$ whenever $s < t < u$ in T and $x \in \Omega_s$.

(ii) Taking $\mu_{\omega a}$ to be the measure on Ω defined from $\omega(t^*)$ and $\langle\nu_{\omega ax}^{(s,t)}\rangle_{s<t,x\in\Omega_t}$ by the method of 455A, then $\{\omega' : \omega' \in \Omega,\ \omega'\restriction D = \omega\restriction D\}$ is $\mu_{\omega a}$-conegligible for every countable $D \subseteq T\cap[t^*,a]$.

(iii) If ω, $\omega' \in \Omega$ and $\omega\restriction[t^*,a] = \omega'\restriction[t^*,a]$ then $\mu_{\omega a} = \mu_{\omega' a}$.

455C Theorem Suppose that T, t^*, $\langle(\Omega_t, \mathrm{T}_t)\rangle_{t\in T}$, Ω, x^*, $\langle\nu_x^{(s,t)}\rangle_{s<t,x\in\Omega_s}$ and μ are as in 455A. Adjoin a point ∞ to T above any point of T, and let $\tau : \Omega \to T \cup \{\infty\}$ be a function taking countably many values and such that $\{\omega : \tau(\omega) \le s\}$ belongs to $\widehat{\bigotimes}_{t\in T}\mathrm{T}_t$ and is determined by coordinates in $[t^*, s]$ for every $s \in T$.

(a) For $\omega \in \Omega$ define $\nu_{\omega,\tau(\omega),x}^{(s,t)}$, for $s < t$ and $x \in \Omega_s$, as in 455Bb, and let $\mu_{\omega,\tau(\omega)}$ be the corresponding measure on Ω. Then $\langle\mu_{\omega,\tau(\omega)}\rangle_{\omega\in\Omega}$ is a disintegration of μ over itself.

(b) Let Σ_τ be the set of those $E \in \widehat{\bigotimes}_{t\in T}\mathrm{T}_t$ such that $E \cap \{\omega : \tau(\omega) \le t\}$ is determined by coordinates in $[t^*, t]$ for every $t \in T$. Then Σ_τ is a σ-subalgebra of $\widehat{\bigotimes}_{t\in T}\mathrm{T}_t$. If f is any μ-integrable real-valued function, and we set $g_f(\omega) = \int f d\mu_{\omega,\tau(\omega)}$ when this is defined in $\mathbb{R}$, then g_f is a conditional expectation of f on Σ_τ.

455E Theorem Let T be a totally ordered set with least element t^*. Let $\langle\Omega_t\rangle_{t\in T}$ be a family of Hausdorff spaces; suppose that we are given an $x^* \in \Omega_{t^*}$ and, for each pair $s < t$ in T, a family $\langle\nu_x^{(s,t)}\rangle_{x\in\Omega_s}$ of Radon probability measures on Ω_t such that

$\langle\nu_y^{(t,u)}\rangle_{y\in\Omega_t}$ is a disintegration of $\nu_x^{(s,u)}$ over $\nu_x^{(s,t)}$ whenever $s < t < u$ in T and $x \in \Omega_s$.

Write $\Omega = \prod_{t\in T}\Omega_t$; for $t \in T$ let $\mathcal{B}(\Omega_t)$ be the Borel σ-algebra of Ω_t, and $X_t : \Omega \to \Omega_t$ the canonical map; for $J \subseteq T$ write π_J for the canonical map from Ω onto $\prod_{t\in J}\Omega_t$. For $t \in T$ and $x \in \Omega_t$ let $\delta_x^{(t)}$ be the Dirac measure on Ω_t concentrated at x.

(a) There is a unique complete probability measure $\hat\mu$ on Ω, inner regular with respect to $\widehat{\bigotimes}_{t\in T}\mathcal{B}(\Omega_t)$, such that, writing $\hat\lambda_J$ for the image measure $\hat\mu\pi_J^{-1}$,

$$\begin{aligned}\int f d\hat\lambda_J &= \int f(\omega(t^*), \omega(t_1), \dots, \omega(t_n))\hat\mu(d\omega)\\ &= \int \dots \iint f(x^*, x_1, \dots, x_n)\nu_{x_{n-1}}^{(t_{n-1},t_n)}(dx_n)\\ &\qquad\qquad \nu_{x_{n-2}}^{(t_{n-2},t_{n-1})}(dx_{n-1})\dots\nu_{x^*}^{(t^*,t_1)}(dx_1)\end{aligned}$$

whenever $t^* < t_1 < \ldots < t_n$ in T, $J = \{t^*, t_1, \dots, t_n\}$ and f is $\hat\lambda_J$-integrable. In particular, the image measure $\hat\mu X_t^{-1}$ is equal to $\nu_{x^*}^{(t^*,t)}$ if $t > t^*$, and to $\delta_{x^*}^{(t^*)}$ if $t = t^*$.

(b)(i) For $\omega \in \Omega$ and $a \in T \cup \{\infty\}$ define $\langle\nu_{\omega ax}^{(s,t)}\rangle_{s<t,x\in\Omega_s}$ by setting

$$\begin{aligned}\nu_{\omega ax}^{(s,t)} &= \nu_x^{(s,t)} \text{ if } a < s,\\ &= \nu_{\omega(a)}^{(a,t)} \text{ if } s \le a < t,\\ &= \delta_{\omega(t)}^{(t)} \text{ if } t \le a.\end{aligned}$$

The family $\langle\nu_{\omega ax}^{(s,t)}\rangle_{s<t,x\in\Omega_s}$, together with the point $\omega(t^*) \in \Omega_{t^*}$, satisfy the conditions of (a), so can be used to define a complete measure $\hat\mu_{\omega a}$ on Ω.

(ii) If $\omega \in \Omega$ and $D \subseteq T \cap [t^*, a]$ is countable, then $\hat\mu_{\omega a}\{\omega' : \omega'\restriction D = \omega\restriction D\} = 1$.

(iii) If ω, $\omega' \in \Omega$ and $\omega'\restriction[t^*, a] = \omega\restriction[t^*, a]$, then $\hat\mu_{\omega' a} = \hat\mu_{\omega a}$.

(c) Let Σ be the domain of $\hat\mu$. Suppose that $\tau : \Omega \to T \cup \{\infty\}$ is a function taking countably many values and such that $\{\omega : \tau(\omega) \le t\}$ belongs to Σ and is determined by coordinates in $[t^*, t]$ for every $t \in T$.

(i) $\langle\hat\mu_{\omega,\tau(\omega)}\rangle_{\omega\in\Omega}$ is a disintegration of $\hat\mu$ over itself.

(ii) Let Σ_τ be the set

$$\{E : E \in \Sigma,\ E \cap \{\omega : \tau(\omega) \le t\} \text{ is determined by coordinates in } [t^*, t] \text{ for every } t \in T\}.$$

Then Σ_τ is a σ-subalgebra of Σ. If f is any $\hat\mu$-integrable real-valued function, and we set $g_f(\omega) = \int f d\hat\mu_{\omega,\tau(\omega)}$ when this is defined in $\mathbb{R}$, then g_f is a conditional expectation of f on Σ_τ.

455F Definitions (a) Let U be a Hausdorff space and $\langle\nu_x^{(s,t)}\rangle_{0\le s<t,x\in U}$ a family of Radon probability measures on U. I will say that $\langle\nu_x^{(s,t)}\rangle_{0\le s<t,x\in U}$ is **narrowly continuous** if it is continuous, as a function

from $\{(s,t) : 0 \le s < t\} \times U$ to the set of Radon probability measures on U, when the latter is given its narrow topology.

(b) Let (U,ρ) be a metric space, and $\langle \nu_x^{(s,t)} \rangle_{0 \le s<t, x\in U}$ a family of Radon probability measures on U. I will say that $\langle \nu_x^{(s,t)} \rangle_{0 \le s<t, x\in U}$ is **uniformly time-continuous on the right** if for every $\epsilon > 0$ there is a $\delta > 0$ such that $\nu_x^{(s,t)} B(x,\epsilon) \ge 1-\epsilon$ whenever $x \in U$ and $0 \le s < t \le s+\delta$.

455G Theorem Let (U,ρ) be a complete metric space and $\langle \nu_x^{(s,t)} \rangle_{0 \le s<t, x\in U}$ a family of Radon probability measures on U, uniformly time-continuous on the right, such that $\langle \nu_y^{(t,u)} \rangle_{y\in U}$ is a disintegration of $\nu_x^{(s,u)}$ over $\nu_x^{(s,t)}$ whenever $0 \le s < t < u$ and $x \in U$. Take a point $\tilde\omega$ in $\Omega = U^{[0,\infty[}$, and $a \in [0,\infty]$. Let $\hat\mu_{\tilde\omega a}$ be the completed probability measure on Ω defined from $\langle \nu_x^{(s,t)} \rangle_{0 \le s<t, x\in U}$, $\tilde\omega$ and a as in 455Eb.

(a) For $\hat\mu_{\tilde\omega a}$-almost every $\omega \in \Omega$, $\lim_{q\in\mathbb{Q}, q\downarrow t} \omega(q)$ and $\lim_{q\in\mathbb{Q}, q\uparrow t} \omega(q)$ are defined in U for every $t > a$.

(b)(i) If $a \le t < \infty$, then $\omega(t) = \lim_{q\in\mathbb{Q}, q\downarrow t} \omega(q)$ for $\hat\mu_{\tilde\omega a}$-almost every $\omega \in \Omega$.

(ii) If $a < t < \infty$, then $\omega(t) = \lim_{q\in\mathbb{Q}, q\uparrow t} \omega(q)$ for $\hat\mu_{\tilde\omega a}$-almost every $\omega \in \Omega$.

(c)(i) Let $C^{\Downarrow}$ be the set of càllàl functions from $[0,\infty[$ to U. If $\tilde\omega \in C^{\Downarrow}$, $C^{\Downarrow}$ has full outer measure for $\hat\mu_{\tilde\omega a}$.

(ii) Let C_{dlg} be the set of càdlàg functions from $[0,\infty[$ to U. If $\tilde\omega \in C_{\mathrm{dlg}}$, C_{dlg} has full outer measure for $\hat\mu_{\tilde\omega a}$.

455H Corollary Let (U,ρ) be a complete metric space and $\langle \nu_x^{(s,t)} \rangle_{0 \le s<t, x\in U}$ a family of Radon probability measures on U, uniformly time-continuous on the right, such that $\langle \nu_y^{(t,u)} \rangle_{y\in U}$ is a disintegration of $\nu_x^{(s,u)}$ over $\nu_x^{(s,t)}$ whenever $0 \le s < t < u$ and $x \in U$. Let $C^{\Downarrow}(U)$ be the set of càllàl functions from $[0,\infty[$ to U. Suppose that $\tilde\omega \in C^{\Downarrow}(U)$, and $a \in [0,\infty]$; let $\hat\mu_{\tilde\omega a}$ be the completed probability measure on $\Omega = U^{[0,\infty[}$ defined from $\tilde\omega$, a and $\langle \nu_x^{(s,t)} \rangle_{0 \le s<t, x\in U}$ as in 455Eb. Then $\hat\mu_{\tilde\omega a}$ has a unique extension to a Radon measure $\tilde\mu_{\tilde\omega a}$ on Ω, and $\tilde\mu_{\tilde\omega a} C^{\Downarrow}(U) = 1$.

455I Lemma Let (U,ρ) be a complete separable metric space and $\langle \nu_x^{(s,t)} \rangle_{0 \le s<t, x\in U}$ a family of Radon probability measures on U, uniformly time-continuous on the right, such that $\langle \nu_y^{(t,u)} \rangle_{y\in U}$ is a disintegration of $\nu_x^{(s,u)}$ over $\nu_x^{(s,t)}$ whenever $0 \le s < t < u$ and $x \in U$. Suppose that $\tilde\omega \in \Omega$, and $a \in [0,\infty]$; let $\hat\mu_{\tilde\omega a}$ be the completed probability measure on $\Omega = U^{[0,\infty[}$ defined from $\langle \nu_x^{(s,t)} \rangle_{0 \le s<t, x\in U}$, $\tilde\omega$ and a as in 455Eb.

(a) Suppose that $0 \le q_0 < q_1$ and $\epsilon > 0$. For $\omega \in \Omega$, I will say that $]q_0, q_1[$ is an **ϵ-shift interval** of ω with **(q_0, q_1, ϵ)-shift point** t if $\rho(\omega(q_0), \omega(q_1)) > 2\epsilon$ and

$$\begin{aligned} t &= \sup\{q : q \in \mathbb{Q} \cap \,]q_0, q_1[\,, \rho(\omega(q), \omega(q_0)) \le \epsilon\} \\ &= \inf\{q : q \in \mathbb{Q} \cap \,]q_0, q_1[\,, \rho(\omega(q), \omega(q_1)) \le \epsilon\}. \end{aligned}$$

Let E be the set of such ω.

(i) $E \in \mathcal{B}\mathfrak{a}(\Omega) = \widehat{\bigotimes}_{[0,\infty[} \mathcal{B}(U)$.

(ii) The function $f : E \to \,]q_0, q_1[$ which takes each $\omega \in E$ to its (q_0, q_1, ϵ)-shift point is $\mathcal{B}\mathfrak{a}(\Omega)$-measurable.

(iii) If $q_0 \ge a$, the set $\{\omega : \omega \in E, f(\omega) = t\}$ is $\hat\mu_{\tilde\omega a}$-negligible for every $t \in \,]q_0, q_1[$.

(iv) If q_0, $q_1 \in \mathbb{Q}$, $\omega \in E$, $\omega' \in \Omega$ and $\omega' \restriction \mathbb{Q} = \omega \restriction \mathbb{Q}$, then $\omega' \in E$ and $f(\omega') = f(\omega)$.

(b) Suppose that $\langle q_i \rangle_{i\le n}$, $\langle q'_i \rangle_{i\le n}$, $\langle \le_i \rangle_{i\le n}$, $\epsilon > 0$, $E \in \mathcal{B}\mathfrak{a}(\Omega)$ and $\langle f_i \rangle_{i\le n}$ are such that, for every $i \le n$,

$$q_i,\, q'_i \in \mathbb{Q}, \quad q_i < q'_i, \quad \le_i \text{ is either } \le \text{ or } \ge,$$

$$]q_i, q'_i[\text{ is an } \epsilon\text{-shift interval of } \omega \text{ with } (q_i, q'_i, \epsilon)\text{-shift point } f_i(\omega), \text{ for every } \omega \in E,$$

and also

$$a \le q_0, \quad q'_i \le q_{i+1} \text{ for every } i < n,$$

$$\text{whenever } \omega, \omega' \in E \text{ there is an } i \le n \text{ such that } f_i(\omega') \le_i f_i(\omega).$$

Then E is $\hat{\mu}_{\tilde{\omega}a}$-negligible.

(c) Suppose that $\langle q_i\rangle_{i\le n}$, $\langle q'_i\rangle_{i\le n}$, $\langle\le_i\rangle_{i\le n}$, $\epsilon>0$, $E\in\mathcal{B}\mathfrak{a}(\Omega)$ and $\langle f_i\rangle_{i\le n}$ are such that, for every $i\le n$,

$$q_i,\,q'_i\in\mathbb{Q},\quad q_i<q'_i,\quad \le_i \text{ is either } \le \text{ or } \ge,$$

$$]q_i,q'_i[\text{ is an } \epsilon\text{-shift interval of } \omega \text{ with } (q_i,q'_i,\epsilon)\text{-shift point } f_i(\omega), \text{ for every } \omega\in E,$$

and also

$$a\le q_0,\quad q'_i\le q_{i+1} \text{ for every } i<n.$$

Then for $\hat{\mu}_{\tilde{\omega}a}$-almost every $\omega\in E$ there is an $\omega'\in E$ such that $f_i(\omega')<_i f_i(\omega)$ for every $i\le n$.

455J Theorem Let (U,ρ) be a complete separable metric space and $\langle\nu_x^{(s,t)}\rangle_{0\le s<t,x\in U}$ a family of Radon probability measures on U, uniformly time-continuous on the right, such that $\langle\nu_y^{(t,u)}\rangle_{y\in U}$ is a disintegration of $\nu_x^{(s,u)}$ over $\nu_x^{(s,t)}$ whenever $0\le s<t<u$ and $x\in U$. Write $C^{\Vert}$ for the set of càllàl functions from $[0,\infty[$ to U. Suppose that $\tilde{\omega}\in C^{\Vert}$, and $a\in[0,\infty]$; let $\hat{\mu}_{\tilde{\omega}a}$ be the completed probability measure on $\Omega=U^{[0,\infty[}$ defined from $\langle\nu_x^{(s,t)}\rangle_{0\le s<t,x\in U}$, $\tilde{\omega}$ and a as in 455Eb, and $\tilde{\mu}_{\tilde{\omega}a}$ its extension to a Radon measure on Ω, as in 455H. Then $\tilde{\mu}_{\tilde{\omega}a}$ is inner regular with respect to sets of the form $F\cap C^{\Vert}$ where $F\subseteq\Omega$ is a zero set.

455K Corollary Suppose, in 455J, that $\tilde{\omega}\in C_{\mathrm{dlg}}$, the space of càdlàg functions from $[0,\infty[$ to U. Then the subspace measure $\ddot{\mu}_{\tilde{\omega}a}$ on C_{dlg} induced by $\hat{\mu}_{\tilde{\omega}a}$ is a completion regular quasi-Radon measure.

455L Stopping times Let Ω be a set, Σ a σ-algebra of subsets of Ω and $\langle\Sigma_t\rangle_{t\ge0}$ a non-decreasing family of σ-subalgebras of Σ. (Such a family is called a **filtration**.) For $t\ge0$, set $\Sigma_t^+=\bigcap_{s>t}\Sigma_s$, so that $\langle\Sigma_t^+\rangle_{t\ge0}$ also is a non-decreasing family of σ-algebras. Of course $\Sigma_t^+=\bigcap_{s>t}\Sigma_s^+$ for every $t\ge0$.

(a) A function $\tau:\Omega\to[0,\infty]$ is a **stopping time adapted to** $\langle\Sigma_t\rangle_{t\ge0}$ if $\{\omega:\omega\in\Omega,\,\tau(\omega)\le t\}$ belongs to Σ_t for every $t\ge0$.

τ will be Σ-measurable.

(b) A function $\tau:\Omega\to[0,\infty]$ is a stopping time adapted to $\langle\Sigma_t^+\rangle_{t\ge0}$ iff $\{\omega:\tau(\omega)<t\}\in\Sigma_t$ for every $t\ge0$.

(c)(i) Constant functions on Ω are stopping times.

(ii) If τ and τ' are stopping times adapted to $\langle\Sigma_t\rangle_{t\ge0}$, so is $\tau+\tau'$.

(iii) If τ is a stopping time adapted to $\langle\Sigma_t\rangle_{t\ge0}$, then

$$\Sigma_\tau=\{E:E\in\Sigma,\,E\cap\{\omega:\tau(\omega)\le t\}\in\Sigma_t \text{ for every } t\ge0\}$$

is a σ-subalgebra of Σ.

(iv) If $\langle\tau_i\rangle_{i\in I}$ is a countable family of stopping times adapted to $\langle\Sigma_t\rangle_{t\ge0}$, then $\tau=\sup_{i\in I}\tau_i$ is adapted to $\langle\Sigma_t\rangle_{t\ge0}$.

(v) If $\langle\tau_i\rangle_{i\in I}$ is a countable family of stopping times adapted to $\langle\Sigma_t^+\rangle_{t\ge0}$, then $\tau=\inf_{i\in I}\tau_i$ is adapted to $\langle\Sigma_t^+\rangle_{t\ge0}$.

(d) Now suppose that Y is a topological space and we have a family $\langle X_t\rangle_{t\ge0}$ of functions from Ω to Y, and that $\tau:\Omega\to[0,\infty]$ is any Σ-measurable function. Set $X_\tau(\omega)=X_{\tau(\omega)}(\omega)$ when $\tau(\omega)<\infty$. If $(t,\omega)\mapsto X_t(\omega):[0,\infty[\times\Omega\to Y$ is $\mathcal{B}([0,\infty[)\widehat{\otimes}\Sigma$-measurable, where $\mathcal{B}([0,\infty[)$ is the Borel σ-algebra of $[0,\infty[$, then $X_\tau:\{\omega:\tau(\omega)<\infty\}\to Y$ is Σ-measurable.

***(e)** Again take a topological space Y, a family $\langle X_t\rangle_{t\ge0}$ of functions from Ω to Y, and a stopping time $\tau:\Omega\to[0,\infty]$ adapted to $\langle\Sigma_t\rangle_{t\ge0}$. This time, suppose that $\langle X_t\rangle_{t\ge0}$ is **progressively measurable**, that is, that $(s,\omega)\mapsto X_s(\omega):[0,t]\times\Omega\to Y$ is $\mathcal{B}([0,t])\widehat{\otimes}\Sigma_t$-measurable for every $t\ge0$, and moreover that Σ_t is closed under Souslin's operation for every t. Then X_τ, as defined in (d), will be Σ_τ-measurable.

***(f)(i)** Suppose that μ is a probability measure with domain Σ and null ideal $\mathcal{N}(\mu)$. Then we can form the completion $\hat{\mu}$ with domain $\hat{\Sigma}$. If we now set $\hat{\Sigma}_t = \{E\triangle A : E \in \Sigma_t,\ A \in \mathcal{N}(\mu)\}$, $\langle\hat{\Sigma}_t\rangle_{t\ge0}$ and $\langle\hat{\Sigma}_t^+\rangle_{t\ge0}$ are filtrations, where $\hat{\Sigma}_t^+ = \bigcap_{s>t}\hat{\Sigma}_s$ for $t \ge 0$.

(ii) We find that $\hat{\Sigma}_t^+ = \{E\triangle A : E \in \Sigma_t^+,\ A \in \mathcal{N}(\mu)\}$ for every $t \ge 0$.

(iii) Of course every stopping time adapted to $\langle\Sigma_t^+\rangle_{t\ge0}$ is adapted to $\langle\hat{\Sigma}_t^+\rangle_{t\ge0}$. Conversely, if $\tau : \Omega \to [0,\infty]$ is a stopping time adapted to $\langle\hat{\Sigma}_t^+\rangle_{t\ge0}$, there is a stopping time τ', adapted to $\langle\Sigma_t^+\rangle_{t\ge0}$, such that $\tau =_{\text{a.e.}} \tau'$.

(iv) Continuing from (iii) just above, we find that, defining $\hat{\Sigma}_\tau^+$ from $\langle\hat{\Sigma}_t^+\rangle_{t\ge0}$ and τ and $\Sigma_{\tau'}^+$ from $\langle\Sigma_t^+\rangle_{t\ge0}$ and τ' by the formula in (c-iii), then $\hat{\Sigma}_\tau^+ = \{F\triangle A : F \in \Sigma_{\tau'}^+,\ A \in \mathcal{N}(\mu)\}$.

455M Hitting times: Proposition Let U be a Polish space and C_{dlg} the set of càdlàg functions from $[0,\infty[$ to U. Let $A \subseteq U$ be an analytic set, and define $\tau : C_{\text{dlg}} \to [0,\infty]$ by setting

$$\tau(\omega) = \inf\{t : \omega(t) \in A\}$$

for $\omega \in C_{\text{dlg}}$, counting $\inf\emptyset$ as ∞.

(a) Let Σ be a σ-algebra of subsets of C_{dlg} closed under Souslin's operation and including the algebra generated by the functionals $\omega \mapsto \omega(t)$ for $t \ge 0$. Then τ is Σ-measurable.

(b) For $t \ge 0$ let Σ_t be

$$\{F : F \in \Sigma,\ \omega' \in F \text{ whenever } \omega,\ \omega' \in C_{\text{dlg}},\ \omega \in F \text{ and } \omega\restriction[0,t] = \omega'\restriction[0,t]\},$$

and $\Sigma_t^+ = \bigcap_{s>t}\Sigma_t$. Then τ is a stopping time adapted to $\langle\Sigma_t^+\rangle_{t\ge0}$.

(c) If A is closed, then τ is adapted to $\langle\Sigma_t\rangle_{t\ge0}$.

455N Lemma Let (U,ρ) be a metric space, $n \in \mathbb{N}$ and $f : U^{n+1} \to \mathbb{R}$ a bounded uniformly continuous function. Let $\langle\nu_x^{(k)}\rangle_{k<n,x\in U}$ be a family of topological probability measures on U such that $x \mapsto \nu_x^{(k)}$ is continuous for the narrow topology for every $k < n$. Then

$$y \mapsto \iint\ldots\int f(y,x_1,\ldots,x_n)\nu_{x_{n-1}}^{(n-1)}(dx_n)\ldots\nu_{x_1}^{(1)}(dx_2)\nu_y^{(0)}(dx_1)$$

is defined everywhere on U and continuous.

455O Theorem Suppose that (U,ρ) is a complete metric space, x^* is a point of U, $\langle\nu_x^{(s,t)}\rangle_{0\le s<t,x\in U}$ is a family of Radon probability measures on U which is both narrowly continuous and uniformly time-continuous on the right, and that $\langle\nu_y^{(t,u)}\rangle_{y\in U}$ is a disintegration of $\nu_x^{(s,u)}$ over $\nu_x^{(s,t)}$ whenever $x \in U$ and $s < t < u$. Let $\hat{\mu}$ be the corresponding completed measure on $\Omega = U^{[0,\infty[}$, as in 455E. Let C_{dlg} be the set of càdlàg functions from $[0,\infty[$ to U, $\ddot{\mu}$ the subspace measure on C_{dlg}, and $\ddot{\Sigma}$ its domain. For $t \ge 0$, let $\ddot{\Sigma}_t$ be

$$\{F : F \in \ddot{\Sigma},\ \omega' \in F \text{ whenever } \omega \in F,\ \omega' \in C_{\text{dlg}} \text{ and } \omega'\restriction[0,t] = \omega\restriction[0,t]\},$$

and $\ddot{\Sigma}_t^+ = \bigcap_{s>t}\ddot{\Sigma}_t$.

For $\omega \in \Omega$ and $a \ge 0$ let $\hat{\mu}_{\omega a}$ be the completed measure on Ω built from $\omega(0)$ and $\langle\nu_{\omega ax}^{(s,t)}\rangle_{0\le s<t,x\in U}$ as in 455Eb; let $\ddot{\mu}_{\omega a}$ be the subspace measure on C_{dlg}. Let $\tau : C_{\text{dlg}} \to [0,\infty]$ be a stopping time adapted to $\langle\ddot{\Sigma}_t^+\rangle_{t\ge0}$.

(a) $\langle\ddot{\mu}_{\omega,\tau(\omega)}\rangle_{\omega\in C_{\text{dlg}}}$ is a disintegration of $\ddot{\mu}$ over itself.

(b) Set

$$\ddot{\Sigma}_\tau^+ = \{F : F \in \ddot{\Sigma},\ F \cap \{\omega : \tau(\omega) \le t\} \in \ddot{\Sigma}_t^+ \text{ for every } t \ge 0\}.$$

Then $\ddot{\Sigma}_\tau^+$ is a σ-algebra of subsets of C_{dlg}. For a $\ddot{\mu}$-integrable function f on C_{dlg}, write $\ddot{g}_f(\omega) = \int_{C_{\text{dlg}}} f d\ddot{\mu}_{\omega,\tau(\omega)}$ when this is defined in $\mathbb{R}$. Then $\ddot{g}_f$ is a conditional expectation of f on $\ddot{\Sigma}_\tau^+$.

(c) If τ is adapted to $\langle\ddot{\Sigma}_t\rangle_{t\ge0}$, set

$$\ddot{\Sigma}_\tau = \{F : F \in \ddot{\Sigma},\ F \cap \{\omega : \tau(\omega) \le t\} \in \ddot{\Sigma}_t \text{ for every } t \ge 0\}.$$

Then $\ddot{\Sigma}_\tau$ is a σ-algebra of subsets of C_{dlg}, and $\ddot{g}_f$ is a conditional expectation of f on $\ddot{\Sigma}_\tau$, for every $f \in \mathcal{L}^1(\ddot{\mu})$.

455P Theorem Let U be a metrizable topological group which is complete under a right-translation-invariant metric ρ inducing its topology. Let $\langle\lambda_t\rangle_{t>0}$ be a family of Radon probability measures on U such that the convolution $\lambda_s * \lambda_t$ is equal to λ_{s+t} for all s, $t > 0$. Suppose that $\lim_{t\downarrow 0}\lambda_t G = 1$ for every open neighbourhood G of the identity in U. For $x \in U$ and $0 \le s < t$, let $\nu_x^{(s,t)}$ be the Radon probability measure on U defined by saying that $\nu_x^{(s,t)}(E) = \lambda_{t-s}(Ex^{-1})$ whenever λ_{t-s} measures Ex^{-1}.

(a) $\langle\nu_y^{(t,u)}\rangle_{y\in U}$ is a disintegration of $\nu_x^{(s,u)}$ over $\nu_x^{(s,t)}$ whenever $x \in U$ and $0 \le s < t < u$.

(b) $\langle\nu_x^{(s,t)}\rangle_{0\le s<t,x\in U}$ is narrowly continuous and uniformly time-continuous on the right.

(c)(i) We can define a complete measure $\hat\mu$ on $U^{[0,\infty[}$ by the method of 455E applied to $x^* = e$ and $\langle\nu_x^{(s,t)}\rangle_{0\le s<t,x\in U}$.

(ii) If C_{dlg} is the space of càdlàg functions from $[0,\infty[$ to U, then $\hat\mu^* C_{\mathrm{dlg}} = 1$, and the subspace measure $\ddot\mu$ on C_{dlg} will have the properties described in 455O, with $\omega(0) = e$ for $\ddot\mu$-almost every $\omega \in C_{\mathrm{dlg}}$.

(iii) $\hat\mu$ has a unique extension to a Radon measure $\tilde\mu$ on $U^{[0,\infty[}$.

455Q Lévy processes Let U be a separable metrizable topological group with identity e, and consider the following list of properties of a family $\langle X_t\rangle_{t\ge 0}$ of U-valued random variables:

$X_0 = e$ almost everywhere,

$\Pr(X_t X_s^{-1} \in F) = \Pr(X_{t-s} \in F)$ whenever $0 \le s < t$ and $F \subseteq U$ is Borel

(the process is **stationary**),

whenever $0 \le t_0 < t_1 < \ldots < t_n$, then $X_{t_1}X_{t_0}^{-1}, X_{t_2}X_{t_1}^{-1}, \ldots, X_{t_n}X_{t_{n-1}}^{-1}$ are independent

(the process has **independent increments**),

$X_t \to e$ in measure as $t \downarrow 0$

(that is, $\lim_{t\downarrow 0}\Pr(X_t \in G) = 1$ for every neighbourhood G of the identity). Such a family I will call a **Lévy process**.

455R Theorem Let U be a Polish group with identity e which is complete under a right-translation-invariant metric inducing its topology. A family $\langle X_t\rangle_{t\ge 0}$ of U-valued random variables is a Lévy process iff there is a family $\langle\lambda_t\rangle_{t>0}$ of Radon probability measures on U, satisfying the conditions of 455P, such that if we start from $x^* = e$ and build the measure $\hat\mu$ on $U^{[0,\infty[}$ as in 455Pc, then

$$\Pr(X_{t_i} \in F_i \text{ for every } i \le n) = \hat\mu\{\omega : \omega(t_i) \in F_i \text{ for every } i \le n\}$$

whenever $t_0, \ldots, t_n \in [0,\infty[$ and $F_i \subseteq U$ is a Borel set for every $i \le n$.

455S Lemma Let U be a metrizable topological group which is complete under a right-translation-invariant metric inducing its topology. Let $\langle\lambda_t\rangle_{t>0}$ be a family of Radon probability measures on U such that $\lambda_s * \lambda_t = \lambda_{s+t}$ for all s, $t > 0$ and $\lim_{t\downarrow 0}\lambda_t G = 1$ for every open neighbourhood G of the identity e in U. For $x \in U$ and $0 \le s < t$, let $\nu_x^{(s,t)}$ be the Radon probability measure on U defined by saying that $\nu_x^{(s,t)}(E) = \lambda_{t-s}(Ex^{-1})$ whenever λ_{t-s} measures Ex^{-1}.

(a) If $0 \le t_0 < t_1 < \ldots < t_n$, $z \in U$ and $f : \mathbb{R}^J \to \mathbb{R}$ is a bounded Borel measurable function, where $J = \{t_0, \ldots, t_n\}$, then

$$\begin{aligned}\iint\ldots\int f(z, x_1, \ldots, x_n)\nu_{x_{n-1}}^{(t_{n-1},t_n)}(dx_n)\ldots\nu_{x_1}^{(t_1,t_2)}(dx_2)\nu_z^{(t_0,t_1)}(dx_1)\\ = \iint\ldots\int f(z, x_1 z, \ldots, x_n z)\nu_{x_{n-1}}^{(t_{n-1},t_n)}(dx_n)\\ \ldots\nu_{x_1}^{(t_1,t_2)}(dx_2)\nu_e^{(t_0,t_1)}(dx_1).\end{aligned}$$

(b) Take $\omega \in U^{[0,\infty[}$ and $a \ge 0$. Let $\hat\mu$ and $\hat\mu_{\omega a}$ be the measures on $U^{[0,\infty[}$ defined from $\langle\nu_x^{(s,t)}\rangle_{s<t,x\in U}$ by the method of 455E, starting from $x^* = e$. Define $\phi_{\omega a} : U^{[0,\infty[} \to U^{[0,\infty[}$ by setting

$$\begin{aligned}\phi_{\omega a}(\omega')(t) &= \omega(t) \text{ if } t < a,\\ &= \omega'(t-a)\omega(a) \text{ if } t \ge a.\end{aligned}$$

Then $\hat{\mu}_{\omega a}$ is the image measure $\hat{\mu}\phi_{\omega a}^{-1}$.

(c) In (b), suppose that ω belongs to the set C_{dlg} of càdlàg functions from $[0,\infty[$ to U. Then $\phi_{\omega a}(\omega') \in C_{\text{dlg}}$ for every $\omega' \in C_{\text{dlg}}$, and $\phi_{\omega a} : C_{\text{dlg}} \to C_{\text{dlg}}$ is inverse-measure-preserving for the subspace measures $\ddot{\mu}$ and $\ddot{\mu}_{\omega a}$ on C_{dlg}.

455T Corollary Let U be a metrizable topological group which is complete under a right-translation-invariant metric inducing its topology. Let $\langle\lambda_t\rangle_{t>0}$ be a family of Radon probability measures on U such that $\lambda_s * \lambda_t = \lambda_{s+t}$ for all s, $t > 0$ and $\lim_{t\downarrow 0} \lambda_t G = 1$ for every open neighbourhood G of the identity e in U; let $\hat{\mu}$ be the measure on $U^{[0,\infty[}$ defined from $\langle\lambda_t\rangle_{t>0}$ by the method of 455Pc. Let C_{dlg} be the set of càdlàg functions from $[0,\infty[$ to U, $\ddot{\mu}$ the subspace measure on C_{dlg} and $\ddot{\Sigma}$ its domain. For $t \ge 0$, let $\ddot{\Sigma}_t$ be

$$\{F : F \in \ddot{\Sigma},\ \omega' \in F \text{ whenever } \omega \in F,\ \omega' \in C_{\text{dlg}} \text{ and } \omega'\restriction[0,t] = \omega\restriction[0,t]\}$$

and $\hat{\ddot{\Sigma}}_t = \{F\triangle A : F \in \ddot{\Sigma}_t,\ \ddot{\mu}A = 0\}$. Then $\hat{\ddot{\Sigma}}_t = \bigcap_{s>t} \hat{\ddot{\Sigma}}_s$ includes $\ddot{\Sigma}_t^+ = \bigcap_{s>t} \ddot{\Sigma}_s$.

455U Theorem Let U be a metrizable topological group which is complete under a right-translation-invariant metric inducing its topology. Let $\langle\lambda_t\rangle_{t>0}$ be a family of Radon probability measures on U such that $\lambda_s * \lambda_t = \lambda_{s+t}$ for all s, $t > 0$ and $\lim_{t\downarrow 0} \lambda_t G = 1$ for every open neighbourhood G of the identity e in U; let $\hat{\mu}$ be the measure on $U^{[0,\infty[}$ defined from $\langle\lambda_t\rangle_{t>0}$ by the method of 455Pc. Let C_{dlg} be the set of càdlàg functions from $[0,\infty[$ to U, $\ddot{\mu}$ the subspace measure on C_{dlg} and $\ddot{\Sigma}$ its domain. For $t \ge 0$, let $\ddot{\Sigma}_t$ be

$$\{F : F \in \ddot{\Sigma},\ \omega' \in F \text{ whenever } \omega \in F,\ \omega' \in C_{\text{dlg}} \text{ and } \omega'\restriction[0,t] = \omega\restriction[0,t]\},$$

and $\hat{\ddot{\Sigma}}_t = \{F\triangle A : F \in \ddot{\Sigma}_t,\ \ddot{\mu}A = 0\}$; let $\tau : C_{\text{dlg}} \to [0,\infty]$ be a stopping time adapted to $\langle\hat{\ddot{\Sigma}}_t\rangle_{t\ge 0}$. Define $\phi_\tau : C_{\text{dlg}} \times C_{\text{dlg}} \to C_{\text{dlg}}$ by setting

$$\begin{aligned}\phi_\tau(\omega,\omega')(t) &= \omega'(t-\tau(\omega))\omega(\tau(\omega)) \text{ if } t \ge \tau(\omega),\\ &= \omega(t) \text{ otherwise.}\end{aligned}$$

Then ϕ_τ is inverse-measure-preserving for the product measure $\ddot{\mu} \times \ddot{\mu}$ on $C_{\text{dlg}} \times C_{\text{dlg}}$ and $\ddot{\mu}$ on C_{dlg}.

Version of 19.5.10

456 Gaussian distributions

Uncountable powers of $\mathbb{R}$ are not as a rule measure-compact. Accordingly distributions, in the sense of 454K, need not be τ-additive. But some, at least, of the distributions most important to us are indeed τ-additive, and therefore have interesting canonical extensions. This section is devoted to a remarkable result, taken from Talagrand 81, concerning a class of distributions which are of great importance in probability theory. It demands a combination of techniques from classical probability theory and from the topological measure theory of this volume. I begin with the definition and fundamental properties of what I call 'centered Gaussian distributions' (456A-456I). These are fairly straightforward adaptations of the classical finite-dimensional theory, and will be useful in §477 when we come to study Brownian motion. Another relatively easy idea is that of 'universal' Gaussian distribution (456J-456L). In 456M we come to a much deeper result, a step towards classifying the ways in which a Gaussian family of n-dimensional random variables can accumulate at 0. The ideas are combined in 456N-456O to complete the proof of Talagrand's theorem that Gaussian distributions on powers of $\mathbb{R}$ are τ-additive.

456A Definitions (a) Write μ_G for the Radon probability measure on $\mathbb{R}$ which is the distribution of a standard normal random variable. For any set I, write $\mu_G^{(I)}$ for the measure on $\mathbb{R}^I$ which is the product of copies of μ_G; this is always quasi-Radon; if I is countable, it is Radon; if $I = n \in \mathbb{N}\setminus\{0\}$, it is the probability distribution with density function $x \mapsto (2\pi)^{-n/2}e^{-x\,.\,x/2}$; if $I = \emptyset$, it is the unique probability measure on the singleton set $\mathbb{R}^\emptyset$.

(b) I will use the phrase **centered Gaussian distribution** to mean a measure μ on a power $\mathbb{R}^I$ of $\mathbb{R}$ such that μ is the completion of a Baire measure and every continuous linear functional $f:\mathbb{R}^I\to\mathbb{R}$ is either zero almost everywhere or is a normal random variable with zero expectation.

(c) If I is a set and μ is a centered Gaussian distribution on $\mathbb{R}^I$, its **covariance matrix** is the family $\langle\sigma_{ij}\rangle_{i,j\in I}$ where $\sigma_{ij}=\int x(i)x(j)\mu(dx)$ for i, $j\in I$.

456B Proposition (a) Suppose that I and J are sets, μ is a centered Gaussian distribution on $\mathbb{R}^I$, and $T:\mathbb{R}^I\to\mathbb{R}^J$ is a continuous linear operator. Then there is a unique centered Gaussian distribution on $\mathbb{R}^J$ for which T is inverse-measure-preserving; if J is countable, this is the image measure μT^{-1}.

(b) Let I be a set, and μ, ν two centered Gaussian distributions on $\mathbb{R}^I$. If they have the same covariance matrices they are equal.

(c) For any set I, $\mu_G^{(I)}$ is the centered Gaussian distribution on $\mathbb{R}^I$ with the identity matrix for its covariance matrix.

(d) Suppose that I is a countable set. Then a measure μ on $\mathbb{R}^I$ is a centered Gaussian distribution iff it is of the form $\mu_G^{(\mathbb{N})}T^{-1}$ where $T:\mathbb{R}^{\mathbb{N}}\to\mathbb{R}^I$ is a continuous linear operator.

(e) Suppose $\langle I_j\rangle_{j\in J}$ is a disjoint family of sets with union I, and that for each $j\in J$ we have a centered Gaussian distribution ν_j on $\mathbb{R}^{I_j}$. Then the product ν of the measures ν_j, regarded as a measure on $\mathbb{R}^I$, is a centered Gaussian distribution.

(f) Let I be any set, μ a centered Gaussian distribution on $\mathbb{R}^I$ and $E\subseteq\mathbb{R}^I$ a set such that μ measures E. Writing $-E=\{-x:x\in E\}$, $\mu(-E)=\mu E$.

456C Theorem Let I be a set and $\langle\sigma_{ij}\rangle_{i,j\in I}$ a family of real numbers. Then the following are equiveridical:

(i) $\langle\sigma_{ij}\rangle_{i,j\in I}$ is the covariance matrix of a centered Gaussian distribution on $\mathbb{R}^I$;

(ii) there are a (real) Hilbert space U and a family $\langle u_i\rangle_{i\in I}$ in U such that $(u_i|u_j)=\sigma_{ij}$ for all i, $j\in I$;

(iii) for every finite $J\subseteq I$, $\langle\sigma_{ij}\rangle_{i,j\in J}$ is the covariance matrix of a centered Gaussian distribution on $\mathbb{R}^J$;

(iv) $\langle\sigma_{ij}\rangle_{i,j\in I}$ is symmetric and positive semi-definite in the sense that $\sigma_{ij}=\sigma_{ji}$ for all i, $j\in I$ and $\sum_{i,j\in J}\alpha_i\alpha_j\sigma_{ij}\ge 0$ whenever $J\subseteq I$ is finite and $\langle\alpha_i\rangle_{i\in J}\in\mathbb{R}^J$.

456D Gaussian processes: Definition A family $\langle X_i\rangle_{i\in I}$ of real-valued random variables on a probability space is a **centered Gaussian process** if its distribution is a centered Gaussian distribution.

456E Independence and correlation: Proposition (a) Let $\langle X_i\rangle_{i\in I}$ be a centered Gaussian process. Then $\langle X_i\rangle_{i\in I}$ is independent iff $\mathbb{E}(X_i\times X_j)=0$ for all distinct i, $j\in I$.

(b) Let $\langle X_i\rangle_{i\in I}$ be a centered Gaussian process on a complete probability space (Ω,Σ,μ), and $\mathcal{J}$ a disjoint family of subsets of I; for $J\in\mathcal{J}$ let Σ_J be the σ-algebra of subsets of Ω generated by $\{X_i^{-1}[F]:i\in J$, $F\subseteq\mathbb{R}$ is Borel$\}$. Suppose that $\mathbb{E}(X_i\times X_j)=0$ whenever J, J' are distinct members of $\mathcal{J}$, $i\in J$ and $j\in J'$. Then $\langle\Sigma_J\rangle_{J\in\mathcal{J}}$ is independent.

456F Proposition Let $\langle X_i\rangle_{i\in I}$ be a family of random variables on a probability space (Ω,Σ,μ). Then the following are equiveridical:

(i) the distribution of $\langle X_i\rangle_{i\in I}$ is a centered Gaussian distribution;

(ii) whenever $i_0,\dots,i_n\in I$ and $\alpha_0,\dots,\alpha_n\in\mathbb{R}$ then $\sum_{r=0}^n\alpha_rX_{i_r}$ is either zero a.e. or a normal random variable with zero expectation;

(iii) whenever $i_0,\dots,i_n\in I$ then the joint distribution of $X_{i_0},\dots,X_{i_n}$ is a centered Gaussian distribution;

(iv) whenever $J\subseteq I$ is finite then there is an independent family $\langle Y_k\rangle_{k\in K}$ of standard normal random variables on Ω such that each X_i, for $i\in J$, is almost everywhere equal to a linear combination of the Y_k.

456G Lemma Let I be a finite set and μ a centered Gaussian distribution on $\mathbb{R}^I$. Suppose that $\gamma\ge 0$ and $\alpha=\mu\{x:\sup_{i\in I}|x(i)|\ge\gamma\}$. Then $\mu\{x:\sup_{i\in I}|x(i)|\ge\frac{1}{2}\gamma\}\ge 2\alpha(1-\alpha)^3$.

456H The support of a Gaussian distribution: Proposition Let I be a set and μ a centered Gaussian distribution on $\mathbb{R}^I$. Write Z for the set of those $x\in\mathbb{R}^I$ such that $f(x)=0$ whenever $f:\mathbb{R}^I\to\mathbb{R}$ is a continuous linear functional and $f=0$ a.e. Then Z is a self-supporting closed linear subspace of $\mathbb{R}^I$ with full outer measure. If I is countable Z is the support of μ.

456I Remarks (a) In the context of 456H, I will call Z the **support** of the centered Gaussian distribution μ, even though μ need not be a topological measure.

(b) If I and J are sets, μ and ν are centered Gaussian distributions on $\mathbb{R}^I$ and $\mathbb{R}^J$ respectively with supports Z and Z', and $T : \mathbb{R}^I \to \mathbb{R}^J$ is an inverse-measure-preserving continuous linear operator, then $Tz \in Z'$ for every $z \in Z$.

456J Universal Gaussian distributions: Definition A centered Gaussian distribution on $\mathbb{R}^I$ is **universal** if its covariance matrix $\langle\sigma_{ij}\rangle_{i,j\in I}$ is the inner product for a Hilbert space structure on I.

456K Proposition Let I be any set, and μ a centered Gaussian distribution on I. Then there are a set J, a universal centered Gaussian distribution ν on $\mathbb{R}^J$, and a continuous inverse-measure-preserving linear operator $T : \mathbb{R}^J \to \mathbb{R}^I$.

456L Lemma Let μ be a universal centered Gaussian distribution on $\mathbb{R}^I$; give I a corresponding Hilbert space structure such that $\int x(i)x(j)\mu(dx) = (i|j)$ for all i, $j \in I$. Let $F \in \operatorname{dom}\mu$ be a set determined by coordinates in J, where $J \subseteq I$ is a closed linear subspace for the Hilbert space structure of I. Let W be the union of all the open subsets of $\mathbb{R}^I$ which meet F in a negligible set, and W' the union of the open subsets of $\mathbb{R}^I$ which meet F in a negligible set and are determined by coordinates in J. If $F \subseteq W$ then $F \subseteq W'$.

456M Cluster sets: Lemma Let I be a countable set, $n \ge 1$ an integer and μ a centered Gaussian distribution on $\mathbb{R}^{I\times n}$. For $\epsilon > 0$ set

$$I_\epsilon = \{i : i \in I,\ \int |x(i,r)|^2\mu(dx) \le \epsilon^2 \text{ for every } r < n\};$$

suppose that no I_ϵ is empty.

(a) There is a closed set $F \subseteq \mathbb{R}^n$ such that

$$F = \bigcap_{\epsilon>0} \overline{\{\langle x(i,r)\rangle_{r<n} : i \in I_\epsilon\}}$$

for almost every $x \in \mathbb{R}^{I\times n}$.

(b) If $z \in F$ and $-1 \le \alpha \le 1$, then $\alpha z \in F$.

(c) If F is bounded, then there is some $\epsilon > 0$ such that $\sup_{i\in I_\epsilon, r<n} |x(i,r)| < \infty$ for almost every $x \in \mathbb{R}^{I\times n}$.

456N Lemma Let J be a set and μ a centered Gaussian distribution on $\mathbb{R}^J$. Let M be the linear subspace of $L^2(\mu)$ generated by $\{\pi_j^\bullet : j \in J\}$, where $\pi_j(x) = x(j)$ for $x \in \mathbb{R}^J$ and $j \in J$. If M is separable then μ is τ-additive.

456O Theorem Every centered Gaussian distribution is τ-additive.

456P Corollary If μ is a centered Gaussian distribution on $\mathbb{R}^I$, there is a unique quasi-Radon measure $\tilde\mu$ on $\mathbb{R}^I$ extending μ. The support of μ as defined in 456H is the support of $\tilde\mu$ as defined in 411N.

456Q Proposition Let I be a set and R the set of functions $\sigma : I \times I \to \mathbb{R}$ which are symmetric and positive semi-definite in the sense of 456C; give R the subspace topology induced by the usual topology of $\mathbb{R}^{I\times I}$. Let $P_{\mathrm{qR}}(\mathbb{R}^I)$ be the space of quasi-Radon probability measures on $\mathbb{R}^I$ with its narrow topology. For $\sigma \in R$, let μ_σ be the centered Gaussian distribution on $\mathbb{R}^I$ with covariance matrix σ, and $\tilde\mu_\sigma$ the quasi-Radon measure extending μ_σ. Then R is a closed subset of $\mathbb{R}^{I\times I}$ and the function $\sigma \mapsto \tilde\mu_\sigma : R \to P_{\mathrm{qR}}(\mathbb{R}^I)$ is continuous.

Version of 18.1.13

457 Simultaneous extension of measures

The questions addressed in §§451, 454 and 455 can all be regarded as special cases of a general class of problems: given a set X and a family $\langle\nu_i\rangle_{i\in I}$ of (probability) measures on X, when can we expect to find a measure on X extending every ν_i? An alternative formulation, superficially more general, is to ask: given a set X, a family $\langle (Y_i, \mathrm{T}_i, \nu_i)\rangle_{i\in I}$ of probability spaces, and functions $\phi_i : X \to Y_i$ for each i, when can we find a measure on X for which every ϕ_i is inverse-measure-preserving? Even the simplest non-trivial case, when $X = \prod_{i\in I} Y_i$ and every ϕ_i is the coordinate map, demands a significant construction (the product measures of Chapter 25). In this section I bring together a handful of important further cases which are accessible by the methods of this chapter. I begin with a discussion of extensions of finitely additive measures (457A-457D), which are much easier, before considering the problems associated with countably additive measures (457E-457G), with examples (457H-457J). In 457K-457M I look at a pair of optimisation problems.

457A Lemma Let $\mathfrak{A}$ be a Boolean algebra and $\langle \mathfrak{B}_i\rangle_{i\in I}$ a non-empty family of subalgebras of $\mathfrak{A}$. For each $i \in I$, we may identify $L^\infty(\mathfrak{B}_i)$ with the closed linear subspace of $L^\infty(\mathfrak{A})$ generated by $\{\chi b : b \in \mathfrak{B}_i\}$. Suppose that for each $i \in I$ we are given a finitely additive functional $\nu_i : \mathfrak{B}_i \to [0, 1]$ such that $\nu_i 1 = 1$; write $⨍ \ldots d\nu_i$ for the corresponding positive linear functional on $L^\infty(\mathfrak{B}_i)$. Then the following are equiveridical:

(i) there is an additive functional $\mu : \mathfrak{A} \to [0, 1]$ extending every ν_i;

(ii) whenever $i_0, \ldots , i_n \in I$, $a_k \in \mathfrak{B}_{i_k}$ for $k \le n$, and $\sum_{k=0}^n \chi a_k \ge m\chi 1$ in $S(\mathfrak{A})$, where $m \in \mathbb{N}$, then $\sum_{k=0}^n \nu_{i_k} a_k \ge m$;

(iii) whenever $i_0, \ldots , i_n \in I$, $a_k \in \mathfrak{B}_{i_k}$ for $k \le n$, and $\sum_{k=0}^n \chi a_k \le m\chi 1$, where $m \in \mathbb{N}$, then $\sum_{k=0}^n \nu_{i_k} a_k \le m$;

(iv) whenever $i_0, \ldots , i_n \in I$ are distinct, $u_k \in L^\infty(\mathfrak{B}_{i_k})$ for every $k \le n$, and $\sum_{k=0}^n u_k \ge \chi 1$, then $\sum_{i=0}^n ⨍ u_k d\nu_{i_k} \ge 1$;

(v) whenever $i_0, \ldots , i_n \in I$ are distinct, $u_k \in L^\infty(\mathfrak{B}_{i_k})$ for every $k \le n$, and $\sum_{k=0}^n u_k \le \chi 1$, then $\sum_{i=0}^n ⨍ u_k d\nu_{i_k} \le 1$.

457B Corollary Let X be a set and $\langle Y_i\rangle_{i\in I}$ a family of sets. Suppose that for each $i \in I$ we have an algebra $\mathcal{E}_i$ of subsets of Y_i, an additive functional $\nu_i : \mathcal{E}_i \to [0, 1]$ such that $\nu_i Y_i = 1$, and a function $f_i : X \to Y_i$. Then the following are equiveridical:

(i) there is an additive functional $\mu : \mathcal{P}X \to [0, 1]$ such that $\mu f_i^{-1}[E] = \nu_i E$ whenever $i \in I$ and $E \in \mathcal{E}_i$;

(ii) whenever $i_0, \ldots , i_n \in I$ and $E_k \in \mathcal{E}_{i_k}$ for $k \le n$, then there is an $x \in X$ such that $\sum_{k=0}^n \nu_{i_k} E_k \le \#(\{k : k \le n,\ f_{i_k}(x) \in E_k\})$.

457C Corollary (a) Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{B}_1$, $\mathfrak{B}_2$ two subalgebras of $\mathfrak{A}$ with finitely additive functionals $\nu_i : \mathfrak{B}_i \to [0, 1]$ such that $\nu_1 1 = \nu_2 1 = 1$. Then the following are equiveridical:

(i) there is an additive functional $\mu : \mathfrak{A} \to [0, 1]$ extending both the ν_i;

(ii) whenever $b_1 \in \mathfrak{B}_1$, $b_2 \in \mathfrak{B}_2$ and $b_1 \cup b_2 = 1$, then $\nu_1 b_1 + \nu_2 b_2 \ge 1$;

(iii) whenever $b_1 \in \mathfrak{B}_1$, $b_2 \in \mathfrak{B}_2$ and $b_1 \cap b_2 = 0$, then $\nu_1 b_1 + \nu_2 b_2 \le 1$.

(b) Let X, Y_1, Y_2 be sets, and for $i \in \{1, 2\}$ let $\mathcal{E}_i$ be an algebra of subsets of Y_i, $\nu_i : \mathcal{E}_i \to [0, 1]$ an additive functional such that $\nu_i Y_i = 1$, and $f_i : X \to Y_i$ a function. Then the following are equiveridical:

(i) there is an additive functional $\mu : \mathcal{P}X \to [0, 1]$ such that $\mu f_i^{-1}[E] = \nu_i E$ whenever $i \in \{1, 2\}$ and $E \in \mathcal{E}_i$;

(ii) $f_1^{-1}[E_1] \cap f_2^{-1}[E_2] \ne \emptyset$ whenever $E_1 \in \mathcal{E}_1$, $E_2 \in \mathcal{E}_2$ and $\nu_1 E_1 + \nu_2 E_2 > 1$;

(iii) $\nu_1 E_1 \le \nu_2 E_2$ whenever $E_1 \in \mathcal{E}_1$, $E_2 \in \mathcal{E}_2$ and $f_1^{-1}[E_1] \subseteq f_2^{-1}[E_2]$.

***457D Proposition** Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{B}_1$, $\mathfrak{B}_2$ two subalgebras of $\mathfrak{A}$. Suppose that $\nu_i : \mathfrak{B}_i \to [0, 1]$ are finitely additive functionals such that $\nu_1 1 = \nu_2 1 = 1$, and $\theta : \mathfrak{A} \to [0, \infty[$ another additive functional. Then the following are equiveridical:

(i) there is an additive functional $\mu : \mathfrak{A} \to [0, \infty[$ extending both the ν_i, and such that $\mu a \le \theta a$ for every $a \in \mathfrak{A}$;

(ii) $\nu_1 b_1 + \nu_2 b_2 \le 1 + \theta(b_1 \cap b_2)$ whenever $b_1 \in \mathfrak{B}_1$ and $b_2 \in \mathfrak{B}_2$.

457E Proposition Let X be a non-empty set and $\langle \nu_i\rangle_{i\in I}$ a family of probability measures on X satisfying the conditions of Lemma 457A, taking $\mathfrak{A} = \mathcal{P}X$ and $\mathfrak{B}_i = \operatorname{dom} \nu_i$ for each i. Suppose that there is a countably compact class $\mathcal{K} \subseteq \mathcal{P}X$ such that every ν_i is inner regular with respect to $\mathcal{K}$. Then there is a probability measure μ on X extending every ν_i.

457F Proposition (a) Let (X,Σ,μ) be a perfect probability space and (Y,T,ν) any probability space. Write $\Sigma\otimes\mathrm{T}$ for the algebra of subsets of $X\times Y$ generated by $\{E\times F : E\in\Sigma,\ F\in\mathrm{T}\}$. Suppose that $Z\subseteq X\times Y$ is such that

(i) Z is expressible as the intersection of a sequence in $\Sigma\otimes\mathrm{T}$,

(ii) $Z\cap(E\times F)\ne\emptyset$ whenever $E\in\Sigma$, $F\in\mathrm{T}$ are such that $\mu E+\nu F>1$.

Then there is a probability measure λ on Z such that the maps $(x,y)\mapsto x : Z\to X$ and $(x,y)\mapsto y : Z\to Y$ are both inverse-measure-preserving.

(b) Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of perfect probability spaces. Write $\bigotimes_{i\in I}\Sigma_i$ for the algebra of subsets of $X=\prod_{i\in I}X_i$ generated by $\{\{x : x\in X,\ x(i)\in E\} : i\in I,\ E\in\Sigma_i\}$. Suppose that $Z\subseteq X$ is such that

(i) Z is expressible as the intersection of a sequence in $\bigotimes_{i\in I}\Sigma_i$,

(ii) whenever $i_0,\dots,i_n\in I$ and $E_k\in\Sigma_{i_k}$ for $k\le n$, there is a $z\in Z$ such that $\#(\{k : k\le n, z(i_k)\in E_k\})\ge\sum_{k=0}^n\mu_{i_k}E_k$.

Then there is a perfect probability measure λ on Z such that $z\mapsto z(i) : Z\to X_i$ is inverse-measure-preserving for every $i\in I$.

457G Theorem Let X be a set and $\langle\mu_i\rangle_{i\in I}$ a family of probability measures on X which is upwards-directed in the sense that for any i, $j\in I$ there is a $k\in I$ such that μ_k extends both μ_i and μ_j. Suppose that for any countable $J\subseteq I$ there is a measure on X extending μ_i for every $i\in J$. Then there is a measure on X extending μ_i for every $i\in I$.

457H Example Set $X=\{(x,y) : 0\le x<y\le 1\}\subseteq[0,1]^2$. Write π_1, $\pi_2 : X\to\mathbb{R}$ for the coordinate maps, and μ_L for Lebesgue measure on $[0,1]$, with Σ_L its domain.

(a) There is a finitely additive functional $\nu:\mathcal{P}X\to[0,1]$ such that $\nu\pi_i^{-1}[E]=\mu_LE$ whenever $i\in\{1,2\}$ and $E\in\Sigma_L$.

(b) However, there is no measure μ on X for which both π_1 and π_2 are inverse-measure-preserving.

457I Example Let μ_L be Lebesgue measure on $[0,1]$ and Σ_L its domain. Set

$$X=\{(\xi_1,\xi_2,\xi_3) : 0\le\xi_i\le 1\text{ for each }i,\ \textstyle\sum_{i=1}^3\xi_i\le\frac{3}{2},\ \sum_{i=1}^3\xi_i^2\le 1\}.$$

For $1\le i\le 3$ set $\pi_i(x)=\xi_i$ for $x=(\xi_1,\xi_2,\xi_3)\in X$.

(a) If $E_i\in\Sigma_L$ for $i\le 3$, then there is an $x\in X$ such that $\#(\{i:\pi_i(x)\in E_i\})\ge\sum_{i=1}^3\mu_LE_i$.

(b) There is no finitely additive functional ν on X such that $\nu\pi_i^{-1}[E]=\mu_LE$ for each i and every $E\in\Sigma_L$.

457J Example There are a set X and a family $\langle\mu_i\rangle_{i\in I}$ of probability measures on X such that (i) for every countable set $J\subseteq I$ there is a measure on X extending μ_i for every $i\in J$ (ii) there is no measure on X extending μ_i for every $i\in I$.

457K Definition Let (X,ρ) be a metric space. For quasi-Radon probability measures μ, ν on X, set

$$\rho_{\mathrm{W}}(\mu,\nu)=\sup\{|\textstyle\int u\,d\mu-\int u\,d\nu| : u:X\to\mathbb{R}\text{ is bounded and 1-Lipschitz}\}.$$

457L Theorem Let (X,ρ) be a metric space and P_{qR} the set of quasi-Radon probability measures on X; define ρ_{W} as in 457K.

(a) For all μ, ν and λ in P_{qR},

$$\rho_{\mathrm{W}}(\mu,\nu)=\rho_{\mathrm{W}}(\nu,\mu),\quad \rho_{\mathrm{W}}(\mu,\lambda)\le\rho_{\mathrm{W}}(\mu,\nu)+\rho_{\mathrm{W}}(\nu,\lambda),$$

$$\rho_{\mathrm{W}}(\mu,\nu)=0\text{ iff }\mu=\nu.$$

(b) If μ, $\nu\in P_{\mathrm{qR}}$, then $\rho_{\mathrm{W}}(\mu,\nu)=\inf_{\lambda\in Q(\mu,\nu)}\int\rho(x,y)\lambda(d(x,y))$, where $Q(\mu,\nu)$ is the set of quasi-Radon probability measures on $X\times X$ with marginal measures μ and ν.

(c) In (b), if μ and ν are Radon measures, $Q(\mu,\nu)$ is included in $P_{\mathrm{R}}(X\times X)$, the space of Radon probability measures on $X\times X$, and is compact for the narrow topology on $P_{\mathrm{R}}(X\times X)$; and there is a $\lambda\in Q(\mu,\nu)$ such that $\rho_{\mathrm{W}}(\mu,\nu)=\int\rho(x,y)\lambda(d(x,y))$.

(d) If ρ is bounded, then ρ_{W} is a metric on P_{qR} inducing the narrow topology.

457M Theorem Let X be a Hausdorff space and $\langle\nu_i\rangle_{i\in I}$ a non-empty finite family of locally finite measures on X all inner regular with respect to the closed sets.

(a) For $A\subseteq X\times[0,\infty[$, set

$$c(A)=\inf\{\sum_{i\in I}\int h_i d\nu_i : h_i : X\to[0,\infty] \text{ is } \operatorname{dom}\nu_i\text{-measurable for each } i\in I,$$

$$\alpha\le\sum_{i\in I}h_i(x) \text{ whenever } (x,\alpha)\in A\}.$$

(i) c is a Choquet capacity.

(ii) For every $A\subseteq X\times[0,\infty[$, the infimum in the definition of $c(A)$ is attained.

(b) Let $f:X\to[0,\infty[$ be a function such that $\{x:f(x)\ge\alpha\}$ is K-analytic for every $\alpha>0$. Then

$$\inf\{\sum_{i\in I}\int h_i d\nu_i : h_i : X\to[0,\infty] \text{ is } \operatorname{dom}\nu_i\text{-measurable for each } i\in I,\ f\le\sum_{i\in I}h_i\}$$

$$=\sup\{\int f\,d\mu : \mu \text{ is a Radon measure on } X \text{ and } \mu\le\nu_i \text{ for every } i\in I\}.$$

457Z Problems Give $[0,1]$ Lebesgue measure.

(a) Characterize the sets $X\subseteq[0,1]^2$ for which there is a measure on X such that both the projections from X to $[0,1]$ are inverse-measure-preserving.

(b) Set $X=\{x:x\in[0,1]^3,\ \|x\|=1\}$. Is there more than one Radon measure on X for which all three coordinate maps from X onto $[0,1]$ are inverse-measure-preserving?

Version of 20.11.17

458 Relative independence and relative products

Stochastic independence is one of the central concepts of probability theory, and pervades measure theory. I come now to a generalization of great importance. If X_1, X_2 and Y are random variables, we may find that X_1 and X_2 are 'relatively independent over Y', or 'independent when conditioned on Y', in the sense that if we know the value of Y, then we learn nothing further about one of the X_i if we are told the value of the other. For any stochastic process, where information comes to us piecemeal, this idea is likely to be fundamental. In this section I set out a general framework for discussion of relative independence (458A), introducing relative distributions (458I) and relative independence in measure algebras (458L-458M). In the last third of the section I look at 'relative product measures' (458N, 458Q), giving the basic existence theorems (458O, 458S, 458T).

458A Relative independence Let (X,Σ,μ) be a probability space and T a σ-subalgebra of Σ.

(a) I say that a family $\langle E_i\rangle_{i\in I}$ in Σ is **relatively (stochastically) independent** over T if whenever $J\subseteq I$ is finite and not empty, and g_i is a conditional expectation of χE_i on T for each $i\in J$, then $\mu(F\cap\bigcap_{i\in J}E_i)=\int_F\prod_{i\in J}g_i d\mu$ for every $F\in\mathrm{T}$; that is, $\prod_{i\in J}g_i$ is a conditional expectation of $\chi(\bigcap_{i\in J}E_i)$ on T. A family $\langle\Sigma_i\rangle_{i\in I}$ of subalgebras of Σ is **relatively independent** over T if $\langle E_i\rangle_{i\in I}$ is relatively independent over T whenever $E_i\in\Sigma_i$ for every $i\in I$.

(b) I say that a family $\langle f_i\rangle_{i\in I}$ in $\mathcal{L}^0(\mu)$ is **relatively independent** over T if $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T with respect to the completion of μ, where Σ_i is the σ-algebra defined by f_i.

(c) I remark at once that a family of subalgebras or random variables is relatively independent iff every finite subfamily is.

(d) If Σ, T are algebras of subsets of a set X, I will write $\Sigma\vee\mathrm{T}$ for the σ-algebra of subsets of X generated by $\Sigma\cup\mathrm{T}$; if $\langle\Sigma_i\rangle_{i\in I}$ is a family of algebras of subsets of X, then $\bigvee_{i\in I}\Sigma_i$ will be the σ-algebra generated by $\bigcup_{i\in I}\Sigma_i$.

458B Lemma Let (X,Σ,μ) be a probability space, T a σ-subalgebra of Σ, and $\langle\Sigma_i\rangle_{i\in I}$ a family of subalgebras of Σ such that $\mathrm{T}\subseteq\bigcup_{i\in I}\Sigma_i$. Suppose that whenever $J\subseteq I$ is finite and not empty, $E_i\in\Sigma_i$ and g_i is a conditional expectation of χE_i on T for each $i\in J$, then $\mu(\bigcap_{i\in J}E_i)=\int\prod_{i\in J}g_i d\mu$. Then $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T.

458C Proposition Let (X,Σ,μ) be a probability space, $\mathbb{T}$ a non-empty upwards-directed family of subalgebras of Σ, and $\langle\Sigma_i\rangle_{i\in I}$ a family of σ-subalgebras of Σ which is relatively independent over T for every $\mathrm{T}\in\mathbb{T}$. Then $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over $\bigvee\mathbb{T}$.

458D Proposition Let (X,Σ,μ) be a probability space, T a σ-subalgebra of Σ and $\langle\Sigma_i\rangle_{i\in I}$ a family of subalgebras of Σ which is relatively independent over T.

(a) If $J\subseteq I$ and Σ'_i is a subalgebra of Σ_i for $i\in J$, then $\langle\Sigma'_j\rangle_{i\in J}$ is relatively independent over T.

(b) Set $\Sigma^*_i=\Sigma_i\vee\mathrm{T}$ for $i\in I$. Then $\langle\Sigma^*_i\rangle_{i\in I}$ is relatively independent over T.

(c) If $\mathcal{E}\subseteq\bigcup_{i\in I}\Sigma_i$, then $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over the σ-algebra generated by $\mathrm{T}\cup\mathcal{E}$.

458E Example Let (X,Σ,μ) be a probability space, $\langle\mathrm{T}_i\rangle_{i\in I}$ an independent family of σ-subalgebras of Σ, and T a σ-subalgebra of Σ which is independent of $\bigvee_{i\in I}\mathrm{T}_i$. For each $i\in I$, let Σ_i be $\mathrm{T}\vee\mathrm{T}_i$. Then $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T.

458F Proposition Let (X,Σ,μ) be a probability space and T a σ-subalgebra of Σ.

(a) Let $\langle f_i\rangle_{i\in I}$ be a family of non-negative μ-integrable functions on X which is relatively independent over T. For each $i\in I$ let g_i be a conditional expectation of f_i on T. Then for any $F\in\mathrm{T}$ and $i_0,\dots,i_n\in I$,

$$\int_F\prod_{j=0}^n g_{i_j}\le\int_F\prod_{j=0}^n f_{i_j}$$

with equality if all the i_j are distinct.

(b) Suppose that Σ_1, Σ_2 are σ-subalgebras of Σ which are relatively independent over T, and that $f\in\mathcal{L}^1(\mu\restriction\Sigma_1)$. If g is a conditional expectation of f on T, then it is a conditional expectation of f on $\mathrm{T}\vee\Sigma_2$.

***458G Lemma** Let (X,Σ,μ) be a probability space, T a σ-subalgebra of Σ, and $\langle\Sigma_i\rangle_{i\in I}$ a family of subalgebras of Σ. Let $\mathbb{T}$ be the family of finite subalgebras of T. For $\Lambda\in\mathbb{T}$ write $\mathcal{A}_\Lambda$ for the set of non-negligible atoms in Λ. For non-empty finite $J\subseteq I$, $\langle E_i\rangle_{i\in J}\in\prod_{i\in J}\Sigma_i$ and $F\in\mathrm{T}$, set

$$\phi_\Lambda(F,\langle E_i\rangle_{i\in J})=\sum_{H\in\mathcal{A}_\Lambda}\mu(H\cap F)\cdot\prod_{i\in J}\frac{\mu(E_i\cap H)}{\mu H}.$$

Then $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T iff $\lim_{\Lambda\in\mathbb{T},\Lambda\uparrow}\phi_\Lambda(F,\langle E_i\rangle_{i\in J})=\mu(F\cap\bigcap_{i\in J}E_i)$ whenever $J\subseteq I$ is finite and not empty, $E_i\in\Sigma_i$ for every $i\in J$ and $F\in\mathrm{T}$.

458H Proposition Let (X,Σ,μ) be a probability space and T a σ-subalgebra of Σ. Let $\langle\Sigma_i\rangle_{i\in I}$ be a family of σ-subalgebras of Σ which is relatively independent over T. Let $\langle I_j\rangle_{j\in J}$ be a partition of I, and for each $j\in J$ let $\tilde\Sigma_j$ be $\bigvee_{i\in I_j}\Sigma_i$.

(a) If $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T, then $\langle\tilde\Sigma_j\rangle_{j\in J}$ is relatively independent over T.

(b) Suppose that $\langle\tilde\Sigma_j\rangle_{j\in J}$ is relatively independent over T and that $\langle\Sigma_i\rangle_{i\in I_j}$ is relatively independent over T for every $j\in J$. Then $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T.

458I Definition Let (X, Σ, μ) be a probability space, T a σ-subalgebra of Σ, and $f \in \mathcal{L}^0(\mu)$. Then a **relative distribution** of f over T will be a family $\langle \nu_x \rangle_{x \in X}$ of Radon probability measures on $\mathbb{R}$ such that $x \mapsto \nu_x H$ is a conditional expectation of $\chi f^{-1}[H]$ on T for every Borel set $H \subseteq \mathbb{R}$.

458J Theorem Let (X, Σ, μ) be a probability space, T a σ-subalgebra of Σ, and $f \in \mathcal{L}^0(\mu)$. Then there is a relative distribution of f over T, which is essentially unique in the sense that if $\langle \nu_x \rangle_{x \in X}$ and $\langle \nu'_x \rangle_{x \in X}$ are two such relative distributions, then $\nu_x = \nu'_x$ for $\mu \restriction \mathrm{T}$-almost every x.

458K Theorem Let (X, Σ, μ) be a probability space, T a σ-subalgebra of Σ, and $\langle f_i \rangle_{i \in I}$ a family in $\mathcal{L}^0(\mu)$. For each $i \in I$, let $\langle \nu_{ix} \rangle_{x \in X}$ be a relative distribution of f_i over T, and $\tilde{f}_i : X \to \mathbb{R}$ an arbitrary extension of f_i to the whole of X. Then the following are equiveridical:

(i) $\langle f_i \rangle_{i \in I}$ is relatively independent over T;

(ii) for any Baire set $W \subseteq \mathbb{R}^I$ and any $F \in \mathrm{T}$,

$$\hat{\mu}(F \cap \boldsymbol{f}^{-1}[W]) = \int_F \lambda_x W \mu(dx),$$

where $\hat{\mu}$ is the completion of μ, $\boldsymbol{f}(x) = \langle \tilde{f}_i(x) \rangle_{i \in I}$ for $x \in X$, and λ_x is the product of $\langle \nu_{ix} \rangle_{i \in I}$ for each x;

(iii) for any non-negative Baire measurable function $h : \mathbb{R}^I \to \mathbb{R}$ and any $F \in \mathrm{T}$,

$$\int_F h\boldsymbol{f} d\mu = \int_F \int h \, d\lambda_x \mu(dx).$$

458L Measure algebras (a) If $a \in \mathfrak{A}$, then we can say that $u \in L^\infty(\mathfrak{C})$ is the conditional expectation of χa on $\mathfrak{C}$ if $\int_c u = \bar{\mu}(c \cap a)$ for every $c \in \mathfrak{C}$. Now we can say that a family $\langle b_i \rangle_{i \in I}$ in $\mathfrak{A}$ is **relatively (stochastically) independent over** $\mathfrak{C}$ if $\bar{\mu}(c \cap \inf_{i \in J} b_i) = \int_c \prod_{i \in J} u_i$ whenever $J \subseteq I$ is a non-empty finite set and u_i is the conditional expectation of χb_i on $\mathfrak{C}$ for every $i \in J$; while a family $\langle \mathfrak{B}_i \rangle_{i \in I}$ of subalgebras of $\mathfrak{A}$ is **relatively (stochastically) independent over** $\mathfrak{C}$ if $\langle b_i \rangle_{i \in I}$ is relatively independent over $\mathfrak{C}$ whenever $b_i \in \mathfrak{B}_i$ for every $i \in I$.

Corresponding to 458Ab, we can say that a family $\langle w_i \rangle_{i \in I}$ in $L^0(\mathfrak{A})$ is **relatively (stochastically) independent over** $\mathfrak{C}$ if $\langle \mathfrak{B}_i \rangle_{i \in I}$ is relatively stochastically independent, where $\mathfrak{B}_i$ is the closed subalgebra of $\mathfrak{A}$ generated by $\{[\![w_i > \alpha]\!] : \alpha \in \mathbb{R}\}$ for each i.

Returning to the original form of these ideas, we say that a family $\langle b_i \rangle_{i \in I}$ in $\mathfrak{A}$ is **(stochastically) independent** if it is relatively independent over $\{0, 1\}$, that is, if $\bar{\mu}(\inf_{i \in J} b_i) = \prod_{i \in J} \bar{\mu} b_i$ whenever $J \subseteq I$ is finite. Similarly, a family $\langle \mathfrak{B}_i \rangle_{i \in I}$ of subalgebras of $\mathfrak{A}$ is (stochastically) independent, in the sense of 325L, iff it is relatively independent over $\{0, 1\}$ in the sense here.

(b) Let (X, Σ, μ) be a probability space and $(\mathfrak{A}, \bar{\mu})$ its measure algebra. Let $\langle E_i \rangle_{i \in I}$, $\langle \Sigma_i \rangle_{i \in I}$ and $\langle f_i \rangle_{i \in I}$ be, respectively, a family in Σ, a family of subalgebras of Σ, and a family of μ-virtually measurable real-valued functions defined almost everywhere on X; let T be a σ-subalgebra of Σ. For $i \in I$, set $a_i = E_i^\bullet \in \mathfrak{A}$, $\mathfrak{B}_i = \{E^\bullet : E \in \Sigma_i\}$, and $w_i = f_i^\bullet \in L^0(\mathfrak{A})$, identified with $L^0(\mu)$. Set $\mathfrak{C} = \{F^\bullet : F \in \mathrm{T}\}$. Then

$\langle a_i \rangle_{i \in I}$ is relatively independent over $\mathfrak{C}$ iff $\langle E_i \rangle_{i \in I}$ is relatively independent over T,

$\langle \mathfrak{B}_i \rangle_{i \in I}$ is relatively independent over $\mathfrak{C}$ iff $\langle \Sigma_i \rangle_{i \in I}$ is relatively independent over T,

$\langle w_i \rangle_{i \in I}$ is relatively independent over $\mathfrak{C}$ iff $\langle f_i \rangle_{i \in I}$ is relatively independent over T.

(c) Corresponding to 458B, we see that if $\langle \mathfrak{A}_i \rangle_{i \in I}$ is a family of subalgebras of $\mathfrak{A}$ such that $\mathfrak{C} \subseteq \bigcup_{i \in I} \mathfrak{A}_i$, and $\int \prod_{i \in J} u_i d\bar{\mu} = \bar{\mu}(\inf_{i \in J} a_i)$ whenever $J \subseteq I$ is finite and not empty and $a_i \in \mathfrak{A}_i$ and $u_i \in L^\infty(\mathfrak{C})$ is a conditional expectation of χa_i on $\mathfrak{C}$ for each $i \in J$, then $\langle \mathfrak{A}_i \rangle_{i \in I}$ is relatively independent over $\mathfrak{C}$.

(d) Corresponding to 458Db, we see that if $\langle \mathfrak{B}_i \rangle_{i \in I}$ is a family of subalgebras of $\mathfrak{A}$ which is relatively independent over $\mathfrak{C}$, and $\mathfrak{B}_i^*$ is the closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{B}_i \cup \mathfrak{C}$ for each i, then $\langle \mathfrak{B}_i^* \rangle_{i \in I}$ is relatively independent over $\mathfrak{C}$.

Corresponding to 458Dc, we see that if $\langle \mathfrak{B}_i \rangle_{i \in I}$ is a family of subalgebras of $\mathfrak{A}$ which is relatively independent over $\mathfrak{C}$, $D_i \subseteq \mathfrak{B}_i$ for every $i \in I$, and $\mathfrak{D}$ is the closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C} \cup \bigcup_{i \in I} D_i$, then $\langle \mathfrak{B}_i \rangle_{i \in I}$ is relatively independent over $\mathfrak{D}$.

(e) Following 458H, we have the result that if $\langle\mathfrak{B}_i\rangle_{i\in I}$ is relatively independent over $\mathfrak{C}$, and $\langle I_j\rangle_{j\in J}$ is a partition of I, and $\tilde{\mathfrak{B}}_j$ is the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{i\in I_j}\mathfrak{B}_i$ for every $j\in J$, then $\langle\tilde{\mathfrak{B}}_j\rangle_{j\in J}$ is relatively independent over $\mathfrak{C}$.

(f) Note that if $\langle\mathfrak{B}_i\rangle_{i\in I}$ is a family of subalgebras of $\mathfrak{A}$ which is relatively independent over $\mathfrak{C}$, and $J\subseteq I$ is finite, and $b_i\in\mathfrak{B}_i$ for each $i\in J$, then $\inf_{i\in J}b_i=0$ iff $\inf_{i\in J}\operatorname{upr}(b_i,\mathfrak{C})=0$.

(g) If $\langle\mathfrak{C}_i\rangle_{i\in I}$ is a stochastically independent family of closed subalgebras of $\mathfrak{A}$, $\mathfrak{C}$ is independent of the algebra generated by $\bigcup_{i\in I}\mathfrak{C}_i$, and $\mathfrak{B}_i$ is the closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C}\cup\mathfrak{C}_i$ for each i, then $\langle\mathfrak{B}_i\rangle_{i\in I}$ is relatively independent over $\mathfrak{C}$.

(h) Let $P:L^1(\mathfrak{A},\bar\mu)\to L^1(\mathfrak{C},\bar\mu\restriction\mathfrak{C})\subseteq L^1(\mathfrak{A},\bar\mu)$ be the conditional expectation operator associated with $\mathfrak{C}$. Suppose that $\langle\mathfrak{B}_i\rangle_{i\in I}$ is a family of closed subalgebras of $\mathfrak{A}$ which is relatively independent over $\mathfrak{C}$. Then

$$\int_c\prod_{j=0}^n Pu_j\le\int_c\prod_{j=0}^n u_j$$

whenever $c\in\mathfrak{C}$, $i_0,\dots,i_n\in I$ and $u_j\in L^1(\mathfrak{B}_{i_j},\bar\mu\restriction\mathfrak{B}_{i_j})^+$ for each $j\le n$, with equality if $i_0,\dots,i_n$ are distinct.

458M Proposition Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and $\mathfrak{B}$, $\mathfrak{C}$ closed subalgebras of $\mathfrak{A}$. Write $P_{\mathfrak{B}}$, $P_{\mathfrak{C}}$ and $P_{\mathfrak{B}\cap\mathfrak{C}}$ for the conditional expectation operators associated with $\mathfrak{B}$, $\mathfrak{C}$ and $\mathfrak{B}\cap\mathfrak{C}$. Then the following are equiveridical:
(i) $\mathfrak{B}$ and $\mathfrak{C}$ are relatively independent over $\mathfrak{B}\cap\mathfrak{C}$;
(ii) $P_{\mathfrak{B}\cap\mathfrak{C}}(v\times w)=P_{\mathfrak{B}\cap\mathfrak{C}}v\times P_{\mathfrak{B}\cap\mathfrak{C}}w$ whenever $v\in L^\infty(\mathfrak{B})$ and $w\in L^\infty(\mathfrak{C})$;
(iii) $P_{\mathfrak{B}}P_{\mathfrak{C}}=P_{\mathfrak{B}\cap\mathfrak{C}}$;
(iv) $P_{\mathfrak{B}}P_{\mathfrak{C}}=P_{\mathfrak{C}}P_{\mathfrak{B}}$;
(v) $P_{\mathfrak{B}}u\in L^0(\mathfrak{C})$ for every $u\in L^1(\mathfrak{C},\bar\mu\restriction\mathfrak{C})$.

458N Relative free products of probability algebras: Definition Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a family of probability algebras and $(\mathfrak{C},\bar\nu)$ a probability algebra, and suppose that we are given a measure-preserving Boolean homomomorphism $\pi_i:\mathfrak{C}\to\mathfrak{A}_i$ for each $i\in I$. A **relative free product** of $\langle(\mathfrak{A}_i,\bar\mu_i,\pi_i)\rangle_{i\in I}$ over $(\mathfrak{C},\bar\nu)$ is a probability algebra $(\mathfrak{A},\bar\mu)$, together with a measure-preserving Boolean homomorphism $\phi_i:\mathfrak{A}_i\to\mathfrak{A}$ for each $i\in I$, such that

$\mathfrak{A}$ is the closed subalgebra of itself generated by $\bigcup_{i\in I}\phi_i[\mathfrak{A}_i]$,
$\phi_i\pi_i=\phi_j\pi_j:\mathfrak{C}\to\mathfrak{A}$ for all i, $j\in I$,
writing $\mathfrak{D}$ for the common value of the $\phi_i[\pi_i[\mathfrak{C}]]$, $\langle\phi_i[\mathfrak{A}_i]\rangle_{i\in I}$ is relatively independent over $\mathfrak{D}$.

458O Theorem Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a family of probability algebras, $(\mathfrak{C},\bar\nu)$ a probability algebra and $\pi_i:\mathfrak{C}\to\mathfrak{A}_i$ a measure-preserving Boolean homomomorphism for each $i\in I$. Then $\langle(\mathfrak{A}_i,\bar\mu_i,\pi_i)\rangle_{i\in I}$ has an essentially unique relative free product over $(\mathfrak{C},\bar\nu)$.

458P Theorem Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$, $\langle(\mathfrak{A}'_i,\bar\mu'_i)\rangle_{i\in I}$ be two families of probability algebras, and $\psi_i:\mathfrak{A}_i\to\mathfrak{A}'_i$ a measure-preserving Boolean homomorphism for each i. Let $(\mathfrak{C},\bar\nu)$, $(\mathfrak{C}',\bar\nu')$ be probability algebras and $\pi_i:\mathfrak{C}\to\mathfrak{A}_i$, $\pi'_i:\mathfrak{C}'\to\mathfrak{A}'_i$ measure-preserving Boolean homomomorphisms for each $i\in I$; suppose that we have a measure-preserving isomorphism $\psi:\mathfrak{C}\to\mathfrak{C}'$ such that $\pi'_i\psi=\psi_i\pi_i:\mathfrak{C}\to\mathfrak{A}'_i$ for each i. Let $(\mathfrak{A},\bar\mu,\langle\phi_i\rangle_{i\in I})$ and $(\mathfrak{A}',\bar\mu',\langle\phi'_i\rangle_{i\in I})$ be relative free products of $\langle(\mathfrak{A}_i,\bar\mu_i,\pi_i)\rangle_{i\in I}$, $\langle(\mathfrak{A}'_i,\bar\mu'_i,\pi'_i)\rangle_{i\in I}$ over $(\mathfrak{C},\bar\nu)$, $(\mathfrak{C}',\bar\nu')$ respectively. Then there is a unique measure-preserving Boolean homomorphism $\hat\psi:\mathfrak{A}\to\mathfrak{A}'$ such that $\hat\psi\phi_i=\phi'_i\pi_i:\mathfrak{A}_i\to\mathfrak{A}'$ for every $i\in I$.

458Q Relative product measures: Definitions (a) Let $\langle X_i\rangle_{i\in I}$ be a family of sets, Y a set, and $\pi_i:X_i\to Y$ a function for each $i\in I$. The **fiber product** of $\langle(X_i,\pi_i)\rangle_{i\in I}$ is the set $\Delta=\{x:x\in\prod_{i\in I}X_i$, $\pi_ix(i)=\pi_jx(j)$ for all i, $j\in I\}$.

(b) Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of probability spaces and (Y,T,ν) a probability space, and suppose that we are given an inverse-measure-preserving function $\pi_i:X_i\to Y$ for each $i\in I$; let Δ be the fiber product of $\langle(X_i,\pi_i)\rangle_{i\in I}$. A **relative product measure** on Δ is a probability measure μ on Δ such that

(†) whenever $J \subseteq I$ is finite and not empty and $E_i \in \Sigma_i$ for $i \in J$, and g_i is a Radon-Nikodým derivative with respect to ν of the functional $F \mapsto \mu_i(E \cap \pi_i^{-1}[F]) : \mathrm{T} \to [0,1]$ for each $i \in J$, then $\mu\{x : x \in \Delta,\, x(i) \in E_i$ for every $i \in J\}$ is defined and equal to $\int \prod_{i\in J} g_i d\nu$;

(‡) for every $W \in \Sigma$ there is a W' in the σ-algebra generated by $\{\{x : x \in \Delta,\, x(i) \in E\} : i \in I,$ $E \in \Sigma_i\}$ such that $\mu(W \triangle W') = 0$.

Remark If μ is a relative product measure of $\langle(\mu_i,\pi_i)\rangle_{i\in I}$ over ν, then all the functions $x \mapsto x(i) : \Delta \to X_i$ are inverse-measure-preserving. It follows that if I is not empty then we have an inverse-measure-preserving function $\pi : \Delta \to Y$ defined by setting $\pi x = \pi_i x(i)$ whenever $x \in \Delta$ and $i \in I$.

458R Proposition Suppose that $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ is a family of probability spaces, (Y,T,ν) a probability space, $\pi_i : X_i \to Y$ an inverse-measure-preserving function for each $i \in I$, Δ the fiber product of $\langle(X_i,\pi_i)\rangle_{i\in I}$ and μ a relative product measure of $\langle(\mu_i,\pi_i)\rangle_{i\in I}$. Let $(\mathfrak{A}_i,\bar\mu_i)$, $(\mathfrak{C},\bar\nu)$ and $(\mathfrak{A},\bar\mu)$ be the measure algebras of μ_i, ν and μ respectively, and for $i \in I$ define $\bar\pi_i : \mathfrak{C} \to \mathfrak{A}_i$ and $\bar\phi_i : \mathfrak{A}_i \to \mathfrak{A}$ by setting $\bar\pi_i F^\bullet = \pi_i^{-1}[F]^\bullet$, $\bar\phi_i E^\bullet = \{x : x \in \Delta,\, x(i) \in E\}^\bullet$ whenever $F \in \mathrm{T}$ and $E \in \Sigma_i$. Then $(\mathfrak{A},\bar\mu,\langle\bar\phi_i\rangle_{i\in I})$ is a relative free product of $\langle(\mathfrak{A}_i,\bar\mu_i,\bar\pi_i)\rangle_{i\in I}$ over $(\mathfrak{C},\bar\nu)$.

458S Proposition Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of probability spaces, (Y,T,ν) a probability space, and $\pi_i : X_i \to Y$ an inverse-measure-preserving function for each i. Suppose that for each i we have a disintegration $\langle\mu_{iy}\rangle_{y\in Y}$ of μ_i such that $\mu_{iy}^*\pi_i^{-1}[\{y\}] = \mu_{iy}X_i = 1$ for every $y \in Y$. Let $\Delta \subseteq \prod_{i\in I} X_i$ be the fiber product of $\langle(X_i,\pi_i)\rangle_{i\in I}$, and Υ the subspace σ-algebra on Δ induced by $\widehat{\bigotimes}_{i\in I}\Sigma_i$. For $y \in Y$, let λ_y be the product of $\langle\mu_{iy}\rangle_{i\in I}$, $(\lambda_y)_\Delta$ the subspace measure on Δ and λ'_y its restriction to Υ. Then $\mu W = \int \lambda'_y W \nu(dy)$ is defined for every $W \in \Upsilon$, and μ is a relative product measure of $\langle(\mu_i,\pi_i)\rangle_{i\in I}$ over ν.

458T Proposition Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of compact Radon probability spaces, $(Y,\mathfrak{S},\mathrm{T},\nu)$ a Radon probability space, and $\pi_i : X_i \to Y$ a continuous inverse-measure-preserving function for each i. Then $\langle(\mu_i,\pi_i)\rangle_{i\in I}$ has a relative product measure μ over ν which is a Radon measure for the topology on the fiber product of $\langle(X_i,\pi_i)\rangle_{i\in I}$ induced by the product topology on $\prod_{i\in I} X_i$.

458U Proposition Let (X_1,Σ_1,μ_1), (X_2,Σ_2,μ_2) and (Y,T,ν) be probability spaces, and $\pi_1 : X_1 \to Y$, $\pi_2 : X_2 \to Y$ inverse-measure-preserving functions. Let Δ be the fiber product of (X_1,π_1) and (X_2,π_2), and suppose that μ is a relative product measure of (μ_1,π_1) and (μ_2,π_2) over ν; set $\pi x = \pi_1 x(1) = \pi_2 x(2)$ for $x \in \Delta$. Take $f_1 \in \mathcal{L}^1(\mu_1)$ and $f_2 \in \mathcal{L}^2(\mu_2)$, and set $(f_1 \otimes f_2)(x) = f_1(x(1))f_2(x(2))$ when $x \in \Delta \cap (\operatorname{dom} f_1 \times \operatorname{dom} f_2)$. For $i = 1$, 2 let $g_i \in \mathcal{L}^1(\nu)$ be a Radon-Nikodým derivative of $H \mapsto \int_{\pi_i^{-1}[H]} f_i d\mu_i : \mathrm{T} \to \mathbb{R}$. Then $\int_F g_1 \times g_2 d\nu = \int_{\pi^{-1}[F]} f_1 \otimes f_2 d\mu$ for every $F \in \mathrm{T}$.

Version of 7.12.10

459 Symmetric measures and exchangeable random variables

Among the relatively independent families of random variables discussed in 458K, it is natural to give extra attention to those which are 'relatively identically distributed'. It turns out that these have a particularly appealing characterization as the 'exchangeable' families (459C). In the same way, among the measures on a product space X^I there is a special place for those which are invariant under permutations of coordinates (459E, 459H). A more abstract kind of permutation-invariance is examined in 495L-495M.

459A Lemma Let (X,Σ,μ) and (Y,T,ν) be probability spaces and $\phi : X \to Y$ an inverse-measure-preserving function; set $\Sigma_0 = \{\phi^{-1}[F] : F \in \mathrm{T}\}$. Let T_1 be a σ-subalgebra of T and $\Sigma_1 = \{f^{-1}[F] : F \in \mathrm{T}_1\}$. If $g \in \mathcal{L}^1(\nu)$ and h is a conditional expectation of g on T_1, then $h\phi$ is a conditional expectation of $g\phi$ on Σ_1.

459B Theorem Let (X,Σ,μ) be a probability space, Z a set, Υ a σ-algebra of subsets of Z and $\langle f_i\rangle_{i\in I}$ an infinite family of (Σ,Υ)-measurable functions from X to Z. For each $i\in I$, set $\Sigma_i=\{f_i^{-1}[H]:H\in\Upsilon\}$. Then the following are equiveridical:

(i) whenever $i_0,\dots,i_r\in I$ are distinct, $j_0,\dots,j_r\in I$ are distinct, and $H_k\in\Upsilon$ for each $k\le r$, then $\mu(\bigcap_{k\le r}f_{i_k}^{-1}[H_k])=\mu(\bigcap_{k\le r}f_{j_k}^{-1}[H_k])$;

(ii) there is a σ-subalgebra T of Σ such that

(α) $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T,

(β) whenever i, $j\in I$, $H\in\Upsilon$ and $F\in\mathrm{T}$, then $\mu(F\cap f_i^{-1}[H])=\mu(F\cap f_j^{-1}[H])$.

Moreover, if I is totally ordered by $\le$, we can add

(iii) whenever $i_0<\ldots<i_r\in I$, $j_0<\ldots<j_r\in I$ and $H_k\in\Upsilon$ for each $k\le r$, then $\mu(\bigcap_{k\le r}f_{i_k}^{-1}[H_k])=\mu(\bigcap_{k\le r}f_{j_k}^{-1}[H_k])$.

459C Exchangeable random variables: De Finetti's theorem Let (X,Σ,μ) be a probability space, and $\langle f_i\rangle_{i\in I}$ an infinite family in $\mathcal{L}^0(\mu)$. Then the following are equiveridical:

(i) the joint distribution of $(f_{i_0},f_{i_1},\dots,f_{i_r})$ is the same as the joint distribution of $(f_{j_0},f_{j_1},\dots,f_{j_r})$ whenever $i_0,\dots,i_r\in I$ are distinct and $j_0,\dots,j_r\in I$ are distinct;

(ii) there is a σ-subalgebra T of Σ such that $\langle f_i\rangle_{i\in I}$ is relatively independent over T and all the f_i have the same relative distribution over T.

Moreover, if I is totally ordered by $\le$, we can add

(iii) the joint distribution of $(f_{i_0},f_{i_1},\dots,f_{i_r})$ is the same as the joint distribution of $(f_{j_0},f_{j_1},\dots,f_{j_r})$ whenever $i_0<\ldots<i_r$ and $j_0<\ldots<j_r$ in I.

459D Proposition Let Z be a set, Υ a σ-algebra of subsets of Z, I an infinite set and μ a measure on Z^I with domain the σ-algebra $\widehat{\bigotimes}_I\Upsilon$ generated by $\{\pi_i^{-1}[H]:i\in I,\,H\in\Upsilon\}$, taking $\pi_i(x)=x(i)$ for $x\in Z^I$ and $i\in I$. For each permutation ρ of I, define $\hat\rho:Z^I\to Z^I$ by setting $\hat\rho(x)=x\rho$ for $x\in Z^I$. Suppose that $\mu=\mu\hat\rho^{-1}$ for every ρ. Let $\mathcal{E}$ be the family of those sets $E\in\widehat{\bigotimes}_I\Upsilon$ such that $\mu(E\triangle\hat\rho^{-1}[E])=0$ for every permutation ρ of I, and $\mathcal{V}$ the family of those sets $V\in\widehat{\bigotimes}_I\Upsilon$ such that V is determined by coordinates in $I\setminus\{i\}$ for every $i\in I$.

(a) $\mathcal{E}$ is a σ-subalgebra of $\widehat{\bigotimes}_I\Upsilon$.

(b) $\mathcal{V}$ is a σ-subalgebra of $\mathcal{E}$.

(c) If $E\in\mathcal{E}$ and $J\subseteq I$ is infinite, then there is a $V\in\mathcal{V}$, determined by coordinates in J, such that $\mu(E\triangle V)=0$.

(d) Setting $\Sigma_i=\{\pi_i^{-1}[H]:H\in\Upsilon\}$ for each $i\in I$,

(α) $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over $\mathcal{E}$,

(β) for every $H\in\Upsilon$ there is an $\mathcal{E}$-measurable function $g_H:Z^I\to[0,1]$ which is a conditional expectation of $\chi(\pi_i^{-1}[H])$ on $\mathcal{E}$ for every $i\in I$.

459E Theorem Let Z be a set, Υ a σ-algebra of subsets of Z, I an infinite set, and μ a countably compact probability measure on Z^I with domain the σ-algebra $\widehat{\bigotimes}_I\Upsilon$. Then the following are equiveridical:

(i) for every permutation ρ of I, $x\mapsto x\rho:Z^I\to Z^I$ is inverse-measure-preserving for μ;

(ii) for every transposition ρ of two elements of I, $x\mapsto x\rho:Z^I\to Z^I$ is inverse-measure-preserving for μ;

(iii) for each $n\in\mathbb{N}$ and any two injective functions p, $q:n\to I$ the maps $x\mapsto xp:Z^I\to Z^n$, $x\mapsto xq:Z^I\to Z^n$ induce the same measure on Z^n;

(iv) there are a probability space (Y,T,ν) and a family $\langle\lambda_y\rangle_{y\in Y}$ of probability measures on Z such that $\langle\lambda_y^I\rangle_{y\in Y}$ is a disintegration of μ over ν, writing λ_y^I for the product of copies of λ_y.

Moreover, if I is totally ordered, we can add

(v) for each $n\in\mathbb{N}$ and any two strictly increasing functions p, $q:n\to I$ the maps $x\mapsto xp:Z^I\to Z^n$, $x\mapsto xq:Z^I\to Z^n$ induce the same measure on Z^n.

If the conditions (i)-(v) are satisfied, then there is a countably compact measure λ, with domain Υ, which is the common marginal measure of μ on every coordinate; and if $\mathcal{K}$ is a countably compact class of subsets of Z, closed under finite unions and countable intersections, such that λ is inner regular with respect to $\mathcal{K}$, then

(iv)′ there are a probability space (Y, T, ν) and a family $\langle\lambda_y\rangle_{y\in Y}$ of complete probability measures on Z, all with domains including $\mathcal{K}$ and inner regular with respect to $\mathcal{K}$, such that $\langle\lambda_y^I\rangle_{y\in Y}$ is a disintegration of μ over ν.

459F Lemma Let X be a Hausdorff space and $P_{\mathrm{R}}(X)$ the space of Radon probability measures on X with its narrow topology. If $\langle K_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence of compact subsets of X, then $A = \{\mu : \mu \in P_R(X), \mu(\bigcup_{n\in\mathbb{N}} K_n) = 1\}$ is a K-analytic subset of $P_R(X)$.

459G Lemma Let X be a topological space, $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ a totally finite quasi-Radon measure space, $y \mapsto \mu_y$ a continuous function from Y to the space $M_{\mathrm{qR}}^+(X)$ of totally finite quasi-Radon measures on X with its narrow topology, and $\mathcal{U}$ a base for the topology of X, containing X and closed under finite intersections. If $\mu \in M_{\mathrm{qR}}^+(X)$ is such that $\mu U = \int \mu_y U\,\nu(dy)$ for every $U \in \mathcal{U}$, then $\langle\mu_y\rangle_{y\in Y}$ is a disintegration of μ over ν.

459H Theorem Let Z be a Hausdorff space, I an infinite set, and $\tilde{\mu}$ a quasi-Radon probability measure on Z^I such that the marginal measures on each copy of Z are Radon measures. Write $P_{\mathrm{R}}(Z)$ for the set of Radon probability measures on Z with its narrow topology. Then the following are equiveridical:

(i) for every permutation ρ of I, $w \mapsto w\rho : Z^I \to Z^I$ is inverse-measure-preserving for $\tilde{\mu}$;

(ii) for every transposition ρ of two elements of I, $w \mapsto w\rho : Z^I \to Z^I$ is inverse-measure-preserving for $\tilde{\mu}$;

(iii) for each $n \in \mathbb{N}$ and any two injective functions p, $q : n \to I$ the maps $w \mapsto wp : Z^I \to Z^n$ and $w \mapsto wq : Z^I \to Z^n$ induce the same measure on Z^n;

(iv) there are a probability space (Y, T, ν) and a family $\langle\mu_y\rangle_{y\in Y}$ of τ-additive Borel probability measures on Z such that $\langle\tilde{\mu}_y^I\rangle_{y\in Y}$ is a disintegration of $\tilde{\mu}$ over ν, writing $\tilde{\mu}_y^I$ for the τ-additive product of copies of μ_y;

(v) there is a Radon probability measure $\tilde{\nu}$ on $P_{\mathrm{R}}(Z)$ such that $\langle\tilde{\theta}^I\rangle_{\theta\in P_{\mathrm{R}}(Z)}$ is disintegration of $\tilde{\mu}$ over $\tilde{\nu}$, writing $\tilde{\theta}^I$ for the quasi-Radon product of copies of θ.

Moreover, if I is totally ordered, we can add

(vi) for each $n \in \mathbb{N}$ and any two strictly increasing functions p, $q : n \to I$ the maps $w \mapsto wp : Z^I \to Z^n$ and $w \mapsto wq : Z^I \to Z^n$ induce the same measure on Z^n.

459I Lemma Let (X, Σ, μ) be a probability space and I a set. For a family $\mathbb{T}$ of subalgebras of $\mathcal{P}X$, write $\bigvee \mathbb{T}$ for the σ-algebra generated by $\bigcup \mathbb{T}$. Let G be the group of permutations ϕ of I such that $\{i : \phi(i) \ne i\}$ is finite. Suppose that $\bullet$ is an action of G on X such that $x \mapsto \phi\bullet x$ is inverse-measure-preserving for each $\phi \in G$; set $\phi\bullet A = \{\phi\bullet x : x \in A\}$ for $\phi \in G$ and $A \subseteq X$. Let $\langle\Sigma_J\rangle_{J\subseteq I}$ be a family of σ-subalgebras of Σ such that

(i) for every $J \subseteq I$, Σ_J is the σ-algebra generated by $\bigcup_{K\subseteq J \text{ is finite}} \Sigma_K$;

(ii) if $J \subseteq I$, $E \in \Sigma_J$ and $\phi \in G$, then $\phi\bullet E \in \Sigma_{\phi[J]}$;

(iii) if $J \subseteq I$, $E \in \Sigma_J$ and $\phi \in G$ is such that $\phi(i) = i$ for every $i \in J$, then $\phi\bullet E = E$.

Suppose that $\mathcal{J}^*$ is a filter on I not containing any infinite set, and that $K \subseteq I$, $\mathcal{K} \subseteq \mathcal{P}I$ and $\mathcal{J} \subseteq \mathcal{J}^*$ are such that for every $K' \in \mathcal{K}$ there is a $J \in \mathcal{J}$ such that $K \cap K' \subseteq J$. Then Σ_K and $\bigvee_{K'\in\mathcal{K}} \Sigma_{K'}$ are relatively independent over $\bigvee_{J\in\mathcal{J}} \Sigma_J$.

459J Corollary Let (X, Σ, μ) be a probability space and I a set. Let G be the group of permutations ϕ of I such that $\{i : \phi(i) \ne i\}$ is finite. Suppose that $\bullet$ is an action of G on X such that $x \mapsto \phi\bullet x$ is inverse-measure-preserving for each $\phi \in G$. Let $\langle\Sigma_J\rangle_{J\subseteq I}$ be a family of σ-subalgebras of Σ such that

(i) for every $J \subseteq I$, Σ_J is the σ-algebra generated by $\bigcup_{K\subseteq J \text{ is finite}} \Sigma_K$;

(ii) if $J \subseteq I$, $E \in \Sigma_J$ and $\phi \in G$, then $\phi\bullet E \in \Sigma_{\phi[J]}$;

(iii) if $J \subseteq I$, $E \in \Sigma_J$ and $\phi \in G$ is such that $\phi(i) = i$ for every $i \in J$, then $\phi\bullet E = E$.

Then if $J \subseteq I$ is infinite and $\langle K_\gamma\rangle_{\gamma\in\Gamma}$ is a family of subsets of I such that $K_\gamma \cap K_\delta \subseteq J$ for all distinct γ, $\delta \in \Gamma$, $\langle\Sigma_{K_\gamma}\rangle_{\gamma\in\Gamma}$ is relatively independent over Σ_J.

459K Example There are a separable metrizable space Z and a quasi-Radon measure on $Z^{\mathbb{N}}$, invariant under permutations of coordinates, which cannot be disintegrated into powers of measures on Z.

Version of 27.2.04

Concordance for Volume 4

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this volume, and which have since been changed.

452I In Fremlin 00 I quote Pachl's result that if (X, Σ, μ) is countably compact, (Y, T, ν) is strictly localizable and $f : X \to Y$ is inverse-measure-preserving, then ν is countably compact; this is now in 452R.

455D The material on Brownian motion in §455, mentioned in König 04 and König 06, has been moved to §477.

458Yd This exercise (on the strong law of large numbers for relatively independent sequences), referred to in the 2008 and 2015 printings of Volume 5, is now 458Ye.

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